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Preface: CERME 12 in virtual Bolzano

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The Conferences of ERME (European society for Research in Mathematics Education) have been held roughly biannually since 1998. The core of the conference is the *thematic working groups* (TWG) in which new, related research studies are discussed, based on papers which participants read in advance. While this format has remained almost unchanged since 1998, the scale and scope of CERME have developed considerably over the years: from 120 participants and 7 TWG at CERME 1, to 915 participants and 27 TWG at CERME 12. Moreover, CERME has gone from being a mainly regional congress (with only a few participants from other continents) to being a truly global event in mathematics education research, known for fostering high quality scientific communication, cooperation and collaboration. At CERME12, no less than 48 nations were represented (Table 1).

At the same time, it is evident that CERME12 was a very special – and historically difficult – congress to organize. It was first scheduled for February 3-7, 2021. In May 2020, the ERME board announced its decision to postpone the congress by one year, due to the then roaring outbreak of the COVID-19 pandemic. Indeed, large parts of the world – and most of Europe – continued to experience lockdowns and restrictions that would have made the scheduled congress impossible. Instead, an online *Pre-CERME12 event* was organized in February 2021, hosted by the Institute of Education at University College London, and made possible by the tireless efforts of the team led by Jeremy Hodgen and Eirini Geraniou (also chair resp. co-chair of the IPC of CERME12). The Pre-CERME12 event allowed the 27 TWGs to meet and prepare for conference, now postponed to 2022.

The biannual General Assembly of ERME was also held during this event. We warmly thank Susanne Prediger for her service as President of ERME from 2017 to 2021! Her leadership also contributed crucially to the organisation of CERME12, and thus to the results presented in these proceedings.

During the summer and fall of 2021, we all continued to plan for CERME12 as an onsite event in Bolzano, Italy. The YESS summer school was held near Bolzano in August, with great success. More than 700 papers and posters were submitted for CERME12 in September. But in November 2021, new and unknown variants of the virus appeared. Their alarming spread forced us to reconsider the situation. Finally, the LOC, the IPC and the ERME board jointly decided that CERME12 would be held as an online congress, as announced in a mail sent to all members of ERME on December 1st:

> It is with great sadness that we must communicate a decision which is forced upon us by the current developments of the COVID epidemic in Europe, and which has been taken by the ERME board in full agreement with us: CERME12 will be organized by the Bolzano team as an online conference, on the same dates as originally foreseen. For a long time we hoped for the much desired possibility of having the first CERME in three years as a normal, face to face event. Organizing a virtual CERME – which we will strive to hold as much "CERME spirit" as possible – will be a very demanding task, both in terms of finding good technical solutions, and in terms of organizing the programme and preparing the many TWG teams in a good way.

Germany	209	Turkey	15	Iceland	3
Norway	85	Ireland	13	Lithuania	3
Italy	81	Czech Rep.	11	New Zealand	3
Spain	62	Slovakia	11	China	2
Sweden	62	Chile	9	Colombia	2
USA	41	Brazil	8	Hong Kong	2
UK	38	Croatia	8	Poland	2
Israel	31	Finland	6	Algeria	1
Netherlands	26	South Africa	6	Egypt	1
Denmark	24	Switzerland	6	Faroe Islands	1
France	21	Australia	5	Malta	1
Austria	20	Malawi	5	Romania	1
Canada	19	Mexico	5	Russia	1
Greece	19	Belgium	4	Thailand	1
Portugal	17	Japan	4	Tunisia	1
Hungary	15	Cyprus	3	Ukraine	1

Table 1: The success of CERME12 in numbers – 915 participants from 48 countries

Indeed, it took a unique *tour de force* for all organisers to prepare – in just two months – an online version of CERME, based as it is on group work and interaction, rather than on one-way presentations (which are relatively easy to transmit online). These effort was crowned by the best success the new conditions could possibly allow: an online congress with more participants than ever, with virtually no technical problems, and not least with a high level of participant satisfaction.

In the history of ERME, CERME12 will be remembered as a scientific highlight during the long and hard pandemic. First of all, that is due to the plenary speakers and panelists, and to the contributors of papers and posters. Your efforts shine through the quality of the scientific texts offered by these proceedings. ERME, as a society of scholars, was not stopped – hardly delayed – by the pandemic, thanks to your ingenuity and unfailing determination to do and share first class research.

The realization of CERME12 was made possible also by the many people who organised the congress, under the difficult conditions alluded to above, namely:

- The Local Organizing committee, led by Giorgio Bolondi and Federica Ferreti, and all of the Bolzano team, including also the technicians who made the online congress run smoothly;
- The International Programme committee, led by Jeremy Hodgen and Eirini Geraniou;
- The leader teams of all 27 Thematic Working Groups.

For all your tireless and unselfish work during the three years between CERME 11 and CERME 12, the community owes you immense and extraordinary gratitude.

And the story goes on: ERME invites all interested researchers to CERME 13 (Budapest , Hungary, July 2023), and after that, to CERME 14 to be held in *real* Bolzano in February 2025.

Introduction to the Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)

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About CERME12

The Eleventh Congress of European Research in Mathematics Education (CERME 12) took place virtually, hosted by the Free University of Bozen-Bolzano, Italy, from 2nd to 6th of February 2022, after a year's delay due to the pandemic. Jeremy Hodgen (UK) and Eirini Geraniou (UK) were chair and co-chair of the International Programme Committee (IPC), which comprised Giorgio Bolondi (Italy), Jason Cooper (Israel), Ana Donevska-Todorova (Germany / North-Macedonia), Çiğdem Haser (Finland / Turkey), Uffe Thomas Jankvist (Denmark), Leander Kempen (Germany), Esther Levenson (Israel), Nuria Planas (Spain) and Michiel Veldhuis (The Netherlands). Giorgio Bolondi and Federica Ferreti were chair and co-chair, respectively, of the Local Organizing Committee (LOC).

CERME12 hosted 27 Thematic Working Groups, listed in the table below. The TWGs 11 and 27 were new TWGs, created following a call launched just after CERME11, and a selection process involving the CERME12 IPC and the ERME board. They have both been very successful. Nine of the TWGs received so many submissions that they had to be split in two – more precisely the TWGs 01, 03, 05, 09, 14, 16, 18, 19 and 20. In the end, CERME12 had 27 TWG leaders and 110 TWG co-leaders.

Thematic Working Group	Leader	Co-Leaders
TWG1: Argumentation and Proof	Andreas Moutsios-Rentzos (Greece)	Orly Buchbinder (USA); Jenny Christine Cramer (Germany); Nicolas Leon (YR) until Aug 2021; and from Sep 2021: Viviane Durand-Guerrier (France); David A. Reid (Norway); Mei Yang

		(British Indian Ocean Territory/UK) YR
TWG2: Arithmetic and Number Systems	Elisabeth Rathgeb-Schnierer (Germany)	Judy Sayers (UK); Beatrice Vargas Dorneles (Brazil) until Sep 2021; Pernille Bødtker Sunde (Denmark) from Sep 2021; Renata Carvalho (Portugal) YR
TWG3: Algebraic Thinking	Dave Hewitt (UK)	Maria Chimoni (Cyprus); Cecilia Kilhamn (Sweden); Luis Radford (Canada) from Sep 2021; Jorunn Reinhardtsen (Norway) YR
TWG4: Geometry Teaching and Learning	Michela Maschietto (Italy)	Alik Palatnik (Israel); Lina Brunheira (Portugal); Chrysi Papadaki (Germany) YR
TWG5: Probability and Statistics Education	Caterina Primi (Italy)	Sibel Kazak (Turkey); Aisling Leavy (Ireland); Orlando Rafael Gonzalez (Thailand); Daniel Frischemeier (Germany) YR
TWG6: Applications and Modelling	Berta Barquero (Spain)	Susana Carreira (Portugal); Jonas Bergman Ärlebäck (Sweden); Katrin Vorhölter (Germany); Gilbert Greefrath (Germany) from Sep 2021; Britta Eyrich Jessen (Denmark) YR
TWG7: Adult Mathematics Education	Kees Hoogland (The Netherlands)	Javíer Díez-Palomar (Spain); Fiona Faulkner (Ireland); Beth Kelly (UK) YR
TWG8: Affect and the Teaching and Learning of Mathematics	Stanislaw Schukajilow (Germany)	Inés Mª Gómez-Chacón (Spain); Çiğdem Haser (Finland); Peter Liljedahl (Canada); Chiara Andrà (Italy); Hanna Viitala (Sweden) YR
TWG9: Mathematics and Language	Jenni Ingram (UK)	Kirstin Erath (Germany); Aurélie Chesnais (France); Ingólfur Gíslason (Iceland) YR

TWG10: Diversity and Mathematics Education: Social, Cultural and Political Challenges	Laura Black (UK)	Anette Bagger (Sweden); Anna Chronaki (Greece); Nina Bohlmann (Germany); Sabrina Bobsin Salazar (Brazil) YR
TWG11: Algorithmics	Christof Weber (Switzerland)	Janka Medova (Slovakia); Ulrich Kortenkamp (Germany); Simon Modeste (France); Piers Saunders (UK) YR until Oct 2021; Maryna Rafalska (France) from Oct 2021
TWG12: History in Mathematics Education	Renaud Chorlay (France)	Antonio M. Oller-Marcén (Spain); Jenneke Krüger (The Netherlands); Tanja Hamann (Germany) YR
TWG13: Early Years Mathematics	Bożena Maj-Tatsis (Poland)	Marianna Tzekaki (Greece); Esther Levenson (Israel); Martin Carlsen (Norway); Andrea Maffia (Italy) YR
TWG14: University Mathematics Education	Alejandro González-Martín (Canada)	Ghislaine Gueudet (France); Olov Viirman (Sweden); Athina Thoma (UK) YR; and from Sep 2021: Irene Biza (United Kingdom); Chris Rasmussen (United States); Ignasi Florensa (Spain) YR
TWG15: Teaching Mathematics with Technology and Other Resources	Alison Clark-Wilson (UK)	Ornella Robutti (Italy); Melih Turgut (Norway); Daniel Thurm (Germany) from Sep 2021; Gülay Bozkurt (Turkey) YR
TWG16: Learning Mathematics with Technology and Other Resources	Paul Drijvers (The Netherlands)	Florian Schacht (Germany); Nathalie Sinclair (Canada); Osama Swidan (Israel); Eleonora Faggiano (Italy) from Sep 2021; Seçil Yemen Karpuzcu (Turkey) YR
TWG17: Theoretical Perspectives and Approaches in Mathematics Education Research	Angelika Bikner-Ahsbahs (Germany)	Heather Johnson (USA); Anna Shvarts (The Netherlands); Abdel Seidouvy (Togo/Sweden) YR

TWG18: Mathematics Teacher Education and Professional Development	Janne Fauskanger (Norway)	Libuse Samkova (Czech Republic); Andreas Ebbelind (Sweden); Marita Eva Friesen (Germany) YR; and from Sep 2021: Tracy Helliwell (UK); Macarena Larrain (YR)
TWG19: Mathematics Teaching and Teacher Practice(s)	Reidar Mosvold (Norway)	Mark Hoover (USA); Siún Nic Mhuiri (Ireland); Edyta Nowinska (Poland/Germany); Helena Grundén (Sweden) YR
TWG20: Mathematics Teacher Knowledge, Beliefs and Identity	Fatma Aslan-Tutak (Turkey)	Miguel Montes (Spain); Francesca Martignone (Italy); Miguel Ribeiro (Brazil); Veronika Hubeňáková (Slovakia) YR
TWG21: Assessment in Mathematics Education	Paola Iannone (UK)	Francesca Morselli (Italy); Michal Ayalon (Israel); Michiel Veldhuis (The Netherlands); Gözde Kaplan Can (Turkey) YR
TWG22: Curricular Resources and Task Design in Mathematics Education	Shai Olsher (Israel)	Sebastian Rezat (Germany); Annalisa Cusi (Italy); Nataly Essonnier (France) YR
TWG23: Implementation of Research Findings in Mathematics Education	Mario Sánchez Aguilar (Mexico)	Boris Koichu (Israel); Morten Misfeldt (Denmark); Rikke Maagaard Gregersen (Denmark) YR until Aug 2021; Linda Marie Ahl (Sweden) YR from Sep 2021
TWG24: Representations in Mathematics Teaching and Learning	Anna Baccaglini-Frank (Italy)	Carla Finesilver (UK); Michal Tabach (Israel); Kate O'Brien (USA/UK) YR
TWG25: Inclusive Mathematics Education – Challenges for Students with Special Needs	Petra Scherer (Germany)	Hana Moraova (Czech Republic); Michael Gaidoschik (Italy); Helena Roos (Sweden) YR
TWG26: Mathematics in the Context of STEM Education	Behiye Ubuz (Turkey)	Michele Stephan (USA); Clelia Cascella (Italy/UK); Nelleke Den Braber (The Netherlands) YR

TWG27: The Professional Practices, Preparation and Support of Mathematics Teacher Educators	Ronnie Karsenty (Israel)	Stefan Zehetmeier (Austria); Hilda Borko (USA); Alf Coles (UK); Bettina Rösken-Winter (Germany); Birte Friedrich-Pöhler (Germany) YR

Editorial information

These proceedings are available as a complete volume online on the ERME website and each individual text is also available on the HAL open archive, where it can be found through keywords, title or author name. This has been the practice since CERME9, to increase the visibility of the huge work done in CERME conferences.

This volume begins with texts corresponding to the three plenary activities of CERME12: the plenary lecture by Susanne Prediger (Germany) on “Enhancing language for developing conceptual understanding: A research journey connecting different research approaches”; the plenary lecture by Jeppe Skott (Sweden / Norway) on “Conceptualizing individual-context relationships in teaching: Developments in research on teachers’ knowledge, beliefs and identity”; and finally the panel discussion “Big Questions in Mathematics Education”. This panel discussion was led by Anna Baccaglini-Frank (Italy), Ingi Højsted (Faroe Islands) and Janka Medova (Slovakia), chaired by Michiel Veldhuis (The Netherlands) and moderated by Eirini Geraniou (UK/Greece). The two plenary speakers, Susanne Prediger and Jeppe Skott, each gave a response to the panel discussion.

After the plenaries, the reader will find 27 chapters corresponding to the work done in the TWGs of CERME12 (with combined introductions from all the split TWGs). These chapters follow a similar structure: they start with an introduction; then the long paper contributions (8-page papers) and the short poster contributions (2 pages) are presented – in alphabetical order by first author’s name.

There are two kinds of introductions to the TWGs, according to the team’s choice: short introductions (4 pages) presenting the contributions; or long introductions (8 pages), which propose, in addition, an analysis of the current research on the theme of the TWG, and perspectives for the future. TWGs 04, 06, 07, 09, 14, 15, 17, 18, 19, 22, 25 and 26 have chosen this form of long introduction.

The publication of these proceedings is the result of a collaborative work, involving the CERME12 IPC, the TWG leaders and co-leaders, the LOC chair and co-chair and the wider team at Bozen-Bolzano. Particular thanks are due to Katrin Lambacher. We warmly thank all these people for their involvement, and hope that this volume will contribute to the development of mathematics education research in Europe and beyond.

Enhancing language for developing conceptual understanding: A research journey connecting different research approaches

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Typical problems in mathematics education require the pursuit of several research goals: not only explaining the problem and its underlying mechanisms, but also designing and evaluating instructional approaches for overcoming them. When the problem is complex, various research approaches need to be combined. In this plenary paper, I will summarize several steps from a 13-year research journey to serve as an example of how different research approaches can be connected to gain deep insights into language-responsive mathematics instruction. On a meta-level, the paper reflects the affordances and challenges of connecting different research approaches and thereby advertises extension of the networking practices that have been well established in ERME research.

Keywords: Connecting research approaches, conceptual understanding, language-responsive instruction, epistemic role of language

Introduction

With respect to typical problems in mathematics education, researchers can choose to pursue different research goals (Mason & Waywood, 1996; Burkhardt & Schoenfeld, 2003):

- *describing* the problem
- *explaining* the problem and its underlying mechanisms
- *designing* instructional approaches for overcoming the problem
- *evaluating* instructional approaches with respect to their effectiveness
- *describing and explaining* challenges in implementing the instructional approaches
- *designing* teacher professional development (PD) and material support for teachers to overcome the challenges

For each of these potential research goals, the need for different research approaches and different background theories (including theories from outside mathematics education) often resulted in research projects in the early years of the academic discipline making a choice between alternative research goals. However, within the last decades, the academic discipline has successively moved towards connecting several research approaches so that the research goals can increasingly be combined in longer chains of projects. In my view, this trend is highly desirable, as it allows research to really make a difference in mathematics classroom practices (Burkhardt & Schoenfeld, 2003).

My message for this conference paper is that we should continue this *important trend of connecting research approaches* while always keeping our feet on the ground, namely, on sound epistemological analyses of the mathematical topic and the topic-specific learning processes in view. I will try to exemplify what this can mean by reporting from the 13-year research journey of our MuM research group in Dortmund (MuM is an abbreviation for “mathematics learning under conditions of language diversity"). While treating a particular problem in the example (in this case, challenges of language in school mathematics learning), the aim of the last section will be to draw some more general conclusions beyond the specific problem being treated.

What I present here is grounded in CERME work in two ways: Our research on language has profited enormously from the ideas exchanged in the CERME Thematic Working Group (TWG) on language throughout several years (Planas et al., 2018), and the meta-perspective from which I reflect on the research journey today is a continuation of ideas emerging from the CERME TWG on theories (Artigue et al., 2005; Kidron et al., 2018), valuing different research approaches and striving to network them (Bikner-Ahsbahs, Prediger, & Networking Theories Group, 2014), not only with respect to using different theoretical lenses on the same data, but with respect to combing through and shifting the research questions, research objects, and perspectives across several studies.

Overview of a 13-year research journey

Our research journey started with recognizing a serious problem occurring in many countries: failure to provide equitable access to mathematics for students with low language proficiency. To address this problem, a chain of research questions and design challenges were pursued:

- What exactly is the problem with language in mathematics? (describing and explaining)
- How can language be enhanced in mathematics classrooms? (designing)
- What language demands are crucial for concept development? (designing and explaining)
- Is language-responsive teaching effective for developing conceptual understanding? (providing evidence for effectiveness)
- Who profits from language-responsive instruction? (extending the target group)
- How does teachers' enactment affect the learning gains? (explaining and connecting to high-quality instruction)

In the following subsections, I try to sketch how these questions were iteratively treated from different perspectives in the 13-year research journey, without accounting for all technical or theoretical details. The section is not a typical research report: Its aim is to exemplify the needs that were identified and the decisions that led to identifying the necessity to connect several research approaches in longer research journeys.

What exactly is the problem with language in mathematics?
Describing and explaining by large-scale assessments

Usually, educational problems are ill-defined in the beginning (Silver & Herbst, 2007). This was also the case for our problem: Large-scale assessments have repeatedly shown that school systems in many countries fail to provide equitable access to mathematics for all students (Braswell, Dion, Daane, & Jin, 2005; Haag et al., 2013; OECD, 2007). Originally, this failure of school systems was often conceptualized as achievement gaps of students with low socioeconomic status, immigrant status, and/or multilingual background (OECD, 2007; Stanat et al., 2012). This led to problematic deficit-oriented policies and approaches for multilingual students (Gutiérrez, 2008; similarly for other challenges of low-attaining students, Hodgen et al., 2021).

The description of the problem, however, depends on the categories used: When not only immigrant status and multilingual background but also the students' *academic language proficiency* in the language of instruction are captured in the large-scale assessment, this background factor (to be further explained below) turns out to be most relevant for students' mathematics achievement (Prediger et al., 2018) and for students' mathematical learning gains throughout a year of schooling in German schools (Ufer & Bochnik, 2020).

From a sociolinguistic perspective, *academic language* is described as a language register between students' everyday language and the technical formal mathematical language (Schleppegrell, 2004; Snow & Uccelli, 2009) and is functionally characterized as the register "used by teachers and students for the purpose of acquiring new knowledge and skills…, imparting new information, describing abstract ideas, and developing students' conceptual understanding" (Chamot & O'Malley, 1994, p. 40). It is seen as important learning medium, both for classroom communication and as a thinking tool in an important epistemic function for mathematics learning (Schleppegrell, 2004). Although the construct of "register" has sometimes been reduced to lexical and grammatical features, Halliday's (1978) definition also entails language use in larger language units above the sentence level. We refer to interactional discourse analysis to capture these language units above sentence level as discourse practices. Discourse practices are conventionalized genres that are interactively co-constructed relying on patterns available in speech communities' knowledge (Heller & Morek, 2015). Academic discourse practices are those optimized for school purposes, in particular *reporting procedures*, *explaining meanings* to convey or construct knowledge, and *arguing* to negotiate divergent validity claims in classrooms.

From a socioeducational perspective, the critical feature of academic language is that it is an unequally distributed learning prerequisite to which some students get more access at home than others, depending on their family backgrounds (Snow & Uccelli, 2009), which is an example of how social background factors coincide with academic language proficiency.

For education practices, the sociolinguistic and socioeducational theoretical embedding of explanative statistical findings that academic language proficiency is a relevant background factor for mathematics achievement leads to changes in responsibilities: Whereas students' family background factors (immigrant status, socioeconomic status, and/or multilingual background) are considered static and cannot be changed by schools, schools can indeed take responsibility for enhancing students' academic language proficiency in order to reduce language-induced achievement gaps, including those occurring in mathematics (Thürmann, Vollmer, & Pieper, 2010).

Large-scale assessments can also support locating the language-induced achievement gaps within different types of mathematical knowledge: In our own study on a high-stakes test of 1,495 10th graders, we expected the language gaps to be largest for items with high reading demands, following the idea of language biases in the item texts (as found, e.g., by Abedi & Lord, 2001). But a differential item functioning analysis revealed that the language gaps were largest for items with high conceptual demands, even if reading demands were low (Prediger et al., 2018). This finding was strengthened by a recent longitudinal study throughout Grade 2 (Ufer & Bochnik, 2020) showing that students' language proficiency had much less impact on gains in procedural skills ($\beta = 0.16$) than gains in conceptual understanding ($\beta = 0.30$ for using concepts for solving word problems and $\beta = 0.29$ for representing operations and concepts in visualizations). In both studies, language gaps were largest for *developing conceptual understanding of mathematical concepts*. These quantitative findings resonate with the sociolinguistic characterization of academic language as learning medium.

As an educational consequence of the identified language-induced achievement gaps, enhancing students' academic language is now considered an important step in achieving more equitable access to higher-order thinking across all subjects (Thürmann, Vollmer, & Pieper, 2010; Chamot & O'Malley, 1994). In Germany, this call for enhancing academic language in all subjects has been integrated into educational policy directives and the syllabus, yet without much support for helping teachers to realize this goal (MSW, 1999).

Summing up, *large-scale assessments* detected an ill-defined problem (there are problems with equity) and helped to locate the problem more exactly, namely, in the relevant background factors (the disparities can be traced back to academic language proficiency) and the type of mathematical knowledge (language gaps are largest for items with conceptual understanding). External *theoretical perspectives* (here from sociolinguistics) helped to understand the findings (academic language proficiency has an important epistemic role for developing higher-order thinking, including conceptual understanding). However, neither large-scale assessment nor the external theoretical backgrounds alone can orient instructional approaches to how to overcome the problem (Sullivan, 2015).

How can language be enhanced in mathematics classrooms? First attempts to learn design principles from good practices

Given the missing support for teachers, it was no surprise that the educational policy directive calling for enhancing students' academic language (MSW, 1999) had only very limited impact in schools. Ten years later, the educational ministry reacted by including language enhancement into the curriculum of initial teacher education of all subject matter teachers. However, this reform risked failure because subject-specific instructional approaches and curriculum materials that could be used in teacher education programs were not available for subject matter teacher educators.

That is why, in 2009, the MuM research group in Dortmund started its work aimed at providing support for mathematics teachers and mathematics teacher educators to enhance students' academic language of school mathematics. The first search for good practices revealed a typical problem that had also been documented in other countries: Many mathematics teachers who start working on enhancing students' language concentrate on simple vocabulary work, mostly concentrated on formal technical mathematical language or grammar (e.g., DFEE, 2000; problematized, e.g., by Moschkovich, 2010), but without an explicit focus on the discourse practices in play. This restriction to vocabulary work was also documented by de Araujo and Smith (2022), who analyzed US algebra textbooks with respect to the hints and materials provided for working with language learners: Reduced cognitive demands and less ambitious learning goals are combined with isolated vocabulary work, but there was no support for engaging students in discourse practices.

Identifying this problem led us to go beyond our attempt to learn from good practices to performing a qualitative classroom video study of 73 mathematics lessons in five Grade 5 classrooms. We studied how teachers engaged their students in discourse practices and what kind of support was given for students with low academic language proficiency (Erath et al., 2018). By coordinating an epistemic perspective on students' participation in collective knowledge constitution processes with the socio-linguistic perspective of interactional discourse analysis (Quasthoff, 2011), we replicated well-known findings that students' mathematical learning opportunities tend to be very significantly shaped by their ability to participate in the discourse (with various theoretical approaches collected by Herbel-Eisenmann et al., 2011), and "content learning is inseparably bound up with language learning and vice versa" (Barwell, 2005, p. 207). In particular, we showed that students' participation in classroom interaction depends on their proficiency in actively engaging in discourse practices such as explaining and arguing, for which almost no learning opportunities were provided in the observed classroom interaction (Erath et al., 2018).

Summing up, the qualitative classroom video study allowed us to confirm the role of academic language proficiency as learning medium and as unequally distributed learning prerequisite, but we could identify hardly any good practices for enhancing students' discourse competences. Although this study substantially contributed to explaining the problem in the discursive dimension, it was again not able contribute to overcoming it.

How can language be enhanced in mathematics classrooms? Second attempt through using design principles imported from language education

As our descriptive, non-interventionist video study in regular mathematics classrooms provided hardly any empirical insight into how language can be enhanced in mathematics classrooms, we changed to an interventionist research approach, namely, to design research. Design research is the research approach that combines the research goals of developing instructional approaches with gaining deep empirical insights for explaining mechanisms of the teaching learning processes (Gravemeijer & Cobb, 2006). In our case, we first had to create the language-responsive teaching learning processes we intended to investigate.

A careful literature review in language education (including second-language education) allowed us to import design principles that have proven effective for providing learning opportunities for learning academic language. We imported three generic design principles for language learning in language lessons:

- *DP1 Pushed output.* Students learn academic language when consistently being pushed to talk and write (Swain, 1995; Walqui, 2006).
- *DP2 Use multiple representations*. Students can construct meanings for new language when moving between multiple representations and language registers (von Kügelgen, 1994).

- *DP3 Sequence language registers*. Students can acquire language when it is carefully sequenced in a language learning trajectory starting from the everyday language and moving through academic language to the technical language of the subject matter (Gibbons, 2002).

These principles gave us important first ideas and strengthened our confidence that good language-responsive classrooms can also be high-quality mathematics classrooms: DP2 has a long tradition for developing conceptual understanding, even before language registers were considered (Lesh, 1979), and DP3 resonates with a similar sequencing principle for mathematical learning opportunities, the level principle in realistic mathematics education (Gravemeijer, 1998). DP1, the emphasis on students' active talk, was advocated in mathematics education before language became an explicit focus (as traced by Austin & Howson, 1979; Pimm, 1987).

What academic language demands are crucial for developing conceptual understanding? Design research for overcoming the specification gap

Research approach

Even if the design principles provided first ideas for the design of an instructional approach that can enhance students' language in mathematics classrooms, we detected a crucial *specification gap*, as the generic design principles imported from other academic disciplines do not help to specify the demands of the learning content in detail (Bailey, 2007). What exactly are the academic language demands that are most crucial for developing conceptual understanding in mathematics?

In our topic-specific design research approach (Prediger & Zindel, 2017), specifying and structuring the learning content is a key working area for treating what-questions (what needs to be learned?) that is also informed empirically by design experiments (van den Heuvel-Panhuizen, 2005) as depicted in Figure 1.

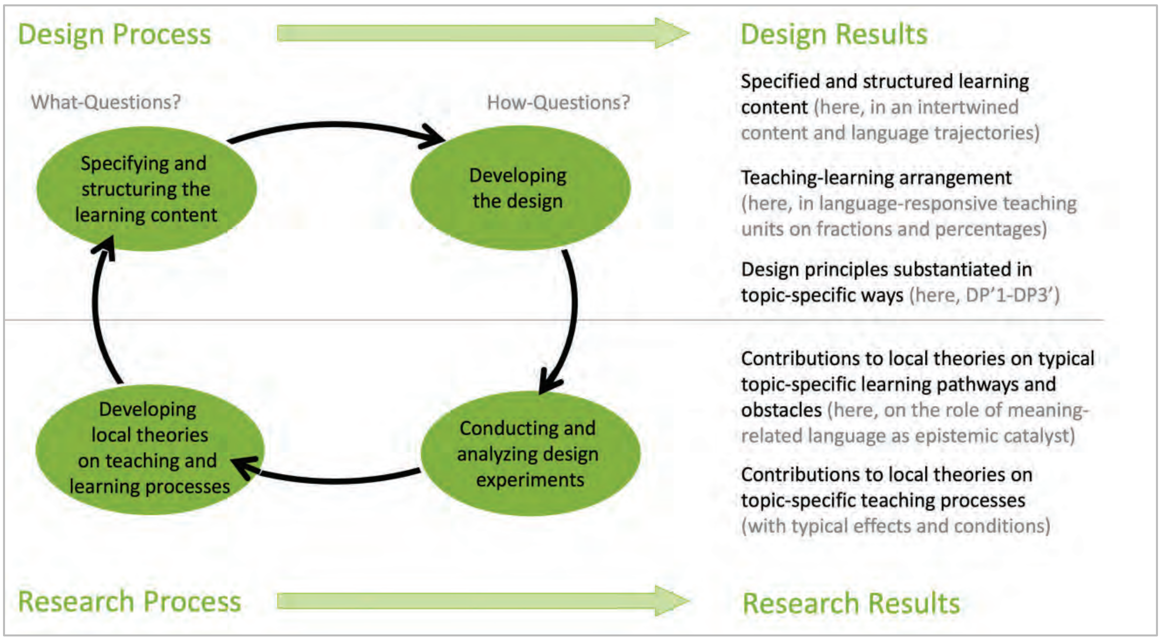


Figure 1. Working areas in topic-specific design research (adapted from Prediger & Zindel, 2017, p. 4168)

In the first design research study (Prediger & Wessel, 2013), we built upon fine-grained existing *specifications* of typical challenges (Behr et al., 1992) and knowledge elements needed for developing conceptual understanding of fractions and overcoming typical mathematical challenges (the part-whole relationship, ordering fractions, and finding equivalent fractions, preferably in a fraction bar model; see Cramer et al., 2009). The language learning content was only vaguely specified in the beginning of our design research study. Developing a first prototype *design of the teaching-learning arrangement* for this learning content is the second working area in the design research cycle (see Figure 1), for which we used the generic design principles DP1, DP2, and DP3.

In the third working area, *design experiments* with monolingual and multilingual seventh graders with low German academic language proficiency were conducted to determine what everyday language resources they bring in and what elements of academic language would support their participation in the process of meaning-making for the mathematical concepts in view.

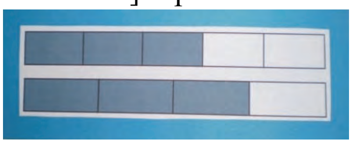
The *qualitative analysis of the design experiments* helped to explain the epistemic role of academic language in more depth. It informed the specification of the crucial academic language demands by acts of empirically grounded theorizing in the fourth working area.

Case study of Cavit and Ismet for specifying the needed academic language

The case of Cavit and Ismet (first analyzed in Prediger, 2013) can illustrate the third and fourth working area, in particular the empirical specification of crucial academic language demands. For this, I provide selected insights into the analysis of one design experiment, conducted in a pair setting with Cavit and Ismet, two sixth graders (12 years old) who were born in Germany and spoke fluent everyday German (and Turkish in their families) but only limited academic language at the beginning of the study. The boys were selected by their teacher as assignments showed incomplete understanding of fractions after their first encounter.

Following the design principle DP2 (using multiple representations), the teacher asks the boys to translate the given fraction bars into symbolic fractions, and they translate easily to 3/4 and 3/5 (see figure in the transcript). Following the design principle DP1 (pushed output) and being aware that students often assign other meanings to graphical representations than the conventional meaning in school arithmetic (Behr et al., 1992), the teacher presses the boys to explain the meaning of the fraction bar.

9 Teacher:	What in this graphical representation [*points to the fraction bars*] represents the fraction?
10 Cavit:	Eh, the... First, you need to count the fields, that is five, and then, the three colored fields, [*silently*] three fifths.
11 Teacher:	Exactly. And which of these two [*points to the fraction bars*] represents the three fifths, actually?
12 Cavit:	First, um, you have to count the fields [*hints at the fraction bars*], here it is one, two, three, four, five [*points at every field of the fraction bar*], and then three, colored [*looks at the teacher*].



The brief transcript (translated from the original German transcript published in Prediger, 2013) documents a particular variant of speechlessness on both sides, both teachers and students, who speak without any lexical or syntactical errors, but the conversation about the mathematical meanings gets stuck. When asked to explain how symbolic fractions and fraction bars are connected (Turn 9), Cavit limits his answer to the discourse practice of reporting the drawing procedure for the graphical fraction bar instead of explaining the meanings of the four fields, the three colored fields and the relationship between them (Turn 10). From his utterances (following a perhaps too implicit teacher question), it is unclear if he understands the part-whole relationship or not. When the teacher asks again, the teacher's speechlessness is indicated by posing nearly the same implicit question as before (Turn 11). Cavit understands the interactive demand for a further explication, but explicates his answer only by counting 1, 2, 3, 4, 5 (Turn 12). In this moment, neither the teacher nor the students seem to have the means to talk about the part-whole relationship, and it is still left unclear if the boys understand the relevant part-whole structure in the bar or not.

At this point in most German mathematics classrooms, the teacher would have given up, assuming in a strength-oriented view that the boys have the right idea even if they cannot communicate it explicitly. This teacher, however, continues to engage the students in making connections between the two representations (following DP2 and DP1) by engaging them in comparing the fractions graphically:

17 Teacher:	How can you see from this picture [*points to the fraction bars*] which fraction is bigger?
…	
24 Ismet:	Here is five [fields] though and there four; from this you see that this [*points to the fraction bar of 3/5*] is big and this is small [*points to the fraction bar of 3/4*].
25 Cavit:	I agree.

When the boys are asked to compare the fractions in the bars, idiosyncratic ideas about the meanings become evident that were not inferable from their earlier utterances. Ismet adopts a typical comparison strategy (Clarke & Roche, 2009) and argues that 3/5 is larger than 3/4 because its fraction bar has 5 instead of 4 fields (Turn 24), and Cavit agrees (Turn 25). It is only in this discourse practice of describing structures (which is more elaborate than reporting a drawing procedure from Turns 10-12) that their ideas about fractions become explicit and discussable. Rather than focusing on the part-whole relationship, they focus on the digits three and five separately without bundling them into one conceptual entity. By this, they are not able to unfold the meaning of fractions as shares appropriately. This observation resonates with a problem previously identified by Steinbring (2005): the "problem of potentially interpreting the given concrete objects and reference contexts for mathematical signs and symbols … as *relational structures*" (p. 5).

A thorough analysis of the boys' discourse practices and their lexical means to realize them exemplifies typical observations: To articulate procedural aspects in the symbolic representation, technical phrases such as nominator and denominator are sufficient. But in this technical language alone, they cannot express conceptual understanding (see Figure 2). The successful translation between graphical and symbolic representation gives a first indication of some conceptual understanding. However, even if the translation is successful, this does not necessarily mean that the connection between two representations involves a deep understanding.

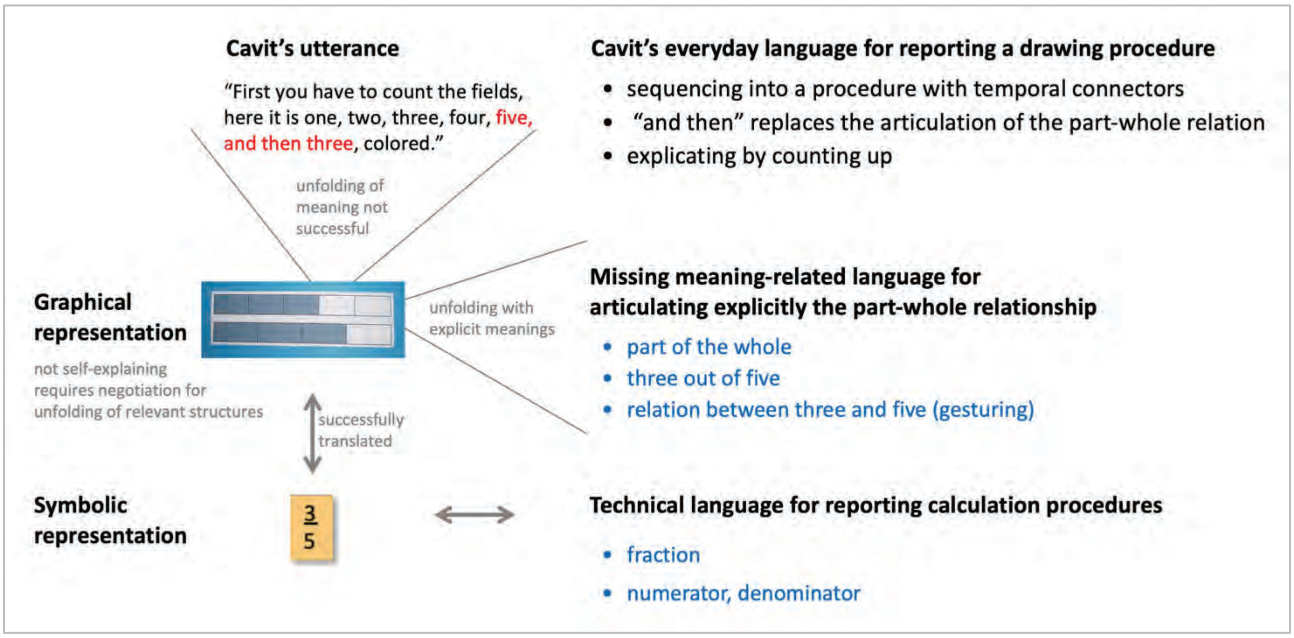


Figure 2. Analysis of involved language registers for the case of Cavit and Ismet

The surface nature of the understanding (sufficient for translating the representation) becomes apparent when the comparison of fractions is to be conducted within the graphical representation (Turn 24), as this requires the unfolding of the fraction concept as a conceptual entity describing the relationship between the whole and the part. From here, it is worth looking back at Turn 12 to analyze Cavit's utterance articulating the surface connection of representations. Rather than explaining the meaning of fractions, Cavit engages in a less demanding discourse practice, namely, reporting a drawing procedure. This proceduralization is an example of sequential reporting also being realized in conceptual contexts, yet without the meaning really being clarified.

Cavit addresses the digits three and five separately; the only connector used between them is "and then," a temporal connector that substitutes the articulation of the part-whole relationship (see Figure 2). The requested further unfolding only leads to counting to five. To articulate the part-whole relationship more explicitly, other meaning-related phrases would have been needed, such as "three out of five" or "part of the same whole." The transcript continues with the teacher offering these meaning-related phrases and filling them with the conventional meaning by heavily gesturing for the relation of five thereof three. By drawing upon gestures for the part-whole relation (indicating the whole, and therein the parts always related to the whole), the teacher attaches the new phrases to students' existing resources and equips them with the meaning-related phrases that enable them to engage in the more demanding discourse practices.

What exactly is the problem with the epistemic role of academic language while developing conceptual understanding? Theorizing by combining epistemic and discursive perspectives

Students' conceptual challenges with fractions as part-whole relationships and comparing fractions have been well documented in the literature (and the existing state of research on student thinking on fractions, e.g., Behr et al., 1992; Clarke & Roche, 2009). The case of Cavit and Ismet's struggle with *articulating* the part-whole relationship adds substantial explanative findings on the role of language in students' conceptual learning pathways to the existing state of research, as it shows typical phenomena we found in many design experiments, for fractions and other mathematical concepts. These phenomena formed the main object of theorizing in the fourth working area of the design research cycle (Figure 1, systematically reflected on in Prediger, 2019). In our research journey, the grounded theorizing based upon these empirically identified phenomena required the combination of epistemic and discursive perspectives.

From an *epistemic perspective*, mathematical concepts are hard to learn as they are usually abstract and relational in nature (Steinbring, 2005). They compact complex relationships (such as the part-whole relationship or functional relationships) into new conceptual entities (such as fractions or functions) that can then be integrated into higher-order networks and relationships. Technical mathematical language is optimized for addressing these abstract concepts in a highly concise and economic manner and allows experts to communicate and think with the language very efficiently. However, for learning the technical language in its compacted form, it must first be unfolded into its meanings (Prediger & Zindel, 2017).

Early work on language in mathematics assumed that all of the unfolding of the formal technical language can be conducted in students' everyday language (e.g., Gallin & Ruf, 1990). But the analyses of multiple design experiments revealed that there is a mediating language register between the everyday language and the technical language that needs to be equally concise and explicit in expressing relationships yet informal, without technical terms and symbolic representations. This is exactly the part of the academic language register that is most crucial for developing conceptual understanding, and we termed it meaning-related language (Pöhler & Prediger, 2015).

From a *discursive perspective*, meaning-related language can be characterized by the discourse practices of explaining meanings and describing mathematical relationships and structures. These discourse practices are typical for the academic language register (Heller & Morek, 2015), and discursively more demanding than narrating or reporting procedures (as already touched by Setati, 2005). In the lexical dimension, reporting procedures mainly requires the less complicated sequential connectors ("first," "and then," etc.), whereas explaining meanings and describing mathematical relationships requires more complex connectors and meaning-related phrases for articulating the key relationships (such as the examples in Figure 5 below). These phrases can also include grammar features in the syntactical dimension (as elaborated in Prediger & Hein, 2017; Prediger & Şahin-Gür, 2020).

For each mathematical topic, these meaning-related phrases need to be identified by epistemological analysis and ideally by empirical investigations of topic-specific learning processes, drawing, of course, upon the existing states of topic-specific research. Consequentially, the research was extended to percentages (Pöhler & Prediger, 2015), logical structures of proofs (Prediger & Hein, 2017), functions (Prediger & Zindel, 2017), qualitative calculus (Prediger & Şahin-Gür, 2020), and other topics. For each mathematical topic, meaning-related phrases for the rich discourse practices of explaining meanings and describing mathematical relationships were identified as crucial.

Summing up, the four practices of theorizing in the first design experiment cycles (fourth working area in Figure 1) led to building a theoretical foundation for specifying the language-learning content (first working area in Figure 1): focusing on the rich discourse practices of explaining meanings and describing mathematical relationships and structures, together with the meaning-related phrases used to realize these discourse practices in processes of developing conceptual understanding. Without the strong epistemological focus on conceptual understanding of mathematical concepts and the epistemic perspective on unfolding their highly compacted meanings, the meaning-related language and its importance as intermediate register would not have been identified.

How can meaning-related academic language be enhanced in mathematics classrooms? Substantiating the design principles with respect to specified learning content

Whereas "how-questions" (How can the learning be enhanced?) and "what-questions" (What has to be learned?) are often treated as independent (like in the beginning of our research journey), the research approach of topic-specific design research has the potential to integrate both questions in subsequent design experiment cycles (Prediger, 2019). The sketched theorizing on the role of the discourse practices of explaining meanings and describing mathematical relationships for unfolding the meaning of mathematical concepts also provided the empirical foundation for substantiating the three generic design principles in more mathematics-specific ways, as shown in Figure 3, so that they can be tailored to enhance those academic language demands that are crucial for developing conceptual understanding of mathematical concepts such as fractions, percentages, and functions.

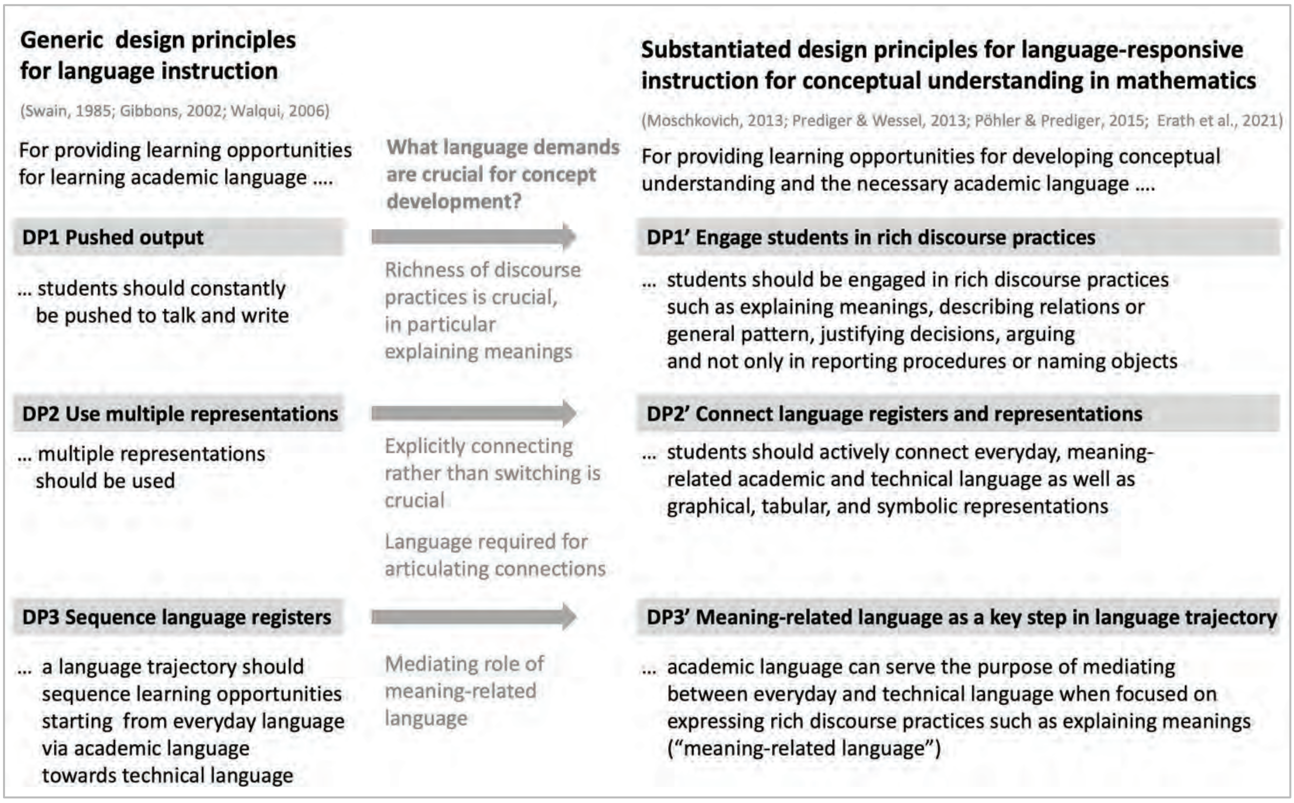
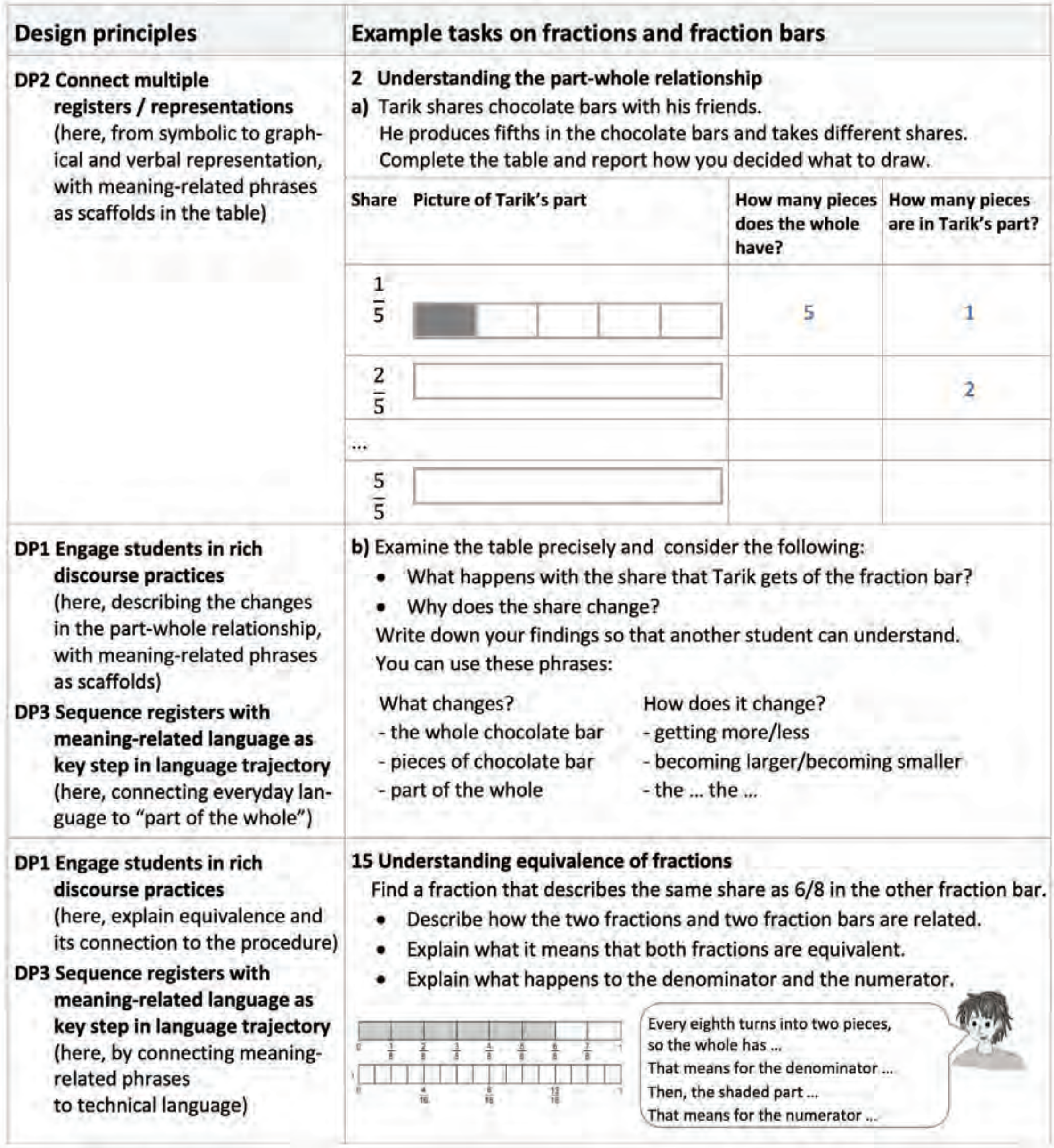


Figure 3. Adapting generic design principles into substantiated design principles for language-responsive instruction of conceptual understanding of mathematical concepts

To refine design principle DP1 (pushed output), we take into account the required *richness* of discourse practices in which students are engaged: Because the discourse practices of explaining meanings or describing mathematical relationships are more relevant for developing conceptual understanding than narrating incidences or reporting procedures, DP1 was refined into "DP1 Engage students in rich discourse practices."



Design principles	Example tasks on fractions and fraction bars
DP2 Connect multiple registers / representations (here, from symbolic to graphical and verbal representation, with meaning-related phrases as scaffolds in the table)	**2 Understanding the part-whole relationship** **a)** Tarik shares chocolate bars with his friends. He produces fifths in the chocolate bars and takes different shares. Complete the table and report how you decided what to draw.
DP1 Engage students in rich discourse practices (here, describing the changes in the part-whole relationship, with meaning-related phrases as scaffolds) **DP3 Sequence registers with meaning-related language as key step in language trajectory** (here, connecting everyday language to "part of the whole")	**b)** Examine the table precisely and consider the following: • What happens with the share that Tarik gets of the fraction bar? • Why does the share change? Write down your findings so that another student can understand. You can use these phrases: What changes? - the whole chocolate bar - pieces of chocolate bar - part of the whole How does it change? - getting more/less - becoming larger/becoming smaller - the ... the ...
DP1 Engage students in rich discourse practices (here, explain equivalence and its connection to the procedure) **DP3 Sequence registers with meaning-related language as key step in language trajectory** (here, by connecting meaning-related phrases to technical language)	**15 Understanding equivalence of fractions** Find a fraction that describes the same share as 6/8 in the other fraction bar. • Describe how the two fractions and two fraction bars are related. • Explain what it means that both fractions are equivalent. • Explain what happens to the denominator and the numerator.

Share	Picture of Tarik's part	How many pieces does the whole have?	How many pieces are in Tarik's part?
$\frac{1}{5}$		5	1
$\frac{2}{5}$			2
...			
$\frac{5}{5}$			

Figure 4. Example tasks realizing the refined design principles DP1-3

For design principle DP2 (use multiple representations), the empirical analyses of many learning processes showed that switching between representations is not sufficient. Instead, explicit connections must be drawn and articulated (as for Cavit and Ismet, who switch successfully but cannot explain the connection). Because this applies not only for graphical, tabular, and symbolic representations, but also for everyday language, meaning-related academic language, and technical language, as their connections also contribute to the meaning-making processes (Gibbons, 2002), DP2 was substantiated into “DP2 Connect language registers and representations.”

For design principle DP3 (sequence language registers), the findings about the mediating role of meaning-related language as a particular part of academic language resulted in articulating “DP3 Emphasize the meaning-related language as a key step in the language trajectory,” because academic language can serve the purpose of mediating between everyday and technical language when focused on expressing rich discourse practices such as explaining meanings (“meaning-related language”). Whereas the refined design principles DP1 and DP2 are present in many instructional approaches (Moschkovich, 2013, and many others; see overview in Erath et al., 2021), the explicit input or elicitation and then consolidation of meaning-related language seems to be the major contribution of the Dortmund research to a vivid field of research and development.

Figure 4 sketches how the design principles were realized in the design of the teaching unit on fractions (Prediger & Wessel, 2013, here cited in the version from Prediger et al., in press). Whereas the tasks have multiple similarities to previous approaches (e.g., Cramer et al., 2003), the consequent demand on explaining meanings is substantially strengthened and supported.

Summing up, the refinement of generic design principles by mathematics-specific and even topic-specific considerations on the epistemic role of meaning-related language in the processes of conceptual understanding led to a substantiation of the design principles for enhancing language not as an end in itself but enhancing *language for mathematics learning*. As observation studies in regular classrooms did not help for this step, design research with its iterative combination of development and qualitative research and of what-questions and how-questions was the research approach of choice in this step.

Within the CERME thematic working group on language in mathematics education, the substantiated design principles found resonance with colleagues and their empirical findings from various countries. Building upon the impressive and complex descriptive and explanative findings on language in mathematics education that have been summarized for the occasion of the ERME birthday book (Planas et al., 2018), the thematic working group has now increasingly also turned to design challenges and their treatment. This extension of common research goals has been expressed in the jointly compiled *ZDM* special issue 53(2) on design principles and teaching practices that enhance students’ language for learning mathematics. The review paper introducing the special issue elaborates six design principles, among them the three from Figure 3 (Erath et al., 2021). The other three are “establish various mathematics language routines,” “include students’ multilingual resources,” and “compare language pieces (form, function, etc.) to raise students’ language awareness,” which are not foregrounded in this paper but have enriched our approach on other occasions. Many working group members have started to combine some of the design principles flexibly, and there is still research to be done on their interdependence.

Is language-responsive teaching effective for developing conceptual understanding? Controlled trials for overcoming the quantitative evidence gap for different target groups

The research review on designs and teaching practices for enhancing language for mathematics learning by Erath et al. (2021) summarizes 26 *qualitative* studies (from many ERME members and beyond) that have provided highly interesting, deep insights into the functioning of some of the language-responsive design principles and conditions of success that need to be considered for their application (and many more studies would have been available for this summary of an impressive body of qualitative research). So far, however, very few studies have provided quantitative evidence for the hypothesized effectiveness of these design principles on students' measurable learning gains. Given the increasing importance of quantitative evidence for policy decisions and research-based professional development, reducing *this quantitative evidence gap* can contribute to two aspects of orienting policies and convincing mathematics teachers and teacher educators: (a) Does it work? (b) For whom does it work? (Cai et al., 2021).

(a) According to methodological standards, quantitative evidence of effectiveness can be gained in controlled trials in pre-post-control-group designs showing that certain instructional approaches result in significantly larger measurable learning gains than mathematics instruction not following these approaches (Styles & Torgerson, 2018). In the MuM research group, we conducted several quasi-experimental and randomized controlled trials to confirm the hypothesis that language-responsive instruction combining the design principles DP1-DP3 can indeed enhance students' conceptual understanding significantly better than in business-as-usual control groups with comparable abilities and teacher backgrounds, first for fractions (Prediger & Wessel, 2013; Prediger et al., in press) then for percentages (Prediger & Neugebauer, 2021).

(b) Beyond confirming hypotheses on effectiveness, controlled trials can also help to deconstruct simplistic ideas about the target group. In the beginning, language-responsive mathematics instruction was mainly planned and offered to a particular target group with a combination of three disadvantaging factors: multilingual background, low academic language proficiency in the language of instruction, and limited school success (see research overview in de Araujo et al., 2018). But from a sociopolitical perspective, the exclusive focus on this particular target group is critical, as it can nurture deficit views on multilingual students, either by the misunderstanding that the three factors always coincide, or by suggesting particularistic approaches rather than inclusive access to good teaching (Barwell, 2005; Moschkovich, 2010). In contrast, the effects of language-responsive instruction on successful multilingual students, on multilingual students with high academic language proficiency, or on monolingual students with diverse language proficiency or school success were hardly investigated, so we decided to conduct three extensions (see Figure 5):

- A first extension of the target group included monolingual students with low academic language proficiency in the language of instruction (measured by standardized cloze tests). They were shown to profit as much as the original target group from language-responsive mathematics instruction (Prediger & Wessel, 2018; Smit & van Eerde, 2013).

- A second extension of the target group involved students with high academic language proficiency: Prediger and Neugebauer (2021) showed in a quasi-experimental field study that monolingual and multilingual students with diverse academic language proficiencies profited equally from a language-responsive intervention on percentages.
- Language-responsive mathematics instruction was, however, still mainly being provided to students with mathematical difficulties at risk of being left behind by the school system. As long as empirical findings about language-responsive teaching concentrate on struggling students, principles of language-responsive mathematics instruction might be misunderstood as being relevant only to overcoming problems. To avoid deficit-oriented policy discourses and practice discourses about language as a problem, it is crucial to provide empirical evidence that not only students with mathematical difficulties profit from language-responsive mathematics instruction, but also successful students. In our recent *JRME* paper, we reported from a randomized controlled trial that confirms exactly this third extension of the target group: Not only students at risk but also successful students in higher tracks can profit significantly more from language-responsive mathematics instruction than from business-as-usual control groups (Prediger et al., in press). Within the language-responsive instruction, low academic language proficiency is less predictive for weaker learning than in the control groups.

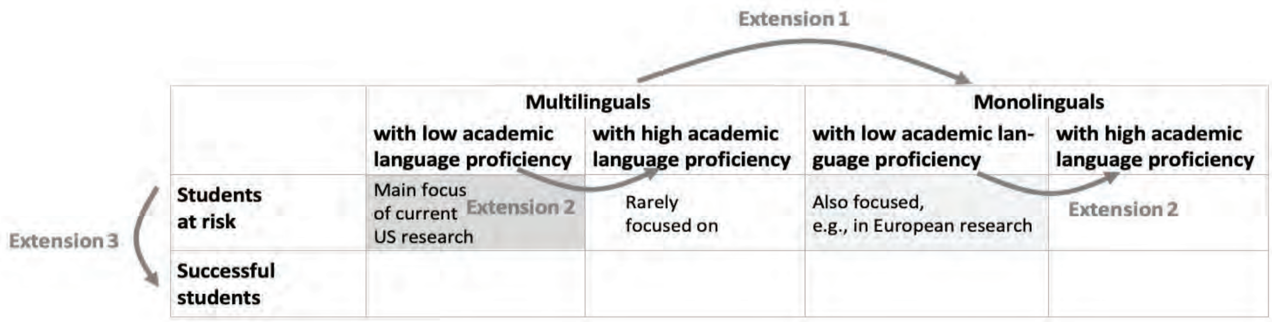


Figure 5. Research base for inclusive language-responsive instruction needs extensions of the target group from multilinguals at risk with low language proficiency to all students (Prediger et al., in press)

Summing up, the quantitative evidence gap can be reduced by findings of a chain of controlled trials showing that students of all backgrounds (monolingual and multilingual backgrounds, students with all degrees of academic language proficiency, and students with the whole range of school success) can profit mathematically from language-responsive mathematics instruction. These indications for effectiveness can strengthen the research base for language-responsive mathematics instruction as being *inclusive for all students* when realized with the strong focus on the design principles from Figure 3. They confirm that enhancing meaning-related language can serve as an epistemic catalyst for developing the conceptual understanding of all students.

This example also shows that every research has a sociopolitical component: As long as our methodological choices (in this case, the choice of our samples) perpetuate deficit orientations or particularistic approaches (in this case, by focusing on a too-specific target group accumulating different challenges or claiming too-specific needs), instructional approaches meeting the needs of particular students' risk continuing to be positioned in a deficit context outside mainstream classrooms. It is our responsibility as researchers to make wise choices in the research questions, methods, and samples we consider so that they can enter mainstream classrooms. Of course, this emphasis on inclusive use of language-responsive teaching for all students does not question the need to allocate more learning time for students at risk who are slower learners. However, it makes it clear that these students do not necessarily need other kinds of instruction.

At least in Germany, the particularistic character of language-responsive mathematics instruction is also fueled by the separateness of different research approaches: Language-responsive instruction is often investigated completely separately from the general academic discourse on high-quality mathematics instruction, which was something we wanted to overcome, as called for by Cai et al. (2020). This challenge guided decisions for the next step in the research journey.

How does the quality of teachers' enactment affect learning gains?
Video quality study for explaining and connecting the approaches to high-quality instruction

Although the overall effectiveness of our instructional approaches was shown in several controlled trials, we could also identify large differences between classes. In the implementation study for the language-responsive teaching unit on percentages, an intraclass correlation coefficient of 0.32 in the 70 intervention classes indicates that 32% of the variance is explained by which class the students were in. As the class compositions were comparable, we assumed this variance could be traced back to different instructional practices the involved teachers used in enacting the language-responsive approach, even though all worked with the same shared language-responsive curriculum materials (presented in Prediger & Pöhler, 2015) and all participated in the same professional development addressing the design principles and the ideas underlying the approach (four sessions of 3 hours each).

Many qualitative case studies have shown how teachers' enactment of instructional practices can promote or hinder the mathematical and language-related learning of the students (e.g., Barwell, 2020). In two research reviews (Walshaw & Anthony, 2008; Erath et al., 2021), a large number of qualitative case studies were summarized that identified typical teaching practices and established classroom norms and hypothesized whether they were productive or unproductive. However, except for one insightful study (Ing et al., 2015), little quantitative evidence has been provided for validating these qualitatively generated hypotheses on the potential effects of these teaching practices.

Within the research logic of controlled trials, the instruction itself is often treated as black box, so researchers have increasingly called for opening the black box and studying the teaching learning processes that occur during the interventions (Styles & Torgerson, 2018). For our research context, the existing qualitative findings suggested strongly that this trend should be followed. Hence, we decided that we need to change the research approach again and open the black box for a video study in a first approach on the following research question: How does the quality of teachers' enactment affect learning gains?

In our research journey, the research question was pursued by analyzing the video-recorded teaching practices from 18 of our heterogeneous intervention classes in a quantitative video-based approach—

a selection that was presumably positive as these 18 teachers volunteered to be videorecorded. Rather than only capturing the realization of the language-responsive design principles, we decided to take into account the large body of research on instructional quality in mathematics (Charalambous & Praetorius, 2018) in order to overcome the separate treatment of instructional quality in general and language-responsive mathematics instruction in particular. We chose (and adapted slightly) Schoenfeld's (2013) Teaching for Robust Understanding (TRU) framework to capture instructional quality in mathematics classrooms, as this framework shares the same ideas of quality with an emphasis on conceptual focus and was optimized in contexts of equity and underprivileged students. The TRU framework has five interconnected dimensions:

- *Mathematical Richness:* To what extent is the mathematics discussed clear, correct, and well justified (tied to conceptual underpinnings)?
- *Cognitive Demand:* To what extent do classroom interactions create and maintain an environment of intellectual challenge?
- *Equitable Access:* To what extent do activity structures invite and support active engagement from the diverse range of students?
- *Agency:* To what extent do students have opportunities to conjecture, explain, and argue, thus to developing agency and authority?
- *Use of Contributions:* To what extent is student reasoning elicited, challenged, and refined?

With respect to language-responsive instruction, we have slightly adapted the dimensions and added two further dimensions (Prediger & Neugebauer, 2021):

- *Discursive Demand:* To what extent do students engage in rich discourse practices? (aligned to our DP1 from Figure 3)
- *Connecting Registers:* To what extent are language registers and representations systematically and explicitly connected? (aligned to DP 2 and DP3)

Our rating of video-recorded lessons revealed that the assessed instructional quality in 18 intervention classes was on the highest level for 79-92% of the time in four dimensions, namely *Mathematical Richness*, *Cognitive Demand*, *Agency*, and *Connecting Registers*. This can be interpreted as a modest indication that the professional development and language-responsive curriculum material (presented in Prediger & Pöhler, 2015) supported the teachers in enacting an impressive instructional quality. In particular, it shows that the refined design principles DP1-DP3 (see Figure 3) are suitable for increasing the *Mathematical Richness* and *Cognitive Demand*, which are the two most content-related quality dimensions. In contrast, for the quality dimensions *Discursive Demand*, *Equitable Access*, and *Use of Contributions*, the 18 teachers showed a high variance in their enactment. These dimensions therefore seem to be less supported by the curriculum material per se and depended more on teachers' moves and strategies in directing classroom discussions (Ing et al., 2015), as has been indicated by a large body of qualitative research (Walshaw & Anthony, 2008).

To validate the hypothesis that these quality dimensions matter for students' learning, we conducted a multi-level analysis to find out how the seven quality dimensions predicted students' measurable learning gains (while controlling for individual background factors). And, indeed, the three dimensions that varied most with teachers' enactment—*Discursive Demand*, *Equitable Access*, and *Use of Contributions*—had an additional significant impact on students' learning gains, but so did *Agency*, in spite of its high-quality realization (Neugebauer & Prediger, submitted). Although the sample of 18 classes is not yet very large, these findings provide promising (and significant) indications for confirming the hypotheses on productive teaching practices by revealing multilevel quantitative evidence for their impact on measurable learning gains.

Summing up, this video quality study shows that language-responsive mathematics instruction can be realized in ways that achieve high instructional quality for *Mathematical Richness* and *Cognitive Demand.* This again can be interpreted as further evidence that enhancing language is an epistemic catalyst in mathematical classrooms if supported by topic-specific curriculum material. At the same time, teaching practices in the classroom interaction can still heavily vary with respect to *Discursive Demand*, *Equitable Access*, and *Use of Contributions*, and their enactment on a high-quality level has additional impact on students' mathematical learning gains. This means that the more the teacher achieves (in engaging all students in rich discourse practices [DP1] and in building upon students' informal ideas and emerging language for the further learning trajectory [DP3] by connecting these elements [DP2] in their moderation of the classroom discussions), the more the students have developed their conceptual understanding on the mathematical topic in view, in this case, percentages. These findings call for integrating more professional learning opportunities for productive teaching practices to raise and maintain discursive demands, provide equitable access, and make use of students' contributions in the discussion about future professional development offers (as, e.g., Kazemi et al., 2021). We might also assume that teachers need some more time to implement these quality dimensions into their practices than we gave them, but this is only one of the several challenges in implementing promising instructional approaches.

Again, this example shows the strength of combining different research approaches, in this case with the aim of including the insights from qualitative research into quantitative designs and validating the generated hypotheses (Creswell, 2015).

Looking back from a meta-level: What does connecting research approaches mean?

In most ERME research, connecting research approaches has been mainly interpreted in two modes:

- connecting methods for a well-articulated research goal (e.g., in the long and rich tradition of mixed methods approaches; see Creswell, 2015) and
- networking theoretical approaches (most often used for analyzing given data from two theoretical perspectives; see Bikner-Ahsbahs, Prediger, & Networking Theories Group, 2014).

Both modes also occur in our research journey: With respect to combining methods, the chain of our projects is characterized by sequential explanatory and exploratory mixed-methods designs. Explanatory mixed-methods designs are used when quantitative findings on statistical connections are investigated deeper in qualitative methods in order to explain the quantitative findings (as in our case, explaining the epistemic role of academic language in more depth after large-scale assessments showed that language gaps are largest for conceptual understanding). Exploratory mixed methods designs explore the generalizability of qualitatively identified phenomena (in our case, generalizing conditions of success in applying the design principles identified in case studies to a larger number of classes and measurable learning gains).

With respect to theoretical approaches, two characteristics make the coordination and integration of theoretical approaches absolutely necessary for the presented chain of (overlapping and interacting) projects:

- Research on language in mathematics learning always requires both an epistemic perspective and a language-related (here, mainly discursive) perspective and an integration of both for explaining how language learning and mathematics learning are related (Barwell, 2005; Moschkovich, 2010). As these networking practices have not been made explicit in this current paper, I refer to three papers from our work that best document different kinds of networking practices (Erath et al., 2018; Prediger & Şahin-Gür, 2020; Prediger & Hein, 2017).
- Design research is always the bricolage of theoretical elements (Alberto, Bakker, et al., 2019; Prediger, 2019). In our case, the integration of the what-questions into the how-questions led to substantiating design principles based on instructional and epistemological foundations (see Figure 3). As in many other cases, topic-specific design research often requires the integration of instructional theories (underlying the design principles) with epistemological theories (underlying the structuring of the content; see, e.g., Alberto et al., 2019).

Beyond these two most discussed modes of connecting research approaches, the summary of our research journey in Table 1 exemplifies that the connection of research approaches in such a chain of overlapping and interacting projects is substantially fueled by *deriving new research questions* from answers to former research questions, and these new research questions become most productive when involving *shifts in the research objects*.

These shifts in research objects are worth reflecting on more systematically, because the choice and the constitution of the research object can have a substantial impact on what can be achieved as a research goal and how research can provide the foundation for design practices (diSessa & Cobb, 2004). For example, as long as only student achievement gaps are the research object, the research hardly contributes to overcoming the problem (Gutiérrez, 2008; Hodgen et al., 2021), and as long as only students with accumulated challenges are investigated, deficit views might be perpetuated without exploring the commonalities to other target groups. Overcoming the problem requires turning to teaching practices and design principles as major research objects (Sullivan, 2015). Additionally, focused mathematical instruction also requires detailed epistemological and empirical insights into the students' learning pathways of particular mathematical topics: This combination of design principles and topic-specific pathways is typical for didactical design research (Gravemeijer & Cobb, 2006; Prediger, 2019). Beyond these, the teaching practices enacted by teachers while realizing a design approach are important research objects for understanding the challenges in implementation processes (Burkhardt & Schoenfeld, 2003), which is also far beyond what we started to understand modestly in our projects.

Table 1. Summary of the research journey in its several steps: shifts in research questions and objects

Method	Research goal	Research question	Research object
Large-scale assessments	Describing and explaining language gaps	What exactly is the problem with language in mathematics?	Student achievement gap Students' background factors Knowledge types in items
Qualitative video study	Identifying good practices	How can language be enhanced in mathematics classrooms?	Teaching practices in regular class-rooms
Import from other disci-pline	Transferring generic design principles	How can language be enhanced in language classrooms?	Design principles for language instruc-tion Tasks and pedagogies
Design research	Explaining learning path-ways and typical obstacles	What is the problem with the epistemic role of academic lan-guage while developing concep-tual understanding?	Students' learning pathways towards conceptual understanding in a mathe-matical topic
Design research	Specifying the mathemati-cally relevant language-learning content	What academic language de-mands are crucial for develop-ing conceptual understanding?	Relevant language in students' mathe-matical learning pathways towards conceptual understanding
Design research	Substantiating language-responsive design princi-ples for math instruction	How can meaning-related aca-demic language be enhanced in mathematics classrooms?	Design principles for enhancing language for developing understanding of mathematical concepts
Controlled trials	Validating effectiveness of approach for different target groups	For whom is language-respon-sive instruction effective for developing understanding?	Effects of language-responsive instruc-tion for learning gains of students with different background factors
Quantitative video study	Disentangling conditions of effective language-responsive teaching	How does the quality of teachers' enactment affect the learning gains?	Teaching practices in language-responsive classrooms and their effects on learning gains

Finally, with every shift in research object, new theoretical perspectives also enter the scene (from epistemological backgrounds of the topic, theories of learning, theories of language and learning, theories of instruction, etc.). In the long run, an integrated theory rather than a chain of unconnected research objects should be the overall research aim. However, this aim will require two more decades and the joint communication, cooperation, and collaboration of various researchers from other European and non-European sites to integrate the different empirically grounded theory elements into a coherent theory on enhancing language as epistemic catalyst for developing mathematical understanding.

It is a particular strength of the ERME community not to be afraid of complex connections of theories and research approaches (Artigue et al., 2005; Bikner-Ahsbahs, Prediger, & Networking Theories Group, 2014): This was established as a meta-theoretical position in the TWG on theoretical approaches and has spread widely to other TWGs in ERME. In the same way that the idea of networking theories was spread among the whole ERME community within the last decade (as documented by Artigue et al., 2018), I hope that this paper can fuel the idea that big problems require complex chains of projects within one research group and more complex networks of projects of several research groups to treat them. This is presumably not only true for language as an epistemic catalyst for mathematics learning, but also for other exciting areas in mathematics education research.

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Conceptualizing individual-context relationships in teaching: Developments in research on teachers' knowledge, beliefs and identity

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Abstract: Research on teachers' knowledge, beliefs and identity have contributed with significant new understandings of their respective fields. In this paper I track developments in each of them over the last few decades and address the question of how individual-context relationships have been conceptualized. Part of the answer is that the two first fields have primarily drawn on the constructivist inspiration that initially gave rise to them, while identity research has been based on more fundamentally social understandings of human functioning. Another part of the answer is that none of the three fields has to any great extent drawn on interactionist approaches. I suggest that doing so may be a fruitful way ahead, if we want to understand the role of the teacher for group life as it emerges in schools and classrooms.

Keywords: Teachers knowledge, teachers' beliefs, teacher identity, the social turn, interactionism.

Research on teachers and teaching

Studies of teachers and teaching have been a prominent part of mathematics education research for decades (Sfard, 2005). This is so not least at CERME conferences, where the research interest seems to have grown consistently over the last 20 years (Skott et al., 2018). Large parts of this research have focused on teachers' knowledge, beliefs and identity, which have often been treated as separate teacher characteristics and dealt with by use of different theoretical frameworks.

Inspired by constructivism, research on teachers' knowledge and beliefs has traditionally considered their respective key constructs mental entities residing within the individual and been premised on the expectation that they have considerable impact on teaching quality and student learning (Cross Francis et al., 2015; Skott, 2015b; Sowder, 2007). In this sense, teachers' knowledge and beliefs have been treated as almost independent variables and semi-causal determinants of classroom practice.

Over the last 10-20 years, this approach to research on and with teachers has been challenged and modified, and perspectives with a stronger social and contextual emphasis have been adopted. This is so in two ways. First, studies of teachers' knowledge and beliefs tend to be less inclined than before to expect that these mental constructs may serve as explanatory principles for practice and to a greater extent adopt – or at least call for – dynamic perspectives on the relationships between knowledge and beliefs on the one hand and classroom practices on the other (Ball, 2018; Skott, 2015c). Second, the field of teachers' professional identities has grown into prominence, generally using more fundamentally social frameworks when studying teaching-learning processes (e.g. Darragh, 2016; Skott, 2013). In general, then, individual-context relationships are viewed differently now than a few decades ago. However, this is so in different ways within and across the three sub-fields.

One aim of the present paper is to outline significant aspects of these developments and elaborate on the theoretical differences involved. This means that rather than discussing specific results on, for instance, teachers' knowledge, beliefs, and identities as they relate to particular mathematical contents (e.g. algebra or problem solving) or particular educational settings (e.g. special categories of schools or teachers), I describe developments from a somewhat different vantage point. Doing so, I am inspired by what Lerman calls "the social turn" in mathematics education research (Lerman, 2000, 2006). The questions I ask are how and to what extent aspects of "the social" have been taken into account in studies of teachers' knowledge, beliefs, and identity, in particular how the individual-context interface is dealt with. It is part of the answer to the latter of these questions that interactionism has played next to no role. It is another aim of the paper to invite considerations of the potential of interactionism, if intentions of research on and with teachers include understanding the meaning teachers make of their professional lives and the role of teachers for the practices that evolve in their classrooms.

I begin the paper with accounts of the developments and main emphases in research on teachers' knowledge, beliefs, and identities over the last few decades. It is beyond the scope of the paper to present a comprehensive review of research in all three fields, and my presentation of the literature is necessarily selective. I seek to present a critical overview that does justice to significant developments within each of them, including their similarities and differences. One of the differences is the extent to which the literature is self-reflective and discusses its own key constructs (i.e. knowledge, beliefs and identity) and the methodological problems involved in researching them. The notions *beliefs* and *identity*, for instance, are recurrent objects of attention, while this is less so with *knowledge*. Another difference is that there are a few established frameworks that dominate research on teachers' knowledge, all of which draw on Shulman's work from the 1980s, while a similar canon of frameworks does not exist in the two other fields. These differences are necessarily reflected in the structure of my presentation.

Following from that, I relate this account to "the social turn", that is, "the emergence into the mathematics education research community of theories that see meaning, thinking and reasoning as products of social activity" (Lerman, 2000, p. 23). I do so by discussing three aspects of research on teachers' professional identities, those of agency, situatedness and structure. Using this triad, I argue that the individual-context relationship is conceived of differently across and to some extent within the three fields. However, I also suggest that interactionist perspectives are conspicuously absent in all of them and that an interactionist complement to current approaches may be useful when seeking to understand the dynamic character of the teacher's role for the emergence of group life in schools and classrooms. In particular, I refer to a framework called *Patterns of Participation* (PoP), a result of networking social practice theory (e.g. Holland et al., 1998; Lave, 2019; Wenger, 1998) with symbolic interactionism (e.g. Blumer, 1969; Prus, 1996) as one possible interactionist approach.

Research on teachers' knowledge

Background and rationale of research on teachers' knowledge

Teachers' knowledge of the subjects they teach has been discussed for decades, including its relation to knowledge of and proficiency with other aspects of the profession (Hill et al., 2007). Some of these studies seek to find positive correlations between teachers' academic proficiency with mathematics and their students' performance. The studies by Begle (1972) and Eisenberg (1977) are classic examples of this. Conducted at the height of the *New Maths* era, Begle's study tested highly qualified teachers' understanding of "modern algebra" (groups, rings and fields) and the real number system and found that any correlation with student learning was so small "as to be educationally insignificant" (p. 13). Eisenberg (1977) did a similar study with a more representative group of teachers, but also found that "teacher knowledge of subject matter had little effect on student performance" (p. 222).

Summing up the results from these and similar studies, *Adding it up*, a report from the National Research Council in USA, argues much later that "proposals to improve mathematics instruction by simply increasing the number of mathematics courses required of teachers are not likely to be successful" (National Research Council, 2001, p. 375). Based on such results, the rationale behind later research on mathematics teachers' knowledge has been to come to grips with what mathematics teachers need to know and how. The results suggest that this knowledge needs to be closely linked to the work of teaching, if it is to improve instruction (e.g. Hoover et al., 2016). The most dominant frameworks that make such links draw on Shulman's work from the 1980s (Shulman, 1986, 1987).

Shulman's knowledge base for teaching

In the 1970s and 1980s, behaviourist process-product studies played a significant role in research on teachers (e.g. Medley, 1977). Shulman challenged this approach and bemoaned its lack of attention to teachers' thinking and to the contents of instruction, which he called a blind spot and a missing paradigm in the field (Shulman, 1986, 2015). In response, he built on a study of novice, secondary teachers of English, biology, social studies, and mathematics to develop a description of teachers' knowledge with seven categories, three of which were immediately related to the contents of instruction: curriculum knowledge, content knowledge, and pedagogical content knowledge (Shulman, 1986, 1987). Especially the last two of these have been discussed extensively in mathematics education and in other educational scholarship.

In relation to *content knowledge*, Shulman says that teachers are members of scholarly communities and "must understand the structures of subject matter, [that is] the principles of conceptual organization, and the principles of inquiry" (Shulman, 1987, p. 9). In essence, this does not differ from the knowledge of others, who know the subject. In contrast, *pedagogical content knowledge* (PCK), the most frequently cited category in Shulman's framework, is specialised to teaching. It is that blend of content and pedagogy that forms "an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (Shulman, 1987, p. 8). In a continental European tradition this resembles methodological or technical aspects of *didaktik* and in the US it was part of the testing of teachers, also before Shulman coined the term of PCK (Hill et al., 2007).

Frameworks on mathematics teachers' knowledge that build on Shulman

A long list of studies have drawn on Shulman's knowledge base for teaching, both in mathematics and in other subjects. Generally, they shift the emphasis from analyses of the subjects themselves to how they are used in teaching. It is a joint outcome that it is not only PCK that is special to the profession, so are aspects of the contents itself. Four of the most prominent of these frameworks in mathematics are Ma's *Profound Understanding of Fundamental Mathematics*; *Teachers' Specialized Knowledge of Mathematics* developed by Carrillo and his colleagues; Rowland et al.'s *Knowledge Quartet*; and Ball et al.'s *Mathematical Knowledge for Teaching*.

Ma (1999) studied the mathematical knowledge of Chinese and U.S. primary teachers. At the time of the study, Chinese teachers were educated in normal schools that recruited grade-9 students for a 2-3 year programme (Li, 2012). Novice teachers, then, were 18-19 years old and had no post-secondary education. In contrast, all American teachers had at least a college degree. In spite of that, the American teachers were outperformed by their Chinese colleagues on tasks from the primary school curriculum, for instance $1\frac{3}{4} \div \frac{1}{2}$, and when asked to suggest educationally relevant situations to represent the meaning of such tasks. Ma concludes that there were significant differences between "the mathematical substance" of the knowledge of the two groups of teachers (p. 92). The Chinese participants were closer to having *Profound Understanding of Fundamental Mathematics* as they focussed on the relationships among the four operations and on longitudinal coherence, while the American teachers had a fragmented understanding of school mathematics that missed important interconnections among different parts of it.

Carrillo and colleagues introduced a framework called *Mathematics Teachers' Specialized Knowledge* (MTSK) (e.g. Carrillo-Yañez et al., 2018; Carrillo, 2021). MTSK, which has been used extensively in recent CERMEs, draws on long-term cooperation between researchers and teachers working at different educational levels. MTSK structures knowledge in terms of *mathematical knowledge* and *pedagogical content knowledge*, which are supplemented with a category of beliefs. Mathematical knowledge is divided in the three sub-categories on topics, practices and structure of the subject and PCK consists of knowledge of teaching, of learning and of standards for learning mathematics. MTSK focuses "exclusively on the knowledge specific to the mathematics teacher" (Carrillo-Yañez et al., 2018, p. 237) and considers neither professional knowledge that may be shared with teachers of other subjects, nor whether parts of the knowledge in the model is shared with other professions. As Carrillo (2021) says, it "is the model in its entirety that is specialized" (p. 90).

Rowland et al. worked with prospective teachers and developed the *Knowledge Quartet* (KQ) as a perspective on the knowledge they need and on the situations in which they need it (Rowland et al., 2005; Rowland, 2008). The four parts of the KQ – foundation, transformation, connections and contingency – describe how the participants' content-related knowledge "contribute to their teaching during the […] the school-based placements" (Rowland et al., 2005, p. 256). In Rowland et al.'s terminology, *foundations* consist of knowledge and beliefs "possessed" and concern "knowledge, understanding and ready recourse to [teachers'] learning *in the academy* […] irrespective of whether it is being put to purposeful use" (Rowland et al., 2005, p. 260, emphasis in original). Transformation and connections are categories of "knowledge in action". *Transformations* describe how "knowledge possessed" needs to be transformed to be educationally powerful, and *connections* concern the coherence of mathematics, that is, how concepts and procedures may be linked and sequenced in education. The last element of the KQ, *contingency*, is "knowledge in interaction" and concerns teachers' capacity to deal with unexpected, content-related issues, for instance when students ask unexpected questions or make unforeseen conjectures. Contingency, then, includes being able to respond to students' ideas as they come up and the willingness and capacity to deviate from the original plan if necessary.

The most referenced perspective on mathematics teachers' knowledge is *Mathematical Knowledge for Teaching* (MKT), developed by Ball and her colleagues (e.g. Ball & Bass, 2000; Ball et al., 2008). Like the frameworks mentioned previously, MKT is an answer to the questions of what knowledge teachers need in instruction and whether and how they need to know it differently from others, who are also proficient with the subject. As mentioned before, Ball et al. build on Shulman, and one contribution of MKT is an elaboration of what may be entailed in PCK in mathematics. Another contribution is a distinction between common content knowledge, that is, knowledge of the contents that teachers share with others who know the subject, and a form of specialised knowledge of the contents itself, "mathematical knowledge that equip [teachers] to navigate […] complex mathematical transactions flexibly and sensitively with diverse students in different lessons" (Ball & Bass, 2000, p. 94). One aspect of specialised content knowledge (SCK) is to be able to "unpack" mathematics, that is, pulling apart compressed concepts and procedures in order to help students understand inherent meanings. SCK is required also if teachers are to deal with tasks such as "responding to students' 'why' questions", "linking representations to underlying ideas and to other representations", and "giving or evaluating mathematical explanations" (Ball et al., 2008, p. 400; the authors list 16 such mathematical tasks of teaching). It is an important aspect of the early work on MKT that SCK be understood in relative isolation from educational issues. It is "a kind of *mathematical* understanding that is pedagogically useful and ready, [but] not bundled in advance with other considerations of students or learning or pedagogy" (p. 88, emphasis added).

Over the last few years, Ball has focused increasingly on the work of teaching itself, rather than on a knowledge base on which it is expected to rest. She deplores that "[s]cholars were [previously] studying classrooms and analysing discourse, tasks, and interactions, but were not unpacking what is involved for the teacher in doing those things" (Ball, 2018, p. 14). This perspective challenges the connotation of knowledge stability, which is apparent in Ball's earlier work, and it is reflected also in a change of terminology, as Ball (like others) shifts her wording from knowledge to knowing and from teacher to teaching. The more dynamic understanding of the teacher-context relationship reflects the recognition that "teaching is co-constructed in classrooms through a dynamic interplay of relationships, situated in broad socio-political, historical, economic, cultural, community, and family environments" (p. 15). The shift of emphasis invites interpretations of teaching and teacher proficiency that align more closely with the social turn. They also seem to question the previous claim that mathematical knowledge is not "bundled" with considerations of students and pedagogy.

Looking across the frameworks on mathematics teachers' knowledge: professional emphases and constructivist inspiration

The frameworks presented above have contributed significantly to shifting our understandings of what teachers need to know by emphasising the professional and specialised aspects of teachers' knowledge. They differ in their empirical grounding as well as with regard to their interpretation of what *specialised* may mean; to the categories of knowledge they use; to whether beliefs are included or not; and to their relative emphasis on knowledge itself relative to the situations in which the categories become professionally relevant.

The frameworks, however, all build on Shulman's work from the 1980s, which, as mentioned earlier, may be seen as a response to process-product studies of teachers and teaching. There are two related aspects to this. First, process-product research had no interest in the contents of instruction. In line with Shulman, the four frameworks refocus attention to the contents. They also take Shulman's emphasis on professionalism a step further by suggesting that it is not only PCK, but also parts of teachers' knowledge of the subject matter itself that is special to teaching. Content knowledge is, then, not merely a matter of knowing mathematics, but a matter knowing what and how mathematics is used in teaching. As Ball and Bass (2002) say the knowledge teachers need "has features that are rooted in the mathematical demands of teaching itself" (p. 4). In this sense, there is a contextual element to the understandings of teachers' knowledge in all of them: knowledge of the contents is at least to some extent contextualised to the profession.

Second, Shulman's knowledge base for teaching was a response to the behaviourist stance of process-product studies. In contrast, he and his team focused on "teacher thinking, teacher knowledge, teacher planning, teacher decision-making, and teachers' conceptions of their subject matter and how that related to how they performed" (Shulman, 2015, p. 6). There is a cognitive emphasis and a focus on individual meaning-making in this (cf. Shulman & Shulman, 2004) that aligns with attempts in mathematics education at the time to initiate "a constructivist revolution […] to countermand the stranglehold that behaviorism had on the field" (Steffe, 2007, p. 281). This "revolution" brought with it new perspectives on students' learning and knowing, by arguing that knowledge and meaning are acquired and possessed individually, although possibly supported or constrained by the social situation. A similar interpretation seems to have guided the frameworks on mathematics teachers' knowledge, at least until fairly recently. Although they are different, the frameworks all originally carried connotations that knowledge is possessed by and resides in the individual, although enactment may be contextually framed. Until recently, they did not draw on other understandings of what it may mean to know.

Research on teachers' beliefs

Background and rationale of belief research

Research on teachers' beliefs also gained momentum in the 1980s, both in mathematics education and in other educational scholarship (Ashton, 2015; Skott, 2015c). There were two aspects to the background for the growing interest (Skott, 2015b). First, some researchers sought to understand classroom processes from the teachers' perspectives (e.g. Nespor, 1987). Elbaz (1983), for instance, explained her own intention of doing so by distancing herself from the general field of research on teachers:

> Too frequently the emphasis was on diagnosing teacher failings and on prescribing improvements, whereas I was interested in seeing and understanding the situation from the teacher's own perspective. (Elbaz, 1983, p. 4)

Second, there was an interest in solving "problems of implementation". The constructivist revolution coincided with a shift of emphasis from subject matter products to processes, in mathematics for instance from algorithms, definitions and proofs to problem solving and later to for instance reasoning and communication. The (sometimes implicit) question in belief research was – and still is – how one may expect this reform to materialise, if its priorities are not shared by the teachers. Therefore, much research has focused on (1) what teachers' believe about mathematics and its teaching and learning, sometimes focussed on particular mathematical topics (e.g. problem solving, probability, algebra) and supplemented with their beliefs about themselves as learners, teachers and doers of mathematics; (2) how beliefs may change; and (3) what the relationships are between beliefs and classroom practice.

Thompson (1984) was one of the first in mathematics to point to belief research as a possible challenge to the behaviourist underpinnings of most research on teachers and teaching at the time. She used the term *conceptions* about a union of teachers' "beliefs, views, and preferences" and argued that "[f]ailure to recognize the role that the teachers' conceptions might play in shaping their behavior is likely to result in misguided efforts to improve the quality of mathematics instruction" (p. 106). Based on a multiple case study of three American junior high school teachers, she addressed the questions of (1) whether there were "incongruities between the teachers' characteristic instructional behavior and their professed conceptions"; (2) how such incongruities may be explained; and (3) whether differences among the teachers may be accounted for with reference to their conceptions (p. 107).

Thompson's second research question suggests that she considered teachers' conceptions the default explanation for classroom practice, as any discrepancy between the two was in need of an explanation. Based on a similar premise, a tremendous amount of research has been conducted on teachers' beliefs ever since, both in mathematics education and beyond (e.g. Clark & Peterson, 1986; Conner & Singletary, 2021; Grossman et al., 1989; Hoffman & Seidel, 2015; Pajares, 1993; Richardsen, 2003; Thompson, 1992; Wilson & Cooney, 2002; Yurekli et al., 2020). In fact, the expectation of belief impact on practice has been the main raison d'être of the field, even to the extent that it has at times been turned into one of direct causality (e.g. Ernest, 1991; Schoenfeld, 1992).

However, belief research has turned out to be a complicated endeavour. Belief change, often seen as an affective counterpart to conceptual change, is difficult to accomplish (Gill & Hardin, 2015); beliefs are not readily observable, and it has proven difficult methodologically to elicit, infer or attribute beliefs, to teachers, based on more readily observable indicators (Abd-el-Khalick & Lederman, 2000; Philipp, 2007); and the congruity thesis has been challenged as much as confirmed in empirical studies (Fives & Buehl, 2012). This last difficulty has led to modifications of the initial emphasis on impact. Before discussing this issue further, I introduce one other challenge, the conceptual problems with the key construct of beliefs.

The concept of beliefs

As mentioned above, Thompson used the term of conceptions about a combination of beliefs, views and preferences. She did so without specifying differences between the three elements or relationships between the higher-level concept of conceptions and each of them. This is not uncommon in the field. As Pajares (1992) said, the construct of beliefs is "messy" and travels under alias. Mason (2004) made a list of related terms, indeed, one or more beginning with every letter of the alphabet – *a*ffect, *b*eliefs, *c*onceptions, …, *z*eal – and commented at the end: "No wonder it is hard to make sense of it all!" (p. 347). This terminological multiplicity goes hand in hand with conceptual confusion and there is little consensus about how *beliefs* – or many of the other terms on offer – may be defined and relate for instance to knowledge (Gill & Fives, 2015; Kagan, 1992; Philipp, 2007; Russ et al., 2016).

Beyond discussions of the concept of beliefs itself, considerable effort has gone into considerations of how beliefs are held and how they function (e.g. Cooney et al., 1998; Pajares, 1992). It is argued, for instance, that they may be unconscious and implicit (e.g. Buehl & Beck, 2015; Kagan, 1992; Rokeach, 1969). Green (1971) suggests that beliefs may have a mutual quasi-logical relationship, as they may be primary or derivative in relation to one another. Further, they may be central or peripheral in terms of psychological significance, and they are held in clusters that are internally coherent, but may be somewhat mutually isolated. As an example, one may consider possible quasi-logical relationships for a teacher among different beliefs about mathematics, students, learning, and about herself in relation to mathematics. One may also wonder what the relative significance is of these beliefs, whether and how they are clustered with each other or other professionally relevant beliefs, and what it matters for belief impact how they are held.

It has also been argued that different beliefs serve different functions. Based on a comprehensive literature review, Fives & Buehl (2012) argue that beliefs may filter information and be important for what a teacher pays attention to for instance in professional development (PD). Beliefs may also structure interpretations of situations and problems, for instance a problematic classroom situation. Finally, they may guide action, for instance when self-efficacy beliefs play a role in the teacher's perseverance in a classroom interaction.

In spite of the confusion about the concept, there does seem to be some consensus about how the term of beliefs is used. A core of the beliefs construct – as used in the literature – may be summed up in four features, namely that beliefs are (1) subjectively true and (2) affectively laden and that they (3) build on substantial prior experiences and (4) have some explanatory power in relation to action and meaning-making (Skott, 2015b). These four characteristics of how the term is used suggest, respectively, that the term is associated with significant degrees of conviction, commitment, stability and impact. The last two of these indicate that the notion of beliefs is an example of what Sfard (2008) calls objectifications, that is, reifications of processes and actions that take on a life of their own as they are interpreted as self-sustained entities with predictive power (Skott, 2015c).

Returning to the quandary of belief impact

As indicated earlier, one finding in belief research is that the impact of teachers' beliefs on classroom practice may not be as direct as initially expected. For the larger part of the field, however, beliefs are still considered the default explanation for classroom practice and as for Thompson (1984) an apparent lack of congruity calls for explanations. Sometimes explanations for lack of documented belief impact refer to methodological problems with accessing these elusive constructs or to the conceptual issues outlined above. Fives & Buehl (2012) argue that if different beliefs may serve different functions (filtering information, framing observations, and guiding action), it is hardly surprising that beliefs inferred from a teacher's response to a questionnaire differ from those accessed when observing how she acts in a possibly conflictual classroom situation. Others have argued that apparent lack of compatibility between espoused and enacted beliefs may reveal that the beliefs are held differently, for instance that beliefs enacted may be unconscious, but more centrally held than those espoused in a research interview (e.g. Cross Francis, 2015). Still others suggest that the problem is caused by methodological difficulties and may be solved by using more specific self-report items or more fine-grained methods of analysis (e.g. Speer, 2008; Yurekli et al., 2020). Finally, Leatham (2006) suggests viewing beliefs as "sensible systems", which – in a situation with apparent inconsistencies – requires the researcher to "look deeper, for we must have either misunderstood the implications of that belief, or some other belief took precedence in that particular situation" (p. 95). Leatham suggests an interpretive stance in which inconsistency is an observer's perspective that does not do justice to the complexities of teaching.

The approaches above come to the rescue of the premise of belief impact by referring to the conceptual or methodological problems of the research process itself. Others have pointed to more substantive issues by suggesting more dynamic interpretations of belief-practice relationships (cf. Skott, 2015b). Schoenfeld (2011) modifies his previous deterministic description of belief impact somewhat and suggests that classroom dynamics may require the teacher to reconsider his/her approach. However, subsuming beliefs under the broader heading of orientations, he maintains that the teacher's "routine and non-routine decision making can be fully characterized as a function of his [mathematics–related] resources, goals and orientations" (p. 13). Sztajn (2003) and Skott (2001) suggest that the significance of mathematics-related beliefs may be challenged and overruled by other educational concerns in a teaching-learning situation, and Lerman (2001) and Hoyles (1992) argue that beliefs are situated and that differences between for instance those espoused in a research interview and observed in a classroom interaction are different by virtue of those situations. The last two sets of approaches do away, respectively, with the premise that *mathematics*-related beliefs impact practice and the expectation of belief stability across contexts. However, they both use beliefs about objectified mental entities, although the significant beliefs in a particular situation may not relate to mathematics and may not be temporally and contextually stable.

One may regret the lack of an agreed-upon definition of beliefs and the methodological difficulties involved in researching them. These problems may be seen as an impediment to growth in the field and to the accumulation of research results. However, developments in belief research indicate that the conceptual core of the term as outlined previously suffices for it to function as what Blumer (1969) calls a sensitizing concept, that is, a concept that does not "provide descriptions of what to see, […] [but] merely suggest directions along which to look" (p. 148)[1]. In combination with the multiplicity of methodological approaches used in the field, this has led to more nuanced and multi-faceted understandings of affective aspects of teaching, including modifications to the field's own initial rationale, the expectation of semi-causal relationships between stable, mathematics-related beliefs and practice. There has, then, been a move from beliefs to dynamic affect systems (cf. Pepin & Roesken-Winther, 2015), a move that has also been apparent at CERMEs (Skott et al., 2018). As suggested above, however, these modifications do not seem to change what is meant by the term *beliefs*. The larger part of the field draws on the original constructivist underpinnings, and even when these are challenged by notions of situatedness (Hoyles, 1992; Lerman, 2001), the perspective on beliefs as objectified mental entities may persist.

Research on teachers' professional identities

Background and rationale of research on professional identities

Since the turn of the century, teacher identity has become a significant field in mathematics education, even though it has only played a minor role at CERMEs (Skott et al., 2018). As with studies of teachers' beliefs, there seem to be two mutually related and often combined aspects to the research interest: One is to understand the lives of prospective or practising teachers in view of cultural and social demands and affordances (e.g. Arslan et al., 2021; Beauchamp & Thomas, 2011; Brown & McNamara, 2011; Cochran-Smith et al., 2012; Hong, 2010; Lutovac, 2020; Skott, 2019); the other is to consider the character and development of identities as they relate to teachers' participation in teacher education or PD programmes (e.g. Bobis et al. 2020; Darragh and Radovic, 2019; Gresalfi & Cobb, 2011; Heyd-Metzuyanim, 2019; Hodgen & Askew, 2007; Horn et al., 2008; Jong, 2016; Ntow and Adler, 2019). Irrespective of which of these interests dominate a particular study, identity research moves beyond cognitive configurations such as knowledge and beliefs when seeking to understand teaching and teacher development. Hodgen and Askew's (2007) wording that their case study of a primary teacher in the UK is based on the premise that professional change "involves at least in part becoming a 'different' teacher and a 'different' person" (p. 474) seems indicative of much research on professional identity. Part of the background to the interest in identity, then, is to challenge

[1] I reluctantly refer to Blumer here. As we shall see later, his theoretical stance is at odds with those adopted in belief research and he argues explicitly against the significance attached to constructs like beliefs. In spite of that, I find the notion of sensitizing concepts useful as a metaphorical description how the term of beliefs has functioned in belief research.

purely cognitive or epistemic approaches and move towards more comprehensive understandings of teachers and teaching, including social and cultural perspectives on identity and identity development. This is reflected in discussions about the concept itself.

The concept of identity

Teacher identity has become a productive line of research, but there is no agreement about a theoretical stance in the field, let alone about a definition of the concept itself (e.g. Beijard et al., 2004; Darragh, 2016; Day et al., 2006; Lutovac & Kaasila, 2018). In fact, the concept of professional identity seems as “messy” as the one of beliefs (cf. the section on beliefs).

Based on her review of the literature, Darragh (2016) lists five different categories of frameworks used in identity research in mathematics education: participative (based on social practice theory), discursive, narrative, psychoanalytic, and performative (based on positioning theory). All of these have also been used in the subset of identity studies on teachers’ identities and sometimes combined in different ways (e.g. Brown & McNamara, 2011; Losano et al., 2018; Mosvold & Bjuland, 2016). However, the participative approach seems to dominate research on mathematics teachers’ professional identities (Lutovac & Kaasila, 2018).

One main difference between the frameworks on offer is the emphasis on local contexts relative to broader structural issues for the character and development of identities. In the first case, the main concern is with whether and how teachers negotiate and identify with normative identities in the locally social, for instance when engaged in PD. In the second case, structures and power relations beyond the current situation are from the beginning part of the conceptual framework.

Across the most frequently used approaches, however, professional identities are considered socially constituted, in either a local or more structural sense, and consequently considered multiple, dynamic and contextually dependent, rather than somewhat stable personality traits. This profound role of “the social” in identity research is evident also in the most frequently used theoretical imports used in the field. Wenger (1998), for instance, talks about identities as negotiated ways of being a person in a context. Looking back on her own work on situated learning, Lave (2019) says that one goal was “to parse a community’s day-today practice with respect to producing ‘old-timers’ from ‘newcomers’” (p. 138), and that the concept of identity was introduced “to insist that knowledgeable skill is only a small part” of that process (p. 137). Holland et al. (1998) say that identities “must be conceptualized as they develop in social practice” (p. 5). And Gee (2000-2001) uses identity about “being recognized as a certain kind of person in a given context” (p. 99). The recurrent references to contexts and practices in these wordings suggest that identities deal with the sense people make of themselves and each other at a particular time and place. They are conceived as individualities that are multiple and contextually dependent.

Aspects of the core of identity

There seem to be three common aspects to and perspectives on lived identities in the definitions above, those of agency, situatedness, and structure (Skott, 2019). In combination with the concern for lived individualities, this identity triad may be seen as the core of the identity concept (figure 1). Like the core of the beliefs construct, it functions as a sensitizing concept that does not determine what to look at, but suggests directions along which to look.

Agency is an individual or communal willingness and capacity to act within social worlds, to influence how they unfold (e.g. Holland et al., 1998). It does not carry connotations of an independent and autonomous actor; rather the premise is that

> [b]oth the immediate and broader social contexts orient teachers' actions, but do so in open-ended ways, leaving space for professional decision-making and agency. Experiences of this agentic space and of the outcomes of manoeuvring within it are aspects of identity. (Skott, 2019, p. 470)

Agency functions both when engaging actively in social practices and when being reluctant to do so. As an example of the latter situation, consider a teacher, who distances herself from the activities in the mathematics department at her school, because she experiences her colleagues as in opposition to the notion of quality instruction promoted by practices at her recent teacher education programme, which still functions an affectively-laden inspiration for her. Both her dissociation from the department and her affective and active affiliation with the teacher education programme are aspects of agency that may affect her professional experiences of herself at the school, that is, her professional identities.

The second core aspect of professional identity is that of situatedness. It acknowledges the local negotiation of the meanings of a practice, including the significance and use of artefacts, reifications and relationships. Such negotiation relates to identity as it both establishes and unfolds by means of mutually acknowledged positions that influence professional experiences. In the example above, the positions of the teacher in the department and at the school more generally are continually renegotiated, for instance positions of being an outsider, being elitist, or being a good mathematician. This negotiation may take place in department meetings, when having lunch in the staff room, and when establishing other – and possibly more productive – collaboration with colleagues beyond the department or with the leadership.

Finally, there is a structural aspect to identity. The way I use the term, *structure* concerns issues stemming from beyond the immediate situation and therefore subject to little agentic control, even if their meaning is negotiated locally. People, reifications, artefacts as well as networks of relationships among them may be subject to such structural influence. A dominant political discourse on teachers and teaching is in this sense structural. Schooling, as constituted globally and locally, is structural with its organisation within and between institutions, formal power relations and divisions of labour, and with the related formal qualification procedures and requirements, timetables, and assessment systems. These structural aspects position teachers and significantly influence lived experience in the profession. Consider again the example above. If a government decision is issued on the subject matter competence of teachers (e.g. all teachers need a Master's degree) or on the introduction of new mathematical contents (e.g. computational thinking) the teacher may be positioned differently at the school, leading to different experiences of herself as (not) valued, as (in-)competent, and possibly as redundant.

As we shall see later, the relative emphasis on and understandings of the relationships among the nodes of the identity triad differ between studies. But to some extent, most identity studies include all three, indicating a more fundamentally social understanding of the individual-context relationship than in research on teachers' knowledge and beliefs.

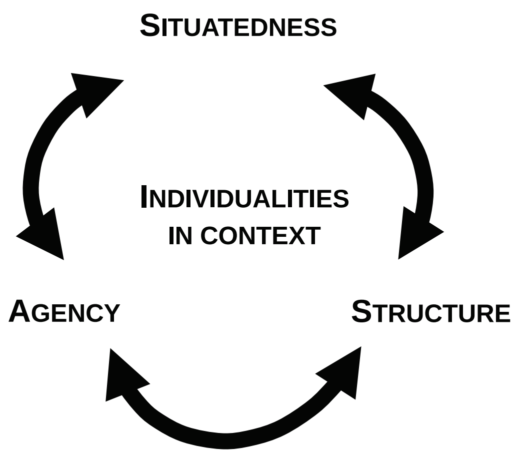

Figure 1: The identity triad: aspects of and perspectives on individualities in context

Summary – so far

It is time to sum up the discussion in order to address the questions of how research on teachers' knowledge, beliefs and identity have developed over the last few decades, in particular how individual-context relationships are conceived and whether and how developments align with "the social turn". I use the answers to set the stage for the following section.

The phrase of the social turn refers to a set of developments in mathematics education research beginning in the second half of the 1980s. This was at the peak of the constructivist revolution, but it was also the period in which the first references were made to theories and frameworks that challenged the exclusive emphasis on individual cognition and conceptualised learning and human functioning in more fundamentally social terms. Ever since, increasing numbers of studies have drawn on theoretical imports to mathematics education such as generations of cultural-historical activity theory (Engeström, 2001; Vygotsky, 1978, 1986; Wertsch, 1985), social practice theory (Holland & Lave, 2001; Holland et al., 1998; Lave, 1988, 2019; Lave & Wenger, 1991; Wenger, 1998, 2010), discourse analysis (Gee, 2000-2001, 2005), and positioning theory (Harré & Van Langenhove, 1999), as well as on Sfard's theory of commognition (Sfard, 2008). They have all fuelled "the strong version" of the social turn, as they describe "learning as development within socio-cultural historical practices and […] see meaning, thinking and reasoning as products of social activity" (Lerman, 2006, p. 172).

My argument so far is that research on teachers' knowledge and teachers' beliefs that developed in the wake of the constructivist revolution were decidedly cognitive in their approach. In both fields, the respective key constructs were considered teacher characteristics that functioned almost as independent variables in relation to instruction and classroom practice. These constructivist underpinnings still seem to inform these lines of research. However, contextuality has gradually been taken into account in both fields, also at CERMEs (Skott et al., 2018). The frameworks developed have explicitly phrased their understandings of teachers' knowledge with reference to the profession, and both fields increasingly consider social challenges to how knowledge and beliefs are enacted, that is, adopt a dynamic acquisitionist stance. This means that to some extent the initial expectation of a semi-causal relationship between context-independent knowledge and beliefs on the one hand and classroom practice on the other has been modified, and that context is considered a possible support to or limitation on the enactment of individuals' knowledge and beliefs. Recently, steps have been taken in research on "knowledge" to adopt more fundamentally social perspectives on human functioning, reflected for instance in the use of the gerund *knowing*. In belief research, the more dynamic understandings do not encompass a reformulation of what it means to believe, for instance if reflected in a similar shift from *beliefs* to *believing* or *affectively relating*. Beliefs are still considered relatively stable mental entities that are acquired and possessed by individuals. In general, then, these fields have not turned social.

The situation is different in research on professional identities. This field took off after references to sociology, anthropology, and cultural psychology had become commonplace in mathematics education research. The emphasis on situatedness and structure and the understandings of agency in identity research indicate that it differs from most research on teachers' knowledge and beliefs with regard to its stance on "the social". In identity research, "the social" is generally not considered merely a set of external constraints on the otherwise autonomous functioning of the individual. Rather, "the social", in the local as well as in the structural sense, significantly influences, and in some interpretations constitute professional identities.

The identity triad – a tool for characterising approaches to identity research

Different understandings of the relationships among the three aspects of the identity triad may serve to characterise approaches to identity research (Losano & Skott, submitted). Darragh and Radovic (2019), for instance, study the effects of long-term PD participation for a group of primary teachers in Chile. They explicitly distance themselves from approaches that do not sufficiently take "the wider social, cultural and political context" into consideration (p. 519). As they use the term, identities are discursive entities existing in a cultural realm beyond the individual and the local situation. Teachers select among and attach themselves to these cultural identities. With the terminology of the present paper, this is a highly structural perspective, and agency is a matter of selecting among predefined identities on offer in cultural worlds (cf. Losano & Skott, submitted).

A different perspective is offered by Westaway (2019). She emphasises agency when focusing on how the experienced teacher in her South African case study enacts her professionalism. This is done in ways that are supported and constrained by local and structural conditions. Not least in South Africa, structures and cultural mechanisms in the form of conflicting systemic roles stemming from the apartheid and post-apartheid eras "condition the way teachers express their […] teacher identities" (p. 484). However, Westaway distances herself from "systemic accounts" of identity, and in her study "the agency of the teacher is re-inserted into understanding why teachers continue to reproduce the old systemic roles of a teacher" (p. 490).

Finally, some research focuses on situatedness. This is often so in studies of whether and how a teacher education or PD programme supports teachers in moving from peripheral to more comprehensive participation in the reform (cf. Lave & Wenger, 1991). In this case,

> reform-oriented practices become the centre of attention [and] the trend is to prioritise a particular set of practices, those related to the PD or teacher education programme, and a related figured world, the reform. (Skott, 2018, p. 608)

Often, tensions between practices promoted by the PD in question and the ones that dominate teachers' school life are acknowledged. Gresalfi and Cobb (2011), for instance, argue that there are normative identities for teaching in both a school context and in a PD, and the question is whether and how participants identify with others' expectations in these contexts, that is, develop personal identities that align with the normative ones.

An interactionist complement to identity research

I suggest that there is a need for a complement to the three approaches to identity research mentioned above, notwithstanding the potentials of each of them. There are two reasons for this, one that relates to studies with significant emphases on agency or structure, and one that refers to studies that emphasise structure or situatedness. My arguments, then, refer less to the nodes of the identity triad, than to approaches located near two of the three axes between them. I refer to these approaches as variable oriented and community/society oriented, respectively. My argument is that in the first case there is a risk of not acknowledging the significance of the locally social that emerges for instance in a classroom or among colleagues at a school. In the second case, the risk is that, somewhat ironically, identity research may lose sight of the individual, as it focuses on pre-established social practices or structures. I suggest that what may be missing in both cases is an interactionist complement that focuses on the emergence of group life, and I argue that Patterns of Participation is one possible framework for doing so. In this section I introduce PoP before returning to possible limitations of the approaches located near the two axes mentioned above.

PoP revitalises symbolic interactionism (SI) when researching learning and lives in schools and classrooms and combines SI with social practice theory (e.g. Skott, 2013). SI seeks to understand the emergence of group life and focuses on how meaning evolves in the locally social when people interact with one another. Interaction, then, is not merely a term for people taking turns in a communicative setting; it is the process through which "people come to fit their activities to one another and to form their own individual conduct" (Blumer, 1969, p. 10). In this process, they instantaneously see themselves and the objects attended to from the perspectives of individual or generalised others and adjust their own contributions accordingly. A teacher may, for instance, see herself and the task or contents attended to in a particular situation through the eyes of the students and anticipate and interpret their verbal and physical reactions to her own conduct, for instance their tone of voice or their lifted eyebrows. She may also – to use a symbolic interactionist phrase – take the attitude to herself of generalised others in the form of practices and figured worlds beyond the current situation such as collaborative settings with colleagues, the discussion at a recent parent meeting, the reform as promoted in a

PD, or a different set of pedagogical concerns that may be unrelated to the contents of instruction (Skott, 2019).

The PoP perspective that I use, define teacher identities as the fluctuating experiences of being, becoming and belonging that evolve as teachers engage with their students, their colleagues, the leadership or others in relation to the profession. Such experiences may include being a good mathematician (or not), becoming recognised as an important adult by the students (or not), and belonging at the school or in the wider professional community (or not). In what follows, I use the identity triad to show how this perspective differs from the approaches to professional identity mentioned previously.

The agency-structure axis: Variable oriented approaches and an interactionist response

Studies that attach primary importance to agency or to structure do not necessarily disregard the role of situatedness for teacher identity. Indeed, Darragh and Radovic (2019) are explicit that teachers select a specific identity “in a particular context and for a particular audience” (p. 518). In spite of that, there is little attention to the role of the locally social and no concern for the possibility that unfolding, local events co-constitute identities, or for what makes the teacher select a particular identity “in a particular context and for a particular audience”. Similarly, but at the other end of the axis, an over-emphasis on agency may overlook aspects of identity that emerge in the locally social. In this sense, neither approach is particularly concerned with the negotiation of identities as teachers engage in local practices.

Studies that highlight agency and structure differ in their perspectives on identity, but in both there is, then, a risk of disregarding the significance of the locally social (see figure 2). If this is the case, they become examples of what Prus (1996) refers to as “variable oriented social science”, that is, approaches that “reduce the study of the human condition to ‘individual properties’ or ‘social structures’” (p. xviii). In contrast and in line with SI, Prus suggests that social science is to understand group life as it unfolds, and

> acknowledge and attend to the ongoing accomplishment of everyday life in the ‘here and now’ (while mindful of the evershifting present within people’s experiences with the past and their anticipations of the future). (Prus, 1996, p. xviii)

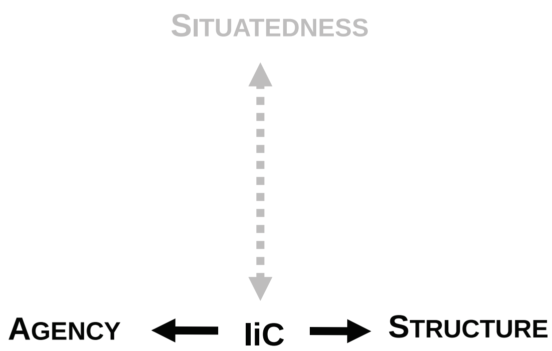

Figure 2: Variable-oriented approaches: *Individualities in Context* (IiC) near the agency-structure axis; little attention to situatedness

The interactionist perspective, then, takes the dynamic view of identities beyond merely being a matter of enacting identities (although possibly constrained by local or structural conditions) or selecting among a number of pre-given ones in some situation. It suggests that identity is itself an emergent phenomenon, "a moving target", one that may be in flux, for instance as classroom processes unfold (Losano & Skott, submitted). It invites a focus on processual identifyings, rather than identities, to avoid the objectifying connotations of the latter term. The criticism of variable-oriented approaches, then, is that they do not sufficiently pay attention to situatedness, that is, life as it emerges locally.

The structure-situatedness axis: Community/society oriented approaches and an interactionist response

Studies that highlight structure or focus on situatedness share their concern for the social constitution of identity and for pre-defined social structures or practices. They focus on what the situated-normative or structural identities on offer are in a particular situation. For ease of communication, I refer to these as community/society oriented.

Lutovac and Kaasila (2018) argue that such approaches may not do justice to key aspects of identity, as they lose sight of the individual. Similarly, I have argued that there is a need to re-centre the individual, rather than a particular practice, if the intention is to understand teacher identities as they evolve in interaction (Skott, 2018). From this perspective, community/society oriented approaches pay limited attention to agency, which is reduced to a matter of selecting among predefined identities on offer or choosing (or not) to identify with and move from peripheral to more comprehensive participation in a pre-established practice (figure 3).

This is at odds with the interactionist approach. As Blumer (1969) says, people act towards the meaning situations have for them, rather than towards structures, and meaning emerges in and from social interaction as an outcome of people taking (each) others' attitude to the situation at hand. As an example, Leticia Losano and I have argued in our study of an experienced, Brazilian primary teacher, that agency is located squarely in the locally social as an aspect of a person-practice interface (Losano & Skott, submitted). Agency is a matter of how the teacher combines and capitalises on the different attitudes she may take to herself in interaction, including how she actively renegotiates the meaning and significance of previous and anticipated future practices and figured worlds in view of interactions as they unfold at the instant. From this perspective, professional identities are not viewed as determined by pre-given structures or practices, but as dynamic experiences of being, becoming and belonging that evolve as teachers participate in group life.

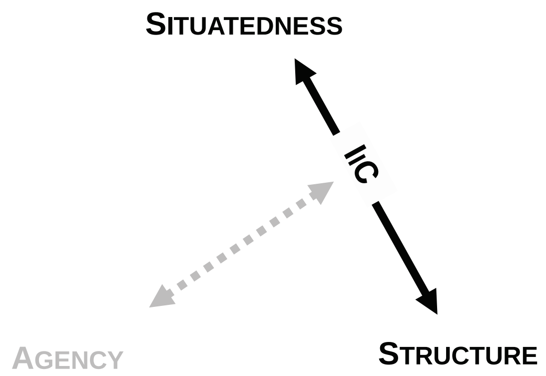

Figure 3: Community/society-oriented approaches: *Individualities in Context* (IiC) near the structure-situatedness axis; possibly with limited attention to agency

The agency-situatedness axis: The location of the interactionist approach

The suggested interactionist complement to other approaches to identity is located along the agency-situatedness axis, that is, an axis oriented towards emerging individualities in the locally social. Neither the variable oriented, nor the community/society oriented approach disregard the aspect of identity located outside their main axes, although these aspects play a minor role. In a somewhat similar sense, approaches located on the locally-social axis, including the interactionist approach, do not disregard structural issues. However, it is not an a priori decision that these issues play a role, but empirical questions whether, how and why this is so. The PoP framework, with its interactionist complement to the other approaches, is based on networking social practice theory and symbolic interactionism (e.g. Skott, 2013, 2018, 2019). The intentions with PoP include developing dynamic and contextual understandings of teachers and teaching, including their professional identities. The attempt is, then, to re-centre the individual, while maintaining the participatory stance of most studies of identity and as part of that to shift the emphasis from identity to fluctuating identifyings (figure 4).

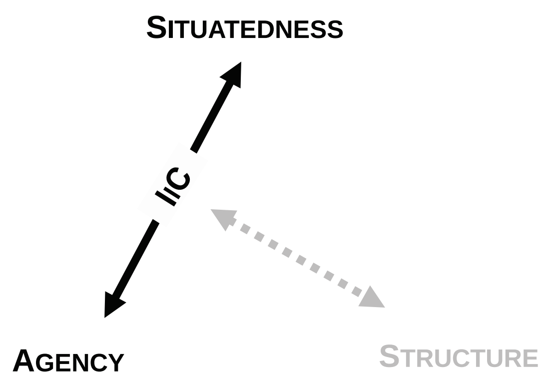

Figure 4: Locally-social approaches: *Individualities in Context* (IiC) near the situatedness-agency axis; acknowledge structure if empirically justified

Extending the participatory stance to studies of teachers' 'knowledge' and 'beliefs'

It was a main point in my discussion of research on teachers' knowledge and beliefs that these fields have "turned social" to a lesser extent than research on professional identity, and, indeed, to a lesser extent than most mathematics education research. Recently, suggestions have been made to do away with the acquisitionist underpinnings of research on teachers' knowledge, but the field does not seem to have moved far in that direction (yet?). To the extent that beliefs and knowledge are still considered objectifications, that is, reified mental entities residing within the individual with substantial impact on practice, agency is a matter of enacting such objectifications, and structure and situatedness are merely considered external constraints. Interpreted in the terms of the identity triad, research on teachers' knowledge and beliefs are located close to the agency node, and there is little in these fields that corresponds to the move from identities to fluctuating identifyings in identity research.

This invites the question of whether it is helpful for understandings how teachers contribute to classroom practice and student learning to shift the emphasis from *Mathematical Knowledge for Teaching* to *Mathematical Knowings When Teaching* and from *Beliefs about mathematics and its teaching and learning* to *Affectively relating to mathematics and students in classrooms* (cf. the section 'Summary – so far'). I suggest that the answer is in the affirmative and that the PoP framework is one possible way of doing so. PoP was initially developed as a participatory challenge to mainstream belief research, and I have argued elsewhere that it is a useful alternative (Skott, 2013, 2015a). It differs significantly from the cognitive underpinnings of belief research and sheds light on the emergent character of how teachers relate affectively to mathematics, to their students' learning of mathematics and to the acts of teaching as classroom processes unfold.

So far little has been done with PoP in relation to teachers' knowings when teaching. My recent study with Despina Potari and Chara Papakanderaki suggests that it may also have some potential in that field, that is, for understanding how teachers' ways of knowing relate to interactions as they unfold for instance in a classroom (Skott et al., submitted). This study brings to the fore relationships between agency, situatedness and structure in the case of an experienced and highly qualified Greek teacher, Elena. In the course of the study, Elena moves from a somewhat traditional secondary school in Athens to an "experimental and model school" that is well known for developing innovative approaches to teaching. The data are from two teaching-learning sequences on functions in grade 10, one from each school. Although they are on the same contents and both taught by Elena, the two sequences are very different. The students' ways of participating differ and so do the mathematical objects of attention, even the object of function. Also, Elena's experiences of belonging differ, and so do the ways of knowing required of her, in particular how she deals with relations between informal and different aspects of formal mathematics. The point in the present context, then, is that it is not only aspects of Elena's professional identifyings that change in and through the interactions; so do her ways of knowing the contents.

Summary and conclusion

Research on and with mathematics teachers has over the last 40 years taken the professional tasks of teaching still more seriously and acknowledged that teaching is relational work conducted in local situations and conditioned by broader structural contexts. After having challenged the behaviourist approach in process-product studies as well as the emphasis on academic mathematics in early mathematics education scholarship, our field changed the emphasis towards (1) understanding teachers' thinking and meaning making as they relate to the profession, with special emphasis on mathematical knowledge for teaching (not necessarily in the MKT sense) and (2) the beliefs teachers hold about the subject and about its teaching and learning. To some extent this takes the social situation of schools and classrooms into account. After the turn of the century, more fundamentally social approaches have been used in the study of teachers' professional identities.

The outline above suggests that research on and with mathematics teachers in general turned social later than other fields of mathematics education, and it may be argued that research on teachers' knowledge and beliefs still has not done so to any great extent. The question I have addressed in the present paper, however, is not merely whether or not specific subfields have turned social, but what it may mean to do so.

To address this last question, I used the identity triad to locate different approaches to research on and with mathematics teachers, and argued that neither research on teachers' knowledge, teachers' beliefs nor teacher identity has adopted an interactionist approach to understanding the emergence of group life. I suggest that this may be needed, if the ambitions of research on and with teachers include understanding the contextual meanings they make of learning and lives in schools and classrooms.

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Big Questions in Mathematics Education: A Panel Discussion

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Three researchers, Anna Baccaglini-Frank, Ingi Højsted and Janka Medová, were invited to present and discuss their proposals for a research agenda for mathematics education over the next 20 years in a panel discussion chaired by Michiel Veldhuis and moderated by Eirini Geraniou.

In this paper, Anna Baccaglini-Frank summarizes the perspective that she presented, as leader of TWG24 "Representations in Mathematics Teaching and Learning", bringing attention to a shift in how groups of researchers in Mathematics Education think and talk about mathematical objects and their representations. She argues that such a shift has theoretical and practical implications that should be taken into account in future research, especially when addressing some "Big Questions in Mathematics Education".

In this paper Ingi Højsted outlines central points from his presentation at the plenary panel "Big Questions in Mathematics Education" of CERME12. He begins by briefly recounting the historical struggles with implementation of digital tools in mathematics education, and refers to the importance of how digital tools are utilized. To provide an example of the nuances and complexity of the research involved on digital technology in mathematics education, he focuses on the specific digital tool of dynamic geometry environments and describes pertinent dimensions of research foci in relation to this software: student learning, task design, the teacher. He considers some of the major challenges with regards to the implementation of dynamic geometry environments as well as other digital resources in mathematics education. Finally, he reiterates the three broad questions posed for the plenary panel debate.

This paper summarizes Janka Medová’s contribution to the plenary panel discussion Big Questions in Mathematics Education and focuses on the role of algorithms in mathematics education. Mathematics and computer science are interrelated from their substance. They share several common concepts including the algorithms, but the way they work with them differ among the two disciplines. Algorithms provide an additional dimension to mathematical knowledge, the deep procedural knowledge. Involvement of coding and work with algorithms in mathematics instruction seems to be a promising activity bridging the two disciplinary approaches. The questions: (i) how the work with algorithms contributes to students’ learning and (ii) how to prepare teachers for this kind of activities should be investigated.

Big Questions in Mathematics Education: A “Representations in Mathematics Teaching and Learning” Perspective

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I summarize the perspective that I presented, as leader of TWG24 “Representations in Mathematics Teaching and Learning”, bringing attention to a shift in how groups of researchers in Mathematics Education think and talk about mathematical objects and their representations. I argue that such a shift has theoretical and practical implications that should be taken into account in future research, especially when addressing some “Big Questions in Mathematics Education”.

Keywords: Digital extension, mathematical objects, representations.

A shift in how we think about mathematical objects and their representations

In a recent paper (Baccaglini-Frank et al., 2022), we discussed what we see as a shift away from a Platonic conception of mathematical objects and their representations, characterizing various current lines of research in Mathematics Education. Such a shift opens new venues to how we think and talk about representations, which, in turn, influences how we use (and study) them in the teaching and learning of mathematics. In Plato’s allegory of the cave, representations of mathematical objects are seen as reflections of natural things; such reflections inspire artificial objects, of which we only get to see shadows; such shadows are the imperfect forms that allow indirect access to the “real” perfect objects behind them. An implication is that whoever seeks mathematical knowledge should strive to obtain *mental representations* as close as possible to the ideal (non-physical) forms. The Platonic philosophical stance is at the basis of much research on mathematical representations. Many theories on learning mathematics claim the importance of understanding mathematical objects by somehow tapping on their “true” meanings that can be constructed by abstracting from their representations. However, researchers in Mathematics education, including members of the CERME12 community, have also started exploring different theoretical perspectives on the meanings and representations of mathematical objects (e.g., Miragliotta & Lisarelli, 2022; Palatnik & Abrahamson, 2022).

A very important shift away from the Platonic perspective has been initiated by Anna Sfard in her Commognitive Framework (2008). Taking a Vygotskian socio-constructivist perspective, and following Wittgenstein, Sfard sees mathematical objects as no longer residing in some hyper-reality, but in discourse itself, being part of an autopoietic system, a system that defines its own objects. Hence, their meanings stem from the ways in which realizations of a mathematical object are used discursively. An implication of such a shift is that the term “representation” is inappropriate: Sfard rejects the Platonic view of mathematical objects existing “out there” and being re-presented in discourse; rather, for her, mathematical objects “come to life” as part of a discourse of certain human communities.

Another perspective, supported by cumulative data from various fields (neurobiology, robotics, kinesiology) is casting doubt on the Platonic view, and in particular on its implication that bodily experiences are separate from the ideal “mental representations” discussed above. Indeed, the “embodied” turn in cognitive science rejects the hierarchical mind–body separation and stresses that perception and action are formatively constitutive of our thinking. In the Mathematics Education field, the embodied paradigm has been taken to suggest that learning new concepts begins with discovering new ways to act in the environment, using new instruments to perform tasks on discovered affordances (e.g., Abrahamson & Bakker, 2016). Working with the things themselves, students develop a capacity to act efficiently; they learn to describe the world mathematically to coordinate collaborative actions; they iteratively encounter more complex problems; and ultimately they modify the environments to solve emergent problems (Abrahamson & Abdu, 2021).

A specific interest of mine concerns learning mathematics with or through digital tools. Turning away from the Platonic perspective, these can be conceived as "extensions" of our mind-and-body selves, and we can explore new ways of thinking, talking and using representations of mathematical objects. This is a line opened by posthuman discourse that describes the blending of human and technology as a "triumphant overcoming" the "natural" limitations of the human body, leading to the fascinating notion of "digital extension" (see the discussion on Merleau-Ponty in Dolezal, 2020).

My three questions

Based on the shift discussed above, on discussions in TWG24 over past meetings, and on my personal research interests, I am particularly interested in thinking about the three following questions.

Question 1: ***How can we (and will we) produce and share representations/realizations of mathematical objects to make teaching-learning processes truly inclusive?*** Indeed, producing and sharing representations/realizations of mathematical objects are fundamental processes to consider and study in the context of inclusive mathematics education. Research in this domain has highlighted the importance of using multi-modal channels of communication, perhaps also supported by Artificial Intelligence artifacts (e.g., Lew & Baccaglini-Frank, 2021). Given the new perspectives on what representations/realizations of mathematical objects might be, we should explore ways to share and appropriate others' thoughts and personal experiences with such representations.

Question 2: ***How does (and will) learning occur (a) through making representations/realizations of mathematical objects or (b) through making artifacts that make these?*** A very interesting (to me) direction of research has been opened by research on "learning as making", in a constructionist perspective, where, for example, shapes in space are constructed using a 3D pen or a 3D printer (e.g., Ng & Sinclair, 2018; Ng & Tsang, 2021), or sketches of figures are produced on the plane by drawing robots that can be programmed by young children using a graphical block coding language (e.g., Baccaglini-Frank et al., 2020; Baccaglini-Frank & Mariotti, 2022).

Question 3: Finally, since learning experiences can be very different and involve many different factors, I ask***: How can we capture and study students' experiences with representations/realizations more holistically?*** I ask this because in most of the research studies I am familiar with, we attend to only a small part of the "whole picture", focusing, for example, either on cognitive aspects or affective ones, or on a certain small "bit", of a student's larger and more complex interaction with an artifact. However, I believe that it would be beneficial to have analytical tools that allow us to see more of the bigger picture. Perhaps we could work harder on trying to integrate results from studies that each looked at a small bit, but that together can provide new insights into students' learning processes.

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Big Questions for Research on Digital Technology in Mathematics Education

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This paper outlines central points from my presentation at the plenary panel "Big Questions in Mathematics Education" of CERME12. I begin by briefly recounting the historical struggles with implementation of digital tools in mathematics education, and refer to the importance of how digital tools are utilized. To provide an example of the nuances and complexity of the research involved on digital technology in mathematics education, I focus on the specific digital tool of dynamic geometry environments and describe pertinent dimensions of research foci in relation to this software: student learning, task design, the teacher. I consider some of the major challenges with regards to the implementation of dynamic geometry environments as well as other digital resources in mathematics education. Finally, I reiterate the three broad questions posed for the plenary panel debate.

Keywords: Digital technology, implementation, DGE, the teacher dimension.

Looking back

Since the introduction of digital technology in mathematics education, extensive research has been conducted in relation to the implementation of different types of digital tools. In fact, already in the mid-eighties, the very first ICMI study (Churchhouse et al., 1986) considered the consequences computers and informatics might have on mathematics and mathematics education, and at that time, researchers were optimistic. However, two decades later, in the 17th ICMI study, Michelle Artigue looked back and reported that "The situation is not so brilliant and no one would claim that the expectations expressed at the time of the first study have been fulfilled" (Artigue, 2009, p. 464). During his CERME11 plenary speech, Paul Drijvers reflected on the low effect of integration of digital technology in mathematics education, acknowledging that "the mathematics education community is still struggling with the integration of digital technology in teaching and learning." (Drijvers, 2019, p. 8), while suggesting that the quality and exploitation of the digital tools are essential ingredients for effective integration. The manner in which digital technologies are utilized seems to be essential (Drijvers, 2019; Højsted & Mariotti, 2021; Jankvist & Misfeldt, 2015). In fact, Niss (2016) proposes that "the very same piece of digital technology can give rise to 'marvels' as well as to 'disasters' in mathematics education. This means that no ICT system, hard or soft, is, in and of itself, good or bad for mathematics education." (p. 247).

Of course, there are many different types of software, and many different software of the same type. If we take the paradigmatic example of dynamic geometry environments (DGE), which has received a lot of research attention, more than 40 DGE had been developed by 2012 (Hollebrands & Lee, 2012). Initially, much of the DGE research focused on students learning with DGEs, for example, Arzarello and colleagues' (2002) seminal work that categorized seven spontaneous ways in which students drag objects in DGE. Another example is Mariotti's (2015) elaboration of the semiotic potential of the dragging tool to introduce conditional statements.

More recently, research attention has shifted towards task design, with authors describing DGE task design principles for specific mathematical aims, for example non-constructability tasks and dependency tasks in relation to conjecture generation and reasoning (Baccaglini-Frank et al., 2017; Højsted & Mariotti, 2021), or developing frameworks to assess task quality (e.g. Trocki & Hollebrands, 2018).

Less research has been conducted in relation to the role of the teacher in facilitating the implementation of DGE (Komatsu & Jones, 2018). The same applies to other digital tools, as described by Traglová and colleagues (2018), who reviewed the research output on digital technology in mathematics education coming from CERME conferences since 1999, "the awareness of the importance of the teacher dimension in research on technology in mathematics education [...] emerged slowly" (p. 154). Although there are several salient contributions on the role of the teacher (e.g., Bartolini-Bussi & Mariotti, 2008; Drijvers et al. 2010) it is my impression, that this dimension still needs the most research attention. Suggesting the same in her PME plenary, Ana Sacristán proposed that teachers are the key players for the successful implementation of digital technologies, calling for "more teacher involvement in both professional development, and as co-constructors and collaborators in the design of technological implementations and resources" (Sacristan, 2017, p. 90).

Indeed, any designed digital mathematical resource may be adopted by their users (teachers and students) to suit their perceived needs in a particular classroom context (e.g., Trgalová & Rousson, 2017). The appropriation process may lead to a use of the resource that is not coherent with the educational intentions of the designer.

In a recent study (Højsted & Mariotti, in press), we found that collaborating with teachers about theoretical aspects of technology implementation can be a complex affair. This intricate issue requires reflecting on how to interface with teachers effectively to accomplish specific educational goals, while taking into account the variety of possible pedagogical paradigms that different teachers may adopt – paradigms that may well be implicit.

Looking forward

Even if much has been accomplished in research on digital technology in mathematics education, it is evident that successful integration of digital technology into teaching and learning mathematics remains a difficult and complex issue – akin to a gordian knot. Therefore, looking forward, I suggest that we reflect on these broad questions in the plenary debate:

1. How can we ensure that in another two decades, we are not still disappointed?
2. Which are the main issues hindering successful implementation of digital technologies in mathematics education and what are the solutions? (The teacher dimension? Design of resources?)
3. How can solid research findings on digital technologies in mathematics education find its way into praxis?

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Algorithms in mathematics education

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This paper summarizes my contribution to the plenary panel discussion Big Questions in Mathematics Education and focuses on the role of algorithms in mathematics education. Mathematics and computer science are interrelated from their substance. They share several common concepts including the algorithms, but the way they work with them differ among the two disciplines. Algorithms provide an additional dimension to mathematical knowledge, the deep procedural knowledge. Involvement of coding and work with algorithms in mathematics instruction seems to be a promising activity bridging the two disciplinary approaches. The questions: (i) how the work with algorithms contributes to students' learning and (ii) how to prepare teachers for this kind of activities should be investigated.

Keywords: Algorithms; mathematics education.

Mathematics and computer science are interrelated since the inchoative stages. Various fields of mathematics, including Boolean algebra, algebraic structures, discrete mathematics and probability provided the theoretical base for the emerging field of computer science. Some years later, in 1976, computer science started to pay off its debt by solving the problem of four colors using the thousands of hours of computing time (Appel and Haken, 1978). The nature of mathematics and natural sciences as they are practiced in the professional world is developing towards computational thinking (Weintrop et al., 2016). Nowadays mathematicians perceive computing as an inherent part of doing mathematics (Lockwood et al., 2019). The use of computers allows processing the calculations in mathematical statistics, even the work with big data.

Algorithms lie on the border between mathematics and computer science. Each discipline looks at algorithms from a different point of view. Computer scientists (e.g., Wirth, 1985) define algorithms as finite, general, deterministic, resultative and elementary which has only limited use in mathematical proofs. Mathematicians investigate the existence and correctness of algorithms, whereas computer scientists look at the algorithms in a more utilitarian way, looking for the real-time instantiations, preferring obtaining the result to its preciseness. Therefore, heuristics were developed for solving problems with high complexity.

Algorithms occur in curricula of both mathematics and computer science. Algorithms for basic mathematical operations were the traditional part of mathematics education. In recent years the algorithms have started to disappear from mathematics curricula as conceptual understanding was in main focus of mathematics educators and procedures were often perceived as the 'rote learning'. On the other hand, Star (2007) sees the particular value in the procedural knowledge itself. Star foregrounds the deep procedural knowledge where the procedure is "known deeply, flexibly, and with critical judgment" (p. 133). Deep knowledge of algorithms, its principles and correctness might be seen as an amalgam between conceptual understanding and procedural fluency, so-called precept (Gray & Tall, 1994). Some studies (Lockwood & De Chenne, 2021) demonstrated the added value involving coding in students' mathematical problem solving. Furthermore, using coding for solving mathematical problems may strengthen the computational thinking of the learners. The questions about how knowledge about algorithms contributes to students' learning of particular mathematical concepts and procedures and how can students profit from generalization of the mathematical procedures in an algorithmic way remain open.

If we want to approach algorithms from both sides, mathematics and computer science, we should decide who should teach about algorithms. There are countries, e.g. Slovakia, where algorithms and programming have had a strong position in curricula since the 1980s, as a part of computer science education. In other countries, e.g., France, algorithms are part of mathematical curricula. Both possibilities have their strengths and weaknesses. Teachers, specialists in mathematics or computer science focus on different characteristics of algorithms, in accordance with the disciplinary practices. Kortenkamp (this proceedings) describe several types of activities usual for work with algorithms in computer science: (i) design of algorithms, (ii) describing the algorithm in a (formal) language; (iii) carrying out algorithms, (iv) proving the correctness of algorithms, and (v) comparing algorithms with respect to complexity, elegance, or simplicity. These activities can be seen also in mathematics lessons and may serve as a bridge between the two disciplinary practices.

The teachers are the agents of any educational change (Kieran et al., 2013) and their work is shaped by their resources, orientations and goals (Schoenfeld, 2010). How the mathematics teachers' predispositions influence their practice regarding the algorithms, what is the knowledge needed to teach algorithms as deep procedural knowledge and whether mathematics teachers see any value in algorithmic approaches to mathematical objects should be examined in further studies.

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TWG01: Argumentation and proof

Introduction to Thematic Working Group 1: Argumentation and Proof

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Introduction

Argumentation and proof continue to attract the growing and wide interest of the mathematics education research community, which was also evident in CERME 12. In Thematic Working Group 1 (TWG01; “Argumentation and proof”), 39 participants from 15 different countries from 5 continents actively engaged in the discussion of the 30 papers and 3 posters. CERME 12 was the first to be organized online; a challenge and an opportunity to investigate ways of practicing and promoting communication, cooperation, and collaboration. For this purpose, the work of TWG01 was organized in both parallel split-group sessions and whole-group sessions, aimed to maximize active participation and to ensure the coherence and unity of the TWG01 identity in the subgroups. Moreover, the online format of the conference proved to be a great opportunity to organize a joint session with Thematic Working Group 9 (TWG09; “Mathematics and Language”). The session allowed the discussion of topics of the common interest of both TWGs, allowing for our practicing the ERME’s spirit of communicating, cooperating and collaborating beyond the conceptual boundaries of our TWG.

The papers of TWG01 were organized in seven themes, which were presented and discussed in split-group or whole group sessions: 1) Theoretical and epistemological perspectives about argumentation and proof, 2) National and international perspectives about argumentation and proof, 3) Argumentation and proof in primary mathematics education, 4) Argumentation and proof in school and university mathematics, 5) Argumentation and proof in teacher education, 6) Argumentation and proof beyond mathematics text and context, and 7) Argumentation, Language and Proof (this was the theme of the joint session with TWG09).

In this introduction chapter, the papers are presented and discussed in three broader topics, in line with the main themes elicited in the discussions of our group: a) Argumentation and proof in school and university, b) Theoretical, epistemological and sociocultural perspectives about argumentation and proof, and c) Argumentation and proof in different texts and contexts.

Discussion of Papers

Argumentation and proof in school and university

In the TWG01 Introduction section of the CERME 11 proceedings, it was noted that one of the areas that the TWG01 participants "would like, and hope, to see more research in future CERMEs was: *The teaching of proof and argumentation* in both school and university settings, including in teacher education with particular emphasis on argumentation and proof at the elementary school level." In CERME 12, the topic of several papers appeared to address this, with a particular focus on primary education. The presented papers investigated diverse aspects of the primary school students' argumentation and proof, including the students' conceptions and understandings about proof (as presented in the papers of Sigrid Iversen and of Merve Dilberoğlu, Erdinç Çakıroğlu and Çiğdem Haser), designs to support the students' proving skills and in highlighting the empirical-deductive gap (as discussed in the papers of Melanie Platz, of Jo Knox and Igor' Kontorovich). Moreover, we had the opportunity to discuss different aspects of reasoning that occur at these educational levels: data-based argumentation as investigated in the paper of Jens Krummenauer and Sebastian Kuntze, and Simone Jablonski's paper about mathematical reasoning outside of the classroom. Furthermore, we discussed topics specific to higher educational levels, including a paper about proofs without words at the secondary education level by Nadav Marco, Alik Palatnik and Baruch Schwarz, as well as Katharina Kirsten's paper about the proving strategies employed by first year university students.

Considering teacher education, a central issue concerned the importance of working with future and in-service teachers to explore ways of efficiently incorporating research findings about argumentation and proving in teacher training and professional development programs. Such efforts were evident in the papers of Orly Buchbinder and Sharon McCrone and of Gabriel Chun-Yeung Lee. At the same time, the papers of Thomas Bauer and Eva Müller-Hill and of Fiene Bredow and Christine Knipping allowed us to gain deeper insight on the teachers' practices through the lenses of different theoretical perspectives. Moreover, Lakatosian ideas were at the crux of three papers concentrating on teachers (the papers of Mei Yang, Andreas Stylianides and Mateja Jamnik, and of Dimitrios Deslis, Andreas Stylianides and Mateja Jamnik) and on teacher educators (as discussed in the paper of Magdalini Lada and Tore Alexander Forbregd).

Overall, the presented papers offered the opportunity for rich discussions about the commonalities and specificities of teaching and learning argumentation and proving at the different educational levels (with a special focus on primary education), as well as about the appropriateness of the respective research approaches. At the same time, the discussion about the implementation of Lakatosian ideas in teacher education raised fruitful deliberations about the way that ideas that have been developed in a specific sociocultural context may (or may not) be applied to educational research, which leads to the second broader topic of the papers discussed in TWG01.

Theoretical, epistemological and sociocultural perspectives about argumentation and proof

The meaning(s) of proof, its relationships with the validity of mathematical knowledge and with the notion of truth, are important epistemological issues that are constantly being re-visited in the TWG01 meetings; question certainties helps to enrich, broaden and alter perspectives. Along these lines, Viviane Durand-Guerrier discussed the dialectical relationships between truth and proof, while the discussion was enriched by Vergnauds' ideas as employed in Nadia Azrou's paper and by Habermas' rationality in the paper of Paolo Boero. Moreover, the role of logic and deductive reasoning is at the crux of argumentation and proof and, hence, in this CERME, logic was again a central theme of our discussions in TWG01. Miglena Asenova's paper, challenged the traditional perspective of classical logic and set-theoretical assumptions, while we had the opportunity to consider the role of unitizing predicates as presented in the paper of Paul Christian Dawkins and Kyeong Hah Roh, as well as the role of deductive reasoning in word problems as investigated in the paper of Rimas Norvaiša.

Considering the complexity of the argumentation and proof phenomena, the discussions about epistemological and theoretical perspectives were explicitly linked with the role of the sociocultural aspects in the teaching and learning of argumentation and proof. For this purpose, we focused on the diverse perspectives and realizations of assessment in different countries (for example, Chile, Hungary, Norway), drawing upon the papers of Kinga Szűcs and of Manuel Goizueta, Constanza Ledermann and Helena Montenegro. Furthermore, the paper of David Reid broadened the discussion to include 'reasoning' in the national curricula and standards in several countries. Moreover, the paper of Karolína Mottlová and Jana Slezáková offered an insight of implementing ideas of the curriculum of one country to another (respectively, from Singapore to Czech Republic), focusing on word problems. At the same time, language appears to be a crucial factor in mastering the logical structure of proofs, as discussed by Kerstin Hein. The latter issues about language are also explicitly linked to the third broad topic of the papers discussed in TWG01.

Argumentation and proof in different texts and contexts

Argumentation and proof have been traditionally linked with language and verbal communication, but in CERME 12 the participants draw the attention to broader conceptualizations of text, including non-verbal and multimodal aspects. Within this context, we discussed the explanation norms expanded to include explanation videos and the explanation norms, as presented in the paper of Jessica Kunsteller. Moreover, in the last CERME, the participants' discussions involved various aspects of the issues of language in argumentation and proof, while it was noted that it might be sensible to address this complexity in collaboration with colleagues of TWG09 and linguists. In this CERME, we addressed this issue by having a joint session with TWG09, where we had the opportunity to discuss language, argumentation and proof. For this purpose, we initiated the inter-TWG collaboration by critically focusing on Toulmin's scheme, which has been a staple tool for analyzing argumentation. The various implementations and extensions to the Toulmin's model were synthesized in the paper of Jenny Cramer and Leander Kempen, while the paper of Andreas Moutsios-Rentzos explicitly acknowledged multimodality in the discussion about argumentation and text. Two papers of colleagues of TWG09 (of Christoph Körner and Michael Meyer, and of Jorge Toro and Walter Castro, which may be found in the TWG09 part of the CERME 12 proceedings) allowed us to reflect upon the commonalities and differences of the two groups with respect to language issues in argumentation and proof.

Furthermore, mathematics is at the crux of the modern scientific, non-mathematical texts, such as physics. The participants of TWG01 investigated aspects of argumentation and proof in historical and physics texts, as presented in the paper of Laura Branchetti, Alessia Cattabriga, Olivia Levrini and Sara Satanassi.

Conclusions and Future Directions for TWG01

We argue that CERME12, the first to be conducted online, offered the participants the opportunity to be engaged in rich, broad and deep discussions about a variety of issues and perspectives. Importantly, we noted a valuable mix of a continuity of topics from previous CERMEs with novel ideas. A series of questions were posed that paint potential paths of future research projects.

Considering the *teaching of argumentation and proof*, we noted the tensions amongst research, intentions and actuality in everyday teaching and the importance of findings ways to bridge the potential divides. Furthermore, we ponder how can we make the teaching of argumentation and proof feasible for the different grades, curricular, educational and sociocultural settings? Should it be incorporated in everyday teaching practices across mathematical contents or should there also be a dedicated section to specific argumentation and proof practices (for example, about logic)?

Moreover, drawing upon the fact that in the modern curricula mathematics is present in non-mathematical courses, we identified a need for investigating *interdisciplinary perspectives about argumentation and proving* in texts, textbooks, and teaching practices.

The rapid technological advances appear to crucially affect the established communication norms and modalities, as they become part of the everyday teaching. Within this context, conceptualizing and investigating *language issues related to argumentation and proof* seems to need to be re-visited, and broadened to include non-verbal (multimodal, embodied, affective etc) and/or implicit aspects.

The *sociocultural aspects of argumentation and proof* seem to be another area of interest that transcends various perspectives, including the implementation of specific theoretical and/or epistemological perspectives in different context (to the one that the perspective originated), as they may entail both cultural and cognitive dimensions.

All these areas of interest intersect in complex ways and derive from this TWG01 meetings and should not be interpreted as prioritizing specific lines of research over others. We are aware that proof and argumentation are approached from different perspectives (and in other TWGs groups) and in TWG01 we are committed to voicing and exploring this diversity.

Questioning the Exclusivity of Classical Logic and Set-Theoretic Assumptions in Analysis of Classroom Argumentation and Proof

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The paper highlights the necessity to question the exclusivity of classical logic, or of approaches that are reducible to it, in the analysis of classroom proof and argumentation processes, as well as the role of the set-theoretic language as intrinsically linked to classical logic. Two examples drawn from mathematics classroom are analysed, recurring to the Ancient Indian empiricist Nyaya logic and to Peirce's non-standard quantification, associating the last to a "free logic", not axiomatizable within an axiomatic system where the specification axiom applies.

Keywords: Logic, Non-standard quantification, Nyaya, Free logic, Set-theoretic language.

Introduction.

The kind of logic students spontaneously resort to when they conjecture, argue or proof in mathematics classroom is often difficult to capture with the formal instruments of propositional logic (Barrier et al., 2009). Some scholars propose natural deduction for First Order Logic (FOL) as useful to reduce the distance between non deductive argumentation schemes and mathematical proof because of the possibility it offers to work on objects rather than on properties (Durand-Guerrier, 2005). To capture reasoning in Mathematics Education (ME) also Hintikka's dialogical logic, in reference to game theoretic semantic, is studied (e.g., Arzarello & Soldano, 2019; Blossier et al., 2009). What is common to these approaches is that they all are classical[1] or are reducible to the classical one.[2] Now, as Lindström's theorems shows, classical logic is intrinsically connected to set-theoretic language (Zalamea, 2021). In classical FOL the variables the quantifiers refer to, range on sets that represent the domains of the predicates. One of the fundamental axioms of set theory[3] is the axiom of specification: given a set A and a formula φ(x), there exists a subset B={a ∈ A: φ(a)}. This axiom is based on Frege's symmetry principle according to which one obtains "an equivalence […] (locally, within the restricted universe A) between φ(a) (intensionality) and a ∈ B (extensionality)" (Zalamea, 2009/2012, p. 324). If this axiom fails, both the law of the excluded middle (thus classical logic) and the standard use of quantifiers fail, because it is not guaranteed that a property univocally determines a set. From the other hand, the domain of reference of the statements during a learning process evolves over time and to grasp this evolution, sets should become "variable" (Lawvere & Rosebgough, 2003). Such sets can be captured by topoi in intuitionistic logic, considering an

[1] Classical logic is the logic where the law of the excluded middle ($A \vee \neg A$) and the law of non-contradiction ($\neg(A \wedge \neg A)$ hold, while non-classical logics are logics where at least one of these two characteristic properties does not hold. Examples of non-classical logics are the paraconsistent logic (the principle of non-contradiction holds only locally but not globally) and the intuitionistic logic (the law of the excluded middle does not hold and consequently also the double negation does not mean in general an assertion: $\neg\neg A \nvdash A$).

[2] The concept of truth in the game-theoretic semantic is different from the classical one (it is based on the logical existence of a choice function that guarantees the existence of a winning strategy for one of the players, called the verifier), but it can be shown that this truth concept is equivalent to the Tarskian one (Arzarello & Soldano, 2019) and thus to the truth conception in classical bivalent logic that follows the Aristotelian tradition.

[3] I refer to the Zermelo-Fraenkel axiomatic system with the axiom of choice (ZFC), as it is the standard axiomatic set–theoretic system within which mathematics usually is developed.

evolution over time, but not by classical sets. Classical sets, and thus classical logic, could be considered as special cases where time collapses into a moment and sets become fix.

Summing up, since classical FOL is exactly tailored to capture classical set theory, restriction to set-theoretical language may not allow different kinds of rationalities, that need to give up some of the constrains of set-theory, to be recognised, and thus it prevents also the investigation of possible shifts between "non-standard" and classical rationalities. Indeed, such "epistemic" rationalities require to consider indeterminacy about the properties that hold or do not hold for an object. I argue, recurring to examples, that non-classical approaches to logic and quantification, which don't require set-theoretic assumptions, could be able to put into evidence these aspects in the analysis of reasoning in mathematics classroom. In this way, (at least) novices' reasoning in ME, even if it does not match to classical logic, could be recognized as *knowledge* within a suitable rationality frame (Boero, 2017), rather than as a lack of rationality.

Theoretical framework.

Nyaya and empiric rationality. In the Western mathematical tradition, the Aristotelian syllogism represents the basis of logical reasoning and for mathematical proofs only the deductive syllogistic inferences are accepted. On the other hand, D'Amore (2005) shows that when dealing with proof, novice students might spontaneously resort to a type of logic very different from the Aristotelian one– the Indian Nyaya logic, a pragmatic and empiricist logic, linked to perception. In the Nyaya induction and deduction are closely interconnected within its "syllogism". Furthermore, the use of examples is not only permitted but is expected by the argumentative model itself and the "formal" and "material" aspects are closely intertwined in it (Sharma, 1962, p. 186), for the inferential model itself is conceived as a proof process of truth. According to D'Amore (2005), the Indian Nyaya philosophical school (1st century BC) recognizes a pre-eminent importance to four *means of knowledge*: testimony, analogy, perception and inference. The inference is what can be considered the Nyaya "syllogism" and has the following structure: (1) the Assertion (what one wants to prove); (2) the Reason; (3) the Thesis (a general proposition followed by an example); (4) the Application; (5) the Conclusion. Finally, one of the fallacies of the "right reasoning" in Nyaya is reasoning on non-existent objects.

Peirce's non-standard quantification. In ME also non-standard quantification, that cannot be framed within classical FOL, is epistemologically accounted (Blossier et al, 2009), with the aim to explain difficulties in managing quantification in classical sense at tertiary level or in the shift from secondary to tertiary level. These authors show that expert students (at tertiary level) spontaneously use different kinds of quantification that often involves temporal aspects and a kind of variation of the variables that often do not fit with the $\exists\forall$-variation as it is known after the introduction of the axiom of choice. They mention within the non-standard approaches to quantification Bolzano's (link between constant and variable quantities) and Cauchy's (link between variable quantity and fixed limit) ones, but they also account for the Peircean one, putting into evidence that it does not rest on logical distinctions but is "inner to the individuum" (Blossier et al., p. 84). I will deepen this last non-standard approach.

According to Peirce, quantification can be general, vague, or precise. Peirce calls generality, vagueness, and determination "the three affections of terms, [which] form a group dividing a category of what Kant calls 'functions of judgment'" (Peirce, CP, 5.450)[4]. *Generality* means absence of distinction of individuals rather than validity for every individual, as it is the case for the classical

[4] Peirce's Collected Papers (CP) are quoted in the usual way: (Peirce, CP, volume number.paragrph number).

universal quantifier that quantifies over sets of individuals; it can be expressed by words like *any*, *whatever*, etc. *Vagueness* means a certain type of existence that does not break the absence of distinction of individuals, but states that there are suitable generic individuals that satisfy a certain property; it can be expressed by words like *some*, *certain*, etc. It is similar to the classical existential quantifier but while the genericity of the latter rests on the proof of independence from the choice of a specific individual, the former rests on the *knowledge of the possibility* to choose individuals that remain indistinct, without a real actualization. *Precision* means effective actualization of possibility; the precise individual represents a rupture of the relationality that distinguishes the vagueness. As Hintikka's logic also the Peircean one is a dialogic logic with a game-theoretic semantic (Pietarinen, 2019), but Peirce's logic is epistemic in a different manner as Hintikka's one. Indeed, as Zalamea (2021) shows, Peirce's logic can be captured by sheaf-logic and sheaf-logic is intuitionistic. Thus, quantification in Peirce's logic does not require the axiom of specification and the symmetry of Frege's abstraction principle fails in general. Furthermore, according to Hintikka (2001) intuitionistic logic is truly epistemic because the crucial notion in it: "is not *knowing that*, but *knowing what* (*which, who, where, …*), in brief, *knowing* + an indirect question, that is, knowledge of objects rather than knowledge of truth" (p. 10) and this *knowing-what*-logic "cannot be analysed in terms of *knowing that* plus the apparatus of received first order logic" (p. 11).

The Nyaya logic is an example of an empiricist logic where reasoning applies on single objects, considered as "existent" by the subject; Peirce's logic with its non-standard quantification can be considered as an example of *free logic,* where the domain the quantifiers range over is not necessarily a closed set but "the class of existing things" (Nolt, 2021). In this sense, these two approaches are compatible and can be combined, at least at the basic level considered in this context.

Methodology of research.

A hermeneutical approach to the text analysis (Palmer, 1969; Bagni, 2009) is adopted. In this approach, the procedure consists in a dialectical back and forth between the meaning of the single parts of a text (oral, written etc.) and its global sense, in a meaning-increasing dialectical interpretation. The begin of the interpretation is always based on the interpreter's presuppositions about the original context of the analysed text (cultural, historical, etc.). The concept of *personal space* (Brown, 1996) of the protagonists (students and teacher) is used to frame the researcher's presuppositions in entering the analysis of the classroom excerpts and in searching for a global meaning, going from the part (examples) to the whole (discussions and conclusions) and vice versa. According to Brown, the *personal space* is the (virtual) space where "an individual sees him or her self acting" (p. 120); it is made by all the aspects, interests, constraints and means that inform the subject's acting in a context and is a source for meaning because "the individual acts in the world he or she imagines to exist" (p. 121). Here it mirrors the students' and teacher's background, inferred by the cultural context they are merged in while making mathematical statements or orchestrating mathematical classroom activities.

Data analysis and discussion.

Example 1: Empiric rationality, Nyaya, and non-standard quantification.

In this section an argumentative text produced by a 15-year-old high school student is discussed. S/he should answer the question: Is it true that *Each number that ends with the digit 1 is a prime number (that means without divisors different from 1 and the number itself) or it is divisible by 3*? The teacher's approach is Aristotelian and her and the student's personal spaces are inferred from information provided by the researcher that collected the data[5]. They are framed by the personal backgrounds (professional and formative), as well as by the classroom context.

In the analysis (Figure 1) classical Aristotelian and Nyaya-lenses are adopted: student's words are marked in **black bold**; the classification based on the Nyaya-scheme in green; the interpretation based on the Nyaya rationality frame in orange and the one within the Aristotelian frame in blue[6].

Assertion. **It is true,**

Statement whose truth is intended to be proven, as well as the expression of an intuition / Expression of an intuition.

Reason. **in fact, if we consider 11, 21, 31, 41, 51, 61, 71, etc. we see that they are all prime numbers or numbers divisible by 3.**

Reason why the statement is true. / Examples.

-------- **On the other hand, it is possible to reason also in general terms.**

Introduction of the general statement / Introduction of the generalization.

Thesis 1. **If a number is prime and greater than 2, it ends with an odd digit**

Shows the existence of the object he is talking about.

Thesis 2. **As for 3, among its multiples there are 21, 51, 81, 111, 141, etc.,**

Shows the existence of the object by showing generic examples.

Reverses thesis and hypothesis.

Application 1. **(and 1 is just an odd digit).**

Application 2. **all numbers ending on the right with the number 1.**

It is shown that the conditions of the Reason are fulfilled. / Wrong reasoning due to the inversion between thesis and hypothesis; tries to prove the inverse statement.

Conclusion. **The thesis has been proven.**

It has been "proven" that given any number that ends with 1, it is either prime or is divisible by 3. / He "proved" the hypothesis.

Figure 1: Interpretation of student's argumentation resorting to the Nyaya approach and to the classical Aristotelian approach

Discussion of example 1.

The student's *personal space* is characterized here by: (1) the experience of the concept of proof in Euclidean geometry; (2) some first explicit information about how a proof is made (thesis, hypothesis, general reasoning, no use of examples); (3) some elements of set-theoretic language in reference to number sets, without deepening of quantification; (4) the interest in showing the own ability (the student was firmly convinced that her/his proof is a good one and s/he wants to prove the truth of the Assertion); (5) the constraint that the text is addressed to the teacher. The teacher's *personal space* is framed at least by the following elements: (I) a valid proof starts from the hypothesis and ends with the thesis; (II) proof is deductive and the use of examples means induction; (III) her spontaneous, implicit, or explicit, use of set-theoretic language as object *language* in mathematical contexts, due

[5] The analysed text was produced with research purposes completely different from the present one; thanks to prof. Paolo Boero from the University of Genoa for having authorized its use for this alternative analysis and for helping to detect the information that was needed to reconstruct the teacher's and the student's personal spaces. We know that the teacher graduated in the mid-1990s at the University of Genoa by a five-years graduation program in Mathematics. In this context, proof is based on classical Aristotelian approach and the object language is always set-theoretic.

[6] The Thesis and the Application of the Nyaya scheme are divided in two parts and the examples are missing.

to her mathematical *forma mentis*. Both *personal spaces* are framed by the assumption that one "uses language in much the same way as everyone else" (Schulz, as quoted by Brown, 1996, p. 121).

The fact that the student has not recognized that the statement is false does not matter; the focus is on her/his reasoning. From the teacher's "classical" point of view, the basis of the student's reasoning could be summed up as follows: The student tries to show that there is a partition of the set of numbers ending with 1 in two subsets: the set A, containing the prime numbers greater than 2 ending with 1, and the set B, containing the multiples of 3 ending with 1. However, s/he does nothing but show that the set of prime numbers ending with 1 is a subset of the set of numbers ending with an odd digit and that there are multiples of 3 ending with 1. Of course, in this way s/he has not proved the existence of the supposed partition, but only the fact that there are two non-empty subsets of the sets A and B, reversing so thesis and hypothesis. Let us now eliminate references to sets in the mathematical sense, that do not belong to everyday reasoning: thinking of the number 3 does not necessarily mean thinking of it as a natural or as a rational number, but as an "object" in itself, in the same way as one thinks of a cup not as an element of the set of all cups, but as an object that falls under the senses.

We see that the student lists some numbers ending with 1, followed by ellipsis, as if this list were to continue. The mathematically shaped thought might interpret this list as the representation of an infinite set. But this list is not necessarily an infinite set in actual sense; it represents probably indeterminacy or vagueness in Peirce's sense or, at most, potential infinity. Indeed, if the student reasons in terms of numerical sets s/he should now try to prove the existence of the supposed partition and s/he does not. But if s/he *does not* reason in terms of numerical sets, what could s/he try to prove? Maybe that given *any* number that ends with 1, *that* number is prime or is a multiple of 3. This reasoning is based on an interpretation of "each" (the universal quantifier) in the sense of "any", that has no meaning in classical FOL but means *generality* in Peirce's sense. The student considers the first numbers listed as *random* cases (any) and finds that they have the required characteristics. This is a not valid generalization both in classical and in Peirce's sense. What the student has shown is that there exist *some* numbers that satisfy the property and so s/he would be able only to quantify recurring to a *vague* existence. This reasoning produces a sort of "fake" generalization by induction. The student knows that the generalization by induction on single cases is not allowed and that s/he must produce a reasoning with general validity (the text is addressed to the teacher). What could *mean* in the student's *personal space* "reasoning that applies in general"? S/he seems simply to produce an *existence* proof, s/he shows that the object being discussed *actually exists* in the sense of the Nyaya logic, and that it is *precise* in Peirce's sense: there are primes (greater than 2) ending with 1 and there are multiples of 3 ending with 1. But the proof is different in the two cases. In the first case s/he shows that the numbers whose existence she wants to prove are a special case of other numbers, "defining" them by next genus (numbers ending with an odd digit) and specific difference (which end with 1). In the second case the proof of existence is made by bringing *examples*. However, s/he does not simply bring examples in the common sense because s/he does not reason on *particular* multiples of 3, but on *some multiples chosen by chance* (they are *vague* in Peirce's sense). To sum up, there seems to be a lack of distinction of vagueness (seen as randomness) and generality (seen as indeterminacy) in Peircean sense. To bridge the gap between every-day-rationality within an empiricist logic (Nyaya) and mathematical rationality, the awareness of this distinction seems to be a necessary condition. Furthermore, the truth concept in the empiricist logic that fits to student's reasoning, seems to be closer to the idea of existence (precise or vague), rather than to the one of generality.

Example 2: Quantification within “blurred” domains.

The second example refers to a classroom argumentation led by the same teacher in another classroom. A worksheet with the argumentation discussed in example 1 is used to show that the proof is not valid. First, the teacher asks to tell if the proof is valid, but the students’ attention is captured by the semantical aspects: they detect two counterexamples (121 and 91) and state that it is false. The teacher brings the attention back to validity by asking what the reasoning on the worksheet is.[7]

9	Student 5:	The reasoning is that the multiples of 3 and the prime numbers end with 1.
10	Student 4:	No, that SOME multiple of 3 and SOME prime numbers end with 1. […]
17	Student 8:	Maybe you want to say that … that for CERTAIN prime numbers or multiples of 3 things are going well because they end with 1, but this doesn’t mean … […]
19	Student 3:	Yes, the reasoning says only that SOME prime numbers or multiples of 3 end with 1.
20	Student 9:	Even, although if ALL prime numbers or multiples of 3 should end with 1, there could be numbers that end with 1 and ARE NOT prime numbers or multiples of 3.
21	Student 6:	It is as if there is a reversal!
22	Teacher:	S6 said something important: “it is as if there is a reversal”. It is an important idea!
23	Student 1:	The hypothesis and the thesis?
24	Student 6:	It seems to me to be of a different matter!
25	Student 4:	To me too, it is a matter … of numbers. Of sets of different numbers. […]
29	Student 9:	I will try to say it again, I don’t know if it is OK: the multiples of 3 and the prime numbers are POSSIBLE numbers that end with 1, but these POSSIBLE numbers do not mean that they are ALL the numbers that end with 1.
30	Teacher:	I would say that’s it.

Discussion of example 2.

In this example the argumentation is carried out by a group of students. Nevertheless, one can state that the elements (1), (2), (3) and (5) of the student’s personal space in example 1 are also elements of the personal spaces of these students because the cultural and formative backgrounds are the same. The element (4) of the student’s personal space in the example 1 is substituted by the following one: (4’) uncertainty about what validity means in a proof and how it can be accessed, beside by bringing of counter examples. This topic is addressed for the first time in this lesson. The teacher’s personal space is the same described in example 1 with the following addition: (IV) intention to focus the discussion on the lack of validity due to a reversal of thesis and hypothesis.[8]

Most of the punctuated words in the transcript are related to quantification but apart from the line 9, the statements show students’ struggle with the determination of the domain of validity of the reasoning expressed on the worksheet and of its relation to the domain of the inverse statement which would be a valid one. The non-standard quantification used by the students express the indeterminacy of that domain: SOME, CERTAIN, NOT ALL, POSSIBLE numbers. For instance, as Student 4 (line 10) sums up the reasoning on the worksheet, s/he uses the term *some* as vague existential quantifier in Peirce’s sense because s/he knows that there are such numbers (the argumentation on the worksheet tells it) but their multitude is indeterminate; s/he is not able to “close” epistemically a set with this property. In line 22 the teacher supports Student 6’s intuition (line 21) that there is a reversal, meaning that the thesis and the hypothesis are reversed, as suggested by Student 1 (line 23). But the students’

[7] In the excerpt we use CAPITAL LETTERS for punctuated words and “…” for pauses longer than 5 seconds.

[8] The points (4’) and (IV) are based on a communication made by the researcher that collected the data.

intuition is not a matter of hypothesis and thesis, it is a matter of "numbers", of "sets of different numbers" (lines 24 and 25): There are numbers that satisfy thesis and hypothesis but also numbers that satisfy only the thesis but not the hypothesis. Thus, the inverse statement of the statement to be proved is not a valid inference. This is quite more than what the teacher wanted to put into evidence (reversal of thesis and hypothesis) although it is logically equivalent to it. As in example 1, students' quantification is suitably captured by the Peircean approach that expresses the epistemic uncertainty as vagueness related to variable sets, but unlike in the example 1, the argumentation produces an insight compatible with the teacher's one, related to classical logic. Thus, an investigation about shifts between different logical frames would be useful to better frame the logical analysis.

Conclusions.

According to the hermeneutical approach, the interpretation of the students' behaviour in the examples is meaningful within the global analysis (discussion) and vice versa. Going on in the interpretation, the analysis shows that students spontaneously resort to non-standard logics and non-standard quantification in Peirce's style and that these kinds of quantification and logic allow to formulate an argumentation that explains in a *reliable way* the lack of validity of a proof resorting to blurred domains, not considered within set-theoretic language. In this sense, further research should examine the shifts between different logical frames and the role of the relation between metalanguage and mathematical object-language not only in mathematics (Asenova, 2019), but also in ME. Furthermore, one can state that: (i) The novice's concept of truth might be related to the concept of *existence* of the objects involved in the statement and not to a predicate that it might satisfy: A statement is true if the objects involved in it actually exist; this kind of existence could be "proven" on different levels: by showing one or more "exemplars" with the required characteristics; by referring to single objects as to randomly chosen examples, in a sort of genericity; by referring to a characterisation of the object by a definition by comparison and contrast; (ii) The concept of "reasoning that applies in general" might be related for the student to the *production of a procedure* of a proof of existence, rather than to reasoning that applies to all cases and therefore to no one in particular. All these aspects join some of the students' most recurrent difficulties concerning proof (Stylianides & Stylianides, 2017) and emerged thanks to the non-standard approaches in the analysis.

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Framing students' difficulties with the concept of proof, with reference to Vergnaud's Conceptual Fields Theory

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By considering proof as a concept, according to Vergnaud's Conceptual Fields Theory, I would like to show how this framing of proof allows describing and interpreting some students' difficulties with proof at the meta-mathematical level. After motivating the choice of the framework and presenting it, I have considered examples of students' proofs, which showed difficulties regarding the mastery of operational invariants, of situations of reference and of the different representations of the concept of proof.

Keywords: Meta-knowledge of proof, conceptual fields, students' difficulties with proof.

Introduction

Researchers have identified several types of students' difficulties with proof and proving; in particular difficulties related to: mathematical logic (Selden & Selden, 1995); language (Boero, Douek, & Ferrari, 2008); writing the proof text (Selden & Selden, 2007; Azrou & Khelladi, 2019); understanding of concepts (Dreyfus, 1999); theorems and definitions (Moore, 1994); strategies of proving (Weber, 2001); argumentation (Balacheff, 1988) and conceptions of proof and related proof schemes (Harel & Sowder, 1998). Also many other researchers dealt with some of the above issues.

I am interested in investigating a type of students' difficulties, which intertwines with some of the above categories (in particular, those related to: conceptions of proof, writing the final proof text, theorems and definitions). My interest concerns students' knowledge of the proof at the meta-level (the logical features of proof, the role of the hypotheses, the requirements for proof texts, etc.). This type of difficulties is related to one of the main differences between students' and mathematicians' proving. Mathematicians would use some aspects of that meta-knowledge when they explain a proof to another mathematician or when they write a proof on the blackboard for students, however most of it remains implicit and it is not the object of any teaching in many university courses of mathematics. In fact, meta-knowledge of proof is so obvious to mathematicians that they are not aware that it might be unseen by students when proofs are written on the blackboard or when students read proofs in mathematics textbooks.

In the literature, the shift to the meta-mathematical level in order to interpret some students' difficulties with proving is proposed by several authors. In particular, Antonini and Mariotti (2008) provided a model to study the proof by contradiction and stated that 'some of the difficulties highlighted in the literature (…) can be described and interpreted in terms of the complexity that the move from the theoretical level to the meta-theoretical level requires' (p. 7).

Turiano and Boero (2019) considered a proof task in Euclidean geometry submitted to 10^{th} grade students, for which an easy proof by absurd (more precisely, by contradiction: see Antonini & Mariotti, 2008) is possible. Students showed difficulties on two points: the fact that the figure (representing the negation of the thesis) was impossible (to see and to imagine) and the managing of the proof by contradiction at the meta-level. Usually, when mathematicians choose the proof by absurd as a way to prove a theorem, they follow *in a rather automatic way* the rule of the game: starting by negating the thesis, in spite of the fact that they are convinced that it is true! Expert mathematicians work like a high school student does, when she applies the distributive property in arithmetic. Vergnaud's notion of "theorem-in-action" in his Theory of Conceptual Fields accounts for this kind of mastery of the distributive property, as a property used many times without necessary reflecting on its origin or interpretation, or on its truth in other situations. These considerations brought me to consider this research problem: Is it possible to move from the level of ordinary concepts (like those of the conceptual fields of additive structures and of the multiplicative structures, considered in Vergnaud, 1990), to the meta-mathematical level by considering proof, theorem, definition as concepts (according to Vergnaud's theory) and, in particular, by trying to describe and interpret some students' difficulties with proving in terms of lack (or misuse) of theorems-in-action for the concept of proof? Moreover: might some students' difficulties be described and interpreted in terms of lack of reference situations for the concept of proof, or lack of mastery of some operational invariants for it (like the negation of a statement), or poor linguistic representation of proof – as results from the analysis of university students' difficulties with proof-text writing in Azrou and Khelladi (2019)? This paper presents a first, partial attempt in this direction.

The idea of considering proof as a concept (according to Vergnaud's Conceptual Fields Theory) has been suggested to me by the elaboration of Durand-Guerrier, Boero, Douek, Epp and Tanguay (2012) on the roots of the mastery of proof, where they considered the use of theorems-in-action[1] in order to promote students' conceptualization of logical principles. Then, I have thought of theorems-in-action for the concept of proof, and I have hypothesized that in order to be competent in proving, students should not only master the definitions and properties related to the mathematical concepts involved in it, but they should also master the meta-knowledge of proof, by mastering proof as a "concept" (according to the Conceptual Fields Theory). The aim of this paper is to show how the extension of Vergnaud's framework to the concept of proof (even if rather limited in this moment) allows to interpret and to classify some student's difficulties.

Theoretical framework

Vergnaud (1990) considers a concept as a triplet of three sets: C= (S, I, $\mathcal{S}$), where:

S: the set of the different situations, which give sense to the concept (the reference).

I: the set of the operational invariants, which are the basis of any operation related to the concept (the meaning).

$\mathcal{S}$: the set of linguistic and non linguistic representations of the concept.

[1] A theorem-in-action is a mathematical property that a person may not be consciously aware of but may use in certain situations, such as to find an answer to a mathematical question. However, because the property may not apply to all the situations in which the person might try to use it, it could lead to an invalid deduction (Durand-Guerrier et al., 2012, p. 377).

For the concept of proof, I have defined its related three sets, respectively, as:

S: the set of the reference situations of the proof (situations of proof in geometry, in Algebra, in arithmetic, etc, different types of proof).

I: the set of the operational invariants (related to the logical structure of proof).

$\mathcal{S}$: the set of the different representations (linguistic; symbolic; geometrical, etc).

As concerns the operational invariants, Vergnaud (1990) considers three types:

- Invariants of propositional type (e.g. a theorem-in-action), which might be true or false, like the fact of considering that if a number of objects is multiplied by 2, 3, …then their price is also multiplied by 2, 3, … (which, is expressed by the theorem-in-action f (nx) = nf (x)). A theorem-in- action may be true or false, depending on the assumptions (e.g. the axioms) and the situation: as an example, many high school students think that every continuous function is derivable, according to the functions usually dealt with by them, and sometimes they even use it without making it explicit;

- Invariants of propositional function type: they are not susceptible to be true or false, they are concepts in action, like the two-arguments propositional function R (x,y) "… is on the right of …"), reported in Vergnaud (1990, p. 143), and the three arguments propositional function "Triangle A'B'C' is symmetrical to triangle ABC in relation to line d" (with reference to the corresponding drawing), reported in Vergnaud (2009, p. 91);

- Invariants of argument type: they are obtained, for instance, by the instantiation of the variables x and y in a propositional functions R (x,y); or in the case: "to be greater than" is an anti-symmetrical relation (Vergnaud, 1990, p.144); or in the case: Symmetry conserves lengths and angles (Vergnaud, 2009, p. 91).

The following examples account for a partial solution of the problem to move from concepts at the mathematical level, to concepts at the meta-mathematical level. I give examples for the three types of invariants for the concept of proof, which result from analogy with the three types of invariants considered by Vergnaud (1990; 2009).

- Examples of invariants of propositional type: a counter example for a general statement is sufficient to prove that the statement is false; to prove that A⇒B it is sufficient to prove that not B⇒not A.

- Examples of invariants of propositional function type: the implication $A \Rightarrow B$, the equivalence $A \Leftrightarrow B$, the negation of A.

- Examples of invariants of argument type: "n is an odd number" is the negation of "n is an even number".

And here is an example of a theorem-in-action about the concept of proof that is false in general but may be true in some situations: if A⇒B then not A ⇒ not B.

I consider that mastering meta-knowledge of proof is mastering it as a concept (with its components, according to Vergnaud' theory). The aim of this paper is to show, in particular and with relatively easy cases, how the partial framing of meta-knowledge of proof, which I have developed till now, works when we would like to analyze students' proof texts and how:

- It is suitable to interpret some students' difficulties with proof in term of the three components of the concept of proof (reference situations, operational invariants and linguistic representations).

- It may be used to distinguish between the sources of two kinds of difficulties put into evidence in the literature: difficulties deriving from the didactic contract (particularly those related to the lack of, or to inappropriate, reference situations); and difficulties inherent in the lack of knowledge of basic notions related to the logical aspects of proof (related to the lack of operational invariants).

The course and the participants

The following examples have been taken from written productions of fourth year students from an Italian university, who attended a 24-hour compulsory course on argumentation and related teaching and learning issues. In the next year, students would take their master's degree, which is necessary to become primary school teachers in Italy. In high school, all students have met mathematical proof in geometry, where "met" means they have been taught several proofs and have been required to learn them and to expose some of them at the blackboard. Usually, they have also met some proofs in arithmetic (e.g., the classical proof of the irrationality of $\sqrt{2}$, or the Euclid's proof of the infinity of natural numbers). Students have never met proofs in previous mathematics and mathematics education courses at the university (only some justifications of the "why" of some arithmetic rules). In the course on argumentation that I am considering here, lectures engage students in filling (during the lecture, and sometimes at home) individual worksheets that concern the topics presented by the teacher. In these worksheets, students are also invited to write down the possible difficulties met by them. Students usually try to do their best in filling the worksheets. Then, some students' productions are chosen by the teacher, for comparison and classroom discussion. The final evaluation is based on the student's detailed self-reflection on, and revision of, her own worksheets, taking what has been discussed in the classroom into account.

The following examples are taken from the worksheets filled by the students in the last part of the course, devoted to argumentation to validate elementary arithmetic statements and to check the validity of proofs produced by other students. In some cases, also audio recordings or field notes of the discussions concerning those worksheets are available. Proof of the truth of general elementary statements in arithmetic is presented as an argumentation with a statement that concerns a specific set of elements (e.g., natural numbers, or couples of consecutive odd numbers), and a reasoning that guarantees the truth of the claim for all the elements of the given set. The reasoning may be a verbal reasoning, or reasoning based on the use of algebraic language. Students already know from high school that checking with a finite number of cases the validity of a statement that concerns an infinite set of elements does not guarantee the truth of that statement, while a single example (counterexample) is sufficient to prove that a general statement is false. Students are invited to reason as mathematics would require them to do; they know that possible educational implications will be discussed later in the classroom.

A-priori analysis of the task

Task 1

Prove that 1 is the only common divisor of two consecutive numbers.

Students have the necessary knowledge about the mathematical concepts involved in the task (divisor, consecutive numbers); they know what "the only" means.

A possible proof is: let n and n' be consecutive numbers (n<n'). Assume that d≠1 is a common divisor of n and n'. Given that n is a multiple of d, the successive multiples of d are n+d, n+2d, n+3d, and so on. Thus n' must be one of those numbers. Thus n' cannot be the consecutive number of n. Thus, it is true that the only common divisor of n and n' is 1.

Theorem in action (for the concept of proof): to prove that A implies B it is sufficient to prove that 'not B' implies 'not A'.

Another possible proof is: let n and n' be consecutive numbers (n<n'). Assume that d≠1 is a common divisor of n and n'. Then d divides n, thus the remainder of the division n:d is 0, and d divides n', thus the remainder of the division n':d is 0. But as n'=n+1, the rest of the division n':d is 1, which results in a contradiction with the previous result (0≠1).

Theorem in action (for the concept of proof): to prove that A implies B it is sufficient to prove that a contradiction may be derived (given A) from 'not B'.

Examples of students' proofs for task 1

Among their productions, we have the following ones:

Example 1

> Any two consecutive numbers have only 1 as common divisor, because if any two consecutive numbers would have common divisors different from 1, then also 1 and 2 would have common divisors different from 1: this is false because they have only 1 as common divisor.

In this case, we find a situation of a rather common lack of mastery of the *theorem- in- action: to negate "P is true for every x" it is sufficient to find one case of x for which P is false.* This student, during her effort to build a proof by absurd, negated "P is true for every x" by assuming "P is false for every x"; then she proved that this assumption is in contradiction with the case of 1 and 2.

Example 2

> Any two consecutive numbers have only 1 as common divisor, because if by absurd one divisor d≠1 of n would be also a divisor of n+1, then n+1 would be a multiple of d, which is false because n and n+1 do not have common divisors greater than 1.

In this case, there is a problem with the mastery of the *theorem-in-action concerning the proof by absurd*: in order to prove that B (d=1) is true under the hypothesis A (if d is a common divisor of n and n+1), the student assumes that B is false (d≠1), then he gets the conclusion that B is true by showing that if B is false it brings to A, which brings to B.

Also, the mastery of *the operational invariant of propositional type "implication"* looks weak for this student, which resulted in the confusion between the hypothesis and the thesis.

Example 3

> Any two consecutive numbers have only 1 as common divisor:
>
> d=1 d\n d\n+1 n+1-n=1
>
> (n+1):d =q+r =1≠ 0 if d≠1. d=1

According to the student-author's intervention in the discussion, his intention was to prove that $d=1$ by considering a divisor d of n and $n+1$. Since $n+1 - n = 1$, if $d\neq 1$ the division of $n+1$ by d has a remainder $r=1$ (and not 0, corresponding to divisibility of $n+1$ by d). Hence $d=1$.

This example is a case of *poor linguistic representation* of a valid reasoning: the sequence of symbols and words (only one word, *if*) does not allow to distinguish between what must be proved ($d=1$ is the only divisor) and what is assumed to build the proof (d as a divisor), and to understand some steps of the proof. In particular, (n+1):d=$q+r=1\neq 0$ *if* $d\neq 1$ is the short writing of a piece of reasoning, where even the algebraic meaning of the '=' sign is twice lost! Similar cases have been considered by Azrou and Khelladi (2019).

Task 2

After the activity on the common divisors of two consecutive numbers, students are requested to prove that the only common divisors of two consecutive even numbers are 1 and 2.

Students have at their disposal, in their worksheet, two kinds of valid verbal proofs concerning the common divisor of two consecutive numbers (these proofs have been produced by two students and then discussed in the classroom):

I) A proof by multiples, based on the consideration of the distance greater than 1 between two consecutive multiples of every number greater than 1 (like the first proof of the a-priori analysis).

II) A proof by remainders (of the division of the second number by the divisors of the first number), like the second proof of the a-priori analysis.

The aim of this task was to engage students in the reflection of what was in common with the first task (the logical structure of reasoning) and what was different (the different hypotheses).

Examples of one student's text for task 2

> I have tried to write a general argumentation starting by the definition I (the proof by multiples) but I was unable to understand the rule that allows getting the proof for two even numbers. I was able to understand the general rule but only in single examples, but when I try to write the argumentation, I am unable to find the right words. I think that the reasoning is based on a formula I need more time to understand, and in this moment, I do not know it.

Firstly, note how the student uses the word "definition" for a "proof" (we might interpret it as a sign of a possible weak mastery of the operational invariants which concern the *meaning* of definition and proof). During the discussion on her production, this student makes a long intervention on her difficulties: according to her, she was not able to make a connection between the proof required in this case, and the proof met in high school (Euclid's theorems in geometry, the proof of the irrationality of $\sqrt{2}$). Even the transition from the proofs discussed in the classroom for the case of consecutive numbers to this new proof situation had been hard for her.

> I was very confused. I had in my mind the proofs in geometry, where clear hypotheses are given and then we must move to the thesis step-by-step. Also, the proof of the irrationality of $\sqrt{2}$ was clear to me: the assumption that $\sqrt{2}$ is rational must bring to an absurd conclusion through formulas. The proofs in the case of consecutive numbers, … I do not see the hypothesis, yes, they look like proofs by absurd, but also the thesis… and the negation of the thesis… Also, not clear for me the use of algebra in one of the proofs that we have seen: I do not see how we must use algebra, the formula that we must use.

This example looks very interesting: the student engaged a lot in previous mathematics and mathematics education courses, with enough good results as concerns the discussion of teaching and learning problems. She looks very committed with this task. Her words reveal a possible lack of *well-interiorized and valid reference situations* of proof. This lack results in a misleading approach to proving in general: "we *must* move to the thesis", "the assumption (…) *must* bring to an absurd". "I do not see how we *must* use algebra". These sentences correspond to what the student wrote in her worksheet: "I was unable to understand the rule that allows to get the proof", "I think that the reasoning is based on a formula (…) that in this moment, I do not know*".*

Like in the case of ordinary concepts, in the case of the concept of proof it is not sufficient to have met some instances of proof and to have learnt some proofs in order to get reference situations for the concept, which are suitable to support problem solving in new proving situations. Moreover, this example helps to focus on what do reference situations mean for the concept of proof: how reference situations for proof are related to the construction of a new proof?

Discussion

As we have seen in the above examples, considering proof as a concept according to Vergnaud's Conceptual Fields Theory is a promising tool to identify and analyze students' difficulties related to the mastery of proof at the meta-level. However, a lot of work is still to be done. Here are three possible directions:

- By deepening the analysis of the components of the concept of proof (particularly as concerns the identification of the operational invariants of the three types), and their role for the development of students' proving competencies and in the interpretation of their difficulties. I think that special attention should be paid for the inferences (among the operational invariants of the concept of proof), since they are a way of considering at the meta-mathematical level what happens at the mathematical level, for instance, when a student writes: "d>1, hence the consecutive of n does not belong to the set n+d, n+2d, n+3d, …" (see task 1).

- By clarifying what the mastery of "reference situations" means in the case of the concept of proof, in terms of how it intervenes in proving (see the last example).

- By identifying the conceptual field, to which the concept of proof belongs, particularly as concerns its three components.

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How preservice teachers enact mathematical argumentation and proof in class – an activity-theoretical perspective

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Argumentation and proof as core activities in mathematics should be staged continuously and meaningfully in mathematics lessons. But to what extent are preservice teachers at the end of their studies able to adequately introduce mathematical argumentation and proof as activities into their classroom planning and staging? Activity theory makes a valuable contribution to answering this question by emphasizing the importance of prospective teachers' development of motives and goals, corresponding modes of action, subjective constructions of meaning and the ability to identify appropriate objects for argumentation and proving activities in the classroom. In this work-in-progress paper, we outline an activity-theoretical framework and present empirical research tools based on it for analyzing prospective teachers' classroom enactments. We apply these to case studies from an exploratory, qualitative study with preservice teachers in their final year of study. We present first results and draw conclusions towards future work.

Keywords: Argumentation, proof, activity theory, preservice teacher education, second discontinuity.

Introduction

In preservice teacher education, one way to address the *issue of developing preservice teachers' ability to adequately introduce mathematical argumentation and proof as activities into their classroom planning and staging* is to foster students' own argumentation competence, e.g., via proving tasks, and to get them acquainted with didactical models and theories on developing argumentation competence in class. However, it is unclear whether this kind of study suffices, within the bounds of possibility, to prepare preservice teachers well for teaching mathematical argumentation. One source of such doubt is the so-called phenomenon of double discontinuity (Klein, 1908), which concerns two difficult transitions: first, the secondary-tertiary transition, when students enter university (see Gueudet, 2008); second, the transition from university to teaching at school. For the latter transition, the crucial question is to what extent teachers are able to make effective use of academic knowledge in their teaching. There is evidence in practice that teachers do not make full use of their content knowledge and pedagogical content knowledge when designing lessons. An activity-theoretical approach can help to theoretically ground such practical impressions regarding the second discontinuity and to substantiate and qualify them through empirical research. It goes beyond the consideration of the role of affect and beliefs for teaching argumentation and proof, but can also function as an interface to it. It offers starting points for the development of suitable formats for preservice teacher training courses in order to effectively address this issue.

Activity-theoretical framework

Activity-theoretic perspectives have already proven helpful in the teaching and learning of mathematical argumentation and proof in connection with the role of tools and cultural artifacts (e.g.

Cerulli, Pedemonte, & Robotti, 2005). In our work we shift the focus to the role of motives, goals, and constructions of meaning in the teaching of mathematical argumentation. We use the conceptual framework of Leontjew (1982), as developed further by Lompscher and Giest (see Bruder & Schmitt, 2016). From the perspective of activity theory, the constitutive elements of human activities are a superordinate *motive*, the *objects* of activity, and *ways or means of action* to act on and with the objects. The motive drives actions directed towards an object of activity, dependent on the repertoire of ways of action and available means. Concrete *goals* of such actions realize the motive in various ways. In individual activity, superordinate motives usually are unconsciously or subconsciously behind consciously set goals for actions. In the process leading from the superordinate motive to the concrete goals of action, *individual constructions of meaning* emerge.

The activity-theoretical framework enables us to analyze differences and commonalities between mathematical practices at university vs. school – they manifest themselves in all components of the activity-theoretical framework: a) While one of the central motives of university practice lies in the argumentative justification and explanation of the deductive derivability of a statement within the framework of a mathematical theory, a related motive in school mathematics would focus on truth or general validity of a statement rather than its derivability. b) While at university the objects of activity are explicitly stated (as in propositions, conjectures or proofs), the objects in school mathematics are often more implicit, "hidden" opportunities for argumentation (such as the comparison of different solutions or the justification of calculation rules). c) At university, the ways and means of action consist (in the context of justification) in valid reasoning within the framework of globally ordered mathematical theories. In school mathematics, we rather find plausible and example-based as well as heuristic and generic argumentation, and (more informal) deductions in locally ordered propositional systems. Prominent goals realizing the motives of argumentation activity in both practices are the well-known "proof functions" according to Villiers (1990).

Research question for the exploratory study

From the perspective of activity-theory just presented, we can now formulate an initial hypothesis with regard to issue raised at the beginning. *We hypothesize that preservice teachers often do not adequately develop suitable motives and corresponding constructions of meaning in their studies, as well as develop an inadequate repertoire of actions and dismiss possible objects for argumentation and proof in mathematics lessons. Consequently, they are often not able to give space and shape to argumentation and proof in their own school teaching in a way that in principle accommodates the profound and multifaceted meaning of these activities for relevant mathematical practices.* The hypothesis is motivated by the observation from university teaching practice that preservice teachers develop a highly reduced image of mathematical argumentation and proof during the mathematics lessons that they experienced at school, which is only put into perspective in de facto little mediated ways at university. Moreover, constructions of meaning for argumentation and proof, developed in university and school mathematics practice, are primarily shaped by actual experience: "Meaning is educated" (Leontjew, 1982). These experiences can be quite one-sided in both of the respective practices. For example, preservice teachers at university increasingly experience that the meaning of proving is systematization, whereas in the school practice they experience it may at best mean verification. Possible objects of argumentation and proof as well as appropriate ways of acting are often not perceived very much. At school, for example, objects of proof appear only singularly, objects of argumentation rather covertly, and generic argumentations are often not (fully) recognized as argumentations. Furthermore, the respective practices are partly experienced under other, more dominant overriding motives, for example as “learning practices” under the motive of solving set tasks according to certain standards. In school, for instance, an emphasis on application, when experienced as dominant, may overshadow meaningful motives of argumentation and proof. We consider such discontinuity experiences of naturally existing differences between mathematical practices on the part of preservice teachers as an additional cause of the circumstances claimed in our hypothesis, and pose the following, *open research question* for our explorative study:

> *In which sense do preservice teachers lack effective motives and corresponding constructions of meaning, appropriate ways of acting or access to suitable objects, in order to stage argumentation and proof activities in a meaningful way in the mathematics lessons they plan and conduct?*

Methodology of the exploratory study

Using the activity-theoretical perspective described above, we deductively developed an observation and analysis framework for teaching productions by preservice teachers. In our study, we applied this instrument in the context of a course in the final year of study, in which six pairs of preservice teachers each plan one classroom session on a mathematical subject of their choice, carry it out as a teaching experiment with their fellow students as peer experts, and receive professional and peer feedback. Hence, we observed and analyzed a total of six different classroom sessions. The chosen topics for the sessions were: *area of triangles* (1), *power functions* (2), *half-life* (3), *zeros* (4), *binomial formulas* (5), *scalar product* (6). The two authors worked independently of each other with a semi-structured observation sheet and compared their observations in follow-up discussions. Descriptions of observed, argumentative or argumentation-related actions of the teacher, of requests for such actions to the learners, as well as related formulations of goals, motives or object designations made by the teacher were noted in the sheet. The time and phase of the lesson or the phase transition were also recorded, as well as optional comments by the observers, both descriptive and interpretive in nature, for example on the actual actions of the learners. In addition to the completed, semi-structured observation sheets, the written plans, the classroom materials and the preservice teachers’ written post-lesson reflections form the data basis of our study.

As a first step of evaluation of the observational data and the planning and reflection documents, we describe stably occurring phenomena and patterns and propose an activity-theoretical analysis and explanation. The framework categories of *motive*, *object*, (*way of) action*, *goal*, and *construction of meaning* we use are obtained deductively from activity-theory. We supplement these inductively with intended or actual motives, goals and ways of action that can be recognized in the data.

Results

In a first review of the observational data, we were able to identify three overarching phenomena and associated stable patterns as specific manifestations of the phenomena in the teaching productions of the preservice teachers, which contribute to further differentiate our initial hypothesis with regard to our research question. In the following, we describe each phenomenon and its patterns, give concrete examples of the patterns and propose an activity-theoretical explanation for the phenomenon.

Phenomenon 1: Missing out on opportunities for argumentation

Results and answers of the learners are not questioned further in class, sometimes not even checked. In addition to "how did you arrive at this result?", questions like "why does it work that way?" and "what is good about this way?" are missing, i.e., questions that are fundamental for mathematical argumentation as an activity.

Phenomenon 1 occurs in three different patterns and shows up both on the situational-spontaneous level of action and on the level of reflexive planning action. As we illustrate in the following, these observations can be understood from an activity theoretical perspective as an indication that the availability or accessibility of *objects of argumentation* has an impact on two essential professional competence areas of teachers: "reflective competence" and "action-related competence", which are defined and measured through the corresponding action (Lindmeier, 2011).

We first describe the patterns and concretize them through examples from the staging observations. Then we add suitable excerpts from the planning observations.

Pattern 1.1: Receiving results and moving on. *Learners' answers are received and rated, but they are not questioned further or confronted with each other.*

Answers and results are not used as potential objects of argumentation activities. This applies both to planning ("symptoms": discussion phases are planned far too briefly, possible variants for solutions are not considered in advance) and to situational ad hoc action in lessons.

Pattern 1.2: Leaving questions from students behind. *Unexpected questions from learners are acknowledged as an element of classroom interaction, but they are left behind as objects of argumentation.*

This pattern primarily concerns situational ad hoc action in class.

Pattern 1.3: Leaving opportunities unused in task construction. *The argumentative potential is not exploited in task construction, the staging does not focus on argumentation.*

This pattern primarily concerns the planning process, when during task construction possible objects of argumentation are not realized and hence do not become effective in staging.

We choose examples for the patterns from session 6 ("scalar product") because all three patterns occur in this session. An overview of the patterns that were recognized in agreement by both observers (regarding all phenomena and sessions) is provided in Table 1. The learning content of session 6 are four basic mental models for the scalar product, relating it to *projection*, *orthogonality*, *product,* and *angle*. Small groups of learners go through four learning stations, each assigned to one of the basic mental models. Patterns 1.1-1.3 can be recognized in the staging observations in the following places:

> [1.1: Receiving results and moving on] At station 1 (*projection*) the learners spend much time with calculations, which are then only looked at. Later in plenary, a pure checking of results is done.
>
> [1.2: Leaving questions from students behind] In plenary after the station work, a learner reports purely procedurally, which the teacher acknowledges with "OK". The question of whether the

scalar product can be negative is raised by a learner and answered by the teacher, but only with the brief mention of an inappropriate technical term. It was obviously not foreseen in the planning.

[1.3: Leaving opportunities unused in task construction] At station 2 (*orthogonality*) the teacher asks about “commonalities” among the given cases of vector pairs. In return to the answer “are perpendicular”, the teacher asks “why”. This could be a good attempt to go into depth argumentatively. Unfortunately, the teacher’s question remains unanswered and is then not pursued further. The conceptual aspect of relative coordinates, which would have been part of such an argumentation, remains excluded throughout the whole session.

Table 1: Phenomena and patterns (X = pattern recognized in class or found in planning documents)

Phenomena	*Patterns*	*Sessions*					
		1	2	3	4	5	6
(1) Missing out on opportunities for argumentation	1.1: Receiving results and moving on	X	X	X	X	X	X
	1.2: Leaving questions from students behind		X			X	X
	1.3: Leaving opportunities unused in task construction	X		X	X		X
(2) Missing focus on content and conceptual core	2.1: Strong emphasis on methodological side of teaching	X		X		X	
	2.2: Missing the conceptual core		X	X	X		X
(3) General structure and discursive character of argumentation not exemplified.	3.1: Teaching by preparing written tasks or task sequences without a (local or global) argumentation-oriented dramaturgy	X	X	X	X	X	X

The additional planning observations can be used on the one hand to support that the occurrence of the three patterns in class is consistent with the planning: The planning statement “The educational content of the lesson is the recognition of a new operation and the application of the arithmetic operation with vectors” is consistent with extended phases of mechanical calculations as described, in contrast with short phases for in-depth comparison of results (Pattern 1.1). The planning statements “The pupils are able to *experience* the scalar product in its various forms and effects in group work” and “In addition, it can have a motivating function to *show* the practical benefits of the new operation” (our emphasis) indicate Pattern 1.3 inasmuch as they focus on the phenomenological rather than on the argumentative aspect. On the other hand, both the staging observations and the planning observations reported so far appear to be in contrast with goals that the preservice teachers set, partly with explicit reference to German core standards K1-6 and levels of cognitive complexity AB1-3:

> “The learning objectives are as follows. [...] The pupils *explain the effect* of the scalar product […] and *explain the connection* between the scalar product and the cosine (K1, K6, AB3). [...] [They] realize that the angle between the vectors plays a *crucial role*. [T]hey *discuss why* zero comes out for the orthogonal vectors and not for the other vectors.” (our emphasis)

We explain phenomenon 1 from an activity-theoretical perspective in a more general way on the level of *objects*: Our findings suggest that, even though goals were set that are appropriate for argumentation activities, in task construction preservice teachers do not succeed in connecting these goals to suitable objects of activity. In our example, they do not recognize the basic mental models of the scalar product as objects of argumentation activity, but rather as phenomena to be experienced.

Phenomenon 2: Missing focus on content and on the conceptual core

A lack of focus on content and on the conceptual core within the planning manifests itself in a conspicuous accumulation of learning activities that are not properly related to the learning content of the session. Accordingly, there is no focused content-related activation of the learners; in particular, activities that can be beneficial to argumentation (e.g. observation) lose the content focus.

Phenomenon 2 occurs in two patterns that fit a distinction developed in Renkl & Atkinson (2007) from the viewpoint of educational psychology: active responding, active processing, and focused processing. *Active responding* merely refers to a visible engagement of the learner with the learning environment. *Active processing* refers to actual processing of the content, beyond overt action and interactivity. Finally, the stance of *focused processing* emphasizes that it may be crucial that learners activities are focused on the central concepts and principles to be learned.

Pattern 2.1: Strong emphasis on the methodological side of teaching. *The staging is methodically (and sometimes technically) overloaded with actions that are not related to the mathematical learning content. As a result, learners' engagement contributes little to their understanding of the content.*

The occurrence of this pattern only leads to active responding of the learners in the sense of Renkl & Atkinson (2007). Pattern 1.1 particularly concerns the level of reflective planning actions.

Pattern 2.2: Missing the conceptual core. *The staging contains mathematical actions related to the mathematical learning content, but these do not reach its mathematical core. As a result, learners' engagement is not focused on the content core. In particular, argumentation-related activities appear not to be "conceived from an explanatory warrant" with a view to foster learners' deeper mathematical understanding.*

In Pattern 2.2 active processing of the learners can be observed, but their mathematical engagement does not constitute focused processing. The pattern concerns planning as well as staging.

The staging in session 3 ("half-life") exemplifies both patterns (see Table 1). The learning content is the half-life in the context of exponential functions, which is concretized regarding the real-life phenomena of beer foam decay and dice throwing. The individual patterns can be recognized in the staging observations in the following places:

> [3.1: Strong emphasis on the methodological side of teaching] The foam measuring activities or the implementation of the dice throwing experiment dominate the staging of the group phases.
>
> [3.2: Missing the conceptual core] In working with the experimental data, learners are asked to plug in and calculate in the first place. Modelling work including discussions about the exponential behavior (as core of the matter) is neither visible during the group phases nor addressed in the follow-up plenary. In both phases we observe active processing, but no focused processing.

The planning documents reveal a certain tension: on the one hand, the aim of the session appears to be the *application* of existing knowledge about exponential processes and half-life to self-conducted, real-life experiments, presupposing that it is *already known* that the core processes involved are exponential. However, the planning of the concrete implementation is geared towards *argumentation* – but it is unclear from which premises and to which conclusion the argument leads:

> “Pupils use the example of an everyday phenomenon to apply their already learned knowledge and skills about half-lives. They determine experimentally the half-life of dice throwing and beer foam decay by conducting experiments.”

> “In the case of the dice experiment, arguments can be made mathematically or with the help of exponential correlations. In the case of beer foam, […] the learners should argue that the half-life does not change. Here, they could, for example, argue with prior knowledge from the previous lesson or the exponential equation. [...]”

All in all, the planning fluctuates between a focus on argumentation about exponential behavior on the one hand and experimentation and application on the other. In the implementation, we saw no argumentation-related activities, but a number of unfocused technical or instrumental activities like measuring, plugging-in, and calculating instead. We explain phenomenon 2 on the level of *motives*: Regarding Pattern 2.1, one possible explanation of our findings could be that “active” learning (in a naive interpretation of “being active” as “doing”) and application to the real world are effective as superordinate motives that override specific motives for mathematical argumentation activities. Regarding Pattern 2.2, the preservice teachers could be guided by rather nonspecific motives like “doing mathematics” (not specifying the content focus) or “doing argumentation” (unaware of concrete assignments of functional roles within the arguments).

Phenomenon 3: General structure and discursive character of argumentation are not exemplified

There is a lack of exemplification of mathematical argumentation by the teacher as a “living model” and as a knowledgeable navigator in argumentative classroom discourse. Such discourse hardly takes place, and if at all, the structural elements of argumentation remain hidden.

We observed one stable pattern which occurred in all six sessions (see Table 1):

Pattern 3.1: Teaching by preparing written tasks or task sequences without a (local or global) argumentation-oriented dramaturgy. *The teacher prepares pre-formulated work assignments for individual or group work, and then largely fades into the background in the production. Neither the work assignments nor the classroom discussion of the results guide argumentation activities in a structural or discursive sense and clarify the argumentative dramaturgy.*

This pattern concerns planning activities in the first place, but can also be instantiated, e.g. in the form of an ad hoc decision of the teacher to withdraw from an active role in a class discussion. Due to limited space, we have to dismiss more detailed example illustrations. We explain phenomenon 3 on the level of *sense constructions*: The planning documents show that the preservice teachers are somehow aware of the general motives of mathematical argumentation activity and also concretize these in part in suitable goals, such as exploration, conjecture and systematization. However, the chosen ways of action are often either not appropriate for pursuing the selected goals or they are not implemented as part of an effective argumentative dramaturgy. A reason for this could be that the preservice teachers lack the corresponding meaningful experience that could serve as a source of meaning constructions. Hence, they lack a basis to link goals and suitable ways of acting in a meaningful way and to concretize the motives of argumentation activity in the lesson.

Summary and Outlook

We showed how an activity-theoretical framework can be used to analyze prospective teachers' classroom enactments. Our analysis of preservice teachers' lesson planning and staging exhibits phenomena and stable patterns that can be explained in terms of motives, objects, goals, ways of action, and meaning construction. Due to the exploratory nature of our study, we obviously cannot draw general conclusions. Regarding specific limitations, we point out that the lessons were conducted in a university seminar (in digital format) with peers as learners. While one could argue that the participants might act differently in a real-life setting, we conjecture that core elements of analysis (i.e., motives and objects) are not affected substantially by the setting. Our observations differentiate the hypothesis formulated at the beginning and illuminate it as a general issue in teacher education from a new perspective. Our activity-theoretical analysis of the three phenomena interprets them as specific variants of the overarching phenomenon of the 'second discontinuity': Developing preservice teachers' own argumentation competence and didactical knowledge alone might not be sufficient for them to successfully enact mathematical argumentation in class. From a developmental perspective, an important objective for future work is the design and exploration of appropriate formats for teacher training that sustainably address the observed discontinuity phenomena. Of particular interest might be in how far the phenomena identified in our observational data can be developed into explicit guiding principles for teaching mathematical reasoning and proof (e.g., Buchbinder & McCrone, 2022) and be incorporated into preservice teacher courses.

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Developing students' rationality by constructing and exercising rational mathematical templates: The case of counter-examples

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The aim of this paper is to put into evidence two aspects of the production of counter-examples during students' approach to it and its development: the epistemic and the teleological (strategic) components of the specific process that brings to the production of a counter-example. By considering two 8th grade classroom discussions (the first one, on counter-examples in arithmetic and the other, in geometry) I will discuss how those aspects are of general interest for the development of students' conscious dealing with the truth of statements and the effectiveness of strategies (i.e. with the development of students' rationality, according to Habermas). Also some affective aspects will be considered, as concerns their effects on the productivity of the discussion.

Keywords: Habermas' rationality, rational mathematical templates, counter-examples.

Introduction

In the perspective of developing rational behavior in mathematics (according to Habermas' elaboration on rationality), a need emerged in our research, suggested by what Habermas (1998) presents as the ideal situation of exercise of communicative rationality:

> Communicative rationality is expressed in the unifying force of speech oriented towards understanding, which secures for the participating speakers an intersubjectively shared lifeworld, thereby securing at the same time the horizon within which everyone can refer to one and the same objective world. (Habermas, 1998, p. 315)

What are the components of "an intersubjectively shared lifeworld"? The definition of rational mathematical template (RMT) (Boero & Turiano, 2020) as *the couple consisting of a mathematical entity (*e.g., the definition*) and a rational process that is purposefully oriented to produce an instance of that mathematical entity* was an attempt to identify and frame (in particular, for mathematics – but the same may be done for other disciplines) some crucial disciplinary components of rational behavior, whose mastery is needed for the interpersonal (and intra-personal too) productive and enjoyable discursive exchange described by Habermas. In Boero & Turiano (2020) we have shown how the RMT construct may work as a tool to analyze the gradual access of students (under the guide of their teacher) to the mastery of the RMT of definition and to productive classroom discussions on definitions. Some open research and educational problems have been briefly presented at the end of that paper. In particular we have considered the need of further investigating the affective dimension of the classroom activities concerning the approach to, and the exercise of, RMTs. That dimension is inherent in the fact that the mastery of RMTs looks as one of the most important conditions for students' and teacher's wellbeing in the mutual exchanges among them and with the Other (the culture). We have also considered the need of identifying didactical choices that are suitable to promote the development of RMTs in the classroom.

In this paper the analysis of two discussions in the same classroom concerning the RMT of counter-example will allow to say more about the development of the RMTs in the classroom: to identify some aspects of the mastery of RMTs on the epistemic and teleological sides, which are relevant for the development of students' rationality, and to provide further evidences concerning the relevance of the affective dimension of classroom interactions for the productive exercise of the RMTs.

Habermas' elaboration on rationality, and rational mathematical templates

Basic notions concerning Habermas' elaboration on rationality are resumed here. Habermas (1998) proposed the construct of rationality (and that of rational behavior) to deal with discursive practices that are characterized by *awareness* when checking the truth of statements and the validity of reasoning according to shared criteria (*epistemic rationality*), evaluating strategies to attain the aim of the activity (*teleological rationality*), and choosing suitable communication tools to reach others in a given social context (*communicative rationality*), the three components being strictly interconnected.

In past years, researchers both in our group and outside it have attempted to adapt Habermas' construct to mathematics teacher education and to plan teaching aimed at developing and analyzing students' rational behaviors (see Boero & Planas, 2014, for a general account about it and a presentation of five studies).

As concerns RMTs, when we have chosen the provisional expression "Rational Mathematical Template" to name (see above) *the couple consisting of a mathematical entity (in our case, the definition) and a rational process (...),* we had in our mind that the rational behavior in the case of the production of a definition (or a mathematical model, or a mathematical proof, etc.) is *specific* for that entity as concerns the three sides of rationality, and also for the whole organization of the process. In particular in Boero & Turiano (2020) we have tried to shed light on how the RMT of definition developed in strict relation to the development of mathematical rationality in the domain of theoretical mathematics: in the second episode, the students took care of criteria that characterize a mathematical definition and that allowed them to decide whether a formulation satisfies them (*epistemic rationality*); they engaged in a goal-oriented process (*teleological rationality)*; during the discussion, the students were concerned with the formulation of the definition and engaged in mutual corrections about it (*communicative rationality*). Still in Boero & Turiano (2020) we have discussed the relationships of our construct in mathematics education with other constructs, in particular with the "process-oriented routines" and the "product-oriented routines" of Lavie, Steiner and Sfard (2019).

Counter-examples in the perspective of rationality

The following ICMI-11 Abstract (Antonini, 2008) well represents the perspective in which a great deal of research work (including Antonini's contributions) has been performed on the generation of examples (including counter-examples) in Mathematics.

> Generating examples of mathematical objects can be very difficult for students and it can be considered a problem solving activity. In literature, some potentialities of such activity are suggested, from different points of view and for different reasons. Our investigation aims to better identify the characteristics and the potentialities of the processes of constructing examples. The analysis, carried out by observing students' processes in Real Analysis, reveals a high complexity of examples generation tasks.(…) The study on these processes highlights the potentialities of

generating examples activity as a tool for researchers in investigating many aspects of students' thinking and for teachers in promoting students' understanding and conceptualization.

In spite of sharing Antonini's point of view on the relevance, difficulty and complexity of dealing with students' production of examples (including counter-examples) as a didactic and a research challenge, the perspective of the research development presented in this paper is different. We want to investigate the classroom production of counter-examples as an instance of the development of students' rationality.

Two discussions in the same classroom

The source of the data

Covid-19 pandemic prevented the teachers who collaborate with me from experimenting classroom situations suitable to construct and exercise RMTs, and by this way to get new elements to corroborate previous hypotheses and answer research questions. However, we have at our disposal, in our group, a lot of transcripts from discussions in primary and lower secondary classes of the past. About 20 years ago we had already elaborated (without any explicit connection with rationality) a germ of idea of "mathematical template" to be developed in the classroom in order to ensure understanding of texts and productive discussions on some mathematical subjects. I have found (under the counter-example label) two transcripts of discussions, with teacher's field notes, which are of interest for the present development of the research. They come from the same 8th grade class of 22 students, in April. Students were 13 and 14 years old; most of them were from middle class. The teacher was Marina Molinari; she was a member of our research team since the eighties.

Underlined words (in the original transcripts) mean that the tune of the voice changed towards putting into evidence what was said; … means: long pauses.

The first discussion: making explicit the notion of counter-example

As we will see later, students had already met some rather easy situations of production of counter-examples (in mathematics, grammar, sciences), without naming them counter-examples.

T: Today we will consider something new (or better: something that we have done several times, but more complex this time – we will see it later!). Let us consider natural numbers that end by 1. Let us explore them, their properties.
S1: like 1, 11, 21, 31?
T: yes! 41, 51, and so on. Let us think for some minutes alone. Silence, please. Try to find some properties of the people of this tribe.
(laughs)
(after about 4 minutes)
T: What have you discovered?
S2: they look …
S3: most of them are prime numbers: 1, 11, 31, 41.
S4: and the others are divisible by 3.
T: thus we may write (she goes to the blackboard, and writes):
"P: all natural numbers that end by 1 are divisible by 3, or prime numbers".
Is it OK for you? Or do you find some exceptions?
S5: not, 21 is also divisible by 7.
T: (writes on the blackboard: 21 is also divisible by 7).
Is 21 an exception for what is written above? (she points at the statement P). It is not an easy answer… Let us try together!

Some voices: Yes… Not…

T: Yes or not? Let us try to know the different positions.

S6: Not, P says that all numbers … all natural numbers that end by 1 are divisible by 3, or prime numbers, while 21 is also divisible by 7.

S4: but it is divisible by 3, thus it is not an exception.

S5: all the students in this school are boys or girls, it is true without any exception. None of us makes an exception (a few hidden laughs in the class). While in the case of P… 21 is an exception.

S7: but P does not exclude the divisibility by 7. Like 81: it is divisible by 3, by 9,… (Other examples are made)

S5: in the case of the students of this school, there are only boys and girls, in the case of P it is like if there is also some special boys who are also another thing (laughs in the classroom). Like 21 or 81, which are divisible by 3, but non only by 3.

S8: it is true that each student in this school is a boy or a girl, but a boy may have blue eyes, this property does not make an exception.

T: perhaps you, S8, have seen the point… May you come back from boys and girls to numbers?

S8: yes, but I do not find the words… I try: 21 is divisible by 3, but it has also other properties… It is divisible also by 7, it is bigger that 20, these properties do not disturb the fact that it is a number ending by 1, not a prime number, but a number divisible by 3. Thus it agrees with P!

S9: and every number that ends by 1 is also divisible by itself… This does not disturb the truth of the statement!

S4: P is true if all numbers that end by 1 are prime or divisible by 3. To be prime or divisible by 3 is sufficient for the truth of P.

S7: if we ascertain it for all numbers.

S5: I start to understand. Yes, P does not say that all numbers are prime or divisible by 3 only! Being divisible by 7 is another property – like for a girl with black hair, like for Samantha, who is a girl with other properties.
(some laughs… Samantha is a very pretty girl! And she looks happy of the reference to her, and smiles)

S6: me too… P does not say “only by 3”, divisibility by 7 is another property, no problem!

T: OK, you have solved together a difficult problem! But the original problem remains: is P true?

S4: uh, 121… it ends by 1 and is neither prime, nor divisible by 3. 121 is equal to 11 by 11.

S9: here we have a true exception!

S8: yes, 121 is equal to 11 by 11, it is not a prime number.

S10: and it is not divisible by 3! Thus P is not true!

T: very well, we have discovered what we call a “counter-example” in mathematics: an example, which shows that the property that we are discussing is not true, in our case: not true for all numbers ending with 1! Your task for the next lecture is to write a letter for a student of another class, to explain what we have discussed today and what we have discovered. We have still some time for us… I ask you: do you remember cases in which we have met some kinds of counter-examples?

S4: yes, in grammar… It is not true that all names of Italian people, which end by a, are female. Andrea is the name of a male!

Other examples follow of situations met in the past: odd numbers that are not prime numbers; dolphins that are mammals which do not live on the ground.

Second discussion: exercising RMT of counter-example in elementary geometry

Two weeks later students are making conjectures on the geometry of triangle. They have already discovered (by exploration in different situations: acute triangles, obtuse-angled triangles, etc.) some general properties, like: in every triangle, each side is shorter than the sum of the other two sides; and: each height of a triangle is no longer that the adjacent sides (i.e. the sides from the same vertex). Students have also discovered that some properties that are true in some situations are not true in other situations (e.g. that one median is not always shorter than the adjacent sides).

The teacher decides to suggest the exploration of the angle between the height and the median of a triangle, which originate from the same vertex. In a few minutes a student (S1) hypothesizes that such angle is smaller than the angle between the adjacent sides. The teacher writes such statement (as statement P) on the blackboard:

P: For every height of a triangle, the angle with the median from the same vertex is smaller than the angle between the adjacent sides.

and invites students to explore if it is true. After a few minutes, the discussion starts

S1: I have proposed that P is true by considering a normal situation: the median and the height inside the angle between the concurrent sides in the same vertex.
S2: yes, it is obvious that in that case P is true, the angle between the median and the height is inside the other angle formed by the sides!
S1: I agree that is obvious that P is true in that case, but I have made two other drawings, with a right-angled triangle and the height that was a cathetus, and an obtuse triangle with the height that was outside the triangle… and I have seen that P was still true!
T: S1, come at the blackboard and show those two cases!
(S1 goes to the blackboard and draws the two situations, with the chalk, the ruler and the square – see Figure 1)

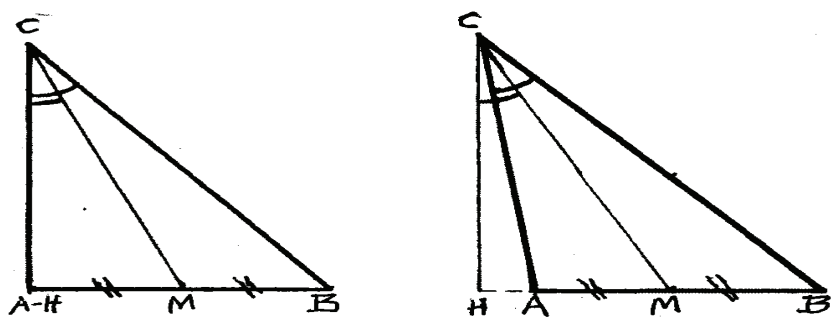

Figure 1: S1's figures

S3: it is clear that P is true also in the case of the right-angled triangle.
S4: because the angle between the height and the median is still inside the other angle…But in another case…
S1: (takes a sheet of paper and by folding it according to the smaller angle she shows that also in the second case P is true).
S4: yes, it is a good example for P also in the case of the height outside the triangle, but…
S5: we need to find if there is a counter-example.
S1: but the property might be true, like in the case of the sum of two sides of a triangle!
S6: (she hints at the blackboard) look at the second drawing… the horizontal side … If I move it rightward…
S7: I see it… the segment walks
S8: or moves like a worm.

S9: not, the worm increases and shortens the length…

S5: or if I shorten it… around the median… the median must remain the same… the angle between the adjacent sides becomes smaller and smaller, …
(S5 makes some drawings corresponding to the shortening of the horizontal side)

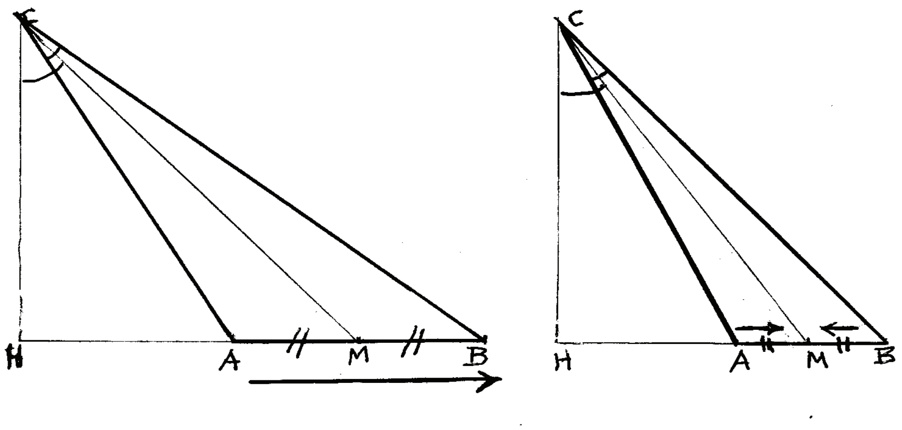

Figure 2: S6's and S5's ways of generating counter-examples

S6: beautiful! The angle between the two sides becomes smaller and smaller, up to zero, while the angle between the median and the height is still the same!

S5: an infinity of counter-examples!

S1: now I realize why I was not able to find the counter-examples: I was stuck to three cases only.

T: which cases?

S1: the height inside the triangle, the height that was one side of the triangle, and one case only of height outside the triangle.

S8: you were not lucky… if you had drawn another figure,…

S11: is it a problem of being lucky or not?

S8: we need a method to find counter-examples!

S5: I see here: the method is to move the figure; but in the case of the numbers ending with 1 the method looks very different!

S6: I am confused with the idea of one method, it is not like to calculate a division!

S1: following previous examples of search of counter-examples, I had a method… which did not work in this case!
(a long silence in the classroom)

T: Let us stop here. I am very satisfied with your work: with the contribution of different people we were able to get a solution for the problem posed by S1 with his … (many voices: conjecture) and to ask ourselves how to produce counter-examples.

Like in the first episode, the teacher asks students to write down a report on the above discussion.

Discussion: Theoretical and educational implications

Now we will consider some aspects of the above discussions. They concern the teleological and the epistemic sides of students' rational behavior, which are specifically inherent in the production of counter-examples, with an eye to educational and theoretical implications related to those sides. We will also consider some affective aspects of the discussions and their relationships with the productivity of the classroom work.

We may identify in the two discussions two different kinds of search for counter-examples: the exploration of individual examples (in the case of the arithmetic statement) and the continuous exploration of the figure (in the case of the geometric statement). What is it better to do, in order to develop students' mastery of the RMT of counter-examples? A solution might be to put into evidence the strategies that have allowed to solve the problems, in the perspective of students' "learning" of them. However Habermas' elaboration on rationality suggests that the a-posteriori analysis of the strategies that result in a failure (or in a success), and of the reasons for that success or failure, is important in order to perform better in future situations. Thus, in an educational perspective, Habermas' position suggests that what counts is not to learn the specific, effective strategies for the different kinds of situations, but to develop the capacity of analyzing them. We find in Habermas (1998), as requirements for teleological rationality in the case of a successful actor:

> he knows why he was successful (or why he could have realized the set goal in normal circumstances); and: this knowledge motivates the actor (at least in part) in such a way that he carries out his action for reasons that at the same time explain its possible success (Habermas, 1998, pp. 313-314)

In the Habermas' rationality perspective, what is relevant is to guide the students to analyze the relationship between the constraints and the opportunities of each experienced situation, on one side, and the goal to achieve, on the other, in the perspective of a-priori making the same kind of analysis in future situations. This remark does not concern only the case of the RMT of counter- example: when students learn to analyze a situation of production of counter-examples they learn to develop a rational behavior (on the teleological side) of more general interest for their rationality.

As concerns the epistemic side of the RMTs, we may observe that in both discussions the full mastery of knowledge inherent in the two problem situations (e.g.: the notions of consecutive numbers, of divisibility, and of prime number, in the first case; and of median and height of a triangle, and the different, possible shapes of triangles, in the second case) was fundamental for the productive development of the discussions. The activation and the productive exercise of a RMT requires a competence which underlies the RMT and that must be specified as a component of epistemic rationality, only implicit in Habermas' elaboration, if we want to adapt the rationality construct to mathematics education needs: the identification and choice of suitable knowledge related to the content of the statements. It includes also the logic-linguistic aspects of the situation at stake, as it is evident in the first discussion (the requirement "divisible by 3, or prime number" does not exclude divisibility by another number). These aspects are relevant in the case of the RMTs related to activities like conjecturing, proving, defining (for the RMT of definition, see Boero&Turiano, 2020).

As concerns affective aspects, we know that the productivity of the discussions (not only for the solution to be achieved, but also for the development of the students' competencies) strongly depends both on long term and on specific educational choices of the teacher. The productive search of counter-examples looks not only as a consequence of the already developed collaborative style of work in the classroom, but also as a promoter of the maintaining and developing of that style of work, of fundamental importance for the development of students' rationality (cf: "an intersubjectively shared lifeworld").

In the reported discussions we have seen how students intervene on their mates' interventions in a very collaborative way – both in the case of an opposition, and in the case of a collaboration to develop their mates' thought, or to make it more precise. Many times the utterance of a student is continued by another student. It is like if the development of ideas is a collective enterprise (which does not contrast the defense of personal ideas). Note also that in the first discussion students S5 and S6 are free to express how they have overcome some wrong ideas: a strong support for those students who (possibly) shared the same ideas but did not express them. Moreover (coming back to the problem of how to develop students' teleological rationality) what S5 and S6 do is precisely what is requested in general in that direction, according to Habermas. And it is also of general interest for the development of students' rationality the shift to the meta-level at the end of the second discussion, a strong contribution to develop students' awareness about their choices and actions (a crucial requirement for rationality).

The work in the classroom is apparently un-stressful: students' minds may move freely from technical mathematical issues to thoughts very far from mathematics, with jokes and double-sense interpretations. Note that this freedom helps (in the first discussion) to understand that some additional properties do not "disturb" the properties to be checked; and (in the second discussion) it allows to get a second infinity of examples through the image of the worm whose length changes. The productive (and friendly) climate in the classroom results from the conscious choices of the teacher since grade 6 (in Italy, the teacher teaches her students from grade 6 to grade 8), within the context of our research team.

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Continuity and rupture between argumentation and proof in historical texts and physics textbooks on parabolic motion

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In this paper, we analyze different presentations in a historical text by Galilei and a textbook for high school of the parabolic motion of a projectile with a lens developed within Mathematics education research on argumentation and proof (cognitive unity; Mariotti et al., 1997; Pedemonte, 2005). The analysis highlights possibilities and problematic issues, with particular attention to the aspects related to continuity and rupture between argumentation and proof in textbooks and the different interdisciplinary relationships between mathematics and physics mirrored by historical sources and textbooks. We discuss how a comparison between them can be exploited to develop a discourse about interdisciplinary that can enlarge the view of the relationship between the two disciplines and didactical implications that can be inferred from this comparison.

Keywords: Interdisciplinary approach, epistemology, proof, cognitive unity, textbook evaluation.

Introduction

To introduce the topic of our contribution, we start from three different representations of the trajectory of a projectile:

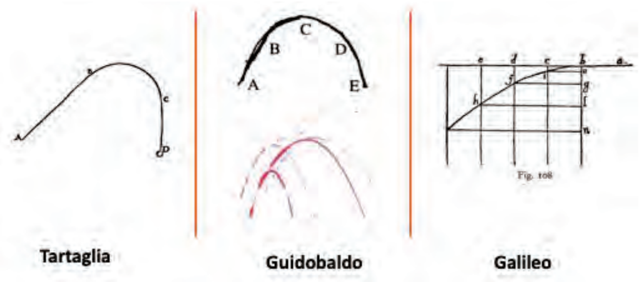


Figure 1: Three different representations of trajectory of projectile in historical texts

In Tartaglia's representation, the trajectory of a projectile consists of three parts: a straight part, followed by an arc of a circle and then ending in a straight vertical line. As stressed in Renn et al. (2000, p. 316): "in the Aristotelian tradition, projectile motion was conceived of as resulting from the contrariety of natural and violent motion, the latter according to medieval tradition acting through an impetus impressed by the mover into the moving body. According to this understanding of projectile motion, the trajectory cannot be symmetrical because the motion of the projectile is determined at the beginning and at the end by quite different causes. At the beginning it is dominated by the impetus impressed into the projectile, at the end by its natural motion towards the center of the earth.". Principles elaborated to interpret motion on the Earth were "embodied" in the form of trajectory, pursuing the aim to provide an axiomatic foundation to the analysis of projectile motion. Guidobaldo's sketch comes from an experiment. The paradigm was slowly changing in science and his transition work was crucial to challenge the medieval perception of motion. As we can see in his representation, the “symmetry” that he had experimentally found in the trajectory (it the ball will take the same path in falling as in rising, and the shape is that which, when inverted under the horizon) led him to corroborate the idea that not necessarily the different kinds of motions are consecutive. This opened the path to new hypotheses compatible with the possibility that motions can compose each other; in this frame, the trajectory could resemble a catenary or hyperbola or parabola. Galilei (1638), as we will show, in *Discourses and Mathematical Demonstration Relating to Two New Science*, completed the process of proving that the trajectory is parabolic, setting up an axiomatic system and grounding reasoning on rigorous proofs inspired to Euclidean ones. These steps were crucial in the birth of Physics and clearly show that the structure of reasonings developed mainly in Geometry, like axiomatics and deductive proofs, from the very beginning played a key role in the development of Physics (Renn et al., 2000). Udhen et al. (2012) stressed that: “the relationship between physics and mathematics has many facets, from the possibility to discover new physics within the mathematical structure to the mathematical nature of basic physical concepts. […] students should not only recognize that mathematics is a valuable tool for physics, but also that it can provide the underlying structure of a physical theory” (p. 493). These historical cases clarify why mathematics is said to play a structural role in physics.

Institutional context and differences between historical texts and textbooks

To promote students’ awareness of the interdisciplinary relationships between mathematics physics and philosophy, in a historical perspective, is a goal of secondary school in the Italian Licei (Mathematics curriculum). In particular teachers are asked to pay attention to these aspects with respect to the XVII century and the birth of modern science. The books by Galilei are the primary sources to consider in order to analyse the topic from the historical-epistemological point of view. In this book the conceptions of disciplines and their relationship differs from today since it is a foundative book, one of pillars of modern scientific method, and an example of rich scientific text that intertwines explicitly many dimensions of knowledge that nowadays are codified in disciplines (mathematics, physics, engineering, philosophy). Physics textbooks for secondary school present a disciplinary didactical transposition that is consistent with the (implicit or explicit) didactical goals of the authors. The topic is not addressed in the same way as Galilei: parabolic motion is presented as a particular case of two-dimensional motion and introduced deserving a lot of space to algebraic passages and formulas, also in the proof. The main differences can be due to the targets (scientific community vs students), the goals (proposing a new theory vs teaching), the development of disciplines and their epistemologies (Euclidean geometry and study of motion vs M&P curriculum at school), interdisciplinarity (scientific discourse intertwining different dimensions vs combination of elements of knowledge taught with a disciplinary perspective).

Literature review and research questions

In this paper, we focus on a specific aspect, epistemologically relevant from the disciplinary and interdisciplinary point of view, that is the way argumentation and proof (A&P) are presented in two texts about parabolic motion: Galilei (1638) and the chapter Two-Dimensional Kinematics in the physics textbook by Walker (2017; high school edition, translated also in Italian). Among the different textbooks used in Italy, we chose that one because it is quite rich from the epistemological point of view (Bagaglini et al., 2021). A first reading of the books showed that in both cases they deal with proving/demonstrating that the trajectory of a projectile is an arc of parabola, but the meanings of the term "proving" seemed to change, as well as the way proof were presented and intertwined with other aspects of the scientific argumentation. We consider A&P key concepts to analyse the structural role of mathematical thinking in physics learning in an interdisciplinary perspective. On one hand awareness about the relationship between mathematical proofs and physical argumentation contribute to developing an authentic picture of the role of mathematics in physics. On the other hand, to trigger a reflection about the meaning of A&P in mathematics and physics (M&P) is an opportunity to investigate the epistemology of such disciplines. With respect to the literature review in mathematics education, our research aims at contributing to address some open questions proposed in the handbook by Durand-Guerrier et al. (2012) about A&P in mathematics and empirical sciences: To what extent should mathematical proofs in the empirical sciences, such as physics, figure as a theme in mathematics teaching so as to provide students with an adequate and authentic picture of the role of mathematics in the world? Could a stronger emphasis on the process of establishing hypotheses (in the empirical sciences) help students better understand the structure of a proof that proceeds from assumptions to consequences and thus the meaning of axiomatics in general? We consider the way the bridge built between mathematical and physical aspects of A&P is presented crucial to address the nature of such a relationship from the didactical point of view. The type of presentation of a proof is already under investigation in mathematics education; open questions we are interested in are: To what extent and how is the presentation of a proof (verbal, visual, formal etc.) (in)dependent on the nature of the proof? Do students perceive different types of proofs as more or less explanatory or convincing?" (Durand-Guerrier et al., 2012). We hypothesized that connecting the notions of A&P in M&P makes this aspect even more important, since the verbal, the visual and the formal aspects of proof might play a different role in explaining and convincing students when "mathematizing" observation and reasonings about empirical phenomena or experiments, and in mirroring the nature of such a kind of proof, whose complexity is evident also in the historical cases briefly resumed in the introduction.

In this paper we analyse the way knowledge belonging to M&P (objects, reasonings, assumptions, epistemological issues) is used in argumentative steps and proof in different texts. We consider the analysis of A&P in texts and the comparison with historical texts a key step to move from the historical-epistemological and cognitive analyses to the classroom practices, in particular considering teacher-students education. This issue has been investigated by papers presented in CERME10 (Stylianides et al., 2018); among the themes discussed, we contribute to highlight the role of language in teaching and writing proofs and to search for analytical frameworks for argumentation and proof in textbook expositions.

Research framework

The didactic value of inserting proof into an argumentative process that involves students in the formulation of conjectures has been highlighted by many studies as a way to move from a reproductive approach to demonstration to a productive one and to focus on proof as a process more than on proof as a product. The construct of cognitive unity has been introduced by Mariotti et al.

(1997) to encode this idea and to stress the need for didactical situations in which the construction of a proof naturally follows from the exploration of a problematic situation by students. In particular we refer to this key aspect: "some aspect of continuity, concerning the production, during the construction of the conjecture, of the elements ("arguments") that are used later during the construction of the proof" (p. 1). This way some elements that characterize the proof (the choice of a statement to refer to, or of the semiotic representation register) are not artificially and suddenly introduced but arise naturally from the exploration, as it happens when statements are proved in research. Otherwise there is a cognitive rupture (Pedemonte, 2005). Proving that the trajectory of a projectile motion is parabolic can be considered a conjecture-proving problem, according to the characterization of Mariotti et al. (1997).

We assume that continuity should be pursued also in physics teaching to guarantee a productive approach of students to proving in this field, in particular when mathematics appears in the statements and semiotic representations of physical entities, since students need to activate resources related to their conception and experience of mathematical processes. What happens to the flow of observation and conjectures about physical phenomena when mathematics enters the discourse? If teachers have to guide a classroom discussion to help the students to include these aspects, is continuity between A&P pursued or do their interventions cause cognitive rupture? As we showed, the issue is critical from the epistemological point of view, so we think teacher-students need examples and meta reflection to guide the students properly in such classroom discussions. The cognitive unity has been developed, and is mainly used, to analyse students' reasonings. We consider texts targeted to non-expert readers as examples of forms of presentation of reasonings, as they were teachers' speeches when they guide students who made observations and conjectures to gradually organize their reasonings. These can be prototypes of different ways the teachers scaffold students' approach to interdisciplinary A&P in the classroom, with possible different impacts on students' learning. We consider thus it useful to carry out analyses with the same lens used with students of the ways the texts guide the readers to move from exploration to A&P.

Methods

The books were analysed at two scales: a global analysis of the organization of the books with epistemological and linguistic lenses (Bagaglini et al., 2021), and zooming in on some excerpts where we could find relevant aspects to analyse in order to identify continuity and rupture between A&P in the texts. In this paper we focus on the second aspect. From the methodological point of view, we referred to the analysis of cognitive unity and rupture proposed by Pedemonte (2005):

– *structural analysis*: refers to the link between the structures of statements used in argumentations and in proofs. There is structural cognitive unity when statements used in the argumentation are also used in the proof. Otherwise, there is structural cognitive rupture.

– *referential analysis*: refers to the systems of reference used in argumentations and in proofs, that is, the systems of signs (drawings, calculations, algebraic expressions, etc.) and systems of knowledge (definitions, theorems, etc.) used. There is referential cognitive unity when some systems of signs or knowledge are used both in the argumentation and the proof. Otherwise, there is referential cognitive rupture. We enlarged it according to our goal (interdisciplinary analysis of prototypes of A&P

connections). We carried out a structural and referential analysis of relevant excerpts from the third and fourth day, concerning the study of local motions in Galilei (1638) and Walker (2017). We identified statements in A&P related to parabolic motion and then systems of representation and knowledge belonging to both mathematics and physics (considered as disciplines taught at school in grades 9-10 in Italy in the textbook's analysis and as historical disciplines analyzing Galilei's excerpts). We organized them on tables reporting on the left the excerpt (statements), on the right the referential analysis. By comparing the A&P steps, thanks to the structural and referential analysis, we detected unity or rupture in both texts. Because of space constraints, we report only a few excerpts to show the analysis of the proof of the statement "the trajectory of a projectile is parabolic" and the previous choices made in the argumentative part.

Main results of the analysis of unity or rupture in Galilei's and Walker's texts

By steady or uniform motion [1], I mean one in which the distances traversed by the moving particle [2] during any equal intervals of time [3], are themselves equal. [D1].	Definition of uniform motion using proportions (equal space in equal time)
A motion is said to be uniformly accelerated [4], when starting from rest, it acquires, during equal time-intervals [3], equal increments of speed.[...] the distances traversed [2] are proportional [D1] to the squares [5] of the times.	Definition of accelerated motion using proportions (equal increments of speed in equal time, space proportional to the square of time)
Imagine any particle projected along a horizontal plane without friction; if the plane is limited and elevated [6] the resulting motion which I call projection [7], is compounded of one which is uniform and horizontal [1] and of another which is vertical and naturally accelerated [4].	Definition of projectile, that incorporates the assumption of composition of motions
Theorem 1 – Proposition 1: A projectile [7] which is carried by a uniform horizontal motion [1] compounded with a naturally accelerated [4] vertical motion describes a path which is a semi-parabola [8].	Theorem formulated using previous definitions
The section of this cone [..] which is called a parabola [8] [..] the square [5] of bd is to the square [5] of fe in the same ratio [9] as the axis ad is to the portion ae.	Definition of parabola
Let us imagine an elevated [6] horizontal line or plane ab along which a body moves with uniform [1] speed from a to b. Suppose this plane to end abruptly at b [6] [..]. Draw the line be along the plane ba to represent the flow, or measure, of time; divide this line into a number of segments, bc, cd, de, representing equal intervals of time [3] [..] in proportion [D1] to the squares [5] of cb, db, eb, or, [..] in the squared ratio [9] of these same lines. [...]the distance traversed [2] by a freely falling body varies as the square [5] of the time; in like manner the space eh traversed [2] during the time be will be nine times [D1] ci; thus it is evident that the distances eh, df, ci will be to one	Proof is presented, where: - the same terms introduced before are used, as well as the same spatial representation (segments/intervals of time) - it is stressed the use of proportional reasoning, that was used to define the kinds of motions that are combined - G. recalls the assumptions about the composition of motions

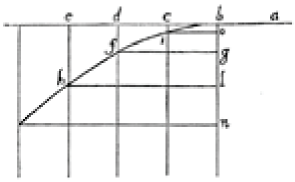
another as the squares [5] of the lines be, bd, bc. The square [5] of hl is to that of fg as the line lb is to bg [D1]; and the square [5] of fg is to that of io as gb is to bo; therefore the points i, f, h, lie on one and the same parabola [8]. 	- G. recalls the setting associated to the definition of projectile with the same words - G. intertwines the definition of parabola and the characterization of accelerated motion in order to exploit the linguistic analogies to stress that the points must lie on a parabola.

Table 1: Analysis of Galilei's excerpts

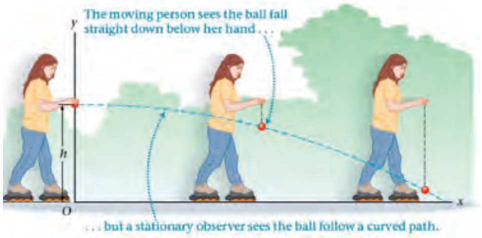
Big Idea 1 Two-dimensional motion is a combination of horizontal and vertical motions. The key concept behind two-dimensional motion is that the horizontal and vertical motions are completely independent of one another; each can be considered separately as one-dimensional motion.	The combination and independence of horizontal and vertical motions are initially introduced in a lateral box as Big Idea. The status of the statement in terms of elements of A&P (axiom, theorem) is not expressed.
Projectile Motion: Basic Equations We now apply the independence of horizontal and vertical motions to projectiles. Just what do we mean by a projectile? Well, a projectile is an object that is thrown, kicked, batted, or otherwise launched into motion and then allowed to follow a path determined solely by the influence of gravity.	The Big Idea is applied to projectile motion to obtain its equations and a phenomenological description of the projectile is presented.
Demonstrating Independence of Motion: A simple demonstration illustrates the independence of horizontal and vertical motions in projectile motion. [..] Notice that the ball goes straight down, lands near your feet, and returns almost to the level of your hand in about a second. [...] To you, its motion looks the same as before. The fact that you were moving in the horizontal direction the whole time had no effect on the ball's vertical motion — the motions are independent. 	A figure represents a moving person with a roller skate and a falling ball; the two combined motions are represented with a reference to real life. The motion is seen also by an external observer and the trajectory is linear and vertical in the system of person and curved in the external system, that is represented through cartesian axes put onto the real life figure. The relativity of motion in different systems is used to demonstrate independence of motions.
To an observer who sees you walking by, the ball follows a curved path, as shown. The precise shape of this curved path—a parabola—is verified in the next section.  (a)	A picture (photo with a camera to a real world phenomenon) is proposed. In the description of the figure, it is mentioned the visualization of concepts and presented as one among other"examples of principle" of independence of motions.

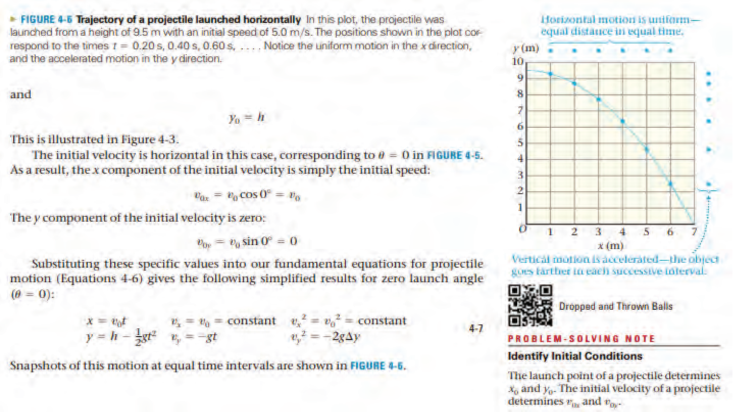
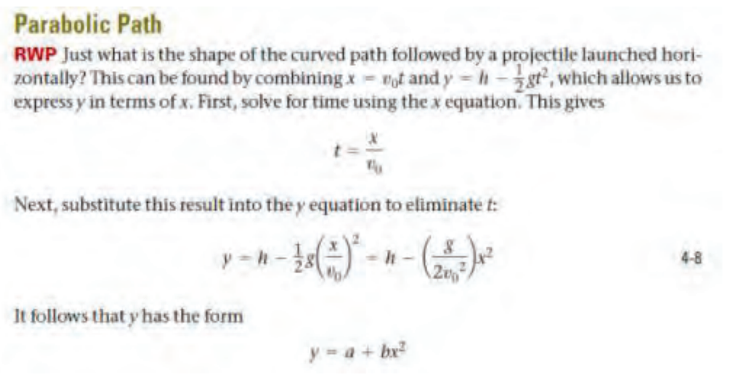
FIGURE 4-4 Visualizing Concepts - Independence of Motion (a) An athlete jumps upward from a moving skateboard.	It is anticipated that the shape is a parabola and that this will be verified later.
 FIGURE 4-5 Trajectory of a projectile launched horizontally In this plot, the projectile was launched from a height of 9.5 m with an initial speed of 5.0 m/s. The positions shown in the plot correspond to the times t = 0.20 s, 0.40 s, 0.60 s, ... Notice the uniform motion in the x direction, and the accelerated motion in the y direction. and $y_0 = h$ This is illustrated in Figure 4-3. The initial velocity is horizontal in this case, corresponding to $\theta = 0$ in FIGURE 4-5. As a result, the x component of the initial velocity is simply the initial speed: $v_{0x} = v_0 \cos 0° = v_0$ The y component of the initial velocity is zero: $v_{0y} = v_0 \sin 0° = 0$ Substituting these specific values into our fundamental equations for projectile motion (Equations 4-6) gives the following simplified results for zero launch angle ($\theta = 0$): $x = v_0 t \quad v_x = v_0 = \text{constant} \quad v_x^2 = v_0^2 = \text{constant}$ $y = h - \frac{1}{2}gt^2 \quad v_y = -gt \quad v_y^2 = -2g\Delta y$ 4-7 Snapshots of this motion at equal time intervals are shown in FIGURE 4-6. PROBLEM-SOLVING NOTE Identify Initial Conditions The launch point of a projectile determines x_0 and y_0. The initial velocity of a projectile determines v_{0x} and v_{0y}.	A graph, resembling the one by Galilei but the use of x,y and units on the axes, is in a lateral box. The horizontal uniform motion is presented using proportions (equal space in equal time) without mentioning the nature of this description as definition. The same happens to vertical accelerated motion. Symbolic expressions are used for the generic case and the Galilei case is obtained substituting a value into equations for projectile motion.
 Parabolic Path RWP Just what is the shape of the curved path followed by a projectile launched horizontally? This can be found by combining $x = v_0 t$ and $y = h - \frac{1}{2}gt^2$, which allows us to express y in terms of x. First, solve for time using the x equation. This gives $t = \frac{x}{v_0}$ Next, substitute this result into the y equation to eliminate t: $y = h - \frac{1}{2}g\left(\frac{x}{v_0}\right)^2 = h - \left(\frac{g}{2v_0^2}\right)x^2$ 4-8 It follows that y has the form $y = a + bx^2$	An algebraic version of the proof is presented (never named proof), with: - reference to a curved path: - the term “found” instead of verify - use of symbolic expression of the two motions combined, as well as the parabolic generic equation - no reference to assumptions about the combination of motions - the use of terms “substitution” and “eliminate” - no mention of the previous graph and the exemplification of principles of independence.

Table 2: Analysis of Walker’s excerpts

Discussion and conclusions

The first analysis shows that Galilei’s text is characterized by structural and referential unity: he mathematized the relationship between space and time with magnitudes and proportions and used always the same objects and properties to merge the observation of phenomena, empirical laws and geometrical properties of conic sections. The mathematization of the experimental setting allowed him to prove, deductively, that the trajectory is a semi-parabola, under the hypothesis that the motion of a projectile results from a composition of independent uniform and accelerated motion. The theory of magnitudes bridges the concrete action of measuring and the theoretical comparison between geometrical magnitudes. The graphic representation plays a crucial role, since the action itself to trace a line/curve with a motion of a point is a sort of ideal machine that draws a trajectory, hybridizing the notion ofs trajectory and geometrical curve to treat the trajectory geometrically. In this case the structural role of mathematics clearly emerges: “importing” the structure of Euclidean proof in the investigation of motion allows to refine and strengthen argumentation.

In Walker's chapter, it is visible the effort to consider the dimension of A&P: there are physical assumptions, a definition of projectile, examples that ground the assumptions about the composition of independent motions on empirical facts, stressing that they are realistic. Some referential choices are consistent: the motion of a projectile is a particular case of a more general motion, equations of evolution are used to derive new equations treating time and space as algebraic variables. However, many elements of rupture are present. both in terms of structural and referential analysis of the relationship between argumentation and proof. Indeed, the presentation of the argument concerning physical principles and entities and the proof are presented with figures and pictures related to real life, while in the derivation of the equation they switch suddenly to algebraic language and analytical reasoning (substituting variables in functions). Moreover definitions, principle, inference, proof are never mentioned. The link between empirical aspects and mathematical knowledge is hard to establish for a reader, because of the strong discontinuity in terms of use of signs and semiotic registers for the expression of the statements.

Our analysis highlighted issues that we consider crucial from the didactical point of view since they connect relevant issues of mathematics education to interdisciplinarity M&P. In particular, from such a comparison prospective teachers can gain awareness about the ruptures that can be found in textbooks and thus adapt their teaching practices to pursue cognitive unity by reflecting on the aspects we stressed with their students and compensating for the weakness of textbooks.

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How teachers address process-product dualities in mathematical argumentation processes

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Teachers play an important role in creating environments for mathematical argumentation processes in classrooms. In the transition from arithmetic to algebra for example, students depend on how teachers address the process-product duality of mathematical argumentation and the duality of mathematical objects. Two empirical examples illustrate that eight-grade students (13-14 years old) struggle with these dualities in mathematical argumentation processes and that they need support of their teachers when creating generally valid arguments in class. This study examines how teachers' actions guide students in handling the process-product dualities and how teachers thus influence and shape the construction of mathematical arguments in classrooms.

Keywords: Process-product duality, mathematics teachers, mathematical argumentation, classroom research.

Introduction

Little research has been done on how teachers act while initiating mathematical argumentations in class discussions (e.g., Conner et al., 2014; Schwarz et al., 2006). It is still uncertain what influence teachers' specific actions have on the quality of the arguments that emerge in classrooms. In addition, there are two process-product dualities students must handle in class when it comes to mathematical argumentations. On the one hand, mathematical argumentations are processes in which participants aim to justify or refute a claim with the intent to formulate a mathematical argument – a product. On the other hand, mathematical arguments are always arguments about mathematical objects and these mathematical objects can be interpreted in a process-oriented or a product-oriented way (Sfard, 1991; Sfard & Linchevski, 1994). Whether the interpretation of the mathematical objects is more process-oriented or product-oriented becomes visible in the arguments produced by the students. How these interpretations of mathematical objects in mathematical arguments are shaped or influenced by teacher actions is still mostly unexplored and is a focus in our project. This paper presents examples of mathematical argumentations in the transition from arithmetic to algebra from two grade 8 classes. The following research questions guide our analysis: How is a process-oriented or a product-oriented interpretation of mathematical objects reflected in the mathematical argumentations and arguments? How does the teacher address these interpretation processes?

Theoretical Framework

Process-product dualities in mathematical argumentations

In school contexts mathematical argumentations in the transition from arithmetic to algebra include two process-product dualities: 1) In classrooms, mathematical argumentation processes are mostly oral and generally distinct from the argument (product) that emerges, which is commonly fixed in a written form (Boero, 1999). This is one duality of mathematical argumentation. 2) At the same time, the mathematical objects in the argumentations can be understood in a dual way: operationally, this means process-oriented, or structurally, which means in a product-oriented way (Sfard, 1991; Sfard & Linchevski, 1994). The algebraic symbolic language, which is itself initially a learning object, can also be interpreted in these two ways. A process-oriented interpretation of an algebraic term means to understand the term as a "sequence of instructions" (Sfard & Linchevski, 1994, p. 191). The term, which one can calculate, represents a computational process. At the same time the term can be interpreted in a product-oriented way as a "product of a computation" (Sfard & Linchevski, 1994, p. 191). In this case, the term is treated as an object itself and mathematical structures of the term are focused. Mathematical argumentations reflect this duality.

These interplays between process-product dualities are significant in our data. For example, if one considers the sum of an even and an odd number, students calculate numerical examples on an operational level and conjecture that the sum is always odd. A process-oriented interpretation of a numerical term here means to calculate a concrete number. At the same time, it is possible to look at these numerical examples from a structural perspective and formulate them as generic examples. When arguing with their mathematical structure, a generally valid mathematical argument arises. Seeing this mathematical structure, requires a change from a process-oriented to a product-oriented interpretation, which students and teachers are often not aware of. A process-oriented interpretation of a term is not sufficient to create a valid argument for a general claim, because a generalization is needed, and this requires arguing with mathematical structures and general characteristics of the mathematical objects. How teachers and their actions contribute to this change of view and support the emergence of valid arguments is of interest here.

Mathematical arguments and their representation

According to Toulmin (1958) an argument consists of different elements. A claim is a statement that participants try to justify or refute. In this study claims are marked with a C. Data (D) are accepted, undoubted facts or shared knowledge of a community, the starting point of an argument. With a warrant (W) the step from the data to the claim is legitimated. Some arguments also contain backings (B) that state why a warrant is generally applicable. To describe the certainty of an argumentation step, a modal qualifier (Q) can be integrated into the argument. Sometimes rebuttals (R) indicate "circumstances in which the general authority of the warrant would have to be set aside." (Toulmin, 1958, p. 101). In addition to these core elements established by Toulmin (1958), Knipping and Reid (2015) introduced refutations (X) as another element of arguments. When parts of an argument are not accepted by a community or are not shared knowledge, these parts can be refuted. But a refutation is not a rebuttal. Refutations refute complete parts of arguments, whereas rebuttals only restrict the conclusion to a locally limited extent (Knipping & Reid, 2015). Therefore, refutations can apply to any element of an argument. In our data, we found statements about the applicability of the warrant in an argumentation step. We call this kind of statements backings B*. These statements specify a characteristic of the datum and relate the warrant, the general rule, to the data. This legitimates the use of the warrant in the argumentation step. These kinds of statements aren't backings in Toulmin's sense because they don't support the authority of a warrant in general but support its applicability in the specific case. On the other hand, they have a different function than data. Toulmin (1958, p. 106) states that data must be explicit in an argument, but these statements are additional facts which characterise the datum and don't need to be made explicit in the argument. Mathematical argumentation processes in mathematics classrooms often include more than one argumentation step. When a conclusion of a previous step is already accepted, it is possible to use this conclusion as data for another step.

Mathematical arguments are represented in different ways in mathematics classrooms. Reid and Knipping (2010) distinguish several types of arguments according to their representation. In school, in the transition from arithmetic to algebra, numeric generic, pictorial generic, narrative, and symbolic arguments are particularly important. Numeric generic arguments use concrete numbers and their inherent structures to justify a general claim. In contrast, pictorial generic arguments, also called visual proofs (Reid & Knipping, 2010, p. 136), practice transformations on visual objects to emphasize the mathematical structures and argue with them. Mathematical arguments which contain only words are called narrative arguments. Symbolic arguments are based on the algebraic symbol language. All these arguments can potentially be read and interpreted in a process- or product-oriented way (although this is already inherent in written narrative arguments). Each interpretation effects the validity of the proposed mathematical argument. Teachers are often only partly aware of these different interpretations when they guide their students in creating generally valid arguments.

Role of the teacher in mathematical argumentations in the transition from arithmetic to algebra

Developing generally valid mathematical arguments in classroom discussions is challenging for students and teachers, especially in the transition from arithmetic to algebra. Teachers have a decisive influence on whether and how mathematical arguments emerge (Schwarz et al., 2006; Conner et al., 2014). In the transition from arithmetic to algebra, mathematical structures are increasingly emphasised, and mathematical properties and relations are used for generally valid arguments. Empirical research documents that symbolic arguments often dominate in mathematic classes (Brunner, 2014). These types of arguments focus mostly on process-oriented transformations of symbols. In this case, the algebraic symbol language often remains misunderstood, and learners demonstrate difficulties in grasping the underlying mathematical structures of the algebraic symbol language used (Kieran, 2020; Pedemonte, 2008). For example, students show difficulties in the interpretation of the equal sign. The equal sign can be interpreted as a "do something signal" or as a "symbol for equivalence" (Kieran, 1981, p. 317). The first, process-oriented interpretation triggers students to calculate the term. The left side of the equation is seen as a task one needs to solve, while the right side is considered as the solution of the calculation. In a product-oriented interpretation, the equal sign can be seen as a symbol between two equivalent terms. In this way, the relation between the terms is focused. Grasping mathematical structures and relations is also crucial when creating valid mathematical arguments for general claims. A process-oriented calculation of a term is not sufficient to grasp inherent mathematical structures and to capture general characteristics of the term. Often a product-oriented interpretation of mathematical objects is required. For example, when arguing with the equation "$2\cdot x+2\cdot y=2\cdot(x+y)$" that the sum of two even numbers is always even, one needs to interpret the algebraic symbols and the equal sign in a product-oriented way. Without an interpretation of the equal sign as a relational symbol (product-oriented) between two terms, one cannot grasp the relation between the terms and cannot argue that the sum of two even number is always even. Students often struggle with these interpretations and need the support of teachers who are aware of these difficulties (Kieran, 2020).

Methods

For researching mathematical argumentation processes in the transition from arithmetic to algebra, a learning environment with 4-5 lessons (90 min each) was designed and implemented in three eight-grade classes (13-14 years old) in Bremen (Germany). The lessons were taught by their teachers, who had different school experiences. Teacher A had been a fully trained teacher for half a year only, teacher B had been a teacher for 10 years and teacher C had been teaching about 5 years. Teacher A and C worked at an "Oberschule" (a German type of comprehensive school), teacher B at a "Gymnasium" (a selective type of school). As a starting point, different representations of mathematical arguments (Reid & Knipping, 2010) were introduced to the learners to facilitate learners' various approaches to mathematical structures and to smooth the transition to algebra. Transcripts of the classroom discussions were made, which included drawings and descriptions of gestures and actions. Based on these transcripts the mathematical arguments that emerged in class were reconstructed with Toulmin's (1958) functional model of arguments, which is well established in mathematics education (Knipping & Reid, 2015).

As the goal was to reconstruct and analyse actual classroom argumentations, our reconstructions include statements and arguments that are not correct from a mathematical point of view. We keep the statements in the reconstructed arguments almost the same as the original utterances of the participants during the classroom discussions. So, it is possible to use the statements in the reconstructed arguments (diagrams) to analyse the participants' interpretations of the mathematical objects involved. For this analysis we use a distinction by Caspi and Sfard (2012), who differentiate between a processual, a granular and an objectified description of a term. A processual description of a term is a process-oriented description of the computational process. In an objectified, product-oriented description, the relationship between the objects is focused, and the actual computational process recedes into the background. Here verbs are nominalized to objects. The granular level is an intermediate step, which will not be focused in our analysis and this article. These analyses allow us to distinguish between a process-oriented and a product-oriented interpretation of mathematical objects in argumentations. For example, a student who describes the term "a+b" as "adding b to a" interprets the term in a process-oriented way, while "the sum of a and b" would be considered as a product-oriented interpretation.

This paper focuses on an argumentation task about "The sum of two even numbers". Students look at several numerical examples and generate a conjecture. Almost all students conjecture that the sum of two even numbers is always even. Afterwards, a reflection is requested if their conjecture is generally valid or if counterexamples exist. Finally, the students are asked to create a mathematical argument. In the following classroom discussion, students present their written arguments to the classroom community. While students' written solutions and arguments are projected on a wall, argumentative elaborations on these products are presented, which constitute further argumentation processes. Products and processes of argumentations are intertwined here. The results of our analyses of these discussions and argumentations are focused in the next section.

Results

In the following two examples, it becomes evident that students struggle with the process-product dualities mentioned above, when they attempt to construct a generally valid argument. The teacher shapes the students' handling of the process-product duality and thereby the creation of mathematical arguments. So, the crucial role of the teacher is apparent in these examples.

Example 1: Numeric generic argument in teacher A's classroom

In the first example, Tabea tries to support the claim that "The sum of two even numbers is always even" with a numeric generic example (figure 1). A projection of Tabea's notes is shown to the class: "14+20=(2·7)+(2·10)=14+20=34". Tabea's first attempt to justify the claim that "The sum of two even numbers is even" is based on a transformation of her numerical example "14+20". Her first transformation "14+20=(2·7)+(2·10)" can be seen as a step towards constructing a generic example. Tabea applies the property that even numbers are divisible by two and thus presents the numbers as a product with the factor two. In this representation a mathematical structure and a product-orientation become visible. In the second step however, Tabea switches to a calculation, "(2·7)+(2·10)=14+20", thus presents a process-oriented perspective. In this step, she doesn't use the represented mathematical structure for a general argument, but she comes back to her starting point. So, Tabea presents an argument that does not contain a generalisation and no connection to her general claim. Therefore, the teacher comments that Tabea "goes in a circle" and refutes Tabea's argument.

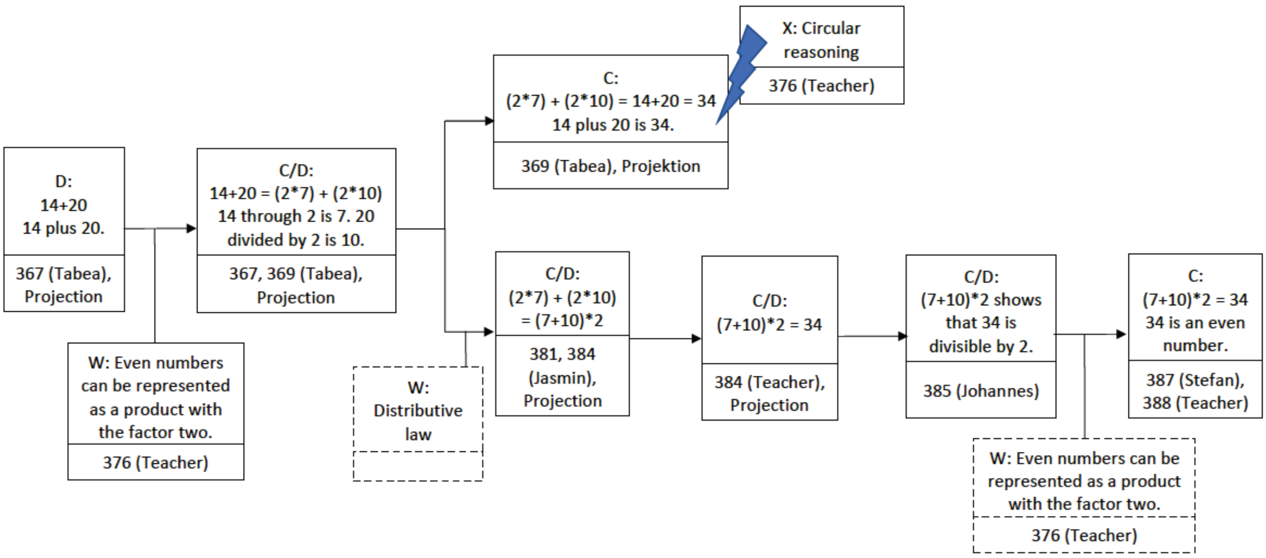

Figure 1: Reconstruction of the numeric generic argument with the Toulmin scheme (Example 1)

Later, the teacher repeats Tabea's first step and asks students to continue with "(2·7)+(2·10)": "What would be the next logical step now?". Jasmin extends Tabea's first step to "(2·7)+(2·10)= (7+10)·2" and uses the opportunity Tabea created. The inherent structure of the numerical example is used and represented explicitly. This requires a structural interpretation of the algebraic term and the equal sign. But now, the teacher pushes towards a calculation, saying that the product is 34. Here, the teacher switches back to a process-oriented interpretation. In the last two steps of the argument, the structure of the numerical example, inherent in the symbolic representation, is used for a conclusion by the students. Johannes finally states that the term "(7+10)·2" shows that 34 is divisible by two, and Stefan concludes that therefore, 34 is an even number. Both emphasise again a structural view. So, in these last two argumentation steps the structure of the generic example is used again. But like in Tabea's attempt, a generalisation is missing - there is no explicit connection to the claim that the sum of two even numbers is even, which the class tries to legitimate.

How does the teacher shape the argument? After Jasmin's transformation of the term, the teacher switches back to a process-oriented calculation of the result. The teacher guides the students with her statement to argue about the concrete number 34. She even asks what this term shows about 34: "What does this seven plus ten times two tell us now, what does it say about thirty-four?". This question invites students to focus on the concrete number, which makes it difficult for the students to conduct a generalisation. The students offer a product-oriented interpretation of the term, but stick to the specific example, instead of justifying the general claim. Therefore, a final conclusion like "The sum of two even numbers is even" is missing in the whole argument. A numeric argument emerges, but its generality is not discussed and addressed by the students and the teacher.

Example 2: Symbolic argument in teacher B's classroom

The second example illustrates the creation of a symbolic argument about the sum of two even numbers in a classroom discussion (figure 2). As a starting point, the teacher asks the students to express an even number with variables by referring to a previous symbolic argument, which also includes even numbers. Levke introduces "2·n" and "2·x" and the teacher writes the sum of them on the blackboard. In these summands, a characteristic of even numbers is visible; each summand is a product with the factor two. But the students' interpretation of the term is ambiguous. The situation allows either a process- or a product-oriented interpretation.

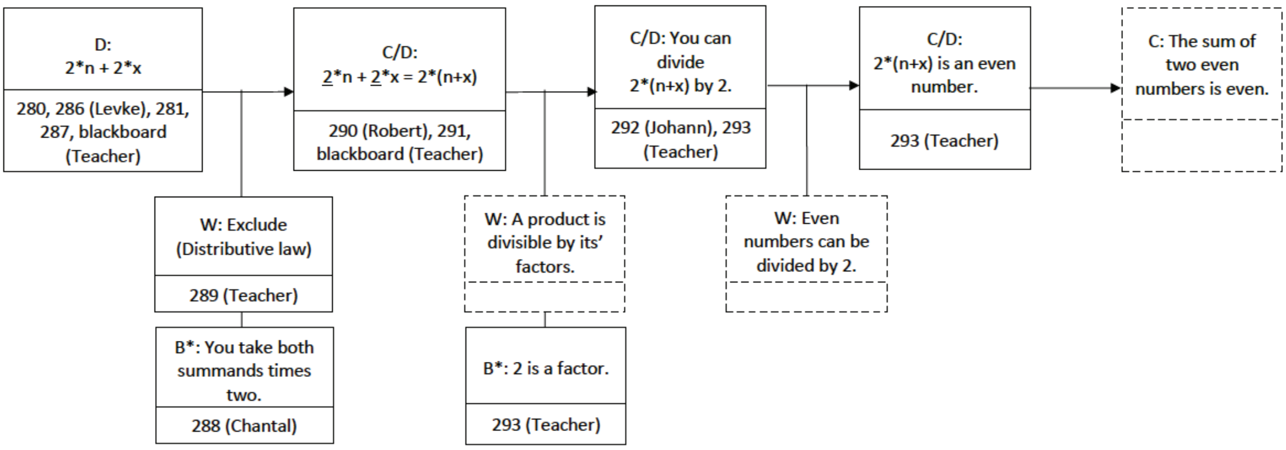

Figure 2: Reconstruction of the symbolic argument with the Toulmin scheme (Example 2)

The teacher asks what is common in both summands. Chantal points out that both are multiplied by two and thus indicates that the distributive law is applicable here. We reconstruct this as a specific kind of backing B*, which justifies not the warrant in general, but the applicability of the warrant in this argumentation step. Chantal's backing B* indicates her process-oriented interpretation of the term. The associated warrant, added by the teacher, is also expressed in a process-oriented way. Consequently, the term is operationally transformed into "2·n+2·x=2·(n+x)". Now "2·(n+x)" must be interpreted. Johann claims that one can divide this term by two. But he does not explicate a warrant and sticks to a process-oriented interpretation. The corresponding warrant “A product is divisible by its’ factors.” is left implicit. The teacher complements Johann’s argumentation step by “2 is a factor”, which is another example of a backing of type B*. This statement B* relates the general warrant to the specific case. “2 is a factor” now supports a product-oriented interpretation. Finally, the teacher concludes that “2·(n+x)” is an even number, and emphasises again a product-oriented interpretation, while the students’ interpretations of the term “2·(n+x)” might be different. If they create a generalisation and therefore connect the argument to the implicit final conclusion is ambiguous.

All in all, the whole argumentation process is strongly guided by questions from the teacher and the students are forced to answer these questions. Students mainly suggest process-oriented statements, while the teacher sometimes switches to a structural level and supplements elements for the argument. The students seem not to grasp the structural ideas proposed by the teacher, but they keep answering and arguing in a process-oriented way.

Summary and Conclusion

In the presented examples of two classrooms, both teachers try to address and shape the handling of the process-product dualities in students’ argumentations but in different ways. In both examples, a product-oriented interpretation is needed in the argumentation process to create a generalisation and produce a generally valid argument. In the first example, the teacher makes it difficult for the students to come up with such a generalisation, because she calculates the result of the generic example. The students use the structure of the created generic example for an argument, but they fail in generalising their claim; the students stick to their numeric example. The second classroom shows a teacher alternating between a process-oriented and a product-oriented interpretation, while students stick to a process-orientation. In two moments, students express a process-oriented statement and the teacher switches to a structural view. However, students do not reproduce this structural view in their contributions. In both classroom episodes a final general conclusion is implicit or even missing in the emerging argument.

All in all, the examples illustrate that a product-oriented interpretation of mathematical objects is often crucial for the creation of generally valid mathematical arguments in classrooms, while it is challenging for students and teachers. The first example also shows that a product-orientation is not sufficient. Especially for generic arguments, a generalisation is required for a complete and generally valid argument. A product-orientation does not produce a generalization by itself. In addition, students must handle the process-product duality of mathematical argumentation. The goal of these classroom discussions (example 1 and 2) is to formulate a written argument. In class, students and teachers switch back and forth between an interpretation of single steps in a written argument and the oral argumentation, where each step is justified and negotiated as reasonable.

It is evident that it matters for students how their teachers address the process-product duality of algebraic symbols. In both examples, it becomes obvious that the interplay between students’ and teachers’ process- or product-orientations is essential. Students need support in reflecting structures inherent in mathematical objects and making generalisations. The comparative analyses of the data in our project illustrate different challenges that teachers encounter in this context. More insights will be gained by further analysis in this project.

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Guiding principles for teaching mathematics via reasoning and proving

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Against the backdrop of policy documents and educational researchers' vision of proof as an essential component of teaching mathematics across content areas and grade levels, teaching of reasoning and proof in mathematics classrooms remains an elusive goal. Teachers and the type of teaching they enact in classrooms are crucial for achieving this goal. This theoretical paper builds on the concept of proof-based teaching and suggest a set of guiding principles for what we call teaching mathematics via reasoning and proving. *These principles were developed as a part of a multi-year design based project, and implemented in an undergraduate course* Mathematical Reasoning and Proving for Secondary Teachers. *We illustrate these principles using examples from proof-oriented lessons plan developed by prospective secondary teachers.*

Keywords: Reasoning and Proof, Classroom Instruction, Secondary Mathematics Teachers

The state of reasoning and proving in mathematics classrooms

Learning mathematics meaningfully entails more than following procedures and recalling facts; rather, it requires understanding, making sense of, and engaging in mathematical practices, such as identifying patterns, conjecturing, exploring with examples and counterexamples, and justifying and critiquing arguments. Collectively, these disciplinary practices are called *reasoning and proving* (Ellis et al., 2012; Stylianides, 2008). Proof serves many roles in mathematics, but in mathematics classrooms, reasoning and proving mainly serve to support students' sense-making by providing insights into *why* something is true, and not just *that* something is true. Importantly, the means for showing why should be through deductive reasoning rather than relying on authority of the teacher or a textbook (Hanna & deVillers, 2012). Student participation in reasoning and proving activities across grade levels and mathematical topics is widely recognized as essential for meaningful learning (NGA & CCSSO, 2010; NCTM, 2009), due to the focus on mathematical sense making. Reasoning and proving are also linked to knowledge retention, enhanced understanding and making mathematics intellectually satisfying for learners (Harel, 2013).

However, the uptake of reasoning and proving as an integral part of school mathematics has been slow and limited (Stylianides et al., 2017; Nardi & Knuth, 2017). Several potential explanations are offered for this phenomenon. Some researchers (e.g., Thompson et al., 2012) locate the problem with the dearth of examples in curricular materials for engaging students with reasoning and proving. Although there are some texts that aim to help teachers to integrate reasoning and proving in their teaching (e.g., Ellis et al., 2012; NCTM, 2009) they are often more appropriate for problem-based curricula rather than what can be considered a traditional secondary mathematics curriculum.

Another reason for the low occurrence of reasoning and proving in mathematics classrooms is that teachers and students tend to interpret it narrowly, as two-column exercises in high school geometry (Herbst, 2002). The didactic contract (Brousseau, 1997) that develops around this genre of exercises seem to be about reproduction of sometimes “obvious” mathematical results in a prescribed format. This leads both teachers and students to perceive proof as intellectually unexciting and a redundant activity (Harel, 2013). The powerful image of proof championed by mathematicians and championed by mathematics educators, does not seem to precipitate into mathematics classrooms. In addition, teachers’ own gaps in mathematical knowledge specific to proof and unproductive beliefs about teaching proof (for example, that only high-performing students can handle proofs) can sometimes pose barriers to classroom integration of reasoning and proving (Stylianides et al., 2017). But the issue runs deeper than that. Studies have shown that even teachers who seem to appreciate proof and acknowledge its importance in mathematics tend to prioritise development of procedural skills over proof-related activities in their own classrooms (e.g., Kotelawala, 2016).

Nardi and Knuth (2017) suggest that “Orchestrating a focus on proof appreciation requires a change in classroom culture with respect to proof and proving, a change that must start with the teacher”, but it would also require “changing classroom culture, curricula, and instruction” (pp. 268-269). As much as we agree with the authors about the need for change and the critical role of teachers in this process, we are doubtful whether such an overhaul approach to classroom teaching is viable. First, many teachers’ personal experiences with proof as learners and their mental image of what it means to engage students in proving may impede them from enacting reasoning and proving in their classrooms. Thus, the role of teacher preparation and professional development in spearheading changes in classroom instruction is critical. Some positive effects of teacher preparation and professional development programs on classroom teaching have been reported in the literature (see Stylianides et. al., for an extensive summary).

The second, and probably more substantial barrier for classroom integration of reasoning and proving, is found in factors external to individual teachers, but nonetheless critical, such as society, institutions and the culture of schooling (Chazan et al., 2016). Cultural environments dominated by standardized testing, pressure from parents, administrators and other stokeholds; a culture in which a teacher’s success is measured by the number of students passing tests, are less conducive change. In such environments, a teacher may find him/herself navigating a complex system of competing expectations and obligations; with the external pressure contributing to teachers’ reluctance to spend time on proof, instead of “covering” the curriculum.

Towards possible solutions

The problem and its plausible causes described above are not new, and several approaches to address it have been proposed over the years. One, already mentioned, is through teacher education and professional development initiatives aiming to enhance future and practicing teachers’ subject matter and pedagogical knowledge specific to proof, affecting their beliefs about proof, and providing ongoing classroom support for teachers enacting reasoning and proving in their classrooms (see Stylianides et al., 2017 for an overview). Such initiatives often rely on researcher-developed instructional materials that teachers try in their classrooms. These efforts are important as they help move the field towards developing theoretical and practical knowledge for making reasoning-and-proving a reality in mathematics classrooms. Towards addressing this knowledge gap, Stylianides et al., (2017) emphasized the need for theoretically sound and well-defined classroom interventions of short duration that support change of students' conceptions.

Another approach, proposed by Reid (2011), is Proof-Based Teaching (PfBT). As opposed to focusing on teaching proof, the PfBT is a way of teaching mathematics in which the teacher guides students through a carefully designed sequence of exploratory activities in which learners develop and prove mathematical results. The PfBT is akin to problem-solving teaching and inquiry-based learning (Ronis, 2008), but with the emphasis on students deductively proving the conjectures they develop during the exploratory activities. The PfBT approach seems to hold much promise for advancing change in teachers' classroom practices related to proof. Changing the discourse from teaching proof towards proof-based teaching, puts student learning of mathematics in focus, potentially helping to alleviate some of the tensions teachers might associate with proof. Far from being a simple change of rhetoric, the PfBT restores the place of proof and proving in mathematics (Hanna & deVillers, 2012), highlighting its role in production of mathematical knowledge.

The key element of PfBT, which guides the work of teaching, is the use of a *framing theory*. This theory outlines a sequence of definitions, statements, and conjectures that students are expected to discover and prove, to develop well-connected knowledge of particular mathematical content. For example, Reid and Vargas (2019) developed a framing theory for learning the operation on integers, called integer tiles theory (ITT), and enacted a PfBT teaching experiment in a 3rd grade classroom based on it. Although these results are encouraging, the PfBT model in its current form might not be easily extendable to secondary classrooms. The current state of educational research is such that many prominent topics in the secondary curriculum, e.g., quadratic functions, logarithms, trigonometry, analytic geometry, and others, do not have well-developed and well-established instructional theories. The lack of such theories may impede the broad application of PfBT. Thus, additional pathways are needed to promote instruction focused on reasoning and proof at the secondary level. In the next section, we outline Teaching Mathematics via Reasoning and Proof (TMvRP) model, which is close in spirit to PfBT, but does not rely heavily on the existence of established instructional theories.

Teaching Mathematics via Reasoning and Proving (TMvRP)

Our method

Guiding principles of the TMvRP model were gradually developed and formulated while engaging in the multi-year design-based research project in which we designed a special course *Mathematical Reasoning and Proving for Secondary Teachers*, and systematically studied its impact on prospective secondary teachers' (PSTs) knowledge, dispositions and practices specific to reasoning and proof. (Buchbinder & McCrone, 2020). In this course, the PSTs designed four proof-oriented lesson plans and taught them in local schools. The course discussions revolved constantly around the role of reasoning and proving in teaching and learning mathematics, and its usefulness and advantages for student learning. Having the PSTs design four proof-oriented lessons and enact these lessons in local schools aimed to convince the PSTs that (1) it is feasible to integrate reasoning and proving with *any* topic from the secondary mathematics curriculum, and (2) students are capable of learning complex concepts pertaining to deductive reasoning, such as conditional statements, indirect reasoning, and quantification. Through the iterative cycles of implementing and studying this course, the guiding principles of TMvRP models emerged.

The Guiding Principles of TMvRP

Guiding Principle 1: Teaching reasoning and proving must be fully embedded within the existing content of mathematics curriculum. When teachers perceive reasoning and proving as practical and not competing with their curriculum, then there is a chance, they will embrace it in their classrooms.

One of the key aspects that PfBT and TMvRP have in common is the emphasis on school mathematics, which we see as essential in order for teachers to get on board with the model. If teachers perceive reasoning and proving as taking time away from "curricular coverage", they will be reluctant to spend class time on it, sending an implicit message of irrelevancy to students. Thus, we integrated reasoning and proving within the regular mathematics curriculum.

Guiding Principle 2: Emphasis on deductive reasoning as a means to produce and validate mathematical knowledge. Proof-oriented tasks should focus on a small subset of deductive reasoning concepts, at the core of conventional mathematical knowledge.

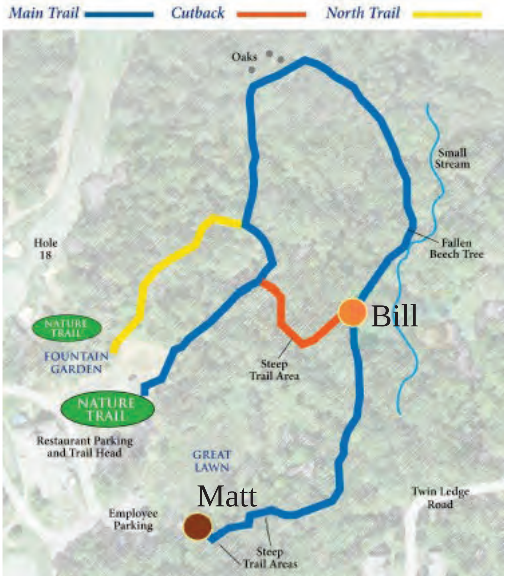
We wanted teachers and students to develop perceptions of reasoning and proving as central to mathematical thinking; processes that help to answer *why* something in mathematics is true (or false). At the same time, in mathematics there are special ways of responding to the question *why*, different from those used in everyday life, sciences, history or other fields. Mathematics requires deductive reasoning, which can be challenging to teach and learn. There TMvRP model allows to address some of the common and persistent proof-related misconceptions (Stylianides & Stylianides, 2017), such as students' overreliance on empirical evidence). Focusing on one such misconception at a time, or on one element of deductive proof can help to maintain the focus on deductive reasoning in the classroom discourse.

Guiding Principle 3: Use language, notation and representations within the conceptual reach of the students. This principle resonates with Stylianides' (2007) definition of proof in school as an argument that uses knowledge, language and representations that are within the conceptual reach of the classroom community. We operationalize this in our TMvRP model by encouraging teachers to *de-emphasize form of the proof* (e.g., two-column or algebra-only) and *avoid unnecessary "logical jargon"* (inverse, contrapositive). Instead, the model encourages multiple proof formats and representations, appropriate to the grade level, to communicate about deductive reasoning.

Illustration of the guiding principles of TMvRP in lesson planning

Following are excerpts of two lesson plans developed and taught by a PST Diane (a pseudonym) in a 9^{th} grade Algebra 1 class. We use these excerpts to illustrate the guiding principles of TMvRP. Lesson plan 1 integrated ideas pertaining the *role of examples in proving* with the mathematical topic of systems of linear equations through a context of the Hiking Exploration Problem (Fig. 1).

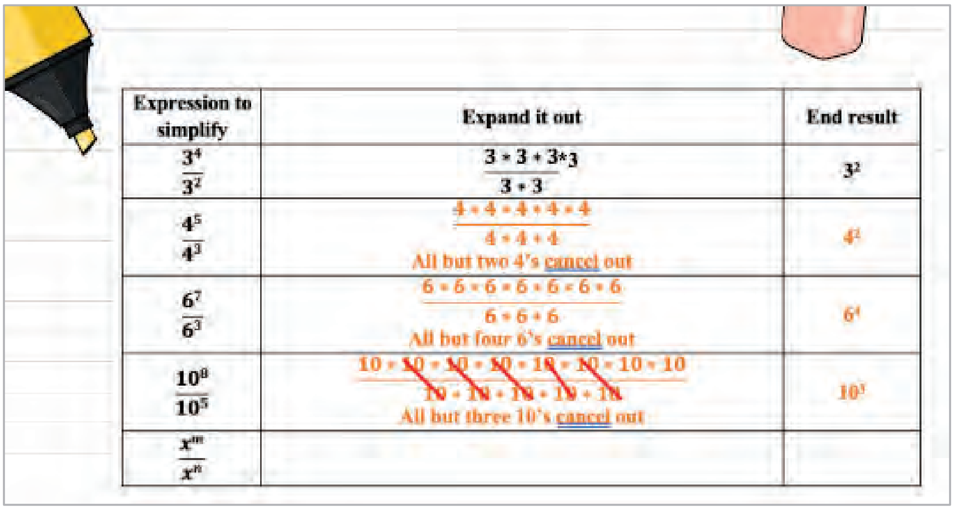
In Lesson 2, Diane integrated ideas about *direct (generic) proof and indirect proof* with the topic of quotient rule for exponents. Diane started by having the students explore patterns of simplifying numeric quotients by expanding the exponents (Fig. 2).



Given a map of a trail with the starting positions of two hikers. Matt starts at the trailhead and walks at a rate of 2 mph; Bill starts 3 miles up the trail and walks at a rate of 1 mile per hour.

1. Represent Matt and Bill's movement along the trail with equations.
2. Find evidence to support or disprove this claim: There exists a time on a 5-hour hike where Matt catches up to Bill.
3. What type of evidence and how much evidence do we need to prove or disprove this claim?
4. Solve the system of equations both graphically and algebraically. What does the solution mean in the context of the problem?
5. Find evidence to support or disprove this claim: Bill is always further ahead on the trail than Matt is.
6. What do we need to show to prove or disprove the claim in #5?
7. Put a point on a graph that represents a counterexample to the claim in #5. What x-value corresponds to it? Use our equations to show algebraically that this x-value is a counterexample.
8. What if the hike was only 2 hours long? Find evidence to support or disprove the claims: (a) There exists a time on the 2 hour hike where Matt catches up to Bill on the trail. (b) Bill is always further ahead on the trail than Matt is.

Figure 1: The Hiking Exploration Problem



Expression to simplify	Expand it out	End result
$\frac{3^4}{3^2}$	$\frac{3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3}$	3^2
$\frac{4^5}{4^3}$	$\frac{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}{4 \cdot 4 \cdot 4}$ All but two 4's cancel out	4^2
$\frac{6^7}{6^3}$	$\frac{6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6}{6 \cdot 6 \cdot 6}$ All but four 6's cancel out	6^4
$\frac{10^8}{10^5}$	$\frac{10 \cdot \cancel{10} \cdot \cancel{10} \cdot \cancel{10} \cdot \cancel{10} \cdot \cancel{10} \cdot 10 \cdot 10}{\cancel{10} \cdot \cancel{10} \cdot \cancel{10} \cdot \cancel{10} \cdot \cancel{10}}$ All but three 10's cancel out	10^3
$\frac{x^m}{x^n}$		

Figure 2: Developing the quotient rule of exponents

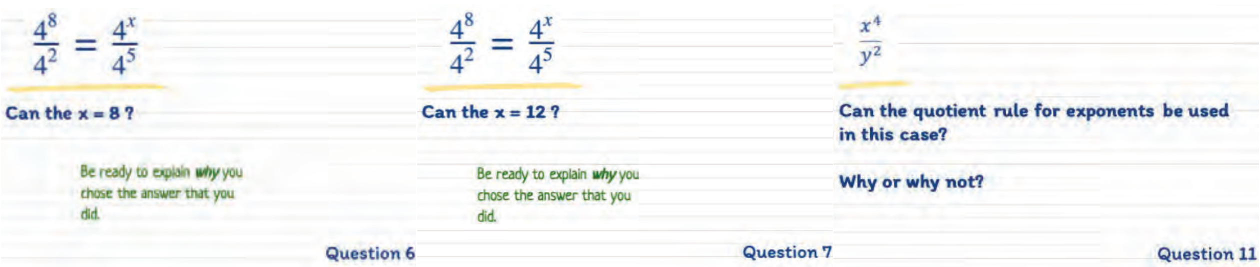


Figure 3: Questions intended for applying indirect reasoning

After students noticed and explained the pattern, Diane guided them through a generic proof (Mason & Pimm, 1984) of the rule. Next, she had students solve a set of tasks, which aimed to test the boundaries of use of the quotient rule (Fig. 3). When solving questions of the type shown in Fig. 3, Diane wanted students to assume that a given x-value is a solution, plug it in and obtain a false statement, which the students would interpret as a contradiction and conclude that the given x-value cannot be a solution. In the lesson plan, Diane wrote: "To summarize the indirect reasoning portion of the lesson, I want to make sure I emphasize that we are finding a contradiction to a statement, and that is why they are false." Note that in the context of this lesson, indirect reasoning has a form: "X cannot be Y, because otherwise we reach a contradiction Z". This does not constitute a proof by contradiction; nevertheless, being enculturated into explicit use of indirect reasoning can help students learn proof by contradiction at a later stage.

The two lessons clearly differ from each other. In the first lesson on the *role of examples in proving* students learned when it is appropriate to use examples to prove existential statements and counterexamples or disprove a universal statement in the context of solving a word problem. The second lesson on *indirect reasoning* did not rely on a real-world context, but it included exploring patterns, conjecturing, using direct generic proof of the quotient rule of exponents. The indirect reasoning came naturally, in the context of testing the boundaries of the quotient rule.

The outlines of these two lesson plans illustrate the guiding principles of TMvRP. First, reasoning and proving are fully integrated with the ongoing topic of the school curriculum. Proof concepts such as generic proof, indirect reasoning, role of examples in proving, and universal and existential statements support the learning of solving systems of equations and of exponent rules. Second, the mathematical topics provide a platform for the teacher to emphasize elements of deductive reasoning in ways that align naturally with the task. The teacher introduces students to the use of deductive reasoning in conventional mathematical ways, e.g., using supportive examples to prove existential statements and counterexamples to disprove universal statements. Third, while the teacher uses precise mathematical language throughout the lesson, she uses language within the students' conceptual reach and makes clear choices about what proof-related vocabulary is or is not critical for students in that lesson. For example, in lesson 1 Diane explained to students what a counterexample is and its role, but in lesson 2 she decided not to introduce the concept of indirect reasoning. In her lesson plan she wrote: "I don't think that using the words "indirect reasoning" is going to be particularly helpful to them [the students]." Regardless of whether we agree with Diane's choice, this example illustrates how a teacher makes instructional decisions regarding their own classroom and adjusts the language and representations to fit the students' perceived conceptual level, while introducing students to the deductive reasoning aligned with conventional mathematical practices.

Conclusions and Future Directions

In the extensive review of the research literature, Stylianides et al., 2017 observe a disproportionally larger number of studies describing difficulties related to teaching and learning of proof compared to studies seeking to address them. The line of research on research-based classroom interventions is gradually emerging, and with it some novel approaches to integrating reasoning and proving in classrooms, e.g. Reid's (2011), and Reid and Vargas' (2019) Proof-Based Teaching framework. In this theoretical paper, we discussed some of the barriers to such integration, some having to do with individual teachers and others reaching beyond to institutional, cultural and societal factors. The analysis of these factors motivated our design-based research project in which we worked to support and empower PSTs to enact reasoning and proving in mathematics teaching. Through this project, we developed the guiding principles for the Teaching Mathematics via Reasoning and Proof (TMvRP) approach: (1) Full curriculum integration, (2) Incremental and continuous emphasis on deductive reasoning, and (3) Adjustment of language, notation and representations to students' conceptual level.

We believe that instructional activities developed in accordance with these principles have a potential to position reasoning and proving as vehicles for learning mathematical content, making these processes appealing and manageable for both teachers and students. The PSTs in our course had four opportunities to apply the guiding principles of TMvRP to their lesson planning and teaching, as the examples of Diane's lesson plans show. Although some PSTs remained skeptical, the overwhelming majority of PSTs developed greater confidence in their ability to enact reasoning and proving in their classrooms. At the same time, we are mindful that the motivation for integrating proof in the lesson plans was external to PSTs as it was a course requirement. Thus, the question to what extent the TMvRP principles became part of their own teaching repertoire remains open. Hence, our new project focuses exactly on that: We follow the graduates of our mathematics education program into their supervised internship and the first two years of autonomous teaching to examine how (if at all) they continue integrating reasoning and proving into their teaching of mathematics. We hope that other researchers will be inspired to test TMvRP principles with prospective and/or practicing teachers to expand the research base behind this approach.

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Toulmin and beyond: structuring and analyzing argumentation

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In recent years, the Toulmin model seems to have become a "silver bullet" when analyzing argumentation (and proof) in mathematics education. While the model and its adaptations are well-fitted especially to reveal the structure of arguments, it is not suitable to grasp all aspects of argumentation which are relevant to the mathematics classroom. In this paper, we outline both, possibilities and limitations of the Toulmin model and present some additional considerations concerning this model undertaken in the last decades. Furthermore, we present and discuss some alternative approaches to argumentation and highlight their additional benefits.

Keywords: Argumentation, Toulmin model, Rationality, Logic of inquiry, Walton's dialogue theory

Introduction

In this article, we would like to follow up on a question that arose in the context of the TWG Argumentation and Proof at CERME11: Which models can be used besides the Toulmin model to analyze (mathematical) argumentation? Indeed, the Toulmin model has proven and established itself for structuring and presenting argumentation processes in a mathematical context. However, the associated disadvantages and 'blind spots' of the model are often ignored or disregarded. Moreover, the model appears to be so widespread that young researchers in particular fail to identify alternatives that might be better suited to address their research questions.

Our first aim in this article is to show which possibilities the original Toulmin model offers for structuring mathematical argumentation processes, which advantages and disadvantages are associated with it, and which conceptual extensions of the model are discussed in mathematics education. Furthermore, we would like to discuss other perspectives on argumentation that go beyond structural analysis and present alternative approaches to investigating argumentation.

The Toulmin-Model

Toulmin (1958) proposed a model for structuring argumentation in general. Inglis et al. (2007, p. 4; emphasis in original) summarize this scheme as follows (see also Figure 1):

> Toulmin's (1985) scheme has six basic types of statement, each of which plays a different role in an argument. The *conclusion* (C) is the statement of which the arguer wishes to convince their audience. The *data* (D) is the foundations on which the argument is based, the relevant evidence for the claim. The *warrant* (W) justifies the connection between data and conclusion by, for example, appealing to a rule, a definition or by making an analogy. The warrant is supported by the *backing* (B) which presents further evidence. The *modal qualifier* (Q) qualifies the conclusion by expressing degrees of confidence; and the *rebuttal* (R) potentially refutes the conclusion by stating the conditions under which it would not hold.

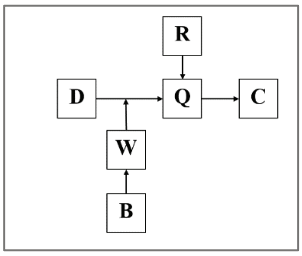

Figure 1: Toulmin's model of a general argument (similar to Inglis et al. (2007, p. 4))

As Nussbaum (2011, p. 86) points out, Toulmin did not originally intend to propose a model of argument. He originally wanted to counter formal logic with an alternative foundation for argument. Nevertheless, the model has been misinterpreted as normative. Toulmin was not concerned with the completeness of arguments, for example, to evaluate their quality. Especially in non-mathematical contexts, the warrants are often not made explicit. Nor was it Toulmin's concern to use the model to teach students how to propose an argument. It has been shown that the Toulmin model can be used to reveal argumentation structures in mathematical contexts (e.g., Inglis et al., 2007; Knipping, 2008). Here, the Toulmin model is often reduced to the somehow core aspects of Data, Warrant, and Claim. However, as stressed by Inglis et al. (2007) and Jahnke (2008), the whole model should be preferred to cover essential aspects of argumentations such as the backing, which specifies the warrant, or likewise the modal qualifier that serves to qualify the conclusion.

Limits of the Toulmin model and opening a discussion

The Toulmin model has its benefits when trying to reveal an argument's structure. However, some limitations of the model have to be considered, too. On the structural level, Nussbaum (2011, p. 86) discusses that in some situations, a reliable distinction between warrant and backing is hard to undertake, and there are some arguments in real life (such as arguments with a number of separate reasons) that do not fit the components and structure of the model. Aberdein (2013) criticizes that the Toulmin model is weak for the representation of instructions which may be part of mathematical arguments in particular. Due to its structural-analytical orientation, the model is less suitable for highlighting how people argue and why, and there is no focus on the correctness of arguments or their acceptance by others. In a collective argumentation setting, the Toulmin model does not specify who produces (parts of) the argument, and the context is not taken into consideration.

The Toulmin model furthermore does not capture the way in which an argument is presented to an audience, although as Gabel and Dreyfus (2019) show with regard to Perelman's new rhetoric, the audience is a major factor in proof teaching. This holds true not only in situations in which a teacher guides the argumentation, but also in collective argumentation settings. Motives and reasons with which intention which person contributes which reasons with which rhetoric are not part of the Toulmin model's structural focus but they are of importance when looking at argumentation and proof in the mathematics classroom.

Extensions of the Toulmin model

Inglis et al. (2007, p. 9–16) distinguish several warrant-types as categories of warrants with similar properties. In an argumentation, the check of some concrete examples or an apparent absence of a counterexample for the statement in focus might reduce the uncertainty about the conclusion of an argument. Such warrants are considered the *inductive warrant type* and might typically lead to modal qualifiers like "it seems that" or "it is plausible that". The *structural-intuitive warrant-type* is about using some kind of observation (it might be of an intuitive type or not) that might persuade the individual to accept the validity of a conclusion. Again, corresponding modal qualifiers might be "it seems that" or "it is plausible that". Finally, the deductive warrant-type is concerned with those warrants used in a valid mathematical proof (axioms, statements, algebraic manipulations, etc.). For professional mathematics, using a *deductive warrant* should normally lead to a modal qualifier like "with necessity". However, it has been shown in the literature that this need not be the case for students (see Reid & Knipping 2010, p. 62 for an overview). Kempen (in press) analyzed high school graduates' proof construction and extended the discussion of the oberserved warrants by the aspect of epistemic value. "The epistemic value is the degree of certainty or conviction assigned to a proposition" (Duval, 1991, p. 254; authors' translation). Accordingly, it can take on values such as obviously, likely, absurd, necessary, etc., and it is closely connected to the individuals' understanding of the content (Duval, 2007, p. 138). This extension is based on the idea that a warrant is linked with an individual level of conviction and validity. Kempen showed that the epistemic values attributed to the warrants involved have an impact on the conclusion's modal qualifiers.

Besides focusing on a single argument leading from data to conclusion, it is also possible to consider longer chains of arguments. Here, a conclusion becomes new data for the application of further warrants. Such chains of arguments have been described as "argumentation stream" (Knipping & Reid, 2019), "Recyclage" (Duval, 1995), or "Sequential" (Aberdein, 2006).

The Toulmin model is not only used as a tool to explore the local structure of an argument, but also its global structure (the structure as a whole) as described by Knipping and Reid (2019) when discussing classroom argumentations. In their analysis of global argumentation structures which transcend individual arguments, these authors added an element called "refutation" and explain:

> A refutation completely negates some part of the argument. In a finished argumentation refuted conclusions would have no place, but as we are concerned with representing the entire argumentation that occurred, it is important for us to include refutations and the arguments they refute, as part of the context of the remainder of the argumentation, even if there is no direct link to be made between the refuted argument and other parts of the argumentation. (Knipping & Reid, 2019, p. 5)

When analyzing complex arguments, e.g., when different people contribute to an argument, it has been valuable to use a schematic representation that enables the description of argumentation at different levels of detail, like using different shades, colours, or forms (see Knipping, 2003). Knipping and Reid (2013) show how teacher actions can shape argumentation processes in the mathematics classroom by identifying different global argumentation structures. They distinguish differently structured discourses that become visible in the shape of the global argumentation structure and thereby show how the Toulmin model may add to a deeper understanding of classroom interaction.

Kopperschmidt (1989) also links the local and the global structure by analyzing the micro and macro structures. The micro structure contains the *functional* analysis (role and functions of the different parts involved in an argument with respect to the Toulmin model), the *material* analysis (linguistic interaction), and the *formal* analysis (structural patterns involved). At the macro level, a distinction is first made between single-step and multi-step argumentation. Here, a multi-step argumentation can be convergent (all elements involved fully support or refute a particular claim) or controversial (the elements involved only partially support or refute a particular claim).

Further models and ideas for considering argumentation

In the following, we will present and discuss other perspectives on argumentation that enable further aspects and views on the topic and thus open up new perspectives for corresponding analyses. A comprehensive overview on developments and different perspectives in argumentation theory in general may be found in Van Eemeren et al. (2014). In selecting the perspectives listed below, we have tried to choose those that have experienced a certain spread in mathematics education and exemplarily focus on various aspects of argumentation beyond structure considerations.

Considering the purpose of argumentation: communicative versus strategic action

Arzarello and Soldano (2019) criticize the jurisprudential nature of the Toulmin model. According to them, in the type of argument considered by Toulmin "the goal is to convince the adversary, not the search/explanation of the truth, which is instead the goal of scientific argument, in particular in mathematics." (Arzarello & Soldano, 2019, p. 227). We do not necessarily share this criticism of the Toulmin model as being too narrowly focused on jurisprudential arguments. However, Arzarello and Soldano's critique raises an important issue: it might be necessary to consider the purpose of an argumentation when looking at argumentation in a classroom context.

One way to distinguish between different purposes in which argumentation may play a role is described by Habermas (1981). He postulates a distinction between two different types of action: communicative action towards reaching agreement, and strategic action focused on securing consent.

> Moreover the action theoretical approaches [i.e. communicative vs. strategic action] differ in whether for coordinating actions they postulate *agreement* [german: „Einverständis"; authors' comment], that means *common* knowledge [german: „gemeinsames Wissen"; authors' comment], or just an external *influence* [german: „Einflussnahme"; authors' comment] on another. "Common" knowledge needs to satisfy challenging conditions. It is not enough if participants share some opinions; not even if they know that they share those opinions. *Common* knowledge is a knowledge that constitutes *agreement*, while agreement terminates in the intersubjective recognition of criticisable claims. Agreement means that participants accept knowledge as valid, i.e. intersubjectively binding. […] The external exertion of influence (in the sense of a causal effect) on the convictions of another participant in the interaction, on the other hand, keeps a one-sided character. (Habermas, 1981, p. 574–575; emphasis in original; authors' translation)

Mathematical argumentation should be a conjoint search for the truth and thus oriented towards reaching common knowledge. However, it is doubtful whether all classroom situations fulfil these noble goals. A study examining the prevalence of communicative and strategic behaviour in the classroom might provide interesting insights.

Within Habermas' framework of communicative action, Boero (2006) identified the construct of rationality as an especially useful tool for grasping the complexity of a classroom situation. Boero (2006, p. 189–190; emphasis in original) paraphrases Habermas' perspective on three types of rationality:

> The *epistemic rationality* is related to the fact that we know something only when we know why the statements about it are true or false (otherwise our knowledge remains at an intuitive or implicit-pragmatic level). […] The *teleologic rationality* is related to the intended character of the activity, and to the awareness in choosing suitable tools to perform the activity and orient it to the aim to be achieved. The *communicative rationality* is related to communication practices in a community whose members can establish communication amongst them.

This threefold view on classroom situations shows that there is more to argumentation than its mere structural components. An individual or a group of individuals producing a mathematical argument need an awareness of shared knowledge which they can presume, they need to choose their tools and strategies wisely, and they need to choose means of communication which can be understood by other participants in the argumentation. Thus, rational behaviour presupposes an orientation towards communicative action.

Boero et al. (2010) have shown how supplementing the structural analysis of argumentation with the Toulmin model with Habermas' perspective on rationality opens up the possibility to consider a speaker's intentions and consciousness within the argumentation. Whether an argument is accepted by a community or not is not simply defined by its logical soundness, its correctness, or its adherence to a certain structure. To put it in a different way: What counts as proof in a 5th year classroom may differ significantly from what is considered an acceptable proof at university level. This is captured well in a quote from Habermas highlighted by Boero and Planas (2014, p. 205): "The rationality of a judgment does not imply its truth but merely its justified acceptability in a given context". Looking at argumentation from an epistemic, a teleological and a communicative perspective simultaneously may help us understand more about how an argument becomes a foundation of newly established common knowledge.

"Logic of inquiry" to account for discoveries in argumentation

Arzarello and Soldano (2019) point out that a perspective on argumentation with a narrow focus on deductive reasoning cannot account well for surprising discoveries. They refer to an example from the story *Silver Blaze* by Sir Arthur Conan Doyle. In this story, the fictional detective Sherlock Holmes concludes that a missing horse had not been stolen as assumed, but had indeed been taken by the stable owner, because the watch dog had not barked in the night. To account for the arising of such new theories and conjectures, Arzarello and Soldano introduce Hintikka's "logic of inquiry", to which they ascribe three characteristics: (i) the dialectic between questions and answers; (ii) the deep link with game theory; (iii) the functional interpretation of connectives and quantifiers (Arzarello & Soldano, 2019, p. 230).

According to the logic of inquiry, deductions such as the one made by Sherlock Holmes in the story can be rewritten into a chain of questions and answers. The argumentation is guided by definitory rules framing the deductive steps on the one hand, and by strategic principles generating the inquiry steps on the other hand. This interplay of rules and strategies forms a connection to game theory. Arzarello and Soldano (2019) show how mathematical proving processes can be regarded as (semantic) games. They introduce an example from a mathematics classroom in which students fulfil roles as verifiers or falsifiers in their joint search for the truth. In this model, the existence of a proof is tantamount to the existence of a strategy with which the verifier will always win.

Teaching and evaluation of argumentation: Walton's Dialogue Theory and a Bayesian approach to argument evaluation

Nussbaum (2011, p. 85) points out that argumentation models may serve three different purposes: analytical (with a focus on revealing the structure of an argument), normative (for judging an argument's quality or to determining what an argument should look like), and descriptive (to identify how people actually argue). He points out that the Toulmin model is primarily analytical, even though its components are not always clearly identifiable. Besides, also the adaptations towards the representation of global argumentation structures by Knipping (2003) serve a descriptive purpose. However, normative aspects are not covered by the Toulmin approach. Nussbaum (2011) therefore suggests two alternative frameworks for looking at argumentation in educational contexts: Walton's Dialogue Theory and Bayesian models of everyday arguments. In this contribution, we only take the former into consideration. The Bayesian model works with probabilistic calculations to help determine posterior odds; it does not help in evaluating mathematical arguments whose premises are usually either correct or incorrect.

Walton's Dialogue Theory, according to Nussbaum (2011), is a complex argumentation framework consisting of several levels: types of argumentation dialogues, specific argument schemes and critical questions, degrees of plausibility, and criteria for moral and aesthetic argumentation. Due to the limited space in this contribution, we focus on two aspects which appear particularly promising to address the shortcomings of the Toulmin model laid out above: argumentation schemes and posing critical questions, which may be applied both to the instruction and evaluation of argumentation.

One way to improve students' argumentation skills could be direct instruction in how to use different argumentation schemes. This has been a strategy in mathematics education, e.g. when teaching students two-column proofs or methods for proving by induction. However, while students might benefit from a scheme on which they can rely, it has been criticized in the past that students tend to apply such schemes without a deeper understanding of why the proof they are creating is, in fact, a proof. Regardless of whether argumentation schemes are explicitly taught to students, they can also serve to evaluate argumentation quality.

A second possible application of Walton's Dialogue Theory, according to Nussbaum (2011), is to use direct instruction on critical questions and *stratagems* useful for argumentation. "A stratagem is a generic type of brief argument expressed in discourse [such as…] I think [POSITION] because [REASON]" (Nussbaum, 2011, p. 93). Both, critical questions and stratagems, serve to inform students of evaluative criteria, which may in turn serve to assess argumentation quality from a teacher or researcher perspective.

Besides schemes and critical questions, Nussbaum (2011, p. 94) also presents a list of criteria for the assessment of argument quality. However, several criteria on the list appear ill-fitted for the mathematics classroom, as they rely on an assessment of the depth of a discussion. The quality of a mathematical argument is not necessarily dependent on the number of arguments brought forth, or by how many of the arguments brought forth were defeated. A closer look at Walton's schemes may, however, be beneficial also for evaluating and assessing mathematical argumentation with regard to quality.

Conclusion: Widen the scope – there is more to arguments than structure

We started out from the question "Which models can be used beside the Toulmin model to analyze (mathematical) argumentation?". Our thorough consultation of alternative approaches to and perspectives on argumentation primarily shows the necessity of specifying the question further: *what exactly do we want to analyze? I.e., what is the focus and the purpose of our research?* The Toulmin model can be suitable or well-fitted to reveal the structure of an argument. This approach, however, makes the structure of an argument (in Toulmin's sense) visible; no more, no less.

As we have seen above, there are several attempts to extend the traditional Toulmin model to gain more information about the different parts involved in the model. Besides, also the connections between these parts have been (theoretically and empirically) analyzed further. It is, however, important to notice that an analysis of the different parts of an argumentation's structure does not imply any information about the soundness of the argumentation steps, the plausibility for the participants in the argumentation, or the generation of common knowledge.

In this paper, other aspects that might be focused on when discussing argumentation became visible that go beyond the (original) intention of the Toulmin model. These aspects can be described as considering the purpose of an argumentation (communicative versus strategic action), accounting for discoveries in the course of the argumentation, and the discussion and conceptualization of normative aspects and educational purposes.

From our point of view, we want to highlight the following conclusions. For structural analysis of the different parts of argumentation, we have not yet discovered a more suitable model than the Toulmin model with its extensions. However, a danger must also be pointed out here: If argumentation is theoretically defined according to the Toulmin model's structural components, the Toulmin model becomes the sole instrument of analysis. One should beware of this redundant relationship because it obscures the view of a field of research that, as we have shown here, is quite broad. Argumentation in the mathematics classroom is more than the sum of its (structural) parts. It involves awareness, validity, intentions, strategies, questions, rhetoric and rationality.

We have tried to broaden the view of the research field of argumentation in mathematics education. The examples we have chosen and brought up are well suited to point out horizons for further research on the topic.

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The role of unitizing predicates in the construction of logic

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Based on data from a teaching experiment with two undergraduate students, we propose the unitizing of predicates *as a construct to describe how students render various mathematical conditions as predicates such that various theorems have the same logical structure. This may be a challenge when conditions are conjunctions, negative, involve auxiliary objects, or are quantified. We observe that unitizing predicates in theorems and proofs seemed necessary for students in our study to see various theorems as having the same structure. Once they had done so, they reiterated an argument for why contrapositive proofs proved their associated theorems, showing the emergence of logical structure.*

Keywords: Mathematical logic, Student reasoning, Undergraduate mathematics education.

Introduction

Theorem 1. "For every integer x, if x is a multiple of 6, then x is a multiple of 3."

Theorem 2. "For any integer x, if x is a multiple of 2 and a multiple of 7, then x is a multiple of 14."

Theorem 3. "For any quadrilateral $\blacksquare ABCD$, if $\blacksquare ABCD$ is a rhombus, then the diagonal $\overline{AC}$ forms two congruent, isosceles triangles ΔABC and ΔCDA."

Theorem 4. "Given any functions f, g that are continuous on the domain $[a, b]$, if $f(a) = g(b)$ and $f(b) = g(a)$, then there exists some c in $[a, b]$ such that $f(c) = g(c)$."

In what ways would we expect students to see these four statements as being "the same"? How would that sameness influence students' reasoning about these statements and related proofs? This is a contextual way of considering the role of logic in students' mathematical reasoning, since the primary analogy between these statements is their logical form: "For all $x \in S$, if $P(x)$, then $Q(x)$." In this study, we investigate how students build such analogies between statements and how it influences their judgments about contrapositive proofs. We propose *the unitizing of predicates* as a construct that describes how students conceptualize the various properties in the statements as entailing properties that each example either has or does not have. This affords a structural analogy between each statement and the corresponding set of objects, which can unify these statements across the mathematical contexts. We provide evidence for how unitizing predicates can be consequential for how students perceive logical structure within each context in the sense of truth conditions (when a statement is true or false) and related types of proof (i.e., direct or contrapositive).

While many studies show that students do not interpret common mathematical statements in a manner compatible with mathematical logic (Epp, 2003; Sellers, Roh, & Parr, 2021; Stylianides, Stylianides, & Phillipou, 2004), our approach to logic learning is to help students construct mathematical logic as they read meaningful mathematical statements. We do not try to use abstract symbols to replace the meaning of statements, which bypasses students' reasoning about the concepts in the statements. Rather, we conducted constructivist teaching experiments (Steffe & Thompson, 2000) to investigate an alternative approach in which students structure their interpretation of each statement (and their proofs) in a manner that affords structural analogies and repeated reasoning. In their survey of research on the teaching and learning of logic, Durand-Guerrier, Boero, Douek, Epp, and Tanguay (2012) argued that "it is important to view logic as dealing with both the syntactic and semantic aspects of the organization of mathematical discourse" (p. 385). To attend to both, logic must capture how Theorems 1-4 refer to mathematical objects, rather than merely how to remove meaning from the statements to render them "the same." The present study contributes to answering the following research question central to our overall agenda of research: How can students abstract logical structure and relationships in a manner that stems from and integrates with their reasoning about various mathematical topics? We propose that *unitizing predicates* provides a construct that characterizes how students can abstract the semantic structure of their mathematical reasoning about particular statements to afford the construction of logical structure that generalizes across contexts.

We focus in this report on a student constructing the principle of *Contrapositive Equivalence*: Contrapositive proofs, which begin "Let not $Q(x)$" and end "Thus, not $P(x)$", prove the given theorem. To understand how students attended to this form of proof and its invariant relationship to the theorems, we designed the theorems they read (which included Theorems 1-4) to vary the relationship between objects and properties in each statement. If we conceptualize a predicate as a truth-function that maps each object to a truth value ("T" if the object has the property or "F" if the object does not), then we can see how the predicates in the statements vary. Both properties in Theorem 1 are somewhat familiar and direct such that the predicate corresponds to a single category of numbers. The antecedent in Theorem 2 is a conjunction, meaning students must somehow combine the properties to yield a single predicate. The consequent in Theorem 3 involves auxiliary objects (triangles), which we may think of as rendering the predicate as a composition of functions (or mental actions). Finally, Theorem 4 is the most complex in that the input consists of pairs of functions, the antecedent is a conjunction, and the consequent is existentially quantified.

Literature on student understanding of conditionals and proof thereof

The mathematical use of conditional statements differs from the everyday meaning. One of the best-supported models for everyday interpretation is the suppositional account that posits people affirm a conditional based on the conditional probability of Q given P (Evans & Over, 2004). This means that people affirm some conditionals in the presence of counterexamples and explicitly think such statements are only about cases where P is true. Inglis (2006) found that mathematicians used a similar criterion, though they sought to know whether Q was certain given P, meaning no counterexamples were allowed. Two important corollaries of this are that the meaning of a conditional is strongly rooted in the meaning of the statement and the semantic links between P and Q. As a result, semantic links are hard to generalize across the statements above since they are context-specific, so to construct logic, students must attend to something more about the conditionals.

There are few studies on how students understand contrapositive equivalence. Stylianides et al. (2004) found that students directly taught the rule often gave answers in conflict with it, especially in mathematical contexts. Yopp (2017) provides one of the only accounts for how students should understand why contrapositives are equivalent. He argues that students should interpret conditionals by *eliminating counterexamples* (cases where P is true and Q is false), similar to how Inglis noted mathematicians reasoning. This supports both an explanation for why direct proofs prove and why contrapositive proofs prove, since a counterexample of the contrapositive statement is equivalent to a counterexample to the original. Hub and Dawkins (2018) observed a student providing a different justification rooted in the subset meaning, which states that a conditional is true whenever the truth set of P is a subset of the truth set of Q. Their student argued for contrapositive equivalence by noting that non-Q cases have no overlap with P cases, which they called the *empty intersection meaning*.

Theoretical framing

Consistent with our interest in how students construct logic as a shared structure across various mathematical statements, we conducted our study using the Radical Constructivist (von Glasersfeld, 1995) notions of assimilation and accommodation. Assimilation occurs when someone interprets a new experience as an instance of something known. Assimilation occurs in the context of goal-oriented activity, and assimilating to a scheme induces some action with an expectation to meet the reasoner's goal. When the goal is not met or something unexpected occurs, the reasoner may experience perturbation and thus engage in accommodation to assimilate the experience to a new scheme or modify the scheme to fit the new experience. This is often the opportunity for learning. Accommodation is often prompted by interlocutors (i.e., teachers) who introduce goals and prompt reflection on previous activities. In particular, we frequently asked the two students in our teaching experiment to compare their reasoning across various theorems and proofs and invited them to repeat their reasoning if possible. Such questions invited them to accommodate so as to construct a shared structure between the theorems and proofs, so as to promote reflexive abstraction by which they could project their images of their own activity onto a higher level where they could re-present the shared structure across the statements and proofs, which would for us constitute logical understandings.

Our notion of *unitizing predicates* is inspired by the work of Steffe (1983) and his colleagues in modeling students' construction of number concepts. They use *unitizing* to describe how children operate on quantities (counting or partitioning) to create new quantities (10 or 1/5) that are simultaneously a new unit and in fixed relation to the original unit (1). We can think of this as closure of numeric units under certain transformations. We use unitizing predicates to describe how students interpret the conjunction of predicates (Theorem 2), the composition of predicates (Theorem 3), and quantified predicates (Theorem 4) as new predicates with the same structure as simple predicates (Theorem 1). We call this unitizing predicates, since it is a form of closure under mental operations.

Methods

The teaching experiment reported here is part of an ongoing sequence of experiments focusing on how students can construct logic by reflecting on their mathematical activity. The participants, whom we call April and Moria, were recruited from a Calculus 3 (multi-variable and vector calculus) course at a public university in the United States. We recruited such students since we anticipated their mathematical understandings would be sufficiently strong, they would not have learned logic before, and they might go on from this course into a proof-based course (depending on their degree program). To further affirm these criteria, we asked them whether they had learned logic and had them complete a logic assessment (Roh & Lee, 2018). April and Moria were computer science majors at the time of participation. We met with them once per week outside of class time for 1-1.5 hours and paid them modestly for their time. We met with April and Moria for six sessions during which they read eight theorems, each paired with 2–4 proofs. Their task in each case was to determine whether each proof proved the statement and, if not, what statement it proved.

Consistent with teaching experiment methodology (Steffe & Thompson, 2000), the first author served as the teacher/researcher and the second author as an outside observer. All sessions were video recorded for iterative and retrospective analysis. During each session, we sought to build second-order models of their reasoning and test those models through questioning. This occurred at multiple levels such as meaning for concepts, construal of proofs, attention to logical structure, etc. Once Moria introduced Euler diagrams to represent the sets of objects referred to in each theorem/proof, we consistently invited the students to produce such diagrams. We hoped such activity would support relating the structure of different statements and re-presenting the logic of conditionals.

For this paper, our analysis focused on the students' construction of analogies between the statements and proofs. We attended to cases when they compared or contrasted theorems/proofs and for the opportunities to assimilate different theorem/proofs that shared the same logical structure (according to mathematical logic). This allowed us to attend carefully to their assimilation and accommodation activity and the particular means by which they connected theorems and proofs across contexts.

Results

We focus in this paper on April and Moria's reasoning about three proofs by contraposition, which proved Theorems 1, 3, and 4 stated above. April and Moria read these proofs during the first, fourth, and sixth sessions, respectively. For brevity, we provide only two of the proofs below in Figure 1 and omit the definitions students saw. "Proof 1.3" denoted the third proof associated with Theorem 1.

Proof 1.3

Regarding Proof 1.3, April produced her first argument for why a contrapositive proof did prove, to which we hoped she might assimilate later such proofs. Based on April's reading of Proofs 1.1 and 1.2, it was clear that she understood that all multiples of 6 are multiples of 3, but that some multiples of 3 were not multiples of 6. Upon reading Proof 1.3, April responded, "I think this is proving the theorem because it's saying if it's not a multiple of 3, then it can't be a multiple of 6." She elaborated in the context that "Because I already know that you have to have 3 times 2 times something to be a multiple of 6… Without being a multiple of 3, it's never going to equal a multiple of 6." She elaborated this argument relative to the proof saying, "Everything that you throw into it is going to give a remainder, how it's set up… you've already eliminated the fact that there's ever going to be a 3 in this, so it just doesn't formulate. It's like making a cake without the flour or the sugar."

We see a few important aspects of April's reasoning that are worth describing in some detail. First, her meaning for "multiple of d" was to factor out a d in an algebraic expression. Hence, she recognized that being a multiple of 6 entailed being "3 times 2 times something." Her meaning for multiple of 6 entailed her meaning for multiple of 3, inducing a sense of necessity between them.

Second, she gave a meaning to the negation of "multiple of 3" in terms of "give a remainder" which she could connect to the various equations in Proof 1.3. We have found that students often want to substitute a positive description for a negative predicate, which greatly facilitates their reasoning. April unitized the negation by giving it a positive meaning.

Third, based on the suppositional account of everyday conditionals (Evans & Over, 2004), we may expect students to think Proof 1.3 is irrelevant to Theorem 1 because it is not about multiples of 6. April explained how not being a multiple of 3 justifies the theorem using the analogy to ingredients. Since the factored out 6 is composed of a 2 and 3, then to try to make a 6 without a 3 is "like making a cake without the flour or the sugar." This argument implicitly draws upon the *empty intersection meaning* in that it shows no number can both be a non-multiple of 3 and be a multiple of 6.

Theorem to be proven 1: For every integer x, if x is a multiple of 6, then x is a multiple of 3.

Proof 1.3: Let x be any integer that is not a multiple of 3.
That means when we divide x by 3, we get a remainder of 1 or 2.
Then there exists some integer k such that $x = k * 3 + 1$ or $x = k * 3 + 2$.
If k is even, then there exists some integer s such that $k = s * 2$.
Substituting into the equations for x, we see:

$$\begin{aligned} x &= (s * 2) * 3 + 1 \\ &= s * 6 + 1 \end{aligned}$$

or

$$\begin{aligned} x &= (s * 2) * 3 + 2 \\ &= s * 6 + 2. \end{aligned}$$

This means x is not a multiple of 6, because x is 1 or 2 greater than a multiple of 6.
If k is odd, then there exists some integer t such that $k = t * 2 + 1$.
Substituting into the equations for x, we see

$$\begin{aligned} x &= (t * 2 + 1) * 3 + 1 \\ &= t * 6 + 4 \end{aligned}$$

or

$$\begin{aligned} x &= (t * 2 + 1) * 3 + 2 \\ &= t * 6 + 5 \end{aligned}$$

This means x is not a multiple of 6, because it is 4 or 5 greater than a multiple of 6.

Theorem to be proven 3: For any quadrilateral $\blacksquare ABCD$, if $\blacksquare ABCD$ is a rhombus, then the diagonal $\overline{AC}$ forms two congruent, isosceles triangles ΔABC and ΔCDA.

Proof 3.3: Let $\blacksquare ABCD$ be a quadrilateral such that when we form the diagonal $\overline{AC}$, the triangles ΔABC and ΔCDA are not both isosceles and congruent.
This means either the triangles are not isosceles, not congruent, or both non-isosceles and non-congruent.
If the triangles are not isosceles, that means that none of their sides are congruent.
This means $AB \neq BC$, which means $\blacksquare ABCD$ is not a rhombus.
If the triangles are not congruent, that means at least one pair of corresponding sides are not congruent.
Clearly, $AC = CA$, so it must be the case that $AB \neq CD$ or $AB \neq DA$.
In both cases, $\blacksquare ABCD$ is not a rhombus.
Therefore, $\blacksquare ABCD$ is not a rhombus.

Figure 1: Theorems 1 and 3 and their contrapositive proofs.

Proof 3.3

Proof 3.3 was the second proof by contraposition that April and Moria read in a new mathematical context and with a more complex predicate in the conclusion. It provided our first opportunity to see whether April would assimilate this proof to her prior argument affirming proof by contraposition. She did not do so initially, but later accommodated once she had unitized predicates relevant to the theorem and developed a strong sense that being a rhombus entailed the theorem's conclusion.

By the time April and Moria read Theorem 3 and its proofs, they had introduced Euler diagrams and the interviewer asked them to produce such diagrams for each proof to portray the relationships between the sets of objects discussed. April and Moria recognized there were two relationships in the early theorems: one set was "nested" in another (proper subset) and one was "nested and exhaustive" in the other (equal sets). Both of these are compatible with the subset meaning for the truth of a conditional, which we intended for them to develop.

Theorem 3 was the first task we introduced with a highly unfamiliar property. After reading Proof 3.1 (Direct Proof), April summarized it saying, "It proves why a rhombus is an isosceles." April thus used the term "isosceles" to stand for the antecedent condition in the theorem. In discussing Proof 3.2 (Disproof of Converse), they used the phrases "quadrilateral" and "rest of the sentence" to refer to that condition. They represented this in an initial Euler diagram with three nested regions: the innermost "rhombus," the outermost "quads," and the middle unlabeled.

At this point, April was searching for an effective way to refer to the consequent condition. She asked if there was a name for this category. The interviewer invited Moria and April to name this middle set, for which April chose the name "fancy." Moria wondered how this category related to parallelograms, and concluded that they are "more conditional than what we are going for in this."

When they then read Proof 3.3 (Proof by Contrapositive), April used the diagram to interpret the proof. She explained, "We're in quadrilaterals [pointing to outer region] and we can't go into fancy quadrilaterals, and therefore we cannot go into rhombuses." However, neither student at this point judged that Proof 3.3 proved Theorem 3. When the interviewer asked them to explain what Proof 3.3 proved, neither one explicitly connected the first line of that proof to "not fancy," despite April's association of the hypothesis with the outer ring of their diagram. Indeed, they seemed to struggle to articulate the hypotheses for Proof 3.3. Moria explained how, if one of the two triangles was scalene, the unequal sides or the triangles having unequal area meant they did not have a rhombus. Thus, while they found multiple ways to conceptualize how Proof 3.3's hypothesis entailed not being a rhombus, they did not judge this as relevant to Theorem 3. They failed to assimilate this proof to the line of reasoning April exhibited for Proof 1.3, though we see their arguments as having the same structure.

Toward the end of the session, the interviewer wrote abbreviated forms of Theorem 3, its converse, its inverse, and its contrapositive on the board (using the phrase "not fancy" in the last two) with a reproduction of their diagram with three nested regions. He asked Moria and April to interpret all four statements using the diagram. They were very comfortable explaining that Theorem 3 corresponded to the subset relation between rhombi and fancy quadrilaterals and that the converse was false because the relationship was "not exhaustive" and "you are going to have some fancy quadrilaterals that are not rhombuses."

Both students agreed that the contrapositive statement (abbreviated as "If not fancy, then not a rhombus") must be true. April explained that non-fancy quadrilaterals were in the outer region (covering it with both hands) and you cannot find a rhombus anywhere except the inner circle (the *empty intersection meaning*). The interviewer then returned to the question of whether Proof 3.3 proved Theorem 3. April now shifted to claiming that it can prove the theorem, explaining "You need to have it so that it forms the isosceles and you need to have it that the sides are congruent. You need to have those properties to have a rhombus." Once again, she was able to use her sense that rhombus entailed fancy to argue why failing to have fancy is sufficient not to be a rhombus.

What allowed April to shift from denying that Proof 3.3 proved Theorem 3 to affirming that it did prove? First, she desired a way to conceptualize the “fancy” condition in Theorem 3 as a predicate of quadrilaterals. By naming the category “fancy” and exploring its relationship to more familiar properties such as parallelograms, she constituted it as a predicate of quadrilaterals.

Regarding Proof 3.3, despite other relevant connections April made, she did not assimilate the hypothesis condition as equivalent to “not fancy,” which we consider a failure to unitize the negation. Once the interviewer wrote down the contrapositive using the phrase “not fancy,” April and Moria were able to assimilate Proof 3.3 to that statement and relate it to their understanding that “rhombus” entailed “fancy.” April’s empty intersection argument (that there are no rhombi among the non-fancy quadrilaterals) closely matched her argument for Proof 1.3. In the same way, having a factor of 2 and 3 was necessary to have a factor of 6, forming isosceles, congruent triangles was necessary to be a rhombus. Thus, April’s abstraction of her argument to the new mathematical context depended upon her ability to unitize the predicate “fancy,” to unitize its negation in Proof 3.3, and to coordinate the entailments among the properties.

Proof 4.3

Due to space limitations, we cannot present the text of Proof 4.3, which proves Theorem 4 from the introduction by contraposition. We will simply note that April initially saw the proof as irrelevant to the statement. Still, as with Proof 3.3 was later able to affirm the proof once she had conceptualized and named the conditions in the theorem. She later paraphrased the theorem as saying that if the endpoints of the functions “swapped” then the functions intersected. She then paraphrased the proof by contraposition saying, “If there is no intersection, there is definitely no endpoint swapping.”

Discussion

We make three claims about April’s ability to assimilate the theorems and proofs to her argument by contraposition. First, she could not assimilate the more complex theorems to the subset meaning (“nested” conditions) until she had unitized each predicate. Regarding Theorem 3, this entailed both seeing the condition on the triangles as a property of the quadrilateral and naming it. Regarding Theorem 4, this entailed conceptualizing the complex conditions and giving them names. Second, she needed to construct a relationship of entailment between the antecedent and the consequent in each theorem. She understood how being a multiple of 3 required being a multiple of 6. She had to construct similar senses of necessity regarding rhombus /fancy and endpoints swapping/intersection. Third, in the latter two cases, we see how unitizing the predicates and constructing a sense of entailment induced a shift from denying that the contrapositive proof proved to affirming it. This suggests that reconstructing each theorem as a relationship between two predicates was necessary to assimilate each new proof by contraposition into her original empty intersection meaning argument.

Based on these claims, we argue that unitizing predicates stands as an essential cognitive precondition for being able to render all of these theorems as having the “same logical structure.” Students must be able to treat more complex conditions as entailing a truth-functional predicate on the relevant set of objects. Furthermore, we note that unitizing predicates is non-trivial. Moria consistently had trouble in the experiment treating conjunctions as single predicates. For her, claims like Theorem 4 were relationships between three sets of objects instead of two, which prevented her from assimilating these theorems to the subset structure of Theorem 1.

We see two primary points of significance to unitizing predicates. First, we hope future work will continue to explore the necessary cognitive work involved in constructing various statements and proofs as having the same logical structure. Second, we hope logic instruction will begin to attend to this critical learning process rather than beginning with logical syntax. We fear that this does not help students like April learn how to unitize predicates and construct relationships of entailment, which we see as central to meaningful learning of mathematical logic.

Acknowledgements

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Justification framework: A tool to analyze reasoning strategies for doing a geometric proof

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Keywords: Congruence, Geometric proof, Reasoning

In 2000, the National Council of Teachers of Mathematics highlighted proof as one of the most important components of mathematical thinking in justifying a mathematical idea in a fluent and formal way. One of the expectations is that students should be able to understand and construct mathematical proofs by the end of secondary school. The purpose of our study was to analyze and classify the justification strategies used by an honors high school student while producing a geometric proof. In this study, we adopted the justification framework (Marrades & Gutiérrez, 2000) to analyze a student's process of doing a congruency of triangles proof task. Data were collected through a cognitive interview and a written task on pencil and paper. The results showed that the student used deductive-formal, deductive-formal-structural, and deductive-failed types of justifications during her proof process.

Theoretical Framework

We used the justification framework developed and elaborated by Marrades and Gutiérrez (2000). The authors established the justification framework by expanding on previous approaches in academic literature. The framework provides an analytical tool to classify and analyze students' strategies in the process of constructing justifications in problem-solving. The authors adopted the concept of justification as "any reason given to convince people of the truth of a statement" (Marrades & Gutierrez, 2000, p. 89). Also, they referred to *proof* as a justification that satisfies the agreed-upon abstraction criteria by expert mathematicians as valid in an axiomatic system. The framework identifies two principal classifications which are empirical and deductive justifications. Within these classifications, there categories and subcategories. The empirical justification has four categories: failed, naïve empiricism (perceptual and inductive), crucial experiment, and generic example (example-based, constructive, analytical, intellectual). The deductive justification has three categories: failed, though experiment (transformative and structural), and formal (transformative and structural).

Method and Data Collection

We used purposive sampling method and the participation was voluntary based. The participant, Nina, was an 11th grade, honors, and a female high school student. Nina took an introductory geometry course previously where she received formal instruction on proofs. This was a crucial aspect of consideration for our study as we were interested in investigating the type of justification strategies the student will use in constructing a proof task. Some criteria we considered were that Nina was in the secondary school level, a level in which congruency of triangles is studied. In a pencil and paper environment and guided by a cognitive interview for 25 minutes, Nina worked on the congruency of triangles proof task.

Research Results

In Table 1, we present Nina´s justifications strategies. By applying the theoretical framework, we determined that the student used diverse strategies in engaging in the geometric proof task of congruency of triangle.

Table 1: Nina’ s justification strategies

Statement	Reason	Justification level	Description
<S ≅ <R	Given	Deductive → Formal	Because the problem gives me this information
XT bisects <SXR	Given	Deductive → Formal	I am given this
<SXT ≅ <RXT	Nina’ s thinking	Deductive →Formal → Structural	The definition of bisector (Congruency of angles)
<STX ≅ <RTX	Nina’ s thinking	Deductive → Failed	XT is a bisector
ST ≅ RT	Nina’ s thinking	Deductive →Failed	XT bisects the side SR in two equal parts
SXT ≅ RXT	Nina’ s demonstration	Deductive →Failed	The congruence theorem ASA

Nina’ s justification process was classified into the deductive main type of justification as she did not make use of any specific examples to do the proof justification, but instead used generic aspects of the problem, mental operations, and logical deductions. Nina´s first two arguments are deductive formal because she used only generic given aspects of the problem. After that, she employed a deductive formal structural argument by applying the definition of bisector to support her justification. Her next justification steps were categorized into the deductive failed strategy as she elaborated correct conjectures but failed in providing the correct justification for them. To summarize, Nina derived part of her justification by using an accepted definition and logical inferences from it. Later, she used an incorrect justification which consequently caused her next steps to be incorrect. Nina made an explicit effort to use rigorous symbolic language to express her thinking in written form while constructing sequences of logical deductions. Nevertheless, at the end she failed to succeed in elaborating the proof.

Limitations, and Future Directions

Because of the limited sample size and the participant’s status as an honors student, our findings cannot be generalized. A future direction of this study would be experimenting with more high school students from different backgrounds by including diverse geometry proof task types.

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Two Primary School Teachers' Mathematical Knowledge of Content, Students, and Teaching Practices relevant to Lakatos-style Investigation of Proof Tasks

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Despite recognition of the importance of Lakatos-style proving activity in the mathematics classroom, we know little about whether teachers' relevant mathematical knowledge is conducive to supporting it in their classrooms. We take a step towards addressing this research gap by reporting the results of an exploration of two primary teachers' mathematical knowledge of content, students, and teaching practices relevant to Lakatos-style investigation of proof tasks. Through vignettes-based, semi-structured interviews, we presented the participants with 19 illustrated classroom episodes covering a range of Lakatosian techniques and a range of student ways of engaging with supportive examples and counterexamples to formulate, validate, refute, and refine conjectures of different types. Participants' responses revealed both productive and counterproductive understandings highlighting that although Lakatos-style proof lies within teachers' reach, supporting their preparation is crucial.

Keywords: Primary school mathematics, Mathematical knowledge for teaching Lakatos-style proof, Conjectures, Supportive examples, Counterexamples.

Introduction

In his book "Proofs and Refutations", Lakatos (1976) reconstructed prominent mathematicians' discussions on Euler's theorem to highlight the key role example use can play in proving and refuting conjectures. As a result of the increasing research interest about the incorporation of proof construction into mathematics classrooms (e.g., Stylianides, 2016), some studies (e.g., Balacheff, 1991; Komatsu et al., 2018) have investigated students' ability to comprehend and employ Lakatosian techniques to solve proof tasks and reported encouraging findings. However, few studies have centred upon primary school level (e.g., Komatsu, 2010; Reid, 2002) and none has had an explicit focus on teachers. Addressing these research gaps, our study sought to answer this research question: In what ways are two primary school teachers' knowledge of content, students, and teaching practices relevant to Lakatos-style investigation of proof tasks similar and different regarding the extent to which they may be able to support potentially their students' engagement with this style of proving?

Theoretical Frameworks

Phases of Lakatos-style Investigation of Proof Tasks

Drawing on Lakatos' (1976) original work and others' account of Lakatos-style proving activity in the mathematics classroom (e.g., Komatsu, 2010; Reid, 2002) we define Lakatos-style proving activity as an iterative and reflective process consisting of four interrelated phases (Deslis, 2020): in *Phase 1* students identify a conjecture which they wish to examine, based on a pattern, an educated guess, or simply following their teacher's suggestion. Then they start examining cases to investigate its validity. The discovery of supportive examples in *Phase 2* may indicate that the conjecture is likely to be true. However, in *Phase 3* counterexamples may also emerge from the examination of cases suggesting that the conjecture is false. In *Phase 4* students reflect on the previously discovered examples to appropriately modify the false conjecture. The domains of the original conjecture can either be restricted or expanded, to exclude the discovered counterexamples, or to transform them into supportive examples, respectively. Once the conjecture has been altered, a new investigation cycle begins aiming to the further refinement of the conjecture. It is important to note that, although our framework is based on Lakatos' work, it focuses on certain aspects of Lakatos-style reasoning and does not purport to reflect the whole complexity of his philosophy. For example, our framework does not focus on the interplay between defining and proving, which is crucial in Lakatos' work.

Conjecture Types

According to the classification of statements proposed by Tsamir et al. (2009), we identify three conjecture types: *Always-True* and *Never-True* conjectures are the two extremes, since only supportive examples and counterexamples emerge during their examination, respectively; in contrast, both example types emerge during the examination of *Sometimes-True* conjectures (see Figure 1).

Justification Schemes (JS) and Refutation Schemes (RS) for Student Understandings

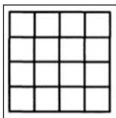
Justification and *Refutation* Schemes are two separate but interrelated three-level classifications describing a rather comprehensive range of student understandings about the interplay between examples and proving. According to Framework JS, which is based on Harel and Sowder's (1998) classification and its adaptation by Stylianides and Stylianides (2009), students at the least advanced level accept generalisations based on supportive evidence coming from a few easy-to-check cases. At the intermediate level they believe that only example-based evidence coming from the examination of representative cases can produce valid generalisations, whereas at the advanced level students are aware of the insufficiency of all types of example-based arguments. By analogy to JS, Framework RS, which we developed based on previous research on students' views around counterexamples (e.g., Balacheff, 1991; Lee, 2016), students at the advanced level consider the discovery a counterexample sufficient to refute a conjecture, while students at the intermediate level question the sufficiency of a single counterexample to refute a conjecture and demand the discovery of more counterexamples, preferably resulting from strategically selected cases. At the least advanced level students treat counterexamples as exceptions, resisting to the idea that the existence of counterexamples can affect the truth of a convincing conjecture. Students who hold the advanced schemes in relation to both example types can potentially also reach a meta-level that is key in the implementation of Phase 4 (Lakatos-style conjecture refinement). On top of the understanding that comes with the acquisition of the two advanced levels, students at this meta-level, which we call "*refinement scheme*", are able to reflect on the previously discovered supportive examples and counterexamples to get insights into how the refuted conjecture can be appropriately modified.

MaKTeLaP: Mathematical Knowledge for Teaching Lakatos-style Proof

Building on previous research on the knowledge needed to teach mathematics (Ball et al., 2008), or specifically proof (Buchbinder & McCrone, 2020), we describe the mathematical knowledge relevant to bringing Lakatos-style activity into the classroom (Deslis et al., 2021). We identify three knowledge components: (1) *CoLaP* which refers to the *Co*ntent knowledge about what constitutes appropriate use of examples in *La*katos-style *P*roof tasks; (2) *StuLaP* which refers to the knowledge of *Stu*dents' typical understandings around the example use in *La*katos-style *P*roving; and (3) *TeLaP* which refers to the knowledge of *Te*aching practices that can appropriately promote students' efforts to productively engage with *La*katos-style *P*roof. Since all Lakatosian phases revolve around two example types, namely supportive examples (SEs) and counterexamples (CEs), we can identify two subcomponents within each component, each focusing on the knowledge around one example type. For example, CoLaP splits into CoLaP [SE] and CoLaP [CE].

Methods

We collected our data through semi-structured interviews based on vignettes (Skilling & Stylianides, 2019), which are contextualised descriptions of classroom situations (Deslis et al., 2021). 10 in-service primary school teachers were presented with 19 classroom episodes with comic-style student characters discussing and exchanging arguments with their peers as they engaged in the phases of Lakatos-style investigation. Three student groups worked on the "Count the Squares" task (Zack, 1997) and each explored a conjecture of a different type (see Figure 1). The student dialogues we used were adapted versions of classroom episodes from fifth grade reported in Zack (1997) and Reid (2002) and covered the whole range of investigation phases and student understandings, as described by our theoretical frameworks. After each episode participants were asked to comment on the validity of arguments, evaluate students' understandings, and discuss how they would respond to each student contribution. We analysed responses for themes and ranked the various teacher understandings relevant to the different MaKTeLaP components and example types according to their level of sophistication (Deslis et al., 2021). In this paper we focus on two teacher participants, identify similarities and differences in their responses, and discuss the degree to which their understandings put them in a good position to support the incorporation of Lakatos-style proving activity into their classrooms. The characteristics of the two teachers were reasonably similar. Alcyone and Nephele (nicknames) were both female, 29 and 27 years old, respectively. At the time of the study, they had 18 and 45 months of teaching experience, respectively, and both held a Bachelor's and a Master's degree in Education. Both participants taught middle-sized (21 and 23 students) fifth-grade classes (ages 10-11) in similar and neighbouring schools in Athens, Greece, with most of their students coming from middle-income households.



How many squares are there in this 4-by-4 grid? Examine other similar grids of your choice. How many squares are there in each of them? Find a general rule that applies to grids of all sizes and prove your answer.

Group 1- Always True Conjecture:	Group 2- Sometimes True Conjecture:	Group 3- Never True Conjecture:
"The number of squares in an n-by-n grid is $1^2+2^2+\ldots+(n-1)^2+n^2$."	"The number of squares in an n-by-n grid is a multiple of five."	"If an n-by-n grid has N squares and an m-by-m grid has M squares, then the m×n-by-m×n grid has M×N squares."

Note. The phrasing of the conjectures has been altered; the student characters in the vignettes presented and discussed these conjectures using language that reflects the mathematical knowledge of students of their age.

Figure 1: The proof task and the conjectures that were investigated by the three student groups

Findings and Discussion

CoLaP: Knowledge of Content

Participants were asked to evaluate the arguments presented in the scenarios regarding their validity. The responses indicate that the two teachers' understandings about the role of examples in the investigation of conjectures had both similarities and differences (see Table 1). Commenting on whether it is appropriate to conclude that a conjecture is true based on the discovery of some supportive examples, Nephele preferred to judge on a case-by-case basis, while Alcyone's responses suggested a blanket rejection of this idea. For example, when the second group (Sometimes-True conjecture) found a few examples that supported their conjecture, the two teachers agreed that this does not allow us to conclude that the conjecture will work for any grid:

Alcyone: The absence of counterexamples so far is not evidence that they do not exist. The examples so far are supportive, but a counterexample can emerge any time.
Nephele: The generalisation is not permissible, because it is only based on two examples.

Alcyone maintained this opinion when commenting on a similar moment during the first group's investigation (Always-True conjecture). Yet, Nephele was convinced that it will work for all grids, although this generalisation would be based on the same number of examples as in the previous case:

Nephele: All the examples so far have confirmed the conjecture, so we can safely conclude that it is correct and works for all grids.

Their comments on arguments against the idea that any example-based argument can lead to safe generalisations of conjectures also highlighted the divergence of views:

Nephele: This is not necessarily true; sometimes even a few examples can provide undeniable evidence and thus enable us to conclude that the conjecture is true for all cases.
Alcyone: The examples may show that there is a possibility for the conjecture to be true, but we cannot be sure about that if the only evidence we have comes from examples.

Overall, Alcyone showed concrete and stable appreciation of the usability and limitations of supportive examples. She valued their role in the investigation of conjectures while being aware that we cannot prove a general statement merely based on examples. In contrast, Nephele's responses varied from one episode to another, indicating weaker understanding which is highlighted by her erroneous belief that in some circumstances examples can be used to prove general statements.

In contrast to the previous example type, the two teachers were found to hold comparable views about the appropriate use of counterexamples in refuting conjectures. Specifically, both participants' responses showed awareness that a single counterexample can sufficiently refute a general statement:

Nephele: Now that we have found a counterexample, we know that the conjecture is false.
Alcyone: The number of supportive examples we have found is irrelevant; one counterexample is enough to show us that the conjecture does not hold.

Table 1: Summary of Alcyone and Nephele's understandings relevant to CoLaP

CoLaP [SE]	*A:* Supportive examples can be used to investigate general statements, but they cannot prove them.	*N:* Supportive examples can be used both to investigate and to prove general statements.
CoLaP [CE]	*A & N:* A single counterexample can sufficiently refute a conjecture.	

STuLaP: Knowledge of Student Understandings

Turning to StuLaP, participants' judgements about the level of student understandings brought to the surface both similarities and differences in teachers' views (see Table 2). Alcyone not only recognised students' common belief that examples can be used to prove general statements is a misconception, but also was aware that the use of representative cases which have been strategically selected does not make the argument any more valid from a mathematical perspective.

> Alcyone: Any student who is happy to accept generalisations that are merely based on examples as proofs (no matter how many examples there are or the process through which they have been identified) has a significantly less advanced level of understanding than students who reject all kinds of example-based proofs.

Unlike Alcyone, Nephele was occasionally favourable towards the use of examples as a means to prove, since she tended to accept arguments that reflected the intermediate JS level as valid:

> Nephele: The student has not realised that the examples so far have covered the whole spectrum of possible grids and therefore proved that the rule is correct. His choice to reject example-based proofs in their entirety signifies weak understanding.

Unlike the case of supportive examples, the two participants had similar views about students' understandings around counterexamples. Specifically, both teachers spoke highly of students who believed that a single counterexample can sufficiently refute a conjecture:

> Nephele: Students who discard a conjecture immediately after the discovery of the first counterexample and consider further checks as unnecessary have a more advanced level of understanding than those who demand a substantial number of counterexamples to be discovered before they refute the statement.

Furthermore, both participants consistently criticised students who continuously treated counterexamples as exceptions and maintained their initial opinion ignoring the evidence against it:

> Alcyone: Students' refusal to reject a conjecture despite the existence of counterexamples and the treatment of counterexamples as exceptions indicates poor understanding.

Yet, Nephele also judged favourably students who were reluctant to reject conjectures after the discovery of one counterexample and instead demanded that a substantial number of counterexamples should be discovered, and characterised this practice as more productive than it actually is:

> Nephele: Students' reluctance to reject a conjecture immediately after the discovery of one counterexample and their need to find additional counterexamples shows a scepticism that is desirable in the classroom of mathematics.

Although Alcyone occasionally criticised this practice, in other episodes she also expressed views that were similar to those of Nephele:

> Alcyone: I like this student's critical attitude! It is always good to be reluctant and demand more evidence.

Overall, both teachers' responses showed a satisfactory degree of awareness about which student understandings about counterexamples signify an advanced level of understanding and which do not, mirroring, to an extent, their good content knowledge on counterexamples. Still, some of their reactions to students who resisted to conjecture refutation unless numerous counterexamples were discovered, were contradictory, thus showing knowledge fragility. The main differences lay in their understandings about student conceptions around supportive examples: unlike Alcyone, Nephele did not consider students' tendency to overrely on empirical arguments as a misconception.

Table 2: Summary of Alcyone and Nephele's understandings relevant to StuLaP

StuLaP [SE]	*A:* Students' belief that conjectures can be proved through supportive examples, even if these are coming from the examination of strategically selected cases, is counterproductive.	*N:* Students' belief that conjectures can be proved through supportive examples is counterproductive unless these are coming from the examination of strategically selected cases.
StuLaP [CE]	*A & N:* Students' belief that a conjecture must be refuted after the discovery of one counterexample is productive, as is students' desire to discover more counterexamples before they refute a statement.	

TeLaP: Knowledge of Teaching Practices

As for the third MaKTeLaP component (see Table 3), participants were asked how they would respond to students' contributions and how they would support students' efforts if they were their teachers. Teachers' responses showed that they promoted a slightly different usage of supportive examples in the exploration of proof tasks. To begin with the commonalities, both appreciated the importance of examining various examples in the beginning of the investigation:

Alcyone: The examination of several different cases is a reasonable thing to do after the formulation of a conjecture, since it can provide clues about how we can prove it.

Nephele's advice regarding which cases students should try first was even more specific:

Nephele: The examination of the easiest-to-check cases can be a convenient way to start.

However, Nephele also thought that it is appropriate to encourage students to terminate the investigation once much confirmatory example-based evidence has been found:

Nephele: The examples have shown that the rule works; now I'd advise students to stop, and I'd give them a new problem to solve.

Taking a different approach, Alcyone appreciated that despite the valuable contribution of supportive examples to the promotion of the investigation, it is inappropriate to conclude an investigation at the stage of the example examination even if several confirmatory cases have been discovered:

Alcyone: I'd tell the students that even if they have found several examples supporting their conjecture, it is prudent to remain cautious and continue the examination of cases in search of counterexamples, which can emerge anytime.

The two teachers' suggestions after the discovery of counterexamples were dissimilar, too. Nephele encouraged students to abandon the refuted conjecture and replace it with a new one:

Nephele: The conjecture clearly doesn't work. I'd encourage students to abandon this idea and try to formulate a new conjecture that is not related to the multiples of five.

Taking a step further, Alcyone not only advised students to replace the faulty conjectures, but also suggested that the new conjecture could be an improved version of the initial conjecture. She also added that reflecting on the characteristics of the previously discovered examples can provide clues for the appropriate modification of the refuted conjecture:

Alcyone: The students can review the examples they found to come up with a refined version of the conjecture. […] There might be a subset of grids for which the conjecture works; for example, for the n-by-n grids where n is a multiple of five.

All in all, both teachers' suggestions showed an appreciation of the important part example examination can play in the investigation of conjectures. However, Alcyone's suggestions in relation to both example types were clearly more sophisticated. Unlike Nephele, Alcyone was fully aware that supportive examples cannot prove general statements and judged the student arguments accordingly. Her suggestions to the students reflected an appreciation of the limitations of example use. A remarkable similarity between the two teachers' responses was that although both recognised that one counterexample can refute a conjecture, both also indulged in judging favourably students who attempted unnecessary checks to discover more. Yet, it was impressive that Alcyone encouraged the use of a technique that is surprisingly consistent with the spirit and the essence of Lakatos-style reasoning. Unlike Nephele who encouraged the replacement of a refuted conjecture with a new one, Alcyone said she would encourage students to use counterexamples not only to refute conjectures, but also to modify and refine them considering the characteristics of the previously examined examples. This practice lies at the heart of Lakatos-style activity and the fact that Alcyone reinvented it despite the lack of any prior relevant instruction in teacher education is encouraging.

Table 3: Summary of Alcyone and Nephele's understandings relevant to TeLaP

TeLaP [SE]	*A:* Use of supportive examples to initiate the conjecture exploration.	*N:* Use of supportive examples to initiate and terminate the conjecture exploration.
TeLaP [CE]	*A:* Use of counterexamples to refute the conjecture and reflection on their characteristics to refine it.	*N:* Use of counterexamples to refute the conjecture and then replace it with a new one.

Conclusion

Research on classrooms of expert teachers (e.g., Zack, 1997) or teachers who worked closely with researchers (e.g., Komatsu et al., 2018) suggests that students can engage productively in Lakatos-style activity and benefit from it, even at primary school level. Still, we know little about how non-expert teachers understand various aspects of this activity and thus whether they would have the knowledge to be able to support it in their classrooms. The present study adds to the increasing literature on the incorporation of Lakatos-style reasoning into school mathematics by exploring ordinary teachers' relevant mathematical knowledge, an area that had previously been unexplored. We analysed and compared two primary school teachers' reactions to a set of 19 illustrated classroom episodes which enabled us, in an explorative way, to shed light on their understandings of content, students, and teaching practices relevant to Lakatos-style activity. Our study offers some encouraging findings while pointing in directions for future research. The case of Alcyone and her many productive intuitive understandings suggest that it is possible for primary school teachers to have necessary (though, arguably, not sufficient) knowledge to bring Lakatos-style proof into their classrooms. Yet, the case of Nephele suggests that it is also important to identify appropriate ways to support the refinement of teachers' understandings and highlights the crucial role teacher education has to play in preparing teachers to effectively engage their students with Lakatosian techniques.

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A sixth-grade student's growth in understanding written proof texts: Pre- and post-interview analyses of a teaching experiment study

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This study aims to describe a 6th-grade student's progress in understanding written proof texts through participating in an individual teaching experiment. The same task was administered to the student twice in an interview setting, before and a year after the teaching experiment. The student was asked to evaluate four arguments aimed to prove a given conjecture. While in the pre-interview, the student accepted all four arguments based on her Naïve Experience; in the post-interview, she rejected empirical arguments and looked for General Procedures (or Abstract Structures) that necessarily apply to the whole set of numbers under investigation. In the post-interview, spontaneous changes occurred in the student's understanding of the given arguments after she constructed her own proof for the same conjecture. Instructional design elements used in the teaching experiment might have facilitated her understanding of the structure of deductive proof texts.

Keywords: Proof comprehension, students' understanding, middle school students, individual teaching experiment.

Introduction

Reliance on empirical reasoning for validating mathematical generalizations is a faulty way of thinking pervasive among students (Harel, 2008). Many students, even after learning about secure methods of proving, are reported to retain an empirical proof scheme (Education Committee of the European Mathematical Society, 2011). Current efforts in mathematics education, therefore, aim to help students realize the limitations of empirical arguments and learn about reasoning deductively at the early grade levels (Stylianides & Stylianides, 2009). Another deficiency in students' learning about proof is holding a ritual proof scheme, in which the student judges the validity of an argument strictly by its appearance, rather than its underlying structure (Harel, 2008). The same proof can be presented in verbal, pictorial, or symbolic forms (Stylianides, 2007). However, it is the logical structure of the argument that determines its validity (Miyazaki et al., 2017). This suggests, one aspect of learning about proof is to distinguish between the structure and form of the argument.

This paper reports on the preliminary findings from an ongoing research study conducted in Turkey. The purpose of the study is to explore the processes by which a 6th grade student (1) comes to understand deductive structure of mathematical proofs, and (2) develops the skills required to prove basic theorems, by use of an individual teaching experiment methodology. Even though the form of argument representation is an inseparable aspect of the instructional design approaches used in the teaching experiment, the main focus of the study is on helping the student understand deductive structure of mathematical proof. The purpose of the research reported here is to describe the student's progress in understanding written proof texts (within and between the two interviews conducted before and a year after the teaching experiment) by using a research-based model, which allows tracking changes in the students' understanding in terms of argument form and structure. Describing the student's dynamic experiences with proof texts might elucidate the strengths and weaknesses of the instructional design approaches used in the teaching experiment study.

The model: Students' ways of understanding a proof

Ahmadpour et al. (2019) put forward a model of students' understanding of a written proof text. The model describes possible *states* of evolving understanding and the *transitions* between them. According to the model, students' understanding of a formally acceptable proof may develop into three different end-states: a Formulated Proof, a Procedural Proof, and a Formulaic Proof. If the reader is aware of the underlying deductive structure of a proof, the proof is said to be "read" as a Formulated Proof. If the reader perceives the proof "as a general procedure that can be applied to any number, as a sort of recipe for producing examples, rather than a deductive structure applicable to all numbers" (Ahmadpour et al., 2019, p. 87), the proof is said to be read as a Procedural Proof. And, if the reader considers only its surface-level form while reading a proof, it is said to be read as a Formulaic Proof. According to the model (Ahmadpour et al., 2019), processes of learning towards the three end-states follow three different theoretical pathways. They are, respectively, the Path of Structure, Path of Procedure, and Path of Form. Switching between the path-ways is possible due to the potential shifts of attention in one's understanding. In this study, the main focus is on the Path of Structure, which describes the processes through which an individual comes to understand the deductive structure of a proof. In this path, through the transitions of *generalization*, *abstraction*, and *formalization*, the student develops from the state of Naïve Experience (in which existence of confirming examples are thought to validate generalizations) to those of General Procedure, Abstract Structure and Formulated Proof sequentially. The state of Abstract Structure was intended by the teaching experiment study reported here.

Ahmadpour et al. (2019) consider understanding as a dynamic process. Accordingly, the model allows description of students' progression over time by linking *form* and *structure*, beyond merely marking the *states* of understanding at fixed points in time. *Transitions* of how one *state* of understanding develops into a next one is a major focus of the model. In this study, the pre-interview captures a fixed, consistent *state* of understanding a 6th grade student demonstrates before participating in a teaching experiment study. The post-interview captures the dynamic changes in her understanding facilitated by her interaction with the interview task.

Method

Context and the participant of the study

In Turkey, students are not explicitly taught the concept of proof at the middle school. The mathematics curriculum (Ministry of National Education [MoNE], 2018) emphasizes students' explaining their reasoning and evaluating others' in the classroom. However, to what extent the abstract structures underlying valid arguments are explicated to the students is questionable, as no detailed prescriptions are provided for teachers. Hence, the 6th grade student, Beren (pseudonym), participated in this study had no previous interaction with the notion of proof. Beren was approached based on her competence in four operations, her ability to express mathematical ideas, and her willingness to learn mathematics. She volunteered for the study along with her parents' consent.

The teaching experiment

The teaching experiment consisted of four major stages (Steffe & Thompson, 2000). The first stage aimed to prepare the student about basic number theory concepts (such as modular structure, parity, and divisibility) that would be the objects of conjectures studied throughout the teaching experiment. The second stage challenged the student's extant source of conviction, Naïve Experience, by creating a cognitive conflict through using Monstrous Counterexample Illustration (Stylianides & Stylianides, 2009). This stage triggered an *intellectual need* (Harel, 2008) in the student for learning about secure methods of proving. The third stage aimed to satisfy this need by designing tasks for her to understand the deductive structure of mathematical proofs. Then, in the fourth stage, the student was encouraged to use this understanding to practice proving a set of theorems.

The first stage introduced to the student a way of representing modular structure of unknown quantities by using a short story, which later provided her a context for explaining mathematical arguments. In this story, an unknown number of cookies evenly distributed in identical cups with some remainders were used to represent the modular structure of an unknown quantity. Based on the story, the algebraic expression defining odd numbers, 2n+1 (where n is a non-negative integer), was represented as two cups containing the same number of cookies and a single cookie. A variation of the representation was used in the fourth stage, as in Figure 1, for the concept of consecutiveness (i.e., a cup of cookies standing for n items, and another cup and one more cookie representing n+1 items).

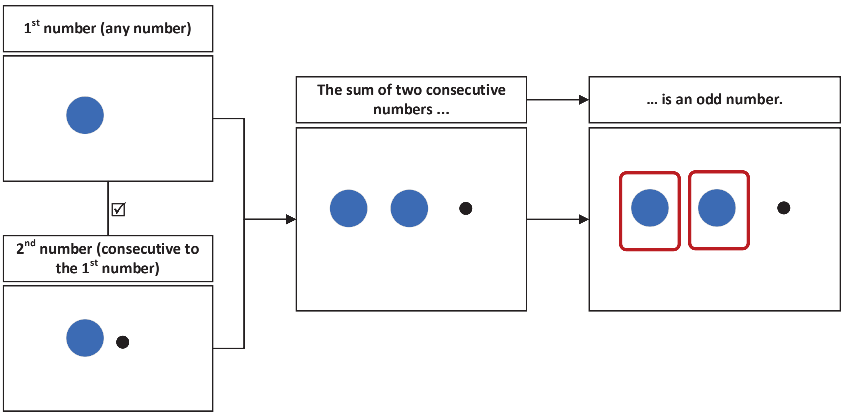
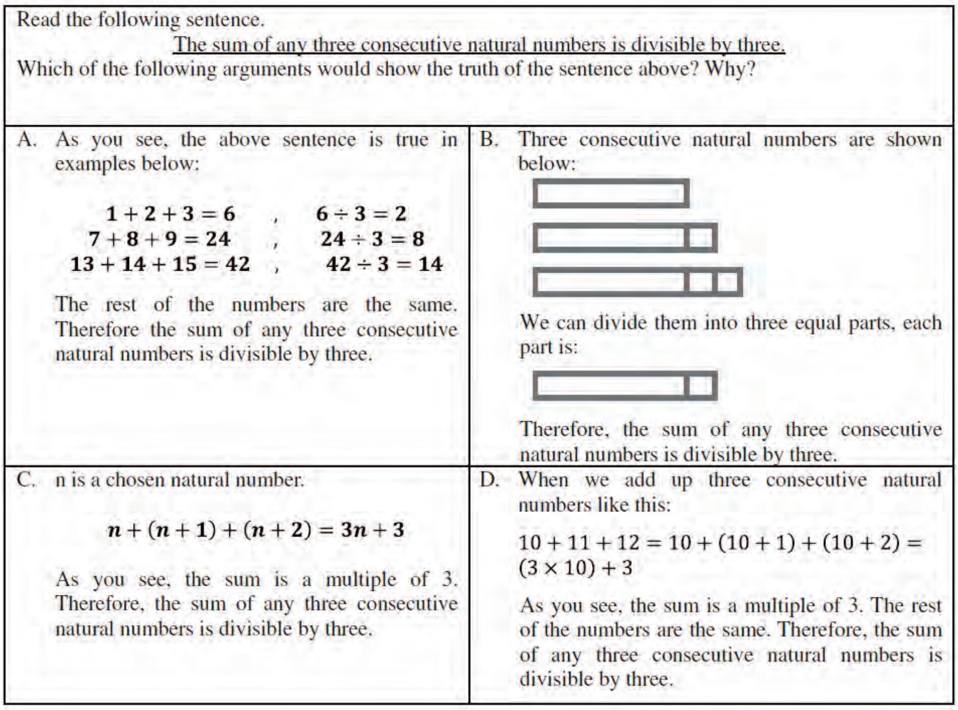


Figure 1: A flow-chart proof of "The sum of two consecutive numbers is an odd number."

Another instructional design component, the flow-chart proof format, adapted from Miyazaki et al. (2017), was introduced to the student in the third stage. The flow-chart proof format was used with the purpose of explicating the structure of deductive arguments. Figure 1 illustrates the flow-chart proof using the specific representation developed in this study. Large circles represent the cups containing an unknown number of cookies, while the small circles represent single cookies. The checkmark between the two leftmost boxes indicates that the two objects shown are related; that is, the cups contain exactly the same number of cookies – stand for the same unknown quantity. In other cases where the two objects are not related (for example in proving that the sum of two odd numbers is an even number) cups of different color or shape are used (invented by the student), and the checkpoints are filled with a cross mark (by the student).

Data collection and analysis

The pre-interview was conducted at the beginning of the teaching experiment study. At the time of the pre-interview, Beren had just completed the 6^{th} grade. The teaching experiment took 8 weeks. The post-interview was conducted a year after the completion of teaching experiment study. The task shown in Figure 2 was translated into Turkish. Beren was asked to think aloud while reading and making decisions about each of the arguments. Probing questions were used with the purpose of capturing details of her understandings. Pre- and post-interview data were analyzed based on the model of students' ways of understanding a proof (Ahmadpour et al., 2019).


Read the following sentence.

The sum of any three consecutive natural numbers is divisible by three.

Which of the following arguments would show the truth of the sentence above? Why?

A. As you see, the above sentence is true in examples below: $1+2+3=6$, $6 \div 3 = 2$; $7+8+9=24$, $24 \div 3 = 8$; $13+14+15=42$, $42 \div 3 = 14$. The rest of the numbers are the same. Therefore the sum of any three consecutive natural numbers is divisible by three.	B. Three consecutive natural numbers are shown below: We can divide them into three equal parts, each part is: Therefore, the sum of any three consecutive natural numbers is divisible by three.
C. n is a chosen natural number. $n+(n+1)+(n+2)=3n+3$ As you see, the sum is a multiple of 3. Therefore, the sum of any three consecutive natural numbers is divisible by three.	D. When we add up three consecutive natural numbers like this: $10+11+12=10+(10+1)+(10+2)=(3\times 10)+3$ As you see, the sum is a multiple of 3. The rest of the numbers are the same. Therefore, the sum of any three consecutive natural numbers is divisible by three.

Figure 2: The task (Ahmadpour et al., 2019, p.89)

At the time of the pre-interview, Beren knew the meaning of algebraic expressions such as "2a+3", and was able to calculate the value of such expressions for specific values of the unknown "a". However, she did not know how to operate on algebraic expressions, which was essential for understanding Argument C. Such syntactical understanding was not among the goals of the teaching experiment and was not part of the instructional design. At the time of the post-interview, however, when Beren completed the 7^{th} grade, she possessed a greater understanding of the algebraic operations used in Argument C, because students learn these skills at the 7^{th} grade in Turkey (MoNE, 2018).

Findings

Two remarks are important to highlight. Although the concepts of divisibility are addressed in Turkish middle school mathematics curriculum, the learning objectives are restricted to the use of divisibility rules within arithmetic. Hence, the conjecture examined in this study was novel to Beren at the time of pre-interview. However, this was not the case in the post-interview. Beren was asked to evaluate validity of the exact same conjecture in the last episode of the teaching experiment study, as part of a proof-production assessment. She was able to produce a valid proof of the statement.

Pre-interview

Each of Beren's evaluations in the pre-interview were based on Naïve Experience. For instance, reading Argument A, she reviewed all the calculations and decided that the given examples were correct. After testing the conjecture for two other sets of three consecutive numbers, she stated:

Beren: I think the sentence is correct. I mean this option A is correct. It shows its truth definitely. It has already given examples, has done the division. I don't think there is a problem with option A.

After going through similar procedures of Naïve Experience for Arguments B, C and D, Beren summarized her thoughts about the task:

Beren: I tried each of them [the four arguments] by doing different operations and three consecutive numbers hold. […] So, there is no reason for this, I find it by trying with numbers, again.

Post-interview

In the post-interview, Beren remembered the task from the last year and sequentially explained the four arguments. Unlike in the pre-interview, she did not accept Argument A as a proof:

Beren: Now, when I read, I do not directly understand because, now, it says 'The rest of the numbers [are the same]'. How can I know without trying it? Who has proven this and according to what? […]. I'm moving on to the other. […] This does not show for sure.

Continuing with Argument B, Beren thought that it could be a proof because it was not based on specific examples. However, she could not make sense of the representations used and hence the underlying ideas communicated. She could not articulate a consistent meaning out of the realistic situations she tried to make up, as she did not consider the relationship between the sizes of the three strips. Although she could not understand Argument B, her preliminary decision was not to eliminate this option. Beren seemed to think that Argument B might be arguing for all consecutive numbers.

Beren: I said it could be a proof because it does not give us a certain number here. No matter how much you divide, it says, the three [strips]. But, still I want to look at the others. According to that… You know, I want to decide whether this is a proof or not and tell its reason based on that.

In Argument C again, Beren could not make sense of the given algebraic expression. Her focus was on the *form* rather than the *structure*. (Note that in Argument C, instead of "n", "a" was used as a variable, for the student's familiarity.)

Beren: Why it says this three? [points to '3a' in 'a + (a+1) + (a+2) = 3a + 3'] … There are three of a's, I see. […] Well, but it again gives us numbers here. It says one, two, … I mean. I think, I cannot prove with this.

Then, an instant shift of attention occurred in her understanding, marking a *transition* towards either a General Procedure or an Abstract Structure in her understanding. She associated the "+1" in the expression "(a+1)" with her previous understanding of how two consecutive numbers were related.

Beren: … like in the logic of a chart [the flow-chart proof]. Well, we were here [points to '(a+1)'] showing [this] 1 more, you know. In here, as well, I wonder, since it says a plus one, it is the extra… I mean we can transfer this into a schema. I, for this reason, think that this could be [a proof.] In fact, I think this shows certainly, as well. I think this has the logic of 'whatever number you try, it will work'.

Beren’s comments about arguments B and C reveal that she was looking for General Procedures (or Abstract Structures) that would apply to all triples of consecutive numbers. She summarized her perception of the first three arguments and then continued with reading Argument D.

Beren: I know that this (Argument A) is not [a] ‘for sure’ [argument], this (Argument B) could be, I said. But I think that this (Argument C) will be for sure.

At her first glance, Beren judged Argument D by the sentence “The rest of the numbers are the same”.

Beren: I think rather than the operations, what is written here is important. Because ‘The rest of the numbers are the same’. We want to proceed by… that shows for everything, any number, here, not one number ten, eleven, twelve. […] I think I would think a lot if it hadn't said that sentence.

Researcher: Well, let's pretend that this sentence is not there.

Beren: Still I don’t think it is [a proof] […] It's not proven for every consecutive number. […] Here you have to try and find it.

The researcher asked Beren how she would “try” in order to see if she would make use of the structure “3 times the smallest number plus 3” illustrated in Argument D, in creating other examples. But, this unintentionally prompted Beren to construct her own proof for the statement, given in Figure 3 (left).

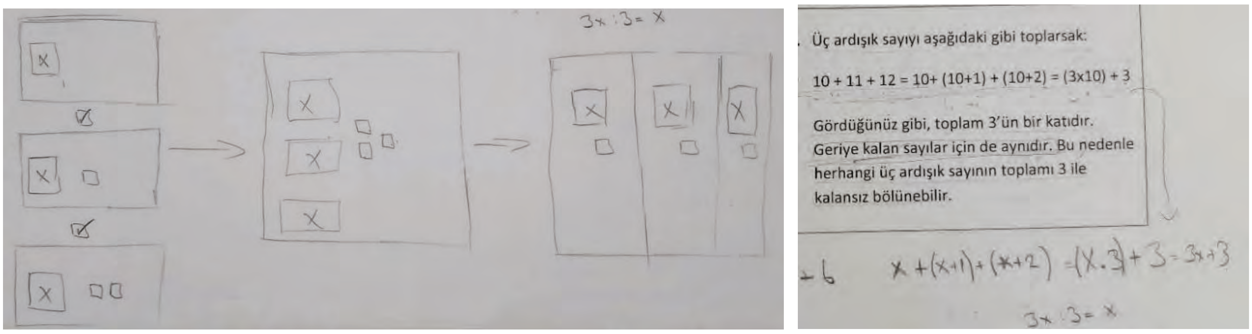


Figure 3: Beren’s flow-chart proof for the statement “The sum of any three consecutive natural numbers is divisible by 3” (left) and her formalized expression for Argument D (right)

While drawing the flow-chart, she explained her ideas in every step. Then she connected the Abstract Structure underlying this proof to that of Argument D.

Beren: Well, three consecutive numbers. In fact, it [her flow-chart proof] is [the same as] the logic in here [Argument D]. But, in here [in Argument D] it evaluates over ten.

Researcher: What if that ten is replaced by another number?

Beren: Again, it will be the same thing, but it’s saying it out of ten seems nonsense to me. For example, if it says x there, would be okay. If it says x, it would have proved for sure. It would have said 'whatever number came there... if any number comes in, it holds.' But I don't think this is okay, it’s being ten.

Then, she produced the algebraic expression in Figure 3 (right). Her explanation of this expression suggests that she perceived 3x+3 as an Abstract Structure subject to the distributive law for division.

Beren: Three x plus three. Three x is already divisible by three. We understand that, it is […] x. When I distribute three also one by one, it comes out one x… x plus one. I can prove it this way.

She also considered Argument D to well emphasize Abstract Structure of three consecutive numbers:

Beren: [Argument D] shows for ten, eleven, twelve, but it says ten plus one and plus two. It even proves more [compares to Argument A] that they are consecutive.

The above two scripts show that Beren understood the Abstract Structure behind Argument D. But, she did not accept this argument as a Formulated Proof, because these structures were not expressed in a formally acceptable way to her.

Beren: I can prove it this way [by using x]. But when I look at here [Argument D] ten, eleven, twelve, since they are consecutive, are summed, divided by three. But, is this same thing valid for two, three, four?

Researcher: Let's start with such a number [instead of ten] that… what is done here is not correct. Is that possible?

Beren: I think, it is not [possible]. Because when I do the same by x, I could show it. It holds anyway.

Note that Beren, when she created the algebraic expression "x + (x+1) + (x+2) = (x.3) +3 = 3x + 3" for Argument D, has just *formalized* the Abstract Structure of her own flow-chart proof (through her independent activity). Also note that an equivalent of the expression was given in Argument C, out of which she was not able to perceive the same structure before. This suggests thinking with the flow-chart proof format helped Beren focus on the Abstract Structure behind Argument D. She then transferred the same understanding of this structure to Arguments C and B.

Beren: I accept [Argument C]. Because I myself expressed it here [in Argument D] by x. Here it uses not x, but a. Again, an algebraic expression it uses, I mean. It is not known what this number is, it could be 1, it could be ten or it could be a hundred. That is why, to me, this is a sufficient proof.

In Argument B, she matched the three strips with the expressions x, x+1 and x+2. She explained the underlying deductive structure of the two arguments and read them as Formulated Proofs.

Discussion

Findings from the pre-interview revealed Beren's reliance on Naïve Experience, which was an expected situation for a 6th grade student who had not received any instruction on proof. On the other hand, analysis of the *states* and *transitions* observed in her understanding of the given arguments during the post-interview highlighted important aspects of the instructional design elements used in the teaching experiment study. First, the student expressed a preference for non-empirical arguments. Unlike many others reported in the literature, who simultaneously possessed deductive and empirical proof schemes after learning about proof (Education Committee of the European Mathematical Society, 2011), she did not retain the empirical proof scheme. This suggests, the cognitive conflict approach used in the teaching experiment (Stylianides & Stylianides, 2009) helped Beren achieve the intended discrimination between valid and invalid modes of reasoning.

Second, in the post-interview, Beren was not able to make sense of the Arguments B and C in her first attempts. She was not familiar with the forms of representation used. After constructing her own flow-chart proof for the given statement, Beren first *formalized* its Abstract Structure through studying Argument D (created her Formulated Proof by using a variable "x") and then, connected this Abstract Structure with the forms of representation used in Arguments B and C. Her understanding of the Arguments B and C reached to the level of Formulated Proof, which would be the ultimate goal of proof comprehension for a mathematics learner. We hypothesize that the flow-chart proofs used in the teaching experiment study helped Beren focus on Abstract Structures behind deductive arguments. For instance, divisibility of 3x+3 by three was visualized in the last action in Beren's proof, which was developed previously from the context of sharing 3 cups of and 3 single cookies among three people. Furthermore, the cups and cookies representation, evolved into the general notion of a collection and some single of the identical items, helped Beren to represent Abstract Structure of the property shared by every set of three consecutive numbers.

Beren demonstrated successful use of the Abstract Structures underlying her flow-chart proof (learned in the teaching experiment) in a novel task. This might be an indication of the strong aspects of the instructional design in supporting young students' learning of the structure of deductive proof. Also, note that Beren did not restrict proofs to necessarily have a flow-chart format or use cups and cookies representation. Examination of the aspect(s) of the teaching experiment (if any) that might have facilitated Beren's discrimination between argument form and structure is an issue of further investigation. Results may provide insights into the ways of preventing the development of a ritual proof scheme (one that is observed frequently among students) while teaching proof. It is also important to note how the administration of the interview task itself, after the teaching experiment, elevated the student's bringing together the *structure* (from the teaching experiment) and *form* (supported by the task) to understand unfamiliar proof texts. Nature of Beren's activity in the post-interview might offer directions in the design of tasks for enhancing student learning about proof.

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On the dialectical relationship between truth and proof: Bolzano, Cauchy and the intermediate value theorem

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Abstract - The question of the nature of the relationship between truth in an interpretation and proof in a mathematical theory is a complex epistemological issue. From a didactic perspective, we argue that it is worthwhile to address this issue in mathematics teaching and learning. In this paper, we illustrate this in the context of the Intermediate Value Theorem, focusing on the approaches of Bolzano and Cauchy, and we discuss possible didactic implications.

Keywords: Truth in an interpretation, proof in a theory, didactic and epistemology of mathematics, Intermediate Value Theorem.

Introduction

A recurrent issue raised by students concerns the need to prove mathematical statements that looks obvious. The prevalence of this issue has increased since the introduction of dynamic geometry software offering strong epistemic conviction through various means (Mariotti, 2006). Therefore, a crucial issue in mathematics education concerns the relationship between proof and truth. Indeed, it is well known in the literature that the motivation for proof and proving is difficult for students to endorse: most of them consider it as a teacher's requirement, not as an epistemic necessity. The need for truth is often considered to come from the possibility of raising doubt, for example because observations or results of actions are counter-intuitive, or surprising. Looking through the history of mathematics at the completeness of the set of real numbers, Bergé (2008, p. 220) points to another issue, namely the necessity, for a thorough study of Analysis, of the theoretical reconstruction of pre-constructed notions of Calculus such as the straight line or the continuity of functions. In this paper we intend to show that the relationship between *truth* and *proof and proving* is not one of subordination, as claimed by some authors considering that only what is proved is true, but rather a dialectic relationship. For this purpose, we will focus on the Intermediate Value Theorem, an existential statement that has long been taken for granted by mathematicians, who have provided proofs based on geometrical arguments.

> One of the most important theorems about the continuum is intuitively obvious: if on a plane a continuous line has one of its extremities on one side of a right line and the other on the other side of the same right line, then the continuous line cuts through the right line (Longo, 2000, p.401).

Bolzano (1817) was the first to explicitly question the validity of proofs of the Intermediate Value Theorem based on geometrical arguments, thus paving the way for the recognition of the need for explicit constructions of the set of real numbers in the second half of the 19th century.

The choice of the intermediate value theorem is motivated 1/ by the epistemological relevance of the discussion on proof and truth; 2/ by the links with the question of $\mathbb{Q}$-incompleteness /$\mathbb{R}$-completeness; 3/ by its presence in many curricula at the transition from secondary to tertiary education, mainly because of its various applications.

In the first section of the paper, we present Bolzano's view that proofs of the Intermediate Value Theorem[1] based on geometrical arguments should not be admissible, while at the same time Cauchy provided such a proof in his course at the Ecole Polytechnique in Paris (Cauchy, 1821). In the second section, we discuss both approaches, with particular reference to a paper by Hourya Benis-Sinaceur. In the last section, we discuss possible didactic implications for proof and proving in mathematics education.

Truth and *proof and proving* in the theory of equation

Bolzano's discussion on geometrical arguments in proofs of the Intermediate Value Theorem

In the preface of his paper "Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege", published in Prague in 1817, Bernard Bolzano wrote[2]:

> There are two propositions in the theory of equations of which it could still be said, until recently, that a completely correct proof was unknown. One is the proposition: that between any two values of the unknown quantity which give results of opposite sign there must always lie at least one real root of the equation. (Russ, 1980, p.159)

It is noticeable that Bolzano does not raise doubt on the truth of this proposition, despite the lack in his view of a fully correct proof. He mentioned several names of mathematicians having provided a proof for this proposition, and claimed: "However, a more careful examination very soon shows that none of these proofs can be viewed as adequate" (Russ, 1980, p. 160). He then provided arguments for this claim.

> The most common kind of proof depends on a truth borrowed from geometry, namely, that every continuous line of simple curvature of which the ordinates are first positive and then negative (or conversely) must necessarily intersect the x-axis somewhere at a point that lies in between those ordinates. There is certainly no question concerning the correctness, nor indeed the obviousness, of this geometrical proposition. But it is clear that it is an intolerable offense against correct method to derive truths of pure (or general) mathematics (i.e., arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part, namely, geometry. (Russ, 1980, p.160).

This discussion by Bolzano brings to the fore the relationship between truth and proof and proving in Mathematics and prefigures Dedekind's concern in his essay "Stetigkeit und irrationale Zahlen" published in 1872 where he writes[3] that in teaching a course in Differential Calculus, he resorted to geometrical evidences "in proving the theorem that every magnitude which grows continuously, but not beyond all limits, must certainly approach a limiting value" (Dedekind, 1963, p. 1). He states that this is useful from a didactic point of view, but cannot be claimed as scientific, that leads him to

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[1] In fact, he deals with the case of two images of opposite signs.

[2] We use the English translation by S.B. Russ, published in Russ (1980, pp.159-181)

[3] We use the English translation by W.W. Beman in Dedekind (1963, pp. 1-27)

search for "a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis." (Dedekind, 1963, p.1)

Bolzano continues, claiming that proofs should not be simply confirmation (Gewissmachungen), but rather justification (Begründungen)[4], and then, he moved back to the Theorem he is considering:

> Consider now the objective reason why a line in the above-mentioned circumstances intersects the x-axis. Everyone will, no doubt, see very soon that this reason lies in nothing other than that general truth, as a result of which every continuous function of x which is positive for one value of x, and negative for another, must be zero for some intermediate value of x. And this is precisely the truth which is to be proved. It is therefore quite wrong to have allowed the latter to be derived from the former (…). Rather, conversely, the former must be derived from the latter if we wish to represent the truths in the science in the same way as they are connected to each other in their objective coherence. (Russ, 1980, pp. 160-161)

These criticisms concern the proofs mentioned by Bolzano at the beginning of the preface, which does not include the proof by Cauchy that was published four years later in 1821, but the criticism apply.

Cauchy's first proof of the Intermediate Value Theorem

In his famous " Cours d' Analyse De L'Ecole Royale Polytechnique", Cauchy provides two proofs of the Intermediate Value Theorem. Nevertheless, as Longo writes, in the first proof of the intermediate value theorem in his Course of 1821, Cauchy "does not go further than the intuition of continuum that comes from strings and curves traced by a pencil and their crossing. (Longo, 2000. p.403)

Later, page 403, Longo provides an English translation of the theorem and of the proof by Cauchy:

Theorem of the mean value:[5] *If the function $f(x)$ is continuous with respect to the variable x between a and b, and if we call c an intermediary value between $f(a)$ and $f(b)$, then we can always satisfy the equation $f(x) = c$ for at least one value of x between a and b.*
Proof (Cauchy, 1821): It is enough to see that the curve which has equation $y = f(x)$ will meet one or more times the line $y = c$ for at least one value of x between a and b.

Figure 1: The theorem and his proof by Cauchy (1821) (English translation in Longo 2000, p. 403)

The history of mathematics has long credited Cauchy's role in developing a solid foundation for Analysis and Calculus, leaving Bolzano's work in the shade. In the second half of the 20th century, comparisons between the work of Cauchy and Bolzano have been discussed in three papers (Grattan-Guiness, 1970; Freudenthal, 1971, Benis-Sinaceur, 1973). In the following section, we present the point of view developed by Hourya Benis-Sinaceur as an epistemological contribution to the reflection on the relationships between truth and proof and proving in a didactic perspective.

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[4] In the French translation by Sebsetik (1964), we have *certifications* (Gewissmachungen) and *fondements* (Begründungen).

[5] We recall that Longo refers to what we have named Intermediate Value Theorem

On a historical perspective between Bolzano and Cauchy

In the abstract in English of her paper of 1973, H. Benis-Sinaceur writes:

> *SUMMARY: The purpose of this article is to show the distance between Bolzano's analytic style with his profound logical tendencies, characteristic of the late called riguor of Weierstrass, and the Cauchy's way which remains, despite the important technical innovations, rooted in traditional geometrism.* (Benis-Sinaceur, 1973, p. 97)

She underlines the desire by Bolzano to found mathematics, and to privilege the proof of a result over the result itself and its application:

> For him [Bolzano], what matters is to express explicitly and to underline the necessity of proving the results, even if they have been known and used for a long time, his own work is to try to make these proofs. (Benis-Sinaceur, 1973, p. 104)[6]

Concerning Cauchy, referring to the preface of the 1821 course, she writes:

> […] there are neither questions of founding, nor of refining the proofs, nor of rejecting geometric intuition (…). The aim is merely of doing mathematics, that is to say of solving problems and improving known solutions, by restriction, rather than by generalization, by specifying under which conditions the formulas are not a empty symbolism. (Benis-Sinaceur, 1973, p. 105)

In her paper, Benis-Sinaceur acknowledges the view of Freudenthal (1971), who against Grattan-Guinness (1970), rejected the idea that Cauchy might have plagiarised Bolzano. She emphasises the differences between the two authors' programmes, which leads to different practises: while Bolzano is primarily concerned with theoretical rigour and the search for purely analytical proofs, rejecting recourse to geometry as vicious circle, the qualities of Cauchy's course are those of synthesis rather than formal rigour (pp. 102-103). She concludes her paper by arguing that the work of Bolzano and Cauchy represent two distinct heterogeneous paths that would not meet until the works by Weierstrass, Cantor and so on, in contrast to the idea of a continuous development through the nineteenth century.

On Bolzano as a precursor of Dedekind

It is a fact that, although it is hard to believe, neither Bolzano nor Cauchy defined the real numbers (Freudenthal 1971, p. 387). As mentioned above, in a certain sense, Bolzano's programme prefigured Dedekind's programme as expressed in the preface of his essay on continuity and irrational numbers. Dedekind uses the intuitive continuity of the line to define the notion of a cut in the set of rational numbers, to prove that some cuts are not operated by a rational number, and by creating for such cuts one and only one irrational number that operate this cut. Having proved that the system composed of all the rational and the irrational numbers is a complete ordered set (namely a continuous set), he was able to prove in a purely analytic way several theorems in Calculus[7]. In a letter to R. Lipschitz of July

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[6] Our translation from French; and the same for other quotations of the paper.

[7] For a presentation of Dedekind construction of the set of real numbers and didactic implication, see Durand-Guerrier (2016).

27th, 1876, Dedekind underlines that nothing is more dangerous than admitting existences without sufficient proof and ask how to recognise the licit and non-licit hypothesises of existences. Returning to Bolzano's geometric concerns, and to the intuitively obvious results mentioned by Longo and used in Cauchy's first proof, Dedekind's work reverses the burden of proof: if the domain of the continuous function at stake is a complete ordered set, then the equation has at least one solution, and thus the curve representing the function f will necessarily intersect the line of the abscissa (in the case of images of opposite signs).

Some didactic implications

Contribution to the discussion on the relationships between *truth* and *proof and proving*

As already mentioned, Bolzano and Dedekind's concern was with the type of proof admissible in the mathematical field in question (Analysis, Differential Calculus). They questioned and rejected as unscientific the use of proofs based on geometrical arguments and considered the need to develop a theory in which it would be possible to prove these theorems within the theory and they did so. From a didactic point of view, this underlines the importance of considering that "what characterises a mathematical theorem is the system of statement, proof and theory" (Mariotti, 2006, p.185). In Durand-Guerrier & Tanguay (2018), we showed the impact of the definitional choices of the real numbers (through Dedekind's cuts; as limits of Cauchy sequences, as unlimited decimal expansions) in the proofs of the completeness of the set of real numbers, and conversely the contribution of working on these proofs to improve the understanding of the nature of the mathematical objects at stake and the topological relationships between rational and real numbers. The R-completeness, compared with the Q-incompleteness, provides strong proof of existential theorems such as the Intermediate Value Theorem, and some fixed-point theorems (for an example, see Durand-Guerrier, 2016 and below in the next section). As argued by Bolzano, for such theorems, the geometric evidence may be misleading. Indeed, in the graphical register, it is not possible to see the difference between a curve representing a function defined on a subset of R that is dense-in-itself but not ordered-complete (e.g. the set of rational numbers), and a curve representing a function defined on an ordered-complete subset of the set of real numbers (e.g. an interval). Therefore, in the theory of equations in the set of rational numbers, the statement corresponding to the Intermediate Value Theorem is false, i.e. the existence of a solution to an equation satisfying the conditions of application is contingent. If we consider a formal axiomatic system of the theory of equations without including a completeness axiom, then there are models of this axiomatic system in which it is not possible to prove the Intermediate Value Theorem, and others in which it is possible to prove it. This was hidden in the proofs based on geometric arguments, such as the first proof of the intermediate value theorem by Cauchy (1821).

This accounts for Bolzano's point of view that what underlies the geometric truth used in the classical proofs that he criticises (the curve intersects the x-axis at a point between x_0 and X) derives from the truth of the Intermediate Values Theorem. Indeed, in a set of numbers that is dense-in-itself but not ordered-complete, the curve will always intersect in the geometric sense the x-axis, but it may not correspond to any point of the set of numbers under consideration. In our view, this supports our claim that the relationship between *truth* and *proof and proving* is not one of subordination, but rather a dialectical relationship. In the case of Bolzano, and Dedekind, a statement can be accepted as true before it can be proven. The challenge is then to elaborate a theory in which it is possible to prove this statement true. As far as Cauchy is concerned, according to Benis-Sinaceur, the requirement for "geometric rigor is not a call to a strict formalism, but a control of conformity to an "intuition" of phenomena occurring in the familiar practice of Analysis" (Benis-Sinaceur, 1973, note 13, p. 107).

Two situations aiming to discuss the relation between Analysis and graphical register

Due to the recurrent difficulties faced by students and teachers for what concerns the teaching and learning of proof and proving in mathematics, we consider that it would be valuable to put on the scene in didactical situations their dialectic relationships with truth. We briefly report two examples involving the context of discreteness-density-in-itself and continuity (completeness) and the associated graphical representations.

A fixed-point problem. In Durand-Guerrier (2016) we have presented and analysed a didactical situation involving a fixed-point problem for an increasing function, in four different cases: 1. the case of a finite segment of the set of natural numbers; 2. the cases of intervals [0,1] 2.1. of the set of finite decimal numbers; 2.2. of the set of rational numbers; 2.3. of the set of real numbers. In case 1 (thanks to discreteness) and in case 2.3 (thanks to completeness), it is always possible to find a fixed-point; it is not the case for cases 2.2 and 2.3, due to incompleteness[8]. We have shown that this situation has the potentiality for questioning with students the relationships between discreteness, density-in-itself and continuity, and the interpretation in the graphical register. For example, after conjecturing that the statement was true in the set of interval [0,1] of decimal numbers and failing to adapt the proof they did in the first case, secondary students (grade 11) solving this problem in an experimental setting raised the following question: "Is it possible that the graph of an increasing function from the set D of finite decimal numbers to itself crosses the first bisector in a point with non-finite decimal coordinates? (Pontille et al., 1996, p. 25)[9]. Prospective secondary teachers faced difficulties very similar to the secondary students for what concerns the case of finite decimal numbers and rational numbers. Moving to the case of the interval [0,1] of the set of real numbers, they generally first considered that the continuity of the function is required, and then solved the problem with the Intermediate Value Theorem. The next step was to understand that the hypothesis of the continuity of the function is not needed, and to move to a proof without this hypothesis (e.g. relying on the existence of Supremum, or by construction of adjacent sequences).

An algebraic definition of exponential. In Durand-Guerrier, Montoya and Vivier (2019), we provided empirical evidence of students' difficulties in distinguishing density-in-itself and continuity, and we hypothesised that the use of digital tools increases the need of conceptual tools to interpreting graphical representations and reconstructing the objects of Analysis. We then proposed a didactical situation with the aim with the aim of providing students with such tools.

The mathematical question consists in searching for the functions $f: E \to R$ satisfying the functional equation $\forall x \in E\ \forall y \in E f(x+y) = f(x)\, f(y)$, where E is a usual set of numbers,

[8] This situation was first presented in Pontille, Feurly-Reynaud and Tisseron (1996)

[9] Our translation from French.

$\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. We assume that $f(1) = 2$ and it can be proved that $f(0) = 1$ and f takes positive values. We ask to compute some values of f (respectively, for each set, value of f at: 3 and -3; 1/3, -1/3, 1.3; -1.3; $\sqrt{3}$ and -$\sqrt{3}$) and to draw graphical representations (on paper and with Geogebra software).

Figure 2: The task posed to students (Durand-Guerrier, Montoya & Vivier, 2019, p. 89)

While in the sets $\mathbb{Z}$ and $\mathbb{Q}$, the function is completely determined by the algebraic relation and the calculation of the image of a yields to 2^a, this is not the case for the set $\mathbb{R}$ of real numbers. In the contexts of dynamic geometry and of paper-and-pencil geometry, the graphical representation of the function in the $\mathbb{Q}$-domain provides the same graph than the exponential function on the $\mathbb{R}$-domain. However, this graphical evidence is misleading: indeed, under the given hypothesis, there are a lot of functions defined on domain $\mathbb{R}$ that satisfy the algebraic property. Therefore, the search for an analytic proof that the function coincides with the usual real exponential function is bound to fail. This should therefore lead students to question the graphical evidence. This situation was experimented in Chile and Peru with prospective teachers. The most advanced students try to prove algebraically that the image of $\sqrt{3}$ is $2^{\sqrt{3}}$, which is not possible without additional assumption about the function (e.g. the function is increasing); others admitted without proof that the function was the exponential function 2^x on the domain $\mathbb{R}$. After being asked to give a value (e.g.1) to the image of $\sqrt{3}$, students provided graphical representations that surprised them. In particular, in some cases there were two points on the same vertical, this making visible the distinction between the real line and the rational line, in other words, the distinction between a continuous (ordered-complete) domain and a dense-in-itself not continuous (not ordered-complete) domain. The first empirical results comfort our conjecture that this problem has the potential to let the need for proof emerge, beyond geometrical evidence, given that the organisation of the didactic situation allow a genuine engagement of students.

Conclusion

In this paper, we have tried to highlight the dialectic relationship between *truth* and *proof and proving* from an epistemological point of view, in the case of the Intermediate Value Theorem. In the last section, we have given two examples of didactical situations that have the potential to stage this issue at the transition from secondary to tertiary level and for teacher training. Designing didactical situation with such potentialities is a challenging issue.

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Logical inferences in the view of first-year students

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Keywords: argumentation, conditionals, deductive reasoning, language.

To control for logical ability in a broader study, the study presented here investigates the understanding of inferences. The tasks consist of a conditional sentence in A, the negated or not negated antecedent (the if-clause) or consequent (main clause) of this conditional in B, and a possible inference from A and B as in the following example: (*A) If the flowers are blue, the car is rolling. (B) The car is rolling. What do A and B imply mathematically-logically, if only the propositions A and B are known and nothing else? Cross the answer with which you agree most.* □ *Given only the propositions A and B, mathematical logic implies the flowers are blue.* □ *…are not blue.* □ *One cannot infer with mathematical logic whether the flowers are blue or not.*

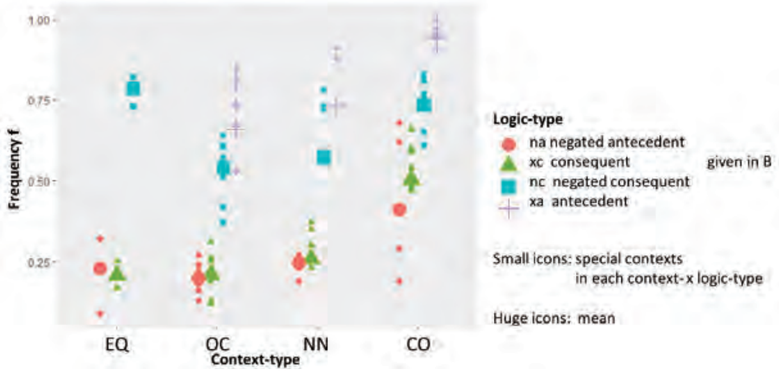
Theoretical background

The analysis of conditionals is immensely broad (von Fintel 2011), being spread out into many disciplines – philosophy, linguistics, logic, and psychology. Perhaps, the root of the matter is the following: With the word “if” one or more possible (or even counterfactual) worlds are possibly entered depending on the context. Therefore, some points have to be considered in the construction of the items: (1) *Plausibility* (Pólya 1990, Vol. 2, Ch. XII & XIII) of antecedent and consequent, the link by the two in its (2) power of (*explanatory) necessity* (Bartelborth, 2007; Müller-Hill, 2017), (3) the formulation of the *question* (v_4 in Durand-Guerrier 2003, p. 18; reasoning *from* or *about* a rule in Wason, 1968; Evans, 1997), (4) the *context* (e.g. deontic or not - Evans, 1997; Valiña & Martín, 2016), (5) implicit *quantification* (propositional, open or bounded sentences - v_7 in Durand-Guerrier, 2003, p. 18), (6) reformulations without conditionals in predicative logic or as *restrictions* (Bartelborth, 2007), (7) *material or logical* implication in A.

Present study

As a consequence of these seven issues, the items are constructed as follows: Plausibility of the antecedent and consequent (1) have to be unknown, the plausibility of the conclusion (1,2) is systematically varied by context-types, three choices for the answer are given (see example above; 3), only reasoning *from* a rule (3) in A, given as material implication (7) by abstract sentences in the present indicative (4), no quantifiers and avoiding the interpretation as open sentences in order to prevent implicit quantification (5, 6). Due to pilot studies, we did not use any negations in the conditional sentence and in a first step only contexts of every-day-life instead of mathematical ones (4, cf. Durand-Guerrier 2003, 18, v_1; also to avoid differences in content knowledge). We used four cases of plausibility in A (*context-type*): CO (cogent) - The direction of the conditional is cogent, the logic converse is not. *If the brakes are broken, the car is out of order.* OC (not cogent) - The direction of the conditional is not cogent, whereas the converse is. Specifically, the converses of the CO-Items are used. *If the car is out of order, the brakes are broken.* EQ (equivalence) - Both directions of the conditional are cogent because it is a material equivalence. *If you are a pupil, you go to school.* NN (neutral) - No available truth value, no probability, arbitrary conditional. *If the train is coming,* *the flower is yellow*. These cases are each combined with the following four logic types, realized by B: xa antecedent given, na negated antecedent, xc consequent, nc negated consequent. The convenience sample is comprised of N=593 students at the beginning of their university studies. Each combination of logic- and context-type used 2-8 contexts. Due to the multi-matrix booklet designs, a Rasch-model (N=593, infit≤1.04, outfit≤1.07, item-total-cor.≥.19, EAP-rel=.36) was applied.

Results



Even if the contexts are extremely specified (see 1-7 above) and the logic-types and negation are controlled, the context has still a great impact on the variance of frequencies. The most abundant answer in each type coincide with those given if material implication in A had been mistaken as an equivalence - in contrast to the findings and interpretation of Durand-Guerrier (2003, p. 22).

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Standardized tests and the development of argumentation competency: Tensions perceived by Chilean mathematics teachers

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Chile's standardized tests SIMCE and PSU have been criticized for their unintended effects in the classroom, for contributing to growing elitism in education, and for perpetuating social inequality. However, teachers' perceptions about these tests and their impact on teaching practice have seldom been considered. We focus on teachers' perceptions about the tension between the development of argumentation competency proposed by the national curriculum and the standardized tests. Resorting to in-depth interviews, we account for the systemic lack of alignment that mathematics teachers perceive between curriculum and tests. Results show that this leads to an impoverished learning experience, suggesting that curriculum and test design might not be enough to foster curricular intended teaching, and that teachers' perceptions must be taken into account.

Keywords: Argumentation, argumentation competency, standardized tests, mathematics curriculum.

Introduction

Argumentation is key to constructing participative, democratic societies. Notions such as *mathematical literacy* (PISA, 2018) express the expectation that the student-citizens engage in argumentation, drawing on scientific knowledge, attitudes, and competencies to understand reality, and participate in their communities in critical, reflective, and value-based ways. This has driven many countries to consider argumentation – under this or other denomination – to be a competency to be developed as part of citizenship education. The Chilean curriculum proposes argumentation as one of four competencies to develop at all levels of mathematics education (MINEDUC, 2019, 2015, 2012).

In the course of the last two decades, citizenship education has gained momentum through international standardized tests, such as the Programme for International Student Assessment (PISA), or the Trends in International Mathematics and Science Study (TIMSS). These tests have stimulated the need to assess competency development in students as part of educational system quality assessment efforts. Argumentation being one of the competencies the Chilean curriculum aims to develop, both the Education Quality Measurement System test (SIMCE) –used nationally to assess the quality of compulsory education at various school levels – and the University Selection Test (PSU) – given nationwide in the 12th grade to rank students for access to higher education, are expressly meant to be measuring it.

However, what is stated in the curriculum and assessment systems does not directly translate into classroom practice. Crucially, it is teachers who interpret them and make the pedagogical decisions that shape students' learning experience. They make these decisions while experiencing the tension between what is explicitly demanded by the educational system and what they perceive could best help students succeed in standardized testing. In contexts of high-stakes standardized testing – such as Chile – the resolution of this tension has enormous consequences.

Regardless of the subject, studies on the impact of high-stakes testing consistently report narrowing of the curriculum, increase of teacher-centered pedagogies, and abandonment of high-quality forms of instruction in favor of routine practices (Au, 2007; Blazar & Pollard, 2017; Meckes, 2007; Valli & Buese, 2007). Poor-quality "teaching to the test" has often been considered a consequence of standardized testing (Darling-Hammond & Adamson, 2014), even though test advocates insist that cognitively demanding tests may motivate teachers to improve and enrich instruction (Au, 2007, Peery, 2013). In the case of mathematics, previous studies at various school levels have found that test preparation activities predict poorer-quality and less ambitious mathematics instruction (Blazar & Pollard, 2017; Cohen & Hill, 2008; Meckes, 2007).

Most of these studies have resourced to classroom observation to evaluate teaching quality. Although such studies are crucial to describing high-stakes testing effects in the classroom, they typically neglect teachers' voices. Besides, they tend to consider classroom activity as a whole and do not focus on specific curricular aspects. In the present study, we draw on in-depth interviews, exploring the perceptions of Chilean mathematics teachers with respect to the tension between development of argumentation as a curriculum-proposed competency and the national standardized tests SIMCE and PSU, and its consequences in classroom teaching practice. We first present aspects of the Chilean curriculum and programs, and of the SIMCE and PSU tests. Then we explain the methodological design of our study. Finally, we present and discuss our results to account for the systemic lack of alignment that teachers perceive between the national curriculum and the standardized tests in terms of the pedagogical work necessary to develop students' argumentation skills in the classroom.

The argumentation competency in the Chilean curriculum

The national Curricular Bases (CCBB) designed by the Ministry of Education (MINEDUC) constitutes the cornerstone of the educational system. They provide a mandatory, standard framework for education in all schools, presenting educational principles, and establishing learning goals (LG) for each level of school instruction. In the case of mathematics, the CCBB establishes specific and progressive LG for the development of argumentation (MINEDUC, 2019, 2015, 2012). Table 1 presents the LG for grades 8 and 11-12.

8th grade (MINEDUC, 2015, p. 112)	11th-12th grade (MINEDUC, 2019, p. 108)
● Describing mathematical relations and situations verbally and using symbols. ● Explaining and justifying: (i) Students' own solutions and procedures; (ii) Results using definitions, axioms, properties, and theorems. ● Backing conjectures using examples. ● Assessing the arguments of others, giving reasons.	● Making decisions based on statistical evidence and/or results obtained from a probabilistic model. ● Using symbolic language and other representations to justify the truth or falsehood of conjectures and assessing the reach and limits of arguments.

Table 1: Learning goals for argumentation, grades 8 and 11-12

Within the CCBB, argumentation is understood to be part of a dialogue, expression of ideas, and collaboration aimed at convincing others of the validity of mathematical results. Students are expected to transit from an informal/intuitive to a formal/mathematical style of argumentation (MINEDUC, 2015). However, the CCBB does not provide teachers with a clear operational definition of argumentation that allows them to identify instances of the competency and to progressively assess its development.

SIMCE, PSU, and assessment of the argumentation competency

The Quality Education Agency (ACE) designs and implements the SIMCE test as part of the National System of Quality Education Assurance (SAC). The objective of SAC is "to ensure access to quality and equity in education for all students in the country, through comprehensive evaluation, pertinent inspection, and constant support and guidance to educational establishments" (CNED, 2021). SIMCE is administered in the 2nd, 4th, 6th, 8th, 10th, and 11th grade, and encompasses several subjects, including mathematics. As a basis for the SIMCE test, the Ministry of Education develops Learning Standards, which "describe the degree of curricular achievement required for evaluation and monitoring of students' performance in order to provide feedback to educational policy and to the school system" (MINEDUC, 2017, p. 5). The Learning Standards should thus allow teachers to monitor and plan for their students' progress following the CCBB. However, and despite declaring that students must develop "problem-solving, representation, modeling and argumentation competencies" (MINEDUC, 2017, p. 50), no specific competency indicators are presented, nor are milestones in expected competency development.

The Department of Educational Assessment, Measurement and Registration (DEMRE) is responsible for developing the PSU test for the process of admission to Chilean universities. On its website, DEMRE declares that the test will assess contents and competencies in line with the current curriculum. The mathematics test "will have 65 multiple-choice questions with a single answer among 4 or 5 choices, (...) it will assess the Understanding, Applying and Analyzing, Synthesizing and Evaluation competencies, (...) and will contain 13 questions that assess the competencies specified in the Curricular Bases: Problem-Solving, Representing, Modeling, and Argumentation" (DEMRE, 2020). On its website, DEMRE refers to argumentation as "the ability of the applicant to evaluate procedures, deductions, solution strategies, and inferences in various problems; to distinguish and detect erroneous arguments; and to understand chains of logical implications". However, no further information is presented about how the test allows argumentation assessment.

We must also consider the consequences that PSU and SIMCE results have, turning them into high-stakes measures for students, schools, and teachers. PSU results determine access to higher education and career options. A university degree is seen as a strong socioeconomic mobility factor in Chile, hence test results are crucial both for students and their families. Although the SIMCE test does not have direct consequences for students, results are clustered by schools and made public on official and media websites, *de facto* functioning as a school ranking tool. This affects families' preferences – hence student enrollment –economically impacting state-financed schools through a per-pupil voucher system based on attendance (Taut et al., 2009). Teachers' income is also often affected through school economic incentives linked to students' test performance (Botella & Ortiz, 2018).

Methods

The study used a qualitative methodology with a multiple-case study approach (Yin, 2009) to explore recurrent themes addressed by participating teachers during semi-structured interviews. A purposive sample of mathematics teachers (Neuman, 2006) was obtained (n=21) using two sampling criteria: i) the subjects were teaching in educational establishments located in Chile's Metropolitan and Valparaíso Regions, and ii) they taught mathematics at the 6th and 12th grade (12 and 17-year-old students respectively), in which the SIMCE and PSU tests are applied. The teachers' ages ranged from 24 to 49, and their teaching experience from 1 to 20 years.

The interview protocol included questions to investigate teachers' perceptions about argumentation and its teaching in the mathematics classroom; decision-making processes aimed at developing the argumentation competency; the relationship between argumentation competency development, and SIMCE and PSU; and the impact of these tests on teaching practice. The interview time with each participant ranged between 32 and 109 minutes. The set of basic questions was common to all, but follow-up questions differed based on answers, explaining the differences in interview duration. All interviews were recorded and transcribed for analysis.

A thematic analysis of the interviews was carried out by two researchers in order to identify recurring patterns and themes in the data and allow for triangulation (Braun & Clarke, 2006). The first stage involved open codification (Strauss & Corbin, 1998) of each of the interview transcripts. Each researcher independently identified conceptually similar codes and interview excerpts, grouping them into preliminary analytical categories. Initial coding was compared and discussed among the researchers for purpose of validation and adjustment, thereby achieving greater reliability. Then axial codification was performed on the whole data corpus (Strauss & Corbin, 1998). Key themes related to main categories were identified through an iterative process of inductive/deductive analysis, enabling elaboration of a thematic map of the analysis (Braun & Clarke, 2006). A systematic data review was conducted in order to identify negative cases, thus avoiding any confirmation bias. Finally, results were prepared in the form of narrative themes (van Manen, 1990) addressing teachers' perceptions about tensions between the development of argumentation competency and SIMCE and PSU tests, and their consequences in the mathematics classroom.

We present three main related themes and include as many excerpts from teachers' interviews as possible to allow their voices to elaborate on them. The original excerpts are in Spanish. We have tried to convey the sense along with idiosyncratic nuances when translating them into English. In the following section, secondary teachers are referred to as ST, while elementary teachers as PT.

Findings

The importance of argumentation in the mathematics classroom

There is consensus about the importance of argumentation in the mathematics classroom among participating teachers. Teachers associate argumentation with self-regulation and understanding, as opposed to mechanical and non-reflective execution of techniques. In the words of ST1:

> ST1: [Argumentation] has to do with realizing how you are thinking. That also allows you to analyze your strategies, to improve and complement them, and to understand others'. Bottom line,

> if I understand how I reason, I am also able to explain it, and being able to explain it means that I have an understanding of what I am doing. [...]. I think that is what we expect of students.

Teachers unanimously emphasize that student's argumentation allows them to assess learning and make pedagogical decisions, since knowledge and reasoning become "visible" in argumentation.

> PT5: When asking the child to explain how he solved a mathematics problem, what he did, the calculations he performed, why he did this or that, that is the way I can become aware of the mistakes he makes. I realize how he is thinking, in what way he is reasoning mathematically. That way, I can address his difficulties.

Obstacles to development of argumentation in the classroom

Despite the importance attributed to argumentation, teachers point out various obstacles to implementing activities for its development related to PSU and SIMCE tests. Their answers can be grouped into two recurring and complementary themes, i) they perceive covering curricular contents in mathematics to be the central requirement of the SIMCE and PSU tests – and of the educational system in general, having priority over development of competencies and attitudes; and ii) they feel pressured to dedicating time to activities aimed at improving students' SIMCE and PSU test scores. These two themes refer to demands perceived by teachers that impact their pedagogical decisions, particularly when selecting learning activities. Below, we elaborate on these themes together, since teachers refer to them when explaining the tensions between the development of argumentation competency and SIMCE/PSU tests, and their consequences in the mathematics classroom.

The pressure to cover curricular mathematics content emerges from teachers' interpretation of the curriculum and from the demands placed on them by senior management staff in schools. Moreover, teachers consider the time devoted to covering mathematics content to be an impediment to planning and implementing learning activities related to argumentation, as expressed by PT5 and ST5.

> PT5: The Ministry mandates delivery of all contents. They say "you have to teach all the contents within the year," which is impossible, but there is no focus on competency development. [...] So, I have always been focused on delivering content, but not on competency development.

> ST5: Both regulations and your school demand content coverage from you, and also that students be prepared to confront the PSU test answering questions correctly and doing well. Thus, one starts cutting away. Are you going to work on argumentation? Really? How much? Once? Twice a month? Damn, twice!? You will be neglecting practice and application. They (senior management staff) are going to question you. Hey, what's going on?

Teachers perceive great importance in PSU and SIMCE results and explain that in a multifactorial way. As already mentioned, student scores are grouped by school and made public. Local media present this information as a measure of a school's quality and of the chances offered for university admission. Teachers perceive that these results are presented publicly as an indicator of the quality of their teaching, impacting their job stability. The perceived importance of these tests translates into pressure to dedicate time to tasks aimed at improving test scores, pressure that is openly or implicitly exerted by school mid-level supervisors (academic coordinators, department heads, etc.). This leads teachers to adapting classroom activity, particularly those years in which students must take the tests.

> ST6: I think the [pressure] mechanisms are not visible or explicit. [...] Because a coordinator won't tell you to privilege [test] results over learning, but one way or another you get the message.

PT7: We have some autonomy in distributing our work and students' work, but always within a framework. Because, at the end we are subject to the standardized assessments, so we make decisions on that basis. One way or another, you lean towards them.

Consequences for classroom mathematical activity

When questioned about the relationship between argumentation and SIMCE/PSU tests, teachers agree that the tests do not measure argumentation competency and that their focus is on memorization, calculation, and mechanization of procedures. Crucially, they believe that in order to improve test scores, it is necessary to implement activities that are similar to the tests in the classroom. This leads them to frequently implement multiple-choice exercises similar to those that appear in the tests, leaving little to no time for other activities. In their view, the tasks they feel compelled to implement do not stimulate argumentation or are even detrimental. In summary, these teachers perceive a systemic lack of alignment between the tests and the curricular requirement to develop argumentation. Moreover, they do not just consider that argumentation is neglected in this process, but also that the overall student learning experience is impoverished.

ST6: The school where I work is focused on scores. Programs are pretty bulky, loaded with content, and focused on standardized tests, SIMCE, PSU. […] For this reason, this competency (argumentation) has not been developed. In the end, mathematics work has been centered on closed-ended question tests. [...] I even feel that this forces teachers to constantly think of multiple-choice tasks, and perhaps also that justification is unimportant.

PT2: I was talking with my colleagues the other day. Why should I care whether students practice argumentation in class? The school is going to assess me just by SIMCE results, and SIMCE does not require argumentation. They just ask you multiple-choice questions.

Discussion

The unintended consequences of Chile's standardized tests have been discussed in previous studies. It has been observed how the use of test results to rank schools within a free-market rationale perpetuates the correlation between socio-economic situation and school achievement, contributing to social inequality (Botella & Ortiz, 2018; Inzunza, 2014; Monarca, 2012). Although teachers are expected to use SIMCE test results as a formative, diagnostic tool to improve their pedagogical practice, Taut et al. (2009) have shown that just "a third of the teachers could accurately tell 'good' results from 'bad', and even fewer were able to accurately evaluate change over time and relative status comparing similar schools" (p. 135). With regard to the impact in the classroom, in line with international research (Au, 2007; Darling-Hammond & Adamson, 2014), previous studies have reported narrowing of the curriculum (Taut et al., 2009) and decreased teaching quality (Botella & Ortiz, 2018; Meckes, 2007). However, such studies have seldom taken into account teachers' perceptions of tests and their impact on argumentation. We focused on the tensions between explicit curricular demands regarding development of argumentation competency in the mathematics classroom and the implicit demands arising from teachers' perceptions of PSU and SIMCE tests.

The lack of alignment between the curriculum and the standardized tests perceived by the participating teachers is expressed in two interrelated but well-differentiated ways. On the one hand, they perceive that SIMCE and PSU tests do not allow the assessment of argumentation competency, since it is not brought into play in the tests in ways that provide evidence of its development. On the other hand, teachers perceive that the best strategy to improve students' scores is to implement multiple-choice tasks similar to those found in the tests, which do not require argumentation. That is, according to this group of teachers, not only is argumentation not measured by the tests, but its development does not help students perform well. Together with the intense pressure to achieve good scores, this results in classroom activities that marginalize argumentation and, in general, impoverish students' educational experience, particularly during the years in which they are tested (Bellei & Muñoz, 2021). Despite being aware of this impoverishment and of the fact that all of them declare that they do so reluctantly, teachers participate in strategies orchestrated by schools to improve test scores. This agrees with other studies that suggest that high-stakes testing often drives teachers "to abandon methods and materials that had been successful with their students" (Bailey, 2000, p.118) and even "engage in practices that were antithetical to their beliefs about a good instructional environment" (Valli & Buese, 2007, p. 552).

It is crucial to note that we did not interview a representative teacher sample, nor we attempted to determine whether SIMCE and PSU tests do measure students' argumentation competency and its development, as declared by test designers. Nevertheless, we have shown that this group of teachers does not perceive this to be the case and that such perception negatively impacts actual teaching practice regarding development of argumentation competency. In line with what Blazar and Pollard (2017), and Cohen and Hill (2008) say, these results suggest that curriculum and test design might not be enough to foster curricular intended teaching, as hoped by test designers and policy makers (Peery, 2013). Our results suggest that teachers' perceptions must be taken into account.

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Language activities related to logical structures of proofs – a theoretical and empirical specification

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This study combines the Toulmin model and processes while proving to analyze the language activities of proving processes concerning logical structures. A case study of Alena and Jannis (both grade 10 and 15 years old) is presented in more detail. The qualitative analysis of the proving processes of Alena, Jannis, and 46 other students (grades 8 to 12) reveals insights into the development of verbal activities from the first verbal sequence to the written product, in particular: 1) The linguistic implicit logical structures in the process of proving, 2) the limited number of language activities, 3) the use of explicit logical language means in the last sequence of writing.

Keywords: Formal proof, Toulmin model, language, proving activities, language activities

Introduction

Language means (words, phrases, and grammatical forms) can have different meanings in mathematics than in everyday language (Duval, 1995; Schleppegrell, 2007); this mainly applies to the language means of proofs and proving (such as conjunctions or other logical connectors). Within the huge learning content of proof and proving, particular challenges were identified in the structural change to deductive reasoning (Fischbein, 1982) and the steps from the process to the product of argumentation, such as proofs (Albano & Dello Iacono, 2019). These challenges are also rooted in linguistic challenges, e.g., with new meanings of the exact words (Duval, 1995) and their necessary connection to the language (Ferrari, 2004). Therefore, the language of deductive reasoning requires more systematic research attention (Durand-Guerrier et al., 2011). To study the language of proving, language must be functionally extended from the words and phrases to the language activities used by these language means (Schleppegrell, 2007; Prediger & Zindel, 2017). This extension resonates with the Vygotskian perspective of this paper, as Vygotsky (1962) regards language as an artifact that is learned by using it in meaningful activities. Hence, the language of deductive reasoning is learned in the language activities involved in proving activities with logical structures. Therefore, specific language challenges have to be figured out for relevant learning contents (Bailey, 2007). This paper aims at specifying the appropriate proving activities and their language representation related to the logical structures of proofs. Therefore, this study focuses on concrete language activities to overcome the logical structures while proving from the first verbal try to the proof text. First, the specification is theoretically derived from the existing state of research and then empirically investigated using students' processes with the following research question: Which language activities do students enact in their transition from verbally arguing to writing proofs?

Theoretical Background

Logical structures

Understanding the logical structures is essential for students' transition from intuitive to deductive proof (Fischbein, 1982). The Toulmin model (1958) is widely used to analyze argumentations, particularly in mathematics education, to model logical structures (e.g., Krummheuer, 1995). In the current paper, the short version of Toulmin's model is adapted so that data, warrant, and claim are described as the logical elements *premise*, *warrant*, and *conclusions* of a proof. Additionally, the logical relations between the logical elements are considered. In line with Duval (1991), the use of a conclusion from a previous step to another as a premise is called *recycling*. In this way, a multiple-step proof can be described. Figure 1 shows the adapted model used in this study to capture students' activities with logical structures.

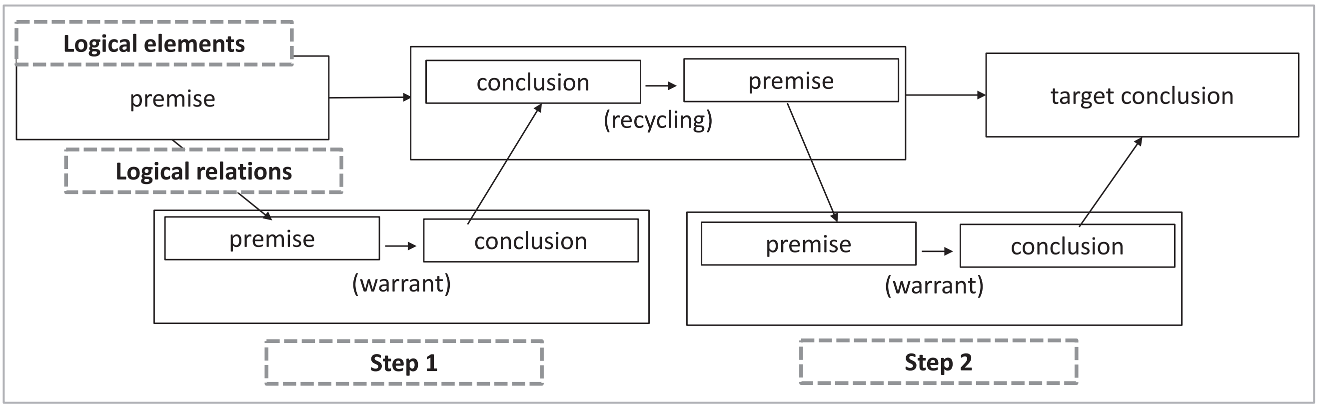


Figure 1: Adapted Toulmin Model about logical structures in proof (here two steps)

Proving activities related to logical structures

Whereas Boero (1999) has specified all proving activities in general, Heinze et al. (2008) have focused mainly on proving activities related to logical structures (for proving activities at proofs with multiple steps). Additionally, Tsujiyama (2011) emphasized the identification of the premise and target conclusion. Based on their work, the following proving activities were specified as relevant concerning the logical structure of proofs (Hein, 2021): 1.) unfolding the information in the task (premise and target conclusion), 2.) identifying warrants and their logical structure, 3.) forming logical relations between logical elements (premise, warrant, conclusion) within the proof steps, 4.) arranging and creating relations between the proof steps, 5.) finding a linear representation of the proof. It should be noted that these activities can emerge in different orders and do not have to be observable.

Language activities related to the logical structures

The state of research on logical structures in the adapted Toulmin (1958) model and the specified proving activities (following Heinze et al., 2008; Tsujiyama, 2011) is the theoretical foundation to identify the language activities:

This paper defines language activities as linguistic representations of concrete (learning) activities. Here, the main emphasis is on the linguistic representations of the proving activities related to logical structures (in the following stated as *language activities*). For every proof activity related to logical structures, a linguistic representation is described and graphically categorized to the Toulmin Model (Figure 2).

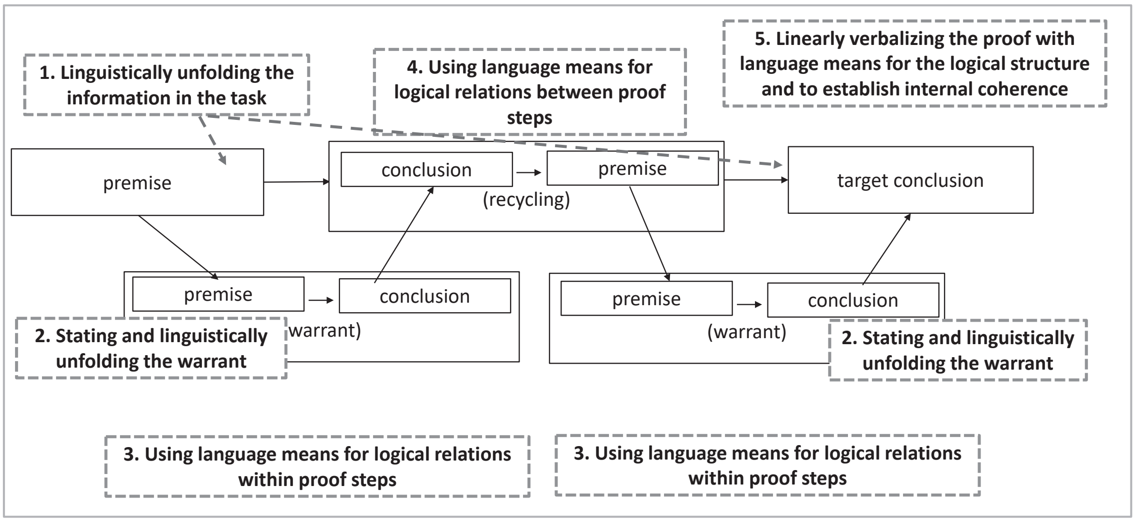


Figure 2: Language activities concerning logical structures (activities in dashed boxes)

Although these activities can be done mentally, this paper describes how these activities can be observed. The underlying assumption is that the students need first to notice the logical structures for their understanding through visible activities.

Methodology for the analysis of proving processes

Data collection

The data corpus for this paper stems from a major design research project on deductive reasoning in grades 8-12 with angle sets. A teaching-learning arrangement with verbal and graphical scaffolds was developed within the project. Whereas other publications investigate students' learning process of the logical structures (see Hein & Prediger, 2017), this paper focuses on identifying the enacted language activities. The complete data corpus consists of transcribed videos from design experiments and written proofs with 24 pairs of students (20 students in grade 8, 6 in grade 9, 4 in grade 10, 18 in grade 12). An example of the analysis is given using the case of Alena and Jannis, both tenth-graders and 15 years old (Hein, 2021). Their process illustrates the typical phenomena for the whole dataset, and the design experiments were still in a laboratory setting to gain deeper insights. The author was the teacher herself in the laboratory setting and interacted with the students to explore their learning (see Cobb & Steffe, 1983).

Methods of qualitative data analysis

The qualitative analysis of the students' language activities while proving draws upon the *Toulmin Model*, the *proving activities*, and the *derived linguistic representations*. First, the related logical structure based on the Toulmin model is coded (Figure 2). Then the proving activities are identified, respectively, as their linguistic representation. Based on the previous steps, the following tool analyzes the language activities in proving and their operationalization within the design experiment: 1.) Linguistically unfolding the information in the task: stating the premise (step 1) and target conclusion (step 2); 2.) Stating (step 1) and linguistically unfolding the warrants (step 2): stating the mathematical propositions that must be used and unfolding the conditional implication structure; 3.) Using language means for logical relations within proof steps: verbalize the logical relations with causal or conditional conjunction respectively prepositions; 4.) Using language means for logical relations between proof steps: verbalize the relation between the proof steps with logical language means; 5.) Linearly verbalizing the proof with language means for the logical structure and making text coherence: primarily through a proof text with an explication of the logical elements in their logical order, and the use of logical language means.

Empirical insights

Alena and Jannis work on their second proof within the teaching-learning arrangement (the alternate interior angle proposition as in Figure 3). They have the necessary warrants (corresponding angle proposition, transitivity proposition, and vertically opposite proposition) available on cards. The whole session, they work as a pair.

For a first overview, Table 1 summarizes the reconstruction of the language activities of Alena and Jannis for each sequence.

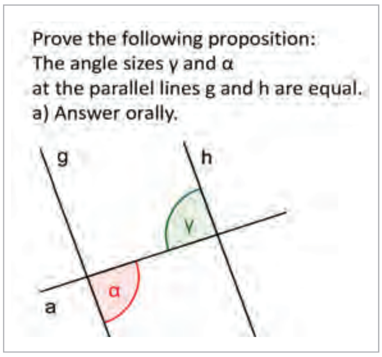


Figure 3: Task

Table 1: Reconstruction of the language activities of Alena and Jannis

Phases of proving	Reconstructed language activities (related to logical structures)
Sequence 1 Verbal reasoning	• No language activity
Sequence 2 Filling in graphical scaffolds	• Linguistically unfolding the information in the task (State the target conclusion (activity 1 step 2) • Linguistically unfolding of the warrants (activity 2 step 2) • using language means for logical relations within proof steps (activity 3)
Sequence 3 Writing the if-then-clause	• Linguistically unfolding the information in the task (activity 1)
Sequence 4 Written products	• Linguistically unfolding information in the task (activity 1) • Stating and linguistically unfolding the warrants (activity 2) • Using language means for logical relations within proof steps (activity 3) • Using language means for logical relations between proof steps (activity 4) • Linearly verbalizing the proof with language means for the logical structure and establishing internal coherence (activity 5)

Sequence 1: Verbal reasoning

In the first sequence, Alena refers to another task with a concrete alternate interior angle task (asking for gamma if alpha is 50 degrees) and points at it (Turn 578). Again, Alena and Jannis perform no language activity here.

578 Alena This is… [*16-sec break*]. Again with this… [*points on the previous task*]

In another task, Alena and Jannis, and the other students state the arguments or point with fingers at the cards with the mathematical propositions, which is possible in the design of the teaching-learning arrangement. Without the teaching-learning arrangement, the students often only state conclusions here.

Sequence 2: Filling in graphical scaffolds

In the second sequence, Alena and Jannis fill in graphical scaffolds to arrange the logical elements (premise, warrant, and conclusion) in the correct order. This process and the graphical scaffolds are described for another student pair in Hein & Prediger (2017, sequence 2). Here the focus is only on the language activities. Through the graphical scaffold, they are demanded to make explicit the target conclusion by writing it down. In the following, a few examples are presented:

581 Alena Yes, mhm. Is gamma equal to alpha?

In this way, the target conclusion from the *task's information is linguistically unfolded* (activity 1.2).

639 Alena […] So this and this is equal [*points on gamma and beta how they have named the vertically opposite angle of gamma*] and this and this is equal [*points on beta and alpha*]. It follows that this and this [*points on gamma and beta*] are equal.

Once a *warrant is linguistically unfolded* (activity 2.2) while applying the transitivity proposition.

Finally, Alena and Jannis use isolated *language means for logical relations within the proof steps* (activity 3), such as in the following example:

620 Jannis So from this [*points on a graphical scaffold*] **follows** for now that […]

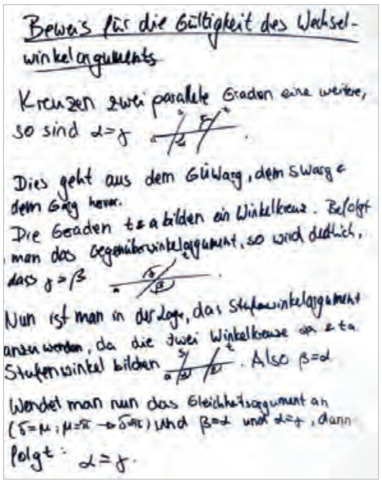
In this sequence, the students use, above all, deictic language means while filling in the graphic scaffolds which represent the logical structures. This sequence is described in detail in Hein (2021).

Sequence 3: Writing the if-then-clause

In this sequence, the students write down the mathematical proposition with "if-then" and differentiate *the premise and the target conclusion* (activity 1).

Sequence 4: Written products

Both students write proof texts independently, based on their filled graphical scaffolds (Figures 4, 5).


Beweis für die Gültigkeit des Wechselwinkelarguments
Kreuzen zwei parallele Geraden eine weitere, so sind α=γ.
Dies geht aus dem GüWarg, dem SWarg & dem GArg hervor.
Die Geraden t & a bilden ein Winkelkreuz. Befolgt man das Gegenüberwinkelargument, so wird deutlich, dass γ=β.
Nun ist man in der Lage, das Stufenwinkelargument anzuwenden, da die zwei Winkelkreuze an s & t a Stufenwinkel bilden. Also β=α
Wendet man nun das Gleichheitsargument an (δ=μ; μ=π → δ=π) und β=α und α=γ, dann folgt: α=γ.

Proof for the validity of alternate interior angle proposition

[If] two parallel lines cross another, **then** is α = γ.

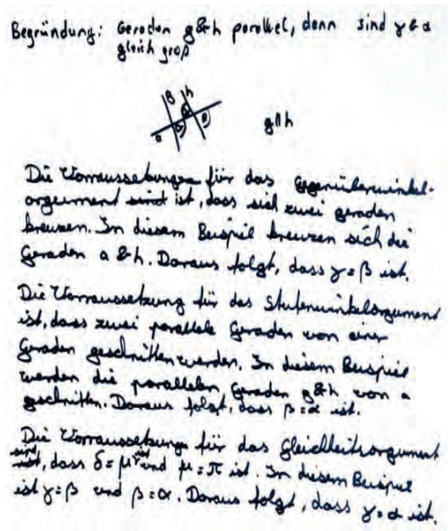
This arises from the vertically opposite proposition, the corresponding angle proposition, and the transitivity proposition [*Alena uses abbreviations chosen by herself for the propositions*].

The lines t & a form an angle intersection. If you follow the vertically opposite proposition, it becomes clear **that** γ=β.

Now, you can apply the corresponding angle proposition because both intersections s/a & t/a form the corresponding angle. **Hence**, β=α.

If you apply the transitivity proposition (δ=μ; μ=π -> δ=π) and β=α and α=γ, **it follows**: α=γ.

Figure 4: Written product of Alena (original (Hein 2021, p. 231) and translation)



Argumentation: Lines g & h parallel, then γ & α are equal.

The **premise** for the vertically opposite angle argument is that two lines cross each other. **In this case**, the lines a & h cross each other. **Thereof it follows that** $\gamma=\beta$.

The **premise** for the corresponding angle argument is that one line crosses two parallel lines. **In this case**, the parallel lines g & h are crossed by a. **Thereof** it follows that $\beta=\alpha$.

The **premises** for the transitivity argument are that $\delta=\mu$ and $\mu=\pi$. **In this case,** 13 $\gamma=\beta$ and $\beta=\alpha$. **Thereof** it follows that $\gamma=\alpha$.

Figure 5: Written product of Jannis (original (Hein 2021, p. 232) and translation)

Both students write first what has to be reasoned (the alternate interior angle proposition) and thereby *linguistically unfold the information in the task* (activity 1).

Both students use *language means for logical relations within proof steps* ("thereof.."; "in that case") (activity 3). To express the relation between the first and second step to the third step, Jannis makes explicit the conclusion of both first steps as a new premise ("The premises for the transitivity proposition are that $\delta=\mu$ and $\mu=\pi$. In this case…") and thereby *use language means for logical relations between proof steps* (activity 4). Finally, both students *linguistically unfold the warrant*, namely the unknown transitivity proposition (activity 2.2). Alena does this in brackets, Jannis, through the explicit application of the transitivity proposition. The other propositions as warrants are not unfolded.

Besides the language means for logical relations Jannis also uses nominalizations ("premise") and makes explicit the logical status of the premise. Alena uses language means of text coherence such as "this" and refers to parts of the text before in this way. She also uses temporal conjunctions ("Now"), which express no logical, but temporal relations.

Both students show *linear verbalization of the proof with language means for the logical structure* and *establish internal coherence* (activity 5), primarily through a proof text with an explication of the logical elements in their logical order and usage of logical language means.

Summary

As summarized in Table 1, Alena and Jannis perform different language activities. However, they can be identified depending on the related logical structures and concrete elements of the teaching-learning arrangement.

To sum up, activity 1 (*the linguistic unfolding of the information in the task*) can be found late, with both premise and conclusion in sequence 3. Although Alena and Jannis state the warrants in sequence 1 (here facilitated through the design of the teaching-learning arrangement), they unfold the warrant while filling in the graphical scaffolds and only make explicit the unknown warrant in their texts in sequence 4. Jannis uses language means of logical relations between proof steps only in his text. The students mainly use language means for the logical relations within and between the proof steps, particularly in the writing products.

For the other 46 students, similar language activities are empirically identified: Many logical structures remain tacit in the first sequence of verbal reasoning. When provided, conclusions are often expressed without further reasoning, and mathematical propositions are stated with the argument cards. While working with the graphical scaffolds, the students perform fewer language activities expected because of deictic language means. Such as "here" can be used instead. The premise and conclusion of the proposition, which must be proven, are stated very explicitly at later stages. The students then use more language means for the logical relations within and between the proof steps in the written products. In the texts, the mathematical content of the proof is represented through language means, with a variable degree of the explicit formulation. Sometimes, unnecessary temporal conjunctions are also used.

In summary, although many opportunities are given within the design experiments, there is a slow increase in language activities related to the logical structure.

Conclusion

In this study, the logical structures and their articulation in language are investigated to specify the language learning content for proving (as demanded by Durand-Guerrier et al., 2011). The main finding in the case study of Alena and Jannis is the following: In line with Albano & Dello Iacono's (2019) results on typical proving processes, some language activities are rarely seen in the early sequence of verbal reasoning. However, there is identifiable progress towards increasing explicitness within the four sequences. Based on previous research on the importance of language activities based on Vygotsky's (1962) theory and this study, the following implications for teaching can be derived: 1.) Increasing the teachers' awareness of the possible explicit language activities may be helpful so that the teachers can consciously demand the language activities. In particular, concerning explicitness for the logical structures (e.g., conjunctions and adverbs for logical relations). 2.) The herein described language activities are a starting point for the students to have opportunities to learn. 3.) Emphasis should be given to having the students write the proof themselves in the classroom.

It should be noted that the teaching-learning arrangement in design experiments causes the data of this study. Therefore, it is likely that students would show even fewer language activities without the teaching-learning arrangement. In addition, not all language activities have to be observable because they can also be only mental. Finally, the study's limitations are the small size of observed students and the limited number of tasks considered.

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Seventh-grade students' perceptions of qualities in a mathematical argument

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This paper reports on a task where seventh-grade students evaluated five pre-written arguments designed to display various proofs and non-proof arguments. The analysis focuses on what students described as qualities of the arguments. The results indicate that students appreciate arguments they perceive to understand, arguments that are short, and arguments containing text. Thus, the task approach holds the potential to unravel what students perceive as the qualities of an argument. However, there is no clear relationship between the features of the arguments and what students perceive to be qualities. Further investigation should occur to see how the task can be improved to better display aspects of valid mathematical arguments to help students appreciate these and, in turn, be able to produce mathematical proofs.

Keywords: Primary education, mathematics, argumentation, proof.

Introduction

Proof holds a prominent role in mathematics, and researchers and policymakers worldwide increasingly appreciate its importance for mathematics learning (Stylianides et al., 2017). In Norway, 'Reasoning and argumentation' is one of six core elements in the new curricula implemented since autumn 2020. It states that the students should prove that their solutions to mathematical tasks are valid (Kunnskapsdepartementet, 2020). This formulation introduces proving to the primary school curricula in Norway (age 6 to 13), but it is not clear how it could be implemented into practice (Valenta & Enge, 2020). Reasoning and proving in primary education (ProPrimEd) is an intervention-based project that aims to answer this call by developing research-based materials to help teachers implement proving into their teaching. A mathematical proof is here defined using Stylianides' (2007) definition of proof: a kind of argument that uses forms of reasoning and expression that are mathematically valid and suitable for a specific classroom community and uses true statements accepted by the same community. By this definition, it is assumed that students at all grade levels can engage meaningfully in the practice of proving.

This paper reports on a lesson in the intervention conducted in grade 7 (age 12-13). The aim was to prompt students to become aware of the qualities of a good mathematical argument as an entry into work with proving. Lannin (2005) recommends that "research should examine the types of tasks that encourage students to examine the variety of justifications and generalisation strategies that other students use" (p. 254). Thus, this study examines the potential of this task approach. In addition, possible connections between students' evaluation and the designed arguments are explored to see whether the task can help students become aware of the features of valid arguments. The research question is: *What are seventh-grade students' perceptions of qualities of a mathematical argument?*

Theoretical framework and related literature

G. Stylianides' (2008) framework defines categories to analyse students' reasoning and proving activities, following A. Stylianides' (2007) definition. The framework distinguishes between proofs and non-proof arguments, where proofs are demonstrations or generic examples. Without a specific example, a demonstration draws on the properties of and relations between mathematical objects to show why a conjecture is true. This can be done using variables or other means of representing mathematical objects. For example, a random even number could be represented as $2n$ or "pairs of shoes", depending on the community. Counterexamples, contradictions, proofs by induction and proofs by exhaustion are also considered demonstrations. A generic example draws on a particular example and explains the underlying mechanisms to show why a conjecture must be valid for all cases. The affordances of using generic examples to help students move from showing that something is true towards showing why it is true is widely recognised (see e. g. Aricha-Metzer & Zaslavsky, 2019). This suggests that generic examples are a promising entry into work with proving at the primary level.

In G. Stylianides' (2008) framework, a non-proof argument is either an empirical argument or a rationale. An empirical argument consists of showing that a conjecture holds in some cases without showing why, hence providing "inconclusive evidence for the truth" (G. Stylianides, 2008, p. 12). A rationale is introduced as a fourth category to capture arguments not covered by the three former types. It is neither an empirical argument nor a proof but an attempt to prove that either lacks reference to accepted statements or uses statements that are not accepted by the community. In this sense, a rationale can be seen as a proof that misses some of the steps or content needed to convince a given community. The categories described in this section provide the backdrop for the five pre-written arguments presented in the Methods section.

How students perceive mathematical arguments have been investigated earlier, for example, by Bieda and Lepak (2014) and Healy and Hoyles (2000). Both studies show that students are likely to accept empirical arguments as proof. In Bieda and Lepak's (2014) study, the students were the same age as those in this study and had no documented proving experience. They were given two examples of arguments to consider, one empirical non-proof argument and one proof, and were instructed to decide which argument they preferred. In addition, they were asked to describe how the one they did not prefer could be amended to be more convincing. The results indicated that students were inclined to prefer examples accompanied by explanatory text. They both had a numeric example to show that a conjecture holds and text explaining why. The students in Healy and Hoyles' (2000) study had undergone teaching of proving and were given several arguments to consider, such as empirical arguments and proofs, using various modes of representation (e. g., everyday language and algebraic symbols). Their results indicated a discrepancy between what kinds of proofs students themselves would produce and what proof they believed would get the best mark by an evaluator. The students in the study had more success evaluating proofs written in words instead of algebraic notation and found them more convincing. The authors inferred that students' informal and narrative argumentation should be exploited to develop their proof competence. The present study draws on these results by 1) using a variety of informal representations such as contexts, drawings, and narrative explanations, and 2) prompting students to reflect on their proof conceptions by asking them to evaluate and choose among a set of arguments. Both studies described above applied interviews and surveys as data collection methods, while this study will take a different approach by observing group work without the presence of a teacher. This difference allows for insight into the potential of the task.

Method

This study is a single instrumental case study, where the researcher focuses on an issue and uses a bounded case to illustrate it (Creswell & Poth, 2018). Here, the case consists of students who work on a proof-related task. The issue explored is the task's potential to increase students' awareness of the qualities of a good mathematical argument. The study was conducted in spring 2021. A class consisting of 19 seventh-grade students participated, and their regular teacher taught the lesson. According to the teacher, who had taught the group for three consecutive years, the students had not met the term argumentation explicitly in their mathematics instruction. Therefore, the study provides insight into their first meeting with this theme, and this case is thus instrumental in exploring this task as an entry into argumentation. However, previous observation and descriptions given by the teacher suggested that the class was in the habit of showing their work, that is, in detailed writing, when they worked on tasks. The data was collected through video recordings of the students working in groups of three to four, giving five groups. Because of limited access to cameras, three out of five groups were chosen to be videotaped based on the level of verbal interaction observed in earlier lessons. The data material consisted of verbatim transcripts of the video recordings of the group discussions and the groups' written responses, including the groups that were not filmed. Therefore, one group was neither recorded nor given any written reasons for their opinions and is not included in the data material. Hence, the number of participating students was 16, whereby 12 were video recorded, and four submitted a shared written response.

The task, shown in Figure 1 below, was presented to the class by the teacher in plenary along with five pre-written arguments, with no additional information given.

> Mira, one of the students in a class, said:
>
> *The five-times table is easy. When you want to figure out what a number multiplied by five is, you can just multiply it with ten and then divide the result by two.*
>
> Several of Mira's classmates were uncertain that this could be true for all possible numbers. Some said that it could be true, but they were uncertain of why. They explored Mira's conjecture and made their own arguments.
>
> Your task is to
>
> 1. Choose the argument that you think is the best one, and write down two reasons why
> 2. Choose the argument that you think is the worst, and write down two reasons why
>
> Be aware that an argument should be such that it makes us more certain that a conjecture is true for all numbers and that we understand why it is true.

Figure 1: The given task

The arguments, shown in Figure 2 below, were crafted to demonstrate different arguments based on G. Stylianides' (2008) framework and were designed to be perceived as the work of a student their age. Abi's argument is a proof in the form of a generic example, while the rest are non-proof arguments. Hannah’s and Inga’s arguments are empirical arguments using a few examples, with the distinction that Hannah gives numeric examples while Inga uses a drawing to show an example. Leo’s argument uses larger numbers and refers to using a calculator. All three are empirical arguments, while Belma’s argument is a rationale, as it contains a part of an argument but lacks logical connections and details to be convincing. None of the arguments is of the form demonstration, which emphasised the difference between generic examples and empirical arguments.

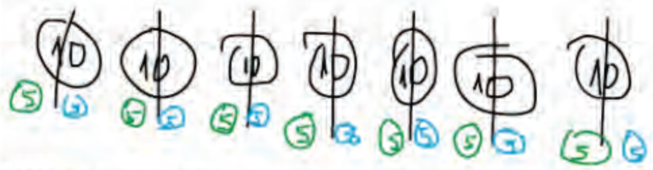
Name	Argument
Hannah (empirical argument)	Yes, it is correct. $2 \cdot 5 = 10 \quad 2 \cdot 10 = 20$ half $\quad 7 \cdot 5 = 35 \quad 7 \cdot 10 = 70$ half $\quad 8 \cdot 5 = 40 \quad 8 \cdot 10 = 80$ half
Belma (rationale)	Yes. $6 \cdot 5 = 30$ $6 \cdot 10 = 60$ It happens because 5 is the half of 10. $5 + 5 = 10 \quad 10 : 2 = 5$
Inga (empirical argument, with drawing)	$4 \cdot 10$  $4 \cdot 5$ half $4 \cdot 5$
Leo (empirical argument, larger numbers)	$3 \cdot 10 = 30 \quad 26 \cdot 20 = 260 \quad 268 \cdot 10 = 2680$ $3 \cdot 5 = 15 \quad 26 \cdot 5 = 130 \quad 268 \cdot 5 = 1340$ It is always this way. I have used my calculator to check many cases.
Abi (generic example)	It is correct, because: For example, if we want to calculate $7 \cdot 5$. That means to find out how much 7 fives are if added together. We can start by working out what 7 tens are:  Next, we can split each ten into two fives:  7 tens = 7 fives + 7 fives $7 \cdot 10 = 7 \cdot 5 + 7 \cdot 5$ Thus, $7 \cdot 5$ is the half of $7 \cdot 10$! If we use another number, not 7, then it will be just the same. It will only be a different number of tens and fives.

Figure 2: The arguments that were given to the students (translated from Norwegian)

The students were given copies of the arguments and the task and worked on it for 20 minutes before being called back to a whole-group discussion. While the groups worked with the task, the teacher and two researchers observed the groups.

The unit of analysis was a student contribution, either written or verbal. The contributions that regarded one of the arguments and described a positive or negative feature were collected. This gave 36 utterances. The further analysis was performed as an inductive qualitative content analysis (Mayring, 2015). The aim was to understand the different perceptions and their magnitude in the data material. The utterances were coded inductively to capture the feature it addressed. Codes describing related features were then collected into overarching categories. For example, the code “short, positive” and the code “long, negative” both belong to the category “short”, as they both suggest the perception that an argument should be short.

Findings

The analysis of the 36 utterances resulted in seven categories, as shown in Table 1 below. In the following, each category is elaborated on in order of appearance in the table.

Table 1: Overview of utterances by category

Categories	Explanation	Short	Text	Order	Drawing	Examples	Warrants
Frequency	15 (42 %)	7 (19 %)	5 (14 %)	3 (8 %)	3 (8 %)	2 (6 %)	1 (3 %)

The most frequent category, explanation, considered utterances related to understanding or explanation. It applied whenever a student stated that an argument was explained well or was easy to understand. Both Hannah’s, Leo’s, and Belma’s arguments were said to be easy to understand, and some students claimed that Hannah “…explained it really well”. One student spoke about Inga: “Really bad explanation. I did not understand what she meant. She just drew.” Abi’s argument is criticised: “…is hard to understand because it is long and messy”. Another negative remark about it is that “It is so much strange going on here at once”, indicating that it was considered complex by the student and might represent something the student was not used to seeing. Other students appreciated Abi’s argument: “Because he explains, for example, that the tens are divided into two”. This utterance indicates that a student noticed an essential feature of the generic argument. Other students said, “It has both writing and drawing. Very good explanation. Everyone can understand this. No difficult words were used”. Thus, Abi’s argument was either valued for its thorough explanation or was not appreciated because of its length and complexity. These examples show that both the short and the more elaborate arguments could be explained well. Hence, what students mean when they say that something explains well or is easy to understand is unclear.

The category short considers utterances about the length of the argument. Hannah’s argument was appreciated because “It is simple and short”, and about Belma’s argument, some said that it was positive “…that she used only one example”. “Leo’s and Hannah’s arguments are good because they did not have too much text to read and understand”, while “Abi’s argument is hard to understand because it is long and messy". The students appeared to value short arguments because they took up little space and took little effort to read.

Several students mentioned the presence of explanatory text, especially regarding Inga's and Abi's arguments. Abi's argument was valued because it had both writing and drawing, as indicated by the quote in the previous section. Inga's argument was the only one that none of the students preferred, and one student said: "The others have written text. That is why hers is the worst because it is easier to explain by writing than by drawing". Other utterances to support this view are: "She does not explain what it is that she has drawn", and "but she does not explain what she does". Hence, a short argument was not necessarily appreciated if the students did not find the content satisfying.

The four least frequent categories are the order of the argument, the use of drawings, the number of examples, and the warrants. Concerning the order of the argument, one student gave all the utterances, for example: "Leo just starts. Now I don't know, if I start with the first, I don't know (if it is true)". The student seemed to believe that the conclusion, whether the conjecture is valid or not, should be stated at the beginning of the argument. Two utterances were about the number of examples: "There are more examples", "checked on many numbers". This indicates an appreciation of examples and is related to the features that the arguments were meant to display. However, the task intended that the students recognise these arguments as mere examples and not convincing arguments. These utterances suggest the opposite outcome of what was intended. The use of drawings is also mentioned by a few students, either saying that it is good to use a drawing or that the quality of the drawing affected the quality of the argument. The last category, warrants, captured an utterance where the student, in a critical tone, said, while reading from Leo's argument: "I have used my calculator to check many cases. Ok?" indicating that this did not strengthen the argument.

There were few direct references to the features that the arguments were designed to display. For instance, no student commented that Hannah's and Leo's arguments only showed that the conjecture was valid for some examples or that Belma's explanation was incomplete. Instead, as shown above, some remarks suggested that it is good to have many examples. The two utterances appreciating Abi's explanation for being thorough are other examples that indicate a possible awareness of how this argument differs from the rest. Except for these few exceptions, the data shows little awareness of the features of the pre-written arguments.

Discussion

This study offers insight into how students perceive mathematical arguments for general conjectures and suggests that features like the explanation, length, and the presence of text are the qualities that the students in this group value most. However, there is no apparent relationship between students' perceptions and the features that the arguments were designed to display. These findings, along with a discussion on the methodological approach and the task's design, are addressed below.

The appreciation of empirical arguments is evident in this study, as in previous studies (Bieda & Lepak, 2014; Healy & Hoyles, 2000). The inclination to prefer explanatory text is also evident, as Bieda and Lepak (2014) also found. However, the data show that students' reactions are more nuanced. The notion of 'explain' seems to hold divergent meanings, where explanation appears to be a feature connected to whether the mere mathematical content of the argument makes sense or is possible for the reader to understand. This discrepancy might be related to the class habits, where there is an emphasis on showing one's work. To clarify how one has found the correct answer to a mathematical task. Thus, explanation, and in its extension conviction, might concern the presentation of a solution. This perspective is not compatible with assessing arguments for general conjectures. There seems to be a gap to fill to bring the students' attention to the difference between evaluating a written task solution and evaluating whether an argument shows that a conjecture must be valid for all cases. It can be understood as a necessary shift in the socio-mathematical norms in the group, concerning what can count as an acceptable mathematical explanation (Yackel, 2002). At the more practical level, the results emphasise the importance of a well-orchestrated classroom discussion where issues like the difference between showing that and explaining why are addressed. Teachers can benefit from exploring teacher moves to support students' argumentation by pressing to justify why something works. Such actions are suggested by Martino and Maher (1999), who describe questioning that can prompt students' justification when they work on mathematical problems.

Methodologically, this study provides a new lens into students' evaluation of arguments by unravelling how students in groups act without the influence of a teacher or a researcher. The results suggest that students can both explore and verbalise what they perceive to be qualities of arguments but that the nature of these qualities is often distant from what would be accepted by the mathematical community. A limitation of this approach is that it makes it impossible to get further insight into the students' meaning of the words 'explain' and 'understand', which frequently occurs in the data material. A follow-up interview where students are asked to elaborate on their conceptions of these notions could therefore be done to enrich the understanding of the case. As discussed in the previous section, this could provide further insight into how the gap between evaluating a written solution and evaluating an argument could be filled.

This study explores the potentials and challenges of a task where students evaluate others' work, which is an approach recommended by Lannin (2005). The results indicate no clear relationship between argument design and the students' evaluation, but that the task offers an entry point into discussing the qualities of a good mathematical argument. Further study should be made to explore improvements in the task. First, one possible approach is to use fewer arguments. In this task, one could reduce the number of arguments to three: one empirical example, one rationale, and one generic example, the distinction between showing that and explaining why could be highlighted in this way. Second, asking the students to argue for the conjecture themselves before being presented with the pre-written arguments should be explored to see if it might influence how they perceive the arguments. Third, one could consider the suitability of the conjecture. Durand-Guerrier et al. (2012) warn that too simple conjectures can obscure the need for proving. Therefore, it should be explored if conjectures of different complexity have different affordances in this task. Last, an extension of this task could be to find ways to highlight the deductive nature of proofs. A possible approach here is to use valid arguments where the order of the steps is altered and ask students to reorganise the steps to make the argument logical and convincing. Exploring these possibilities could be further steps toward finding fruitful ways to engage primary school students in proving.

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Mathematical reasoning outside the classroom – A case study with primary school students solving math trail tasks

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Research findings underline the potentials of outdoor mathematics – leaving the classroom presents the students with every-day situations in which they can apply mathematics. By example, a math trail guides students along a path with several mathematical tasks to be solved on-site, e.g. by measuring or counting. Apart from long-term learning benefits and motivational aspects, it remains unclear how this form of experiential learning involves fundamental mathematics working methods and competences, i.e. mathematical reasoning. In this article, the method and results of a case study with 15 primary school students are presented concerning the frequency and qualitative use of mathematical reasoning during a math trail with eight different mathematical tasks from the areas of arithmetic, combinatory and geometry. The results show that reasoning is involved in the students' solution processes, mainly in developing a solution plan and in geometry tasks, i.e. for estimations.

Keywords: Reasoning Skills, Outdoor Education, Mathematics Education, Experiential Learning

The Experiential Learning Theory and outdoor education

The learning theories by Lewin, Dewey and Piaget have – among other aspects – the importance of experiences with the environment in common (Kolb, 1984). Still, "learning was primarily a personal, internal process requiring only the limited environment of books, teacher, and classroom. Indeed, the wider 'real world' environment at times seems to be actively rejected by educational systems at all levels." (Kolb, 1984, p. 34). In contrast, the Experiential Learning Theory (ELT) emphasizes the "central role that experience plays in the learning process" (Kolb, Boyatzis & Mainemelis, 2000, p. 1). According to the authors, activities such as concrete experiences, reflective observation, abstract conceptualization and active experimentation are essential for the learning process and, in particular, the acquisition of mathematics concepts and skills. Also, Hattie et al. (1997) highlight the importance of *out-of-class experiences* for education, i.e. in the sense of first-hand experiences and through the embodiment of "abstract mathematical concepts in concrete terms, using ideas and modes of reasoning grounded in the sensory-motor system" (Lakoff & Nuñez, 2009, p. 5).

In the context of mathematics education, the term *outdoor mathematics* describes the teaching and learning of mathematics outside the classroom in interaction with the environment. One approach is the *math trail* which describes a route with mathematics tasks to be discovered in and solved with the real-world environment (Zender & Ludwig, 2019). In the educational context, during a math trail, the students cooperate in groups of three and solve the tasks by means of mathematical activities. Furthermore, through first-hand, out-of-class experiences of mathematical concepts, math trails have the potential to foster the acquisition of mathematics skills and competencies, e.g. modelling, problem solving and reasoning (Buchholtz & Armbrust, 2018). The potential of the latter is focused on in the following.

Mathematical reasoning in the context of outdoor mathematics

Mathematical reasoning shows high relevance in the teaching and learning of mathematics at different ages, e.g. concerning the question of how task design can foster reasoning processes (Stylianides et al., 2019). In this paper, the focus is on reasoning in primary school. Therefore, a broad definition of reasoning is chosen, whereby both – arguing and proving – are regarded as subareas of reasoning. This definition is comparable to Stylianides' term *reasoning-and-proving* (Stylianides, 2008) and results in the description of reasoning through several activities, i.e. to identify patterns, make conjectures and provide arguments (Arnesen, Enge, Rø & Valenta, 2019). With the study being located in Germany, the following activities are highlighted in the curriculum for mathematics education in primary school by the Conference of Ministers of Education (KMK, 2004, p. 8): (a) question mathematical statements and check their correctness, (b) recognize mathematical relations and develop assumptions, and (c) search for and understand reasons.

Being listed as a possible potential of math trails, it is examined on a theoretical level how far outdoor mathematics and math trails, in particular, can foster mathematical reasoning in the sense of the three mathematical reasoning activities.

1. Out-of-class experiences: From ELT, it can be assumed that experiences play a major role in learning processes. In the setting of a math trail, the students have to interact with objects and situations in the real world – hereby, they collect experiences in this context. Still, it is not only the collection of real-world experiences. To solve a task of a math trail, the students have to reflect on their experiences by formulating mathematical statements and assumptions. In this reflection process, mathematical reasoning is mainly necessary in the sense of the reasoning activity (a) and (b).
2. Group Interaction: During a math trail, the students work in small groups. For the different activities, i.e. searching the task's object, planning the solving process, exchanging ideas, collecting data and validating the results, it can be assumed that the students interact with each other and reason for and against proposals and ideas. In this social process, mathematical reasoning is mainly necessary for reasoning activities (b) and (c).
3. Transfer of Mathematical Knowledge: Being outside the classroom, the students have to choose the data to be collected from all available data – whereby this number might be higher in the real world than in a school book. Hereby, the students have to transfer the mathematical knowledge that they acquired inside the classroom to a new context that was not primarily created for educational purposes. In contrast to calculation tasks where the mathematical content is often straightforward, the students have to decide (and reason) which mathematical characteristics and relations can be found in the real-world situation. In this reflection process, mathematical reasoning is necessary for the reasoning activities (a), (b) and (c).

State of the art and research question

From quantitative empirical studies, it results that math trails have the potential for positive (long-term) learning outcomes (Zender & Ludwig, 2019), motivational aspects (Gurjanow, Oliviera, Zender, Ludwig & Santos, 2019) and individual opportunities for the support of strengths and weaknesses (Buchholtz & Armbrust, 2018). With reference to ELT, math trails are a promising approach for mathematics education. Still, there has been a lack of studies to examine the process-related mathematical competences when solving math trail tasks, i.e. mathematical reasoning. From the theoretical potentials, it can be assumed that math trails foster reasoning on different levels. Still, the considerations need empirical validation. In order to focus on the aspect of mathematical reasoning in more detail, the paper focuses on the research question:

To what extent and for what purpose do primary students use mathematical reasoning while solving math trail tasks outdoors?

In particular, the focus is on the question of what characterizes the solution processes of math trail tasks concerning mathematical reasoning. From this perspective, it can finally be concluded to what extent math trail tasks are suitable for learning mathematical content and fostering mathematical reasoning.

Methodology

In June 2021, a case study with primary students was conducted to answer the research question. In total, 14 students aged 9-11 years were divided into six groups of two, three or four students. In these groups, the students followed the route of a math trail located at Goethe University Frankfurt with eight different tasks, including combinatory, geometry and arithmetic tasks (see Figure 1). The tasks do not include an explicit question that initiates reasoning.

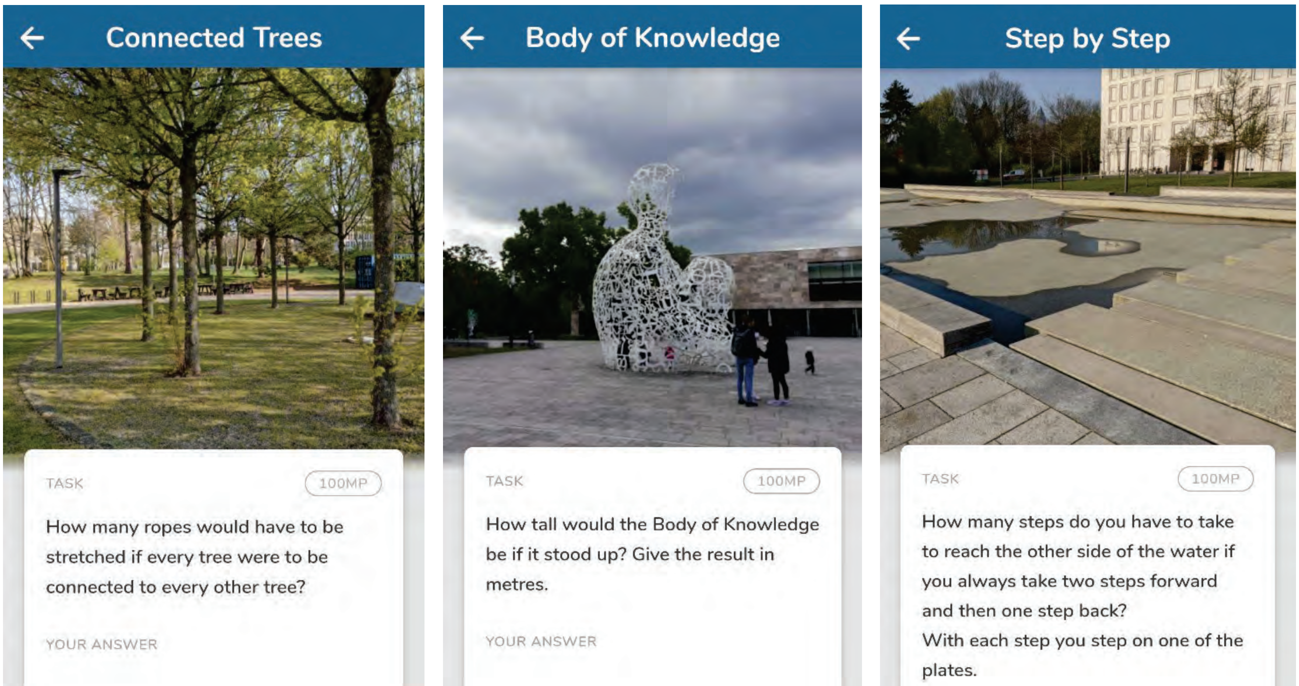

Figure 1: Three tasks from the study for combinatory (left), geometry (middle), arithmetic (right)

For the entire route, the groups needed between 45 and 70 minutes (M = 62 minutes). Hereby, the students were supported by the MathCityMap app through a map, hints and an answer validation (for more information see Zender & Ludwig, 2019). In addition, they were accompanied by a university student who filmed their solving processes and interacted with the students if clarification was necessary. This setting resembles the methodological adaptation of the *narrative walk-in-real-time interview* that Buchholtz, Orey & Rosa (2020) adapted to the context of math trails.

In total, the video material contains about 370 minutes. Especially the students' conversations during the actual solution processes are considered to analyze their way of reasoning. Therefore, the conversations are transcribed and analyzed in two different ways. First, on a quantitative level, the different mathematical activities are coded in accordance to Pólya's phases of problem solving: 'Understand Task', 'Develop Solution Plan', 'Solve Task' and 'Task Validation' (Pólya, 2004). In addition, the mathematical reasoning activities are coded and added to the respective activities in the solution processes. Through this, it is possible to specify the quantity of the students' reasoning in relation to the activities during the solution process. This is visualized by means of activity diagrams (Ärlebäck & Albarracín, 2019; see Figure 2). Afterwards, on a qualitative level, the students' identified reasoning activities are analyzed by means of a qualitative content analysis according to Mayring (2000). With this, different categories of mathematical reasoning can be identified inductively from the empirical material. The categories created in this process are to be considered disjoint. The results of the analyses on both levels are presented in the results section.

Results

Frequency of reasoning activities during the solution processes

Figure 2 shows the activity diagram of the solution processes during the math trail by one of the participating groups. The diagram includes the mathematical activities of the group according to Pólya and specifies, in addition, the activities in which mathematical reasoning is relevant through shading. The diagram gives an overview of the eight tasks of the trail, whereby the tasks from Figure 1 are Task 3 (Connected Trees), Task 6 (Body of Knowledge) and Task 7 (Step by Step). The time for navigation from one task to another is excluded in this presentation.

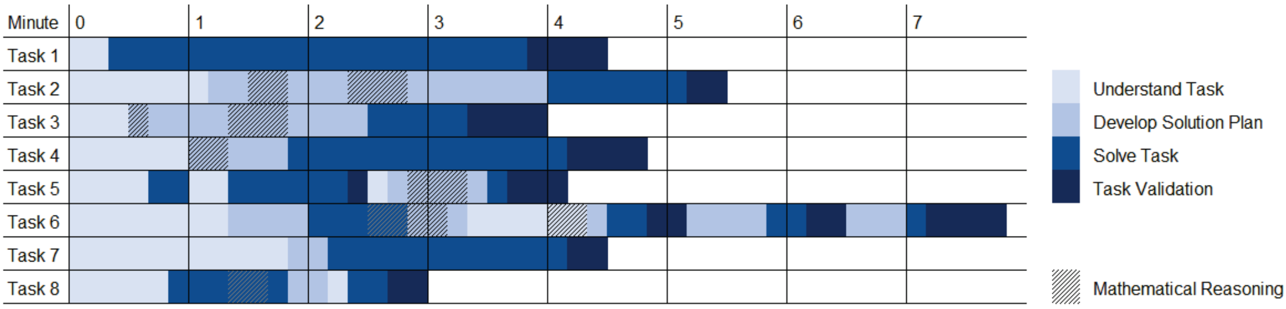

Figure 2: Activity diagram of solution processes and mathematical reasoning activities

In total, the amount of reasoning activities in this group is about 10 % of the time of the math trail task solution processes. Again, this number should be interpreted in the context of the tasks which do not include explicit claims for reasoning. During 25 % of the time, the students understand the tasks. This activity is relevant in all eight tasks. It happens mainly at the beginning of the solution processes through the actual task formulation and an analysis of the object and/or situation. In 4 % of the understanding activities, the students reasoned. The activity "Develop Solution Plan" is relevant in 26 % of the group's solution processes and – despite Task 1 – relevant in every task. In this activity, reasoning is coded most frequently, namely in 28 %. The students spend 35 % of the solution process on the actual task solving. About 5 % of this activity can be coded as reasoning activities. The group's task validation has a relative duration of 14 %. In this activity, they do not use any reasoning.

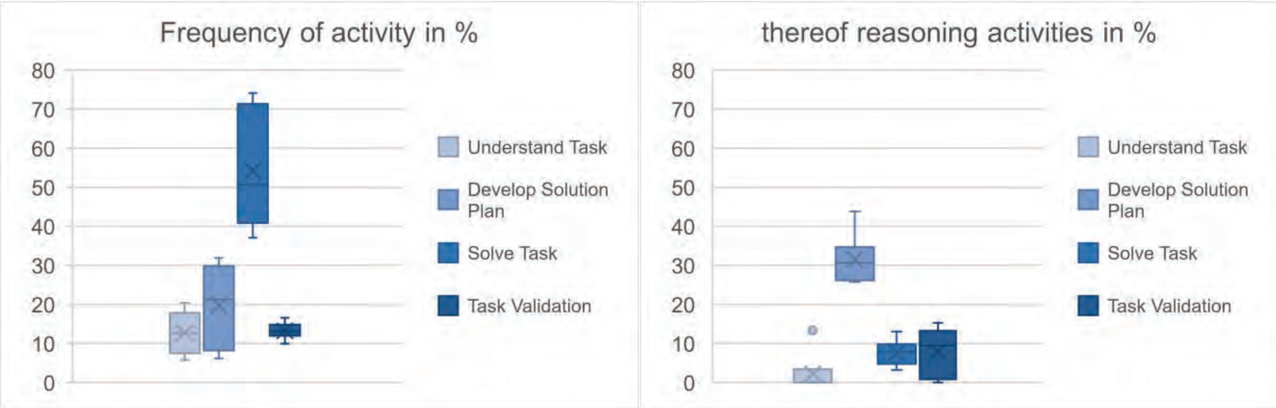

Figure 3: Quantitative overview of all solution processes and reasoning activities

Focusing on all six groups, in about 12 % of the math trail solution processes – excluding navigation – reasoning activities are coded. Figure 3 gives an overview of the frequency of the solution processes and the reasoning activities of all six groups. As a result of this, the beforehand described observations can be summarized for the sample as follows: The actual task solving activity is the most frequent activity in the solution process with more than 50 % on average. It is followed by the development of a solution plan whereby the high deviation shows differences between the groups for both activities. The activities of understanding and validating the task take both about 13 % on average. As presented in the example group, the students reason most frequently in developing a solution plan. From the 20 % of development activities, nearly one third involves reasoning activities. It is followed by the solving and validation of the task, in which 8 % of the activities involve reasoning activities. In the understanding of the task, reasoning plays a minor role.

With tasks from three different mathematical topics being involved in the study, it is possible to analyze the frequency of reasoning activities with regards to the different actions, i.e. counting in the arithmetic tasks, measuring and estimating in the geometry tasks and trying and sketching in the combinatory tasks. Hereby, it can be observed that the geometry tasks involve about two reasoning activities per task. Despite the development of a solution plan, the students reason frequently on their estimations in the activity 'Solve Task'. In the combinatory tasks, about one and a half reasoning activities are coded on average, mainly in the development of a solution plan. In the arithmetic tasks, the amount is about one reasoning activity per task. This might be explained by the comparably low duration of the activities in which the students develop a solution plan. As the activity is mostly counting, the students tend to skip the development of a plan in which reasoning is most frequent.

Categorization of the reasoning activities

The qualitative content analysis results in different categories that describe the reasoning activities for the different steps in the solution process according to Pólya (2004) in more detail. Due to the low frequency of the activity 'Understand Task', it is excluded from the qualitative analysis.

With this analysis, it is possible to identify different ways of the reasoning activities the students used. Table 1 gives an overview of the categories with a representative example from the study (translated by the author from German to English).

Table 1: Qualitative categories describing the purpose of the reasoning activities in the different steps of the solution process

	Definition	Example
Reasoning during Development of a Solution Plan		
Choose a strategy	The students reason about a strategy they might use to solve the task. Aspects such as precision and efficiency are included where appropriate.	“We should count the number of stones in one row, because there are four [rows] […] and it should be nearly the same [number of stones] in each row.”
Identify object/situation characteristics	The students use the local settings and refer to characteristics of the task object/the task situation to reason about a solution plan.	“We have to count with six people. Because it [the bench] has three parts where two persons can sit on each.”
Reasoning during Solving of Task		
Do measurements/ estimations/counts	The students reason about their approach to measuring, estimating or counting. In particular, considerations of the correct procedure are included.	*Student shows a way on a map (see Figure 4 left).* “No, this is not the shortest way. You have to go here and here and only measure up to this building.”
Perform a calculation	The students reason how to calculate, referring to known formulae and mathematical relations where appropriate.	“The first tree has 14 ropes. And then the second [tree] has only 13 [ropes], because it is already connected to the first one. […] So we have 14 plus 13 plus 12 […] and so on.” *(see Figure 4 middle)*
Reasoning during Task Validation		
Check result’s correctness	The students reason for or against a statement whether the achieved result might be correct.	*Student counts 16 regular elements on the task’s object (see Figure 4 right).* “I think it is possible with two [colors] because 16 is an even number.”
Make changes in the solution plan	The students reason why an incorrect result was obtained and how the solution plan should be changed.	“11 is too low, because we estimated 2 meters, but I think it is more. Can you stand next to the figure so that we can estimate with you?”

Figure 4: Impressions of the solving processes during the math trail

Discussion

With regard to the question to what extent and for what purpose primary students use mathematical reasoning while solving math trail tasks outdoors, the following conclusions are drawn. Math trails provoke reasoning activities without explicit questions for reasoning: The students reason especially during the activities in which they decide on a solution plan, i.e. reason about a strategy and by means of characteristics. From the theoretical considerations, in particular Kolb (1984), especially the *out-of-class experiences* and the *group interaction* seem to be relevant for the involvement of mathematical reasoning hereby. Still, the activities 'Solve Task' and 'Task Validation' involve reasoning. During the solving activities, they reason about measurements and calculations. From the theoretical considerations, especially the *transfer of mathematical knowledge* seems to be relevant, i.e. when choosing a suitable formula. During the task validation, again, the *out-of-class experience* seems to be an important factor, especially when it comes to the reflection of the solution process in the context of being outdoors.

The results are in-line with the theoretical considerations – mainly derived from Kolb (1984) and Lakoff & Nuñez (2009) – concerning the relevance of mathematical reasoning in the outdoor setting of a math trail. Still, the indicator that the tasks involve reasoning to a different amount, i.e. in relation to the mathematical topic, raises further questions. Also, the deviation between the groups indicates that the groups might follow different patterns of reasoning along the tasks. Finally, the question arises to what extent the reasoning outdoors differs from the reasoning inside the classroom in a comparable setting – on both, a qualitative and a quantitative level. Comparable data are not available at the current time. In a follow-up study, these questions will be answered to examine the potential of math trails to foster mathematical reasoning. This will be part of the MAP-Study "Modelling, Arguing and Problem Solving in Outdoor Mathematics" funded by Dr. Hans Messer Stiftung.

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The use of a deductive reasoning when solving a word problem

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The response of problem solvers to a request like 'Explain your answer' does not necessarily provide the assurance that the mathematical representation describing a word problem situation is correct. We suggest to compliment a traditional algebraic word problem with a request to establish a mathematical representation generalizing the problem situation and then to justify its derivation of this mathematical representation. Next, we suggest the construction of a valid deductive argument and evaluate the justification. An example executing the suggestions is given.

Keywords: Word problems, deductive argument, proof validation, magnitudes.

Introduction

Recognition of the importance of proof in school mathematics has increased during recent decades (Balacheff, 1988; Hanna, 2000; Harel & Sowder, 2007; A.J. Stylianides, 2007; G.J. Stylianides, 2008). There are several books presenting new forms of problems that are designed to engage teachers in the practice of reasoning and proving (Arbaugh et al., 2019; A.J. Stylianides, 2016). However, despite these trends in the mathematical research area, the choice of ways to learn to prove in a classroom seem to be quite poor when compared with the available resources for traditional tasks.

Usually word problems have no relationship to reasoning or proving. Their aim is to learn to apply mathematics in contexts outside of mathematics. Word problems are designed just to find an answer to the formulated question. In itself, solving word problems is a great challenge for students as well as for teachers (Verschaffel et al., 2000). By modifying word problems from a plain search for an answer to a tool for reasoning, we hope to help students to make better progress in this type of task. We therefore suggest practising mathematical reasoning in the process of solving word problems. Uncovering logical relationships in the search for mathematical representations of word problems is a way to understand solutions and to make sense of them.

Our research question is how can deductive reasoning be taught and learnt when solving word problems?

The logical connections between statements constitute an important knowledge which students should learn about a deductive reasoning in school mathematics. For example, the rules for drawing the Modus Ponens inferences appear well-developed by early adolescence according to various studies from psychological research (Stylianides & Stylianides, 2008). Therefore the modified word problems considered below are appropriate for students of lower secondary education. This immensely increase opportunities to engage in reasoning-and-proving activities (also see Chapter 5 in Arbaugh et al., 2019, for other arguments).

The discourse of a classroom should be central to the teaching and learning when using modified word problems. This facilitate the exchange of ideas between students and a teacher. Combining different levels of discourse could benefit students' deductive reasoning (p. 22 in Blanke, 2018).

Algebraic word problems

To explain the idea we comment on the supposed phases and components of solving a word problem. The solution is obtained by the following six parts of a *problem solving process* (Verschaffel et al., 2020, pp. 909–910):

> (a) the construction of an internal model of the problem situation, reflecting an understanding of the elements and relations in the problem situation; (b) the transformation of this situation model into a mathematical model of the elements and relations that are essential for the solution; (c) working through the mathematical model to derive mathematical result(s); (d) interpreting the outcome of the computational work; (e) evaluating whether the interpreted mathematical outcome is computationally correct and reasonable; and (f) communicating the obtained solution.

The question is what makes us sure that the solution that is obtained is correct with respect to the original problem. We argue that the evaluation suggested by part (e) does not provide this assurance. Part (e) refers to the answer obtained from part (c) and the interpretation made in part (d). However, the question that remains is whether the mathematical model constructed in part (b) is correct. In other words, if the solver's understanding of the problem or the transformation to the mathematical model are incorrect then the actions suggested by part (e) do not help. We therefore suggest using reasoning-and-proving activity to make sure that the mathematical model in part (b) is correct.

From now on we replace the phrase 'mathematical model' in the description of the problem solving process by the phrase 'symbolic representation', for two reasons. First, we consider traditional word problems as a tool for learning the abstract concepts of school mathematics rather than as a tool for relating mathematics to the real world. In this paper a *symbolic representation* is a generalization of the relationships in a problem expressed in terms of mathematical symbols, such as an equation, a system of equations, inequalities, and so on.

The second reason for the change in terminology is the discovery that the symbolic representations of the problem situations examined in this article are based on a suitable mathematical structure. We hypothesize that such a situation is typical for traditional algebraic word problems. The mathematical structure comes from the hidden properties of the magnitudes describing the word problem. These properties surface when we search for the premises of deductive reasoning. Thus, when considering word problems we pay special attention to magnitudes and their properties.

Now we can specify the above mentioned discovery that a mathematical representation of an algebraic word problem is based on a suitable mathematical structure. Let the three magnitudes A, B and U be given. Let $[A : U]$ and $[B : U]$ be numerical values of A and B with respect to the unit magnitude U. Then the numerical value $[A + B : U]$ of the (non-arithmetical) sum $A + B$ of magnitudes is the arithmetical sum of the numerical values $[A : U]$ and $[B : U]$, or in symbols we have the equality:

$$[A + B : U] = [A : U] + [B : U]. \quad (1)$$

For example, if A is a length and B is a length then one can take two line segments with lengths A and B, respectively. Suppose that $[A : U]$ and $[B : U]$ are the corresponding numerical values measured with respect to a unit length U. Then the numerical value of the length $A + B$ of the two end-to-end concatenated segments is obtained by the equality (1).

Magnitudes usually appear in a contextual representation of the word problem contained in part (a) of the problem solving process. Some of them are unknown and have to be found. By part (b), the contextual representation is transformed into a symbolic representation by means of the relationships between the numerical values of the magnitudes. For example, all symbolic representations of the word problems considered in this paper are equations relating numerical values of magnitudes to each other. The unit names are added back to the numerical values when the result of the computational work is interpreted, as listed in part (d) of the problem solving process.

We are now ready to name the class of word problems studied in this paper. An *algebraic word problem* (AWP) is a verbal description of a problem in which one or more questions are raised, the answers to which can be obtained by establishing and solving an equation with respect to a numerical value of unknown magnitude. This is a modification of the description of an arithmetic word problem given by Verschaffel et al. (2000).

The primary goal of our suggestion is to teach and learn mathematical reasoning in school mathematics. With this aim, in addition to the usual question in an algebraic word problem we have two tasks. The first of these is to justify a symbolic representation that answers the question, and the second is to prove the stated representation by constructing a valid deductive argument. The second task is required since in this paper a 'justification' is a broader term than a 'proof': a *justification* means a set of arguments used to give reasons why a conjecture is true. We use the symbol AWP+2T to denote an AWP together with the two added tasks. The proof, in the form of deductive reasoning, assures us that the overall solution is correct with respect to the original problem. In sum, a problem solving process of an AWP+2T has a new part, (b'), consisting of proving the symbolic representation, while parts (a) and (b) now provide a response to the first task.

Framework for reasoning-and-proving

In this paper, reasoning-and-proving activities are considered when solving algebraic word problems. Here we recall the analytic framework describing the meaning of reasoning-and-proving given by G.J. Stylianides (2008) (see also Arbaugh et al., 2019).

In mathematics, a proof of a new piece of knowledge is the final step of a work researching a mathematical phenomenon. It is preceded by asking questions, searching for patterns, making conjectures, and going back and forth. We use the mathematical component of the *analytic framework* that integrates three activities: identifying patterns, making conjectures, and providing arguments which may or may not qualify as proofs.

How does solving word problems with reasoned judgement fit into the analytic framework? Briefly, the three activities of the mathematical component correspond to parts (a), (b) and (b') of the problem solving process of AWP+2T. In other words, identification of patterns corresponds to searching for mathematical relationships in the problem, making conjectures corresponds to formulating a symbolic representation, and giving arguments should prove the correctness of the symbolic representation. Analysis of the symbolic representation of the word problem and the computational work are therefore not part of the analytic framework, and nor are they a focus of the present paper.

An AWP problem is usually described in terms of magnitudes. We interpret an attempt to understand the mathematical relationships between the magnitudes as a search for patterns in the context of the analytic framework. On what basis does a problem solver choose one possible mathematical relationship over other possible relationships? This question corresponds in the framework to the separation between plausible patterns and definite patterns. The patterning activity is called 'definite' if it is mathematically possible for a problem solver to provide conclusive evidence for the selection of one specific pattern over other patterns that also fit the data. In some tasks, the necessary information for the definiteness of a pattern is given explicitly (G.J. Stylianides, 2008; Arbaugh et al., 2019). How can we make definite choices between possible patterns when solving AWP+2T? The basis for this choice is given by the mathematical structure describing the properties of magnitudes. The structure we use below is relation (1) as explained in the preceding section.

The third and final framework activity is the construction of arguments which may or may not qualify as proofs. The term 'proof' is used with the meaning suggested by A.J. Stylianides (2007) – a valid argument based on accepted truths for or against a mathematical claim. The term 'valid' indicates that the assertions making up the argument are connected by means of accepted canons of correct inference such as modus ponens and modus tollens. The term 'accepted truth' refers to a class of statements like axioms, theorems, definitions, modes of reasoning and representational tools that a classroom community may take as shared at a given time. We call such statements mathematical. An argument that qualifies as a proof makes explicit reference to the accepted truths that it uses. Next, we elaborate on the meaning of accepted truth, since it is too narrow when solving word problems with reasoned judgement.

In the process of solving an AWP+2T problem, one needs to construct arguments proving a justification of the symbolic representation. The proof is based on a deductive argument, which is a series of statements consisting of premises and a conclusion. Clearly, the conclusion must be the symbolic representation of the problem. To specify a class of possible premises, we note that mathematical statements may not be sufficient. Statements about a word problem may refer to a contextual representation of facts about an imaginary world and may not be a part of reality. The truth of such statements is therefore of a different kind. The new term 'accepted truth in context' will refer to a class of mathematical statements as well as statements about a word problem that a classroom community may take as shared at a given time.

Definition 1. *Proof by context* is a valid deductive argument with the premises being accepted truth in context and with the conclusion being a symbolic representation of the problem.

A deductive argument is said to be valid if and only if it takes a form that makes it impossible for the premises to be true and the conclusion nevertheless to be false. Otherwise, a deductive argument is said to be invalid.

Having a precise meaning of the phrase ‘proof by context’, we can work on a definition of the term ‘justification’ by adopting the ‘proof schemes’ approach of Harel and Sowder (1998). Similarly, we suggest using three levels of justification: by external sources, as a collection of facts, and proof by context. The changes reflect the specific features of word problems. Justifications by external sources are those in which problem solvers give reasons based on (a) the ritual or the form of the appearance of the argument, (b) the word of an authority, such as a teacher, and (c) some symbolic manipulation without reference to meaning. For example, the reason is ritual if there is no mention of constant speed in situations discussing the movement of objects. Justification as a collection of facts appears when there is no mention of the logical relations between statements leading to a hypothetical conclusion. Logical connections ensure the validity of a deductive argument, leading to the third level of justification.

An example of AWP+2T

This section answers the first research question of how deductive reasoning can be taught and learnt when solving word problems. We take *any* algebraic word problem (AWP) and complement it with two tasks. The first task is to justify a symbolic representation that answers the question, and the second is to prove the justification of the symbolic representation. The symbol AWP+2T denotes the resulting modified AWP.

A solution of AWP+2T is obtained by the following modified problem solving process:

I. Justifying a symbolic representation of the problem situation.
II. Answering the word problem question.
III. Constructing and validating a deductive argument proving the stated representation.

Relating this to the traditional problem solving process described at the beginning of Section 2, Part I corresponds to (a) and (b), part II corresponds to (c), (d), (e), (f), and part III is the new (b’).

Recall the concept of a *constant speed* (Wu, 2011).

Definition 2. An object moves at a constant speed along a straight line if there is a real number v such that for each real number $t \geq 0$, the distance $s(t)$ (measured in kilometres, metres, ...) covered by the moving object during the time period from 0 to t (measured in hours, seconds, ...) is equal to the product $v \cdot t$. This number v is called the speed of the motion, and it is the derived magnitude (measured in kilometres per hour, metres per second,…, respectively) related to the fundamental magnitudes of distance and duration by the equality of numerical values

$$v = \frac{s(t)}{t}, \qquad \text{for each } t > 0.$$

This example of an AWP is taken from a Lithuanian book for teachers. The original formulation has no hypothesis of the constancy of speed. We use a justification taken from this book.

Word Problem. *Tourists walking at a constant speed planned to cover a distance between a river and a tourist camp in 6 hours. However, after 2 hours’ walking they slowed down their initial speed by 0.5 km/h and were 30 minutes late arriving at the camp.*

1. *What was the initial speed of the tourists?*

2. *Justify a symbolic representation of the problem situation.*
3. *Create and validate a deductive argument proving the justification.*

Solution I. Justification. **(1)** Suppose that the initial speed of the tourists is x km/h. **(2)** Since distance = speed × time and since the tourists intended to walk from the start to the end in 6 hours, the distance between the river and the camp is equal to $6x$ km. **(3)** In fact the tourists walked at different speeds: for the first 2 hours at the initial speed of x km/h and for 4.5 hours (since they were 30 minutes late) at a speed of (x-0.5) km/h. **(4)** The distances they travelled are therefore equal to $2x$ km and $4.5(x-0.5)$ km, respectively. **(5)** Combining the resulting distances we obtain the desired equation:

$$2x + 4.5(x - 0.5) = 6x. \tag{2}$$

II. Finding an answer. Solving this equation one obtains $x = 4.5$. Thus the initial speed is 4.5 km/h, which is the answer to the problem.

III. Deductive argument and validation.

The figure depicts a mathematical model of the trip described by the word problem. The point A denotes the location near the river where the tourists began their walk. The point C denotes the location of the tourist camp, the final destination of the trip. In between, the point B is the place where the tourists slowed down from their initial speed. The trip itself can be imagined as a moving point along the line segment AC at the speed described by the word problem.

Next we construct the deductive argument

$$P_1, P_2, P_3, P_4, P_5 \ \vdash \ \text{equation (2)}, \tag{3}$$

with the following premises.

P_1 If the length of the line segment AB is S_1 km and the length of the line segment BC is S_2 ~~km~~ then the length of the line segment AC is $S_1 + S_2$ ~~km.~~

P_2 If an object moves at a constant speed v km/h during the time duration t h then the distance covered $s = v{\cdot}t$ km.

P_3 Fact I: the distance between points A and B is travelled at the speed of x km/h in a time of 2 hours.

P_4 Fact II: the distance between points B and C is travelled at the speed of $x - 0.5$ km/h in a time of 4.5 hours.

P_5 Fact III: the distance between points A and C is travelled at the speed of x km/h in a time of 6 hours.

Now we show that the deductive argument (3) is valid (that is, that the truth of all the premises entails the truth of the conclusion).

P_6 According to premises P_2 and P_3 and the inference rule modus ponens, $S_1 = 2x$.

P_7 According to premises P_2 and P_4 and the inference rule modus ponens, $S_2 = 4.5(x - 0.5)$.

P_8 According to premises P_2 and P_5 and the inference rule modus ponens, the distance between points A and C is equal to $6x$ km.

P_9 According to premise P_1, statements P_6, P_7 and P_8 and the inference rule modus ponens, equation (2) holds true.

Evaluation of the level of justification of (2). This is done by interpreting each sentence of the justification.

(1) The unknown initial speed of tourists is called the unknown variable and is denoted by the symbol x. This is used to represent the distances travelled by the tourists. **(2)** Fact III is obtained using the hypothesis of constancy of speed. **(3)** The value of the reduced speed is obtained using the relations described by the problem statement. **(4)** Facts I and II are obtained using the hypothesis of constancy of speed. **(5)** While it is not stated explicitly, the stated equation (2) is obtained by the property (1) of measurements.

In conclusion, all the facts of the deductive argument are mentioned in the justification with no mention of logical links between them. Therefore the justification is at the second level.

Conclusions

The meaning of a symbolic representation of an algebraic word problem lies in the mathematical system of the properties of continuous magnitudes. This conclusion is confirmed by examples of solutions of modified algebraic word problems using reasoning-and-proving. The deductive argument appears to be a necessary instrument for this discovery. We do not need additional instruments to see a link between an arithmetical operation and a relationship between discrete magnitudes in an arithmetical word problem.

A stepping stone to success in solving a modified word problem is the question of how to construct a deductive argument having built a non-proof argument. For example, having built a justification at the second level, to obtain a valid deductive argument one needs to find proper logical links between statements presumably implying a symbolic representation. This is the kind of task that is familiar in propositional logic. In conclusion, solving word problems becomes an exercise in logic.

The most promising conclusion is that solving word problems with reasoned judgement in the sense of the present essay provides an unlimited source of reasoning-and-proving activities.

More examples of AWP+2T and advantages of using reasoning to solve word problems are discussed in Kilienė & Norvaiša (2021).

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Mathematical reasoning at the age of four

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Keywords: Mathematics education, reasoning skills, word problems (mathematics).

Word problems for developing mathematical reasoning

Mathematical reasoning is a widely analyzed topic amongst secondary and high school students. It is widely believed that mathematical reasoning is only possible for students once they are aware of various mathematical concepts, so we decided to check whether 4-year-olds are capable of mathematical reasoning. We used word problems that have been shown to develop mathematical reasoning for 8–10-year-old students in Schliemann et al. (2002). In this study, we adapted the methodology to test the word problem with a 4-year-old child and older solvers.

We analyze ides about mathematical reasoning from Thompson et al. (2012), Arbaugh et al. (2018) and Stylianides et al. (2016) and define word problem solving characteristics that develop mathematical reasoning.

A word problem develops mathematical reasoning, if solving includes argumentation, justification or patterns identification and if it uses the statements known to a presenter and a class community.

Experiment with a 4-year-old student

We adapt the word problem for 4-year-old students, who may not know how to read. We illustrate the problem and change the numbers to smaller ones. Our aim is to check if we can develop mathematical reasoning from early years and to see how students of different ages solve the same problem.

A word problem from Schliemann et al. (2002) is used:

> Table A has 3 candy bars and 2 chairs. Table B has 6 candy bars and 4 chairs. Table C has 5 candy bars and 2 chairs. Show how you would divide the candy bars equally among those at each table. Which table would you sit at? Convince us that you chose the best table. Are any tables the same?

The numbers in the original problem are such that at the first and second tables the subject gets 1 and a half candy bars, and at the third table there are 2 and a half candy bars. Our word problem (illustrated in Figure 1), using smaller numbers so younger kids do not make mistakes in computation. The solver must choose between tables where the bear character gets less than 1, more than 1 or 1 cupcake. The question is formulated in the following way: All bears are going to sit on chairs. If the bear wants to eat as many cupcakes as possible, at which table he should sit? Could you explain why?

A 4-year-old girl attending kindergarten solved this word problem. During the interview, she expressed ideas such as:

Girl:	At the first table, the bear would get one cupcake.
Interviewer:	Really? How many bears will sit there?
Girl:	Four bears. Then we need to cut this cupcake into two parts. This bear will get one part; another bear will get second part (points to chairs). Let us cut another cupcake

into two parts. This bear will get one part; another bear will get the second part (points to chairs).

Interviewer: How much would each teddy bear get?
Girl: By four parts[1].
Interviewer: How much would each bear get sitting at the middle table?
Girl: He would get two.
Interviewer: Even two? Why?
Girl: There are two chairs, three cupcakes.
Interviewer: One would get two, and the other?
Girl: Then we can cut one cupcake. One would get two and another the other two.
Interviewer: But not two whole ones?
Girl: By one and part[2].
Interviewer: And at the third table.
Girl: They will get one each.
Interviewer: Where is it best to sit?
Girl: At this table (shows to the middle table), there they will get two cupcakes each.

Figure 1: Mathematical reasoning word problem in image

Results

We interviewed different age groups with the same task. The justification was different, but correct amongst all interviewees, and we can identify that the argumentation changes during the learning period. The 4-year-old solver makes mistakes in the concepts she used, but shows correct reasoning. Our results show that we can use word problems that develop mathematical reasoning for younger kids.

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[1] The 4-year-old student indicated that each bear would receive ¼ of two cupcakes. She did not say this explicitly because she has not learned fractions and could therefore not articulate the answer in those terms.

[2] Same comment as above. The 4-year-old student indicated each bear would receive 1 ½ cupcakes.

Procedures in proof construction: five proving strategies of first-year university students

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As proof construction typically presents many difficulties to university students, there is an increasing interest in student-centred forms of in-process support. This is accompanied by a need for research on how students – effectively or non-effectively – proceed in proof construction. This study therefore examines the proof-construction behaviour of first-year university students in 24 cases. Following the principles of empirical type construction, the author identified five proving strategies, each of which is characterised by a specific combination of sub-processes. Comparing these strategies to each other provides new insights into both different approaches to proving and typical obstacles. However, a comparison between successful and non-successful students did not reveal a clear relationship between the strategy used and performance. Our results suggest that the crucial aspects of proof construction are related to implementation at the microscopic rather than macroscopic level.

Keywords: Reasoning and proof, proving strategies, tertiary education, type construction.

Introduction

Dealing with mathematical proof is a complex and demanding activity that causes great difficulties for many university students (Selden & Selden, 2008). One possible reason for these difficulties is related to a lack of transparency between proof as a product and its process of development (Hemmi, 2008). Therefore, there are many reasons to place more emphasis on underlying proving processes and to specifically train proving strategies in university teaching (Karunakaran, 2018; Schoenfeld, 1985). However, so far there has been little research on which such fostering programs could be based. This study, therefore, takes a process-oriented perspective on proving, and examines how university students behave in constructing proof. By comparing successful and non-successful students, the study, moreover, aims to identify procedures that are more effective than others.

A process-oriented perspective on proving

Although proof-writing processes run differently for each individual, previous research revealed some invariances occurring across different cases. On the one hand, there are efforts to describe proving processes as a combination of recurring sub-processes. Each sub-process focuses on different cognitive demands and pursues a specific sub-goal of proof construction. Another approach refers to general proving strategies that describe typical behaviour patterns within proof construction. The paper first gives a brief overview of both approaches before connecting them.

Sub-processes of proving

In the field of reasoning and proof, various models have been proposed to describe relevant sub-processes of proof construction (G. Stylianides et al., 2017). As the models differ in their target group (school students, university students and mathematicians), they consider different demands in terms of formalisation and rigor. While some models emphasise the creative part of proving and highlight processes of exploration and discovery (Hsieh et al., 2012; Schwarz et al., 2010), others include precising parts in the same way (Boero, 1999; Stein, 1986). However, from a synthesis of existing models four sub-processes emerge that are likely to be relevant in different contexts (Kirsten, 2018):

- Understanding the problem situation represented in the proving task,
- Identifying (informal) arguments that support the purported assertion,
- Structuring single proving steps in a logical chain,
- Formulating a proof according to the requirements of the community.

Taking the process of proving from the problem-solving perspective, it is reasonable to add a fifth sub-process, that is, checking the proof's validity (Carlson & Bloom, 2005; Schoenfeld, 1985). Although validation might be implicitly included in the sub-processes listed above, it seems useful for teaching and research to make validation visible in a separate sub-process. In fact, the assumption of a total of five sub-processes proved to be empirically valid in initial studies examining university students' proving processes (Kirsten, 2018). To that effect, there is empirical evidence that the five sub-processes display basic elements of proof construction and constitute an individual's proving process. However, there is limited research on how these sub-processes interact in proof construction and how, i.e., in what order and weighting, they can be fruitfully combined.

Proving strategies of university students

Proving strategies describe behavioural patterns that are characteristic for a proving process' overall structure. A proving strategy is thus characterised by features such as dealing with impasses, incorporating metacognitive processes, or planning an approach based on understanding (Zazkis et al., 2015). Examining the problem-solving behaviour of college students, Schoenfeld (1992) identified the *wild goose chase* strategy as a frequently used but highly inefficient approach to proving: "roughly 60% of the solution attempts are of the 'read, make a decision quickly, and pursue that direction come hell or high water' variety" (p. 356). Consistent with this, Karunakaran (2018) reports that first-year students often adhere to the linear structure of their proving process and therefore have difficulty with overcoming impasses. Both strategies reported here are likely to fail, because little time is spent on carefully analysing the problem situation or students do not adapt their process to the impasses occurred. Against this background, Zazkis et al. (2015) explicitly analysed the proving behaviours of six highly successful university students with reference to Pólya's four stages of problem-solving. Their analyses revealed two promising strategies, namely the *target strategy* and the *shotgun strategy*. The target strategy describes a highly systematic behaviour, where students chose their approach on the basis of a careful problem analysis, and continuously monitor their own progress. Students, on the other hand, who use the shotgun strategy, implement many different approaches in a short period of time. They spend little time on understanding the given statement or setting up a plan, but adjust their approach as soon as it proves to be of little use. In contrast to the wild goose chase approach, the shotgun strategy thus includes a higher level of process control and is, therefore, considered to be more effective (Zazkis et al., 2015).

Combining both frameworks, it can be assumed, that each proving strategy is reflected in a characteristic combination of sub-processes. For example, the target strategy proceeds in a straight line, while the shotgun strategy moves back and forth between identifying arguments and validating.

The current study

Summarising previous findings, process-oriented research provides initial evidence that proving processes consist of at least five different sub-processes, each of which makes its own contribution to proof construction. How students combine these sub-processes though proving strategies, however, has been studied scarcely, and predominantly in relation to episodes of problem solving (Schoenfeld 1985; D. Zazkis et al. 2015). To address this research gap, the current study aims to provide a holistic analysis of student proving processes that examines the interplay of individual sub-processes.

Research Questions

The current study pursues two main goals: first, the study aims to identify recurring proving strategies, each of which is characterised by a specific combination of sub-processes in terms of frequency, duration and sequence. Given the findings of Schoenfeld (1992) and Zazkis et al. (2015), it is reasonable that some proving strategies are more purposeful than others. Therefore, in a second step, we investigate the relationship between proving strategies and students' proof construction performance. Here, the aim is to identify effective and non-effective aspects of proving strategies in order to replicate and enhance previous findings. In particular, the research is guided by the following questions: (1) Which proving strategies do university students use in proof construction? (2) Is there a relationship between the proving strategy used and the students' performance? To address these questions, the study relies on an exploratory-descriptive design that allows fine-grained analyses.

Participants and data collection

To gain insights into students' proof-construction behaviour, the author conducted task-based interviews with first-year university students, including preservice teachers (Goldin, 1997). As part of a regular exercise, students were asked to complete the two proving tasks given in Figure 1. If students agreed, one of these tasks was solved under interview conditions. As is customary in the course, students were allowed to work in groups of two or three. In the interview, however, they received no support other than a standardised version of lecture notes (Schoenfeld, 1985). Following this procedure, the author and one of her colleagues conducted a total of 97 interviews, all of which were videotaped. From this total sample, we selected those cases for fine-grained analysis that were deemed representative of the entirety of cases. For this purpose, we first assessed the students' performance in proof construction by rating their final product on a four-point scale, with a score of 4 indicating a complete and valid proof and a score of 0 describing no substantial progress (Recio & Godino, 2001). Interrater reliability confirmed very good agreement for this rating ($\kappa = .82$). Based on the assessment of proving performance, the final sample was drawn by applying a qualitative sampling plan that follows the principles of maximum variation and homogenous sampling (Patton, 1990). Thus, the final sample includes 24 proving processes that are almost evenly distributed among the different performance scores and that cover both proving tasks equally.

Task 1 Let $f : [0,1] \to [0,1]$ be a continuous function. Show that f has a fixed point, that is, there exists a $x \in [0,1]$ with $f(x) = x$.

Task 2 Consider a function $f : \mathbb{R} \to \mathbb{R}$ whicht is twice differentiable. Let $x_1 < x_2 < x_3$. Given $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$, show that there exists a $y \in \mathbb{R}$ with $f''(y) \geq 0$.

Figure 1: Proving tasks used in the tasked-based interviews

Data analysis

In accordance with the research questions, data analysis was conducted in two stages. First, all proving processes were fully transcribed and coded leaning on Schoenfeld's (1985) method of protocol analysis. The proving protocols were partitioned into coherent episodes, each of which is assigned to one of the sub-processes listed above (Kirsten, 2018). The coding thus led to a macroscopic description of the students' processes, providing data on the frequency, duration and sequence of the sub-processes that occur (κ = .92). Based on this information, an empirical type construction was conducted in which processes with similar qualitative characteristics are grouped into types (Kluge, 2000). In particular, the transitions between sub-processes were considered in order to identify recurring sequences and empirical patterns that represent specific proving strategies. Each proving strategy thus describes a particular combination of sub-processes in terms of sequence and frequency. To identify particularly promising proving strategies, the second stage of analysis compared successful (score 3 or 4) and non-successful cases (score 0 or 1) in terms of strategy use.

Results

Our analyses of 24 proving processes revealed five different proving strategies that first-year students apply in the field of real analysis. The following section gives an overview about the proving strategies and its characteristics. Each strategy is illustrated by an exemplary process and its schematic representation.

Proving strategies and sub-processes used by first-year students

Proof constructions using the *step-by-step strategy* are characterised by a linear structure in which the individual sub-processes are run through in the natural sequence listed above. Doing so, all sub-processes are systematically built on one another, that is, each sub-process is worked through conclusively before the students move on to the next. In this strategy, the proving process is based on a careful analysis of the given statement and professes throughout. In particular, the choice of a proving approach is well-planned and highly target-oriented.

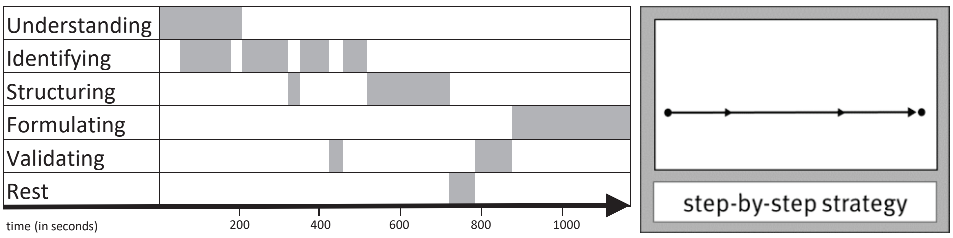

Figure 2: Exemplary sequence of sub-processes when using the step-by-step strategy (Alina & George)

The *scorekeeping strategy* is represented by proving processes in which a complete sequence of sub-processes is followed by an additional loop of brainstorming. This loop occurs rather accidentally and is initiated by a formulation process in which the students write down a response that is not satisfactory for them but serves to achieve partial points in the exercise. While formulating the response, a process of insight takes place and the students generate new ideas for proof construction. However, the students do not pursue the new approaches consistently and therefore end proof construction with a process of identifying arguments (see the case of Lisa, Pia & Laura in Figure 3).

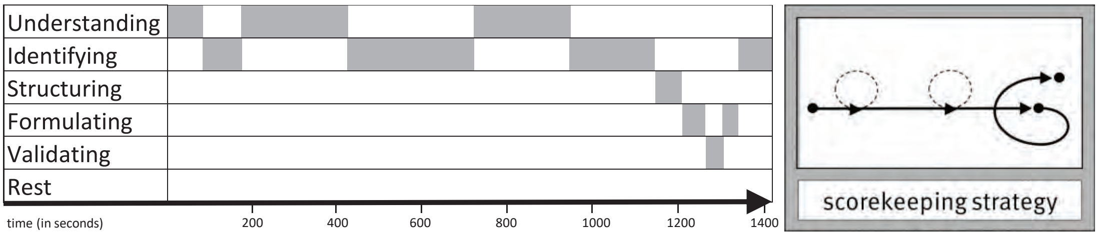

Figure 3: Exemplary sequence of sub-processes when using the scorekeeping strategy

Proving processes that are assigned to the *rebooting strategy* are characterised by a cyclical structure in which several approaches are implemented one after the other. Each cycle contains a linear, self-contained sequence of sub-processes such as understanding, identifying (informal) arguments and organising single proving steps. Thus, the rebooting strategy initially follows the step-by-step-strategy until the current approach proves to be of little use and a new cycle is initiated. The validation and rejection of an approach necessitates a restart which is characteristic for this strategy.

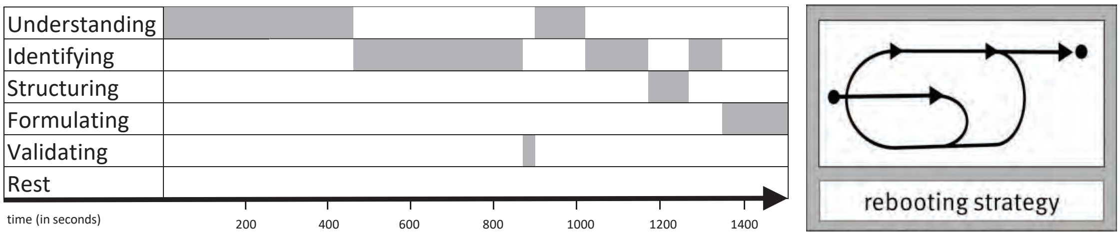

Figure 4: Exemplary sequence of sub-processes when using the rebooting strategy (Lukas & Tim)

When applying the *decomposing strategy*, a proving process shows a repeating pattern of development, elaboration and formulation, which is depicted in a staircase pattern in the schematic representation of the process. In contrast to the rebooting strategy, each cycle of development contributes to the final product, so there is a continuous progress. In our sample, students use the decomposing strategy in two different ways depending on the structure of the statement to be shown. Some proving tasks allow, for example, a distinction by cases which naturally divides the proving process into different parts. By considering sub-proofs, the complexity of proof construction is reduced and each part can be dealt with in a self-contained step-by-step approach (see the process of Andreas & Ibrahim in Figure 5). If no content or structural subdivision is created in the task, students perform a chronological division by writing down previous results from time to time. In some cases, formulating parts of the final response allows new insights and thus supports further progress.

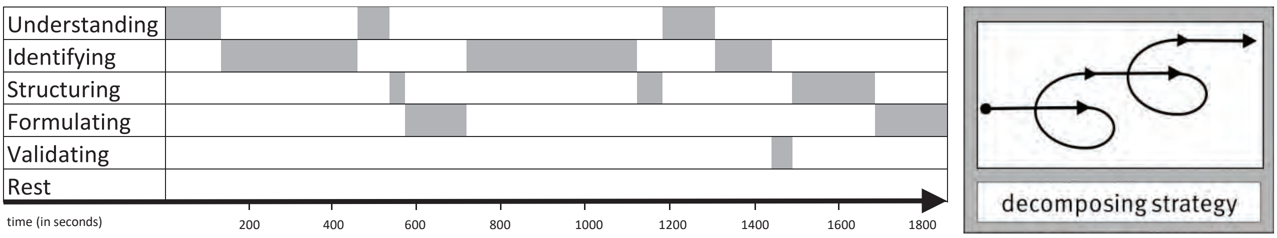

Figure 5: Exemplary sequence of sub-processes when using the decomposing strategy

The *looping strategy* describes a proof-writing behavior in which students frequently move back and forth between sub-processes. Due to an occurring uncertainty or impasse, students first return to the previous sub-process and then continue with their proving process based on the newly gained knowledge. Since the switches mainly occur between successive sub-processes such as understanding and identifying or structuring and formulating, the looping strategy basically follows a linear approach but is interspersed with mini-cycles. In contrast to the step-by-step strategy, the looping strategy thus provides several deepening loops. Doing so, the individual sub-processes are worked out in interaction with each other and are thus more closely linked.

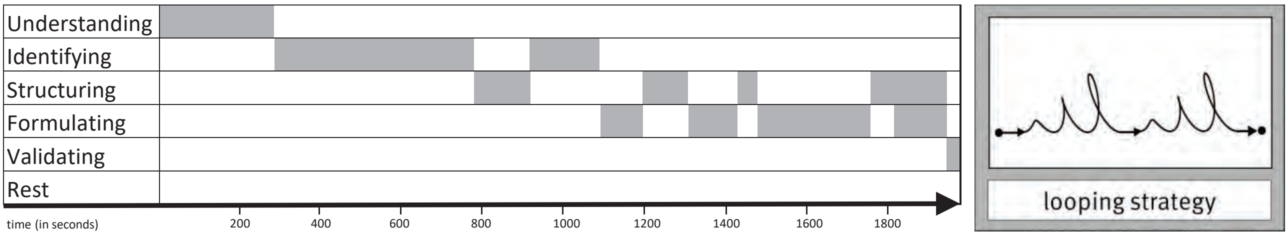

Figure 6: Exemplary sequence of sub-processes when using the looping strategy (Alina & Sascha)

Comparing the different descriptions, a basically linear structure emerges across all strategies. While the step-by-step strategy represents an idealized linear procedure, the decomposing and the rebooting strategies divide the process into linear subsequences and the looping strategy uses mini-cycles to deepen individual sub-processes within a mainly progressive procedure. Due to these modifications, each strategy places a different emphasis on the individual sub-processes.

Proving strategies and performance

To identify particularly effective strategies of proving, we compared successful and non-successful processes regarding the proving strategies used. Table 1 shows how the examined cases are distributed among the different strategies and performance categories. Each letter combination represents one proving process and refers to the initials of the students involved.

Performance	step-by-step	rebooting	decomposing	scorekeeping	looping
successful	ML, AG		FT		AS, TL
non-successful		LT, DP	AI, SDH	TH, LS, JH, OJ, LPL	MFY, JIA, LUM, LKH, DJ

Table 1: Distribution of the analysed processes among the different strategies and performances (successful = score 3 & 4, non-successful = score 0 & 1, score 2 excluded)

The distribution in Table 1 shows that with the step-by-step strategy, the rebooting and the scorekeeping strategy proof-writing behaviours exist, which are exclusively associated with a low or high performance. However, the observed pattern is only conditionally suitable for a hypothesis generation. While the low proving performance of scorekeeping is already anchored in its definition, the rebooting strategy describes an approach that, from a theoretical perspective, does have effective components because an unsuitable approach is rejected in favour of a new idea. A detailed review of the assigned cases suggests that the difficulties encountered here arise less from restarting, but rather are the result of a lack of conceptual, procedural or strategical knowledge. The decomposing and the looping strategy, on the other hand, occur across different scores. It is likely that these behavioural patterns are not effective per se, but that it is the specific implementation that matters.

Discussion and conclusion

In the presented study, we analysed the proof construction behaviour of first-year university students by looking at the frequency, duration and sequence of sub-processes typical for proving (Kirsten, 2018). Our research revealed five different proving strategies that replicate and complement previous findings. The step-by-step and the rebooting strategy are comparable to the target and the shotgun strategy reported by Zazkis et al. (2015). In addition, we identified two other strategies, namely the decomposing and the looping strategy, which can be considered as variants of the target strategy. Both strategies proceed target-oriented, but include several cycles of sub-processes in order to reduce complexity, handle impasses or create flexibility. The scorekeeping strategy is similar to wild goose chase processes in the way that students do not make progress over an extended period of time (Schoenfeld, 1985). Nevertheless, the students' efforts in this study go beyond a wild goose chase as there is a slight process control through the formulation of ideas. One possible reason for these differences is that proof construction in the university context is often associated with performance. Formulating partial solutions is a common strategy for coping with studies in the first year. Efforts in teaching should therefore aim at making this kind of strategy fruitful by valuing the late insights. In general, fostering programs such as heuristic worked examples, heuristic trainings, and learning videos should consider different types of proving strategies to provide authentic insights into proving.

Comparing successful and non-successful students, we could not establish a clear correlation between proving strategies and performance. Specifically, our results show that a proving strategy in which different approaches are implemented in the sense of the shotgun strategy is not necessarily associated with a high level of performance. As the study by Zazkis et al. (2015) only considers successful students, our results usefully complement the state of research and open the view for further aspects. On the one hand, it remains open whether proving strategies reflect an individual’s disposition or whether they are task-, situation- or proof-specific. On the other hand, our results suggest that students might fail because they have difficulties with strategy implementation. Further research should therefore analyse students’ proof-construction behaviour at a microscopic level. Especially those strategies that occur independently of performance should be analysed with particular care, as they can provide insights into effective and dysfunctional aspects of implementation.

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Capitalizing on interdiscursivity to support primary school students to bridge the empirical-deductive gap: The case of parity of numbers

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Previous research has made calls for interventions to assist students in bridging the empirical-deductive gap. We respond to this call in a larger commognitive study that involved small groups of primary school students who discussed parity of natural numbers with a teacher-researcher. As part of this study, a learning environment was created based on the construct of interdiscursivity – elements of task design that afford students to draw on and expand their familiar empirical approaches in situations where deductive arguments are needed. The study aims to explore how interdiscursivity can support students in taking their first deductive steps. In this paper, we analyze the activity of two 8-year-olds to learn how the interdiscursive elements contributed to their discursive growth. The central finding pertains to the activity of the teacher-researcher, who played a key role in realizing the interdiscursive potential of the designed environment.

Keywords: Proof and proving, primary school students, discourse, commognitive framework

Introduction

Research has shown that primary school students are capable of engaging in proving and generating valid mathematical arguments (Stylianides, 2007). However, a wealth of research corroborates what Stylianides and Stylianides (2017) describe as "key and persistent problems" (p. 124)—students' reliance on a few confirming cases as proof and the difficulty of transitioning from empirical to deductive arguments. To address these problems, Stylianides and Stylianides (2017) call for studies into classroom-based interventions, highlighting a particular need for research at the primary school level. The researchers foreground three desired characteristics of such studies: an explanatory theoretical framework to identify key features to which the impact of the intervention could be attributed, a narrow and well-defined scope, and an appropriate mechanism to trigger and support changes in students' learning to prove.

In her doctoral research, Knox (2021) has been addressing this call by using the commognitive framework, focusing on parity of natural numbers, and designing a learning environment to support primary school students in bridging the empirical-deductive junction. This paper presents a snippet of this research. We begin with the commognitive conceptualization of this junction, describe the designed environment, and analyze two students' collaborative activity in this environment.

Commognitive framing of the study

The commognitive framework (Sfard, 2008) views *mathematics* as a historically established discourse, when *mathematics learning* is construed as a lasting change in the ways one participates in this discourse. A discourse about natural numbers (or a *numerical discourse*) is in our focus, when we distinguish between its two versions—*empirical* and *deductive*. The main objects of the empirical discourse are specific numbers, and it can be characterized with *keywords* that signify them verbally (e.g., “six”), *mediators* that point at them visually (e.g., the symbol “6”, Numicon tiles in Figure 1), *narratives* about them and their properties (e.g., “6 is even”), and *routines* that involve them (e.g., substantiating that 6 is even). Deductive numerical discourse enables communicating about numbers without specifying their quantities. This communication becomes possible through the usage of such keywords as “an even number” and such symbolic mediators as “2n + 1”. Deductive numerical discourse also allows for generating universal statements, such as “the sum of two odds is even”.

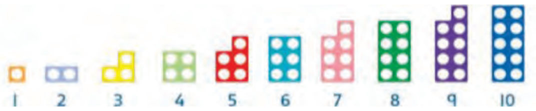


Figure 1: Numicon tiles

In commognitive terms, the deductive version of numerical discourse *subsumes* the empirical one since everything that can be said and done in the latter holds in the former. But deductive discourse affords much more. This is useful in situations that can be handled within both discourses. For instances, the narrative “3 + 5 is an even number” can be substantiated through either executing the addition and consequent arguing that the sum is even, or through resorting to the universal narrative stating that “odd + odd = even”.

The deductive numerical discourse summons specific ways for substantiating universal narratives. Here, showing that such a narrative holds for particular numbers is usually not viewed as an acceptable substantiation. Moreover, the deductive and empirical numerical discourses differ in their usage of some keywords. For instance, in the former, “odd” and “even” are conventionally interpreted as “all odds/even” or “the set of odd/even numbers”; in the empirical version, these words act as adjectives that describe properties of specific numbers. In commognition, discourses that differ in their underpinning rules and the use of the same keywords are called *incommensurable*.

Like many theories informed by Vygotsky’s work, commognition suggests that learners’ transitioning from a familiar into a new and incommensurable discourse, requires the support of a more knowledgeable other. The commognitive stance has traditionally assumed that at such junctions, a learner’s motivation to adopt the rules of a new discourse would initially be to meet the approval of the teacher and the learner would focus on repeating the steps of a shown procedure (*ritualistic participation* in terms of Sfard, 2008). Yet, the ultimate goal is for the learner to eventually realize the productiveness of the new discourse and participate in it for themselves.

Cooper and Lavie (2021) argue that carefully designed tasks can support learners to by-pass the ritualistic stage in their transition between incommensurable discourses. By drawing on the much-explored construct of intersubjectivity (e.g., Wertsch, 1984), Cooper and Lavie (2021) propose *interdiscursivity* as a discursive mechanism whereby learners may be eased into a new and incommensurable discourse via “the blending of discursive elements from different discourses” (p. 1). Specifically, the researchers hypothesize that a change in learners’ routines might be facilitated by tasks that capitalize on elements of the learner’s existing discourse in such a way that these elements “simultaneously [hold] meaning in an emerging discourse” (p. 8). Accordingly, in this paper we address the research question: “How can interdiscursivity support primary school students in taking their first steps in a deductive numerical discourse?”

The study

To pursue our research aim, a set of *concept cartoons* (Keogh & Naylor, 1999) with three characters was designed, each with a speech bubble featuring contradicting statements on parity. In each cartoon, two statements were universal, while the third statement foregrounded the "sometimeness" of the previous statements (see Figure 2); we refer to the statements of the latter type as *bilateral*. In empirical numerical discourse, refuting one statement with a counterexample would be considered sufficient to endorse the other statement without the need to substantiate it for all relevant cases. Hence, the inclusion of bilateral statements was part of the interdiscursive design to support students' move away from empiricism.

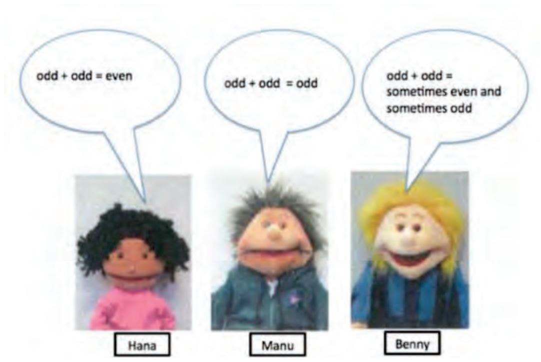


Figure 2: The "odd + odd" concept cartoon

Another element of the designed environment were *generic strips* – long and folded paper bands with both ends exposed (Figure 3). Generic strips bear an interdiscursive potential in the sense that they can visually mediate numbers in both empirical and deductive discourses. In the former, the strips can stand for specific numbers, in a way which is not very different from Numicon tiles (Figure 1) that the students were familiar with from their school studies. On the other hand, the strips can also be treated as a generic odd or a generic even number, by focusing on the exposed edges and visually concealing the precise number of dots by folding the strip. We envisaged the introduction of the generic strips might provide the students with the material means for communicating about numbers and their parity in a nonspecific way. For instance, by drawing on the students' familiar deeds with Numicon tiles, they may be able to use the generic strips to show that "*nonspecific* odd number + *nonspecific* odd number = *nonspecific* even number".

Figure 3: Odd and even generic strips

Twenty-eight Year 4 students (8- and 9-years old) from two New Zealand schools participated in the larger research. The participants were selected by their teachers to be withdrawn from their class to work in groups of four with the first author as a teacher-researcher (*TR*). The group work was video-recorded with two cameras and students' written artefacts were added to the data corpus. Each group session was transcribed in its entirety and analyzed with commognitive tools to identify instances of students' discursive development (i.e., changes in the use of keywords, visual mediators, narratives, or routines). The question underpinning the analysis was whether and how the designed interdiscursive potential of the designed environment was realized to support the students in their substantiation of universal statements in the deductive numerical discourse.

Findings

In this paper, we present a case of a single group, which was selected due to its affordances to illustrate the role of the interdiscursive environment in students' learning to substantiate universal statements. We provide excerpts to show three episodes where the group made their first steps towards deductive discourse: (1) facilitating dissatisfaction with an empirical discourse, (2) interdiscursivity in action, and (3) the students' uptake of these new discursive features in a deductive substantiation. The collected data affords focusing on two students in the group: Jane and Zara.

Prior to the first episode, Jane and Zara, classified numbers presented as Numicon tiles and written numerals as odd or even. Here, they had provided substantiations that referred to the generic structure or properties of odd and even numbers. For instance, they used the terminology of "in twos" and "doubles" to substantiate that 6 is even. When substantiating that Numicon 5 is odd, Zara said, "Instead of adding two on, you add on one and then it wouldn't be even… every time you have an even number it has to be in twos." The students' attention to numbers' structures thus placed them favourably for considering the structure of generic odd and even strips.

Dissatisfaction with an empirical substantiation of a universal narrative

In this episode, TR has just introduced the "odd + odd" concept cartoon to the students.

240	Jane:	Yeah, that's right [referring to Hana's narrative "odd + odd = even"]. […]
242	TR:	What about Manu ["odd + odd = odd"]?
244	Jane:	It's wrong. […]
246	TR:	Whereas Benny says…
247	Jane:	Odd plus odd equals sometimes even and sometimes odd [reading Benny's statement in the cartoon].
248	Danny:	No wrong. See watch this [holds up two Numicon tiles of "3" and fits them together to make "6"]. 
249	Jane:	I sometimes think that's right [referring to Danny's demonstration in 248] because, um it might be— 'cos three and three—but what if there's another number and we add them together? […]
253	Jane:	So, because an odd and odd, like a three and three, equals odd [probably meant even] but if you put it together, they're a small number... But some odd numbers are like really big—we don't, we can't even count them—they might not, they might be odd but can't go even." […]
282	Zara:	Okay, so two odd numbers can sometimes make an even number.

In what appears as an attempt to convince Jane that "odd + odd = even", Danny visually mediates the even sum using two Numicon tiles of "3" [248]. This is an implementation of an empirical substantiation since it shows that the universal statement is correct for this example without accounting for all pairs of odd numbers. Jane, in turn, appears dissatisfied with Danny's choice of "small" numbers rather than with him resorting to an example in the first place. On the other hand, her use of the word "sometimes" [249], mimics the bilateral narrative and may be interpreted as a realization that Danny's demonstration does not account for the universal scope of the statement, and especially for numbers that are "really big, we don't, can't even count them" [253]. This utterance provides an opportunity for TR to offer discursive tools for students to communicate about numbers of this sort as part of ushering them into deductive discourse. Notably, the window is not open indefinitely as Zara seems to capitalize on Jane's doubt [253] to endorse the bilateral statement [282].

Interdiscursivity in action

To take advantage of Jane's uncertainty, TR introduces an odd generic strip.

298	TR:	Is it an odd number or an even number [holds up an odd generic strip]?
299	Jane:	Odd. […]
303	TR:	So, Jane, you think it's an odd number. […]
305	Zara, Jane:	[together] Yeah.
306	TR:	How do you know it's an odd number?
307		[Jane, and Zara point to the unpaired circle at the end.]
308	TR:	What are you pointing at Jane?
309	Jane:	'Cos, that. It's like that [traces the unpaired circle].
		[Then counts the circles visible on the top layer] It's nine. And nine's an odd number.
310	TR:	It's nine. Did you need to count it to recognize it's odd?
311	Jane:	No.
312	TR:	What were you looking at, that showed you it's odd?
313	Jane:	Cos if it's like that, it would be even [covers the unpaired circle with her finger]. But since there's one more [removes her finger].
314	TR:	Ah, so it's like the extra one that makes it odd, is that right?
315	Jane:	Yeah. […]
328	TR:	Would that be an odd number [points to the even generic strip]?
329	Jane, Zara:	[together] No.
330	TR:	No. How did you know straight away?
331	Jane:	Because there're no extra bits [points to the ends of the even generic strip].
332	TR:	Ah [holds up the folded odd generic strip]. Okay so we'd need two that kinda look like this [i.e., two odd generic strips to represent "odd + odd"]? Zara, are you happy with that?
333	Zara:	[nods.]
334	Jane:	Yes and … there's the same amount of dots on each one [side], [points to the pairs on the even generic strip] but that one there [points to the "extra one" on the odd generic strip], there's one more on one side. […]

This episode presents the full span of moves that TR undertakes to capitalize on generic strips and equip the group with discursive tools to communicate about generic (rather than particular) odd and even numbers. She starts with introducing each strip as a number that is either odd or even [298; 306; 328] and presses the students to engage with the structural features of the strip and draw on them in their substantiations. In [309], we see Jane harking back to empirical discourse, aiming to base her substantiation of the strip's oddness on counting; just as she did a short time ago with specific numbers. Then, TR steers Jane away from counting [310] towards substantiations based on the strip's edges [310, 312, 330]. Notably, TR does not enable students to "get away" with just pointing at the structural features of the strips but she asks them to generate verbal narratives [308; 312; 330]: she also rephrases students' narratives and models substantiations [314].

This teaching seems to work for Jane and Zara as their substantiations gradually change to become in tune with the rules of a new "discursive game". Indeed, the students cooperate with the underpinning assumption that a long strip, not all parts of which are visible, stands for a number, and that its parity can be determined by edges only [299; 305; 307; 311]. Jane shifts from providing gestural responses to communicating verbally [e.g., from 307 to 309]. Moreover, she does not only abandon her counting substantiation [309], but also independently generates substantiating narratives through introducing new phrases ("extra bits" [331], "same amount of dots", and "one more" [334]).

The students' uptake of new discursive features in a deductive substantiation

In the following episode, TR prompts the group to draw on the previous conversation regarding a second nonspecific odd having "one more" [Episode 2, 334], to substantiate "odd + odd = even".

335 TR — So … She [Jane] said that with odd numbers there's one more and with another odd number is there going to be an extra one again? […]

337 Jane: — …if it was like three then we could put it there [picks up Numicon 3-tile and connects it with the odd generic strip].
And then it's not gonna be like that and there's gonna be one left no more [puts the Numicon tile down and runs her finger around the "extra one" of the strip].

338 TR: — Is it like a jigsaw puzzle so they kind of fit together?

339 Jane, Zara: — [together] Yeah. […]

341 Zara: — Yes, so if you have something like a square. If you have something like this [draws a rectangle].

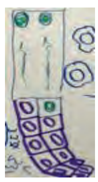
342 Jane: — A rectangle.

343 Zara: — Yes, it's an oblong. So, if you have like two circles on each then it will be even [draws two circles in the rectangle]. And just keep on going down [draws two lines going down from each of the circles]. But if you added on an extra one here, then it wouldn't be even [draws the extra circle at the bottom-right]. […]

345 TR: — Can we use another colour to show how another odd number would fit with that [referring to Zara's drawing in the previous episode, line 343]?

346 Jane: — Yes [picks up a pen to add onto Zara's drawing].

347 Zara: — So, then if you put like another one there ["one" is taken here to mean another "odd"] then it would be even [referring to her



drawing]. [Jane draws one extra purple circle in a square to form a rectangle and continues to add pairs of purple circles to extend the rectangle.]

In this episode, we see changes in Zara and Jane's communication. First, they now endorse the "odd + odd = even"-statement, which marks a shift from where they were in the first episode. Second, the episode captures a development in their substantiations. Initially, Jane chooses a numeric example (Numicon 3-tile) to "add to" the odd generic strip, and, by running her finger around the exterior of the latter, she visually mediates the generic "even" rectangular shape created by the two odds. This gesture can be viewed as Jane divorcing from the "counting-the-dots" approach that was previously evident [309]. In turn, Zara introduces the generic word "square" [341] and instantly replaces it with Jane's "rectangle" [342]—two nouns that had not been previously used by TR. These nouns, coupled with Zara's sketches [341-343] offer even more abstract realizations of even and odd numbers than the generic strips. Zara's narratives underscore that the structure of the sketched rectangle determines its parity (i.e., "if you have like two circles on each side" and "if you added an extra one"), when the precise number of paired circles is irrelevant and can "just keep on going down". This may be interpreted as a marker of her first independent steps in the generic talk on numbers' parity that is necessary for deductive discourse.

Note that TR still has a role to play in leading students' discourses. She asks the students to draw "another odd number" [345] so that the two figures "fit together…like a jigsaw puzzle" [338]. Here, she signals to the students that they can use the structural features of odd (i.e., "an extra one") to substantiate their endorsement of "odd + odd = even", rather than perform a calculation.

Summary and Concluding Remarks

This paper offers a snippet of Knox's (2021) larger research into young students' first steps in a deductive version of a numerical discourse. This research contributes a New Zealand case to the international body of knowledge on how young students generate deductive arguments to substantiate universal statements (e.g., Stylianides, 2007). Moreover, with commognition as its explanatory theoretical framework to identify key features to which the impact of the intervention could be attributed, a narrow and well-defined scope, and with interdiscursivity as a mechanism to trigger and support changes in students' learning; this research specifically addresses Stylianides and Stylianides (2017) call for research into proving at the primary school level.

Knox (2021) takes Cooper and Lavie's (2021) theoretical proposal to an empirical test drive by investigating how interdiscursivity can be mobilized to assist students to by-pass ritualistic participation in a new discourse that is incommensurable with their familiar one. The two distinctive features of the designed environment are bilateral narratives and generic strips. In the case of a single group of students, we illustrated how the bilateral narrative played a role in triggering one student's dissatisfaction with empirical substantiations with "small numbers" and interest in accounting for numbers that are "really big […] we can't even count them". This interest opened the door for introducing a way of talking about parity that does not rely on counting. Generic strips were designed to have meaningful interpretations in an empirical and a deductive version of the discourse. These artefacts laid the grounds for providing students with the discursive apparatus to communicate about generic odds. In the analysis, we highlighted the roles that these features had in the development of Jane and Zara's substantiations of "odd + odd = even".

It seems unlikely that the potential of the two abovementioned features would be fully realized without the guidance of the knowledgeable other. In accord with many researchers (Cooper & Lavie, 2021; Sfard, 2008; Stylianides, 2007), our analysis stressed the crucial role of the teacher in facilitating students' discursive developments. Conceptualizing the empirical-deductive gap as an instance where substantial discursive change is required, teachers' actions emerge as a key feature of an interdiscursive environment. To put it differently, we suggest that pre-designed features can set up the stage for interdiscursivity, but it is the teacher who can make the environment interdiscursive.

The presented case of Jane and Zara was relatively successful and concise, but it is silent about the other two members of the group. Furthermore, Knox's (2021) larger research showed that some students' journeys were longer and more complicated, while some did not seem to reach the target. Moreover, even when some students appeared to participate in a midway discourse, they still often fluxed between it and the more familiar empirical arguments. Hence, metaphorically speaking, interdiscursivity does not appear to us as a train that takes learners from an empirical to a deductive station, but a bridge on which students can walk in both directions. Nevertheless, we propose that interdiscursivity may be of interest for further research to explore how students can be assisted in bridging the empirical-deductive gap.

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Building up Professional Knowledge on Fostering Primary Students' Data-based Argumentation – an Intervention Study

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Engaging students in argumentation based on statistical data is seen as an important aim of statistics education from primary school on. For being able to provide related learning opportunities to students and to help students to further develop in data-based argumentation, teachers need corresponding professional knowledge. As prior research had shown that teachers often do not have such professional knowledge, we investigated in a quasi-experimental intervention study with N = 57 student teachers whether it is possible to build up professional knowledge on fostering data-based argumentation. The results provide evidence for the efficacy of the intervention approach and point to implications for its implementation in future courses.

Keywords: Argumentation, professional knowledge, statistical literacy, awareness, intervention research.

Introduction

Fostering students' argumentation is emphasised frequently as a key element of the mathematics classroom. In the past decades, a wide body of research into different aspects of teaching and learning of argumentation (for an overview see e.g. Stylianides et al., 2016) has emerged, for instance with a focus on engaging students in argumentation (e.g. Bezold, 2012; Jablonski & Ludwig, 2021) or on teachers' professional knowledge related to fostering students' argumentation (e.g. Ayalon & Nama, 2021). However, argumentation based on *statistical data* has been largely unexamined so far, in particular with a specific focus on children in primary school. Addressing this gap in research, a series of studies (e.g. Krummenauer et al., 2020; Krummenauer & Kuntze, 2018) has been conducted. The findings provided evidence that argumentation based on statistical data is possible for many students in primary school and that students' data-based argumentation can be fostered in the primary mathematics classroom. However, creating learning opportunities related to students' data-based argumentation requires that teachers have corresponding professional knowledge; an awareness of possible learning opportunities related to data-based argumentation can be expected to be a key element of such professional knowledge. First empirical research (Krummenauer & Kuntze, 2021a, 2021b) has implied that many primary school teachers do not have such an awareness, which points to a need for research on how a basis of professional knowledge related to fostering students' data-based argumentation can be build up during student teachers' university education. The intervention study reported on in this contribution addresses this research need by evaluating the efficacy of an intervention approach with a particular focus on raising student teachers' awareness of learning opportunities for fostering primary students' data-based argumentation.

In the following section, we present the theoretical framework of the study and give a brief overview of existing research. Subsequently, we specify the research questions and outline the design of the intervention study, before presenting results and discussing implications as well as limitations of the study.

Theoretical Framework

Data-based Argumentation as a Key Idea of Statistics Education

Within the wide body of literature on statistics education, the relevance of processes of argumentation for dealing with statistical data is highlighted frequently. Ben-Zvi and Garfield (2004), for instance, describe a key goal of statistics education as "Being able to properly evaluate evidence (data) and claims based on data […]" (p. 3). Gal (2002), to give only one further example, defines two core elements of statistical literacy as "[…] people's ability to *interpret and critically evaluate* statistical information, data-related arguments, or stochastic phenomena […]" (p. 2) and " […] their ability to *discuss or communicate* their reactions to such statistical information […]" (p. 3). Although the importance of argumentation for dealing with statistical data is emphasised in numerous publications, there has hardly been any systematic research with a specific analysis focus on primary students' data-based argumentation. In response to this research need, a series of interrelated empirical studies (e.g. Krummenauer et al., 2020; Krummenauer & Kuntze, 2018, 2021c) has been conducted, specifically focusing on primary students' data-based argumentation, which we consider as a specific case of argumentation in which statistical data is used for substantiating certain statements (e.g. Krummenauer et al., 2020). A first key finding of these studies was that many students in primary school are able to evaluate data-related claims and can develop corresponding data-based arguments in appropriate task contexts. Further, the studies have shown that primary students' data-based argumentation can be fostered effectively in the mathematics classroom. However, in order to foster students in further developing their data-based argumentation in the mathematics classroom, it is necessary that teachers provide appropriate learning opportunities to their students, such as appropriate statistical data and corresponding impulses facilitating processes of data-based argumentation; for this, teachers need corresponding professional knowledge.

Teachers' Professional Knowledge on Fostering Students' Data-based Argumentation

Such professional knowledge may include, for instance, knowledge about possible learning scenarios (e.g. what kind of data sets could be provided to students as a basis of argumentation), knowledge on how to initiate processes of data-based argumentation in the classroom, or knowledge about possible difficulties of students as well as corresponding possibilities of supporting students' learning (see e.g. Krummenauer et al., 2020; Krummenauer & Kuntze, 2018). Beyond only possessing (potentially inactive) knowledge, the extent to which teachers are able to *use* such professional knowledge can be assessed when teachers *analyse* in profession-related requirement contexts, such as classroom scenarios or responses of students (Kuntze & Friesen, 2018). The term *analysing* has been defined by Kuntze, Dreher, and Friesen (2015) as an "awareness-driven, knowledge-based process which connects the subject of analysis with relevant criterion knowledge and is marked by criteria-based explanation and argumentation" (p. 3214).

Consequently, whether or not professional knowledge is actually used by teachers in their analysis largely depends on teachers' *awareness* of relevant criteria (Kuntze & Friesen, 2018). As defined by Kuntze and Dreher (2015), we consider criterion awareness as "a part of professional knowledge which influences the readiness and ability of teachers to use related professional knowledge elements in instruction-related contexts" (Kuntze & Dreher, 2015, p. 298). Related to fostering students' data-based argumentation, this includes, for instance, to be aware of potential opportunities for students to engage in data-based argumentation through textbook material or in classroom situations. As described in the model by Kuntze and Friesen (2018), teachers' criterion awareness is crucial for processes of analysing, as it can activate relevant professional knowledge for teachers' analysis. Conversely, a lack of criterion awareness would have the consequence that a teacher would not apply corresponding professional knowledge in the analysis (and in subsequent actions based on such an analysis), even if the teacher held relevant knowledge (Kuntze & Friesen, 2018). Therefore, a particular focus on fostering teachers' awareness related to students' data-based argumentation appears to be a promising intervention approach with high relevance for teachers' classroom practice.

Existing Research on Professional Knowledge related to Fostering Students' Argumentation

Several studies exist with a focus on student teachers' and in-service teachers' professional knowledge related to fostering argumentation in the mathematics classroom (e.g. Ayalon & Nama, 2021; Bleiler et al., 2013; Brunner, 2019). As reviewed by Stylianides et al. (2016), many of these publications report on weaknesses in teachers' professional knowledge in this regard. This is also the case in a prior empirical study with $N = 44$ in-service primary school teachers (Krummenauer & Kuntze, 2021a), in which the participants did not show an awareness of learning potentials related to data-based argumentation in most of their analyses.

Further, there are some publications reporting research on building up professional knowledge in in-service and student teachers related to argumentation. Brunner (2019), for instance, reports on an intervention with kindergarten teachers. The author describes a positive development of their professional knowledge during the intervention and concludes that it is possible to successfully establish argumentation practices through professional development activities. In contrast, Melhuish and colleagues (2020) report on a long-term intervention study with the aim of fostering teachers' noticing related to argumentation processes, which had only small effects on teachers' professional knowledge. The authors highlight, that they found a large discrepancy between teachers' self-reports and teachers' actual performance regarding their argumentation-related noticing.

To our knowledge, however, there are hardly any studies with a specific focus on building up professional knowledge related to fostering primary students' *data-based* argumentation during teacher university education. It is therefore largely unclear what effects can be reached by means of an intervention focussing on building up awareness of learning opportunities related to data-based argumentation and to what extent possible effects are stable after the intervention.

Research Questions

Consequently and responding to this research need, the study reported in this contribution addresses this research need with a particular focus on the following research questions:

RQ1: Is it possible to increase student teachers' awareness of learning opportunities related to fostering primary students' data-based argumentation by means of an intervention?

RQ2: How stable are possible intervention effects?

Methods

Study Design and Sample

The study is designed as a quasi-experimental intervention study with three points of measurement. The first two tests were conducted right before and after the intervention, which was implemented in two university courses with different course formats: the first course had a regular, weekly seminar format with weekly sessions of 90 minutes, whereas the second course was conducted as a compact course taking place on 4 days within a period of 7 weeks. As a third group, a non-treatment control group was included in order to control for possible re-test effects. In the intervention groups, the intervention was implemented in 3 parts (each approx. 90 minutes, excl. testing time). Approximately five weeks after the intervention, a follow-up test was administered to the participants of the intervention groups.

The sample consisted of $N = 57$ student teachers; $n = 28$ of them participated in the weekly course, $n = 10$ in the compact course, and $n = 19$ in the control group. Most participants were in their third semester (about 70 %; average: $M = 4$ semesters). About 50 % of the participants studied mathematics as a minor subject, the other half of the sample studied mathematics as a major subject.

Data Collection

For assessing the effects of the intervention, a questionnaire instrument was used, which had been already successfully used and validated (inter-rater reliability: $\kappa = .84$) in a prior study (Krummenauer & Kuntze, 2021a). The questionnaire consists of eight items, which are different types of vignettes to be analysed by the participating student teachers. The vignettes represent different kinds of profession-related requirement contexts (e.g. Buchbinder & Kuntze, 2018) which require teachers to be aware of learning potentials related to data-based argumentation in order to generate appropriate learning opportunities for students. In a dichotomous, criteria-based top-down rating, it was analysed whether the student teachers' analyses indicated that the participants were aware of learning potentials related to fostering data-based argumentation. For being rated as "successful analysis", an answer had to contain a reference to relevant aspects of the data contained in the vignette, and a corresponding learning potential related to data-based argumentation had to be described. Further details about the design of the instrument are described in Krummenauer and Kuntze (2021a, 2021b).

Design of the Intervention

In accordance with the theoretical framework introduced above, the main focus of the intervention is to build up awareness of learning potentials related to fostering students' data-based argumentation. In the intervention, this was combined with building up practical knowledge on possible learning environments, such as knowledge about what kind of data can be used for data-based argumentation in primary school, or what impulses can be given by teachers in order to engage students in data-based argumentation.

In the first part of the course, the student teachers were asked to develop data sets which they consider as suitable for primary students and to develop corresponding impulses they would provide to their students (e.g. questions or claims to be evaluated based on the data by students). This first part of the course is intended to activate and formatively assess the prior knowledge of the participants, and to gain a basis for the further seminar work. In a subsequent step, the data-sets and corresponding questions were discussed in the seminar according to their learning potential for students' data-based argumentation. After a presentation phase focussing on characteristics of data sets which facilitate data-based argumentation and possibilities of providing stimuli to engage students in data-based argumentation, the course participants jointly discussed and further developed their data-sets and corresponding stimuli in several steps. After a joint discussion of the further developed material, further classroom material was presented and discussed with the participants. The participants were asked to think of corresponding stimuli (and, if necessary, to modify the material) in order to develop awareness of learning potentials related to fostering data-based argumentation in different contexts.

Results

Figure 1 shows the development of the mean percentage of answers rated as a successful analysis for each group. The performance in the pre-test of all three groups does not differ significantly from each other, which was assessed by means of a Kruskal-Wallis test ($H(2) = 2.331$, $p = .312$). The spread was also relatively low ($SD_{\text{pre-test}} = 0.93$), so that all three groups entered the study with a relatively low and relatively equal awareness related to learning potentials to foster students' data-based argumentation.

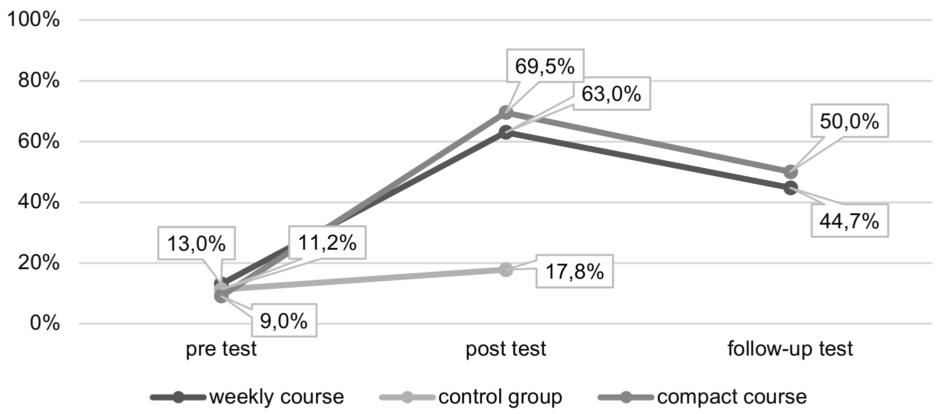

Figure 1: Sub-group comparison of mean percentages of answers rated as a successful analysis

In order to assess whether the changes between pre-test and post-test within the groups are statistically significant, we analysed the difference of answers rated as a successful analysis between pre-test and post-test ($\Delta_{t0,\ t1}$) by means of a further Kruskal-Wallis test. The test revealed significant group differences ($H(2) = 49.398$, $p < .001$). Corresponding post-hoc tests showed that both intervention groups differed significantly from the control group (weekly course: $Z = 19.291$, $p < .001$; compact course: $Z = 21.840$, $p < .001$). The small increase in the control group is statistically non-significant ($Z = -.504$, $p = .614$). The effect size between pre-test and post-test is $d = 4.66$; this large effect size parameter (Lenhard & Lenhard, 2016) has to be interpreted against the relatively low and homogeneous ($SD_{\text{pre-test}} = 0.93$) pre-test performance. In the follow-up test, the effect size dropped around one-third from $d = 4.66$ to $d = 3.19$.

In order to further explore the individual development of the participants of the intervention groups, we further analysed the differences between pre-test and post-test ($\Delta_{t0,\ t1}$) (see Figure 2) and between the post-test and follow-up test ($\Delta_{t1,\,t2}$) (Figure 3). As shown in Figure 2, the participants solved in the post-test between one and seven items (out of eight items) more than in the pre-test (M = 3.9 items). From the post-test to the follow-up-test, the performance of 58.1 % of the participants of the intervention groups was reduced (between 1 and 6 items); 29 % of the participants in the intervention groups had the same test result, and 12.9 % solved one item more than in the post-test (Figure 3).

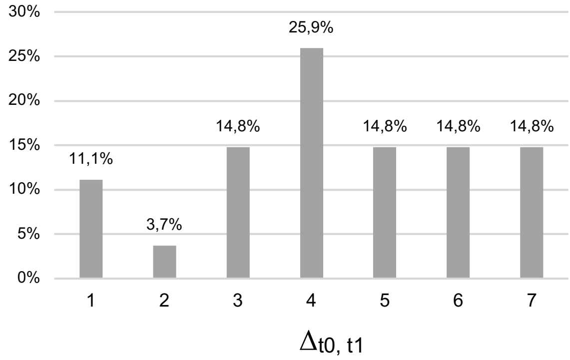

Figure 2: Individual development ($\Delta_{t0,\,t1}$) of the participants of the intervention groups

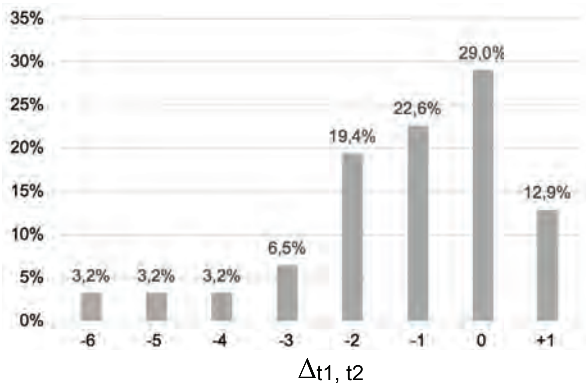

Figure 3: Individual development ($\Delta_{t1,\,t2}$) of the participants of the intervention groups

Discussion and Conclusions

Before discussing the key results of the study, we would like to consider limitations of the study. A first limitation is that the sample of student teachers is not representative for a larger population, and it was not possible to randomise the assignment of the participants to the different groups, so that generalisations to other groups of students need to be reflected carefully. Additionally, although the overall sample size is satisfactory, the size of the sub-group of the compact seminar is relatively small, which needs to be considered when comparing the results of the sub-groups.

Regarding the first research question, the results have shown that it was possible to increase student teachers' criterion awareness in the domain of fostering data-based argumentation substantially in both courses by means of the intervention. The intervention implemented in the compact course appears to be similarly effective as the implementation in the weekly course. The effect size of the intervention corresponds to a large effect (e.g. Lenhard & Lenhard, 2016), and all students improved at least for one item after the intervention. Regarding research question 2, the results show that the effect of the intervention decreased about one-third in effect size by the follow-up test a few weeks after the intervention. This implies that there is still a substantial effect, but that the effects of such an intervention should be stabilised over a longer period, which could be addressed in future implementations of the intervention. However, the decrease of the performance pertains not the whole intervention group, as about 40 % of the student teachers participating in the intervention successfully analysed the same or even a higher number of items in the follow-up test.

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Children's norms of explanations in explanation videos

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Explanation videos are becoming more popular in mathematics education. Most of them are made by teachers who are explaining mathematical topics. In this project, primary school students create their own videos addressing other children who have the same age. This paper will give a brief introduction into the project and first observations of children's norms of explanations in explanation videos will be presented.

Keywords: Explanation videos, norms, explanation, (meta-)discursive rules, primary school

Introduction

In recent years, digital media, such as explanation videos, have moved into the focus of mathematics-education research (Ball, Ladel & Siller, 2018). Explanation videos (e.g., Kulgemeyer, 2020 in physics), video podcasts (e.g., Kay, 2014 in mathematics in university) or instructional videos (e.g., Sharma, 2018 in mathematics in college) are primarily used by teachers to impart knowledge in order to explain mathematical content. So far, there have been few empirical studies showing that learners create videos themselves. However, initial studies emphasize that the independent production of explanation videos has a positive effect on learning mathematics (Kay, 2014). In this paper, a project is presented in which the learners themselves create so-called *explanation videos* (Kulgemeyer, 2020) for other learners in their age. For this purpose, an interview study was carried out with children from the third and fourth grade of primary school (aged 8-10 years). In the interviews, they work in tandems on tasks which contain arithmetic tasks. Based on these tasks, they then plan and produce a video for other learners their age. According to Prediger, Erath & Opitz (2019) learning mathematics is led by language. Learners negotiate mathematical content through language (Voigt, 1995). Tiedemann (2017) shows that the use of language in mathematics lessons is also guided by *norms*. Erath (2017) points out that explaining is a practice which is characterized by regularities. Experts for explanation videos have also set up criteria for explanation videos (e.g., Kulgemeyer, 2020). But which rules and norms do children use in an explanation video? One request of the project's research is to elaborate *rules* (Sfard, 2008) and *norms* (Voigt, 1995) which the learners use while explaining mathematical content during the planning and producing of the videos.

(Meta-)discursive rules and norms

Human interaction is shaped by rules (Sfard, 2008). For instance, when you greet a person, they return it by greeting back. Or, if you get a present, you say "thank you". The examples given are based on rules of courtesy. As Voigt (1985) pointed out classroom interaction is also led by patterns and routines. Sfard (2008) distinguishes between two sorts of rules: "object-level rules" (pp. 201f) and "meta(-discursive) rules" (ibid.). Object-level rules are based on the mathematical content, like mathematical rules and definitions (ibid.). Metarules relate to the adaptation in the respective situation. According to Sfard (2008) they are usually not formulated explicitly, but take place in classroom conversations:

> More generally, object-level rules are narratives about regularities in the behavior of objects of the discourse, whereas metarules define patterns in the activity of the discursants trying to produce and substantiate object-level narratives. […] Metarules are also made distinct by being mainly tacit, and by being perceived as normative and value-laden whenever made explicit. (pp. 201f)

Sfard (2008) points out that metarules could become norms by referring to the works of Voigt (1995) and Yackel & Cobb (1996). Rules and Norms have in common that they both lead to an interaction with observable regularities (Sfard 2008). But they differ due to their commitment (ibid.). Due to Sfard (2008) for a rule to become a norm, it must fulfill two conditions: The rule must be widely known within the respective social group. And the rule must be accepted within the group. In this study, interviews are carried out with tandems of the same school class (aged 8 to 10). The interviewer and the children can be viewed as a social group. Furthermore, the children come from a social group, their school class. Thus, the first condition is fulfilled if both of the children are listening. The second condition is fulfilled if both of them accept the rule. In other words, if one student establishes a rule and the other student does agree to the prior statement, the rule becomes a norm.

Explanations and Explanation Videos

Explanations and their importance are often emphasized in mathematics education (e.g., Erath, 2017; Hanna, 2000). To describe a scientific *explanation*, Hempel & Oppenheim (1948) highlight the indispensability of causality. For example, in the “Discovering Pac” (Nührenbörger et al., 2019, p. 76) in Figure 1 (left) learners could observe the phenomenon that the result of both calculations is 12. It could then be questionable why the result is 12. Thus, the equality of the sums has to be explained. It can be explained by, for example, changing the first term by minus two and changing the second term by plus two (see ibid.). This change makes it clear that the tasks are the same and therefore also have the same result ((10 - 2) + (2 + 2) = 8 + 4). The underlying law here is the constancy of the sum ($\forall\ a, b, x \in \mathbb{N}: (a - x) + (b + x) = a + b$). The stated explanation is using an example and a general form, the equality of totals to explain the equality of the sums.

Following Kulgemeyer (2020) *explanations videos* are “short videos (usually between 5 and 10 min) with the goal to explain a particular content area” (p. 2444). In this sense the intention of *explanation videos* is to explain a mathematical content. Kulgemeyer (2020) points out a “framework for effective explanation videos” (p. 2450) for teachers who teach physics to plan and optimize their explanation videos. Similarly, Kay (2014) formulates “key components” (p. 24) for video podcasts. In terms of Sfard (2008) one can say that these topics are norms for explanation videos in the context of physics education. As Erath (2017) pointed out explanations are also led by implicit regularities. But which norms of explanations do children establish while planning explanation videos? First observations will be elaborated at the empirical part of the paper.

Methodology and Methods

The research concern of this article is to reconstruct and categorize students’ norms of explanations in explanation videos. As part of a first pilot, eight semi-standardized interviews were carried out with four tandems each from the third and fourth grade of primary school (students at the age of 8-10) whose learning level in learning mathematics are similar to each other. Three sessions (à 45 minutes) were held with each tandem. At first, they learn how explanation videos are constructed to get familiar with the method:

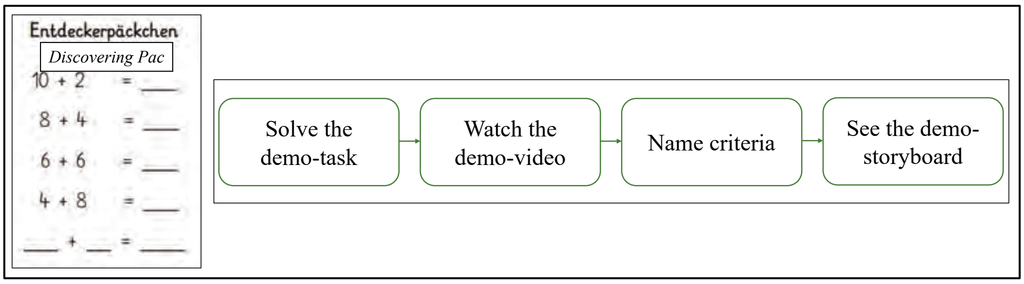

Figure 1: Left: The demo-task; Right: Procedure for introducing explanation videos.

Primarily, they solve a Discovering Pac (see Figure 1, left), which is familiar to them. Subsequently, they watch an explanation video of this task to emphasize the most important facts about explanation videos (e.g., only the hands can be seen, images and writings are used). This task was chosen because it is not too demanding in terms of the mathematical content so that the learners focus ever more on the method. Nevertheless, the task discovery and explanation. Then they get a *storyboard* in which the images and the spoken text of the explanation video are planned (see Figure 2). The interviewer explains that the pictures are outlined and the text of the corresponding pictures is prescribed.

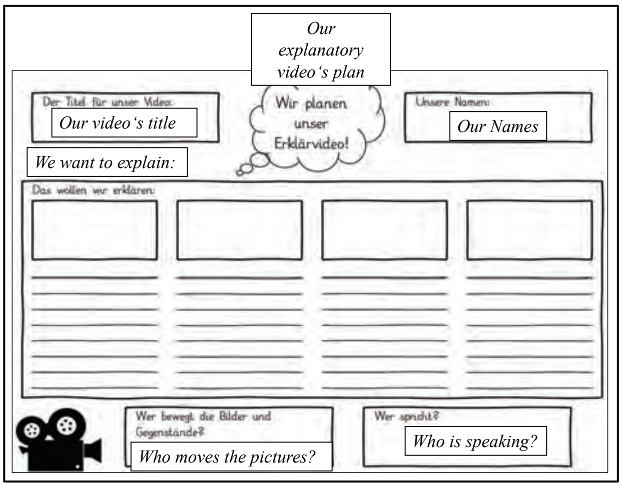

Figure 2: The storyboard, which the students use to plan their video

After this introduction they solve the first task. The learners are asked to discover and explain the mathematical content together. The mathematical tasks differ between the third and fourth grade because of the tasks' complexity.

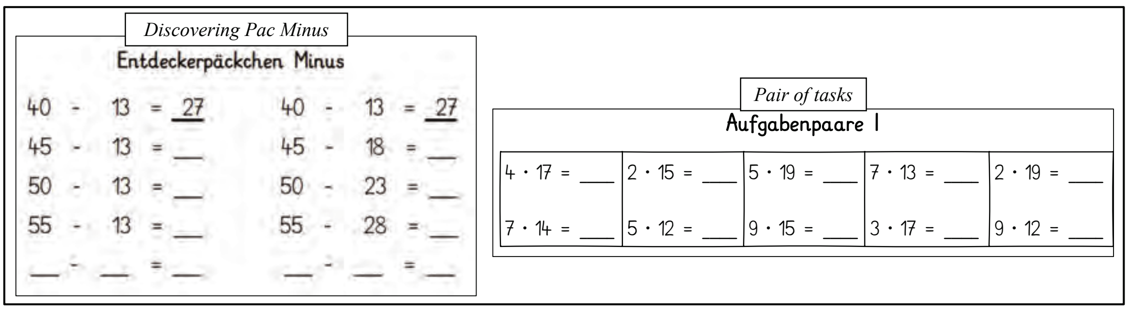

Figure 3: Left: Second task for the third grade; Right: First task for the fourth grade

After the editing, the tandems write a storyboard together to plan their video. The learners are told that they would record their video for children their age who are learning at home because of the Covid-19 pandemic. The video should help the fictional children to solve such tasks. After planning the video, the learners record their video and revise it at their own request. This first shooting cycle, consisting of developing the content, writing the storyboard as well as the video production and revision, was repeated by solving a similarly structured task. The purpose of the repetition is to focus more on the content, not on the method.

In this paper children's norms of explanations in explanation videos will be categorized. The qualitative analysis is based on an interactionist point of view (see Voigt, 1995). The rules and norms are reconstructed with regard to Sfard (2008). The interviews were videotaped and transcribed (rules of transcription, see Kunsteller, 2018). The following students' quotations were translated and transcribed by the author.

Empirical examples

First observations of children's norms of explanations in explanation videos will be expounded. Due to the shortness of the paper, they will only shortly be stated. For a better understanding, only excerpts of interviews by two tandems are shown. Nevertheless, the shown norms could also be confirmed in other interviews. The given examples do not claim to be a complete list of norms. Consequently, they can be expanded. Furthermore, they could be similar to the criteria that are set on explanation videos (e.g., Kulgemeyer, 2020).

Norms concerning the video's layout

After Hülya and Yao (third grade) watched the demo-video they had to tell if the person in the video explained the mathematical content well. Both agreed and Hülya added:

53	Hülya:	"yes she explained that well ... I would make the writings a little bigger' because if you always take a camera like that, you don't always see the writings so well."

Hülya's statement can be interpreted that if a person explains a content well the presentation of the explanation is also important. Since Yao does not disagree, they implement a norm which concerns the video's layout. It can be argued that an explanation in an explanation video should not only be good in terms of content but also in terms of layout. Hülya could mention this because the content could also be influenced by legibility. Norms of this kind concern the external shape of the explanation video. For instance, in a different scene, Hülya and Yao wanted the numbers to be written more properly. Nevertheless, these norms have an influence on the quality of the explanation. Yet, this seems to be relevant for the children. One reason could be because they have to learn to write correctly in primary school, so they talk about this while planning the video. It could be assumed that children would set this kind of norms as well for explanations in explanation videos in a different subject (e.g., biology) because the norm concerns aesthetics in a broader sense.

Norms concerning mathematics education

Once Hülya and Yao have processed the discovering pac of subtraction, they plan their video. First of all, they talk about the discovering pac which they use for their explanation. Meanwhile, the following situation arises:

727	Hülya:	"I think we could make an image with balloons (*brings hands together and apart*) because that <u>explains</u> it well too"
728	I:	"mhm. you could think about it."
729	Yao:	"or with candies"
(Afterwards they write the minuend of their discovering pac down)		
752	Yao:	"yes. and if we do this (*points at the third column of the storyboard, see Figure 2*) with balloons then it would be funny then three could pop. broken."
753	Hülya:	"(*laughs*) good"

Hülya says that they could use an "image with balloons" (727) to explain the content. She points out that the image would explain it "well" (ibid.). "It" (ibid.) could refer to the discovering pac of subtraction which they want to explain. Yao seems to like that idea and suggests to use "candies" (729). Since Yao agrees with Hülya's idea and even suggests a new idea, it can be argued that they are establishing a norm. That Yao likes her idea is also confirmed during the further process, when he picks up the example again and also extends it. Furthermore, Hülya laughs afterwards and classifies the statement as "good" (753), she seems to agree with Yao's suggestion. According to Hülya and Yao, an explanation in an explanation video should be funny and demonstrative. In addition, it can be assumed that they base their explanations on examples from math class. In German mathematics books, subtraction is often represented by the disappearance of objects (e.g., the bursting of balloons). This example could be used because they could assume that other children also know these examples. Consequently, their examples would ease the understanding of the content. Norms of this kind concern examples which are used in math books which are led by mathematics education's ideas. Furthermore, they implemented a norm that an explanation in an explanation video should be interesting or funny. As the interview progresses, the children develop their own discovering pac (see Figure 4 without the circles). At the end of writing the discovering pac, the following situation occurs:

909	Yao:	"should we *(shows at their own discovering pac)* circle it here so that the children could better- ... *"*
910	Hülya:	"<u>maybe</u>'"
911	Yao:	"should I do it'"
912	Hülya:	"<u>yes</u> but <u>please</u> use this color here *(gives him a turquoise pen. Yao circles the operands, see Figure 5, left)* […]"

Yao suggests to circle numbers in the discovering pac. One reason why he suggests the circles could be that the circles could simplify the understanding of the discovering pac for the other "children" (909). In Turn 912 Hülya agrees and suggests a special color for the circles. Due to the fact that she does not intervene how Yao circles the numbers she seems to agree with the way it does it. Thus, it can be argued that they implement a norm that circles and colors are important to express or underline an explanation in an explanation video. Like in the example this norm is led by mathematics education's ideas. In Germany it is common that students learn to use colors to highlight their discoveries (Nührenbörger et al., 2019). Furthermore, the highlighting is guided by the idea that the other children should understand their explanation. Thus, it can be claimed that they implement a norm that explanations in explanation videos should be understandable for (other) children. A different interpretation that the students are highlighting operands is that they could transfer it from the demo-video, in which the operands are outlined as well. In accordance with this interpretation, this would lead to the norm that components of the demo-video should be included in the video. Nevertheless, this norm is led by mathematics education's ideas. Since the students seem to identify this highlighting as relevant for their video and they could know it from German mathematics books.

In the following an image is shown which Hülya and Yao use in their video. The image is showing some learning materials made out of wood which are used in primary school to explain arithmetic tasks. Hülya and Yao use these materials to explain their discovering pac. Again, this usage underlines the fact that their explanations are led by mathematics education's ideas.

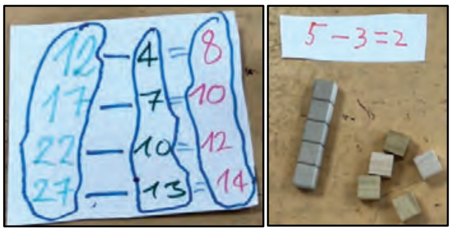

Figure 4: Extracts of images in Hülya's and Yao's explanation video

Norms concerning the mathematical content

Jasper and Bastian (fourth grade) want to explain the pairs of tasks (see Figure 3, right). They already decided to push the sheet of paper with the pairs of tasks in the focus of the video. Jasper had written the calculation for the first pair of tasks. At the end of writing the calculations he suggests:

460	Jasper:	"we could do that *(points to the first pair of tasks)* with pushing this in here *(points to his calculation),* this one without calculations *(points to the second pair of tasks)*, because actually they are simple, # this one *(points to the third pair of tasks)* again with"
461	Bastian:	" #yes but .. it doesnt matter if its easy or not, we should just explain it, for the difficult things, so that they understand you understand'"
462	Jasper:	"yes *(nods)"*

Jasper indicates that he wants to push his calculations of the first pair of tasks into the focus of the video as well. Then he states that the calculations for the second pair of tasks are not necessary because "they are simple" (460). At this point Bastian intervenes and states that not only "difficult things" (461) should be explained because it is important that the other children ("they", ibd.) understand their explanations. Since Jasper agrees (462) it can be argued that they implement the norm that even when the person itself understands the mathematical content it should be explained for another person. This example shows that you can explain or rather prove mathematical contents in different ways (Hanna, 2000) and that the acceptation of the explanation depends on the addressee (Heintz, 2003). Like the previous norm this kind of norm seems to be special for mathematics. Since it focuses different ways of explanations.

Statements and Outlook

Experts usually create videos with the intention that learners understand the mathematical content better. In this project, the children are the ones who explain the mathematical content. However, they are not experts, so the explanations are not necessarily complete and correct from a mathematical point of view. Nevertheless, the brief insight shows that the learners establish different norms on their explanations. Mostly, these norms are led by the desire that the other children understand their explanations. The following norms were elaborated:

- Norms concerning the video’s layout
- Norms concerning mathematics education
- Norms concerning the mathematical content

Tiedemann (2017) elaborates norms of the language in mathematics lessons. She points out that a “grammatical norm” (ibid, p. 51) is set when a fixed linguistic structure has to be used by the students. Similarly, to this grammatical norm the children underline the external shape or the layout for the explanation in the videos (e.g., height of numbers, readability). Still, these aesthetic norms are also guided by the wish that their explanation is better transported through the external shape. Likewise, Kay (2014) mentions the topics “readability” (p. 24) and “layout” (p. 24) for creating video podcasts as well. As already pointed out norms concerning the video’s layout could be named as well in other subjects. In contrast, the last two norms seem to be specific to mathematics or mathematics education. Norms concerning mathematics education accentuate that the children’s explanations are led by their mathematical socialization. Kay (ibid.) and Kulgemeyer (2020) also state the topic “highlighting” (p. 2450) but they use it in order to emphasize important aspects. Whereas the students use the highlighting of several numbers as a part of the explanation. It can be interpreted that the highlighting helps them to formulate their explanations. Furthermore, those norms give indications what seems to be a good explanation for the children. These indications as well as the norms for explanations will be elaborated in further publications. As a next step it is possible to investigate how students evaluate explanations of self-made videos by other students. By asking children if and why they like or dislike the video one could get a deeper insight of children’s norms of explanations in explanation videos.

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Student teachers' proofs and refutations on cyclic quadrilaterals

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We follow and analyze three teacher students' behavior while they engage in a task with the goal of the reinvention of the characterization of cyclic quadrilaterals in terms of their opposite angles. We show how, during the exploration of the problem, the teacher students go through cycles of proofs and refutations, in the sense of Lakatos, and we discuss the role of the teacher educator in supporting and conceptualizing these processes. Our results may have the potential to be used in designing mathematical activities aiming at scaffolding teacher students´ knowledge for teaching proof.

Keywords: Proof, counterexample, conjecture, task design, teacher knowledge.

Introduction.

Lakatos' *Proofs and refutations* (1976) marks the turn that Philosophy of Mathematics took towards fallibilism, the view that mathematical knowledge is corrigible and advances through a process of criticism and revision (Van Bendegem, 2018). Lakatos used the history of Euler's formula for polyhedra and its proofs as a basis for describing the road to mathematical discovery and at the same time presenting different views, over time, on what a satisfactory result and a valid proof consist of.

Besides its philosophical importance, Lakatos' work is also significant from a pedagogical point of view since it provides us with a window to original mathematicians' work presented in an educational setting. The idea of using Lakatosian notions in designing, implementing, and analyzing authentic mathematical activities is not new (Lampert, 1990). Several researchers have used Lakatos' ideas to develop frameworks that can be used for analyzing students' behavior in proving activities in various levels of mathematics education from primary (Komatsu, 2010), to secondary (Komatsu, 2012) and into undergraduate level (Larsen & Zandieh, 2008). Other studies have focused on designing activities that foster methods of proofs and refutations (Komatsu & Jones, 2017), or on teachers' mathematical knowledge for teaching Lakatosian proof (Deslis et al., 2021). Our study adds to this growing body of research-based literature first by providing some more examples of Lakatosian notions and processes, this time from teacher education, and second, by suggesting elements of instructional design aiming to build teachers students' mathematical knowledge for teaching Lakatosian proof. For the purpose of this article, we consider Lakatosian proof as it is described in (Deslis et al., 2021). However, discussed in the last section, authentic Lakatosian proving activities need to be further understood and defined. We seek to answer the following questions: 1) In which ways do teacher students' attempts to reinvent a characterization of cyclic quadrilaterals parallel the methods of Lakatos' proofs and refutations? And 2) How can teacher educators design and implement tasks that support teacher students' mathematical knowledge for teaching Lakatosian proof?

Theoretical background.

With starting point the naive conjecture "All polyhedra satisfy the formula $V - E + F = 2$", where V stands for vertices, E for edges and F for faces, and Cauchy's proof (Lakatos, 1976, p. 7), and after counterexamples are discovered, Lakatos described techniques for improving the conjecture and its proof. In this process crucial role is played by the type of counterexamples. Lakatos referred to counterexamples refuting the initial naive conjecture as *global* and to those refuting at least one step of the proof as *local* (ibid., p.11). According to the type of the encountered counterexample, a method was suggested that would lead to an improved conjecture or/and proof. We only mention here those methods that are needed for analyzing our findings.

One of the first methods, *exception barring,* was proposed when the global and local counterexample of a picture-frame polyhedron was discovered and was initially described as restricting the domain of the conjecture to exclude the counterexample, without reference to the proof (ibid., p. 28). As opposed to this, *lemma incorporation* was suggested as investigating the proof to identify the step which is refuted by the counterexample and incorporating it as a condition in the domain of the conjecture (ibid., p. 35.) Later in the discussion the exception barring method is generalized to include "a whole continuum of exception barring attitudes" (ibid., p. 39), according to how much the proof is used to determine the domain of the conjecture, with lemma incorporation being the limiting case. Lemma incorporation can also be used in the opposite way to expand the domain of a conjecture that has been previously restricted by exception barring. According to Lakatos this is the approach of the *best exception-barrers* and it involves starting with a restricted *safe domain*, devising a proof, examining the proof for imposed conditions that were not used, and generalizing the initial modest theorem by incorporating only those conditions that proof relies on (ibid., p. 40).

Mathematicians' views on *satisfactory* results are voiced via Lakatos' fictional student Omega at a point where different proofs, corresponding to different domains had been proposed, like Gergonne's (ibid., p. 63) and Legendre's (ibid., p. 64) proofs, none of which was including the *local but not global* counterexample of the great stellated dodecahedron (ibid. p. 65). According to Omega, "a proof should explain the phenomenon of Eulerianness in its entire range" (ibid., p.67). Omega suggests that when encountering a local but not global counterexample, it might be needed to invent a "completely different, more embracing, *deeper* proof" (ibid., p. 62).

Two opposite paths towards mathematical discovery are discussed in (Lakatos, 1976). The first one, which Lakatos calls *analysis,* involves starting with a naive conjecture, to which one arrives by trial and error, and testing it by drawing consequences (ibid., p. 80). This path we learn later, can be followed even without a conjecture to start with. One can pretend that "the result is there and device an analysis" (ibid., p. 83). The other direction, which Lakatos called *synthesis* (ibid., p. 81), is what we normally call a deductive proof. For this direction too, starting with a conjecture is not necessary. One can immediately "device a synthesis […] from a related proposition that is known to be true" (ibid., p. 83). In this way, synthesis corresponds to another term introduced by Lakatos, *deductive guessing,* as opposed to naive guessing, and to what Herbst (2004) refers to as building reasoned conjectures. As an aid to support students' deductive guessing, Herbst proposes the *generative mode* of interaction between a student and a diagram which involves, among others, drawing new features on the diagram, transforming its shape and size, and changing location or orientation.

Method.

The course and the participants.

This study was conducted within the context of a course titled Historical and Philosophical Aspects of Mathematics aimed at student teachers in their 4th year of a five-year master program for grades 5-10 in a Norwegian university. Our informants are three of the courses participants and have gotten the fictional names Marie, Lars, and Siri. Our data consists of their solutions to a task included in one of the course's mandatory assignments supplemented by semi-structured follow-up interviews. The assignment was given after the last teaching session, which had focus on proof and proving. During the session, after an introduction to Lakatos' proofs and refutations were concepts like conjecture, proof analysis, global and local counterexample, and their role in proving were presented, the students discussed how Lakatos' ideas could be used in the mathematics classroom and how a teacher can support pupils in engaging in Lakatosian proof. Their discussions were based on the papers by Komatsu (2012) and Larsen & Zandieh (2008) that the students had read in advance as preparation.

Task selection.

The task included in the students' assignments was meant to function as a starting point for involving them in genuine mathematical activity by reinventing a mathematical result. For this purpose, we considered the following characteristics that the target result should be: 1) new for teacher students but within reach using tools from school mathematics, 2) analogous to previously established result known to teacher students, and 3) valid under certain conditions – necessary or/and sufficient.

The characterization of quadrilaterals in terms of their opposite angles, which can be formulated as "A quadrilateral is cyclic if and only if its opposite angles are supplementary", has all the above characteristics. It is not normally included in the school curriculum in Norway, but its proof is based on the inscribed angle theorem which is. Also, our students are familiar with the circumcircle of a triangle which can serve as the motivation for investigating a similar question for quadrilaterals. Nevertheless, we decided to include this as the first part of the task, which was formulated as follows:

a. Prove that the three perpendicular bisectors on the side of a triangle meet at one point and that this point is the center of a circle that goes through the triangle's vertices.
b. Investigate if a similar result holds for quadrilaterals. State a conjecture and try to prove it.

Three students, all having delivered different solutions, were chosen for the follow-up interviews.

The interviews.

The three students were interviewed after the end of the course and before the exam. The interviews varied from 40 to 50 minutes and took place in a digital environment. The interviewers were the authors of the paper where the first author was, in addition, the teacher of the course. With the students' different solutions as a starting point, we intended first to get an insight on how each one had arrived at their solution and then try to stimulate them to continue the investigation and reach a deeper result. Following Lakatos' heuristic methods, we aimed to offer, or lead the students to discover, counterexamples to their claims or arguments, anticipating that this would lead them to modify the proposed claim or argument. The initial goal was to lead the students to formulate and prove the characterization of cyclic quadrilaterals as those that where opposite angles are supplementary. This goal soon showed to be too optimistic, so we ended all interviews once a necessary condition was conjectured by the students, namely "If a quadrilateral is cyclic, then its opposite angles are supplementary". As a tool to assist in exploring the problem, we used the dynamic geometry software Geogebra on a shared screen. In Marie's interview, the researcher controls the software and shares his screen, while in the other two interviews, the students do so.

Analysis.

In what follows, the first author is referred to as the teacher, the second as the researcher. The analysis is based on and structured over Marie's interview, but supplementary comments based on data coming from the other two students are also included.

Surrendering.

Marie presents in her solution a random quadrilateral with two circles drawn, each one going through two of the vertices. She concludes that "it will therefore not work to make a circle that goes through all vertices at the same time, but one can make a set of circles so that all vertices lie on one of them".

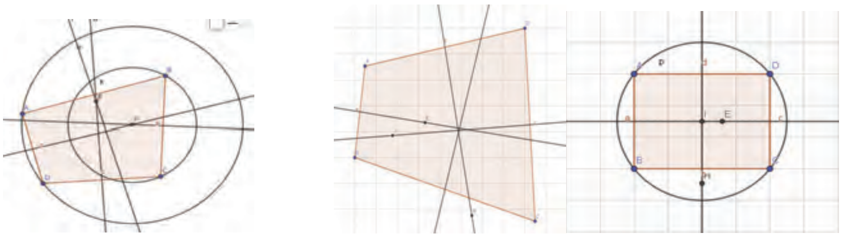

Figure 1: Marie's multiple circles and Lars' two examples

When asked if she had an initial feeling that led to her strategy the student replied that she thought it should not work (to have a circumscribed circle for a quadrilateral) since she had never heard of it. The drawing she presented was her first move towards dealing with the task and since it confirmed her suspicion, she decided to end the investigation following the method of surrender (Lakatos, p. 14). She seemed to think that this was enough for what the problem was asking: "I got a result and I thought, ok, I found it out".

Another student, Lars, that had managed to come a bit further in his investigation and had concluded that some quadrilaterals are cyclic while others are not, explained in the interview that if he was to go any further and try to find out what all cyclic quadrilaterals have in common, this should have been explicitly asked for as the next part of the problem. He also felt that his conclusion was enough.

The two ways of using lemma incorporation.

In the interview, Marie admitted that she could have worked more with the task, which was the starting point for us to trigger her to investigate a bit further. The teacher asked the student if she could think of any quadrilateral that could be inscribed in a circle.

Marie: If I had drawn a quadrilateral with right angles, and maybe equal sides, a square, then I would have gotten a center if I am right […] I just see that when the sides are equal you will get this diagonal's cross and you can have the center there.

Teacher: What is it that makes it work for squares? Can we find more classes?

Marie: All rectangles then? Because then you will get that diagonal's cross […] which is equally far [from the vertices]

Marie retreats to quadrilaterals with right angles and equal sides, which includes only squares, as the safe domain of her theorem. She argues that, in this case, the diagonals' intersection point can be used as a center for a circle that goes through all vertices. This argument can, of course, work for any quadrilateral having right angles, that is, any rectangle. It can thus be incorporated as a condition expanding the domain of the theorem. Marie manages to do so as soon as the teacher turns the focus on the argument. This process parallels what Lakatos (1976) described as a combination of exception barring and lemma incorporation.

Siri, another of our students, used lemma incorporation to restrict the domain of her claim. She had overgeneralized from rectangles the property that the perpendicular bisectors of opposite sides coincide and stated that "all parallelograms are cyclic" because of this reason. When encountering a local and global counterexample offered by the researcher - a random non-rectangular parallelogram - she restricted the domain to include this property as a condition ending up with only rectangles as the domain of the theorem.

More local counterexamples and the need of a deeper proof.

Recognizing that the argument concerning the intersection point of the quadrilateral cannot be used to include other classes than rectangles the educators suggest that we look at the problem in another way, namely starting with a quadrilateral that lies on a circle. All drawings and their manipulations are performed and shared by the researcher as the discussion goes on.

Researcher: If you start at the opposite end can you draw a quadrilateral that lies on a circle?
Marie: … (seems confused)
Teacher: so if you start by drawing a circle
Marie: then I can set two random…, if you want to have a rectangle then, parallel lines anywhere on the circle so that you get four intersection points

Marie aims at constructing an inscribed rectangle, but her instructions lead to a non-rectangular isosceles trapezoid. This is a local counterexample since it refutes her argument but still supports the conclusion of her theorem. At first, Marie gets surprised with the outcome and resists in expanding the domain of the theorem.

Researcher: Was this a rectangle now?
Marie: No, it wasn't ... (laughs)
Researcher: What does this tell us?
Marie: That it is possible to draw a quadrilateral in a circle. But it doesn't say anything about the center, the intersection point of the perpendicular bisectors.

The researcher draws the perpendicular bisectors to confirm that the intersection point coincides with the circle's center. This seems to confuse her even more.

Marie: Now I am so confused that I don't know anymore
Researcher: What did you get confused about?
Marie: About what I have done in the problem

The researcher explains the steps that led us from starting with a circle to drawing an inscribed isosceles trapezoid and eventually Marie accepts that the domain should be expanded.

Marie:	So, while it works for trapezoids and rectangles and squares, it is just for my odd quadrilateral that it doesn't work…that it doesn't have any parallel sides.

Here, there is a missed opportunity of offering or leading the student to discover another counterexample. Maria has not noticed that the trapezoids need to be isosceles. This could have led to focusing on investigating earlier and more naturally the properties of the angles of cyclic quadrilaterals. Instead, the researcher chooses to focus on the parallel sides' condition imposed by the student and draws a random inscribed quadrilateral with no pair of parallel sides to offer as a counterexample. The student gets surprised once more.

Marie:	It works here too
Researcher:	You seem surprised
Marie:	I am confused
Researcher:	Why are you so surprised? If you think what we have done until now…
Marie:	Here it makes sense because we have kept us in a circle, but when I tried to do it the opposite way it didn't work, or at least I couldn't make it work

Assuming that the conclusion is true for a random quadrilateral – that is, the quadrilateral is cyclic - and drawing consequences was particularly challenging for all three students that we interviewed. This heuristic method, which parallels what Lakatos called analysis (Lakatos, 1976, p. 80), can, in our case lead to discovering the necessary and sufficient conditions so that a quadrilateral is cyclic, and synthesizing the proof arriving in a final theorem. However, the student here does not seem to be aware of this possibility. Similar reactions were observed with the other two students we interviewed. The teacher educator, at this point, paused and explained how taking this direction can help us explore the properties that all cyclic quadrilaterals have in common.

Arriving to a conjecture by generative interaction with diagrams.

The researcher now sets it as a goal to describe all quadrilaterals inscribed in the same circle and asks the student how she would go on towards this goal.

Marie:	it was easy to start with the squares because all vertices are equally far from a center…but in such rare quadrilaterals it is hard to find a center.

Because of symmetry, it is easy to visualize the center for the case of squares and rectangles. The student knows that the diagonals' intersection point is equally far from the vertices of a square and reasoned from this to get her initial conjecture/theorem. Even with the diagram showing a "rare" quadrilateral inscribed in a circle the student hesitates to take further action.

At this point the researcher interacts with the diagrams to lead the student to arrive at a new conjecture. In what follows, we describe some of the actions the researcher took.

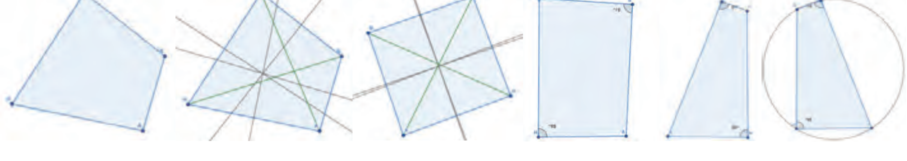

Figure 2: Generative mode of interaction

The researcher removes both the circle and circle's center from the drawing to move the focus away from the circle's center. To show to the student that the argument must be about something different from the diagonals the researcher draws the diagonals and the perpendicular bisectors and shows that it is only in the case of the rectangles that the intersection points of those coincide. To turn the focus to the angles, he marks and measures a pair of opposite angles, moves around the vertices, and lets the student observe what happens to the angles and make a hypothesis. After discussing what changes and what remains the same the student manages to arrive at the conjecture: "if a quadrilateral is cyclic their opposite angles are supplementary", which ended our investigation.

Although the mode of interaction with the diagram is generative, the way the student reaches the conclusion is rather empirical. The conjecture lays on observations, while the reason it works is still hidden and, to get closer to what Lakatos would call deductive guessing, one needs to make it visible.

Discussion and conclusions.

Several methods Lakatos (1976) described were observed during our students' explorations. A difference between our task and the problem the fictional students of Lakatos were investigating is that while for the Euler formula the number of supportive examples, together with what was then thought to be a polyhedron, made it natural to consider all polyhedra as the domain of the naive conjecture, in the case of quadrilaterals one quickly realizes that most of them are not cyclic. This resulted in our students restricting the domains of their theorems too much. Then, counterexamples like the isosceles trapezoid that Marie came upon or a random quadrilateral inscribed in a given circle, although local in the sense that they refuted her main argument, cannot be considered as non-global since they were not included in the domain of the conjecture/theorem. For the same reason, the method of analysis as exemplified in our investigation which lacks a broad naive conjecture, would be starting by assuming that a quadrilateral is cyclic and drawing consequences. Another deviation is that in Lakatos' classroom the counterexamples stem from the definition of a polyhedron and how this changed over time while theorizing. In our activity, this interplay between defining and proving, is absent. Although the criteria under which our task was designed were adequate for capturing some Lakatosian processes, authentic Lakatosian activities should also invite for exploring the role of definitions in proving and, in the opposite way, the role of proving in concept formation.

Our students' solutions showed that they stopped their investigation before reaching a mathematically satisfactory result. From the interviews, it became clear that presenting a partial solution like Marie's and Lars', as shown in figure 2, was experienced by the students as satisfactory enough for the purpose of the problem. Besides the students' reluctance to further investigate the problem, it was also revealed that they might lack the techniques for doing so. This became apparent when we suggested working with the problem in the opposite way, starting with a circle and drawing an inscribed quadrilateral to determine necessary conditions for a random quadrilateral to be cyclic. The students seemed confused and could not anticipate what this would lead to. These findings, call for negotiating the didactical contract and offering more tasks that demand taking on responsibility in mathematical discovery in general and in exploring necessary conditions in particular. The same conclusion regarding instructional norms is also supported by analyzing how students interacted with the diagrams. The dynamic software environment that we used during the interviews, and which the students also used spontaneously when working independently with the problem, proved to be a useful tool for finding examples, counterexamples, and possible arguments. But just offering this as a tool was not enough to build reasoned conjectures. As Herbst (2004) also concluded, norms have to be breeched in engaging students in generative interaction. A reformulation of the task may also be needed to assist students in deductive guessing. Last, conducting the study in a physical classroom environment may enable the students to interact more independently with the diagrams.

Our analysis gave insight into our students' mathematical knowledge for teaching Lakatosian proof, particularly the component Deslis et al. (2021) called knowledge of content. It revealed that the students could not always use examples in ways that would lead them to a satisfactory mathematical result, without further guidance. This, in turn, might translate into students' low competency in guiding pupils in using examples in productive ways, which is associated with the students' knowledge of teaching (Deslis et al., 2021). These observations suggest that besides exposing students to tasks that would naturally engage them in Lakatosian proof, instruction that aims to build their mathematical knowledge for teaching should also include discussions centered on specific Lakatosian techniques that can be followed after a supportive/counter example has been encountered.

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“I would (not) teach proof, because it is (not) relevant to exams”: changing beliefs about teaching proof

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Experts in mathematics education agree that reasoning and proof are essential and should be made central to learning mathematics. However, some school teachers tend to focus on procedural skills because of different beliefs unfavourable for the teaching of proof. To encourage teachers to teach proof, I developed and studied an intervention for preservice teachers in Hong Kong. In this paper, I report the findings of this study regarding changing Hong Kong preservice teachers’ beliefs about teaching proof, particularly their beliefs about the relevance of proof to examinations.

Keywords: Beliefs, intervention, preservice teachers, proof.

Introduction

Proof can play different important roles in learning mathematics, including, but not limited to, justifying, communicating, and explaining mathematical knowledge (Knuth, 2002). Consequently, many experts in mathematics education suggest that learning activities involving proof and proof-related reasoning should spread through school mathematics at different levels and in different content areas. Noting that teachers are the key decision-makers for what students will experience in classrooms, researchers have carried out studies on teachers’ proof-related knowledge and beliefs (e.g., Knuth, 2002) and have identified different difficulties that the teachers often have with (teaching) proof (e.g., Stylianides et al., 2013). Few intervention studies have been conducted to explore a resolution to these difficulties (e.g., Buchbinder & McCrone, 2020). To address the need to encourage teachers to teach proof, I conducted a 4-cycle design-based research study (The Design-Based Research Collective, 2003) that aimed to develop an intervention to change Hong Kong preservice teachers’ beliefs about teaching proof, through iterations of implementation, evaluation, and revision (McKenney & Reeves, 2014). In this paper, I report the findings of the fourth research cycle, focusing on the change in the participants’ beliefs about teaching proof, particularly their beliefs about relevance of proof to examinations.

The context: Hong Kong

Depending on different factors (e.g., classroom culture, curriculum & policy), students’ opportunities to learn proof can vary in different contexts. In Hong Kong, the place of proof in school mathematics is ambiguous. On the one hand, Hong Kong students seem to have more opportunities to be exposed to proof than their counterparts in other countries (Leung, 2005). For example, proof is introduced early in junior secondary mathematics, under the topics of “Pythagorean theorem” and “deductive geometry” in Year 8 (CDC, 2017). On the other hand, there are indications that proof has a marginal place in Hong Kong classrooms. The majority of the tasks that appear in textbooks or exam papers are non-proof-related and focus on students’ routine procedures, for example, applying formulae and solving equations. Few tasks focus on developing and/or assessing students’ proof and argumentation skills (e.g., Lee, 2021; Wong & Sutherland, 2018). Moreover, some Hong Kong teachers prioritise the development of students' skills in routine procedures over proof (e.g., Lee, 2019). The observed hesitation of Hong Kong teachers in teaching proof indicates that there is a need for interventions with (preservice) teachers. As the findings regarding the place of proof in the Hong Kong mathematics curriculum are also consistent with the results of studies conducted in other countries, it is believed that a study in developing an intervention that changes preservice teachers' beliefs about teaching proof is in the international community's interest.

Theoretical background

Beliefs refer to a statement of relation among ideas or objects that an individual holds to be true (Philipp, 2007), which might have influences on the way a teacher teaches mathematics (Furinghetti & Morselli, 2011). Beliefs about teaching proof are multidimensional. They include, but are not limited to, beliefs about the nature and roles of proof in mathematics and in education, beliefs about teaching, and beliefs about students (e.g., ability, motivation). Different dimensions of beliefs are interrelated and related to emotions, such as anxiety about producing proof and interest in proof; the more positively a teacher views proof, the more likely s/he values proof and provides students with related activities (e.g., Fraiser, 2010; Kotelawala, 2016). Research studies reported that many teachers agree on the importance of proof in mathematics but they often have difficulty in teaching proof because: (a) they, themselves, have cognitive and/or attitudinal difficulties with proof (e.g., Stylianides et al., 2017), (b) they believe that proof is not accessible to *all* students (e.g., Knuth, 2002), and/or (c) they believe that proof is not indispensable in school mathematics (e.g., Lee, 2019).

Preservice teachers enter teacher education programmes with pre-existing beliefs about mathematics and its teaching, which are often based on their previous learning experiences. Although beliefs are often considered to be resistant or slow to change, there is evidence (e.g., Yoo, 2008) that preservice teachers' beliefs about teaching proof can be changed after a programme, a course or even an intervention of relatively short duration, if they are provided with alternative learning experiences (Liljedahl et al., 2021). In this study, I developed an intervention that provided preservice teachers with alternative experiences of learning and teaching proof, which were different from the teacher-talk approach they had mainly experienced in the past.

Intervention design

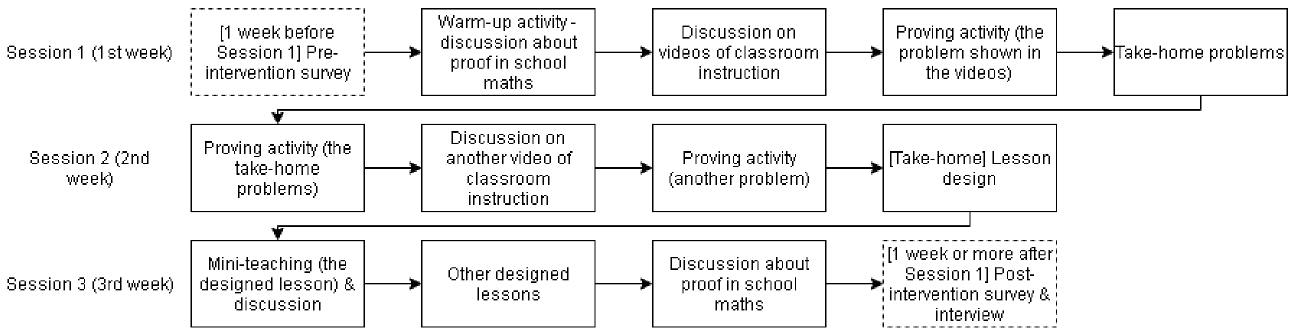

Figure 1: Design of the intervention

In this study, an intervention was designed for Hong Kong preservice teachers. The intervention consisted of three weekly extracurricular workshops. Each of the workshops lasted two hours. In the first three research cycles (July 2018–August 2019), the intervention design was trialled with different groups of preservice teachers. The trials enabled evaluations of the intervention design, informing revisions of the design. The intervention design in the fourth research cycle (Figure 1) differed from that in the first research cycle; in general, the participants (a) engaged in proving activities that emphasised learning mathematics through proof and argumentation, (b) discussed the role of proof in school mathematics, (c) watched and discussed videos of classroom instruction involving proof and argumentation, and (d) planned a lesson and trialled it with peers (Table 1).

Table 1: Examples of activities of the intervention

Type of activities	Example
Proving activities	What is the divisor of the sum of any *three* consecutive integers? What about *four*, *five*, and so on?
Discussions on the role of proof	When and how does proof appear in the previous task? What is the role of proof in this task?
Videos of classroom instruction	[*TIMSS 1999 Video Study – HK3 Polygons*] What did the teacher do in the video? Why? When did proof appear?
Lesson design and mini-teaching	Use what you have learnt in the workshops so far to develop a less on "Area of a circle"

Methods

The fourth research cycle took place between September and November 2019. The intervention was not conducted with one single group, but multiple small groups. Twenty-seven Hong Kong preservice teachers attended the intervention. Among them, 18 participants attended all sessions of the intervention whereas 6 attended only one session.

Table 2: Examples of questionnaire items

Type of items	Example
Likert items: Beliefs about the importance of proof	Making proofs improves mathematical thinking.
Likert items: Beliefs about the relevance of proof to examinations	Knowing how to make proofs is very important in excelling an examination/a test in school mathematics.
Likert items: Enjoyment and interest in proof	Making proofs arouses my curiosity.
Likert items: Anxiety about proof	I feel myself under pressure when I make proofs in mathematics lessons.
Likert items: Negative reactions when failing to produce proofs	Not being able to prove upsets me.
Open-ended questions	Should we implement tasks of proof and proving in mathematics class? Please briefly explain.

A questionnaire, consisting of Likert items and open-ended questions (Table 2), was designed, and used to gather information about preservice teachers' beliefs about teaching proof before and after the intervention. A subset of the teachers was invited to attend individual interviews after the intervention so that they could explain their beliefs about teaching proof, elaborate on their responses to the questionnaire and give their feedback about the intervention. The intervention was also audio-recorded and fieldnotes were made as supplementary data. Among all participants, 23 completed the pre-intervention questionnaire, 7 completed the post-intervention questionnaire, and 12 attended the post-intervention interviews. Pre- and post-intervention data were coded separately according to different dimensions of teachers' beliefs about teaching proof. The coded data were then compared to identify differences and, in turn, changes in their beliefs about teaching proof after the intervention.

More data were expected, but due to pro-democracy protests since June 2019 and the coronavirus pandemic since December 2019, several planned interventions were deferred and eventually cancelled. However, despite a reduced amount of data, analysis of the available data was sufficient to show the effects of the intervention on the participants' beliefs about teaching proof.

Results

I analysed 12 sets of pre-post data, indicating the corresponding 12 participants' beliefs about teaching proof and their changes after the intervention. In this paper, I first discuss the general findings in relation to the changes in the participants' beliefs about teaching proof, particularly their beliefs about the relevance of proof to examinations, then use selected data excerpts to illustrate the changes in the participants' beliefs about the relevance of proof to examinations and discuss how the intervention facilitated such belief changes.

In general, the participants developed more positive beliefs about teaching proof after the intervention. The analysis of the Likert items showed the participants maintained agreement that proof is essential to mathematics and developed more positive and less negative emotional dispositions towards proof, and more participants reported agreeing that proof is related to examinations. The analysis of the open-ended questions also showed that more participants reported valuing the explanatory power of proof and being willing to teach proof regularly. Having said that, a few participants reported and maintained some of their worries about teaching proof (e.g., students lacked ability and interest, class time was not enough). Worries about students' ability to learn proof remained influential in the participants' beliefs about teaching proof. After the intervention, however, these worries no longer discouraged these participants from teaching proof, but rather prompted them to consider different approaches for teaching proof.

In the following, I use Participant 4's data as an example of my analysis. Before the intervention, although Participant 4 was able to relate proof to the development of mathematical understanding, he believed that teaching proof was not related to examinations and was therefore not essential:

> Participant 4: […] I think, for secondary school students, proof is not so important to them. In relation to exams, [proof] is not so important, but proof can allow them to learn why this [mathematical statement/ concept/ idea] is true, making them not to learn by rote, but to understand a theorem and its principle, for applying the theorem more easily or knowing when to apply it.

At the beginning of the intervention, Participant 4 also revealed his belief about the relationships between school mathematics, proof, students, and examinations. He believed that a number of students learn mathematics merely because it is a compulsory subject for all students in Hong Kong and prefer learning by rote over learning proof, so he believed that these students are relatively weak with proof:

> Participant 4: I think there must be a portion of students who are not interested in mathematics. To them, learning mathematics is for taking [the Hong Kong public exam]. So, there must actually be a portion of students who rely on [learning] formulae by rote [and learning] examples by rote, for taking exams. Therefore, when seeing non-proof tasks, and when [seeing] tasks that can be solved by simply applying methods [that

they] have learnt by rote, s/he [the student] can easily solve [the tasks], answer correctly. However, when [seeing] proof tasks, [particularly] those that are not standard, they [the students] do not know how to apply [the methods] to solve [the tasks], so […] these students might be weaker with proof tasks.

Yet, after the intervention, Participant 4 developed a deeper understanding of the relationship between proof and the development of mathematical understanding and became aware that proof can play important roles in preparing students for examinations. Particularly, he developed a belief that learning proof is helpful for students to develop knowledge and skills in solving proving and non-proving tasks:

Participant 4: [Before,] I thought there were not many proving tasks in [exams], so [students] could still achieve good results without proof. However, [now] I think [proof] can not only be [helpful in] proving tasks, but also helpful in other tasks. I think proof can be related to other domains.

Participant 4 attributed his learning in and changes after the intervention to the discussion about proof and the videos of classroom instruction, which allowed him to explore and reflect on different views about proof and teaching. Particularly, Participant 4's experience of the intervention provided alternatives to his pre-existing beliefs about what proof is and about approaches for teaching proof. Moreover, after seeing how primary students could learn mathematics through argumentation and proof in a video, Participant 4 became aware that students have the ability to learn mathematics through proof and argumentation. To him, the ideas of teaching proof conveyed in the intervention were positive and he wanted to apply them into his future teaching:

Interviewer: What did you experience and learn in [the intervention]?

Participant 4: In fact… at the beginning [of the intervention], [the instructor] asked everyone's views about proof. Using [the discussion about which tasks in exams are proof-related] as an example, it could be seen that there are some people [who hold views] that are different from my view. This might, in turn, reflect [an idea that] students [have different views about what proof is]. Second, video… [I] watched two videos; [I] saw more, different approaches for teaching, [for example,] that [video] of teaching children even and odd [numbers]. That teaching approach [was something that] I have never experienced. I [started to] thinking about my teaching approach, and whether some of [my] lessons can involve that approach. [I also] saw that they [the primary students] were able to think [argue the meanings of even and odd numbers], [indicating that] probably older, secondary students should have this ability as well. [After the intervention, I started to] consider more about this aspect.

Participant 4's change exemplifies one possible way how the intervention facilitated changes in the participants' beliefs about teaching proof, particularly their beliefs about the relevance of proof to examinations. The intervention, through discussions about roles of proof in learning mathematics and videos of classroom instruction, provided the participants with alternative experiences of learning and teaching proof. These experiences challenged the participants' pre-existing beliefs about teaching proof (e.g., "proof is always difficult, algebraic and formally presented", "proof is separate from other domains of school mathematics") and broadened their horizons by conveying that proof can be accessible to most (if not all) students and can be communicated via different representations as long as the representations are accessible to and accepted by the students. Consequently, the revised ideas of proof and teaching mathematics allowed the participants to discover more connections between proof, learning mathematics and examinations, and in turn to develop more positive beliefs about the relevance of proof to examinations.

I do not claim that all participants became convinced of the relevance of proof to examinations after the intervention. Rather, I assert that the participants gained *more positive* beliefs about the relevance of proof to examinations. In other words, after the intervention, whilst there were participants becoming convinced that proof is relevant to examinations (as exemplified by the case of Participant 4), others continued to believe that proof is not indispensable but helpful to students when preparing for and taking an exam. For the latter case, since the majority of tasks appearing in exam papers require students to apply formulae and solve equations and only a few require students to produce a proof or an argument (Lee, 2021), the participants continued to believe that proof is not necessary for students to pass an exam. Yet, after the intervention, they became aware that proof is somewhat related to examinations: proof can promote students' mathematical understanding and reasoning, implying that teaching proof can help students do better in examinations.

In summary, during the intervention, the participants were provided with alternative experiences of proof (different from their past experiences), which challenged their pre-existing beliefs that discourage teaching proof (e.g., proof is not related to examinations, proof confuses students). The participants had positive experiences with proof (e.g., they deepened mathematical understanding via proof) and developed positive emotions about proof (e.g., excitement, interest), helping them replace their pre-existing, discouraging beliefs by beliefs that encourage teaching proof (e.g., proof can promote students' mathematical understanding and reasoning, proof can help students prepare for examinations). Having said that, beliefs that have basis in external factors, particularly in quantitative information (e.g., the number of proving tasks in exam papers and textbooks), seemed to be difficult to change (e.g., it is possible to pass an exam without proof).

Summary and discussion

There is some evidence that teachers' beliefs about and practices of teaching proof are affected by examinations (e.g., Frasier & Panasuk, 2013; Lee, 2019; Nyaumwe & Buzuzi, 2007), in Hong Kong and in other countries. This paper demonstrates that whilst (a) some counterproductive beliefs of preservice teachers about the relevance of proof to examinations (e.g., "Students can pass exams without proof") are difficult to change if the composition of tasks appearing in exam papers is not changed (curricular aspect), (b) more positive beliefs (e.g., "Proof can promote students' mathematical understanding, which in turn helps preparing for and doing better in exams") can be developed after an intervention that is carefully designed to provide preservice teachers with positive experiences of proof. In other words, for an effective intervention that aims to change preservice teachers' beliefs about teaching proof, it is important to create alternative experiences of learning and teaching proof in which preservice teachers can deepen their mathematical understanding through proof and can translate such experiences into their future teaching.

It is believed that this study is in the international community's interest for three reasons. First, the findings of this study are consistent with other studies that involved interventions with preservice teachers in which the participants developed more positive beliefs and emotional dispositions towards (teaching) proof (e.g., Buchbinder & McCrone, 2020; Yoo, 2008). These consistent findings in different settings not only provide evidence that it is possible to change preservice teachers' beliefs about teaching proof (which are often considered difficult to change) during teacher training, but also show that it is possible to generalise each finding to other settings.

Second, in contrast with other interventions that were implemented in regular courses and lasted about a semester, the intervention of this study was designed as extracurricular workshops and took place within a month. The short-duration format allowed the intervention design to be tested and revised multiple times within a relatively short period of time, thereby enhancing the validity of the study's findings. This study also explored and showed the possibility of changing preservice teachers' beliefs about teaching proof after an intervention of a short duration, responding to a "challenging but important question for mathematics education researchers: *Would it be possible to design classroom-based interventions of short duration in mathematics classrooms that could help alleviate significant problems of students' learning in mathematics?*" (Stylianides & Stylianides, 2013, p. 339).

Third, I reported different changes in the participants' beliefs about teaching proof after the intervention in this study. The findings also demonstrated why some beliefs (e.g., belief about roles of proof in learning mathematics) changed and some (e.g., belief that students can pass exams without proof) remained unchanged. In other words, some beliefs about teaching proof are dependent on external factors (e.g., the proportion of proof to tasks that require only procedural skills in exam papers) and often remain unchanged if the corresponding external factors remain unchanged, and some are dependent on one's past experiences and can be revised if positive and inspiring experiences of proof are provided (e.g., during courses of university mathematics and mathematics education).

Future work should continue to design (and improve) positive and inspiring experiences of proof for (preservice) teachers and explore and validate the effects of different designs on beliefs about teaching proof (ideally, both immediate and delayed effects) with larger samples (ideally, together with comparison or control groups). In order that dimensions of beliefs about teaching proof can be quantitatively measured for pre-post analysis and statistical modelling, it is also important to conduct studies to develop practical instruments.

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Redesigning Proofs Without Words for secondary level mathematics

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This study seeks to capitalize on the pedagogical potential of visual proof documents called Proof Without Words (PWWs) to advance proof learning in secondary mathematics. The data is drawn from design-based research (DBR), in which a PWW document was iteratively redesigned to lead students to generate more detailed and rigorous proof attempts. The gap-filling theoretical framework for proofs in mathematics education was used to evaluate students' written proof products when working on initial vs. redesigned PWW versions. The results reveal a substantial positive effect of the PWW redesigned version on the quality of students' products. They suggest the potentiality of five proposed PWW design principles in promoting secondary students' proficiency in proof-related activities.

Keywords: Proofs Without Words, gap-filling, design-based research, secondary level proving.

Introduction

Mathematical proofs are fundamental in mathematics as they verify the truth of mathematical statements. Nevertheless, the importance of proofs goes beyond this function as they carry mathematical knowledge, ideas, and methods to propel further scientific developments (Rav, 1999). Notwithstanding the importance of proofs in mathematics, we witness a decline in proof-related activities in schools (Kotelawala, 2016), partly due to the difficulties students at the secondary level encounter in proof activities (e.g., Miyazaki, Fujita & Jones, 2017). This abdication of proofs at the secondary level widens the gap between school and university mathematics and may further reduce students' chances of pursuing and persisting in postsecondary STEM studies (Clark & Lovric, 2009). It is then imperative to develop new approaches to foster students' interest and success in learning mathematical proofs.

The approach used in this study relies on Proofs Without Words (PWWs) – diagrammatic learning resources that allude to a proof process and scaffold its discovery. In PWW-based activities, students are given a PWW and requested to discover a proof and write it down. This paper focuses on a particular geometry PWW task of the Pythagorean Theorem, shown in Figure 1 (Garfield PWW - adapted from Nelsen, 1993, p. 7):

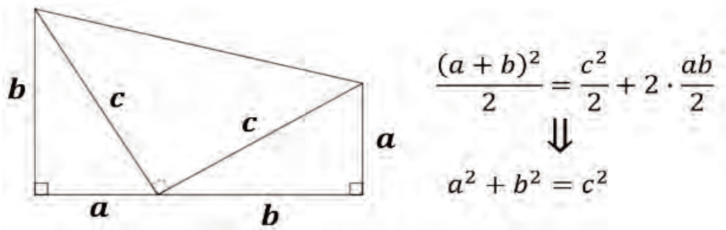

Figure 1: "discover and write down the proof implied by this diagram"

Theoretical underpinnings

Proof Without Words (PWWs) activity at the secondary level

Ever since the seventies, PWWs have started to be published in journals like the *Mathematics Magazine* and *The College Mathematics Journal*, designated primarily for mathematicians. Given a PWW, an experienced mathematician can efficiently perform some mental actions to develop a formal proof: Translate diagrammatical information into verbal conjectures, construct a chain of justified arguments while filling in necessary gaps using prior knowledge. Without these mental actions, many mathematicians would not consider a PWW a proof in and of itself (Biehler & Kempen, 2016). Therefore, in this paper, we refer to PWWs as learning resources and not as proofs. Still, mathematicians value PWWs because of their elegance, mathematical beauty, and the insights they encapsulate (Arcavi, 2003). Mentioning these qualities, Nelsen (1993) advises math teachers to share PWWs with their students.

Nevertheless, are PWWs as accessible for secondary students as they are for experienced mathematicians? In other words, can secondary students develop proofs based on the visual clues given in a PWW? A previous exploratory case study demonstrated that the answer is not straightforward positive (Marco, Palatnik & Schwarz, 2021). On the one hand, providing a PWW led most students to generate proof attempts containing the proof's key idea(s). However, students' written proof attempts were meager, lacking details such as justifications, articulation of constructional procedures, and generality arguments. Figure 2 presents such a proof attempt that most mathematics educators will probably not accept as valid proof for the Pythagorean theorem. Therefore, we launched a design-based research program to find new ways to redesign PWWs to make students produce more detailed and rigorous proof attempts.

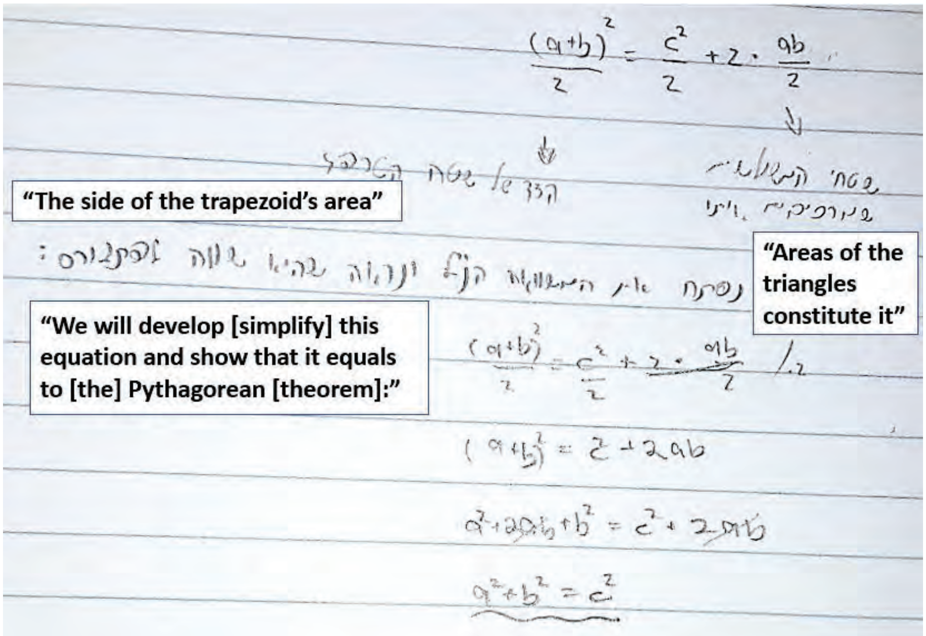

Figure 2: A student's proof attempt based on Garfield PWW

Gap-filling theoretical framework

The idea of *gap-filling* was introduced in literary theory. Gap-filling is a reader-oriented theory that emphasizes the reader's role in sense-making when reading a text. It conceptualizes any text as a system of gaps, which the reader constantly needs to fill by adding information to construct meaning (Perry & Sternberg, 1986). In a passage from literary theory to mathematics education, we suggested gap-filling as a theoretical framework for activities around mathematical proof-document (Marco et al., 2021). We defined a gap in a proof document as missing information essential for a specific reader's understanding. Accordingly, gap-filling is any action the reader takes to add information to complete what she identifies as a gap. For example, in Figure 1, the connection between the diagram and the equation $\frac{(a+b)^2}{2} = \frac{c^2}{2} + 2\frac{ab}{2}$ is not indicated and constitutes a gap. In Figure 2, the student takes action to fill this gap – she brings in the notion of area calculations only implicitly represented in the PWW. After a first DBR iteration, we listed nine gap-filling actions, presented in Table 1, that we expected students to perform while working on the Garfield PWW in consequent iterations (Marco, Palatnik & Schwarz, under review):

Table 1: The gap-filling actions we expected students to perform in the Garfield PWW

#	Description of the gap-filling action(s)
G1	Identifying what is given (an arbitrary right triangle with sides a, b, and c) and what should be proved ($a^2 + b^2 = c^2$).
G2	Specifying the construction procedures through which the trapezoid is obtained.
G3	Justifying the congruence of the two triangles with sides a, b, and c (SAS congruence theorem)
G4	Verifying that the third middle triangle is isosceles and right-angled (by angle calculations).
G5	Proving that the whole figure is a right trapezoid (by definition).
G6	Recognizing the theorem can be derived from calculating the trapezoid's areas in two different ways.
G7	Calculating the areas of all the different figures (using area formulas) and writing an equation such as $\frac{(a+b)^2}{2} = \frac{c^2}{2} + 2\frac{ab}{2}$.
G8	Simplifying the equation to $a^2 + b^2 = c^2$ (algebraic manipulations)
G9	Explaining why the proof that is constructed based on a particular case can be seen as general proof.

Note that not all the gap-filling actions in Table 1 are of the same nature. We divided the gap-filling actions into four categories: *Constructional* (G2), *justification of a figure's properties* (G3, G4, and G5), *key-idea* (G6, G7, and G8), and *generalization frame* gaps (G1 and G9). As mentioned above, in early DBR iterations, almost all students filled key-idea gaps, but only a few identified or filled gaps from the other three categories (Marco, 2021; Marco et al., 2021; Marco et al., under review).

The five PWW design principles

In our three-year DBR, we probed for practical design principles that invite the filling of subtler gaps other than the key-idea gaps (Marco et al., under review). After three DBR iterations, we came up with five tentative PWW design principles with which we redesigned Garfield PWW (Figure 3). The process through which these principles were recognized and established cannot be expounded within the limits of this paper (for more details, see Marco et al. under review). Also, we do not claim these five principles are exhaustive and are yet to be considered hypothetical (Van den Akker, 2013; Bakker, 2018). We investigate if when these principles are enacted in a PWW's redesign, more gap-filling from Table 1 takes place. The five principles are:

1. *Key idea discoverability* – Ensure the discovery of the proof's key-idea(s) with minimal scaffolding. We showed that students tend to fill key-idea gaps of PWWs even if no adjustments are inserted. This finding is an encouraging one that we wish not to impair. So, all changes in the PWW must not risk the discoverability of the key ideas.
2. *Theorem's conditions distinctiveness* – Distinguish theorem's givens from other elements of the proof. Herbst (2004) observed that students are not used for producing a proof "unless the conclusion to be proved, and the conditions under which that conclusion is true, are stated for them" (p. 133). Therefore, the visual grammar should indicate which parts of the diagram are given and conceived through construction. In this manner, the distinction creates a timeline (Dimmel & Herbst, 2015) in which the given parts precede the constructed ones.
3. *Constructional visibility* – Present construction procedures. Construction procedures have a significant epistemic role in geometry proofs. Without verifying how a diagram is conceived, no general truth can be established. If a teacher aspires students to generate rigorous proofs based on PWWs, we suggest the diagram to tell the construction story. In line with Dimmel and Herbst (2015) and Alshwaikh (2018), we found dashed lines and arrows to be well understood as representing constructional procedures.
4. *Figure's properties concealment* – Avoid marking figures' properties. Hewitt (1999) warns teachers not to inform students with necessary mathematical properties that can be deduced. In our study, when a property was marked, most students perceived it as a given and did not justify it. So, if the diagram conceals a figure's property, it prompts students to conjecture it is true. Students are then more likely to gap-fill it by constructing a sub-proof. In this manner, the PWW-based activity combines conjecturing and proving that are regularly applied by geometry teachers as separate proof-related activities (Aaron & Herbst, 2019).
5. *Human agency* – Present the diagram as obtained by human activity. Morgan (2016) upholds that presenting mathematics as abstract, symbolic, and the absence of human agency may prevent students from accessing mathematics. Following this line, Alshwaikh (2018) argues that when diagrams tell a story and include human agency, they communicate better with learners.

Research questions

What impact does a PWW, redesigned according to these five design principles, have on students' gap-filling actions?

Method

This section introduces the redesign version of Garfield PWW and explains how the design principles are implemented. Taking a quantitative approach, we seek to evaluate and compare students' gap-filling actions when working on the initial PWW version (Figure 3, left) vs. on the final redesigned version (Figure 3, right). We then present the participants and describe how we collected and analyzed the data for this study during the second and fourth DBR iterations.

Material

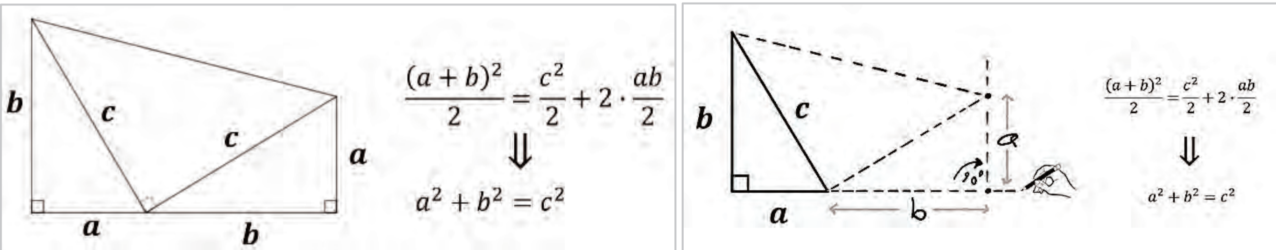

Figure 3: Garfield PWW initial version (left) and redesigned version (right)

How does this redesign implement the five design principles? Due to space restrictions, we will only focus on the third and the fourth principles, which are dominant in this redesign.

Constructional visibility: For denoting construction procedures, we used dashed lines, handmade inscriptions, gray-color arrows, and a drawing hand icon. We used continuous bold strokes only for the leftmost arbitrary triangle and printed letters *a*, *b*, and *c*, to indicate that these are the only givens in this diagram. We prolonged the perpendicular dashed lines more than needed to appear as rays created with a straightedge. The newly assigned segments a and b are denoted in gray handwritten letters and arrows on these perpendicular dashed lines. We escaped using the square notation for the rightmost right angle because it could lead students to perceive it as one of the theorem's givens. Instead, we used a circular arrow with a handmade inscription of "90°". This notation signifies that this measure stems from a deliberate human construction activity.

Figure's properties concealment: We left the middle right angle unmarked and omitted the "c" notation from the rightmost triangle hypotenuse since both can be inferred from triangles' congruence.

Participants

144 Israeli students of ages 15-16 participated in four DBR iterations. In the second DBR iteration, 37 Grade 10 students from two mathematically advanced classes from the same school participated. In the fourth DBR iteration, 72 Grade 10 students from another school participated. The fourth DBR iteration occurred at the beginning of the school year before students were grouped into different mathematics streams. All students were familiar with the Pythagorean Theorem since Grade 8 and had experience solving proof-related exercises in geometry. Most of the students in our study were not introduced to any proof of the Pythagorean theorem when they were eighth-graders, and the vast majority were not familiar with PWWs.

Procedure

The PWW-activity consisted of three phases: (1) Students collaborated in small groups to discover a proof while having the Garfield PWW at hand. (2) Each student individually wrote and submitted a proof attempt and (3) completed a proof comprehension test. In this paper, we only report on the students' submitted written proofs produced in phase (2). To assess students' written proofs, we used the notion of gap-filling. The quality of proofs was based on the number of gaps, from Table 1, that students identified and correctly filled. When grading students' written proofs, we marked 1 for any identified and correctly filled gap, 0 for overlooked gaps, and 0.5 for identified gaps that were not adequately filled (i.e., partially or inaccurately filled). We then calculated the mean gap-filling rate (GFR) for each gap in each version. To answer our research question and assess the impact of the redesigned version on students' gap-filling actions, we undertook a two-tailed t-test assuming unequal variance. The null hypothesis was that there would be no differences in gap-filling rates between the initial and redesign version of Garfield PWW.

Results

Table 2 presents the GFRs in the initial and redesigned version of the Garfield PWW.

Table 2: The average gap-filling rates (GFR) in the initial (I-V) and redesigned (R-V) versions

	G2: Explaining the construction	G3: Why are the triangles congruent	G4: Why the middle triangle right	G5: Why the whole figure a right trapezoid	G6: Calculating area in two different ways	G7: Assigning an area formula for each figure	G8: Algebraic operation to obtain the theorem
I-V GFR (SD), N=37	.02 (.34)	.17 (.23)	.18 (.3)	.24 (.43)	.92 (0.28)	.89 (.24)	.78 (.4)
R-V GFR (SD), N=72	.34 (.48)	.61 (.48)	.50 (.47)	.45 (.48)	.94 (0.22)	.93 (0.24)	.91 (.25)
t-test TE2 vs. TE4	$p < .0001$	$p < .0001$	$p < .001$	$p < .05$	N/S	N/S	N/S
Effect size Hedge's g	0.887	1.002	0.756	0.462	/	/	/

In the three gaps from the category of proof's key ideas, we see no significant differences between the two PWW versions. However, in the categories of *constructional* gaps (G2) and *justification of a figure's properties* gaps (G3, G4, and G5), the GFR are significantly distinct, with the redesigned version having much higher GFRs with substantial effect sizes. Note that increasing GFR in G2 stemmed from more information about the construction that the redesigned version provides. Contrastingly, each justification of a figure's properties gaps, G3, G4, and G5, increased even though the clues about this property were omitted. For instance, the middle triangle was marked right-angled in the initial version, and in the redesigned version, it was not. Remarkably, significantly more students filled G4 given the redesigned version.

Discussion

Testing the redesigned version of the Garfield PWW, we found that more secondary students filled constructional gaps if the construction procedure was displayed in the diagram. Contrastingly, students were more likely to fill gaps associated with the justification of a figure's properties when the property was not explicitly evident in the diagram. We assume that behind the latter finding lies students' need to remove doubts about figure properties when interacting with the diagram. If a PWW leaves doubt about the truth of a figure's property (i.e., not marking a right-angle mark), students will try to verify it by constructing a sub-proof. Removing a mark that reassures a figure's property invites students to conjecture this property is true. After conjecturing, they naturally turn to prove it out of their own epistemic need for certainty (Marco et al., 2021). By doing so, they engage in an authentic mathematical inquiry in which conjecturing leads to formal proving (Aaron & Herbst, 2019).

In our redesigned version of Garfield PWW, we implement five design principles that we gleaned from the data of a broader DBR (Marco et al., under review). We elaborated here on two of these principles and showed their unequivocal effect on students' written proof attempts. As usual, the list of principles remains hypothetical and non-exhaustive (Van den Akker, 2013). Further research with students from different age groups and various PWWs could lead to their development. These design principles have proved beneficial in leading students to fill more gaps in the case of Garfield PWW and are likely to be generalized for other geometry PWW-activities. However, they can serve as a starting point for more research on the use of PWWs in mathematics education. Investigating to what extent these principles are helpful in redesigning PWWs in other domains of mathematics (i.e., progressions, algebra, and calculus) is an exciting avenue for future research.

We used the notion of gap-filling to assess students' written products and compare the effectiveness of two versions of the same PWW. Gap-filling theory can change our perspective about how students learn from mathematical texts and shift our focus when designing them. Instead of exposing all the information we wish the student to engage with and understand, we need to carefully present the minimal information that still enables identifying and filling certain gaps. So, designing a proof document is not just about designing what it contains but also designing what it lacks. Well-adjusted gaps in proof-documents may help rekindle proving activities as central in mathematics education.

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What number makes sense? Standard word problems with non-standard wording

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In the reported experiment, we focus on word problems of the type What number makes sense, which comes from Singapore textbooks. This problem was set to primary school 5th graders. In the analysis of the experiment, we focus on mathematical reasoning involving argumentation and on communication observed in pupils' discussions while solving the problem. We comment on pupils' reactions in terms of three approaches (mathematical, linguistic, real contextual) of which we have kept record also in our previous experiments.

Keywords: Argumentation, word problems, declension, multiplicative relations, Singapore math.

Introduction

Word problems represent a problematic area for both pupils and teachers. Pupils are not always motivated to solve them, as in word problems they practice stereotypical solving strategies they do not always understand. Teachers see pupils' problems in solving word problems mainly in the lack of logical thinking or insufficient level of reading literacy, which is manifested, for example, by incorrect understanding of the text or the meaning of the words. This is why pupils often produce a wrong solution and use incorrect or inappropriate argumentation (Vondrová, 2013).

Different approaches to word problems can be observed in different countries. Within the frame of the Czech project "Support of the integration of mathematical, reading and language literacy in primary school pupils" (TA ČR, 2020), Singapore approach to word problems was used. "Singapore outperforms the rest of the world in the OECD's latest PISA survey, which evaluates the quality, equity and efficiency of school systems" (OECD, 2016). This, for example, leads to "to the promotion of approaches from Shanghai and Singapore in England, with 'Singapore maths' gaining considerable popularity" (Hough et al., 2019, p. 4523).

Various types of problems from Singapore textbooks (created with the help of researchers from the Université du Québec à Montréal) support the development of many skills in pupils as well as different approaches to the problem, which are the essence of their argumentation (Savard & Polotskaia, 2017). "Mathematical reasoning and communication are two key process skills in the framework of the Singapore school mathematics curriculum (Ministry of Education (Singapore), 2012) that have been advocated for a long time" (Chua, 2017, p. 115). Mathematical reasoning is an inherent part of solving word problems of the type What number makes sense? (Kaur & Har, 2009) and communication takes place, among other, in the discussions of the solution, which was used in our experiments. This approach to word problems has not yet been researched in the Czech Republic, which motivated us to commence research using this approach in one type of word problems.

Theoretical framework

We work with the following definition of a word problem: A word problems is a "verbal description of problematic situations that give rise to one or more questions whose answers can be obtained by applying mathematical operations to the numerical data present in the problem" (Verschaffel et al., 2000, p. 641). We agree with this definition. When starting the research with the type of word problem "What number makes sense?" (Fig. 1) we regarded it as a non-standard word problem (Slezáková et al., 2021). Having studied the definition of a non-standard word problem (Jimenez & Verschaffel, 2014), however, we classify this type of a problem as standard with non-standard wording. What makes the problem look 'non-standard' is the way the text is organized.

Do the number make sense? | 32 2 80 5 |

There were __ children at a party. Ms Sally bought __ packets of sweets. There were __ sweets in each packet. During the party, she gave __ sweets to each child and had none left.

Figure 1: What number makes sense?

This type of word problems has two parts. The first part is the instruction: "Fill in the missing information in the word problem." and the data that will be filled in. The second part is a problem situation with missing numerical data. These are replaced by underscores. The order of the numbers given in the first part does not have to correspond to the order of blanks in the problem. The pupil's task is to fill in the numbers in such a way that the problem makes sense from the point of view of

a) **mathematics**, i.e. the filled in numbers make a problem situation that is mathematically meaningful, in other words the numerical data are in additive or multiplicative relation (e.g. the following holds for the numerical data 2, 3, 4, 5: $2 + 5 = 3 + 4$). Pupils prefer to look for relations between numbers, which comes out of their experience and beliefs gained while solving word problems, which is that we get a result with the help of one or more arithmetic operations with numbers from the assignment while ignoring connections to real-life experience (De Corte & Verschaffel, 1985; Nunes et al., 2016).

b) **language**, i.e. the filled in data are grammatically correct (in Czech e.g. declension). With respect to language, researchers focus especially on simplification of sentence formulation (Plath & Leiss, 2018) or the length of words and sentences (Bergqvist et al., 2018). The issue of grammatical correctness with respect to declination characteristic of Slavonic languages has not yet been subject to research in the area of word problems,

c) **real-life context**, i.e. the filled in data make a meaningful real-life situation. Pupils tend to use their real-life experience more often while solving word problems (Van Dooren et al., 2019) and thus get a clearer picture of the problem situation better (Cooper & Harries, 2002). The problem situation can be mathematically meaningful but nonsensical in a real-life context.

Research questions

The main goal of our experiment was to analyse pupils' reactions when solving a word problem and to find out: What arguments for completing the text of the word problem with the given numbers will pupils use? Specifically, will they be sensitive to the linguistic aspect of the problem? Further partial goals that arose while conducting the experiment were: to compare approaches from two different classes from different schools, to gain evidence on whether this problem is suitable for a discussion leading to cooperation or involving conflicting opinions, which reinforces argumentation, to find out whether pupils are aware that the task has more than one solution.

Methodology

Let us now introduce the studied word problem (see Figure 2).

Number 10, 5, 4 and 2 fled from the text. Put them back in their places and check that they make sense. Children brought pheasants to a rescue station. They brought __ pheasants. They bought grain in __ packages. There were __ kg of grains in each of the packages. That means that if divided fairly among them, each pheasant would get __ kg of grain.

Figure 2: The problem 'Pheasants', translated from Czech

From the mathematical point of view, the problem has four solutions that are meaningful as the numbers are in a multiplicative relation (e.g. 4 x 5 = 2 x 10). With respect to real-life context, this situation is not a conflicting one since all four solutions make a possible real-life situation. However, when we look at the problem from the linguistic point of view in Czech, the problem has only two solutions, namely 5, 2, 10, 4, or 10, 4, 5, 2. Otherwise the text is not grammatically correct (declension in Czech). We are aware that each language is specific (Daroczy, 2015). The impact of a country's language and culture on the way pupils argument, verify and prove while solving problems was also discussed in the CERME11 TWG1. (Stylianides et al., 2019)

Let us now explain readers who do not speak Czech the linguistic reasons for selecting the right numbers in the blanks. The first, third and fourth blanks are in the Czech original in a sentence with "be" (was, were "*bylo*" or "*byly*"). What matters is the last letter in the word: "o" or "y". The word "*bylo*" works only with the numbers 5 or 10. The word "*byly*" needs the word 4 or 2. In case of the second blank, the relevant word is the preposition "in" (in Czech "*v*" or "*ve*"). The form "*v*" can be followed only by the numbers 5 or 10. The form "*ve*" can be followed only by the numbers 2 or 4.

Research participants

Two fifth grade classes (A, B) took part in the experiment in 2021. There were 18 pupils in class A and 24 in class B (10 to 12-year-old pupils). Pupils from each class were working together online for 45 minutes. The lesson included a discussion with the pupils' argumentation on the solutions. The online lessons were recorded and transcribed. The pupils' personal data were anonymized.

Analysis of pupils' work

We present the most interesting pupils' reactions in the lessons that point out their approach to the problem. Both individual and mutual reactions in the episodes are grouped with respect to their nature: 1. Pupils' first immediate reactions, 2. Reactions that show different approaches to problem solving, 3. Incorrect or inappropriate argumentation in a pupil's reasoning, 4. Correct argumentation in a pupil's reasoning. Each is followed by our commentary. The pupils are coded A or B, depending in the class they are from. The number specifies the individual pupil.

1. The first immediate pupils' reaction

Pupil A1 — I'm ready. It's as if there were 10 shoppers. They bought 5 packages. The children brought pheasants to the rescue station. I thought it was 10 but I'm not really sure.

Commentary: Pupil A1 already had experience with this type of problem because he had solved a simpler one at the beginning of the lesson. He tried to fill in the first two numbers, but he was not sure if he had understood the wording of the problem.

Pupil B1 — I know. I think it's in pairs, it's vice versa 2, 4 ,5 … no, not this way. Wrong.

Commentary: The impulsive reaction of pupil B1 could have stemmed from his need to show off in front of the teacher and other classmates. That might be the reason why he did not comment on the numbers in the text very meaningfully. He decided about numbers 2, 4 that the numbers went in pairs but soon realized the mistake.

2. Reactions that show different approaches to problem solving

Pupil A1 — I think instead of 10 there will be 2. Because 10 must be down there. 10 must be the kilograms of grain, I found out because a pheasant is quite heavy in fact.

Teacher A — So the picture showing the size of the pheasant helped you?

Pupil A1 — Yes.

Pupil A2 — Well, pupil A1 said there must be 2. But it doesn't make sense when I read it *´bylo jich 2´*. [there were two - problem in declension]

Pupil A3 — There can be *´bylo jich 5´* [there were five], that makes sense. It can't be 10, because 10 is a lot.

Commentary: Pupil A1 approached the problem from the point of view of real context, which was strengthened by the fact that his classmates showed him what a pheasant looks like in a picture from the Internet. Thus, the picture was the basis for his argumentation, namely that the pheasant needs a lot of grain because it is large (in his words 'heavy'). Pupil A2 approached the task from the linguistic point of view. He realized the correct placement of numbers depended on declension. His argument was important for the thinking of the other classmate A3. They presented arguments with other possible numbers corresponding to the declension (5 and 10). Pupil A3 rejects the number 10 from the given numbers, from the point of view of the real-life context - 10 pheasants brought to the rescue station are too many and therefore it would be necessary to prepare far more grains for them.

Pupil B2 — I've solved it. I realized it can't be '*bylo jich* 4' [there were four] because it's not right from the point of Czech. It must be *´bylo jich 10 5´* [there were 10 5].

Pupil B1 — You know what? '*Bylo jich*', it must be there 100%. Grammatically it must be 5 or 10. Because it can't be '*bylo jich 2 nebo 4'* [there were two or four].

Pupil B3 — I think the second number could be a two because 10 and 5 are grammatically correct in the previous case … aha … they are both divisi … no, 10 is a divisible number. And two is also divisible and I think that could be it.

Pupil B1 — No, that wouldn't make sense grammar wise. It's definitely wrong.

Commentary: Pupil B2 approached the problem from the linguistic point of view. She focused on the verb ‘to be’. This motivated her classmates to the same approach. Pupil B1 reacted by listing all the possibilities, which he classified according to declension. Pupil B3 added an argument based on mathematics to the declension argument. She focused on common characteristics of numbers 2 and 10 as numbers divisible by two, which did not bring her to the solution.

3. Incorrect or inappropriate argumentation in a pupil’s reasoning

Pupil A5	I was now calculating for myself. I decided there could be 5 pheasants. Then the 2 kilograms, [unsure], in fact that it would be the 4 kilograms and I calculated that 1 pheasant gets 2 kilograms, another one also 2 kilograms and then I got 5 pheasants and 10 kilograms altogether.
Pupil A6	Those 2 packages and 4 kilograms of grains in each of them. It means it’s 8 altogether. 2 times 4 equals 8. And it wouldn’t make sense that each gets 10 kilograms if there are only 8.

Commentary: Pupil A5 approached the problem from the mathematical point of view. Intuitively she was aware of the multiplicative relation between the numbers 2 and 5. There is a multiplicative relation between the numbers but not between these two. Pupil A6 then applied the multiplicative relation on the numbers used by pupil A5, which led to a number that is not offered. He based his argumentation contradicting pupil A5 and her solution on this.

Teacher B	[the filled in numbers are 5, 2, 10, 4]. How much grain did they have altogether?
Pupil B1	Four. No, ten. Ten.
Pupil B4	No, there were two packages, so 2 x 10 …
Pupil B5	There were two packages and, in each package, there were 10 kilograms of grain. 2 x 10 is twenty.
Pupil B1	Yes, this made me confused. They had 20 kilograms. You’re right.
Teacher B	And how did they divide it fairly among the pheasants?
Pupil B1	Well, they had each pheasant have 4, because there are 10 and to make it fair.

Commentary: Teacher B invited the pupils to check whether the previously proposed solution was correct. Pupil B1 was not able to answer the teacher’s question. Pupil B4 wanted to formulate the correct solution but was interrupted by pupil B5, who completed the argument in the same way as pupil B4 was planning to. Probably thanks to the safe environment in the classroom, pupil B1 acknowledged his mistake. The teacher wanted to check if pupil B1 really understood the problem, so he asked the pupil to argue the right solution. Pupil B1 was not able to do that.

Pupil A7	There were 4 pheasants, they bought for them grains in 2 packages and, in each package, there were 5 kilograms of grain so together in the 2 there were 10 kg of grain.
Teacher A	Why 4 pheasants?

Pupil A7	Because it works like that. 4 pheasants, they scatter the grain on the ground and they can divide it somehow. This is how it works on farms, that they scatter the grain on the ground. They don't let them peck it to pieces. They don't put the package in the coop. [muted laughter]
Teacher A	How many pheasants were there?
Pupil A7	There were 4 pheasants and each had 2 packages with 5 kg of grain.
Teacher A	And now each has 2 packages or they have 2 packages together?
Pupil A7	They have them together. So, each pheasant couldn't have 10 kilograms, in fact. [enthusiastic reaction] I get it. There were 5 packages and, in each package, there were 2 kilograms of grain.

Commentary: As in the situation in class B, there was communication between one pupil and the teacher, who tried to use questions to make the pupil argue the solution. Pupil A7 first tried to find a solution by trial and error. After the teacher's first question, he tried to argue from the point of view of the real-life context, which could have also been his attempt to make the class laugh (it certainly made him laugh). Unlike pupil B1, he perceived the context in the problem better, so after the third question from the teacher he became aware of the mistake and enthusiastically came up with a new solution, which, however, was not correct.

4. Correct argumentation in a pupil's reasoning

Pupil A6	I think there are 10 kg of grains in each package. Because there are 2 packages and in each package there are 10, which is 20 and these 20 kg can be divided among 5 pheasants. That is 4.

Commentary: Pupils A6 felt no need to express explicitly that the numbers are multiplied or divided but obviously he understood the multiplicative relations.

Pupil B2	The first is 5, they brought 5 pheasants. They bought 2 packages for them. Each package was 10 kg. 10 x 2 is 20. And each had 4 and there were 5. Because 5 x 4 is 20.

Commentary: Pupil B2 stated explicitly in her argumentation that the numbers are multiplied. She did not use division. She was aware of the multiplicative relation between the four numbers.

Conclusion

These two experiments showed that pupils from two fifth grades react similarly to this type of word problems. First came positive reactions of pupils facing a new type of problem. However, both of these reactions contained incorrect or meaningless arguments that were not related to the problem itself. Examples of reactions in different approaches to the problem show that these pupils had richer experience with language than younger school pupils (Slezáková et al., 2021) and therefore they focused on the language aspect of the problem, which helped them solve it. The choice of the number was argued with the declension rule. In this case, the linguistic aspect was helpful, reducing the number of possible solutions. On the other hand, we are aware of limitations of language in a word problem. Still, it can bring new possibilities in the area of interdisciplinary relations between mathematics and language and it also gives more space for argumentation. Other arguments for filling in the numbers were the mathematical and the real-life context aspect. Extracts from the episodes in the 2nd and the 3rd points prove that the problem could provide potential opportunities for, among other, two key processual skills, mathematical reasoning (i.e. also argumentation) and communication, which is formulated in the Singapore curriculum. All this happened despite the unfavourable conditions of online education. Pupils actively listened to each other and used the arguments of others. The work of the whole class led to discussion and mutual support while searching for various solving strategies. In the final extracts of correct argumentation of the solution, we see similar reactions showing pupils' insight into the multiplicative relations between the numbers. What makes these reactions different is that the pupil does not feel the need to express explicitly that it is either multiplication or division.

These experiments showed that problems of the type What number makes sense? can be the opportunity to the development of pupils' argumentation skills and moreover they connect mathematical and language skills. While solving this type of problems, pupils naturally reason and justify their thinking processes. We are at the beginning of our research and in the future will conduct experiments with a wider set of problems, greater research sample and longer intervention. We would like to focus on the following questions: Will pupils use previous experience with this type of problems to solve new problems? How will pupils' approaches vary depending on their age? Will language aspects in other countries project into pupils' argumentation in problem solving?

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Proving as multimodal argumentation: an investigation based on Toulmin's scheme

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In this theoretical paper, we discuss mathematical argumentation and proving as multimodal text. It is argued that such a perspective is in line with the everyday experiences of mathematics and school mathematics. The PROving-as-MultimOdal-TExt analytical tool (ProMoTe) is presented, which draws upon Toulmin's scheme to address proving and argumentation as multimodal text. Two exemplary implementations and their implications on research and practice are discussed.

Keywords: Argumentation, proof, Toulmin, multimodal argumentation, multimodality.

Arguments, proofs, mathematics, and school mathematics in a multimodal reality

Mathematical argumentation is at the crux of everyday mathematical practice, whilst the notion of mathematical proof lies at the heart of modern mathematics. Specific forms and norms of reasoning about well-defined objects with fundamental properties constitute axiomatic systems that need to be consistent, independent, and complete. Within these constructions, mathematical proof serves as the gatekeeper, assigning to a mathematical conjecture the status of mathematical knowledge (Mariotti et al., 2018). It seems reasonable to expect that the importance of both mathematical argumentation and proof would be also evident in the school mathematics curricula and the mathematics teacher education programmes. Nevertheless, mathematics education researchers reveal that opportunities for reasoning and proving are limited in the mathematics textbooks (Davis, 2013; Hanna, 2000; Stylianides & Silver, 2004). Importantly, in school mathematics, proof seems to act as a dichotomizing, rather than a unifying, practice that crucially affects the way that each mathematics learner positions himself/herself with respect to mathematics. For example, it seems that these opportunities are often linked with those tasks that are more demanding or with proofs to (already known) theorems (Hanna & de Bruyn, 1999; Nordström & Löfwall, 2005). Moreover, everyday mathematics teaching practices, include an amalgam of verbal, non-verbal, emotional, and embodied aspects, which are also part of mathematical argumentation and proving. Diverse semiotic systems (for example, linguistic, visual, audio, gestural etc) are present in the mathematics classroom and the students are required to make meaning from and amongst them, as they are engaged with mathematics. The complex relationships amongst the meanings related to the different semiotic choices (for example, mutually reinforcing, antagonistic, or complementary) that sometimes co-exist during the utterance of a mathematical argument are of the interest of this paper.

In this paper, mathematics argumentation is conceptualised to be communicated as a multimodal text, in the sense that it involves the employment of more than one semiotic system (cf. Kress, 2010; O'Halloran, 2008); including, verbal and non-verbal aspects in diverse semiotic systems, as well as implicit, explicit, cognitive and affective aspects. This led to the development of an analytical tool (Proof-as-Multimodal-Text; ProMoTe), with the purpose to gain deeper understanding of the complexity of the multifaceted communicational interactions that are at work when teachers and students are engaged in argumentation and proving.

Toulmin’s scheme in mathematics education

Toulmin, in his seminal book *The Uses of Argument* (Toulmin, 1958), developed a scheme to analyse the logical micro-structure of an argument. According to the scheme, each argument includes the following components (see Figure 1): Data, Warrant, Backing, Qualifier, Rebuttal, Claim. A Claim is drawn upon some facts (the Data), based on a rule (a hypothetical statement; the Warrant) that associate such facts to this claim. This relationship is valid to a degree of certainty (the Qualifier) unless there is a case of refuting this relationship (the Rebuttal). The applicability of employing a warrant in the specific argument is supported by a categorical statement (the Backing) that identifies the broader system within this warrant may be utilised.

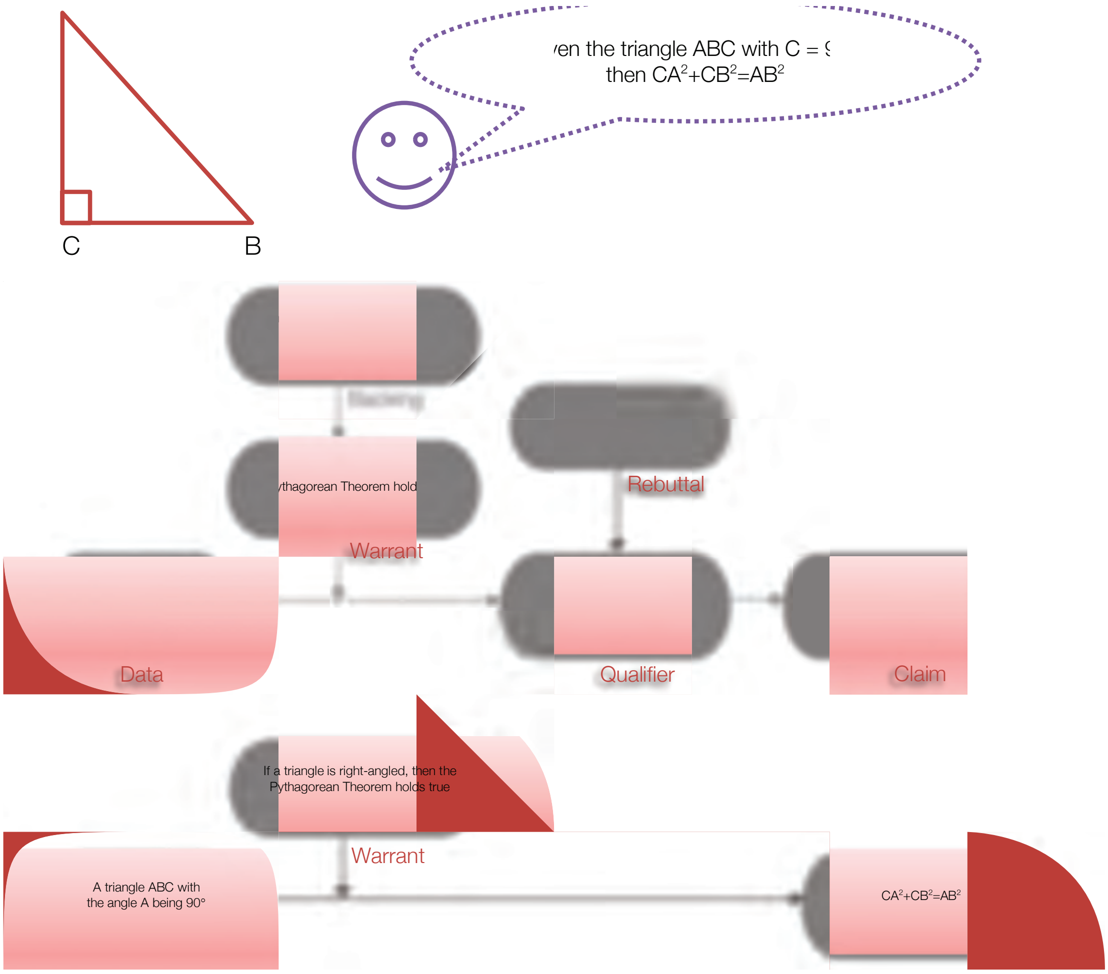

Figure 1: An example of applying Toulmin’s scheme; full (up) and a restricted version (below)

In Figure 1, two versions of Toulmin’s scheme are applied to analyse the argument posed by a student who saw the red triangle: “Given a triangle ABC with C = 90°, then $CA^2+CB^2=AB^2$.” Considering the full scheme, the Claim is an algebraic relationship with geometrical meaning that links the lengths of the sides of the right-angled triangle ABC, given the Warrant of the Pythagorean Theorem which holds true for all (no Rebuttal, absolute Qualifier) right-angled triangles within the system of the Euclidean Geometry (the Backing). Note that not all components of the analysis are explicit (verbally or else); the Warrant, the Backing, the Qualifier, and the Rebuttal are usually implicit and are hypothesised by the analyst (for example, the researcher or the teacher). Moreover, in our example, the data provided to this student draw upon different semiotic systems (numerical, figural), whilst the students' answer combines (natural) language with algebraic symbolism.

Though the scheme was not designed specifically for mathematical arguments, it has been widely utilised in mathematics education research. Initially, it was employed in restricted versions (Krummheuer, 1995). Nevertheless, Inglis et al. (2007) argued for the necessity to utilise the full version of the scheme to gain deeper understanding of in-class mathematical argumentation and its development towards a proof. They showed that students faced difficulties due to their inappropriately linking qualifiers and warrants (for example, an inductive warrant with an absolute qualifier). Various mathematics education researchers agree with employing the full version of the scheme, whilst others propose expansions of the scheme drawing upon other theoretical tools (Aberdein, 2005; Conner et al., 2014; Krummheuer, 1995; Pedemonte & Balacheff, 2016; Simpson, 2015). Moreover, researchers have employed Toulmin's scheme to investigate collective argumentation (Knipping & Reid, 2013), the teachers' supporting the students' mathematical argumentation (Conner et al., 2014), and the development of teaching tools about argumentation and proof (Hein & Prediger, 2017; Moutsios-Rentzos & Micha, 2018).

The fact that mathematical argumentation and proving involves both verbal and non-verbal aspects in diverse semiotic systems, as well as implicit and explicit aspects led to the development an analytical tool (ProMoTe) that acknowledges the mathematical argument as a multimodal text.

The PROof-as-MultimOdal-TExt analytical tool

A series of small-scale studies have been conducted to incorporate Toulmin's scheme within a tool to analyse the micro-structure of multimodal mathematical argumentation. The Proof-as-Multimodal-Text (ProMoTe) tool was designed to analyse multimodal arguments according to Toulmin's scheme, also identifying the *explicit* and the *implicit* elements of the arguments (see Figure 2).

ProMoTe Elements	Data	Warrant	Claim	Qualifier	Rebuttal	Backing
Explicit						
Verbal						
natural language						
symbols (numbers, symbols etc)						
Non-verbal						
written (figures etc)						
embodied -sensory (gestures etc)						
Implicit						
Verbal						
mental (non-explicitly communicated verbal warrants etc)						
Non-verbal						
mental (non-explicitly communicated non-verbal warrants etc)						
embodied -bio-metric (heart-rate etc)						

Figure 2: The Proof-As-Multimodal-Text (ProMoTe) analytical tool.

The explicit/implicit differentiation reflects what is considered to be overtly present for those engaged in the mathematical argumentation. Explicit elements include those verbal and non-verbal aspects of the mathematical argumentation that are in some way sensory-present to those who are engaged in the communication. Explicit verbal elements may include natural or mathematical language, whilst explicit non-verbal elements may be classified as written or as embodied-sensory (for example, gestures, posture, voice pitch etc). On the other hand, implicit elements include aspects of the argumentation that are only mentally present (for example, a non-explicitly communicated warrant; the Pythagorean Theorem in Figure 1), or they are not consciously present (for example, biometric data, such as heart rate or salivary cortisol measurements). Moreover, element types that are hypothesised by the analyst (for example, researcher or a reflective practitioner) may be also included in this category. Notably, ProMoTe allows for the identification of both cognitive and affective aspects of mathematical argumentation.

The analysis of each argument according to ProMoTe is designed to be conducted in three complementary stages/layers. First, the argument is decomposed according to Toulmin's scheme with respect to its explicit verbal elements. Subsequently, the non-verbal explicit elements are considered. The third stage/layer concerns the identification and/or assumption of the implicit elements.

In this paper, two exemplary implementations of ProMoTe are presented from a study[1] conducted in 2018 in a Greek high school (14 years old). Video-recorded data (only the teachers were visible; informed consents were obtained) of eight teaching hours were collected and analysed. Two restricted versions of ProMoTe were utilised (respectively, one including Data, Warrant and Claim, and one including Data, Warrant, Claim and Qualifier) to analyse one teacher's multimodal mathematical argumentation focusing on both cognitive and affective aspects.

The first example (outlined in Figure 3) concerns a teacher's argument regarding the slope of a linear function. He attempts to exemplify the property that when the slope of a linear function is positive, then the function is strictly increasing. He utilizes the example of $f(x)=3x$ to argue that "when α is positive [referring to the general form $f(x)=\alpha x$], the line goes upwards". The analysis reveals that the teacher's arguments are essentially two completely different ones that are co-developed simultaneously: a) a deductive argument discussing the graph of the specific function as a special case of the general rule he had presented a few moments earlier, and b) an inductive argument about the general rule that is verified by the specific example. At the same time, the teacher explicitly non-verbally shows the specific example and verbally utters the general rule. The students are implicitly required to differentiate between these two arguments, which warrants link which data with which claim. Importantly, not all the elements of the ongoing argumentation are legitimatised to be temporary stable; that is to be written on the blackboard.

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[1] Stamataki, A. (2019). *Non-verbal affective aspects in education* [Mi Lektikes Opseis tou Thymikou sti Didaktiki] (Unpublished Master's dissertation). Athens, Greece: National and Kapodistrian University of Athens.

ProMoTe Elements	Data	Warrant	Claim
Explicit			
Verbal			
natural language	*"…when α is positive…"*	-	*"… the line goes upwards"*
symbols			
Non-verbal			
written	α=3 of the graph of the function f(x)=3x as depicted on the blackboard	Visual inspection of the graph	The graph of the function f(x)=3x as depicted on the blackboard
embodied -sensory			Upwards gestures over the straight line Followed by a movement of the hands from left to right on the horizontal axis
Implicit			
Verbal			
mental	The graph of f(x)=3x is a line going upwards, with α=3>0	Inductive	In a function f(x)=αx, when α>0 then the graph is an increasing straight line
	α=3>0	When the slope of a straight line is positive, the angle of the line with the horizontal axis is less than 90°	The line is strictly increasing
Non-verbal			
mental			

Figure 3: Implementing the ProMoTe analytical tool (example 1)

In the second example, the same teacher, teaching in another cohort of students of the same grade, presents a textbook exercise and asks the students to create the graph of the function y=3x. He argues: "The function y=3x is of the type [raises his voice a little, makes intense hand-movements, his eyes were bulging out] … y= αx, then it is a straight line". The application of ProMoTe (see Figure 4) contrasts a standard verbal analysis which would identify only one argument, by revealing two arguments that work in parallel: the first is "The function y=3x is of the type [raises his voice a little, makes intense hand-movements, his eyes were bulging out] … y= αx" and the second argument is "then it is a straight line".

The decomposition of the second argument is in line with the standard analysis, as it involves only verbal explicit and implicit elements (see Figure 4, below). However, regarding the first argument, ProMoTe allowed us to identify the role of non-verbal affective elements in argumentation in the micro-structure of argumentation (see Figure 4, up). It appears that the teacher amplifies the presence (cf. Perelman & Olbrechts-Tyteca, 1969) of the whole argument by providing three implicit non-verbal affective warrants, as depicted in the ProMoTe tool. At the same time, the argument is also based on the implicit data that "Raised voice, intense hand-movements and eyes bulging out imply that emphasis is given and attention is required", as well as on the implicit warrant of the teacher's authority, on the implicit claim that "When the teacher emphasises something and/or draws attention to something, then this is correct and/or important", and on the implicit qualifier "certainly".

Notice that the teacher's argument draws simultaneously on cognitive and affective aspects, though they appear to inform different aspects of the mathematical argumentation. There seems to be an implicit dimension, essentially forming an implicit argument about the affective warrants. Explicitly and implicitly the teacher's verbal argumentation seems to promote deductive argumentation. At the same time, his implicit and explicit non-verbal argumentation appears to legitimatise his authority as means for the validation of mathematical argument, as the embodied-sensory authority-derived warrants together with the deductive implicit warrants are linked with the qualifier "certainly", which may communicate inappropriate warrant-qualifier couples as acceptable (Inglis et al., 2007).

ProMoTe Elements	Data	Warrant	Claim	Qualifier
Explicit				
Verbal				
natural language *symbols*	*"The function y=3x is of the type"*	-	*"y= αx"*	
Non-verbal				
written	-	-	-	-
embodied -sensory	-	raised voice intense hand-movements eyes bulging out	-	-
Implicit				
Verbal				
mental		The graph of the function f(x)=αx is a straight line The teacher's authority		certainly
Non-verbal				
mental	Raised voice, intense hand-movements and eyes bulging out imply that emphasis is given and attention is required		When the teacher emphasises something and/or draws attention to something, then this is correct and/or important	certainly

ProMoTe Elements	Data	Warrant	Claim	Qualifier
Explicit				
Verbal				
natural language *symbols*	-	-	"then it is a straight line"	-
Non-verbal				
written	-	-	-	-
embodied -sensory	-	-	-	-
Implicit				
Verbal				
mental	The function y=3x is of the type	The graph of the function f(x)=αx is a straight line		certainly
Non-verbal				
mental				

Figure 4: Implementing the ProMoTe analytical tool (example 2)

Concluding remarks

In this paper, we approached argumentation and proof as multimodal text, with the purpose to design a tool that may prove to be helpful in analysing the complexities of everyday mathematics argumentation and proving. We presented the implementation of the ProMoTe tool on a teacher's multimodal argumentation. The ProMoTe tool appeared to be helpful in identifying the specific ways that different verbal and non-verbal semiotic systems are explicitly and implicitly orchestrated in the different elements of mathematical argumentation as decomposed according to Toulmin's scheme. The PRoMoTe tool helped in revealing aspects of the teachers' argumentation that may allow for the construction of inappropriate sociomathematical norms (Yackel & Cobb, 1996) about argumentation and proving (cf. Moutsios-Rentzos et al, 2019). Though researchers have considered multimodal aspects of argumentation (Krummheuer, 1995), ProMoTe allows for the different lines of argumentation that may be developed within and/or across different semiotic systems to be visible (cf. O'Halloran, 2008). Considering the rapidly evolving and technologically expanded mathematics classroom (cf. Hanna et al., 2019), which includes the digital, the hybrid and the virtual, and the relevant radical transformations of the educational processes, settings, and power-relationships, it is posited the multimodal mathematical argumentation (further amplified by technology) is at the heart of mathematical practices, thus rendering crucial the development of appropriate analytical tools that explicitly address multimodality to gain deeper understanding of the complex phenomena. Ongoing research is focusing on utilising the ProMoTe tool in mathematical multimodal argumentation about tasks concerning measurement and area, on including implicit non-verbal embodied-biometric and digital data, as well as on investigating the incorporation of ProMoTe in a teacher education programme.

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On the implementation of proving in primary school

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The project Prim-E-Proof aims to generate substantial learning environments to support proving skills at the primary level. The learning environment "The Pythagoreans" on the theorem "the sum of two odd numbers is always even" is constantly further developed through design science research. The evaluation showed that the majority of the primary school pupils use empirical justifications and that the solutions of younger primary school children do not seem to differ much from those of older ones. A first concept for the reflection of justifications is supposed to support learners in developing valid justifications and teachers in implementing proof in primary school.

Keywords: Primary education, "inhaltlich-anschaulich" proving, reasoning, instruction support.

Introduction

"Reasoning" can be understood as a generic term for forms of argumentation and proof (Brunner, 2014, p. 31). However:

> A formal-deductive understanding of proof is not suitable to describe the aspect of reasoning in primary school [in Germany]. The same is valid for the "inhaltlich-anschaulich" [a loose translation could be "content-visual"] proving, although its use in primary education is undoubtedly possible. However, a content-visual proof also aims at proving the validity or non-validity of general mathematical facts such as rules, conspicuousness, relationships, or the like. The detachment from the individual case, the generalization, represents an indispensable aspect of content-visual proofs. In contrast, mathematics lessons often require reasoning to lend validity to a very concrete example or a particular assertion. (Peterßen, 2012, p. 20)

Stylianides (2016) formulates two crucial goals from a pedagogical standpoint that can be reached through a more central place of proving in the mathematical experiences of children:

> First, elementary students will have more opportunities to learn deeply in mathematics and engage with it as a sense-making activity. Second, when students get to secondary school they will not only be better prepared to engage with proving, which they will see as a natural extension of their prior mathematical experiences, but also better able to reason mathematically in disciplined ways […]. (Stylianides, 2016, p. 10)

In the context of primary school, especially the research perspective of Blum and Kirsch (among others 1991), together with Wittmann and Müller (among others 1990), which Reid and Knipping (2010, p. 43) call a "preformalist" research perspective, seems suitable. In content-visual proofs, preformalists loosen the requirement that proofs must be formal. Still, they expect deductive reasoning to be used and that proofs based on intuitive or obvious foundations are as convincing (or even more convincing) as formal proofs (ibid.). A *content-visual proof* is based on a concrete, visually perceptible object, on which something general is proven and which is usually presented in an iconic way. It can be understood as an object of a more general kind by trained observation. To produce a content-visual proof, the more general in the special of an example must be mentally seen

(Krumsdorf, 2015). But in particular, to disengage from examples, i.e., to generalize the effects of operations, confronts primary school children with a problem that they cannot solve spontaneously and without support (Sturm, 2018). Another challenge for primary school learners is that the *content-visual proof* must be verbalized. What is initially subjectively found to be universally valid can be socially shared and recognized by others (Wittmann & Ziegenbalg, 2007). The pupils face the requirement of generalizing without knowledge of the algebraic language. They interpret a general structure into the given mathematical signs and use the signs they know from other contexts to describe the general patterns and structures (Akinwunmi, 2012). Besides the challenges for learners, proving in primary school brings challenges to teachers that make an implementation into mathematics teaching difficult. These are:

> […] the weak knowledge that many elementary teachers have about proof (factor 1) and their presumed beliefs that proving is an advanced mathematical topic beyond the reach of elementary students (factor 2); the high pedagogical demands placed on elementary teachers who strive to engage their students in proving (factor 3); and the inadequate instructional support offered or available to elementary teachers about how to achieve that goal in their classrooms (factor 4). (Stylianides, 2016, p. 21)

Melhuish et al. (2020) evaluated the efficacy of a 3-year teacher professional development program with participants from 25 elementary schools. They concluded: "Even when teachers are engaged in a PD designed to promote high-level reasoning forms (in our case justifying and generalization) and are provided with relevant definitions, teachers may continue to revert to nonstandard views of those reasoning forms." (p. 63). This emphasizes the complexity of designing support measures for teachers.

The *Prim-E-Proof* project's objective is to develop *substantial learning environments* for primary school mathematics lessons, among others inspired by the tasks stimulating *operative proving* in the primary school book *Zahlenbuch* (based on Wittmann & Müller, 1990), to support proving skills with a particular focus on the factors identified by Stylianides (2016). Wittmann (1998) characterizes substantial learning environments based on four criteria. They must (1) represent central goals, content, and principles of mathematics education; (2) provide rich opportunities for students' mathematical activities; (3) be flexible and easily adaptable to the particular circumstances of a given class; (4) integrate mathematical, psychological, and pedagogical aspects of teaching and learning holistically and therefore offer a broad potential for empirical research. The starting point is the theorem: *the sum of two odd numbers is always even*.

The learning environment "The Pythagoreans"

The tasks within the learning environment are as follows:

(1) *What was again an even number, and what was an odd number?* With this question, the childrens' previous knowledge is linked up.

(2) *In the past, calculation stones were used for calculating instead of reversible tiles. The Pythagoreans alienated the calculating stones to perform mathematical proofs – among other things about even and odd numbers. These calculation stones of the Pythagoreans were further developed*

in this applet [besides the applet also analog material is provided]. *Let me show you what the applet can do* [the functions of the applet (www.melanie-platz.com/Steinchen-Applet/Steinchen.html, see Fig. 1) are presented]. *How might the Pythagoreans have represented even and odd numbers with stones? Use the applet and represent an even and an odd number.* To avoid a defensive reaction (Platz, 2019) and to be able to arouse a need for proof and thus provide rich opportunities for mathematical activities for pupils, a historical digression (Krauthausen, 2018) is initiated.

(3) *Always add two odd numbers. What do you notice?* Bezold (2009, p. 37) describes four activities or steps in an argumentation chain. The first two activities are stimulated by this task: the discovery of mathematical features as *prerequisites* for argumentative activities. *Step 1* represents the description of discoveries (ibid.) or a reference to common prior knowledge (Neumann et al., 2014).

(4) *Why is this so? Give reasons!* With this task, *step 2* of the argumentation chain, according to Bezold (2009), the questioning of discoveries, and *step 3,* the finding of justifications or ideas for justification of mathematical regularities and connections, are stimulated.

Research Questions

In this paper, the following research questions are targeted: *RQ(1) Which level of proving do learners working on the learning environment "The Pythagoreans" reach when providing support measures to support and supplement the learners' activities?*; *RQ(2) Which consequences can be drawn for the further development of the learning environment?*

Methods and procedures for collecting and analyzing data & obtained data

A multiple case study (Yin, 2018) using qualitative data collection and analysis methods was conducted. A clinical interview with only key questions defined and the requirement to follow children's thinking is analogous in principle to classroom management in implementing a substantial learning environment (Wittmann, 1998). The collected data can provide information "[...] about teaching/learning processes, thinking processes and learning progress of students [...]. On the other hand, they help to evaluate and revise the learning environments to design teaching/learning processes even more effectively." (ibid., p. 339).

A course concept for student teachers, who did not have experiences with proving in primary school before, was developed: the students elaborated various support options for the learners that were applied (if necessary) depending on the proof phase, e.g., variation of suitable materials or representations (Krauthausen, 2001), variation of appropriate example-based proof processes (approaches) (Krumsdorf, 2015), use of Big Numbers (Martin & Harel, 1989) or merely presented examples (Krumsdorf, 2015) or use of pre-exercises for proof (Brunner, 2014). Before the students performed the clinical interviews, the learning instruction was focused which should help to enable them to support and supplement the children's activities with suitable impulses that do not go too far and to be a rich source of reliable factual information for the children (Wollring, 2008).

The learning environment was tested by student teachers at the University of Münster in the winter semester 2020/21 and the University College of Teacher Education Tyrol in the summer semester 2021 in 40-minute clinical interviews with primary school children (grades 1-4). The interviews were videotaped and transcribed. The solutions of 17 pupils were analyzed regarding the handling of examples (abstraction) and generalization, the level of justification, the need for support during proving and classified in proving levels. Almeida (2001, p. 55) adds *'proof' by authority* (level 1) and *'proof' by intuition* (level 2) implied by Cobb (1986) to Balacheff's (1991) widely accepted descriptions of the levels of proving: (3) *'proof' by naive empiricism*, (4) *'proof' by crucial experiment*, (5) *proof by generic example* and (6) *proof by thought experiment*. Proofs on levels 5 and 6 can be regarded as valid.

Results & Discussion

RQ(1)

Analyzing the proving level of the pupils from grades 3 and 4 (8-12 years), seven pupils used empirical justifications, three *generic examples*, and one a *proof by thought experiment*. Two could not be convinced that the assertion is true. This draws a similar picture to the results of a study of the proof perceptions and practices of "nineteen year ten pupils" (p. 55): Almeida (2001) observed that the majority of pupils preferred empirical justifications. In our case, four younger children (grade 1 & 2, 6-7 years) were interviewed, and the approach to proving did not seem to differ that much from that of the older children. Neumann et al. (2014), who assessed students' mathematical reasoning skills from two 3rd grades, five 4th grades, and five 6th grades, observed something similar. The comparison of the groups showed no significant differences between grades 3, 4, and 6 (n=243). Focusing on the proving skills of the youngest pupils, one interesting case is the following: Emma (grade 1, 7 years) was asked by the interviewer, what pattern all the odd numbers have in common and how she can tell straight away that she laid an odd number with the tiles. Emma explains: "[…] because you can't always count two together with them. For example, you can't count to four now, when in steps of two, two, four, then one tile is missing." (see Fig. 1, middle). Then this scene follows:

1	Interviewer:	And if you then put two such numbers together? (...) What happens then (...) with the steps of two?
2	Emma:	(...) Then it works.
3	Interviewer:	Then it works.
4	Emma:	Yes
5	Interviewer:	Um (hesitating). Why is this always the case?
6	Emma:	(...) Hm (thoughtful) (...) Because (...) the odd numbers are odd numbers?

Stimulated by the interviewer, Emma uses the self-chosen example 3+3=6 to show that the sum of 6 is even by counting in steps of two (see Fig. 1, right).

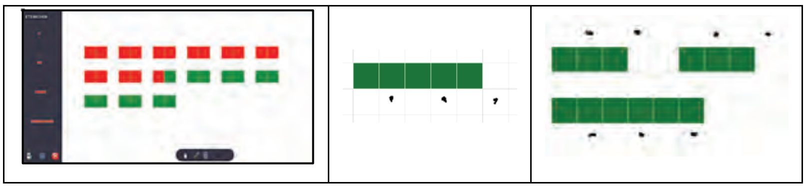

Figure 1: Screenshot of the "Steinchen-Applet" and visualization of Emma's solution

Emma describes odd numbers through the basic idea of dividing by grouping (pairing) the tiles (counting in steps of two). If no tile is missing to form a pair, it is an even number. If there is a tile missing, it is an odd number. Emma does not express the proof idea (that the two individual tiles can be joined together and that an even number can be produced when forming the sum by pairing). She just states "then it works" (*turn 2*) and responds to the *why-question* of the interviewer (*turn 5*)

"Because the odd numbers are odd numbers?" (*turn 6*). Afterward, Emma verifies with an example that her procedure works. The reaction of Emma to the *why-question* could be compared to the "I see it this way" (cf. *proof by intuition*) described as an answer of children to the question "Why? How do you know?" by Freudenthal (1978, p. 262) focusing on a geometric context. This can be transferred to the geometric-visualized pattern structuring in our case: "One thing I know for sure today: the answer is not an excuse, it is true. It is not a symptom of guessing, nor is it a confession of powerlessness in linguistic expression. On the contrary: that the child sees it prevents the linguistic effort." (ibid., p. 263). Consequently, Emma seems to start with a *proof by thought experiment,* because she verifies the statement by appealing to the structural properties of mathematics independent of examples. In *turn 6,* she seems to switch to a *proof by intuition* and is based on stimulation by the interviewer directed to a *proof by a generic example* (which is not completed).

RQ(2)

The mathematical reasoning competencies of the students mostly do not correspond to the curricular requirements of the educational standards, which can also be attributed to a lack of meta-knowledge of the students regarding mathematics-specific reasoning (Vogt, 2020). But a targeted further development of mathematical reasoning skills in primary school mathematics lessons seems to be possible in principle (Bezold, 2009; Peterßen, 2012; Vogt, 2020). Peterßen (2012) reports on a study where most grade 3 pupils (n=20) used valid justifications in the categories *generic example* and pure *thought experiment* to validate a claim; empirical justifications accounted for only about one-quarter of the students' justifications. "One possible explanation could be the culture of justification already embedded in the class. The students were used to being asked *wh*y over and over again. Furthermore, the quality of pupils' explanations was also the subject of the lesson." (ibid., p. 63). Understanding proof in the following way makes the negotiation for a common argumentation basis apparent: "A set of propositions that are accepted as true, together with inferences that are accepted as admissible, shall be called a basis of argumentation. Reasoning based on a given argumentation basis be called a proof concerning that argumentation basis." (Fischer & Malle, 1985, p. 180). In terms of a common argumentation basis, Peterßen (2012, p. 348) emphasizes the *reflection of justifications* with pupils. Pupils need to understand what is required of a justification, i.e., which justification is accepted or rejected, when, and why. Justifications in themselves and their respective persuasiveness should be made the subject of teaching. Such a reflection of justifications was not stimulated in the learning environment *the Pythogoreans*: The tasks enable to pass through the four activities or steps in an argumentation chain described by Bezold (2009) as mentioned above. Comparing this argumentation chain with the *process model of the proof process of experts* (the seven phases do not necessarily have to be passed through in the order given here; rather, a frequent alternation between these phases is to be expected, especially for experienced mathematicians), the first three phases seem to coincide partly: (1) *Finding a conjecture from the mathematical problem field*. (2) *Formulate the conjecture according to common standards*. (3) *Exploring the conjecture with the limits of its validity; making connections to mathematical framework theory; identifying suitable arguments to support the conjecture* (Reiss & Ufer, 2009, p. 162). The phases (4)-(6) are not stimulated with the tasks: (4) *Selecting arguments that can be organized in a deductive chain to form a proof*. (5) *Fixing the chain of arguments according to current mathematical standards*. (6) *Approximation to a formal proof*

(Phase (6) is only realized in a few cases. Usually, the proof found in phase (5) is reviewed and, if necessary, published by the mathematical community at conferences and in the context of peer review.) (ibid.). (7) *Acceptance by the mathematical community* (ibid.), whereby in the school context, this role of the mathematical community is usually performed by the teacher. As the model is based on an expert's (optimal) proof process, it cannot directly be transferred to purposes of teaching (ibid., p. 163), but it can be used as a starting point for the development of a teaching concept. With this intention, the reflection of justification is focused for helping students understand what is required of a justification (Peterßen, 2012) and to find a common argumentation base (Fischer & Malle, 1985) to be able to select arguments that can be organized in a deductive chain to form a proof (phase (4)). In the next step of the project, the reflection of justifications is focused on, which is supposed to be done via analyzing authentic pupils' solutions. These solutions can include errors or not. What should be emphasized is that most of the pupils' answers are not wrong; the pupils just had another understanding of the argumentation base. The *reflection of errors or justifications that are to be rejected* can lead to negative knowledge (Oser et al., 1999). This includes two main components: the differentiation knowledge (To what extent does something not belong to a specific item, concept, or procedure?) and the error knowledge (What must not be done in a particular situation?). The use of solutions of other (anonymous) pupils can be justified because not each error has to be done by oneself to learn from it, and pupils usually prefer to work on mistakes from others (Prediger & Wittmann, 2009). The aim of *reflecting "good" solutions or justifications that are to be accepted* is not for every child to learn and use as many ways of proving as possible. It is essential that they find ways to prove according to their individual preferences. This way, the lack of familiarity with adequate tools (Krauthausen, 2001) and the interpretative openness of figural representations (Krumsdorf, 2015) in tile-proofs can be counteracted. Training in the flexible perception of different states (one representation, different perspectives) can be supportive. It can be helpful if the same situation can be experienced and explicitly compared using different representations (ibid.). For analyzing the students' solutions, the teacher can guide the pupils through stages with increasing metacognitive demand similar to those proposed by Prediger & Wittmann (2009, p. 8) for analyzing errors: (1) *Find calculation errors or errors in the procedure of the student* (What is wrong here? Justify why.; What rule was broken?). (2) *Compare two solutions* (Here are two different solutions. Which one is better?). (3) *Explain the procedure of a student* (How did the student think here?). (4) *Reconstruction of the reason why a justification was given that must be rejected/accepted* (What idea is behind the solution?; Explain why the justification is rejected/accepted.). (5) *Development of optimization measures* (How would you help the student?; What would you do same wise/different?). The following scene could be used as starting point for a discussion (especially items (3)–(5); in comparison with Emma's solution also (2)): After checking on several examples, Ben (grade 3, 8 years) was asked why his procedure (pairing) always works:

Ben: If you can partition the little tiles exactly. Then it must be an even number. But. (5) Two odd numbers make an even one. (...) Because it remains. If you divide them up, there is always one remaining. And if you combine the one leftover with the other one/ one is left over. If you then put them together, you have two again. And (..) if you had three odd numbers now, they would have remained odd.

Conclusion & further Development

Peterßen (2012) identifies the following aspects that positively affect the development of reasoning competence and a subjective need for reasoning: discovery learning as a teaching principle, good tasks, regularity, pupil-focused communication, and subject-specific competent teachers. By developing substantial learning environments that should be implementable regularly, discovery learning is realized, and good tasks are designed. Focusing on the reflection of justifications, informative scenes from the clinical interviews will be extracted and prepared for use in the primary school classroom. Melhuish et al. (2020) suspect that

> […] the ability to notice complex mathematical-reasoning forms (a necessary prerequisite for fostering them in classrooms) may rely not only on providing teachers with mathematical definitions and rich examples but also on promoting explicit connections between definitions and the examples themselves. (p. 64)

With the focus of promoting explicit connections between definitions and the example scenes, teacher training courses will be developed, building on the experiences of Melhuish et al. (2020) and the concept *professional vision* (Sherin, 2007) to support teachers subject-specifically and in pupil-focused communication.

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‘Reasoning’ in national curricula and standards

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I examine the use of the word ‘reasoning’ in the 2020 Norwegian national mathematics curriculum, in the 2000 National Council of Teachers of Mathematics (NCTM) Standards and in the 2003 Education Standards of the German Kultusminister Konferenz (KMK). I identify differences in usage, make comparisons to the classification of aspects of reasoning proposed by Jeannotte and Kieran (2017), and suggest expanding their framework by addressing the distinction between the activity of reasoning and the process of reasoning, and also addressing the goal of reasoning. Specifically, the use of reasoning to explain is neglected in their framework.

Keywords: Reasoning, Argumentation, Proof, Mathematics curriculum, Language

Introduction

The differing meanings and usages of terms related to argumentation and proof have been discussed in the literature and related to differences in language (e.g., Sekiguchi & Miyazaki, 2000), professional context (e.g., Godino & Recio, 1997), and epistemological perspectives (e.g., Balacheff, 2008). Here I will contribute to this literature by examining the use of the word ‘reasoning’ in the 2020 Norwegian national mathematics curriculum, in the 2000 National Council of Teachers of Mathematics (NCTM) Standards and in the 2003 Education Standards of the German Kultusminister Konferenz (KMK). I will identify differences in usage, make connections to related terms such as ‘argumentation’ and ‘proof’, and suggest a framework for further discussion of these differences.

Related literature

Jeannotte and Kieran (2017) conducted a thorough survey of the ways mathematical reasoning is described in the mathematics education literature and they propose a conceptual model, based in a within a commognitive theoretical framework, to describe mathematical reasoning. In their model they distinguish between structural and process aspects. The structural aspect refers to the form of the reasoning: deductive, inductive or abductive. The process aspect is more complex, and is divided into three processes related to the search for similarities and differences, validating, and exemplifying. The first two processes are further divided into subprocesses.

The search for similarities and differences includes generalizing, conjecturing, identifying a pattern, comparing, and classifying. By ‘generalizing’ they mean inferring something from a given set that applies to a larger set containing it. Conjecturing is characterized by the epistemic value it assigns to an inference: probable or likely. These two processes can (and perhaps often) occur together, but they are distinct. One can generalize with making a claim that the generalization is probable, and one can make a claim that an inference is probable that does not involve generalizing from a subset to a set. Jeannotte and Kieran use ‘identifying a pattern’ to refer to the process of identifying a relation among a set of objects, but this relation need not be extended to a larger set (as in generalizing) nor given a probable epistemic value (as in conjecturing). Comparing can occur along with the other process, and is necessary for identifying a pattern. Comparing, however, refers only to the observation of similarities and differences, without identification of a relation between the objects. Classifying involves making a class of objects based on shared properties.

Processes related to validating include justifying, proving and formal proving. Validating refers to any process that is directed towards changing epistemic value, towards higher or lower likelihood. Justifying is validating that includes a search for data, warrants and backing to modify epistemic value. Proving is validating that specifically modifies epistemic value to truth. It is linked to a set of accepted truths, the use of deductive reasoning (at least in its final stages), and particular, socially accepted, forms of expression. Formal proving is proving that meets stricter criteria for the accepted truths used and the final forms of expression. As Jeannotte and Kieran put it, "formal proving relies on mathematical theory built a priori and on formalized realizations (axioms and theorems)." (2017, p. 13)

Jeannotte and Kieran list a final process, exemplifying, which they say supports the other processes. It consists of producing examples which can then allow patterns to be identified, conjectures and generalizations to be made, and which can be used to justify claims.

It should be noted that Jeannotte and Kieran considered other aspects of mathematical reasoning that were not directly included in their final model, as they felt that the distinctions that they had made captured these other distinctions. One distinction they encountered in the literature but did not explicitly include is the activity/product dichotomy, which separates the product of reasoning from the inaccessible mental activity that gives rise to it. They also considered the inferential nature of mathematical reasoning, that is, the origin of novel ideas through reasoning, and the goal and functions of mathematical reasoning, which refers to the purpose of reasoning which is often verification but might also be exploration or explanation.

Jeannotte and Kieran found that there is no universal agreement on the meaning of 'mathematical reasoning' in the research literature, but they were able to, by considering carefully the ways this term is used, to identify key aspects of it. They note that policy documents like curricula and standards around the world emphasize mathematical reasoning as a goal, but that the description of it in such policy documents "tends to be vague, unsystematic, and even contradictory from one document to the other" (p. 2). Nonetheless, such policy documents seem likely to have a stronger influence on what teachers think mathematical reasoning is, and hence what goes on in classrooms with the goal of fostering mathematical reasoning, than the research literature. Hence, I have chosen to examine one such curriculum document and two national standards documents to explore the use, and hence the meaning, of 'reasoning' in them. I have been informed by, but have not strictly applied, Jeannotte and Kieran's framework, in order to allow for the possibility that distinctions are made in policy documents that differ significantly from those included in Jeannotte and Kieran's framework.

The Norwegian mathematics curriculum

The current Norwegian curriculum (Kunnskapsdepartementet 2019a, b) lists *Reasoning and argumentation* as one of six "core elements", each of which is described in a paragraph. The other core elements are: *Exploration and problem solving, Modelling and applications, Representation and communication, Abstraction and generalization,* and *Mathematical fields of knowledge* (which includes number, algebra, functions, geometry, statistics and probability).

The official English translation of the *Reasoning and argumentation* core element is:

> **Reasoning** in mathematics means the ability to follow, assess and understand mathematical chains of thought. It means that the pupils shall understand that mathematical rules and results are not random, but have clear and logical grounds. The pupils shall formulate their own **reasoning** to understand and to solve problems. Argumentation in mathematics means that the pupils give grounds for their methods, **reasoning** and solutions, and prove that these are valid. (2019b, p. 3, bold added)

An important distinction in the original is missing from this translation. The original text is headed "Resonnering og argumentasjon" and reads:

> **Resonnering** i matematikk handlar om å kunne følgje, vurdere og forstå matematiske tankerekkjer. Det inneber at elevane skal forstå at matematiske reglar og resultat ikkje er tilfeldige, men har klare grunngivingar. Elevane skal utforme eigne **resonnement** både for å forstå og for å løyse problem. Argumentasjon i matematikk handlar om at elevane grunngir framgangsmåtar, **resonnement** og løysingar og beviser at dei er gyldige. (2019a, p. 2, bold added)

Notice that the word 'reasoning' in the English translation is used for two Norwegian words, 'resonnering' and 'resonnement'. The word 'resonnering' does not have a dictionary entry of its own. It is formed from the verb "resonnere" (to reason) with the suffix "-ing" to make it a noun. The word 'resonnement' is a noun means 'thinking, way of thinking, concluding'. Both are nouns, but 'resonnering' is closer to the verb form and is used to refer to the process of reasoning, while 'resonnement' refers to the product of reasoning. Recall that the activity/product dichotomy is a distinction that Jeannotte and Kieran chose not to specifically include in their model. However, in this case, where the language allows this distinction to be explicitly marked, the authors of the Norwegian curriculum have chosen to do so. This suggests that the dichotomy is an important one to them, and that teachers being guided by the curriculum might make a similar distinction. It also reminds us that this distinction is harder to observe in languages like English that have only a single word for reasoning.

Sources

One of the influences on the Norwegian curriculum is the work of Mogens Niss and his colleagues (e.g., Niss & Jensen, 2002; Niss & Højgaard, 2019). The idea of 'chains of thought' originates from Niss's work. For example, Niss and Jensen (2002) write:

> [The reasoning] competence consists, on the one hand, in being able to follow and assess a mathematical reasoning, i.e. a chain of arguments put forward by others in writing or in speech in support of a statement (p. 54, my translation)

However, Niss and Jensen also say that the competence involves:

> understanding what a mathematical proof is and how it differs from other forms of mathematical reasoning, e.g., heuristic reasoning resting on intuition or on consideration of specific cases, and to be able to determine when a mathematical reasoning actually constitutes a proof and when not. (p. 54, my translation)

and

> consists of being able to devise and implement informal and formal reasoning (on the basis of intuition), including transforming heuristic reasoning into actual (valid) proofs. (p. 54, my translation)

The 'reasoning' in the Norwegian curriculum does not mention intuition, consideration of specific cases, or heuristic reasoning, and its emphasis on "clear and logical grounds" suggests that such reasoning is not included.

Summary

The description of reasoning in the Norwegian curriculum touches on several characteristics: the process of reasoning involves following, assessing and understanding mathematical chains of thought; it is related to the grounds of mathematics; the product of reasoning can be formulated; and giving grounds for that product is central to argumentation. It excludes heuristic reasoning and reasoning based on specific cases and intuition, which Niss and his colleagues include as mathematical 'reasoning'.

The NCTM Standards

The 2000 NCTM *Principles and Standards for School Mathematics* have influenced curricula both in the United States and internationally. The structuring of the Norwegian curriculum into core elements may have been influenced by the structure of the NCTM *Standards.* Three of the core elements (*Exploration and problem solving, Reasoning and argumentation,* and *Representation and communication*), have, at least in part, the same names as four of the NCTM 'Process Standards' (*Problem solving, Reasoning and Proof, Communication,* and *Representation*). A fourth (*Modelling and applications*) is very similar in content to the NCTM's *Connections* standard.

The NCTM standard *Reasoning and Proof* is described in a four page long general section, as well as in separate sections for the Pre-K–2, 3–5, 6–8 and 9–12 grade bands. The general section begins with a listing of four goals:

> Instructional programs from prekindergarten through grade 12 should enable all students to—
>
> - recognize reasoning and proof as fundamental aspects of mathematics;
> - make and investigate mathematical conjectures;
> - develop and evaluate mathematical arguments and proofs;
> - select and use various types of reasoning and methods of proof. (p. 56)

The goal "develop and evaluate mathematical arguments and proofs" is also the central focus of the Norwegian core element *Reasoning and argumentation.* However, the NCTM standard is broader as it includes making and investigating mathematical conjectures. Conjecturing is not mentioned in the Norwegian curriculum but the core element *Exploration and problem solving* includes "searching for patterns, finding relationships" (p. 2). This parallels the NCTM's "note patterns, structure, or regularities in both real-world situations and symbolic objects" (p. 56) in the *Reasoning and Proof* standard.

The NCTM Standards also includes the ability to "select and use various types of reasoning and methods of proof." (p. 56). This suggests that, like Niss, the NCTM sees mathematical reasoning as including several kinds of reasoning. One distinction the NCTM makes may be similar to Niss's distinction between heuristic reasoning and proofs:

> At all levels, students will reason inductively from patterns and specific cases. Increasingly over the grades, they should also learn to make effective deductive arguments based on the mathematical truths they are establishing in class. (NCTM, 2000, p. 59)

Several characteristics of reasoning are evident in the description given.

> People who reason and think analytically tend to note patterns, structure, or regularities in both real-world situations and symbolic objects; they ask if those patterns are accidental or if they occur for a reason; and they conjecture and prove. Ultimately, a mathematical proof is a formal way of expressing particular kinds of reasoning and justification. (p. 56)

The asking if patterns are accidental or if they occur for a reason parallels the Norwegian curriculum's "pupils shall understand that mathematical rules and results are not random, but have clear and logical grounds" (Kunnskapsdepartementet 2019b, p. 3) but for the NCTM reasoning includes deciding if a pattern occurs for a reason, while the Norwegian curriculum is more narrowly focused on rules and results that have reasons.

The NCTM associates reasoning with analytic thinking, though what the distinction is between them is not clear. Later the *Reasoning and Proof* standard states "Classrooms in which students are encouraged to present their thinking and in which everyone contributes by evaluating one another's thinking provide rich environments for learning mathematical reasoning" (NCTM, 2000, p. 57). This further suggests a link to a mental activity.

The NCTM is also explicit about what 'proof' means; the Norwegian curriculum does not mention 'proof'. For the NCTM there is an emphasis on proofs being formulated and expressing a particular kind of reasoning, which the passage quoted earlier (from p. 59) suggests is deductive reasoning.

For the NCTM, reasoning is related to understanding.

> Being able to reason is essential to understanding mathematics. By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas and—with different expectations of sophistication—at all grade levels, students should see and expect that mathematics makes sense. (NCTM 2000, p. 56)

Similarly, the Norwegian curriculum states "The pupils shall formulate their own reasoning to understand" (Kunnskapsdepartementet 2019b, p. 3).

Summary

The word 'reasoning' in the NCTM Standards is used more broadly than in the Norwegian curriculum. It includes not only giving reasons, but also making conjectures. These two activities are associated with deductive and inductive reasoning, respectively. Proof is formulated deductive reasoning. As in the Norwegian curriculum, in the NCTM Standards, 'reasoning' is connected to a mental activity, to understanding, and to finding the reasons underlying a pattern, rule or result.

The KMK Standards

In 2003 and 2004 The Standing Conference of the Ministers of Education and Cultural Affairs of the Länder in the Federal Republic of Germany (KMK) issued educational standards for mathematics for the different German school forms. The first one, issued in 2003 for middle schools ending at Grade 10, is my focus here.

The KMK Standards are structured into competences, analogous to the NCTM's Standards and the Norwegian curriculum's core elements. They are *Argumentation, Problem solving, Modelling, Using representations, Dealing with symbolic, formal and technical elements,* and *Communication*. The names of these strongly parallel the names of the NCTM Standards with the exception of *Modelling* (which the NCTM calls *Connections*) and *Argumentation* (which the NCTM calls *Reasoning and Proof*).

The naming of the *Argumentation* competence reveals an interesting linguistic difference between German, English and Norwegian. German has not adopted a word based on the French *raisonner,* and it seems to lack a direct equivalent. Possible translations for the verb 'to reason' include *schlussfolgern* (to conclude) and *begründen* (to give reasons for, to justify). The noun form 'reasoning' can be translated as *logisches Denken* (logical thinking), or *Argumentation* (argumentation). It is interesting that the question "How is the word 'reasoning' used in the KMK Standards?" can be answered briefly "It isn't", and also that the question cannot even be asked in German.

However, we can compare the use of *Argumentation* in the KMK Standards to the use of 'reasoning' elsewhere. The *Argumentation* competence states that:

> Mathematical argumentation … includes:
>
> - Posing questions that are characteristic of mathematics ("Does there exist ...?", "What changes if...?", "Is that always so?") and expressing justified conjectures,
> - Developing mathematical arguments (such as explanations, justifications and proofs)
> - Describing and justifying solution methods. (p. 8, my translation, original in Appendix)

Conjecturing, or at least expressing and justifying conjectures, is included under *Argumentation*, similarly to the NCTM's *Reasoning and Proof*. Developing mathematical arguments and describing and justifying solution methods are both found in both the Norwegian curriculum and the NCTM Standards. Those documents, however, include references to "chains of thought" (Kunnskapsdepartementet, 2019b, p.3) and students presenting "their thinking" (NCTM, 2000, p. 58). These refer to the mental process of reasoning. The KMK Standards, however, do not mention this process and instead focuses on the observable social product of reasoning.

The three kinds of argument named: explanations, justifications and proofs ("Erläuterungen, Begründungen, Beweise" in the original, KMK, 2003, p. 8), suggest different goals for an argument: explaining, justifying, and doing so in a way acceptable to the mathematical community. However, these terms are themselves not defined, and so this connection to goals might not have been what the authors had in mind. If this is what is intended, then these goals overlap with those expressed in the

Norwegian curriculum. which refers to “understanding”, ”give grounds” and “prove” (Kunnskapsdepartementet, 2019b, p. 3). Similarly, the NCTM Standards (2000) state that “being able to reason is essential to understanding mathematics” and that “a mathematical proof is a formal way of expressing particular kinds of reasoning and justification” (p. 56).

Conclusion

Examining the Norwegian curriculum and the NCTM and KMK Standards shows that one distinction Jeannotte and Kieran made, between processes related to the search for similarities and differences and processes related to validating, is also useful for describing a key difference between the Norwegian curriculum and the two Standards documents. In the Norwegian curriculum only processes related to validating are included under the core element *Reasoning and argumentation.* Processes related to the search for similarities and differences, such as conjecturing, are instead included under *Exploration and problem solving*.

Jeannotte and Kieran’s wider distinction, between processes and structures is also interesting, as only the NCTM Standards mentions different structures of reasoning, specifically inductive and deductive reasoning. This may be simply because the other two documents are much shorter.

One difference that Jeannotte and Kieran’s framework does not capture is that between the mental activity of reasoning and the social product of reasoning. This distinction is important in the Norwegian curriculum and reflected in the words used. Though the NCTM Standards uses ‘reasoning’ to refer to both the activity and the product, both are included in the descriptions. The KMK Standards, however, refer only to the product, as ‘Argumentation’. This activity/process distinction reflects most strongly the possible influence of language on the mathematics curriculum in this area. As Jeannotte and Kieran were working within a commognitive theoretical framework, which denies the existence of a split between mental activity and social discourse, it is not surprising that this distinction is not captured in their framework. It does, however, seem to be an aspect of the use of the word ‘reasoning’ in policy documents, and therefore worth attending to.

A final distinction not captured in Jeannotte and Kieran’s framework, but important in the policy documents, is the distinction between different goals of reasoning. The explicit listing of explanations, justifications and proofs as three kinds of arguments in the KMK Standards is also reflected in the other two documents, but seems impossible to capture in the categories listed by Jeannotte and Kieran. The goal of explanation, which is perhaps the most important in educational settings (Hanna, 1989) seems not to be included at all in Jeannotte and Kieran concept of ‘reasoning’.

To capture the way the word ‘reasoning’ is used in policy documents, it would be useful to add a fourth process, *explaining*, to Jeannotte and Kieran’s search for similarities and differences, validating, and exemplifying. Furthermore, some way to make a distinction between mental activities and social discourse seems to be needed. It is not evident how this could be added to Jeannotte and Kieran’s framework, except perhaps as a third aspect.

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Appendix

The original text of the Argumentation competence in the KMK Standards is as follows:

Mathematisch argumentieren

Dazu gehört:

– Fragen stellen, die für die Mathematik charakteristisch sind („Gibt es ...?“, „Wie verändert sich...?“, „Ist das immer so ...?“) und Vermutungen begründet äußern,

– mathematische Argumentationen entwickeln (wie Erläuterungen, Begründungen, Beweise),

– Lösungswege beschreiben und begründen. (KMK, 2003, p. 8)

Changing the significance of argumentation and proof in final secondary school examinations – a comparison between Hungary and Thuringia

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Argumentation and especially proofs play a crucial role in mathematics as a science. To convey an authentic view of mathematics in school, arguments and proofs should play an important role in mathematics classrooms, too. Analysis of textbooks, curricula and examination tasks provide relevant insight into the role that proofs play in mathematics education. This paper compares the weight and the content of arguing and proving tasks in final secondary school examinations in two educational administrations which share similar recent political, social, and educational history.

Keywords: Secondary school mathematics, exit examinations, mathematics tests.

Introduction

Because of verifying mathematical knowledge, proofs play, without any doubt, an essential role in mathematics as a science (Hanna & Barbeau, 2008). Moreover, Rav (1999) emphasizes the unique value of proofs in mathematics, based on their function as a means for gaining and systemizing mathematical knowledge. He concludes: “Proofs, I maintain, are the heart of mathematics, the royal road to creating analytic tools and catalysing growth.” (ibid., p. 6). In addition, Heintz (2000) declares, proofs would constitute mathematics and, mathematics would define itself by proofs. In any case, demonstrating proofs must be only seen as one of the main activities in mathematics.

Among mathematics-education researchers, there is a consensus about the key role of proofs both in mathematics classrooms and curricula (e.g., Hanna, 2000; Harel, 2008; Mariotti, 2006; Nardi & Knuth, 2017). The reasoning behind this, however, can fundamentally differ. While Harel (2008), for instance, underlines the way of thinking which becomes manifest in proofs, Hanna (2000) emphasizes the function of promoting mathematical understanding. Beyond that, Nardi and Knuth (2017) point out an active view of learners and plead for proofs in school because they are “critically important to knowing and doing mathematics” (p. 267). Different reasoning, indeed, leads to pursuing various objectives as well as to varying implementations when teaching proofs in mathematics classrooms.

In particular, cross-national research on curricular documents, such as educational standards, curricula, and textbooks, can provide an essential contribution to detecting the above-mentioned differences in regard to the objectives and implementations of proofs in mathematics education. This approach was taken by Jones and Fujita (2013), who investigated the implementations of national curricula in the geometry chapters of textbooks in England and Japan. Despite noticing many similarities between the geometry curricula, they identified differences in the two countries regarding the treatment of proofs. Whereas in England a low ratio (6.9 %) of proof-related tasks was found, the ratio was much higher (26.2 %) in Japanese textbooks. This result can be explained by the fact that the Japanese curriculum explicitly stipulates proofs only for geometry, whereas the English curriculum requires them also in the domains of numbers and algebra. Based on a comparative analysis of textbooks, teacher guides of textbooks and curricula, Miyakawa (2017) identifies differences between France and Japan regarding the nature of proofs to be taught in geometry. He finds that implicit differences in geometry theory as well as in the principal function of proof related to that theory influence the nature of proof in geometry education. In addition, there is a wide range of national analyses of textbooks related to the role of proof, for example, in Australia (Stacey & Vincent, 2009), in the USA (Stylianides, 2009), and in Hong Kong (Wong & Sutherland, 2018).

According to Karp and Shkolnyi (2021), not only textbooks and curricula, but also final exams have a high impact on mathematics education. They also state that the scholarly literature related to final exams in mathematics is not extensive. Moreover, none of the reported studies in their very recent article is explicitly linked to argumentation or proof. With the cross-national study presented in this paper, the author aims to contribute to this research gap.

The final secondary school examination in Thuringia and Hungary

Even if during the last decades, emphases have changed several times in mathematics education, proofs still play an important role in Hungarian mathematics education. A similar development can be seen in Thuringia, these days a German federal state. Hungary and Thuringia share a very analogous recent history on the one hand, while having both experienced substantial changes to their educational systems about the same time, on the other. Following the political changes in 1989, the democratization of the school system took place in the 1990s in both places. For different reasons, a second transformation took place around 2005. In Germany, the Standards for the General Certificate of Secondary Education in Mathematics (Kultusministerkonferenz, 2004) were established, and all federal states started to implement them in their curricula. Meanwhile, Hungary established a new, modernised final secondary school exam in mathematics at that time (Lukács, 2006).

For better understanding of the current research, I describe in the following passage some essential changes, which were applied to the final secondary examination in mathematics in both places. Note that the final secondary exam is carried out in each place at different levels of ability and in different kinds of secondary schools. This paper focuses on the main group addressed by this type of examination: learners in grammar school (Gymnasium, gimnázium) taking a basic course in mathematics and taking the secondary school exam between 2001 and 2020.

Table 1 shows the changing of the main exam characteristics in both places over time. The data demonstrate different tendencies in Hungary and Thuringia: in Hungary, one main structural change took place in 2005, while the characteristics of the examination in Thuringia were changed stepwise. Both places, however, share the characteristics that at the beginning of the 2000s only complex tasks were assigned, whereas in 2020 a mixture of elementary and complex tasks characterize the examination in mathematics, the latter being (partly) elective. Elementary tasks (such as calculating the first derivative of a polynomial function) require only a few cognitive steps. In contrast, complex tasks (such as sketching a curve that is models an every-day problem) not only need the use of several cognitive steps but also often require combining different kinds of information. In addition, in Thuringia over the last twenty years, not only have the total exam points available increased twice, but also the length of time for the exam has been significantly extended.

Table 1: main characteristics of the final secondary school examination in mathematics

<table>
<tr><th>time period</th><th>Hungary</th><th>Thuringia</th></tr>
<tr><td>2001</td><td rowspan="2">180 minutes
total score: 80
6 complex tasks and 1 additional proving task</td><td>210 minutes, total score: 60
1 (out of 2) and additional 2 (out of 3) complex tasks</td></tr>
<tr><td>2002 – 2004</td><td rowspan="2">210 minutes (270´ from 2011), total score: 60
4 – 5 elementary tasks and 2 (each out of 2) complex tasks</td></tr>
<tr><td>2005 – 2013</td><td rowspan="3">180 minutes
total score: 100
12 elementary tasks and 2 (out of 3) additional complex tasks</td></tr>
<tr><td>2014 – 2016</td><td>270 minutes, total score: 60
7 – 8 elementary tasks and 2 (each out of 2) complex tasks</td></tr>
<tr><td>2017 – 2020</td><td>(270 in 2017) 300 minutes, total score: 120
7 – 8 elementary tasks 1 complex task and 1 (out of 2) additional complex task</td></tr>
</table>

Research questions

Against the background of the high significance of proofs in mathematics as a science as well as in mathematics education, this paper investigates, to what extent has been changed their weight and content in mathematics education in these two states. Such an analysis can give insight into the claims implemented in mathematics education and addressed to the learners at the end of their secondary education. The notions "argumentation" and "proof" are used in different ways in the related literature. However, in this paper both, mathematical argumentation and proof, are understood as realizations of reasoning in mathematics, based on Brunner (2014). The research presented in this paper was led by the following question: To what extent have the expectations at the end of secondary education related to mathematical argumentation and proof in Hungary and Thuringia been changed in the last two decades? Are the tendencies in those places similar? Which conclusions can be drawn about the role of proofs, based on the identified tendencies?

Methodology

The methodology used in the study was already successfully applied and detailed described in Szűcs (2021). A short summary of this is to identify temporal alterations based on existing tasks and guidance material for marking and evaluating. Documentary research was chosen as an appropriate method for data collection. In addition, qualitative content analysis, which allows for the systematic and theory-based processing of big textual data, was selected for data examination.

Documentary research

Original examination tasks, as well as the relevant guidelines for marking and assessment, are the primary sources of the study. *The nature of the source material* in the places that were researched is insignificantly different. Whereas examination tasks, including guidance material, are in the public domain in Hungary, this is not the case in Thuringia. However, materials for the whole period under analysis could be reconstructed in both places based on the public sources of the Hungarian Ministry of Education, on the unpublished sources of the Thuringian Ministry of Education and, on the information available from task developers (Fried, 2004; Skorsetz, 2005). However, based on the authority of the providers, we can conclude, that all sources are authentic, reliable, and trustworthy.

Structuring qualitative content analysis

Since models and categories related to argumentation and proof in mathematics already exist, a special type of qualitative content analysis, the so-called structuring qualitative content analysis, was deemed to be appropriate. In addition, this method is particularly useful for identifying temporal tendencies, which is extraordinarily relevant for the current research questions.

Table 2: excerpts from the encoding manual

category	**reasoning with mathematical tools**
definition	Argumentation is based on mathematics, but not necessarily on deductive steps.
standard example	Three books were taken from a bookcase and put back arbitrarily. […] Demonstrate, that it is not possible that exactly two of the three books are in the right place. (2009, part C, task e)
explanation	Systematic testing of all ways is possible.
encoding rule	Reasoning based on concrete examples is possible and productive.
category	**logical reasoning with mathematical tools**
definition	Deductive steps are identifiable, but they are not necessarily formal.
standard example	Three points, A (3;0;5), B (6;4;3) and D (0;4;3) are given, in a Cartesian coordinate system. Show that A, B and C explicitly determine a plane ε. (2001, part 2.1, task a)
explanation	Definition of linear dependence and vectors must be deductively applied.
encoding rule	Deductive application of definitions, theorems and methods is needed, but no formal proof.
category	**formal-deductive proof**
Definition	formal chain of deductive steps
Standard example	(a_n) is a sequence of numbers with $a_n = \frac{n+1}{n}$, $n \in \mathbb{N}$ and $n \geq 1$. Show that (a_n) is not a geometric sequence. (2002, part C, task a)
explanation	Formal notation and, the generality of the claim suggest that a formal proof is expected.
encoding rule	Requirement of a formal proof must be recognized.

To determine *basic material,* it is necessary to define the terms “argumentation” and “proof”. A viable definition of argumentation, which includes the notion “proof”, was given by Schwarzkopf (2000). It means a social interaction in school, in which a need for reasoning is indicated, and afterwards, this need is tested to be satisfied. According to this, all examination tasks have been rated as argumentation tasks, in which a need for reasoning was clearly indicated, meaning tasks contained verbs such as “show”, “reason”, “prove” etc. Note that searching for keywords is a popular method to identify argumentation tasks when analysing textbooks (Mariotti et al., 2018). The set of tasks containing at least one of the listed verbs is called the basic material. A *set of categories* has been specified, according to Brunner (2014). She identifies four categories based on the cognitive level needed for reasoning in mathematics classrooms, which enables a qualitative differentiation between the expected arguments. Excluding the type “everyday-arguing”, the following three types have been applied to the data: reasoning with mathematical tools, logical reasoning with mathematical tools and, formal-deductive proving. The *encoding* was carried out as follows: Each coded text passage was mapped to one of the three categories. This mapping process was supported by the information given in the guidance material. Table 2 shows excerpts from the encoding manual; the tasks have been translated into English by the author of this paper.

After encoding, *results were prepared* as follows: Based on the marking instructions, the score of each argumentation task, and its related mathematical domain were recorded. The total score of each category was calculated for each year and, afterwards, their proportion of the whole exam was determined. Tasks of choice, in the meaning of alternatives, were noted. Scores of the three categories added up to a total argumentation score for each year. Within this total score, the proportion of each mathematical domain was also calculated. Table 3 gives insight into this process, based on the Thuringian data from 2002.

Table 3: Encoding the arguing tasks from Thuringia in 2002

year	task	category	score		proportion of the exam	domain
2002	part A1, task b	log. reasoning with math. tools	1	average: 1.5 (A1 and A2 alternatives)	2.5 %	calculus
	part A2, task a	log. reasoning with math. tools	2			calculus
	part C, task a	formal-deductive proof	2		3.3 %	calculus
	total of argumentation tasks		3.5		5.8 %	

Results

Comparison of the changing of quantitative aspects in the final secondary school examination

Tendencies related to the volume of argumentation tasks and the type of reasoning are shown in Figure 1. The data suggest – similarly to the findings of Miyakawa (2017) – that there are differences in the nature of argumentation tasks between Hungary and Thuringia, which can be traced back to different implicit views on mathematics. In Hungary, before the new examinations in 2005, a high proportion (15.31 %) of argumentation tasks was present and, those tasks required only formal-deductive proofs. However, the introduction of the new exam led not only to the halving of the proportion of these tasks (7.75 %) but also to the complete absence of formal-deductive proofs. Hungary experimented between 2005 and 2016 by addressing argumentation tasks, which do not require deductive reasoning but gave up on that in 2017. Hungary seems to be a country, in which proofs in school mainly have the function of demonstrating a specific, deductive way of thinking (Harel, 2008). In Thuringia, in contrast, formal-deductive proofs did not play an important role in the period under investigation. However, the proportion of argumentation tasks was relatively constant over the time in question (12.7 %–15.8 %) and increased in the last four years to over 20 %. The proportion of tasks requiring logical reasoning with mathematical tools varies between 5.8%–14.5%, but they are complemented by an increasing number of tasks requiring reasoning with mathematical tools. Thus, it could be inferred that argumentation plays an important role in Thuringia, too, but the focus is more on applied mathematics.

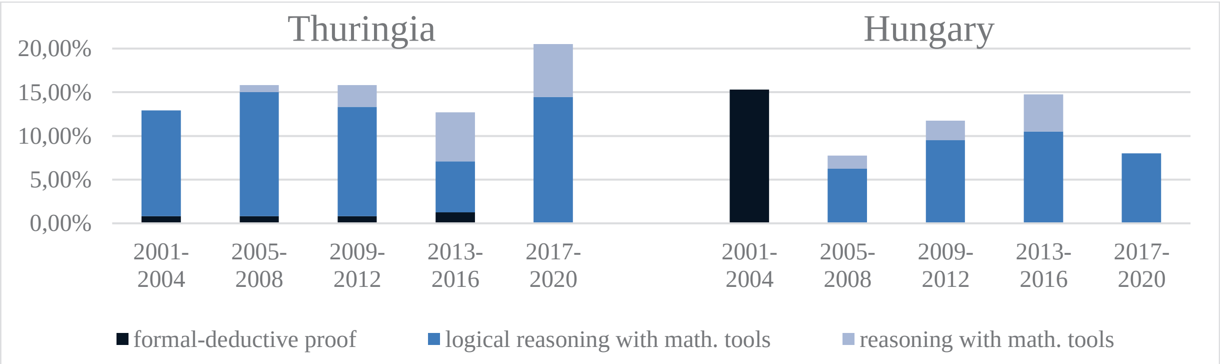

Figure 1: changing of the types of reasoning in final secondary school examinations over time

Comparison of the changing of qualitative aspects in the final secondary school examination

The above-described quantitative tendencies regarding the type of reasoning also have a qualitative component. Changes related to the specific mathematical domains are presented in Figure 2. Each percentage expresses the proportion of the score of the domain related to the total score of argumentation tasks.

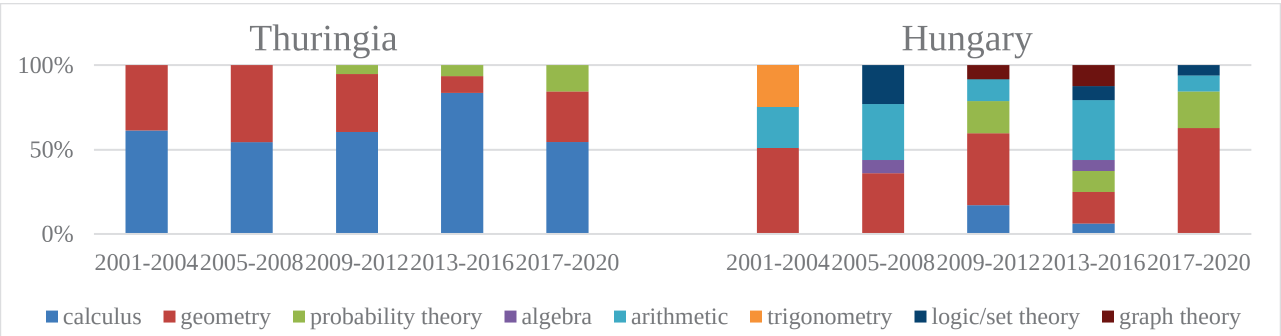

Figure 2: changing of the mathematical field of reasoning tasks over time

Significant differences between the two places in the study can also be observed regarding domains. Whereas only three mathematical domains are used in Thuringia, the argumenation tasks in the final secondary school examination in Hungary become more varied over time, covering up to seven different domains. Furthermore, calculus dominates constantly in Thuringia, but it is barely evident in Hungary. In addition, while geometry is noticeable but not dominant in Thuringia, it is the main mathematical domain of argumentation in Hungary. Mathematical reasoning is more spread across various domains in Hungary than it is in Thuringia. These findings are similar to the results which compare England and Japan (Jones & Fujita, 2013) and can traced back to different values of the final secondary school exam in those two places: Whereas this type of examination rounds upper secondary school education in Thuringia, it finishes the entire secondary education in Hungary.

Summary and open questions

Even if argumentation and proof form the main parts of mathematics as a science leading to the view that they should play a key role in mathematics education, they are only moderately included in final secondary school examinations in Hungary and in Thuringia. Especially alarming is the fact that formal-deductive proofs currently play no role in those examinations. However, slightly different routes led to the current situation: In Hungary, formal-deductive proofs disappeared after the structural change of the exam in 2005, while they have never been focused on in Thuringia. Moreover, mathematical reasoning is spread across more various mathematical domains in Hungary than it is in Thuringia. These results may allow us to infer different views of mathematics in the two places. However, further analysis of curricula, textbooks and classroom activities would be needed to investigate those views and confirm this inference.

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Chinese teachers' professional noticing of students' reasoning in the context of Lakatos-style proving activity

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Despite the pedagogical values of Lakatos-style proving activity in school mathematics, little is known about how teachers react to students' reasoning in this kind of activity. As an attempt to unpack the underlying decision-making processes, this study examines teachers' noticing of students' justification and refutation of conjectures in the context of Lakatos-style proving activity. Twelve Chinese pre-service and in-service secondary mathematics teachers participated in semi-structured, vignette-based interviews where they were presented with realistic classroom scenarios. These were based on actual classroom episodes reported in the literature and covered various aspects of Lakatos-style reasoning in the context of geometry. Findings show that teachers can better notice students' justifications than students' refutations, and better notice students' valid arguments than invalid arguments. Key themes are identified to characterise their noticing of various student arguments.

Keywords: Proof, Lakatos, Justification, Refutation, Professional noticing.

Introduction

In the form of fictional classroom discussion, Lakatos (1976) described how mathematicians constructed and utilised mathematical knowledge through a zig-zag reasoning process. Some aspects of Lakatos-style reasoning, such as conscious guessing and the zig-zag path of reasoning, were suggested by mathematical educators (e.g., Lampert, 1990) to be applied in some school mathematical activities for engaging students in authentic mathematics. Some empirical studies also showed that school-age students can perform in line with Lakatos-style reasoning, but most of them paid attention to students' proving processes (e.g., Komatsu, 2016; Reid, 2002), while few studies focused on the role of teachers. It remains unclear how teachers deal with various types of students' responses throughout different phases of Lakatos-style proving. To address this research gap, one promising attempt is to investigate teachers' professional noticing by focusing on how they pay attention to and make sense of particular instructional situations (Jacobs et al., 2010). We explore teachers' professional noticing of students' reasoning in the context of Lakatos-style proving activity. In this paper, we focus specifically on how teachers notice students' uses of examples and counterexamples during the justification and refutation of conjectures, which potentially can inform students' further refinement of the conjectures or proofs thereof.

Theoretical framework

Lakatos-style proving process

Using Lakatos' (1976) book and some mathematics education studies that discussed the implementation of his approach in school mathematics (e.g., Reid, 2002; Komatsu, 2016; Deslis et al., 2021), we identified five phases of the Lakatos-style proving process to capture some aspects of Lakatos-style reasoning: First, a conjecture is formulated through conscious guessing (Phase 1). Then the conjecture is tested through examination of supportive examples (Phase 2). A proof may be constructed to further validate the conjecture (Phase 3). Yet, counterexamples may emerge that refute the conjecture or the respective proof (Phase 4), thus necessitating the refinement of the conjecture or the proof (Phase 5). Note that Lakatos' philosophy is more complex than this 5-phase framework. It also emphasises other aspects (e.g., the crucial role of the interplay between defining and proving).

Justification and refutation schemes

We utilised a framework of students' justification and refutation schemes to conceptualise how students justify and refute conjectures in diverse ways, reflecting different degrees of mathematical sophistication. This framework was constructed drawing on previous research on students' proof schemes (Balacheff, 1988; Harel & Sowder, 1998; K. Lee, 2016; Stylianides & Stylianides, 2009; Deslis et al., 2021). There are four levels of justification schemes. Students at the lowest level (Naïve empirical justification) believe that examining a few examples which are easy to check (e.g., examining the example "x = 1" for validating the conjecture "for any natural number, 2x is an even number") can prove a mathematical generalisation. Students at a higher level (Crucial experiment justification) also accept example-based proofs, but they believe that the examples need to be strategically identified following some rationale (e.g., examining a set of odd numbers "x = 1, 3, 5, 7, 9…" for the above-mentioned conjecture). Students at the next level (Nonempirical justification) believe that it is not sufficient to validate a conjecture based on a subset of examples, but unlike students at the most advanced level (Deductive justification), they may not recognise the role of deductive inferences in proof. There are also four levels of refutation schemes. Students at the least advanced level (Naïve refutation) regard counterexample(s) as exception(s), and still consider a conjecture to be true regardless of the existence of counterexample(s). Students at the next level (Empirical refutation) think it is insufficient to refute a conjecture based on a single counterexample and need to see more counterexamples to be convinced that the conjecture is false. Students at the next level (Single counterexample refutation) believe that it is sufficient to refute a conjecture based on a single counterexample. Students at the most advanced level (General counterexample refutation) accept the sufficiency of a single counterexample in refuting a conjecture and recognise further that identifying the common properties of counterexamples can support the refinement of the conjecture.

Noticing

Despite various conceptualisations of teacher noticing, it is generally considered to involve at least two components, *Attending* and *Interpreting* (e.g., Es & Sherin, 2008). Jacobs et al. (2010) additionally introduced a third component, *Deciding*, which works with the other two components in integrated ways to lay the foundation for teachers' responses to students' mathematical thinking. This idea has been used in much later research (e.g., M. Y. Lee & Francis, 2018). Following Jacobs et al.'s (2010) conceptualisation, we define teacher professional noticing as an integrated set of three key processes: (1) selectively *attending* to noteworthy students' strategies in particular instructional events; (2) *interpreting* students' understanding reflected in these strategies; and (3) *deciding intended responses* to students (as opposed to executing actual responses).

Research methods

Data were drawn from semi-structured interviews with twelve Chinese teachers. For a diversity of teacher profiles, these participants included four pre-service teachers, four novice teachers with an average of 2.75 years of teaching experience, and four experienced teachers with an average of 17.75 years of teaching experience in junior high school for students aged 12-15. They were recruited through convenience sampling.

The one-hour interviews were conducted online. The participants were presented with a set of classroom vignette episodes showing how students solved a geometric proof task (see Figure 1). These episodes were adapted from actual classroom scenarios reported in Komatsu et al.'s (2014) research on Lakatos-style proving, guided by the above-mentioned theoretical frameworks. They reflect the five phases of Lakatos-style proving and diverse students' understandings in a format of classroom discussion, trying to make the episodes sufficiently realistic to elicit teachers' responses in the context of Lakatos-style activity (Skilling & Stylianides, 2020). We chose to present teachers with comic-style episodes because Lakatos-style proving activity is believed to be scarce in existing Chinese classrooms. Also, comic-style episodes allowed us to provide participants with sufficiently realistic but also abstract enough information, aiming to direct their attention to critical aspects of the classroom practices of interest, and allow them to form their interpretations of such context (Herbst et al., 2011). Note that we did not study what teachers actually noticed in real classrooms. Instead, we attempted to explore what they might notice in the context of Lakatos-style proving activity.

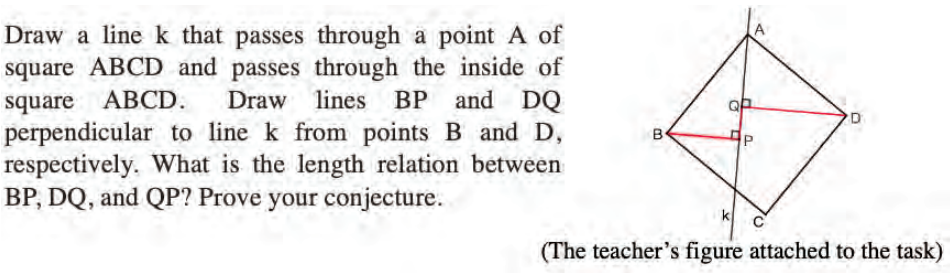

Figure 1: The geometric proof task (Komatsu et al., 2014) presented in the vignette episodes

Based on the above rationales, we designed eleven episodes around this proof task, among which there were four episodes about justifications and four episodes about refutations of the conjecture (i.e., irrespective of the position of line k, PQ = DQ − BP). These eight episodes respectively reflected each level of Justification and Refutation schemes, covering Phases 2-4 in the Lakatos-style proving process. Figure 2 shows two sample episodes translated from Chinese to English.

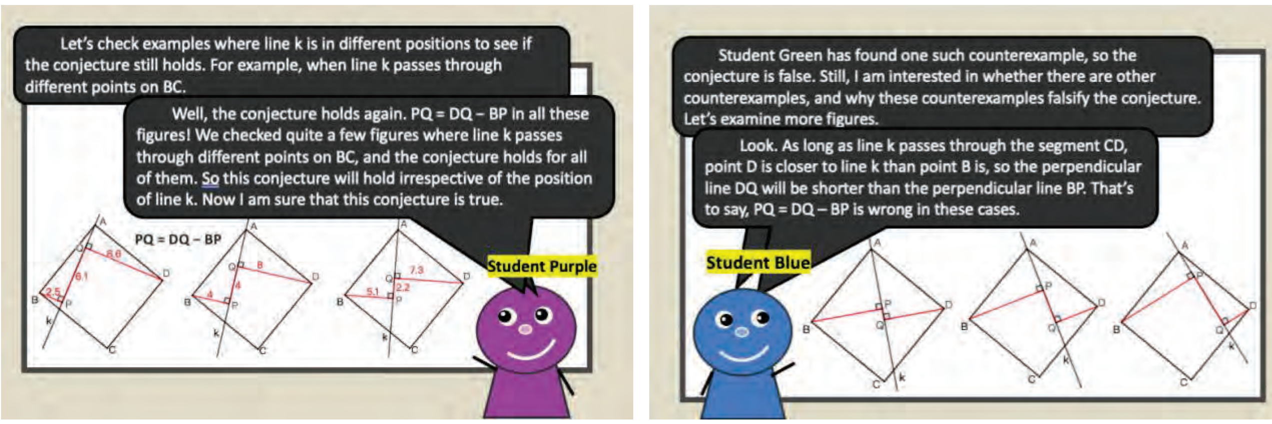

Figure 2: Crucial experiment justification (Left) and General counterexample refutation (Right)

After seeing each episode, participants were asked to describe (i) the students' thinking and/or actions that they attended to, (ii) how they interpreted the students' understandings, and (iii) how would they respond to the students, corresponding to the *Attending*, *Interpreting*, and *Deciding* aspects of teachers' professional noticing. Following Jacobs et al.'s (2010) coding scheme of teacher noticing, teachers' responses were coded based on the extent of evidence that teachers demonstrated in their consideration of students' reasoning. Specifically, responses about the *Attending* aspect were coded on a 2-point scale: Evidence (1) and Lack of evidence (0), and responses about the *Interpreting* and *Deciding* aspects were coded on a 3-point scale: Robust evidence (2), Limited evidence (1), and Lack of evidence (0) (Jacobs et al., 2010). After coding teachers' responses into the various categories, emerging themes were identified for each category to capture its characteristics (Corbin & Strauss, 1990). Double counting was applied for responses that related to more than one theme.

Findings

Overview of teachers' professional noticing

To capture participants' noticing of students' justifications and refutations, mean scores of the *Attending, Interpreting,* and *Deciding* aspects were calculated for each of 8 episodes. Given that the two lower levels and the two higher levels of Justification and Refutation schemes describe students' invalid and valid reasoning, respectively, we calculated the average scores of each pair of levels as a more stable measure of teachers' noticing of (in)valid justification/refutation arguments (Table 1).

Participants had on average higher scores in noticing students' justifications than refutations, across each aspect of teacher noticing – Attending (0.75 vs. 0.54), Interpreting (1.29 vs. 1.13), and Deciding (1.31 vs. 1.29). Although participants showed the same averages (1.29) in interpreting students' valid and invalid justifications, they showed higher averages in attending to (0.96 vs. 0.54) and deciding how to respond to (1.42 vs. 1.17) students' valid justifications than invalid justifications. A similar pattern emerged from participants noticing of students' refutations.

Table 1: Means (SD) for teaches' scores of noticing students' justification and refutation

Component skill (Scale)	Justification			Refutation		
	Invalid	Valid	Overall	Invalid	Valid	Overall
Attending (0-1)	0.54 (0.51)	0.96 (0.20)	0.75 (0.44)	0.46 (0.51)	0.63 (0.49)	0.54 (0.50)
Interpreting (0-2)	1.29 (0.55)	1.29 (0.46)	1.29 (0.50)	1.13 (0.68)	1.13 (0.54)	1.13 (0.61)
Deciding (0-2)	1.17 (0.56)	1.42 (0.65)	1.31 (0.59)	1.08 (0.65)	1.50 (0.66)	1.29 (0.68)

To supplement the average scores and give a more complete picture of teacher noticing, Table 2 shows the number of teacher responses which were coded as showing different extents of evidence.

Attending to students' strategies

Regarding the *Attending* aspect, responses with evidence mentioned exact mathematically important details of specific students' strategies, as described by our theoretical framework. Most participants gave evidence of attending to students' justifications, except to a student's Crucial experiment justification. Meanwhile, most participants demonstrated evidence of attending to Single counterexample refutation, but they did not demonstrate evidence of attending to other types of refutation. This may explain partly why participants had higher averages in attending to students' justifications, especially valid justifications, than other types of student reasoning.

Table 2: Number of teacher responses which were coded as showing the different extent of evidence

	Justification (Total: 48 teacher responses)		**Refutation** (Total: 48 teacher responses)	
	Invalid (Total: 24)	**Valid** (Total: 24)	**Invalid** (Total: 24)	**Valid** (Total: 24)
Attending	Evidence (N=13) Lack of evidence (N=11)	Evidence (N=23) Lack of evidence (N=1)	Evidence (N=11) Lack of evidence (N=13)	Evidence (N=15) Lack of evidence (N=9)
Interpreting	Robust evidence (N=8) Limited evidence (N=15) Lack of evidence (N=1)	Robust evidence (N=7) Limited evidence (N=17)	Robust evidence (N=7) Limited evidence (N=13) Lack of evidence (N=4)	Robust evidence (N=5) Limited evidence (N=17) Lack of evidence (N=2)
Deciding	Robust evidence (N=6) Limited evidence (N=16) Lack of evidence (N=2)	Robust evidence (N=12) Limited evidence (N=10) Lack of evidence (N=2)	Robust evidence (N=6) Limited evidence (N=14) Lack of evidence (N=4)	Robust evidence (N=13) Limited evidence (N=9) Lack of evidence (N=2)

Among the lack-of-evidence responses, some of them omitted mathematically important details of the students' strategies. This characteristic was evident in 8/12 teacher responses to a student's Crucial experiment justification. To illustrate, 7 teachers mentioned that this student tested *more examples* compared to another student who only tested one example, but they did not comment on the unique feature of this student's reasoning, that is, the *strategic selection* of examples. Some other lack-of-evidence responses included information that was inconsistent with the students' strategies. This was notable in teachers' responses to students' invalid refutations (9/24). For example, regarding the episode about Empirical refutation, 3 teachers wrongly noted that the student wanted to find *all* counterexamples, whereas the student just suggested finding *more* counterexamples.

Interpreting students' understandings

Concerning the *Interpreting* aspect, robust-evidence responses exactly described what the students did and did not understand (as suggested by our theoretical framework), citing details of the students' mathematically important strategies. Eight participants gave robust-evidence responses for interpreting a student's Naïve empirical justification. Yet, when interpreting other types of student arguments, no more than 4 participants provided robust evidence. This may partially explain teachers' slightly higher average scores in interpreting justifications (1.29) versus refutations (1.13).

The majority of limited-evidence responses omitted some details of students' (mis)understandings. This characteristic existed in more than half of teachers' responses, except those for interpreting Naïve empirical justification and Empirical refutation. For example, when interpreting a student's General counterexample refutation, a teacher expressed appreciation of this student's idea of comparing three counterexamples to examine the conjecture, but she did not elaborate on this idea.

Some limited-evidence responses contained descriptions inconsistent with specific student strategies, although they precisely mentioned some (mis)understandings of these students. To illustrate, commenting on a student's Empirical refutation, a teacher said this student's idea (i.e., finding more counterexamples so that we can be convinced) was acceptable in refutations, although she also mentioned that using one counterexample can already refute the conjecture. By contrast, another teacher misinterpreted that this student would like to test *all* counterexamples, and all descriptions this teacher provided were inconsistent with the student's strategy details. This was unclear whether this teacher understood the student's strategy, so this response was coded as "Lack of evidence".

Some teachers' limited-evidence responses contained descriptions as if based on teachers' assumptions rather than on the vignette provided. For example, a teacher assumed that the student with Naïve refutation ignored the counterexample because "this student was stubborn and was not willing to hear others' ideas", but this was hard to justify based on what the student said.

Deciding how to respond based on students' understanding

Around half of the teacher responses for valid justifications (12/24) and refutations (13/24) gave robust evidence when deciding how to respond to a student, while much fewer responses for invalid justifications (6/24) and refutations (6/24) showed robust evidence. In these robust-evidence responses, teachers demonstrated explicit consideration of the students' reasoning and how their proposed responses could further these students' thinking (as suggested by our theoretical framework). For example, in a robust-evidence response for a student's Crucial experiment justification, a teacher made use of one mathematically important aspect of this student's strategy (i.e., asking the student to analyse common properties of these strategically identified examples) to facilitate the student's progression from testing examples to giving a proof:

> Teacher: I will appreciate the student's spirit of exploration, and I will remind him these are only some examples…The student needs to learn how to analyse…whether there are common properties among different figures (i.e., examples). If there are, can we start proving this conclusion with geometric proof? He has drawn a lot of figures, and we need to find their common properties.

Limited-evidence responses demonstrated teachers' uses of students' reasoning in deciding how to respond but in a general or unproductive way. To illustrate, 4/12 teachers proposed very similar responses (e.g., simply asking students to give a proof) for students' Naïve empirical justification and Crucial experiment justification, even though both types of student reasoning indicated different levels of mathematical sophistication (as reflected in students' way of identifying examples to validate the conjecture). Some teachers suggested to the student who ignored the counterexample (Naive refutation) to find more counterexamples or accepted the idea of the student with Empirical refutation to find more counterexamples, in order to convince both students that this conjecture was refuted. But they did not remind the students that one counterexample can already refute the conjecture, and this was not conducive to students' development of refutation ability.

Unlike responses with robust or limited evidence, lack-of-evidence responses cited few or no details of the specific students' reasoning (e.g., "I will let the student think more and then examine based on his idea…I think it will be better if I allow students to inquire rather than directly telling them the answers"), leaving open the question of whether teachers consider these students' reasoning.

Discussion and Conclusion

To conclude, our results show that teachers may be less able to notice students' refutations than justifications. When the students in our vignettes gave invalid arguments (regardless of whether for justification or refutation), teachers were less likely to attend to and decide how to respond to their thinking. Noticing with non-robust-evidence may hinder teachers from making use of students' reasoning to engage students in Lakatos-style zig-zag processes of conjectures, proofs, and refutations.

For instance, many participants noticed that compared to the student with a Naïve empirical justification, the student with a Crucial experiment justification tested more examples. Yet, they did not point out that these examples were strategically identified. In other words, teachers may be sensitive to whether students' reasoning is example-based, but they may pay limited attention to and interpretation of students' ways of identifying supportive examples. Their neglect of such a mathematically important strategy may partly explain why some teachers suggested to both students a very similar next step (i.e., directly asking them to switch from example-based reasoning to giving a proof), without using each student's existing reasoning as a starting point for his/her further development. By contrast, like a teacher's robust-evidence response that we quoted above, one productive use of a student's Crucial experiment justification can be letting this student analyse the common properties of the strategically-identified examples for the further construction of a proof.

For students' refutations, despite the wide recognition of the role of a single counterexample in refutation, some teachers still allowed or even encouraged students who had a Naïve refutation or an Empirical refutation scheme to examine more counterexamples to confirm the conjecture was false. This may not be conducive to students' understandings of the minimally necessary and sufficient way of refutations. Yet if teachers can emphasise the role of a single counterexample and meanwhile support students' investigation of more counterexamples to find out their common properties that refute the conjecture (as described by the General counterexample refutation), students may have opportunities to experience the process of refining a conjecture based on analyses of counterexamples.

Overall, this study constructs a picture of teachers' professional noticing of students' thinking in justification and refutation in the context of Lakatos-style activity, which was seldom investigated in previous research. This can help us (as a field) to better understand in such context how teachers attend to and make sense of students' justifications and refutations in different ways, which in turn may shape what learning opportunities teachers offer to students in the follow-up refinement of the proof or the conjecture. To better prepare teachers to implement Lakatos-style proving activity, further training on their professional noticing, especially their noticing of students' refutations and invalid justifications, is needed. Finally, we acknowledge Sherin and Star's (2011) critique of research on teacher noticing that we indirectly learn what teachers notice from what they express in their comments, but the underlying mechanism of teacher noticing is still unclear. In a further study, we will try to unpack such a mechanism by considering how teachers' views (e.g., their views of proofs, teaching proof, and noticing) condition teachers to notice.

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TWG02: Arithmetic and number systems

Introduction to the work of TWG 2: Arithmetic and number systems

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Introduction

Learning and teaching arithmetic and number systems through activities in kindergarten and school is a relevant and broad field in mathematics education. In kindergarten, children start to develop a number concept of natural numbers with counting and subitizing. Primary school aims at evolving a deep conceptual understanding of numbers and basic arithmetic operations. The transition from natural numbers to rational numbers is a key challenge for mathematics education in the first years of secondary school. Only if students have a deep understanding of natural numbers, will they be able to develop conceptual understanding and procedural skills in rational numbers. Even if a great variety of topics comes with learning and teaching arithmetic and number systems there are common goals, such as developing conceptual understanding and building number- and structure-sense as foundations for flexibility in mental and written arithmetic operations. Research in arithmetic and number systems does not only focus on typical content, but also on models for teaching and learning, approaches for heterogeneous and inclusive classrooms, analogue and digital tools to support understanding, and, not to forget, teachers' competencies and cultural practices.

Working Group 2 was formed in 2011 and has developed as a forum for discussing theoretical and empirical research on the teaching and learning of arithmetic and number systems. Over the last decade, our work has aimed to acquire and enhance knowledge about students' understanding and meaningful learning in this content area regarding different ages and achievement levels. The scope of the TWG comprises kindergarten to 12th grade and emphasizes research-based specifications of domain specific frameworks, concepts and goals, analysis of learning processes and learning outcomes in different classroom cultures, as well as innovative teaching and diagnostic approaches that attend to both procedural and conceptual knowledge.

This year, we faced the same variety of topics as in previous conferences, but even more diversity in theoretical approaches and measurements. This was a great opportunity to negotiate our understanding of terms and clarify our theoretical and methodological approaches. The group intensively discussed 15 papers and 6 posters in the plenary whole group as well as in small groups.

Discussed papers and posters

The presented and discussed papers and posters can be pooled in four thematic groups: Number sense, understanding of basic ideas, strategies in mental arithmetic, teaching approaches and methodological approaches.

Number sense

Nathalie Bisaillon explores the role of groupitizing (the ability to recognize structured quantities without counting) in building internal representations of quantities to enhance students number sense. Three activities are presented to highlight the links between groupitizing skills, the construction of dynamic and imagistic mental representations and the comprehension of numeration. It is hypothesized that the use of these activities with 7–8-year-olds would enhance their number sense.

Brumm Leonie and Elisabeth Rathgeb-Schnierer present their initial study on comparing strategies in numerosity estimation in grade 3 students. Their interest focused on using two and three-dimensional tasks, and the relationship between strategies and task characteristics. The aim of the study is to reveal effective strategies which lead to high estimation accuracy according to the type of task. (poster)

Zübeyde Er and Perihan Dinç Artut investigate the strategies of gifted students in grade 5 in number sense problems. Twenty-one students undertook twenty-five questions in a number sense test where both opinions and answers were analysed. The results of this test revealed that the majority of strategies used are rule-based strategies not number sense based.

Elvira Fernandez-Ahumada, Natividad Adamuz-Povedano, Enrique Martinez-Jimenez and Jesús Montejo-Gámez present a review of instruments on assessment of number sense and mature number sense. The authors highlighted the existence of 10 instruments, of which the majority focused on the assessment of early number sense, mainly for ages between 3 and 8 years, while there are far fewer instruments for the assessment of the so-called mature number sense. (poster)

Astrid Junker investigates counting strategies of one first-grade student with a high proficiency in foundational number-sense (FoNS). A task-based, semi-structured counting interview was conducted with the aim to qualitatively analyze counting strategies. The results suggest that a student with proficiency in number-sense does not necessarily exhibit flexible counting strategies in the number range 10-20.

Pernille Bødtker Sunde and Judy Sayers investigate perspectives of six Danish first-grade teachers on teaching and learning number and addition. Data was gathered by a semi-structured interview with open-ended questions that allow to reveal teachers' emphasis in teaching on number. Data was analyzed based on the FoNS framework. The results show that teachers hardly mention estimation, quantity discrimination and number pattern when they report on their teaching practice in first grade.

Understanding of basic ideas

Marei Fetzer and Kerstin Tiedemann developed a theoretical approach in how to introduce multiplication to support children's understanding of this operation. The theoretical approach makes the distinction between basic ideas (Grundvorstellungen), strategies and representations of multiplication. The authors illustrate their approach through a German textbook example.

Aurelien Ovide, Lalina Coulange and Grégory Train examine the potentials of a specific approach for teaching and learning fractions. Data was collected throughout teaching experiments, one designed by Brousseau for 5th graders and others designed by the author for 8th graders. The results highlight the potential of commensuration meaning to enable students to access a broad conceptualization of multiplicative comparison relationships between fractions.

Strategies in mental arithmetic and flexibility

Claudia Corriveau, Doris Jeannotte and Sandrine Michot investigate how the way the manipulatives are used on a fraction task influence the students' reasoning. The methodological and analytical work is based on the concepts of affordance and didactic variables. The analysis shows that introducing familiar manipulatives in new tasks forces the students to develop new ways of doing and thinking.

Timo Flückiger and Elisabeth Rathgeb-Schnierer present the development and piloting of a standardized, semi-structured interview guide designed to capture second and third graders abilities in flexible mental calculations. This interview guide will allow for more generalizable conclusions on the cognitive elements sustaining the solution process based on students' explanations and justifications of solution methods. (poster)

Ioannis Papadopoulos and Michail Karakostas investigate students' strategy choices when performing mental and written fraction arithmetic. They find that the written algorithm was the preferred strategy in both mental and written calculations. The students' arguments for their choice of the algorithm was accuracy, speed and easiness, although the algorithm was in fact the most time-consuming strategy for all items.

Maria Pericleous investigates how a number line can be used as a vehicle for mathematical understanding. The author uses concept cartoons as a tool to support the role of the number line to foster and develop conceptual understanding of simple calculation strategies. Nineteen participants, ages 7-8 years old, engaged in whole class discussion drawing on their perceptions of constructing and reading open number lines.

Anders Månsson addresses the inter-coder reliability of three researchers when categorizing mental computation strategies of prospective elementary teachers (PETs). PETs' mental computation strategies were captured questionnaires (15 two-digit addition problems) and categorized by strategies described in literature. Based on the high consensus in categorization, the author concludes that the questionnaire and the categories are appropriate and reliable to investigate PETs' strategies.

Steven Van Vaerenbergh, Irene Polo-Blanco, Lara González-de Cos and Juncal Goñi-Cerveradeveloped investigate the strategies used by students with an autism spectrum disorder diagnosis when solving Cartesian product problems. An exploratory and descriptive investigation was conducted with 26 students (6-12 years). Results show a low success in solving problems by the participants, but a variety of correct strategies were found, predominantly operation strategies.

Cristina Zorrilla, Pedro Ivars, Ceneida Fernández and Salvador Llinares present a study in progress that aims to examine characteristics of the transition from natural to rational numbers when grade 6 students (11-12 years old) solve multiplicative structure problems. Data was collected throughout a teaching experiment (with three phases), a pre-test, an instruction, and a post-test. Preliminary results show different levels of success according to the numerical set used and the type of problem solved.

Teaching approaches

Silke Friedrich and Elisabeth Rathgeb-Schnierer investigate whether students work on open-ended tasks according to their learning and performance levels. This initial study compares 160 grade 3 students' attitudes and achievement levels in mathematics, compared to the complexity of the equations individuals created. Results suggest that those who scored highly in achievement tests invent particularly complicated equations. (poster)

Laura Korten presents the development of a diagnosis-guided support programme for primary school students with mathematical learning difficulties. The programme will be conducted by university students, which will stimulate, develop and refine their own fostering and supporting competences. Initial results suggest university students need specific criteria to enable them to support children's strategies more effectively. (poster)

Sze Man Yeung and Taro Fujita introduce and provides some examples of productive practices in the context of a study that aims to investigate how learning environments of productive practices can be embedded into the daily lessons as a part of the curriculum for basic skills and higher-order skills training. Number pyramids in a grade 2 classroom will be used, and students' mathematical thinking processes while doing productive practices will be analyzed. (poster)

Methodological approaches and tools

Mayu Akoi and Carl Winsløw pursue the aim of elaborating a large-scale model of an arithmetic curriculum. Based on the Anthropological Theory of Didactic they created an epistemological reference model for the entire domain of arithmetic in a Japanese primary school from grades 1 to 6. The model allows observed lessons to be evaluated in the context of the entire curriculum as well as the comparison of different curricula.

Einat Heyd-Metzuyanim, Avital Elbaum-Cohen and Michal Tabach introduce a tool for analyzing the arithmetic discourse of students. This tool allows to map students' participation in the discourse on a continuum between ritual and explorative. Therefore, the individual performance of a student is assessed based on eight characteristics (e.g., objectification, flexibility or focus on process or procedure) that enable to construct a ritual/explorative ratio.

Anna Lisa Simon, Benjamin Rott and Maike Schindler introduce the use of eye-tracking analysis to explore students' strategy use when naming and locating numbers on a marked number line. When measuring response time, they found that the use of reference points, e.g. gazing at the nearest reference point, was more efficient than strategies based on counting procedures, such as counting from the beginning of the number line.

Summary

The discussed theoretical and empirical projects show a huge variety, but also common goals for current and future research: We aim to investigae students' development of conceptual knowledge, number sense, flexibility and adaptive expertise in arithmetic. Additionally, we intend to develop and evaluate teaching approaches that lead to a deep understanding of arithmetic and number systems.

How to map larger parts of the mathematics curriculum? The case of primary school arithmetic in Japan

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We present a methodological discussion of how to elaborate different scale epistemological reference models and in particular, present a large-scale model for the domain of arithmetic in Japanese primary school, based on the national program and selected textbooks. Our theoretical base is the Anthropological Theory of Didactic (ATD), specifically the notion of praxeology and the levels of didactic codetermination. We also briefly discuss our motivation for creating such models, namely to analyse how students experience shifts between mathematics in different school systems.

Keywords: Japanese arithmetic curriculum, An epistemological reference model, The Anthropological Theory of Didactic

Introduction

When we observe mathematics teaching in a classroom, we observe teachers and students interacting on tasks like “Let us think ways of calculating 12+3” or “Let us calculate 29 × 3” etc., normally introduced by the teacher; we may want to understand how the activity promotes various types of learning. Other questions may be difficult to answer from what we observe in the class, like: where do the specific mathematical tasks come from? Why do the students encounter difficulties? etc. In my doctoral research project, I am interested in pupils who attend (or attended) mathematics teaching in different school systems (Japanese, Swedish and Danish). For these and many other questions in educational research, we need to include information on conditions and constraints in a wider context, even if we just want to analyse the phenomena observed within certain classroom situations.

A metaphor may be useful here. The world map gives us a prices idea of the location of the countries we want to look for. For instance, we can see that Germany is located on the European continent and Kenya on the African continent. With a more detailed map, we can find out more details (e.g., what cities are there in Germany?). In other words, we are able to look up and compare since we have a map. Even when preparing a holiday in a small beach town, we need to know how it is situated in relation to other locations like airports and so on.

In the same way, when analysing a classroom episode on the teaching of addition of fractions in grade 3, to understand the actions of teachers and students, a wider “map” of the knowledge at stake in the subject (or even in the school institution) may be needed. The main point of the present paper is to show how the anthropological theory of the didactic (ATD) can be used to define different scales of such a “school knowledge maps” (also called epistemological reference models), and to develop a methodology for the specific purpose of creating “large scale” models that shows how a mathematical domain is structured over several school years within a specific school institution. Concretely, we present an epistemological reference model for the domain of arithmetic in Japanese primary school, along with a methodological discussion of how that model was produced, based on certain official documents along with textbook material. It is also a main point that the ATD allows us to connect such large-scale models directly with more detailed models of specific parts of the domain which are more commonly used in ATD analysis of classroom episodes.

Theoretical Framework and research question

Our study adopts the anthropological theory of the didactics (ATD) proposed by Yves Chevallard from the early 1980s (Bosch & Gascòn, 2006), more precisely, the notion of praxeology and the levels of didactic codetermination. According to ATD, any human knowledge and practice can be described by using the notion of praxeology. Praxeologies are made of a practical block: a type of task (T) and a technique (τ) and a knowledge block: a technology (θ) and a theory (Θ). In addition, Chevallard proposed the levels of didactic codetermination for researchers to identify conditions and constraints at different levels. The scale of levels of didactic codetermination are constituted of nine different levels: 1. *Subject* 2. *Theme* 3. *Sector* 4. *Domain* 5. *Discipline* 6. *Pedagogy* 7. *School* 8. *Society* 9. *Civilisation* (Bosch & Gascòn, 2006). Concretely, for instance, the quadratic formula, polynomial equations, polynomials, and algebra constitute a subject, theme, sector and domain, respectively (Artigue & Winsløw, 2010). The school subject of mathematics is a discipline. Higher levels determine all disciplines and will not be modeled here. The first three levels (1-3) are defined by praxeologies (Artigue & Winsløw, 2010). A subject corresponds to a few praxis blocks, while a theme and a sector are determined by a shared technology and a shared theory, respectively.

We can now formulate our research question for this study: How to elaborate an epistemological reference model for an entire domain of school mathematics, in the sense of the official domain to be taught in a given school system, when the model is required to cover several years of teaching, and still with a level of detail that would make it useable to situate a concrete teaching situation and to compare with how the domain is taught in other school systems? As a case (considered here), how to do this for the domain of arithmetic and grade 1-6 in Japanese primary school?

Note that this paper is mainly about how to develop a method based on a strong theoretical paradigm, and methodology is therefore more important in this paper than the products we exhibit from the concrete case mentioned in the research questions, in order to illustrate it our methodological approach.

Methodology 1: choice of data (and how to get the data)

Before describing the process of elaborating an epistemological reference model, we explain the selection of materials for the case, which could be similar (yet different) according to the context. There are two levels of official programs in Japan: a general course of study for primary school (SHOGAKKO GAKUSHU SHIDO YORYO) and a primary school *teaching guide* for the Japanese course of study in mathematics (SHOGAKKO GAKUSHU SHIDO YORYO KAISETSU SANSU-HEN). Both are issued by the Ministry of Education, Culture, Sports, Science and Technology (MEXT). The former includes the basic act of education, general educational goals, and an outline of contents for teaching in each discipline. By contrast, the latter is published in each discipline and contains more detail than the course of study. When more detail is sought, we consider also mathematics textbooks authorized by MEXT. These are published by commercial textbook companies, based on the first mentioned documents, and are selected and distributed by schools. The three levels of material are interconnected, and tightly aligned. The primary sources of our modeling approach are the teaching guide for mathematics (refered to in the sequel as the *program*), and one selected textbook series. The textbook is published by TOKYO SHYOSEKI publishing company, and is widely used both in Japan, and in the supplementary schools that will later be a main focus of our research. From the program, we mainly use a table that summarizes mathematical contents for teaching in each grade. This table is extremely useful for elaborating large-scale models since it specifies that, for instance, teaching addition and subtraction of 2-digit numbers occurs in grade 1. One can say that the table specifies the location of themes and sectors. Then, the textbook can be used to develop more detailed, local models (example given in table 2), in other words, at the subject level in terms of types of task and techniques. This is why we use the program rather than the general course of study to develop our models. It is also easier to model because what is shown in the general course of study school is similarly explained in the programme, with additional detail. In fact, we can elaborate the large scale model based on the program to some extent; however, textbooks are needed to specify more closely the types of task and techniques to be taught.

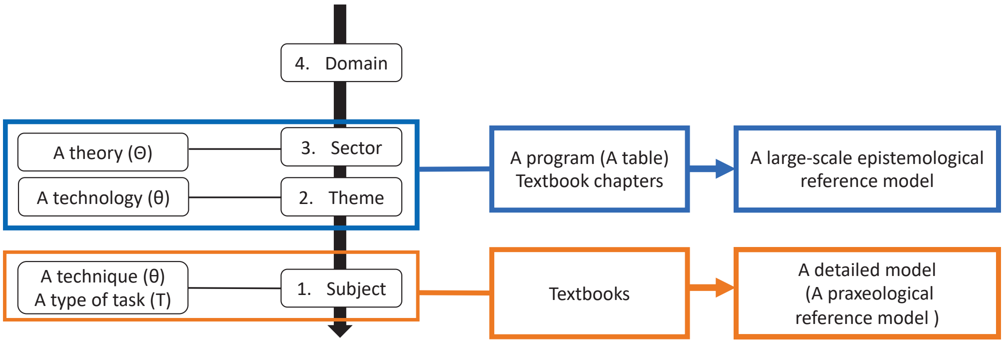


Figure 1: How to elaborate different scale models

The process of elaborating a large-scale epistemological reference model is then as follow:

1. Browse through the table for the domain of arithmetic, dividing it into sectors.
2. Identify themes within each sector from the table, and to clarify and substantiate these further, refer to the detailed description in the program and to practices exhibited in relevant textbook chapters.

The method of constructing detailed models of themes is then as follow:

1. Browse through relevant textbook chapters, and analyse all examples and exercises to identify types of tasks and the corresponding techniques; in conjunction with that, when we identify types of tasks and analyse techniques. (In Japan, as textbooks are strongly aligned with the program, it is justified for many purposes to base the model on just one textbook system.)
2. Whenever a task is encountered which does not belong to a type of task already identified, a new type is added to the model. The end result covers the theme in question.

The second part of our approach is similar to the one employed in the study by Wijayanti and Winsløw (2017). They elaborate praxeological reference models for certain themes related to proportion, as they appear in a range of different Indonesian textbooks. However, our method to categorize and situate different themes and sectors, is new.

Methodology 2: analysis of data (examples and overall outcomes)

We now present a large-scale epistemological reference model for the domain of arithmetic in Japanese primary school in grades 1-6 (Table 1), based on the above methodology. After the long table presenting the result, we provide further details of how the method was applied.

Table 1: An epistemological reference model 1 for the domain of arithmetic

Sector$_{1}$: Representation of (positive) rational numbers	Grade					
	G1	G2	G3	G4	G5	G6
Theme$_{1-1}$: Compare the size given objects	■					
Th.$_{1-2}$: Count the size given objects	■	■				
Th.$_{1-3}$: Decompositions and compositions of a given numbers	■					
Th.$_{1-4}$: Cardinal numbers	■					
Th.$_{1-5}$: Ordinal numbers	■					
Th.$_{1-6}$: Representation of 2-digit numbers	■					
Th.$_{1-7}$: Representation of 3-digit numbers	■	■				
Th. $_{1-8}$: The principle of the base-10 numeration system	■	■				
Th. $_{1-9}$: Representation of 4-digit numbers		■				
Th. $_{1-10}$: Representation of unit fractions		■	■			
Th. $_{1-11}$: Representation of unit of ten thousand ≦ N ≦one hundred million			■			
Th. $_{1-12}$: Representation of fraction			■		■	
Th. $_{1-13}$: 10 times, 100 times, 1000 times and 1/10 times of whole numbers			■			
Th. $_{1-14}$: Representation of decimal numbers (0.1)			■			
Th. $_{1-15}$: Representation of unit one hundred million and trillion				■		
Th. $_{1-16}$: Representation of approximate numbers				■		
Th.$_{1-17}$: Representation of proper fraction, improper fraction, and mixed fraction				■		
Th. $_{1-18}$: Representation of decimal numbers (0.01, 0.001)				■		

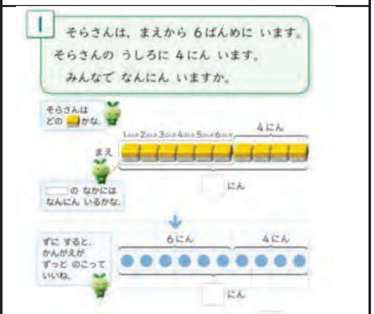
Th. $_{1\text{-}19}$: Representation of even and odd numbers					■	
Th. $_{1\text{-}20}$: Representation of divisors and multiples					■	
Th. $_{1\text{-}21}$: 10 times, 100 times, 1000 times, 1/10 times and 1/100 times of decimal numbers					■	
Sector$_{2}$: Operations with (positive) rational numbers	G1	G2	G3	G4	G5	G6
Theme$_{2\text{-}1}$: Addition of 1-digit numbers	■					
Th.$_{2\text{-}2}$: Addition of simple 2-digit numbers	■					
Th.$_{2\text{-}3}$: Addition of 2-digit numbers		■				
Th.$_{2\text{-}4}$: Addition of simple 3-digit numbers		■				
Th.$_{2\text{-}5}$: Addition of 3-digit numbers			■			
Th.$_{2\text{-}6}$: Addition of 4-digit numbers			■			
Th.$_{2\text{-}7}$: Addition with decimal numbers (until 1/10)			■			
Th.$_{2\text{-}8}$: Addition with decimal numbers (until 1/100)				■		
Th.$_{2\text{-}9}$: Addition with fractions of the same denominator (total is less than 1)			■			
Th.$_{2\text{-}10}$: Addition with fractions of the same denominator (total is more that 1)				■		
Th.$_{2\text{-}11}$: Addition with fractions of the different denominator					■	
Th.$_{2\text{-}12}$: Addition using approximate numbers				■		
Th.$_{2\text{-}13}$: Addition using letters such as a, x						■
Th.$_{2\text{-}14}$: Subtraction of 1-digit numbers	■					
Th.$_{2\text{-}15}$: Subtraction of simple 2-digit numbers	■					
Th.$_{2\text{-}16}$: Subtraction of 2-digit numbers		■				
Th.$_{2\text{-}17}$: Subtraction of simple 3-digit numbers		■				
Th.$_{2\text{-}18}$: Subtraction of 3-digit numbers			■			
Th.$_{2\text{-}19}$: Subtraction of 4-digit numbers			■			
Th.$_{2\text{-}20}$: Subtraction with decimal numbers (until 1/10)			■			

Th$_{.2-21}$: Subtraction with decimal numbers (until 1/100)						
Th$_{.2-22}$: Subtraction with fractions with the same denominator (total is less than 1)						
Th$_{.2-23}$: Subtraction with fractions with the same denominator (total is more than 1)						
Th$_{.2-24}$: Subtraction with fractions of the different denominator						
Th$_{.2-25}$: Subtraction using approximate numbers						
Th$_{.2-26}$: Subtraction using letter such as a, x						
Th$_{.2-27}$: Multiplication of 1-digit numbers (multiplication table)						
Th$_{.2-28}$: Multiplication of simple 2-digit numbers						
Th$_{.2-29}$: Multiplication (2-digit numbers or 3-digit numbers $\times$ 1-digit numbers or 2-digit numbers)						
Th$_{.2-30}$: Multiplication using approximate numbers						
Th$_{.2-31}$: Multiplication of decimal numbers in which a multiplier is whole numbers						
Th$_{.2-32}$: Multiplication of decimal numbers						
Th$_{.2-33}$: Multiplication of fractions						
Th$_{.2-34}$: Multiplication using letters such as a, x						
Th$_{.2-35}$: Division in which a divisor and a quotient are 1-digit numbers						
Th$_{.2-36}$: Division (simple calculation in which a divisor is 1-digit numbers, and a quotient is 2-digit numbers)						
Th$_{.2-37}$: Division using approximate numbers						
Th$_{.2-38}$: Division (calculation in which a divisor is 1-digit numbers or 2-digit numbers, and a dividend is 2-digit numbers or 3- digit numbers)						
Th$_{.2-39}$: Division of decimal numbers in which a divisor is whole numbers						
Th$_{.2-40}$: Division of decimal numbers						
Th$_{.2-41}$: Division of fractions						
Th$_{.2-42}$: Division using letters such as a, x						

In our model, representation with rationale numbers and operation with rational numbers were identified as the two sectors, corresponding to theories developed throughout the six years of primary school. This is because the domain of arithmetic is roughly divided into two parts in the table of the program: number and operations, and it is also justified by the details (roughly, themes) appearing in these parts. In Japan, negative numbers are taught only in secondary school. Hence, rational numbers do not include negative numbers in this context. Within the sector1 and 2, we identified twenty-one themes (for instance, representation of 3-digit numbers, Representation of fractions) and forty-two themes (for instance, addition with fractions of the different denominator), respectively, based on the table of the program. Table 1 shows in which grade(s) a given theme is to be taught.

We finally show a more detailed model of one particular theme: Addition of 1-digit numbers (Table 2), with types of tasks (T) and technique (τ), along with examples of tasks from the textbooks. This is the kind of model presented and used (more extensively) by Wijayanti and Winsløw (2017).

Table 2: A praxeological reference model of one theme: Addition of 1-digit numbers

Type of tasks	Technique	Example of task	
T_1: The addition of 1-digit numbers (cardinal numbers, cases of combination)	τ_1: Put the two sets or quantities together and count the result	t_1: Write the equation. We put 2 pencils and 5 pencils in the box. How many pencils do we have in total?	
T_2: The addition of 1-digit numbers (cardinal numbers, cases of increase)	τ_2: Counting up from an augend by the addend	t_2: Write the equation. There are 4 flowers in the vase. The girl puts 3 more flowers in there. How many flowers are there in total?	
T_3: The addition of 1-digit numbers (ordinal numbers).	τ_3: Converting a given ordinal number into cardinal number based on a figure, then use τ_2.	t_3: Sora is 6th from the front. Behind Sora, there are 4 people. How many people are there in total?	

Conclusion

In this paper, we have shown how ATD can be used to answer a methodological question: how to elaborate a large-scale model of a curriculum. As a concrete case we have shown different scale models for the domain of arithmetic in Japanese primary school. Such a model can be used, for instance, to precisely situate observations from a lesson in a larger context.More generally, the levels of codetermination (subject, theme, sector) and praxeologies allow us to elaborate different scale models and relate them to each other. We acknowledge that the form and characteristics of study programmes and textbooks may vary from country to country. For instance, our next step will be to develop similar models for the domain of arithmetic in primary school in Denmark. However, the Danish curriculum is (roughly speaking) less explicitly structured than the Japanese one; therefore, the precision of the model will be different. We will later use the model to address the main questions of my PhD project: how the domain of Japanese arithmetic, defined by MEXT, transposes to the Japanese supplementary school in Denmark (held on Saturdays), and how the students there experience it, as they attend Danish school on working days.

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Taking advantage of groupitizing skills to better understand numeration

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Number sense is one of the pillars of arithmetic learning. Among other things, its acquisition allows to understand our numeration system. Understanding this numeration system remains a challenge for many students. The purpose of this theoretical essay is to explore the role that groupitizing might have in building internal imagistic and dynamic representations of quantities to help students better understand our decimal numeration system. Three activities issued from a didactic sequence will be presented. These activities are intended for students at the end of Grade 2 and the beginning of Grade 3 (ages 7-8) who are having difficulty with the concept of numeration. They were developed as part of our thesis.

Keywords: Number sense, Groupitizing, Internal representations, Numeration, Arithmetic learning.

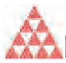
Introduction

Number sense is a key part of learning arithmetic in elementary school. Sayers et al. (2016) differentiate three different conceptions of number sense. The first is an innate or preverbal conception. It considered the ability to perceive accurately small numbers (from 0 to 4 elements) or, roughly (without counting each element), quantities that are in the range of single to double, for example 4 and 8 or 10 and 20 (Dehaene, 2001; Jordan & Levine, 2009). The second is what grade one children need to understand about number and number relationships. This understanding requires instruction. Sayers et al. (2016) call it the foundational number sense (FoNs). They identified eight distinct and not unrelated FoNs components : number recognition, systematic counting, awareness of the relationship between number and quantity, quantity discrimination, understanding different representations of numbers, estimation, simple arithmetic competence and awareness of number patterns (Sayers & Andrews, 2015; Sayers et al., 2016). The third conception is called applied number sense and it refers to “those core number-related understandings that permeate all mathematical learning” (Sayers et al., 2016, p. 473). It is characterized by a flexibility in number manipulation that reflects a true understanding of the concept of numeration that goes beyond the correct execution of techniques or routines (Bobis, 2007; Reys, 1994). A good acquisition of number sense is necessary to manipulate numbers according to the operations requested, to estimate an answer to invent calculation procedures (Reys, 1994). Yet, some 7–8-year-old students still seem to have some difficulties acquiring this level of understanding. The present study considers more the applied number sense but refers to some elements of the FoNs, particularly the awareness of the relationship between number and quantity, the understanding of different representations of numbers and the awareness of number patterns.

In Quebec, Koudogbo (2013, 2017) studied 7–8-year-old students from a regular classroom. She wanted to assess their understanding of numeration, an important component of number sense. Her results suggest that the underlying principles of the number system are difficult to grasp and understand by students of this age, similar to those identified in research conducted in the 1980s.

Indeed, Bednarz and Janvier-Dufour (1984a; 1984b; 1986) noted that many students did not perceive the multiplicative role of the grouping principle in quantity processing. However, the work of Bednarz and Janvier-Dufour (1984a; 1984b; 1986) shows that it is possible to support students so that they develop and demonstrate a true understanding of numeration. In addition, the studies by Jordan and Dyson (2016) on number sense suggest that developing FoNs by taking advantage of students' ability to count globally when items are organized in groups of 4 or fewer, which is called groupitizing (Starkey and McCandliss, 2014; Wender and Rothkegel, 2000), would lead to a better understanding of the number system.

Therefore, we wanted to understand the links between groupitizing and numeration. To do this, we first defined what the decimal numbering system is. We then defined what it means to understand this system. We have retained a key element in the understanding of this system that many students struggle with (Bednarz & Janvier-Dufour, 1984a, 1984b, 1986; Koudogbo, 2013; Koudogbo et al., 2017), namely the perceived relevance of groupings, and detailed it. Links to groupitizing skills and perceived relevance of groupings were then established. These concepts have also been detailed. Finally, activities developed within the framework of our thesis are presented to show the role that groupitizing skills could have in the construction of internal imagistic and dynamic representations of quantities, facilitating the understanding of the number system.

Numeration is a "system comprising symbols and rules for using these symbols to write, represent and name various numbers" (Patnaude & Mathieu, 2019). Understanding our numeration system requires being able to make and dissolve groups, groups of groups or exchange a group of a certain order for a higher order unit or vice versa (Bednarz & Janvier-Dufour, 1984a, 1984b, 1986). Before understanding the concept of equivalence, it is necessary to understand the role of groups, and more particularly, groups of ten. The grouping strategy is triggered by a task using large numbers of objects to count. Figure 1 is a good example of this. To be able to count a large quantity at a glance, it is necessary to form groups of ten (tens) and ten groups of ten (hundreds). This makes it easier to access the cardinal (or number of elements) of this collection thus organized.

counting one by one the organized collection

Figure 1: Example of visual and global organization of 324 elements

Grouping simplifies memorization more than the tedious task of one-to-one counting of items. As mentioned previously, despite its capital importance, few students see the relevance of using grouping and few students can explain how grouping helps organize a collection (Bednarz & Janvier-Dufour, 1984a, 1984b, 1986). Also, few students are able to form dynamic and imagistic mental representations of groups and groups of groups (Thomas et al., 2002).

What if we took advantage of groupitizing skills to help students build dynamic mental representations of quantities? Learning to place items correctly for small sets or elementary constellations could help students group large quantities to see them better. Research by Wender and Rothkegel (2000), Starkey and McCandliss (2014) and Clements (1999; 2019) has provided insight into this strategy by finding that items placed in small groups of 2 or 3 are counted faster than if they are placed in a disorganized array. This ability is called groupitizing.

In the 2000s, researchers in cognitive science questioned the impact of quantity organization on counting skills. Wender and Rothkegel (2000) measured the time adults took to count small amounts. They found that this time is shorter when items are organized in small “subitizable” groups, whether they are in classic configurations, such as dots on a die, or other subsets that are easy to count at a glance. Their conclusion is that structuring the quantity has a major impact on counting performance. They proposed the term groupitizing to designate this global recognition strategy. These authors conclude that the idea of groupitizing would support the development of the desired flexibility in addition and could also lead to a better understanding that larger numbers are formed by nestling smaller numbers together. In other words, counting with small groups would later facilitate the perception of the role of grouping in our numeration system.

Somewhat more recently, Starkey and McCandliss (2014) looked at the impact of small group organizations that respect the limits of perceptual subitizing (ability to instantly recognize numbers less than four (Clements, 1999)), on the effectiveness of counting in students, from preschool to third grade, which led them to the groupitizing strategy proposed by Wender and Rothkegel (2000). They found counting speed is increased when small groups are present, especially when the groups meet the limits of perceptual subitizing. Images displaying 5, 6, or 7 dots, of equal size, were shown to students. The dots could be 3 or 6 mm in diameter and the space they occupied ranged from 5 cm by 5 cm to 10 cm by 10 cm. Starkey and McCandliss’ (2014) results show that counting speed is influenced by whether the quantity is organized in small groups or not. They also compared students' scores on groupitizing tests with standard arithmetic tests and found that students who performed better on groupitizing also performed better on arithmetic. They conclude that groupitizing contributes to the understanding of our symbolic numeration system.

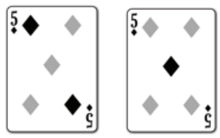


Figure 2: Example of global recognition

Groupitizing is what allows global recognition, the ability to recognize quantities without needing to point out each of the individual elements (Brissiaud, 2005). This ability is called conceptual subitizing by Clements (1999). The quantity is organized and structured, respecting the sensory limit of our perceptions, which is three or four elements. This skill helps develop number abstraction and arithmetic strategies (Clements, 1999). For example, in figure 2, it is possible to see on the card on the left that 3 + 2 = 5. It is also possible to see on the card on the right that 2 + 2 = 4 and 4 + 1 = 5. This global recognition relies on perceptual subitizing skills. It is also about a global count which makes it possible to perceive the figures as a whole, without going through sequential counting, enumeration. This involves finding the quantity of a collection from the composition or the decomposition of a constellation (Clements, 1999; Hunting, 2003; Van Nes & De Lange, 2007). This flexibility of representation enriches number sense (Brissiaud, 2005). It is not simply a matter of recognizing constellations, but playing with different arrangements and spatial structures (Battista et al., 1998; Braconne-Michoux and Marchand, 2021). This will eventually support calculation skills (Brissiaud, 2005; Van Nes & De Lange, 2007).

This flexibility will alleviate some frequently encountered difficulties by students, such as the recall of numerical facts in addition (Brissiaud, 2005; Fayol, 2012). Indeed, they will have a mental representation of the quantity.

Thus, our perception of the role of groupitizing in our numeration system may be linked to our ability to "groupitize". In this paper, we have chosen to use the word *groupitizing* rather than *conceptual subitizing* to better emphasize the idea of small group counting based on subitizing skills. Global counting and groupitizing are unfortunately undervalued in our school culture which is frantically focused on sequential counting. Twomey and Dolk (2011) insist that the ability to group is the crux of the matter, as students must give up these one-to-one counting strategies.

The proposal in this article is to first allow students to see, through problem-solving activities, that when placing items to be counted in small subitizable groups, it is easier and faster to count them. Then, through other activities, giving the students the opportunity to create a dynamic and imagistic mental representation of quantity. Finally, to allow them to reuse these observations and these mental representations to better understand the numeration system.

We wanted to verify this hypothesis within the framework of our doctoral thesis. We have designed a didactic sequence allowing students to develop a better understanding of number sense. This sequence gives them the opportunity to develop dynamic and imagistic mental representations thus taking advantage of their groupitizing skills. This gives the students the opportunity to experience activities related to their development of number sense. We present three activities in this article to highlight the links between groupitizing skills, the construction of dynamic and imagistic mental representations and the comprehension of numeration.

Activities

The didactic sequence created within the framework of our thesis consists of ten 60-minutes periods. Two activities are offered during these periods: an activity to train subitizing and groupitizing skills and an activity aimed at a central concept of the number system. An a priori analysis was carried out for each of these activities and links were made between each activity to allow students to reuse what they have learned in a following activity.

Through this sequence, students first have the opportunity to become aware of the role of their groupitizing skills in counting. When the quantities are arranged in a subitizable manner, they are easier to see and count. Students can practice using these skills and, in the process, build a dynamic and imagistic mental representation of quantities. Then follow-up activities allow them to reuse their groupitizing skills with larger quantities encouraging a shift to the grouping principle. Arranging the elements in subitizable patterns makes it possible to perceive larger and larger sets of elements as a whole. Students then can practice using grouping to count groups and groups of groups. Other activities then build on these visual skills to introduce counting and equivalence. Here, too, students are given opportunities to practice. There is a back and forth between problem solving, allowing students to discover concepts (groupitizing, grouping, numeration) and activities in which it is possible to use the acquired knowledge. Three of the twenty activities developed are briefly presented in this article: Baby Birds, Jewels of her Majesty and The Shepherdess and the Witch (Lyons & Bisaillon, 2011).

Baby Birds

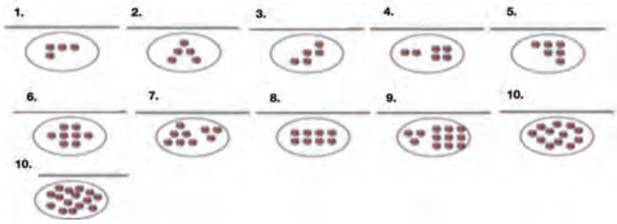


Figure 3: Case of the Baby Birds activity

The goal of this activity is to get students to establish the relevance of groupitizing for counting, memorizing, and reproducing a quantity. This activity is a problem-solving activity: A magpie cries over the fate of her young:" At each meal, it's the same crisis! If I do not put exactly the same amount of food in front of each of my young, they squabble and scramble. One day, some of them will end up falling from the nest. I do not know what to do to avoid that". The instructions are as follows: "Help the mother Magpie to feed her young. Observe the nest carefully. How could you make sure that you have the correct amount of fruit "? Pictures of baby birds in a nest are shown on an IWB for exactly two seconds (see Figure 3). Students must place the tokens on the model nest in front of them, respecting the layout of the baby birds shown on the screen. Students can explore and manipulate the material. They also have the opportunity to share with their peers the strategies they used. When returning to the large group, at the end of the activity, all the nests are shown on the screen and the teacher leads a discussion about the more difficult arrays.

In this activity, students become aware of the role of their groupitizing skills. Various constellations have been chosen, allowing them to start from their basic subitizing skills to then process larger and larger and larger quantities. The chosen constellations are not prototypical since the goal is to get the students to see the importance of correctly placing lots of tokens in order to count them better. For the last two cases, the baby birds are placed in a representation that is not subitizable, that is, one that is not organized into small noticeable groups of four elements or less. The goal is to provoke a reaction in the students and get them to see that when the items to be counted are well arranged, it is easier to form a mental representation of them.

Jewels of her Majesty

The goal of this activity is to make students aware of the relevance of grouping to count large quantities. It is also about getting them to build on the development of their groupitizing skills to see the usefulness of placing groups in a subitizable arrangement. This activity provides the context for the activity and initiates a discussion with students about their strategies associated with grouping. It also allows students to develop their awareness of number pattern.

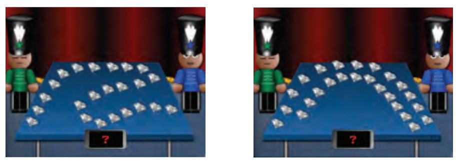


Figure 4: The two introductory scenarios for the Jewels game

This activity is a problem-solving activity: “Greengard and Blueguard are responsible for guarding the jewels of the Kingdom. Greenguard often dozes off and something tells him that his colleague is taking advantage of this. But how to prove it? Greenguard does not know how to count, but he has a good eye... He rearranges the diamonds: "If things are moved, I will know!". Figure 4 shows the two jewel arrangements of jewels proposed to students. In both cases, the arrangement of the jewels is not subitizable, to make students realize the limitation of the strategies used by Greenguard (a long line does not allow you to "see" rapidly see the quantity and to initiate a discussion about better counting strategies.

The Shepherdess and the Witch

The purpose of this activity is for students to reinvest the fruits of their learning in the activities they have done to date. They have had the opportunity to see the role of subitizable collections and their usefulness in forming a mental representation of quantities and the usefulness of counting instead of sequentially. This activity is a bridge between enumeration and numeration. A representation using the principle of enumeration shows all the elements to be counted and it is always possible to count them one by one. A representation using the principle of numeration shows higher order units resulting from the substitutions of the different groups (such as tens, hundreds, etc.), which no longer allows direct visual access to all the elements of the set to be counted (see Figure 5 presented below). This activity also allows students to develop their awareness of the relationships between number and quantity and number pattern. It takes place in two parts and is based on a scenario that is presented to the students on the IWB.

Figure 5: Example of one group of ten and six sheep

Episode 1

This first episode is a problem-solving activity. Shepherdess Bella is responsible for looking after the kingdom's sheep and she must make sure to bring them all back at the end of the winter or else Emperor Notsokind will be very unhappy. She knows that she will have to count the sheep every day, but this is made more difficult by the sheep continuously moving to have access to fresher and greener grass to eat. There are 16, a group of ten and a group of six (students use tokens). Students are asked the following question: "Do you have any ideas on how to help Bella count the sheep faster?". This is a way for the students to use the knowledge acquired in the Jewels activity, namely the relevance of forming groups. The expected answer here is to place them in groups so that they so that they can be counted more easily and circumvent the mobile sheep problem. Astuzia, her sister the witch, comes to her aid by giving her a magic flute that precisely follows these recommendations. When she plays the flute, the sheep stand in groups of ten and the sheep that are not in groups of ten place themselves in a subitizable constellation (Figure 5). They remain motionless long enough to be counted. Bella embodies the strategic boundaries of enumeration and Astuzia helps demonstrate the basics of numeration. The students have time to count the ten sheep grouped together one by one, for. This episode introduces an important kind of group, groups of 10; it also serves as an introduction to the next episode.

Episode 2

Figure 6: Triangles of 10 and mixted notation

When winter comes, Bella brings all her sheep back to Emperor Notsokind, who is very satisfied. The following spring, Bella is still responsible for the sheep, but this time there are sheep and despite using the magic flute, she cannot count the sheep as they are too numerous to count. Students are asked the same question: "Do you have any ideas on how to help Bella count the sheep faster?" This is a way to reuse the knowledge and skills acquired in Jewels, or the usefulness of counting. Astuzia comes to the rescue and introduces the ten-triangle, where we no longer see 10 sheep, which is an unconventional configuration for groups of ten. She's going to show Bella that she can count groups of ten (count by leaps). Mixed notation is also introduced to allow Bella and the students to keep track of the number of sheep (Figure 6). The end of this second episode leads to an activity in which students are asked to practice using the knowledge acquired in the first and second episode.

Conclusion

Classroom use of the activities proposed in this sequence by teachers with 7–8-year-olds would enhance the importance of number sense for elementary school students. The role of imagistic and dynamic mental representations in the understanding of these concepts would also be highlighted.

Furthermore, this implementation would confirm that interventions allowing the construction of various mental representations, by relying, among other things, on the aptitudes of groupitizing as early as preschool age and would contribute to a reduction in the number of children having difficulties with the construction of number sense. This research focuses on 7–8-year-old students, but these numerical teaching tools have the possibility of greatly improving number sense in all elementary school-aged students.

The pandemic made it impossible for us to try-out these activities and didactic sequence. We hope to do an in-class experimentation of these number-teaching sequences and strategies very soon so that we can present the results in the near future.

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Strategies in numerosity estimation
Comparison of students with high and low accuracy

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Keywords: Estimation strategy, numerosity estimation, elementary school students.

Introduction and theoretical framework

Estimation is an important part of everyday activities and is present in everybody's life. Generally, an estimation encourages methods that lead to reasonable, not necessarily accurate results. Estimation is also defined as mental comparison and measurement (Schipper, 2009). Consequently, it includes a varied set of processes which differ depending on the task and type of estimation. In recent English literature, four types of estimation are distinguished: Measurement, computational, numerosity and numberline estimation (Sayers et al., 2020). This project refers to numerosity estimation. It requires translating a non-numerical quantitative representation into a number (Siegler & Booth, 2005).

Accordingly, estimation is conceived as problem-solving process. Every common problem requires different mathematical knowledge and flexible ways. Adaptive problem solving is one crucial goal of mathematics education (Siegler & Booth, 2005). Furthermore, research results suggest that students who are gifted estimators show better arithmetic skills in terms of counting, number sense, mental computation, strategy flexibility, and conceptual understanding (e.g., Booth & Siegler, 2006; Crites, 1992). Particularly, numerosity estimation influences the development of number knowledge with increasing numbers (Wessolowski, 2014). So, another reason for fostering estimation abilities is the great impact for the development of arithmetic skills (e.g., Luwel et al., 2005; Siegler & Booth, 2005).

Despite the importance of numerosity estimation, there is a lot more known about other basic numerical processes (Booth & Siegler, 2006). However, previous research shows various strategies in numerosity estimation. Furthermore, the accuracy in estimating quantities increases with age. The performance as well as the adaption to task characteristics seemed to increase with age. In general, former studies show that strategy choices in numerosity estimation depend on specific problem characteristics. (e.g., Crites, 1992; Siegler & Booth, 2005).

Aim

According to previous research, the present study focuses on comparing strategies in numerosity estimation in two- and three-dimensional tasks of students (third grade) with high and low accuracy in estimation. Another emphasis is on investigating the exhibited strategies in relation to task characteristics. Further, the project targets to reveal strategies which lead to high estimation accuracy according to the type of task. We are also interested in correlations of arithmetic skills and numerosity estimation abilities.

Method

Data will be collected by two discretely developed instruments. We developed a numerosity estimation test to assess estimation accuracy. The test results facilitate to select the students for semi-structured interviews. The semi-structured interviews intend to reveal estimation strategies.

The test is digitally implemented in an online survey software. The final test will include 21 estimation tasks with different characteristics. A task consists of a number of elements that is to be estimated. After structuring different types of tasks, seven tasks were chosen for the estimation test. So, following characteristics will be considered in the test: Dimension (2D/3D), arrangement (structured/unstructured) and elements (equal/unequal). It is important to regard that on a screen it is only possible to see a representation of a three-dimensional quantity. Amongst others, that is one reason why not all summarized task characteristics were chosen. Every of the chosen seven tasks will appear three times in a different number range. The number range up to 50, from 50 to 100 and from 100 to 150 is covered that way. Overall, the students can see the picture for a certain time. After this time expired, the students still have time to adjust their result with a slider. Before the test starts, an introduction of estimation as well as an instruction how to use the test are intended. Students who made accurate and less accurate estimates in the test will be part of the interviews. A semi-structured guideline will be developed. It will mainly contain tasks of the estimation test. For evaluating the interview, it will be videotaped. The use of both instruments enables new perspectives on numerosity estimation. Besides, other constructs like linguistic and arithmetic skills will be surveyed.

The pilot study of the numerosity estimation test (N = 31) shows that the test is reliable. The Cronbach’s Alpha of the 2D subscale is $\alpha = .81$ and $\alpha = .80$ for the 3D subscale.

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Didactic variables related to the use of manipulatives and their impact on students' activity: an illustrative case around fractions

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This study investigates the use of manipulatives by a pair of students working on a fraction task. Extending previous work on the role played by the manipulatives in students' activity, we now focus on the linkage of 1) different choices made around the task and the manipulatives and 2) students' activity. The theoretical underpinnings allow envisioning the manipulatives through the concept of affordance. It also helps to picture which didactic variables are at stake when using manipulatives. The analysis of Tim and Nick's mathematical activity allows us to better understand how the way the manipulatives are used serve as breaching elements and lead them to reason differently.

Keywords: Manipulatives, Number sense, Fractions, Affordance, Reasoning.

Introduction

In this paper, we draw on our previous research presented at the 10th CERME meeting (Jeannotte & Corriveau, 2020) about the role played by the manipulatives in students' activity when solving an arithmetical task. We aim to show how some choices made around different variables impact the students' activity. The paper reports more precisely on how subtle changes in the conditions put into place around the use of manipulatives can destabilize enough students to doubt their ways of reasoning. In the first section, a problem statement specifies what we intend to study. The concepts of affordance and didactic variables are then theoretically explicated to frame our methodological and analytical work. The third section presents an illustrative case.

The use of manipulatives

The use of manipulatives when teaching and learning arithmetic and number sense at the elementary level is recommended, even prescribed in education internationally (e.g., OECD, 2019). However, simply using manipulatives will not necessarily increase student success in mathematics. Carbonneau, Marley and Selig (2013, p. 397) mention that "instructional variables either suppress or increase the efficacy of manipulatives," but argue there is a lack of research that identifies these variables. Research on manipulatives mainly focuses on whether the manipulatives support student achievement in mathematics (e.g., Moyer, 2001). However, the way it is used is rarely made explicit. What were the choices regarding its use? When it comes to manipulatives, it is as if the only variable considered was its use or non-use. Nonetheless, we consider that many variables are at stake and play different roles in students' activities.

The few research that studies students' mathematical reasoning when they use manipulatives shed light, indirectly, on some of these variables. For example, the use of familiar manipulatives may mislead students (McNeil & Jarvin, 2007) or, for the same task, two different materials lead students to engage in different reasonings (Jeannotte & Corriveau, 2020). The familiarity and the type of manipulatives are two examples of variables among others that can shape the students' mathematical activity. These variables can therefore be of different natures.

In this paper, we present a fraction task that students solve using Pattern blocks. We aim to show how the choices made around different variables introduce breaching elements that impact the students' activity and promote the development of different ways of doing.

Conceptual framework

With the aim of better understanding how the use of manipulative impacts students' activity and what different variables are at stake when introducing manipulatives in a mathematical task, our theoretical underpinnings convoke the concepts of affordance and of didactic variables.

We call **affordances** the relational properties of an object within a certain environment that are bound to certain actions. For example, a pen will generally be seen as an object to write with. Moreover, "[w]hether or not the affordance is perceived or attended to will change as the need of the observer changes, but being invariant, it is always there to be perceived" (Gibson, 1977, as cited in Brown et al., 2004, p. 120).

In a mathematics class, each manipulative has its own properties and, with pupils and the whole class actions, generates different affordances (Jeannotte & Corriveau, 2020). In other words, the affordance realized in the classroom relies not only on the tool itself, but rather on the exploitation of it, on the educational context and how it is driven by the teacher (Drijvers, 2003). In that regard, we observed that students mainly rely on what they are used to do with a specific kind of manipulatives to solve a task rather than what they do without it or what they must do according to the task. Thereby, working with familiar manipulatives for novel purposes might drive the attention of pupils in an "undesired" direction (McNeil & Jarvin, 2007). This does not mean that a same type of manipulative should always serve the same kind of task. On the contrary, it is by breaching these familiar ways of working with it that students will reason and make new mathematical meaning of its use. The concept of breaching (from ethnomethodology) refers to a subtle breakage of what is familiar. By varying different aspects related to the use of manipulatives, it is possible to place the students in conditions familiar enough for them to engage in the task but sufficiently unfamiliar so it resonates as a conflictual use of the manipulatives. In a way, it plays into how students perceived the affordances of the manipulative and leads them to explicit usual ways of reasoning with it and renewing these.

Didactic variables are defined as parameters, linked to the design of a teaching situation, which can vary according to the teacher's choices, and which can lead to a change in the students' mathematical activity (Brousseau, 1981; Grugnetti & Jaquet, 2005). There are three categories of didactic variables: those related to conditions (e.g., the choice to have students work in groups of 3), those related to the use of tools (e.g., the choice to use a single sheet of paper in a group) and those related to the problem (e.g., the choice to use fractions in a problem, rather than whole numbers, to modify its difficulty).

The concept of didactic variables aims at designing teaching situation. However, it might also be a powerful methodological tool. Varying these didactic variables related to manipulative may act as breaching elements as discussed above.

Methodology

We use an illustrative case that emerged from data collected within MathéRealiser-project, a collaborative research project undertaken with elementary teachers from first to sixth grades. In collaboration with five of them, we have created a task (titled "twelfth") that aims to develop different reasoning to represent fractions using pattern blocks. We experimented "Twelfth" into two fourth-, two fifth-, and one sixth-grade classes of about twenty students each. The task was implemented with different choices according to teachers' rationales (mainly based on students' abilities and familiarity with fractions). In this paper, we present an analysis of students' activity regarding the use of manipulatives when solving the task.

The task "Twelfth"

"Twelfth" is, at first sight, a common task in which students are asked to represent a certain fraction with pattern blocks. Students work in pairs; they receive a kit of Pattern blocks (Figure 1) and are asked several questions. These can be divided into three categories:

1) The given fraction is a twelfth and students must represent a third (the piece representing a twelfth varies): e.g., *the green triangle (blue rhombus, red trapezoid, etc.) is worth 1/12, represent 1/3 of the same whole.*
2) The given fraction is a twelfth and students must represent other varied fractions (unit and non-unit): e.g., *the blue rhombus is worth 1/12, represent 1/24 of the same whole; the red trapezoid is worth 1/12, represent 5/6 of the same whole.*
3) The given fraction varies, and students must represent varied fractions: e.g., t*he yellow hexagon is worth 2/5, represent 1/15 of the same whole.*

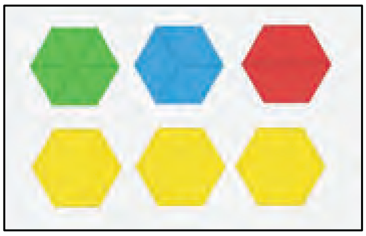

Figure 1: kit of pattern blocks available to each pair of students

The didactic variables

While elaborating the task with the teacher, we had to make several choices. For example, the choice of manipulatives was discussed. Because of their availability and familiarity, we chose to use Pattern blocks. These manipulatives have different characteristics that can lead to different affordances.

Another example is the limitation of the number of pieces given to encourage varied strategies. We purposely did not want the student to be able to represent the whole with the same piece repeated as many times as the inverse of its value (e.g., 12 times for a twelfth). When a question such as "if a red trapezoid represents 1/12, how can you represent 1/3?" was asked, the whole could not be represented with twelve red trapezoids. For some questions, it was not possible to represent the whole at all. Bellow, we present a list of choices made regarding the design of the task. We indicate the corresponding didactic variable.

- Didactic variables related to conditions:
  - Students were asked to work in pairs (to foster communication)
  - Questions were asked during their work (to encourage substantiation)
- Didactic variables related to tools:
  - Choice of pattern blocks (known, diversity of affordances possible)
  - Students shared a single kit (to foster communication)
  - Limit number of pieces in a kit (to encourage the renewal of strategies)
  - Allowing or not the use of paper and pencil (to keep track of work and answers)
- Didactic variables related to the problem:
  - Choice of fractions (to encourage the renewal of strategies)
  - Asking about finding a fraction from a given fraction (as a breaching element)

Description of the analysis

To analyze, we listened to the videos with the transcripts in support. From this new listening, we reconstructed a table showing the different phases of the activity. The table contains three columns. The first column gathers the time of the video. In the second column, which is descriptive in nature, we portray in detail students' actions and their important comments and moves. We also used a colour code to identify four key actions related to manipulatives in the students' mathematical activity: the **representation** of fractions or of the whole (in yellow), the **resolution** which corresponds to solving (in green), the **validation** used to convince themselves (in cyan) and finally the **explanation** when they explain their solution to the teacher or researcher (in pink). In the third column, of an analytical nature, we pointed out two types of information: breaching elements (highlighted in bold in the table) and an interpretation of the use of the manipulatives for each key action.

Results: The illustrative case of Tim and Nick

The analysis presents the case of Tim (10 y.o.) and Nick (11 y.o.), two fifth-grade students who are both perceived by their teacher as quick-witted, particularly in mathematics. They generally succeed very well in mathematics. We present the analysis of their work below. We take up the salient elements in terms of affordance and breaching, and we relate it to the use of the manipulatives.

Q1. "If a triangle represents one twelfth, can you represent the third of the same whole?"

Before the question is even asked, Tim says they got the answer. He uses two hexagons as a whole. While Tim raises his hand to present their solution, the teacher asks: "can you represent the third of the same whole?" This question acts as a breaching element as Tim stops and starts to reflect. He had anticipated the question would be about finding the whole. Nick and Tim think aloud, "one third, one third…" It seems Tim calculated mentally and then, adds two rhombi on top of the two hexagons. The student perceives that covering one part of the whole by overlaying pieces (property of manipulatives) makes the fraction visible. Nick wants to validate. To do so, he counts "one third, two thirds and three thirds" pointing with his finger over the whole but without moving any pieces. When a researcher asks them to explain their solution, they argue that one-twelfth means there are twelve triangles in the whole. "The third of twelve is four." Therefore, the answer is four triangles. Their explanation relies on calculation, but they also overlay four triangles on the two hexagons using once more a property of Pattern blocks (Figure 2). When asked to validate, they answer that four triangles “fit” three times in two hexagons. They complete the validation by effectively making three-thirds “fit” on top of the two hexagons. The limitation of the pieces is not a problem as they use four triangles or any equivalence (two rhombi or one trapezoid and a triangle) to represent each third. They are able to completely cover the whole. The ways of using manipulatives are familiar to Tim and Nick.

Figure 2: Overlaying pieces on top of the whole

Q2. “If a rhombus represents one twelfth, can you represent the third of the same whole?”

Tim and Nick use three hexagons as a whole. When they count the number of rhombi fitting in the whole to validate (“3, 6, 9...”), they realize this is not enough. “There are not enough hexagons,” Tim says, surprised. Based on Tim’s reaction, we hypothesize that when they work with Pattern blocks, the hexagon, or some hexagons, is commonly used as a whole (affordance). To solve this impasse, Nick uses two trapezoids to complete the whole (Figure 3a).

Then, the resolution seems to result from mental calculations. Tim says they need four rhombi. They overlay the three rhombi and two triangles on top of the whole (Figure 3b). Again, overlaying is employed, and the limited number of pieces necessitated the use of different pieces (affordance).

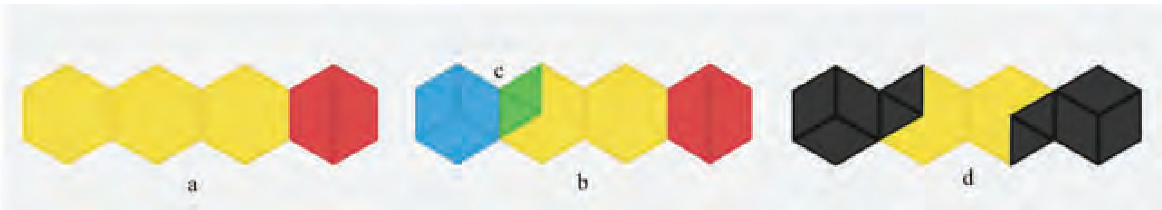

Figure 3: Tim and Nick’s actions during question 2

They try to validate using the manipulatives. However, this is difficult. The properties of the manipulatives make impossible to perceive the same “pattern” of a third displayed in the represented whole. Indeed, the format of a third initially represented (Figure 3c) cannot be used three times to complete the whole (Figure 3d). The limited number of pieces makes impossible to completely cover the whole, as they did previously (question 1). Nick tries to move the pieces to see if he can display the same pattern (Figure 3c) three times. He has difficulties doing so. Both Tim and Nick know this is not easy and could engender mistakes, so they use a marker (hatched triangle in Figure 4a) to realize the validation. However, they count the marker as being part of the second third (Figure 4b). They represent a third, but it laps the one originally represented (see Figure 4c for a right possibility). We hypothesize this confusion is linked to the need to move the pieces and the wish to use the same pattern (we hypothesize this is based on the usual ways of doing). Because they cannot validate their solution this way, they try to resolve it another way. Tim decides to rely on triangles. He perceives that a rhombus is formed with two triangles. Twenty-four triangles constitute the whole. “One-third of twenty-four is eight” (Tim). This comes back to three rhombi and two triangles. Tim and Nick are happy. Nick tries to validate using the manipulatives. However, they cannot complete the validation.

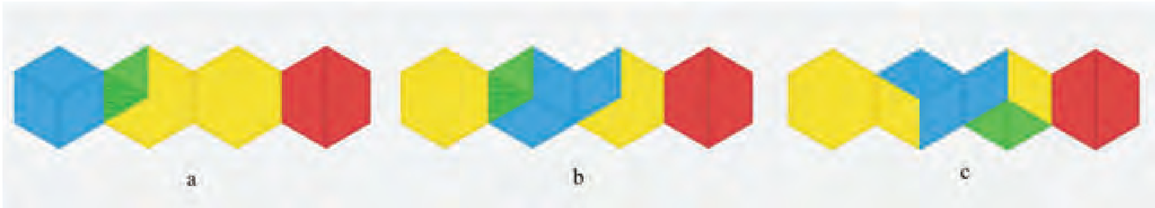

Figure 4: Using the triangle (hatched) as a marker

Q3. “If a trapezoid represents one twelfth, can you represent the third of the same whole?”

Tim and Nick put three hexagons in front of them. Tim picks up the trapezoid and says: “it must fit twelve times in the whole.” While Tim uses triangles and rhombi to complete the whole, Nick argues they need three other hexagons to represent the whole (familiar way of representing wholes). Tim tells Nick they need to represent the whole, but both realize it is impossible to use all the pieces, especially because the trapezoids are used as indicators of 1/12, placed on top of one hexagon (familiar ways of representing fractions). From the start (questions 1 & 2), they have been using pieces to represent the whole. Then, they overlay other pieces to represent their solution. We hypothesize this way of using the manipulative is familiar within this class (affordance). The limited amount of manipulative combined with the choice of fractions (variable) acts as a breaching element.

Nick suggests: “Let’s imagine there’re three more hexagons.” He suggests working with imaginary Pattern blocks. This is possible because there are physical Pattern blocks they can rely on as well. Tim says: “the third of twelve is…, the third of twelve is… four.” They conclude they need four trapezoids. Although they have a solution, they feel they must validate using the manipulative (affordance). Nick’s fingers point to the first two hexagons, then to two imaginary hexagons and so on. Tim convinces Nick they must represent the whole and “steals” some Pattern blocks left on the teacher’s desk. “It’s important not to mix them with our kit though,” Nick cautions. They complete the whole with three more hexagons, overlay four trapezoids and show their response to the teacher.

The teacher validates the answer but reminds them that they need to work only with their kit. The students feel they must start over. To validate their reasoning, they change the way of resolving (“representing the whole” strategy). Tim, therefore, proposes to “reduce the fraction.” He says: “Instead of four twelfths, we do two sixths… (thinks)… and a third.” He shows a third with the manipulatives. Using the manipulatives, he changes the whole as he “reduces” the fraction. According to him, 1/3 means they must use “three” hexagons as a whole. Nick says, “this is not one third, but think again saying: “it’s the same thing, we just reduced…” Both seem aware that something is not working but cannot go further from this proportional reasoning.

A researcher comes to ask about their solution. Nick tells her: “It's the two of them (showing two yellow hexagons overlaid by two trapezoids) but should be full of red (meaning with four trapezoids overlaid).” During their attempt to explain why their solution works, the researcher sees they struggle to represent the whole and make the three-thirds visible. The identification of the fraction with overlaid pieces, in that case, is their only way to act. She asks: “why are you keeping the red trapezoids on top?” She takes the two red trapezoids. She combines three pairs of hexagons with the pieces to show the three thirds and says “that makes 1/3, that 1/3 and that another third” (Figure 5). They are stunned. “Euh!… Well… yes!”, says Tim. “We did not succeed from the start,” adds Nick.

Figure 5: validation from the researcher

Discussion and conclusion

This analysis of students' action drifts us away from the belief that using manipulatives necessarily helps students (Ball, 1992). Tim and Nick are able to resolve the questions without any manipulatives. But using it makes them act differently. Thus, this illustration is coherent with previous observations that using manipulative is not a concrete version of doing mathematics without it, it is a completely different activity (Corriveau & Jeannotte, 2015). Moreover, we move from viewing manipulatives as an aid to facilitate the resolution of a task, to considering it as a constraint that stimulates new ways of doing and thinking. This is visible especially when Tom and Nick want to validate.

Through the lens of affordance, we see from the actions of Tim and Nick, that the use of Pattern blocks is bound to certain actions: representing the whole, usually with hexagons and representing the fraction by overlapping pieces, using the same partition (or pattern) for each part. Articulating the calculation and the validation with this manipulative, and showing the result, command this way of using it. We could interpret this use as a ritual from a commognitive perspective (Lavie, Stein & Sfard, 2019). However, in the progression of the three questions, we show how certain choices based on didactic variables create breaching elements that play an important role in students' activities. Even if Tim and Nick are able to resolve the questions by calculation, the introduction of manipulatives commands renewal of familiar ways of using it. They need to complete the whole using other pieces than hexagons, they deal with the impossibility to overlay pieces (using all the pieces to represent the whole) and even, in some other questions, they manage the impossibility to represent the whole. From a commognitive lens, the way the task is constituted acts as deritualization, a process that transforms a ritual into an exploration (Sfard, 2008). The students have no specification regarding the way in which the manipulatives must be used. They are the ones in charge of the endorsement. Mathematically, this task, and all the choices made around it, put the focus not only on one way to envision the concept of fraction (1/12 means there are 12 equal parts and a third means they are divided on three) but also put the focus on the equivalence of fractions (4/12 and 1/3) and on the process of validation. It emphasizes on rules that are necessary to communicate mathematically. Tim and Nick's activity indicates somehow that a solution and its validation must me "readable." It is as if oral explanations and calculations are not recognized as a way of communicating in mathematics for them. We hypothesize that this is an important aspect of the classroom culture. Finally, the way the manipulative is used nuances what is mathematically important when using it and what is not: e.g., the overlapping is a nice way to compare the fractions but is not necessary.

Acknowledgment

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Multiplicative comparison of fractions: potentials and limits of commensuration meaning

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We aim at examining the potentials of the commensuration meaning for teaching and learning fractions, in particular "multiplicative comparison relationships" between fractions. We focus on students' reasoning based on commensuration meaning by analyzing a variety of data collected through teaching experiments: on the one hand, old data from experiments conducted at the Michelet school by Brousseau in primary school in the eighties and on the other hand, recent experiments that we have conducted in a form of continuity with college students in a new context (one experiment designed by Brousseau for students of grade 5 and another we designed for students of grade 8). The results of our research highlight the potential of commensuration meaning to enable students to access to a broad conceptualization of multiplicative comparison relationships between fractions.

Keywords: Multiplicative comparison, fractions, commensuration

Introduction

Developing a deep understanding of fractions requires a conceptual understanding of different meanings of fractions and relationships between them (Clarke et al., 2008). In French primary school, fractions are studied as splitting the unit meaning. The French curriculum describes this meaning of fractions as follows: you split a unit by a natural number of equal parts and you take a number of those parts, potentially larger than the numbers of parts included in the unit. Brousseau's work (1987) highlights another original meaning of fraction, called commensuration. However, commensuration has not found its place in the French curriculum. In previous work, Chambris et al. (2021) and Coulange and Train (2021) have shown how splitting the unit meaning of fraction does not easily build multiplicative relations between quantities in primary school. Our goal is to examine the potential of the commensuration meaning for teaching and learning fractions, in particular "multiplicative comparison" (Greer, 1992) relationships between fractions. In this study, we focus on students' reasoning relying on multiplicative comparison relationships between different fractions with commensuration meaning.

Theoretical framework and methodology

Multiplicative comparison relationships, "flexible units" and fractions

In the domain of models of multiplication and division, Greer (1992) distinguished two main types of multiplicative word problems: equal group and multiplicative comparison. Relating to Greer's theoretical viewpoint about multiplicative field, we distinguish multiplicative comparison relationships between measures or numbers of units (seen as n more…than or n less…than) from inclusive relationships. Inclusive relationships between measures or numbers of units (seen as n times in) refer more to equal groups of objects considered in the context of measurement or counting. Such multiplicative comparisons seem necessary to develop a high-level conceptualization of magnitude

(Steffe & Olive, 2010; Thomson et al., 2014) relying on “flexible units” linked by multiplicative relationships to each other (Chambris, 2021; Chambris et al., 2021). Thus, they may provide a flexibility of reasoning about both measures and numbers in the field of arithmetic (Chambris, 2021).

Returning to our main subject, we assume that a broad conceptualization of fractions requires multiplicative comparisons, following a “fraction as comparer” approach (Freudenthal, 1983). Cortina et al. (2014) referring to Norton and Hackenberg (2010) also emphasizes the conception of fraction as “a size of an attribute, instead of as an actual piece of an object, or as a discrete element in a set (p.84)” in such an approach. From our point of view, this assumption is related to the distinction between “equal groups” and “multiplicative comparison” mentioned above.

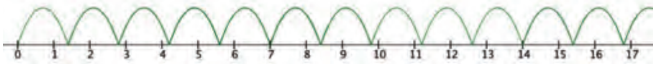
In our study, according to this theoretical background, we aim at examining the potential of commensuration (Brousseau, 2002) for teaching and learning multiplicative comparison between different fractions.

Commensuration meaning of fractions

Commensuration fraction meaning is distant from “splitting the unit” meaning that is usually considered in French curriculum. In the context of commensuration, $\frac{a}{b}$ is the relation between two physical quantities, such as b times the first quantity gives a times the second one. Classroom experiments related to teaching and learning this meaning of fraction were conducted in primary school (grade 5) during the eighties (Brousseau, 2002; Brousseau & Brousseau; 1987, Brousseau et al., 2014). In particular, Brousseau and Brousseau (1987) designed the “thickness of a sheet of paper” situation in which children had to find an optimal measurement strategy in situations of measuring something so thin that previously known techniques of direct measurement could not be applied. The situation involved measuring and comparing the thicknesses of various types of paper. The aim was to characterize the thickness of a sheet first by a pair of whole numbers (for example, 56 sheets – 8 mm) and then by a fraction (for example $\frac{8}{56}$ mm). Thus, the thickness of a sheet is $\frac{8}{56}$ mm means that 56 sheets of paper measure 8 mm. Building on this previous research work, we designed mathematical tasks based on the motion of bounds of robots along a number line intended to promote teaching and learning of the commensuration meaning with students in secondary school. These tasks contextualize commensuration: some robots make regular bounds on the number line, reaching a natural number of units in several bounds. The aim is to characterize the length of robot’s bound by a pair of whole numbers (for example 15 units and 7 bounds) and then by a fraction ($\frac{15}{7}$ units). The table below summarizes the two types of contextualization of commensuration which rely on two kinds of magnitude and measurement:

Table 1: Two types of contextualization of commensuration

Sheets of paper	Robot
Number of (regular) paper sheets: q sheets of paper	Number of (regular) bounds of a robot: q bounds

Measure of the thickness of a pile of (q) sheets measured in mm: p mm	Length reached on the number line measured in units (represented on the line): p units
Measure of the thickness of a sheet: $\frac{p}{q}$ mm but also $\frac{p\times a}{q\times a}$ mm, etc.	Length of the bound of a robot: $\frac{p}{q}$ units but also $\frac{p\times a}{q\times a}$ units, etc.
	

Methodology

Two types of data were used to study the potential of commensuration in the teaching and learning of multiplicative relationships between fractions.

On the one hand, the examination of old data relating to the engineering conceived by Brousseau (Brouseau & Brousseau, 1987) and experimented within COREM during the 1980s: all the sessions conducted during 1987 to implement the engineering in a class of 5th Grade (17 students) were the subject of a priori and a posteriori analyses (Artigue, 1988), based on all the videos of the sessions (archived in the VISA database) and the students' productions (archived and available for consultation at the "CRDM - Guy Brousseau).

On the other hand, building on this initial analysis, we designed and experimented with a resource based on a new context - that of fictitious robots and the study of the length of their bounds - a context that takes up the essential variables of the initial engineering proposed by Brousseau. In particular, our attention has been focused on tasks involving multiplicative comparison relationships between fractions. Experiments were conducted during 2021 in the 8th Grade (25 students) in accordance with the current mathematics curriculum. All the sessions were systematically video-recorded and the work produced by the students was collected in full.

Tasks of "multiplicative comparison" between fractions with commensuration

The background on the tasks under study

The examination of "old" data (Coulange et al., in press) of the teaching experiment of "sheets of paper" (Brousseau & Brousseau, 1987; Brousseau, 2002) emphasized some potential of commensuration as a means to build multiplicative comparison relationships between integers related to measures or of numbers of units. Indeed, in order to express the length of the thickness of a sheet of paper by equivalent fractions, students have to rely on such comparisons between thicknesses of a stack of paper sheets and between numbers of paper sheets. Our analyses of data related to the teaching experiment of "paper sheets" led us to consider that commensuration meaning could therefore offer strong potentialities related to the conceptualization of multiplicative comparison between fractions. Besides, the original sequence of "paper sheets" aimed at teaching such multiplicative comparison relationships (Brousseau & Brousseau, 1987). For this purpose, Brousseau designed mathematical tasks centered on the thickness of a cardboard sheet comprised of a given number of paper sheets. For example, (Figure 1, extracted from CRDM – Guy Brousseau) students had to answer the question: "A cardboard sheet is comprised of 5 paper sheets – the thickness of each paper sheet being $\frac{3}{25}$mm, what is the thickness of a cardboard sheet?"

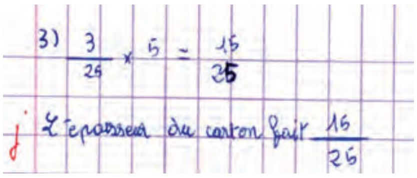


Figure 1: Student's work

We identified two kinds of possible reasoning supported by multiplicative comparisons:

- The thickness of a cardboard sheet is 5 times thicker than a paper sheet, so the pile of 25 cardboard sheets will be 5 times thicker than a pile of 25 paper sheets. Indeed, the thickness of a cardboard is: $\frac{3}{25}mm \times 5 = \frac{3\times 5}{25}mm = \frac{15}{25}mm.$

- The thickness of a cardboard sheet is 5 times thicker than a paper sheet, so we need 5 times less cardboard sheets than 25 paper sheets to obtain a pile of cardboard sheets which will have the same thickness of 3 mm. Indeed, the thickness of a cardboard is: $\frac{3}{25}mm \times 5 = \frac{3}{25:5}mm = \frac{3}{5}mm.$

Similar reasoning could also be conducted about dividing a fraction by a natural number to answer another type of questions: "The thickness of a cardboard sheet measures $\frac{7}{25}mm$. This cardboard sheet is made up of 8 similar paper sheets. What is the thickness of a cardboard sheet?"

Commensuration meaning of fractions may promote the emergence of these kinds of reasoning relying on multiplicative comparison between measurements or numbers of units. Analysis of "old" video data from the VISA video data base show that it is not so easy to promote such reasoning for students of grade 5. Students had difficulties in answering questions about the measurement of the thickness of a cardboard sheet. The teacher tried to help them by using an additive model: considering that the thickness of a cardboard sheet equals to the iterative addition[1] of the thicknesses of paper sheets. Referring to an additive model avowed students to solve the problem given the right answer. From our point of view, this may have moved them away from the two types of reasoning described above. In fact, the iterative addition model seems to be closer to an "equal groups"'s point of view (of objects as paper and cardboard sheets) than to a "multiplicative comparison"'s point of view (comparing the thicknesses of paper and cardboard sheets). Therefore, we aimed at investigating further the potential of commensuration meaning for the conceptualization of multiplicative comparison between different fractions through a new teaching experiment with older students. Thus, we designed new mathematical tasks related to commensuration and based on the motion of bounds of robots (see above) for students of grade 8. We experimented these mathematical tasks with a class of 25 students during the month of June 2021.

1 The thickness of a paper sheet being $\frac{a}{b}$ mm, the thickness of a cardboard comprised of n paper sheets equals to $\frac{a}{b}$ mm + $\frac{a}{b}$ mm + …. + $\frac{a}{b}$ mm + $\frac{a}{b}$ mm, n times.

A new experiment about commensuration and multiplicative comparison of fractions (grade 8)

In this new experiment, we have, on the one hand, taken up tasks close to those described above, involving the multiplication and division of a fraction by an integer. In order to permit the strong emergence of reasoning relating on commensuration meaning, we did not use, in a first step, the common notations of fractions but coded the length of the bound of a robot which reaches *a* units in *b* bounds by the couple (a u; b s)[2]. In a second step, the link between this notation and the usual notation of fractions was made. Such a choice allowed us to see a variety of reasonings, and not only reasonings based on "splitting the unit" meaning of fractions.

For example, the question asked to students was: "The robot A reaches 5 units in 12 bounds. The robot B makes bounds 4 times longer than those of the robot A. What is the length of a bound of the robot B?". The students' works below (Figure 2) show that they appear to have employed both types of reasoning mentioned above.

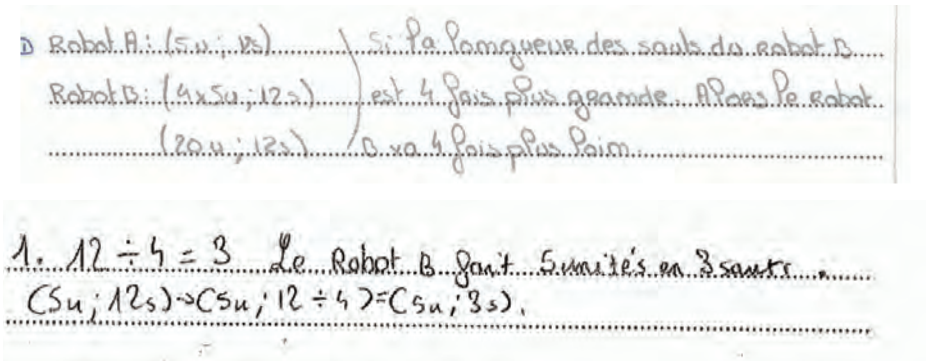


Figure 2: Two kinds of reasoning

The video data shows that students formulated these reasonings relying on multiplicative comparison between measurements and numbers of units. For example, to answer the question: "The robot A reaches 8 units in 21 bounds. The robot B makes bounds 14 times longer than bounds of the robot A. What is the length of a bound of the robot B?", one student spontaneously gave the correct answer:

Student: The robot A does 8 units in 21 bounds and the robot B make a bound 14 times longer than the length of the bound, of a bound of the robot A. So we have 8 times 14. It gives 112 units.

In addition, some of the wrong answers given by students in the class were able to be invalidated by themselves through interactions in the classroom. In the case of the episode below, a student was able to give the correct answer relying on multiplicative comparison, after a discussion with the teacher and another student.

Teacher: Why could it not be 21 times 14, actually?
Student 1: No because it is 21 bounds times 14 bounds. What I'm saying makes no sense.
Teacher: So why do you multiply the number of units by 14 and you don't multiply the number of bounds by 14. Is it correct?
Student 2: Because it is the length of a bound so it is the units which are multiplied and not the number of bounds.

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[2] We made the assumption that we had used standard notation

Our analysis of this new teaching experiment confirms the potential of commensuration meaning to conceptualize multiplicative comparison of fractions, even though students sometimes partially state their own reasoning. For example, in the episode transcribed above, students did not formulate the argument of "with the same number of bounds" which remained implicit. However, these results led us to design more complex mathematical tasks addressed to students in the same class.

A new task: "composition of multiplicative comparison" between fractions with commensuration

Using the same experiment, students had to answer the following questions: "we consider the robot A and its bound (48u;9s) and the robot B and its bound (24u;18s). Which one of these two robots make the bigger bounds? Can you find a multiplicative relationship between the lengths of the two bounds?"

To answer the first question, we will not detail students' reasoning which appeared during the lessons (concerning the longest bound). Such reasoning has already been presented in Coulange and Train, (2021). Instead, we will focus on the second question centered on multiplicative comparison between fractions. This mathematical task is different from the tasks addressed to students before (and mentioned above). To answer the question, students must perform a multiplicative comparison between two given lengths of bounds of robots. A general description of the given mathematical task would be: "Giving two lengths of bounds, the length of the robot A (e u; f s) and the length of the robot B (g u; h s), the robot A reaching a distance n times greater than the distance reaches by robot B in m times less bounds (i.e., $e = n \times g$ and $h = m \times f$), find the multiplicative comparison relationship between these two lengths?". In the particular case of the tasks mentioned above, n=m. However, such a mathematical task requires reasoning based on composition of multiplicative comparison of relationships between measurements of distances (reached by robots) and numbers of bounds (made by robots). The robot A reaches a distance, n times greater than the one reached by the robot B, with m times less bounds than the number of bounds of the robot B (to reach the initial given distance related to this robot). So, the length of a bound of the robot A will be $n \ \times m$ longer than a bound of the robot B. In the particular case of the mathematical task addressed to students, it gives: a distance 2 times longer reached with 2 times less bounds mean bounds are 4 times longer.

Classroom experiments bring to light both difficulties to find the meaning of this composition of multiplicative relations but also levers to support the conceptualization of such compositions. For example, students seemed to mistake "a distance 2 times longer reached with 2 times less bounds" with the length of "a bound 2 times longer" $([m \times k\ u; n \times k\ b]$ and $[m\ u; n\ b] \times k)$. Such confusions may refer to previous tasks in which students had to find different designations of a same robot. Qualitative strategies of comparisons (for more details, see [Coulange & Train, 2021]) helped students to overcome such difficulties:

Teacher:	The robot A goes further than robot B? Is that true?
Student:	Yes.
Teacher:	And, concerning the number of bounds?
Student:	Smaller, but it goes further.
Teacher:	It goes further, it reaches a greater distance in fewer bounds? is that true?
Student:	Yes.

Concerning the composition of multiplicative comparison, the introduction of a third robot (reaching a distance two times greater than the distance reached by the robot B in the same number of bounds) allowed students to note that the length of robot A is not two times bigger than the length of robot B.

Teacher:	Robot A makes bounds two times larger than Robot B? I suggest you check it out! I want to find a robot C that makes bounds two times bigger than robot B. I would like to know if this robot is robot A […]
Teacher:	Is it robot A?
Students:	No!
Teacher:	It is not robot A. Is that true that Robot A makes bounds two times larger than robot B?
Student:	No.

According to these episodes, the didactical variable's choice, especially m=n (a distance 2 times longer reached with 2 times less bounds) seems not to be the best choice: it fostered the emergence of wrong answers given by students (the length of a bound of the robot A is 2 times larger than the length of a bound of the robot B). In the same time, the "third" robot introduced by the teacher to invalidate these first propositions played another function. It finally helped students to access the composition of the two multiplicative comparison relationships: they identified the length of a bound of this "third" robot as an intermediate measurement which is both two times longer than a bound of the robot B and two times less long than a bound of the robot A. Indeed, at the end of the session some students formulated the right answer taken up by the teacher:

Teacher:	The A makes bounds two times longer than C. […]
Student:	Two times this one (Robot C) which is two times this one (Robot B). […]

Conclusion

The findings highlight the potentials of commensuration meaning to access to a broad conceptualization of multiplicative comparison relationships between fractions. In our view, such multiplicative relationships are distinct from multiplicative inclusive relationships that could be constructed with an "equal groups" approach. Thus, the composition of multiplicative comparison relations as "*n* times more…than *m* times more…than" is more difficult to conceptualize than "*n* sets of *m* sets of". We assume that our new experiment did not allow all the students to easily access such conceptualization. However, this experiment shows that students, under teacher's guidance, finally mobilized such mathematical knowledge. We believe that these first results will lead us to improve the design of mathematical tasks devoted to the teaching and learning of multiplicative comparison relationships between fractions.

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Investigation of Number Sense Strategies Used By 5th Grade Gifted Students in Turkey

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The purpose of this research is to determine the strategies which 5th grade gifted students use in number sense problems. This study has been designed as a case study, which is one of the qualitative research designs. The sample of the study consisted of 21 fifth-grade students who were diagnosed as gifted in a centre of a southern city in Turkey. The number sense test was used as the data collection tool. This test consisted of 25 items, which were prepared in line with the five basic number sense components. The data obtained from the data collection tool were analysedthroughqualitative analysis methods. In addition, the students' opinions and answers to the questions were taken. The research findings presented that 25.90% of the students' solutions are number sense-based, 23.42% of them are partially number sense-based, and 43.61% of them are rule-based strategies.

Keywords: Number sense, Number sense components, Number sense strategies, Gifted students.

Introduction and Theoretical Framework

Number sense refers to a person's general understanding about numbers and operations in addition tohis ability and tendency to use this understanding to make mathematical judgments in flexible ways and to develop useful strategies for coping with numbers and operations. It reflects the tendency and ability to use numbers and quantitative methods as a tool for conveying, processing and interpreting information (McIntosh, Reys&Reys, 1992).

According to the standards of National Council of Teachers of Mathematics (NCTM,2000), students from pre-school to the end of the secondary education period must have understood numbers, the ways of representing numbers, the relationships between numbers and number systems, the meaning of operations and their interrelationships, and they must have been able to calculate smoothly and make appropriate estimations. On the other hand, it is emphasized that students must have developed the concept of number sensebased on these standards.Within the educational context, not every individual is the same and they have different learning rates, so there are individual differences among students. In this context, gifted students can be considered in one of the student groups that show individual differences.

Special talent is defined as a person's high level of performance in abstract thinking and reasoning skills and being above the normal intelligence age (Gagne, 2004). Renzulli (1978), on the other hand, stated that individuals with special talent have a high level of task awareness and creativity skills and they have academic skills above average.

Due to their different cognitive, personal and emotional characteristics compared to students with normal development, the question of how well gifted students have mastery of number sense strategies comes to mind. Foustana, Luwel, &Verschaffel (2017) stated that there are individual differences in the choice of strategy while solving the problems. In addition, stated that the choice of strategy depends on talent.In the related literature, there are some studies conducted about number sense with a group of gifted students, one of the group of students which show individual difference, (Doğan&Paydar, 2020;Tunalı, 2018; Wang, Halberda&Feigenson, 2017). Available resources were reviewed and no study investigating the strategies that are used by gifted fifth grade students in the number sense test was found. It becomes important to determine the number sense strategies they use when solving number sense problems of gifted students, a group of students with individual differences, since it produces significant results to be able to meet the educational needs of these individuals. Moreover, the study might have some contributions to create awareness about the importance of number sense of gifted and talented students in mathematics education.In this context, this study aimed to determine the strategies used by the fifth grade-gifted students while solving the number sense problems.

Method

Design of the Study

This study was designed as a case study, qualitative research designs. In case studies, generally more than one data collection methods are used. The purpose is to reach a rich variety of data that will confirm each other. In this study, the answers given by the students to the Number Sense Test (NST) were investigated as documents. In addition, the students' opinions and answers to the questions were takeninto account.

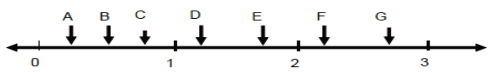
Participants

In Turkey, there are Science and Art Centres (SAC) in order to provide the most appropriate education and training environment for the gifted preschool, primary, secondary and high school individuals who have individual differences.Students who are diagnosed as gifted by standard tests receive education in line with their needs in Science and Art Centres outside of formal education hours.This research was conducted with 9 female and 12 male students, 21 in total, who were diagnosed as gifted and studying at fifth grade in the centre of a city in the southern part of Turkey. The research sample was determined on a voluntary basis among the students studying at the Science and Art Centre.

Data Collection Tool

The data collection tool consisted of some questions in the number sense scale adapted by Singh (2009) from McIntosh, Reys, Reys, Bana, and Farrell, (1997). Following the field experts' suggestions, some of the questions in the original form were excluded from NST, as they were not considered appropriate for the levels of the fifth grade students. Moreover, the views of a language specialist were taken for controlling the translated version of NST. In line with the advice given, some measurement units such as miles and gallons, which were not used in Turkey, was taken out. Then, the test was finalized. There are 5 components in the test as understanding the concept of number, using the multiple representations of numbers, understanding the effect of operations, using equivalent expressions, using calculation and counting strategies. Understanding number concepts of component is about understanding the value that the number represented and the size that the number indicated. They understand and use the numbers and relationships completely (Harç, 2010). For example, the skill of knowing that there are indefinite decimal number between 0.7 and 0.8 is a sign of this component. Using multiple representations is related to knowing different representations of the number or the value the number represented. Understanding the effect of operations is related to realize how the result can be affected when a number or the value of the operation is changed in an operation. Using equivalent expression is related to knowing the numbers in different expressions, that is, their equivalences. For example, understanding that two out of five can also be represented by an equivalence or understanding that 80 x 0.5 and 80:2 are equivalent symbolize this component (İymen, 2012). Using counting and computation strategies is related to knowing the result of the operation without using pen and pencil. The components of Number Sense Test (NST), the distribution of the questions about these components and some sample items are presented in Table 1.

Table 1:The components of Number Sense Test, the distribution of the questions about these components and some sample items

Components	Items	Sample Item
Understanding the concept of number	Number of Items: 6 Items: 1,6,11,16,21,25	Item 1:Is there a fraction between $\frac{2}{5}$ and $\frac{3}{5}$? If yes, how many?
Using multiple representation	Number of Items:5 Items:2, 7,12,17, 22	Item 7:  Some letters are given on the numerical axis. Please, make up a fraction of letters in which the numerator is about two times the denominator.
Understanding the effect of the operation	Number of Items:5 Items:3, 8,13,18,23	Item13: Circle the appropriate choice for the solution of 87 x 0.09. A) much bigger than 87 B)a little smaller than 87 C) a little bigger than 87 D) much bigger than 87
Using the equivalence representation	Number of Items:4 Items:4,9,14,19	Item 9: Which of the following operations' result is bigger? A) 145 x 4 B) 144 + 146 + 148 + 150
Using calculation and counting strategies	Number of Items:5 Items:5,10,15,20,24	Item 24:Circle the appropriate choice forthe result of [6 x 347] ÷ 43. A) About 30 B) About 50

		C) About 80 D) about 100
Total	25 items	

Data Collection and Analysis

The data of the research wereobtained by means of the interviews and document analysis technique. The students' answers on the number sense test were used as documents in this research. The interviews were conductedwith the same students in order to determine the way they thought while solving the questions in the number sense test. The NST wasadministered to each student individually. After the students finished answering the questions in the test, they were interviewed to find out the strategy they used while solving the questions. The interviews were recorded by a recorder after necessary permissions had been obtained from the students.

The data of the research was analyzed by making use of qualitative analysis methods. The audio recordings obtained from the interviews were transcribed and they were analyzed descriptively. Descriptive analysis is interpreting and summarizing the research data according to the themes which had been determined before (Yıldırım&Şimşek, 1999). Codes such as S1, S2, … were assigned to the students who were interviewed so as to keep their identities confidential. The strategies used while solving the number sense problems are *no answer(Unanswered) or explanation (Unanswered), an answer containing no explanation (Unexplained), rule-based strategy (RBS), number sense based strategy (NSBS) and partially number sense based strategy (PNSBS).* These strategies were explainedin below.

No answer or explanation (Unanswered): This category contains situations in which there are no answers or explanations,

An answer containing no explanation (Unexplained): This category contains answers without any explanation,

Rule-based strategy (RBS): This category contains finding the result by grounding on the operations or rules. For example; finding the result by equalizing denominators while adding two fractions with different denominators to each other,

Number sense based strategy (NSBS): This category contains understanding the numbers, knowing the relative size of numbers, using reference point, estimating the result and being able to evaluate its appropriateness, knowing the effect of numbers on operations (Şengül, 2013). For example; being able to judge that the fraction 4/7 is bigger than the fraction 2/5 without using any algorithms,

Partially number sense based strategy (PNSBS): This category contains situations in which both rule-based and number sense based strategies are used together. For example,using the points of 1 and 0.5 as reference points while comparing numbers. Feeling obliged to convert the number into a decimal number while deciding if the number 8/15 is bigger than 0.5 and doing this by using paper and pencil algorithm when needed (Şengül, 2013).

Findings

In this part, findings obtained from the research data and interpretations are presented.

The Five Components of the Numbers Sense Strategies

The strategies that 21 gifted students used for the problems about each number sense component were investigated and findings approached on the basis of components.525 answers that was used related to the strategies (25x21=525) were obtained from the students‟ solutions in the NST of 25 items.The distribution of the strategies that the students used in the answers to the questions in NST is presented in Table 2.

Table 2: The distribution concerning the strategies about the five the component theNCT

Components	Strategies	*F*	%
Understanding the concept of number	NSBS	30	23.80
	PNSBS	17	13.49
	RBS	61	48.41
	Unexplained	3	2.38
	Unanswered	15	11.90
	Total	126	100
Using multiple representation	NSBS	15	14.28
	PNSBS	36	34.28
	RBS	44	41.90
	Unexplained	2	1.90
	Unanswered	8	7.61
	Total	105	100
Understanding the effect of the operation	NSBS	47	44.76
	PNSBS	19	18.09
	RBS	38	36.19
	Unexplained	-	-
	Unanswered	1	0.95
	Total	105	100
Using the equivalence representation	NSBS	35	41.66
	PNSBS	13	15.47
	RBS	32	38.09
	Unexplained	3	3.57
	Unanswered	1	1.19
	Total	84	100
Using calculation and	NSBS	9	8.57
	PNSBS	34	32.38

counting strategies	RBS	58	55.23
	Unexplained	1	0.95
	Unanswered	3	2.85
	Total	105	100
Total	NSBS	136	25.90
	PNSBS	123	23.42
	RBS	229	43.61
	Unexplained	10	1.90
	Unanswered	27	5.14
	Total	525	100

As seen in Table 2, according to the component of understanding the concept of number, 48.61% of the strategies that the students used in the answers to the items in NST are rule-based strategies..Moreover, some examples from the interviews with students and some sample answers to the items in the component of understanding the number concept are given below (Figure 1).

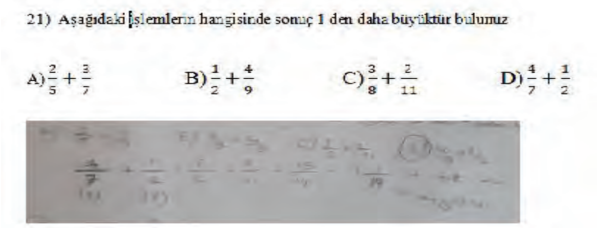


Figure 1:**Solution by using rule-based strategy**

Item 21 in NST is "In which of the following operations is the result bigger than 1? Please, find?". The explanation of the student who gave the answer in Figure 1 is as follows;*"I equalized the denominators as I did here. When I equalized the denominators, I reached the result in Choice D Choice D is bigger than 1 (S9)."*When the solution in Figure 1 and the opinions of the student are considered together, it can be said that the student used rule-based strategy. While S9 was doing addition in fractions, he solved the problem by grounding on the rules of equalizing the denominators. The explanation of S4 for the item 21 in NST is as follows; *"I considered the choices whether they were bigger or smaller than a half. Both fractions in choice A were smaller than a half so they were also smaller than 1. In choice B, there are two fractions. One of them is a half and the other one is smaller than a half. In choice C, both fractions are smaller than a half. In choice D, there is the addition of a fraction bigger than a half and a half so the number is bigger than 1 (S4)*". Thus, it can be interpreted that from the explanation of S4 that he used the number sense based strategy.

The findings about the strategies that are used in the problems of the component of using multiple representations are given in Table 2.Table 2 shows that 41.09% of the strategies that the students used in the answers to the items in NST are rule-based strategies. In addition, some excerpts from the interviews with students and some sample answers to the items in the component of using multiple representations are given below (Figure 2).

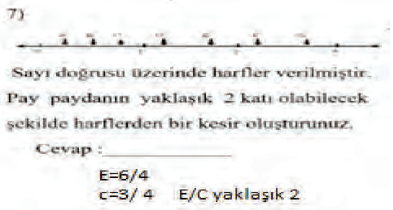


Figure 2: Solution by using rule-based strategy

Item 7 in NST is “Some letters are given on the numerical axis. Please, make up a fraction in which the numerator is about two times the denominator”. The explanation of the student who gave the answer in Figure 2 is as follows;*“I assigned values to the letters on the numerical axis. I assigned letter E as 6/4 and letter C as 3/4. I found out that the result would be about two times when I took the first fraction as it was, turned the second faction upside down and multiplied them (S15)”*.When the solution in Figure 2 and the opinions of the student are considered together, it can be interpreted that the student used rule-based strategy in his solution. The explanation of S7 about his answer for item 7 is as follows;*“As number G is bigger than 2 andnumber E is bigger than 1, I thought G/E would be about 2 (S7)”*. Thus, it can be interpreted S7 used number sense based strategy in his solution.

As seen in Table 2, according to the component of understanding the effect of the operation 44.76% of the strategies that the students used in the answers to the items in NST are number sense based strategies. According to the component of using equivalence representations 41.66% of the strategies that the students used in the answers to the items in NST are number sense based strategies. Also, according the component of using calculation and counting strategies 8.57% of the strategies that are used by the students in the answers to the items about using calculation and counting strategies in NST are number sense based strategies and 55.23% of them are rule-based strategies.

When Table 2 is considered, it is seen that 43.61% of the strategies that the students used while answering the items in NST are rule-based strategies. Table 2 presents that majority of the students used rule-based strategies while solving the problems in NST.

Conclusion andRecommendations

This study, in which it is aimed to determine the strategies that are used by fifth grade students in solving number sense problems, grounds on the findings received from 21 students.When all the solutions obtained from the students (without considering whether they are right or wrong) are considered, it is seen that 25.90% of the students' solutions are number sense-based, 23.42% of them are partially number sense-based, and 43.61% of them are rule-based strategies.

Er and Artut (2018), in the studythey conducted with 8thgrade students with normal developmentwith the same data collectiontool, stated that the students mostly (55, 57 %) used the rule-basedstrategy in their answers.Although the percentage of gifted individuals using rule-based strategies is high, it can be said that they prefer number sense-based strategies when compared to students with normal development.This may be due to the characteristics of gifted students.Tunalı (2018), in his study,found that the number sense levels ofgifted students were betterthan those with normal development.In addition, in his study, it was determined that gifted students preferred number sense strategies more, while students with normal development preferred rule-based approaches.In this context, it can be said that the results obtained from this study are parallel to the results obtained in Er and Artut (2018) andTunalı (2018).

When the importance of number senseis taken into consideration, the number of activities about the concept of number sense should be increased in the curriculum. Moreover, these activities should be administered in the lessons and gifted students should be encouraged to use their number senses.Furthermore, training should be given to gifted students to develop their mental calculation skills and prediction skills, so students' sense of number are developed.

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How is number sense assessed in the early years of mathematical learning?

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Keywords: Assessment, number sense, numeracy, test.

Introduction

The importance of fostering number sense from the early years of learning has been highlighted by both universal references and normative frameworks for decades. However, despite this long history, no consensus has been reached on the definition of this object of study and how it should be measured and evaluated. This paper presents a revision of instruments dealing with number sense assessment.

Given that early numeracy skills are vital for later mathematics learning (Aunio, 2019; Jordan et al., 2009) it is necessary to have appropriate assessment tools that provide detailed information about children's performance and development (Purpura and Lonigan, 2015) and allow teachers to plan targeted instruction and interventions (Aunio, 2019).

Among the typology of existing instruments, Purpura et al. (2015) distinguish between those that perform discrete measures, assessing individual components and specific mathematical skills, versus those that perform broad content measures focusing on multiple mathematical components. Foegen et al. (2007) indicate that a tool covering a broader range of content might be a more suitable means of assessing early mathematical skills in children as these, particularly in the early years, develop as a sequence of connected concepts and skills (National Mathematics Advisory Panel, 2008).

The instruments studied in this paper are focused on the assessment of number sense, considering both early number sense (ENS) and mature number sense (MNS), according to the approach of Whitacre et al. (2020). ENS includes learned skills that involve explicit knowledge of numbers, such as number recognition abilities, counting, number pattern recognition, number comparison, performing arithmetic operations, measurement, and estimation concepts. For its part, MNS focuses on habits of mind and ways of behaving mathematically that are considered desirable, such as the flexible manipulation of numbers and operations or the ability to use numbers and quantitative methods as a means of communicating, processing, and interpreting information (McIntosh et al. 1992).

In this paper, we focus on these two components since we consider them to be of greatest interest in the field of mathematics education, precisely because they can be acquired and learned. Some of the instruments revised here are not specific to number sense, but address some of its dimensions within a broader assessment.

The instruments analyzed were found in documents obtained after searching the SCOPUS databases with the descriptors "tool" AND "numeracy" AND "primary education", "test" AND "mathema*" AND "primary education", "numeracy test" AND "primary education", and "number sense" AND evaluation. This search generated an initial sample of 187 articles. Subsequently, an abstract review was performed to select papers that focused on the use of instruments to measure number sense or related mathematical skills.

The analysis of these works leads us to highlight the existence of 10 instruments, among which there is a proliferation of those focused on the assessment of early number sense, mainly for ages between 3 and 8 years, while there are far fewer instruments for the assessment of the so-called mature number sense.

It is also observed that all the instruments analyzed approach the assessment of number sense from a quantitative perspective. However, given that there is consensus in considering number sense as something easily observable, it seems reasonable to address in future work instruments that attempt to assess number sense from a qualitative approach.

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Multiplication as a matter of Grundvorstellungen, strategies and representations

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This paper is about the initial part of our research concerning the question of how to introduce multiplication in mathematics classes and how to support children's understanding of it. For that purpose, we developed a theoretical approach first which makes a distinction between basic ideas (*Grundvorstellungen*), strategies and representations of multiplication. On this basis, we are now able to investigate more closely how different aspects of the multiplication teaching fit together and which difficulties might arise in their interplay. In this paper, an example from a German mathematics textbook illustrates our theoretical reflections and distinctions.

Keywords: Multiplication, Grundvorstellungen, strategies, representations, textbooks.

Introduction

It is multiplication that, for primary school children, opens the door to larger whole numbers. In many countries, the action of repeated addition is commonly used to introduce multiplication. This initial approach often shapes the idea of multiplication and, thus, tends to become the dominant frame for interpreting multiplicative situations – for students as well as for primary teachers (Askew, 2018). However, relying exclusively on repeated addition proves to be critical for sustainable understanding of multiplication (Bakos & Sinclair, 2019; Askew, 2018). In relation to these findings, we plan a larger empirical study to have a closer look on the question of how multiplication is actually introduced in everyday classroom communication. But for now – and in this paper –, we focus on clarifying our theoretical perspective and, thus, differentiate between basic ideas in the sense of *Grundvorstellungen* (GVs), strategies and representations of multiplication. This way, we have become able to investigate more closely how different aspects of teaching multiplication fit together and which difficulties might potentially arise in their interplay.

Thus, in this paper, we present part of our theoretical framework and approach the following questions: 1) What are basic ideas of the mathematical concept of multiplication? 2) What are appropriate strategies for obtaining correct solutions to multiplication problems? 3) Which representations can help to teach and learn about those basic ideas and strategies?

We will present the example of a German textbook in order to illustrate the use of our theoretical reflections: What suggestions of handling the complex interaction of basic ideas, strategies and representations can be found in the textbooks?

On Grundvorstellungen

In order to be able to support children's understanding of multiplication, we have to ask what multiplication is all about: What does it actually mean to multiply? For our study, we follow vom

Hofe & Blum (2016, p. 226) and use the German term *Grundvorstellung* (GV; pural 'Grundvorstellungen'). Grundvorstellungen (GVs) characterize mathematical concepts and their applications to real-life situations. On a primarily normative level, they are descriptions of the relationship "between mathematical structures, individual-psychological processes, and subject-related contexts, or, in short: the relationships between mathematics, the individual, and reality." (ibid, p. 213). With regard to elementary school, the real contexts are, above all, everyday contexts of action. For the case of multiplication, then, the question is which everyday actions have an inherent multiplicative structure.

According to the GVs concept, three aspects can be distinguished: The *first* aspect is the "*constitution of meaning*" of a mathematical concept (ibid, p. 230). In our context, this means that children get to understand multiplication by linking the mathematical procedure to real-life contexts, situations, and actions. The *second* aspect is the "*generation of a corresponding mental representation*" (ibid, p. 230). In our context, this means that children approach multiplication by constructing mental representations which include exactly those aspects of the real-life context that are relevant from a mathematical perspective. The *third* aspect is the "*ability to apply*" the mathematical concept to real-life situations by recognizing a corresponding structure (ibid, p. 230). In our context, this means that children become able to apply the mathematical concept of multiplication to real-life contexts by recognizing multiplicative structures in the complexity of real-life situations.

We chose the GV concept for our research on multiplication for at least two reasons: *First*, the concept highlights that teachers (and mathematics educators) have to make decisions on a normative level: From an expert's point of view, which everyday actions have an inherent multiplicative structure and can be, for this reason, a suitable starting point for individual constitution of meaning? Thus, the GV concept focuses very clearly on the connection to real-life situations. *Second*, the GV concept allows us to differentiate between this normative level on the one hand and an empirical level on the other hand. Vom Hofe & Blum (ibid, p. 232) speak about a normative and a descriptive way of using the GV concept. When we use the concept in a descriptive way, we try to get as much information as possible about the mental representations that individual students *actuall*y have developed. Those mental representations might correspond to the intended ones more or less, but they are crucial when it comes to the actual processes of teaching and learning.

In the following, we will first use the GV concept in a normative way. Thus, we present two essential GVs of multiplication that can be found in the literature. Both of them ground on the activity of building units or – a bit shorter – of *unitizing* (Götze & Baiker, 2020; Lamon, 1994). This initial selection might need to be complemented and adapted in our ongoing research work.

Repeated addition

In Germany as in many other countries, repeated addition is the most common approach to introduce the basic idea of multiplication. Tasks then focus on *sequential situations* in which someone performs a specific *action* several times. An example from the German textbook "Zahlenbuch 2": A boy carries 3 books from a box behind him to a table in front and he does it exactly 7 times (figure 1, left). The boy says: "Always 3 books on a pile.". Two aspects are characteristic of the GV repeated addition: Someone bundles units and performs, one after another, a specific action on each of those units.

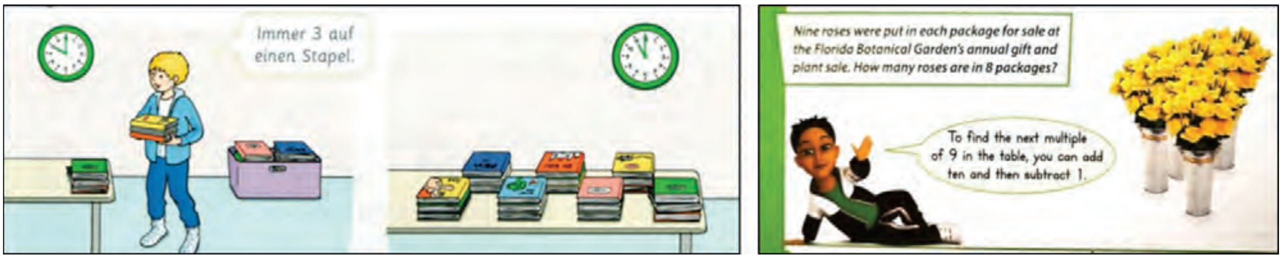

Figure 1: Dynamic and static situations

The story about carrying books from a box to a table can be told mathematically in two different ways. It can be described as a process of repeated addition (3+3+3+3+3+3+3) or, in order to shorten the story, as a process of multiplication (7·3). The different meanings of multiplicand and multiplier fit perfectly well to the real-life situation (3 books on each pile, carrying 7 piles one after another) (Götze & Baiker, 2020). Thus, multiplication is introduced as a 'shortcut' of a repeated addition. However, is multiplication really nothing more than a certain form of adding?

In other tasks, the chronological sequence is not highlighted that much. For example, nine roses were put in each package for sale at the "Florida Botanical Garden's annual gift and plant sale" (figure 1, right, from the US-American textbook "enVision Mathematics Grade 3"). This is a rather static situation; all activities are already completed. The only question that brings us to repeated addition is the question of the total number: "How many roses are in 8 packages?" In order to find an answer to that question, the boy suggests to add (and subtract) repeatedly: "To find the next multiple of 9 in the table, you can add ten and subtract 1." Thus, repeated addition is mainly presented as a *strategy* to obtain a correct solution.

In fact, there are mathematics educators that come up with doubts about the GV of repeated addition and its potential for understanding multiplication. For example, Nunes & Bryant (2009, p. 9) summarize: "Finally, it is assumed that, in spite of the procedural links between addition and multiplication, these two forms of reasoning are distinct enough to be considered as separate conceptual domains." Similarly, Bakos and Sinclair (2019) state, together with Akew (2018), that the exclusive reference to repeated addition implies "limiting access to opportunities through which functionally thinking can emerge" (Akew 2018, p.1). Since several years, this position is supported from different sides. First, there are empirical studies that report a correlation between the use of addition strategies and the underachievement on multiplication problems (Baroody, 1999; Park & Nunes, 2001). Second, studies focus on successful forms of teaching multiplication which do not introduce multiplication as a 'shortcut' of addition, but as a mathematical operation in its own right (Park & Nunes, 2001). Third, some researchers stress that the GV of repeated addition does not allow to recognize the *functional* relation between multiplicand and multiplier (Askew, 2018). This last reference leads us directly to the second GV.

One-to-many correspondence

The concept of one-to-many correspondence refers to the activity of comparing quantities in a certain way (Vergnaud, 1983). A first way of comparing quantities is to compare them additively. For example, Sara has 7 playing cards more than Jonathan has. There are two sets of cards and we can determine the difference between these two sets by adding: If Jonathan collects 7 additional playing cards, he will have as many cards as Sara. Thus, additive reasoning stems from the (mental) action of joining and placing sets in one-to-one correspondence. A second way of comparing quantities is to compare them multiplicatively (Sinclair & Bakos, 2019). For example, “Amy’s Mum is making 2 pots of tomato soup. She wants to put 3 tomatoes in each pot of soup. How many tomatoes does she need?” In this case, there are more than two quantities and these quantities are not compared additively. Instead, the action is rather putting two variables in one-to-many correspondence (Nunes & Bryant, 2009, p. 11). Two aspects are characteristic of the GV of one-to-many correspondence: Someone performs an action that keeps the ratio between two variables (tomatoes, pots) constant and that leads to bundled units – at least in the end.

It is noteworthy that the concrete action can actually be performed in one way or another. For example, Amy’s Mum can put one tomato in each pot until there are 3 tomatoes in both pots *or* she can always put 3 tomatoes at once in a pot. Both actions lead to the same result, to a constant ratio between tomatoes and pots. Thus, in the GV of one-to-many correspondence, the focus is put on relations between quantities. In other words, the basic idea of one-to-many correspondences puts particular emphasis on the relation between the multiplicand and the multiplier. Accordingly, this GV stresses the asymmetry of multiplication as well as repeated addition does.

The example of cooking tomato soup (with surprisingly few tomatoes) is taken from a study conducted by Park & Nunes (2001, p. 768). In this intervention study, the researchers compare two treatment conditions: teaching of multiplication through repeated addition and teaching through one-to-many correspondence. Both groups made significant progress from pre- to posttest. But, at posttest, the group taught by one-to-many correspondence performed significantly better than the repeated addition group in multiplicative problems even after controlling for level of performance at pretest (Park & Nunes, 2001, p. 770). On this basis, the researchers come to the conclusion that teaching of multiplication should not be grounded in repeated addition, but in one-to-many correspondence (ibid, p. 772). Further research results strengthen this position, namely those about children’s informal knowledge about multiplication. Several studies report that many children already start school with a remarkable understanding of one-to-many correspondence and that this informal knowledge seems to be quite resistant (Nunes & Bryant, 2009, p. 12, 21). Moreover, many children, who have not been taught about multiplication yet, quite successfully use correspondence strategies in order to solve multiplicative reasoning problems (Kouba, 1989; Carpenter et al., 1993).

On strategies

Independent of the focussed GV, it is another important aspect of teaching multiplication to provide strategies for children which allow them to obtain correct solutions to multiplication problems in a flexible and efficient way (Nunes & Bryant, 2009). However, strategies taught in mathematics classes seem to be different in different countries. To start with, we draw on the German perspective and refer to core strategies (Götze & Baiker, 2020).

Knowing by heart

From our perspective, knowing by heart is not actually a *calculating* strategy. Thus, you will not have to calculate anymore if you know the solution to a multiplication problem by heart. Still, it can serve as a very helpful “tool for solution” – for example as part of addition strategies as we will see in the next paragraph (Rathgeb-Schnierer & Green, 2013, p. 354). Knowing by heart is not necessarily linked to (any) Grundvorstellung of multiplication.

Repeated addition

Repeated addition was introduced as a GV above. Although some researchers argue that it is not really a separate idea of multiplication, it is understood as an appropriate strategy to solve multiplication problems anyway (Götze & Baiker, 2020).

One possibility to realize repeated addition is to add every single unit: $3+3+3+3+3+3+3 = 21$. Alternatively, you can start from a result that you know by heart and add or subtract the 'missing' units: $7 \cdot 3 = 15+3+3 = 21$. However, the concrete calculation process may look like, the "procedural links between addition and multiplication" become obvious (Nunes & Bryant, 2009, p. 9).

Changing order

Changing the order of the factors is a helpful strategy when solving multiplication tasks. This strategy is based on the mathematical structure of commutativity. Nevertheless, it is quite difficult to link this strategy to real-life contexts and, thereby, to the intended GVs. It is much easier, but takes longer to carry 7 times 3 books from a box to a table than it would be to carry 3 times 7 books. If Amy's Mum put 2 instead of 3 tomatoes in each pot and took 3 instead of 2 pots, the soup might still taste the same, but there would be more of dishwashing to do.

On representations

As we can see so far, the interplay of GVs and strategies might be rather difficult in detail. In this regard, it is particularly relevant that all of them require representations in order to be accessible to children in mathematics classes (Kuhnke, 2013). For this very reason, representations are the third part of our theoretical perspective.

Real-life or didactical

Kuhnke (2013, p. 42) differentiates between real-life and didactical representations. Real-life representations take up real-life contexts that children probably already know, whereas didactical representations are specially made for teaching purposes and, therefore, are strongly adapted to the intended mathematical structure. With a view to GVs, this distinction is important because real-life representations are much more helpful for linking the mathematical concept of multiplication to typical application situations that children might know from their everyday lives outside school.

Real-life: Picture sequences

Picture sequences are real-life representations. They are well-known and widespread representations of multiplication and usually consist of two or more pictures telling a story of repeated actions. Thus, this representation is closely connected to the GV and the strategy of repeated addition. As we see in the story of the boy and the books above, the story-line itself is reduced to a minimum. Thus, links to every experiences are supposed to be realized and, at the same time, processes of abstracting and seeing the mathematical aspect within that story are meant to be enabled.

Real-life or didactical: Unstructured and structured quantities

Pictures of structured or unstructured quantities might be either realized in real-life or in didactical representations. This way of representing multiplication is based on the discrimination of multiplicand and multiplier. In the case of unstructured items, the action of unitizing is highlighted. The one-to many correspondences may be visualized as well as the concept of repeated addition. Strategies supported by this way of representing are unitizing and repeated addition.

Mainly didactical: Rectangular arrangements

In Germany, rectangular arrangements as a particular form of structured quantities are quite common. They might be either real-life or didactical representations and support a close link between geometry and arithmetic. Rectangular arrangements especially enable the visualization of the commutative structure of multiplication. Such structured arrangements can represent both GVs: The focus can be on repeated addition or on one-to-many correspondences. Strategies supported by this way of representing are unitizing and counting units, repeated addition and changing order.

5. First insights: Textbooks

How is multiplication introduced in textbooks? Do representations align with certain GVs? What strategies are introduced and supported? In the following, we present an example from our textbook analyses in order to illustrate the use of our theoretical distinction between GVs, strategies, and representations as an analytic framework. Thus, we ask for 1) representations in order of their appearance and analyze on this basis 2) which GVs are addressed and 3) which strategies are supported.

In the German textbook „eins zwei drei Mathematik 2“ (one two three mathematics 2), the introduction of multiplication is to be found on pages 72-73.

Context: A common classroom.

Representations: Rectangular arrangements embedded in the classroom situation, didactical rectangular arrangement, pre-structured representations of quantities

Tasks: Talk about the picture, find multiplicative structures in your own classroom, talk about quantities and about amounts of units, write multiplication tasks according to the representations given, draw representations

Addressed GVs: repeated addition

Potentially supported strategies: knowing by heart, changing order, repeated addition

In this introduction, the focus is exclusively on the GV of repeated addition, although the given everyday situation of a classroom would support a much wider spectrum. Such pictures basically offer the opportunity to include unstructured quantities which require the process of unitizing and support the concept of one-to-many correspondences. Besides, the tasks reduce the potentially wide range of supported concepts. Accordingly, this textbook conceptualizes multiplication solely in the context of repeated addition. In particular, multiplication is reduced to a certain way of writing and speaking. It is understood as a ‘shortcut’ for addition.

The next pages 74-75 focus on rectangular arrangements.

Context: Children working in a (math) class room
Representations: rectangular arrangements (one embedded in a story line showing the sequence of progression, one row after the other is uncovered)
Tasks: write the addition and the multiplication problem, show the multiplication problem on the hundred board, draw the multiplication problem and write the matching addition problem.
Addressed GVs: repeated addition
Potentially supported strategies: knowing by heart, repeated addition

On these pages, the focus is put on rectangular arrangements. These didactical representations offer the opportunity to refer to the commutative structure of multiplication and to introduce the changing of order as an appropriate strategy to solve multiplication tasks. Interesting enough, this potential is not used. Instead, children should 'translate' between one type of representation (pictures of rectangular arrangements) and another (symbolic representations, multiplication and addition problems). Accordingly, there is a focus on the structure accomplished by a differentiation in rows and columns. Again, repeated addition is the only addressed GV and the only addressed strategy as well. The misleading idea of multiplication being a different way of writing additions is strengthened.

6. Discussion

There are at least two GVs of multiplication: repeated addition and one-to-many correspondences. Both rely on the basic mathematical activity of unitizing (Lamon, 1994). However, repeated addition is the dominant approach in many countries including Germany, Italy, Taiwan, the US and Canada. First (German) textbook analyses confirm that this way of introducing multiplication provides a good basis to repeated addition as a *strategy*. Nevertheless, we found *representations* that might be used for addressing a much wider range of multiplicative situations. We can think of comparing quantities multiplicatively, of stressing the functional relation between multiplicand and multiplier and, in this way, of focusing on one-to-many correspondences. But, as first analyses indicate, these potentials concerning the *Grundvorstellungen* don't seem to be exhausted in the textbooks. Instead, we found the introductions of multiplication being mainly restricted to the GV and the strategy of repeated addition. This is a surprising result, especially as many researchers agree on the GV of one-to-many correspondence as very promising for supporting children's understanding of multiplication and their performance on multiplication problems.

On the basis of these first results, we regard the theoretical discrimination of GVs, strategies and representations as helpful for our work on the question of how to introduce multiplication in a meaningful and consistent way. Thus, we resume that these elements do not always complement each other in a useful way, but can actually be in conflict.

How do we plan to proceed in our larger project? On the one hand, we want to find out how teachers are supported in their teaching of multiplication in different national contexts in order to prove, deepen and complete our findings from the textbook analyses. On the other hand, we are in process to do research in mathematical classrooms to get insight into the ways introduction of multiplication is empirically realized in everyday classrooms. We have identified remarkable challenges from a theoretical perspective. Thus, the question arises how teachers actually face these challenges of introducing multiplication. At the moment, we observe (German) mathematics classes of grade 2 in order to reconstruct empirically how teachers actually work with the offers of their textbooks and meet the challenge of introducing multiplication in class discussions. This way, we hope to contribute to the scientific discussion about competing approaches to multiplication.

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Capturing flexibility in mental calculation

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Keywords: Flexibility in mental calculation, addition and subtraction, elementary arithmetic.

Introduction

Flexibility in mental calculation is a central goal of elementary mathematics education and there was an increasing research interest in children's approaches and flexible strategies in the last two decades (e.g. Green & Rathgeb-Schnierer, 2020). Empirical findings show that primary students rarely solve problems flexibly and adaptively, especially after learning standard written algorithm (e.g. Selter, 2001). Accordingly, students are often not capable to refer to number patterns and relationships as well as specific problem features for solving one or multi-digit problems flexibly (ibid.). Research results suggest that students' creative and flexible use and combination of different strategic means (Rathgeb-Schnierer & Green, 2013) decrease after introducing respective strategies as a sample solution in the classroom (Benz, 2005). Thus, specific approaches for learning arithmetic seem to have a negative impact on students' abilities in flexible mental calculation. On the other hand, studies show that students' flexible mental calculation can be supported by appropriate instructional approaches (e.g. Rechtsteiner & Rathgeb-Schnierer, 2017). For the development of perceiving and using number patterns and relationships as well as flexible mental calculation Schütte emphasizes the importance of the approach “Zahlenblickschulung" (2004, p. 142; see also Rechtsteiner & Rathgeb-Schnierer, 2017). In general, teaching approaches have a crucial impact on students' abilities in flexible and adaptive arithmetic. Although flexibility in mental calculation is consensually considered a relevant ability, the concept is not consistently defined and operationalized in different studies. Rechtsteiner-Merz (2013) has analyzed the existing approaches. In this study, we refer to the approach which connects the notion of flexibility to cognitive elements that sustain the solution process (e.g. Rathgeb-Schnierer & Green, 2013). Cognitive elements are defined in the context of this study according to Rathgeb-Schnierer & Green “as specific mental actions that sustain a solution process […]. These can be learned *procedures* (such as computing algorithms) or recognition of *number characteristics* (such as number patterns and relations)” (2019, p. 5, emphasis in original). In this vein, we define flexible mental calculation according to Rathgeb-Schnierer and Green: "Only if the tools of solution are linked in a dynamic way to problem characteristics, number patterns, and relationships would we consider as evidence of flexibility in mental calculation" (2013, p. 357).

Aim of the study

There are research projects that investigate flexibility in mental arithmetic with regard to the cognitive elements that sustain the solution process (e.g. Green & Rathgeb-Schnierer, 2020; Rechtsteiner & Rathgeb-Schnierer, 2017). However, conclusions about these cognitive elements cannot be drawn directly based on manifest characteristics, such as the obtained solution or the described methods of solution. Valid conclusions can only be derived from students' explanations and justifications. So far, no standardized instrument exists for capturing the cognitive elements objectively and reliably.

Furthermore, the existing instruments are not sufficiently validated. According to this, already published research results only allow limited generalizable conclusions. Our study pursues innovative approaches: It targets to develop and evaluate a standardized, semi-structured interview for second and third graders which allows capturing abilities in flexible mental calculation by revealing the cognitive elements that sustain the solution process. Therefore, we developed a semi-structured interview guideline regarding the quality criteria of objectivity, reliability, and validity. The interview contains activities of sorting problems (prompt: "Which problem is easy/hard for you?"), reasoning about sorting as well as comparing (prompt: "Which problem is easier for you?") and solving problems (e.g. Rathgeb-Schnierer & Green, 2013; Rechtsteiner & Rathgeb-Schnierer, 2017). For evaluation, we will conduct and videotape approx. 100 interviews with students from the end of second grade and beginning of third grade. Additionally, we survey other constructs, such as arithmetic skills and linguistic abilities. We aim to enhance existing methods to capture flexibility in mental calculation by providing an interview instrument with a different perspective that allows to be applied to a large sample. This interview offers new options regarding quantitative and qualitative analyses as well as triangulation analysis. We completed the pilot run of the semi-structured interview guideline with eleven second graders. Some of the interviews were also conducted by an assistant to see how well the semi-structured guideline worked.

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Open-ended tasks as approach for learning arithmetic in heterogeneous elementary classrooms

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Keywords: heterogeneity, open-ended tasks, equations

Introduction

This study explores a teaching approach for heterogeneous learning groups in mathematics instruction. In elementary school the diversity of students is a key challenge. Since elementary schools draw their student body from residential areas, students exhibit a wide range of abilities and achievement levels. In order to cope with heterogeneous learning groups, various approaches have been developed within mathematics education. In the German context, these approaches base on the concept of differentiation through open-ended tasks that allow multiple solutions on different achievement levels. This concept is specifically labeled as ‘natural differentiation’ (literally translated) (Scherer, 2013). This concept assumes that students work on differentiating open-ended tasks according to their learning and performance level. Whether this assumption is also confirmed in practice has not yet been systematically empirically tested. This is the aim of the presented research project. The study will compare the individual learning condition of students and the performance level when working on an open-ended task that allows multiple solutions (naturally differentiation).

Theoretical framework

The theoretical framework is the offer/take-up model (Angebots-Nutzungs-Modell) (Göbel & Helmke, 2010) developed by Helmke. This model describes the complex processes of teaching and learning and all influencing factors in a simplified way. To explain the effects of teaching and learning success, the model gives a compact overview of the most important variable clusters.

In this project we offer students an open-ended task. Open-ended tasks allow multiple solutions at different levels and thus provide learning opportunities for students with different levels of proficiency. The use of open-ended tasks that allow multiple solutions on different achievement levels represents an innovative approach of teaching and learning arithmetic in heterogeneous classes and inclusive education (Lindenskov & Lindhardt, 2020).

Aim of the study

The study aims to determine to what extent the use of an open-ended task depends on prior knowledge and other variables of learning condition. Our research question is: Do students process the open-ended task "Kombi-Gleichungen" (invention of equations with multiple operations) according to their prior knowledge and individual learning conditions?

Methodology

The sample includes 160 third graders from different heterogeneous classrooms. Firstly, we assessed the learning condition of the students by a questionnaire to analyze students' attitudes towards learning mathematics and a standardized test to measure the achievement level in mathematics. The chosen instruments allow to survey cognitive learning prerequisites as well as motivational and volitional learning prerequisites. Secondly, we conducted two math lessons and offered all students the open-ended task to invent equations in which the equal sign is relationally used (e.g., 2 + 7 – 3 = 10 – 4). This task allows multiple solutions, enables students to work at their individual performance level. The students' worksheets provide the database for our analysis. To assess the level of students' solution process, we use a category system (Friedrich & Rathgeb-Schnierer, 2020). Finally, we link level of the solution process to the learning prerequisites of the students.

Preliminary outcomes

After analyzing part of the data, it appears the highest processing level is characterized by invention of equations with different operations, multi-digit numbers, several calculation steps, and transitions with a systematic procedure. Inventing an equation system including two or even more complex equations is a characteristic of a high level of processing. All students were able to handle the open-ended task: Those who scored highly in the achievement test attempted to invent particularly complicated equations. Students with a low score invented many equations with few calculation steps.

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The aRithmetic Discourse Profile as a tool for evaluating students' discourse according to the ritual to explorative continuum

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We propose a new tool – the aRithmetic Discourse Profile (RDP) for analyzing the arithmetic discourse of students, based on the commognitive theory. The tool maps students' discourse on a continuum between ritual and explorative participation. We apply the tool to the discourse of 12 7th grade students and exemplify the analysis on three tasks: one in addition of whole numbers, one in multiplication and one with fractions. Tasks were taken from regular curricular materials of elementary school mathematics. We discuss the affordances of the tool in relation to existing diagnostic tools.

Keywords: Arithmetic, discourse, assessment, commognition.

Introduction

Assessing students' mastery of arithmetic has multiple potential benefits. It can assist teachers in evaluating students' performance in relation to curricular expectations and provide researchers with measures of pre- and post- interventions. Two major forms of assessment currently exist for assessing students' arithmetic discourse. One is the common school-based exams (or standardized tests). The other are "diagnostical tools" usually used by educational psychologists to assess students' "arithmetic abilities" (or disabilities) (e.g. Chinn, 2020). School-based exams are effective for efficiently assessing students' outcomes. Diagnostic psychological tools are effective for identifying certain common deficits (such as problems with fact retrieval) (e.g. Dowker, 2012). However, both these types of tools do not provide a fine-grained profile of students' arithmetic knowledge and skills.

Previous commognitive studies (Ben-Yehuda, 2003; Ben-Yehuda et al., 2005) have started offering an alternative to the above two types of assessment, by proposing a form of "profiling" a students' arithmetic discourse, based on transcriptions of think-aloud interviews. However, these studies were quite preliminary in their conceptualization. Their categories were difficult to replicate and not sufficiently connected to a broad theory of learning. In the past two decades, the commognitive theory has evolved and sharpened its tools of analysis. One useful conceptualization that has come out of commognitive studies is that of the "ritual-exploration" dyad, which qualifies participation in discourses both according to development (learners starting from ritual and moving to explorations) (Lavie & Sfard, 2019) and according to achievements (learners experiencing difficulties performing more ritually than successful learners) (Heyd-Metzuyanim, 2015). Armed with these new conceptualizations our goal is to develop and test a tool for mapping students arithmetic discourse based on the continuum of ritual to explorative participation.

Theoretical Background

Commognition theorizes the learning of mathematics as a process whereby learners gradually become participants in the mathematical discourse. This process is characterized by movement from performance of ritual routines to explorative routines. Routines are pairs of *task* (what it is that the performer feels obliged to do) and *procedure* (the steps to achieve the goal). Ritual routines are aimed primarily at pleasing the masters of the discourse, usually teachers or parents. As such, they are mostly characterized by thoughtful imitation (Lavie et al., 2019). The learner attempts to repeat actions that she saw the expert perform in situations that she perceives as similar to the present task. As the learner gains experience with similar tasks, her routines gradually become more explorative (Sfard & Lavie, 2005). Explorative participation is characterized by flexible usage of different procedures to achieve a single task, and by the learner initiating these procedures to achieve her own goals. That is, the learner performs the routines to produce narratives about the world, not to please an external authority. Explorative participation is thus characterized by agentivity. In addition, explorative routines are *bonded*, that is, the output of each step serves as an input to the next step in the procedure. In contrast, ritual routines may often be unbonded since they are made up of steps imitated of others, without the performer being aware of how or why each step is bonded with the former ones.

In mathematics, routines are mostly concerned with producing narratives about mathematics objects. For example, the routine of multiplying 5×3 leads, eventually (after, for example, adding three to itself five times), to the narrative "5 times 3 is 15". Yet for such routines to be enacted with ease, learners need to *objectify* numbers. The process of objectification, according to commognition, is the main process that underlies what is often termed as "mathematical understanding". Within this process, learners come to view processes (such as counting or adding) as products. For example, the process of counting "one, two, three, four" eventually gets reified into an object ("four") that can be treated as existing on its own, regardless of the process of counting that led to it. The process of objectification, however, takes time, and ritual performers often treat the signifiers of mathematical objects (such as the digits of a number in the realm of tens) only *syntactically,* that is, without relation to the object that they are supposed to signify. The processes of objectification are never-ending in the learning of mathematics since mathematical discourses are hierarchical. Thus, for example, once the learner masters (or becomes explorative in) the discourse of natural numbers, they are expected to enter the discourse of fractions (or rationale numbers). Once this has been mastered, the expectations for participation are raised to the discourse of real numbers, algebra, etc.

School exams are ill-suited to examine the extent to which students perform explorative routines in a certain discourse, as they mostly give a picture of the final answer produced by the students' routines, not the procedure by which the answer was produced. Therefore, students may be identified as "successful" (or achieve highly on exams) in school mathematics, while many of their mathematical routines are still ritual (Heyd-Metzuyanim, 2015). Another problem with standard school assessment is that they can miss certain explorative routines that students who err in their final answers do perform. This is especially relevant to low achieving students (Ben-Yehuda et al., 2005; Heyd-Metzuyanim, 2013). To overcome these limitations, early commognitive research came up with a tool called "Arithmetic Discourse Profiling" (Ben-Yehuda, 2003). This tool, based on a form of cognitive interviewing, attempts to map the student's discourse in ways that highlight not just what the student does "wrong" (or non-canonically) but also (and more importantly) the actions the student does perform when attempting to solve mathematical tasks. In this work, we aim to extend the early attempts at profiling arithmetic discourse by using the more recent definitions and operationalizations of ritual vs. explorative routines. Our research question is: Based on tasks taken from elementary school curriculum, how can students' arithmetic discourse be characterized on the continuum between ritual and explorative participation?

Method

Our analytical tool was developed as a secondary data analysis, on data collected by the first author (Heyd-Metzuyanim, 2011). The data consists of arithmetic interviews conducted with 12 7th graders during 2008. The interviews were conducted as part of a larger study where the author taught the 12 students in three groups over a period of 5 months, in an out of school "course". The course was broadly defined as "mathematics enrichment" and the original study was aimed at examining interactions between emotional and cognitive aspects of learning. The students were placed in three groups (each of 4): Low achieving, middle-high achieving, and excelling. The excelling group consisted of 4 boys who came from an accelerated mathematics and science classroom (to which entrance was based on very high grades in elementary school mathematics and other subjects). Placement in the low and middle-high achieving group was based on reports coming from the school and parents of the students regarding their grades in elementary school mathematics and their achievements in the beginning of 7th grade.

All the interviews were conducted by the first author, in a one-on-one setting. Students were asked to think out loud as they solved the problems (Young, 2005). When they were not successful and asked for help, the interviewer attempted to scaffold the task, yet mostly she refrained from giving feedback on the correctness of the results. Interviews were videotaped with two cameras, one pointing at the student's face and one on his writing. Written artifacts were collected too.

The interview protocol was based on Ben-Yehuda's (2003) ADP protocol and consisted of 24 tasks (some very short) that had potential to illuminate various aspects of students' arithmetic discourse. For the present study, which goal was mainly to create analytical tools, we chose three tasks for analysis (the full protocol appears in (Heyd-Metzuyanim, 2011)). These tasks were (1) Add 96+7936 (if possible, "in your head") (2) Multiply 25×99 and (3) $\frac{2}{3}\times 9$. We chose these tasks as they covered a relatively large domain of arithmetic skills, including manipulating whole numbers and fractions.

Analysis: Analysis was done on full verbatim transcriptions of the interviews and proceeded in two steps. First, we divided each routine (task + procedure) into sub-routines (for example, adding 96+7935 often consisted of sub-routines such as adding 5+6, adding 11 to 8020, etc). Next, we determined whether the routine (or sub-routines) were ritual or explorative according to eight categories, taken from the literature on ritual and explorations. These are explained in Table 1.

Table 1 - Criteria of analysis for ritual-explorative routines

	Criterion	Analytical actions	Characteristics of an explorative routine	Characteristics of a ritual routine
1	Objectified /syntactic mediation.	Searching for evidence that the nouns signify numbers/quantities and	In whole numbers, relating to the place value of the numeral. In fractions: relating to different realizations of the fraction as the same, including fraction as	In whole numbers: relating to operations as signaling procedures on digits rather than on the whole number. In

		not just the signifier of the number	operator, part of whole, part of quantity, etc.	fractions, relating separately to the numerator and denominator.
2	Flexibility	Look for multiple procedures that are associated with the same task.	More than one procedure is associated with the main task OR a non-standard procedure is applied to the task.	Relying on only one procedure while showing rigidity and reluctance to use any other procedure.
3	Agency/External authority	Look for subjectifying discourse; examine verbs/pronouns and non-verbal signals that indicate the confidence.	Mathematizing with high confidence (no hesitations, question marks, no looking for approval). Spontaneously articulating mathematical narratives	Talking with question marks; Verbally or Non-verbally seeking approval from the interviewer; Relating to external authority for justification (e.g. "that's what I learned in school")
4	Focus on goal or on procedure	Look for verbs indicating doing (e.g. "I add") vs. being verbs indicating the result ("it is…");	Talking about the result, checking it, or explaining it spontaneously	Talking about the actions of the procedure. Ending the procedure without relating to the reasonableness of the result.
5	Bondedness	Examine the procedure; sub-procedures and the bonds between them.	Each sub-procedure feeds the next sub-procedure. The narrative of the result of sub-procedure N serves as the input of sub-procedure N+1.	There is a disconnect between a certain sub-procedure and its following one OR sub-procedures using different realizations are not treated as the same.

The final three characteristics were *Canonical procedures* that were coded as explorative if all the steps in the procedure aligned with standard mathematical procedures; *Canonical narratives* were coded as explorative if the end result of a procedure (e.g. the end result for the task "two thirds of 9" was "six") was canonical; and *Mediation* was coded as explorative if the procedure was fully initiated and enacted by the student, and ritual if some parts of it were mediated by the interviewer.

The first stages of constructing the coding criteria (exemplified in the findings section) were created by the first and third authors, through mutual agreement. After that, the second author was taught the coding scheme, blindly replicated it on 50% of the data, and created a coding manual.

After coding was complete, a "ritual/exploration" ratio was calculated for each student, on each of the tasks. The highest explorative ratio could be 0/8 (0 ritual, 8 explorative characteristics), whereas the most ritual performance could by 8/0. Notably some of the characteristics (agentivity, bondedness, canonical and non-canonical procedure/narrative) could be coded *both* as ritual and as explorative. This was necessary since we wanted to account for explorative elements of sub-routines. Thus ratios such as 7/3 or 5/6 were also possible. In general, ratios close to 1 showed "mixed" performance.

Findings

Table 2 - Ritual-Exploration ratios and relative placement of 12 students

Student	Achievement group	96+7935	25×99	$\frac{2}{3}\times 9$
Dana	Low	6/4	8/2	7/3
Hili	Low	7/2	Not attempted	Not attempted

Hila	Low	7/3	7/1	Not attempted
Naor	Low	2/6	8/3	4/5
Edna	Middle-high	8/3	6/7	Not attempted
Idit	Middle-high	0/7	6/5	7/4
Dan	Middle-high	1/7	7/6	2/6
Ziv	Middle-high	3/6	3/8	2/6
Ram	Excelling	0/6	0/8	1/7
Gabby	Excelling	0/7	0/8	1/7
Yoram	Excelling	0/7	0/8	0/7
Amir	Excelling	0/7	1/8	0/8

Table 2 summarizes the ritual/explorative ratios of the 12 students' (all pseudonymed) performance on the three tasks. Before we delve into the exemplification of how these ratios were determined, there are a few observations worth mentioning regarding this table.

Our first observation is that the ritual/exploration tool seems to capture a wide range of routine enactments, from those very high in exploration (0/8) to those almost only featuring ritual characteristics (8/2). Secondly, we see the low achieving group very high in ritual characteristics (or not attempting tasks at all); the moderate-high group is fluctuating widely, between 0 and 8; and the excelling group is quite consistent around the 0 – meaning high exploration. This lends validity to the tool as capturing features that reflect students' success in school mathematics. A third observation is that students can be inconsistent with respect to different tasks. For example, Dan's performance is explorative in the addition task, ritual in the multiplication task, and relatively explorative again in the fractions task. We do see, however, that in the excelling group these fluctuations do not exist.

Next, we demonstrate our analysis on two episodes from students' interviews, showing different types of routine enactment, associated with different ritual/explorative ratios.

Episode I: Hila, 96+7935 – Dominantly ritual performance

	What is said (what is done)	Writing
1	Hila: Ah, ninety-six plus seven, seven... seven thousand, seven thousand ninety and thirty, um, can I calculate?	96 + 7935 = 8021
2	Interviewer: Is there a way that you can do it in your head?	
3	Hila: I, um, it's so difficult for me, I-	
4	Interviewer: Try it	
5	Hila: OK. Ah, wow, three plus (..) nine, twelve. Um twelve, um twelve and five plus six is eleven. Eleven plus twelve, equals thirty-three (..), yes, ah no, twenty-three. So, it's eight thousand ninety and thirty, Ah no. Twenty three. OK.	
6	Interviewer: OK, look at the answer and tell me if it looks alright to you?	1 11 7935 + 96 8021
7	Hila: Oops.	
8	Interviewer: Wait, oops. Why oops? What's not good about it?	
9	Hila: 'cause I had to do plus	
10	Interviewer: And what did you do?	
11	Hila: Aah... (points at the paper) I didn't combine like... here the...	
12	Interviewer: Now do you want to do it vertically?	

13	Hila: (Writes). Eighty, three. Five – six, nine, yeah twelve, ten, yeah aah (erases) eighty, eighty (erases).	
14	Interviewer: And that’s a better way?	
15	Hila: Yes.	

We see in Hila’s performance 7 characteristics of ritual and 3 or explorative participation, thus her ritual/explorative ratio is 7/3. Following is the characteristics analysis: A. Non-objectified discourse. The digits in Hila’s discourse are treated as independent entities, to be combined and manipulated in some form, but not as indicative of a whole number or as place value (see line 5, where the digits are first added horizontally, and then line 13). B. Rigid performance: Hila sticks only to procedures of adding the digits separately. Even when asked to do it “out loud”, she tries (albeit unsuccessfully) to reconstruct some sort of procedure for adding the numbers digit by digit. C. External authority: Hila’s discourse is hesitant, and she relies on the interviewer to encourage her to try out the task to begin with [4], to monitor her answer [6, 8] and to suggest an alternative procedure [12]. D. Focus on procedure: Hila focuses only on the procedure (“I had to do plus” [9] “I didn’t combine” [11]). She does not check her answer showing no interest in the result. E. Canonical procedures: Some of the procedures Hila enacts are non-canonical (e.g. adding the sums of the unit digits and tens digits: 11+12). Other procedures are canonical (e.g. writing the “carry on” digits in the appropriate places in the vertical solution). Therefore, she got 1 on both “canonical” and “non-canonical” sides of the ritual/explorative table. F. Canonical narratives: Similar to E. some of Hila’s narratives are canonical (e.g. 5+6=11) while others are not (the overall sum is non-canonical). G. Mediation: Hila’s performance is mediated by several comments of the interviewer, especially the suggestion to “do it vertically” [12]. H. Bondedness: all of Hila’s sub-procedures feed one into the other. Therefore, she received a ‘1’ on the bonded criteria (explorative).

Episode II: Idit solving 25×99 - Example of mixed performance

1	Idit:	(Reads the task, sighs) I usually get messed up with such exercises
2	interviewer:	Is there something that you can do with the... ninety-nine? (..) That is very close to another number?
3	Idit:	A hundred (Interviewer: OK) You can round it (up) to like... (looks at Interviewer), no. First, I’ll round it to 25 times 100. Can I do (that)? (Interviewer: OK) Which turns out two thousand and five hundred (Interviewer: Umhmm) and then take off one. But (comments on failing on such exercises. Interviewer encourages her to try, nevertheless). So, it turns out a thousand… No (erases)… (writes 1049) wait, no (erases, mumbles, writes 2499.
4	interviewer:	Two thousand four hundred and ninety-nine
5	Idit:	that’s what seems most reasonable
6	Interviewer:	But do you think it’s correct? (Idit shrugs “I guess so” and smiles). If I tell you that this is 99. I tell you that this is 99 times (intrusion outside). If I tell you this as an exercise of 99 times 25 would it help?
7	Idit:	(giggles) No

Idit’s ritual/explorative ratio was determined as 6/5 for the following reasons: A. Objectified discourse: we only find indications of objectified discourse in Idit’s solution. She treats 25 times 100 as objects (“which turns out two thousand and five hundred” and “I’ll round *it* to 25 times 100” [3]). B. Rigid performance: Idit does not have an alternative procedure for multiplying 25×100 (e.g. by long multiplication), even though she is unhappy with the result [7] C. Both Agentivity and External authority: in some points, Idit makes independent choices and seeks no guidance (e.g. decides 1049 is an error and instead writes 2499). In other parts, she seeks approval from the interviewer (e.g. “Can I do that?” [3]) D. Focus on product. There are no indications that Idit is focused on the procedure and she does comment on the final product (“That’s what seems most reasonable” [5]) E. The overall procedure is non-canonical since she sames 25×99 with 25×100 -1 F. Canonical and non-canonical narratives: all the sub-narratives (25×100=2500; 2500-1=2499) are canonical however the overall narrative 25×99=2499 is non-canonical G. Mediation: Idit’s performance is mediated by the interviewer suggesting the similarity between 99 and 100 [2] H. Bondedness and non-bondedness: all of Idit’s sub-procedures are bonded, yet the overall procedure is not bonded to the overall task.

Discussion

In this study we asked how can students’ arithmetic discourse be characterized on the continuum between ritual and explorative participation? Our results indicate that the eight characteristics of ritual/explorative participation are useful for locating students on such a continuum. These characteristics are: Objectification, Flexibility, Agentivity, Focus on procedure/product, Canonical procedures, Canonical narratives, Mediation and Bondedness. We used these characteristics to construct a ritual/explorative ratio, where ratios close to 0 indicated high explorative performance, whereas ratios nearing 8/0 indicate ritual performance. Ratios around 1 indicate mixed performance, which mostly show parts of the sub-procedures are ritual while others are explorative.

The validity of this tool is stemming from three sources. First, we were able to achieve blind inter-rater reliability showing that our criteria of analysis were operational. Second, we found general coherence between the ratios and students school achievements (as indicated by placement into the achievement groups in the study, see table 1). Third, the ratios generally cohered with the first author’s experience with the students, gained through teaching them for five months.

Our findings generally support the commognitive theory of the development of students’ mathematical discourse in several respects. First, they show that students generally achieving higher (and thus, presumably more fluent in the arithmetic discourse) indeed perform more exploratively, while low achieving students (who have not mastered the discourse) perform ritually (Heyd-Metzuyanim, 2015). Second, we see from table 1 that students generally performed more exploratively in discourses that are primary (e.g. natural numbers) while in a newer discourse (fractions) the performance was more ritual. This is consonant with the theory that states higher level discourses are built upon primary discourses that are subsumed by them (Sfard, 2008). However, we also found some anomalies, such as a student whose performance on the multiplication task was ritual while his discourse on fractions was explorative. Future studies should look more into the question of whether these are anomalies that are characteristic of “transition” phases (such as at the level of beginning middle school) or whether they can be seen also in later stages of mathematical learning.

The study of course has several limitations, the main one of which is the relatively small number of participants. This is a limitation related to the high work-intensiveness of commognitive analysis and may be overcome in future studies by finding ways to make the analysis more efficient. Nevertheless, the method suggested in this study is the first form of quantifying mathematical performance according to the ritual-explorative continuum. As such, it opens up multiple avenues for further inquiry. For example, future studies can examine whether the pattern of "being around 1 ratio" in students who are identified as middle-high achieving repeats itself, while the pattern of "close to 0 ratio" is common in students identified as excelling. The latter question is especially important since we note that the curriculum mostly advances according to the assumption that students have mastered primary discourses. For example, middle-school curriculum is based on the assumption that students are explorative participants in the arithmetic discourse (Karsenty & Arcavi, 2003). Whether and how students "fill in" formerly learned discourses is an open question, which the tool proposed in this study may help to answer.

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Counting strategies in a proficient grade 1 student: the case of Petra

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A case study design was used to explore a foundational number-sense (FoNS) proficient student's strategy use in counting. The student, referred to as Petra, participated in a task-based semi-structured systematic counting interview. The interview was analysed qualitatively exploring part-whole reasoning in counting strategies in interrelation with FoNS, which was measured using a digital assessment. The interrelations were discussed as reflecting strands that Kilpatrick et al. (2001) found important for mathematical proficiency. Petra showed the ability to use combinations of one-to-one and relational strategies and high levels of adaptive reasoning and productive disposition. Petra also showed that a number-sense proficient student may be flexible in counting in the 0-10 range but not necessarily in the 10-20 range. Results are discussed as part-whole reasoning on the mental number line.

Keywords: Part-whole reasoning, mental number line, counting strategies, number sense, grade 1.

Introduction

Counting is a central to number sense, which is the ability to work flexibly with numbers and quantities predicting students' mathematical proficiency (Andrews & Sayers, 2015). Flexibility is essential in definitions of number sense. Developing mental representations of numbers and strategies represented on a mental number line enables flexible mental calculations, part-whole reasoning and relational strategies (Dehaene, 2011; Hunting, 2003; Rathgeb-Schnierer & Green, 2019).

Variation in counting and counting strategies is traditionally described in terms of Gelman and Gallistel's (1986) three how-to-count principles: the one-to-one principle of correspondence between numerals and items, the stable-order principle of numerals in any count, and the cardinal principle of the final counted item's numeral representing the number of items in the set. Students with mathematical difficulties are found to exhibit immature, inflexible and inefficient counting strategies and having problems shifting between strategies compared to proficient students (Gelman & Gallistel, 1986). Alternatively, or complementarily, variation in counting strategies can be considered as included in arithmetical problems as part-whole structures supporting relational strategies and part-whole reasoning (e.g. Hunting, 2003).

Mental flexibility is critical to the efficiency of mental strategies (Rathgeb-Schnierer & Green, 2019). Carpenter and his colleagues (e.g. Carpenter et al., 1996) has emphasised understanding students' thinking as a knowledge base for cognitively guided instruction promoting mathematical development. However, what mental processes contribute to mental flexibility? Kilpatrick et al. (2001) claimed that five interwoven and interdependent strands represent the flexible interrelations needed for mathematical proficiency. Extended knowledge of the continuum of number-sense proficiency and flexibility in counting is needed to develop characteristics of various proficiency, and improve the discovery and teaching of typical and atypical variations to support mathematical development in all students. The paper discuss flexibility as part-whole reasoning by exploring a grade 1 number-sense proficient student's systematic counting strategies considered in interrelation with foundational number sense (FoNS) concepts. The paper interprets interrelations to promote counting flexibility and raises the following research questions:

RQI: What strategies does a number-sense proficient grade 1 student use in counting?

RQII: How may strategy use be interrelated with part-whole reasoning on the mental number line?

The paper reports on data from a Ph.D. project studying 75 Norwegian grade 1 students' variations in counting and patterning strategies in relation to FoNS and verbal and nonverbal reasoning.

Theoretical framework

Andrews and Sayers' (2015) described systematic counting as being able to count forwards and backwards between 0 and 20, understanding ordinality and being able to start counting from an arbitrary starting point between 0 and 20. Systematic counting is integrated into Andrews and Sayers' (2015) FoNS definition and interrelates with the seven other components: number identification, number and quantity, quantity discrimination, representing numbers, estimation, arithmetic competence, and number patterns (Andrews & Sayers, 2015). Counting strategies develop from count-all one-to-one correspondence strategies to the more efficient retrieval strategies, such as count-on, which partly depend on one-to-one strategies and relational strategies. Finally, interrelations between counting and arithmetic emerge in the most abstract derived fact relational strategies based on commutativity and the inverse principle. Such mental representations enable part-whole reasoning about the relations between parts constituting the whole and decomposition of the whole into parts of the mental number line and efficient relational strategies (Dowker, 2014; Hunting, 2003).

Flexible counting strategies, conceptual understanding of numbers and the ability to count systematically have been found to depend on estimation abilities and the ability to compare small quantities without counting, also called subitising (Andrews & Sayers, 2015; Dehaene, 2011). Derived fact strategies might be a combination of counting and subitising (Dowker, 2014).

Understanding structures of patterns develops generalisations and promotes part-whole reasoning, which is critical for commutativity, number sense, algebra, and counting (Hunting, 2003). Counting repeating patterns (e.g. ABABAB, □ΔΔ□ΔΔ□ΔΔ) and growing patterns (e.g. 1 3 5, □ □□ □□□, ABAABAAAB) enable the use of both one-to-one building-up strategies and more advanced unit factor and scalar relational strategies, which consider a repeating unit or multiplicative relations between quantities in two or more measure spaces, but with different advanced levels.

Concrete counting operations with counting blocks or other manipulatives develop flexible and efficient mental counting and calculations (Rathgeb-Schnierer & Green, 2019). The mental number line enables the use of part-whole reasoning and is the cognitive system underlying number sense (Dehaene, 2011) serving mental representations of the ordinality, cardinality and magnitudes of numbers as well as the relations between numbers, thus enabling flexible operation with numbers.

Kilpatrick et al. (2001) holistically described flexibility as the following five strands that make up mathematical proficiency: (1) conceptual understanding interrelations between mathematical ideas and their mental representations, including knowledge of why and when mathematical ideas are important and useful, (2) strategic competence to formulate, represent and solve mathematical problems, (3) procedural fluency and knowing when procedures are appropriate, which is shown as flexible, accurate, and efficient procedures, (4) adaptive reasoning as the capacity to think logically about interrelations between concepts and situations, and (5) productive disposition involving the ability to see mathematics as useful and understanding that steady effort will pay off in combination with a view of oneself as an effective learner and doer of mathematics (Kilpatrick et al., 2001). The strands include students' attitudes and metacognition as it concerns knowledge and reflection on one's reasoning and problem-solving abilities.

Methods

A task-based semi-structured interview was designed to explore Petra's strategies for systematic counting: forwards and backwards in counting on 1s, 2s and 3s in the 0-20 range. Petra's counting and counting strategies were explored using a case study design (Yin, 2014).

Recruitment, sample, and case

Following informed parental consent, Petra, aged 5 years and 11 months, was purposefully chosen from the larger study sample as the student with the highest score on the digital number sense test of 75 grade 1 students. Petra is thus referred to as a number-sense proficient student.

Assessments and procedures

A task-based semi-structured counting interview was developed based on Andrews and Sayers' (2015) definition of systematic counting. A toy frog named Mr Minus asked ten questions to facilitate observation of Petra's counting strategies. Counting blocks were made available. If needed, Mr Minus modelled counting by jumping on a number line made of counting blocks. No time limits were given, the interview was video-recorded and lasted approximately 10 minutes. The author was Mr Minus's voice and initiated a dialogue by asking the following questions: (1) 'What number do we start counting at?' (2) 'How far is it possible to count?' (3) 'Is it possible to count backwards as well?' (4) 'Do we need to start counting at 1? Is it possible to count forwards from 7?' (5) 'Is it possible to count backwards from 9?' (6) 'Can we count from 20 to 0 too?' (7) 'I have heard some adults count in a weird way. They counted 2-4-6… Have you heard such counting? I am wondering how to continue to count if I am to continue to count the same way. Can you help me?' (8) 'Is it also possible to count backwards this way?' (9) 'I have heard some adults count in a weird way. They counted 1-3-5… Have you heard such counting? I am wondering how to continue to count if I am to continue to count the same way. Can you help me?' and (10) 'Is it also possible to count backwards this way?'. Mr Minus (test-administrator) gave verbal and nonverbal prompts if needed.

A digital FoNS assessment (Saksvik-Raanes & Solstad, in press) measured all FoNS components except for representing numbers, following Andrews and Sayers' (2015) number sense definition. Subitising was included as an eight component: (1) number identification, (2) systematic counting, (3) number and quantity, (4) quantity discrimination, (5) estimation, (6) arithmetic competence, (7) subitising, and (8) number patterns. Subitising tasks were timed.

In a 30-minute session, Petra dragged, dropped and organised objects on the screen or tapped the appropriate multiple-choice item to solve tasks. Figure 1 provides an example of each of the different task designs and the verbal instructions given in the systematic counting component tasks. Petra scored 62 out of 69 possible points.

Figure 1. The tree task design of the systematic counting component

A) Put the numbers in order, B) Place the star on the third star in the line, and C) Place the correct number into the box.

Analytical procedures

Petra's counting response and strategies in tasks 1 to 6 (see description in the "assessments and procedures" section) were coded and qualitatively analysed according to Andrews and Sayers' (2015) description of systematic counting and Gelman and Gallistel's (1986) how-to-count principles using NVivo. In addition, Petra's strategy use in tasks 7 to 10 was coded as one-to-one building-up/correspondence strategies, and as part-whole relational strategies depending on multiplicative relations between two or three units in counting. Number sense was measured using a digital assessment based on Andrews and Sayers' (2015) eight components of FoNS on an individual tablet. Petra's responses to the tasks in the digital FoNS assessment were scored dichotomously.

Petra's strategy use in relation to her FoNS was discussed to reflect part-whole reasoning on her mental number line, using Kilpatrick et al.'s (2001) strands of mathematical proficiency as guidelines.

Results and analyses

Semi-structured systematic counting interview

Mr Minus: What is the smallest number we can start to count from?
Petra: 1.
Mr Minus: How far is it possible to count?
Petra: Am I to count to hundred? 1-2-3-4-5-6-7-8-9-10-11-12-13-14-15-16-17-(…)-100.

At every -9 from 29 to 99, she prolonged pronunciation of the /i:/ vowel before the 10 transitions.

Petra: I also manage to count to hundred like this: 10-20-30-40-50-60-70-80-90-100. Now I counted only the tens.

Her thumb represented 10 and her index finger represented 20. No fingers were used for subsequent counting.

Mr Minus: Is it possible to start counting backwards?
Petra: I manage to count from hundred to eighty, I think. 99-98-97-96-95-94-93-92-91-80-89-88-87-86-85-84-83-82-81.

She prolonged the pronunciation of the vowels in 95 and 91 and quickly said 80 without self-correction.

Mr Minus: What comes next, after 81?
Petra: Seventy!

Petra finger-counted on her left hand and correspondently whispered every 10th number word from 10 in the additive direction for each finger, which was the opposite direction from the subtracting counting she had started using. She said 70 loudly and stopped counting.

Mr Minus:	Can we count from twenty to zero too?
Petra:	I do not think I know much about that! Sometimes I need to do like this.

She illustrated finger-counting corresponding to lip movements pronouncing 1-2-3-4-5.

Mr Minus:	That is okay if you need to.
Petra:	20-90-no-19-17-15-14-13-12-11-10-9-8-7-6-5-4-3-2-1-0.

Petra self-corrected 90 to 19. The three fist phonemes of Norwegian number words for 90 and 19 are identical and perhaps phonologically distracted her. After she said 19, she started finger-counting on her left thumb and whispered 1 before she said 10 while still holding up only her thumb, representing the 10th addend. As such, she combined a retrieval relational and a one-to-one strategy, as adjacent fingers one-by-one represented the consecutive addends while she whispered 11-12-13-14-15-16 before answering 17 loudly. She counted forwards from 10 and stopped when she saw there was one less finger on her hand than the number of fingers she remembered seeing on her hand from the preceding answer. Then she repeated her strategy of the thump representing 10. Because she whispered 11 without showing the index fingers before she whispered 11, she answered 15. She used the same strategy to find the subsequent answers, including 11. Petra stopped using her fingers when she continued to count down from 10 to 0.

Mr Minus:	High five! You made it! Is it possible to counting forwards from seven to twenty?
Petra:	I think so. That is almost what I already counted. 8-9-10-11-12-13-14-15-16-17-18-19-20. That was easy, you only start at eight.
Mr Minus:	Cool! Is it possible to start counting from 14 as well?
Petra:	14-15-16-17-18-19-20.
Mr Minus:	Is it so that we can count backwards from an arbitrary point as well, say from 9?
Petra:	9-8-7-6-5-4-3-2-1-0.
Mr Minus:	I have heard some adults count in a weird way. They counted 2-4-6… Have you heard that? I wonder how to count on from 6 if I were to continue counting the same way. Do you know?
Petra:	Yes, 1-2-3-4. And that’s only two! The next numbers are 8-10-12. And you know what? I also manage from three to nine.

Fingers one-to-one corresponded to the number words. Then she grouped two and two fingers on her left hand. She counted 3-6-9 while she showed one and one finger.

Mr Minus:	WOW! What is the next number when counting like that?
Petra:	12-15-18-20-23-26-29.

Petra one-to-one correspondently finger-counted and whispered numbers between 9 and 12. She continued to whisper-count with the same three fingers and said the number on the third finger loudly. Incorrect one-to-one correspondence made her answer 20 without self-correction.

Mr Minus:	You teach me lots of things! Is it possible to count backwards in the same way as you counted 2-4-6 forwards?
Petra:	Yes, I can try that. 10-8-6-4-2-0.
Mr Minus:	Okay. I have heard some adults count in a weird way. They counted 1-3-5… Have you heard that? I wonder how to count on from 5 if I am to continue counting the same way. Do you know?
Petra:	1-… Am I to do addition?

Mr Minus: I do not know. I just heard someone count that way. 1-3-5, and I do not understand what to say next if I am to continue in the same pattern.
Petra: 1-3-5…? 6?
Mr Minus: Maybe? I want to try something! I want to jump like they counted. 1-3-5… Now I jump once more. What am I to say when I land here? And here?

The interviewer lined up counting blocks, making a number line from 1-10. The frog jumped on every other block and said the number names corresponding to the number line position on which he landed on. The frog jumped on every other block and said the number names corresponding to the number line position on which he landed.

Petra: 7-10

This was Petra' answer for the two positions Mr Minus jumped on after 5.

Mr Minus: How did you do that?
Petra: I counted inside myself. I counted 2-4-6-8-10.

She correspondently pointed at the second, fourth, sixth, eight and tenth block.

Mr Minus: Okay. But I said 1-3-5 and you said seven was the next number. How did you know that?
Petra: I added two more.
Mr Minus: Okay, that was what happened! Is it possible to jump back and count in this way as well?
Petra: Do you mean 7-5-3?

She tapped her fingers on the table in the opposite downwards or subtractive direction.

Mr Minus: Yes! And now I think you showed me that it is possible!
Petra: I know how to make a rocket with my fingers! I do like this. Do you manage?

She ended the dialogue, making a rocket with both hands and laughed.

Foundational number sense (FoNS)

Petra mastered all number sense components in the digital assessment, expect for estimation and arithmetic. For numbers between 10 and 20, she showed difficulties estimating a number's position on the number line and doing arithmetical operations.

One-to-one and relational strategies

For tasks 1 to 6, Petra successfully used the required one-to-one strategy with correspondence between the number name and the counting procedure. Tasks 7 to 10 served as opportunities for part-whole reasoning about multiplicative relations between numbers and processes on the mental number line. Therefore, the tasks enabled the use of both one-to- one and relational strategies. As the results showed, Petra successfully and unsuccessfully used a combination of counting-all, counting-on and retrieval strategies. That is, both one-to-one and relational strategies. Success seemed to depend on the number range in which she counted. Petra did not use the one-to-one count-all strategy alone. Despite various success, she considered the appropriateness of more advanced relational strategies and showed emergent part-whole reasoning (Dowker, 2014). She used these strategies in combination with one-to-one backup strategies for support. The digital assessment also supported the assumption that she used relational fact retrieval strategies for numbers between 1 and 10 in forwards counting but used a combination of one-to-one and relational strategies in backwards counting above 10.

Discussion

Indications of conceptual understanding, procedural fluency, and strategic competence

Petra met some of Andrews and Sayers' (2015) systematic counting criteria. She counted correctly forwards to 100 and met their criteria of forwards counting from 0-20, but she did not meet the criteria of backwards counting from 20 to 0. She mastered forwards and backwards counting in patterns of 2s on even numbers in the 0–20 range but mastered only backwards counting on odd numbers in the 10–0 range. She successfully counted backwards from an arbitrary starting point between 10 and 0, but not between 20 and 10. There is compliance between Petra's counting, estimation, and arithmetic competence. Estimation is important for mentally representing the number line (Dehaene, 2011). Petra met the one-to-one principle of Gelman and Gallistel's (1986) in a count-all backup strategy but not in a combined backup and retrieval strategy counting-on from 10. The stable order principle was met in forwards counting in 1s and in patterns of 2s and 3s. She was challenged in backwards counting by 1s in the 20-10 range, omitting two numbers but not in the 10-0 range. The cardinal principle was confirmed in the number and quantity component of the digital FoNS assessment. Petra's strong subitising abilities may have played a role when her thumb represented 10 and when she reasoned in multiplicative relations, counted on 2s and explored counting on 3s.

Self-correction and efficient strategy use were implemented as procedural fluency. Self-correction occurred inconsistently. It is unknown whether incorrect counting from 99-80 resulted from number name or ordinality problems and a not fully established mental number line for the eighth or nineth 10 range, something her strategy use in the 20-10 range supports. If so, subtracting perhaps distracted her. These findings underline the interrelations between conceptual understanding and procedural fluency. Petra used efficient strategies, that is, when she counted to 100 by counting by 10s and when she counted in 2s and tried to count in 3s. Additionally, she showed a high level of strategic competence as she represented and solved problems verbally, explained her mental strategies and utilised manipulatives. The findings show the mutually supportive interrelations between strategic competence, conceptual understanding and procedural fluency (Kilpatrick et al., 2001).

Indications of metacognition through adaptive reasoning and productive disposition

Petra searched for similarities and differences or patterns in the counting and gave generalised explanations to show accountability of her strategy use in other situations. For example, she argued for transferring the use of the strategy, which made her master counting from 1 to 20 when she was asked to count-on from 7 to 20, and that she recognised relational patterns and explained she had to count by 2s to correctly continue the number sequence 2-4-6. Petra initiated, argued, explored, and demonstrated different ways and patterns of counting. For example, she counted by 10s to 100 and she tried to count by 3s after counting by 2s. Her interest in exploring unknown counting overcame her insecurity of succeeding or meeting an expectation of mastery without finger-counting.

Kilpatrick et al. (2001) claimed that knowledge about one's own thinking and ability to monitor one's own understanding and problem-solving contribute to strategic competence and adaptive reasoning, and is known as metacognition. Petra showed indications of a high metacognitive level. One is that she explained how she was strategically going to approach new tasks, which, according to Kilpatrick et al. (2001), reflected her motivation or productive disposition, which is important for learning.

Counting strategies as part-whole reasoning on the mental number line?

Exploring Petra's counting raises the question of whether and how mental number lines representing the different 10s interrelate and how to develop an understanding of the pattern regularity linking them. Petra seemed to have mental representations for the tens' group structure and the positioning system of -1 to -9 range. Still, ten-transitioning was challenging in backwards counting and in the 10-20 range. The phonological structure perhaps distracted her. Exploring and considering strategy use in an integrated approach of the area model part-whole reasoning and the linear model the mental number line and, as such, combined the mental map of numerosity and the mental number line needs further examination (Dehaene, 2011).

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MS-Math – A design research project to support university students' fostering-competencies and primary school students' learning with the focus on perceiving, understanding, and using patterns and structures

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Keywords: mathematical patterns and structures, flexible calculation, fostering-competencies

Introduction

The poster presents the German project "MS-Math – Muster und Strukturen der Mathematik wahrnehmen, verstehen und nutzen lernen" ('Learning to perceive, understand and use patterns and structures in mathematics'). In this design research project, a support-program will be iteratively designed and implemented at the University of Muenster. It aims to provide diagnosis-guided support for primary school students with difficulties in learning mathematics. The support will be planned, realised and reflected by university students (student teachers). By doing this the university students will be stimulated to develop and improve their fostering-competencies at the same time.

Design and research interest

As the full name of the MS-Math project (see above) implies, the support-program focuses on patterns and structures in mathematics. For the area of arithmetic, this means that children get specifically supported in perceiving, understanding, and using number and problem relations for flexible calculation. Regarding this, a differentiation between 'inter-modal' problem relations (between problems) and 'intra-modal' problem relations (within problems) is defined in the project. MS-Math pursues the following three goals, which also reflect the design and research interest:

(1) Practice-oriented research – Gaining new research insights: In the project, in-depth insights of arithmetical learning processes of the supported children as well as of the supporting students regarding their fostering-competencies will be gained. The analysis of the learning processes and learning outcomes on both levels will contribute to a research-based specification of a fostering concept for primary school students with difficulties in learning mathematics. This concept has its focus on perceiving, understanding, and using arithmetical patterns and structures. The project also contributes to a deeper understanding about university students' development of fostering-competencies.

(2) Research-based design – Developing a university support-program in a research-based and practice-oriented manner: The university students are given the opportunity to develop and deepen fostering-competencies regarding support-oriented diagnosis & diagnosis-guided support in a case-based manner. The analysis of the learning processes and outcomes of the supported children and of the supporting university students (see (1)) will allow a research-based development and an empirical assessment of the support-program as a meaningful and rich learning environment for both, university students and primary school children.

(3) Research-based cooperation – Supporting teaching and learning at schools in Muenster: Teachers receive support in fostering children with difficulties in learning. The participating children receive weekly support based on their individual competencies and on latest research findings.

Theoretical background and design research methodology

Regarding primary school students' learning processes, with its outlined focus on perceiving, understanding, and using arithmetical patterns and structures, the project is following promising research findings (e.g. by Rechtsteiner-Merz & Rathgeb-Schnierer 2016; Verschaffel et al., 2009; Mulligan & Mitchelmore, 2013). On this basis the support gets planned, realized, and analysed. Regarding university student' fostering-competencies the approach of „professional noticing of children´s mathematical thinking" by Jacobs et al. (2010, p.172f.) is leading, to conceptualize three aspects: 1) attending to children´s strategies, 2) interpreting children´s understandings and 3) deciding to respond on the basis of children´s understanding (ebd., p.172ff.).

Following a design research approach, a support-program as a meaningful and rich learning environment for university and primary school students will be designed, tested, and improved by conducting design experiments in iterative design research cycles (Prediger et al., 2015). Each design experiment – a support session of two supporting university students and one supported child – gets video recorded. Their learning processes will be analysed using interpretative tools, which allow to gain new research insights (see (1)) and a research-based development of the support-program (see (2)).

First results: University students need specific criteria to be able to attend to children´s arithmetical strategies. Two criteria could be derived so far: Criterion of 'informativeness' and 'process reference'. The awareness and reflection of 'macro and micro cycles of diagnosis and support' helps them to respond more and more adaptively on the basis of children´s arithmetical understanding.

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Categorization reliability of preservice elementary teachers' use of mental computation addition strategies on natural numbers using a written questionnaire

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Preservice elementary teachers' use of mental computation strategies in addition on two-digit natural numbers is analyzed using a written questionnaire. The inter-coder reliability of three researchers in their categorization of the preservice elementary teachers' explanations as mental computation strategies is investigated. The results show that the coders are to a large degree in agreement, strengthening the reliability of the method.

Keywords: Mental computation, addition, questionnaires, interrater reliability.

Introduction and theory

There exist variations in the definition of mental computation in the research literature, but a general trait among the definitions is calculating without using any equipment (Baranyai et al, 2019; Hartnett, 2007; Heirdsfield, 2011; Maclellan, 2001; Thompson, 1999). There are many advantages of becoming better at mental computation; for example, it improves number sense, it gives a better understanding of the place value system and elementary calculation rules, and it is often involved in everyday use of mathematics (Hope & Sherill, 1987; Maclellan, 2001; Thompson, 2010). Mental computation is part of the elementary curricular content globally, and in several countries there have in recent years been an increased focus on mental computation in teacher education (Hartnett, 2007).

Mental computation strategies are different ways that arithmetic problems are solved mentally (Hartnett, 2007; Threlfall, 2000). To do mental computation efficiently, one needs to be flexible, learn several different strategies, and know when to use which strategy (Rechtsteiner, 2019). Some strategies are more general, and others are more dependent on coincidences in the calculation. Thompson (2009) stresses the importance of teaching and using mental calculation strategies, since the traditional methods are generally not effective enough to improve students' numeracy proficiency. However, even though mental strategies are a desired focus for computational instruction in schools, Hartnett (2007) suggests that teachers have been slow to adopt such changes in their classrooms, and that a possible block to adopting this approach is the teachers' lack of knowledge about possible computation strategies. There is evidence to suggest that pupils are often not directly exposed to mental computation strategies in school, but rather are left to devise for themselves more or less efficient strategies (McIntosh et al, 1995). Many mental computation strategies are possible for pupils to discover on their own, but one cannot presume that all pupils will be able to do so (Murphy, 2004). Some pupils get stuck in unwieldly mental computation strategies, such as doing the standard algorithm mentally, and therefore need to learn more efficient strategies in an organized and systematic way (Heirdsfield & Cooper, 2004).

Mental computation strategies in connection to elementary pupils has been well researched, but less so in connection to preservice elementary teachers (PETs). Since PETs are the next generation of teachers, it is important that they know and master mental computation strategies. They need a strong foundation of the mathematics of mental computation including an ability to use efficient strategies of their own (Heirdsfield & Cooper, 2004; Thompson, 2009). It is therefore important to know the current knowledge and proficiency base of PETs on mental computation. Knowing which strategies PETs are aware of and use, gives valuable information for continuing professional development on how to address mental computation strategies in teacher education and related research, to improve teacher content knowledge on mental computation (Hartnett, 2007; Thompson, 2009; Valenta & Enge, 2013).

To investigate the mental computation strategy use of PETs, researchers need to categorize the PETs' current strategies. In research literature a common method to do this is through interviews, where one can ask the PETs follow up questions, an alternative that is not available in the same way for a written questionnaire. However, a written questionnaire is a more efficient tool than an interview when it comes to gathering larger sets of data in statistical investigations (Mastrothanasis et al, 2018).

There is not a single, consensus approach regarding how to categorize mental computation strategies (Whitacre, 2015). In this paper, I focus on strategy categorization using as a theoretical lens a comprehensive list of mental computation strategies on addition on two-digit natural numbers (Table 1). The list is the result of an exhaustive search of the mental computation strategies that occur in the literature. After each strategy is given a reference to where the definition can be found in the literature.

Table 1: Mental computation strategies for addition on two-digit natural numbers

Strategy	Definition	Source
1010 [ten-ten]	$46+23 \to 40+20=60 \to 6+3=9$ $\to 60+9=69$	(Beishuizen, 1993)
10s [1010 stepwise]	$45+39=\big((40+30)+5\big)+9$ $=(70+5)+9=75+9=84$	(Reys et al, 1995)
A10 [adding-on]	$35+29=(35+5)+24=40+24=64$	(Blöte et al, 2000)
AUTO [Automatic calculation]	Retrieve the answer automatically or from memory.	(Lucangeli et al, 2003)
B [balancing]	$89+24=(89+1)+(24-1)=90+23$ $=113$	(Heirdsfield & Cooper, 1997)
Counting	$3+5$: $4, 5, 6, 7, \underline{8}$	(McIntosh & Dole, 2005)
Doubles and near doubles	$6 + 7$: $6+6$ is 12, so it is one more.	(McIntosh & Dole, 2005)
N10 [stringing]	$46+23 \to 46+20=66 \to 66+3=69$	(Beishuizen, 1993)

N10C [stringing with compensation]	$52 + 79 \rightarrow 52 + 80 = 132 \rightarrow 132 - 1 = 131$	(Baranyai et al, 2019)
Round one or both addends to multiple of ten, then adjust	$79 + 26 \rightarrow 80 + 30 = 110 \rightarrow 110 - 1 - 4 = 105$	(Reys et al, 1995)
Round to multiples of five	$79 + 26 = (75 + 25) + 4 + 1 = 100 + 5 = 105$	(Reys et al, 1995)
SA [standard algorithm done mentally]	Mental image of pen and paper algorithm, placing numbers under each other, as on paper, and carrying out the operation, right to left.	(Heirdsfield, 2001)
u-1010 [1010 right to left]	$46 + 23 \rightarrow 6 + 3 = 9 \rightarrow 40 + 20 = 60 \rightarrow 60 + 9 = 69$	(Beishuizen, 1993)
u-N10 [N10 right to left]	$46 + 23 \rightarrow 46 + 3 = 49 \rightarrow 49 + 20 = 69$	(Beishuizen, 1993)
Using tens as the unit	$80 + 50 = 8$ tens $+ 5$ tens $= 13$ tens $= 130$	(McIntosh & Dole, 2005)

When categorizing there is the question of inter-coder reliability (Lange, 2011), that is if different researchers agree in their categorizations. Evaluating the inter-coder reliability is recommended as good practice in qualitative analysis, although this is a somewhat controversial topic in the qualitative research community, with some arguing that it is an inappropriate or unnecessary step within the goals of qualitative analysis (O'Connor & Joffe, 2020). Team coding is a good inter-coder reliability check (Miles & Huberman, 1994), and a standard way of doing this is using percentage agreement or Cohen's Kappa (Cohen, 1960).

Method

Purpose of article and research question

This paper is the first article in a planned series of articles where I utilize written questionnaires to conduct research into PETs' mental computation strategy use, making use of the comprehensive list of strategies in Table 1. In general, research articles categorizing students' and PETs' strategy use only draw on a subset of these strategies and different subsets in each article, thus limiting the degree of comparisons that can be done across the research literature. Before using the comprehensive list of strategies in Table 1 to categorize data from the PET questionnaires, I propose that it is important to first investigate the inter-coder reliability of researchers. (Some researchers may not think this is necessary in qualitative research, so alternatively the investigation can also be seen as out of curiosity or interest to understand how different teacher educator colleagues categorize the same data.) If different researchers' categorizations are in agreement that strengthens the reliability of the method. Therefore, a relevant and interesting research question is:

Using a written questionnaire and a comprehensive list of mental computation strategies occurring in the literature, how do different researchers differ in their categorizations of preservice elementary teachers' mental computation strategy use in addition on two-digit natural numbers?

To limit the scope of the investigation only addition on two-digit natural numbers is considered. If the categorizations are in agreement in this particular case, that increases the confidence in using written questionnaires to analyze PETs' mental computation strategy use in general.

Research participants

A written mental computation strategy questionnaire was in 2020 given to two different classes of PETs at a mid-sized university in Norway:

I. 31 second year PETs with 30 ECTS credits of university mathematics.
II. 15 third year PETs with a mix of 0, 30 or 60 ECTS credits of university mathematics.

Measures

The PETs' use of mental computation strategies was measured with a written questionnaire consisting of fifteen exercises on addition on two-digit natural numbers. Figure 1 shows the questionnaire instructions and how each exercise was presented to the PETs.

Important instructions! For every exercise do these steps in order:

1. **Calculate the exercise in your head**.
   - The calculation must be done solely in your head. You are **not** allowed to write anything on the paper.
   - If you do not know the answer after you have been thinking for a while, do not write anything and instead go to the next exercise.
2. **Write down the answer**.
3. **Explain mathematically how you were thinking.**

Exercise 1

$11 + 13 =$______

Explanation:______________________________

____

Figure 1: Questionnaire instructions together with the first exercise

The exercises were constructed by the author so that many different strategies would be used by the PETs:

1. $11 + 13$, **2.** $17 + 18$, **3.** $62 + 27$, **4.** $80 + 50$, **5.** $76 + 58$, **6.** $44 + 33$, **7.** $38 + 76$, **8.** $60 + 37$, **9.** $47 + 45$, **10.** $64 + 46$, **11.** $70 + 67$, **12.** $97 + 86$, **13.** $88 + 88$, **14.** $45 + 79$, **15.** $99 + 9$

The PETs' written explanations were categorized as mental computation strategies by three different researchers (of which one was the author) according to the list of strategy definitions given in Table 1. The explanations could also be categorized as "Other" (O) or as an "Unclear strategy" (US). In addition to being instructed to use the list of mental computation strategies, the researchers were directed to base their categorization solely on a PET's own written explanation (not speculating on how a PET was "really thinking" when calculating).

The categorization inter-coder reliability was measured by comparing (1) the percentage agreement (that is the proportion of the exercises that the coders agreed on) and (2) the average strategy distribution of the coders. Note that in (1) percentage agreement was chosen instead of Cohen's Kappa (Cohen, 1960), because the difference between them is negligible since there are many categories of strategies and the probability for random agreement when categorizing is thus small.

Data collection

The questionnaire was administered as part of a normal lecture in two classes (referred to here as Class I and Class II) at the university. The PETs' participation in the questionnaire was voluntary and anonymous. They were not informed beforehand that they would take a questionnaire, so they had no way of preparing for it. There was no time limit to the questionnaire.

The PETs' explanations in Class I were categorized as strategies by the author and another mathematics teacher educator (called instructor A). Both the author and instructor A are associate professors in mathematic didactics. The PETs' explanations in Class II were categorized by the author and another mathematics teacher educator (called instructor B). Instructor B is a university lecturer in mathematic didactics with several years of experience as an elementary school teacher.

Results

Class I categorized by author and instructor A

The author and instructor A were in agreement in $89.9\ \%$ and disagreement in $4.7\ \%$ of the $465(= 31 \times 15)$ exercises. In $5.4\ \%$ of the exercises their categorizations were overlapping (but not inclusion in set theory terms). One can conclude that the two persons categorizations are to a high degree in agreement, strengthening the reliability of the method.

Figures 1 & 2 show the averages of the 15 exercises for Class I as categorized by the author (Figure 1) and instructor A (Figure 2), where similar strategies have been grouped. (The strategies were consider separately when considering if categorizations were in agreement. Each strategy in the category Remaining have a small relative frequency ($\leq 1\ \%$).)

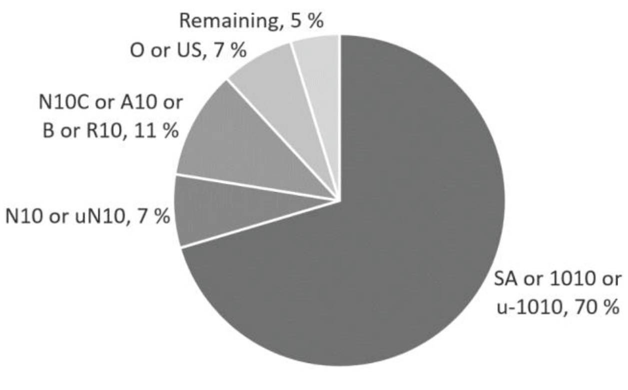

Figure 1: Class I categorized by author

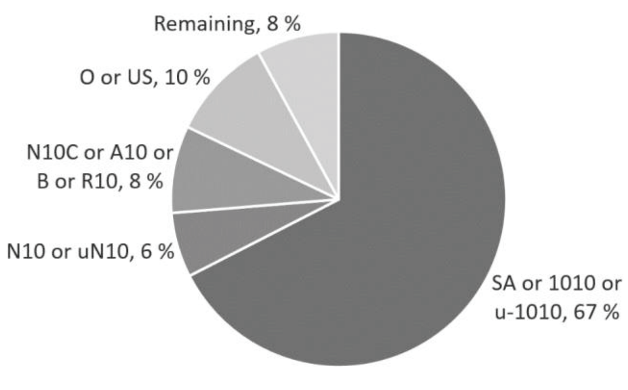

Figure 2: Class I categorized by instructor A

Generally seen the averages in Figures 1 & 2 reflect the relative frequencies in each of the 15 exercises. One exception is exercise 4 (that is $80 + 50$). Most of the PETs have calculated exercise 4 by adding $8 + 5$ and then attaching a zero. The author has categorized this strategy as "Other" (O) whereas instructor A categorized it as an "Unclear strategy" (US).

Class II categorized by author and instructor B

The author and instructor B were in agreement in 80.4 % and disagreement in 11.6 % of the $225(= 15 \times 15)$ exercises. In 8,0 % of the exercises the categorizations were overlapping (but not inclusion in set theory terms). Although the author's agreement with instructor B was somewhat smaller than with instructor A, the overall agreement of two categorizations is good also in this case.

The diagrams in Figures 3 & 4 show the averages of the 15 exercises for Class II as categorized by the author (Figure 3) and instructor B (Figure 4). The grouping of the categorizations is here the same as for Class I.

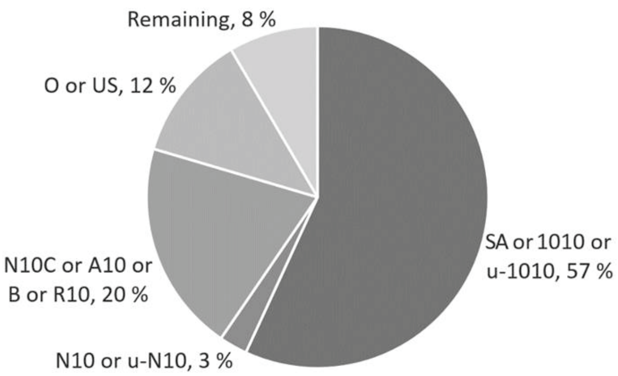

Figure 3: Class II categorized by author

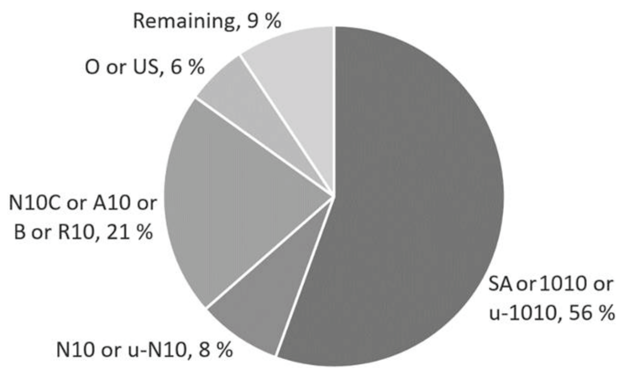

Figure 4: Class II categorized by instructor B

Discussions and conclusions

Exploring how different researchers categorize preservice elementary teachers' mental computation strategy use, utilizing the comprehensive list of strategies as a theoretical lens to analyze their work, is an important first step for me before moving into the all-important task of understanding more, in general, about PETs' strategy use. We have in this paper seen that, using a written questionnaire and a comprehensive list of mental computation strategies occurring in literature, there is good agreement between how three different researchers categorize PETs' explanations as mental computation strategies. This is a valuable result since it means that written questionnaires and the comprehensive list of strategies in table 1 can be used more reliably to analyze PETs' use of mental computation strategies.

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Mental calculations with rational numbers: strategies' range and strategy selection

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In this paper primary and secondary school students are followed while they try to perform operations with rational numbers first mentally and then in paper-and-pencil. The same tasks were given to the participants of both groups. The findings give evidence that no matter the context (mental or written calculations) the dominance of the mental form of the written algorithm is unquestionable for both groups. Moreover, most of the student were not able to explain their decision to use the algorithm in their mental calculations. The students who were able to justify their choice claimed that their criterion was the accuracy, speed and easiness provided by the algorithm which creates a paradox since for all the items on the study the algorithm is the most time-consuming strategy.

Keywords: Mental calculations, rational numbers, written algorithm.

Introduction.

Mental calculations with rational numbers have been the focus of several studies (Caney & Watson, 2003; Rezat, 2011; Carvalho & Ponte, 2019). The ability to perform mentally such calculations is considered significant. According to the Principles and Standards of the NCTM (2000), "students should develop and adapt procedures for mental calculations and computational estimation with fractions, decimals, and integers" (p. 220). In the last CERME, Papadopoulos et al. (2019) highlighted the dominance of the use of mental form of the written algorithm as a way to calculate mentally the outcome of certain operations with rational numbers. The discussion that followed the presentation was fruitful and raised questions concerning the possible reason the participants choose this strategy, or whether they know alternative strategies and if yes why they do not use them. Moreover, the audience in the conference was wondering whether the results would be the same in case the students were asked to solve the same tasks in paper-and-pencil.

In this setting the current study attempts to give some answers to the above-mentioned concerns. So, the research questions are as follows:

(i) What is the range of the strategies employed by primary and secondary education students when they execute operations (first mentally and then in a written form) that involve rational numbers?

(ii) On what criterion do they choose their mental calculation strategy?

Theoretical background

Most of the studies on mental calculation focus on whole numbers and their four operations (Rezat, 2011). However, in recent years there is an ongoing interest on mental calculations with rational numbers examining issues such as the comparison of rational numbers (Yang et al., 2009) and the operations with them (Caney & Watson, 2003; Papadopoulos et al., 2019). Caney and Watson (2003) recorded the strategies the participants used while trying to calculate mentally. They ended with a series of different strategies: *'changing operation'* (e.g., subtraction to addition), *'changing representation'* (e.g., fractions to decimals and vice versa), *'use of equivalents', 'use of known facts'* (e.g., times table), *'repeated addition/multiplication'* (e.g., by doubling, halving), *'use of bridging'* (bridged to one/whole),*'working with parts of a second number'*(split the second number by place value or by parts), *'working from the left/right'*, *'using mental picture'*, *'using mental form of written algorithm'* and *'using memorized rules'*. They group these strategies in two main categories: instrumental and conceptual strategies. The first refer to strategies based on the use of procedural paths learned by rote. The second occurs when students make use of their knowledge on the specific set of numbers and operations in order to calculate mentally. Callingham and Watson (2004) found that in the realm of rational numbers it is easier to work mentally with fractions rather than with decimals or percentages and that division is more demanding compared to addition and subtraction (which holds equally for the set of whole numbers). Moreover, it seems that in the context of the addition and subtraction of decimal numbers the students preferred strategies they already used for mental calculations with whole numbers (Rezat, 2011). In their study on mental calculations with rational numbers by primary school students Carvalho and Ponte (2019) highlight two main findings: In their effort to work mentally the students (i) tend to use the strategy of 'changing representation' (from fractions to decimals), and (ii) 'use numerical relationships strategies supported by propositional representations' (p. 393). Another related issue is the ability of flexibility in mental calculations that has been raised by scholars such as Rathgeb-Schnierer and Green (2015) and the lack of which has been recorded not only for students but for teachers also (Lemonidis et al., 2018). Finally, Papadopoulos et al. (2019) who tried to record and compare strategies used by participants across all educational levels for the same tasks found that the mental form of the written algorithm dominated in the participants choices no matter the educational level (almost 43% for primary school students, 31% for secondary school students, 46% for University students from the Department of Primary Education and 59% for University students from the Mathematics and Engineering Departments). All the other strategies' percentages were close to 1%-2%. This raises the issue of how the solvers choose strategies in mental calculations. Indeed, the choice of the proper strategy is of critical importance. Threlfall (2000) described the process of choosing strategies as a series of certain steps: (i) prior analysis of the problem to recognize features that are associated to different possible strategies, (ii) decide which ones of these strategies are viable in terms of knowledge and skills, (iii) decide between the viable strategies the one that is (possibly) the easiest, and (iv) carry through the decision in practice. But as he admits 'neither children nor adults actually calculate in that way' (p. 84). The idea of teaching criteria for deciding in advance which strategy to use is rather not feasible and perhaps this explains why there are no suggestions in the relevant research literature for direct teaching on how to decide strategically the proper way of mental calculation (Threlfall, 2002). So, there is no proof on the way students choose to calculate mentally and this is why in this paper we chose to examine this issue. Threlfall (2002) claims that mental calculation strategies are not purposefully selected by the students, but they occur since the students are interested in finding the solution rather than the method. More precisely, he claims that "They are ways of thinking about mental calculations that do not describe the whole sequence to the solution, but concern just some of the steps, for example ways of beginning, ways of thinking about the numbers, and ways of relating the numbers to other knowledge" (p. 42).

Setting of the study

The total number of the participants was 127 students from primary and secondary education. More precisely there were 65 students from grades 5 and 6, and 62 students from grades 10, 11 and 12 in a rural area of Northern Greece.

Five tasks in total were given to the students (Fig. 1)

$\frac{3}{6}+\frac{4}{8}$	$4 \times \frac{6}{12}$	$1\frac{1}{2}-\frac{1}{4}$	$2\frac{5}{10} \times 2$	$8 : \frac{4}{8}$
Item 1	Item 2	Item 3	Item 4	Item 5

Figure 1: Tasks posed to the students

The tasks' design followed two principles. First, there are more than one ways to calculate each item. Second, all the items can be calculated in a fast and easy way if the solvers notice the quantities involved in each operation. Therefore, item 1 includes the sum of two halves and therefore the answer is 1. Item 2 is about four halves, therefore 2. In item 3 if the solver sees $\frac{1}{2}$ as $\frac{1}{4}+\frac{1}{4}$ it is easy then to get the result $1\frac{1}{4}$. For item 4 the result should be the double of 2.5, therefore 5. Finally, for item 5 the solver must think how many halves are needed to get 8. Therefore, the answer is 16. This small collection of items involves all the four operations and a variety of combination of numbers (fractions, whole numbers, mixed numbers).

The whole study consisted of a two-step process. Initially all the participants were interviewed individually. They were asked to solve the tasks one-by-one mentally vocalizing their thoughts while solving them. They were not allowed to make written calculations or to keep some notes. There was no time restriction, and they could skip tasks in case they felt they couldn't solve them. No feedback was provided to them during their effort. All interviews were recorded and transcribed.

For the second part, the students were invited to work individually on paper-and-pencil. Now they had to solve each item with as many different ways as they could. After completing the solutions for each item, the students had also to write down their answer for the question: Can you explain why from all these different ways of solving this item, did you choose this specific strategy for your mental calculation?

The transcribed protocols (first part) and the students' worksheets (second part) constituted our data. For the first part of the study the data were analyzed in two levels. First, the answers were categorized according to whether they were correct, incorrect, unanswered or not codable. Second, the correct answers were distributed to the different strategies of Caney and Watson (2003). It must be said however that not all the strategies appeared in the students' answers and that some new strategies emerged. For the second part, the data analysis took place at a qualitative level on the basis of content analysis following a more deductive sort of thematic analysis (Mayring, 2014). The data were coded independently by the authors and validity and reliability were established by comparing sets of independent results, clarifying codes and re-coding data until agreement.

Results and Discussion

Table 1 summarizes the distribution of the total number of answers (635 answers for the first part of mental calculation and 721 for the second part of written calculations) across the different strategies.

Table 1: Distribution of answers across different strategies for both parts of the study

	1. Written algorithm	2. Equivalent	3. Parts of a number	4. Change representation	5. Reduction
Mental calculation	**229/635 (36.06%)**	6/635 (0.94%)	5/635 (0.78%)	8/635 (1.25%)	4/635 (0.62%)
Written calculation	**296/721 (41.05%)**	26/721 (3.6%)	7/721 (0.97%)	20/721 (2.77%)	10/721 (1.38%)
	6. Algebraic	7.Combine	8. Incorrect	9.Unanswered	10.No-codable
Mental calculation	0/635 (0%)	7/635 (1.1%)	**263/635 (41.41%)**	**112/635 (17.64%)**	1/635 (0.15%)
Written calculation	2/721 (0.27%)	37/721 (5.13%)	**252/721 (34.95%)**	**68/721 (9.43%)**	3/721 (0.41%)

Examining the different strategies, it seems to be an overlapping between the use of *Equivalent* and use of *Reduction* since from the mathematical point of view it is the same. Reduction results always to equivalent fractions. We consider them different based on the wording of the participating students who treated them as such. It was the solver's aim as this was expressed verbally that made us to decide if the solution is associated with one strategy or another. Another new strategy is the 'algebraic' one (despite its limited presence) that was not included in the list of Caney and Watson (2003). An example of this approach collected from answers in item 1 is:

$$\frac{3}{6}+\frac{4}{8}=x \Rightarrow 48\cdot\frac{3}{6}+48\cdot\frac{4}{8}=48x \Rightarrow 8\cdot 3+6\cdot 4=48x \Rightarrow 48x=48 \Rightarrow x=1.$$

The category of 'combination' has been also added to include answers that combine more than one strategy at the same calculation. For example, in the following answer for item 5 the student combined the *Reduction* and *Written algorithm* strategies: $8\div\frac{4}{8}=8\div\frac{4\div 4}{8\div 4}=8\div\frac{1}{2}=8\cdot\frac{2}{1}=16$. For the same item, another response combined the *Equivalent* and *Change representation* strategies: $8\div\frac{4}{8}=8\div\frac{1}{2}=\frac{8}{0.5}=16$.

Two interesting observations can be made based on the arithmetical data of Table 1. The first is related with the range of the strategies employed by the participants in their mental and written calculations. It can be said that the results are more or less the same in both cases. Papadopoulos et al. (2019) highlighted the dominance of the mental form of the written algorithm in mental calculations with rational numbers. But it seems now that the situation is the same no matter the way of calculation (mental or written). Therefore, it is not the context of the calculation that promotes the use of the algorithm. The second observation is that almost the total number of the collected answers is around the triplet algorithm-incorrect-unanswered (almost 94% and 86% for the mental and written calculation respectively). This means that the main option for a correct answer is to use the algorithm. Otherwise, the most possible is to get an incorrect answer or skip the task.

The situation remains the same when the data from Table 1 are distributed across the two samples in Table 2. The algorithm percentages for primary education students for the mental and written calculations are 29.84% and 37.73% respectively. Interestingly no other strategies are employed by the primary school students except 2 answers using the ‘parts of numbers’ in the mental part of the study, and 2 cases in the written part (‘parts of numbers’ and ‘change representation). All the students (almost 100%) are gathered around the same triplet mentioned earlier. The secondary education students employed several strategies in their mental and written calculations. However, the frequency of these strategies is small compared with the use of the algorithm. So, in the mental part of the study 42,58% of the correct answers were based on the use of algorithm while only 28 out of 310 answers (9,03%) employed other strategies. The situation is improved in the written part of the study since the percentage of the correct answers that use other strategies (except from algorithm) is increased to 25,31% (100 out of 395 answers). But again the dominance of the algorithm (43.79%) is unquestioned.

Table 2: Distribution of strategies across the two samples for both parts of the study

	Mental calculation		Written calculation	
	Primary Education	Secondary Education	Primary Education	Secondary Education
Written Algorithm	**97 (29.84%)**	**132 (42.58%)**	**123 (37.73%)**	**173 (43.79%)**
Equivalent		6		26
Parts of number	2	3	1	6
Change represent		8	1	19
Reduction		4		10
Algebraically				2
Combination		7		37
Incorrect	**138 (42.46%)**	**125 (40.32%)**	**148 (45.39%)**	**104 (26.32%)**
Unanswered	**88 (27.07%)**	**24 (7.74%)**	**53 (16.25%)**	**15 (3.79%)**
Non-codable		1		3
TOTAL	325	310	326	395

An alternative way to organize the data for the written phase of the study is according to the number of different strategies per item employed by the participants (Table 3). From the 325 answers given by the primary school students (65 students × 5 items) 123 employed one strategy (i.e., the mental form of the written algorithm) and the rest of them but one did not employ any strategy. Only one student was able to use 2 different strategies for one item. The situation is slightly better for the secondary education students (62 × 5 = 310 answers). There was a small number of answers that employed 2, 3, and 4 strategies (49, 13, and 3 answers respectively). But again, the main characteristic is that the majority of the answers are around 0 and 1 strategies. So, our claim that it is not the context that imposed the use of the algorithmic approach is rather strengthened.

Table 3: Number of strategies for the same task

Number of different strategies	Primary Education	Secondary Education	TOTAL
0 strategies	**201 (61.84%)**	**122 (39.35%)**	**323**
1 Strategy	123 (37.84%)	123 (39.67%)	246
2 strategies	1	49	50
3 strategies		13	13
4 strategies		3	3
TOTAL	325	310	635

Therefore, almost half of the attempts represent lack of any strategy while in the case of the successful attempts almost 8 out of ten were based on the use of just one strategy, that is the use of the mental form of the written algorithm. So, the research question about the range of the strategies employed in these two different contexts can be answered in a rather clear manner: No matter the context, the students can merely use the algorithm in their calculations, or they are unable to respond successfully.

Table 4: Reason for choosing a strategy

Reason of choosing a strategy	Primary Education	Secondary Education	TOTAL
Easy / fast / efficient	7 (2.15%)	80 (25.8%)	87 (13.7%)
Teaching practice	1	2	3
Certainty	1	28 (9.03%)	29
Lack of another knowledge	8	5	13
I don’t know	308 (94.76%)	195 (62.90%)	503 (79.21%)
TOTAL	325	310	635

The second research question aims to reveal the criterion the students use to choose their mental calculation strategy. Our hope was to contribute to the issue raised by Threlfall (2002) that there is no proof on the way students choose to calculate mentally. The analysis of the collected answers resulted in five categories (Table 4). The students explained that they chose the specific approach in their mental calculation because (i) they think that this was the easier, faster, and more efficient way for the calculation, (ii) this is the way they were taught to calculate with rational numbers, (iii) they felt secure with the specific strategy, (iv) this was the only strategy they knew, and (v) they were unable to provide any explanation for their decision. For the primary school students what is impressive is the very small number of responses. In (only) 17 (out of 325) cases the students were able to explain the way they worked. This means that for almost 95 out of 100 cases they were not able either to justify their choice or to find any strategy that would serve their purpose (unanswered items). For those who were able to provide an explanation this was mainly the easiness/fastness/efficiency of the algorithmic approach. For the secondary education students, for almost 63 (out of 100) cases the students were unable to justify their choices. From those who gave explanations, again most of them justified their choice on the basis of the easiness/fastness/efficiency of the algorithmic approach. In comparison to the primary school students the only difference here was the increased number of answers (28 vs 1) referring to the issue of the certainty the students felt with using the specific strategy for mental calculations. Actually, focusing on the total sample what is evident is the students' inability to justify their choices. But what is especially interesting is a paradox that seems to appear after analysing all the responses. From Tables 1-3 it can be seen that the students exhibited an almost exclusive preference to the mental form of the written algorithm. This choice was later justified by them as the most easy, fast, and efficient way of calculating. Its efficiency is unquestionable. Indeed, the correct application of the algorithm guarantees the correct result. But it is interesting that they consider it easy and fast. For example, for item 1, the sum $\frac{3}{6}+\frac{4}{8}$ can be immediately (in an easy and fast way) be seen as the sum of two halves which is equal to 1. On the contrary, the participants preferred to make the fractions having the same denominator ($\frac{3}{6}+\frac{4}{8}=\frac{12}{24}+\frac{12}{24}$), to add them ($\frac{3}{6}+\frac{4}{8}=\frac{12}{24}+\frac{12}{24}=\frac{24}{24}$), to reduce the sum, to find 1 ($\frac{3}{6}+\frac{4}{8}=\frac{12}{24}+\frac{12}{24}=\frac{24}{24}=1$) and they considered this process as the most easy and fast.

Conclusions

The ability for mental calculations and the selection of the most suitable strategy for mental calculations are considered especially significant by the research community. In this paper we followed primary and secondary school students in their attempt to calculate (mentally and in paper-and-pencil) the same collection of tasks. It seems that the algorithm was the only option for primary school students no matter whether they calculated mentally or in paper-and-pencil. The secondary school students exhibited an ability to use alternative strategies, but the presence of these strategies was very small compared to the use of the algorithm. Most of the participants used just one strategy for their calculations in paper-and-pencil, which is indicative of limited flexibility (Heinze et al., 2009). Finally, most of the students justified their choice of the algorithmic approach considering it as being easy, fast, and efficient. From the mathematics point of view this creates a paradox since the use of algorithm provokes an increased cognitive load compared to many other mental approaches in calculating. The findings reveal some interesting aspects about mental calculations, but these findings cannot be generalized due to the relatively small number of participants. However, they deepen our understanding of the topic and challenge us for a future study to strengthen our arguments made here.

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Using Concept Cartoons to support the number line as a vehicle for mathematical understanding

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This contribution draws on a research study the aim of which was to investigate the number line as a vehicle for mathematical understanding in the naturalistic setting of the elementary classroom. This paper focuses on one aspect of this research; the use of Concept Cartoons as an educational tool to support the role of the number line as a tool for fostering the development of conceptual understanding. Evidence from whole class discussion and the pupils' own productions points to the role of Concept Cartoons in supporting pupils' sense making, the elaboration of informal strategies, and the development of more sophisticated ones.

Keywords: Concept Cartoons, number line, mental calculations, primary education.

Background.

The understanding of and ability to use the linear, mathematical number line constitutes an essential facet of pupils' mathematics education (Lemonidis, 2016). The number line representation plays an important role in facilitating pupils to develop flexibility in mental arithmetic as they actively construct mathematical meaning, number sense, and understandings of number relationships (Frykholm, 2010). The number line is not a simple representation. According to Herbst (1997), a number line is formed by the consecutive translation of a specified segment U, as a unit from zero that can be partitioned in an infinite number of ways. He suggests that the number line is a metaphor of the number system. The number line is also considered a geometrical model, involving a continuous interchange between a geometrical and an arithmetic representation (Gagatsis et al., 2003). Within the literature, two major types of number lines can be identified; the structured number line and the empty number line (Diezman et al., 2010). There is a large body of literature that discusses the number line and its crucial role in teaching and learning elementary mathematics (Beishuizen, 2010; Gravemeijer, 2020). Whilst generally effective, research findings often raise doubts about the usefulness of the number line as a didactical model (Van den Heuvel-Panhuizen, 2008). These studies point towards a coherent treatment of the number line throughout the years of compulsory education and presentation of the number line in the school official documentation in a developmentally appropriate manner, by focusing on the simultaneous presence of the geometric and the arithmetic conceptualization of number on the number line. It is acknowledged that it is superficial to simply recommend the use of the number line for the students' mathematical development or include it in curriculum materials and other recourses. If emphasis is given on the nature of the number line and its use as a representation of sophisticated ideas, then a conceptual way of teaching and learning is being encouraged, contributing to addressing students' difficulties (Van den Heuvel-Panhuizen, 2008). In promoting a non-threatening classroom culture that encourages all pupils to create, formulate, extend and express their mathematical understanding, and simultaneously provides the teacher an insight into the pupils' understanding, a didactical tool that can be utilised alongside the number line is Concept Cartoon.

Concept cartoons are cognitive drawings or visual engagements that use a cartoon-style design to present mathematical conversations inside speech bubbles (Dabell et al., 2008). The integrated written text is characterized by minimal use of written language, in order to be accessible by all pupils, regardless of their literacy skills. The cartoon characters are engaged in a dialogue or an argument presenting alternative statements, concepts or questions that are related to a central content-related topic. All alternative viewpoints presented by the cartoon characters have equal status and are grounded on research related with pupils' understanding in order to appear credible (Naylor & Keogh, 2013). Thus, the legitimized argument presented in the concept cartoon provides a comfortable and non-threatening environment inviting pupils to engage in and extent the characters' argument. Concept cartoons are considered a valuable learning and teaching tool that provides the pupils the opportunity to interpret and understand mathematical concepts (Naylor & Keogh, 2013). As the pupils' ideas are articulated by the cartoon characters, even reluctant or less confident pupils are encouraged to express their personal views as responses to the characters and justify their reasoning and thus, engage in the discussion (Sexton et al., 2009). Comparing and contrasting ideas facilitates learning and interpretation of knowledge, reveals and eliminates misconceptions (Dabell, 2008). Adding to the above, research studies with a focus on engaging pupils with concept cartoons, also argue that the use of concept cartoons has the potential of increasing levels of motivation, involvement and interest towards lessons and have a positive effect on mathematical achievement (Naylor & Keogh, 2013).

Keeping in mind the aforementioned, this paper focuses on Concept Cartoons, as an educational tool to support the number line as a vehicle for mathematical understanding. To be more precise, this paper examines how Concept Cartoons can be utilized in a Year 2 mathematics classroom to provide a fruitful context for appreciating the nature of the number line and its use as a representation of sophisticated ideas.

Method.

The results presented in this contribution form part of a broader study that constitutes an attempt to translate the idea of the number line as a vehicle for mathematical understanding in the naturalistic setting of the elementary classroom (Pericleous, 2022). To be more descriptive, this study sought to examine the use of the structured and empty number line as a tool to support and develop Year 2 and Year 3 pupils' sense making and calculation strategies specific to addition and subtraction in the number domain 0-1000. The study employed a design based research methodology (Cobb et al., 2003) that was informed by Realistic Mathematics Education and a socio-constructivist approach to teaching and learning (Gravemeijer, 2020). Thus, pupils' learning process was viewed from both the individual perspective and the social perspective. In this learning process, where pupils pass through various levels of understanding, models such as the number line function as a bridging device between informal to more formal mathematics, supporting a shift from a 'model' of pupils' informal solution strategies to a 'model for' mathematical reasoning (Gravemeijer, 2020). The study was conducted in a public primary school in Cyprus. The participants, as relevant for this paper, were 19 pupils (7-8 years old) of a wide range of abilities. The instructional sequences were conducted throughout a school year and were carried out as a part of the ordinary mathematics classroom with the teacher as the researcher and author of this paper. Starting from pupils' embodiment of the number line, and explicitly giving emphasis on the nature of the number line, the instructional approach was organized around number sequence and recognition, addition and subtraction in the domain from 1 through 100. The overall aim of the instructional approach was to create opportunities for the pupils to build connections or relations between representations of mathematical ideas, to have the freedom to come up with their own notations, find their own ways to decompose quantities and regroup/recombine them and express and discuss their ideas. In doing so, the pupils had the opportunity to understand the nature of the structured and empty number line before acting on them and modelling their solution strategies.

The data collection process as relevant to this paper included video data from the teaching sequences, the field notes from the teacher and pupils' written work. The overall process of analysis of the collected data drew upon progressive focusing and the constructs 'account-of' and 'accounting-for'. According to Stake (2004), progressive focusing is achieved in various stages; first observing, then further inquiry, beginning to focus on relevant issues, and then seeking to explain. The analysis of collected data started simultaneously with the data collection process, in order to further organise the instructional sequences. The data were treated both qualitatively and quantitatively, aiming at identifying pupils' thinking, strategies and procedures and their development. The ongoing analysis being conducted while the study was in progress led to a focus on several issues and events; accounts-of, which were then placed in a broader theoretical context by conducting a retrospective analysis (accounting-for). Creating the Concept Cartoons aimed at providing pupils the opportunity to appreciate the number line as a rich model that can have different manifestations, by giving them as much initiative as possible, and simultaneously reducing the leading role of the teacher. This paper focuses on Concept Cartoons that were employed as a way to support pupils' understanding of the number line and gain access to their calculation strategies. They present addition situations up to 100 without bridging. The pupils would explore a specific Concept Cartoon either individually or in small groups. After exploring a concept cartoon, pupils would draw a star next to the cartoon they felt more close to, as a way for the teacher to gain access to their thinking and understanding and adequately build on their existing knowledge. Classroom discussion would follow.

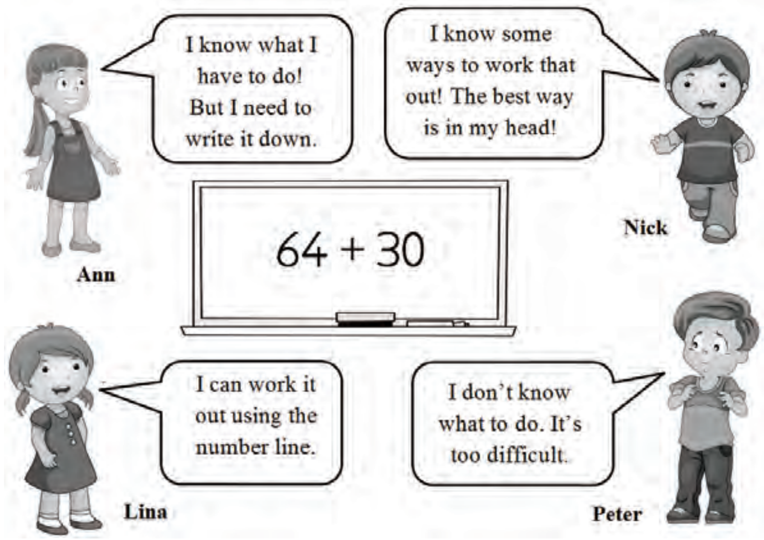

Figure 1: Gaining insight into the pupils' calculation strategies

Findings.

Supporting pupils' understanding of the number line.

There were instances where the Concept Cartoons were related with defining the structured and empty number line and developing key understandings underpinning the conventions considered to interpret, create and use number lines. For instance, in defining the structured number line, the Concept Cartoon was created by taking into consideration pupils' embodiment of the number line, requesting pupils' opinions regarding the cartoon characters' diagrams. The speech bubbles included two structured number lines (0-15, 0-30), a line with no marked numbers and a number line with no equal intervals between the marked numbers. All pupils drew a star on the structured number lines.

Student1: The child's *(cartoon character)* number line is wrong, because the numbers are at some parts very close to each other and at other places they are away from each other. They do not have the same distance from each other.

Student2: And the other one has no points for the numbers.

Student1: Yes.

Student3: It is a number line *(0-15)* because it begins from zero, the numbers have equal spaces from each other, and they are marked.

Student4: And the numbers are in order.

Student5: I agree. It is the same with the one that goes up to 30.

Teacher: Keeping in mind this discussion, how can we define the number line?

Student5: It is a straight line that has numbers that are placed in order; they are shown on the line with a mark and have the same distance from each other.

Student6: And it goes to infinity.

While, at first the pupils would refer to the perceptual features of the number line, progressively, they would make reference to order, continuity and the variety of numbers that could be represented on it. The discussion led to the conclusion that a number line shows the order of specific numbers, it includes points in equal intervals that show the place of the number and that the difference of a number to the next is constant. The classroom constructed a definition of the number line. This definition was often revisited throughout the year to support the development of a global perspective on the number system, by focusing on the unit interval and the partition of this interval.

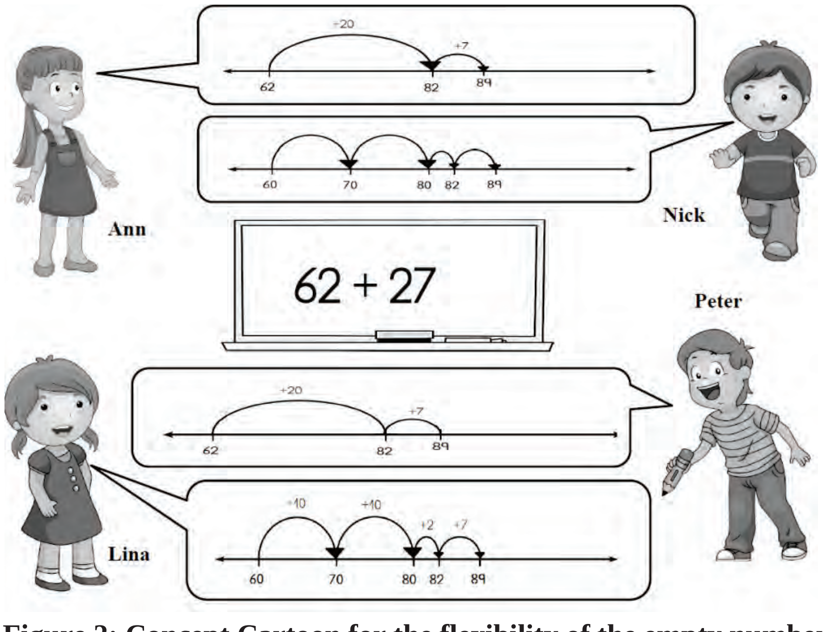

Figure 2: Concept Cartoon for the flexibility of the empty number line

Furthermore, Concept Cartoons were created so as to demonstrate the flexibility of the number line. For instance, the Concept Cartoon in Figure 2 was explored after the classroom was introduced to the empty number line as a tool to record one's thinking. By drawing upon the pupils' own informal strategies, Figure 2 provided the classroom the opportunity to discuss calculation strategies and the flexibility in the ways of recording results (for example including arrows or writing on top of the jumps made) and in the jumps made on the number line to solve a computation task. In this Concept Cartoon, the pupils' preferences were divided between the cartoon characters of Ann, Lina and Peter.

Student1: Ann and Peter added the tens and then the ones. They worked in the same way but Peter did not put the arrows.
Student2: Yes, but do we have to? It's addition, we know it from the plus sign.
Teacher: The empty number line gives us a lot of freedom in the ways of recording results.
Student2: Thus, we can decide what to do.
Teacher: Yes.

The pupils argued that it was more difficult to follow Nick's way of recording results because of the missing symbols above the jumps. The classroom established that the written symbols that accompany the jumps show the steps one follows to calculate the result, as well as possible errors. Furthermore, one pupil stated that even though he chose a cartoon character, his choice was related with the way of recording results and not with the calculation strategy presented by the cartoon character, explaining *"I would start from 60, add the tens and then the ones. It is easier this way"*. At this point, it should be noted that even pupils that struggled to model their calculation strategy on the empty number line, they understood the conventions used in interpreting the diagram, and thus, reading and identifying the strategy being modelled on the empty number line. Folllowing this Concept Cartoon, the pupils solved a calculation task, using their own strategies (see Figure 3).

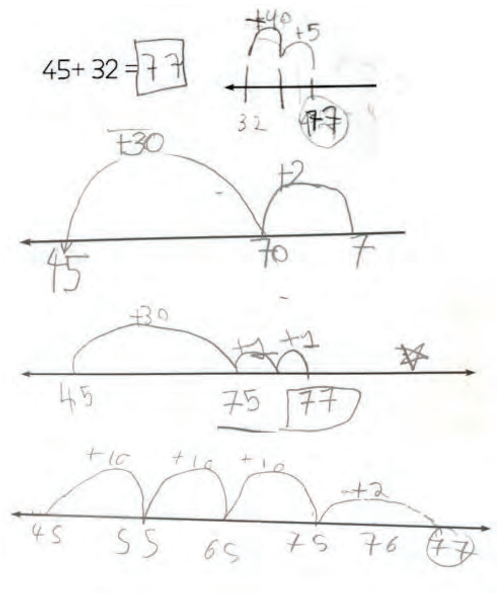

Figure 3: Pupils' calculation strategies

Gaining insight into pupils' calculation strategies.

Another type of Concept Cartoons created and employed in the classroom aimed at gaining insight into pupils' calculation strategies, as well as providing pupils with the opportunity to experience and develop a range of mental calculation strategies. Speech bubbles proposing various solution procedures encouraged pupils to identify the name of the character that best matched their personal strategy choice for calculating the result, and providing reasons for choosing the specific strategy.

For instance, in Figure 1, the Concept Cartoon was exploited as an introduction to the addition of two-digit numbers, where the second number is multiple of 10. The strategies used by the cartoon characters would enable the teacher to determine whether the pupils perceive the calculation as too hard, can solve the calculation in different ways, but recognise that a mental strategy is most efficient, would solve the calculation using auxiliary means, can remember a strategy, but would need to write it to perform the calculation (Sexton et al., 2009). The discussion led to a repertoire of proposed methods and procedures to calculate the result. According to the pupils the result could be calculated using arithmetic blocks, making jumps on a structured number line (counting by ones or tens), with a drawing (iconic representation of tens and ones, money model) or mentally. The pupils' mental reasoning strategy was to split 64 into tens and ones and processed separately. It was noticed that writing on paper to solve the calculation meant making a drawing and not a written calculation strategy. This can be explained by the fact that the pupils had not been introduced to written calculation strategies yet. This is also an indicator of pupils engaging in and extending the cartoon character’s argument. In addition, while no pupil perceived the calculation as difficult, the reliance on arithmetic blocks was evident, suggesting the need for the pupils to be provided with more opportunities to develop mental calculation strategies. Nonetheless, all pupils, using their preferred models, method and procedure reached the same result.

Adding to the above, in Figure 4, the methods used by the cartoon characters would enable the class to discuss various ways of computation as well as enable the teacher to determine the ways the pupils preferred to solve the calculation. The speech bubbles included splitting (Ann) where the numbers are divided by multiples of ten and units and processed separately when operations are carried out, stringing or compensation strategy (Nick) which refers to keeping the first number intact while splitting the second number into tens and ones, which are then added separately from the first number, the conventional paper and pencil algorithm (Lina) and arithmetic blocks (Peter). Even though the algorithm had not been introduced in the classroom, it was mentioned previously by some pupils, as knowledge they had acquired outside school. Thus, it was included in the Concept Cartoon, so as to be explicitly discussed, but without expecting pupils to use it as a way of working. Indeed, two pupils chose Lina. One pupil chose Peter, revealing that pupils would gradually rely less on models to solve a calculation task. The other pupils’ preferences were divided between Ann and Nick, with four pupils commenting that they did not need to write it down because they worked the calculation mentally. Figure 5 illustrates pupils’ calculation strategies when solving similar calculation tasks. Comparison of the pupils’ calculation strategies modelled on the empty number line that followed the exploration of Concept Cartoons presented in Figure 3 and 5 accordingly, also revealed that pupils’ strategies involved fewer steps.

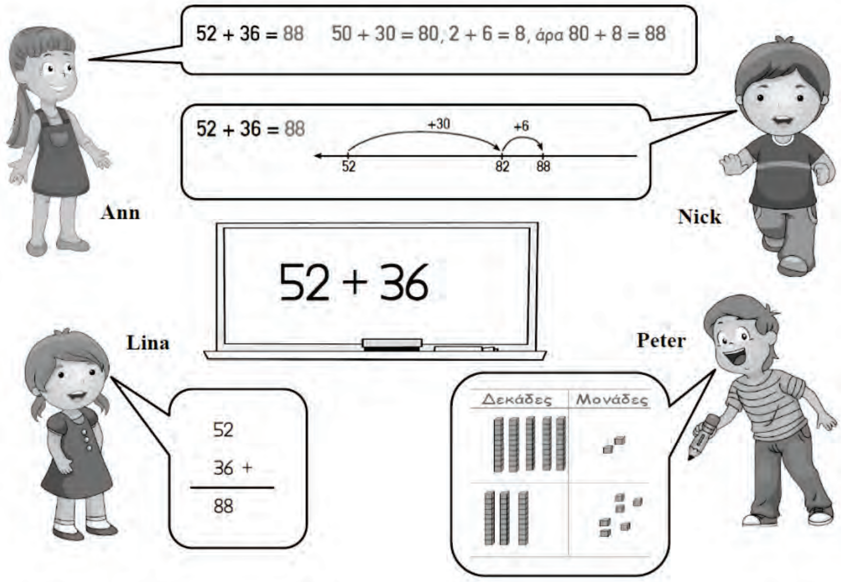

Figure 4: Gaining insight into the pupils’ calculation strategies

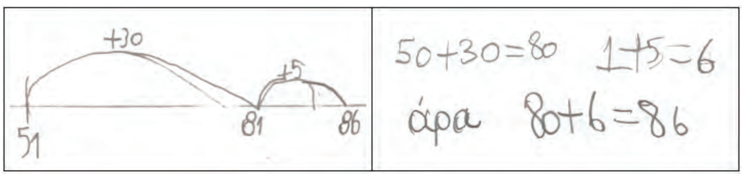

Figure 5: Pupil's calculation strategies

Discussion and Conclusion.

The purpose of this paper was to examine Concept Cartoons as an educational tool to support the number line as a vehicle for mathematical understanding. This study confirms findings from other research studies that Concept Cartoons act as an engaging and beneficial tool that supports pupils in interpreting and understanding concepts (Naylor & Keogh, 2013; Sexton et al., 2009). The Concept

Cartoons presented in this paper constitute an illustration of the attempt made to treat in a coherent way the number line by focusing on the simultaneous presence of the geometric and the arithmetic conceptualization of number on the number line, and fostering the transition from a model of pupils' informal solution strategies to a model for mathematical reasoning. Empirical evidence from the classroom indicates that Concept Cartoons created opportunities for the pupils to build connections and relations between representations of mathematical ideas, to have the freedom to come up with their own notations, as well as express and discuss their ideas, pointing to the usefulness of the number line as a didactical model (Van den Heuvel-Panhuizen, 2008). Initially, the Concept Cartoons provided the pupils with a platform from which, by drawing upon experience, they reached a definition, as suggested by Herbst (1997). Even though the instructional approach followed in this study was focused on positive whole numbers, the pupils' statements and work point to the number line being used as a metaphor to support thinking. Furthermore, by supporting pupils in defining and understanding the number line, the pupils were able to understand the conventions used in interpreting diagrams and thus, reading and identifying the strategy being modelled on the empty number line (see Figures 2 and 4). Adding to the above, findings from this study, also show, that the Concept Cartoons containing procedures in their bubbles (as a text or image), provoked discussion, reasoning and reflection regarding strategy choice and comparison of strategies, supporting pupils in developing mental calculation strategies (see Figures 3 and 5). As the Concept Cartoons were carefully designed by taking into account pupils' backgrounds, language literacy skills, as well as their level of mathematical understanding, pupils were kept motivated and engaged. This was also evident in the pupils' mathematical journal where they shared their thoughts stating for example *"I like concept cartoons because the characters are like us"*, *"The lesson is easier with the concept cartoons"*.

Concept Cartoons also proved a valuable instrument in providing the teacher with insight concerning pupils' understanding of the number line and the repertoire of strategies pupils use to perform an addition calculation. At this point it should be noted that this was not a straightforward process. For the effective use of the Concept Cartoons and supporting pupils' understanding of the number line and development of strategies and procedures, the classroom environment encouraged communication, exploration, discussion and reasoning. The social and sociomathematical norms negotiated and established in the classroom (Kilpatrick et al., 2001), offered pupils the freedom to develop, express and share their thinking. However, this does not tell the whole story. Concept Cartoons containing bubbles with proposed various results or with empty bubbles were also created and employed in the classroom both in addition and subtraction in the number domain 0-100. Through reflection and classroom discussion supported pupils progressing towards more elegant and higher-level strategies (Gravemeijer, 2020). Interpreting the number line, associating actions with it and communicating mathematical meaning may contribute towards a comprehensive picture of its conceptual structure and complete development of understanding of the number system. Additionally, employing Concept Cartoons alongside the number line in the teaching and learning process, points to the role of the teacher in taking into consideration their affordances, as well as the difficulties and limitations in their use depending on the mathematical content and the cognitive level of the pupil.

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Strategy use in number line tasks: An exploratory eye-tracking study

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The number line is an important external representation in primary education and research has shown that performance in number line tasks correlates with general mathematical achievement. A promising tool for gaining insights into students' strategies—in number line and other tasks—is eye tracking. However, previous eye-tracking research has predominantly addressed students' use of strategies on an empty number line, whereas students' strategy uses and eye movements on the marked number line have hardly been explored. This paper presents an exploratory study that investigates student strategies in locating numbers on a marked number line. In this paper, we present students' strategies in naming and placing numbers on the number line as well as differences in the use of strategies between students with short and long response times, indicating the use of more or less efficient strategies.

Keywords: Number line tasks, eye tracking, strategies.

Introduction

It is of fundamental interest for mathematics education research to examine students' mathematical performance levels and their individual strategies when working on mathematical tasks to meet the individual needs of students (Anthony & Walshaw, 2009). The basis for many mathematical learning processes is laid in mathematics learning at primary level. Lacks in basic mathematical knowledge can lead to difficulties at secondary level (Moser Opitz et al., 2017). One essential aspect for learning mathematics is to develop the concept of numbers, which requires various aspects, including numbers in their ordinal aspect (Fuson, 1988). To build up and deepen an ordinal understanding of numbers, external representations with linear arrangement of numbers, such as the number line, are often used (Diezmann & Lowrie, 2007). Furthermore, number line tasks are "widely used to investigate mathematical learning and development" (Schneider et al., 2018, p. 1467) and students' number line estimation performance is of predictive nature for their mathematical development (Booth & Siegler, 2008). Therefore, it seems crucial to examine students' strategies in locating numbers on the number line. Previous research has indicated that eye tracking holds potential for the analysis of strategies in empty number line tasks (e.g., van't Noordende et al., 2016). In this paper, unlike previous ET studies, we investigate the use of strategies on the marked number line and show an exploratory first excerpt from a larger study.

Number line tasks

The number line is one of the essential external representations in mathematics education to address the ordinal number aspect (Diezmann & Lowrie, 2007). On the number line, numbers are arranged linearly and represented by their position in relation to other numbers (Schulz & Wartha, 2021). Students performance in number line estimation tasks correlates with broader mathematical competence (for a meta analysis, see Schneider et al., 2018). There are several types of number lines, for example, the marked number line or the empty number line. For the marked number line, there are different presentation options. The number line can be completely or only partially marked with hatch marks and labelled with numbers. Hatch marks can represent visual reference points, also without being numbered. Also, different ranges of numbers can be represented. Thus, on different scales the same marks and distances must be interpreted differently (Schulz & Wartha, 2021).

Often, the empty number line (where only beginning and endpoints are labelled) is used to study children's number sense. In studies using the empty number line, children are often asked to estimate the spatial position of numbers (e.g., van't Noordende et al., 2016). These studies indicate that children with mathematical difficulties (MD) show less precise estimations as compared to children without MD (Landerl et al., 2017). Further, eye-tracking studies have indicated that differences in the estimation of numbers between children with and without MD are related to the different use of strategies for estimating numbers (van der Weijden et al., 2018), that students with MD are less flexible in strategy use, and that student strategies may involve, for example, the use of reference points such as the given beginning, end- and midpoint (van't Noordende et al., 2016).

However, so far only little research has investigated students' strategies on the marked number line and previous works on the empty number line (e.g., Barth & Paladino, 2011) suggest that strategies may differ between the two representations—due to, for example, hatch marks, etc. Differences in strategy use on the marked number line may be related to a different use of reference points—as compared to the empty number line—and, furthermore, to the use of additional marks, which may be indications of counting strategies, as Diezmann and colleagues showed in an interview study (2010). This calls for studies investigating students' strategy use on the marked number line.

Eye tracking

Eye tracking (ET) is the technique to record a person's eye movements (Holmqvist et al., 2011). It has proven to provide insights into children's mental processes and their strategies when working on mathematical tasks—through domain-specific interpretations (Schindler & Lilienthal, 2019). There are some ET studies addressing number line tasks in the field of natural numbers—to investigate strategies in solving these tasks—both in adults (e.g., Sullivan et al., 2011; with MD, e.g., van der Weijden et al. 2018) and in children (e.g., Schneider et al., 2008; with MD, e.g., van't Noordende et al., 2016).

ET research has already addressed students' diversity with respect to their competence levels, for example, addressing children with and without MD. These ET studies use the empty number line with varying ranges of numbers (usually 0-100, e.g., Schneider et al., 2008; and 0-1000, e.g., Sullivan et al., 2011). Yet, little is known about how students deal with the marked number line and there are—to the best of our knowledge—no ET studies investigating the use of strategies in locating numbers in marked number lines tasks. As mentioned earlier, differences in students' locating of numbers on the number line could be due to different use of strategies. Therefore, it seems worthwhile to investigate students' strategies locating numbers on the marked number line.

We ask the following research question: *What strategies do students use in locating numbers on a marked number line, and how does students' strategy use differ?*

This study

Tasks and procedure

In this study, we used a number line with labelled beginning and endpoint and hatch marks—with distance between adjacent marks of a unit of 10. We used this type of number line, which is an intermediate form between a completely marked and an empty number line (Schulz & Wartha, 2021), to give students some help and orientation (steps of 10) but not to display all of the information (i.e., every single number) on the number line. We used two different task types. In the “number-to-position-task” (NP), we presented a symbolic number to the students and then asked them to place it on the number line. This task type is common in ET studies of number line tasks (e.g., Sullivan et al., 2011). In the other task type, the “position-to-number-task” (PN) (Figure 1), students were shown a position (red cross) on the number line and asked to name the number that corresponded to it. This task type avoids falsification of results due to possible motoric difficulties (Gomez et al., 2017).

Figure 1: Position-to-number-task

In individual sessions in a quiet room in their school, the students worked on the tasks. There were three items for each task format. Before the tasks, there was one practice task each (with number 10 each) to introduce the number line and get the students acquainted with the task format. In the NP, the numbers 70, 30, 90 (in that order) were to be placed on the number line. The numbers were presented in the upper left corner of the screen. The students were asked to read the number aloud to ensure they had perceived it correctly before the number line appeared. Then students were asked to point at the corresponding place, fixate it with their eyes, and let us know when they were done. In the PN, the position (red cross) of the numbers 80, 40, 60 (in that order) were shown and the students were asked to say what number was indicated. In between the tasks, the students were instructed to fixate a star in the upper left corner of the screen before the next task appeared, so that students’ gazes started from the same place. The students received no response as to whether their answers were correct. Verbal answers were recorded through an audio-recorder.

Participants and eye tracker

A total number of 186 German fifth graders (165 students of a comprehensive school, 21 students of a special school for learning difficulties) worked on the tasks. We chose fifth graders since at the end of primary school (i.e., grade 4), students may still have difficulties with the number line (Rodriguez et al., 2001). From the entire group of participants, ten students each were selected for the NP and PN tasks (mean age: 10.11 years) based on response times in the respective task type: To get insights into a diverse set of strategies, we included five students (both tasks: three students from special school and two students from comprehensive school) whose response times were particularly long (LRT-group), and five (from comprehensive school; except for NP: one student from special school) with particularly short response times (SRT-group). This choice was made since long and short response times typically reflect different strategies (e.g., Schindler et al., 2019). Additionally, we analyzed only trials where students solved the tasks successfully, to rule out that the students just guessed.

The students’ eye movements were recorded with the remote eye tracker Tobii Pro X3-120 (120 Hz, binocular, infrared). This eye tracker is attached to the bottom frame of a monitor and therefore very unobtrusive. The average accuracy for the students in this study was 0.7°. Tasks were presented on a 24” Full HD computer screen. The distance of the students to the screen was about 50 cm.

Data and data analysis

We used response times and videos, which were provided by Tobii Pro Lab software. The existing ET studies on number line tasks examine fixations. In our study, however, we use gaze-overlaid videos (where student gazes are augmented through a dot wandering around)—like, for example, Schindler and Lilienthal (2019) did. These videos are unfiltered and contain all captured information, that is, all eye movements are considered (not only fixations), which is advantageous when studying strategies. Analysis of these data has already proven to be useful in identifying children’s strategies in other mathematical domains and tasks (for quantity recognition, see e.g., Schindler et al., 2019). We analyzed the gaze-overlaid videos in an inductive manner based on Mayring’s (2014) qualitative content analysis: For each task, the eye movements were first looked at, described, interpreted, and then paraphrased. In a subsequent strategy finding process, commonalities between the eye movements of the students were sought, categories of strategies were found and each assigned with corresponding descriptions. Finally, the strategies used by the students in the two groups (with long and short response times) were compared qualitatively.

Results

In the following, we will pursue the research question: *What strategies do students use in locating numbers on a marked number line, and how does students’ strategy use differ?* We will do so by elaborating on the strategies used by the LRT-group and the SRT-group respectively. For visualizing gaze patterns, we use gaze plots, although we used gaze-overlaid videos for the data analysis.

Position-to-number-task

Task 80: The five students of the SRT-group showed an orientation to the end of the number line (100), this means they used counting from 100 backwards to the red cross (80) (Figure 2).

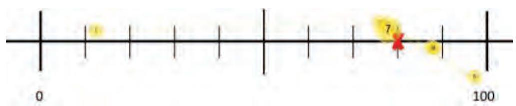

Figure 2: Gaze plot SRT-group—identifying 80 on the number line (PN)

The students of the LRT-group mainly used counting from the beginning of the number line forward and were partially additionally oriented to the whole number line (i.e., gazes were at beginning and endpoint) (Figure 3, left). One student also showed an orientation of 100 backwards after first appearing to want to count from the beginning (Figure 3, right).

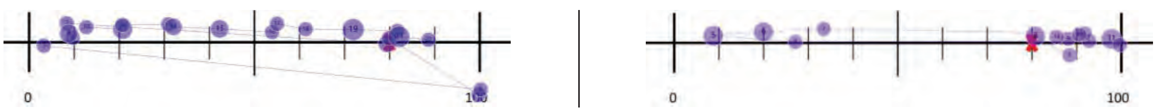

Figure 3: Gaze plots LRT-group—identifying 80 on the number line (PN)

Task 40: The students of the SRT-group named the number without looking at reference points (i.e., direct identification), or oriented themselves to the middle (50) of the number line (Figure 4).

Figure 4: Gaze plot SRT-group—identifying 40 on the number line (PN)

The LRT-group students mainly used counting from the beginning of the number line forward (Figure 5, left). One student additionally used the middle of the number line as orientation (Figure 5, right).

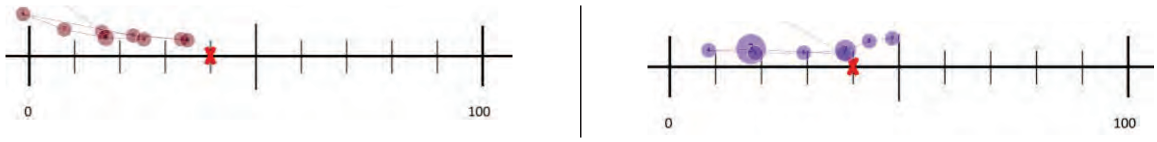

Figure 5: Gaze plots LRT-group—identifying 40 on the number line (PN)

Task 60: Both groups predominantly used the middle of the number line as orientation (Figure 6, left), while the students in the LRT-group partly also paid attention to the entire number line (i.e., beginning and endpoint) (Figure 6, right).

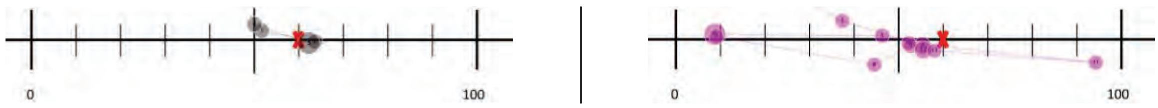

Figure 6: Gaze plots—identifying 60 on the number line (PN)

In sum, we found that students with long response times predominantly used counting procedures, which are time-consuming—especially from 0 forward for high numbers—or a combined use of reference points (e.g., beginning and endpoint). In contrast, students with short response times tended to use more efficient strategies such as looking directly at the red cross or the nearest reference point.

Number-to-position-task

Task 70: In this task, there was predominantly an orientation to the endpoint (100) for the SRT-group. Two examples of orientation to the endpoint are shown in Figure 7: from 100 stepwise (counting) to 70 (left) and from 100 directly to 70 (right).

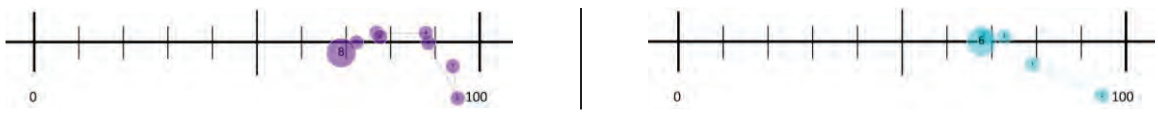

Figure 7: Gaze plots SRT-group—placing 70 on the number line (NP)

The orientation to 100 was also frequently evident in the LRT-group, but here students sometimes additionally showed an orientation to the midpoint (50) (Figure 8, left). One student of the LRT-group used counting from 0 forward (Figure 8, right).

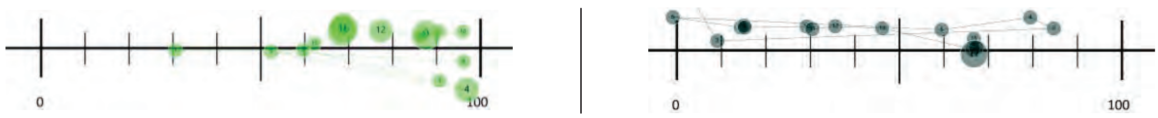

Figure 8: Gaze plots LRT-group—placing 70 on the number line (NP)

Task 30: The students of the SRT-group either counted from the beginning (Figure 9, left) or used no reference points (Figure 9, right).

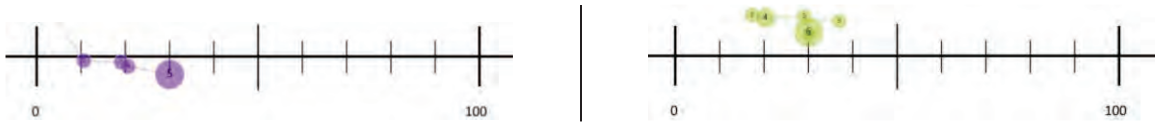

Figure 9: Gaze plots SRT-group—placing 30 on the number line (NP)

One student of the students of the LRT-group counted from the beginning but additionally had many gazes at the end of the number line (Figure 10, left). Another one was more oriented towards the middle (50) and counted backwards (Figure 10, right).

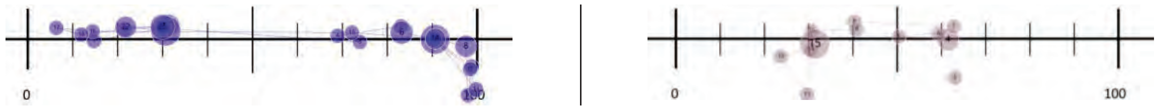

Figure 10: Gaze plots LRT-group—placing 30 on the number line (NP)

Task 90: Here, the strategy use was similar for the two groups. The placing of the number happened either directly (Figure 11, left) or in orientation to the endpoint (Figure 11, right).

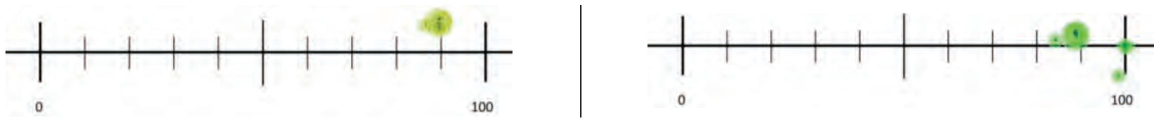

Figure 11: Gaze plots—placing 90 on the number line (NP)

In sum, for the first two tasks of the NP, students who had short response times appeared to use more efficient strategies than students with long response times, such as using a near reference point only. In contrast, in the group of students with long response times, for example, combination of strategies (reference points) or counting from the beginning (time-consuming for high numbers) occurred.

Discussion

Our study revealed different strategies for tasks on the marked number line—indicated by the students' eye movements. We found that the students appeared to use different reference points (beginning, midpoint, or endpoint) and that they partially combined reference points. We found counting strategies starting from different reference points and we also found strategies of direct orientation starting from different reference points or without looking at any reference point. Furthermore, our analyses indicate differences in the use of strategies between students who had long response times (LRT) and those with short response times (SRT). For the naming of numbers on the number line (PN), there were apparent differences between the two groups: The LRT-group used time-consuming counting procedures along the number line and the combined use of reference points. They used the relation of given marks (Schulz & Wartha, 2021), for example, that the number 40 is next to the number 50, the marked midpoint, less often than students with shorter response times. SRT-group students had almost exclusive gazes at the marked place and the nearest reference point. Students with longer response times used less efficient strategies. This is in line with previous ET research (e.g., Schindler et al., 2019), including studies on the empty number line, which showed differences in the efficiency of different students' use of strategies (e.g., van't Noordende et al., 2016). Beyond that, for placing numbers on the number line (NP), similar differences in the use of strategies were found. However, these differences in the use of strategies only showed for two out of three tasks. This could be due to the chosen number of the third task—90 is close to 100 and probably therefore, there was little variation in this task.

Our insights relate to the results found by Rodriguez et al. (2001) that students in the transition from primary to secondary level may still have difficulties with the number line. This implies that naming and placing numbers on the number line partially needs to be supported at the beginning of secondary school—even in the range up to 100. With respect to the whole sample, future studies should explore the tasks on the marked number line further to see if the results found here are transferable and if the trends of our exploratory study can be found in larger samples. Looking at more students, might show a greater variation of strategies. It would also be interesting to investigate if there are differences between children with and without MD in their strategy use to locate numbers on the marked number line—as shown for the empty number line (e.g., van't Noordende et al., 2016). We believe that our exploratory study contributes to gain fine-grained insights into students' strategies when working on number line tasks—an important tool for the development of the number concept and its ordinal aspect (Diezmann & Lowrie, 2007). Gaining such insights is an important step for being able to address students' individual needs in this mathematical content (Anthony & Walshaw, 2009).

Acknowledgment

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Teachers’ perspectives on number and addition in year one: application of FoNS framework in interview analyses

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This study investigates Danish teachers’ perspectives on teaching and learning number and addition in year one through analyses of semi-structured interviews with six year one teachers. The teachers’ perspectives are analysed through an established year one framework for number sense: Foundational Number Sense (FoNS). The analysis indicates that the FoNS framework is a useful tool to identify factors that teachers privilege, but important foundational factors, estimation, quantity discrimination and number patterns were only superficially discussed by the teachers in their interviews.

Keywords: Foundational Number Sense, teacher perspectives, simple addition, year one students.

Introduction

Learning and teaching of arithmetic has been a major topic for educational research for decades (Nunes et al., 2016) and the importance of students’ arithmetic competence for mathematical development in general is widely acknowledged. Thus, numerical and arithmetic competence has been linked to development of mathematics achievement and difficulties (e.g. Feigenson et al., 2013; Geary et al., 2013; Ostad, 1997). Teaching of arithmetic in primary school relies on the implementation of key knowledge to set up strong foundations for a successful development of arithmetic competence, e.g. adaptive flexibility, strategies for mental calculation, and number knowledge. Therefore, it is important to get insight into teachers’ perspectives on the teaching of number and arithmetic in the early years of school, where number and basic arithmetic is the primary focus, and research suggests is a crucial stage in children’s development of number competencies.

In this paper, we explore six Danish teachers’ perspectives on the teaching and learning of number and addition and analyse whether this is aligned with the FoNS framework.

Numerical and arithmetic competence

Development of arithmetic competence rely on several components of numerical competence or number sense (Desoete & Grégoire, 2006; Fuson & Burghardt, 2003), for example: symbolic knowledge and number words (e.g. Chu et al., 2015), mapping symbols to quantity (e.g. Geary, 2013), basic counting skills (e.g. Jordan et al., 2009), number comparison skills (e.g. De Smedt et al., 2013), estimation skills (e.g. Booth & Siegler, 2008; Gilmore et al., 2007), and knowledge of base-ten number structure (Laski et al., 2014). The individual components of number sense are all important for further mathematical development, however, the links between them are essential for students’ development (Gersten et al., 2005).

Recently, Andrews and Sayers (2015) proposed a framework for identifying students’ opportunities to acquire foundational number sense, FoNS, and demonstrated its strength when analysing lessons and textbooks in different cultural contexts (Andrews & Sayers, 2015; Löwenhielm et al., 2019;

Sayers & Andrews, 2015). The framework is derived from an extensive literature study and comprises eight categories of FoNS (for a thorough description see Andrews and Sayers, 2015): 1) Number recognition, 2) Systematic counting, 3) Relationship between number and quantity, 4) Quantity discrimination, 5) Different representations of number, 6) Estimation, 7) Simple arithmetic competence, and 8) Awareness of number patterns. Each of these categories has been shown to play an important role in students' development in mathematics (Andrews & Sayers, 2015).

The current study

This study investigates six Danish year one teachers' perspectives on the teaching and learning of number and addition in year one through analyses of semi-structured interviews. The aim of the study is to analyse whether teachers' perspectives are aligned with established mathematical knowledge of numerical and arithmetic competence. Our research questions are: 1) Can the FoNS framework be applied to teacher interviews effectively? 2) Can the FoNS framework inform us what teachers privilege when discussing their lessons on number and arithmetic?

Methodology and Methods

Drawing on exploratory Case study methodology (Yin, 2013), this small qualitative study is based on semi-structured interviews and lesson observations with six Danish year one teachers. The teachers' utterances related to different aspects of the teaching and learning of number and addition were analysed and categorised using the FoNS framework (Andrews & Sayers, 2015).

The selection of participants ensured an equal number of male and female teachers, and a cross section of professional experience (2-24 yrs.) and age (30-49 yrs.). The teachers were informed about the project both in writing and at an introductory meeting prior to the interviews took place. By the end of the project, the teachers were offered to read and approve the transcripts of the interview. None of the teachers took advantage of this offer. Pseudonyms have been used throughout.

Teacher interviews

In accordance with an exploratory Case study investigation a series of open ended interview questions were used to elicit teachers' perspectives on what they emphasise in the teaching on number. Each interview lasted between 45 and 60 minutes, and the interview focused on the teachers' perspectives on teaching and learning of arithmetic in year one. The semi-structured interview was guided by questions related to the teacher's plans for and reflections on a specific observed lesson on number and arithmetic as well as general questions on the teaching and learning of number and arithmetic (addition and subtraction) in year one. The questions on the observed lesson was related to planning of the lesson: "Why did you choose these specific activities?", carrying out the lesson: "How did you experience the lesson? Did it proceed as you had expected?", Progression: "How will you follow up on this lesson? What will be the next step?", and characteristics of a 'good activity': "what is a good activity and what makes it good? What do the students learn in these activities?".

General questions about learning were asked specifically about teaching addition, for example: "how do you introduce the students to addition" and "what aspects do you emphasize?", prerequisites for learning addition: "What is the prerequisites for learning addition, how do you ensure the students have the prerequisites?", and "Are there aspects of learning addition the students' find especially difficult?", Furthermore, questions were asked about children's progression in teaching addition, for example: "How do you see the progression in teaching addition?" and "What do you expect your students to know or be able to work on by the end of year one?".

Throughout the interview, if the teacher primarily referred to practical aspects, e.g. "a good activity is easy to explain for the students" or "the lesson went well because many students participated in the activity", the teachers were asked additional questions related to the mathematics of the activities and lesson, e.g. "what aspects of number and addition do you think the students learn through that (particular) activity?". According to Bryman (2016), this was to ensure teachers had an opportunity to reflect on the aspects of mathematics that were of interest in this project. Only utterances with mathematical content related to number and arithmetic were categorised in the analysis.

To analyse how six teachers, describe their approaches to their lessons we transcribed the interviews using NVivo. Excerpts presented here were all translated into English, a process that included transforming Danish idioms into equivalent English expressions without losing the speaker's intended meaning (Brinkmann & Kvale, 2015). We have chosen to scrutinise these teachers' utterances to identify what components of number sense they privileged over others in their teaching of number and addition. In so doing we used a deductive approach (Bryman, 2016; Yin, 2013) where the first author read each transcript repeatedly, to determine which FoNS components were addressed. To minimize the need for translating the interviews, the second author then read the collective excerpts of the different categories to ensure consistent categorisation. In the following, we present the results of our application of the FoNS framework to the interviews.

Results

In the following we provide examples of utterances by teachers that were mapped directly to each of the FoNS categories. However, a statement or description of an activity can contain several FoNS categories and is then assigned to all relevant categories.

FoNS categories in the interviews

Number recognition: Knowing number symbols and number names were emphasised by all teachers but explicated very differently. The teachers expressed very different levels of necessary knowledge for the students. One teacher, Else, said that the students "need to know the numbers, their value and be able to write and read them, and recognise them". Frida exemplifies by saying "they should know what twelve looks like". Allan specifies a number range: "the students need to learn the numbers to 100" and Dan explicated that the students do not have to know the number name as long as they "can write it" although he also emphasised knowing the names of the tens. Knowing the names of the tens was also mentioned by Carl as a help to find the number names of two-digit numbers. Bettina, talking about the base ten number system, explained how she focused on "enhancing the students' competencies of naming number". Naming two-digit numbers are something that many students find difficult because of the Danish number names. Carl mentioned this and explains how he addressed this in activities where the students "have to find the number 13 or 17 so they practise finding the correct symbol for the correct number name".

Systematic counting: Counting skills are addressed by all teachers. Dan said that “they need to know the number sequence” and Carl stated “Early maths is mostly about counting”. Else said, when asked what she thought should be the focus in year 0 (a preschool class) she said “it’s important that they just count and count and count”. All the teachers provided many examples of counting procedures, often performed by the use of manipulatives or other representations e.g. a number line. The number line was mentioned by all the teachers in relation to activities of ordering numbers or “find the number” and when performing counting procedures. However, Carl and Else were the only teachers directly referring to knowledge of “the number before and after”. All teachers referred to skip counting, often by ten. Skip counting was used together with references to times tables and the teachers thus referred to “knowing the ten times table” when they taught the students to count in tens in order to find the name of a two-digit number.

Number and quantity: The relationship between number and quantity was addressed by four of the teachers but with different levels of articulation. Allan emphasised that “they need to recognise that quantity and number kind of go together”. Bettina said “the students need to understand the symbols and the naming of quantity”. Carl, elaborating on “the translation between number and quantity”, underlined that “they need to understand quantity; the symbol 4 equals four things”. Likewise, Else emphasised the understanding of the relationship between number and quantity. “They need to have an idea of what value is and what is worth more (…) so many dots or centicubes, what is the size and quantity of that”.

Quantity discrimination: Comparing quantity was addressed very briefly by only two teachers and only in a single statement from each. Else described an activity where students construct two-digit numbers by combining two playing cards, write the number and compare with the next number they construct. She reflects: “I don’t know if they just write some numbers or if they actually understand which is bigger and which is smaller”. Allan also addressed comparing numbers in relation to doing addition and comparing possible results: ”seeing this result is one bigger and this is one smaller than the other”.

Representations of number: All teachers mentioned several different representations of numbers, both concrete materials like money, centicubes and fingers, but also partitioning in tens and ones, friends of ten and the number line. In two of the classes all students had a tablet, and the two teachers also mentioned and app, Number Pieces, where students can represent numbers using ones, tens etc. and partition numbers.

Estimation was only mentioned by one teacher, Else. She very briefly referred to estimation of quantity by mentioning an activity of “how many in the jar”. This is an activity, where the students have different containers with an unknown number of items. The students then guess how many items are in the container and afterwards they count the exact number of items. However, she did not explicitly use the expression estimation or to estimate.

Simple arithmetic: Given the teaching and learning of number and arithmetic in year one was the focus of the interviews, simple arithmetic was mentioned by all teachers. However, the teachers differed substantially with regard to their focus on different calculation methods and strategies, the number range, bridging ten and level of fluency with single digit addition.

Number patterns: Several teachers talked about knowing the sequence of numbers (categorised as systematic counting), but only two explicitly referred to putting in correct order. Else referred to a specific evaluation activity where the students have to put some number cards in the correct order. Carl described an equivalent activity where students are given a number card and then has to line up according to the number sequence.

Summarising Results

The FoNS categories were well presented in the teachers' responses to questions. Table 1 provides an overview of how the categories are distributed over the six teachers' interviews.

Table 1: Overview of the categories of Foundational Number Sense (Andrews & Sayers, 2015) and their presence in the interviews with the six teachers indicated by X. A: Allan, B: Bettina, C: Carl, D: Dan, E: Else and F: Frida.

Category	A	B	C	D	E	F
Number recognition	X	X	X	X	X	X
Systematic counting	X	X	X	X	X	X
Number and quantity	X	X	X	X	X	
Quantity discrimination	X				X	
Different representations	X	X	X	X	X	X
Estimation					X	
Simple arithmetic competence	X	X	X	X	X	X
Number patterns			X		X	

Discussion

The aim of this paper was to investigate whether teachers' perspectives are aligned with the FoNS framework in year one. The discussion provides insights into how year one teachers in six Danish schools perceive key teaching and learning number and arithmetic attributes, but also how a simple framework can be used to identify these.

The analysis of the interviews revealed that all components of FoNS were addressed explicitly or implicitly in the interviews, but not equally by all teachers. A single teacher, Else, addressed all eight categories, whereas the teacher Frida only addressed four. The remaining four teachers addressed five or six of the categories. However, what is perhaps more important is to what extent the different components of FoNS were addressed by the teachers.

Although all teachers explicitly emphasized number symbols and number names (Chu et al., 2015) as the most important aspects of "knowing number" not all teachers mentioned quantity and creating the link between symbols and quantity (Geary, 2013). However, all teachers mentioned the importance and relevance of using manipulatives, which implicitly provides learning opportunities for students to create the link between number and quantity, and other number representations. Counting skills (Jordan et al., 2009) was explicitly mentioned by all teachers. Thus, some of the basic components of FoNS and prerequisites for doing arithmetic is explicitly or implicitly part of the teachers' perspectives on the teaching and learning of number in year one.

Three of the components found to be of special importance in the early years, quantity discrimination (De Smedt et al., 2013), estimation (Booth & Siegler, 2008; Gersten et al., 2005; Gilmore et al., 2007) and number patterns (Gersten et al., 2005), was only addressed implicitly by one (estimation) or two teachers (quantity discrimination and number patterns), and in all cases these categories was only addressed superficially and implicitly by the teachers. These findings resonate with a cross-cultural study by Sayers and Andrews (2015) on the opportunities to learn different aspects of FoNS in different activities observed in six different European classrooms. Across countries, they found no episodes where teachers encouraged students to estimate and only 2 of 18 episodes where a single teacher introduced quantity discrimination.

Although estimation is considered to be one of the most important mathematical competences along with proportional reasoning and problem solving (Sriraman & Knott, 2009) it is remarkably absent in both classrooms (Andrews & Sayers, 2015; Sayers et al., 2016), textbooks (Sayers et al., 2021), and curricula (Andrews et al., 2021; Sunde et al., 2021). In this study we have shown that teachers do not explicate estimation or estimation related activities as an important part of their year one teaching.

With regard to the application of the FoNS framework to teacher interviews we found it successful on two key points: 1) The framework provided an easy to use categorisation of components of number sense known to be of importance for further development in mathematics (Andrews & Sayers, 2015). 2) The use of the framework also highlighted the differences in the number sense components the different teachers addressed.

In conclusion, the findings indicate that teachers are cognisant of a wide range of the important foundations for developing FoNS and arithmetic competence. However, it is also apparent that three crucial aspects, quantity discrimination, estimation skills and number patterns, were only mentioned implicitly by one or two teachers in the interviews.

We have shown that the FoNS framework is easy to apply in analysing teacher interviews, and can successfully reveal patterns of teachers' perspectives on the learning of number in year one. The analysis shows the differences between teachers with respect to the number of FoNS categories and it highlights the underrepresented categories.

This study cannot provide insight in how teachers actually teach. The analysis can only give an indication of what teachers emphasise in their classroom practice. A teacher's description of an activity cannot provide the full picture of the complete range of FoNS categories that the activity would cover when actually performed by the teacher in interaction with students in the classroom. Thus, the actual learning opportunities for the students might be richer than the interview would suggest. However, it would be reasonable to expect that what the teachers emphasise in the interviews is what they would also emphasise during teaching. Further research on video observations will show to what extend the findings of the lack of awareness on quantity discrimination, number patterns and estimation skills are accentuated in the actual teaching and learning in the classroom.

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Informal strategies of students with autism spectrum disorder in solving Cartesian product problems

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The present work focuses on arithmetic word problem solving and explores the strategies used by 26 students diagnosed with autism spectrum disorder when solving multiplicative Cartesian product problems. The students solved two outfit problems involving small and large numbers, respectively. The success in both problems was low. We found a variety of correct strategies, predominantly operation strategies. Most incorrect strategies were based on additive relations with modelling. We detail the difficulties observed during the problem-solving process, and implications for teaching students diagnosed with the disorder are drawn.

Keywords: Primary education, combinatorial thinking, cartesian product problems, solution strategies, autism spectrum disorder.

Introduction

Combinatorics constitutes a significant component of the mathematics curriculum, building on a rich structure of principles that underlie several other areas, such as counting, numeration, computation and probability. While developing their combinatorial thinking skills, children learn key mathematical skills such as constructing meaningful representations, reasoning mathematically, and generalizing mathematical concepts (English, 1991, 2005).

An important process in the development of combinatorial thinking skills is the acquisition of combinatorial strategies. A standard task to help children acquire these strategies is the Cartesian product problem (English, 1991), which consists in finding all possible combinations of two items, taken out of two different sets of items. Mulligan and Mitchelmore (1997) found that children in grades 2 and 3 used three main intuitive strategies for solving different types of problems with multiplicative structure, and all three are encountered among the correct resolutions of Cartesian product problems. These strategies are: i) direct modeling with counting strategies (when concrete manipulatives or drawings are used to model the problem situation, and objects are counted with no obvious reference to the multiplicative structure); ii) counting strategies (when the same actions are performed as in the previous level, but without the use of manipulatives); and iii) operation strategies (when multiplications are used). Several studies have shown that children develop these strategies intuitively, and that they acquire increasingly more sophisticated strategies for this type of problem, depending on age and experience (English, 1991; Maher & Martino, 1996). Mulligan and Mitchelmore (1997) note, though, that the Cartesian product problems are considered very difficult by the children, and most of the responses obtained in their study were incorrect. Furthermore, the majority of these incorrect responses were based on applying an inappropriate additive strategy, in which the numbers were added instead of multiplied. The prevalence of this incorrect strategy in the resolution of Cartesian product problems is confirmed by Nesher (1992), for students in grades 3 to

6, and by Ivars and Fernández (2016), for students in grades 1 to 6. The latter study additionally performs a more detailed analysis of the incorrect responses, identifying strategies such as one-to-one combinations (when elements are combined one to one without repetition) and nonsensical strategies (which include blank responses).

The focus of our research is on students with autism spectrum disorder (ASD). This disorder is characterized by deficits in social development, communication, and restrictive and repetitive behaviors or interests (American Psychiatric Association, 2013). These characteristics may lead to poor problem-solving capabilities, in particular since they often result in low reading comprehension and difficulties in thinking ahead or planning tasks. In order to improve problem-solving capabilities in ASD students, adapted instruction is required. To that end, there has been a growing interest in researching mathematical learning in this group (Bullen et al., 2020; Polo-Blanco et al., in press a; Polo-Blanco et al., in press b) and, in particular, in the strategies they employ when solving mathematical problems (Polo-Blanco et al., 2019, 2021). This research is especially relevant since students with ASD are increasingly incorporated into mainstream educational settings at all levels of education (Roberts & Webster, 2020).

The literature on probabilistic thinking, in particular combinatorial thinking, in students with ASD is very scarce. To our knowledge, only the work by López-Mojica, e.g. (2013), analyzes the resolution of combinatorial tasks in one student with ASD. The author highlights the need to explore combinatorial activities in order to introduce the idea of probability (López Mojica, 2013). At the same time, the importance of combinatorial thinking in students in general is clear, as emphasized by several authors (e.g., Eizenberg & Zaslavsky, 2004; English, 1991, 2005): First, as mentioned before, it allows them to acquire the mathematical skills that are present in the educational curricula. Second, people use basic principles of combinatorics in many everyday situations, for instance by enumerating all possible ways an event can occur, which is key to making informed decisions (Yee, 2009). Combinatorics therefore develops skills needed in daily life, and we consider this aspect to be especially relevant for ASD students, whom it helps to be more autonomous in their adult life.

For these reasons, in this paper we set out to investigate the strategies used by students with an ASD diagnosis when solving Cartesian product problems. In particular, we study the strategies they use when solving two “outfit problems”, which require a multiplication to obtain all possible combinations. Based on the results in previous studies with students of typical development, we anticipate that the students with ASD will also experience difficulties in the task, and that they will use basic strategies in their resolution.

Our research questions are:

- What strategies do students with ASD employ to solve multiplicative Cartesian product problems?
- What are the main difficulties they encounter during the process of solving Cartesian product problems?

Methodology

We conducted an exploratory and descriptive investigation (Yin, 2017) in which we detailed the solving strategies of 26 students with ASD, as well as the main difficulties identified, when solving combination problems with multiplicative structure.

Participants

The participants were 26 students aged 6 to 12 years (23 males and 3 females), diagnosed with ASD according to DSM-5 (American Psychiatric Association, 2013), with minimum IQ of 70 on the WISC-V (Wechsler, 2014), and minimum equivalent mathematical age of 5.5 years. All of them were attending primary education in 19 ordinary schools in Cantabria (Spain). The mean chronological age of participants at the time was 9.35 years, with a standard deviation of 2.06. The mean IQ of the participants was 89.88, with a standard deviation of 11.77.

Data collection instrument

Based on Mulligan and Mitchelmore (1997), we designed a questionnaire with 16 multiplication and division problems of the types: equal groups, multiplicative comparison and Cartesian product. Of these 16 problems, students first solved 8 problems involving small numbers. Then, the students who had provided the correct solution for a problem were asked to solve the corresponding large-number problem. In this study we analyze the two Cartesian product problems that required a multiplication for their resolution, one with small numbers and one with large numbers. These problems are:

- Outfits Problem, Small (OPS): I have 3 shirts of different colors and 4 different pairs of pants. If I wear one shirt and one pair of pants each time, in how many ways can I dress?
- Outfits Problem, Large (OPL): I have 8 shirts of different colors and 3 different pairs of pants. If I wear one shirt and one pair of pants each time, in how many ways can I dress?

The students solved these problems individually, in one session of approximately 25 minutes and in a classroom free of distractions, with only the interviewer and the student present. Before starting to solve the problems, the interviewer explained what the test consisted of, and made sure that he or she understood the statements, reading them with him or her in cases where the student was confused. The student was told that he or she could write, use manipulatives (interlocking blocks) or answer orally. All sessions were videotaped, and the solutions were transcribed for later analysis. The students' strategies were coded by the fourth author. An experienced mathematics education teacher, who was blind to the hypotheses of the study, recoded 30% of the students' strategies. The mean interobserver reliability for strategy categorization was 94%, calculated as the number of agreements divided by the number of agreements plus disagreements and multiplied by 100.

Analysis categories

We adhered to the following system for classifying the strategies used to solve multiplicative structure problems (Ivars & Fernández, 2016; Mulligan & Mitchelmore, 1997): incorrect strategies (level 0), direct modeling with counting (level 1), counting (level 2) and operation strategies (level 3). The incorrect strategies (level 0) considered were inappropriate additive relationships, one-to-one combinations, and given number (when one of the numbers in the problem is given as the answer).

Results

Table 1 shows the strategies followed by the students on the Cartesian product problems with small numbers, OPS. Eight out of the 26 students followed correct resolution strategies, the most frequent one being operations based (six students). Two students (S7 and S15) represented this strategy symbolically in the form of a horizontal algorithm, while another two (S13 and S26) expressed the multiplication verbally ("Three times four"). The last two students (S32 and S35) started by manipulating cubes and then gave the answer, one verbally and the other symbolically. In Figure 1, we can see that S32 used the orange and purple blocks to create structures of different heights, representing respectively the three T-shirts and the four pants. He then selected an orange structure and hit it against each of the purple ones, saying aloud the numbers "*one*" till "*four*". Finally, he said "*four times three*" and wrote the number 12 as the solution. S35 joined four blocks and then another three, and wrote the number "7", as we can see in Figure 1. He then corrected "*ah, but it asks you how many ways... Seven is the total*". He wrote the multiplication in the form of a vertical algorithm as the result and said, "*Twelve ways. I think 12 ways*", and he crossed out the number seven he wrote earlier.

Table 1: Strategies followed for the OPS problem

Correct strategies			Incorrect strategies (level 0)			
Direct modeling with counting (level 1)	Counting (level 2)	Operation strategies (level 3)	Inappropriate additive relationships	One-to-one combinations	Given number	Other
S10	S19	S7, S13, S15, S26, S32, S35	S3, S4, S11, S12, S16, S17, S20, S21, S24, S25, S27, S31	S8, S29	S28, S34	S14, S30

One student (S10) demonstrated a matching strategy that he expressed through drawings of all possible combinations of shirts with pants. As shown in Figure 1, he assigned a number to each shirt and pair of pants, and used the symbol "+" to express the pairing. After finishing the drawing, S10 counted the pairs obtained and provided the answer.

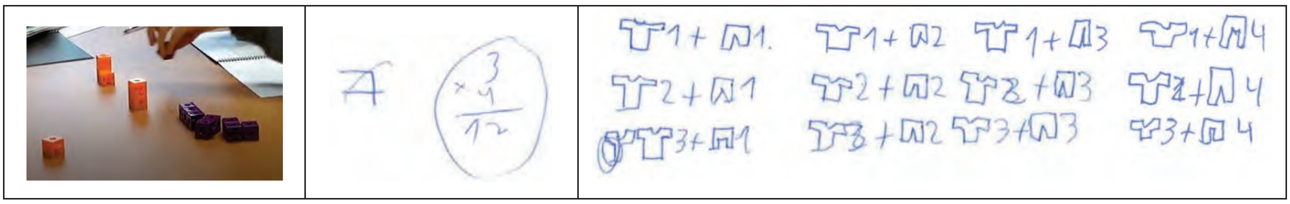

Figure 1: Examples of correct solution strategies, by S32 (left), S35 (middle), and S10 (right)

Another student (S19) used a correct counting strategy, although he made a calculation error when executing it. In particular, his strategy consisted in the repeated addition of the same number ("*four*"). He performed mental calculation to keep track of the running total, while using his fingers to represent the amount of times he had added this number. Eventually, however, he got confused and raised an additional, fourth finger, answering: "*I would say sixteen*".

The most often encountered incorrect solution strategy for the OPS problem was the application of inappropriate additive relations (12 students). In this case, the students added the quantities given in the statement instead of multiplying them, obtaining “7” as a result. Out of them, five students responded verbally: three stated this orally and two (S17 and S31) simply wrote the result without further explanation. In Figure 2, we can see that two students responded by expressing the sum symbolically, one in the form of a vertical algorithm (S16) and one in the form of a horizontal algorithm (S20). One student (S3) used drawings to help him perform the additive strategy. After reading the problem, he drew a boy wearing pants and a T-shirt, and an additional two T-shirts and three pairs of pants around it. Interestingly, one of the T-shirts resembled very much the one he was wearing at the moment, and from his remarks he was imagining these were his clothes: *“Okay, I always wear this one... ah, no, only on one day I wear this one*”. Finally, he said “*It would be one plus one equals two*”, and concluded “*seven*”, which he wrote down as the result. Another student (S4) used cubes to calculate the result, as we can see in Figure 2. He picked up three cubes with one hand and placed them on the problem sheet, and then put four more cubes, concluding that there were “*seven*” different shapes.

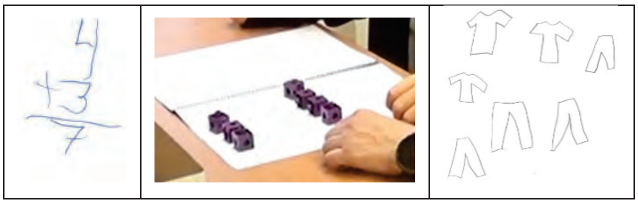

Figure 2: Examples of incorrect solution strategies, by S16 (left), S4 (middle), and S12 (right)

Three students (S11, S12 and S27) tried multiple representations to solve the problem. S11 initially took three red marbles with his left hand and four orange blocks with his right hand, said *“seven”*, and wrote the number “7”. He argued to the interviewer that this was the result by saying, “*Because I added the t-shirts [shows his right hand with four orange blocks] and also the pants and in total it would give... [starts singing, playing with the chips]*.” S12 initially answered, “*Three and four*, *seven.”* After the interviewer asked him what that “seven” was, S12 began to draw the 3 shirts and the 4 pants, as we can see in Figure 2. After the interviewer insisted “*how many ways can I dress?”* S12 repeated, “*Seven”*. S27 made arguments apparently unrelated to the task and first said that the answer was “14”, writing down “14 and 30” and finally ended up saying that it was “7”:

S27: I got it, seven.
Interviewer: And how do you know it's seven?
S27: The first one you put the shirt on, then socks and pants and lastly combing our hair and brushing our teeth. Okay? That's it.
Interviewer: So you... you count seven things that you do. But why do you know it's seven?
S27: Because I do, because three plus four is seven.

Two students (S8 and S29) performed an incorrect one-to-one combination strategy, by matching each garment from one set with one from the other set, without repetition. Both students expressed this verbally. For instance, S29 wrote “*three ways”* and argued *“because there are four pants, I can only use three because... [he thinks] because I have one pair of pants left over*”.

Two of the students (S28 and S34) responded a number already given in the statement. S28 verbally expressed that the solution was “*three*”, and argued that “*because he had heard it*”. S34 answered several times as a result “*many*”, and, after the interviewer requested that he specify how many, he said “*three or four*”, which are the number of shirts and pants given in the statement, respectively.

One student (S14) performed a strategy that could not be identified as any of the previous. After reading the problem, S14 said “*Three, four... Ouch! Let's see...*” and wrote the number “5”, and argued: “*Three, four, five*”. We interpret that he provided “5” as the answer because it was the next number in the numerical sequence. A final case of a strategy classified as “other” is that of S30, who drew a picture of a boy wearing a tracksuit, copying some letters from his own jacket. Although the interviewer insisted that he continue, S30 was tired and distracted and did not answer anything else.

All students who obtained a correct answer in the OPS problem went on to solve the large-number multiplication problem, OPL. These students, seven in total, again used a correct strategy to solve this second problem, as summarized in Table 2. Specifically, most of them used operation strategies (5 students), which three of them (S7, S15 and S32) represented symbolically in the form of a horizontal algorithm, and the other two (S13 and S26) expressed verbally. For instance, S26 read the problem and said “*I think I am going to multiply eight by three*”, and then wrote “24” as a result.

Table 2: Strategies followed for the OPL problem

Correct strategies		
Direct modeling with counting (level 1)	Counting (level 2)	Operation strategies (level 3)
S10, S35		S7, S13, S15, S26, S32

Student S10 repeated the matching strategy he had applied successfully in the OPS problem, drawing all possible combinations of shirts and pants. This time, he represented them by the letters “C” (from “*camiseta*”, in Spanish) and “P” (from “*pantalones*”) accompanied by numbers, as shown in Figure 3. When finished drawing, he counted the pairs obtained and provided the answer.

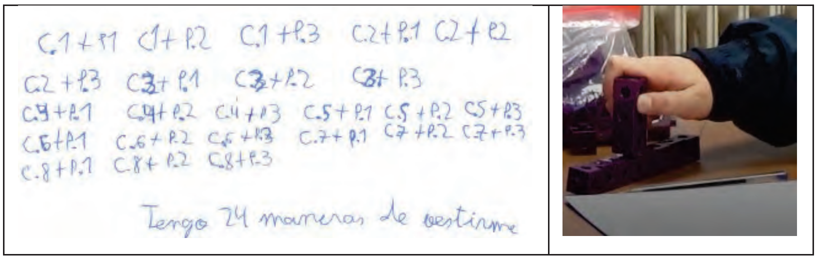

Figure 3: Examples of correct solution strategies for OPL, by S10 (left) and S35 (right)

Finally, student S35 used a modeling strategy with a manipulative type of representation making use of blocks. He first joined eight blocks, and then another three blocks, after which he combined both groups forming an inverted “T”, as we can see in Figure 3. Following this, he touched each of the blocks in the row of eight and repeated this step three times. He then said “Twenty-four”.

Interviewer:	Okay, how did you know?
S35:	By counting per pair of pants how many shirts there are.
Interviewer:	And what did you count?
S35:	Well I counted [touching the blocks in the row where there are eight]: one, two, three, four, five, six, seven, eight [counts the row again] nine, ten, eleven,...
Interviewer:	Okay okay.

Discussion and conclusion

This work contributes to the area of problem solving in students with ASD. Specifically, we have analyzed the strategies used by students with ASD when solving Cartesian product problems that involve multiplications. Most of the students failed to solve the problems correctly, and a variety of strategies were found in the analysis of their solutions. The most frequently used correct approach consisted of operation strategies based on internalized calculations. In line with the results found in the literature for students of typical development (e.g., Ivars & Fernández, 2016), the most frequent incorrect strategy was the use of additive relations, carried out on many occasions through modeling.

The results show significant difficulties in understanding the problems, confirming previous studies on problem solving in ASD students (Polo-Blanco et al., 2019), which could be related to the language difficulties characteristic of the disorder. In order to facilitate the understanding of Cartesian product problem solving, the problems could be contextualized to topics familiar to the student, in line with previous work (Polo-Blanco et al., 2021). In addition, basic modeling strategies could help the student understand the situation and the combinations posed in the problem. In order to move from modeling and counting strategies to operation strategies, it is advisable to adapt the instruction to the needs observed, and to start from the strategy used by the student. For instance, if the student uses a table to list the combinations, it may be useful to help them see that the number of combinations coincides with the result of the multiplication. In general, teaching methodologies adapted to the characteristics of ASD students should be designed for the resolution of these problems (Polo-Blanco et al., in press a), for instance, by including self-instruction lists with the support of visual guides.

The results of this work allow us to further explore the elements that hinder the learning of students with ASD, in order to offer effective instructions to achieve an improvement in academic performance and, ultimately, a greater autonomy and quality of life in adulthood.

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Make practice more productive – Introducing productive practice

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Keywords: Procedural learning, productive practice, mathematical thinking.

Repositioning the role of practice in mathematics learning

There is a growing concern about effectively carrying out practices for enhancing the students' procedural fluency (Codding et al., 2011). Nonetheless, the function of practices is often inadvertently reduced to a way of mechanical skill training. It is believed that the didactic potential of practices is far more than this. Wittmann and Müller (2017) carefully categorise three types of practice that give us a clear purpose of doing practice: Introductory practice, Basic practice, and Productive practice. Introductory practice helps students to familiarise themselves with the new knowledge; Basic practice focuses on the small set of skills which should be mastered automatically; Productive practice integrates the practice of skills with the training of higher-order skills (e.g., exploration and explanation of patterns, problem-solving) (Wittmann, 2019). These three types of practice have their own unique functions in the learning process, so they are important and irreplaceable.

This study aims on investigating how the learning environments of productive practices can be embedded into the daily lessons as a part of the curriculum for basic skills and higher-order skills training. The aim of this poster is to introduce and provide some examples of productive practice.

Productive practice

Wittmann (2019) emphasises that productive practices are mathematically rich and well-structured small tasks, which provide unique opportunities for exploring and explaining the mathematical patterns while encouraging students to have plenty of basic skill practice. Two examples of productive practices (Wittmann & Müller, 2017) are shown below:

Schöne Päckchen (Pretty Packages)

Pretty Packages (Figure 1) are deliberately arranged in columns with flexible addends that either ascend or descend. While students are having plenty of practice time on addition, they also have the chance to explore the patterns and understand the concept of particular arithmetic laws (in this case, the associative law). They can make use of what they discovered and solve the questions effectively.

12+15=?	12+15=?	12+15=?
12+16=?	13+16=?	13+14=?
12+17=?	14+17=?	14+13=?
12+18=?	15+18=?	15+12=?
:	:	:

Figure 1: Schöne Päckchen (Pretty Packages)

Number pyramid

Number pyramid is commonly used in school for practising addition and subtraction and itself is based on Pascal's triangle, which is a source of rich mathematical properties in mathematics. It can become a good example of productive practice because the numbers in the bricks can be arranged deliberately to create different circumstances for exploring. For example, set A in Figure 2 with the number in the right bottom one increases by 1 and set B in Figure 2 with the number in the middle one increases by 1 create different effects to brick A. During the process of dealing with the actual numbers and finishing a collection of additional exercises, students can observe how the pattern of the given numbers at the bottom might affect the later answers and explain it mathematically. Students are not expected to explain the pattern in terms of algebraic expression, but they can have an important pre-algebra experience.

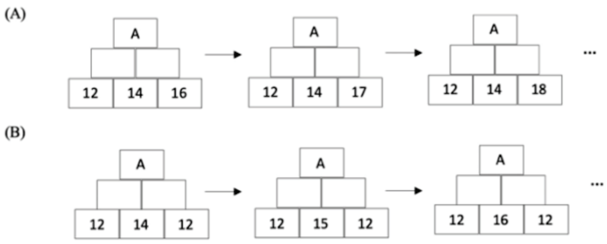

Figure 2: The bottom bricks of number pyramids are intentionally arranged

A problem-solving task with various solutions can be created by arranging the number pyramid in another way (Figure 3). While practising the arithmetic, students also have a chance to enhance their problem-solving skills through the process of observing, conjecturing, justifying, and reasoning.

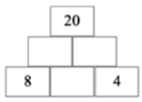

Figure 3: Productive practice which nurtures problem-solving skills

What's next?

Productive practices are not some separated tasks for specific problem-solving skills; instead, they are well-designed packages of learning environment which are fully merged with the curriculum. They can prompt deep procedural learning, meanwhile create opportunities for students to understand phenomena in a mathematical way and enhance their high order thinking skills. To evaluate and further develop the design of the learning environment with the use of productive practices, a design research study of using number pyramids in grade 2 classroom will be conducted. In the study, the lessons will be observed and some of the students will be invited for interviews afterwards; thereby analysing their mathematical thinking process while doing productive practices.

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Solving multiplicative structure problems with fractions in Primary School: Prior strategies to a teaching experiment

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In this paper, we present the study in progress that is being carried out to examine characteristics of the transition from natural to rational numbers when Primary School students solve multiplicative structure problems. A teaching experiment has been designed, including a pre-test, an instruction, and a post-test. In these pages, we focus on the first two phases of this teaching experiment and the beginning of phase three with a preliminary analysis of the pre-test. Results show the levels of success and strategies that students use when they solve multiplicative structure problems before participating in the instruction.

Keywords: Numbers, fractions, multiplication, division, elementary education.

Introduction

In the 80s and early 90s, a line of research focused on students' difficulties in solving multiplicative structure problems (e.g., De Corte et al., 1988; Fischbein et al., 1985; Levain, 1992) emerged. This research showed that although solving multiplicative structure problems involves reasoning about the structure of the problems independently of the number set involved, primary and secondary school students had difficulties solving multiplicative structure problems when natural numbers were replaced with rational numbers (e.g., Fischbein et al., 1985).

In light of these results, there was a need to provide mental strategies to students to counteract these difficulties (Fischbein et al., 1985) and present the different structure problems, systematically alternating natural and rational numbers as numerical sets (Levain, 1992). De Corte et al. (1988) analysed whether 11-12-year-old students had other informal strategies, different from the algorithm, and to what extent these led them to the correct answer. For this purpose, they designed a test with problems with natural and rational numbers, including eight multiplication problems with an asymmetrical structure. This test was solved twice: once with a multiple-choice format, which required the choice of the appropriate arithmetic operation among six options; and once with a free-response format, which required the use of any strategy that provided an answer to the problem. They found that sixth-grade students were more successful in solving multiplication problems as free-response tasks than multiple-choice tasks, where the correct strategy was reduced to the algorithm. However, as far as we know, there is no research focused on how primary school students progress from natural to rational numbers when they solve multiplicative structure problems, although students' difficulties in solving these problems with rational numbers persist (Zorrilla et al., in press).

This study examines characteristics of the transition from natural to rational numbers in primary school students when they solve multiplicative structure problems. For this purpose, we design a teaching experiment (Stylianides & Stylianides, 2013) that theoretically focuses on teaching with variation and on developing students' relational thinking during instruction. This methodological approach distinguishes three stages (Cobb & Gravemeijer, 2008; Gravemeijer & Prediger, 2019): preparation (the design of a teaching module), implementation, and analysis. In this paper, we present the design of the teaching module that consists of a pre-test, instruction and a post-test, a description of the implementation, and a preliminary analysis focused on the pre-test. With the pre-test analysis, we can answer the research questions: What are the levels of success of sixth-grade Primary School students (11-12-years-old) when they solve multiplicative structure problems with natural and rational numbers before the instruction? Furthermore, which strategies do sixth-grade Primary School students use to solve them?

Conceptual framework

Teaching with variation

Learning is related to developing a particular way of seeing (Marton & Pang, 2006): discerning. Students must discern the object of learning, which implies discerning the critical features. To discern each critical feature, students must experience variation in one of its dimensions while other features remain invariant (Marton & Pang, 2006).

Teaching with variation is an approach widely shared by Chinese teachers (Cai & Nie, 2007), being even imperceptible within Chinese culture (Sun, 2011). In contrast, in Occident, imperceptibility is not associated with its popularity but with its unfamiliarity (Sun, 2011). This teaching with "indigenous" variation is called *Bianshi* teaching in Chinese, whose translation would be "changing form" in English (Sun, 2011; 2019). *Bianshi* practice is not only an approach in mathematics education (Sun, 2011) but also it is an efficient way of working on problem-solving (Cai & Nie, 2007). *Bianshi* variation practice incorporates three types of widespread activities (e.g., Cai & Nie, 2007; Sun, 2019):

- One problem, multiple changes (OPMC). Presenting an initial problem and, once it is solved, presenting and solving variations of the initial problem.
- One problem, multiple solutions (OPMS). Presenting a problem and providing the opportunity to solve it using different strategies to promote flexible ways of thinking in choosing/designing a strategy to solve the problem.
- Multiple problems, one solution (MPOS). Using the same strategy to solve a set of problems of identical structure.

Student participation in these *Bianshi* activities promotes meaningful connections (Cai & Nie, 2007; Sun, 2011; 2013). Furthermore, *Bianshi* activity allows students to develop a confident attitude towards unfamiliar problems, advance problem-solving skills and obtain flexible thinking (Cai & Nie, 2007). The latter benefit could be closely related to developing relational thinking strategies, which we discuss below.

Relational thinking: Strategies for multiplicative structure problems

In this study, we focused on the isomorphism of measures problems (Vergnaud, 1997), whose structure is a proportion between two measure spaces (M1 and M2), each containing two quantities. In these problems, one of the quantities is reduced to 1, so three types of problems arise depending on which of the other three quantities is the unknown (Greer, 1992): multiplication, where the unknown is the total quantity; partitive division, where the unknown is the quantity per group; and quotitive division, where the unknown is the number of groups.

Relational thinking involves using the fundamental properties of operations and equality (Empson & Levi, 2011). Multiplicative structure problems involving fractions support the emergence of relational thinking about operations as students begin to find their own more efficient strategies, relating operations and quantities and representing and structuring their thinking (Empson et al., 2011).

Empson and Levi (2011) identified different strategies that students produce with multiplicative structure problems. The strategies develop from a basic way of thinking (*direct modeling* and *repeated addition*) to more sophisticated ways of thinking (*grouping and combining strategies* and *multiplicative strategies*):

- Direct modeling. Students represent all the quantities by drawing. Then, they count, add or subtract until they get the answer.
- Repeated addition. In this strategy, students also count, add or subtract; however, unlike direct modeling, they take mathematical symbols to represent mathematical relationships.
- Grouping and combining strategies. Unlike the previous strategies, students only represent quantities that they consider necessary. They start grouping quantities until they reach what Empson and Levi (2011, p. 57) call "friendlier amounts", usually natural numbers, which they then work with.
- Multiplicative strategies. Students show multiplicative thinking through the formation of groupings, but which, unlike the previous strategy, are linked multiplicatively.

The latter two strategies show a greater understanding of the relationships between quantities, as students begin to simplify their computations. Below, in Table 1, we show these strategies by solving a quotitive division problem.

Table 1: Children's strategies for a quotitive division problem

Children's strategies for multiplicative structure problems	Problem: My mother has made 2 litres of orange juice. If she has distributed it in $\frac{1}{4}$ litre glasses, how many glasses has she filled?
Direct modeling	
Repeated addition	$\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{8}{4}=2$ She has filled 8 glasses
Grouping and combining strategies	$\frac{1}{4}+\frac{1}{4}=\frac{2}{4}$ litres → 2 glasses $\frac{2}{4}+\frac{2}{4}=\frac{4}{4}$ litres → 4 glasses $1+1=2$ litres → 8 glasses
Multiplicative strategies	4 groups of $\frac{1}{4}$ is 1. There are 2 ones in 2, so there are 2 times 4 or 8 groups of $\frac{1}{4}$ in 2 so she has filled 8 glasses

Method

We have designed a teaching experiment that consists of three stages (Cobb & Gravemeijer, 2008; Gravemeijer & Prediger, 2019): preparation, implementation, and analysis. The first phase consisted of the design of a teaching module. First, we defined the learning objective: favouring the transition from natural numbers to fractions in solving multiplicative structure problems using relational thinking strategies (Empson & Levi, 2011), and the use of teaching with variation (e.g., Sun, 2011). The teaching module consisted of eight sessions of 50 minutes approximately: a pre-test (one session), the instruction (six sessions) and a post-test (one session). The instruction is based on the three types of *Bianshi* activities and the development of the different strategies introduced above:

- OPMC. Each session starts with an initial problem, and variations are made according to a feature, while other features remain unchanged. For instance, in Figure 1, the initial problem has the total quantity as unknown (multiplication problem), while the variations have as unknown the quantity per group (partitive division problem) or the number of groups (quotitive division problem). All other characteristics remain unchanged (for instance, the number used). Another example of variation is shown in Figure 2. The quantities of the initial problem are natural numbers, while the following problems vary according to the quantities considered (natural numbers, N; unit fractions, UF; non-unit proper fractions, PF; and improper fractions, IF). All other characteristics remain unchanged.
- OPMS. Students can solve the problems using different strategies (students could use the strategies described above to solve the problems from Figures 1 and 2). The instructor (one of the researchers) guides them in progressing from basic to more sophisticated strategies.
- MPOS. The problems and their variations allow students to use the same strategy as they are problems with the same structure.

The pre-test and post-test were designed to analyse changes in both the students' levels of success and the strategies used by them in solving multiplication, partitive division and quotitive division problems with natural numbers and fractions. Both tests had three problems of each typology.

In the second phase, we implemented the teaching module with three different groups of primary school students (a total of 61 6^{th} graders, 11-12 years old). The students solved the pre-test and post-test individually during the first and the last session of the teaching module. During the instruction, first, students worked in small groups, solving three problems in each session and then, the different strategies used were discussed with the whole class. Data collection is necessary to document students' reasoning' progress and their evolution during the teaching module (Cobb & Gravemeijer, 2008; Cobb et al., 2016). Therefore, all sessions were videotaped, we collected students' worksheets during all the sessions and students' discussions in small groups were also voice recorded.

Currently, we are at the beginning of phase 3, analysing the data collected. The research data are primary school students answers to the pre-test and post-test, the videos and audio transcriptions of the sessions and students' worksheets collected during the instruction. In this study, we present a preliminary analysis of the students' answers to the pre-test to explore the levels of success and the strategies that sixth-grade Primary School students' use when solving multiplicative structure problems with natural and rational numbers before participating in the instruction. This gives us information about the starting point in our instruction, and later it will allow us to identify changes along with the teaching module and the post-test to identify characteristics of the transition from natural to rational numbers in primary school students.

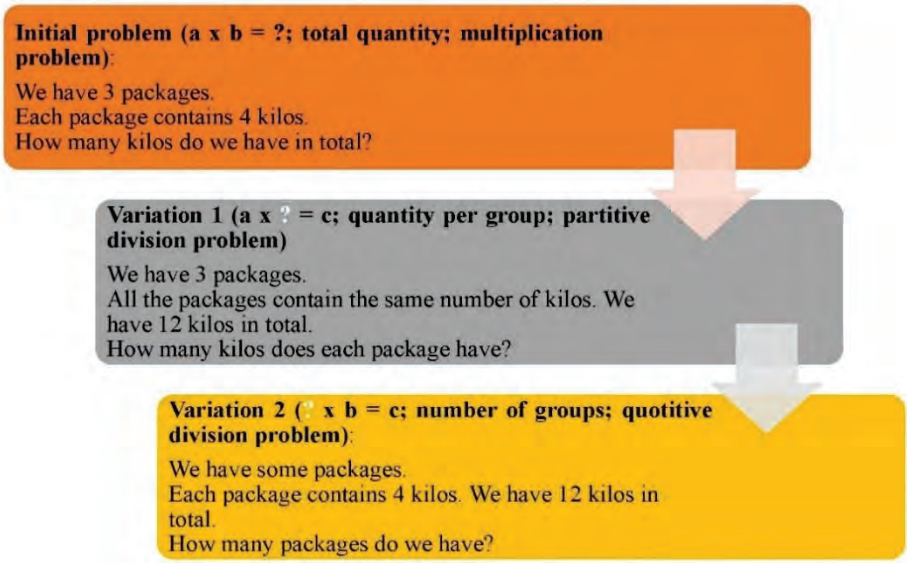

Figure 1: An example of variation: the unknown quantity as a critical feature

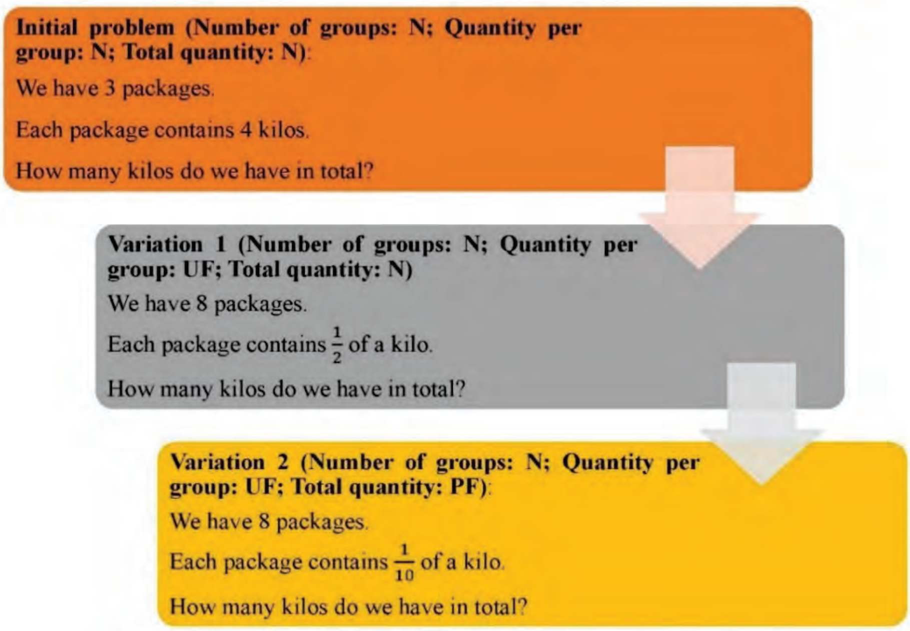

Figure 2: An example of variation: the quantity used (N, UF, PF, IF) as a critical feature

Pre-test analysis

The analysis is being carried out in two phases. In the first phase, students' success levels in each problem were analysed. In the second phase, we focused on students' strategies. To illustrate the analysis process, we will use a quotitive division problem from the pre-test. The problem is: *My mother has made 2 litres of orange juice. If she has distributed it in* $\frac{1}{4}$ *litre glasses, how many glasses has she filled?*

Phase 1. Analysis of students' success levels

Each problem is codified as "1", whether the procedure is correct (independently of computation errors), or as "0", whether the procedure is incorrect (Table 2) to obtain students' success levels.

Table 2: Example of success level analysis

Student answer	Code	Description
$\frac{1}{4} \times 2 = \frac{2}{4}$	0	The student uses a multiplication algorithm correctly but incorrectly according to the problem statement
Ha llenado 8 vasos 2 litros = 2000 ml $\frac{1}{4}$ litro = 250 ml 2000 ÷ 250 = 8 vasos	1	The student converts the number of litres to millilitres and uses the division algorithm correctly to get the number of glasses she has filled

Phase 2. Analysis of students' strategies

Secondly, the analysis focuses on the strategies used by the students. Currently, four researchers are analysing a sample of different problems to generate descriptors of the strategies used in each problem. We have begun to discuss the similarities and differences in the strategies used by the students. Below are some examples of the strategies that have emerged during this first approach to the analysis process.

In Figure 3, to solve the problem, the student (P52) graphically represents the total quantity (two litres) in two jugs which he/she divides into fourths (which is the amount of juice in a glass). To give the result, the student counts the number of $\frac{1}{4}$ he/she has drawn and answers, "she has filled 8 glasses".

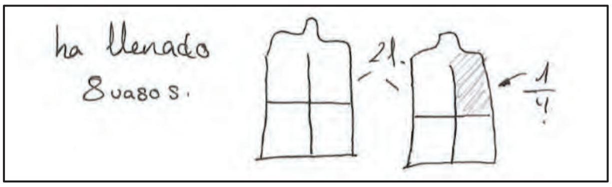

Figure 3: Strategy based on a graphical representation (P52)

In Figure 4, the student (P58) does not need to represent the quantities graphically and makes groupings. First, the student identifies that four times $\frac{1}{4}$ is a litre of juice. As the total quantity is two litres, to obtain the answer he/she doubles the number of glasses he/she fills with one litre, obtaining eight glasses. In Figure 5, the student (P12) converts two litres into 2000 millilitres and $\frac{1}{4}$ litre into 250 millilitres to divide with natural numbers.

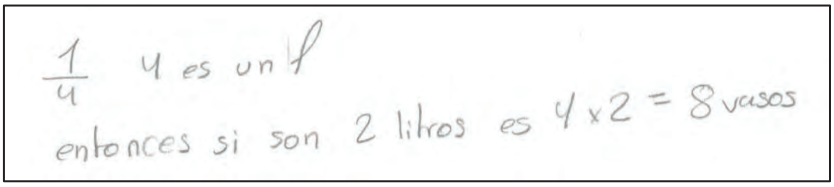

Figure 4: Strategy based on groupings (P58)

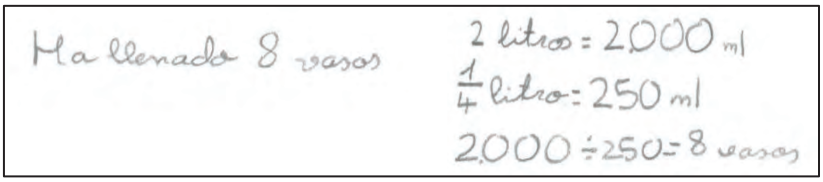

Figure 5: Strategy based on a conversion to natural numbers (P12)

First results and conclusions

We have identified different levels of success according to the numerical set used and the type of problem solved. On the one hand, concerning the numerical set, 94% of the participants correctly solved problems with natural numbers while 46% correctly solved problems with fractions. On the other hand, according to the type of problem, results of the pre-test showed that 68.3% of the participants correctly solved the multiplication problems, 52.5% the partitive division problems and 65% the quotitive division problems.

Regarding the strategies, although we are in the process of analysis, the descriptors of strategies identified are closely related to the general categories identified by Empson and Levi (2011; e.g., drawing-based strategies such as *direct modeling* or grouping-based strategies such as *grouping and combining strategies*). Furthermore, we have identified incorrect strategies which suggest the need for teachers to focus students' attention on the invariance of the structure of the problem regardless of the numerical set involved. Our results provide details regarding the participants' starting point prior to the instruction and will allow us to identify changes along the teaching module to identify characteristics of the transition from natural to rational numbers.

Acknowledgement

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TWG03: Algebraic Thinking

Algebraic Thinking

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Keywords: Algebra, early algebra, functions, research frameworks, teaching and learning algebra.

The Working Group

In CERME12, the Thematic Working Group 3, 'Algebraic Thinking', continued its work from previous CERME conferences. We had a total of 27 papers and three posters with a total of 39 people in the group. Participants represented countries from Europe and other continents: Canada, Cyprus, Denmark, Finland, France, Germany, Greece, Hungary, Italy, Norway, Spain, Sweden, The Netherlands, United Kingdom, and the United States of America. The papers loosely centred around six themes. These were: Generalisation and Pattern, Structure, Equations and Variables, Theoretical, Functional Thinking, and Algebraic Thinking. We discuss each of these themes below.

Generalisation and Pattern

Generalisation is a key aspect of algebraic thinking and there were a number of papers which looked at how students might be helped in developing their generalisation skills further. Figural patterns continue to feature as a significant research tool to explore students' generalization skills, although perhaps less common during this conference than the last. Mazza et al. looked at Grade 10 students as they considered proofs related to figurative patterns. Students were presented with 'visual proofs' and asked to justify theorems based upon those pictures. They found that there was a close match between the way in which students described the figural patterns in terms of mathematical properties and their explanations of the theorems. Goñi-Cervera et al. used a well-known growth pattern involving chairs placed around a certain number of tables. Their particular focus was students with autism spectrum disorder (AS) compared with students they describe as 'typically-developing' (TD). Although there was more success gained by TD students, it was found that the most frequent strategies for both groups were the same. Lócska and Kovács found that generalisation and reasoning strategies were supported by an intervention with 7th grade Hungarian students. This intervention used numerical tricks based around the array of numbers in a month found in calendars. Reinhardtsen and Carlsen's study involved students approaching introductory algebra through a calculational perspective. They found that teaching norms of emphasising procedures and products remained and that students could evaluate letters but struggled to use them to express generalisation and structure. Kilhamn reported a case study where computer programming was used when working with pattern generalizations. Tinkering with the code sparked students' curiosity in new ways, and when the computer did all the arithmetic, the students were free to look for pattern and structure.

Structure

A key aspect of algebra which was prominent in the previous CERME was that of structure. The continuum between operational and structural thinking was present in a number of papers. Lenz's study offered situations which included boxes with unknown or indeterminant numbers of marbles in boxes. She looked at kindergarten and elementary school children's approaches to establishing relationships within the continuum of number-orientated and structure-orientated approaches. The need to break away from concrete representation and perceive the variable as a thought object was argued. Unteregge's study involved slightly older children in Grade 4 and looked at the way they justified equalities within the result-orientated and structure-orientated continuum. It was found that students with a clear understanding of equality used different rationales across this range and could easily switch between them. Sencindiver (see Wladis et al. with Sencinder underlined) presented a paper concerning college students doing an algebra course, also using a framework of operational and structural thinking. Students were asked to justify algebraic transformations of expressions; an analysis of their attempts was conducted in terms of the students' understanding of equivalence, substitution, and substitution equivalence. Wladis et al. (paper with Wladis underlined) also used the operation and structural continuum but along with another dimension of extracted vs. stipulated to look at college students' thinking about equivalence. They found that this two-dimensional framework was useful to analyse students' definitions of equivalence. Their finding was that although students noticed 'sameness', they struggled with articulating a more standard definition of equivalence. Vlachos investigated students' understanding of what constituted a set. The sample included students from Grades 6, 9 and 12. Various prevalent misconceptions were identified. Grade 12 students did considerably better but there was no significant difference found between Grade 6 and Grade 9 students.

Equations and Variables

Dealing with equations is a standard part of the algebra curriculum. Roos and Kempen looked at the bar model as a pedagogical tool to assist with solving algebraic equations. Their study involved two cycles with low attaining Grade 10 students followed by Grade 8 students. They found that there was a need for students to develop conceptual understanding of which operations were illustrated within the model, rather than focusing on the numbers involved. López Centella et al. studied Grade 5 students to see whether they could relate a given algebraic equation to five different contextual situations. They found that the students had different forms of justification depending upon the contextual situation, and that the students were able to infer mathematical truths which had not been explicitly taught to them. Korntreff and Prediger's study involved Grade 7 and 8 students. They focused on the variable which appears within certain algebraic activities. Variables can play the role of 'generalizers' or unknowns. Their teaching experiment showed that students could construct both meanings for a variable, but only some students were able to distinguish explicitly the distinction between the two. They also found that the meaning students had of a variable was related to the algebraic activity within which the variable was used. Tondorf and Prediger studied Grade 5 students who were asked to justify the transformation from one arithmetic expression to another, both of which could be represented by the same geometric image. When making sense of the transformation of an expression, many students linked this with the geometric representation. They concluded that the dynamic transformation of a graphical representation could be used ahead of the dynamic transformation of symbolic expressions.

Theoretical

There were a number of papers which were based around theoretical arguments or developing models related to the teaching and learning of mathematics. Eriksson carried out a literature review on algebraic thinking related to students of 5-12 years old. She considered three perspectives in respect to teaching approaches which were based upon whether arithmetic thinking or algebraic thinking was developed first or whether they were developed at the same time. It was found that more research is needed on the perspective of developing algebraic thinking before arithmetic thinking. Weigand et al. came from the German tradition of *Grundvorstellungen*, which they translated as Basic Mental Models. They compared four different models for an equation and explored the relationship between these models and solving equations, particularly in relation to the use of technology. Three other papers used a similar framework to each other in their studies. Strømskag and Chevallard analysed textbooks from different countries, along with other publications from influential authors. They found that the evolution of curricula related to the notion of functions has seen a decline in algebra being taught as a modelling tool, and the reduced inclusion of parameters in algebraic equations. Hällback et al. analysed the algebra content of two Swedish upper secondary programmes and found that the programme for vocational education and training was more focused on *know-how* aspects rather than *know-why*. In contrast, the programme for higher educational preparation was more evenly balanced between these two aspects. Finally, Tonnesen constructed a praxeological reference model to develop a diagnostic test tool. This was used to examine students' technical and theoretical knowledge of basic algebra.

Functional thinking

Functional thinking is a significant aspect of algebraic thinking. Frey et al. interviewed 35 educational experts across five countries about their views of what constitutes functional thinking. In their preliminary results from two German interviewees and one interviewee from the Netherlands, it was found that there were different views across those experts. Sterner's study involved contextual growth patterns being presented to Grade 1 students and found that graphical representation, along with well-thought-through terminology, was significant in developing students' reasoning about recursive and also covariational relationships. Pittalis et al. focused more widely on algebraic thinking. They proposed and empirically validated a framework describing algebraic thinking abilities of Grade 3 students. Functional thinking, along with two other abilities, were found to form an index of those students' capacity to respond to algebraic tasks.

Algebraic thinking

Algebraic thinking is, of course, the focus of our Thematic Working Group. In that sense this is a theme which pervades all the papers. There are papers, though, where this is a more explicit focus. Radford presented his own conception of what constitutes algebraic thinking before then going on to analyse Grade 3 students' engagement in tasks presented in concrete and iconic semiotic systems. He found that the students generated two key algebraic ideas: that of 'removing' (from both sides) and 'separating' (effectively reducing the coefficient of the unknow to 1). Bräuer's study also avoided use of symbolic language by using the image of balance scales to support learning of linear equation systems, involving multiple unknowns, with Grade 3 and 4 students. It was found that the students could use algebraic strategies in an informal way without the need for an explicit intervention programme. Akinwunmi and Steinweg made a distinction between focusing on patterns and focusing on structure in their study with Grade 3 and 4 students. They argued that awareness of structure was important for students to engage in reasoning about patterns and made a case for the importance of material representations to help develop more structural arguments. Chimoni (Pitta-Pantazi et al.) presented a paper where they also made the case that seeing structure is important in early algebraic thinking. They found that students who solved arithmetical tasks using algebraic strategies were also able to solve algebraic tasks. They concluded that generalised arithmetic abilities underpin manipulation of algebraic expressions. Lastly, Fred et al. saw algebraic thinking as a resource or tool for action. They argued that explicit connections should be made between algebraic thinking and ways in which algebra plays a role in addressing challenges such as sustainability and climate change. In their analysis of two major research review books, they found that most of the 'powerful algebraic ideas' had a logical or psychological focus.

Discussions and further directions for TWG3

There was much discussion about the relationship between arithmetic thinking and algebraic thinking. Sometimes these were mentioned as a transition from one to the other. If that were to be the case, then is there a type of continuum between the two or is there an abrupt shift from one to the other? What constitutes the beginning of algebraic thinking? Alternatively, are these seem as running in parallel with each other, or are they actually more separate than sometimes imagined? For example, algebraic thinking can exist within a non-arithmetic context. We feel the link between these two is an important area for future work, perhaps considering the idea of *advanced arithmetic thinking* (which looks forward from arithmetic, rather than *pre-algebra*, which looks back from algebra). We also noted that terms, such as arithmetic thinking, functional thinking, relational thinking, and algebraic thinking were used without them always being defined. Different ways in which such terms are used can result in quite different analysis and conclusions being drawn from studies. The use of representations/models was a feature in a number of papers, and we felt that more work could be focused on how the abstraction process might involve gradual moving away from the use of representations/models. Algebra was often presented as a desired endpoint within some empirical studies, and we felt that more research could be done around the way in which algebra can be used, such as a modelling tool. We were aware of some important areas which were not represented so much in the papers and posters at this CERME. This included the use of technology. Also, there was no reference to the aesthetic aspect of algebraic activity. This is far from a dry area of mathematics and can bring many insights and Aha! moments which have an affective impact. We are aware that some papers may be presented within other TWGs in relation to both these aspects, but we feel that research around both these would be a very welcome addition to the next CERME in this thematic working group. Lastly, we noted the relative lack of discussion about the role of teacher interactions with students whilst involved with algebraic activities. Often focus was on the tasks presented to students but the way in which tasks are introduced and the nature of the teacher-student interactions which follow can be just as, or even more, significant as the tasks themselves.

Analysis of children's generalisations with a focus on patterns and with a focus on structures

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Generalisation processes belong to the core of algebraic thinking. The development of these processes begins as early as the primary school classroom, when children describe and reason about mathematical patterns. Our theoretical framework includes a distinction between patterns and structures. The focus on structures is important for children to engage in reasoning about patterns rather than just describing them on the surface. In an exploratory study with 45 third and fourth graders, we use an epistemological perspective to examine how this distinction is reflected in children's generalisations. The results create awareness of the algebraic quality of justifications.

Keywords: Early algebra, generalisation, description, explanation.

Introduction

Algebraic thinking can evolve around a wide range of mathematical topics. Blanton's research group addresses equations, generalised arithmetic, and function (Blanton et al., 2019). Twohill (2020) collapses the first two into one branch. In line with various researchers, we regard arithmetic as a starting point and interpret arithmetic tasks as potentially algebraic (Malara & Navarra, 2003; Britt & Irwin, 2008; Steinweg, 2017). This potential can emerge as early as arithmetic lessons in primary school if the underlying mathematical structure is focused on. Kieran et al. (2017) indicate structure as "one of the central pillars in the development of early algebraic thinking" (p. 423). The relation between structure and pattern, therefore, is essential and focussed on in the first part of the theory.

Structural thinking in the outlined sense expects generalisations which can be characterised as "seeing the general in the particular" (Mason & Pimm, 1984). The process of generalisation includes a certain shift of attention (Mason, 1989) from calculating given examples to thinking about the objects and their properties and relations (Mason et al, 2009). This paper provides a suggestion for a more in-depth analysis of generalisation processes and gives insight into a pilot study.

Theoretical framework

First, we provide an overview of key ideas of algebra arising from arithmetic and thereby clarify the difference we assume between mathematical structures and patterns. Taking a closer look at generalisations, this distinction leads to two categories of generalisations. Finally, the research questions of the study presented here are outlined.

Our research is embedded in a specific perspective on the relationship between *patterns* and *structures* (Steinweg et al., 2018). In line with Schifter (2018) and Mason et al. (2009) *structure* is defined by the mathematical properties and relations. In primary school arithmetic four realms of objects with specific properties can be identified and ordered in key ideas. The earlier used key ideas of algebraic thinking (Steinweg, 2017; Steinweg et al., 2018) have been fundamentally revised and enhanced based on theoretical and empirical findings. We consider properties and relations of numbers, operations, equations, and functions as key ideas (Figure 1).

Each key idea holds very specific mathematical structure to be explored, even though the realms are interwoven, e.g., equations require operations and numbers. The properties that are explored in one key idea differ from the ones that are focused on and explored in another. The focus of attention enables different generalisations of specific structural aspects. For example, working on the key idea of operations requires specially tailored tasks and questions that allow to recognise commutativity, distributivity, et cetera.

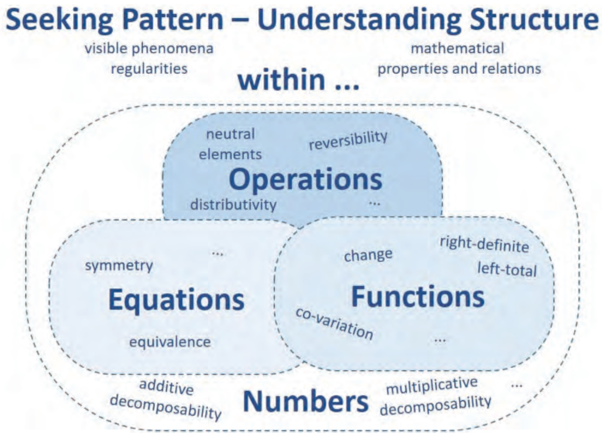

Figure 1: Key ideas of algebraic thinking

The mathematical cores of the objects associated with the key ideas are the abstract *structures*. Understanding structures and reasoning with these properties requires advanced relational or structural knowledge (Schwarzkopf et al., 2018) in the sense of profound background information. Access to mathematical structures is not easily gained and should be mediated. *Patterns* can literally function as mediators and thus as door openers to the respective mathematical background. The abstract mathematical structure can be made accessible and particularly visible in patterns and, vice versa, the mathematical structure reveals itself in patterns (Figure 2).

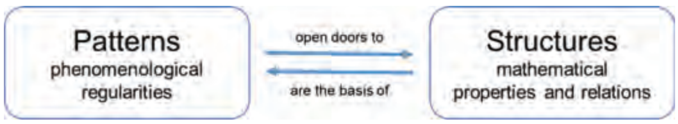

Figure 2: The distinction and relation of patterns and structures

Patterns offer perceptible regularities in visible phenomena (Schwarzkopf et al., 2018; Mulligan & Mitchelmore, 2009). Seeking for pattern is the first step to find the door behind the scenes. Pattern as representations hold information about the underlying structure (Küchemann, 2020). The visibility of pattern can be realised in various representations, i.e., tables, graphs, number sentences, and so forth.

Pattern tasks that aim for understanding structure allow for four different activities summarized as ReCoDE (Recognise-Continue-Describe-Explain) (Steinweg, 2019). Descriptions and explanations of a certain pattern require expressions, which can be verbal or symbolic. Even when the use of variables is not available or not yet developed, research documents various effective linguistic means such as referring to examples, phrasing conditional sentences, quasi-variables (Akinwunmi, 2012; Bastable & Schifter, 2008; Cooper & Warren, 2011; Schliemann et al., 2007).

Children’s reactions to given patterns may include both descriptive and explanatory parts. More importantly, *generalisations* may occur in both parts. Strømskag (2011) provides theoretical foundation for regarding algebra as generalisation of patterns. Blanton et al. (2019) differentiate between four types of thinking: generalise, represent generalisations, justify generalisations, reason with generalisations. Although this distinction enables teachers to focus on different aspects of generalisations activities, these four aspects are not disjunctive. For example, representations may support justification, or justification may include reasoning with the newly gained objects (properties, relations). Mason et al. (2009) identifies a disposition to think structural and to integrate the structural ideas into one’s mathematical thinking. Therefore, there is an opportunity to open the doors to the abstract mathematics behind a pattern, to realise structures by exploring patterns. “Generalization of a phenomenon involves the analysis of visible instances of the phenomenon, and the application of conclusions to cases that are not observable” (Twohill, 2018, p. 216). The step through the pattern door into the mathematical realm –towards mathematical thinking and away from given examples– does not emerge automatically.

In our view, *generalisations* in *descriptions* generally focus on the visible regularities, i.e., the patterns. *Explanations* provide the possibility to focus on the structure behind a pattern. However, this possibility is not automatically perceived by every child. Explanations justify certain regularities and may refer either to patterns or to structures.

Our research project addresses qualitative analysis of children’s responses, considering the outlined distinction between patterns and structures. The analysis categorises:

- *generalisations* which focus on the *pattern*, i.e., on the visible regularities, and
- *generalisations* which focus on the *structure*, i.e., on the mathematical properties or relations.

Furthermore, the *means of generalisations* in both categories are studied on. The processes of generalisation –starting from local findings and combining them into more general arguments– appear to be linguistically similar at first. Some national studies document that children –without further impulses of the teacher– typically recognize regularities only on the surface of the given patterns and justify their findings with empirical knowledge without looking beyond the pattern (Häsel-Weide, 2016; Link, 2012). However, a closer look at the two categories reveals that the objects of generalisation, and thus the awareness, differs substantially. Accordingly, the distinction between pattern and structure as two kinds of context of reference (Steinbring, 2005) may shed light on different –if any– forms of generalisations.

Finally, the in-depth analysis can be beneficial to the classroom interaction and provide adequate and helpful ideas to facilitate approaches to mathematical structures. Supporting children's development of algebraic thinking depends on deliberate attention of the teachers. The teachers need awareness of the given references in explanations and should invoke appropriate impulses and further questions.

According to this research desideratum described above, we addressed the following research questions:

- Which focus of generalisation can be identified in children’s explanations in pattern tasks?
- Which impulses do support the development of generalisations that focus on structure?

Methodology and Design

In this paper we present results from an interview study with 45 participating 3^{rd} to 4^{th} grade primary school children. The one-on-one interviews are conducted in separated rooms at the school, video-taped and transcribed. The content focuses on patterns based on the properties of operations.

In the first part of the interviews the participants are confronted with patterns in symbolic representation (e.g., left side of Figure 3; task type A). The arithmetic context is deliberately no challenge to calculate for 3^{rd} and 4^{th} graders, as the study focuses on generalisations. These kind of pattern tasks are common in German textbooks. Children are, therefore, used to the typical classroom questions to continue and describe the pattern.

In the second part, a special impulse is designed: A picture showing a child using counters in a common material (field of 20) to solve the second task of the given pattern (the right side of Figure 3 shows a simplified version of the picture which is presented to the pupils; task type B). Counters are on the table throughout the interview. The aim is that the change of representation from symbolic to iconic stimulates the children to change the focus of generalisations.

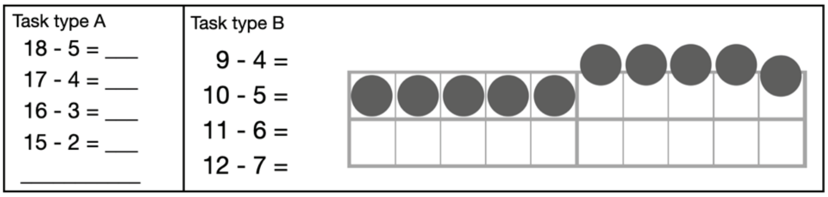

Figure 3: Task sheets on exploring the constant difference

The qualitative analysis of the data follows the interaction analysis according to Steinbring (2005). Following this method in children's interaction and verbal responses mediation processes between the given *signs/symbols* to be interpreted and the *reference contexts* that the children use for interpretation are identified. The revealed mediation in turn provides information about the children's so called epistemological basis of mathematical knowledge, i.e., the *concepts*. The triadic relationship, usually represented in a triangle (cf. Fig. 4–6), thus allows the re-construction of the child's focus in the generalisation process in order to answer the raised research question.

Results and Discussion

Due to place restrictions, only a summary of the overall findings of the qualitative analysis of all 45 cases can be given at the end of this section. The in-depth analysis is exemplarily set out in two cases, Katie and Nicole. These cases are chosen because they exemplify two substantially different arguments used in the generalisation processes.

Katie is asked to explain the pattern (task type A) and then to justify the constant result. Like all the participating children in our study, Katie generalises the symbolic patterns in the first part of the interview by describing the variation of the numbers separately.

Katie: Here, the numbers are getting smaller and here they are also getting smaller. […] For plus [addition tasks] these would have to get bigger. For minus [subtraction tasks] it is also getting smaller, because otherwise it wouldn't work.

Katie refers to the given numbers as if minuends and subtrahends were separate sequences ['here' and 'here']. In the further comment, she describes the increase and decrease for all addition or subtraction tasks in a more general term. The use of the subjunctive indicates the process of abstraction away from given (numerical) examples towards general statements. Katie gives evidence to her generalisation referring to (classroom) pattern experiences and, therefore, empirical knowledge (Schwarzkopf et al., 2018). Hence, her justification does not yet include a structural argument. Using the epistemological triangle (Steinbring, 2005), Katie's generalisation process, her interpretation of the signs mediated through a reference context, can be expressed as follows (Figure 4):

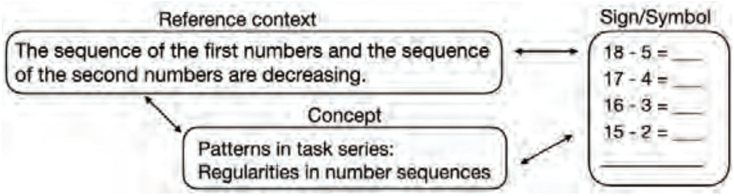

Figure 4: Katie's interpretation of the task series

Given the iconic impulse in the second part (task type B), Katie immediately decides to reconstruct the tasks using counters and lines up 9 counters in front of her.

Katie: So, 9 minus 4. 1, 2, 3, 4 (counts aloud while removing 4 counters successively) is 5. Now we have one more here (takes another counter from the material box and fills the line up to 10 counters), because now it's 10 (points to the task sheet), but in return we are allowed to take away one more (removes 5 counters). Again it's 5.
(Fills up the line to 10 counters.) Now again we take one more (adds one counter from the material box to the line) and we are allowed to take away one more. 2, 3, 4, 5, 6 (counts aloud while removing 6 counters then regarding the remaining counters). And therefore, the result is always the same.

Working with the counters, Katie's arguments change fundamentally. It is the change of representation that creates a new interpretive challenge that leads Katie to consider and generalise the effects of her actions. She now focuses on the effect of changing both numbers simultaneously and she reasons, that 'in return' to add one more (increasing the minuend), one is allowed to also take away one more (increasing the subtrahend). This expression focuses on the reversibility of addition and subtraction and the context of reference in her generalisation is, therefore, now one of the important properties of those operations. Her reasoning with structural knowledge exceeds the numerical examples and includes a valid argument for an indeterminate 'we'. She intends not only to convince the interview partner, but points to a general validity. Her conclusion ['therefore'] holds universally ['always']. The difference in arguments in her generalisation process is also visible in the epistemological triangle (Figure 5).

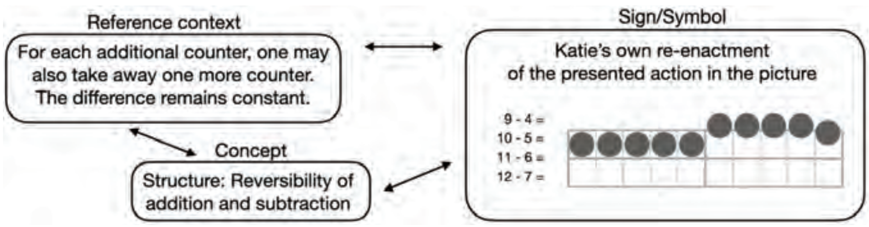

Figure 5: Katie's interpretation of her own re-enactment of the action in the picture

In the second case, Nicole's reaction in the first part of the interview is very similar to Katie's reaction and is therefore not presented here. In the second part (task type B) her reasoning is quite different from Katie’s, although she also decides to use counters to explain why the result remains constant.

Nicole: Now if you had five minus three (puts 5 blue counters in a first row and 3 red counters in a second row without one-to-one correspondence), that would be 2 and that is the starting task. And if you then remove this one here (takes away one from the blue counters) and this (takes away one from the red counters), then immediately you see, that both are getting smaller. And for minus it has to be smaller on each side (looks up thoughtfully as she speaks), so that the result stays the same.

The analysis of Nicole’s argument shows some generalisations with the focus on patterns. First, she presents a self-selected task 5 – 3 that is not part of the task sheet. She describes the regularities of the pattern in a general way using some general expressions ["both" and "each side"]. Nicole is convinced that the simultaneous decrease of the two numbers will always lead to the same results. Her generalisation refers to known empirical facts [‘it has to be’] without giving evidence, why this rule holds. In her justification, she does not focus on the properties of the operations. It can neither clearly be said, whether she interprets the space of the missing counters in the second row as the difference of the two amounts, nor does she express the effect (the shift) on the space by decreasing minuend and subtrahend simultaneously. Moreover, she does not mention the reversibility of the two operations. Instead, she refers to her existing empirical knowledge to justify the pattern, as illustrated in Figure 6. The lack of reference to the properties of the operations in the given explanation in this second case gives no discernible indication that Nicole understands the underlying structure that generates the pattern.

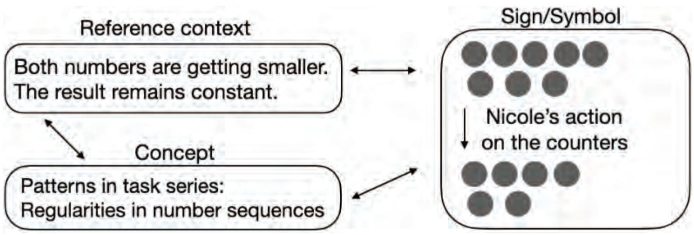

Figure 6: Nicole’s interpretation of her own action on the counters

Through such analyses as exemplified above, insights could be gained into the generalisation processes of the 45 participating children, some of which we will summarize here:

- Both foci of generalisation processes with the focus on patterns and with the focus on structure were identified in the interviews but not in equal proportions during both interview sections (with task type A and B).
- In the first encounter (task type A) none of the children focused on the properties (of the operations) that are the structural basis for the emergence of the pattern. Instead, they used generalisations of patterns by referring to empirical knowledge (classroom experiences or known facts) or they used one discovered regularity in the pattern to justify another (see Nicole’s case).

- In the second part of the interview (task type B), the iconic impulse and the counters initiated reasoning processes in which almost all children started to focus on structure, i.e., the properties of operations (see Katie's case as an example).

Closing Remarks

Working on pattern tasks is an established approach to algebraic thinking in primary school. The distinction between *patterns* as visible regularities and *structures* as abstract properties and relations that generate the patterns provides opportunities for deeper analysis of children's reactions to pattern tasks.

In our view, it is useful to distinguish between these generalisations theoretically as well as empirically. Researchers and even more teachers need to be aware of both kinds of arguments in order to provide appropriate follow-up-questions or stimuli –such as changing representations– to help children access the mathematical background behind patterns.

> Teachers who are themselves explicitly aware of structural relationships, who are aware of perceiving situations as instances of properties (rather than as surprising and unique events), are in a position to promote similar awareness in their learners (Mason et al., 2009, p. 29).

To identify children's understanding of structure is a complex endeavour. With the theoretical framework and the methodology presented in this paper we contribute a possible approach to deeper analysis of generalisations in algebraic thinking processes and provide some exemplary insights. In particular, generalisation processes with the focus on structure can be identified as references to properties of mathematical objects.

Generalisations do not necessarily imply or include structural justifications. The context of reference provides information about the concept focused on and the depth of a generalisation. The focus on patterns can pave the way to the underlying structures, i.e., the mathematical background. Sticking to the surface of patterns only and teacher's acceptance (or expectation) of solely generalisations of patterns hinders deeper understanding and ultimately algebraic argumentations.

This study also shed light on the impulses which support the development of generalisations that focus on structure. It became apparent that, paradoxically, symbolic representations of tasks entice children to remain on the surface of the pattern and stick to concrete (numerical) examples, meanwhile iconic or material-based representations provide the opportunity to find more structural arguments. Therefore, representations and processes of representing play a crucial role for generalisations with the focus on structure. However, they are not a sufficient criterion for structural arguments as can be seen in Nicole's case.

The qualitative results of our study are of course limited to first indications but also pave way for further studies. They are only a first step, and further studies on the other key aspects of algebraic thinking (numbers, equalities, and functions) or different age groups could be conducted.

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Dealing with multiple unknowns and their relationships using the balance scales model at primary school age

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The present study design focuses on primary school students' abilities in early algebraic thinking, using a task format with balance scales, which intends to provide a material-supported introduction to dealing with linear equation systems with multiple unknowns and their relationships. This offers an approach not demanding formal symbolic language, i.e. without the use of letters. In this paper, first insights from case studies on students' approaches are presented.

Keywords: Algebraic thinking, unknowns, linear equations, balance model, relational thinking.

Introduction

This article focuses on how primary school students aged 9–11 years old (grade 3 to 4) deal with linear equation systems as well as how they use and describe the contained relationships between multiple unknowns, as a core of algebraic thinking. To circumvent the well-known difficulties of learning algebra in later grades (e.g., Kieran, 2004), a task format was developed, which consists of iconic representations of balance scales and colored geometric shapes. This omits, for example, the equal sign and formal symbolic language in the form of letter variables. The aim of the study is to investigate how students use and describe relationships between multiple unknowns in linear equation systems. The ongoing study takes place before the introduction to algebra in the classroom context as well as without a previous teaching program. Therefore, the research questions are the following: *To what extent do primary school students show different approaches and strategies for dealing with multiple unknowns, as a core of algebraic thinking, in linear equation systems represented as multiple balance scales? How do the children use and describe the relationships between the unknowns represented in the tasks?*

Theoretical framework

Early algebraic thinking

So far, there is no unified definition of early algebraic thinking. Kieran, for instance, made the following definitional attempt:

> "Algebraic thinking in the early grades involves the development of ways of thinking within activities for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all, such as, analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving and predicting" (2004, p. 149).

Furthermore, to conceptualize early algebraic thinking from a multidimensional perspective, the following components of algebraic thinking in early grades can be cited: *Handling operations (as objects) and their inverses; establishing relationships between numbers, sets, and relations* *(relational thinking); generalizing; dealing with changes; dealing with unknowns; using symbolic representations* (Fritzlar, 2011, pp. 37–38).

Starting from this characterization of algebraic thinking this paper focuses primarily on the component “Dealing with unknowns”, which is particularly important and means the handling and analysis of known and unknown quantities as well as establishing relationships and structures between them (e.g., equivalence transformations). In addition to this aspect by dealing with unknown and known quantities, students have to operate on the unknowns which comes along with an analytically thinking where logical deductions are to be made, without necessarily using letters. This characterization of algebra’s analytic nature distinguishes algebraic from arithmetic thinking (e.g., Radford, 2015).

Balance scales model

The balance model is “one of the most popular didactic artefacts used to teach the solution of linear equations” (Lins, 1992, p. 208) which also underlies many studies in algebraic thinking. Previous studies have shown that even students in the early grades can solve linear equations where the unknowns can occur on both sides of the equation (e.g., Brizuela & Schliemann, 2004). To enable students to deal with linear equations and develop an understanding of equality, the balance model is often used in early grades because it “can assist students in providing a language base for solving problems'' (Otten et al., 2019, p. 2). Otten et al. (2019) found in an intervention study with fifth-graders using embodied learning environments that the balance model can promote students’ ability to solve linear equations, to further illustrate equivalent transformations using conventional algebraic equation solving strategies (restructuring, simplification, isolation by elimination and substitution) in an informal way. Regarding the fifth-grade students’ algebraic reasoning, Otten (2020) investigated different types of representation and states that algebraic reasoning was better developed by the balance model than by informal linear equations. Warren and Cooper (2005) conducted an interview study with third-grade students supported by a teaching program with the aim to investigate the extent to which students can solve tasks with an unknown using a concrete physical balance scale model. They found out that this expands the understanding of equality and equal sign to a more comprehensive relational perspective.

While previous studies are usually associated with a teaching program, the following study will forego this to be able to analyze the students’ approaches and capabilities of algebraic reasoning. Previous studies investigated just one linear equation with often only one unknown, thus focusing on just determining the unknowns or moving to a formal symbolic language rather than, as in this case here, on the process itself as well as dealing with multiple unknowns and their relationships. In relation to the component “Dealing with unknowns”, this means that relations between unknowns must be established and used and operating with the unknowns as known mathematical objects is necessary. Moreover, not only the unknown should be determined, but the relationships between the multiple unknowns are in the focus. Furthermore, it can be assumed that the presence of multiple unknowns leads to a more complex task structure, and structure of relations to be established, so that early algebraic thinking can be investigated particularly well in such situations, since arithmetic-based procedures seem less obvious. The aim of the present study is firstly to investigate how students work on tasks with two or more linear equations represented by drawn two-plate balance scales, secondly which relationships between unknowns were constructed and used, and thirdly how they deal with the unknowns (in the sense of equivalence transformation). Thereby students' reasoning regarding algebraic strategies solving systems of informal linear equations should be analyzed. In addition, first impressions and, above all, the potentials of the developed research approach, are to be highlighted to show to what extent the study design allows an investigation of early algebraic thinking at all and makes a reconstruction of this possible and worthwhile.

Methodological approach

To be able to reconstruct the approaches as well as students' description of their procedures regarding algebraic ways of thinking in as much detail as possible, an ongoing interview study of my PhD project is currently being conducted. This study focuses on the mathematics education before the introduction of linear equations and algebra in general, therefore students of different grades of primary school in Germany were selected for the thirty-to-forty-five-minute interviews.

The assignment is to work on the tasks and to explain and justify how the students proceed and think about, in which everything can be used that is on the table (paper, pens and laying material designed analogously to the tasks presented below). The "Thinking-aloud" Method is also used to try to gain insight into the students' ways of thinking, so that they are also encouraged to verbalize these ways of thinking and proceeding. This also results in the role of the interviewer, who is only to encourage, with as few inquiries as possible, that the procedure and the considerations are verbalized by the students.

To investigate the approaches for dealing with linear equation systems containing multiple unknowns in terms of algebraic thinking in primary school age, the following task design was developed. Based on the number of unknowns (in the form of geometric shapes of different colors) and the underlying task structure, different types of balance scale tasks emerged. In Figure 1, we can see an introductory task, firstly supporting students to become familiar with the task format and secondly, if necessary, to avoid misconceptions due to less existing experience with this type of scale from everyday life, and the task types. This will now be explained in more detail.

The number of unknowns per se remains constant within the four described task types A to D. The task structure can involve scales and relationships between unknowns, for both the given and the asked scale pans. They contain each either one unknown per scale pan (unmixed) or partially multiple unknowns per scale pan (mixed). Students are asked to construct and formulate a relation between two unknowns (unmixed) or a requested new relation between multiple unknowns to one unknown (mixed).

In task type A each given scale pan contains only one unknown. Students are asked – illustrated by a third scale – to construct another relation between two unknowns (unmixed–unmixed). The specifications of the given scale pans in task type B are like task type A, however the last scale asks for a relation between multiple unknowns (e.g., a triangle and a red square) on the one side to one unknown on the other side (unmixed–mixed). The task structure is changed in task type C because the given scale pans are partly each based on one or more unknowns and students are asked for a relation between two unknowns (mixed–unmixed). For task type D, the one or more given scale pans contain multiple unknowns and a relation between multiple unknowns is asked (mixed–mixed).

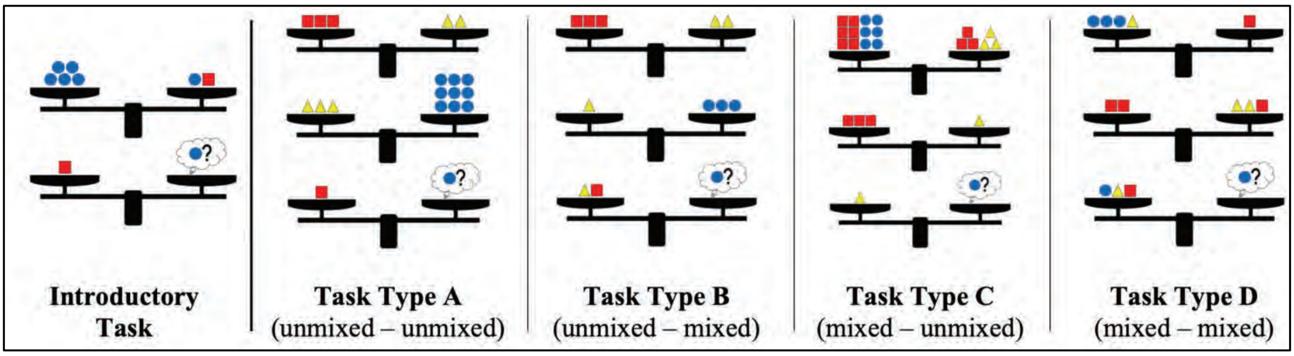

Figure 1: Overview of examples of the task types used in this study[1]

Likewise, the task types can be extended with respect to more unknowns like four (e.g., additional green pentagons). Tasks on this have also been worked on by this age group but are not considered in this paper. Within the task types, further distinctions can be made with respect to difficulty-increasing factors (e.g., whether substitution is possible), but these are not examined in this paper.

The individual interviews, consisting of tasks of all task types (see above), were audiovisually recorded, transcribed, and afterwards analyzed using the qualitative content analysis according to Kuckartz (2018). The categories for analysis were obtained inductively from this data material, as well as deductively based on theoretical considerations, for example regarding the algebraic strategies for solving systems of linear equations.

First results

Within this paper, the task types A, C and D (see tasks worked on in Figure 1) are considered, because the most different approaches can be expected here. Subsequently, to show a beginning spectrum of different approaches in terms of strategies and use of materials etc., which is supported due to the good verbalization ability of the students, I refer to the three case studies of Caspar, John (both at the end of fourth grade) and Lenny (at the end of third grade) and their answers.

Insight into the approaches for task type A and use of material

In the processing of this task example Caspar uses the material (Figure 2). He represents the second scale of the task sheet (step 1), counts the circles and then removes three circles on the right side and one triangle on the left side of the scale (step 2). He apparently recognized that three circles correspond to a triangle, which is challenging, because this relation is not explicitly given. Caspar replaces the two remaining triangles with three squares in the next step (step 3). He has thus recognized that the left side of the second scale corresponds to the right side of the first scale. Then Caspar replaces one square and two circles. Based on the now two laid squares on the one hand and

[1] These given examples of task types, related to examples with three unknowns (blue circles, red squares, and yellow triangles), will be referred to when presenting the results of the case studies.

the four laid circles on the other hand, he can determine that one square corresponds to two circles (step 4). At this point, for example, Caspar logically deduced, a characteristic of algebraic thinking.

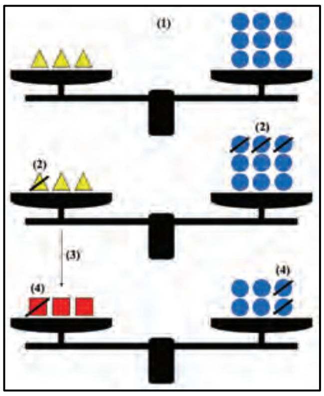

Figure 2: Caspar's use of material in example task of task type A[2]

After Caspar was able to determine the solution with the help of the material, he explained his approach as follows:

Caspar: Because (…) I took away three here [points to the circles, as indicated by the second scale in Figure 2] so that this is as heavy as two [points to the triangles (second scale)].

Interviewer: Yes.

Caspar: And then I had six here [points to the circles on the right side (second scale)] and three here [points to the triangles on the left (first scale)] and then I had taken away one here [points to the square (third scale)] and then I had taken away four here [points to the four circles on the right side (third scale)] and then one here [points to the square on the left side (third scale)]. Then I had four here and two here [points to both sides (third scale)] and then I had taken away one more [points to left side] and here again two [points to right side] and then I had one to two.

The assumption described above about Caspar's approach is confirmed by his explanations. He thus used an explicitly represented relation to replace one unknown with another unknown, making use of substitution, and constructed two non-explicitly represented relations (e.g., the two triangles he is pointing are equivalent to six circles) and used them to simplify. While Caspar proceeds step by step using the material to remove elements of the same value on both sides, John verbalizes some steps, so that the recognized and apparently used relations only become implicitly clear. The description of John's procedure shows that he clearly refers to the calculation he performs when verbalizing it. John interchanges the dividend and divisor:

John: Ah, one square is two circles, because three squares are two triangles and three triangles are nine circles and if you still, so three divided by nine is three and, if you subtract one of those three (..) so from the triangles, (…) that would actually be two circles one. Yes, two circles are one square.

Lenny solves the task and verbalizes his procedure in a highly abbreviated computational description:

Lenny: Two.

Interviewer: Explain why.

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[2] The strokes visualize the laying away of the figures using the material.

Lenny: That‘s nine and that’s three. Because nine divided by three is three [points to the second scale of the task sheet (Figure 1)]. Two times three is six [points to the first scale]. Six divided by three is two. One equals two circles [points to the third scale].

In contrast to Caspar, Lenny presumably does not simplify, but uses the non-explicit relation that one triangle corresponds to three circles in his procedure by determining how many circles correspond to the two triangles on the right scale pan of the first scale, as well as the three squares on the left and then the relation between the one square and the circles he is looking for.

Insight into the approaches for task type C

Caspar, using the relationship between the triangle and squares on the second scale, replaced the squares with triangles on the upper scale, which points to the substitution strategy. Based on the understanding of the balance of the upper scale, i.e., matching the elements on the two scales, Caspar can correctly determine that one triangle corresponds to three circles (left side of Figure 3).

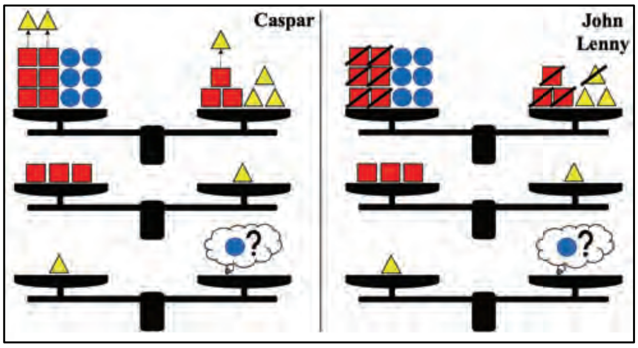

Figure 3: Approaches of Caspar, John & Lenny for the example task of task type C

John and Lenny each describe the same procedure, but in reversed order of the following steps: In one step, three squares on each side of the upper scale equalize with each other, and in another step, a triangle on the right side equalizes with three squares on the left side of the upper scale. In doing so, both John and Lenny explicitly recognize and use the relationships and choose ‘‘balancing’’ (as so named by John) as an isolation strategy, in contrast to Caspar, for completing the task, by removing elements with same value on both sides (right side of Figure 3):

John: (38 sec pause) With the one triangle we already balance the one half, that is three squares [points to three squares of the left side of the first scale]. With the other triangle with the three squares [points to the squares on the right side of the first scale] we balance the second row of squares on the left side and then there are two triangles left for a total of six circles. Therefore, the two triangles weigh exactly as much as six circles, so one triangle weighs exactly as much as three circles.

Lenny’s description also makes it clear that he is aware that the balance must meanwhile continue to exist:

Lenny: I cross those off [cross out the three squares on the right side of the first scale on the task sheet] and cross those off [cross out three of the six squares on the left side].
Interviewer: Ok.
Lenny: But they are one here [points to triangles]. If you cross out one of them [cross out a triangle on the right side], you cross out these again [cross out the three remaining squares on the left side] and now you know pretty much how many these are [points to the six circles], namely exactly as many as these [points to the two remaining triangles].

Interviewer: Ah.

Lenny: Because... namely those remain [circles the triangles on the right] and those remain [circles the circles on the left], because they must weigh the same, otherwise the scale would not be in the balance. So (..) three.

Insight into the approaches for task type D

If we look at the individual processing steps of the three children, there are no differences. The three children refer to the second scale first (Figure 1). While John and Lenny continue working in the next step from the relationship that one triangle corresponds to half a square, John names this relationship incorrectly at the beginning (see the following transcript), Caspar uses the reverse relationship, namely that one square corresponds to two triangles. Based on this consideration, the next step concerning the left side of the first scale turns out to be obviously more difficult for John and Lenny. While Lenny starts from the triangles as one half and the three circles as the other half of the square on the right side, John relates circle and triangle inversely, so that one circle corresponds to one third of the triangle. Verbalizing this relationship is especially difficult for John and the editing process is much longer than Caspar's:

John: So two triangles are half a square (..) because here are two squares [points to the left side of the second scale] and there [points to the right side] are two triangles and one square and that is balanced. (20 sec pause) And three circles are also as heavy as half a square. (8 sec pause) So a circle is as heavy as a third of a square. (17 sec pause) That's why there must be at least six circles here [points to the right side of the last scale] (5 sec pause) and in addition there are three and one, so there are a total of seven, no, ten circles here.

Caspar: I cross off this and that [points to one square each on both sides of the second scale], two yellow triangles are as heavy as one red square. (13 sec pause) They are [points to the circles of upper scale] like a yellow triangle. [8 sec pause, takes a pen and starts a tally sheet next to the last scale]. I think you need ten circles, because a square is as heavy as six circles, because three circles are as heavy as a yellow and two triangles are as heavy as a red square. [5 sec pause] So a triangle is as heavy as three circles and a blue circle is as heavy as one blue circle and that's now ten circles [points to thought bubble on task sheet].

Conclusions

It should be taken for granted that, based on the case studies presented, only initial conclusions can be drawn, which do not aim to make generalizable statements. First, it has been shown that it seems possible that the children are able to cope with the developed task format of the balance scales, as well as with the material provided with it. The solving processes can be made accessible by means of study design (also through possible independent use of material) and subsequently described in a differentiated manner. Already the few case studies could reveal that a diversity of approaches between and within the tasks can be identified, but needs to be investigated in further research, so only a first insight can be given here. This wide variety of approaches could be shown within the processing of one task, for example, Caspar making use of substitution and John and Lenny of isolation strategy in task example of task type C. The approaches and reasoning of the presented students show that they often logically deduce and use algebraic strategies in an informal way even without an intervention program – a characteristic of thinking algebraically.

Possible changes of students' approaches working on the different tasks can also be analyzed. This becomes clear in the case of Caspar, for example, who worked with and without materials, but also used different strategies for working on the linear equation systems (see above). In addition, it was shown that the children of the presented study without having passed a special teaching program, in principle, can deal with multiple unknowns, recognize, and describe the associated relationships between the unknowns. Such a handling would hardly be realizable at this age level with a formal representation in the form of linear equation systems with letter variables.

The qualitative interview study will continue, and further research will also investigate difficulty-increasing factors, which, in addition to the number of unknowns etc., may influence task difficulty, approaches and the success by solving the tasks as well as the possible transfer of these results to other task types and the implementation in the classroom context.

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Teaching algebraic thinking within early algebra – a literature review

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There is a lack of overview regarding previous empirical studies within the early algebra research field. Consequently, the aim of this study is to propose one way to organise how algebraic thinking can be operationalised when teaching students five to twelve years old. The study is conducted as a literature review. The results show six categories of operationalising algebraic thinking with these young students. These categories can briefly be organised as three traditions: (1) arithmetic thinking tradition developing arithmetic thinking first, (2) developing arithmetic and algebra at the same time, or (3) algebraic thinking tradition developing algebraic thinking first. This method of organisation highlights one tradition of algebraic thinking where more research is needed - the tradition in which algebraic thinking is developed first. This tradition, as stated in the results, includes the category algebraic work.

Keywords: Early algebra, algebraic thinking, literature review, primary school students.

Introduction

In the early algebra research field, authors attempt to explain student opportunities to explore and discern mathematical relations, patterns and arithmetical structures through processes of noting, conjecturing, generalising, representing, justifying and communicating (Kieran et al., 2016). Early algebra is then manifested using symbols other than numbers only including geometrical figures, verbal and written language and gestures (Kaput, 2008; Kieran, 2004, 2018; Kieran et al., 2016). One problem within this research field is the age of the students and which level of the school system that is referred to as "early". In the literature included in this review, early algebra can refer to the youngest students' work on structures in mathematics, the introduction of algebra in secondary school or intermediate algebra, for example, as preparation for college-level mathematics (Katz, 2007). This broad focus on the level of schooling makes it difficult to navigate this field of research. Additionally, early algebra focusing only on the youngest students has been operationalised in different ways in different empirical studies (Blanton & Kaput, 2011; Kieran, 2004). Parts of this research concerns issues about how algebraic thinking should best be introduced to the youngest students. Hodgen, Oldenburg, and Strömskag (2018) argue, in a discussion on the last twenty years of developing research in mathematics education, that there is a need of an overview of this large number of different empirical studies regarding early algebra. Thus, it is difficult to navigate in this research field. The aim of this paper is to propose a method of categorising research regarding the teaching traditions of, or for, algebraic thinking in the age group five to twelve years old. The research question guiding the literature review is: According to which different traditions can algebraic thinking be operationalised within early algebra for students five to twelve years old?

Early algebra research and ways of operationalising algebraic thinking

This section briefly highlights different traditions of operationalising student algebraic thinking.

Research on algebraic thinking within early algebra concerns student actions related to ways of doing, thinking and talking about algebra (Hodgen, Oldenburg & Strömskag, 2019). This research also includes ways of operationalising algebra in teaching (Hodgen, Oldenburg & Strömskag, 2019). Kaput (2008) suggests two core aspects of algebraic thinking that may briefly be described as: (1) algebra as generalisations and expressions of the generalisations, and (2) algebra as guided actions on symbols within conventional symbol systems. Kaput (2008) further describes three strands of an embodiment of these core aspects: algebra as the study of structures, algebra as the study of functions, relations and statements, and algebra as the application of modelling languages. Kieran (2004) has proposed that algebraic thinking is connected to three interrelated activities for teaching school algebra broadly described as: (1) generational activities, for example forming equations, (2) transformational activities, for example rule-based operations, and (3) meta-level activities, as for example problem-solving in which algebra can be used as a tool. Further, Radford (2014) describes three ways of manifesting algebraic thinking as; (1) factual algebraic thinking, when students use their daily life language, (2) contextual algebraic thinking, when the symbols and language the students use are related to the specific context or situation, and (3) symbolic algebraic thinking, when the students use formal algebraic symbols. Davydov (2008) provides a fourth way of describing algebraic thinking related to the youngest students that is theoretically grounded on the idea that algebraic thinking develops if students can work with, and reflect on, arithmetical generalisations and that the youngest students are able to carry out such generalisations. Davydov (2008) argues that the young students should be introduced to algebraic work from the very beginning of their schooling and that students need to jointly take part in the work of identifying mathematical problems, choosing tools to work with, developing models to reflect on solutions and mathematical concepts, and lastly reflecting on whether the models developed are general and will work when solving other types of mathematical problems. The suggested tools when constructing these models include most algebraic symbols and geometrical figures (Schmittau, 2003).

Concerning the youngest students in the school system, van Oers (2001) describes three different traditions of mathematics teaching: (1) arithmetic thinking first – an arithmetic tradition in which teaching focuses on operations with numerical examples, (2) arithmetic and algebra at the same time – a problem-solving tradition in which teaching focuses on arithmetic and algebra as methods for solving tasks, or (3) algebraic thinking first – a tradition of algebraic thinking in which teaching challenges the students to identify mathematical problems and focus on what tools to use when solving these problems. In this third tradition van Oers (2001) suggest algebra to be used as a tool when teaching the youngest students.

Methods

This systematic literature review was conducted using the keywords; early algebra and algebraic thinking in the *Education Resource Information Centre (ERIC)* database. Early algebra was searched on 12 March 2018 and yielded 206 articles, algebraic thinking was searched on 26 October 2018, and yielded 331 articles. Fifty-one articles were identified in both searches. One observation due to the date of the search is that any articles of a later date can be deductively organised into the categories described below.

While reading the titles and the abstracts of the 486 articles, a process of elimination was conducted in two steps. In a first reading, 274 articles were taken for further analyses including studies of students five to twelve years old. In a second reading, still based on the titles and the abstracts, 147 articles focusing on operationalising algebraic thinking in teaching were selected for further analyses. Articles concerning teachers or teacher students and articles about, for example, students with less ability in mathematics not focusing on teaching were omitted. They were omitted because these studies focused on how the participants understood algebra not on how to operationalise algebraic thinking. In total 147 articles were included in the extended analyses.

The next step in the analyses was to identify the descriptions of student opportunities to think algebraically. This was achieved by constructing thematic categories regarding operationalising algebraic thinking in early algebra according to the analysis question; Who is doing what with what tools and with what aim? (Eriksson & Eriksson, 2021). Here, the categories were inductively identified related to in which way the teaching of algebraic thinking was described in the studies. And finally, the categories found in this step of the analyses were interpreted and grouped into more overall traditions inspired by van Oers' (2001) suggestions concerning different traditions for mathematics teaching; (1) as an arithmetic thinking tradition or arithmetic first (2) as a tradition of arithmetic and algebra at the same time or (3) as an algebraic thinking tradition or algebra first.

Results

The results of the literature review are presented in Table 1. This table includes a presentation of the six categories regarding operationalising algebraic thinking within early algebra.

Table 1:

The traditions, categories, and teaching examples given in the articles

Tradition	Category	Examples of focus in teaching
1. Arithmetic thinking tradition (arithmetic first)	1.a) Algebraized elementary mathematics	47-18+18=47
	1.b) Pre-algebra	Numerical answers to unknowns
2. Algebra and arithmetic at the same time	2.a) Early algebraization	Operating with unknowns, equalities
	2.b) Arithmetico-algebraic thinking	Relationships between different tools and notations
	2.c) Emergent algebraic thinking	Geometrical patterns
3. Algebraic thinking tradition (algebra first)	3.a) Algebraic work	Structures between concepts jointly reflected using algebra, geometrical figures and language

1. Arithmetic thinking tradition

Category 1.a) is termed algebraized elementary mathematics and is related to the *arithmetic thinking* tradition. In this category, algebraic thinking is built on details manifested as arithmetic (Britt & Irwin, 2008; Lins & Kaput, 2004). Algebraic thinking could be developed by making structures visible using arithmetical examples. As one example, a statement such as 47+18-18=47 visualises that whatever is added and then subtracted entails that the original number does not change (Lins & Kaput, 2004). To summarize this category: algebraic thinking is related to the idea that algebra is about generalisations that can, or need to be, developed from arithmetical examples (Britt & Irwin, 2008; Lins & Kaput, 2004). Thus, algebra is introduced after the students have developed their arithmetic abilities.

The category 1.b) includes studies presented as pre-algebra. These studies describe teaching that aims to prepare for algebraic teaching grounded in arithmetic (e.g., Carraher & Schliemann, 2007). Arithmetic is thus seen as operations with numbers, separated from algebra that is about generalisations. The teaching of pre-algebra in these studies is proposed to be positioned after arithmetic but before teaching algebra. The students are, for example, supposed to work with (1) counting, grouping and sorting artefacts, (2) numbers and numbers of objects, (3) comparisons of quantities and values, (4) organisation of sequences and (5) sums, differences, quotas, and products of quantities and values. To summarize this category: the focus is on numerical values, thus the arithmetic aspect of mathematics.

2. Algebra and arithmetic at the same time

Category 2.a), related to the *algebra and arithmetic at the same time* tradition and is termed early algebraization (Blanton & Kaput, 2011; Kieran, 2004). Early algebraization is based on the idea that arithmetic can be more than counting and by-heart knowledge (Blanton & Kaput, 2011; Kieran, 2004). The teaching described often focuses on student opportunities to analyse relationships between quantities, identify structures, generalise, solve problems, model, argue, prove and predict (Blanton & Kaput, 2011; Kieran, 2004). The studies representing this category state that the differences between arithmetic and algebra are not completely distinct, but the differences can be presented according to what is specific for algebra, thus: a focus on relationships, not counting numerically, a focus on operations and their inverses, a focus on the process of problem-solving, not only the answer, a focus on symbols such as numbers and letters and a focus on the meaning of the equals sign (Kieran, 2004). To summarize this category: algebra is used to analyse arithmetical relationships beyond numerical answers using numerical symbols.

In the category 2.b), teaching is focused on an arithmetico-algebraic way of thinking. The teaching within these studies describes, for example, a modelling process focusing on both arithmetic and algebra (Hitt et al., 2016; Pittalis, 2018). This type of teaching is categorised by student actions related to arithmetic, visible arithmetic processes, their transformations to algebra and their inverses. Students are supposed to develop arithmetic and algebra at the same time, consequently teaching focuses on relationships between different notations, tools and student actions (Hitt et al., 2016). To summarize this category: the studies identify points of contact between arithmetic and algebra instead of describing differences.

Category 2.c) is termed emergent algebraic thinking and is suggested by, for example, Radford (2000, 2014) and Zazkis and Liljedahl (2002). Research interest concerns teaching focused on student ability to generalise and to then symbolise generalisations, thus students are allowed to express generalisations verbally using gestures and symbols without any requirement for students to note generalisations in a purely correct algebraic manner. Emergent algebraic thinking can thus be developed without using common mathematical nomenclature. The studies included are based on student opportunities to work algebraically by, for example, representing solutions to mathematical problems verbally, by using written language, drawings, symbols, models and gestures. These actions form the observable data used to analyse the algebraic thinking developed in teaching (Roth & Radford, 2011). Problems are often manifested by geometrical pattern development. To summarize this category: emergent algebraic thinking focuses on different symbols, verbal language and gestures to operationalise algebraic thinking.

3. Algebraic thinking tradition

The final category 3.a), and the only category that is related to the *algebraic thinking* tradition, is termed algebraic work in which researchers often refer to the El'konin-Davydov Curriculum (Kozulin, 2003; Schmittau, 2003; Sophian, 2002; Venenciano & Dougherty, 2014). Here, teaching the youngest students begins with measuring and comparing quantities related to lengths, volumes, areas, weights and numerical values. These quantities are often noted using algebraic symbols and geometrical length segments. This type of teaching is operationalised as collective analyses of mathematical concepts, their derivation from measurement and their representation by schematic models. Relationships between quantities are identified by the students in collective problem situations. For example, a whole class may collaborate together with a teacher to develop models that visualise how a relationship noted as $A = B + C$ can also be represented as $C = A – B$ or $B = A – B$. In order to be able to discuss such statements, the students and the teacher often use length segment models and algebraic symbols as in Figure 1.

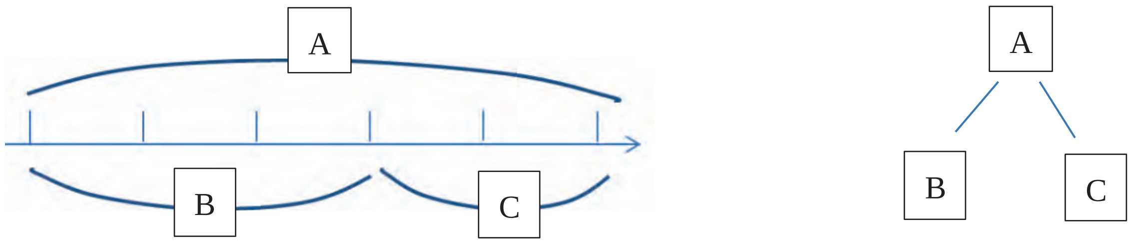

Figure 1: The relationship A = B + C as it is presented in Davydov (2008)

Here, Algebra is used as a tool to discuss general structures of mathematical concepts (van Oers, 2001). The relationships depicted in Figure 1 can be used by students as a means for reflection on the essence of mathematics as scientific knowledge of quantity and relationships. In a next step students can compare quantities that are almost the same, quantities that do not differ significantly and thus need to be measured to be compared. Students may also compare quantities of lengths to quantities of weights to discuss if they are possible to compare. A third task may be to compare quantities of, for example, volume in containers of different shapes. Students then must identify that an intermediary unit is necessary in order to compare the different quantities. Such tasks can be designed without using numerical examples. The sets of progressively more difficult problems are not organised with different content but as problems in which previous solution methods are inadequate but give guidance. Students are supposed to identify the need for new methods, tools and conceptual knowledge. Increasingly difficult and complex problems are designed for the students to solve (e.g., Schmittau, 2003). It is proposed that this algebraic way of teaching is introduced to students from about five years old. The studies categorised as algebraic work under the *algebraic thinking* tradition often focus on student agency, their opportunities to initiate and take part in discussions on mathematical content and reflect on structures and relations. Summarising this category, the students begin with algebra before arithmetic in joint activities.

Discussion

As the results above describe, the research field concerning early algebra is multifaceted in how algebraic thinking can be operationalised in teaching. Researchers in the literature review argue that it is not easy to obtain an overview of what algebra is and what arithmetic is when describing teaching in the different types of teaching traditions. This is managed here by categorising different teaching according to how algebraic thinking is operationalised. However, the same researchers argue that it is important to grasp differences and similarities between these two contents in order to better understand what is in focus and what is possible for the students to distinguish (Radford, 2000; Kieran, 2004, 2018). In this overview, these categories have been organised as; beginning with arithmetic and then introducing algebra, working with arithmetic and algebra at the same time and beginning with algebra to develop algebraic thinking as well as arithmetic thinking. Some concluding remarks are given in relation to the first and the third tradition.

In the arithmetic thinking tradition, arithmetic abilities are supposed to be developed before the students are expected to develop algebraic thinking. Within this tradition, algebra is usually introduced in middle school when the students have worked with arithmetic for a while and supposedly have developed basic arithmetic abilities. Even though these studies are included in the research field of early algebra, it is thus not the youngest students who are referred to in the research literature. This teaching tradition empowers students to communicate generalisations by using numerical examples as in, for example, pre-algebra (Carraher & Schliemann, 2007) and algebraized elementary mathematics (Lins & Kaput, 2004). According to Kaput's (2008) description of algebraic thinking, this teaching can be understood in relation to generalised arithmetic, but less in relation to syntactical, guided manipulations of symbols. Based on Kieran (2016) and Kieran et al. (2018), this kind of teaching tradition can be understood as generalisation activities despite the symbols used being numerical. However, referring to Radford's (2014) descriptions regarding manifestations of algebraic thinking, it is doubtful whether this is to be considered as algebraic thinking when the content is related to numerical values only.

In contrast to the arithmetic thinking tradition, the algebraic thinking tradition emphasis that algebraic thinking needs to be developed first in order to develop arithmetic thinking. Here algebraic thinking is operationalised in the earliest grades in elementary and primary school. Algebraic structures and relationships are worked with as a foundation for arithmetical work (Davydov, 2008; Schmittau, 2003; Sophian, 2002). The development of mathematical abilities using algebraic symbols and line segments is suggested as a means in a collective, problem-solving activity (Kozulin, 2003; Schmittau, 2003; Sophian, 2002; Venenciano & Dougherty, 2014). Teaching should focus on relationships between mathematical concepts and structures within arithmetic such as relationships between quantities (Schmittau, 2003). One important difference between this *algebraic thinking* tradition and the *arithmetic thinking* tradition is the idea that theoretical knowledge is developed by ascending from the abstract to the concrete (Davydov, 2008). In order to enable this process among the youngest students as well, algebraic symbols and algebraic ways of thinking are essential in the algebraic tradition. Comparing this way of operationalising algebraic thinking to Kaput's (2008) three strands, this can be seen as: a) algebra as the study of structure and systems abstracted from computation, b) algebra as the study of relationships and structures and c) algebra as collective modelling. Referring to Kieran (2004), the *algebra first* tradition is focusing generalisation work in which students and a teacher jointly construct equations and reflect on structures as they identify problems and explore mathematical tools.

The result of this literature review indicates that there is a lack of studies within the third tradition - starting with algebra. This gap is also identified by Coles (2021) in a discussion in Educational Studies in Mathematics, in which he states that ”[t]he manner in which symbols arise from activity within Davydov’s work potentially offers huge advantages in multi-lingual classrooms, and I would see this as a rich area of future research” (p. 475). Consequently, this review confirms Coles’ (2021) argument that more studies in this field are necessary, specifically studies that could expand our knowledge of the *algebraic thinking* tradition.

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Powerful algebraic ideas: Early algebraic thinking and active citizenship

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How can powerful algebraic ideas be understood for the combined purpose of developing students' emergent algebraic thinking and fostering future active citizens? To address this question, we have examined two major research review books on early algebra to investigate the interpretations of "powerful algebraic ideas" that are present in the books as a whole. Skovsmose and Valero's (2008) four interpretations of powerful mathematical ideas (which focus on the logical, psychological, cultural, and sociological power of mathematics) were used. We show that in the books and book chapters there is a dominance of the logical and psychological interpretations of the power of algebraic. Furthermore, the cultural and sociological interpretations appear connected to algebraic thinking as a resource or tool for action in "society". Advancing new possibilities of expanding the ways in which early algebraic thinking is made powerful for students is a challenge to research.

Keywords: Early algebra, algebraic thinking, powerful algebraic ideas, socio-political perspective.

A socio-political perspective on early algebra?

In the last decades, early algebra research has stressed the importance of students' introduction to algebraic practices in the lower grades to support students' emergent algebraic thinking (e.g. Kaput, 2008; Radford, 2014). Such an early introduction lays the ground for further algebraic thinking. Since algebra has a special position in mathematics, through its applicability in other areas and for its role in supporting general reasoning, conclusions, and proofs, an early introduction to algebraic practices and thinking is considered a foundation for realizing the intentions of the mathematics curriculum with respect to students' overall mathematical learning (Cai & Knuth, 2011). Simultaneously, interest on the socio-political dimensions of mathematics education has grown among researchers and practitioners (Gutiérrez, 2013; Planas & Valero, 2016). A socio-political approach to mathematics education considers the development of mathematical thinking and learning as an aim tightly connected with the overall societal intention of providing students with tools to become active citizens. In other words, mathematics education should offer clear opportunities to deploy mathematical thinking to consider and act on the problems and concerns of students as members of a society (e.g., Skovsmose, 1994). When bringing these two lines of research together, one could think that the aim of developing students' algebraic thinking is not only to advance the learning of further mathematics, but also more explicitly to empower children's critical reflection and democratic participation in communities of peers and in society (Hauge & Barwell, 2017). As it appears from the analysis of the basic works in our study, more often than not this combined aim is not so clearly articulated in research. Thus, bringing together students' emergent algebraic thinking and fostering future active citizens becomes a relevant challenge to advance a socio-political research work on early school algebra.

This paper has the intention of examining the way in which the aims of early algebra are expressed in research. Identifying how such aims are articulated is a first step in finding ways to connect them with the intentions of empowering students as future citizens, and in opening possibilities for new design of curricula and pedagogies that promote algebraic thinking.

We took inspiration in Skovsmose and Valero (2008), who reviewed how research in mathematics education views mathematics as being powerful and learning mathematics as empowering students and giving them access to participation in society and democracy. They reviewed research with three questions in mind: a) What is the understanding of "power" present in research? b) What is the source of the power of mathematical ideas? and c) What are the consequences of that power? As a result, they identified four distinct interpretations of the notion of "powerful mathematical ideas" that are present in the field. Mathematics and mathematics education are powerful in a *logical* sense when the focus is on the internal characteristics of mathematics. The learning of mathematics is then justified "for the sake of the internal characteristics of mathematics" (Skovsmose & Valero, 2008, p.15). Powerful mathematics education is powerful in a *psychological* sense when the focus is on the individual's possibility of acquiring mathematical knowledge. Powerful mathematical ideas are then defined in relation to the mental operations involved in the learning of mathematics, rather than the internal logics of algebra. Mathematics education is powerful in a *cultural* sense when the focus is on students' interpretations of their life possibilities in a context and reflects a situated learner's perspective, addressing students' background, but also their foreground, or how students perceive themselves as mathematicians in their future live. Mathematics education is powerful in a *sociological* sense when considering mathematics as a central tool for larger social action and organization. Powerful mathematical ideas, then, are "defined in relation to the extent to which they are used as a resource for action in society" (Skovsmose & Valero, 2008, p. 21).

In our case, these four interpretations invited us to inquire in which ways democratic access to early algebraic thinking can be considered as powerful for the purpose of connecting the development of algebraic thinking with the intention of education for an active citizenship. The question we want to discuss in this paper is: How can we understand powerful algebraic ideas for the combined purpose of developing students' emergent algebraic thinking and fostering future active citizens?

Methodology

To answer this question, we examined two major research review books in the field of early algebra (Cai & Knuth, 2011; Kieran, 2018) to investigate the interpretations of "powerful algebraic ideas" in the books as a whole. This process was done in three steps. First, an analytic tool was created (Table 1) where Skovsmose and Valero's (2008) three questions (a) What is the understanding of "power" present in research? b) What is the source of the power of mathematical ideas? c) What are the consequences of that power?) helped guiding the description of each interpretation. Second, we performed an exercise of *researching research* (Pais & Valero, 2012) on the two books to identify statements present in the texts related to each of the four interpretations. This was done by scanning for keywords associated to each interpretation, identifying excerpts that expressed these ideas, and articulating the overall sense of powerful algebraic ideas present in the books. In researching research as an analytical approach, we were not interested in pointing the particular author of an idea. Rather, we identified the regularities of what is being said about powerful algebraic ideas. Therefore, the excerpts that illustrate the interpretations are coded “C&K” and “K” plus the page number to point the source of the citations in Cai and Knuth (2011) and Kieran (2018) respectively.

Table 1: Summarised view of Skovsmose and Valero’s (2008) Powerful mathematical ideas…

Logically speaking…	**Psychologically speaking…**
Mathematics is seen as objects. The focus is on understanding mathematical concepts and doing mathematics for the sake of the internal characteristics of mathematics. The goal is to establish mathematical knowledge, its ways of working and to provide new insight into a different set of concepts. Being able to make abstractions is essential. Mathematics empowers through peoples’ enculturation in it.	Mathematics is conceived primarily as a learning process. The focus is on capturing and facilitating the developmental nature of mathematical thinking towards higher levels of abstraction and formalization. The mathematical power is situated in its developmental potentialities.
Culturally speaking…	**Sociologically speaking…**
Mathematics is seen as a tool to relate to people’s context and life conditions, for making decisions, participating in different practices, and envisioning future life possibilities. Mathematics is powerful as it allows the understanding and transformation of who learners can become.	Mathematics is seen as a descriptive and prescriptive resource and tool for action that formats society. Mathematics is powerful as it allows planning and decision making as an integrated part of technological actions. It also allows to recognize the harms that mathematics, as a resource in technological action, can create.

Powerful algebraic ideas

Table 2 shows the approximate distribution of excerpts in the texts. From this distribution, it is clear that most of the terms appear in connection with the logical and psychological focus.

Table 2: Distribution of number of excerpts

Interpretation	Keywords	Nr. of excerpts
Logically speaking...	abstraction, concept, generalization, mathematical structures, origin	More than 200
Psychologically speaking…	argumentation, mathematical thinking, mediating tools, modelling, reflection,	More than 100
Culturally speaking…	critical decisions, culturally, family, student´s background, student´s foreground	Less than 20
Sociologically speaking…	agency, empowerment, citizenship, critical thinking, critical reflection, democratic, real, society	Less than 10

In what follows we characterize the four different interpretations of powerful algebraic ideas as they appear in the two books. We acknowledge that even though we try to keep the four interpretations as distinct perspectives and emphases, they are not strict, clear-cut categories. Rather there are fine lines of distinction and similarity between them. Therefore, we tried to capture what can be foregrounded within each interpretation.

… Logically speaking

Algebraic ideas are presented in the books as powerful logically speaking when stress is placed on the features of algebra such as abstraction, mathematical structures and generalization. This dimension of powerful algebraic ideas connects to the multiple perspectives one needs to understand concepts and by linking these concepts to one another. For example, generalizing is described as powerful in relation to processes of identifying mathematical structures and relationships in mathematical situations. Seeing and describing mathematical structures and relationships are also seen as powerful in relation to constructing meaning. For example, understanding the "multiple meanings of variables and the ability to employ variables to express mathematical relationships or situations" (K, p. 144) are also expressed in terms of powerful algebraic ideas. The power of thinking algebraically is described as empowering students to analyse relationships, to notice structures, to generalize, to problem-solve, to model, to justify, to prove, and to predict (K, p. 408). Furthermore, "structure in terms of an agreed list of properties" is seen as a powerful algebraic idea when using them as axioms for deducing other properties (K, p. 287). In sum, the power in this category is strongly related to the internal logics of algebra.

… Psychologically speaking

From a psychological point of view, powerful algebraic ideas seem primarily to be defined in relation to what students can grasp and give meaning to in the learning processes of algebra. For example, symbolic notions are emphasized as tools in relation to students' thinking regarding generalizations and relationships between variables: "in particular, to recognizing the varying nature of variables" (K, p. 276). The algebraic symbols are also emphasized as powerful to describe and reason with overall mathematical ideas (C&K, p. 19). Schematizing, discovering patterns, "to imagine and to express, to specialize and to generalize, to conjecture and to convince" is also mentioned in terms of being powerful algebraic ideas (K, p. 334). This in relation to recognise "situations as instances of a class of similar situations, which constitute a person's example space" (K, p 334). "Observing examples to find regularities, noticing structure and relationships, forming conjectures about the observations, and then proving and concluding general statements" was also mentioned in terms of being powerful (K, p. 356). Mental models are mentioned as ways of thinking about abstract concepts (e.g., balance for equivalence) and/or represent abstract concepts (e.g., physical balances, balance diagrams, balance language, equations as balance). Transforming processes, the recognition of mathematical ideas and the use of mathematical strategies and analytical tools were mentioned (C&K, p. 332). Abilities such as meaningful symbolic reasoning are seen as powerful because they prepare for the abstraction of more advanced concepts and thinking in later grades. (C&K, p. 14). In contrast to powerful mathematical ideas in a logical sense, mathematical ideas are powerful because they relate primarily to students' *learning* of algebra rather than to a predominant focus on the internal characteristics of algebra.

… Culturally speaking

We interpret this dimension as emphasizing informal notions of algebraic concepts where language plays an important role. Competencies as to going from informal notions to more formal ways of mathematical thinking are emphasized (K, p. 29). Algebra is described as a cluster of modelling languages both in and out of mathematics (C&K, p. 492). The importance of tasks for students' lives is also emphasized for students to engage and participate (C&K, p. 430). Argumentation competences is attributed a crucial role (C&K, p. 469) even though algebraic thinking not always is mentioned directly; rather, activities with a special focus on argumentation. Further, is algebraic thinking highlighted as creating marginalization of students in schools and society as well as "a gateway to academic and economic success" and in that sense is algebra seen as valuable for the students. Culturally speaking powerful algebraic ideas recognize the *situated perspective* of learning algebra and the culturally loaded meaning associated to learning algebra.

… Sociologically speaking

The role of algebraic thinking is sometimes backgrounded, and sometimes it is foregrounded. Furthermore, "the use of symbols (letters) to express relationships (to model) and thereby to resolve problems" are mentioned in terms of powerful algebraic ideas (C&K, p. 561). "Real" problems are used as starting points where transforming processes, reorganizing mathematical ideas, schematizing, discovering relations and patterns, symbolizing, using analytic tools, and refining existing models are deployed as tools for students to have agency over their life situations and to critically scrutinize their environment interpreted as powerful algebraic ideas (C&K, 2011 p. 332). Students' abilities to make sense of data is mentioned, this in relation to whether and how different quantities relate to each other. In relation to this seem tables, graphs, and the use of symbols to work as tools or models to invite the students to describe and reason about mathematics ideas (C&K, 2011 p. 19). Algebraic reasoning is also highlighted, this in terms of being a powerful algebraic idea (K, p.380). In some excerpts is algebraic thinking expressed powerful in relation to work as a "fluid domain of thinking", "habit of mind" and as a particular resource to be used and integrated in every topic (C&K, p.18).

Discussion and conclusions

As described earlier on, we set us the task of discussing how to understand powerful algebraic ideas for the combined purpose of developing students' emergent algebraic thinking and fostering future active citizens. That is, we try to move early algebra education towards a practice at intersection of fostering future democratic citizens and the emergence of algebraic thinking for future mathematical learning. The main result, in screening two major research review books, is that there is a dominance of the logical and psychological interpretations of "powerful algebraic ideas". This result resonates with Skovsmose and Valero (2008) who, for the general field of mathematics education, could see a large number of publications adhering to these two interpretations of power. To keep assuming that the logical and psychological interpretations will lead to an empowerment of students may be problematic. As we know, the connection between mathematical knowledge and competence and relevant problems that people face in their lives does not happen "naturally" (Hauge & Barwell, 2017). As society gets more complex and children face the challenges of sustainability and climate change, it becomes necessary to make explicit connections between algebraic capacities/thinking and the ways in which algebra plays a role in addressing such shaky situations.

Our analysis of powerful algebraic ideas culturally and sociologically speaking can guide us in the attempt to enlarge the idea of algebraic practices where students are involved in more socially relevant issues, as for example climate change and sustainability, and where algebraic thinking becomes a resource or a tool for concrete critical thinking and action. One way of inviting students to be involved in this kind of activities is to work with models of situations that go beyond so called realistic or semi-real references for a controlled problem. For example, we can challenge students to read or model "reality" and then by raising certain questions/issues implicit make algebraic thinking becoming an analytical tool in the exploration of those models. Further, asking questions/making statements that invite the students to becoming aware of models' potential of visualizing only certain "things" and leaving other things unnoticed and invite them to reflect on what kind of consequences that may have. In this kind of work, algebraic thinking can become a language tool/resource as well as an analytical resource/tool (see also Blanton 2008). Another way of inviting students to this kind of work is in relation to technological action, where algebraic thinking can work both as a tool to create technological solutions as well as enable the students to detect that all things that are created with mathematics are not all good. Thinking in this direction, the contextualization becomes crucial, this in the establishment of an arena where algebraic thinking can operate both as a source of power but at the same time inviting students to critical examinations of mathematics itself. This implies a need to uncover the contextualization of a certain structure and at the same time creating another kind of need for algebraic thinking.

The remaining question is then, which could be new possibilities to build those kinds of arenas with the combined aim for early algebraic thinking? We believe that as a community of researchers, we could embrace the challenge of designing algebraic practices that place algebraic thinking in a larger context. In contrast to the usual work with problems and contexts tailored for algebra, we see that *wicked problems* (Block et al., 2018; Hauge & Barwell, 2017; Jurdak, 2016; Rittel & Webbers, 1973; Steffensen, 2021) can be a fruitful idea to both empowering the contextualization as well as creating another kind of need for algebraic thinking. Wicked problems can help us bridging the gap between school algebra and more socially relevant issues (e.g., Jurdak, 2016; Steffensen, 2021). Wicked problems can be described as complex problems with no definite formulation of what the problem and its solution actually is. Then the problem-formulations are vague and involve different interests and/or perspectives which encourage to negotiate disagreements that open up for different framing of the problem. Thus, if we want to go beyond providing students with tools to solve problems encountered in real life and instead invite the students to work as mathematicians handling conflicting stakes, complexity, decisions, and uncertainty, exploring algebraic activity to address wicked problem can be rewarding.

Neither of the above, however, comes without challenges. There is a risk that the students' either disregard mathematical (in this case algebraic) aspects or disregard socio-political aspects or disassociate them. Our further research intends to explore this further by imagining wicked problems in an iterative and collaborative process including pre-service teachers, teachers, researchers, and teacher educators, where we also stage these imagined wicked problems with 6–9-year-old students.

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What is *functional thinking*? Theoretical considerations and first results of an international interview study

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In this paper, we present the first results from the Erasmus+ project FunThink which focuses on enhancing functional thinking from primary to upper secondary school. In an international interview study (in Cyprus, Germany, Netherlands, Poland, and Slovakia) we investigated 35 educational experts' views on what they consider functional thinking to be. From each country between six and nine experts were interviewed. We analyze these semi-structured interviews using qualitative content analysis, with both deductive and inductive categories, related to different conceptualizations of functions, mathematization, activities supporting functional thinking, and cognitive aspects related to functions. These analyses are currently underway; therefore, we present our theoretical background, our coding scheme which is under construction, and excerpts of three interviews in this proposal.

Keywords: Functional thinking, expert interviews, empirical study.

Introduction

Functional thinking is required when relating two or more quantities, e.g., when understanding scientific laws such as the dependency between speed, distance, and time or when modelling something we read about in every newspaper such as the spread of a virus. Hence, it is not only a key element of (school) mathematics but also relevant for other disciplines and everyday situations (e.g. Selden & Selden, 1992; Vollrath, 1989). However, there is no consensus in the international literature on what exactly encompasses functional thinking and, hence, educators might also understand this notion differently, with different implications for teaching practice. This paper presents first findings of the Erasmus+ project *FunThink- Enhancing functional thinking from primary to upper secondary school*. The overarching goal of this project is to improve the teaching and learning of functional thinking across all school grades. As a basis for further steps in the project, the project members, inter alia, conducted a corresponding literature review, charted national curricular situations, and interviewed mathematics education experts[1] in order to portray their individual perspectives on functional thinking. Altogether, the interview study was conducted in five countries, yet, in this paper only interview excerpts from Germany and the Netherlands are presented regarding the question *what educational experts consider functional thinking to be*. To relate these empirical insights to relevant theoretical considerations on functional thinking, we present in the following section the corresponding theoretical background.

[1] Further partners in the interview study are Martina Geisen, Veronika Hubeňáková, Monika Krišáková, Edyta Nowińska, Marios Pittalis, and Miroslawa Sajka.

Theoretical background

Based on the concept of function which reaches back to Bernoulli (1667 – 1748, Büchter & Henn, 2010), the notion of functional thinking was introduced over 100 years ago during the reforms of Meran in 1905. At that time, functional thinking was understood as conceptual interpretation of the mathematical object of function and was considered a “guiding category for teaching mathematics in order to concentrate, unify and structure different areas of mathematics taught in schools” (Krüger, 2019, p. 35). Since then, it has developed in different ways in the international context which led to a variation in definitions. In the following, we present three main strands in the understanding of functional thinking.

First of all, functional thinking can be seen as a major component of algebraic thinking (Warren & Cooper, 2005). More precisely, Pittalis et al. (2020) describe functional thinking “as the process of building, describing, and reasoning with and about functions” (p. 632) and relate this rather broad definition to Blanton and Kaput (2011), Stephens et al. (2017), and others.

The definitions by Markworth (2012) and Smith (2008) rather focus on the aspects of representation and generalization of functional thinking. They see functional thinking as a type of

> […] representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances. (Smith, 2008, p. 143)

Besides those two strands, Cañadas et al. (2016) describe functional thinking in a general sense composed of topics, methods, and relationships concerning functions. Moreover, these authors show examples that fit into the two previously outlined strands: Functional thinking includes functional relationships between quantities, the generalization, and representation, which all support the understanding of function behavior (Blanton & Kaput, 2011). Moreover, it is linked to the ideas of change, more explicitly to qualitative and quantitative change, the relationship between changes and the ability to use these relationships for solving problems (Warren & Cooper, 2005).

These three definitions illustrate that there is no clear consensus about what functional thinking entails. Although they appear disparate, they do share the idea that functional thinking involves reasoning about the relationship between quantities. Considering that, one could ask how functional thinking can be developed by learners and how teachers can support this process. Functional thinking cannot be learned as an independent topic but has to be considered in close connection to the concept of function (cf. Vollrath, 1989). With this regard, the literature describes four perspectives on functions that play an important role when dealing with concrete function tasks or preliminary activities. These so-called *function aspects* include characteristics of functions and can form a basis for the design and implementation of tasks in mathematics education. In the international context, usually four main aspects of functions are distinguished: input-output, covariation, correspondence, and mathematical object (e.g. Doorman et al., 2012; Pittalis et al., 2020).

Function as an input-output assignment stresses the operational and computational character of the function concept; in this sense, it is not necessary to be aware of the causal relation between the in- and output (Pittalis et al., 2020). It is for example relevant when dealing with patterns and structures:

within a sequence of values, *recursive patterning* describes the existing variation and indicates how a next element can be determined if the previous element or a number of elements is provided (Stephens et al., 2017).

The aspect of covariation emphasizes the simultaneous variation of two quantities, often a dependent and an independent variable, and relates to Thompson and Carlson (2017). In their work, they offer a definition of a function with a focus on covariational reasoning:

> A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other. (p. 436)

This definition highlights the connection of two variables and their interdependency without using the terms dependent and independent variable. Similarly, Confrey and Smith (1995) describe the covariational approach as comprehending, analyzing, and manipulating the relation between two changing quantities. The change in one quantity appears if a change in the related quantity occurs.

The view on a function as a correspondence relation focuses on the relation of the independent and dependent variable and on how this relation can be represented (Smith, 2008). In more formal definitions of functions this view is expressed as ordered pairs:

> […] a *function* from a set S to a set T is a rule that assigns to each element x of set S a unique element of set T. The set S is called the *domain* of the function. If f is the name of the function, then the unique element in T corresponding to an element x in S is denoted $f(x)$ […] and is called *image* of x. The set $\{f(x) \mid x \in S\}$ is called the *range* of the function. (Yandl, 1991, p. 72)

To conclude, the fourth aspect focuses on a function as a mathematical object with its own specific representations and properties which can be dealt with. This perspective is needed to compare a function with another function or with another mathematical object. Higher-order processes like differentiation or concatenation require this view of a function (Lichti & Roth, 2019).

Different to the international context, in Germany, only three aspects are commonly discussed. The aspect of input-output assignment is omitted as a separate aspect. It is rather included in the other aspects. For example, using a function machine where something is put in, which then results in an output relates to the aspect of correspondence due to the direct assignment. Moreover, considering the covariation between inputs and outputs can help finding the underlying rule. At the same time, the input-output assignment can refer to the object aspect if the calculation does not happen within a function but with the whole function (e.g. addition of two functions) which then results in a new output. This difference in the distinction of the aspects of functions might be due to country-related particularities or the historical development as in Germany the notion of functional thinking is clearly associated to Vollrath (1989) who only distinguishes these three aspects of functions.

The set of four aspects can be considered to show an increasing level of sophistication. Studies report a gradual development from a process view which is similar to the input-output-assignment aspect to a more structural view which can be compared to the function as a mathematical object aspect (Sfard, 1991). Activities with a focus on input-output assignment are often already included in primary school (e.g., Leinhardt et al., 1990; Lichti & Roth, 2019; Pittalis et al., 2020; Stephens et al., 2017).

Studies show that young students are able to reach sophisticated ways of reasoning with functions, or algebraically, if rich tasks are provided accompanied by fitting instruction (e.g., Blanton et al., 2015; Stephens et al., 2016; Stephens et al., 2017). The implementation of the three other aspects often follows later in the curricula, whereas the object aspect appears to be the most abstract one.

As stated above, functional thinking is closely intertwined with the concept of function and cannot be considered on its own. The topic of function has been found to cause difficulties for many secondary school students (Sproesser et al., 2020). Reasons for these difficulties might be found in the abstract character of functions which makes the concept only accessible through modelling in representations and focusing on the changes between such representations (cf. Duval, 2006). Tables, algebraic expressions, graphs, and verbal descriptions are the most common representations used in school. Each of these types of representation has advantages and disadvantages depending on the specific situation and task at hand. A flexible use of representation and changes between representations, can support students' learning and understanding of functions and therefore of functional thinking (e.g. Adu-Gyamfi, 2007).

Returning to what was stated at the beginning, functional thinking is considered a key aspect in mathematics and relates to many other disciplines, and everyday life. It is present in many situations even if we are not aware of it. The second part of this paper, which describes excerpts of an international interview study, provides insight into how international educational experts see functional thinking. This is particularly important in how they frame the development of students' functional thinking. Similarities and differences to the above-mentioned definitions of functional thinking will become visible from our analysis of the interviews.

Research question and methodology

The interview study was carried out in order to collect views and experiences of educational experts on functional thinking and to get insight of which elements described in the literature are particularly relevant for them. The research question for the main study is: *what do educational experts in Cyprus, Germany, the Netherlands, Poland, and Slovakia consider functional thinking to be?* In this paper, only exemplary results from Germany and the Netherlands are presented.

Sample

Experts of mathematics education in all five partner countries (Netherlands, Poland, Cyprus, Slovakia, and Germany) were informally approached by project members to participate in this study. The interviewees ranged from professionals for mathematics education from primary to tertiary education working at universities to experienced mathematics teachers for primary and secondary schools and curriculum developers. They were chosen in order to gather views from different professional perspectives but all were considered as experts referring to functional thinking in their embeddings. Between six and nine interviews were conducted in each partner country which led to a total of 35 interviews. In this paper, only excerpts from two interviews in Germany and one interview in the Netherlands are presented. A more detailed description of these three interviewees can be found in the results and discussion section.

Procedure and interview guideline

Prior to the interviews, a semi-structured interview guideline was created to answer, inter alia, the questions of what the experts understand by functional thinking and how it can be addressed in the classroom. Further questions included what students should learn to develop functional thinking in the interviewee's opinion and what exemplary tasks could look like. Moreover, some information was gathered about the interviewees' professional background. The interviews took place virtually or in person depending on the current situation (mostly related to COVID-19 restrictions) in each country. A recording, video tape (together with corresponding transcripts) or a detailed protocol of each interview was used for the analysis. The analysis is currently still in progress. The analysis methodology we use is qualitative content analysis according to Mayring (2014). This is used for building a coding scheme with inductive and deductive categories.

Coding scheme

As our coding scheme is currently under development, we only refer to the main categories we are working with. In a first step, we code educators' ideas in the perspective on functional thinking they referred to. Here, the four aspects of considering functions (input-output, covariation, correspondence, mathematical object) play the main role. Secondly, we code how functions are used for mathematization described by educators, which can take place inside (from informal to more formal mathematics, i.e., vertical mathematization) and outside (modelling a meaningful situation with mathematical tools, i.e., horizontal mathematization) of mathematics. In a third step, we code the activities educators described which they thought could support or require functional thinking. This especially addresses patterning and dealing with representations. Finally, we code semantic and syntactic elements and concepts related to functions and functional thinking, other related fields and counterexamples. As these codes are still under construction, in the following, we only show a first sketch of the analysis of interview excerpts.

Results and discussion

The first interviewee from Germany (G1) works at the transition from university to licensed teachers (a part-time seminary, where graduated college students gain their teaching license) with a focus in mathematics education. Interviewee G1 answered the question of what he considers functional thinking to be in the following way:

> Functional thinking [..] is everything that has to do with the dependence of two quantities, of two variables. […] It is so the upper goal, the upper principle, so on the one hand the one variable has a value, that affects the value of another variable that dependents on it. It would so rather be the static side, so the allocation, then also the change, if one variable changes, what consequences does it have for the other variable. Yes, and the third would be so basically the course that you can conclude, the overall picture of the dependency. […] It already goes in the direction of the idea of using mental representations of mathematical concepts (*Grundvorstellungen*), but above these basic ideas stands the consciousness of dependence and everything that is around it or what is subordinate, the calculating that must actually, that leans on this principle. […]

The description of functional thinking by interviewee G1 is rather broad and highlights the dependency of two variables. Concerning his perspective on functions, the aspects of covariation, correspondence, and mathematical object are clearly mentioned and described as basic ideas. According to the interviewee, everything that follows, like calculations, can be derived from these principles. Due to the prominence of the aspects according to Vollrath (1989) in Germany, it is not surprising that the aspect of input-output assignment is not mentioned.

Another German interviewee (G2), a teacher from primary school (Grade 1-4), answered the same question. The interviewee is a longtime teacher who initially studied education for primary and lower secondary school with a focus in general studies, German, and mathematics. Besides the degree in education, the interviewee also has a postgraduate degree in pedagogy.

> […] what do they actually want with that in elementary school? […] it's about relationships for me in functional thinking, so not just functions according to the motto of a value is assigned to another value, but about relationships, about the discovery of relationships, and then again a bit of the science lesson plays into it for me, which then says laws of nature, you can make discoveries, you can observe them, you can explore them, you can measure something, math comes into it again […].

Interviewee G2 is rather general in her definition of functional thinking. G2 sees functional thinking in a wider sense than just related to functions. The focus is on relationships and connection to real life. The elements of discovery, observing, exploring, and measuring of relationships show this close connection to the real world. G2's description of functions indicates the aspect of correspondence. Later in the interview, as example, she mentions collections of tasks with continuous elements where students can recognize patterns (*starke Päckchen*) as an activity for addressing functional thinking which includes elements of the input-output assignment. In general, G2 seems less aware of the aspects of functions and functional thinking. In contrast to G1, G2 only mentions some aspects and does not address them explicitly.

An interviewee from the Netherlands (N1) has been a teacher for 16 years, mainly in the upper primary school grades (Grade 5 and 6). When asked about her definition of functional thinking, she mentioned "that must be about relating mathematics to a context and its utility." This is related to our code on horizontal mathematization (modelling extra-mathematical situations with mathematical tools), which is rather well established in the Netherlands, due to the implementation of realistic mathematics education. When prompted by the interviewer that functions could also be interpreted in a more mathematical sense, she referred to patterning tasks in the early grades of primary school, doing rows of calculations and observing what remains fixed and what changes, graphing activities, and summarized all these as "reasoning about relations." In this she clearly related to the covariational view of functional thinking while describing useful activities for eliciting it. Interestingly, she connected this reasoning about relations also to an attitude that students should develop in society, seeing relations, experimenting, encountering obstacles, and systematically try to deal with them.

These first excerpts indicate a clear difference in views between experts. Functional thinking is mostly understood in a way that is somewhat similar to one of or a mixture of the definitions mentioned in the theoretical background. Yet, the descriptions provided by the interviewees are less detailed and some lack a complete description of all aspects of functions and functional thinking. The detailed analysis which is to follow will provide more insights, from all the partner countries, into the extant conceptualizations of functional thinking in practice.

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Strategies exhibited by students with autism spectrum disorder and their typically-developing peers when solving a generalization task: an exploratory study

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This paper focuses on algebraic learning and explores the strategies followed by 17 students with Autism Spectrum Disorder and 17 typically-developing students to solve a generalization task in a functional thinking context. We design and administer a questionnaire with different questions about specific (consecutive and non-consecutive) cases and the general case. Success in the proposed task was higher in the group of typically-developing students. Different strategies were encountered, predominantly (1) modeling with drawing and (2) operations, with neither group having much success obtaining the general term. We discuss the implications for teaching students with autism.

Keywords: Autism spectrum disorder, functional thinking, generalization, primary education, strategies.

Introduction

In the context of mathematics education in Spain, the official primary school curriculum establishes the requirement for students to successfully solve situations involving both numbers and their relationships (BOE, 2014). Along these lines, the early algebra presented is intended to introduce algebraic thinking in levels before secondary school (Blanton et al., 2015).

Some studies on functional thinking have focused on studying the strategies shown by students when solving generalization tasks that involve functional relationships. For example, Blanton and Kaput (2004) noted differences between the various strategies used depending on the age of the participants. In Early Childhood Education, students used counting and addition strategies, while some fifth graders managed to establish multiplicative patterns through words and symbols. Cañadas and Fuentes (2015) also studied the strategies shown when solving a functional relationship and concluded that the six- and seven-year-old students participating in their study responded by using strategies such as counting with drawings, direct answer, associating elements in groups and others.

Although the official curriculum in Spain is standardized and intends for all students to achieve the same goals by the end of the primary school, the reality is that classrooms are increasingly heterogeneous in terms of their students' characteristics. This diversity of students includes those with Autism Spectrum Disorder (ASD), which is a developmental neurobiological disorder that manifests itself during the first years of life and lasts throughout the entire life cycle. Its main symptoms are: (a) persistent deficits in social interaction and communication and (b) restrictive and repetitive patterns of behavior, interests or activities (APA, 2013). In addition, people with ASD may exhibit resistance to change, a tendency to maintain routines, deficits in executive functions, and difficulties in understanding spoken and figurative language, and in inferring the mental states of other people (Ozonoff & Schetter, 2007).

Students with ASD are increasingly enrolled in mainstream schools alongside their typically-developing (TD) peers (Barnett & Cleary, 2019). Some of the deficits of the disorder, such as those in executive functions or verbal comprehension, may interfere with learning, particularly in learning mathematical concepts (e.g., Chen et al., 2019; Polo-Blanco et al., 2019; in press). Specifically, deficits in abstract reasoning may limit the ability to generalize (Minshew & Goldstein, 2002; Ozonoff & Schetter). Although there are studies that analyze the understanding of pre-algebraic functional tasks by TD students (Cañadas & Fuentes, 2015; Morales et al., 2018), there are few analogues for ASD students (Barnett & Cleary, 2019; Goñi-Cervera et al., 2021; Polo-Blanco et al., under review).

Considering the above, we expect that students with ASD will rely often on basic strategies like the use of modeling with manipulatives when solving a task in a functional context, and that they will show difficulties when generalizing the functional relationship.

Objectives

This study is part of a larger one whose main objective is to describe the mathematical abilities of students with ASD and their relationship with cognitive variables, such as executive functions. This work focuses on algebraic learning and explores the strategies employed by TD and ASD students to solve a generalization task in a functional thinking context in order to observe similarities and differences between the two groups of students in terms of the strategy use, and to describe possible difficulties in the ASD group.

Methodology

This research of an exploratory nature (Yin, 2017) relied on cases and controls, and compared the solution strategies of students with ASD matched with their TD controls when solving a task that involves the functional relationship $f(x) = 2x + 2$.

Participants

The participants were 34 students in grades 1 through 4 (6 to 9 years old) from 12 schools in Cantabria (Spain). Of these, 17 had ASD diagnosis and 17 were TD at the time of the study. The students were paired: each ASD with each TD control from the same school and grade as his/her ASD student pair. An inclusion criterion for participants from both groups was to have IQ equal to or greater than 70 as measured by WISC-V. Ten of the participants were enrolled in 1st grade (A1, A2, A3, A4, A5, and their respective TD pairs T1, T2, T3, T4 and T5), two (A6 and T6) in 2nd grade, six (A7, A8, A9, T7, T8, T9) in 3rd grade and 16 (A10, A11, A12, A13, A14, A15, A16, A17, T10, T11, T12, T13, T14, T15, T16 and T17) in 4th grade. The arithmetic mean and standard deviation of the IQs for the ASD group were 87.35 and 10.22, and 103 and 13.12 for the TD group.

The participants with ASD were recruited in different ways, by advertising the project through social media, the press, associations, guidance staff in schools and hospitals. Once an ASD participant was recruited, a TD student from the same school and class to act as a control was sought via family members or the school's guidance staff.

Information gathering tool

A task used in Carraher et al. (2008) and Merino et al. (2013) whose structure involves the function $f(x) = 2x + 2$ was adapted and implemented. The adaptation consisted of using simple language, helping the students read the statement and guiding their work. The task began by presenting, as an example, a square table with four people around it and two square tables together with six people (Figure 1). They were then asked how many people could be seated if 3, 4, and 5 tables (consecutive terms) were joined, if 8, 18, and 100 tables (non-consecutive terms) were joined, and if any number of tables (general term) were joined. The task was given to the students in printed form so they could solve it individually in writing, or both in writing and orally, and they had manipulatives (blocks) that they could use if they wanted.

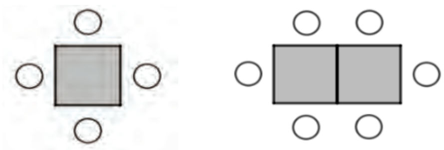


Figure 1: Introduction of the task with people arranged at one and two tables

The students answered the task individually in a classroom free of distractions, with only the interviewer present. Prior to this task, the students had solved three other tasks with the same interviewer in a previous session. First, the interviewer created a climate of trust with the students, letting them play with the blocks or drawing. If the student was not comfortable, the session was postponed. The solution process was videotaped and transcribed for later analysis. Both the written and oral responses were analyzed.

Analysis categories

Based on the strategies defined by Morales et al. (2018), the following categories of answers were established: (a) no answer: if the student does not provide a response, either verbal or written, or does not know the answer; (b) direct answer: if the student provides an answer that is difficult to justify or has no apparent relation to the task; (c) given number: if the student provides as an answer the number given in the statement; (d) modeling with manipulatives and counting: if the student models the situation using the available manipulatives; (e) modeling with drawing and counting: if the student models the situation using drawings; (f) counting: if the student performs the same actions as in modeling, but without using manipulatives or drawings; and (g) operations: if the student performs additive or multiplicative calculations orally or in writing.

Results

Next, and to simplify, we show the frequency of the strategies used by the participants in those questions that involve consecutive, non-consecutive and general terms (see Table 1). The results show how often a certain strategy is used by ASD and TD students. The numbers in parentheses indicate the number of right answers. The column "students" shows the number of students who used this strategy at least once.

Table 1: Frequency of Strategies and Success in Solving the Task f(x) = 2x + 2

	Autism spectrum disorder students				Typically-developing students			
	Case							
Strategy	Consecutive	Non-consecutive	General	Students	Consecutive	Non-consecutive	General	Students
No answer	0(0)	8(0)	4(0)	5	0(0)	8(0)	2(0)	4
Direct answer	9(2)	7(1)	6(0)	10	0(0)	1(0)	0(0)	1
Given number	3(0)	3(0)	1(0)	1	0(0)	2(0)	0(0)	2
Modeling with manipulatives and counting	0(0)	0(0)	0(0)	0	6(3)	4(1)	2(0)	4
Modeling with drawing and counting	15(12)	7(2)	2(0)	6	21(17)	14(7)	3(0)	9
Counting without manipulatives	3(3)	2(0)	0(0)	1	0(0)	0(0)	1(0)	1
Operations	21(0)	24(0)	4(0)	11	24(6)	22(5)	9(1)	14
Total number of strategies	51 (17)	51 (3)	17 (0)		51 (26)	51 (13)	17 (1)	

In what follows, the strategies are exemplified, focusing on the two most frequent in both groups. We also show some examples concerning other less frequent strategies.

As Table 1 shows, the *operations* strategy was the most frequent, in both groups of participants with ASD (49 occasions among all the sections) and the TD students (55 occasions among all the sections). However, no student with ASD used this strategy successfully, compared to three 4th-grade TD students (T10, T12, and T15) who correctly answered twelve times.

The students with ASD who used the *operations* strategy chose, for the most part, additive operations. For example, A1 exhibited the strategy by using an incorrect additive operation, answering for 100 tables: "200. Because 100 + 100 is 200". Student A8 also used an incorrect additive strategy, answering all the sections of the task by adding six to the number of tables given. Among the ASD students, only A13 resorted to multiplication (for example, for 18 tables, he multiplied 10 by 18 and for the general term, answered: "you have to multiply it by 10"). Two other students with ASD used multiplicative reasoning, although without explicitly stating the multiplicative operation. For example, A15 answered for the general term, "I counted by twos", and A11 used expressions such as "four times four" or "you add ten times four".

In turn, 14 TD students used the *operations* strategy. They proposed correct and incorrect additive operations. For example, T12 responded correctly to the consecutive terms, looking at his previous answer and adding two. On the contrary, T1 used an incorrect strategy, because when asked "How many people can be seated if 100 tables are joined?" he described being unable to draw the 100 tables and referred to how "3 + 2" people sit at each table. A transcript of the conversation between the interviewer and the student (Figure 2) is provided below:

Interviewer: What if we have 100 tables?

Student T1: Well… two and three. Two and three and two and three and two and three

Interviewer: How would we put that?

Student T1: We have to put: three, two, three, two, three, two, three, two… in all of them. […]

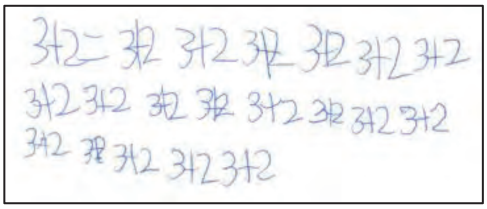
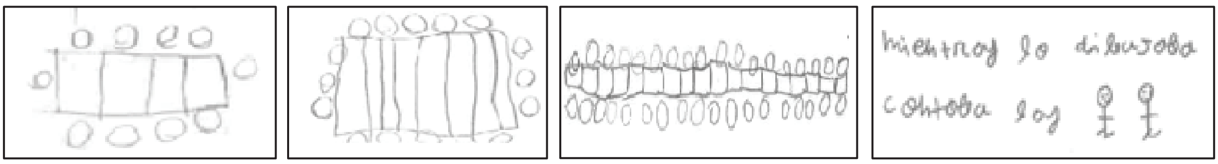


Figure 2: Operations Strategy for 100 Tables (T1)

Among the 14 TD students who used *operations*, 10 of them used some multiplication throughout all the sections, and of these, 7 used multiplications exclusively. Thus, T10 used a multiplicative operations strategy by reasoning "you have to multiply the number of tables by 2 and then add 2 for those at the ends", when asked for the general term.

The *modeling with drawing and counting* strategy was the second most frequent strategy, both for ASD (24 times) and TD (38 times) students. In addition, this strategy was the one that led to the most correct answers, especially in the consecutive cases.

Figure 3 (a and b) shows the solution for four and eight tables given by student A7. The first solution is correct, while the second is incorrect, as it places more guests than there should be at the ends. Figure 3 (c and d) shows the solution for 18 tables and the general term for student T7. The solution for 18 tables is incorrect, as he forgot to draw the people at the ends. The solution of the general term shows that T7 determined the number of people by drawing and counting them.



While drawing, I counted the

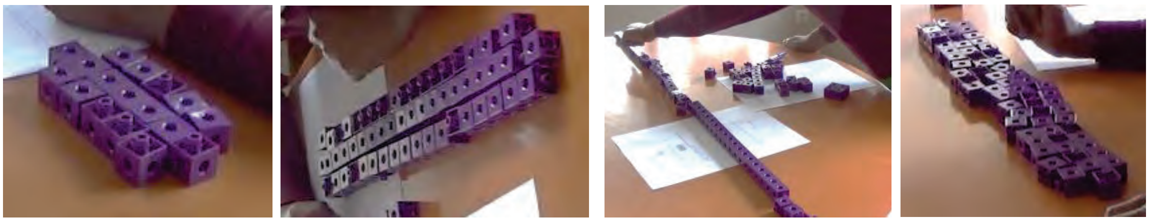
(a) (b) (c) (d)

Figure 3: Modeling Strategy with Drawing

Other less frequent strategies

The remaining strategies were less frequent in the two groups of students. For example, on up to 12 occasions, four TD students used the *modeling with manipulatives and counting* strategy, which was not used by any student with ASD. Student T16 used the blocks made available during the task, modeling the situation for the different numbers of tables. The modeling was correct for 3, 4, 5, 8 and 18 tables (Figure 4, a and b), and incomplete due to insufficient blocks for 100 tables (Figure 4, c).

In addition, T16 also used blocks, particularizing to represent "any number of tables" and respond to the general term (Figure 4, d). He said: “counting these pieces... The middle thing was the tables and those who are here next door were the people”.



(a) (b) (c) (d)

Figure 4: Modeling with manipulatives strategy for 5, 18, 100 tables and general term (T16)

Another infrequent strategy in ASD students was *counting without manipulatives*. Two students (A1 and T4) used a counting strategy for some questions. Thus, to determine how many people could be seated if 4 tables were joined, A1 swayed to the right and left, and each time his body tilted to one side, he counted one. Each time his body titled to the middle he said “a gap here” meaning it was a table and no one sits in a table. He said: “one, a gap here, two, a gap here, three, a gap here…". Student T4 also employed a counting strategy by responding to the general term by writing "using the tables and counting them".

Regarding the *direct answer* strategy, it was only used by one TD student and for just one question, while 13 students with ASD gave these answers on 22 occasions. After being unable to draw 100 tables, T4 replied: “A thousand… I don't know. Or 50 or so... A thousand".

In the group of students with ASD, an example of the *given number* strategy was provided by student A12, who did not justify his answer to the problem in any section. However, his answers show a relationship p = t, where p is the number of people and t the number of tables. His answers for the terms t = 3, 4, 5, 8, 18, and 100 were 3, 4, 5, 8, 18, and 100, respectively. In addition, for the general term he wrote the number zero. This could be related to the absence of a specific number of tables or people in the statement. Students T6 and T16 also responded with the same number of tables to the question that involved 100 tables.

Conclusions

With this exploratory study, we are contributing to the start of an investigation comparing algebraic thinking in both TD and ASD students in primary education. Success in the proposed task was higher in the TD group, although it was accessible to some of the students with ASD.

In both groups of students, the predominant strategies were *modeling with drawing and counting* and *operations*. The task increased in difficulty as the number of tables rose, with the question involving the general term proving very difficult. Contrary to what we expected, *modeling with manipulatives* was not used by any student with ASD. One reason that could explain this is that the material provided (blocks) was not adequate to represent the situation. Given the type of literal thinking common in ASD students (Happé, 1995), they may have had difficulty imagining that the blocks represented the tables and people around them. However, they did not show difficulties in implementing the modeling strategy through drawings. The frequent use of a *direct answer* strategy among students with ASD and given that in most cases it did not lead them to the correct answer, could be associated with a poor understanding of the task. The preference in ASD students for following additive strategies agrees with previous studies carried out with TD students and enrolled in lower grades (Blanton and Kaput, 2004). In addition, the choice of strategies based on drawings agrees with previous research involving 5- and 6-year-old TD students (Cañadas & Fuentes, 2015) and it is in line with other works about problem solving by ASD students (Polo-Blanco et al., 2019).

As a future line of research, the sample could be expanded to delve into possible differences within the group with ASD, and to see if subgroups of normal-performing and low-performing students are identified, in line with previous work (Chen et al. 2019). The aim of this study is to enhance the research on mathematical learning in students with ASD. In particular, this study serves as an aid to those teachers who work with ASD students enrolled in mainstream classrooms who follow the official curriculum, alongside their TD peers.

Acknowledgments

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A praxeological comparison of algebra content in vocational and academic preparatory programmes in Sweden

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We compare the algebra content in the Swedish upper secondary syllabi for higher educational preparatory (HEP) and vocational educational and training (VET) programmes. The study is theoretically embedded in the Anthropological Theory of the Didactic (ATD), where praxeology is used as an analytical tool. The results reveal that the algebra content in the VET programmes is more focused on praxis ('know-how') aspects compared to the HEP programmes, in which the balance between praxis and logos ('know-why') aspects is more even. We discuss the results in view of students' opportunities to develop algebraic knowledge within the framework of the given syllabi.

Keywords: Algebra, ATD, praxeology, syllabus.

In 2011, Swedish upper secondary school underwent a major reform, with the implementation of new curriculum documents and school organisational structures (Swedish National Agency for Education, 2011). An essential consequence of this was a stronger separation between vocational educational and training (VET) and higher educational preparatory (HEP) programmes (Lindberg & Grevholm, 2013), which has raised questions regarding social justice and purposes of education (Nylund et al., 2017). As part of the reform, three alignments in mathematics were introduced: for vocational, social science-oriented, and natural science-oriented programmes. Before 2011, all students, regardless of programme, took the same first course in mathematics. In this paper, we look more closely at the mathematical content in the syllabi for the three alignments in the 2011 curriculum, particularly the algebra content in the first two courses for each alignment.[1]

Algebra is often referred to as a gatekeeper, not only to more advanced studies in mathematics and science (Blanton et al., 2015) but also to participating in society and gaining full access to civil rights (Moses & Cobb, 2001). At the same time, algebra appears to be a problematic topic for students in several countries (Hemmi et al., 2021). One reason for this may be that algebra has traditionally not been introduced until secondary school, creating a gap between arithmetic and algebra (Linchevski & Herscovics, 1996). However, in recent decades, research has repeatedly shown that students benefit from being gradually introduced to algebra already from the earliest grades (Blanton et al., 2015). These findings have slowly made their way into the educational system (Hemmi et al., 2021).

In Sweden, algebra has been a difficult topic for students to manage for many years (Hemmi et al., 2021). In the international evaluation TIMSS (Trends in International Mathematics and Science Study), Swedish students' results in algebra have been below international average since the 1960s (Bråting, 2021). Research has revealed that generalised arithmetic is virtually absent in the last three Swedish curricula for compulsory school (Bråting, 2021), and Swedish students have trouble understanding the different roles of variables (Kilhamn, 2014) as well as using the relational property

[1] Since the analyses conducted in this paper, small revisions have been made to the syllabi. The analyses in this paper are based on the syllabi that were valid until 30 June 2021.

of the equal sign (Madej, 2021). However, this research, like most of the recent Swedish research on algebra learning, was conducted at compulsory school level. Research on algebra learning at upper secondary level in Sweden is lacking, particularly since the 2011 reform. One exception is Gustafsson's (2019) thesis on upper secondary students' difficulties with algebra, revealing that students still struggle to understand the different roles of variables and the invisible multiplication sign in expressions such as 4x (see also Hewitt, 2012).

This study is part of a larger project exploring socioeconomic aspects of the didactic transposition of algebra in Sweden. In the present study, the focus is on the transposition of algebra from scholarly produced knowledge to the educational system (Chevallard, 2006). Hence, the study is theoretically embedded in the Anthropological Theory of the Didactic (ATD), where praxeology is used as an analytical tool. The aim is to unpack differences in how algebraic content is formulated in the syllabi for VET and HEP programmes, and to discuss how these differences may affect students' opportunities to learn algebra. In forthcoming studies, the focus will shift towards aspects regarding social justice and purposes of education. We pose the following research question: *From a praxeological perspective, what characterises the algebra content in syllabi for VET and HEP programmes and what are the main differences?*

A praxeological perspective on school algebra

The Anthropological Theory of the Didactic (ATD) is particularly useful for studying teaching content, such as school algebra, from an institutional perspective. It acknowledges that humans and human activity are involved in the teaching and learning process, that syllabi do not appear ex nihilo, and that the process is an "exogeneous production" (Bosch & Gascón, 2006: p. 54). In other words, the process is "something generated outside school that is moved […] to school out of a social need of education and diffusion" (ibid.). Within the ATD, this process is expressed in terms of the *didactic transposition*, which describes how (mathematical) knowledge is transposed between different institutions (Figure 1). The outcomes of the teaching and learning process depend on the humans involved in it – starting with scholars developing and determining the content at one end, and ending with the students' learned knowledge on the other (Bosch & Gascón, 2006). From this perspective, it is thus possible as a researcher to study a content from an unbiased position. In this study, we are interested in investigating the transposition of algebra from 'Scholarly knowledge' to 'Knowledge to be taught' (Figure 1). This corresponds to the algebra content developed by professional mathematicians, selected by the educational system, and transferred into so-called school algebra (Bråting, 2021; Hemmi et al., 2021; Kilhamn, 2014).

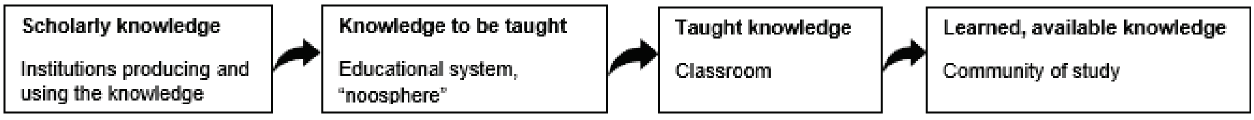

Figure 1: The didactic transposition from Bosch and Gascón (2006, p. 56)

To describe mathematical (indeed, any human) activity, ATD employs the notion of *praxeology* (Bosch & Gascón, 2006). Any praxeology is divided into two blocks: praxis ('know-how') and logos ('know-why'). The praxis block thus contains the practical part and consists of types of *tasks*, (mathematical) tasks or exercises to be done, and *techniques*, the method(s) connected to the type of task. The second block, logos, consisting of *technology* and *theory*, refers to human thinking and reasoning and is the 'explaining' part of the praxeology (Chevallard, 2006) (see Figure 2). The suffix -*logy* in technology indicates that this is a discourse on a given technique. This discourse is expected to justify the technique as a valid way of not only solving a particular type of task but also of clarifying the logic behind it. Theory can therefore be explained as an overarching theory or a set of underlying principles that justifies the technology (e.g. Chevallard, 2006).

PRAXIS	TASK
	TECHNIQUE
LOGOS	TECHNOLOGY
	THEORY

Figure 2: A model of the praxeology concept (Chevallard, 2006)

School algebra has been studied from an ATD perspective for many years. Summarising findings from such research, Bosch (2015) describes a situation in which a formal approach to algebra predominates, to the detriment of a functional view. In today's secondary schools, Bosch claims, algebra is largely identified with equation solving, and 'the language of algebra' is reduced to a formal structure whereby students are asked to manipulate algebraic expressions with little regard for what they represent. In contrast to this, she outlines a view of school algebra building on ATD principles, where it is instead interpreted as "a *process of algebraization* of already existing mathematical praxeologies" (ibid., p. 61, emph. in original). In other words, instead of being yet another piece of mathematical content, algebra appears as a general tool for modelling any school mathematical praxeology.

Method, material, and procedure

In this study, we have conducted a praxeological analysis of the syllabi for the first two mathematics courses in the national curriculum for upper secondary level in Sweden, Lgy11[2]. We view curriculum documents as indicative of the transposition from scholarly knowledge to knowledge to be taught (Figure 1); through this analysis we have been able to discern nuances in the written language and unpack implicit meanings behind the formulations in the syllabi, thus contributing knowledge of the transposition processes behind the algebra content in the upper secondary curriculum.

The first two mathematics courses in Lgy11 each have three alignments: a, b, and c, aimed at vocational (VET), social sciences-oriented (HEP), and natural sciences-oriented (HEP) programmes, respectively. In vocational programmes, Course 1a is compulsory and 2a is optional. In social sciences-oriented programmes, 1b and 2b are compulsory and 3b is optional (compulsory in economy). In natural sciences-oriented programmes, 1c, 2c, and 3c are generally compulsory, while 4 and 5 are optional (4 is compulsory in some programmes). The subject plan starts with an introduction to the subject and a formulation of aims. The syllabus for each course (1a, 1b, 1c etc.) consists of core content and knowledge requirements based on mathematical competencies. In this study, the sole focus of the analysis has been on the core content. This is organised in categories

[2] In the rest of the paper, we will use the abbreviation Lgy11 (Läroplan för gymnasieskolan [Syllabus for the upper secondary school], Swedish National Agency for Education, 2012).

written in bullet points, which constitute our unit of analysis. Before beginning the analysis, we extracted the algebra content from the core content, using Blanton et al.'s (2015) so-called big ideas of algebra consisting of I) variables, II) equivalence, expressions, equations, and inequalities, III) generalised arithmetic, IV) proportional reasoning, and V) functional thinking. The units of analysis containing algebra content were then classified in praxeological terms. Here it should be pointed out that Swedish school syllabi contain no explicitly formulated tasks, meaning that this part of the praxis block did not appear in the analysis.

In the classification, we made use of what we have called 'determining words'; that is, words that carry information on how the core content can be interpreted praxeologically. For instance, the core content in Course 1a regarding problem solving is formulated: "Strategies for mathematical problem solving including the use of digital media and tools" (Swedish National Agency for Education, 2012, p. 5). The word 'strategies' indicates an emphasis on different methods, which we interpret as *technique* (praxis). As another example, in the core content of Course 3b in the category 'Algebra', we find: "The concepts of polynomial and rational expressions, and generalisation of the laws of arithmetic for dealing with these concepts" (ibid., p. 27). The term 'concept' suggests that the teaching situation should offer a wider perspective on polynomials and rational expressions, i.e. to learn *why* they work. In learning a mathematical concept, definitions and underlying principles are conceivably offered, which we interpret as *theory* (logos). Contrastingly, the term 'generalisation' suggests drawing on established rules to justify new algebraic rules, and we have interpreted this as indicative of *technology* (logos).

Following the classification, the pieces of algebra content were sorted into tables with one column for each alignment. To help identify patterns in the content of the different courses and to recognise differences and similarities between the alignments, as a last step the praxeological 'determining words' were picked out and placed in a new summarising table. The analysis was done by the first author, but was discussed and reviewed by all authors throughout the process.

Results

We begin by presenting the results of the analysis of algebra content in all three alignments in the core content categories "Understanding of numbers, arithmetic, and algebra" and "Relationships and change". In order to be as transparent as possible, we provide the tables used in our analysis for Course 1 in both categories. Due to lack of space, the tables connected to Course 2 are not shown. In the following tables, green, purple, and yellow mark technique, technology and theory, respectively.

Table 1: The category 'Understanding of numbers, arithmetic, and algebra' in Course 1

Ma1a (VET)	Ma1b (HEP, social sciences)	Ma1c (HEP, natural sciences)
Handling algebraic expressions and formulae relevant in subjects typical of a programme, […]	Handling algebraic expressions and formulae relevant to subjects typical of programmes.	Generalisation of the rules of arithmetic to handle algebraic expressions.
	The concept of linear inequality.	The concept of linear inequality.
[…] as well as methods for solving linear equations.	Algebraic and graphical methods for solving linear equations and inequalities and exponential equations.	Algebraic and graphical methods for solving linear equations and inequalities and exponential equations.

As Table 1 shows, in the first category in Course 1 alignment *a* merely contains determining words indicating a focus on technique (praxis), whereas alignment *b* also contains wordings indicative of theory (logos). Words interpreted as indicative of technology (logos) were only found in alignment *c*. Furthermore, it is notable that the term 'generalised arithmetic' (technology) only occurs in alignment *c* (top right in Table 1) and that inequalities are only referred to in the HEP alignments.

Overall, Course 2 contains more algebraic topics in this category than Course 1 does. Some of the content focusing on praxis aspects is exactly the same in all three alignments, for instance certain methods for solving equations. However, the way linear equations are described differs between the alignments: While alignment *a* contains the phrase "use of linear equations in problem solving situations", in alignments *b* and *c* it is formulated as "the concept of linear equations". A similar distinction appears regarding logarithms: While alignment *c* introduces the *concept* of logarithms (theory), in alignment *b* the concept of logarithms is connected with *solving* exponential equations (technology). In alignment *a*, logarithms are not included at all.

Table 2: The category 'Relationships and change' in Course 1

Ma1a	Ma1b	Ma1c
The concepts of ratio and proportionality in reasoning, calculations, measurements, and constructions.		
Differences between linear and exponential processes.	The concept of a function, domain, and range of a definition, and also properties of linear functions and exponential functions.	The concept of a function, domain, and range of a definition, and also properties of linear functions and exponential functions.
	Representations of functions, e.g. in the form of words, shapes, functional expressions, tables, and graphs.	Representations of functions in the form of words, functional expressions, tables, and graphs.
	Differences between the concepts of equation, algebraic expression, and function.	Differences between the concepts of equation, algebraic expression, and function.

In the category "Relationships and change", wordings suggest a greater emphasis on logos compared to the previous one. This holds for both Courses 1 and 2, and for all three alignments. In Course 1, phrases indicative of technique (praxis) are only found in alignment *a* (in connection with ratio and proportionality; see top left in Table 2), while technology appears in all three alignments, either in the form of concepts in connection to specific techniques (in alignment *a*) or specific aspects of concepts, such as properties of or representations of functions (alignments *b* and *c*). Furthermore, in Course 1, a focus on theory can only be found in alignments *b* and *c*, in connection with the function concept. However, here it is important to emphasise that in alignment *a*, functions are not introduced until Course 2. The syllabus for Course 2a describes the function concept in almost the same way as in alignments *a* and *b*, except that "applications of functions" are added in Course 2a, indicating an emphasis on praxis. Finally, it is worth mentioning that in Course 2 the content regarding graph construction is phrased identically in all three alignments.

In Table 3 we have compiled all determining words found in the syllabi, and ordered them according to whether they are indicative of technique, technology, or theory. As the analysis revealed only small differences between the *b* and *c* alignments, in the table we have merged the two into the same column. In conclusion, Table 3 indicates that the praxeological organisation of the algebra content of the first two courses in the VET and HEP alignments differs in focus. The VET praxeology has a strong emphasis on praxis, while logos, in particular theory, is largely absent. The HEP praxeology, on the other hand, is more even in its emphases, with a fairly strong theoretical component.

Table 3: A compilation of the determining words that indicate technique, technology, or theory

	Ma1a	**Ma2a**	**Ma1bc**	**Ma2bc**
Technique	Methods Handling Strategies Calculations Solution of Construction Measurements	Methods Handling Strategies Use of Solving Construction Applications	Methods Handling Solving	Methods Handling Solving Construction Applying
Technology	 Reasoning Differences	Representations Reasoning Properties	Representations Properties	 Reasoning Properties
Theory	The concept	The concept Differences between concepts	The concept Differences between concepts Motivation Generalisation	The concept Differences between concepts Motivation Extension

Discussion

In response to our research question, through a praxeological analysis of the mathematics syllabi for the first two courses in the Swedish national curriculum for upper secondary school, we have concluded that the praxeological organisation of the algebra content in the VET alignment emphasises praxis (techniques), whereas the praxeological organisation in the HEP alignments is more evenly balanced between the praxis and logos blocks. However, although theory and technology (logos) are emphasised more in the HEP alignments, they are typically not connected to particular techniques. Indeed, connections between techniques and technologies are rarely explicit in the syllabi. The content regarding ratio and proportionality in Course 1a, generalisation of arithmetic laws in Course 1c, and complex numbers in Courses 2b and 2c are the only pieces of core content with formulations of technologies supporting techniques in the same core content. This lack of connection between technique and technology, as well as the limited role of theory in alignment *a*, makes it difficult to view the praxeologies outlined by the syllabi as complete, even allowing for the lack of explicit task formulations.

One of our main findings was that the praxeological organisation of school algebra in VET programmes syllabi emphasises several techniques, with little or no technology or theory justifying them. This resonates with the formal and technical approach that Bosch (2015) claims is characteristic of school algebra internationally: The 'language of algebra' is reduced to different techniques of equation solving and manipulation of algebraic expressions. It is not *per se* a problem that praxis is more emphasised in the VET programmes syllabi; given that the overall purpose of VET programmes is to act as preparational for the various future vocations, a more practical focus is needed. However, what is problematic is the inadequate connection between techniques and technologies in the core content. As technologies serve to justify the given techniques (see Figure 2), one apprehension regarding this – which is supported by Bosch's (2015) description of school algebra – is that the techniques are turned into mere 'recipes' for solving given tasks, without being grounded in knowledge of the underlying concepts. Such knowledge might, for instance, enable students to select the most efficient technique for solving a particular task. In the core content for HEP programmes, on the other hand, we saw more formulations that were only connected to the logos block (mostly theory), but with little or no connection to particular techniques.

As Gustafsson (2019), Madej (2021) and Hewitt (2012) have already stressed, students have difficulty understanding the meaning of different algebraic representations, such as the meaning of the equal sign and variables. From this study, we know that the algebraic knowledge to be taught at upper secondary level has a fairly insufficient organisation of praxis and logos, regardless of alignment. While it might be possible to build teaching situations merely around logos or praxis aspects of a concept, for a praxeology to hold and make sense, both aspects are needed, as "[praxis] entails logos, which in turn backs up praxis" (Chevallard, 2006, p. 3). Thus, one might ask whether the insufficient organisation observed in this study is connected to students' understanding of different algebraic representations, and whether the differences detected between alignments persist throughout the didactic transposition. Hence, as this small study focused only on formulations in the syllabi and not on students' knowledge, textbook content, or teaching situations, our next step is to examine how textbooks and teachers emphasise and organise algebra praxeologically. These forthcoming studies will also help us to dig more deeply into questions regarding social justice and purposes of education (Lindberg & Grevholm, 2013; Nylund et al., 2017).

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Tinkering in algebra – the case of John

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Finding a general rule for a figural pattern is a common type of task in early algebra, intended to enhance the ability to express generalization. In light of the incorporation of programming in school mathematics, this paper reports on one teacher's experience of using tinkering as a didactic strategy for patterning tasks, in comparison with a traditional approach. The discussion centers around possible benefits of integrating programming and algebra. An affordance of working with a computer program was that the general expression became relevant for the students and changed from being the end point of a patterning task to function as the starting point for mathematical tinkering.

Keywords: Algebra, figural patterns, computational thinking, programming, tinkering.

Introduction

The current world-wide incorporation of programming in school curricula (Brown et al., 2014; Mannila et al., 2014) raises questions of how these new ideas and technologies could enhance mathematics education. In Sweden, as in many other countries, *programming* has been included in the mathematics syllabus (Bocconi et al., 2018). Uniquely for Sweden, programming has also specifically been connected to algebra, although it is not a well-defined concept in the curriculum documents (Kilhamn & Bråting, 2019; Swedish National Agency of Education, 2018). Programming could be broadly interpreted as a pedagogical tool for developing students' digital competence and computational thinking, or in a narrow sense as a set of computer coding activities. Many Swedish teachers struggle to understand and shape what programming in school mathematics is and what it might be in relation to mathematics education (Kilhamn et al., 2021; Misfeldt et al., 2019). In this paper programming is seen in the broad sense, defining computational thinking (CT) in line with Aho (2012) as "the thought processes involved in formulating problems so their solutions can be represented as computational steps and algorithms" (p. 832). The aim of this paper is to contribute to a discussion about how algebra learning could benefit from programming activities and computational thinking practices such as tinkering. By describing a teacher's experience of a lesson on figural patterns based on tinkering with code, the paper seeks to induce a discussion in the spirit of Gadanidis et al. (2017), who write that "We need many more cases of what *might be* in mathematics and CT integration to better understand the role CT affordances might play in disrupting and improving mathematics education" (p. 94).

Figural patterns

Algebra is sometimes described as the study of structure, where algebraic thinking emphasizes relations and structure over processes. Hewitt (2019) defines algebraic structure as a combination of recognizable parts and recognizable patterns connecting the parts. Structure is important to discern in order to describe generalizations across and between specific instances. Patterning tasks, i.e. working with figural patterns, can be used to enhance students' sense of structure (Hewitt, 2019), and to promote students' understanding of functional relationships (Friel & Markworth, 2009). In this paper, figural patterns are seen as a way into generalization.

A traditional teaching trajectory for patterning tasks starts with a figural pattern where the first three figures are visually exposed. The students are then asked to find the number corresponding to the next figure, then perhaps to figure number 7, then 10, then 20, then 100. Typically, students insert the numbers in a table to find a functional relationship between a figure number and the total number of elements in the figure, and finally write a general expression for figure *n*.

This teaching trajectory has been widely reported in research literature (e.g. Blanton et al, 2015; Kieran, 2018). Some researchers stress the importance of drawing (Friel & Markworth, 2009) or using words and gestures (Radford, 2014) before symbolizing. However, sometimes the teaching trajectory quickly moves away from the visual figures into a table of values, where a total number of elements for each figure is calculated and students are asked to look for patterns in the table. Hewitt (2019) describes the danger of students losing touch with the context of the figural pattern by spending time calculating and looking at the table of values created from the figures. In order to see structure, he suggests students should stay focused on the figures and avoid doing the arithmetic, since the total number hides the structure that reveals the generalization. He argues that “learners should never do any arithmetic, just write down the arithmetic they would do”(ibid, p 563). El Mouhayar (2018) showed that students who focused on numbers were more likely to take a recursive approach, going from one figure to the next, whereas Strømskag (2015) found that students who stayed within the figural context more often found a general expression

In contrast to the traditional learning trajectory for figural patterns, a more challenging problem could be to present a pattern and ask for the figure number of a large number of parts. One such problem about a matchstick figure, taken from TIMSS07, was used in a study of problem solving in small groups (Kilhamn, 2012). When the study was presented at a conference in Cambridge, participants in the audience criticized the problem, advocating instead the traditional step-by-step instruction described above, claiming it to be the ‘right way to teach patterns’. The reaction pinpointed the fact that this type of instruction is a well-established didactic strategy. However, as the case of John presented below will show, it may not always be the most prosperous strategy.

Computational thinking and tinkering

Computational thinking (CT) is a fairly new concept in educational research, first introduced by Papert in 1996. The term involves the kind of thinking skills needed to understand and capitalize on digital technology, and practices used by programmers. In recent years, researchers in computational science as well as mathematics education have attempted to define CT or create frameworks that describe it (e.g. Aho, 2012; Grover & Pea 2013; Kotsopoulos et al., 2017). One of several commonly accepted elements of CT is pattern generalization (Grover & Pea, 2013), which is also an important ingredient in algebraic thinking. In this paper we will focus on the practice of tinkering brought up more or less explicitly in all CT frameworks.

Dictionaries often define tinkering as the act of improving something by making changes to it, and a tinkerer as a person who enjoys experimenting with and repairing machines. In a CT perspective, tinkering experiences promote engagement in changes and modifications of existing objects (Kotsopoulos et al., 2017). Brennan and Resnick (2012) describe several CT practices such as testing, debugging, reusing and remixing, that could be seen as aspects of tinkering. Tinkering involves exploration, modification and reflection, it is about trying, adjusting and trying again. Fundamentally, it builds on learning from failures, turning mistakes into triumphs in the spirit of Thomas Edison, who is said to have uttered “I have not failed, I have just found 10 000 ways that won’t work”.[1] In mathematics education, tinkering could mean to explore a mathematical idea or mathematical relationship by making small and purposeful changes and reflecting on these. It is not the same as a guess-and-check problem solving strategy, since the main objective when tinkering is not to find a correct answer but to explore and change something already present. In a description of a teaching intervention in a Grade 6-7 algebra class, Boaler and Sengupta-Irving (2016) used the term *mathematical tinkering* to describe how the students played around with a problem and its solution, going beyond the problem at hand to explore the effect of small changes, challenging themselves to make the problem harder.

Method

The case study presented here emerged from a larger set of data collected when studying the transposition of knowledge that followed the recent inclusion of programming into school mathematics in Sweden (Bråting et al., 2021). A research question for the larger study was: *What opportunities, challenges and pitfalls related to the learning of algebra can be identified in the didactical choices teachers make when implementing programming in mathematics?* Interviews were initially made with 20 teachers identified as early adopters (Kilhamn et al., 2021). They were all enthusiastic about the challenge, had some previous experience in teaching programming, and many were responsible for implementation of digital technology in their schools. The audio recorded interviews took approximately 30 minutes and were semi-structured around questions that had been supplied in advance. Following four background questions the interview guide included the following topics: What is the role of programming in mathematics? Where do you find inspiration and ideas? Can you give an example of a good programming activity that you have tried? What programming concepts are important to bring up in mathematics?

Many of the teachers described programming activities with little connection to mathematics, and when there was a mathematics content, it was most commonly geometry, arithmetic or probability. One teacher was different in that he said he tried to incorporate some Python programming into every topic, in almost every lesson. When asked to describe a good lesson he chose an algebra lesson. He claimed that the incorporation of programming had changed his teaching and created better learning opportunities for his students. His story is reported here as the case of John.

The case of John

The interview with John was made in October 2019. John had by then been teaching mathematics, science and technology in grades 7-9 for over 25 years. In 2017 he was appointed head teacher with a specific responsibility to coach his colleagues in the use of digital tools and programming, mainly in technology. In mathematics he used an interactive whiteboard, but struggled to find ways to engage students in programming activities that were compatible with the mathematics curriculum. In 2018,

[1] https://www.goodreads.com/author/quotes/3091287.Thomas_A_Edison

when programming was officially included in the mathematics curriculum, he started to use the net-based platform Google Collaboratory, coding in Python. John described a lesson about figural patterns in Grade 7 (students' age 13), claiming that it was the first time ever that his students truly engaged in finding a general expression for a pattern. When he started working with these students two months earlier, they had no previous coding experience through school. Every student in his classroom has a personal laptop, so he can use the platform both in class and for homework, where he can see what the students do, give personal feedback and pick up student solutions for whole class discussions. John's ambition is to get students into the habit of using Python code every lesson so that they become accustomed to programming, learning syntax through experience more than direct teaching. John stood out from the other early adopters in that he included programming in almost every content topic, rather than making it a topic in itself. He said he did not teach programming per se, but he told his students to use programming, treating it as a mathematical tool. Typically, he would give them an example code to tinker with and modify. In addition to the interview, python codes as well as written reflections that John shared with his colleagues were collected.

The lesson

Below, the lesson is described in three parts and analysed in relation to aspects that were in some way different from the traditional teaching trajectory on figural patterns described above.

Part 1: Introducing the general expression

During the previous lesson, the class had worked with a figural pattern where a number of markers in a figural arrangement increased by three for each figure, starting with five. They had explored and discussed what features of the pattern were important in order to find a general expression: change and starting number. Together they now wrote the following basic code in Python, which everyone copied into their personal computer and tried out.

```
figure_number = int(input("What figure do you want the number of markers for? "))
number_markers = figure_number*3 + 2
print("Figure number ", figure_number , "has ", number_markers, "markers")
```

The code gives a sequence of instructions that computes and prints the figure number and its corresponding number of markers. John expressed that he did not expect all students to comprehend the code in detail, but for this part of the lesson it was enough to copy the code and run it to confirm that it worked for that pattern. John wrote the following reflection: "Many students found this difficult and did not seem to see the benefit of this program. The next step is to get the students to program."

Part 2: Transferring to other patterns

In contrast to a standard lesson where the general expression for the pattern comes at the end, the programming activity used the general expression as the starting point for exploration through tinkering. After introducing the above piece of code, the students worked with similar figural patterns presented as tasks in their textbook, all visual configurations of growing patterns with linear solutions. The students were encouraged to make use of the code they had produced but with access to sticks and markers if they needed. All students chose to work with the code. When they knew that one pattern was described by `figure_number*3+2` they could start to tinker with the expression to explore the effect of small changes. A further challenge came when the question was posed the other way around, for example “what figure number would need 64 sticks?” The code did not produce the answer and needed to be modified. John explained:

> “Then I showed the class how to modify this code so that instead of asking for a certain figure it prints the first 50 figures with the number of sticks in. And then you can see that: ‘Well there are 55 sticks in figure number 17’. And this – when they suddenly saw – they had it all there! It was like ‘wow’ they could do anything, all information was there in this table that was printed.”

In this section of the lesson the power of the computer was put to use, so that patterns could be investigated using larger numbers and more data without having to do all the calculations. The students discussed large numbers and compared different outputs. The most confident students started to think about what needed to be changed in the code to get the inverse operation instead.

Part 3: Homework assignment

Finally, John assigned the following task as homework:

Make a program that calculates the number of white squares in the following pattern. The program should ask what figure you want to know the number of white squares for, and then show you the result. If you want, you can modify and improve the program.

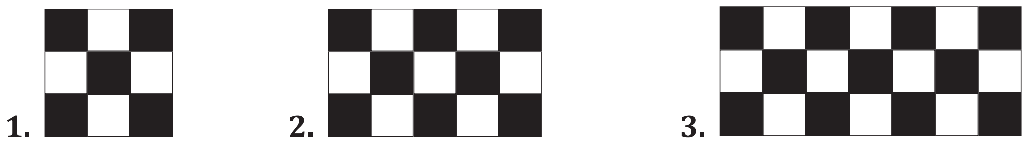

Figure 1: Figural pattern used for homework: the first three figures.

All students managed to solve the task although the pattern on surface value looked quite different from previous patterns. Many explained that they had copied the original code and tinkered with it until they got the correct result when they ran it (`figure_number*3+1`), checking with the given Figures 1–3. Some modified the code, adding features. Some came up with quite different solutions, with more or less efficient codes. Others came up with creative expansions, such as making a loop that kept asking for another figure number or commands that could handle an invalid input.

In the interview, John expressed that his students better understood the importance of finding the formula for a pattern because it is used in the program. It was the first time he saw that students found the general expression useful. In traditional patterning tasks many students would do the arithmetic for the first three figures and perhaps a few more, but they would lose interest before they got to the general expression, not seeing the point of finding it. Now they were dazzled with what the computer could do once they inserted the expression and were curious to see what happened with the numbers produced when small changes were made. It was, John noted, easier for them to relate to the figure *n*, because it was there, in the program, with the descriptive name `figure_number`.

Discussion

The case of John brought out several possible affordances of introducing programming when dealing with figural patterns. One was that students, as suggested by Hewitt (2019), could explore the pattern without doing a lot of arithmetic. In fact, the power of using a computer code to do the arithmetic enabled students to look for patterns in larger numbers and to find structure in a table with many more entries. Iteratively moving back and forth between the code, the printed output and the original pattern figures enhanced students' focus on structural aspects of the pattern while downplaying arithmetic procedures. However, according to John, the main benefit had a more psychological character addressing the fact that students found the general expression relevant and useful as a way to communicate with the computer. Instead of being the end point of the task, as in traditional patterning tasks, the expression became a starting point, something to tinker with that sparked curiosity. Furthermore, the expression became more visible and made more sense when the variable was represented by a descriptive word, not as the abstract letter *n*. This could potentially be a useful intermediate step before dealing with variables in symbolic algebra, well worth further investigation.

Tinkering proved to be a valuable practice for these students when looking for a general expression for a pattern. The algebraic question "What is the general expression for the *n*th figure of the pattern?" changed into a computational question "What do we need to change in the command given to the computer so that it describes or generates the pattern?", leading up to questions about the effects of different changes made in the code. While this is in line with how Aho (2012) defines CT, it also uncovers opportunities for students to rise to yet another level of generalisation that further develops their algebraic thinking, namely that very different patterns can be described by the same or a similar algebraic expression. Tinkering, or what Brennan and Resnick (2012) describes as reusing and remixing predefined code, empowered students to see similarities between patterns with quite different surface features.

Another benefit was the opportunity of expansion, providing more challenge for students who needed it. Many of the students in John's class started posing new questions, such as what to do if the input is invalid or how to write the code so that the figure number is the output when the total number is the input. Furthermore, it would be possible to tackle the pattern the other way round by asking students to change the expression in the code slightly and then try to create a pattern to match the new expression and the numbers printed when the program was run.

The various affordances of programming in John's algebra lesson suggest that we should be open to the use of programming is school mathematics. Further design research with lessons like this would therefore be valuable. However, a teaching approach that involves tinkering with code is only possible when the teachers' coding proficiency is high, otherwise the work is bound to get stuck on syntax issues. This case study describes a teacher with many years of experience and good programming skills, which is not always the case. Much work is still needed to help teachers develop necessary programming competence (Kilhamn et al., 2021; Misfeldt et al., 2019), but in the meantime teachers could embrace a tinkering approach to teaching and start using mathematical tinkering, in the sense of Boaler & Sengupta-Irving (2016), for mathematics in general and patterning tasks in particular.

Why not let the lesson start with an intricate algebraic expression, perhaps graphically represented using dynamic software, and then explore what happens when small changes are made?

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Students' ideas about variables as generalizers and unknowns: Design Research calling for more explicit comparisons of purposes

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Students' challenges with variables are well documented. Although instructional approaches have been developed to support students' conceptual understanding of variables as generalizers, many students confound variables as generalizers (e.g. in generalization activities) with a second meaning, variables as unknowns (e.g. in equation solving activities). In our design research project, we developed a teaching learning arrangement in which both meanings could be constructed. The design experiments reveal first qualitative evidence that students were able to construct both meanings, but only some students developed explicit awareness of the distinctions of meanings. This calls for more explicit comparison activities with a focus on different algebraic activities and their underlying epistemic purposes, as well as the development of a meaning-related language in which students can express both meanings.

Keywords: Variables as generalizers, variables as unknowns, variation principle, learning from comparison, design research.

Theoretical background

Learning from comparison has proven effective in many contexts for raising awareness of critical differences (Alfieri et al., 2013). In our design research study, we use this design principle to increase students' awareness about critical differences between meanings of the variable (Usiskin, 1988; Malisani & Spagnolo, 2009). The research question for this paper is: *Which ideas do students articulate about variables as generalizers and as unknowns and how do they compare the conceptions?* The qualitative case of a larger design research study reveals that not only local meanings, but also underlying epistemic purposes need to be compared. We outline the theoretical background, the design principle and the methodological framework, before unfolding the case of two students.

Variables as generalizers and unknowns

The concept of variable lies at the heart of most algebraic activities, yet various studies reveal many students' difficulties with understanding variables in all their meanings (e.g., Malisani & Spagnolo, 2009; Küchemann, 1981). Whereas the meaning of *variables as unknowns* (a fixed set of hidden numbers that have to be disclosed, e.g. in equation solving activities) seem to be constructed more easily (Küchemann, 1981), the meaning of the *variable as a generalizer* reveals a critical obstacle in many students' understanding of the variable (Bardini et al., 2005; Malisani & Spagnolo, 2009). The conception of the *variable as generalizer* draws its relevance from generalization activities, in which the variable is used to express general relationships without any further specification of the numbers for which they apply (Usiskin, 1988).

Although various instructional approaches were established to enhance students' conceptual understanding of the variable as generalizer, students still tend to confound the two conceptions of variable as unknown and as generalizer, even after going through these approaches (Bardini et al., 2005; Prediger & Krägeloh, 2016). At least three reasons can be identified for these confoundations of conceptions: *First*, variables as generalizers and as unknowns share crucial similarities on the surface. Both represent numbers, both are signified by letters, and the same syntactical operations can be performed with them during algebraic transformations. A *second* reason for these confoundations was identified in the unprecise language by which students express their ideas about variables (Prediger & Krägeloh, 2016). Generalization activities require certain phrases to *think* and express generalizations (see Table 1). Hence, to allow students to grasp relevant differences in meanings, contrasting relevant phrases might be useful. *Third*, students do not seem to be aware about the different natures of the algebraic activities in which they are used. They naïvely assume that all variables are used in problem solving activities, in which a variable denotes a hidden number, which has to be disclosed by solving an equation and, thus, assign this meaning also to variables in generalization activities. As Bardini et al. (2005) showed, many of these students impose the meaning of unknowns to the generalization activities by interpreting them as a "*temporally indeterminate number* whose fate is to become determinate at a certain point" (p. 129, emphasis in original). That means, students use variables in activities, which require a generalizer, but act as if they were disclosing unknowns.

Table 1 summarizes similarities and differences of the two conceptions that are potentially overlooked by students. Other authors (Usiskin, 1988; Bardini et al., 2005; Prediger & Krägeloh, 2016) have already identified the surface level, the meanings and the algebraic activities as crucial for students' understanding. Additionally, the underlying epistemic purpose (i.e., what kind of knowledge gain is intended by its use) became apparent during our empirical analysis. Thus, the last row serves as an advance organizer for the later analytical outcomes.

Table 1: Similarities and differences of variables as generalizers and unknowns (of increasing depth)

	Variable as generalizer	**Variable as unknown**
Similarities on surface level	Both represent numbers Both are signified by letters and are object of equal syntactical operations	
Differences in meanings in examples (Küchemann, 1981)	Variable stands for all possible numbers of a relevant domain (e.g., in equality 3 (x+5) = 3x + 15)	Variable usually stands for a (concrete set of) hidden number of a relevant domain (e.g., in equation 3x+5 = 13)
Meaning-related phrases for expressing the meaning (Prediger & Krägeloh, 2016)	An arbitrary number, always a different number, the number always changes, …	The number we are looking for, number is already fixed
Differences in underlying purposes		

Algebraic activity (Usiskin, 1988)	Generalization activities in which variable is used to express general relationships	Problem solving activities in which variable is used as a hidden number that needs to be disclosed
Epistemic purposes (Usiskin, 1988, Bardini et al., 2005)	One wants to express that no further determination of an algebraic statement is needed – there is nothing more to know	One wants to express that something is not yet known, but shall be known in the future

Design Principles: Creating Rich Experiences and Learning from Comparison

The presented research is embedded in a larger design research project (Gravemeijer & Cobb, 2006) that aims at designing a teaching-learning arrangement to enhance students' conceptual understanding of variables as generalizers and as unknowns and at developing an empirically grounded local instruction theory. Here, we focus on sequences in which student explicate the meanings and the purpose of variables, first in rich experiences with generalization and problem solving activities, and later in reflective sequences by the design principle of learning from comparison.

Learning from comparison has proven to be effective in different domains for increasing students' conceptual and procedural knowledge and awareness (cf. meta-analysis by Alfieri et al., 2013). When comparing different uses of a concept or procedure, students have been shown to detect relevant structural features, and to learn to distinguish them from irrelevant surface features (ibid.). In algebra education, learning from comparison proved effective for increasing students' flexibility with different solutions procedures, in particular the Chinese Bianshi tradition also applied it for conceptual learning, e.g. for geometry concepts (Guo & Pang, 2011). Here, we adapt it for our purpose of raising students' awareness for the difference in meanings between generalizer and unknown.

This adaptation is not self-evident as conceptual differences of the generalizer and the unknown seem to lie especially deep, because their meanings ground in different underlying purposes which seem to be confounded by the students (Table 1). Thus, the differences of these two meanings of variables cannot be reduced to pointing out relevant definitional features and contrasting them. In addition, language challenges can be expected in the design experiments (Prediger & Krägeloh, 2016).

Methodological framework of the design research study

Design experiments as method of data gathering

In this paper, we focus on snapshots of students' ideas when *learning from comparison about variables as generalizers and unknowns* in a first design of a teaching-learning arrangement that engages students in generalization activities as well as informal equation solving activities. The data was gathered in design experiments in an online setting, taught by the first author of this paper. For each of the six pairs of students (n = 12) with varying mathematical prior knowledge, usually 4-5 sessions were conducted. In total, 33 hours of video were recorded and partly transcribed. This paper focuses on the case of two girls from a German-speaking school for higher tracked students: Anna (Grade 7) was a high achieving student who had not encountered variables signified by letters in her formal education before. Nora (Grade 8) already worked with variables in various algebraic problems (transforming expressions etc.), but Nora's teacher considered her in need to refresh and deepen her understanding of variables.

Methods of data analysis

For the qualitative analysis of students' ideas about the variable as generalizer and unknown, the transcripts were segmented and situations were selected in which students explicitly related the two meanings of variables to another. Then, we applied an open coding procedure to code students' utterances with respect to their ideas about main differences and similarities, starting from codes in Table 1. The last line of Table 1 emerged during this coding procedure.

Empirical insights into students' ideas about generalizers and unknowns

Anna and Nora's understanding of the variable as generalizer

In the first three design experiment sessions (of 90 minutes each), Anna and Nora receive rich possibilities to engage in generalization activities in the context of an e-scooter rental (0.15 € per minute riding time, 1 € to unlock the scooter). The students conduct repeated calculations in the table in Figure 1 and discover that variables can represent all relevant numbers. In Session 1 after filling the table, the teacher asks for an explanation:

Date of e-scooter ride	Driving time (in minutes)	Costs for pure driving time (in Euro)	Expression for complete costs (incl. 1 € unlock)	Total price (in Euro)
May, 16	20	20 x 0.15	20 x 0.15 + 1	4.00 €
May, 19	12	12 x 0.15	12 x 0.15 + 1	2.80 €
May, 24	27	27 x 0.15	27 x 0.15 + 1	5.05 €
May, 25	18	18 x 0.15	18 x 0.15 + 1	3.70 €
General	x	x x 0.15	x x 0.15 + 1	Not calculatable

Figure 1: Table for re-inventing variables as generalizers in e-scooter context

1.601 Teacher: Um, now we have written *x* there suddenly. What does the *x* mean in this calculation? Can you explain that?
1.602 Nora: Um, that is an *x*-arbitrary number.
1.603 Teacher: Mhm, and what does that mean, "*x*-arbitrary number"?
1.604 Nora: Uh, that every number can stand there.
1.605 Teacher: Mhm, very nice. Anna, any extensions?
1.606 Anna: Yes, *x*-arbitrary just means that only this one number in this calculation, which is represented by this letter x, though, can be every number. This number, 0.15 or 15 Cent, they are not changeable, they stay always. Only *x* is changeable […].

Nora describes the meaning of the variable as generalizer by "*x*-arbitrary number" and "every number", which does not necessarily guarantee that she has firmly grasped the idea of a variable as an indeterminate generalizer for generalizing expressions (as the analysis of comparable cases in Prediger & Krägeloh (2016) reveals). In Turn 1.606, Anna compares the changeable nature of *x* with the unchangeable nature of constants and thereby implies that the variable represents different numbers. Later on, Nora also expresses her understanding by describing that different driving times influence the total cost of the ride:

1.649 Nora: […] if one drives 10 minutes now, then it is cheaper, as if one drives 20 minutes.

Anna and Nora's understanding of the epistemic purpose of the generalizer

Even if both girls have built a solid understanding of the meaning of the variable as generalizer, their ideas substantially differ when asked to explain the purpose of describing situations with variables:

2.029 Teacher: So, why do we do that, writing down such a general expression [with variables]?
2.030 Anna: Yeah, to represent, not so much to know a total price in Euro. Rather, for representation. Okay, um, this information here, the 0.15 and the 1, these are fixed numbers which stay like this in every calculation, but the *x*, this is the changeable, variable number.
[…]
2.034 Nora: I don't know. Maybe if one calculates this mentally, and then one forgets it, one has to start from the beginning. And here, one has written it down?
[…]
2.039 Teacher: But what is the advantage if I […] remember this expression [*points at* $33 \times 0.15+1$] or this one [*points at* $x \times 0.15+1$]? So, what is the difference?
2.040 Nora: Uh, that for the one below [*refers to* $x \times 0.15+1$] we don't know how long he was driving, and then, we actually don't have to calculate it.
2.041 Teacher: […] Anna, any extensions?
2.042 Anna: Yeah, then one just has to look: Okay, I know for sure that the 0.15 and the 1 stay. Thus, I only have to look which information *x* has been in this calculation. And this would be 19, for example. Then I only have to look: Okay, 19, and then I can calculate again.

In Turn 2.030, Anna describes the purpose of using variables (represented by letters) from a representation perspective: Variables represented by letters allow to differentiate between changeable and constant numbers in a series of similar calculations. Nora, in contrast, points out in Turn 2.034, that algebraic expressions with letters give relief to one's memory capacity. Anna takes up this idea in Turn 2.042 and further develops it into explaining the purpose typical for formulas: Algebraic expressions establish a general sequence of calculations, which guarantees the determination of a number depending on another number. Thus, Anna articulates a purpose that is typical for a "calculator's mind" (Radford, 1996, p. 50): one formula can substitute many similar calculations. In sum, Anna assigns two purposes to variables as generalizers: (a) It signifies the changeable numbers and therefore distinguishes them (visually) from constants, and (b) it allows to establish a formula, which reduces complexity for the evaluation of new numbers.

For Nora, Turn 2.034 does not yet provide evidence that she also thought about Anna's purpose (b) or simply referred to the general idea that written language allows to relief memory capacity. However, in Session 3, after extensive work in a spreadsheet environment, she gives the following answer when asked to describe the use of the variable in algebraic expressions:

3.752 Nora: So that we can use the formula for every single number and immediately know how much the solution is.

Thus, at latest the work in the spreadsheet environment allowed Nora to establish the perspective of the calculator's mind and to express Anna's purpose (b). Yet, there might be a crucial difference between Anna and Nora concerning the *epistemic purpose* (i.e. the intended knowledge gain) with which they use the variable. For Nora, the immediate knowledge of a solution seems to be the main intention to establish a formula using variables. Thus, she ascribes an epistemic purpose to generalizers that is more typical for unknowns, namely that the variable needs to be known in the future (Bardini et al., 2005). In contrast, Anna explicitly articulates in Turn 2.022 that the calculation of "a total price in Euro" is not the main goal. Anna's idea (a) allows her to see that the algebraic expression has a function beyond quick calculation and the reduction of complexity, namely to contrast changing quantities from constants. This suggests that Anna is aware of the crucial epistemic purpose of the variable as generalizer, namely to see the variable not just as a temporal misfortune that needs to be overcome at a certain point (Bardini et al., 2005), but as something that has a value in its own right.

We infer that for these students (and also other students in our design experiments) the understanding of generalizers needs to be expressed also by the underlying epistemic purposes for the use of variables. Therefore, epistemic purposes can reveal subtle but important differences in students' ideas.

Nora's struggle to compare the generalizer and the unknown

In Session 4 of the design experiment series, Nora and Anna are engaged in problem solving activities in which they had to find unknown numbers. For example, they find the unknown price per minute x by informally solving the formal equation $x \times 11+1 = 3.20$ that identified two descriptions of the total cost of an e-scooter ride, namely one as a general algebraic expression and the other as a concrete price. Nora manages to interpret both sides of the equation as the total cost of the ride and she also describes the differences of these two descriptions as follows: "one is the total costs in a calculation and the other is the sum of the total costs" (Turn 4.133). Yet, when the teacher points out a difference between x in the general expression $x \times 11+1$ and in the equation $x \times 11+1 = 3.20$, Nora struggles:

4.136	Teacher:	[…] If I only have the expression [*writes* $x \times 11+1$], Nora, and it stands for the total costs, does it always result in 3.20?
4.137	Nora:	Um, no, because the x, it can be replaced by any arbitrary number.
4.138	Teacher:	Okay, very nice. And here, what is it like [*points at equation* $x \times 11+1 = 3.20$]? What does x stand for, here?
4.139	Nora:	Um, what [what do you mean]?
4.140	Teacher:	Okay, does this x still stand for all numbers, somehow arbitrary numbers, like you have said before?
4.141	Nora:	Yes? [*hesitant, asking*].

Afterwards the teacher checks again if Nora really believes that x in the equation $x \times 11+1 = 3.20$ still represents any arbitrary number, and Nora again confirms this. It seems that Nora struggles to give the letter x a different meaning than that of an arbitrary number, even though she has profound experiences with tasks on disclosing unknowns from her previous classroom experiences and from the task just solved. For example, she found one concrete number that solves the equation, interpreted this number correctly as the driving price per minute, and, furthermore, in Turn 4.137 explicitly stated that the expression $x \times 11+1$ does not always equal 3.20. Hence, for Nora it is not at all obvious that letters need to be reinterpreted in certain algebraic situations.

A possible explanation for Nora's struggle to reinterpret the letter is that she acts like the students observed by Bardini et al. (2005) and understands it as a temporal generalizer that eventually needs to be disclosed (e.g., by transforming the equation to get a solution). Hence, for Nora the determination of an unknown number is part of the *epistemic purpose* that she connects with letters that represent any arbitrary number. If this explanation of Nora's struggle is accurate, it shows that she could profoundly profit from a more explicit comparison of the generalizer

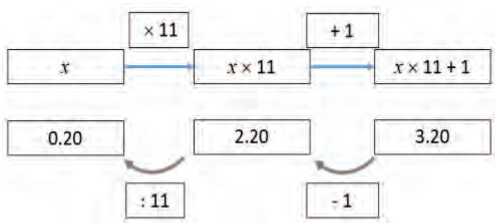

Figure 2: Working backwards for informally solving x × 11 + 1 = 3.20

and the unknown, not only comparing their meanings, but also their purposes, attached to their use in generational and problem solving activities.

Anna's intuitive comparison of the epistemic purposes of generalizer and unknown

In Session 5 of the design experiment series, Nora and Anna are again engaged in informal equation solving activities in which they have to find unknown numbers by undoing, i.e. working backwards in arrow schemes (Figure 2). During a discussion about how the arrow scheme has to be interpreted, Anna provides exactly the comparison of epistemic purposes that Nora needed:

5.101 Anna: If I had to describe this by and large, these are two different cases. One is someone who may have to do this calculation several times, because he is somewhat like a salesman or something and has different prices per minute [*x represents the price per minute in the expression* $x \times 11+1$] […]. Then of course such a calculation [$x \times 11+1$] is a good idea, because one can then simply always insert the x, right? So, whether it is 0.1, or 0.2 or 0.3.
And the other one is someone who might want to know exactly, uh how - why am I paying 3.20 now? So, he calculates backwards again, looks, uh, what price is coming from where, how is it put together?

In this remarkable statement, Anna describes the upper and the lower part of the arrow scheme as two different cases which represent different meanings of the variable. She explicitly compares the different epistemic purposes of the generalizer and the unknown: For the generalizer, she refers to the calculational ease that comes along with a formula – her purpose (b). Moreover, she enriches it by a meaningful story in which a general expression by means of variable is a "good idea" in itself. Hence, it is quite likely that Anna again uses the variable with the epistemic purpose typical for the generalizer. For the unknown, in contrast, Anna constructs a scenario where someone wants to know a certain unknown quantity. Thus, she explicates the typical epistemic purpose of the unknown.

Discussion and Outlook

In this paper, we investigated which ideas students articulate about variables as generalizers and unknowns before and when being prompted to explicit comparisons. As the presented case study indicates, prompting students to compare the different conceptions of variables seems to be a fruitful approach to explore and develop students' ideas about variables. In the case of Nora, for example, such a comparison-prompt revealed her difficulties in switching from one conception to another. The analysis revealed that Nora's difficulties with confounding meanings might be explained by a misunderstanding located in the purpose that she attributes to the variable as generalizer: Her understanding of the generalizer still used the epistemic purpose of the unknown, namely that variables are only temporally indeterminate and shall be disclosed in the future (Bardini et al., 2005).

From these observations (in the case of Nora and other students in our sample), we infer that the level of comparison should be deepened: *not just the meaning* of generalizer and unknown need to be compared to gain an elaborate and distinctive understanding of these two conceptions, but also the different *underlying epistemic purposes of their use in two different algebraic activities* (what resonates with the epistemological analysis by Usiskin, 1988). The case of Anna showed that some students intuitively compare the epistemic purposes by themselves. Interestingly, Anna embedded such comparisons in everyday scenarios that made the different algebraic activities and the underlying purposes accessible for her. Since the explication of epistemic purposes proved to enhance the students' deep understanding of the conceptions of variables, we will incorporate this focus of comparison in future design experiment cycles.

In sum, this case study underlines and explicates the diagnosis of Bardini et al. (2005) that students' understanding of the meaning of variable depends on their deep lying ideas about the algebraic activities in which the variable is used, and we add an explicit awareness of the epistemic purpose of this use. Therefore, it seems that a reflection on the purpose of different meanings of variables and, thus, a comparison of different algebraic activities as presented by Usiskin (1988) could also be promising to foster students' understanding of variable.

Acknowledgment

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Norwegian and Swedish student teachers’ explanations of the solution of a linear equation: A qualitative approach

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Keywords: Linear equation, explanation, student teacher.

The purpose of this study is to explore how student teachers explain the solution to a linear equation. Previous analyses with a quantitative approach have reported what aspects the participants included in their explanations (Larson & Larsson, 2021), applying a set of low inference codes developed by constant comparison (Andrews & Larson, 2019). This report focuses qualitative aspects in these explanations. The participants were 146 Norwegian and 161 Swedish student teachers for compulsory school. They were given a correct but deficit three step solution to a linear equation:

$x + 5 = 4x – 1$; $5 = 3x – 1$; $6 = 3x$; $2 = x$

The participants were invited to explain the solution to a fictive friend, who was absent when the topic was introduced to the class. The analytic tool (Andrews & Larson, 2019) included codes of how the operations in the solution were explained, and ‘explanatory codes’ identifying e.g. if the student teacher clarified that the purpose of solving an equation is to find the value of *x*, and that you in the solving process want to separate unknowns from knowns. The quantitative analyses showed that less than half of the participants clarified that the purpose of solving an equation is to find the value of *x* (the code ‘conceptual objective’), while a majority explained that you want to separate unknowns from knowns in the solving procedure (‘procedural objective’). Even though previous reports revealed relevant results, including differences between the two countries (Larson & Larsson, 2021), quantitative analyses produce only rough descriptions of the participants’ replies. If a script was coded as ‘conceptual objective’, there might still be differences in how this was expressed. Identifying such differences requires a qualitative approach, where the wording that yielded a specific code is scrutinised. The purpose of this report is to highlight some initial results from this qualitative analysis.

Conceptual objective

This code deals with the purpose of solving an equation, that is to find the value of *x*. Since the task was to explain how to solve the current equation, it is a strength if the purpose of the solution was stated early in the reply. One good example is a script that was well organised in bullet points. The first point said: “When you want to solve an equation, you want to find the value of *x*.” This immediately enlightened the purpose of the solving process for the fictive friend or a potential student. Two other examples were: “In the task above we want to find the answer to *x*. That is what number *x* is.” and “Here, we have to find out which *x*-value makes the equation work out and the values of each side to become equal.” Despite the wording in the former excerpt might not be the best, it still explains that the purpose is to find the value of *x*. That is also true for the latter, which in addition mentions the balance property of an equation.

The excerpt “*x* is a variable we want to find out.” includes the notion of variable. It might, however, be unclear because it does not explain what it means to ‘find out a variable’. This also applies to the student teacher who as first sentence wrote “What is sought is the symbol ‘*x*’.”, and later in the script wrote “To get *x* …”. These excerpts deal with the purpose of solving an equation, but the descriptions are imprecise and might not help a student who does not already know what it means to ‘get *x*’.

An example of lower quality is where the ‘conceptual objective’ first is mentioned in the last assertion, after several operational steps: “We do <u>not</u> wish to be left with $3x = 6$, we wish to be left with how much $\underline{x}$ is. That is $x = 2$.” Despite this is a fair explanation of the last step, the reply does not provide the fictive friend an initial information of the purpose of all the operational steps.

Procedural objective

This will be illustrated by just one example. The first sentence said: “Collect all *x* on one side.” The following assertions were numbered. Point 1 began with: “Move the *x* which is easiest to move (the one that stands alone).” In addition to handling the separation of unknowns and knowns, this assertion also touched how this should be accomplished. Point 2 began: “Collect all detached numbers on one side.” And point 3 said: “To get *x* alone, divide both sides by the number in front of $x \rightarrow 3$.” Overall, this is a thorough description of the procedure in the solution, that is to separate unknowns from knowns. Several scripts included only the third point from the example above. In spite all these scripts were coded as ‘procedural objective’, providing this description at the end only is an explanation of lower quality than the one providing the idea of separating unknowns from knowns in each step of its explanation.

Finally, there were excerpts mentioning both the ‘conceptual’ and the ‘procedural’ objective, as the one starting “Equations mean to find out what the unknown is, that is *x*. We do this by getting *x* to stand alone on one side of the equals sign.” Although the language has some shortcomings, this explanation early highlights both the purpose and the process of the solution, which is likely to be beneficial for the student.

Summary and implications

This paper suggests it is important as well *what* is included in the explanation of the solution, as *where* in the solution it appears. It highlights, that before the operational steps you should stress that the purpose of solving an equation is to find the value of *x*, and to do that you want to separate unknown terms from known. That will justify the operational steps and is hence likely to be beneficial for the learner. This means the results presented might be useful for teachers, and in teacher education.

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Relational thinking with indefinite quantities - case studies from elementary school children

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Early algebra is becoming more and more important in research of mathematics education. Relational thinking and variables are emphasized as essential sub-areas of algebraic thinking. The article provides an insight into kindergarten and elementary school children's abilities to establish relationships between quantities. The children's approaches can be described in a continuum between number-oriented and structure-oriented approaches. The influence of indefinite quantities can be shown by comparing three case studies.

Keywords: Elementary school mathematics, early algebra, relational thinking, variable, task design.

Introduction

The present study is about early algebra, which importance is steadily increasing for mathematics teaching in elementary schools (e.g. Kieran, 2018; Cai & Knuth, 2011). Studies show that an algebraization of mathematics lessons in elementary school is beneficial for itself as well as for later algebra learning (e.g. Mason, 2008; Kaput, 2008). Compared to arithmetic thinking, algebraic thinking is characterized by a structural rather than an operational view of mathematical objects (Steinweg, 2013; Kieran, 2004). Kieran (2004) named the consideration of relational aspects of operations instead of their computation as an essential way of thinking in early algebra. Likewise, comparing mathematical expressions and a relational understanding of the equal sign plays an essential role in early algebraic thinking. The creation of relationships between mathematical expressions defines algebra and can be described in particular in the aspect of relational thinking, which forms the theoretical framework of the present study.

Theoretical framework

Relational thinking is to be regarded as a sub-area of early algebra and can be defined as follows:

> "Sentences have to be considered as wholes instead of as processes to do step by step. When students analyze expressions, they compare elements on one side of the equal sign to elements on the other side of the equal sign or they look for relations between elements on one side of the equation" (Molina & Ambrose, 2008; p.64).

Fostering relational thinking with the help of equations is a possibility and has already been studied (e.g. Carpenter et al., 2003; Molina et al., 2005; Molina et al., 2008). Nonetheless, non-formal representations can also be used to establish relationships between elements (Schliemann et al., 2007). They offer the advantage of granting access to younger children as well and thus capturing pre-school starting points for mathematical learning. Furthermore, problems of the formal representation are avoided. This includes the relational understanding of equal signs, which can be seen as a prerequisite for using relational thinking with regard to equations (cf. Molina et al., 2005).

In addition to relational thinking, dealing with variables is an essential aspect of algebraic thinking. In principle, three aspects of variables can be identified: variables as unknowns (e.g. $5 + x = 8$), variables as indeterminate like in functional relationships or as a general number, for example to describe the laws of calculation (e.g. Freudenthal, 1983). The influence of various variables on relational thinking is illustrated by a study by Stephens and Wang (2008). They showed that the inclusion of multiple variables supports the use of relational thinking. Secondary school students had to put numbers in the boxes of the equation $18 + \square = 20 + \square$, establishing relationships between two indeterminate values. Compared to placeholder problems with only one unknown, these equations could stimulate students to think relationally rather than using operational solutions.

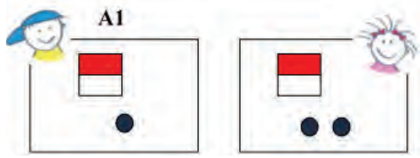
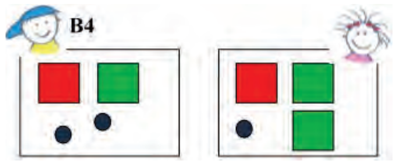
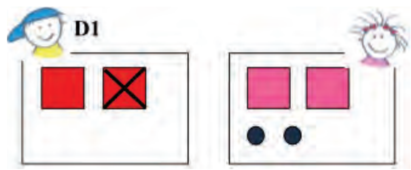
Relational thinking as the creation of relationships between mathematical expressions - be it equations, operations or relationships in factual relationships - is of great importance in early algebra. In particular, the connection between variable concepts and the use of the relationships between mathematical elements is challenging and requires further investigation, particularly with regard to young children of preschool and elementary school age. Therefore, the research questions examined in this article are the following: *How do elementary school children establish relationships between known, unknown and indefinite quantities that are represented by real material? What influence does the different use of known, unknown and indefinite quantities have on the use of relational ways of thinking?*

Method

The aim of the study is an exploratory recording of the abilities of preschool and elementary school children to think relationally[1]. As Stephens and Wang's (2008) studies have shown, the use of multiple variables within equations is helpful in stimulating relational thinking. With regard to the age group of kindergarten and elementary school children, a design was chosen in which equations with known and unknown quantities were represented with the help of material. Based on Melzig (2013), tasks were created in which boxes and marbles represent known and unknown quantities. Melzig showed that boxes open up a first access to a sustainable understanding of variables. The non-formal task design with real material (boxes and marbles) should also enable preschool children to show their relational thinking and can serve as a starting point for developing sustainable ideas about variables. Various equations with one or more variables have been translated into an arrangement of different colored boxes and single lying marbles. A story was told: two children play with marbles, some of which they keep in colorful boxes. Within a task there are the same number of marbles in boxes of the same color. Boxes of the same color may contain a different number of marbles in a different task. There were 12 tasks in 4 different task types.

[1] Other studies used the data to examine the conceptualization of the variables (e.g. Lenz, 2021).

Table 1: Overview of the tasks.

Task type	Example of the tasks
Type A: “The same?” Two given sets of quantities have to be compared with each other. The content of the boxes is not known and does not need to be determined to answer the question. The boxes can be seen as a variable as an indefinite.	 Task A1: “Does both children have the same amount of marbles? How did you get that?”
Type B: “How many?” The contents of a box must be determined. The amount of marbles asked for can clearly be determined and so appears as an unknown. Exercises B1 and B4 also contain additional boxes (the red ones), which content is not known and does not necessarily have to be determined in order to find a solution.	 Task B4: How many marbles have to be in a green box, so that both children have the same amount of marbles? How did you get that?
Type C: “How many?” Both children have boxes of unknown content. In contrast to the previous type, no specific quantities can be given for them. In order to answer the question, children can state a relationship between the amount of marbles in the boxes. Since the contents of the boxes cannot be clearly determined in comparison to task type B, it can be seen as indefinite.	 Task C1: How many marbles have to be in the boy’s box so that both children have the same amounts of marbles? How did you get that?
Type D: “Make them equal” Both children have the same amounts of marbles. The interviewer makes a transformation by removing or adding a box by one child in the task. Children have to decide what amounts of marbles they have to give to the other child in the task or take away from the child to the interviewer, to make the quantities equal again. Tasks D1 could be answered with a specific number of marbles, while in task D2 a relationship between two indefinite quantities had to be established.	 Task D1: Both children have the same amount of marbles. Now, the boy gives one of his boxes to the interviewer. How much does the girl have to give to the interviewer, so that both children have the same amount of marbles again?

To get insight into children’s ways of thinking and their use of relational thinking, a qualitative survey method using interviews was chosen. The study follows a diagnostic approach and is not to be understood as an intervention. The tasks were dealt with in video-recorded, semi-standardized individual interviews with 80 children in three age groups. 5-6 years old kindergarten children (N = 25), 7-8 years old second-graders (N = 29) and 9-10 years old fourth-graders (N = 26) took part. The children were asked to explain their approach. The transcribed interviews formed the data basis for the subsequent qualitative content analysis (cf. Mayring, 2010) and the method for analyzing interviews according to Schmidt (2005). Categories were formed out of all of the interview-transcripts which described children's approaches to answer the tasks. They were created deductively based on preliminary theoretical considerations and inductively obtained from the data. The categories were recorded in a coding guide. After re-coding all the material, an overview of the categories of the entire data material was given in frequency tables. These were used for further analysis by pointing out possible relationships that need to be checked in individual cases. In a last step, in-depth case analyzes were made (Schmidt, 2005).

Findings

After answering the tasks, the children were asked how they got their answer. Based on the analysis of the interviews, this article focuses on comparing the approaches described by the primary school children. First, the evaluation dimensions are presented and related to the theoretical background. Then insights into three case studies from primary school children are given.

Number-oriented and structure-oriented approaches

The approaches described by the children for processing the tasks can be described across all tasks in a continuum between number-oriented and structure-oriented approaches.

Number-oriented approach

The number-oriented approach focuses on the specific amounts of the marbles. Because the calculation of sums instead of relating quantities predominates, it can be characterized as an arithmetic way of thinking.

Structure-oriented approach

In a structure-oriented approach, children make gestural or linguistically clear that equal subsets are related to each other. The focus is on the quantities themselves and not on their value ("they are the same" instead of naming the specific number). Children take a structural perspective on the task and make connections between sub-structures of the task. According to Molina and Ambrose (2008), this approach can be characterized as relational thinking.

Three case studies

The distinction between number-oriented and structure-oriented approaches is compared using the example of the processing of tasks B4 and C1 (see Table 1) by three children. But they also show the influence of adding another, indefinite variable to exercise C1. It should be noted that the change in task types was not communicated to the children. In task type B, it was possible to answer with specific numerical values. Immediately afterwards, task C1 was set, whereby no specific numerical values could be given. Thus, some problems for the children are to be expected. But the answers to task C1 also reflect the spontaneous approaches of the children without being influenced by learning effects.

Leonie, 4th- grader

In task B4, Leonie states that there must be a marble in each of the red and green boxes. She justifies this with the fact that both children then each have four marbles. When asked whether she needs to know the contents of the red boxes, she seemed unsettled. This shows that in addition to the number-oriented approach described, no structure-oriented approach is conceivable for her, in which the content of the red boxes is not determined.

In task C1, Leonie spontaneously gives numerical values for the two boxes given. The interviewer then names various other numerical values as the contents of the girl's box. Leonie can give the correct amount of marbles for the boy's box. This shows that she does not accept the aforementioned numerical values as fixed actual content, but can imagine different values. She is able to deal with changing numerical values and recognizes the dependency. However, she does not explicitly succeed in generalizing the relationship between the indefinite quantities.

Matteo, 4th-grader

In task B4, Matteo structures subsets as equivalent to one another without having to determine their specific content. Based on the individually lying marbles, he deduces the contents of the green boxes. This approach can be characterized as structural:

Matteo: There must be one marble in a green box, because here (*points to the red boxes*) there are the same numbers and here (*points to a green box on the boy's side and on the girl's side*) ... In order to make this one marble difference (*points to the girl's marble*), one must also be in here (*points to the girl's front green box*).

In task C1 Matteo states that there is a dependency between the indefinite quantities of marbles in the two differently colored boxes. He says correctly that the contents of the boy's box cannot be determined because the content of the girl's box is not known. When asked by the interviewer, Matteo can then state the general relationship between the quantities of marbles in both boxes and refers to the marble lying individually in his argumentation. In addition, a structuring of the given quantities becomes clear:

Matteo: Here (*points to the boy's box*) there is one more marble than here (*points to the girl's box*), because here (*points to the girl's marble*) there is a single marble.

Julius, 2nd grader

In task B4, the second grader Julius names the value of one marble as the content of the green box. He claims to have checked this by calculating subtotals: the contents of the green boxes and the individually lying marbles. He doesn't say anything about the red boxes. He may have excluded these because they contain the same amount of marbles. Although his approach by calculating partial sums is to be regarded as number-oriented, this already represents a transition to the structure-oriented approach. He has recognized partial amounts of equal amount (the red boxes) that will be disregarded for further consideration.

In task C1 Julius succeeds in specifying the relationship between the indefinite quantities:

Interviewer: How many marbles must there be in a green box so that both children have the same amount of marbles?[…]

Julius: Because one thing (*taps the girl's marble*) and then you still have to calculate that here (*taps the girl's box*), there can be one, two or three (*waving his hand rhythmically in the air*) and there would always have to be one more (*taps on the boy's box*) to be in it than in the box (*points to the girl's box*) then it would be right.

Julius example shows impressively that he is able to establish relationships and to support this with gestures. At this point he shows a structure-oriented view of the task.

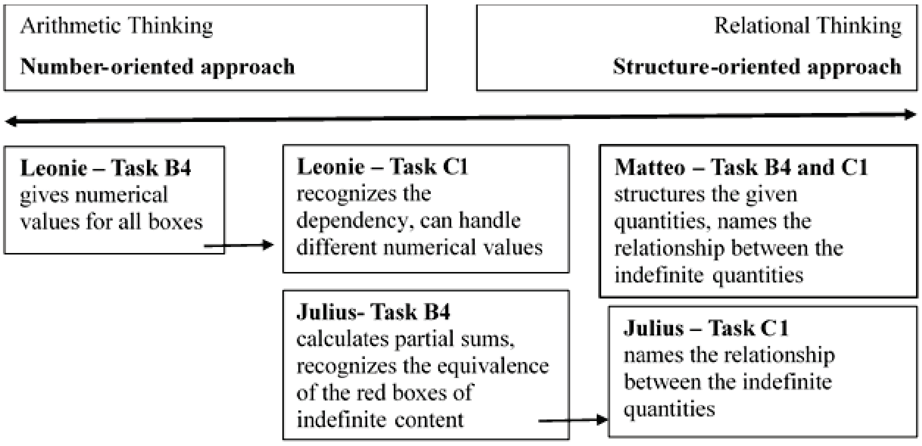


Figure 1: Approaches of the three children in comparison of tasks B4 and C1

Figure 1 shows how Leonie and Julius move from a number-oriented approach to a structure-oriented approach in the transition from task types B to C. This suggests that the task design, and especially task type C, can encourage children to think relationally. Since not all values are given, the children have no way of calculating specific numbers. They are encouraged to take a look at the whole and to establish relationships between the indefinite quantities. Thus, the results of the study by Stephens and Wang (2008) can also be confirmed with regard to the handling of real material and with primary school children.

Conclusion

The analysis of the entire interviews showed that there are children who mainly proceed in a number-oriented approach. To do this, they name numbers for additional boxes in task type B (like the red boxes in task B4) and give arithmetic reasoning. In task type C, these children are primarily tied to numerical values in order to find access to the task. This is shown in the case study of Leonie. However, she already manages to deal with various numerical examples in task C1 but she cannot indicate a static relationship between the indefinite quantities.

In contrast, there are children who show mainly structure-oriented approaches across all of the 12 tasks. Matteo's explanations of tasks B4 and C1 serve as an example. He is able to take a structure-oriented view of the whole from above across all tasks. This can be characterized as relational thinking (e.g. Molina & Ambrose, 2008).

In addition to these cases, however, there are also children who switch approaches. Second grader Julius described in task type B mainly number-oriented procedures. This may be due to the fact that it is possible to operate with specific numerical values. But also in his processing of task B4 it should be noted that he is already on the way to a structure-oriented approach in that he only calculates partial sums instead of the whole sum of marbles, as Leonie did. In task C1, Julius shows that he is able to describe relationships between the sets. The tasks of type C thus stimulated him to move from a number-oriented approach to structure-oriented approach and thus relational thinking.

Discussion

The empirical study shows that relational thinking as a sub-area of algebraic thinking can be stimulated in elementary school children. The study also showed that some kindergarten children are also capable of relational thinking, which, due to the length of this article, could not be taken up. The skills of the kindergarten children are to be regarded as prior knowledge and at the same time the starting point for further mathematical learning with regard to early algebra. With the help of the task design in its non-formal representation, instead of starting from procedures and calculating result values, many children succeed in recognizing relationships between the elements with the help of a look at the whole and using them to find solutions. It is precisely this “view from above” on the elements of a mathematical situation and their relationship to one another that constitutes relational thinking and thus also an essential aspect of algebraic thinking.

In particular, the type C tasks, which contain indefinite quantities, encourage children of primary school age to use relational thinking. This is already indicated by the small insight into the processing by the second grader Julius within the case studies and also corresponds with the explanations of Stephens and Wang (2008). As a result, the inclusion of unknown and indefinite quantities in the sense of a spiral curriculum also appears profitable in elementary school lessons.

The analyses give a little insight into the differences in the approaches of the children. With regard to the use of the task design in school, very different approaches by the children are to be expected. A classroom discussion about children’s ways of thinking used can lead the children to focus on both number-oriented approaches and structure-oriented approaches. The real material can be used to clarify your own ways of thinking to others - in particular the established relationships between the quantities.

Nevertheless, it is important to break away from a real representation and later perceive the variable as a thought object instead of a real object in the form of the box.

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The “Sense-Making-Algebra” project for Hungarian seventh graders

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According to the Hungarian National Curriculum, the systematic study of algebra starts in the seventh grade. In the initial phase, traditional methods focus less on generalization as a cornerstone of algebra and more on algebraic operations. Matched with the traditional syllabus, the authors have incorporated lessons into the teaching process at this early stage of algebra learning that focus on generalization activities and the meaningful use of symbols. The authors investigated how generalization activities manifest themselves during these lessons.

Keywords: Algebraic reasoning, generalizations, using symbols, algebra in the early grades, lower secondary level education.

Introduction

The cognitive ability to abstract is given special attention in the Hungarian National Core Curriculum in grades 7-8 (Government Decree, 2012, rev. 2020). The generalization of experiences and the need to justify relationships are also expected. The curriculum also requires the learner to formalize the mathematical content of everyday problems and use letters to denote unknown quantities. To begin teaching using symbols, a procedural approach based on “letter arithmetic,” i.e., practicing operations with algebraic expressions based on definitions and rules, is a traditional teaching strategy also presented in Hungarian textbooks. However, less emphasis is on generalization and the meaningful use of symbols in that early stage. Discrepancies between development goals and the practice in teaching early algebra were the impetus of our research. Improving this situation requires recognizing that algebraic thinking, which involves the meaningful use of symbols, is a form of mathematical sense-making related to symbolization (Schoenfeld, 2007). Many authors explain the sense-making process of a mathematical concept by connecting it to prior knowledge (Scheiner, 2016). Palatnik & Koichu (2017) found that the criterion for successful implementation of the sense-making process is that students are adequately engaged in the task and that the learning environment, which in their scenario was project work, is supportive. Inspired by these ideas, the authors of this paper added mathematical problem situations to the traditional “letter arithmetic” to support the generalization process and the meaningful use of variables. This paradigm is named the “Sense-Making-Algebra” teaching strategy by the authors of this paper. A rationale behind this teaching strategy is to allow students to see the power of symbolization as early as possible in the algebra learning process. Furthermore, with a problem-oriented teaching approach (Kónya & Kovács, 2021), the authors aimed to create a supportive learning environment.

The authors’ research questions are, (1) how do seventh-grade students’ generalization activities manifest themselves during the “Sense-Making-Algebra” intervention, and (2) what resources do they employ to create their generalization? We hypothesize that most learners can generalize from experience and argue for generalization using natural language or symbols after the teaching unit.

This paper describes a part of the broader experiment: the content and analysis of the first experimental lesson and the related follow-up test. In this lesson, we used the so-called "calendar problem," which has been discussed in various places in the literature, as an effective initiative to encourage generalization (Friedlander & Hershkowitz, 1997; Huang et al., 2014).

The first author performed the action research reported in this paper in her three seventh-grade classes. The second author acted as a critical friend during the teaching experiment. The study provides an example of how to put into practice the sense-making use of variables. Although there are some well-prepared collections on teaching strategies and recommendations on algebra learning (e.g., Friedlander & Arcavi, 2017), these works are focused on equations. The authors found fewer compendiums of a similar style at the pre-equations stage of algebra learning, with generalization and reasoning in focus, which also motivated this research.

Theoretical underpinning

Several scholars attempted to investigate the nature and content of algebraic thinking. There are several perspectives on what defines algebraic thinking, but many agree that generalization, or the ability to discern the general in the specific, is a crucial element (Pittalis & Zacharias, 2019). The present article focuses mainly on generalization, which may be accomplished using natural language and symbols. While outlining the theoretical basis, we emphasize this perspective, and in this regard, the work of Kaput (2007) serves as the theoretical framework for our research.

Freudenthal (1977) made an early attempt to conceptualize school algebra and included algebraic thinking in the subject. Freudenthal highlighted that the ability to describe relations and solve procedures in a general way is part of algebra. In this novel perspective, algebra is seen as a human cognitive activity. Thus, those who conceive algebra as reasoning prefer to examine how students think and speak about it. Arcavi (1994) considers the dichotomy of generalization and symbol usage in school algebra and claims that many students who master algebraic methods often fail to perceive algebra as a tool for comprehending, expressing, conveying generalizations, and constructing mathematical arguments. According to Kieran's (2004) model for conceptualizing algebraic activity, algebra is a multidimensional activity that includes numerous ways of thinking. Algebraic thinking approaches quantitative situations that aim to find relationships and structure using not strictly letters-symbolic approaches. Kaput (2007) emphasizes two core aspects of algebraic reasoning. One of them is generalization, and expressing generalizations in increasingly coherent symbol usage (Core Aspect A). "Increasingly" means that initially, the students use their resources, typically natural language, but later switch to conventional representational forms. Core Aspect B means the syntactically guided action on symbols. Kaput argues that the "Core Aspect B" should come later than the "Core Aspect A" since rule-based actions on symbols depend on knowing the allowed combinations of symbols, particularly which combinations are equivalent to others.

Method

The action research took place from December 2020 till March 2021 in a Hungarian practicing school for teacher training. A total of 68 seventh-grade students aged 13-14 years (hereafter S01-S68) from three classes (hereafter classes b, d, z) participated in the experiment. The groups have three maths lessons a week, each 40 minutes long. The teacher was the same in all three classes. Based on the textbook used in the school, the teacher's syllabus allocated 28 lessons to algebra instruction. The intervention took place in the first teaching unit (Algebraic expressions and operations with algebraic expressions), for which the curriculum allocated 11 lessons. The teacher conducted four experimental lessons in this teaching unit. The authors only report on the first experimental lesson in this paper and the follow-up test relating to it. The authors used the following tools for data collection and analysis: (1) classroom observation through a visiting teacher's field notes; (2) the teacher's research journal; (3) copies of students' written outputs.

The first lesson was inspired by Huang et al. (2014), where the authors report a lesson on investigating patterns in calendars, focusing on improving empirical reasoning towards deductive proof. The planning of lessons followed a problem-oriented teaching approach (Kónya & Kovács, 2021). It means that students: (1) analyzed mathematical problems; (2) were allowed to reflect on their own and their classmates' thinking critically in classroom discussions; (3) were encouraged to explain and justify their thinking.

The plan of the first lesson is as follows. Part 1: pattern finding. Students looked for patterns in the December 2020 calendar during the preparatory phase of the lesson. First, the teacher chose the diagonal 1-9-17-25 (Figure 1, left). Then, after determining that the numbers in the sequence increased by 8, students worked in pairs to explore similar patterns (Figure 1, right). The purpose of the pattern-finding task was to make students aware of the mathematical structure of the calendar, which is already familiar from everyday life.

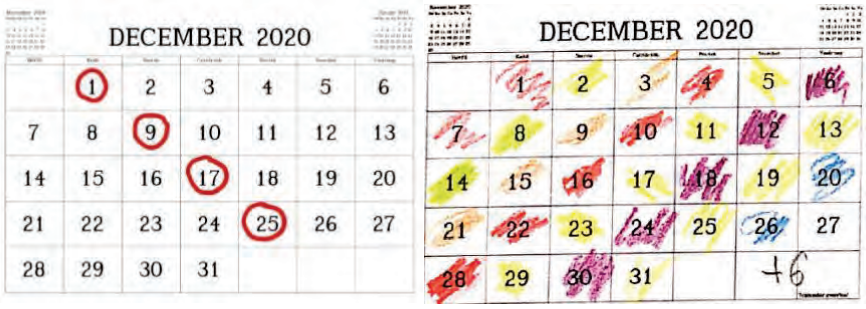

Figure 1. The December 2020 calendar with diagonal patterns

Part 2: magician's trick. The principal part of the lesson was based on a magician's trick. "Choose a number with a top right and a bottom left neighbor. Then, you tell me the sum of the three numbers, and I will guess the chosen number!" The processing of the magician's trick had three stages: (1) Numerical experience. The number said by the student and the number thought of was recorded on the board. (2) Formulating the rule: how did the magician do the calculations? (3) Explanation of the rule. The theoretical consideration behind placing the problem in a playful context was to engage as many students as possible in the task and maintain motivation.

In the follow-up test, two compulsory and one optional assignment were given to the students (Table 1). The first and second problem was close to the calendar task in the lesson. These two tasks were chosen to require generalization and make it easy to argue without using symbols. On the other hand, the third task was assumed to be much more difficult to explain without symbols.

Table 1: The follow-up test (for March calendar)

Task 1. (Compulsory) Nándi selected a number in the calendar and added its left and right neighbors. When he halved the sum, he got exactly the number he had chosen. Do you think this is always true if you choose a number with a left and a right neighbor? Justify!	
Task 2. (Compulsory) Csenge chose a number from the calendar with both a lower and an upper neighbor. She added the two neighbors together and then halved the sum. What has she experienced? Will this always be true if you choose a number that has a lower and an upper neighbor? Justify!	
Task 3. (Optional) Laci had a magician trick. If someone told him the sum of three numbers in the calendar in the shape of a V, he would guess the middle number. (For example, if someone told him that the sum is 22, he would know that the middle number is 12.) However, unfortunately, he forgot how the trick worked. Help him!	

To assess the follow-up test, the authors examined the students’ solutions, for which they created the following code system (Table 2).

Table 2: The coding system to evaluate students’ outcomes

Code	Description	Definition
NA	No Answer	One of the following elements: The student has not submitted a solution. No explanation. The student did not understand the text. The student tries to give reasons but does not give a relevant explanation.
AR	Arithmetic level	The student makes relevant calculations only on some numbers.
ALG	Algebraic level, without symbol	The student argues generally and correctly, using natural language without using symbols.
ALG+	Algebraic level, using a symbol	Three criteria must be fulfilled. (1) A symbol represents a number in the configuration. (2) Student correctly expresses the relationships between the numbers, and (3) student performs algebraic operations.

Findings and analysis

During the pattern-finding task (part 1 of the lesson), students have skillfully formulated the rules in the natural language during classroom discussions. Moreover, the generalization process emerged naturally:

S45: Is what we are talking about a feature unique to December or not?

S32 [Enthusiastically] There are seven numbers next to each other, that is, they are arranged by week, so for each month, it is [true].

While working on the magician's trick (part 2 of the lesson), the experience was written on the whiteboard (Table 3), and the one who figured out the trick could be the next magician.

Table 3: Numerical experience in class *d*

The magician	Teacher	Teacher	Teacher	S31	S31
Sum	33	66	69	73	75
Number thought	11	22	23	"You made a mistake."	25

Finally, students formulated the rule in their own words; they had to divide the sum of three numbers by three to get the number they had chosen.

When questioned why the trick worked, a faulty analogy emerged. Several students suggested that it should be divided by three because three numbers were added. The teacher took advantage of this situation and involved the class in discussing the mistake. The following dialogue is from class *b*, although almost the same happened in class *z*.

Teacher: [Adressing the faulty analogy.] Look at the diagonal patterns in the calendar! [She points to the corresponding pattern, see Figure *1* (right) that she deliberately left on the board at the beginning of the lesson.]

S03: [Uncertainly] The [number] six is the key.

S14: One number is six less and the other six more [than the number one thinks].

T: Let us write what he said! Let, for example, the number thought be 11. Thus $(11-6)+11+(11+6)=3\cdot 11=33$.

T: What if the thought number is not 11?

S: [Unrecognized student from the class] It can be anything

S14: Let it be x!

First, S14 tried to explain the rule with his natural resource, which was the basis of the teacher's general example. They concluded with the deductive proof $(x-6)+x+(x+6)=3\cdot x$. It is worth noting that the students explained the rule of merging starting from the problem situation; this is a point where the advantage of the "Sense-Making Algebra" approach is very apparent.

In class d, after the general examples, S46 proposed denoting the numbers with a, b, c, d, i.e., $a + b + c = d$. Although the need to use letters emerged in the student, she did not express the relationships between the symbols. The student's typical failure provided an appropriate opportunity for the teacher to draw attention to the principle that symbols must express the mathematical relations extracted from the situation.

In two classes, the students wanted to create their own magic tricks. The need for generalization was raised naturally in all three classes: do the series apply in other months, does the magic trick work in other directions, or other months? The use of letters and symbols occurs spontaneously in two of the three classes, with minimal teacher guidance. However, more teacher guidance and questions were needed in the third class.

The result of the coding of the follow-up test is shown in Table 4.

Table 4: Result of the follow-up test

	Task 1	Task 2	Task 3
NA	23	28	61
AR	6	5	0
ALG	25	23	0
ALG+	14	12	7
SUM	68	68	68

39 students gave algebraic level answers (ALG and ALG+) in the first task, i.e., 57% of all students. The result is similar in the second task, 52%, 35 out of 68 students. 21% of all students used symbols to express their reasoning (ALG+) in the first task, while 18% in the second task. The following examples are representative of each category.

NA (Task 1., S12) "Yes, that is true. Because if a number has a left and a right neighbor, both will be odd or even, and if you add two odd or even numbers together and divide by two, you always get an integer." What student writes is true in itself, but it does not explain the experience.

AR (Task 1, S22) See Figure 2. The student refers to only one particular case and concludes that the rule is always valid. (Translation of the text in the figure: "always true").

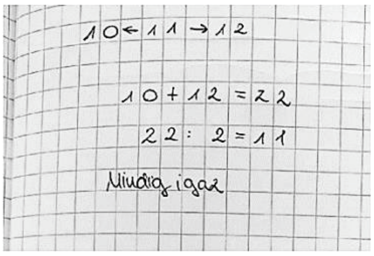

Figure 2: AR solution of Task 1 by S22

ALG (Task 1. S1) "The rule is valid because if you add a number one greater than the number you have chosen and a number one less than the number you have chosen, you will always end up with twice the chosen number.” The student correctly expresses the relationship between numbers in natural language; his reasoning is general, but he does not use symbols.

ALG+ (Task 3. S4) See Figure 3. The student represented the central element of the V-shape by z, correctly determined the other two elements ($z-8, z-6$), and performed the merge correctly ($3z-14$). The number needed is then determined by thinking backward from the sum. Translation of the text: “Add fourteen to the total and divide by three.”

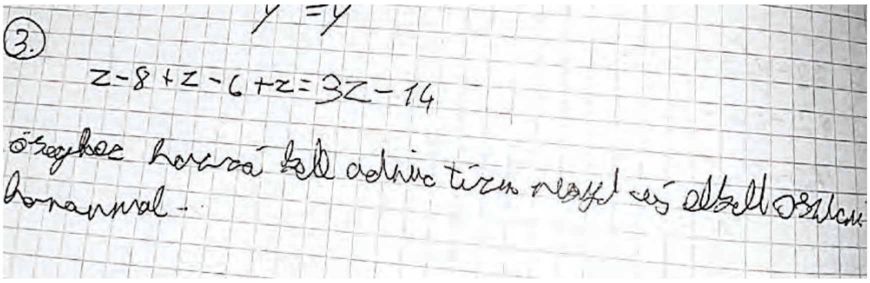

Figure 3: ALG+ solution for Task 3 by S4

The follow-up test also showed an example of an unexpected generalization: S15 wrote: “Conclusion: if you add the neighbors of a number (it can be second neighbors, third neighbors, fourth neighbors), you always get twice the number. It is because both neighbors differ by the same difference from the number.”

Discussion and pedagogical implications

The authors’ first research question was: how do seventh-grade students’ generalization activities manifest themselves during the “Sense-Making-Algebra” intervention? Consistent with Friedlander & Hershkowitz (1997), our research confirmed that students generalize and reason willingly and satisfactorily at the early algebra learning stage. A possible good choice for this process is the “calendar problem,” as presented by Huang et al., (2014). The mathematical structure is simple, based on addition, so it can be used to pose problems accessible to a wide range of learners. Furthermore, we found that making and justifying conjectures occurs naturally using this type of exploration task. Moreover, this study found that the “problem-oriented” teaching approach supported generalization and reasoning activities, providing a good space for classroom discourse (Kónya & Kovács, 2021). Generalization, in many cases, did not occur as an individual activity but as a consequence of group discourse. The starting point for generalization could be the students’ questioning in addition to the teacher’s initiative.

The second research question was: what resources do students employ to create their generalization? Students expressed their ideas verbally in natural language resources in classroom discourse, which facilitated generalization. The predominance of textual formulation remained in written work until the end of the learning cycle. However, about 20% of the students were already confident using symbols in problem situations in our experiment. Although we did not use project work as a learning environment, we saw an identical process described by Palatnik & Koichu (2017), who reported that students developed and justified claims, made generalizations, addressed why-questions, and established coherence among the explored objects.

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Recognition of an equation as an algebraic description of contextualised situations by fifth graders

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In this study we qualitatively explore the conceptual and semantic understanding of equations of elementary school pupils who were introduced to solving basic linear equations. We analyse the individual answers of 38 fifth grade students in a Czech public primary school to a paper-based task consisting of justifying whether the equation 5+2x=13 algebraically describes each one of 5 contextualised situations presented. Under a grounded theory approach, we provide a system of categories of the students' strategies to address this question. Our findings show the students' abilities to deal with the task and to infer true mathematical facts about equations, the wide variety and nature of students' strategies and the apparent dependency of the strategy on the system of representation of the situation.

Keywords: Equations, contextualised situations, elementary students, strategies.

Introduction

Gradually, algebraic activity is finding its place in primary school classrooms, following the guidelines of international organisations of Mathematics Education (National Council of Teachers of Mathematics [NCTM], 2000) and the evidence and conclusions from multiple research works on early algebra (e.g., Kaput, 1999). Getting acquainted with the notions of unknown amounts and variables, establishing dependency relationships between variables, developing functional thinking, representing information in different systems and transferring it from one to another, symbolising and using meaningfully algebraic notation are part of these algebraic activities. In this scenario, equations bring a great chance to deal with several of them at once. In the context of a task involving a linear equation and five contextualised situations verbally and pictorially presented (Figure 1) to elementary students of fifth grade, the aims of this research study are:

(a) Explore the students' abilities to identify and justify the given equation as an algebraic transcription of a particular situation,

(b) Describe the strategies used by students to address this question.

Background and theoretical framework

A branch of research studies in algebraic thinking have focused on epistemological analysis of scholar work with equations and the exploration of different aspects of children's performances when working with them, discerning in such a way a demarcation between arithmetic and algebra. Filloy and Rojano (1989) suggest the terms *arithmetic equations* to refer to equations with an unknown on only one side of the equal sign (so that undoing the operation at hand is enough to know its value), and *algebraic equations* to those where the unknown appears in both sides and therefore must be operated on. Balacheff (2001) understands that the shift from an arithmetic to an algebraic interpretation of equality corresponds to a shift of emphasis in the validation of the problem solution: from a *pragmatic control*, where the solution is validated arithmetically with reference to the initial context of the problem; to a *theoretical control*, where the solution is validated with reference to mathematical principles. Concerning the ways to formulate equations from verbal data, Herscovics (1989) recognises *syntactic* and *semantic translations* as different procedures, referring respectively to direct translation of key words to symbols, and the attempt to express the meaning of the problem. According to Hoch and Dreyfus (2004), "any algebraic expression represents an algebraic structure. The external appearance or shape reveals, or if necessary can be transformed to reveal, an internal order. The internal order is determined by the relationships between the quantities and operations that are the component parts of the structure" (p. 50). In this paper we refer to these notions as *external structure* and *internal structure* of an equation. We say that an equation *algebraically describes* a contextualised situation if there is a correspondence relationship between the equation terms and a set of elements of the context in such a way that the equality established between the relationships of the equation terms is also fulfilled between the homologous relationships of the corresponding elements of the context.

Other works explore the processes of mathematical reasoning of equations in primary mathematics lessons through operations with structures of computing-terms (Nührenbörger & Schwarzkopf, 2016) and how students can come to understand and use the syntactic rules of algebra based on their understanding about relations between quantities (Brizuela & Schliemann, 2014). These studies show that dealing with equations is not beyond ten years old students' mathematical understanding and that much more could be achieved if algebraic activities became part of the daily mathematics classes offered to elementary school children. In this sense, Figueira-Sampaio et al. (2009) propose a constructivist computational tool to assist in learning equations of first degree in primary school, to illustrate the idea of equilibrium and properties of the equality. Otten et al. (2019) report that the algebraic strategies such as restructuring, isolation and substitution that primary school students used when working with a hanging-mobile during their teaching experiment were later used by these students for solving linear equations in new contexts.

Research methodology

This study is part of a larger qualitative and descriptive research project, based on classroom interventions to explore the understanding of elementary students in algebraic activities.

Participants and design of the task

Thirty-eight fifth grade elementary students (10-11 years old) of mixed abilities in mathematics from the same public primary school in the city of Prague (Czech Republic) participated in the study. They were two class groups, both taught by the same teacher. Although equations are included in the second cycle (6-9 grades) of elementary school in the Czech curriculum (VÚP, 2017, pp. 31-34), the participant students had been briefly introduced by their teacher to symbolic letters and solving simple linear equations in the classroom. They had been taught by a method partially inspired in Hejný Method (Hejný, 2012). They had played with numbers and arithmetic operations subject to certain conditions and worked with unknown amounts. They had no prior experience with activities as the involved in our present study.

According to our research aims, we presented to the students the task shown in Figure 1.

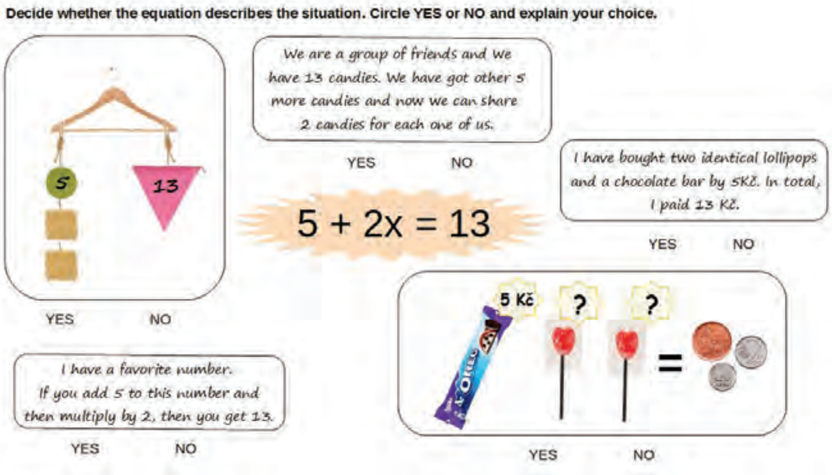

Figure 1: English version of the worksheet presented to the students

The worksheets for them were drafted in Czech language, the students' mother language in which they also wrote down their answers. Table 1 specifies the task variables considered.

Table 1: Task variables and features of the worksheet situations

	Friends	Lollipops	Labels	Number	Hanger
Representation	Verbal	Verbal	Pictorial	Verbal	Pictorial
Unknown	Number of friends	Price of lollipop	Price of lollipop	Favorite number	Weight of square
Order of appear.	13, 5, 2	2, 5, 13	5, 1, 1, 13	5, 2, 13	5, 1, 1, 13
Possible equa.	(13+5)/x=2	2x+5=13	5+x+x=13	(x+5)2=13	5+x+x=13
Described?	No	Yes	Yes	No	Yes

Collection of data and data analysis

The data collection was performed in one session at the beginning of the first semester of the Czech school year 2019-2020, during normal mathematics class time of the participant groups. The worksheets were given and administered by the teacher to the students with the firm instruction of doing the task individually and writing down their explanations with pen on the sheet (and its back if they needed it). During their work, the students did not receive any feedback about the correction of their answers or suitability of their strategies. Upon completion, students' worksheets were delivered to the research team and constitute the data for this qualitative research.

We considered as units of analysis the students' written answers to the different items of the task: selection of yes/no and corresponding explanation for each situation. We transcribed and translated the answers from Czech to English. After thorough reviews, we categorised the data following the principles of Grounded Theory (Corbin & Strauss, 1990). In agreement with our research goals, we establish categories of the focus of their performances and of their strategies, as shown below. Students' anonymity was ensured by assigning each a label: Si, i=1, …,38.

Focus of performance

• Resolution. Ignoring the given equation, the student tries to find a solution for each situation. If the student can solve it, she/he circles “Yes”. Otherwise, she/he circles “No”.

• Discrimination. Considering the given equation, the student decides—based on one or more strategies—whether this algebraically describes each situation. If the student thinks so, she/he circles “Yes”. Otherwise, she/he circles “No”.

Strategies

• Identification. The student establishes a one-to-one correspondence between the contextual elements of the situation and the terms of the given equation.

• Equation solving. ‣ Operational (O). The student follows the algebraic rules and standard solving equation process. ‣ Trial and error (T&E). The student assigns a value to the unknown and checks whether the equality of the equation holds, repeating the process until finding a solution or stopping after some trials. ‣ Memory and mental calculation (MC). The student does not annotate anything about the solving process, using own record of numerical facts and mental calculation.

• Divisibility. The student discusses the existence of solution relying on divisibility arguments.

• Solution checking. The student arithmetically checks whether the solution of the given equation is also a solution for the situation at hand.

• Equations comparison. The student formulates an equation for the situation at hand and compares it—without solving it—with the given equation.

• Solutions comparison. The student formulates an equation for the situation at hand and solves it, numerically comparing its solution with the solution obtained by solving the given equation.

• Analogy. The student recognises a situation as an analogy of another, copying the answer provided in it.

• Numbers comparison. The student compares the numerical data appearing in the statement/picture of the situation at hand with those of the given equation.

Note that, in contrast to the solutions comparison strategy, in the solution checking the student does not formulate any equation of the situation at hand.

Results

Focus of performance

As shown in Table 2, students’ participation in the task was notably high: 186 out of potential 190 answers were provided. More than the half of the students (20 of 38) focused their performances on discrimination—as requested by the task—, while the remaining students (18) did on resolution. Restricting ourselves to the first ones, it is remarkable the high rate of right answers. Interestingly enough, there were those who deliberately circled both “Yes” and “No” for the same situation. This is the case of S_{11} for the Hanger situation:

S_{11}: Yes, since 5+4=9+4=13. No, since 5+?+?=15 [he writes down two 5’s and two arrows from each one of them to each question mark].

He argues similarly in the Labels situation:

S_{11}: Yes, 5+2x=13|-5, 2x=8|:2 [vertical bars are part of his symbolism], 1x=4 [he writes 4 next to each question mark of the lollipop price labels]. No, because the price of lollipop could be 4.50.

After formulating an equation for it, getting its solution and applying the solutions comparison strategy, S_{11} also thinks in a different way: he considers another value for the unknown and checks that the equality of the equation does not hold for that value, what he offers as evidence that the equation might not describe the situation. The answer of S_{37} (focused on resolution) to Friends situation also illustrates this: “Yes: How many people were there? 9 people. No: 1 to 8 people”.

Table 2: Total number of students’ answers of each type

		Friends	Lollipops	Labels	Number	Hanger
Resolution (18)	Yes	12	13	14	4	10
	No	4	4	3	14	6
	Yes & No	2	1	1	0	1
	No answer	0	0	0	0	1
Discrimination (20)	Yes	0	19	18	1	17
	No	19	1	0	17	2
	Yes & No	0	0	1	1	1
	No answer	1	0	1	1	0

The lack of an explicit question in the statements prevented some students to identify an unknown susceptible of being involved in an equation. This is the case of S_{13} in Friends situation:

S_{13}: We do not have any unknowns. It is not a task, because there is no question to answer (it is all information).

Order of operations potentially matters. It turns interesting to analyse the only “Yes” answer in Number situation (restricting to discrimination as focus):

S_{17}: 5+2x=13, 5+2x=13|-5, 2x8=13|:2, x=4 [each step is reproduced as originally written by the student, typos included]. If you add 5 and multiply 2x, you get 13. 13-5=8, 8:2=4.

On one hand, she solves the given equation (getting x=4). On the other hand, she formulates the Number situation in her own words. Then, apparently in order to find the unknown favorite number, she tries to undo the operations that, applied on this number, give 13. She replaces addition by subtraction and multiplication by division, but she does not reverse the application order of the operations (she calculate (13-5):2=4 instead of 13:2-5). In this way she gets 4 as outcome and, comparing the solutions, she circles “Yes”.

Strategies

Table 3 shows the frequencies of use of each strategy by the students who focused their performances on discrimination (20), as requested by the task.

Table 3: Frequencies of the strategies used by students focused on discrimination

	Friends	Lollipops	Labels	Number	Hanger
Identification	0	2	2	0	3
(O, T&E, MC)	(6, 0, 0)	(3, 0, 3)	(6, 0, 0)	(3, 1, 0)	(4, 0, 0)
Divisibility	0	0	0	7	0
Solution checking	0	0	6	1	6
Equations comparison	6	6	1	6	3
Solutions comparison	6	7	3	5	3
Analogy	2	2	9	0	4
Numbers comparison	2	0	0	0	0
Total	22	23	27	23	23

As evidenced by the total number of strategies used in each situation, students often involved more than one strategy in their answers. The frequency of each strategy seems to be influenced by the representation system of each situation. In verbal situations, equations comparison together with solutions comparison and equation solving were the most used strategies, while in pictorial situations the most frequent were analogy and solution checking. Verbal statements awakened the students' need to formulate an equation describing the situation, while pictures let them more easily perceive analogies with other situations and they offered a scenario (the "blanks" of the pieces of cloth and of the lollipops price labels) to implement solution checking. In Friends situation (where the unknown is the number of friends), some students misunderstood the role of the number of candies, as they compared or related it to the value (4) of the unknown for the given equation:

S_{18}: 5 + two x = thirteen → x=4, 13-5=8, 8:2=4 [she solves the given equation]. No. Everybody would have to get 4 c.

S_8: No. This equation shows that one man would get 4 candies [referring to the solution of the given equation].

In the same situation, some students' answers reveal how important the external structure and numerical data were for them, obviating the internal relations:

S_5: No. It will not work because there should be 13 of anything.

S_{33}: No. 13+5=18. This task does not work because it comes out of 18 candies, but we need 13 candies in total.

S_{23}: Yes, because 1,2 [referring to Friends and Lollipops situations] are the same because the numbers are the same.

In contrast, other students referred to mathematical properties when comparing equations. In Lollipops situation, S_{13} alludes to commutativity (involving symbolic letters and numbers):

S_{13}: Yes. This equation would look 2x+5=13, but we can switch 5 and 2x [comparing it with the given equation 2x+5=13].

Both disambiguation and interpretation of the statements also become crucial. For the same situation, S_{10} justifies her negative answer as follows:

S_{10}: I say no, because it says "two identical lollipops and chocolate for 5 Kč" so this all together written "for 5 Kč" and then total costs 13 Kč. It is an illogical task. "My choice".

Some students used the same letter to symbolise different quantities. This is the case of S_{30} in his attempt of syntactic translation of the Number statement:

S_{30}: No because it would be: x+5= x, x·2 = 13 but it is not: 5+2x=13 and also does not happen x·2=13 since 13 is not a divisor of two. [He verbally exchanges multiple by divisor.]

In Hanger situation, this same student does not interpret the need for the numbers in the squares to be equal, but to add up to the difference (8) between the quantities in both sides (13 and 5):

S_{30}: Yes, because the two numbers under 5 can be any way, so there could be 2 times 4, or 5 and 3 etc.

Discussion and conclusions

In this work we show elementary students' abilities to recognise whether a linear equation describes different contextualised situations. We shed light on the individual strategies that fifth grade students newly initiated in solving equations used to discern this. We highlight the wide variety and nature of their strategies as well as the different frequencies of their use depending on the system of representation of the situations. Faced with verbally presented situations, students tended to formulate an equation based on syntactic and semantic translation of the statement, to compare it with the given equation. Faced with pictorially presented situations, their tendency was to allude to an analogy between the situation at hand and another one of the tasks, concluding then the same as they did in the analogous situation. Alternatively, in this case, they also preferred to check whether the solution of the given equation was a solution for the situation at hand. This finding can be used with teaching purposes. It is also remarkable that students used more than one strategy to justify their decisions and that they did not systematise their strategies but adapting them at each situation. This is illustrated by the fact that seven used divisibility arguments to discuss the Number situation.

The students' performances indicate that they were able to infer mathematical truths not explicitly taught to them when briefly introduced to equations in the classroom. Behind their use of solution checking and solutions comparison strategies are the following facts, connected to the internal structure of equations (related to the equivalence of linear equations and uniqueness of solution): (a) If the solution of a linear equation is a solution for a situation, then this equation can algebraically describe that situation; (b) if two linear equations have the same solution, then they can algebraically describe the same situations. In contrast, most of the students who used the equations comparison strategy based their strategy on the external structure of the equations. Although this did not prevent them to get right answers in this task, it is not a safe practice, since two equivalent equations can have different external structures. Another practice that stands out in the students' performances is the frequent mathematical checking of their answers (e.g., S_{18}: 1 lollipop costs 4Kc, 13-5=8, 8:2=4, 4+4+5, 8+5=13). This validation was mostly pragmatic (Balacheff, 2001), although also a few students referred to mathematical rules in their justifications (see S_{13} above). Based on our findings, as many other authors (e.g., Brizuela & Schliemann, 2004), we conclude that algebraic activities and tasks in which the notion of linear equations is implicitly or explicitly involved could be worked at this educational level, serving to enrich students' mathematical thinking.

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Proofs without words: focus on argumentation as a tool to investigate the link between visualization and generalization processes enacted by students

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The study presented in this paper is based on data collected within a teaching experiment aimed at the implementation of a didactical path focused on proofs without words. Our research aim is to investigate the link between visualization and generalization processes enacted by students during their work on the tasks that constitute the didactical path. To pursue our aim, we analyzed students' argumentative processes, documented within their written protocols and within excerpts from the video-recordings of small group activities and classroom discussions. Through this analysis, developed by means of Radford's theory of objectification and Duval's levels of apprehension, we highlight the role played by visualization in supporting generalization.

Keywords: Argumentation, explanation, generalization, proofs without words, visualization.

Background

A proof without words is a proof that makes use of a graphical artifact, a picture or other visual mean, to reconstruct a deductive process that justifies a given statement or an equation (Nelsen, 1993). There is debate around whether a proof without words really qualify as a proof (Gierdien, 2007, p.1); anyway research in Mathematics Education highlighted the powerful role of proofs without words in learning and teaching mathematics. Hanna (1989) showed that involving students in interpretation and analysis of activities on proofs without words could shift the focus of teaching from proofs that prove to proofs that explain.

In 2019 we designed and implemented a teaching experiment focused on proofs without words. It was a pilot study that involved three 10th grade classes of a scientific high school near Rome, with a total of 66 students. In this paper we will focus on the second stage of this experiment trying to highlight the link between the visualization process and the generalization one through the tool of argumentation. By argumentation we mean, according to Stylianides et al. (2016), the discourse or rhetorical means used by an individual or a group to convince others that a statement is true or false. It may involve exploration of examples, generation or refinement of conjectures, and production of arguments for these conjectures.

Theoretical framework

The analytical framework we refer to is made up of two main components, the apprehension levels described by Duval and the theory of objectification introduced by Radford.

Following Duval (2006, p.107) "Mathematical objects, in contrast to phenomena of astronomy, physics, chemistry, biology, etc., are never accessible by perception or by instruments". He goes on to state that “the only way to have access to them [mathematical objects] and deal with them is using signs and semiotic representations”. This is the reason why “representation and visualization are at the core of understanding in mathematics” (Duval, 1999, p.1). Furthermore visualization, in the Duval framework, is a process that goes over the vision. “Unlike vision, which provides a direct access to the object, visualization is based on the production of a semiotic representation” (Duval, 1999, p.7). A semiotic representation does not show things as they are, it “shows relations or, better, organization of relations between representational units” (Duval, 1999, p.7). In our work students are asked to explain the relations they see between images of proofs. The *perceptual apprehension* refers to the recognition of figures and subfigures; *sequential apprehension*, is used when describing a construction of a figure based on mathematical properties; the ability to recognize such mathematical properties and explain them refers to the *discursive apprehension* while the *operative apprehension* is a process by which students, operating on the figure, identify the solution of a mathematical problem.

Following Radford (2001, p.81), generalization must be considered as “one of the more natural human semiotic process”; looking into the way students “deploy and mobilize signs to accomplish mathematical generalization”, Radford points out three levels of generalization. *Factual generalization*, that is, “a generalization of numerical actions in the form of an operational scheme that remains bound to the numerical level” (p.82), at this level the semiotic tools for the objectification are the perceptual semiosis, the generative functions of language and the operational schemes. The factual generalization is achieved by students by means of rhythm in the utterances, ostensive gestures, adverbs like always, every, etc. and making reference to space and time (i.e. *the next figure*…). The second step in generalization is *contextual generalization*, at this level students are able to write explanations regarding their conjectures and mathematical operations, in these explanations they perform actions on abstract objects (*you take that square…*) but still have a *perspectival view* of mathematical objects. The last level is *symbolic generalization*, it occurs when students talk about mathematical objects in an impersonal way (*the sum of the first n squares…*, *the n-th figure…*) and using a more specific terminology, these aspects show that the symbolic generalization is reached when students succeed in a de-subjectification process.

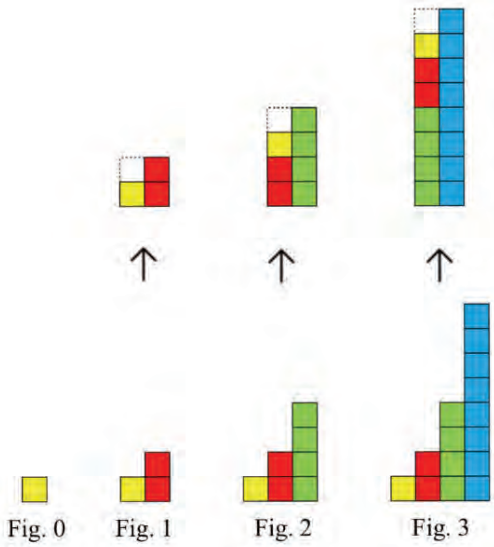
These two theorical frameworks start from different systems of principles since Duval takes constructivism as his point of departure, while Radford’s theory stems from a sociocultural perspective on learning. Nevertheless, we decided to implement a *networking* (Prediger et al., 2008) of the two theories, integrating them locally in order to study the visualization process, by means of the apprehension levels, and the generalization process referring to Radford theory. This decision is motivated by the fact that, beyond the effectiveness of the level of apprehension in analysing the visualization processes, we noticed that students, explaining and justifying on the way they look at the figure, use the figure itself as an artifact of communication and signification. This is in line with the sociocultural perspective that grounded the theory of *semiotic mediation* in the sense of Vygotsky (Cole et al. 1978) and is one of the starting points for Radford’s theory.

Context of the study

The study presented in this paper took place in Spring 2021 (totally 8 hours) and involved a 10^{th} grade class of an Italian high school (26 students). Due to the pandemic period, the presence of students in the classroom was limited to 50%. Half of the students were present in the classroom, while the other half was at home connected using a video-communication service (Google Meet) and a Multimedia Interactive Whiteboard.

The students were divided into 6 non-homogeneous groups, in agreement with their teacher. Each meeting was articulated in an initial phase of small group activity, followed by a classroom discussion. The students were invited to discuss the questions as a group and to write down the answers that would be later collected by the teacher. After the small groups' work, the students were involved by the researchers (the authors of this paper) and by their teacher in a discussion regarding the strategies used by the different groups to deal with the tasks they had to face.

The didactical path was designed as a sequence of worksheets focused on questions based on examples of proofs without words. One of the worksheets is represented in Figure 1.



The number of squares in the previous figures can be represented as follows:

Figure	Number of squares
Fig. 0	$1 = 2^0$
Fig. 1	$1 + 2 = 2^0 + 2^1 = 3$
Fig. 2	$1 + 2 + 4 = 2^0 + 2^1 + 2^2 = 7$
Fig. 3	$1 + 2 + 4 + 8 = 2^0 + 2^1 + 2^2 + 2^3 = \ldots$

1. Imagine to continue the sequence of figures, what will be the number of squares (not considering the dashed one) in Figure 5? And in Figure 9?
2. Explain how to determine the total number of squares in any figure in the previous sequence.
3. Explain which features of the figures helped you in constructing answers to the previous questions.
4. These figures aim to justify a theorem on the sum of the powers of two. Try formulating this theorem.
5. Write a reasoning to justify the theorem you formulated in question 4.

Figure 1: Worksheet 3

All the worksheets included questions about a number sequence, as the one in Figure 1. The students were asked to find some elements of the number sequence under investigation (question 1, Figure 1) and to explain how to find out a generic element of the sequence (question 2, Figure 1). To make them reflect on the visualization processes they enacted, the students were asked to explain which features of the figures helped them in formulating their answers (question 3, Figure 1). Nevertheless we add a table with numbers (see figure 1) to let the students be free to choose the semiotic representation they felt more comfortable with: the picture that recalls a proof without words or the table that recalls a more traditional approach. From the analysis of data collected we could see that only one group preferred the table and did not use the picture in answering questions 1 and 2.

The last two questions aimed to prompt students identify the theorems behind the proofs without words proposed in the worksheets and reflect on the role played by figures in supporting the formulation of conjectures and in proving theorems (questions 4 and 5, Figure 1).

Research questions and methodology

Our research hypothesis is that argumentation could represent an effective tool through which the link between visualization and generalization could be explored. The analysis we developed was aimed at investigating this link. The research question that guided our analysis is: referring to the level of generalization and of apprehension reached by students, what characterizes the link between visualization and generalization when students face activities focused on proofs without words?

In order to answer this question, we collected the protocols written by students during the group work and the audio and video-recordings of all the activities (both face-to-face activities and activities carried out at distance), including group work and classroom discussions. We then analyzed the whole material, transcribing some excerpts we considered as interesting in order to establish a link between visualization and generalization.

For our analysis, we referred to the main components of our theoretical framework: Duval’s levels of apprehension and the generalization categorization proposed by Radford.

In this paper we will analyse the work of only two groups. The reason for this choice is that the students in these groups felt the need to explain in depth their written protocols during the class discussion even if they didn’t grasp the whole solution of the task. This didn’t happen in other groups, probably because students were not used to argumentation neither written nor oral. Sometimes (it is the case of one of the groups of our experiment) the solution of the task was easily grasped, and the visualization process remained implicit.

We follow the evolution of each group along three steps: (1) the discussion in the small group, (2) the students’ written protocols, and (3) the classroom discussion with the teacher and the authors. We chose to focus our analysis on the excerpts regarding questions 2 and 3 of the worksheet in Figure 1, since they enable us to focus on the way students formulate their generalizations (question 2) and explain how they looked at the figures, that is the visualization processes that supported their generalizations (question 3).

Regarding step (2), we should note that the condition under which the experiment was conducted (remote meeting) strongly limited the semiotic potential of the interaction between students. In particular, it was not possible to collect any data about students’ use of gestures or facial expressions during the small group discussions.

The final discussion in class which is reported here was led by one of the authors with the aim to foster students’ explication of their visualization and generalization processes.

Data analysis

Analysis of data taken from group 1

The first excerpt is taken from the discussion of group 1. They are working together during a remote meeting. One of the participants share the screen with the figures of worksheet 3 (Figure 1). The group is trying to answer the number 2 question while they look at the figures on the screen.

20	Giada:	…But…I think I understood that the next figure is always equal to twice the previous figure plus one.
21	Giorgia:	Yes, it’s true.
22	Lea:	Giada, can you repeat what you said?
23	Giada:	So… for instance figure 5 is equal to twice the figure 4 plus 1.
24	Lea:	Ah, ok, I got it, so you suggest writing down this instead of the whole formula?

In line 20 Giada comes up with an idea, “the next figure is always equal to twice the previous figure plus one”, and she shares the idea with the group. The group discusses and they write it in their protocol. From these data it emerges, according to Radford theoretical framework, a *contextual generalization*, it goes over the numerical level; they find out a property that holds for each figure. However, we argue that they still have a *perspectival view* of the mathematical objects, seen in the expression “the next figure …the previous figure”, so the generalization process is not complete, they did not reach to the symbolic level.

In their written answer to question 2, the students wrote: “*The number of squares of the next figure, with respect to the figure we are considering, is doubled plus one*”. And when they are asked to clarify the visualization process (question 3) they discuss and then write: “*Colours that helps to differentiate the different columns, the increasing shape, the figure number*”.

During the discussion in class, we tried to push forward the investigation about the visualization, asking the students for explanations about the written answers. This can be seen in the following excerpt involving one of the authors and a student from group 1.

13	Giorgia:	Given a figure of the sequence, to get the number of squares of the following figure, we doubled the number of squares of the given figure and add 1.
14	Author:	Where did you see it in the figure?
15	Giorgia:	… for instance [*pointing at figure 0 of the worksheet displayed on the whiteboard*] the figure 0… 1 times 2 is 2, plus 1 is 3, and so on…
16	Author:	Did the figure help you in discovering that?
17	Giorgia:	Yes.
18	Author:	How?
19	Giorgia:	It helped us because we saw that … this one [*pointing at the yellow square*]… in order to get to the red one [*pointing at the two red squares*] … we multiply times two and then we add the one that we had before.
20	Author:	Which one?

21	Giorgia:	The yellow one.
22	Author:	And what about the next steps?
23	Giorgia:	… all this stuff [*pointing at figure 1, lower part*] times 2 is 6, the same as 1,2,3,4,5,6… the red and green squares [*pointing at the red and green squares of figure 2, lower part*] plus 1 that is, once again, the yellow one.

We think that, in the previous excerpt, Giorgia has carried out a very peculiar visualization process. Even if she didn't use the hint given by the upper part of the sequence of figures of the worksheet, she is able to perform an advanced process of visualization. She actually doesn't stop herself to the visual perception of the figures, she goes over, seeing the "relations", and the "organization of relations between representational units" (Duval, 1999). In this case the representational units are the squares. She highlighted the relation between the yellow square of figure 0 and the two red squares of figure 1 (the yellow square doubled and "became" the two red squares of figure 1) and did the same for each figure in relation with the next one. Moreover, in line 19, Giorgia describes the construction of any figure of the sequence starting from the previous one (*sequential apprehension*).

Analysis of data taken from group 2

We now move to the analysis of the material of group 2. We will notice that, even if the two groups obtained two different formulas and looked at the figures in two different ways, the characterization of the link between the visualization and the generalization process is similar.

7	Francesco:	…Anyway… these figures…are simply increasing…every next column is one plus the previous one and then…I didn't get why… they … put them on top of each other...
8	Federica:	I think they go like 2 x 2 is 4, next 4 x 2 is 8, the last column, so 8 x 2 I think … so figure 4 will be 16, the last column and so…
9	Francesco:	Are you talking about…the one that has been added?
10	Federica:	Yes, the blue one...
11	Francesco:	Ok, yes, it is multiplied by 2
…		
16	Francesco:	… anyway… the column that has been added is a number… and the other columns equal that number minus 1, so… for instance, in figure 5 they will be 32 plus 31…
17	Federica:	Yes, is it so.

Francesco, in line 16, makes an effort towards generalization. In this moment he is contributing to the work of the group with an idea that goes over the numerical level. After the discussion, the group wrote down their idea in the written answer to question 2 as follows:

Last column times 2, minus 1

We notice that this protocol is different from the formula given by group 1. Even the generalization process is slightly different because group 2 explained how to compute the number of squares of any figure without any reference to the previous figure, so they go over the *perspectival view*. Anyway, the generalization level of group 2 is still a *contextual generalization* because they did not proceed to the *des-embodiment* in their mathematical description; they actually need to know the number of squares of the *last column* in order to compute the number of squares of the whole figure. Since we want to investigate the link between this generalization process and the visualization, we analyze the written answer to question 3 given by group 2. It was:

It helped us the way the squares are arranged in the upper part of the figures.

Therefore, the way students in group 2 look at the figure is also different with respect to group 1; specifically, students of group 2 look at the upper part of the figures while students from group 1 look at the lower part. In the next excerpt, taken from the class discussion with one of the authors, the visualization process fielded by the students in answering question 2 emerges.

41	Francesco:	A way to find the number of squares is taking the last column, multiply such a column times 2 and subtract 1.
42	Author:	What tells you that?
43	Francesco:	We can argue that from the figure here above, i mean … you see that it always misses a square… is this one times 2 minus 1 [*pointing at figure 1*]
44	Author:	Ok
45	Francesco:	…Or this times 2 minus 1 [*pointing at figure 3*]…because…I mean…you can tell it by this figure [*pointing at the upper part of figure 3*]

In line 41 Francesco describes an action on abstract objects (*taking the last column*); this kind of operation, characteristic of the *contextual generalization*, is suggested by the figure, as Francesco explained in line 43. Furthermore, from line 45 we argue that the description of Francesco is indeed the description of the construction of the whole sequence of mathematical figures, as pointed out by Duval describing the *sequential apprehension*.

Conclusions

Generalization and visualization are two essential components of the mathematics learning and teaching process. Often the link between these two components is obscured by many factors. We refer to the attitude of students toward argumentation (as we already discussed) and to the peculiar conditions in which the experiment took place (remote meetings) which affect the social-communicative dimension of the interaction between students. In the small-group activities mediated by screens, students' semiotic assets are restricted. The fact that *implicit and mutual agreement of face-to-face interaction must be replaced by objective elements of social interaction*, according to Radford (2001), guides the activity toward the generalization. Thus, the balance between visualization and generalization it's unlikely to arise. However, the two examples discussed in this paper show that pushing forward the argumentation process is an effective way to clarify the connection between the visualization and the generalization processes fielded by students when they approach a mathematical problem. The link between visualization and generalization process was outlined in other studies (e.g., Barbosa, 2011). The contribution of this paper is to characterize, in terms of level of apprehension, the generalization achieved by 10th grade students who do not succeed in the whole task. In the analysis we outlined that, even if the two groups looked at the figure in different ways and produced different formulas, their contextual generalization was guided by a sequential apprehension. So, in these two cases, the link between the visualization and the generalization process is characterized by the match of sequential apprehension and contextual generalization. It seems to us to be worthy of further study to investigate the visualization process of groups of students that reach a symbolic generalization in solving similar tasks.

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The role of generalized arithmetic in the development of early algebraic thinking

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The purpose of this study is to characterize early algebraic thinking in relation to students' understanding of generalized arithmetic concepts and procedures. The participants of the study were 203 Grade 6 students. Data were collected from two tests which focused on number, operations, equivalence, and equality properties. The first test included arithmetic tasks which could be solved by either strategies based on calculations or strategies based on arithmetic structure, while the second test included algebraic tasks which involved letters-symbolic representations. The results indicate that students who were able to solve arithmetic tasks using arithmetic structure were also able to solve algebraic tasks. The findings imply that generalized arithmetic abilities underpin the syntactical manipulation of algebraic expressions and exemplify why seeing structure is a highly important process of early algebraic thinking.

Keywords: Generalized arithmetic, seeing structure, early algebraic thinking.

Introduction

Generalized arithmetic has been described by Kaput (2008) as a basic algebra content strand in the K-12 curriculum. In the context of early algebra, generalized arithmetic provides students with the opportunity to work with properties that are inherent to number and arithmetic operations and understand that these properties are generalizable (Kieran et al., 2016). This kind of activity entails the process of generalizing, but also the complementary process of seeing structure.

Placing attention on structure means turning attention away from carrying calculations (Hewitt, 2019). Looking for and becoming aware of structure pertains to the manipulation of number and arithmetic operations as objects which could be decomposed and recomposed in various ways (Kieran, 2018). While the process of generalizing has been explored in extend, more research is still needed in respect to the process of seeing structure, especially within generalized arithmetic contexts (Kieran, 2018). This study aims to contribute towards a better understanding of the role of generalized arithmetic and arithmetic structure awareness in the development of early algebraic thinking. Specifically, the study aims to exemplify why and how Grade 6 students' abilities in generalized arithmetic may be related to their understanding of algebra symbolic language and the meaningful manipulation of algebraic expressions.

Literature review

Generalized arithmetic

Generalized arithmetic abilities involve understanding of (a) the properties of numbers, such as odd and even numbers (Bastable & Schifter, 2008), (b) the way operations relate to each other and the use of properties of operations for transforming arithmetic expressions (Britt & Irwin, 2011), and (c) the relational conception of the equals sign, which is not merely a signal for performing operations, but also a signal for presenting the relationship of equivalent expressions and preserving the same value on both sides of an equation (Jones et al., 2012). These abilities could be developed within activities that are abstracted from computations and do not address the use of letters (Kieran et al, 2016). They facilitate the development of structural awareness which in turn enables the interpretation of arithmetic expressions as encoding the operational composition of numbers (Subramaniam & Banerjee, 2011). For example, the number 989 could be decomposed and recomposed into various structures depending on different relations, i.e., $9 \times 109 + 8$ or $9 \times 110 - 1$ or $989 = 9 \times 10^2 + 8 \times 10^1 + 9 \times 10^0$) (Kieran, 2018).

Several studies explored the significance of structural awareness within generalized arithmetic contexts for students' successful engagement in algebraic activity. Warren (2003) indicated that understanding the use of properties of operations, like the commutative and associative property of addition, is critical for interpreting an operation as a general procedure that reflects a specific relationship between quantities, either known or unknown. Similarly, it was found that the use of opposite and inverse relations for simplifying arithmetic expressions enables students to understand that performing calculations in a serial way is not always necessary and that similar procedures could be applied for manipulating algebraic expressions (Linchevski & Livneh, 1999).

Britt and Irwin (2011) also indicated that helping students to understand the relational function of the equals symbol supports their meaningful engagement with algebra as a formal symbolic language. For example, using compensation in addition to transform $27 + 6$ into $30 + 3$ provides a foundation for generalizing that $a + b$ is an equivalent expression to $(a + c) + (b - c)$. Likewise, students can more easily understand that the transformations performed for solving an equation, like $4x - 8 = 3x + 1$, should always preserve the equivalence of the two sides (Knuth et al., 2005).

Another example concerns the result of operations with classes of numbers that have specific properties, like odd and even numbers. This kind of activity was found supportive for developing generalization, since multiple instances are compressed into a single, unitary form (Blanton et al., 2018). At the same time, the process of seeing structure is triggered, since the result of operations with specific classes of numbers is justified by their additive and multiplicative structure. At a later stage, students become able to represent the structure of operations with specific classes of numbers using letters i.e., odd + odd = even as $(2n + 1) + (2m + 1) = 2n + 2m + 2 = 2(n + m + 1)$.

Aim of the study

The current paper aims to extend prior research by providing insight into students' behavior when they confront a variety of generalized arithmetic tasks and whether this behavior influences their abilities in solving algebraic tasks. Therefore, this paper aims to contribute into further characterizing abilities in generalized arithmetic and exemplifying its role in developing early algebraic thinking abilities. Specifically, this paper focuses on describing Grade 6 students' abilities in solving generalized arithmetic tasks that involve (a) properties of operations, (b) equivalence and equality properties, and (c) properties of numbers. Moreover, this paper aims to illustrate why and how these abilities may be related to students' capability for solving respective algebraic tasks.

Method

Participants

A group of 203 6th grade students from 4 urban schools (7 classrooms) participated in the study. The schools were chosen through convenience sampling, but attempt was made to invite schools from various socio-economic backgrounds to participate in our study. All students had received similar instruction in respect to algebra which was based on the content of their mathematics textbooks. Mathematics textbooks are the same for all students and are used at a national and compulsory level. These lessons involved the use of properties of numbers for simplifying calculations and finding the value of numerical expressions, the use of variables for representing unknown quantities, the formulation of algebraic expressions, the concept of equality and equality properties, equation solving (one-step and two-step equations), the concept of function, the use of different representations for representing functional relationships, and modeling problems.

Procedure and materials

Two tests were administered to the students. To distinguish more easily the two tests, we named the first test "arichthmetic" and the second "algebraic". The first test included 10 tasks that involved numbers, whereas the second test included 10 corresponding tasks that involved letter-symbolic representations. By corresponding tasks, we mean that for each task in the first test, there was a corresponding task in the second test which involved the same concept (Table 1). Each test consisted of three categories of tasks: (a) properties of operations (3 tasks), (b) equivalence and equality properties (4 tasks), and (c) properties of numbers (3 tasks). The tasks in the arithmetic test could be solved based entirely on calculation procedures, but they could also be solved through procedures based on the identification of general properties. The tasks in the algebraic test could only be solved through algebraic procedures.

Table 1: Examples from the arithmetica nd algebraic tests

	Arithmetic tasks	Algebraic tasks
Properties of operations	Calculate the result. Show your work. (a) $27 + 89 + 262 - 92 + 92 - 89 =$ (b) $846 \div 28 \cdot 28 \div 2 =$	Vasiliki claims that she can find the value of the following expressions. Do you agree or disagree with Vasiliki. Explain your answer (a) $420 - 3\chi - 4\psi + 3\chi + 3\psi + \psi =$ (b) $35 \cdot 3\chi \div 4\psi \div 3\chi \cdot 4\psi =$
Equivalence and equality properties	Find the missing number. Explain your thinking. $\square + 17 = 15 + 24$	$18 + A = 20 + B$ Based on the above equality, express the relationship between A and B.
Properties of numbers	Name number y, using one of the following labels: odd number, multiple of 10, 2-digit number. $(a)\ y = (10 \times 3) + 8$ $(b)\ y = 246\,779\,568 + 1$ $(c)\ y = 10 \times 1245$	Name number y, using one of the following labels: even number, multiple of 10, odd number. $(a)\ y = 2a$ $(b)\ y = 2b + 1$ $(c)\ y = 10c$

Research questions

To fulfill the general aim of the study, the following research questions were addressed:

(1) Is there a statistically significant difference between students' performance in the arithmetic and algebraic tasks?

(2) Is there a statistically significant difference between students' performance in the tasks concerning (a) properties of operations, (b) equivalence and equality properties, and (c) properties of numbers, when these tasks include only numbers and when they include letters?

(3) Are there any differences in the strategies that low, medium and high achievers employ to respond to these types of tasks?

Analysis

The first step of the analysis was to mark the tests and give 1 point to correct solutions and 0 points to erroneous solutions. The second step was to assess the type of strategy that students used to respond to each task in the first test. Researchers categorized students' strategies as (a) strategy based on claculations, (b) strategy based on arithmetic structure, and (c) no strategy. By "strategy based on calculations" we mean the application of calculation procedures in a serial way or trial and error strategies. By "strategy based on arithmetic structure" we mean the identification of general properties, as these are instantiated as relationships between mathematical objects (Mason et al., 2009). No strategy was used to characterize responses that did not include a description of the way an answer was reached. To ensure reliability of the categorization of students' strategies, a subset of responses was randomly selected and coded by each author individually. Then, the authors compared their coding. Disagreements were resolved through discussion, until consensus was reached. The data analyses were conducted with IBM SPSS Statistics 27. Descriptive analysis was conducted for the variables and paired sample t-test to compare the students' results in the two tests.

Presentation of Results

To respond to the first research question, we analyzed the means and standard deviations of students' responses in the two tests. Students had a mean performance of .777 in the first test and .664 in the second. The difference in the means of these two tests was statistically significant ($p<0.001$).

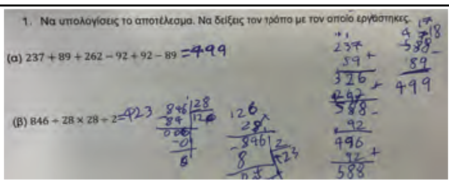
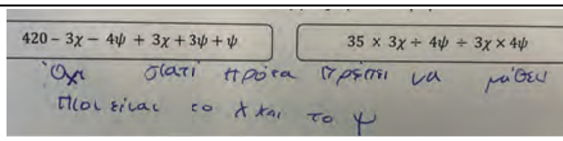
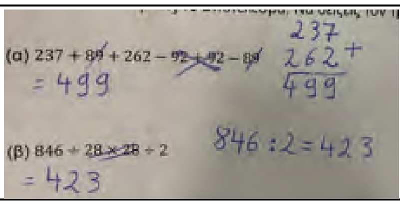
To respond to the second research question, Table 2 shows the means and standard deviations in the three different categories of tasks and the results of the paired t-tests. The mean performance in all the categories of the arithmetic tasks was always higher than in the respective algebraic tasks. Paired t-test analysis showed that this difference was always statistically significant.

Table 2: Students' performance in the three categories of tasks in the two tests

	n	Mean	SD	t	df	p
Properties of operations (arithmetic test)	203	.757	.202	7.465	202	<0.01
Properties of operations (algebraic test)	203	.654	.241			
Equivalence/equality properties (arithmetic test)	203	.787	.214	8.104	202	<0.01
Equivalence/equality properties (algebraic test)	203	.674	.239			
Properties of numbers (arithmetic test)	203	.789	.293	5.590	202	<0.01
Properties of numbers (algebraic test)	203	.664	.294			

To respond to the third question, we first examined the strategies used by students to solve the arithmetic tasks. Table 3 illustrates examples of students' responses to an arithmetic task that involved the use of opposite and inverse operations. Students' responses to the respective algebraic task are also presented. Christos was one of the students that used a strategy based on calculations for solving the arithmetic task; he performed calculations in a serial way, without noticing the components of the expression and the relationship between them. In the respective algebraic task, he answered that he could not find the value of the expression, because he needed first to know the values of the variables. Dora was one of the students that used a strategy based on arithmetic structure for solving the same arithmetic task. Dora seems to have immediately looked at the arithmetic expression and noticed the relationship between opposite and inverse numbers. She was also successful in solving the algebraic task. As Mason et al. (2009) have highlighted "to decide without any calculation is a form of relational thinking, of appreciating arithmetic structure" (p.21).

Table 3: Examples of students' responses to arithmetic and algebraic tasks

	Arithmetic task	Algebraic task
	Calculate the result. Show your work. $(a)\ 27 + 89 + 262 - 92 + 92 - 89$ $(b)\ 846 \div 28 \cdot 28 \div 2$	Vasiliki claims that she can find the value of the following expressions. Explain your answer. $(a)\ 420 - 3\chi - 4\psi + 3\chi + 3\psi + \psi$ $(b)\ 35 \cdot 3\chi \div 4\psi \div 3\chi \cdot 4\psi$
Arithmetic strategy (Christos)		 *"No, because you first need to know the value of χ and ψ."*
Arithmetic structure strategy (Dora)		 "Yes, it is correct because if you add a number and then subtract it, the number stays the same. The same stands for multiplication and division."

To better understand the profile of students who used each type of strategy, we identified three groups of students, based on students' mean score in the two tests, We labeled as (a) low achievers those with a mean score in the two tests equal or below 0.654 (N=58, 28.6% of the population), (b) middle achievers those with a mean score between 0.655-0.854 (N=88, 43.3% of the population), and (c) high achievers those with a mean score 0.855-1 (N=57, 28.1% of the population). Then, we examined the type of strategy that low, middle, and high achievers used to solve the arithmetic tasks in the first test. Tables 4, 5, and 6 show the percentage of students who used strategies based on arithmetic structure (S.S), strategies based on calculations (S.C), and no strategy (NoS.).

The percentages in Tables 4, 5, and 6 tell the same story. In the three categories of tasks, low achievers, consistently used more strategies based on calculations. High achievers systematically used strategies based on arithmetic structure. Middle achievers used more frequently strategies based on arithmetic structure than the low achievers but less frequently than the high achievers. However, we should note that the percentage of strategies based on arithmetic structure in solving properties of numbers tasks was much smaller in relation to the other two categories.

Table 4: Percentages of different strategies in properties of operations tasks

Item	Concept	Low Achievers N=58 (28.6%)			Middle Achievers N=88 (43.3%)			High Achievers N=57 (28.1%)		
		S.S	S.C	No S.	S.S	S.C	No S.	S.S	S.C	No S.
Ar1a	Opposite operations	15.5	72.4	12.1	58.0	38.6	3.4	86.0	14.0	0
Ar1b	Inverse operations	20.7	46.6.	32.8	70.5	22.7	6.8	94.7	3.5	1.8
Ar2a	Commutative property	19	53.4	27.6	56.8	35.2	8.0	70.2	29.8	0
Ar2b	Commutative property	10.3	55.2	34.5	39.8	53.4	6.8	56.1	43.9	0
Ar2c	Commutative property	20.7	43.1	36.2	56.8	35.2	8.0	66.7	31.6	1.8
Ar2d	Commutative property	12.1	48.3	39.7	45.5	43.2	11.4	52.6	43.9	3.5
Ar3a	Distributive property	3.4	63.8	32.8	6.8	81.8	11.4	21.1	73.7	5.3
Ar3b	Distributive property	1.7	60.3	37.9	5.7	75.0	19.3	22.8	66.7	10.5
Ar3c	Distributive property	1.7	55.2	43.1	3.4	71.6	25	22.1	68.4	10.5

Table 5: Percentages of different strategies in equivalence/ equality properties tasks

Item	Concept	Low Achievers N=58 (28.6%)			Middle Achievers N=88 (43.3%)			High Achievers N=57 (28.1%)		
		S.S	S.C	NoS.	S.S	S.C	NoS.	S.S	S.C	NoS.
Ar4a	Equality -additive structure	5.2	43.1	51.7	9.1	69.3	21.6	26.3	52.6	21.1
Ar4b	Equality -additive structure	3.4	39.7	56.9	11.4	63.6	25.0	22.8	50.9	26.3
Ar4c	Equality -multiplicative structure	5.2	46.6	48.3	14.8	63.6	21.6	28.1	49.1	22.8
Ar4d	Equality -multiplicative structure	8.6	8.6	82.8	25.0	33.0	42.0	36.8	31.6	31.6
Ar5	Equality - scale representation	46.6	20.7	32.8	70.5	22.7	6.8	94.7	5.3	0
Ar6	Equality – additive structure	6.9	48.3	44.8	10.2	77.3	12.5	10.5	84.2	5.3
Ar7a	One-step equation	0	0	100	9.1	0	90.9	17.5	0	82.5
Ar7b	One-step equation	0	0	100	11.4	0	88.6	22.8	0	77.2
Ar7c	Two-step equation	0	0	100	10.2	1.1	88.6	24.6	0	75.4

Table 6: Percentages of different strategies in properties of numbers of tasks

Item	Concept	Low Achievers N=58 (28.6%)			Middle Achievers N=88 (43.3%)			High Achievers N=57 (28.1%)		
		S.S	S.C	NoS.	S.S	S.C	NoS.	S.S	S.C	NoS.
Ar8	Consecutive/odd numbers	3.4	29.3	67.2	13.6	60.2	26.1	15.8	64.9	19.3
Ar9	Consecutive numbers	3.4	32.8	63.8	1.1	63.6	35.2	0	71.9	28.1
Ar10	Multiples of 3	5.2	25.9	69.0	4.5	67	28.4	12.3	78.9	8.8

Discussion and conclusion

The purpose of this study was to characterize students' early algebraic thinking in respect to generalized arithmetic abilities. The results showed that students who were able to manipulate structure within arithmetic expressions were also able to manipulate the syntactical structure of algebraic expressions where calculations are impossible. Therefore, this study offers empirical evidence to theoretical postulations which addressed that seeing structure is significant for the emergence of early algebraic thinking, equally important to generalizing (Kieran, 2018).

The results revealed that some students used strategies based on calcilations for solving the arithmetic tasks, whereas others used strategies based on arithmetic structure. It was found that low achievers more often used strategies based on calculations to solve the arithmetic tasks. In contrast, high achievers did not perform calculations right away, but they analyzed and made use of the embedded structure. In a similar way, they approached respective algebraic tasks. This result shows that to achieve the full potential of generalized arithmetic in enhancing early algebraic thinking, instruction needs to support all students to look for structure.

The tasks in the two tests captured three basic categories of generalized arithmetic, as these were described by previous theoretical and research studies: (a) properties of operations, (b) equivalence and equality properties, and (c) properties of numbers. In each category, students' awareness and use of the embedded properties seem to have enabled the manipulation of both arithmetic and algebraic expressions as objects that can be decomposed and recomposed in flexible ways (Kieran, 2018; Subramaniam & Banerjee, 2011). The fact that the algebraic expressions involved letters did not seem to obscure their manipulation. This finding is also supportive of the idea that understanding of the formal and symbolic language of algebra might be rooted in students' understanding of the irrelevance of performing calculations and the fact that expressions, even if they involve only numbers, could remain uncalculated to declare a kind of relationship between two quantities or more.

Further analysis of the data revealed that the lowest percentage of strategies based on arithmetic structure was applied to tasks that involved properties of numbers, implying that these tasks might be more difficult or less emphasized in schools. This kind of strategies was more requently used in tasks that involved equivalence and equality properties, implying that the number of students in the sample who could understand the equals sign relationally was higher. The highest percentage of strategies based on arithmetic structure was observed in tasks that involved properties of operations. These findings denote that the instruction that these students received might have placed an emphasis on specific generalized arithmetic concepts and procedures and less attention was paid to others.

Summing up, this study illustrates through empirical evidence how and why generalized arithmetic comprises a critical content strand of early algebra lessons. As to how, the results imply that generalized arithmetic offered students the opportunity to decompose and compose arithmetic expressions and transfer this process when the format of an expression changed from numerical to letter-symbolic. As to why, it seems that generalized arithmetic provided a concrete foundation for placing attention to structure and not to calculations. Future research may examine students' behavior through qualitative methods to clearer describe the transfer of working from arithmetic to algebraic contexts and describing pedagogical instructional environments which establish this transfer for all students.

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Grade 3 Students' Algebraic Thinking Abilities: An empirically Validated Theoretical Framework

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The present study proposed and empirically validated a theoretical framework describing algebraic thinking abilities for Grade 3 students. Four algebraic abilities were incorporated in this framework: 'relational manipulation of equalities/inequalities', 'compose and decompose number and arithmetical expressions', 'functional thinking' and 'representing-modeling'. The study involved 124 students. Analysis showed that the three first constructs compose a general thinking ability that could be considered as an index of Grade 3 students' capacity to respond to a variety of algebraic thinking tasks. This general ability was a strong predictive factor of students' representing-modeling ability.

Keywords: Early-algebra, functional thinking, generalized arithmetic, modeling.

Introduction

Initiatives worldwide have underlined the significance of early algebra in mathematics education and stressed that to meet the goal of developing a fundamental algebraic understanding, students in elementary school should be involved in activities that prepare them for algebra in later grades (National Council of Teachers of Mathematics, 2000; Stephens, et al., 2017). Early algebraic thinking can be coherently conceptualized as a synthesis of different content strands, concepts, processes or forms of reasoning that relate to the ideas of equivalence, properties of numbers and operations, variable, proportional reasoning, modelling, and functional thinking (Chimoni, et al., 2018; Kaput, 2008; Kieran, et al., 2016). Research studies suggest that elementary school children could engage in sophisticated ways of algebraic thinking, such as generalizing, representing, justifying, and reasoning with mathematical structure and relationships (Stephens et al., 2017). However, the existence of different approaches to the notion of algebraic thinking, in terms of content strands, concepts, processes and reasoning forms makes it challenging to provide a coherent description of the nature of algebraic thinking. The relation between young students' algebraic thinking abilities, remains under-researched. In this study, as a first step, we explore Grade 3 students' thinking algebraic abilities to identify their relations based on empirical data.

Theoretical Considerations

Several research efforts concentrated on the analysis of the nature and content of algebraic thinking and provided a list of characteristics of algebraic thinking in all grades (e.g., Blanton et al., 2011). An overarching definition suggested that algebraic thinking is a 'habit of mind' that enables students to identify and express mathematical structure and relationships (Blanton & Kaput, 2005). Researchers claimed that a fundamental element of algebraic thinking is generalization, that is the ability to see the general in the particular (Kaput, 2008; Kieran, 2007). Chimoni et al. (2018) provided a synthesis of the literature, suggesting four basic dimensions in terms of (a) content strands, (b) concepts, (c) processes, and (d) reasoning forms: A number of theoretical frameworks conceptualized algebra in terms of content strands and concepts. For instance, Kaput's (2008) model asserts that generalizing and symbolizing are tightly linked in that symbols allow generalizations to be expressed in a stable and compact form, throughout generalized arithmetic, functional thinking, and modeling languages. Generalized arithmetic entails noticing relationships between numbers, the manipulations of operations and their properties, and the transformation and solution of equations (Chimoni et al., 2018). Functional thinking has generally been defined as the process of building, describing, and reasoning with and about functions (Stephens, et al., 2017). Modeling languages involves the use of symbols for developing models, and re-translating between models and situations. Kaput's framework was empirically validated by Pitta-Pantazi et al. (2020) for grades 8 and 9.

Kieran (2007) conceptualized algebra as a multidimensional activity that encompasses various types of tasks and ways of thinking. She suggested three types of activities: *generational, transformational*, and *global, meta-level*. The generational activities refer to the generation of equations and expressions in various situations and involve exploration of problem situations and numerical and geometrical patterns that lead to the formulation of generalizations, and exploring numerical relations. The transformational activities refer to the transformation of expressions by applying specific rules and involve conceptual understanding of algebraic objects. The global, meta-level activities are not strictly algebraic in nature, but algebraic tools are needed to be investigated and involve general mathematical processes, such as proving, studying functional relations, and identifying structure. Further, Kieran (2007) suggested that the smooth transition from arithmetic to algebra could be achieved by focusing on (a) the relations of numbers and not only on calculations; (b) relations between operations; (c) representation and solution of problems; (d) the use of numbers and letters; (e) the meaning of the equal sign. This list of activities provides a comprehensive lens to examine algebraic thinking in terms of content strands and specific algebraic concepts. In addition, Drijvers et al., (2011) distinguished four strands in algebra teaching: patterns and formulas; restrictions; functions; and language. Patterns and formulas involve searching for regularity, patterns and structures and embeds generalization. Restrictions entail manipulating equations or inequalities, such as finding what value of the unknown variable satisfies the required conditions. Functions reflect algebra as the study of relations and functions. Language concerns algebra as a symbolic system. Driscoll (1999) provided a description of algebraic thinking in terms of habits of mind that fits with the frameworks suggested by Kieran (2007) and Drijvers et al. (2011). He suggested 'doing and undoing mathematical processes', 'identifying and representing functional rules', and 'thinking about computations independently of particular numbers'.

In terms of processes and reasoning forms, a number of research studies proposed noticing, representing, and justifying with mathematical structure and relations as core processes for searching similarities and differences (Blanton et al., 2011; Jeannotte & Kieran, 2017). Abductive, inductive and deductive reasoning were proposed as important types of reasoning forms in algebraic situations (Chimoni et al., 2018). For instance, abductive reasoning is necessary at the stage where individuals develop a prediction about a plausible generalization (Rivera & Becker, 2007), while inductive reasoning is required to grasp a generality through noticing how a local commonality hold across all terms. The variety of the frameworks described above shows the complexity in defining early algebraic thinking by clarifying its differences from arithmetical thinking and defining well-accepted algebraic abilities for young students (Chimoni et al., 2018). Furthermore, to date, these frameworks have not been extensively validated based on empirical data for young students.

The Present Study

The main purpose of the study is to examine, based on a synthesis of well-accepted theoretical frameworks, Grade 3 students' algebraic thinking abilities. We define algebraic thinking abilities as the capacity of the individuals to perform various early algebra tasks and include both relevant knowledge, reasoning skills, and algebraic processes, such as generalizing, representing, justifying and reasoning with mathematical relationships (Blanton et al., 2011). We propose algebraic thinking abilities in an attempt to describe the structure of algebraic thinking by amalgamating various types of algebraic tasks, processes and practices. The proposed framework is based on (a) Kaput's (2008) algebra core content areas, (b) Blanton et al.'s, (2011) description of algebraic processes, (c) Kieran's (2007) algebraic types of activities, (d) Drijvers et al.'s, (2011) algebra strands, and (e) Driscoll's (1999) algebraic habits of mind. The framework (see Figure 1) involves four distinct but correlated factors and defines a measurement model of young student's algebraic thinking abilities. The innovative aspect of the proposed framework is the fact that it integrates aspects of the fore-mentioned frameworks that meet Grade 3 students' early-algebraic experiences and needs. It is grounded on embedding early algebraic processes and practices in specific content areas to define algebraic thinking abilities that Grade 3 students should develop based on well-accepted frameworks, contemporary curricula, and policy documents. The proposed thinking abilities can be used as measurement indicators of Grade 3 algebraic thinking.

The first factor, *relational manipulation of equalities/inequalities*, corresponds to students' capacity to manipulate equations and equalities, find the value of the unknown in equations and inequalities that are represented in the form of an empty box to be filled or in balance scale equalities and inequalities that are suitable for Grade 3. It embeds Driscoll's (1999) idea of doing and undoing mathematical procedures. The second factor, *compose and decompose numbers and arithmetical expressions*, includes students' ability to conceptualize the relations and properties of numbers, relations between operations, to reflect and make predictions about computations and solve problems independently of particular numbers, based on number-property generalizations. In addition, it includes the ability to interpret equalities expressed in different forms (Blanton et al., 2011). Students need to decide as a result of noticing an arithmetic or computational structure, before proceeding to a second action. Conclusively, it conceptualizes students' capacity to make numeric and arithmetical computations, solve numeric related problems, and interpret equivalence expressions by grasping mathematical structure and relations in these situations, abstracting from arithmetic properties and objectifying the generalized properties. The third factor, *functional thinking*, adopts the definition proposed by Pittalis et al. (2020) for young students, suggesting that it encompasses student's capacity to notice, generalize, and abstract relations between covarying quantities/variables, represent these relations through natural language, symbols, and appropriate representations and use the generalized representations in problem solving situations. Finally, *representing-modeling languages*, represents students' capacity to represent word-problem situations that embed relations among quantities using a variety of representations, such as number sentences, literal symbols, and figural models (Kaput, 2008; Kieran et al., 2016). The research questions of the study were: Could different algebraic thinking tasks be categorized on the basis of the factors of the proposed theoretical framework? What is the structure and the relations between the proposed algebraic thinking abilities, as they are projected through Grade 3 students' responses?

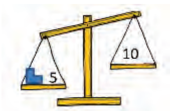
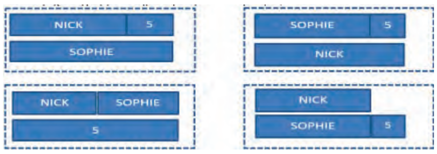
Measures

The test items were adopted or developed based on previous research studies (Blanton et al. 2011; Chimoni et al., 2018). The multiple-choice tasks were considered as correct or incorrect, while the open tasks were given partial marks. The first factor, *relational manipulation of equalities/inequalities*, was measured by three types of tasks: completing the missing value in an equity, finding the value of the unknown in an equation that was presented in a balance-scale form, and finding one possible value of an unknow variable in an inequality that was presented in a pictorial form (Drijvers et al., 2011). Variables in balance-scale equations and inequalities were represented by pictorial symbols (see Table 1). The second factor, *compose and decompose numbers and arithmetical expressions*, was measured by four types of tasks. The first type required students to solve number-property problems, such as conceptualizing odd and even numbers. The second type of tasks provided students the result of an addition or subtraction and asked students to find the sum or difference based on the derived facts (Driscoll, 1999). Generalization of the derived fact was necessary to conceptualize the structure of the provided and the given calculations and grasp the differences. The third type required students to predict whether 'big calculations' result to odd or even numbers, without making any calculations but noticing the odd/even property of the numbers and generalizing the result of adding/subtracting odd/even numbers. The fourth type provided students two equalities in the form of balance-scale situation. Student had to find the heaviest toy by interpreting the equalities and making assumptions regarding the relations of the involved objects. The *representing-modeling languages* factor was measured by three types of tasks that required translating a word-problem situation by noticing the relation between the involved quantities. In the first type students had to represent the problem in a number sentence form, in the second one in a model form (Kieran et al., 2016), and the third one in literal symbols. *Functional thinking* was measured by tasks that entail four modes of thinking (Pittalis et al., 2020): recursive patterning (14 tasks), covariational thinking (5 tasks), correspondence-particular (7 tasks), and correspondence-general (7 tasks). Due to space limitations, we do not present the functional thinking tasks.

Participants, Procedure and Data Analysis

Consent forms for students to participate in the study were distributed to two urban primary schools in Cyprus. The schools, teachers and students involved participated voluntarily, thus our sampling was a convenient one. Parents' consent forms were returned for 95% of the students, resulting in a sample of 124 Grade 3, 9-year-old students (65 males and 59 females).

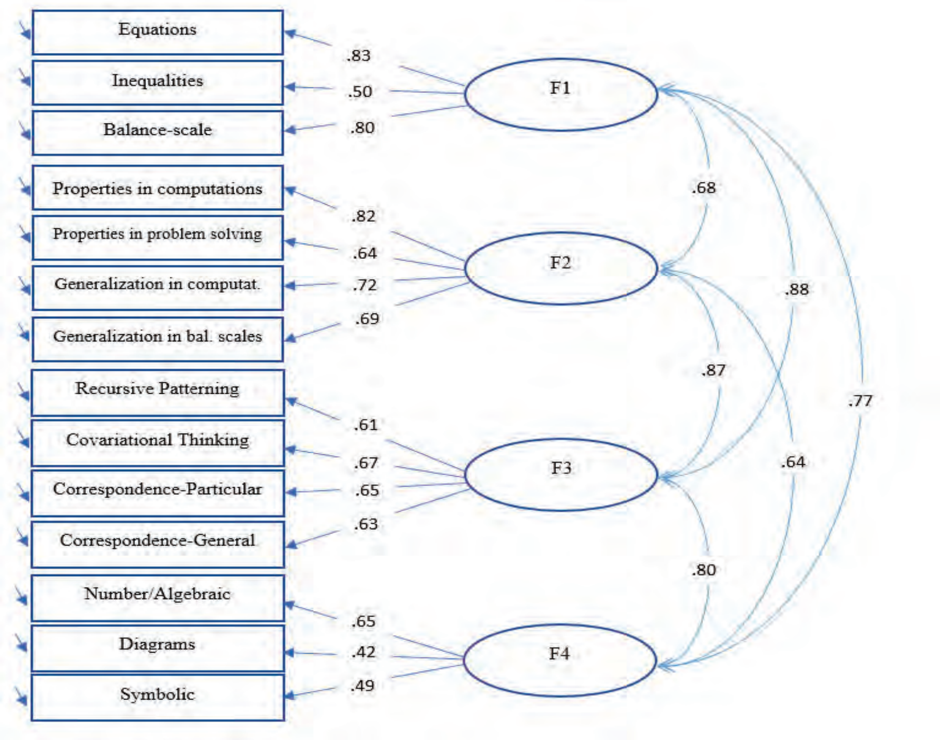
Table 1: Examples of tasks

Factor	Type of task	Example
Relational manipulation of equalities and inequalities	(a) Completing equalities (b) Solving equations in balance scale form (c) Solving inequalities	(a) Fill in the missing number: $12 + 4 = \square + 6$, $3 \times \square = 15$ (b) Find the value of the symbol. (c) Find a possible value for the symbol.  
Compose and decompose numbers and arithmetical expressions	(a) Solving number-property problems (b) Make calculations based on derived facts (c) Make predictions based on number-properties (d) Interpret equities	(a) I had in my pocket more than 12, but less than 15 euro. I spent all my money to buy pencils that cost 2 euro each. How much money I had in my pocket? (b) Find the following if you know that 118+8=126. 128+8, 118+18, 118+9, 218+8, 119+18 (c) Does the following give even result (without making calculations)? 122+18, 15+45, 478+222+444, 333-115 (d) Which toy is the heaviest? 
Representing-modeling languages	(a) Number sentence (b) Model (c) Literal Symbols	(a) Which number sentence represents the problem? *I had 4 pencils. I gave 3 pencils to my brother. How many pencils have I got now?* $3 + 3 + 4 = \square$ $3 \times 4 = \square$ $4 + 3 = \square$ $4 - 3 = \square$ (b) Which model represents the situation? *Nick has 5 stamps less than Sophie.*  (c) Which relation represents the situation? *Katy (K) and Lia (L) have altogether 3 stamps less than Mary (M).* $K + L - 3 = M$ $K + L + 3 = M$ $K + L = M + 3$ $K + L + M = 3$

The tasks of the study were split into two parts. Consideration was given to the number of tasks and time required to be approximately the same. Each part was administered in the form of a written test during one school period. The two parts were administered in two successive weeks. The instructions were provided in written and verbal form. Confirmatory factor analysis was used to examine the validity of our proposed, a priori model by using MPLUS 8.0 (Muthén & Muthén, 1998-2007). To evaluate model fit, three widely accepted fit indices were computed: χ^2/df should be <2; the Comparative Fit Index should be >.9; and the root mean-square error of approximation (RMSEA) should be <.08. The Cronbach's alpha index of internal consistency was very good (α=.81).

Results

Confirmatory factor analysis (CFA) was used to evaluate the construct validity of the model. Based on the results, the a-priori model matched the data set of the present study and determined the "goodness of fit" of the hypothesized latent construct. Analysis showed that the fit-indices of the hypothesized model were excellent (χ^2/df=1.27, CFI=.97, and RMSEA=.05), validating empirically the fit of the structure of the model to the empirical data. Figure 1 presents the observed variables (different types of tasks in Table 1), the corresponding factor loadings and the factor correlations. CFA showed that the factor loadings of the tasks were statistically significant and most of them were rather large, ranging from .42 to .83, giving support to the assumption that all latent factors were adequately measured by the observed variables. In accordance with our theoretical assumption, all measures were clustered into four first-order factors in the expected factor-loading pattern. Thus, analysis showed that algebraic thinking consists of four interrelated abilities that is (a) relational manipulation of equalities/inequalities, (b) compose and decompose numbers and arithmetical expressions, (c) functional thinking, and (d) representing-modeling languages. The correlations between the four abilities were significant (see Figure 1) and ranged from .64 to .88.



p<.05, **F1:** Relational Manipulation of Equalities/Inequalities, **F2**: Compose and Decompose Numbers and Arithmetical Expressions, **F3**: Functional thinking, **F4**: Representing-modeling Languages

Figure 1: The standardized solution of the proposed framework

To investigate the relations between the four algebraic thinking abilities, we examined the fit to the data of alternative structural models, hypothesizing a direct sequential path between the four factors or the existence of a higher-order thinking ability. The model that had the best fitting indices (χ^2/df=1.25, CFI=.97, and RMSEA=.04) showed that F1, F2, and F3 compose a higher-order factor, F100, reflecting students' general algebraic ability to manipulate relationally equalities and inequalities, compose and decompose numbers and arithmetic structures, generalize and abstract functional relations (see Figure 2). Analysis showed that F100 could accurately explain students' variances in F1, F2 and F3, suggesting that young students develop these abilities in parallel, their development is interrelated, and they share equally important contribution in building up a general functional thinking ability. F100 could be considered as an index of students' readiness to engage in algebraic thinking situations by noticing, interpreting, abstracting, and effectively using structure in arithmetic situations, number and operations properties, and quantity relations. It is the consequence of the correlations between F1, F2 and F3 that underlies a synthesis of specific arithmetic and algebraic thinking abilities. The regression coefficient of F100 on F4 was 0.76 ($p<0.05$), indicating that F100 is a strong predictive factor of students' modeling languages ability using a variety of representational forms. It could be supported that F100, as a general arithmetic-algebraic ability predicts rather accurately 'representing-modeling languages' that has a salient algebraic-nature.

Figure 2: The relations between algebraic thinking abilities

Discussion

The contribution of the study lies on the empirical evaluation of a proposed model that unpacks the dimensions of Grade 3 students' algebraic thinking. The results of the study showed that Grade 3 students' variances in algebraic situations might be modelled by four distinct and interrelated algebraic thinking abilities; relational manipulation of equalities and inequalities, compose and decompose numbers and arithmetical expressions, functional thinking. and representing-modeling languages. Structural analysis showed that the three abilities, *relational manipulation of equalities and inequalities*, *compose and decompose numbers and arithmetical expressions*, and *functional thinking* compose a general algebraic thinking ability that can be considered as an index of Grade 3 students' capacity to respond adequately in a variety of arithmetic-algebraic thinking tasks (Blanton et al., 2011, Driscoll, 1999). This general ability proved to be a strong predictive factor of students' ability to represent word problems and situations using number sentence, models or symbols. It could be supported that *compose and decompose numbers and arithmetical expressions* facilitates grasping the numerical relations in a problem and conceiving how the quantities set an equivalence or an equation, *functional thinking* contributes in noticing the relations between the involved quantities in the word situation, and *relational manipulation of equalities and inequalities* to discern the known and the unknown in the formed expression. The results of the present study are important in terms of teaching implications. The framework helps teachers to get a better understanding of algebraic thinking and the specific type of abilities that Grade 3 students should develop. Furthermore, the description of algebraic thinking abilities may inform teachers about students' potential difficulties and thinking requirements in a variety of situations.

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Early algebra: Simplifying equations

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Placed within the early algebra research field (Blanton et al., 2017; Cai & Knuth, 2011; Kieran, 2018; Kilhamn & Säljö, 2019), this article focuses on young students' understanding of basic algebraic ideas around equations. The article seeks to contribute to the field by shedding light on Grade 3 students' meaning-making processes underpinning the simplification of equations and the algebraic operations involved. In the first part, I present a theoretical conception of algebraic thinking. I also describe two non-alphanumeric semiotic systems that played an important role in the students' dealings with algebra. In the second part, I discuss two episodes of students simplifying $ax + b = cx + d$ *equations.*

Keywords: Algebra, equations, isolating the unknown, semiotics.

Algebraic thinking

One of the most enduring problems with which mathematics educators have been confronted is the problem of characterizing algebra and clarifying what makes it different from arithmetic. Two main solutions have been suggested. One consists in equating algebra with the *use of letters*. The other consists in conceiving of algebra as focused on *operations* rather than on results. While the first solution offers a very narrow conception of algebra—impeding teachers from recognizing algebraic thinking in activities based on types of mathematical representations different from letters—the second one offers a very narrow conception of arithmetic, which becomes demoted to simple computation.

In previous work (Radford, 2014) I have suggested three elements to characterize algebraic thinking:

(1) *Indeterminacy of magnitudes*: algebraic thinking involves *indeterminate* magnitudes. These can be unknowns, variables, parameters, etc.

(2) *Denotation*: the indeterminate quantities involved must be named or symbolized. This symbolization can be carried out in several ways. Alphanumeric signs can be used, but not necessarily. The denotation of indeterminate quantities can also be symbolized by means of natural language, gestures, unconventional signs, or even a mixture of them.

3) *Analyticity*: algebraic thinking (a) calculates/operates with indeterminate magnitudes *as if they were known* and (b) treats the mathematical relations featuring determinate and indeterminate magnitudes (equations, formulas, expressions, etc.) in a *deductive* manner.

Simplifying equations

Drawing on the aforementioned conception of algebraic thinking, in what follows, I report on the results of a teaching-learning activity in a Grade 3 class (8-9-year-old students). The activity was based on the use of two non-alphanumeric semiotic systems: a *Concrete Semiotic System* (CSS) and an *Iconic Semiotic System* (ISS) through which students could translate simple word-problems into linear equations.[1] The CSS is comprised of material objects: a) paper envelopes that each contain the same unknown number of cardboard cards; b) cardboard cards, and c) the equal sign. The envelopes played the role of unknowns while the card played the role of concrete numbers (constants). The ISS is derived from the CSS: it replaces concrete objects with iconic drawings {, , =, ↑}. The additional "arrow" sign replaces *actions* performed on concrete cards or envelopes of the CSS during the process of simplifying equations. The students could substitute the arrow by simple lines indicating that a card or envelope (or sets of) are removed. The range of problems that can be formulated in natural language and translated into the CSS and ISS is very limited, but it is enough to ensure that young students have their first encounter with algebraic thinking.

The research question that this paper seeks to address is about the identification of the teacher and students' meaning-making processes underpinning the understanding of algebraic techniques of isolating the unknown in one-unknown linear equations. Following the methodology of the theory of objectification (Radford, 2021), the data analysis involves a multimodal investigation of teaching-learning activity where students work in small groups and participate in collective discussions.

In Grade 2 the students started being familiarized with the isolating-the-unknown procedure using the CSS (Radford, 2017). At the beginning of the teaching-learning activity that I investigate here (which was the first Grade 3 activity on equations), the teacher organized a general discussion around the equation $3 + x = 7$. (Of course, no alphanumeric symbolism, was used in Grades 2 and 3). The students discussed various solving procedures: trial and error, comparison of terms (more on this below), and the isolating-the-unknown procedure. In Grade 3 the isolating-the-unknown procedure was not yet the students' first choice. The teacher had to ask, referring to what they had learned in Grade 2: "What do we mean by isolate? If I tell you, I'd like to isolate the envelope . . . " Cyr, one of the students, answered: "Does that mean like putting it alone?" When the teacher asked Cyr to articulate the idea, Cyr went to the blackboard and removed one card after another from each side of the equation, showing the procedure. The isolating-the-unknown procedure remained *shown* with actions rather than articulated with words. The teacher rephrased Cyr's actions: "If you remove one [card] on this side, what do you do?" Cyr answered: "I remove another one from there (the other side of the equation). Isolating-the-unknown procedure was a key aspect in the systematization of algebra conducted by Arab mathematicians in the 8th and 9th centuries (Al-Khwārizmī and others; see Oaks & Alkhateeb, 2007). It involves operations with known and unknown magnitudes to simplify equations. Mathematicians called these simplifying operations *al-gabr* and *al-muqābala*, and it is from the former that our modern term algebra borrows its name. By working with Cyr and by thematizing actions through language, the teacher strives to enable the students to reach a deeper level of understanding of the ideas underpinning the algebraic procedure. In the next sections, I discuss the work of one small group, focusing on two $ax + b = cx + d$ equations.

[1] Here is an example of a simple word problem: "Sylvain and Chantal have some hockey cards. Chantal has three cards and Sylvain has two cards. Their mother puts some cards in three envelopes and makes sure to put the same number of cards in each envelope. She gives one envelope to Chantal and two to Sylvain. Now the two children have the same number of hockey cards. How many hockey cards are inside each envelope?" (Radford, 2017, p. 18).

The equation $2x + 1 = x + 6$ in the CSS and the ISS

The two equations were given in the ISS. They are translations of a story in which two children have cards and envelopes. Each envelope has the same unknown number of cards, and both children have in total the same number of cards (see footnote 1). The exercise of translating stories of this type into the ISS was done in Grade 2 and continued in Grade 3. In this section I discuss the students' dealings with the equation $2x + 1 = x + 6$ (Figure 1.1) and in the next section I discuss the equation $3x + 1 = 5 + x$ (Figure 1.2).

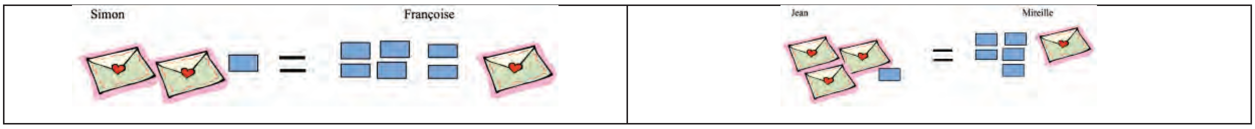

Figure 1. The equations $ax + b = cx + d$ as presented to the students in the ISS

Using a kit of envelopes and cards, the students were asked to make an equation and solve it, then draw their procedure. The idea was, hence, to have the students solve the equation first in the CSS, then using the ISS. The students made the equation in the CSS (Figure 2.1). Then, they drew the equation in the ISS. Elsa says: "We must remove that (*she circled the card on the left side of the equation*) so that there are just envelopes, do you remember? (*Then she removes one card on the other side)* 1, 1." *(Figure 2.2).* The answer is found by the *comparison method* (i.e., the students compare the equal to the equal and associate the remaining parts of the equation: in this case, one envelope on the left side is equal to the envelope on the right; hence, the other envelope is equal to the five remaining cards).

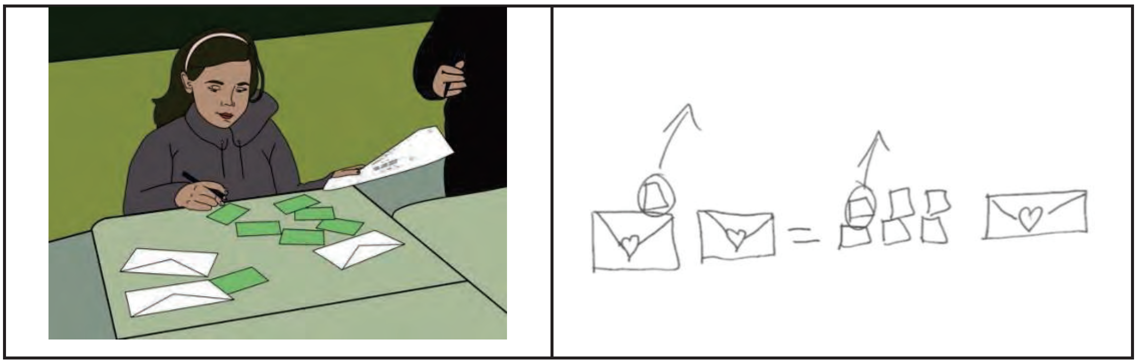

Figure 2. Solving the equation $2x + 1 = 6 + x$ in the CSS and the ISS

The teacher arrives and asks the students to explain their procedure. The students construct again the equation in the CSS. They remove one card from each side of the equation. The teacher says: "You are in the process of isolating! . . . How many envelopes do you want on one side?" Puzzled by the question, the students look at each other. One moment ago, Elsa mentioned the idea of having envelopes on one side. The teacher's intervention pushes the conversation further. On the one hand, the teacher acknowledges that the students are in the process of isolating the unknown. On the other hand, she raises a question that deals with something that has not been considered by the students. It is this unconsidered aspect of the simplification of the equation—a mathematical operation that would lead from $2x = x + 5$ to x equal to something—that puzzled the students.

1 Teacher: You want to know how many cards there are in ONE envelope (*she points to the envelope several times when she says ONE*) . . . First of all, you did this (*she removes a card from*

each side) you removed a card . . . Okay, what happens now? There are 2 envelopes (*pointing to the envelopes on one side of the equation*), then (*pointing to the objects on the other side of the equation*) 1 envelope and 5 cards.

2 Cora: We counted all these (*points to the cards*). It's 5. So, it (*pointing to 1 of the envelopes*) should have 5 too (*see Figure 3.1*).

3 Teacher: How do you know?

4 Elsa: We are going to remove (*she removes 1 envelope from the left side; see Figure 3.2*).

5 Teacher: You're removing 1 envelope?

6 Elsa and Cora: Yes. (*Elsa removes an envelope from the other side as well; Figure 3.3*).

7 Teacher: Why did you choose to do that?

8 Cora: Because this (*the sides of the equation*) must be equal.

9 Elsa: because we must remove; because there must be only 1 envelope left (*she takes the envelope that is left*)

10 Teacher: Is it okay to remove 1 envelope and then 1 envelope? Is your equation still equal?

11 Cora: Yes!

Figure 3. The students and the teacher discussing the equation $2x + 1 = x + 6$

In Line 1 the teacher starts simplifying the equation as the students did. She says: “First of all, you did this” and removes one card from each side. Then, in an encouraging tone, she asks “What happens now?” In Line 2 Cora resorts to the comparison method, but the verbal articulation of ideas leaves important relations unaccounted for. These are the relations that the teacher asks for in Line 3. In Line 4 Elsa starts removing one envelope from each side. The teacher wants to make sure that the students understand the idea behind the “removing” operation. So, in Line 7 she asks for reasons. In Lines 8 and 9 the students offer two answers: Cora’s focuses on the conservation of the equality between both sides of the equation; Elsa’s focuses on the idea of ending up with one envelope. In Line 10 the teacher wants again to make sure that there is a clear understanding of the actions that are carried out to simplify the equation. When the teacher leaves, the students come back to the equation in the ISS and remove one envelope from each side (Figure 3.4).

So far, the isolating-the-unknown procedure has necessitated the application of a key operation: *removing* equal things from both sides of the equation. In the next equation an additional mathematical operation is required. Let’s turn to the students’ investigation of this equation.

The equation $3x + 1 = 5 + x$ in the CSS and the ISS

The students tackle the equation $3x + 1 = 5 + x$. They construct the equation in the CSS and, instead of solving it with the help of concrete materials, they draw the equation.

Cora starts by removing one envelope from each side. After that, she removes one card from each side (Figure 4.1).

12 Elsa: You only removed 1, but there must be only 1 envelope left. That's a problem. (*They think for a while; then Elsa continues*). Four [cards], but there's not another envelope here (*points to the right side of the equation*).

13 Cora: There are 4 cards left, that's 4, we must remove these cards (*she circles the 4 remaining cards on the right side of the equation*) . . . And here (*she points to 1 of the remaining envelopes on the left side of the equation*) there are 0 [cards].

14 Elsa: Yes but look! If there is 0 [cards] in the envelope, this (*pointing to the envelope on the right side of the equation*) will be 4 and this (*pointing to an envelope on the left side*) will be 1 [meaning perhaps 0]. But the 2 [envelopes] must have the same exact (*she points to the drawing*), the 2 [envelopes] must have the same number [of cards].

15 Cora: (*Explaining the idea again*) We removed that (*the 4 cards*).

16 Elsa: Then, there are 0, but there must be some cards [in the envelope].

17 Cora: Why?

18 Elsa: Here you have to remove this, here you remove this (*points with her pen to her drawing*) and you can't remove that [the 4 cards on the right side], because there are not 4 other [cards] here [on the left side] that you can remove . . .

Here, the students find themselves in a new situation. While in the previous problem, removing the same number of cards and envelopes was sufficient to isolate the unknown, in this problem the “removing” operation is not enough. They end up with two envelopes on the left side of the equation and four cards on the right side. They cannot continue removing envelopes for, as Elsa notes in Line 12, there are no more envelopes to remove on the right side. And “That's a problem.” Cora suggests removing the four cards on the left side, which will lead them to zero cards. She then assigns zero cards to one of the two envelopes on the left side, which means that there are four cards in the other envelope. Elsa points out two problems with Cora’s suggestion. First, she argues that all envelopes must have the same number of cards (Line 14). Second, simplifying entails removing the same things *on both sides* of the equation (Line 18). This requirement or condition is violated.

The students reach an impasse. “On est en train de se chicaner pour la réponse” [“We are having an altercation over the response”]. They tried to call the teacher, but she was busy discussing with another group. I was videotaping this group; I removed my headphones and went to talk to the students. I suggested that they use the concrete material (envelopes and cards). The students constructed the equation again and proceeded to remove one card and one envelope on each side.

19 Elsa: There are still 2 envelopes left (*see Figure 4.2*).

20 Mia: Then, there are 2 (*pointing to 2 cards*) here (*pointing to 1 of the envelopes*) and 2 (*pointing to the 2 remaining cards*) here (*pointing to the other envelope; see Figure 4.3*).

21 Cora: There must be 1 envelope!

22 Elsa: (*She removes 1 envelope and moves the cards to the other side of the equation; see Figure 4.4*)

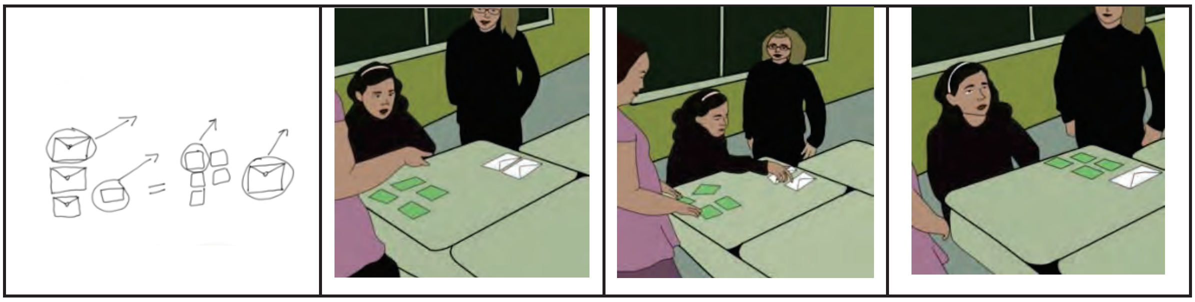

Figure 4. Discussing the solution of $3x + 1 = 5 + x$ in the CSS

In Line 20 Mia suggests an idea. However, the idea is not taken into consideration by the other students. Perhaps because the idea is not framed within the kind of actions that the students recognize as legitimate in solving the equation. Yet, we see in Figure 4.4 that Elsa, in despair, removes one envelope and transfers the cards to the other side of the equation, placing them underneath the envelope, twice breaking the "do the same on both sides" rule. Not finding a convincing way to proceed, Elsa (like Cora in Line 13) steps outside the boundaries of the algebraically thinkable that they have established so far. The situation once again became very tense as we saw in Line 18. Elsa says that they are still altercating and laughs. Cora says: “OK. We’ll do it again!” They remove one card and one envelope from each side of the equation.

23 Elsa: There are 4 [cards]. We must have just 1 envelope remaining. So, we must remove 1 [envelope]; we don’t have a choice (*she removes the envelope*).

24 Cora: Yes, but if we remove 1 … we must remove something else (*she points to the other side of the equation*).

They discuss for a while and come back to the simplified equation ($2x = 4$). After looking attentively at the 4 cards and the 2 envelopes, Elsa says that she has an idea:

25 Elsa: Wait, wait. Here's my idea. Because we have 2 [cards] here (*with each hand, she takes 2 cards from the bunch of 4 cards; then, she slowly moves the 2 hands holding the cards and puts them in front of each of the envelopes; see Figure 5.1. When the cards arrive at their destination, she says*) 2 in each envelope.
She immediately starts the explanation again: she slides the 4 envelopes as she did before, on one side of the equation. She says:

26 Elsa: Separate this [the 4 cards] into 2 (*as she says this, she separates the envelopes; see Figure 5.2. She then slides them in front of each envelope*); there are 2 in each envelope.

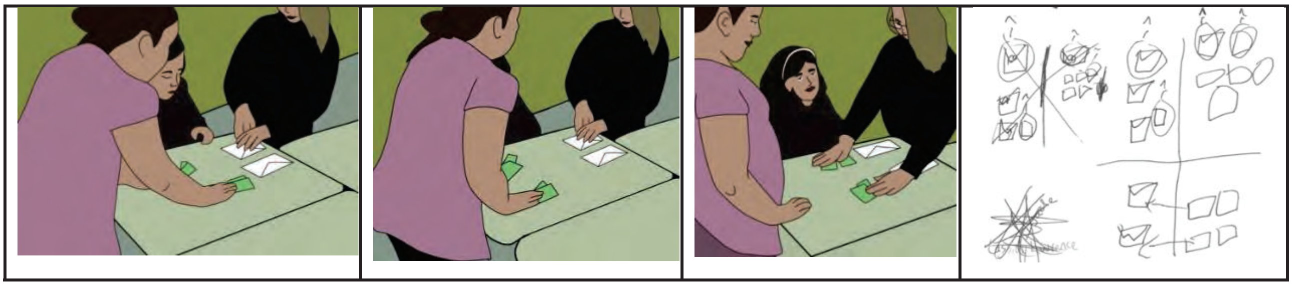

Figure 5. Finding (again) how to solve the $2x = 4$ equation

Elsa’s demonstration is followed by Mia’s reaction:

27 Mia: This is what I said before, but you, you were . . .

28 Elsa: (*completing Mia’s sentence*) . . . altercating!

29 Mia: . . . you said, no, no . . .

30 Elsa: I am sorry, Mia!

Cora makes the equation again and goes through the steps to isolate the unknown. When she reaches the equation $2x = 4$, she says:

31 Cora: We are going to separate . . . (*and slides the 2 cards towards 1 envelope and 2 cards towards the other envelope; see Figure 5.3*).

Mia is right in arguing that she suggested long before (Line 20, Figure 4.3) that each envelope has two cards. However, her suggestion was not articulated in terms of a separation of cards. In Elsa’s case, the solution appears first in an embodied way: “Wait, wait. Here's my idea. Because we have 2 [cards] here . . . 2 in each envelope.” The few uttered words are accompanied by a complex set of grabbing and sliding actions that remain unqualified linguistically. The linguistic articulation appears when she starts again the process of solving the problem. She says: “Separate this into 2, there are 2 in each envelope.” Although the importance of the kinesthetic dimension that accompanied the problem-solving procedure does not disappear, the thematic articulation in language is much more sophisticated. The new mathematical operation is named “to separate.” This new operation is a precursor of what will later be known as the algebraic operation of division. The Arab mathematicians had a term for it: *al-radd*, decreasing the coefficient of the unknown to 1.

In previous research we have found that at the precise moment of learning something, the students undergo a process where mathematical thinking becomes reorganized; what previously took many words and actions becomes reorganized and contracted: the students filter the necessary from the unnecessary and their semiotic activity becomes contracted. There is a *semiotic contraction* (Radford, 2021). Here, we see the opposite process: in Line 26 Elsa adds actions and words to signify the emerging operation. There is a *semiotic expansion* that allows her and her teammates to better notice the operation and endow it with meaning.

The students kept solving with their hands the equation in the CSS several times. It seems that seeing was not enough and that feeling with their hands and their bodies was necessary. Then, they drew their solution in the ISS. The new operation requires a sign to be expressed. Figure 5.4 shows that the students chose an arrow, which is reminiscent of the sliding action that makes the two cards correspond to each envelope. The sign is an icon of the action.

Concluding remarks

This article dealt with the topic of equations in early algebra. It focused on the way Grade 3 students dealt with some of the key algebraic ideas that underpin the simplification of equations. In the first part, I suggested that the characterization of algebra (a) as calculation with letters or (b) as focused on operations rather than on their results are both unsatisfactory. In the first case, the characterization falls short by limiting the scope of algebra; in the second case it fails by downplaying the complexities of arithmetic thinking (which is reduced to trivial calculations). Based on historical-epistemological considerations (Radford, 1995; 2001), I suggested a conception of algebra that stress the authenticity of denotating unknown magnitudes in various ways and emphasizes the analytic-deductive nature that underpins algebraic inquiries. If we know that second degree equations have at most two solutions, it is not because we guessed the solutions, it is because they were *deduced*.

Starting from these premises, the Grade 3 teaching-learning activity was didactically organized around the use of two semiotic systems: the CSS and the ISS. The excerpts analyzed here started with a classroom general discussion around different methods to solve the equation $3 + x = 7$. According to the definition of algebra suggested in the first section of this paper, the solution of equations $ax + b = c$ does not include the operation of the unknown. As a result, in solving those equations the students have not stepped yet into the realm of algebra (Filloy & Rojano, 1987). However, the investigation of the equation $3 + x = 7$ provided the students with an opportunity to continue familiarizing themselves with the isolating-the-unknown procedure that they encountered in Grade 2. In this sense the equation $3 + x = 7$ was envisioned rather as a propaedeutic step towards tackling equations of the type $ax + b = cx + d$ algebraically, something that the students did in the second part of the teaching-learning activity. We can see in Figures 3.2 and 3.3 the moment at which Elsa applies the *al-muquabāla* or removing operation that was previously applied to the constants in solving the equation $3 + x = 7$ to the equation $2x + 1 = x + 6$. The “removing” operation now acquires a new and more developed meaning. It requires seeing the unknown and the equation under a new light. It is this new aspect of the mathematical activity that leads the teacher, in Line 10, to ask two fundamental questions: "Is it okay to remove 1 envelope and then 1 envelope? Is your equation still equal?" More generally, the CSS- and ISS-based teaching-and-learning activity made room for meaning-making processes out of which the Grade 3 students to generate, in their work with the teacher, two important algebraic ideas that underpin the simplification of equations: "removing" (removing equal terms from both sides of the equation) and "separating" (i.e., reducing the coefficient of the unknown to 1), those operational ideas that Arab mathematicians referred to as *al-gabr* / *al-muqābala and al-radd*, respectively (Oaks & Alkhateeb, 2007). The emergence of these sensuous and embodied operations served as foundational blocks for the students' encounter with algebraic alphanumeric symbolism, which happened one year later, when they were in Grade 4.

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Calculational and analytical perspectives in introductory algebra: a theoretical contribution

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In this theoretical paper we discuss what a calculational perspective on mathematical activity entails and discuss its relevance in terms of related speech genres in the introductory algebra classroom. Learning in the introductory algebra classroom is multifaceted because major shifts in form (the algebraic syntax versus arithmetic syntax) and function (analytic rather than calculational) are expected to take place. This theoretical argument is exemplified through student utterances from a Californian 6th grade classroom discussion, in which students' calculational perspective on mathematical activity is apparent. The data reveal traits and intentions from both traditional and alternative instructional genres. Our main argument is that the students' approaches to generalize provided patterns were limited to their perceptual field and thus fail to include a deductive argument, the key feature of algebraic reasoning.

Keywords: Algebraic reasoning, analytic perspective, calculational perspective, introductory algebra, speech genre.

Introduction

This theoretical paper discusses two perspectives on mathematical activity, *calculational* and *analytic*, that are discernible in introductory algebra classrooms. Different frames of reference, or speech genres (Bakhtin, 1986), can present communicational challenges and limit possibilities for algebra learning. We take these two perspectives as points of departure and discuss these in terms of previous research. A calculational perspective is exemplified through discussions of patterning tasks in a US 6th grade classroom.

Hewitt (2019), in a theoretical contribution, suggested that students in school ought to focus on "structure within complex examples" (p. 558) to develop algebraic thinking. Moreover, the students should be encouraged to make their reasoning explicit through expressing what they see rather than performing any mathematical calculations. Montenegro et al. (2018) argued in a similar manner from an empirical point of view. They found that students succeeded in obtaining a functional relationship in a figural pattern when the students turned away from the patterns in numbers and turned towards structure in the figural pattern engaged with. Both Hewitt (2019) and Montenegro et al. (2018) thus argued in favour of an analytic approach over a calculational one.

Although the goal of algebra teaching is for students to engage in analytical thinking, the calculational discourse persists in many classrooms. We argue that research has to acknowledge and investigate the role that this discourse plays in students' participation in algebraic activity. Learning in the introductory algebra classroom is multifaceted because major shifts in form (the algebraic syntax versus arithmetic syntax) and function (analytic rather than calculational) are expected to take place. Sfard (2007) addresses the bewilderment involved in the shift in function of mathematical activity algebra teaching may cause for students through her heading “When the rules of discourse change, but nobody tells you”. We would like to contribute theoretical insights into the characteristics of students’ calculational perspective in introductory algebra classrooms

A sociocultural approach to mathematical thinking

To address students’ perspective on mathematical activity in classroom interactions we draw on Radford (2014) who defines thinking as cultural, embodied and material. Students mathematical thinking is not a separated and ideal internal process but occur as they use cultural artifacts (function table, algebraic syntax, etc.) and other semiotic means in social and goal-oriented activity. The semiotic resources the students use to solve patterning problems, such as gestures, spoken and written words, numbers, function tables and algebraic symbols, shape the form and generality of their mathematical thinking (Radford, 2018). Furthermore, the patterning activity itself and the cultural artifacts employed gain meaning from cultural practices such as algebraic reasoning and schooling, of which language is a prominent aspect (Sfard, 2007).

We draw on Bakhtin’s (1986) notion of *speech genres* when considering the nature of different perspectives on mathematical activity and their role in shaping the classroom discourse. According to Bakhtin, speech genres are extremely heterogeneous, and examples varies from everyday small talk and narratives to sports commentary and different kinds of scientific statements. As an inclusive concept, speech genre is not generally defined. Instead, we conceptualize it in terms of the specific speech genres discussed in this study. For Bakhtin (1986), the utterance, i.e. the unit of language in use and in context, was the basic unit of analysis and he argued that *addressivity* is the key to understanding it. Addressivity refers to the dialogic nature of the utterance, as it is shaped to address a particular listener (real or imagined) foreseeing his response. Furthermore, Bakhtin found our utterances relatively stable in forms of construction and argued that no utterance is given that does not belong to a speech genre, emphasizing that we are not always conscious of it: “[we] speak in diverse genres without suspecting that they exist.” (Bakhtin, 1986, p. 78).

Related speech genres in the introductory algebra classroom

In the introductory algebra classroom, we argue that there are several related speech genres with deep cultural roots at play. A genre form may include linguistically coded intentions and rhetoric that a current speaker may not be aware of (Gerofsky, 1999). Gerofsky argued for the importance of intertextuality between related genres as “a genre which, in its form and addressivity, recalls other familiar genres may bring to consciousness the hidden ground and intentions embedded within [these]” (p. 38). Genres speak to students telling them what to expect and what is expected of them. Radford (2001) argued that algebra as a way of reasoning involves dealing with indeterminate quantities in an analytic and deductive manner with the intensions of explaining and arguing about general relationships and methods for solving problems. Thus, algebra is a specific genre form.

Another speech genre at play is instructional discourse. Mehan (1979) showed that utterances in the classroom (mathematics as well as other subjects) followed a distinct pattern of initiation-reply-evaluation (I-R-E). Thus, it can be seen as a rather standard and rigid speech genre, also found to dominate mathematics teaching in American classrooms (Stigler & Hiebert, 1999).

Experimental studies have developed alternative programs to open up classroom discourse, in which students explore new problems, pose conjectures and argue (Lampert, 1990). Reform documents, such as Standards 2000, aim to bring these ideas into the ordinary classroom (Hiebert, 1999). The speech genre involved shares some ground and intentions with algebraic reasoning since algebra grew out of a sociocultural context of explaining and arguing (Radford, 2001).

Hiebert (1999) explained that the implementation of alternative programs in school districts have not been effective for a simple but unappreciated reason, since it "is hard to change the way we teach" (p. 15). Furthermore, Hiebert argued that there is overwhelming evidence that what students learn in traditional teaching is unsatisfactory and consists of the following: "In most classrooms, students have more opportunities to learn simple calculation procedures, terms, and definitions than to learn more complex procedures and why they work or to engage in mathematical processes other than calculation and memorization" (p. 12). This supports the notion of students developing a calculational perspective on mathematical activity in mathematics classrooms. Furthermore, this genre can be seen to emerge as students' uptake of teachers' intentions in traditional instruction. A genre analysis of classroom interactions can shed new light on the communicational processes that shape the introductory algebra classroom discourse. In this paper we limit the further discussion to explore the nature of a calculational perspective and contrast it to an analytic perspective.

Calculational and analytical perspectives in introductory algebra

We propose a new theoretical stance on student activity in the introductory algebra classroom, i.e. a calculational perspective. Based on previous research on algebra learning, we single out as key demarcations between algebraic and calculational perspectives: (1) ways of interpreting signs and operations; (2) strategies for solving problems; and (3) ways of justifying solutions. This is contrasted to an analytic perspective, which is a central feature of algebraic reasoning (Radford, 2012).

As regards the first demarcation, Booth (1984) and Kieran (1981) found that students working with numbers and their operations tended to interpret the operational signs and the equal sign as signaling types of actions rather than structure. Additionally, Kieran showed that students developed a limited view of what constitutes a mathematical solution and were likely to search for numerical answers (lack of closure). Radford (2018), upon repeatedly observing that 4th, 5th, and 6th grade students included an equal sign in their generalizations (i.e. $n + n = x$), argued that the conceptual challenge for students was not necessarily the alphanumerical symbols themselves, but a need for reconceptualizing numerical operations.

Students holding a calculational perspective are limited to a processual view of signs and operations. In contrast, an analytic perspective includes a structural view of operations: (1) signaling mathematical structures that are relevant and can be used in transformations of expressions (Kieran, 2018); and (2) seeing expressions as objects in themselves that can be operated upon, substituted, or classified (Sfard, 2007). These are interdependent of a developing awareness of mathematical generalities (Kieran, 2018) and an acceptance of indeterminacy, both in operations (operating with indeterminacy as if it were a number) and solutions (Radford, 2018).

Concerning students' problem-solving strategies, the second demarcation, students often use a guess-and-check strategy to solve new algebraic problems, including patterning tasks (Bednarz & Janvier, 1996; Lannin, 2005; Radford, 2018). Bednarz and Janvier (1996) found that students using a guess-and-check strategy were not able to accept indeterminacy and remained dependent on performing calculations on numbers. In contrast, students that used a strategy of determining the underlying structure of the problem (which included the linking of different mathematical structures such as for example additive and multiplicative to determine how many equal parts) were able to explain an algebraic equation when shown such a solution.

Addressing the third demarcation, students' justifications, Lannin (2005) showed that 6th grade students mainly used empirical results to determine and justify their pattern generalizations when left to themselves. They tended to focus on particular values to determine and verify generalizations, rather than a general relation. Radford (2012) argued that algebra is an analytic art. Formulas must be deduced and not guessed for students to engage in algebraic reasoning. They have to come up with a deductive argument. A deductive argument, for example in a patterning activity, includes three components: (1) noticing a commonality between the terms p_1, p_2, p_3, …, p_k; (2) generalizing this to all subsequent terms p_{k+1}, p_{k+2}, p_{k+3}, …; and (3) using the communality to formulate an expression for any term in the sequence, i.e. an explicit strategy to generalize a sequence (Radford, 2008). However, students more often pursue a recursive strategy where each term (p_k) is determined from the previous one (p_{k-1}) (Lannin et al., 2006). Radford (2012) argued that such a generalization is of an arithmetic nature, as opposed to algebraic, as it is limited to include terms within the perceptual field. It does not include the third component of a deductive argument. Nevertheless, guess-and-check and recursive strategies might be helpful in generalizing a pattern. Particularly, a recursive strategy can give insight into the pattern's rate of change, which again can be helpful in developing an explicit expression. However, research shows that students struggle to see the connection between the two ways of generalizing a pattern (Lannin et al., 2006).

Ellis (2007) showed that the processes of generalization and justification were interwoven in students' activity. 7th grade students' generalizing acts were directly linked to the justifications they gave. On the other hand, their view of what counts as an acceptable justification influenced their generalization process and students following a deductive justification scheme developed more sophisticated generalizations from initially limited or unhelpful generalizations.

In sum, a calculational perspective on mathematical activity includes (but is not limited to): 1) viewing numbers as the main objects of activity and other signs and operations as actions to perform; 2) employing strategies rooted in calculational processes in problem solving, and 3) relying on empirical results for justification. These aspects may confirm and reinforce each other as Ellis (2007) pointed out, and they may function as barriers for students' engagement in algebraic, analytic reasoning. The intention of the calculational genre, as opposed to the algebraic genre, is to perform calculations following known procedures to produce correct numerical solutions.

Students' calculational perspective in a 6th grade classroom

To empirically exemplify our theoretical contribution, we draw on data from a Californian 6th grade classroom where students were working with patterning tasks. The background for this paper is an international algebra project called VIDEOMAT (see Kilhamn & Säljö, 2019). An in-depth analysis of students' discourse in classroom patterning activity showed that students across countries pursued similar solving strategies. Reinhardtsen and Givvin (2019) found that students focused on processes of calculations rather than structure and quantitative relationships in their work, i.e. they drew on operations on numbers rather than introduced algebraic ideas and symbols.

The teacher's approach to working with every patterning task involved three main phases of activity, classroom exploration of a pattern, making a function table and developing an explicit generalization from a recursive one when possible, and extending the generalization to all subsequent terms and creating an algebraic expression. The teacher first asked for students' observations and suggestions before proceeding, allowing students to pose conjectures and ideas.

We structure the following section according to the three main characteristics of the calculational perspective outlined above. The first two episodes exemplify students' view of numbers as the main objects of activity and other signs and operations as actions to perform. The third episode exemplify students use of strategies rooted in calculational processes, thus relying on empirical justification.

Explicit and recursive generalizations: "Multiplying by 4 or they're adding 4"

In the first lesson the teacher introduced the metaphor of function machine, emphasizing an explicit relationship between sets of numbers in the second phase of activity. In the first phase of activity the class discussed the numerical sequence 4, 8, 12, 16, __, __, __. Students suggested both recursive generalizations: "Like it's plus 4, 4 plus 4 equals 8, and 8 plus 4 equals 12, and so on", and explicit ones: "I did multiples of 4, 20, 24, 28".

In phase two, the teacher drew a function chart and a function machine and then asked the students to discuss shortly with their seat-partner: "And if I put a 2 in, what's going on in here, so that I get an 8 out. What's happening in this machine that represents this number sequence?". One student, Mara, made a link to the previous discussion and said to her partner: "Multiplying by 4 or they're adding 4". Another student, Ivy, offers a more literal explanation: "And then it came out in a different number, so it's a function machine".

The ideas presented by students in the whole class discussion in the first phase are mainly concerned with how to calculate the terms in the sequence. Mara's uptake of the calculational ideas as presented in phase two shows that these are accessible to other students. However, making sense of the difference between the two generalizations, recursive ($p_k= p_{k-1}+4$) and explicit ($p_k=4k$) and the relationship between them, requires looking at the numbers and operations analytically.

Neither Ivy nor Mara picked up on the significant changes in the perspective that the teacher was emphasizing. Ivy offered a literal interpretation and referred to a transformation of a number, as if by magic. Mara linked the teacher's question to the previous discussion but did not notice that the operation of "adding 4" no longer was appropriate, i.e. the horizontal relationship between numbers in the two columns only corresponds to an explicit generalization of the sequence.

Using letters: "Because you can do *4n* … *n* can be any number, so multiply it"

In the third phase of working with the sequence above after having determined the 10^{th} and 20^{th} terms of the sequence together, the teacher asked the students to discuss the general expression with their seat-partner "How do I show 4 times any number. We're just calling it *n* for right now".

Trace explained his expression to his partner: "*4n*. Because you can do *4n*, yea, *4n*. *n* can be any number, so multiply it, *4 dot n*, *4n*, stuff like that". Trace made sense of his expression on the terms of being able to perform a calculation when replacing *n* with a number: "Because you can do *4n*… *n* can be any number, so multiply it". Thus, Trace was able to respond to the teacher's question of how to show "4 times any number". However, as Küchemann (1978) pointed out, there is a leap between being able to evaluate an expression and using a letter as a variable to express mathematical structures and relationships.

In the third lesson the class worked with a figural pattern. In phase one the whole class discussion centered around rate of change. However, in the second phase of creating a function table guided by the teacher, the students no longer made references to the figures but quickly noticed a familiar (quadratic) relationship between the numbers in the two columns. Luna set up multiple numerical expressions such as $1 \cdot 1=1$, $2 \cdot 2=4$ and $3 \cdot 3=9$, and in the third phase she suggested: "It's called *n* times *n* equals *x*". Her generalization was developed and verified through calculating numerical expressions, and her algebraic expression also resembled these.

Guess-an-check strategy: "I tried the next number which is 6"

The teacher emphasized the relationship between a recursive and an explicit generalization as a strategy to determine the explicit expression throughout the classroom work. However, the students mainly used a guess-and-check strategy when working independently with function charts.

In the third lesson the teacher gave the students a function chart in which the left column included the numbers 1-8, with *x* as the last entry, and the right column included the four first entries: 5, 11, 17, 23. The instructions were to fill in missing numbers, describe patterns and write an expression.

The teacher, in a whole class setting, asked Liam to explain what he noticed. He explained that he first looked at "1 times something gets to 5", deciding this being 5. He then tried the next pair of numbers in the chart (2, 11), but found that $2 \cdot 6 \neq 11$: "So, I realized that you could do, I tried the next number which is 6". He then explained he did minus one to get the number he needed; five. He then tried the sequence of operations regarding the next pair of numbers in the chart (2,11): $6 \cdot \mathbf{2}=12$, $12-1=\mathbf{11}$, and found that it worked. Liam demonstrates how the students approached the number sequences through guess-and-check. Also when working with figural sequences, the students used this strategy. Lisa, in the fourth lesson, explained her approach: "Um, I tried it, I kind of did, um. I don't know how to explain it, but I started with doing multiplying it by 1 and adding 2, but it didn't work. So, I tried multiplying by 2 and subtracting 1". The students working with function charts increasingly made explicit generalizations, but these were mainly based on numerical schemes rather than analytical ones. This finding coincides with Ellis' (2007) argument.

Synthesizes and concluding remarks

In this paper we set out to come up with theoretical insights into the characteristics of students' calculational perspective in introductory algebra classrooms. We have argued that developing algebraic reasoning is a challenging process as previously met calculational perspectives interfere when trying to reason analytically about number patterns and figural patterns. The empirical examples demonstrate that students approach problems in introductory algebra through a calculational perspective. They face difficulties making sense of the difference between a recursive and an explicit generalization (cf. Lannin, 2005; Lannin et al., 2006), they evaluate letters but struggle using them to express structure and generalities (cf. Küchemann, 1978; Radford, 2018), and they use a guess-and-check strategy (cf. Bednarz & Janvier, 1996; Ellis, 2007).

The classroom data reveal traits and intentions from both traditional and alternative instructional genres. Elements such as letting students explore sequences, notice patterns, and suggest ideas invited students to engage in generalization processes. However, the use of a function chart and algebraic notation as a step-by-step process heavily supported by the teacher share traits with the traditional instructional discourse and may have informed students' participation in the activity. Deeply rooted norms (cf. Stigler & Hiebert, 1999) of emphasizing procedures and products (solutions, facts, etc.) rather than mathematical processes (problem solving, generalizing, etc.) came to the fore in the classroom, despite efforts by the teacher to focus on the latter in discussions.

We argue in accordance with Radford (2012) that the students' calculational perspective is of an arithmetic nature, as opposed to an algebraic nature. The students' approaches to generalize the patterns are limited to their perceptual field, and thus fail to include a deductive argument, the key feature of algebraic reasoning (Ellis, 2007; Radford, 2012). Despite the teacher's efforts, the students do not make sense of the shifts in form (the algebraic syntax versus arithmetic syntax) and function (analytic rather than calculational) in this classroom. These expected shifts unfortunately do not take place. More research is needed to investigate the role that the calculational perspective play in students' participation in algebraic activity.

Radford (2018) found that symbolic thinking, in which letters are used to develop a generalization, took a long time for students to develop. Furthermore, it developed in line with an increasingly analytic approach to patterning activity. A genre analysis of the introductory algebra classroom supports this finding. A genre form includes many aspects that reinforce each other, as we have pointed out concerning calculational and analytical genres. Our discussion suggests the importance of addressing the elements of the algebraic genre as a consorted effort.

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Using the bar model to ease the transition from transforming arithmetic-numerical to algebraic equations: theoretical considerations and possible obstacles

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We discuss an approach to transforming and solving algebraic equations via the so-called bar model, based on the strategy of transposing. After developing a learning environment, we conducted design experiments to get insights into how students work with it. First, this paper aims to present the core idea of our learning environment. Second, we highlight the following difficulties that students face when working with the bar model: (1) the model itself, with its translation processes between graphical and symbolic representations (such as numbers, variables or operation signs), turned out to be a considerable learning content, (2) the transition from arithmetic-numerical contexts to general algebraic equations in the bar model seems to bear distinct conceptual obstacles which may even lead to misconceptions based on over-generalizations resulting from the bar model. We point to theoretical insights and implications for enhancing our learning environment.

Keywords: Bar model, equations, algebra, design research.

Introduction

In the transition from arithmetic to algebra, students face a variety of difficulties (e.g., Warren, 2003). Especially for low-achieving students, it is important to fill the concepts and procedures introduced in arithmetic-numerical contexts with meaning for transferring them into the field of algebra. One example of such an idea that originates from numerical considerations is the concept of equivalence with its strong connections to the procedure of equivalence transformations regarding solving equations. This process, however, is known to bear several problems for learners. Kieran (2006) gives a short overview of students' errors, like ignoring the minus sign or reduction errors. To work against these procedural errors, a deep understanding of the idea of equivalence could be helpful. The idea of equivalence represents a central part of solving equations in school algebra and can be characterized by three different perspectives (Prediger & Roos, in press): (1) equations are equivalent if they have the same set of solutions, (2) equations are equivalent if there exists an equivalence transformation that transforms one into the other[1], and (3) equations are equivalent if they describe the same situation. While the first and second characterizations stay close to the formal mathematical definition, the third can be used for giving meaning to equivalence transformations in a rather intuitive and visual way. Thus, especially for low-achieving students, this could be a promising approach to build up conceptual understanding for the mathematical procedure of solving equations.

Our focus is on developing a learning environment, i.e. teaching material, to ease the transition from transforming arithmetic-numerical to algebraic equations. With the help of the so-called bar model,

[1] An equivalence transformation can be seen as an application of a bijective function on both sides of the equal sign.

we want to emphasize the meaning of equivalent equations in order to develop the strategy of transposing for solving equations. Firstly, we summarize the theoretical background for creating our learning environment. Secondly, we report results of interview studies used to evaluate the learning environment to derive theoretical and practical conclusions.

Theoretical Background

Regarding the process of solving algebraic equations, Selter et al. (2012) differentiate between two formal strategies: *performing the same operation on both sides* ($B + C = A \Leftrightarrow B + C - C = A - C$) and *transposing* (put an expression on the other side of the equal sign by applying the respective inverse operation: $B + C = A \Leftrightarrow B = A - C$; compare Figure 1) (see also Kieran, 1992). While solving equations by performing the same operation on both sides should depict one main objective (Malle, 1993), especially at the beginning of the learning process, the strategy of transposing is considered more intuitive for students (Mason et al., 2005). Selter et al. (2012) particularly emphasize the close relationship of the idea of transposing to former arithmetical experiences. Therefore, the strategy of transposing can be linked back on the one hand to the concept of equivalent equations as equations describing the same situation and, on the other hand, to experiences that were made in arithmetical contexts. We explain both shortly.

Equivalent equations as describing the same situation – a graphical model

According to Malle (1993), object relationships represented in drawings (such as relationships between line segments) can be used to give variables, expressions, and formulae a meaningful interpretation. Such graphical models can also be used later on for making sense of transposing when solving equations (ibid.; see Figure 1).

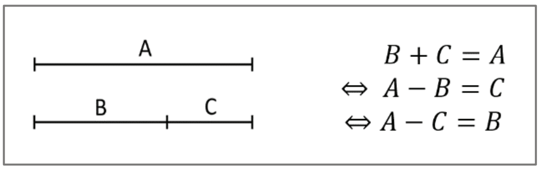

Figure 1: Bar model for representing transposing; based on Malle (1993, p. 220)

The Singapore bar model (e.g., Fong Ng & Lee, 2009) works similar to this approach. This model is used widely among others in Singapore's primary schools (e.g., Kaur, 2019) and discussed as a method for supporting students to get good results in problem solving activities for instance in international assessment studies like TIMSS (e.g., Beckmann, 2004).

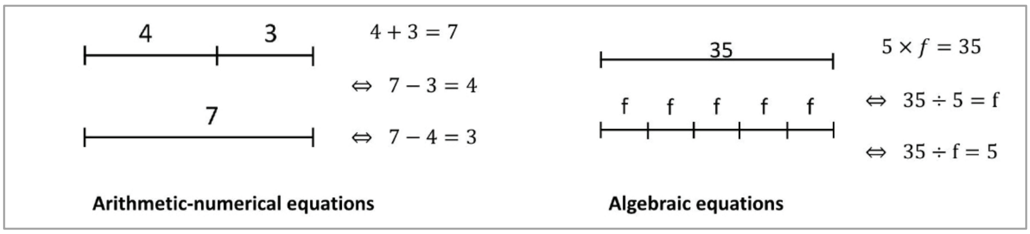

Figure 2: The bar model as a visual representation of equivalent equations

One bar model serves to describe three equations (see Figure 2) and that is why these equations are called equivalent. Following a semiotic point of view, the bar model can be considered a representational system. As with every representational system, working with the bar model presupposes specific knowledge being implicit (see Kempen and Biehler (2020) for a more detailed description) to work with. For example, two numbers are added by placing corresponding line segments next to each other, subtraction by crossing out or erasing. Multiplication is traced back to counting units in this model and is thus done by bundling line segments with equal length (see the five fs in Figure 2); division can be done by laying out a line segment with smaller segments of equal length. The equality of two values, expressions, etc., results from the phenomenon that two resulting line segments are equal. It becomes clear that one needs both an understanding of the mathematical operations and how they are represented or performed in the context of the bar model. To communicate such incidents, a distinct language is required that refers to the operations performed in the bar model and to the model itself. This language is closely related to the meaning-related language needed to describe basic operations (see Table 1).

Linking back the idea of equivalent equations to arithmetical contexts

For the strategy of *transposing* the relationship between the operations – the inverse operation of addition is subtraction etc. – is one key component that must be transmitted from arithmetic to algebra. To accomplish this transmission, it seems helpful to focus on the meanings of the operations involved as well as the related language that comes along with these meanings (see Table 1). The use of such meaning-related language seems also helpful when working with the bar model (see above).

Table 1: meanings and meaning-related language (see also Prediger & Roos, in press)

Meanings of the basic operations	Examples for meaning-related language
• Addition as putting together • Subtraction as taking away or determining the difference	• I have 2 and I **put it together** with 3, then I obtain **in total** 5. • I have 5, and I **take away** 3, so 2 remains. • I have 3, so I **need 2 more to reach** 5.
• Multiplication as counting in units • Division as sharing (partitive model) • Division as measuring (quotative model)	• I count in groups: 3 **groups / sets / units of** 2 are 6. • I **share** 6 **among** 3 people, so everybody gets 2. • 2 fits 3 times into the 6.

The learning environment

Based on the considerations above, we developed a learning environment to prepare the transition from arithmetic to algebra for transforming equations using the bar model (Prediger & Roos, in press). The steps in the learning environment are shown in Figure 3, although the paper focusses on step II and V.

Different representa-tions for equations **(I)**	The bar model in numerical contexts **(II)**	Equivalent equations in numerical contexts **(III)**	The bar model in algebraic contexts **(IV)**	Equivalent equations in algebraic contexts **(V)**

Figure 3: Steps in the learning environment towards transforming algebraic equations

In *step II (The bar model in numerical contexts)*, students get to know the bar model and start working with it. The key component here is the understanding of the bar model itself and how to perform and understand basic (arithmetic) operations in it (see Table 1, Figure 1). Learners need to connect the bar model as a graphical representation of equations with equations represented symbolically, and formulate verbally corresponding relationships. In *step V (Equivalent equations in algebraic contexts),* the objective is to detach the ideas of transforming equations from the use of the bar model so that students can transform equations also in rule-based procedures. Here, abstraction processes and the idea of reverse operation play a decisive role.

Research Question

Our focus is on developing a learning environment to ease the transition from transforming arithmetic-numerical to algebraic equations by using the bar model. The design research project (Gravemeijer & Cobb, 2006) builds and enhances the design based on empirical insights into students' learning processes. In this paper, we focus on the following research question: *Which obstacles become apparent when transforming and solving algebraic equations with the aid of the bar model?*

Methodology

Within the applied design research approach (Gravemeijer & Cobb, 2006), the learning environment created was sequenced into five steps (Figure 3). The target group of our learning environment consists of low-performing students who need a second chance to develop an understanding of the mathematical process of solving equations. The first two design experiment cycles addressed three low-achieving tenth graders in remediating mathematics classes, aiming to pass their Grade ten exam in a prevocational setting (Cycle 1) and three eighth graders of a German comprehensive school (Cycle 2). Data was collected while the students worked on equations in zoom-sessions in addition to their 'normal' math classes during the pandemic in January – May 2021. In total, 960 minutes of video data were collected and partially transcribed.

During the sessions, the design experiment leader watched the student work with the material. Whenever the student's work stayed unclear, she asked the students to explain their approach, thoughts, and solutions. Also, when students needed additional help or had questions, she explained the tasks in more detail.

In our analysis, we watched the videos and selected places where difficulties with the bar model or the transformation of equations occurred. For these places, we took a closer look into the corresponding transcripts. Two of the typical conceptual challenges appear when working in step II and step V of the learning environment. They will now be discussed based on the cases of Vivien and Anno. The problems discussed below can be considered prototypical for our sample in terms of their characteristics.

Tentative Results

Case of Vivien

The case of Vivien was already presented in Prediger & Roos (in press). She is an 18-year old girl in grade 10 participating in a remediating mathematics class. In this episode, the design experiment leader (DEL) talks with Vivien about the task in Figure 4 located in step II of the learning environment.

Three different running programs and their drawings

- John goes running every Saturday to train for a half marathon. He runs 21 km every time.
- Eva runs three times a week 7 km and
- Max runs every day 3 km from his place to his grandparents.

Assign Eva, Max and John's runs to the matching drawings.

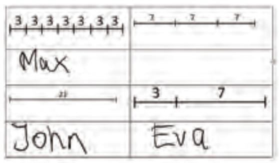

Figure 4: Vivien's solution to the running program task

1 Vivien: And Eva beneath, the one next to it [*refers to the lower right bar*]. Where there is written 3, then the middle line and then 7.
2 DEL: Here?
3 Vivien: Yes.
4 DEL: And can you explain how you came up with that or why?
5 Vivien: Because it says, “Eva runs three times a week 7 km”. And then I would say, the 3 stands for “three times a week” and the 7 for “7 km”.

Vivien focused on the numbers while missing to connect the bar model with the intended mathematical operation. This problem is also mirrored in her explanation: Rather than grasping the additive structure of 3 and 7 in this part of the bar model (e.g., with meaning-related language of addition like “putting together”, see Table 1), she only articulates the numbers, not joint lengths: “Where there is written 3, then the middle line and then 7” (Line 1). Vivien shows difficulties with the distinction of additive and multiplicative structures in connection with the bar model. This is also reflected in her rather simple use of language: “And then” is the only connective between 3 and 7 that she uses, which does not allow her to distinguish an additive structure from a multiplicative structure. Moreover, Vivien fails in realizing the idea of multiplication displayed in the bar model on the upper right. Here, multiplication is shown as counting units. It becomes evident that a learner has to combine two facets of knowledge: First, the conceptual understanding of multiplication as counting units is needed for mathematizing the text on Eva’s run. Second, the representation of this multiplication (as several units consisting of 7 km each) in the context of the bar model has to be realized (three line segments of length seven are meant to represent "3 · 7"). In this sense, the conceptual understanding of multiplication serves as a prerequisite for choosing the adequate bar model. However, based on this conceptual understanding, the corresponding representation in the bar model (the juxtaposition of three line segments [addition] of equal length [leading to multiplication]) has to be understood, too, as a matter of implicit knowledge.

Case of Anno: Anno is a 14-year old German 8th grader with average achievement in mathematics classes at the comprehensive school. In the beginning of his learning process, he displayed a good understanding of the bar model by explaining the meanings of the underlying operations. When talking about the bar model, he finds the correct corresponding symbolic equations and gives a correct explanation (see Figure 5).

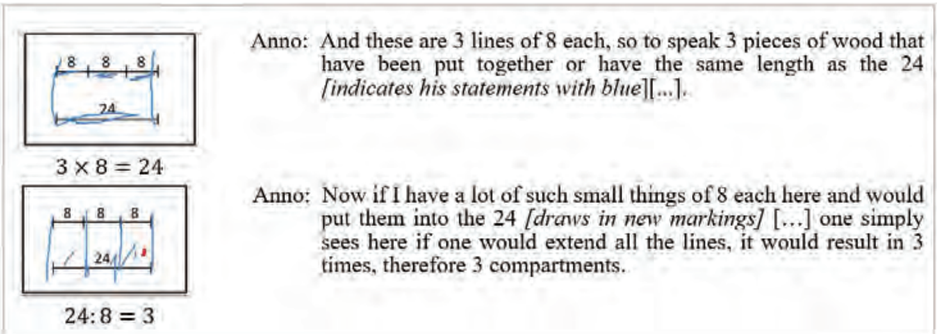

Figure 5: Anno explaining corresponding equations in the bar model

Anno refers directly to the significance of the operations when he speaks of pieces that have been "put together" or of the "three compartments" that have been placed in the 24. Thus, although the bar model is new to him, he seems to have a good intuitive understanding of the operations and the corresponding connections between the bar model and the symbolic representation. In the following episode, the design experiment leader and Anno discuss the task shown in Figure 6 (step V, see Figure 3). This task was designed to initiate a detachment from the bar model with which the students had worked before. Detaching is necessary because the multiplication with the rate r can hardly be visualized as counting in units in the bar model (and scaling up and down is not known by Anno). Therefore, from the tasks, which were solved before with the bar model, the idea of the reverse operation is to be transferred.

You know an example of such a formula from percentage calculation:

$\underbrace{amount}_{a} = \underbrace{rate}_{r} \times \underbrace{base}_{b}$

i) What are the equivalent equations here?

The same is now possible with every formula. So I finally don't have to learn so many by heart…

Figure 6: Anno's task for detaching from the bar model due to other meanings of multiplication

After Anno finds the correct equivalent equations ($a = r \times b; r = a \div b; b = a \div r$) the interviewer asks how he came up with his solutions:

Anno: Yes, because it was always in the beginning [*referring to former tasks in the learning environment*] larger [value] was always calculated by the two smaller ones. If you now assume that somehow these are the two smaller ones, like 3 times 2 or something, the larger value is calculated by these two [*marks the rate and the base*]. This [*referring to a*] you can then divide by the two, by the rate and the base, I suppose.

Also in the following, Anno continues explaining his strategy of transposing equations by using ideas of smaller or bigger numbers. For him, the number or variable that stands alone opposite the multiplication on one side of the equation has to be the largest. If one wants to obtain an equivalent equation, this only makes sense if one divides the larger number by a smaller number. He shows here – similar to Vivien – a non-sufficient focus on the underlying operations with respect to their structures. Although Anno described the meaning of multiplication and division as inverse operations in earlier tasks, he can only superficially exploit this idea when developing a strategy without the bar model. He uses a method that is based on the magnitude of numbers ("if one now assumes […] these are the two smaller […] the larger value is calculated by these two"). This strategy might be considered an over-generalization resulting from the bar model: The one number alone on one side of the equation is always bigger than the two numbers on the other side. In fact, multiplication in the bar model is based on the aspect that at least one of the factors is a natural number. Albeit Anno's strategy can be helpful when working with equations with natural numbers, the strategy fails when multiplying with factors smaller than 1.

Discussion and Conclusion

The insights we gained in our design experiments are the following:

Regarding the understanding of the bar model, some students have considerable difficulties connecting representations (iconic-symbolic-verbal) relying on conceptual understanding of basic operations. For such students, it is hard to use the model to make sense of equivalence transformations of equations. Even the step before, representing one side of the equal sign in the bar model already depicts challenges. These students focus on the given numbers in the model instead of on the underlying mathematical structures. This phenomenon has also been reported in the context of multiplication for grade 5 students (Prediger, 2019). However, our sample consists of students in grade 10. Other students who have acquired an adequate understanding of the meaning of equivalent equations within the bar model are not necessarily able to develop an appropriate strategy (like transposing) when prompted to detach it from the bar model. Especially regarding algebraic equations with a multiplicative structure, the concept of inverse operations should be used, not an idea concerning the magnitude of numbers that are not transmissible to decimal numbers below 1.

Based on these findings regarding students' obstacles when using the bar model for solving equations, we want to highlight the following theoretical implications. First, when working with the bar model, a profound understanding of basic mathematical operations has to be considered essential. This understanding is not only necessary for performing respective operations when solving equations in arithmetical or algebraic contexts; learners need to have an appropriate conceptual understanding to understand which operations are illustrated in the bar model or perform operations in the bar model themselves. Although Koleza (2015) found that third graders already understand multiplication after short instruction with the bar model, our preliminary results show that this does not have to be the case even for students in grade 8. Moreover, the bar model must first be seen as a learning object in its own right before it can aid learning. In this sense, the bar model is neither self-evident nor self-explanatory, as learners need specific knowledge to work with it. Besides, the student's attention must be directed from a focus on the numbers to a focus on the underlying operations (see also Prediger (2019)) and its representation in the bar model. In the case of Anno, the extensive work with the bar model led to an over-generalization of a respective strategy for working with equations. Furthermore, this overgeneralization might be considered a considerable misconception. Accordingly, the idea of inverse operation has to be highlighted in the bar model to focus rather on the operations than on the 'length' in the bar model. Following these theoretical insights, we draw the following related practical implications: (1) Especially for low achieving students, the repetition of the meanings of the basic operations seems necessary. Accordingly, we plan to extend our learning environment with a new part at the beginning to ensure respective prerequisites. (2) The conceptual meanings of the basic operations need to be more closely related to the bar model so that the bar model is brought in more explicitly as an independent object of learning (step II, Figure 3). (3) Tasks must be added that focus on the application of the idea of inverse operations; this idea should be followed throughout the whole learning environment. In addition, tasks have to be incorporated to help students detach from the bar model and thus work against overgeneralizations and misconceptions.

Based on the results of our analysis, we are planning the next cycle of our research project for the end of 2021.

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Using the graph when talking about functional relations in Grade 1: The importance of terminology

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This study investigates how the graph representation creates opportunities for young students to develop an understanding of functional relationships in pattern generalizations. The empirical data is from an educational teacher-focused classroom design research focusing on generalizations in arithmetical growing patterns in Grade 1. The results show that the students in Grade 1 are given an opportunity to reason mathematically in both recursive- and covariational thinking. The results also show how the teaching provided opportunities for the students to use multiple representations of functional thinking and how oral language is a common representation to describe relationships. However, using a well-thought-out terminology to exploit the potential of the graph representation when discussing functional relationships and generalizations appears to be important.

Keywords: Generalizations, functional relationships, graph representation, elementary school, terminology.

Introduction

The concept of *generalization* is used in various ways and in relation to multiple mathematical contents, for example, generalized arithmetic or generalized relationships between quantities. Generalizing the functional relationship between two variables in growing patterns has been used successfully, even with young students, and research indicates that young students can develop their functional thinking (e.g., Blanton & Kaput, 2004; Blanton et al., 2015; Stephens et al., 2017). However, teachers as well as students have difficulties in working with generalizations and students are often not given opportunities to develop mathematical generalizations (e.g., Stylianides & Silver, 2009). In the Algebraic Thinking Group at CERME, there is an ongoing discussion about figural patterns and how they can support young students' understanding of algebraic generalizations. Nevertheless, what is unsolved is how teaching can contribute to students' shifting their focus "from the visual structure to the algebraic structure" (Chimoni et al., 2019, p. 530) and how teaching can contribute to young students' overcoming the difficulties described by Stylianides and Silver (2009) as a way to support their understanding of functional relationships. In an attempt to contribute insights about teaching functional relationships, an educational teacher-focused classroom design research with three teachers was conducted (Sterner, 2019; 2021). This paper builds on the teaching of one Grade 1 teacher who used graphs to talk about functional relationships and generalizations and includes reflections from the three teachers in the design research (intervention). The research question in the paper is, "In what way does the graph representation support teaching and provide learning opportunities about functional relationships and generalizations in Grade 1?". What is interesting in relation to young students' learning about functional relationships is the teachers' reflections on their challenges.

Background and conceptual frames for the intervention

Functional thinking is a part of algebraic thinking that, focuses on the relationship between two (or more) quantities. The definition central for this study is from Wilkie (2016): "Functional thinking relates to understanding the notion of *change* and how varying quantities (or "variables") relate to each other" (p. 247). Research indicates that teaching functional thinking in elementary school is dominated by identifying recursive patterns, where the relationship between quantities is often missing (Blanton et al., 2015; Carraher et al., 2008). In addition, young students are often introduced to variable notation as a quantity with a "missing value", and thus the relationship between the independent and dependent variables and proportional reasoning is lacking in elementary school (Blanton et al., 2015; Carraher et al., 2008).

In Swedish primary schools (Grades 1–6), it is not common to teach functional relationships; the concepts of "double" and "a half" are used in lower grades when talking about proportional relations. In this study, design principles (DPs) are used as a theoretical frame to focus on generalizations and functional relationships. As can be seen from the Methodology section, the research project is intended as an educational design research. The content of the design principles (DPs) used in this study are formulated based on previous research on generalizations and functional relationships in teaching in elementary school. I interpret these principles as theoretical frames for the intervention as well as a guide for the content of the teachers' teaching, and the following design principles have been used. The second (DP2) is particularly central in this study.

DP1: The students should be given opportunities to identify a pattern, structure the pattern, and generalize the pattern (Mulligan et al., 2013; Stylianides & Silver, 2009).

DP2: The students should be given opportunities to work with algebraic reasoning, including functional thinking and proportional relationships and determining relations between two or more varying quantities (Beckman & Izák, 2015; Blanton et al., 2015; Blanton & Kaput, 2004).

Theoretical frames in the analysis

The analysis was done in three steps. In the first step, the definition from Wilkie (2016) was used in relation to the content of the design principles. The first analysis resulted in the empirical data, including students' and the teachers' discussions about the content of the design principles. In the second step, the second strand of Kaput's (2008) three algebra content strands was used, *Functions, Relations, and Joint Variation*. This strand provided an opportunity to highlight sequences in the empirical material, including functions, relations, and joint variation. In the third and last step of the analysis, I explored how the graph representation supports teaching and young students' learning of functional relationships and generalizations. Based on Kaput's (2008) algebraic thinking and the core aspects, Blanton et al., (2018) and Blanton et al, (2019) worked on generalization and functional thinking. Blanton et al., (2019) suggested four essential algebraic thinking practices: *generalizing, representing, justifying,* and *reasoning with* mathematical structure and relationships. In the last analysis, I used Blanton et al.'s (2019) essential algebraic thinking practices of generalizing as a theoretical frame. The second and the third practices, representing and justifying, became the most central practices to explore how teaching can be formulated to contribute to young students' learning of functional relationships and generalization in arithmetical growing patterns. The combination of representing generalizations and justifying generalizations made visible what opportunities the graph representation provides to students to justify generalizations.

Methodology

The study outlined in this paper is a part of an educational teacher-focused classroom design project inspired by *classroom design with teachers* (Stephan, 2015). One aspect of mathematical reasoning central in this paper is generalizing the functional relationship between two variables. In addition to the author, three mathematics teachers from different schools in Sweden, one from Grade 1 (Jonna) and two from Grade 6 (Clara and Irma), participated and collaborated in three recurring design cycles, an intervention, over a period of nine months (Sterner, 2019). The intervention is guided by design principles (Greeno, 2006; McKenney & Reeves, 2012), which are then used as the theoretical frame for the mathematical content of the intervention and as a guide for the content of the teaching (see "Background and conceptual frames for the intervention" section).

The empirical data in this paper include video recordings from various parts of the design process: five common meetings from the design– and refining phase and three lessons from Grade 1. Other data material includes field notes from the Grade 1 classroom, observations, copies of student work, and tape recordings of reflections with the teacher directly after teaching. Jonna, as well as the other teachers are experienced and development-oriented. The students are 6 or 7 years old, and the study is conducted with half the class (twelve students). The teacher (Jonna) describes functional relationships as a new and unknown way to learn pattern generalization in Grade 1. As mentioned, the design principles served as a theoretical frame for the intervention and teaching, and thus empirical data from both the teaching in the classroom and discussions from the design process were necessary. The empirical data was analyzed in three steps and three different frameworks were used (Blanton et al., 2018; Blanton et al., 2019; Kaput, 2008; Wilkie, 2016).

Ethical considerations

The intention was not to generalize the results but rather to exemplify the opportunities and challenges in teaching functional relationships in Grade 1. In the intervention, it was important to consider the teachers' and the students' participation and the chosen mathematical content.

The intervention in Grade 1

In the intervention, the three teachers (Clara, Irma, and Jonna) designed various tasks to support students in Grade 1 to *identify*, *structure*, and *generalize* different patterns (DP1). In the initial stage of the teaching in Grade 1, the students were given the opportunity to identify and structure different sequences of repeating patterns (e.g., AABAABAAB…) as well as conceptualize the idea of a unit of a repeating pattern.

One task concerning a certain number of dogs and the corresponding number of tails, ears, and legs was used to exemplify direct proportionality and to support functional relationships, as well as the importance of how quantities vary in relation to each other (DP2). This growing pattern task was used in teaching in both Grades 1 and 6 and was taken from an earlier study inspired by Blanton and Kaput (2004) and Mulligan et al. (2013). Students in Grade 1 counted and structured the tails, ears, and legs and represented the data and their functional thinking in different ways, for example, by drawing pictures, putting matches in piles, and making simple tables. The whole class talked about the patterns, for instance, those in the relationship between the number of dogs and the number of tails, ears, and legs (i.e., 1 dog has 2 ears…2 dogs have 4 ears…3 dogs have 6 ears… Each dog has 2 ears, and each dog has 4 legs). In the whole-class discussion, Jonna, the teacher in Grade 1, structured the data in function tables and made a graphical representation in a Cartesian coordinate system. They made graphical representations of the three relationships, where the number of dogs was illustrated as a function of the number of tails, ears, or legs respectively ($y = x,\ y = 2x,$ or $y = 4x$). Finally, Jonna and the students discussed the three different graphs. In the class, questions were asked like the following, "Which of these graphs represent the tails of the dogs and how do you know that?", "How many dogs do we need for 6 tails?", or "How many dogs do we need for 6 ears?" Some of the students counted, and some used the graph to recognize the patterns and the functional relationship.

Result and analysis

Despite that the teachers found it challenging to teach about functional relationships and pattern generalizations in the classroom, it turned out that a significant part of the classroom conversation worked well as an entry to discussing functional relationships. There was a learning potential to work with the graph representation and discuss the relation between quantities in Grade 1.

In the reflection directly after the teaching, Jonna talked about her perceived challenges in the teaching. Jonna talked both about the lack of words to describe the functional relationships and the challenges in explaining generalizations in Grade 1. Similar challenges arose in the discussions in the design process. The transcript below is from the teachers' discussion in the design process after the teaching had been completed, whereby the teachers refer to the previously mentioned tasks with dogs and the number of legs.

Clara: I don't have the words to talk about the functions the relation between dogs and legs. I don't know what questions I should ask the students to give them opportunities to develop generalizations.

Jonna: To be honest… I am not sure what a generalization includes. Is it enough when the students say, "Every time I add 4 more…"?

The transcript highlights two thoughts that often arose in the intervention when the teachers discussed generalizations in patterns and functional thinking. First, the teachers talk about challenges in how to talk about functional relationships. They talk about the lack of words to describe and express how quantities vary in relation to each other. In the intervention, the concept of a functional relationship is new and unknown in relation to pattern generalization the teachers used before ("double" and "a half") when working with relationships in elementary school. Now, the teachers are searching for words to describe and explain, for example, coordinates, independent- and dependent variables, proportional relationships, graphs, functions, and the x- and y-axes. The teachers are searching for the terminology that they have not previously used in their teaching.

Secondly, the teachers expressed uncertainty about when the students' arguments are enough to represent a generalization. In the dialogue, Jonna's reflections about whether it is enough when the students say, "Every time, I add 4 more." This can be interpreted as the student having identified a pattern and is somehow about to generalize the pattern. In the design process, the teachers discuss "When can a generalization be seen as a generalization?" in Grade 1.

At the end of the intervention, the teachers planned activities that would allow the students to identify functional relationships and patterns from what they called the "opposite way". They start from the graph representation, the function, and ask the students to find patterns. They begin an activity like the previously mentioned activity involving functional relationships with discrete numbers, that is the dogs and the relationships between the numbers of tails, ears, or legs ($y = x, y = 2x$ or $y = 4x$). The graph representation creates opportunities to talk about functional relationships in discrete values in Grade 1. However, in the intervention, the participants problematize the graph's visualization of continuous values.

Before the lesson in Grade 1, Jonna expressed worry about explaining and talking about functional relationships to the students. She mentions again the missing terminology for talking about how quantities vary in relation to each other. Jonna starts to give all the students a Cartesian coordinate system with two graphical representations and an empty table. The two graphs (the green and the red graph, see Fig. 1) illustrate the functions $y = x$ and $y = 3x$.

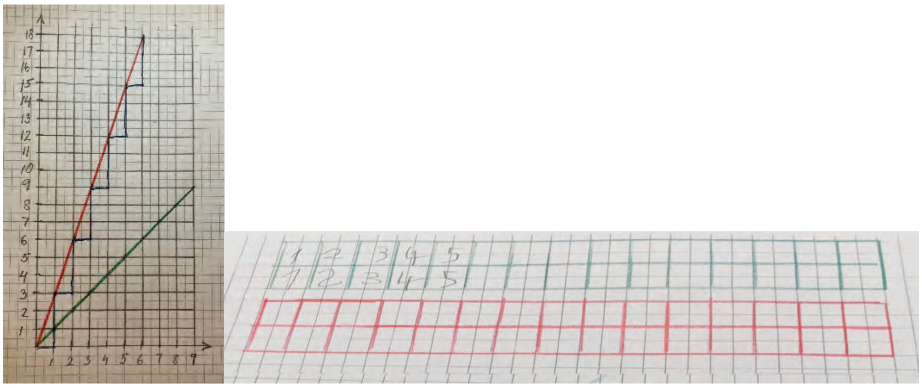

Figure 1: The Cartesian coordinate system and Karim's table,

illustrating the functions $y = x$ and $y = 3x$

The students used the graphs, wrote the independent and the dependent values in the tables, and then identified the pattern. Some of the students had to write many coordinates in the tables before they identified the pattern and tried justifying the pattern with oral language. Some students found the pattern quite fast and tried to express the pattern illustrating the function $y = 3x$ in the classroom. One of the students, Karim, talks to the teacher; he points to the camera stand with three legs and says, "Every time we have a new camera stand, we have three more legs."

In the class discussion, Jonna, the teacher, initiated talks about the identified patterns and used the tables and the graphs to talk about predicting near data. The results show that the teacher-led-class discussions included recursive- and covariational reasoning. The teacher and the students spoke about what emerges in a single sequence, namely, what happed to the dependent value? This short transcript is an example of a conversation in the discussion relating to one of the graphs in Figure 1, that of the "red" graph illustrating, $y = 3x$.

1	Teacher, Jonna:	Can someone describe the identified pattern?
2	Student, Sara:	The pattern increases by 3.
3	Teacher, Jonna:	How do you know that the pattern increases by 3?
4	Student, Ida:	In the table, it is "3–jumps".
5	Teacher, Jonna:	Yes, what can the red graph illustrate?
6	Student, Alex:	A snowman with 3 snowballs.
7	Student, Tanesha:	A tricycle with 3 wheels.
8	Teacher, Jonna:	Yes, how many wheels do we need for 2 tricycles?
9	Student, Frank:	We need 6 wheels.
10	Teacher, Jonna	Tell me, how do you know that?
11	Student, Frank:	3 wheels are needed for each tricycle.

The observed teaching offers several opportunities to represent pattern generalizations in multiple ways, such as using pictures, tables, graphs, or oral language. The students used the Cartesian coordinate system and structured coordinates in a table. The transcript indicates that the students only have their oral language to describe the patterns, as they have no access to variable notation. However, the most common way to talk about generalization and functional relationships was in a recursive pattern, which means describing the variation and the relation in the dependent value, for example, line 2 ("… increases by 3") and line 4 ("… 3-jumps"). It continues like this, and the teacher tries to initiate discussions, including those centered around the relationship between the independent and dependent variables. An initial reasoning, including the covariational relationship, is visible at the end of the transcript in lines 8-11: "3 wheels are needed for each tricycle."

The class discussion continues and the teacher, Jonna, points to the Cartesian coordinate system and asks the students to say something about the green and red functions ($y = x$ and $y = 3x$) . One student notes that "the red line goes straight up compared to the green one." Jonna, the teacher, agrees and asks the students if anyone wants to justify why this may be so. One of the students points to the red graph and shows what he calls the "small tents" [the tent is much bigger in the red graph]. The teacher says, "Yes … that's right … and how can we explain this big and small tent?". In the class, the silence is noticeable before one of the students points to the red graph and says, "The red graph could be the tricycle, 1 tricycle has 3 wheels". The student points to the "tent" in the y-axis and counts "1, 2, 3…". The student then goes on to point to the x-axis and y-axis and the coordinate (2.6) and says, "2 tricycles have 6 wheels…".

The discussion ends with the teacher asking what the green graph could illustrate. Several students recognize the graph and the table from the last lesson about the dogs, and their corresponding tails. Another student connects this to the tricycle and says, "…The green line shows a unicycle."

Discussion

This study sheds light on the role of the graph representation and how it can develop students' functional thinking. The results provide insights about the opportunities the graph representation invites, in terms of both the teachers' and students' reasoning about functional relationships when working with pattern generalizations in Grade 1. Although the graph representation first and foremost invites discussions of recursive reasoning, both the teacher and the students in Grade 1 also succeeded in discussing covariational relationships.

Despite Jonna's worries before the lesson, it seems that functional thinking is visible in the class discussions. For example, Jonna's teaching visualizes one way to try to get closer and talk about how quantities vary in relation to each other, which could be compared to what Wilkie (2016) claims, "the notion of *change* and how varying quantities (or "variables") relate to each other" (p. 247). The graph representation supports the teacher and the students to justify the relationship between independent- and dependent variables in the oral language without variable notations. This suggests that my results show that the graph representation can be seen as a tool to use in elementary school when transforming what Chimoni et al. (2019) call shifting focus "from the visual structure to algebraic structure." However, Jonna and the other teachers in the intervention are still searching for the terminology to talk about functional relationships in elementary school.

The results also indicate what the teacher in the study expresses "I am not sure what a generalization includes. Is it enough when the students say, Every time I add 4 more…?". This phrase is a recursion and could be interpreted as a generalization in Grade 1. One may wonder what else could be done if there was some big number of wheels and one more tricycle was added and in what way would symbols for the unknown number of wheels and tricycles be introduced. Maybe the discussion about functional relationships with the support of the graph representation can be seen as the first step towards variable notations, which does not just support a missing value. This falls in line with both Carraher et al., (2008) and Stephens et al. (2017), who stress the importance of using coordinated changes in quantities when teaching generalizations. However, the study shows the importance of using a well-thought-out terminology to exploit the potential of the graph representation when discussing functional relationships and generalizations. What is interesting in this example is how the teacher's reflections enable the teacher to find new and unknown "ways" to teach generalization.

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Conditions for revitalizing the elementary algebra curriculum

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Elementary algebra has been the bedrock of science and technology for centuries. But taught algebra today is much more a set of formal exercises than a modelling tool. The present study focuses on the notion of formula and how its curricular evolution appears to be a witness and a cause of the degradation of taught algebra as a modelling tool. This examination involves careful analyses of curricular facts that seem not to have attracted the full attention of researchers, such as the vanishing of parameters from algebraic equations. Drawing on the anthropological theory of the didactic (ATD), we outline the perspective of an imperative revitalization of the elementary algebra curriculum. Data used in the study include curriculum materials such as textbooks from different countries and various types of publications on school algebra by authors influential in their time.

Keywords: Curriculum, didactic transposition, elementary algebra, formulas, parameters.

Introduction

The passing of time changes curricular contents. This universal process of "curricular aging" can lead any subject matter to lose a large part of its instrumental value and to deteriorate to the point that its study at school becomes but a rite of passage imposed on the younger generations. In this study, we try to set out conditions favourable to making *elementary algebra*, understood as the algebra taught in secondary schools, an effective tool for understanding many "facts" of both the mathematical and the extra-mathematical world. By studying how the possibilities offered by elementary algebra have been greatly reduced by the evolution of the algebra curriculum over the last century, we will highlight the potential of elementary algebra to become a tool for understanding the world around us. Our research question is: *What key conditions should secondary school algebra meet to become an effective modelling tool for our time?*

On the epistemology and methodology of the study

In the framework of the ATD (Chevallard, 2019, 2020), the modelling of didactic phenomena rests on the notions of *person*, *institution*, and *institutional position*.

All human individuals are persons. Any "instituted" reality is an institution, such as a family, a class, a couple, a school, a ministry, the Norwegian society, and the French society. Any institution is organized into a set of institutional positions: In a classroom, there is the teacher position and the position of student; in mathematics education, there are the positions of textbook author, of teacher educator, of "great mathematician", etc. An institutional position is occupied by persons who thus become "subjects" of the institution. Persons are shaped by the set of institutional positions they occupy and have occupied. Persons are thus singular representatives of a position to which they are subjected. At the same time, persons can change the positions they occupy; there is thus a dialectic between persons and institutions in the making of a society.

Consequently, to study any institutional position, one studies the persons who are or have been its subjects; and, conversely, in order to study a person, one studies the positions he or she occupies or has occupied. Hence, in this study, we have studied persons' publications that allow us to enlighten the historical evolution of the teacher and student positions regarding elementary algebra. This has enabled us to identify conditions and constraints that have determined the current didactic transposition of elementary algebra and, most importantly, the vanishing of parameters from it. Our methodology is essentially that of *didactic transposition analysis* (Chevallard, 1991).

The take-off of algebra: From rules to formulas

We shall first highlight the key points of the changes that have affected the algebra curriculum, in order to identify core requirements for revitalizing secondary school algebra. We will focus on an aspect little considered: the role played (or not played), in the algebra curriculum, by the notion of *formula*, seen both as a symptom and as a cause of the impoverishment of school algebra. Here is a typical rule found on the Internet for how to find the area of a trapezoid (How to Find the Area, n.d., Example Question #1 section): "To find the area of a trapezoid, multiply the sum of the bases (the parallel sides) by the height (the perpendicular distance between the bases), and then divide by 2." Once fully algebraized, this rule becomes a formula: $A=\frac{1}{2}(b_1+b_2)h$, where b_1 and b_2 are the lengths of the parallel sides and h the distance between them.

Parameters: From implicit to explicit

To distinguish between "arithmetical rules" and "algebraic formulas," we must use the essential notion of *parameter*. In the formula $A=l\times b$ for the area of a rectangle with length l and breadth b, the letters l and b are parameters specifying the rectangle. While a rule (in words) contains parameters *implicitly* (like "length" and "breadth"), a formula contains *explicit* parameters. Consider the following rule given by Percival Abbott (1869–1954) in his book *Algebra* (Abbott, 1942/1971):

> The area of a rectangle in square metres is equal to the length in metres multiplied by the breadth in metres. This rule is shortened in Algebra by employing letters as symbols, to represent the quantities… [With the letters l, b, and A representing respectively the length, breadth, and area (in metres and square metres)] the above rule can now be written in the form: $A=l\times b$." (pp. 13–14)

A rule thus expressed is therefore called a formula. With this, a question arises. According to Abbott (p. 13), the "classical" doctrine on the arithmetic-algebra divide consists in the fact that, in arithmetic, one employs *definite numbers*, whereas in algebra, "we are, in the main, concerned with *general expression* and *general results*, in which letters or other symbols represent numbers not named or specified." Now this statement is subtly contradictory to the notion of "arithmetical rule": In the rule for the rectangle, we have, not "definite numbers," but "implicit parameters," the length and the breadth, not represented by *letters*.

How can this seeming discrepancy be explained? A quick answer is: In arithmetic, students are *given* a rule—they only have to *apply* it when the (implicit) parameters in the rule take definite numerical values. *Deriving* a formula (relying on some basic, given rules or formulas) is usually rather easy when done algebraically. In contrast, if you do it "arithmetically," it usually becomes more complex, and beyond the reach of beginners. This can be linked to the notion of *topos* of an institutional position, that is the set of *types of tasks* that persons in that position may have to perform (Chevallard, 2019). Let us repeat: in arithmetic, the teacher provides the rules ready-made, and the students simply apply them to numerical values.

In his *History of Mathematics,* David E. Smith (1925/1958, p. 437) made this comment: "To the student of today, having a good symbolism at his disposal, it seems impossible that the world should ever have been troubled by an equation like $ax+b=0$." Such however was the case. The algebraization of arithmetic was a huge step forward, a game changer for the sciences. But an almost surreptitious drawback happened that greatly reduced the power of the algebra actually taught.

A Pyrrhic victory

The algebraic modelling of arithmetical rules might pertain to the student *topos,* as the exercise set of Abbott's *Algebra* (1942/1971) seems to show. Here are some of Abbott's exercises (pp. 20–21): "Write down expressions for: (1) The number of pence in £x; (2) The number of pounds in n pence… (19) A train travels at v km/h. How far does it go in x hours and how long does it take to go y km?"

The implicit parameters of arithmetical rules are here translated into explicit parameters: n, v, x, and y. However, in the history of elementary algebra, explicit parameters will tend to be replaced by "definite numbers." Here is an example taken from the chapter "Simple Equations" of Abbott's book:

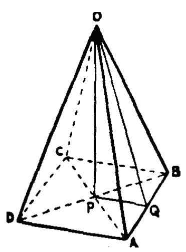
> A motorist travels from town A to town B at an average speed of 64 km/h. On his return journey his average speed is 80 km/h. He takes 9 hours for the double journey (not including stops). How far is it from A to B? (Abbott, 1942/1971, p. 66)

Abbott (1942/1971, p. 76) describes another type of equations "in which the values of the unknown quantities will be found in terms of letters which occur in the equation." He calls them *literal equations* and states that they can be solved by the same methods as simple equations. The examples he gives of literal equations are $5x-a=2x-b$, and $a(x-2)=5x-(a+b)$. The 14 equations in the related exercise set are similar to these: they are quite simple, and nobody knows what they claim to model—they are a mere didactic device, related to the need for tasks for repetitive training. We will come back to this phenomenon in the section "A turning point in the didactic transposition process."

Transformation versus evaluation of formulas: Reaching a demarcation line

The paucity of the material thus presented by Abbott is in striking contrast to the chapter's introduction, which begins with this promising statement (Abbott, 1942/1971):

> One of the most important applications of elementary Algebra is to the use of formulae. In every form of applied science and mathematics… formulae are constantly employed, and their interpretation and manipulation are essential. (p. 69)



The author explains that formulas "involve three operations: (1) Construction; (2) manipulation; (3) evaluation" (p. 69). The construction of a formula does not start from scratch: it relies on formulas previously established, either theoretically or empirically. The first "worked example" given by Abbott is typical: "Find a formula for the total area (A) of the surface of a square pyramid as in Fig. 10 [see figure opposite] when AB$=a$ and OQ$=d$" (p. 70). Here, the use of algebra is genuine but minimalist. By contrast, the second example and all the exercises given by Abbott are just *evaluation* tasks, as this exercise: “The volume of a cone, V, is given by the formula $V=\frac{1}{3}\pi r^2 h$, where r=radius of base, h=height of cone. Find V when $r= 3{\cdot}5$, $h=12$, $\pi=\frac{22}{7}$” (Abbott, 1942/1971, p. 70).

The *manipulation* of formulas is only present in the section entitled “Transformation of Formulae.” About the volume of a cone ($V=\frac{1}{3}\pi r^2 h$), Abbott (1942/1971) writes:

> It may be necessary to express the height of the cone in terms of the volume and the radius of the base. In that case we would write the formula in the form: $h=\frac{3V}{\pi r^2}$, that is, the formula has been transformed. When one quantity is expressed in terms of others, as in $V=\frac{1}{3}\pi r^2 h$, the quantity thus expressed, in this case V, is sometimes called the *subject of the formula*… This process of transformation has been termed by Prof. Sir Percy Nunn *“changing the subject of the formula.”* (pp. 71–72)

About such a change, Abbott (1942/1971) adds this caveat: “The transformation of formulae often requires skill and experience in algebraical manipulation” (p. 72). He then illustrates the “methods” to be followed by five “worked examples.” In one of them (p. 73), he transforms the formula $L=l+\frac{8d^2}{3l}$ to find d in terms of L and l and arrives at $d=\sqrt{\frac{3l(L-l)}{8}}=\sqrt{\frac{3lL-3l^2}{8}}$. Readers are not asked to find the expression of l in terms of L and d—the answers are $l=\frac{L}{2}\pm\sqrt{\frac{L^2}{4}-\frac{8d^2}{3}}$—, which would require solving the *quadratic* equation $l^2 - Ll + \frac{8d^2}{3} = 0$. Here we reach the demarcation line drawn by the traditional didactic transposition of elementary algebra.

This line draws a curricular curiosity. Firstly, the quadratic equations with parameters considered have only one parameter. Secondly, students are not asked to give the expression of their solutions (which, in the general case, would include the parameter), but simply to specify, according to the value of the parameter, when they have 0, 1 or 2 roots. This sudden change of didactic contract (Brousseau, 1997)—an equation is no longer “something to be solved” but to be “studied” or “discussed”—was (and still is) a source of difficulty for students. In spite of this, the question of the “manipulation” and transformation of formulas, alongside their “construction” and “evaluation,” which are much less problematic, is at the heart of what algebra can consist of. In Abbott’s Example 5 (p. 74), readers are asked to find the length l of a simple pendulum in terms of the other quantities when its time of vibration is given by $t=2\pi\sqrt{\frac{l}{g}}$. Exercise 13 (No. 2) is about expressing the radius r of a sphere in terms of its volume V (p. 74). The usefulness of these transformations seems obvious. Now the big problem is that the type of tasks in question— “changing the subject of a formula”—has become marginalized in most secondary curriculums. A study of Norwegian textbooks used in recent decades is a clear testimony to this fact (Strømskag & Chevallard, 2021). In one of the textbooks for Grade 11 (Sandvold et al., 2006, p. 26), the authors consider the formula $v=\frac{d}{t}$ (where v is the speed, d is the distance travelled, and t is the time) and explain how to “solve the formula with respect to the time t" as if the students were complete beginners in elementary algebra. The authors then explain how to solve for c the (fabricated) formula $p=a+\frac{1}{2}bc^2$ to arrive at $c=\pm\sqrt{\frac{2-2a}{b}}$. Then follow tasks that ask to solve for t these formulas: $d=vt$; $d=\frac{1}{2}at^2$; $v=v_0+at$; $d=\frac{(v_0+v)t}{2}$. This is a limited viaticum (e.g., there are no quadratic equations with parameters) for a further journey into elementary algebra. The same phenomenon prevails in French textbooks (Chevallard & Bosch, 2012) and in German textbooks (e.g., Brandt & Reinelt, 2009). The latter is in line with the German Mathematics Standards for the expected level by the end of upper secondary education (Kultusministerkonferenz, 2012), where parameters are present in some of the example tasks but never in equations to be solved.

The pitfalls of didactic transposition

The marginalization of formula transformation

Let us consider the following exercise proposed by Abbott (1942/1971, p. 75): "There is an electrical formula $I=\frac{V}{R}$. Express this (1) as a formula for V and (2) as a formula for R. Find I if $V=2$ and $R=20$." A number of teaching institutions choose to "spare" their subjects the algebraic "work" needed to go from the formula $I=\frac{V}{R}$ to the formulas $V=R\cdot I$ and $R=\frac{V}{I}$. One of the most widespread techniques, it seems, consists in substituting to the algebra needed a graphic "mnemonic trick" which takes the form of a triangle in which the parameters V, I, R are displayed (see example in Figure 1).

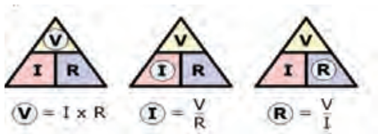


Figure 1: A graphic mnemonic trick (retrieved from Nimar_geek, 2020)

This "triangle technique" is pushed forward by institutions. The institutional enforcement of this technique seems to send the following message: "You don't need to know algebra *at all*."

There is also another, widespread technique, which is implemented more by people—students, in particular—than by institutions. This technique consists in avoiding any literal calculation. Suppose we are given the formula $I=\frac{V}{R}$ and values for I and R, and are asked to calculate V. If $I=1.2$ and $R=20$, the formula gives rise to the equality $1.2=\frac{V}{20}$, which is a linear equation in V that the student can therefore easily solve. This technique consists in first transforming a formula into a "simple" numerical equation.

A turning point in the didactic transposition process

How has this demarcation line been drawn? The answer must involve the conditions and constraints that have historically determined the didactic transposition of elementary algebra. Two influential textbook authors who have taken part in this transpositive work are Abbott and Nunn. About the phrase "Change the subject of a formula," Nunn writes in his book *The Teaching of Algebra* (1914):

He [the author himself] believes that it was used for the first time in his lectures to teachers of mathematics in 1909. It was subsequently adopted in the Report on the Teaching of Algebra by the Committee of the Mathematical Association. (p. 78)

Why did Nunn introduce this way of saying, which was adopted by Abbott and others, when what is required is simply to "solve the equation $A=\pi r^2$ for r?" Nunn seems to have been quite aware of the change he wanted to popularize. Thus, he launches an attack against the position of strength given to equations, which he calls "conundrums" that "the school tradition has not lifted … to a much higher level of intellectual dignity" (Nunn, 1914, p. 77).

Nunn's degradation of equations leads to the coming apart of two distinct topics: equations and formulas. Formulas such as $V = R \cdot I$, which become equations once an unknown has been chosen (we can solve $V = R \cdot I$ for I for example), are indispensable in many fields of science and technology. Paradoxically, they were going to be marginalized by their very "promotion."

This detail of the didactic transposition process is linked to two great constraints. The first constraint is that of *simplicity*: the transposed content must be "simple" enough to offer students a *topos* that they can actually occupy. In France, in the early 1960s, a demanding exercise textbook for Grade 10 still proposed the following, highly artificial exercise (what exactly is it modelling?): "Solve the equation $(a+b)^2x^2 - (a-b)(a^2-b^2)\,x - 2ab(a^2+b^2) = 0$" (Combes, 1961, p. 124). But parameters in equations were officially deemed "undesirable" in 1981 (Chevallard & Bosch, 2012, p. 16). The second constraint is that, for didactic reasons of repetitive training, the teacher must be able to produce *at will* tasks of any type he or she has to teach. However, because of their origin in specific domains (geometry, physics, technology, etc.), it seems that the list of formulas to be "solved" is limited. In order to make it easier to create formula transformation tasks, it is accepted to break the link between a formula and what it models. The constraints mentioned therefore contributed to making the algebra taught a separate field, almost foreign to the other fields of mathematics and science.

Systems and models: Algebra for the future

So, what should elementary algebra consist of? To answer this question, we must first introduce two basic notions of the ATD: the notions of *system* and *model*. A system $\mathcal{S}$ is any entity subject to laws of its own. For example, a (geometric) sphere is a system whose "laws" are generally called the *properties* of the sphere, such as the following: "A great circle… of a sphere is the intersection of the sphere and a plane that passes through the centre point of the sphere" ("Great Circle," 2021). Any formula is a system as well. The formulas for the volume and the surface area of a sphere of radius r, that is, $V=\frac{4}{3}\pi r^3$ and $A=4\pi r^2$, are systems in their own right, which themselves have properties (we have $V=A\times\frac{r}{3}$ or $A=\frac{3V}{r}$ or $3V-rA = 0$, etc.). Given a system $\mathcal{S}$, a system $\mathcal{S}'$ is said to be a model of $\mathcal{S}$ if, by studying $\mathcal{S}'$, one can answer certain questions Q about $\mathcal{S}$. In practice, given a question Q relating to $\mathcal{S}$ which one wants to answer, one tries to build up a model $\mathcal{S}'$ of $\mathcal{S}$ (or choose one already existing) whose study with respect to the question Q is easier, safer, quicker than by a "direct" study of $\mathcal{S}$. For example, if the radius r of a sphere increases by 20%, the new surface area A' will be $4\pi r'^2=4\pi(1.2r)^2=1.44A$, so that the surface area will increase by 44%—a tricky result to obtain experimentally.

The great catastrophe which historically disorganized and denatured elementary algebra resulted from the generalized rupture of the link between the systems $\mathcal{S}$ to be modelled algebraically and their algebraic models $\mathcal{S}'$ relating to some question Q about $\mathcal{S}$. In most textbooks, this link has disappeared entirely. The vanishing of parameters from algebraic expressions goes together with the purely formal existence of algebraic expressions, which consequently lose their functional role, that is, the role of elements of a model of a system.

Conclusion: Conditions for a more authentic algebra curriculum

How can we pave the way towards a curricular reconstruction that revitalizes elementary algebra? The answer rests on two components: 1) the notions of system and model explained above and 2) rescuing the notion of formula and reintroducing expressions in several indeterminates in order to be able to model a diversity of mathematical or extra-mathematical systems. In a mathematics class, it is essential to study triples ($\mathcal{S}$, Q, $\mathcal{S}'$) composed of a mathematical or extra-mathematical system $\mathcal{S}$, a question Q raised about $\mathcal{S}$, and a model $\mathcal{S}'$ (related to $\mathcal{S}$ and Q) which contains mathematical elements that are key to constructing an answer to Q.

Mathematics education is therefore potentially concerned with all situations in which mathematics is or can be used to better understand the situation in question. In this respect, let us remind the reader that, from about 1600 to 1800, mathematics was divided into two branches, that of *pure* mathematics and the widely embracing branch of *mixed* mathematics (see e.g., Bacon, 1605/1901, pp. 172–174).

So, what key conditions should school algebra meet to be an effective modelling tool for our time? By way of a conclusion, we shall sum up the core of a more "authentic" study and use of algebra identified in the course of this inquiry:

> 1) The students start from a system $\mathcal{S}$ and a question Q raised about it, whose adequate treatment seems to involve mathematical elements; 2) These students build up a model $\mathcal{S}'$ of $\mathcal{S}$, relative to the question Q, which will be built with elementary algebra (and will include as many parameters as seems useful); 3) They work on $\mathcal{S}'$ to derive an answer A deemed adequate to the question Q; 4) At the same time, prompted by this process of inquiring about $\mathcal{S}$, they discover the resources of algebra, study or restudy them in order to make an efficient use of the tools thus garnered.

A brief example is in order here. The starting point is the theorem which says that when the sum of three numbers a, b, and c is constant, then the expression $ab+bc+ca$ is maximal when $a=b=c$. What can this result be used for? One answer concerns the prices of diamonds, when assumed to be proportional to the *square* of their weight. If the price of a diamond of weight w is equal to kw^2, where $k>0$, and if a diamond is broken into three pieces of weight a, b, and c, respectively, the price of each of these pieces is ka^2, kb^2, and kc^2 while the price of the original diamond of weight w_0 was $kw_0{}^2=k(a+b+c)^2$. We have: $(a+b+c)^2-(a^2+b^2+c^2)=2(ab+bc+ca)>0$. The price of the original diamond is therefore greater than the sum of the prices of the three diamonds obtained. As a consequence of the equality $3(ab+bc+ca)=(a+b+c)^2-\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]=w_0{}^2-\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$, the loss of value caused by the breaking of the diamond into three parts is maximal when $a=b=c$, that is when the three pieces have the same weight. Here, the system $\mathcal{S}$ is a diamond and its selling price, and a key element of the model $\mathcal{S}'$ is the equality

$3(ab+bc+ca)=(a+b+c)^2-\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$ with three parameters. For a more detailed discussion of this example, see Strømskag and Chevallard (2021). The four points listed above outline a research and innovation programme to which the present study is a contribution in order to help develop, in the decade to come, the full collaboration of researchers, teachers, and teacher educators.

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Programming tasks as an instrument for helping students make meaning of methods for solving quadratic equations

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Keywords: Mathematics education, algebra, programming.

Poster summary

Although there has been a good deal of research into the problems students often encounter when solving and trying to understand first-degree equations, there has been relatively little comparable research involving quadratic equations. A number of authors have identified difficulties with quadratic equations which overlap with those known for first-degree equations. For example, some students were unable to solve quadratic equations because of difficulties with working with fractions, negative numbers and basic computational skills, as well as difficulties with understanding the concept of variable (e.g., Didiş & Erbas, 2015; Hu et al., 2021; O'Connor & Norton, 2016). However, a number of problems specific to solving quadratic equations have also been identified, including using the quadratic formula, completing the square, factoring second-degree polynomials, and understanding the zero product property (e.g., Hu et al., 2021; O'Connor & Norton, 2016; Tall et al., 2014).

Research into how programming can help students understand algebra has been rather limited so far. Most of the connections to algebra that have been looked at are related to built-in features of the chosen programming environment, such as coordinate systems and variables (e.g., Germia & Panorkou, 2020). Much recent research has focused on using Scratch to introduce programming to younger students, both with the goal of teaching computational thinking as well as teaching concepts within mathematics (e.g., Benton et al., 2016).

Although the zero product property is unlikely to be useful for solving an arbitrary quadratic equation, since it is unlikely to be factorable (Bossé & Nandakumar, 2005), it has clear uses within the context of school mathematics, as well as being important for understanding more advanced mathematics such as the roots of polynomials and the partial fraction method of integration. Therefore, this poster aimed to initiate a discussion in the TWG3 about the use of programming tasks as an instrument to teach and develop algebraic thinking, in particular with regard to using the zero product property to solve quadratic equations. In addition to background information, the poster includes information about the intended design of the project as well as example tasks.

In cooperation with secondary school teachers, pairs of students will be given a series of programming tasks designed to elicit a deeper understanding of the zero product property and how to apply it when solving quadratic equations. The tasks will be developed together with the teachers, tapping into their knowledge of their own students' current understanding of mathematics. The students will be recorded while working on the tasks and interviewed afterwards in order to gain insight into the effectiveness of the programming tasks. The tasks and results will be designed and analyzed using the theoretical framework Abstraction in Context (AiC), which builds on the works of Freudenthal and Davydov and combines both cognitive and sociocultural ideas (Dreyfus et al., 2015; Dreyfus & Kidron, 2014). AiC relies on a detailed analysis of student discussions during the tasks as well as their interview responses in order to identify the formation of new mathematical constructs. This method of analysis does not depend on comparing the results of a pre-test and post-test.

This poster presents the research project in a graphical format in order to more clearly show how the various phases fit together.

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Diagnostic test tool based on a praxeological reference model to examine students' technical and theoretical algebra knowledge

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The main aim of this paper is theoretical and methodological: to show how The Antropological Theory of the Didactics – in particular, the construction of a praxeological reference model – can be used as a foundation for developing a diagnostic test tool which examines students technical and theoretical knowledge of basic algebra (and related arithmetical knowledge on which initial algebra is based). The point is to provide explicit control on what is being tested, for instance, in relation to a given national curriculum, and in relation to teaching interventions.

Keywords: Anthropological theory of the didactics, school algebra, praxeological reference model, diagnostic test.

School algebra

In secondary school, basic algebra is a crucial bridge between arithmetic and more advanced subjects involving functions and analytic geometry. To set up algebraic models, it is necessary to master arithmetic and algebraic techniques by hand, at to know associated theory like the distributive law. But it is a longstanding and widespread problem that large groups of students seem to get stuck at this bridge between arithmetic and algebra (Herscovics & Linchevski, 1994). This has major personal and societal consequences because basic algebra as taught in lower secondary school plays a crucial role in upper secondary mathematics and hence for access to attractive higher education programmes. In that way school algebra is often described as a central gatekeeper (Loveless, 2013).

The last decades of algebra research has given more attention to the theoretical foundations of students' work. Algebraic transformations are not viewed only as procedures, but also as theoretical entities (Kieran, 2007). Kieran divides research in algebraic transformations into theoretical, technical, and practical elements. These elements are closely connected, and different institutions within the educational system manage them in subtly different ways. Early research on school algebra tended to make a sharp distinction among procedural and conceptual approaches. This dichotomy is currently challenged, and the potential of new theoretical and methodological approaches become essential to investigate the crucial *connections* between techniques and theory (Schneider & Stern, 2010). The Anthropological Theory of the Didactic offers, in particular, a promising new approach to this task.

To investigate the transition from arithmetic to algebra and to gain knowledge about what algebraic techniques and theory are particularly problematic for students in Danish lower secondary school, we ask the research question: "How can the Anthropological Theory of the Didactics (ATD) and the construction of a praxeological reference model (PRM) be used as foundation for developing a diagnostic test tool, to examine students technical and theoretical algebra knowledge?"

This paper will concentrate on the construction of the PRM and the derived diagnostic test tool with preliminary results from the pilot test. The PRM and the results of the diagnostic test will be used in a latter study to select research-based resources and design teaching interventions to support teachers teaching basic algebra.

ATD as theoretical foundation

The Anthropological Theory of Didactics, subsequently noted as ATD, has emerged as a theory of mathematics education. A central feature in ATD is the use of praxeology to model school mathematics activity. A praxeology compromise types of task, techniques, technologies, and theories (Bosch, 2015). The "practical block" or *praxis* is formed by the types of task and by the techniques used to solve them (Barbé et al., 2005). The "theoretical block" or *logos* consists of technology (discourse on techniques) and theory (more general discourse, based on deductive reasoning from definitions and the like). This means the techniques for carrying out tasks are explained and justified by a 'discourse on the technique' called technology; taking this discourse to a more abstract level yields mathematical theory, to validate the technological discourse and to connect entire praxeologies (Bosch, 2015).

The use of praxeology to analyse school algebra has been particularly successful, since the birth of ATD as a theoretical foundation for mathematics education research (Bosch, 2015). Bosch argues that the explicit reference praxeological models (PRM) concerning school algebra provides opportunities to ask research questions that go beyond the assumptions held by the school institution itself. At the international level, ATD research has led to significant new insights on the algebra problem, including the frequent disconnectedness of praxeologies taught and learnt.

According to Chevallard (2019), praxeologies are not static, but a dynamic system of institutionally situated activities. The explicit construction of a PRM will enable us to analyse what arithmetic and algebraic praxeologies are currently taught in the Danish lower secondary school according to curriculum, textbook material, and written examination. The PRM will also form the foundation for developing a diagnostic test tool, which can "diagnose" what algebraic techniques and theory are problematic for students (in our case, Danish grade 7). The result of the diagnostic test is analysed in terms of the PRM and may lead to revise the tool (e.g., if unexpected techniques appear). In a later study the diagnostic tool and the associated PRM will be used to design the intervention based on resources and to analyze the effects of the interventions. Thus, the model is the researchers' explicit reference throughout the four-step research process shown in Figure 1.

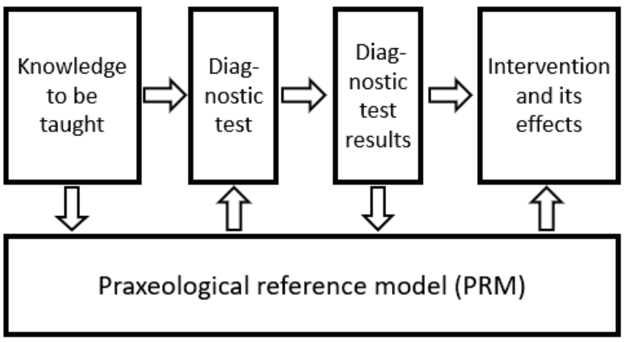


Figure 1. Construction and use of the praxeological reference model

Methods to construct the praxeological reference model

In Denmark, the so-called "common goals" for mathematics (Danish Ministry of Education, 2019), constitute the official directives for all primary and lower secondary school. They are divided into four parts: competences, number and algebra, geometry and measurement, statistics, and probability. The overall goal for algebra is that "the student can apply real numbers and algebraic expressions in mathematical investigations". It is up to textbook authors and teachers to transpose the common goals into teaching practice; in addition to the official goals, they can also find some direction in the exercises appearing the national exam after grade 9.

Our first step in building the PRM was to identify types of tasks appearing in the 2019 national exam (Prøvebanken, 2021) and in the textbook series Kontext+ grade 7-9 (Lindhardt et al., 2021). We began by identifying the tasks in the material associated to arithmetic and algebra, understood as tasks solely focused on operations, equations and order relations involving numbers (arithmetic), or numbers and literal symbols (algebra). At the level of theory, operations are in these two cases obeying the axioms of ordered integral domains or fields. In this paper, we do not consider the use of CAS tools and instrumented techniques.

The second step is to analyse in terms of task type T_i and corresponding techniques τ_i used to solve T_i. Let us first take an example from arithmetic where students are asked to calculate the following tasks (Lindhardt et al., 2021, p.104).

a. $6+(-7)$
b. $5-(-7)$
c. $-3-(-6)+(-4)-2$

Tasks a. and b. are simple (they can be solved by one technique). Task a. is what we have named type T_4: addition of negative integer to a positive integer, with corresponding technique τ_4: $a+(-b)=a-b$; where b. is a task of type T_6: subtraction of negative integer from negative integer, with corresponding technique τ_6: $a-(-b)=a+b$. Task c. requires both techniques τ_4 and τ_6 and is thus a combination of more elementary tasks.

An example of a type of task from algebra is T_{15}: solve a first-degree equation. Tasks of this type appear for instance in the written national 2019 exam (Prøvebanken, 2021)

Solve the equations

d. $5x+9=34$ $\qquad x=$ _______
e. $3x+4=6x-5$ $\qquad x=$ _______

They can be solved by the technique τ_{15}: involving addition, subtraction, multiplication, and division on both sides of the equal sign.

In general, exercises can contain several questions, where not all questions can be answered by a single technique. This means, that once a model of types of tasks and corresponding techniques has been established, more complex questions must be decomposed in tasks of the types established (Winsløw et al., 2013). It is relatively straightforward to identify types of tasks and techniques in arithmetic and algebra as shown above (cf. Wijayanti & Winsløw, 2017). The next step is to identify *themes* (groups of practice blocks unified by a technology) and *sectors* (groups of themes unified by a theory).

Example of PRM theme and sector

The problem of ordering two given fractions can, according to special cases, necessitate different techniques. It thus leads to group of practices which are taught together and are unified by a shared discourse about techniques, involving characteristics of the special cases, and descriptions of the techniques. Table 1 is type of task T_i and corresponding techniques τ_i from the analysed textbook material and written exam and gives an overview of the theme of PRM according to fractions.

Table 1. Theme of PRM based on textbooks (Kontext) and written exam (FSA) according to fractions

Type of task	Techniques	Kon-text+ 5	Kon-text+ 6	Kon-text+ 7	Kon-text+ 8	Kon-text+ 9	FSA 2020 Dec	FSA 2021 May
T_{19}: Given two unit fractions $\frac{1}{a}$ and $\frac{1}{b}$, which is largest	τ_{19}: The fraction with the lowest denominator is largest i.e., $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$							
T_{20}: Examine which fraction with like denominator and different numerators, $\frac{a}{c}$ and $\frac{b}{c}$ is largest.	τ_{20}: The fraction with the highest numerator is largest i.e., $if\ a < b \implies \frac{a}{c} < \frac{b}{c}$							
T_{21}: Examine which fraction with like numeration and different denominators fraction $\frac{a}{b}$ and $\frac{a}{c}$ is largest.	τ_{21}: The fraction with lowest denominator is largest i.e., $if\ a < b \implies \frac{a}{b} > \frac{a}{c}$							

It is characteristic of Danish textbooks that the same types of tasks reappear year after year, while the theme as a whole may still be relatively disconnected. This theme is a part of a larger "fraction sector", unified by a theory which involves both informal and more formal representations and properties of fractions, from addition of simple fractions with pizza diagrams to calculation rules given with algebraic symbolism.

Methods to construct the diagnostic test tool

The aim of the diagnostic test tool is to detect what arithmetic and algebraic techniques are particularly problematic for Danish students at early lower secondary school, and to get insight in the students' theoretical knowledge especially in relation to basic algebra. The development of the diagnostic test tool is based on the PRM and was inspired by an earlier project by Cosan (2021) on middle school arithmetic.

Items designed to test techniques are straightforward to construct. And several items can be included to investigate how variations influence on success rates. For example, the item "Compute $6 + (-5)$" represents the type of task T_4: addition of negative integer to a positive integer, as defined above.

It is more difficult to design items that detect students' theoretical knowledge which includes technology to describe and explain techniques. The item "Explain why $a - (-a) = 2a$" requests from students a piece of discourse justifying the algebraic rule, which may appeal to more or less formal theoretical principles. For instance, some students may refer to "two minuses can be replaced by a plus" as an overarching principle in such contexts; this could, in fact, be part of a theory that some students hold. In the ATD sense such theory elements are empirical objects, to be discovered and traced.

Items can also simply request a description of a technique (i.e., a technology), e. g. "Explain how you would determine which of the fractions $\frac{2}{5}$ *and* $\frac{3}{5}$ is largest"

Results from the pilot test of the diagnostic test tool

The diagnostic test has been pilot tested by 25 grade 7 students (12–13-year-old) in lower secondary school in the capital of Denmark. The students got 45 min. to do the 67-item paper and pencil test.

Table 2. Sum of type of answers in the test

Correct answer	Incorrect answer	No answer	Sum
455	306	914	1675

Table 2 shows that more than half of the items has not been answered by the students. Despite this, it is possible to give some preliminary results. The following are examples of analyses of test answers in relation to the previously selected examples from the PRM.

Table 3. Item and associated sum of answers in the test

Item	Item number	Correct answer	Incorrect	No answer
$6 + (-5) =$	1.4	20	2	3
$7 - (-9) =$	1.5	8	16	1

Table 3 shows that almost all the students can solve the item of task type T_4: addition of negative integer to a positive integer. But only a third of the students could solve the item of task type T_6: subtraction of negative integer from negative integer. This result indicates that students know that adding opposite is the same as subtraction, but they cannot apply the rule $-(-a) = a$.

By varying the items given for a specific type of task, we can also discover specific features of type of task. For example, the test contains variations of the task type T_{15}: solve a first-degree equation (discussed above). These variations result in hugely different success rates.

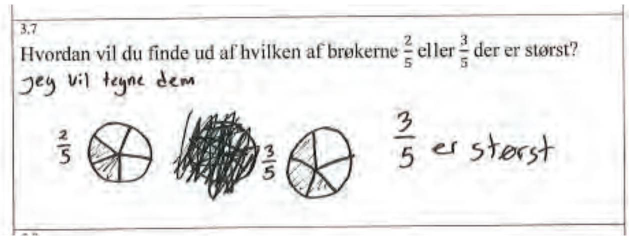
Table 4. Test results of variations of the task type T_{15}

Item	Item number	Correct answer	In-correct	No answer
$36 - \_\_\_\_ = 29$	1.1	21	2	2
$8 + 4 = \_\_\_\_ + 5$	1.6	4	18	3
$\_\_\_\_ - 32 = 45$	2.1	23	1	1
$2x = 10$	2.5	10	1	14
$7x - 7 = 13 - 3x$	4.7	8	9	13

When comparing the results from item number 1.1, 1.6 and 2.1, item number 1.6 has significantly fewer correct answers. One possibly crucial difference between 1.6 and the other questions is the location of the unknown. In 1.6 the unknown is located on the right side of the equal sign. That this could make a big difference is confirmed by a longitudinal examination of how middle school students understand the equal sign and equivalence of equations (Alibali et al, 2007).

The purpose of item number 3.7 is to get insights in students' argumentation for which fraction is largest. And we get answers like: "If the denominators are the same, I just look at the numerators which are the highest" and "I look at the denominators and if the fractions have the same denominator, then I will look at the numerator which one is the largest". With such items, we can detect not only a (correct) technique but also what it is, and a level of technology.

In the following two examples, the student's argumentation is based on pizza representations of fractions, which are also extensively used in Danish textbooks. Figure 2. "I want to draw". Figure 3. "By seeing it as a circle and look where there are most fields that are filled".



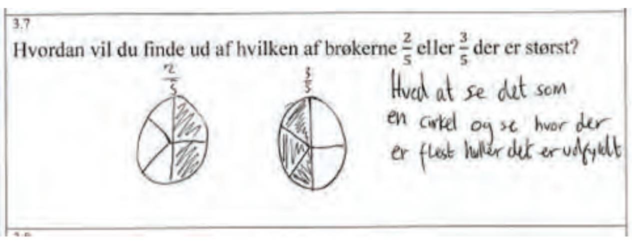


Figure 2. Student answer to question 3.7 **Figure 3. Student answer to question 3.7**

In Figure 2, we can see the representation serves as an argument but in Figure 3, this wordless technology fails because of missing the usual convention, that the circle must be divided into equal parts. The diagrammatic representation has evident forces in giving meaning to fractions (between 0 and 1) for young children; but the division of a circle into five equal parts is also a task which contain other meanings (including angles etc.) that are in some sense irrelevant to the task. Varying this task to include fractions with large denominators or nominators would evidently also make this technology fail.

Conclusion

In this paper, we have shown by a few examples, how ATD and the construction of a PRM, can be used as a foundation for developing a diagnostic test tool, to examine students technical and theoretical knowledge. By first constructing the PRM based on textbook material, written examination, and common core. Then grouping the task and corresponding techniques in themes and sectors according to shared technology and theory. The PRM is then used to design the diagnostic test items in line with the tasks, techniques, and level of theory in the PRM. In that way the diagnostic test is aligned with the knowledge to be taught and provide explicit control on what is being tested.

Among the examples from the pilot test, we discussed the students' difficulties with relating subtraction and additive inverse, and especially with repeated additive inversion. These examples indicate that the diagnostic test, based on the PRM, can be used to identify significant obstacles. This is crucial for the next steps in our doctoral project.

The next step in the project is to apply the results from the pilot-test to inform and strengthen the PRM. Then to revise the diagnostic test tool and complete the final test in four classes, before and after the teaching intervention. The overall aim is to investigate the transition from arithmetic to algebra and to explore if and how interventions with research-based material can support teachers' efforts to teach basic algebra.

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Primary school children's justifications of equalities

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A focus on mathematical structures and relationships in mathematical activity is important for the development of algebraic thinking. A comprehensive and flexible understanding of equality, which in particular includes a relational perspective on equality, is an important prerequisite for this. In the project, that forms the basis of this paper, substantive learning environments were further developed by use of Term fields and tested in teaching-learning experiments. In the video-based qualitative analyses, the focus is on the justification processes of fourth graders in the context of equalities. This paper presents and discusses the main results of the study, four different ways of justifying equalities and their characteristics.

Keywords: Early algebra, understanding of equalities, equal sign, mathematical reasoning.

Introduction: understanding of equalities in primary school

A call for greater support of algebraic thinking in the early grades is made in current research (e.g., Steinweg, 2017). Particular emphasis is placed on mathematical structures, found to be central to the development of early algebraic thinking (Hewitt, 2019; Steinweg, 2017). According to Steinweg (2017), *equivalence structures* are an algebraic key idea that focuses on the relation of given numbers, sums, products, etc. in equations. However, in primary school the equal sign is often introduced as an operational sign, prompting calculation.

Studies have found that students interpret the equal sign predominantly operationally and have difficulty taking a relational and structural view when needed (e.g., Eichhorn et al., 2018; Stephens et al., 2013). Some authors further differentiate between views of the equal sign. For example, regarding the relational understanding of the equal sign, Stephens et al. (2013) distinguish between a *relational-computational* view and a *relational-structural* view. A relational view is crucial for flexibility in mental calculation (Rechtsteiner & Rathgeb-Schnierer, 2017; Steinweg, 2017). This is an important skill in elementary school arithmetic and also essential as preparation for algebra (Jones et al., 2012).

In addition to the numerous studies concerning the use of the equal sign and the solving of formal equations, approaches have also investigated the content-related understanding of equalities in primary school (Mayer, 2019; Nührenbörger & Schwarzkopf, 2015). In these studies, substantial learning environments were developed, incorporating task formats that children are familiar with from textbooks, such as *Number walls* and *Computing chains*. These task formats share a specific external structure, combining numbers and operations in a way that allows numerous basic mathematical activities to explore structures and relationships. Thus, the learning environments focus on equality but without the use of the equal sign. The results of these studies show that primary school children can indeed take a relational view of equalities and interpret them structurally.

Comparing the results of the studies described above using quite different learning environments, it is evident that children interpret equalities differently depending on the context and form in which they are presented. Seo and Ginsburg (2003) use the term *pseudo-flexibility* to describe children's context-dependent interpretation. Children can take an operational view or different relational views of equalities, but these views appear to be entirely determined by the context and are mostly not linked to each other.

The project, from which data is excerpted and analyzed in this paper, has two objectives. On the one hand, a teaching-learning arrangement was developed that can stimulate a comprehensive and flexible understanding of equality in children by using *Term fields* (see Figure 2). On the other hand, the children's justifications of equalities are analyzed qualitatively. The concepts from the literature described above serve as a starting point for analyzing the justifications provided by the children in this project. This paper focuses on the research question: How do children justify discovered equalities in the context of the developed learning environments?

Methodology and design

In line with the aim of the project, qualitative interview studies about German primary school children's conceptions of mathematical equality were conducted. The studies focus on two learning environments that were planned on the basis of already existing well known substantial task formats such as *Arithmetic triangles* and *Number sequences* (see Figure 1). The latter were enriched by using design principles developed from theory and study results.

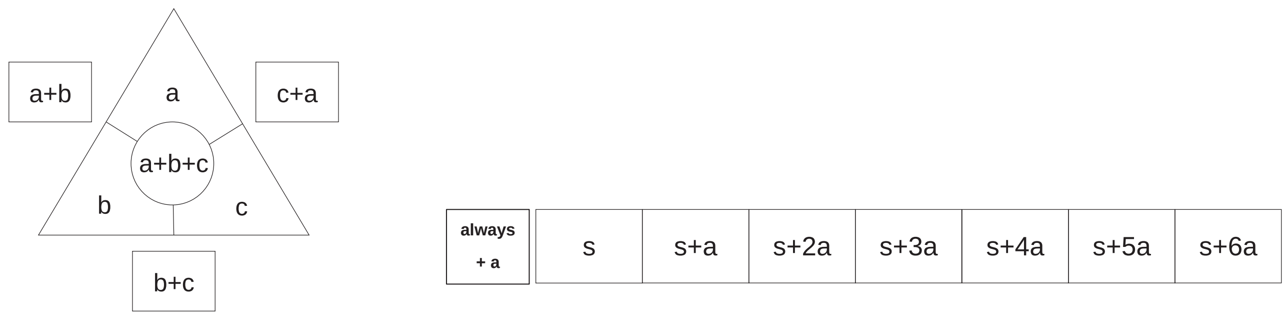
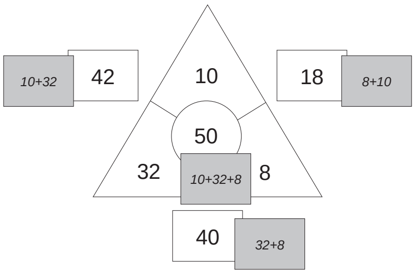


Figure 1: General structure of the task formats *Arithmetic triangles* and *Number sequences*

The central feature of both learning environments lays in the combination of a substantial task format and formal representation of appropriate equations. The children first work on tasks that encourage them to explore equalities in the context of the task formats and ask them to describe and justify these equalities (e.g., "Calculate the sum of the inner numbers and the sum of the outer numbers of the arithmetic triangle."). The goal is to develop a content-related relational concept of equality. Content-related does not refer to a factual context, but to the task formats, e.g. the arithmetic triangles. The relationship between the numbers of an arithmetic triangle is supported by the visual structure of the task format. Similar to the way Nührenbörger and Schwarzkopf (2015) used "Term walls" with notated calculations instead of results in the context of "Number walls" in their study, the learning environments in the present study were enriched by using Term fields, i.e. small sticky notes with terms (e. g. 10+32) related to number fields (e. g., 42) of the task format (see Figure 2).



always + 4	10	14	18	22	26	30	34
	10	10 +4	10 +4+4	10 +4+4+4	10 +4+4+4 +4	10 +4+4+4 +4+4	10 +4+4+4 +4+4+4

always + 8	10	18	26	34	42	50	58
	10	10 +8	10 +8+8	10 +8+8+8	10 +8+8+8 +8	10 +8+8+8 +8+8	10 +8+8+8 +8+8+8

Figure 2: Term fields that explicate calculations behind the numbers

Term fields are meant to serve as a kind of exploration tool to make explicit the essential structure of the task formats. They should be used as a basis for explaining relationships between number fields of the task format. In this way, children have an opportunity to focus on structures between terms in addition to compute them. Based on students' work with the Term fields, the students are given appropriate tasks for assigning equations to arithmetic triangles and number sequences, as well as tasks for evaluating, completing, and correcting equations. The equations always consist of terms that arise or potentially arise from the task format (Figure 3, see associated arithmetic triangle and number sequence in Figure 2). The children's attention is to be focused on the structure of the terms by means of certain guiding questions that are repeatedly asked during the interview (e. g., "From which triangle/number sequence was the equation formed?", "Why is the equation (not) correct?").

$$\boxed{2 \cdot 10+32+8 = 10+32 + 32+8 + 8+10}$$

$$\boxed{10 + 8+8+8 = 10 + 4+4+4+4+4+4}$$

Figure 3: Equations consisting of terms that arise from an arithmetic triangle and a number sequence

A total of 42 collaborative peer interviews, each with two fourth grade children, were conducted on the described learning environments. A pre-test was administered to each student at the beginning of the first interview. Students were given a set of equations to sort into two boxes ("true" and "not true"). Some equations were incomplete and had to be completed first. The pre-test was used to determine the children's interpretation of the equal sign before working in the learning environments. The children were randomly selected by the teacher and had neither previously encountered the Term field learning environments nor Arithmetic triangles or Number sequences in class. At the beginning of the interview series, the task formats were worked out together. The interviews were videotaped and largely transcribed. The analysis of the data follows the interpretative paradigm (Krummheuer & Naujok, 1999; Voigt, 1991). In particular, Steinbring's theory of the construction of new mathematical knowledge in classroom interaction is used to reconstruct the respective reference contexts of the children while they were justifying equalities (Steinbring, 2005). Thus, different ways of justifying were inductively reconstructed. These are presented and discussed below.

Empirical results and discussion: ways of justifying equalities in a range between result orientation and structure orientation

Equalities can be interpreted in different ways and, accordingly, justified differently, depending on which aspects of equality are focused on. The data analysis showed that children develop and use different justifications for equalities in the context of the learning environments, which can be located in a range between result orientation and structure orientation (see Figure 4). Previous findings from the literature described above were incorporated and extended in the development of the four categories. For example, the analyses confirmed the two relational views of equality, relational-computational and relational-structural, suggested by Stephens et al. (2013). In the study presented here, it was found that these views of equalities could be further differentiated by distinguishing between determination of overall and intermediate results (category 1 and 2), and using structure with or without generalized approaches (category 3 and 4). Category 1 and 2 correspond to a relational-computational view as defined by Stephens et al. (2013), while category 3 and 4 correspond to a relational-structural one (see Figure 4). The children in the study didn't show a purely operational view of equalities in the interviews. However, students' operational view was evident in the separate pre-test. The learning environments were intentionally designed to stimulate a relational interpretation of equalities by including the task formats Arithmetic triangles and Number sequences.

Justifications by
1. determining **overall results**

2. determining **intermediate results**

3. using a **structure** of the arithmetic terms

4. using a **structure** of the arithmetic terms
with generalized approaches

Figure 4: Justifications for equalities in the range between result orientation and structure orientation

In contrast to the study by Stephens et al. (2013), the study presented here reconstructed the particular situational view behind a justification, rather than investigate a child's fundamental concept of equality. In the following, the four categories of justifications are explained using selected episodes from the interviews.

Justifications by determining overall results

One way of justifying equality, evident in the present study, is based on determining the overall results of the terms. Each term in an arithmetic triangle or a number sequence is perceived as a separate task and their respective values are determined independently, and then finally compared. Charlie and Fabian evaluated equations formed from arithmetical triangles and corrected them if necessary. The following equation was placed on the table: 42 + 8 = 10 + 32 + 8

Interviewer: Why is that true (points to the equation)?
Charlie: (Pulls the equation towards him.) Forty-two (points to 42 on the left side of the equation) plus eight (points to 8 on the left) is fifty. Ten (points to 10 on the right) plus thirty-two (points to 32 on the right) is forty-two plus eight (points to 8 on the right) is fifty.

Charlie determined the overall results of both sides of the equation separately (50) by adding the summands linearly from left to right.

$$\underbrace{42 + 8}_{50} = \underbrace{\underbrace{10 + 32}_{42} + 8}_{50}$$

Figure 5: Reconstruction of Charlie's justification by determining overall results on each side

Justifications in this category suggest that the children make a strongly result-oriented interpretation of the terms. They do not direct their attention to the mathematical regularities underlying the terms but focus on the results. In the range between result and structure orientation, such a way of reasoning is to be located correspondingly far to the left (see Figure 4).

Justifications by determining intermediate results

Another way of justifying relies on the determination of intermediate results. This involves determining selected term components of arithmetic triangles and number sequences and comparing the resulting partial results. Characteristic for such justifications is the omission of obviously equal term components occurring on both sides of the equation (e.g., equal starting numbers in number sequences or equal inner numbers in arithmetic triangles). Sometimes longer components of the terms are left out of the evaluation of equality, sometimes only single numbers, as in the following example. Lara and Grace evaluated equations which could have been formed from number sequences: 10 + 3 + 3 + 3 + 3 = 10 + 6 + 6

Interviewer:	Why does it fit?
…	…
Lara	That's both twelve (points to the equation), always.
…	…
Lara	So that together are twelve (taps the sixes on the right side) and three plus three plus three plus three (taps the threes on the left side) are also twelve again.

The summand *10*, which occurred on both sides of the equation (starting number in the task format Number sequences), was omitted by Lara when evaluating the equality. She calculated only the non-identical term components (12 each), compared the intermediate results and concluded based on their agreement that the equation is correct.

$$10 + \underbrace{3 + 3 + 3 + 3}_{12} = 10 + \underbrace{6 + 6}_{12}$$

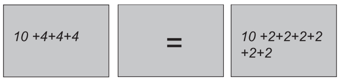
Figure 6: Reconstruction of Lara's justification by determining intermediate results on each side

Justifications in this category, point to a predominantly result-oriented understanding of equality. Compared to the previous category, however, such justifications are still somewhat more relational, since equal values are compared and deliberately disregarded. The location of these justifications in the range between result orientation and structure orientation is therefore on the result-oriented side, although not quite as strongly to the left as that of the first category (see Figure 4).

Justifications by using a structure of the arithmetic terms

Another way of reasoning is based on focusing and comparing the two terms of an equation with respect to their structure and their relation to each other. The justifications refer to the concrete numerical values. There are many differences in how the children relate the structures of the terms. This depends on which mathematical laws they (implicitly) refer to. Rachel and Kate each wrote down different plus numbers and the corresponding term fields for a number sequence with starting number 10 and target number 22. As part of the task, the interviewer combined one of Rachel's term fields (10+4+4+4) and one of Kate's (10+2+2+2+2+2) into an equation. The children agreed with it and justified their decision:

Kate: Here are (points to term field 10+2+2+2+2+2) uh, wait a minute (.) one two three four five six times (taps on the twos one after the other) and here are three times (points to term field 10+4+4+4) and that (points to the left term field) is the double of it (points to the right term field) so that- and here (points to the right term field) are six times and there are three times (points to the left term field) and three is half of six



… …

Rachel: And two and four is also the - so two $\#_1$ is half of four.

Kate: $\#_1$ Exactly. I just said that.

… …

Rachel: Yes, because here are six t- oh (points to term field 10+2+2+2+2 and moves it by mistake) six times (points to the term field again) and there are three times (points to term field 10+4+4+4) and then it must work.

The terms were interpreted multiplicatively. The size and the number of the summands of one side (size: 2; number: 6) were related to the size and number of the summands of the other side (size: 4; number: 3). The reasoning implicitly referred to the law of constancy of the product.

$$\begin{aligned} 10 + 4 + 4 + 4 &= 10 + 2 + 2 + 2 + 2 + 2 + 2 \\ \boldsymbol{10 + 3 \bullet 4} &= \boldsymbol{10 + 6 \bullet 2} \\ &= \boldsymbol{10 + (6:2) \bullet (2 \bullet 2)} \end{aligned}$$

Figure 7: Reconstruction of Rachel's and Kate's justification by using a structure of the arithmetic terms (based on the law of the constancy of the product)

Justifications in this category point to a predominantly structure-oriented interpretation. The interpretation implies mathematical regularities that underlie the structure of the terms. Such justifications are accordingly on the structure-oriented side.

Justifications by using a structure of the arithmetic terms with generalizing approaches

The fourth way of justification, like the third one, refers to the comparison of the structure of terms and their relation to each other. The justifications are also based on mathematical laws. However, such a reasoning is not only based on the concrete numerical values, but is detached from them by making generalizable statements. The degree of generalization can vary. Lara and Grace used term fields to justify why 34 occurred in both the +8-sequence and the +4-sequences. Lara's reasoning was similar to Rachel's and Kate's, but showed clearly generalizing approaches:

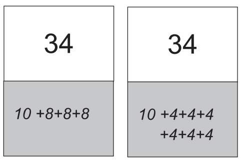


Figure 8: Term fields for the 34 from the +8 sequence and the +4 sequence to which the children refer

Lara: Then you have to take those two times, that's six and then six times (writes down her idea on a piece of paper)

$3 \cdot 8 = 24$
$2 \cdot 3 = 6$
$6 \cdot 4 = 24$

… …

Lara: So, I first have three times eight are twenty-four, uh but because I have half of the eight, uh you have to take the double of the front, so what you, with which number

	you multiplied that. So we have to- theoretically you have to halve here (points to "8" in "3•8=24" on her note) and double here (points to "3" in "3•8=24" on her note).
…	…
Lara	So, you you you double the three (points from "3" in "3•8=24" to "6" in "6•4=24") and halve the eight (points from "8" in "3•8=24" to "4" in "6•4=24"). Then a four will be written there and, you know what I mean? (Looks at Grace.)
…	…
Lara	You halve, you dou- halve the front number (points briefly to her note) or you halve the back number, you halve one number and you double the other one and then the same result comes out.

Lara justified the equality of the terms based on her written representation of her idea. She shortened the terms of the relevant term fields by multiplicative representation of the same summands and by omitting the identical starting numbers. Her justification contained generalized approaches. She replaced the number of summands, now represented by the first factor in each case in her representation, with a general description ("with which number you multiplied that"). When Grace didn't understand, she first concretized her idea using the exemplary numbers of the task ("you double the three and half the eight"). In terms of content, Lara's idea referred to the constancy of the product. Later she expressed her idea again in a highly generalized way. She replaced both the first and the second factor with a general description ("You (...) halve the front number or you halve the back number, you halve one number and you double the other one and then the same result comes out."). In addition, Lara clarified that it doesn't matter which of the numbers is halved and which is doubled. She moved away from concrete numerical values and instead used descriptive word variables.

$$
\begin{aligned}
10 + 8 + 8 + 8 &= 10 + 4 + 4 + 4 + 4 + 4 + 4 \\
\mathbf{10 + 3 \bullet 8} &= \mathbf{10 + 6 \bullet 4} \\
&= \mathbf{10 + (2 \bullet 3) \bullet (8:2)}
\end{aligned}
$$

(one number : 2) • (2 • other number) = one number • other number

Figure 9: Reconstruction of Lara's justification by using a structure of the arithmetic terms with generalized approaches (based on the law of the constancy of the product)

Justifications of this category suggest that the children have a strong structure-oriented understanding of equality. The relationship between the terms is interpreted and explained on a more general level with regard to the mathematical regularities on which it is based. The classification of such generalizing justifications takes place far on the structure-oriented side.

Closing remarks

The analyses showed that children use different justifications in the range between result orientation and structure orientation, (implicitly) refer to appropriate mathematical laws and generalize these to different degrees. The way of justification depends on different factors, e.g. on the numerical values, the suggestions of the other child or the impulses of the interviewer. This connection will be examined in more detail in further data analysis.

The presented learning environments, which combine content understanding and formal representation of equations with the help of term fields and focus on sharing discovered equalities and possible justifications of these, can contribute to stimulate a flexible understanding of equalities.

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Students' intuitive conceptions regarding the concept of set

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Abstract: *In this study, a questionnaire was administered to different-aged students in order to explore the intuitive conceptions they hold regarding the concept of set. A number of different conceptions emerged. Students' scores per grade were compared in order to investigate the effect Mathematics education has on their conceptions.*

Keywords: Set, intuitive conceptions, infinity.

Introduction

The concept of set is fundamental in modern mathematics. In school mathematics, the concept of set is used in various contexts, generally in an inconsistent manner (Fischbein & Baltsan, 1998). In the curriculum of most countries, sets are included in Algebra as a means of defining the concept of function, express the solution of an inequality etc.

The set concept is accepted in mathematics as a primitive, undefined concept. Instead of a formal definition one starts with an intuitive model, the idea of a collection of objects (Fischbein & Baltsan, 1998). In order to grasp the essence of a set, students need to be taught the formal properties and aspects of sets in school mathematics. If that doesn't happen, students can only rely to their intuitive conceptions regarding the set concept. Spontaneous conceptions are the result of generalization of everyday experience in the absence of systematic instruction and can lead to the creation of "spontaneous concepts" (Karpov, 2003). Such concepts are not conscious and often differ from scientific ones. Scientific concepts, once acquired by students, begin to mediate their thinking and problem solving. Thus, the instruction of such concepts plays a decisive role in their mental development facilitating their ability to operate at the level of formal-logical thought (Karpov, 2003).

Theoretical framework

Spontaneous conceptions that are contradictory to scientific conceptions (conceptions that experts have consensus on) are named misconceptions. Fischbein and Baltsan (1998) identified a number of students' misconceptions. The most profound being the following: (a) there must be at least one common property or a relationship between elements of a set; (b) a set must contain at least one element; (c) a set can contain repeated (not discrete) elements; (d) two sets are equal if they contain the same number of objects; and (e) a set cannot contain an infinite amount of elements. Bagni (2006) showed that students might be unable to distinguish between the concepts of "inclusion" and "belonging" especially when they are dealing with various semiotic systems (verbal, diagrammatic and symbolic). Lastly, Tsamir (2001) found that students hold contradictory ideas when they are asked to compare the number of elements of infinite sets. The most prominent categories of ideas were: (a) all infinite sets are equivalent; (b) infinite sets are incomparable; (c) if set A is a subset of set B then A has fewer elements than B; and (d) 2 sets are equivalent if there is a one-to-one correspondence between their elements.

Fischbein and Baltsan found that the formal mathematical properties of sets, which usually contradict the corresponding intuitive ones, are "forgotten as an effect of age" (1998, pp. 20). In Greece, students are using the concept of set in various contexts, even before it is formally introduced to them. They learn about number sets and they define a circle as a set of points on a plane that are equidistant from a given point. The concept of set is first taught in grade 9 in the course of Probabilities and then in grade 10 in the course of Algebra. In both grades the curriculum suggests that the teacher will spend two lessons teaching the concept of set and the operations between sets. In most cases, even those two lessons are neglected and the set is just being seen as a supplementary concept in different contexts like geometry, algebra, analysis, or probabilities. The research questions of this study are: (1) What conceptions do different-aged students have regarding the mathematical concept of set?; and (2) What effect does age have in the understanding of the set concept and its properties?

Methodology

The research tool was a questionnaire, consisting of 11 questions, which were constructed based on the aforementioned findings of previous research regarding the concept of set. Questions regarding the concept of set and equality between sets (1 to 7) were included in the questionnaire given to all groups, while questions regarding union, interception, inclusion, belonging and comparison of infinite sets (8 to 11) were only included in the questionnaire of groups 2 and 3. The questionnaire was administered to 116 grade 6 students (group 1, primary school), 127 grade 9 students (group 2, junior high school) and 154 grade 12 students (group 3, high school), from various schools around Greece. Students from the first two groups had never been explicitly taught anything regarding sets. Some of the students of the third group had been provided with a definition for the concept of set in grade 10 (two years prior to this study), but all of them use sets regularly, in the form of "the domain" or "the range" of a function. Data were analyzed with IBM SPSS Statistics 26. To answer the first research question we calculated the percentages of students that held each conception (questions 1 – 7, 11). We will present and discuss the results from some of the questions that we find particularly interesting. As far as the second research question is concerned, we conducted an ANOVA test to check the hypothesis that, no differences were found between the scores of the three groups.

Results

First research question

We begin by addressing our first research question, namely, what conceptions students have regarding the mathematical concept of set. For that purpose, we are going to present some of the questions from our questionnaire and we will discuss the results.

Question 1.

What do you think a set is in mathematics?

In the first question of our questionnaire we found a number of different conceptions students have regarding the set concept. We will provide some examples of answers and elaborate on how we coded them. A student answered that "a set is the collection of similar objects and their classification into a group”. In our understanding, this student saw sets as both a *collection* and as a *group* (the Greek word for group can also be translated as team). As expected, there were answers like “a set is any collection of objects” or “a set is a group of objects”, for which the coding is obvious. Another student described a set as “a collection of numbers”. This student thought of sets as a collection of *numbers*, not accepting non numerical elements. A student wrote that “a set is a group or a category of similar objects, rational numbers for example”. That student thought of sets as both *groups* and *categories* and gave the example of rational numbers. In our estimation, this is not enough evidence to assume that he only accepts numbers as the elements of a set. There was a student that gave us the simple answer “the whole”. Others wrote that a set is “the whole or a group with some common element” or “something complete”. These students were considered to hold the conception that a set is *the whole*. Some students used the Greek word “plethos” saying for example that a set is “a plethos of elements that meet certain conditions”. The word plethos can be translated as “a number of”, “a multitude”, or in some cases “a crowd”. We chose the code *multitude* for those answers. Lastly, a lot of students provided answers like “for me, a set is the outcome”, or “I perceive this in two ways, as the outcome of an addition or as a group”. Those students seemed to conceptualize a set as *the outcome* or the sum of certain numbers. One would expect that every student that holds that conception accepts only numbers as the elements of a set. However, this was definitely not the case in our survey, as there were different conceptions for sets, sometimes contrasting ones, coexisting in students’ minds.

There were also students whose answers were out of context or formulated in a way that didn’t allow for a proper coding or they were simply unintelligible, for example “I think that the elements of a set must have a common property” or “something that is concentrated”.

Table 1: Words used by students in order to define the set concept

What do students think a set is in Mathematics	Percentages of students holding this conception
A collection	2.3%
A group	28.7%
A category	2.3%
The whole	5.5%
A multitude	6.9%
The outcome	24.2%
A cluster of numbers	5.6%
Out of context/ unintelligible	21.2%
Missing values	19.1%

Since a student can use more than one of the aforementioned words, the percentages do not add up to 100%. One thing that stands out is the fact that 24.2% of the students think of a set as the outcome of an operation (or the sum of certain numbers). If we include the number of students that chose (a) or (b) on question 4 (see below) and those that answered question 7c (see below) incorrectly (both implying that those students are adding up the elements) then we find that 49.4% of students think of sets as the outcome of an operation or a sum. That is probably due to the fact that, in Greek the word for "set" is commonly used as "total". It is also interesting that 28.7% of students view a set as a "group" or a "team" of elements, probably implying a uniformity between the elements of a set, while only 2.3% of them think of a set as a collection of objects (which is the term used to "formally" describe a set).

Question 3.

Do you think that the elements of a set must have a certain common property?

The students' answers to the third question (Table 2) show that most students thought that the elements of a set must have a certain common property or conform to a certain criterion that allows them to be included in the set. This finding is consistent for all three groups. Fischbein and Baltsan (1998) found that the proportion of wrong answers increases generally with age. This doesn't seem to be the case in our study since the percentage of wrong answer is almost the same for all groups. This misconception is common among prospective teachers as well (Fischbein & Baltsan, 1998). It is worth noting that the description of a set in the Greek textbook used in grade 9 highlights that we use to collect or choose certain objects and sort them into groups or categories, for example, numbers, letters of the alphabet, books in a library depending on their content, and that, groups or categories like those, are called sets. Albeit not wrong per se, such a description can easily foster the misconception that the elements of a set must be of the same kind.

Table 2: Group * Question 3 Crosstabulation

Group	Yes	No
1	77.7%	22.3%
2	76.8%	23.2%
3	80.7%	21.4%
Total	78.6%	21.4%

Question 4.

Mark each of the following answers that seem correct to you. The set of the letters of the word "TENNIS" is:

(a) {5}
(b) {6}
(c) {T,E,N,I,S}
(d) {T,E,N,N,I,S}

On the 4th question (table 3) we examine whether students think that a set can contain repeated elements (those that answered {6} or {T,E,N,N,I,S}) or not. We found that most students think that it is acceptable for a set to contain repeated elements. That is especially true for groups 1 and 2 (younger students). It is also important to note that only 27.7% of the students marked only the correct answer (students were allowed to mark more than one answers in this question), which is {T,E,N,I,S}.

Table 3: Question 4 – Percentages of students accepting repeated elements in a set

Group	Repeated elements are allowed.	Elements of a set must be discrete.
1	78.3%	21.7%
2	76.0%	24.0%
3	56.5%	43.5%
Total	69.0%	31.0%

Question 5.

Do the following collections define a set? Choose the correct answer.

(a) Natural numbers greater than 8 and smaller than 10.
  i. Yes, {9}
  ii. No, a set must have more than 1 element.

(b) Natural numbers greater than 8 and smaller than 9.
  i. Yes, {}
  ii. No, a set must contain at least 1 element.

On the table below, we examine whether students accept the notion of a "unit set" (question 5a) and the notion of an "empty set" (question 5b). We found that almost half the students think that a set must have two or more elements, while 63.9% of the students think that a set must contain at least one element.

Table 4: Group * Question 5 Crosstabulation

	Question 5a		Question 5b	
Group	Correct	Wrong	Correct	Wrong
1	53.6%	46.4%	36.0%	64.0%
2	37.8%	62.2%	32.3%	67.7%
3	58.4%	41.6%	39.2%	60.8%
Total	50.4%	49.6%	36.1%	63.9%

Question 7.

Describe the following statements as true (T) or false (F).

(a) {1,2,3,4}={4,3,2,1}
(b) {1,2,3,4}={a,b,c,d}
(c) {1,2,3,4}={1,2,7}

This question is an extension of Fischbein's and Baltsan's (1998) observation that, two sets are equal if they contain the same number of objects. We presumed that some students were going to think of a set as "an ordered n-tuple", thus answering that {1,2,3,4}≠{4,3,2,1}, or as "the sum of all elements", thus answering that {1,2,3,4}={1,2,7}. From the following table we infer that: (a) 22.8% of the students think that {1,2,3,4}≠{4,3,2,1} which implies that the order of the elements play a role when describing a set; (b) 57.7% of the students believe that {1,2,3,4}={a,b,c,d}, which means that two sets are equal when they have the same number of elements; and (c) 21.9% of the students answered that {1,2,3,4}={1,2,7}, which could imply that two sets are equal if their elements add up to the same number.

Table 5: Group * Question 7 Crosstabulation

	Question 7(a)		Question 7(b)		Question 7(c)	
Group	Correct	Wrong	Correct	Wrong	Correct	Wrong
1	77.9%	22.1%	37.1%	62.9%	73.7%	26.3%
2	77.4%	22.6%	41.1%	58.9%	74.6%	25.4%
3	76.5%	23.5%	47.4%	52.6%	84.3%	15.7%
Total	77.2%	22,8%	42.3%	57.7%	78.1%	21.9%

Question 11.

Describe the following statements as true (T) or false (F).

(a) The number of elements in an infinite set is larger than the number of elements in any of its proper subsets.
(b) All infinite sets have the same number of elements.
(c) Infinite sets are incomparable.

In question 10 (not listed here), which was included only in the questionnaire for groups 2 and 3, we asked the students to compare some infinite sets. We used different representations, known to elicit specific answers (Maria et al., 2009, Tirosh & Tsamir, 1996, Tsamir, 2001). For example, we gave them the sets A={1,2,3,4,5,…} and B={1,4,9,16,25,…} and asked them whether the number of elements in set A is larger, smaller, or equal to the number of elements in set B, or if it the two sets are incomparable. Then we gave the students the sets A={1,2,3,4,5,…} and B={1^2,2^2,3^2,4^2,5^2,…}, and asked them the same question. The first representation is known to lead students to answer in terms of "inclusion", while the second is supposed to lead them to consider "one-to-one correspondence" as the appropriate way to answer the question (Maria et al., 2009, Tirosh & Tsamir, 1996, Tsamir, 2001). The main reason for using this question was to engage students with a task regarding the cardinality of infinite sets and then, explicitly ask them about their ideas, in question 11.

In question 11 (table 6), which was also included only in the questionnaire for groups 2 and 3, we verified Tsamir's (2001) findings that some students come up with contradictory ideas when they are asked to compare the number of elements of infinite sets. Specifically, 32 junior high and high school students marked both Q11a and Q11b as true (“true” being the wrong answer), 38 of them marked both Q11c and Q11a as true (also wrong) and 29 marked both Q11b and Q11c as true (also wrong). It is important to note that 15 students didn’t answer Q11a, 17 didn’t answer Q11b and 15 didn’t answer Q11c, and so they are excluded from the following table. We should also point out that, Tsamir (2001) used the phrase “number of elements” in her study. In our questionnaire, written in Greek, we actually used the word “plethos” instead of “number”. The Greek word for “cardinality” is derived from the word “plethos”.

Table 6: Question 11a, 11b, 11c – Crosstabs

		Q11b				Q11a				Q11c	
		Cor.	Wr.			Cor.	Wr.			Cor.	Wr.
Q11a	Cor.	64	113	Q11c	Cor.	114	46	Q11b	Cor.	44	74
	Wr.	52	32		Wr.	64	38		Wr.	116	29
	Total	116	116			178	84			160	103

Second research question

Fischbein and Baltsan (1998) found that the formal mathematical properties of set are “forgotten as an effect of age”. To check that hypothesis, we decided to use our questionnaire as a test, thus, calculating the total score of each student for questions 1-7 (those that were included in the questionnaires of all three groups) and then, to compare the mean scores of the three groups (Table 7 and Table 8, SUM1_7). What we found was that group 3 (high school students) scored significantly higher than the other two groups. That finding could be explained due to the fact that, in Greek schools the concept of set is almost never taught explicitly even though students are using it in various contexts. It is impossible to forget something that you have never been taught, so the effect of age cannot be explained in terms of forgetfulness.

Table 7: Oneway ANOVA for Sum1_7

	Sum of Squares	df	Mean Square	F	Sig
Between Groups	95.246	2	47.623	11.335	0.000
Within Groups	1655.384	394	4.201		
Total	1750.630	396			

Table 8: ANOVA – Post Hoc Tests – Homogeneous Subsets – Tukey Ba,b – Sum1_7

Tukey B[a,b]		Subset for alpha = 0.05	
Group	N	1	2
1	116	6.9397	
2	127	6.9843	

3	154		7.9675

Means for groups in homogeneous subsets are displayed

a. Uses Harmonic Mean Sample Size = 130.502

b. The group sizes are unequal. The harmonic mean of the group sizes is used.

Discussion

All misconceptions found by previous research (Fischbein & Baltsan, 1998, Tsamir, 2001) were verified by our study and some new ones were added in the mix. The most prevalent conceptions that differ from the formal use of the set concept were: (1) a set is an outcome, (2) a set is a group of objects, (3) elements of a set must conform to a certain criterion, (4) a set must have at least one element, (5) a set can contain repeated elements, (6) a set is an “ordered n-tuple”, (7) two sets are equal if they contain the same number of elements, (8) two sets are equal if their elements add up to the same number (only apply to number sets), (9) the number of elements in an infinite set is larger than the number of elements in any of its proper subsets, (10) all infinite sets have the same number of elements and (11) infinite sets are incomparable.

When we compared the mean score of each group, we found no significant difference between groups 1 (primary school) and 2 (junior high school). High school students (group 3) scored significantly higher than their younger counterparts. This discrepancy may be due to the fact that 12th graders have gained more experience in sets as a result of using them in Algebra and Analysis courses in High School. Fischbein and Baltsan (1998) found that the percentage of wrong answers increases as age increases. This wasn’t the case in any question of our survey. On the other hand, Tsamir’s (2001) findings that some students hold contradictory ideas when they are asked to compare the number of elements of infinite sets were in accordance with our findings.

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Basic Mental Models of Equations - Theoretical Conception and Practical Meaning

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A Basic Mental Model (BMM) of a mathematical concept is a content-related interpretation that gives meaning to this concept. There exist many suggestions for BMMs in mathematics education, e.g., for natural numbers, functions, derivatives or integrals. In this article we present four BMMs of an equation and discuss the relation between these BMMs and the solving of equations. Particularly, we are interested in the interrelationship between these BMMs and the use of digital technologies in this solving process. These theoretical considerations bring up research questions which should be answered to achieve a better understanding of the concept of equation and the solving processes.

Keywords: Equation, solving of equations, Basic Mental Models, digital technologies.

Basic Mental Models (BMM)

The concept of Basic Mental Model (in German: Grundvorstellungen) has been well established in German-speaking didactics of mathematics for many years (Hofe v. & Blum, 2016). *A BMM of a mathematical concept is a content-related interpretation that gives meaning to this concept, providing relations to meaningful contexts* (see Greefrath et al., 2016). The *meaning* of a concept is constituted by its interpretation in scientific mathematics, in school mathematics and in the (historical) development of the concept (Kilpatrick et al., 2005). BMMs can be conceived as prerequisites for students to deal with mathematical concepts in an insightful way.

The concept of BMM can be used both in a *normative* (prescriptive) and an *individual* (descriptive) sense (see Hofe et al., 2005).

- *Normative* BMMs are the answer to the question: How should students generally and *ideally* think of a given mathematical concept? They are identified by didactical analyses of the mathematical concept. They can be used as educational guidelines and to specify learning objectives for mathematics lessons. The determination of BMMs is a didactical challenge for researchers and requires a subject-oriented classification of mathematical and real-life situations of the concept.
- *Individual* BMMs are individual mental models or concepts students *actually* develop in learning processes and problem-solving situations. They can vary from or represent only part of normative BMMs, they even can be based on misconceptions. Individual BMMs are the result of the personal development of meaning and the integration of the concept into an individual's personal worldview.

BMMs of mathematical concepts can be considered within the theoretical framework of “Concept Image – Concept Definition” (Tall & Vinner, 1981). They are parts or subsets of the “Concept Image” of a mathematical concept. While “Concept Image” refers to all individual mental images identified with the concept, BMMs are the *core* or *central components* of these images.

The development of BMMs is a creative task for mathematics educator. Based on the concept definition, a critical investigation of the use of this concept and the involved perceptions in problem solving situations in scientific mathematics, in university mathematics lessons and in the school curriculum give hypotheses of the existence of different BMMs. These have to be empirically verified, like Greefrath et al. (2021) have done this for the concepts of derivative and integral.

A mathematical spotlight: equations and equivalence of equations

Equations with only numbers or quantities are statements which can be true or false. E.g., the equation $3 + 5 = 8$ is a true, $3 + 5 = 7$ is a false statement. Equations with unquantified variables, e.g., $T_1(x) = T_2(x)$, T_1, T_2 algebraic terms, are predicates, which will turn into statement if the variables are quantified, e.g., for $x = 4$, $3 \cdot x + 1 = 7$ is a false statement, for $x = 2$ it is a true statement. Two equations are *equivalent*, if

- their sets of solutions are the same or
- there is an equivalence transformation which transforms one equation into the other.

Each injective function applied to both expressions of an equation is an equivalence transformation. This applies especially for the elementary transformations like addition and multiplication with real numbers (for more details see Arcavi et al., 2017).

BMMs of an equation

BMMs of an equation form a content basis for what students normatively need to work with equations. They are central and important in developing perceptions and (mental) representation about basic activities while solving equations. BMMs are based on mathematical aspects of equations: the definition of an equation, the use of the equals sign and the relationship between the equation concept to the concepts of algebraic expressions and functions.

- **Operational BMM**: An equation is understood as a calculation or transformation. The equal sign is seen as an operational sign, which indicates a reading direction of the equation in the sense of a "resulting-in” sign.
  - Arithmetic calculations: The equation $3 + 4 = 7$ can be expressed as "$3 + 4 \rightarrow 7$" or verbally by "3 plus 4 results in 7". It is the predominant BMM of an equation in primary school.
  - Transformations of algebraic expressions: Equations like $(a+b)^2 = a^2 + 2ab + b^2$ can be expressed by $(a+b)^2 \rightarrow a^2 + 2ab + b^2$ or verbally by “expanding $(a+b)^2$ results in $a^2 + 2ab + b^2$”. "Results in" can also be read—with “factorizing” instead of “expanding”—in the other direction.

- **Relational BMM**: An equation is understood as a task to determine numbers or quantities for the expressions on both sides of the equation to get the same value or quantity on both sides. The equal sign is seen as a relation sign. The variable here is understood as an unknown which has to be determined.[1]

The visualization for or a representation of this BMM is the model of an equal arm balance. In this model, each expression is represented as a combination of weights. Numerical values are represented by unit weights and each variable value is assigned by a fixed—initially unknown to the spectator—weight. The equality of values of both terms is expressed in the balance of the scales. Like in Figure 1, the balance model is suitable especially for linear equations. These are carried out as actions of removing the same weights from or adding the same weights on both scales.

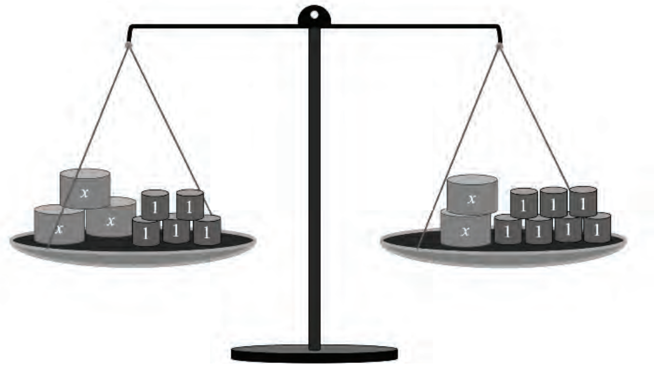


Figure 1. An equal arm balance

- **Functional BMM**: An equation $T_1(x) = T_2(x)$ is a comparison of two expressions which are understood as functions with $y = T_1(x)$ and $y = T_2(x)$. Here, too, the equals sign is understood relationally, but the BMM of the variable is that of a changing number or quantity.

In this functional interpretation, the variable x "passes through" the values of the definition range of the two terms $T_1(x)$ and $T_2(x)$. This results in the equation becoming a false or true statement, depending on the values of $T_1(x)$ and $T_2(x)$. However, we want to determine the value(s) x_o with the same function values $T_1(x_o) = T_2(x_o)$. In a graphical representation of the two functions with $y = T_1(x)$ and $y = T_2(x)$, solving an equation means determine the intersection points of the two function graphs. This is possible for nearly all types of functions. Figure 2 shows an example of a quadratic equation.

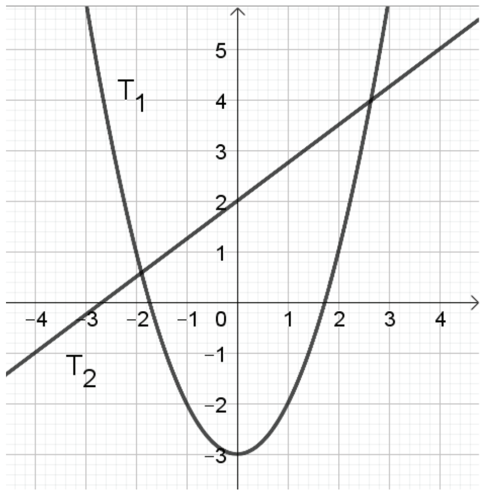


Figure 2. The equation $x^2 - 3 = 0.75x + 2$ leads to the two functions with $T_1(x) = x^2 - 3$ and $T_2(x) = 0.75x + 2$

This BMM can be transferred to a system of equations with 2 variables if a function is defined by each of the two given equations whose common function values are to be determined.

- **Object-BMM**: An equation is regarded as a mathematical object that has characteristic

[1] We refer to the three central BMMs for variables without explaining their background (see e.g. MacGregor and Stacey (1997) or Oldenburg (2019) for details): The variable as a general number, the variable as an unknown number and the variable as changing number or quantity.

properties, such as the number of possible solutions, the definition range or special solution algorithms. E. g.

- The quadratic equation $x^2 + x - 3 = 0$ has exactly two solutions. There exist several methods to calculate these solutions.
- The equation $x^2 + y^2 = z^2$, $x, y, z \in \mathbb{N}$ is a Diophantine equation whose solutions are called Pythagorean number triples.
- The equation $x^2 + y^2 = r^2$ with $x, y \in \mathbb{R}$, $r \in \mathbb{R}^+$ is an equation of a circle with the coordinate origin as the center and the radius r.

The first two BMMs are based on the operational and relational meaning of the equal sign and thus mark the transition from the arithmetic-dominated mathematics of the primary level to the algebra of the secondary level. The third and fourth BMM represent an understanding of equations on (lower and upper) secondary level. They will be especially important while working with digital technologies. There is a hierarchy in the appearance of these BMMs from primary to upper secondary school: Operational BMM → Relational BMM → Functional BMM → Object-BMM .

Central activities and working with equations

The meaningful working with and the flexible use of concepts require a wider view of the concept, especially seeing the concept in relation to different applications and to different representations (see e.g., Freudenthal (1983)). At least since the works of Piaget and Aebli, and further those of Vygotsky and Leont'ev, real hands-on activities are seen as central and fundamental for the development of mental structures or (thinking) operations. The development of BMMs must therefore be seen in close relation to activities and related tasks with this concept.

Referring to the quite elementary "Input-Operation-Output"-model concerning the working with tasks (see e.g., Günster and Weigand (2020)), this model can be transferred to three *central activities* while working with equations (in mathematics lessons). These *central activities* are:

- **Input or setting up an equation**: Setting up an equation in such a way that the central relations presented in a problem are expressed appropriately in the equation. It is therefore a matter of translating a problem situation into the language of mathematics.
- **Operate on or anticipating and applying transformations**: Developing strategies which can be flexibly used to solve equations in a problem-adequate way.
- **Output or interpreting**: Gaining insights into an issue by means of an equation, appropriately transformed equations or the solution(s) of an equation.

These central activities require *syntactic* knowledge concerning the structure of expressions that determine the equation and knowledge of solution strategies concerning different equation types. Furthermore, flexibility and symbol sense are required, and the ability to transfer equation to other (equivalent) equations (see Kieran (1992), Oleksik (2019)).

Semantic knowledge is required with regard to the meaning of equations in relation to environmental situations, inner-mathematical models and representations, and the interpretation of the solution(s) in relation to these situations.

Digital technologies—nowadays and in the future

For solving equations, there is already a wider range of digital technologies that can be roughly divided—not non-overlapping—into two categories: on the one hand, *tools* for solving equations and, on the other hand, *learning environments*. *Tools* such as spreadsheets or computer algebra systems allow the user alternative ways of using them. *Learning environments* are structured learning arrangements designed with a teaching objective, using tools and visual aids. Figure 3 shows an interactive digital simulation of a beam balance and the solving of a linear equation. Moreover, there are digital assessment systems such as STACK (Sangwin, 2013) or learning apps like *Photomath*[2], *Chegg Math Solver*[3], *Cymath*[4] or *Mathway*[5]. A didactic discussion of such programs can be found in Arcavi et al. (2017, p. 106). These systems will certainly be further developed in the coming years.

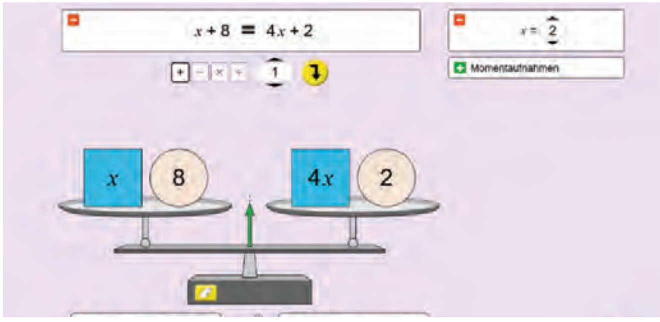


Figure 3. Simulation of a balance (University of Colorado Boulder)

BMMs and the solving of equations with digital technologies (DT)

The four BMMs allow meaningful explanations for different algebraic methods of solving equations. Since the main focus here is on the significance of the BMMs and the use of DT, the operational BMM, which essentially plays the significant role in primary school and lower secondary school, will not be discussed (cf. Arcavi et al. (2017)).

The Object-BMM

A CAS solves equations on the numeric and symbolic levels, and it is a formulary that offers formulas for linear, quadratic, like in Figure 4, and special types of polynomial functions of higher order than 2.

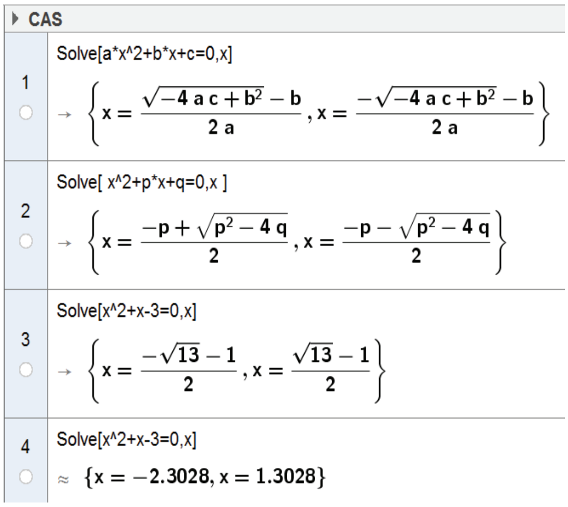
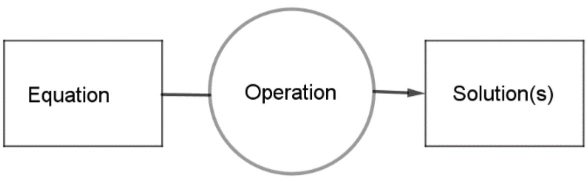


Figure 4. Solving quadratic equations with a CAS



2 https://photomath.app/
3 https://www.chegg.com/math-solver
4 www.cymath.com/
5 www.mathway.com

The Object-BMM can be integrated into an Input-Operation-Output model:

“Operation” may be seen as a “black box”, if the learner does not know the algorithm used in this “box”. But it might also be a “white box”, if the CAS is “only” used as a tool to get the result quicker than with paper and pencil. Furthermore, to turn the “black box” into a “white box” it might be helpful to explain “Operation” and the Object-BMM—in dependence of the type of the equation—by referring to the relational and the functional BMM.

The functional BMM

A CAS allows the solving of equations on different representation levels. It calculates the zeros of (special types of) functions on the symbolic and numeric level by only pressing one button and it visualizes the solutions on the graphic level. The solutions are the zeros of the function. Figure 5 shows an equation with a parameter, the solutions are given dynamically by varying the parameter b with a slider.

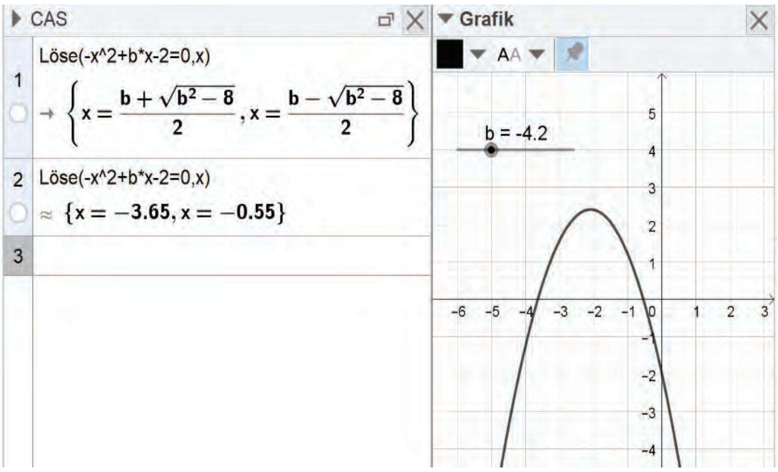


Figure 5. Solving quadratic equations (with a parameter b) and the functional BMM

Special equations of higher degree, particularly equations of degree 3 and e.g., trigonometric and exponential equations can be solved—quite often—numerically, sometimes symbolically, and nearly always graphically.

Solving (more complex) equations with a CAS is not only a “pressing a button”-activity. Basic knowledge of solving different types of equations and different strategies for solving of equations are necessary, especially if an approach that had been used did not lead to a successful solution. Figure 6 shows an example of the solving of the equation $1 + sin(x) = 2^x$. In this case, trying to get a symbolic or numeric solution with a CAS is not possible.

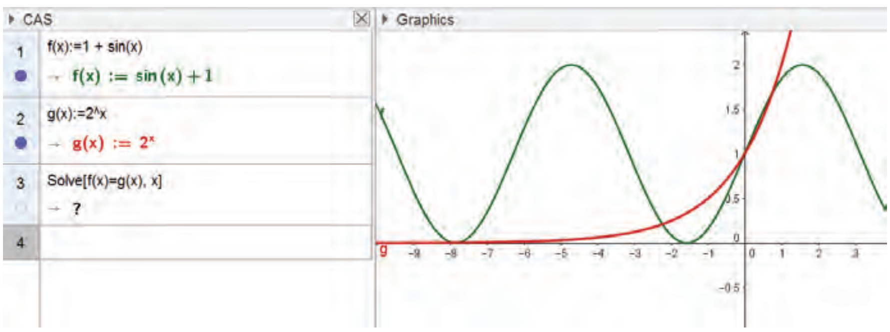


Figure 6. Graphical solution of the equation $1 + sin(x) = 2^x$

The relational BMM

Concerning the functional BMM, a CAS is a *tool* for solving equations. However, a CAS can also be used as a *teaching-learning system* and integrated into a *learning environment*. An example is the step-by-step execution of (equivalence) transformations for an equation. For this, arithmetic operations are applied to an equation as a whole while the algebraic transformations can be visualized graphically. The rather confusing static Figure 7 has to be seen in a dynamic step-by-step presentation to show more clearly the invariance of the x-coordinate of the intersection point of the graphs and hence the consistency of the solution of each transformed equation.

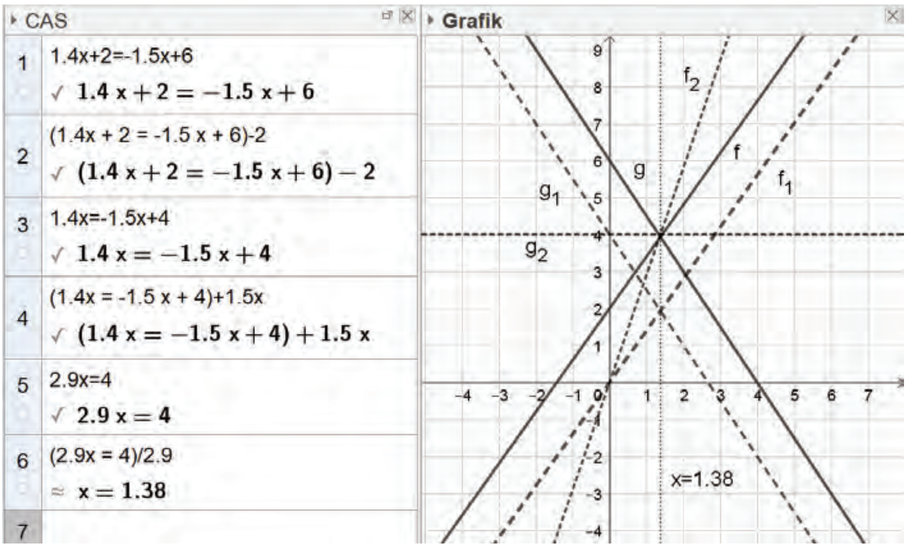


Figure 7. Step-by-step transformations of an equation

Conclusion and research questions

Solving of equations—in this article—is seen in relation to the BMMs of equations (operational, relational, functional BMM and Object BMM) and the use of DT, mainly CAS, but with different kinds of representations. The first research question is the substantial question about the existence and the kind of individual BMMs:

1. *Are the four BMMs represented in students' thinking if they solve equations and how is the relationship between BMM and the way of solving an equation?*

We suspect that the BMMs have to be distinguished concerning different types of equation, especially linear, quadratic and "other" equations, and different methods of solving equations (especially symbolic, graphic, numeric).

The second research question is about the three *central activities* while working with equations and their relation to the BMMs?

2. *How do the BMMs of equation interact with the central activities while solving equations?*

Concerning the solving of equations with DT, the interrelationship between three dimensions—BMMs, DT and central activities—seems to be a crucial point. The third research question aims to get criteria for the development of tasks for constructively generating BMMs of equations.

3. *Which tasks and activities—especially with the use of DT—support the development of BMMs for equations (depending on the different types of equations)?*

The analyses in this article are the theoretical basis for answering these research questions empirically. This will be the next step in this research project.

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Student definitions of equivalence: structural vs. operational conceptions, and extracted vs. stipulated definition construction

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In this study, we describe a model of student thinking around equivalence (conceptualized as any type of equivalence relation), presenting vignettes from student conceptions from various college courses ranging from developmental to linear algebra. In this model, we conceptualize student definitions along a continuous plane with two-dimensions: the extent to which definitions are extracted vs. stipulated; and the extent to which conceptions of equivalence are operational or structural. We present examples to illustrate how this model may help us to recognize ill-defined or operational thinking on the part of students even when they appear to be able to provide "standard" definitions of equivalence, as well as to highlight cases in which students are providing mathematically valid, if non-standard, definitions of equivalence. We hope that this framework will serve as a useful tool for analyzing student work and exploring instructional and curricular handling of equivalence.

Keywords: Equivalence, solution set, operational-views, structural-views, definitions.

Equivalence is central to mathematics at all levels, and across all domains. In mathematics education, much research has focused on studying how students think about the equals sign in primary (Knuth et al., 2006) through post-secondary (Fyfe et al., 2020) school, because student conceptions of the equals sign are related to their arithmetic and algebraic calculations. However, equality is just one example of the larger concept of equivalence—other types of equivalence occur extensively throughout the K-16 curriculum, but are rarely, if ever, taught under one unifying idea called equivalence (Wladis et al., 2020). On the other hand, multiple types of equivalence (e.g., similar/congruent figures, function types, equations with the "same form") are contained in the *U.S. Common Core Mathematics Standards* but are never explicitly labeled as a type of equivalence. When equivalence is not explicitly defined, students may extract their own non-standard, ill-defined, or unstable definitions, or they may inappropriately use the definition of equivalence from one area (e.g., expressions) in another area where it cannot be directly applied to obtain the "standard" definition expected of them (e.g., equations). In this paper we will illustrate this problem by presenting examples of student definitions around equivalence and a model for analyzing student definitions, focusing on college students' definitions of equivalent equations. Examples of student work will be used as vignettes to illustrate the model. Our aim in presenting this model is to start a conversation about student definitions of equivalence and to present an initial framework that can then be further tested, refined, and revised by future empirical work.

Theoretical framework

We frame the analysis of student definitions of concepts in terms of Tall and Vinner's (1981) concept image and concept definition constructs. A person’s *concept image* describes the “total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (p.152). Their *concept definition* describes a “form of words that a learner uses for his own explanation of his (evoked) concept image” (p.152). Hence an individual’s *personal* concept definition is idiosyncratic to the individual, may vary based on the context it is evoked, and may deviate from the larger mathematical community (the *formal* concept definition). When we refer to a student’s ‘definition’, we are referring to their personal concept definition.

In this paper, we define the *formal* concept definition of equivalence as an equivalence relation. The formal definition of an equivalence relation most often given in advanced mathematics classes is that of a binary relation that follows the identity, symmetry and transitive properties. However, another equivalent but more accessible definition of an *equivalence relation* is that of a partition on a set, or more informally: If we have a set of objects and a mathematically well-defined rule for sorting objects into sets so that each goes into one and only one set, then this “sorting” is an equivalence relation, and two objects are equivalent if they belong to the same set. Using this definition as an analysis tool allows us to account for many types of equivalence, with many different mathematical objects (e.g., numbers, algebraic expressions, algebraic equations) and equivalence relations (e.g., equality of expressions; insertion equivalence of equations; Wladis et al., 2020; Zwetzschler & Prediger, 2013). (We note that this definition is an analysis tool that is not necessarily intended to be given to students.)

From this perspective, a student’s personal definition of equivalence in a given context is valid in so far as it is an equivalence relation and can be expressed in a mathematically well-defined way by the individual. When students have no explicit definitions of equivalence, this presents several potential problems: (1) students may incorrectly apply one definition to another context where it fails to produce the “standard” definition (e.g., definition of equivalent expressions to equations); (2) they may have only ill-defined or operational definitions of equivalence which inhibit their ability to reason through problems; or (3) they may use valid but non-standard definitions of equivalence, in which case they are being penalized for not knowing certain socio-mathematical norms (Yackel & Cobb, 1996) even when they are reasoning correctly. We argue that the model presented here allows us to better recognize when these three situations (as well as others) might be occurring with students.

Model of equivalence

Our model of student thinking about equivalence conceptualizes student definitions as existing on a two-dimensional plane with two axes: operational vs. structural conceptions of equivalence (Sfard, 1992), and extracted vs. stipulated definitions of equivalence (Edwards & Ward, 2004). A student with an *operational* conception thinks of mathematical entities as a process of computation, while a student with a *structural* conception thinks of them as abstract objects which can then be acted on by even higher-order processes. A student with a structural conception sees objects as reified processes (e.g., $6x$ is seen as an object itself, and not just as the process of multiplying x by 6), however when students view something as an object which is not the reification of any process, this is called a *pseudostructural* conception (p.75, Sfard, 1992). We see Sfard’s constructs as related to the computational/relational distinction made in research on the equals sign, where the computational view is a cue to calculate, and the relational view focuses on equality as a relationship (e.g., Knuth et al., 2006). Our model can be seen as a generalization of the computational/relational distinction made in research on the equals sign, where equivalence structures (the equivalence relationships) are core objects that justify computation. This is in contrast to Sfard's description of structures (e.g., algebraic expressions) being viewed as a normative process which is reified into an object.

Extracted definitions are created to describe actual observed usage of a term (e.g., a student may extract a meaning for equivalence from their instructional experiences, whether or not they have encountered an explicit definition). In contrast, *stipulated* definitions are those definitions that are stated explicitly—to determine if something fits the definition one must consult the definition directly (Edwards & Ward, 2004). We note that in our model, a stipulated definition may be stipulated by the student or an authority—the key features we use to determine if a definition is stipulated in our framework is whether it appears to be explicit, well-defined, and stable across contexts. We note that while we have displayed our model in Table 1 as a two-by-two grid for the sake of simplicity, but we conceptualize these categories as a spectrum (thus, Table 1 is actually a continuous 2D plane).

Table 1: Model of student thinking about equivalence

	Extracted Definition	**Stipulated Definition**
Operational Conception of Equivalence	**Pseudo-process view:** Students see equivalence as a computational process, and their approaches to those processes are dictated by prior experience in ways that are extracted rather than stipulated. Definitions of equivalence are typically non-standard, ill-defined, and/or unstable.	**Process view:** Students see equivalence as a process, but do process computations by referring to stipulated rules or properties. Students with this view may be able to perform calculations correctly but this does not necessarily translate to being able to use stipulated definitions to recognize equivalent objects.
Structural Conception of Equivalence	**Pseudo-object view:** The student is able to consider whether two objects are equivalent without reverting to an explicit computation, perhaps by considering the structure of the objects; but definitions of equivalence are typically extracted in some way from experience rather than based on stipulated definitions of equivalence, and as a result are typically non-standard, ill-defined, and/or unstable	**Object view:** The student is able to consider whether two objects are equivalent without reverting to an explicit computation, perhaps by considering the structure of the objects; definitions of equivalence used to determine equivalence are stipulated. The student conceptualizes equivalence classes (or solution sets) as objects, although they need not do this formally.

Methods

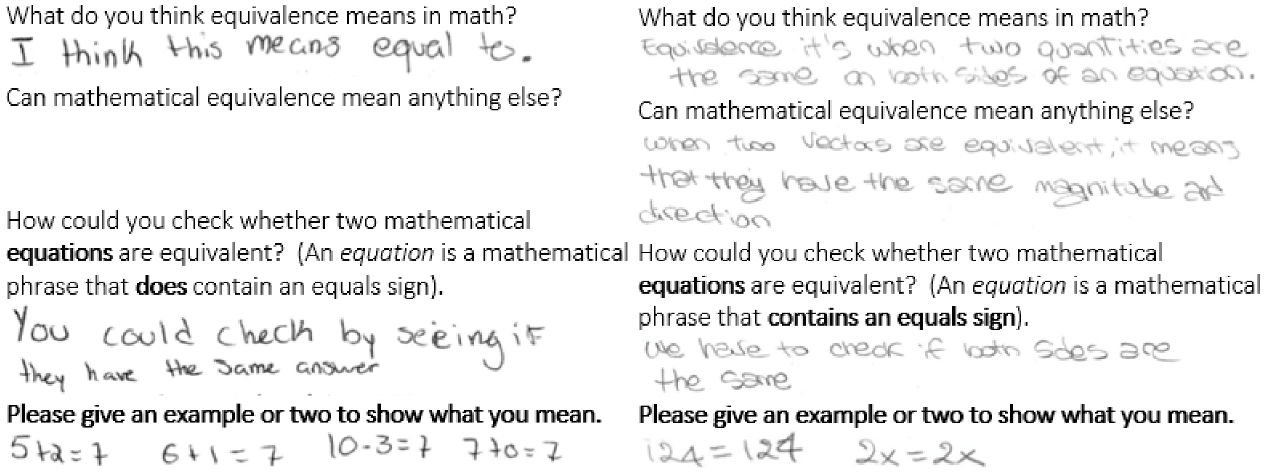
Data for this study were collected from 124 students at an urban community college in the US through open-ended questions in 18 different courses, from developmental elementary algebra (similar to Algebra I in secondary school) to linear algebra. Student responses were analyzed using thematic analysis (Braun & Clarke, 2006), combining codes from the model above with an emergent coding scheme. Multiple coders participated in several rounds of coding until consensus was reached. Responses coded as indicative of an operational view of equivalence provided evidence of thinking of equivalence as an algorithm; those coded as indicative of a structural view of equivalence provided evidence of thinking of equivalence as a fixed trait of an object, or reasoning about equivalence via its general properties. Further coding details are described below.

Results

Students often struggled to provide definitions of equivalent equations for several different reasons.

One issue appears to be that students attempted to apply the definition of equivalent expressions to that of equivalent equations.

Pseudo-process view (extracted and operational): For example, in Figure 1, we see the work of two students, one in elementary algebra, and one in linear algebra, both of whom give somewhat similar definitions of equivalent equations. The elementary algebra student gives a more ill-defined definition ("same answer") but we see from the examples that they provide that they appear to be thinking about equivalent arithmetic expressions. We would classify this response as a pseudo-process view, as the definition is not well-defined and appears to center around arithmetic calculation.


What do you think equivalence means in math?
I think this means equal to.
Can mathematical equivalence mean anything else?
How could you check whether two mathematical **equations** are equivalent? (An *equation* is a mathematical phrase that **does** contain an equals sign).
You could check by seeing if they have the same answer
Please give an example or two to show what you mean.
5+2=7 6+1=7 10-3=7 7+0=7

What do you think equivalence means in math?
Equivalence it's when two quantities are the same on both sides of an equation.
Can mathematical equivalence mean anything else?
when two vectors are equivalent, it means that they have the same magnitude and direction
How could you check whether two mathematical **equations** are equivalent? (An *equation* is a mathematical phrase that **contains an equals sign**).
We have to check if both sides are the same
Please give an example or two to show what you mean.
124=124 2x=2x

Figure 1: Definitions from an elementary algebra student, pseudo-process view (on left) and a linear algebra student, extracted view (on right)

Operational view: We see similar work by the linear algebra student in Figure 1, with some differences; they give broader examples of equivalence (describing also vectors) and their definition is more detailed ("when two quantities are the same on both sides of an equation"). But like the elementary algebra student in Figure 1, they conflate the definition of equivalent expressions with equations (they include an algebra example, but only show identical expressions as equal). Their definition of equivalent equations is also not fully well-defined ("check if both sides are the same"), because the word "same" is not well-defined. While their answer shows signs of having seen more examples of mathematical equivalence, this does not appear to have positively impacted their definition of equivalent equations; we classify their definition as extracted, because it is ill-defined.

Structural view: Students who apply the definition of equivalent expressions to equations may even do this in a way that is mathematically valid (i.e., fits the definition of an equivalence relation), even though it is not a "standard" definitions of equivalent equations (e.g., same solution set). Consider Figure 2, where a precalculus student has defined equivalent equations as two equations where "the result or the number after the equal sign are equivalent". Based on their examples, they seem to be suggesting that any equations of the form $expression = n$ for fixed n would be equivalent. This is similar to definitions given by other students in other research (Wladis et al., 2020). This example is particularly interesting, because the two equations given here also happen to have the same solution set, so it is unclear if this is also an implied part of the student's definition. Whether the definition includes this feature or not, we would classify it as structural even though it is a "non-standard" definition, because the student has given what could be a well-defined but alternate definition of equivalence (whether or not their definition is fully well-defined is unclear)[1].

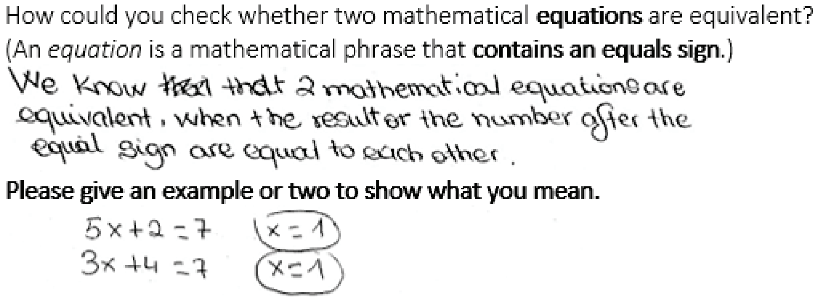

How could you check whether two mathematical **equations** are equivalent?
(An *equation* is a mathematical phrase that **contains an equals sign**.)
We Know that 2 mathematical equations are equivalent, when the result or the number after the equal sign are equal to each other.
Please give an example or two to show what you mean.
$5x+2=7$ $x=1$
$3x+4=7$ $x=1$

Figure 2. Precalculus student's non-standard structural definition of equivalent equations

In contrast to the previous examples, some students did draw in some way on the notion of "solving" equations or the solution sets of equations when defining equivalence. However, the ways in which students drew on notions of "solving" also fell into different areas of our framework. Simply talking about the "solution" of an equation was not sufficient to classify work as either stipulated or structural even though it sounds like it is related to the standard insertional equivalence definition of equations (i.e., same solution set).

Structural view: In Figure 3(a), we see the work of a Calculus III student, who appears to have a well-defined and structural view of equivalent equations: they define equivalent equations as having the same solution set (seeming to conceptualize the solution set as a fixed object); and their definition appears to be well-defined, not just because of their stated definition, but also because the example they give which shows that their interpretation of "same solution" appears to be the "standard" one. We note that this is critical, as many students used the language of "same solution" but actually meant it to describe equivalent sides of an equation (equivalent expressions) rather than solution set.

Pseudo-process views (operational and extracted): See, for example, the work of an introductory statistics student in Figure 3(b). This student wrote that two equations are equivalent if you "substitute the value in for x and the solution is the same for both equations": this sounds like the standard definition of equivalent equations (if an incomplete one that does not account for the possibility that x may have more than one value), however, looking at the example they provided, we see that to them "solution" denotes the quantity resulting from simplifying one side of an equation (not the solution set of an equation). In this sense, the student's definition is ill-defined, because the vocabulary that they are using appears to be ill-defined and has multiple, perhaps vague, meanings. For these reasons, we would classify this work in (b) as a pseudo-process view, even though on the surface the definition initially looked similar to the one in (a). The student work in Figure 3(c) shows another common approach that students used, in which they drew on notions of solving when asked about equivalent equations, but struggled to relate these notions to any well-defined definition of equivalence. This

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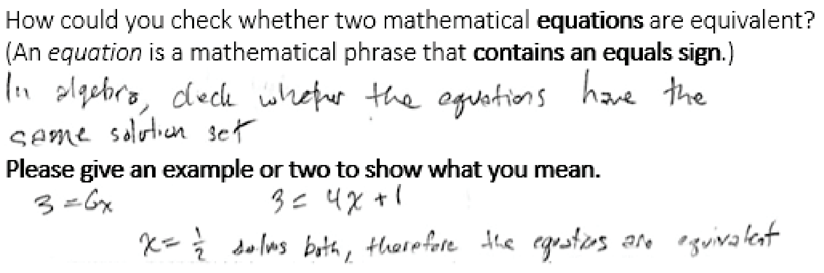
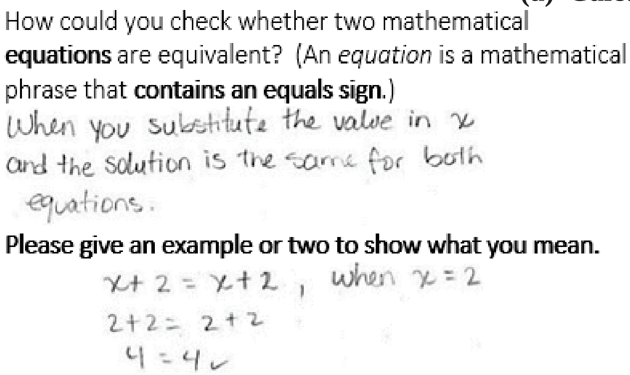
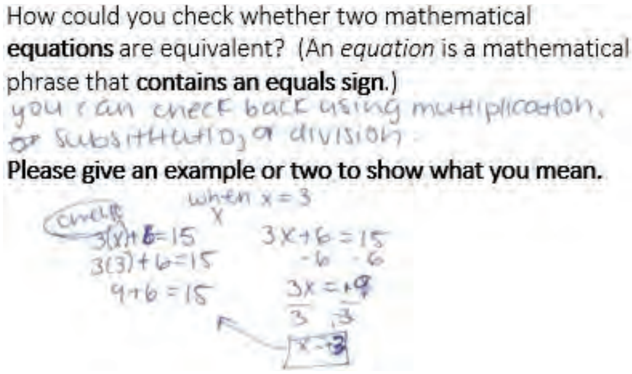
[1] This student may be drawing on notions of equations with the "same form" (e.g., $y = mx + b$, $ax^2 + bx + c = 0$) which is another type of equivalence that is commonly used in the algebra curriculum, even if it is not called equivalence in the curriculum (however, "same form" could in fact be codified as a formal equivalence relation, and students may be noticing this when they draw on it in their equivalence definitions (Wladis et al., 2020)).

student has solved an equation and checked the solution by substituting it back into the original equation; however, it is unclear what the definition of equivalent equations is, or even which two objects the student is claiming are equivalent (perhaps equivalence for them is not about the relationship between two objects, but instead names a process of checking the solution of an equation). Thus, we classify this as a pseudo-process view—there is no well-defined stated definition, and the student's focus is on computation.



(a) Calculus III student





(b) introductory stats student **(c) intermediate algebra/precalculus student**

Figure 3: Examples of different ways that students used "solving" in defining equivalent equations

Students also gave other non-standard definitions of equivalence that might have been well-defined equivalence relations (e.g., equivalent arithmetic equations as ones that express the same additive relationship; equivalent algebraic equations as ones that express the same relationship between the variables). However, we note that by de-coupling our categorization of student definitions of equivalence from what is "standard" and thinking more carefully about the extent to which student definitions of equivalence are stipulated (and an equivalence relation); and the extent to which student conceptions of equivalence are structural or operational, we may be able to achieve two critical goals more effectively: (1) we may be able to better identify student thinking which "sounds right", but is actually ill-defined; and (2) we may be able to identify valid student thinking that simply does not adhere to "standard" definitions. Both of these goals may better help us to tailor instruction to students.

We now briefly describe some overall trends we found in coding responses to open-ended questions on definitions of equivalence (Table 2). Students primarily associated equivalence with equality, and rarely cited other forms (e.g., equivalent equations), although the incidence of non-equality examples rose somewhat with course level. Similarly, students at all levels were extremely likely to give ill-defined or vague definitions of equivalence when asked. When asked about their definitions of equivalent equations, most students conflated this with the definition of equivalent expressions; this did not appear to improve with course level, suggesting that the lack of explicit definitions of equivalent equations in textbooks and curricula (Wladis et al., 2020) may well be contributing to student difficulties in understanding how definitions of equivalence vary in different contexts. Some of these definitions, while non-standard, may have been equivalence relations, and therefore reflect mathematically valid reasoning—the prevalence of this was not correlated with course level, suggesting that students at all levels may sometimes be generating valid but non-standard definitions. Many students associated equivalent equations with solving, but this was rarely done in a well-defined way: roughly one quarter of all students at all course levels solved an equation but did not relate this in any well-defined way to the definition of equivalent equations (most commonly they solved a single equation and then checked the answer, with no mention of which two things were equivalent); a smaller percentage of students did this at levels of precalculus and above, but the differences by course level were small. Some students interpreted equivalent equations as equations with the same solution set, and did so in a well-defined way; this was slightly more common as course levels went up; however, the vast majority of these students did so in an operational way (i.e., solved two equations and said they were equivalent, without discussing the solution set in a more general or structural way). This is perhaps to be expected, given the operational way in which the question itself was phrased, however, this does follow patterns observed in questions without this more operational wording, such as the more general question about the definition of equivalence given on this set of questions (although the tendency to use structural rather than operational definitions did increase with course level). However, we note that overall, structural and well-defined definitions were rare among all students, suggesting that instruction which specifically includes explicit stipulated definitions, and which encourages structural reasoning, is needed at all levels.

Table 2. Summary of student definitions of equivalence

	elementary alg. or below	intermediate alg. or 100-level	200-level or above
General definition of equivalence			
ill-defined or vague	67%	71%	60%
cited equality	94%	87%	80%
other valid definition	0%	3%	16%
operational definition	41%	18%	17%
structural definition	0%	2%	17%
How to tell if two equations are equivalent			
conflated w/ equiv. expressions	44%	48%	44%
of these, possible well-defined defn.	19%	6%	16%
finding solution set, operational	0%	3%	8%
related to "solving" but ill-defined	22%	29%	16%
solution set, structural	0%	2%	4%
total *n*	36	62	25

Discussion and conclusion

Our model of student definitions of equivalence aims to refocus our attention from whether definitions look “standard” to whether student definitions are well-defined equivalence relations, and whether their definitions are structural vs. operational. Using this lens allows us to pinpoint when students appear to understand a standard definition, but upon deeper analysis we find that their definition is ill-defined or wholly operational, limiting their ability to use it. On the other hand, this model also allows us to recognize students’ mathematically valid definitions even when they are nonstandard or students are not able to explain them fully formally. Evidence from examples of student work suggests that students do notice many kinds of “sameness”, yet struggle to articulate this in mathematically well-defined ways, just as they struggle to articulate “standard” definitions of equivalence in well-defined ways. This suggests that students are capable of noticing and using more generalized notions of equivalence, but need more explicit definitions and language in order to be able to do this rigorously. Future research is necessary to better understand what kinds of explicit definitions of equivalence work best for students in different contexts, and the extent to which discussions of the more general notion of an equivalence relation might be helpful in instruction. This framework may also be able to serve as a framework for instruction and curricula, to assess how the concept of equivalence is presented to students as they are learning at various levels in the curriculum.

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A model of how students' definitions of substitution and equivalence may relate to their conceptualizations of algebraic transformation

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This empirical paper explores students' conceptions of transformation as substitution equivalence by linking it to their definitions of substitution and equivalence. This work draws on Sfard's (1995) framework to conceptualize conceptions of substitution equivalence and its components, equivalence and substitution, each on a spectrum from computational to structural. We provide examples of student work to illustrate how students' understandings of substitution, equivalence, and substitution equivalence as an approach to justifying transformation may relate to one another.

Keywords: Equivalence, substitution, substitution equivalence, transformation, definitions.

Transformation has been framed as a core mathematical activity (Kieran, 2004), and all mathematical calculation can be viewed as a process of transformation. Researchers such as Kirschner & Awtry (2004) have found that students' thinking about transforming symbols tends to be rooted in visual patterns of symbols rather than a deeper understanding of mathematical structures. Since algebraic transformation is often taught procedurally, there is a need to frame these manipulations in a structural way. By exploring the core mathematical ideas that justify why particular transformations are mathematically valid, we conceptualize transformation as a process of replacing one symbolic object with an equivalent one, and name this process substitution equivalence (Wladis et al., 2020). This includes the process of identifying sub-objects and replacing them with equivalent ones in order to generate a new equivalent object. This identification process is non-trivial for many students, and we hypothesize that thinking around substitution equivalence may be intimately connected to many of the struggles that students have with transforming symbolic mathematics in various contexts, yet this idea has rarely been explored in research. Here we present a model of students' thinking around substitution equivalence and illustrate potential affordances it might have in analyzing student work.

Conceptual framework and Prior Research

Substitution equivalence as a lens for mathematical transformation

In this paper, we focus on students' thinking around *substitution equivalence*, or the notion that two expressions, equations, or other mathematical objects are equivalent if one can be generated from the other through a sequence of substitutions carried out using standard interpretations of syntactic structure and mathematically valid uses of mathematical properties (Wladis et al., 2020).

Definition of substitution

In order to see how all mathematical activity could be viewed through the lens of substitution equivalence, we define *substitution* more broadly than has been done explicitly in much existing research and curricula. Jones and colleagues (2012) describe substitution as "the replacement of one representation with another" (p.167). Our definition builds on this idea by requiring the equivalence of mathematical objects being replaced and extending to other mathematical objects as well, such as equations. For us, substitution is the process of replacing *any* mathematical object (or any unified subpart of an object) with any *equivalent* object, regardless of complexity. This includes not only the replacement of x in $2x^2 - 2x + 1$ with -3, but also the replacement of $x^2 - 6x = 1$ with the equivalent equation $x^2 - 6x - 1 = 0$ during solving.

Definition of equivalence

Substitution equivalence is dependent upon an underlying equivalence relation. This may be a context-specific definition of equivalence (e.g., insertion equivalence in Zwetzschler and Prediger, 2013), or a more generalized concept of equivalence (e.g., an equivalence relation). Indeed, any definition of equivalence that satisfies equivalence relation criteria could be used.

Definition of substitution equivalence

Despite the importance of substitution equivalence to algebraic justification, little research has focused explicitly on substitution equivalence (see e.g., Pinkernell et al., 2017). A search in ERIC (the education research database maintained by the US Institute of Education Sciences, https://eric.ed.gov/) yields no results for *substitution equivalence*, *substitutional equivalence* or *substitution property of equality*. Other researchers have explored the "substitution principle", which refers to the structural sameness preserved when replacing a variable with a compound term, or vice versa (Musgrave et al., 2015). While this is related to our definition, as both rely on the substitution property of equality, this is not how we use this term. We define the domain of substitution equivalence as composed of two main ideas:

1. The general notion of substitution equivalence: A student understands that we can replace an object with any other equivalent object when problem-solving.
2. The notion that substitution of unified sub-objects preserves equivalence: A student understands that objects can be broken into unified sub-objects, and that replacing any unified sub-object with an equivalent unified sub-object produces an object that is equivalent to the original one (as long as substitution leaves the rest of the structure of that object unchanged).

The second notion leads us to another core definition: We use the term *subexpression* (or sub-object, more generally) to denote a substring of an expression (or other object) that can be treated as a unified object without changing the syntactic meaning of the original expression (or object). For example, $a - b$ is a subexpression of $a - b - c$, but $b - c$ is not (because putting brackets around $b - c$ would change the syntactic meaning of the whole expression). This different from, but related to, what Kieran (1989) refers to as surface structure (identifying the syntactic meaning of a symbolic algebraic representation) and what Malle (1993) refers to as *Termstrukturen* ("expression structuring") (identifying all algebraic expressions with the same syntactic meaning).

In this paper, we use Sfard's (1995) work to frame our thinking about student conceptions, where student thinking about a concept may be operational (as a process, often of computation) or structural (as an abstract object in and of itself). We conceptualize students' definitions on a continuum that can be primarily structural, primarily operational, or somewhere in between. A student may vary along the spectrum flexibly, but the ability to think structurally, at least some of the time, is necessary in order to progress to some higher order processes (Sfard, 1995).

Aside from Wladis et al. (2020), we have found little (if any) work on student conceptions of substitution equivalence, although there has been substantial work around equality. One common strand focuses on conceptions of the equal sign, where students see the symbol either operationally (as a ‘do something symbol), or relationally (as a relationship between two entities) (Knuth et al., 2006). In terms of substitution, relatively little work has been done, although some research has explored student notions of substitution equivalence in the context of arithmetic (Jones et al., 2012).

Model of operational and structural view of substitution equivalence

Wladis et al (2020) described key features of students’ thinking around substitution equivalence on a spectrum from structural to operational approaches. This paper aims to take this further by drawing on empirical data to explicitly describe how students’ conceptions of substitution equivalence may be dependent upon their definitions of substitution and equivalence (see Figure 1).

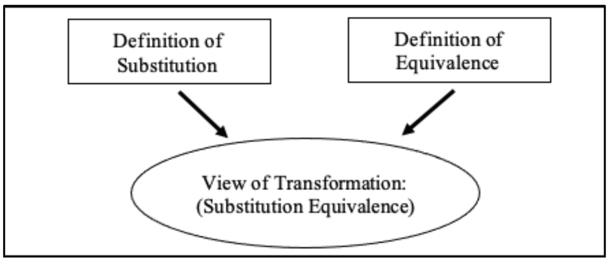


Figure 1: Model of student thinking about substitution equivalence

Method

This work draws on data collected from multiple classes across six years at an urban community college in the US, including cognitive interviews and open-ended questionnaires. Open-ended questionnaires and cognitive interviews were distributed to participants in courses from elementary algebra through linear algebra within a larger data collection process in efforts to develop an algebra concept inventory. These data were analyzed using grounded theory (Strauss & Corbin, 1990) to generate and refine models of students’ conveyed meanings to explain their written and spoken work. Categories developed during analysis were heavily influenced by the work of Sfard (1995), and existing literature about students’ understanding of the equals sign (e.g., Knuth et al., 2006). An initial coding scheme was developed by a single coder, and then in subsequent rounds, multiple coders revised the scheme until consensus was reached; coders included mathematicians, mathematics education researchers, and elementary algebra instructors.

The Model

Table 1: Components of substitution equivalence model

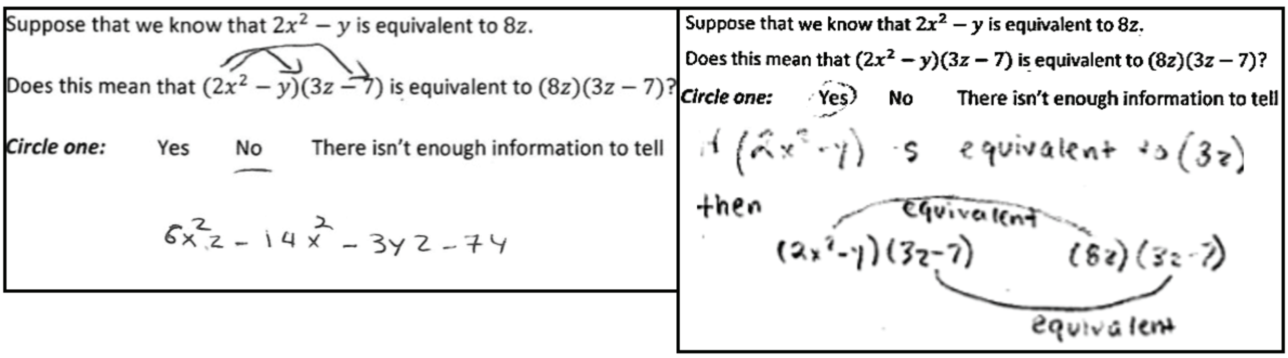
	Operational Thinking	**Structural Thinking**
View of Transformation	Students see transformations of expressions and equations (or other objects) as a process of “operating on” the original object itself. They may or may not see this as linked to any notion of equivalence.	Students see each step in a transformation as the process of replacing one object with an equivalent one through substitution, using properties and existing syntactic structure. They appear to have some notion of an equivalence class as an object (which need not be formally defined).

Definition of Equivalence	Students either ignore the notion of equivalence entirely, or appear to have only vague, ill-defined, or unstable notions of equivalence, or try to apply one definition of equivalence that works only in one context to another context.	Students have a well-defined and relatively stable definition of equivalence, and recognize that it is context-dependent. They recognize that equivalence is a fixed trait (two objects are either equivalent under a particular definition or not—they do not "become" equivalent).
Definition of Substitution	Students see substitution only as plugging a number in for a variable (and then computing the result). They see variables as representing only numbers.	Students see replacement of any object (or sub-object) with an equivalent one as substitution. They see variables as representing any valid mathematical object, including numbers or (potentially complex) expressions.

In the model in Figure 1, holding well-defined and standard definitions of both substitution and equivalence are necessary but not sufficient conditions for students to develop a view of transformation justified by substitution equivalence. A student may have trouble thinking of transformation as substitution equivalence because (a) their definitions of substitution are too narrow; (b) their definitions of equivalence are ill-defined, unstable, or mathematically invalid; (c) they do not draw on their knowledge of substitution and/or equivalence when performing transformation; or a combination of all of these. We conceptualize students' views of substitution, equivalence, and transformation as being on a continuum from operational to structural (Table 1). This model is based on the notion that the *ability* to conceptualize transformation as a process of substitution equivalence may be useful for students in developing deeper understanding of the justification behind their transformation work (and a way of checking the validity of that work).

Vignettes: A model in action

We now provide examples of students' written work from our dataset to illustrate how one might use the model we present here. These are intended to highlight the continuum of operational and structural views. To see how students' views of transformation as substitution equivalence can vary along this spectrum, we present two developmental algebra (a non-credit course with similar content to Algebra I from secondary school) students' responses about assessing whether two expressions are equivalent (Figure 2), where the first response (Figure 2a) exemplifies an operational view and the second response (Figure 2b) exemplifies a structural view.



(a) **(b)**

Figure 2: Examples of responses rooted in an operational (a) and structural view (b) of equivalence

The first student's response (Figure 2a) foregrounds computation and symbolic manipulation, so we classify it as an operational view of transformation. In cognitive interviews, students on similar problems have provided similar work and explained that they can only tell if two expressions are equivalent if they both simplify to the same final "answer". Hence we see the approach taken in

Figure 2a as indicative of having an internal computational definition of equivalence of “expressions that simplify to the same thing”. In contrast, the response in Figure 2b illustrates exactly how the two equivalent subexpressions are substituted into the larger expressions using arrows to indicate the relationship between each piece and to highlight the structure of the two expressions. This student mapped each unified subexpression in the first expression to an equivalent unified subexpression in the same place in the second, in order to illustrate why the two expressions are equivalent. Though the student didn’t use the word “substitution”, we see evidence that they were depicting a replacement or exchange of one equivalent sub-part with another.

Student definitions of equivalence

To see how students’ definitions of equivalence can vary, we refer to the previous examples and consider the definitions of equivalence the students seem to be evoking. These responses exemplify operational and structural definitions of equivalence, respectively. In Figure 2a, the student attempted to simplify the expressions to determine whether they are equivalent, and then appeared to decide that the expressions were not equivalent after they could not simplify them further. This definition (“two expressions are equivalent only if they simplify to the same thing”) of equivalence appears to be computational, and their work doesn’t seem to explicitly acknowledge equivalence relationships which justify their work. Because the student abandoned the attempt after simplifying did not work, this suggests that they did not see a way to use the structure of the given expressions to determine equivalence beyond simplifying both sides to see if the results are the same. In contrast, the response in Figure 2b suggests that the student may have a structural definition of equivalence. They drew on the structure of two complex expressions to show how they map to one another in such a way that each subexpression is either the same or equivalent, and they leveraged that structure to show that the final result is equivalent. This definition of equivalence appears to be well-defined and to be identifying fixed traits of the expressions.

Student definitions of substitution

To demonstrate differences along this spectrum, we look at two students’ definitions of substitution evoked from the prompt “In math, what is substitution? (Or what does it mean to substitute?)”. One student wrote “To substitute is to replace a number with a variable”, and further provided an example “$2x + 3 = 9$; $2(3) + 3 = 9$; $x = 3$”. This response (“putting a number in for a letter”) was one of the most common given by students at all levels, from elementary algebra through linear algebra. We classify this narrow definition of substitution as operational, whereas the response “To replace one number, variable, or expression for another” (a response from another student) was classified as structural definition of substitution. This is because their definition affords a greater variety of terms to be replaced for one another, which involves conceptualizing complex subexpressions as objects.

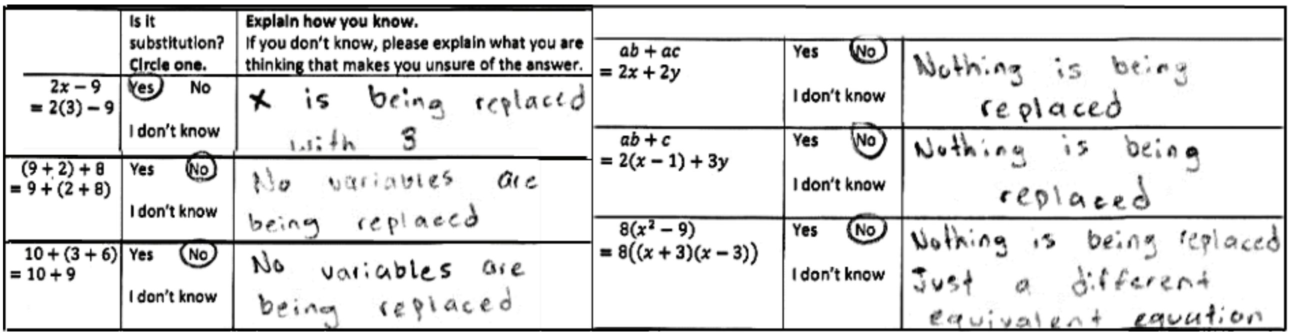
In order to see how students’ views of substitution may impact their view of transformation of expressions, we further examined their responses to a task to identify instances of substitution, and found that their responses were typically consistent with their definitions (e.g., only recognizing transformation as substitution when it involved a number being substituted in for a letter if that was their stated definition); we include one such example of this in the next section.

Using the framework to analyze student work longitudinally

In order to illustrate the potential of this model for deeper analysis, we consider examples from a single Algebra I student (whom we call Epsilon, like ε) across multiple tasks and points in time.

Substitution

We first consider Epsilon's definition of substitution, who gave the response that classified as an operational definition of substitution in the prior section. This correlates with the extent to which they identify different computations as substitution in the following work (Figure 3).



	Is it substitution? Circle one.	Explain how you know. If you don't know, please explain what you are thinking that makes you unsure of the answer.
$2x-9 = 2(3)-9$	Yes (circled) No I don't know	x is being replaced with 3
$(9+2)+8 = 9+(2+8)$	Yes No (circled) I don't know	No variables are being replaced
$10+(3+6) = 10+9$	Yes No (circled) I don't know	No variables are being replaced
$ab+ac = 2x+2y$	Yes No (circled) I don't know	Nothing is being replaced
$ab+c = 2(x-1)+3y$	Yes No (circled) I don't know	Nothing is being replaced
$8(x^2-9) = 8((x+3)(x-3))$	Yes No (circled) I don't know	Nothing is being replaced Just a different equivalent equation

Figure 3: Epsilon's interpretations of substitution in specific contexts

We can see in Figure 3 that Epsilon rarely identified computation as substitution when it was more complex or generalized. They noticed, for example, that the expressions in the last question in Figure 3 are equivalent, but they did not see replacement of the subexpression $x^2 - 9$ with $(x+3)(x-3)$ as an instance of substitution ("nothing is being replaced"), which is consistent with the more limited operational definition of substitution that they provided in the previous section.

Equivalence

Now we consider Epsilon's definition of equivalent expressions. When given the prompt "How could you check whether two mathematical expressions are equivalent? (An expression is a mathematical phrase that does not contain an equals or inequality sign)", Epsilon wrote "If they both have the same correct answer". Epsilon provided a seemingly correct (if perhaps incomplete or ill-defined) definition of equivalent expressions. We cannot be sure whether they understand that expressions must have the same value for every possible combination of variable values or that this applies to algebraic and not just arithmetic expressions, and the word "answer" is ill-defined; however, their definition is in line with the standard definition used in algebra, and they correctly identified that the algebraic expressions in the last question in Figure 3 were equivalent (as well as in other questions not shown here). Their definition also appears to be operational, as it is rooted in computations with expressions.

Substitution equivalence

Now we consider the extent to which Epsilon recognized instances of substitution equivalence in certain algebra examples. Epsilon was given the following two questions: 1.) "Suppose we know that $2x^2 - y$ is equivalent to $8z$. Does that mean that $(2x^2 - y)(3z - 7)$ is equivalent to $(8z)(3x - 7)$?" and "Suppose we know that $3a + b$ is equivalent to $42a$. Does that mean that $7a - 5 + (3a + b) + b^2 - 3a^2$ is equivalent to $7a - 5 + 42a + b^2 - 3a^2$?". Epsilon did not recognize either example as substitution equivalence, given that his response was "I don't know". To the first prompt, they wrote

"$24z^2 - 64z$" seemingly to simplify the expression "$(8z)(3x - 7)$" (in line with their definition of equivalence in the prior section), but this did not help them to identify whether the two expressions are equivalent. They did not appear to draw on the given fact that $2x^2 - y$ is equivalent to $8z$ when attempting to determine if the two larger expressions are equivalent. They provided no additional inscriptions in response to the second question.

This suggests that they may not have a notion of substitution equivalence or may be unable to draw on it in this problem context. Epsilon's operational approach when attempting to determine if the two expressions are equivalent suggests that their operational conception of equivalence may be limiting their ability to recognize and use substitution equivalence when performing mathematical transformations. Another potential barrier to Epsilon developing a robust notion of substitution equivalence and linking this to their transformation work may be their narrow notion of substitution itself. They likely did not recognize the transformations in the questions presented here as substitution just like they did not recognize most of the transformations in Figure 3 as substitution.

Potential impacts of instruction

Epsilon was part of a cohort that took part in a semester-long classroom intervention in which students were taught broader structural definitions of substitution, equivalence, and how to view transformation as substitution equivalence explicitly (as well as other concepts). After the intervention, Epsilon was not able to identify substitution equivalence in all cases, but they were able to recognize it in cases similar to those in the prior section. When given the prompt "Suppose that $3x = 2y + 1$. Does that mean that $5x^2 - (3x) + 7 = 5x^2 - (2y + 1) + 7$?", Epsilon wrote "Yes because $3x = 2y + 1$; its plugged in correctly". From this response, we see how they were able to recognize a complex equation as an equivalence relationship between two structurally identical expressions where one equivalent subexpression could be conceptualized as having been substituted for another. Epsilon's use of the words "plugged in" is a common phrase often used by students to indicate substitution. We do note, however, that this language still suggests a computational approach. However, Epsilon is drawing on structural features of equivalent algebraic expressions through the lens of substitution equivalence, even if their approach is still partially operational. We have insufficient space to discuss the intervention at length here—we simply include this short example to demonstrate that more structural and well-defined definitions of substitution, equivalence, and substitution equivalence approaches to transformation can all be learned by students, even those in developmental mathematics courses, when students are given the right supports.

Conclusion

We have presented a model that describes how students' definitions of substitution and equivalence may related to their ability to justify transformation through the lens of substitution equivalence. Using students' work, we have illustrated some of the affordances of this lens. We have demonstrated that students may struggle with substitution equivalence for different reasons, which may then require different instructional approaches. For example, if a student's definition of equivalence is ill-defined, it may be important to find ways for them to improve their personal definition; whereas if a student has broad and well-defined definitions of substitution and equivalence, a more effective approach may be to help them to see connections between this existing knowledge and their computational work when performing transformations. These are very different approaches to solving what might on the surface look like similar errors, but which stem from very different underlying patterns of how students think about mathematics. Thus, we hope that this model may aid us to better tailor instruction to respond to how to students think, and to better think about how definitions of substitution and equivalence are presented in instruction. We have also shown through one student's work that, with the right kind of instructional approaches, students can learn to think about transformation through a substitution equivalence lens. Further research is needed to investigate which ways of thinking may be most productive for students in different contexts.

Acknowledgments

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A dynamic early algebra approach for transitions from describing figures to transforming expressions

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Early algebra approaches have prepared the transition from arithmetic to algebra for concepts such as variables and the equal sign. In this paper, we focus on necessary transitions for the concept of equivalence of expressions. The design research project shows how fifth graders can develop an understanding of the transformations of expressions by dynamically restructuring figures.

Keywords: Early algebra, equivalence of expressions, connecting multiple representations.

Introduction: Early algebra for preparing transitions from arithmetic to algebraic concepts – the case of the concept equivalence of expressions

Students' challenges in the transition from arithmetic to algebra are widely documented empirically (Kieran, 2004). In different theoretical frameworks, two main instructional strategies have been applied to prepare these transitions: (a) *Making differences explicit,* e.g. between the operational and the relational meaning of the equal sign (Knuth et al., 2006), (b) *Preparing algebraic concepts* already in arithmetic in early algebra approaches, e.g. generalization activities for the later formal introduction of variables (Mason et al., 1985), but also the early confrontation with relational meanings of the equal sign (McNeill et al., 2019). These approaches can be effective in enhancing students' understanding that is fundamental for the later symbolic formalism for variable and equal sign. But as transitions are also necessary for a third algebraic concept, equivalence of expressions (equalities), we pursue the research question: *How to enhance the connection between different characterizations of equivalence to prepare the transition to algebra?*

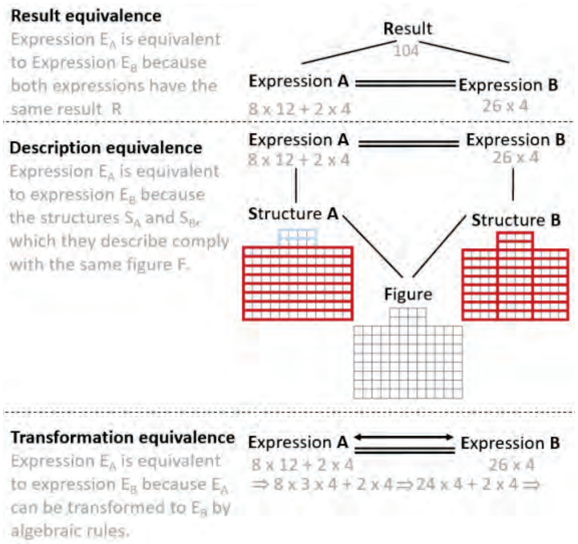


Figure 1: Three characterizations of equivalence of expressions (adapted from Tondorf & Prediger, submitted)

Background: Existing approaches and our design approach

Existing approaches for enhancing students' understanding of algebraic equivalence

For refining the research question, we adopt an epistemological perspective and first specify different characterizations of equivalence (Kieran, 2004; Zwetzschler & Prediger, 2013). Students usually start with an operational characterization of equivalence, *result equivalence* (defined as two numeric expressions are equivalent if they have the same result, see example in Figure 1). However, students also need to become aware of a relational characterization as *description equivalence*. Description equivalence is defined as two expressions are equivalent if they describe the same situation or figure, however, our learning environment, focuses on geometric figures, see example in Figure 1. In a later phase, the transition (or flexible move forward and backward) to a transformational characterization of *transformation equivalence* is needed (defined as two expressions are equivalent if they can be transformed into each other according to transformation rules) (Kieran & Sfard, 1999). Strategy (b) was already realized to facilitate a transition from result to description equivalence (Mason et al., 1985; Kieran & Sfard, 1999). However, the connection between description and transformation equivalence has hardly explicitly enhanced, so far. Based on the existing approaches, we conduct a design research project for developing and investigating an early algebra approach for bridging this gap for expressions without variables. On this background, we refine our research question as follows: *How can early algebra students' connection between description and transformation equivalence for expressions be enhanced?*

Based on a literature review, we briefly sketch two existing approaches for establishing these two characterizations: (1) Early algebra approaches can provide first opportunities for students to experience how different expressions can describe the same situation or geometric figure through connecting different representations (e.g. Mason et al., 1995). Some algebra approaches also make explicit that these pairs of expressions are then called equivalent, thus fostering the characterization of description equivalence (Zwetzschler & Prediger, 2013; Sfard & Kieran, 1999). However, the transition to transformation equivalence is hardly supported. (2) If the learning of the characterization of transformation equivalence is supported, then starting from result equivalence, not from description equivalence (e.g. Schwarzkopf et al., 2018). Usually without reference to a described situation or figure, these approaches engage students in discussing why two expressions have the same result by investigating involved expressions and the structure of their sub-expressions, e.g. by using the distributive law intuitively to summarise sub-expressions. As they do not yet sufficiently involve the connection of multiple representations, the representations will be a focus of our qualitative analysis.

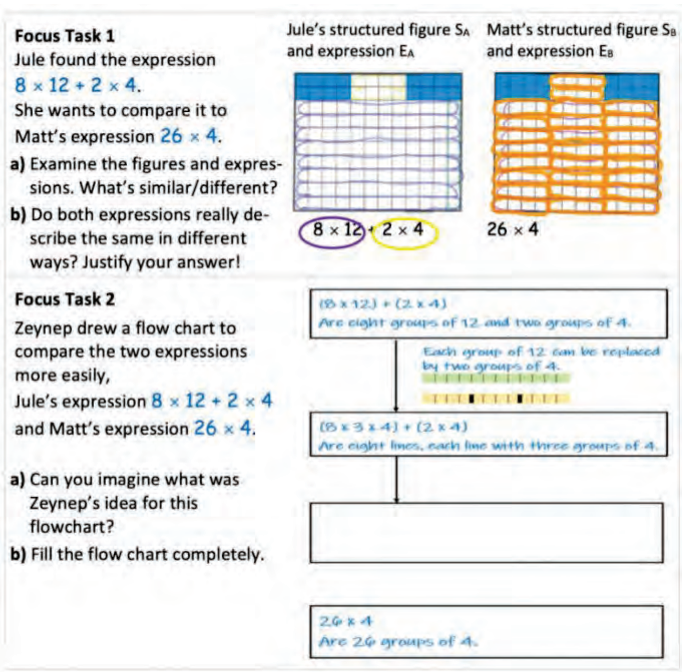


Figure 2: Task 1 for description equivalence, Task 2 for transcription to transformation equivalence

Our first design approach for connecting description and transformation equivalence

As the state of research shows, establishing a strong fundamental understanding of equivalence of expressions by the relational characterization as description equivalence is crucial, e.g. by connecting multiple representations

(Mason et al., 1995). However, existing approaches do not sufficiently support the connection of description and transformation equivalence. We designed a learning environment to engage students in activities that could lead to a recognition of this aspect of equivalence. Figure 2 shows two tasks focussed on in this paper: Students worked out the connection of structures of expressions and subexpressions to geometric structure in the figures and the characterization of description equivalence in previous tasks. Focus Task 1 then asks in a first approach to compare two different geometric structures and also their respective expressions.

Focus Task 2 then was the preliminary version designed to engage students in the connection to transformation equivalence by asking to connect each step of transformation to the geometric figures in a way that each expression can be replaced by a description-equivalent-one.

Methodology of the design research study

Within the chosen Design Research methodology (Gravemeijer & Cobb, 2006), we developed the initial design (from Figure 2) and investigated the research question mentioned above.

Methods of data gathering. Up to now, design experiments have been conducted in laboratory settings with seven pairs of fifth graders. In total, 27.5 hours of video were recorded and partially transcribed. In this paper, we analyze a case of two girls, Lea and Linda (10-11 years old), described as high-achieving by their teacher. This case was chosen due to its significance in illustrating two complementary phenomena concerning the task potential with respect to the research question.

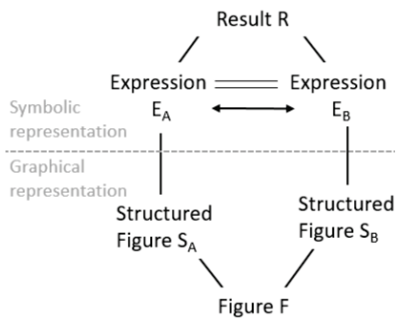


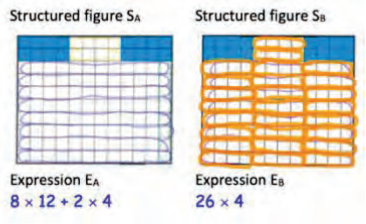
Figure 3: Analytic framework for capturing representations and connections addressed in students' utterances

Methods of data analysis. According to the analytic procedure developed in Tondorf & Prediger (submitted) for other cases (Figure 3), the transcript was qualitatively analyzed in two steps: In step 1, the students' utterances are coded according to the addressed components: result R and the expressions E_A and E_B, in symbolic representation, or in the graphical representation the structured figures S_A or S_B, and the figure F. In step 2, students' utterances were coded regarding the implicitly or explicitly drawn links between the components and marked in the analytic framework (═ marks the conclusion that E_A and E_B are equivalent, ↔ marks a transformation between E_A and E_B , — marks the connection of two concerned / marked objects. The coding was conducted by the first author, checked by the second author, missing intercoder agreement was solved by consensual discussion.

Empirical insights into students' transition pathways

Episode 1: Students' invention of restructuring from one structured figure to the next

Episode 1 illustrates students' emerging idea of a *dynamic comparison* of the given structures that we identified empirically: By dynamic we mean a comparison which results from a stepwise modification of one object into the other (e.g. E_A to E_B in transformation equivalence). We distinguish it from *static comparison*, which is bound to a third stable object of comparison (like result equivalence comes from connecting both expressions E_A and E_B independently of one another to result R). In Task 1 of Figure 2, Linda and Lea are asked to compare the two structured figures (named S_A and S_B for easier analysis) and the expressions E_A and E_B with respect to similarities and differences. The girls had already allocated the structure of the expressions to the structure of the figures by explained marking of S_A coming from E_A or rather by finding E_B coming from S_B, described how both expressions are description equivalent, and named same-structured and different-structured areas of S_A and S_B. When asked to transfer their discovery onto the symbolic representations, Linda focused on S_A and S_B and starts to describe the relation between 8×12 in E_A and groups of fours in E_B.



49	Linda:	Here [gestures along the rows of structured figure S_B with 3 groups of 4], when you take that together
50	Teacher:	Um, that is a great idea, can you also.
51	Linda:	four, eight, twelve

Linda described the relation between three groups of four in one row of structured figure S_B and one complete row of 12 (like in S_A). Even though she used merely the vague "here" to refer to the units (Turn 49) and a language of symbolic representation for the process ("take together" in Turn 49), she hinted at the structured figure S_B (Turn 49) and explained how the 3 groups of 4 in a row are related by counting up in units of fours (Turn 51). We interpret her dynamic notion ("take together") and the counting in units as her explanation of how a row with 3 groups of 4 can be dynamically modified into 1 group of 12. Within our analytic framework in Figure 4, we thereby inserted an additional edge between S_A and S_B that we had not found in the literature prior to our analysis.

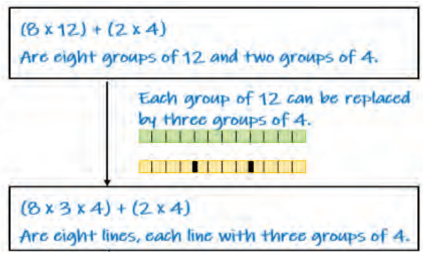
Figure 4: Additional edge for students' restructuring process from S_B to

The case of Lea and Linda resonates with other cases analysed in Tondorf & Prediger (submitted), in which we identified similar dynamic descriptions of the structured figures. With different technical or everyday phrases ("break apart", "draw a line"), students express emerging ideas of restructuring the figures. So, the extension of the analytic framework by a new edge (not anticipated from the literature) is not only a technical analytic detail, but an important finding of a forth characterization of equivalence that the first group of students invented. It can be seen as the graphical counterpart of the symbolic transformation equivalence. We call this dynamic relation between structured figures *restructuring equivalence* (represented by the double arrow between S_A and S_B).

Episode 2: The connection of transforming and restructuring expressions

Linda and Lea continued to build upon the dynamic connection between the structured figures S_A and S_B (that they invented in Episode 1): Episode 2, documented in the next transcript, took place in the beginning of their discussion of Task 2 (from Figure 2), when they tried to retrace the ideas of the fictitious student Zeynep and connected her idea to their previous idea from Turn 49/51:

154	Linda:	She [the fictitious student Zeynep in Task 2] always means, um, she writes always what is here [points at “Each group of 12 can be replaced by 3 groups of 4.”], above.
155	Lea:	Uh-huh, she describes the expressions, always.
156	Linda:	[After 10 sec break] Ok. [reads out loud] “Each group of 12 can be replaced by two groups of 4.” That works, because four, eight, twelve [gestures along the yellow line in the picture]. Just like it was done by Matt [hints at S_B]. And then, he has done also always, uhm uh-huh, groups of 12 always into groups of 4. Just like here, also here [hints at the green and yellow line]. She [referring to Zeynep] says, what you could do.
157	Lea:	Yes, every [group of] 12 can be, by three fours
158	Linda:	be replaced
159	Lea:	Can be replaced.
160	Linda:	Yes, and that works, actually
	…	
162	Linda:	And here [unclear where exactly she points at]. Three times – um – eight times three plus – no: eight times three times four. Two fours
163	Lea:	#plus two times four. This is here, that, though [shows the upper lines in S_A/S_B which only two groups of four squares are marked] two times four.
164	Linda:	Yes, she has only splitted them, here [hints at $8 \times 3 \times 4$], hasn’t she?



When Linda read the given explanation of a replacement process in Turn 156, she immediately connected it to the dynamic modification she conducted in Turn 51 in Episode 1 (when taking together the 3 groups of 4 into 12) and interpreted this modification backwards, as splitting the 12 into 3 groups of 4. Whereas the phrase “can be replaced” in the task is more static (i.e. not involving dynamic restructuring processes), her interpretation was expressed dynamically as “done […] groups of 12” (Turn 156). This approach of dynamic modification of the figures is then transferred to making sense of the symbolic transformations in the following lines: Examining the expression $8 \times 3 \times 4 + 2 \times 4$, Linda mixes language of symbolic representation (Turn 162 “eight times three time four”) and referring to the graphical representation of multiplication (Turn 162 “two fours”). We interpret this as an indication that she mentally connects the graphical and symbolical representations to one object. Lea strengthened this connection by articulating how both representations match each other (Turn 163). Apparently, Linda and (maybe also) Lea succeed to connect the known characterization of description equivalence with their emerging characterization of restructuring.

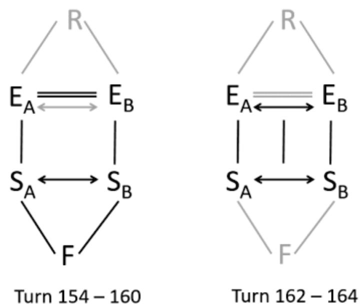


Figure 5: Students’ process of argumentation and connecting of restructuring and transformation

Based on this connection, Linda started a first attempt to explain the change within the symbolic expression (Turn 164). She addresseed the transformation of 8×12 into $8 \times 3 \times 4$ (Turn 164), again expressed with reference to the graphical representation (“splitted”), which refers to her underlying characterization of restructuring. Although her description of the process is not completely explicit, her emergent meaning-making of the transformation process is recognizable. Both girls constructed the meaning of transformations starting from the geometric characterization of description equivalence given in the task. However, they used a further bridging characterization for this transition, their dynamic characterization of restructuring equivalence.

Other students in our design experiments also used this newly emerging idea of restructuring as a bridge from description equivalence to transformation, so again, the identified phenomenon was not only specific to Linda and Lea. Like for Task 1, students used various ways of articulating their ideas to bridge the gap and explain the connection. Also in Task 2, they varied in the degree of explicitness when describing and explaining the transformation of expressions. However, they had in common to connect both processes, restructuring and transforming, in their meaning-making processes (for further cases see Tondorf & Prediger, submitted). The empirically identified phenomenon lead to another extension of our analytical and epistemological framework in Figure 5: the additional vertical edge between the double arrows links the restructuring from S_A to S_B and the transforming between E_A and E_B. This extension signifies the mentally constructed connection between both characterizations.

Discussion and conclusion

Summary and embedding of findings

Our research question, *how students' mental connections of description equivalence to transformation equivalence for expressions can be enhanced*, was addressed in a design research methodology (Gravemeijer & Cobb, 2006). Our preliminary design answer developed in the first design experiments was to present expressions $8 \times 12 + 2 \times 4$ and 26×4 that describe the same geometric figure structured in two different ways, and to engage the students not only in discussing similarities and differences of the structured figures (Task 1 in Figure 2) but also to stepwisely compare bridging expressions from $8 \times 12 + 2 \times 4$ via $8 \times 3 \times 4 + 2 \times 4$ and $24 \times 4 + 2 \times 4$ to 26×4.

The empirical analysis of the presented case (and further cases in Tondorf & Prediger, submitted) reveals that students can even go beyond the stepwise static comparisons initiated in the task (e.g. between the subexpressions 8×12 and $8 \times 3 \times 4$): several (but not all) students were found to adopt a dynamic approach to find the modifications made in the structured subfigures and explain how to restructure one subfigure into the other. They articulate these restructuring activities in dynamic wording ("take together", "break apart", "split"). Whereas a dynamic approach is usually only used for the transformation of symbolic expressions, we learned from these students that a dynamic approach can be used already earlier, when working with the graphical representations. Thereby, they constructed a fourth bridging characterization that we named *restructuring equivalence*, for which we see a huge potential to prepare the meaningful ground for transitions to later algebraic transformations.

The activity of restructuring figures is used by mathematicians in so-called proofs without words for proving the equivalence of symbolic expressions by restructuring figures, of course while acknowledging that an explicit articulation of connection is also required (Nelsen, 1993). Our study reveals a possibility for students to discover the underlying restructuring equivalence when working with geometric figures and we argue that this has important theoretical implications. We further develop this empirically grounded theorisation in Tondorf & Prediger (submitted).

In our data, we found a second important phenomenon that contributes to the theorisation: Students use the restructuring equivalence to make sense of transformation equivalence. When trying to make sense of the transformation of the expression, many students connected the symbolic and graphical representations intensely, e.g. by combining references to symbolic representation and graphical representation to describe the expression or figure (like Lea in Turn 162 of Episode 2). In these contexts, students built upon their dynamic approach (constructed for comparing structured figures) to explain the transformation steps between the bridging expressions. Thus, they made sense of transformation equivalence by explicitly connecting it to restructuring equivalence. The restructuring equivalence thereby takes the role of an important new bridging characterization in a dynamic approach (Figure 6) that prepares not only the shift from graphical to symbolic representations, but also the transition from the static approach of description equivalence to the dynamic approach of transformation equivalence. With the new bridging characterization, we can engage students in connecting multiple representations which has often been shown to be effective for learning (e.g. Mason et al., 1995). Moreover, we can engage them in connecting concepts and procedures, which is of major importance for the successful transition from arithmetic to algebra (Kieran, 2004).

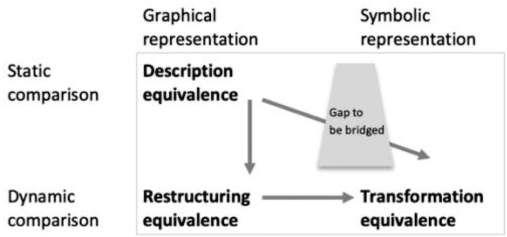


Figure 6: Restructuring equivalence as bridging characterization in early algebra

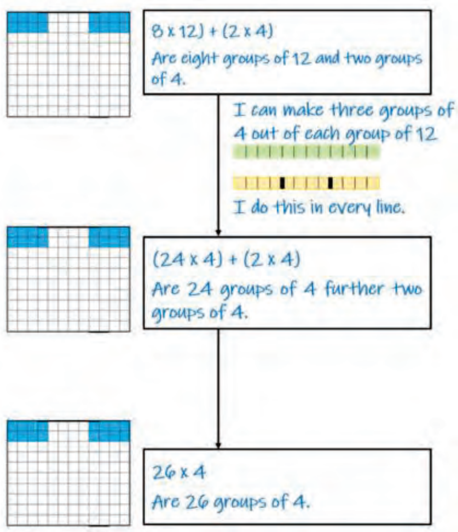


Figure 7: Revised Task with a more dynamic approach

Introducing bridging characterizations might be another instructional strategy (c) that can enrich strategy (a), making differences explicit, and (b), treating core ideas of algebraic concepts before the formal symbolism of variables is introduced, (see introduction).

Consequences for our instructional design

We used the empirical findings for revising the task in our second design experiment cycle: In order to enhance *all* students' mental connections of graphical and symbolic representations as identified for Lea and Linda, figures were added in each step of transformation and restructuring. Additionally, the explanation in the first step was phrased in a more dynamic way to support students' dynamic thinking (Figure 7). As the analysis of processes initiated by the refined Task 2 reveals (Tondorf & Prediger, submitted), these revisions allowed also students who could not adopt a dynamic approach in Task 1 to take over the offered dynamic language to describe and partially explain the transformation by restructuring. With these additional supports, the bridging characterization seems to be in reach for many students.

Methodological limitations and concluding remarks

The presented design experiments provide promising insights into how one may enhance students' understanding of transformation equivalence. However, there are still many gaps to fill. Firstly, we have not yet transferred the idea to other expressions and other transformation processes. Secondly, the presented task and the learning environment do not include the transition to generality which is crucial in early algebra as well. Thus, students' learning is not yet enhanced beyond the concrete case given in the task. There is need for further research to develop learning opportunities in which the contextual ideas can evolve into general concepts.

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TWG04: Geometry Teaching and Learning

Introduction to the papers of TWG04: geometry teaching and learning

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Keywords: Geometry education, tools, teacher training, geometry task.

Introduction

The TWG04 is concerned with research on geometry teaching and learning from kindergarten to university, including teacher education. In particular, it aims to contribute to the congress and research by discussing new and upcoming matters regarding the teaching and learning of geometry and, at the same time, continuing the discussions on the issues raised in the last CERME.

At virtual pre-CERME12 in 2021, two topics were discussed (*The specific aspects of mathematical activity in geometry* and *Teaching and learning geometry*), around questions such as "Which competencies (visual, reasoning, operational, and figural) should students/pupils have acquired/developed at the end of a specific school level?", "Which is the impact of online teaching and learning geometry?", and "What geometry our students should know when they move from primary to secondary to tertiary education?". Crucial points highlighted by several contributions are the relevance of language from a very early age, the use of material and digital tools, and the focus on visualization. At CERME12, in TWG04, we intended to continue discussing the two issues proposed for pre-CERME12 and relaunch a third issue, *Teacher education in geometry*.

Around 25 researchers from South Africa, the Middle East, East/Central/West Europe, the USA, and South America participated in TWG04 during the online sessions. The TWG04 work was stimulated by and organized around nine research papers and three posters. This group's exciting and relevant feature was its polyphony: a blend of experienced and young researchers discussing geometry teaching and learning. In the concluding survey, the participants indicated that participation in the TWG04 was an enriching experience.

Organization of the TWG04 at the congress

The papers and posters were divided into three topics, concerning their primary focus on *Teacher education in geometry*, *Use of tools in geometry teaching and learning*, and *Students*.

The fundamental elements of all the papers and posters were presented by the authors and discussed in groups. In particular, in each session, two papers/posters were introduced by a participant, and then a collective discussion followed. In turn, one participant was in charge of writing down the questions and comments arising during the discussion on an open document, as well as each participant could contribute by writing questions/comments on the same open document. At the end of the work on the set of papers/posters concerning a topic, the participants were divided into three subgroups to discuss related questions, which were formulated by the team of group co-leaders to guide them towards a synthesis of the previous collective discussion. Finally, the last session was dedicated to sharing the synthesis of all the discussions and small group works to prepare the final report of this working group.

Focus on teacher education in geometry

This topic includes contributions that have teacher training and reflections on teachers' beliefs on geometry as crucial elements.

In her paper, Kuzle investigates the value of geometry in mathematics instructions through a questionnaire proposed to a sample of 120 German in-service teachers. She analyses two items of that questionnaire: the first item aims to collect teachers' opinions on the role of geometry instruction in elementary school, and the second item asks to mark the most significant reasons for neglecting geometry education. A result of this research is that geometry is still insufficiently taught compared to other areas of mathematics, in particular arithmetic, even if positive changes are observable. In general, the predominance of arithmetic seems related to time-practical reasons rather than a disregard for geometry. Nevertheless, there seems to be a difference between prospective and in-service teachers, related to the insecurity that prospective teachers have with respect to geometry content knowledge. Two reasons could explain this difference: a lack of mathematics knowledge from school as learners and insufficient preparation at the university level as teachers. From this research emerges the question of knowing if there are some geometry topics teachers feel more insecure about teaching them. Further research could provide elements to plan teacher education from different perspectives.

Finally, the discussion on the paper confirmed the relevance of the investigated question "What role does geometry instruction have nowadays in school mathematics?" for all the teachers, and the relevance of taking into account how textbooks convey an image of geometry with respect to the other parts of mathematics curriculum (i.e., starting from the organization of chapters in the textbook).

Hausler and Kuzle's poster contributes to the discussion on teachers' images of geometry. By "Imagine you are an artist. A good friend asks you what geometry is. Draw a picture in which you explain to him or her what geometry is for you. Be creative in your ideas", prospective primary teachers attending a geometry course at the university were invited to perform two drawings, one at the beginning of the course and the other at the end. The two authors analyse the drawings to detect changes due to the attendance of the course taught by one of the authors.

Although research is carried out on teachers' beliefs (e.g., see other CERMEs groups), these two contributions emphasize that in teacher education, it is vital to consider not only primary teachers' beliefs about mathematics but also about particular fields of mathematics and its teaching, as in this case geometry. From this point of view, it might become fundamental to investigate which activities/tasks affect prospective teachers' attitudes towards geometry and its teaching and define criteria for identifying their impact on teachers.

The questions concerning the relation between teachers' content knowledge, their difficulty in teaching geometry, and didactical tasks are also investigated in other papers from different perspectives.

Ratnayake et al.'s paper focuses on secondary school prospective teachers and their difficulty in enacting geometry in the classroom. In the trend of research focusing on teachers' mathematical knowledge, their research project aims to offer a set of criteria for developing aspects of examples in tasks to analyse teachers' knowledge in geometry. In particular, with this work, the analysis of figures and their attributes in solving tasks is discussed.

Several questions emerged, such as which tasks can be considered emblematic for prospective teacher education in geometry. The papers presented by Bernabeu et al., Giménez et al., and Brunheira et al. contributed to the discussion on primary school teacher education. In particular, Bernabeu et al. and Giménez et al. propose classification tasks, the former on quadrilaterals and the latter on 3D figures. Brunheira et al. focus on the reasoning process in the context of a task on 3D figures. In all these papers, the features of the tasks allow highlighting the knowledge that students possess and the knowledge that teachers need to develop for geometric thinking.

Bernabeu et al. analyse quadrilateral classifications to bring out the definitions possessed by prospective primary teachers and to identify prospective teacher profiles. Although their research focuses on the mathematical knowledge of prospective teachers, the task analysed here can be interpreted between specialized content knowledge and pedagogical knowledge (Ball, Thames & Phelps, 2008). The researchers do not collect quadrilateral definitions and classifications with direct questions but as responses to a professional task given to prospective teachers. Specifically, they analyse teachers' knowledge through their answer to the request to anticipate children's answers for a classification task. The analysis shows that only around 30% of the teachers use hierarchical definitions involving the transitivity of inclusion relations (although with some non-economic definitions). The discussion of this paper also led to a comparison of quadrilateral classifications and definitions in the primary school of participants' countries, showing how classifications can be relevant – referring to Usinkin and Griffin's work (2008) – for comparing and examining curricula and textbooks (for instance, see the definition of rhomboid/parallelogram or trapezoid/trapezium).

Giménez et al. analyse the answers to the task of constructing polyhedra using material half-cubes and classifying them; they aim to identify which types of knowledge prospective teachers possess. As in Bernabeu et al.'s paper, they are interested in mathematical content knowledge; nevertheless, the authors also consider the meta-didactical knowledge on the value of the classification process in learning geometry. Indeed, the questions they ask students include important mathematical knowledge and invite discussion of the relevance of the processes activated in these tasks (for instance, "Why do you consider that when we study various types of figures, a fundamental process to work with is classification?"). The analysis highlights that even at the undergraduate level, the perceptual appearance dominates over certain conceptual aspects: most of the classification criteria are based on a description of the object with a low level of structuring. Despite having manipulative elements which could support them, prospective teachers are not capable of expressing mathematically well-formulated explanations. For instance, they use terms from plane geometry in their discourse on spatial geometry, and their arguments are very similar to those that pupils use. Thus, the question of what role the construction of tangible models plays in developing prospective teachers' geometrical thinking and knowledge for teaching geometry remains relevant.

In Brunheira et al.'s paper, 3D geometry is the context in which the researchers study prospective (primary and secondary) teachers' knowledge of reasoning process within the framework of specialized mathematics knowledge for teaching. The task for prospective teachers proposed in the paper consists of two parts: in the first part, the prospective teachers have to identify the reasoning processes involved in the task which will be proposed to students; in the second part, prospective teachers are asked to discuss on students' reasoning that emerging in an excerpt of a dialogue between students and teacher. The analysis shows that this kind of task and context promotes the knowledge of reasoning processes.

The discussion which developed in the working group proposed additional issues, such as:

- It is significant to bring together the mathematical properties and visualization processes involved in solving a geometric problem.
- There is a need for more collaboration between researchers and teachers in the research field of geometry education.
- The role embodied context plays in geometry teaching and learning, particularly the use of tools, types of tasks, and activities.
- Research could focus on the learning trajectory between primary and secondary education and its relation to the education of prospective teachers.

Finally, two suggestions emerged for further research contributions. The first suggestion is the following research questions: "Do teacher training programs in different countries have geometry/ teaching geometry courses (and to what extent)?" and "How can this choice influence prospective teachers' efficiency/preparation for geometry instruction?". The second suggestion concerns the development of research on secondary teacher education in geometry in connection with primary teacher education.

As pointed out in the discussion of some of the papers in this section, tasks on 3D geometry seem to be a good context for exploring geometrical thinking. The papers in the next section also focus on this content using DGS and material (tangible) models.

Focus on the use of tools in geometry teaching and learning

The group discussed this topic based on three contributions: one paper concerns using the 3D environment of GeoGebra DGS, and the other papers report two experiments with material (tangible) models of 3D figures made with different tools (3D pens and construction kits). All these papers refer to secondary school. They are based on different theoretical frameworks (i.e., the theory of semiotic mediation and embodied learning).

In Sua et al.'s paper, 3D DGE mediates the relationships between 2D space and 3D space. The task described in the paper involves geometrical constructions and their meaning in plane geometry and pays attention to the meaning of correct constructions (corresponding to several solutions) for constructions in 3D space. The dimension of space is the only variable of the task: the request for constructing triangles (isosceles and equilateral ones) is the same in the two spaces, 2D and 3D, starting from the same kind of initial object (the side of the triangles). While adapting the 2D procedure, the students consider the new dimension of the space passing from circles to spheres in their constructions, but they do not control the intersections between the drawn spheres as they do with the circles. On the other hand, the syntax of some commands in 3D DGE does not match 2D procedures; the transition between environments also requires a different and new conceptualization of 2D objects by students. This paper opens several research questions on: 3D visualization mediated by the DGE, cognitive and epistemological control in the transition from 2D to 3D spaces, awareness of this transition for students (it seems to observe a misattributing of 2D properties to 3D objects). Finally, in general, which is the challenge of this kind of task for secondary students?

The two other papers investigate 3D geometry using 3D pens and construction kits.

Palatnik and Abrahamson develop an enactivist argument for learning 3D geometry by constructing tangible models in their paper. They describe the gap between the geometry taught at school, mainly 2D geometry with 2D tools, and the 3D space where students live and experience 3D objects. From an embodied perspective in mathematics education, the authors investigate students' cognitive processes in 3D tasks with tools for constructing 3D models of solids (i.e., cube and tetrahedron) in small and medium sizes (hand-held and human-scale models) (Herbst, Fujita, Halverscheid & Weiss, 2017). These tasks foster the transition between 2D and 3D figures and models and, in general, between 2D and 3D space differently from that analysed in Sua et al.'s paper. Students' physical actions led to shifts in perceptual-motor attention, which involve a refinement of geometric reasoning.

Rosenski et al.'s research focuses on emotional aspects and the influence of affect in 3D problem solving using 3D pens. Their results show the relevance of those aspects in learning geometry with tools and the potential of the 3D pen for engaging students in spatial geometry tasks. Compared to the construction kits, the 3D pen is an emerging technology for mathematics education. Subsequent group work discussed the role of the novelty of material and digital tools in student engagement, particularly when the novelty of a tool is neutralized by regular use in the classroom.

The discussion in the group raised several questions, such as: "What is the added value of tangible and/or digital/virtual manipulatives in geometry instruction?", "Is it important to start a teaching experiment from material manipulatives with students?", "What are the affordances and constraints of using material manipulatives?". For students in different age groups, we asked: "What constitutes learning of geometry when they use physical models?", "When is it useful to propose material tools, and when is it necessary to let students work only with their mental images?", "What are principles of task design with manipulatives?", "What makes work with manipulatives (material and virtual) exciting for students (except the novelty of medium)?". Furthermore, "Why is 3D DSE especially important as a tool of semiotic mediation?". These questions remain open for research.

Our group agreed on the importance of using material tools to teach geometry in elementary school, high school, and even at university level. Nevertheless, the idea that manipulatives and digital tools are age-sensitive emerged: some tools (or some ways of using them) are adequate for younger or older students; so, while moving between different manipulatives or a manipulative and symbolic forms, the meanings that need to be constructed have to be in focus. Material and digital tools offer a combination of representations, which is very important in geometry teaching and learning (Herbst, Fujita, Halverscheid & Weiss, 2017). In some cases, as with 3D pens, boundaries between material and digital tools, novel and traditional mediums, 2D and 3D, are constantly crossed; thus, it becomes interesting to study them. Although the novelty of a tool is a factor of motivation and reinforcement in the learning process, it seems that students' motivation in activities with tools cannot be ascribed to their novelty alone; the creative and collaborative nature of activities could maintain student's interest because they made something together, they have to communicate, and so on.

The discussion proposed additional issues, such as:

- Taking into account instrumental genesis for students and teachers and the importance of students' experience with tools.
- Identifying the tool's novelty effect on students' engagement and motivation.
- Considering the change of didactical contract when a new tool is introduced in the class for planning activities and identifying students' engagement.
- Emphasizing connections between geometry education and its applications in the real world (e.g., architecture, mechanical engineering, and art).

Focus on students

The discussion on the third topic was based on three papers concerning students in grades 5, 6, and 9. These papers are developed within different theoretical frameworks (e.g., embodied cognition, realistic geometry, van Hiele Levels, PISA framework for competences) and focus respectively on abstraction in geometry (Boonstra), problem solving requested by online DGS tasks in the time of COVID (Edamus et al.), and assessment of geometrical competencies (Bočková et al.).

The contributions of Boonstra and Edamus et al. discuss the same mathematical content, the geometric reflection, and pay attention to static and dynamic aspects concerning the reflection in two different environments. Grounded in the realistic mathematics education approach, Boonstra focuses on the processes activated by 5-6-grade pupils facing two problems on reflection through the use of a mirror ("Move the chess towers in front and behind the see-through mirror and let them meet in the mirror"; "Place a see-through mirror in the correct position on the line") to investigate how embodied activities support abstraction. This paper suggests discussing which characteristics and properties the tasks should have to support abstraction and which is the role of the tool (e.g., tasks with a mirror and tasks without a mirror) within the embodied research design.

Edamus et al. analyse the resolution of a task on the composition of two reflections with parallel axes to study students' understanding of reflection from a functional point of view. In particular, after constructing the symmetrical figures, two students have to collaborate online to solve the task of moving axes and describing/explaining what happens to the symmetrical figures. In the discussion on students' processes, the questions on why students must move between static and dynamic aspects for understanding reflection and how this helps them in their learning process are asked. The setting of the experiment is interesting for fostering communication between the students, the emergence and development of mathematical language: in pair work, one student gets access to the online DGS file and shares his screen; only this student can manipulate the DGS figure while the other one is forced to verbalize his ideas and give instructions.

The third contribution brought in the discussion on the assessment of geometrical competencies. The heart of the matter is which strategies and methods researchers and teachers have at their disposal to assess students’ competencies in geometry. The assessment involves choosing what is to be evaluated, a framework for geometry learning and competencies, and a format in which questions are asked. The crucial question arises as to a suitable framework for constructing a good test to measure geometric thinking.

In the discussion, additional issues emerged, such as:

- Different types of tasks and content (i.e., the inclusion of non-Euclidean geometry; topology; task with manipulatives) can surface different aspects of students’ geometric thinking, reasoning, behaviour, argumentation, …
- Engaging students in collective visualization is challenging in the absence of physical manipulatives.

Finally, other questions emerged for further research contributions, for instance, concerning social aspects of students learning of geometry (i.e., gender differences in participation)

Future Directions for TWG04

The discussions in this TWG were vibrant and, as a good discussion, posed more questions than answers. In this paper, we leave some other suggestions for the next TWG on geometry teaching and learning.

In the frame of the geometry education research community considering the possible international study on the geometry content knowledge necessary for (primary/secondary) teachers:

- What research questions should be considered?
- Which differences and similarities of national curricula should be taken into account?
- Which theoretical perspectives on geometry teaching and learning should be taken into account?
- What tools does the current research provide for teacher education in geometry?

In the paper and poster presentations, we observed three essential processes involved in students’ work when they engage with a geometric problem/task: visualization, argumentation, and the transitions between different representations (e.g., 3D geometrical objects and their 2D representation). As researchers and/or teachers, how can we become aware of/observe when our students engage in those different processes? Finally, how can we foster/promote the activation of those processes?

We hope that this TWG on geometry teaching and learning continues to advance the research on geometry education, paying attention to the changes in scholarly contexts and the new educational needs that arise.

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Preservice primary teachers’ curricular reasoning when anticipating primary students’ answers to geometrical figure classification tasks

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This study aims to characterize the preservice primary teachers’ curriculum reasoning when anticipating primary students’ answers to figure classification tasks. Anticipating answers to figure classification tasks involves considering the relationship between the definition and classification and the characteristics of geometrical thinking progression. We analyzed the definitions of geometric figures posed by twenty-eight preservice primary teachers to quadrilateral classification tasks. The results show three preservice primary teachers' profiles taking into account the specialization/generalization of the quadrilateral definitions provided and the transitivity of inclusion relations. These results indicate features of curricular reasoning in anticipating tasks as part of preservice primary teachers’ curricular noticing.

Keywords: curricular noticing, anticipating students’ answers, geometric reasoning, hierarchical classification, definition of geometric figures.

Introduction

Previous researches (Jones & Tzekaki, 2016; Sinclair et al., 2016) argue that it is important for teachers to understand the elements that condition understanding of inclusive relationships to plan geometry instruction. This generates the need to know how preservice primary teachers (PPT) reason when anticipating student answers as an aspect of *curricular noticing* (Dietiker et al., 2018). This construct describes a set of professional practices that allow teachers or preservice teachers to recognize, interpret, and generate learning opportunities from teaching-learning situations when they interact with curricular materials. In these situations, they had to carry out the *curricular reasoning* (Breyfogle et al., 2010), which has been defined as the thought processes in which teachers or preservice teachers engage when working with curricular documents or materials to plan, implement and reflect on instruction.

Theoretical Background and Framework

A key aspect in the progression of geometric thinking, whatever the educational level, is understanding a geometric figure as an example of a class of figures (hierarchical classification). The understanding of hierarchical classifications is linked to the definition of the figure. These hierarchical classifications imply that, for example, a figure (square) in a class A (rectangle) has all the properties of the class B (parallelograms) that includes it. There are two processes for defining these figures. On the one hand, the *specialization* process, in which to get an example of A, attributes have been added to the definition of B (more general class); and, on the other hand, the *generalization* process, by suppressing attributes from the definition of an example of A to create a more general class, like B. The process of including an example within a more general class, and so on (A is an example of B, B is an example of C, so A is an example of C) is known as *transitivity* (de Villiers, 1994) (Figure 1).

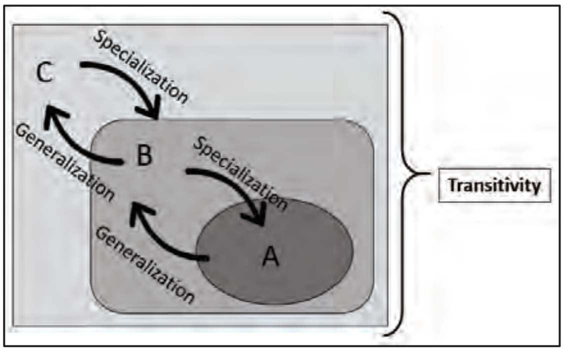

Figure 1: The specialization and generalization of the definitions and the transitivity of the inclusion relations

However, understanding the *specialization* and the *transitivity* in the hierarchical relationships between figures is difficult (Fujita, 2012; Fujita & Jones, 2007). Previous research (Brunheira & da Ponte, 2019) showed that prospective teachers, after a teacher education experiment that includes this kind of classifications, had a limited figural concept, and therefore, difficulties in the hierarchical classifications using the transitivity. These difficulties may be a problem for prospective teacher when they will have to plan the teaching/instruction.

For planning the teaching of hierarchical relationships of geometric figures in primary education, PPT should anticipate the thinking processes that the tasks may promote in students considering the cognitive abilities of the students (both mathematical and didactic contents). For example, PPT should consider, as key aspects in learning, the process of *specialization* of the definition (adding properties to a general class) as well as the *transitivity* of the inclusion relations (A→B and B→C, then A→C) (de Villiers, 1994). The way in which PPT use these two features of inclusion relations between geometric figures could provide information on how they reason about the hierarchical classification tasks. In this sense, anticipating students' answers to hierarchical classification tasks in a lesson is a skill linked to *curricular reasoning* (Breyfogle et al., 2010) as an aspect of *curricular noticing* (Dietiker et al., 2018). Thus, this study aims to find out:

- How do PPT reason when anticipating correct answers in hierarchical classification tasks as an aspect of curricular noticing?

Method

This study involved 28 PPT who participated in a learning environment (4 sessions of 2 hours) aimed at developing *curricular noticing*. The content of the learning environment includes the characteristics of geometric thinking development in primary education and the types of tasks for enhancing the students' geometric thinking along the lesson plan. The PPT, during the learning environment, analysed primary school students' answers, anticipated answers from primary school students to the tasks of classifying geometric figures and shapes, and analysed teaching activities to support understanding of inclusion relationships (Lehrer et al., 2014). These actions related with the *curricular noticing* are what the PPT would put into practice during their teaching practices at school and in their future as teachers. At the end of the learning environment, PPT solved a task on anticipating primary school students' answers to an activity in inclusive relations between geometric figures (Usiskin & Griffin, 2008). In this study, we analysed the PPT's answers given to this last task through a qualitative analysis (Figure 1).

The task

The task is contextualised in the planning of a lesson with the learning objective: *inclusive relations between geometric objects*, for pupils aged 10-12 years. The focus was the relation between definitions of geometric objects and classification. Three versions of the same task were proposed in the domain of quadrilaterals (Figure 2). The task highlights the *curricular reasoning* processes (Breyfogle et al., 2010) of PPT by anticipating possible student answers considering mathematical knowledge and the development of geometric thinking in primary education (Sánchez-Matamoros et al., 2019). PPT should take into account the process of specialization (add conditions to generate subclasses) and the transitivity of the inclusion relationships of quadrilaterals. For example, in version 1b task, the rhombus must have *four equal sides* to be a particular example of the rhomboid class (*quadrilateral with two pairs of opposite sides parallel*) (Table 1).

Some of these quadrilaterals often appear in the primary school curriculum with partitive (non-inclusive) definitions, such as defining trapezium as a quadrilateral with only two parallel sides. Thus, this task has a high cognitive demand for PPT, as it requires modifying and overcoming the definitions associated with prototypical images and partitive definitions.

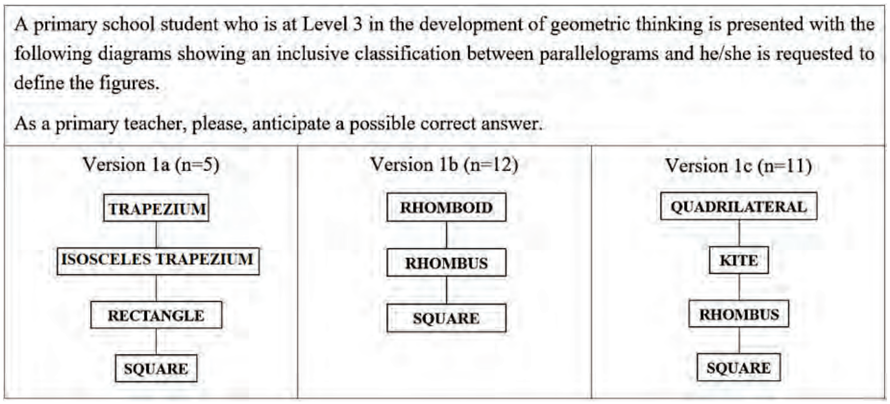

Figure 2: Task of anticipating student answer as part of teaching planning

Version 1a	Version 1b	Version 1c
Trapezium: quadrilateral with two parallel sides.	Rhomboid: quadrilateral with two pairs of opposite sides parallel	Quadrilateral: polygon with four sides

Isosceles trapezium: trapezium with congruent diagonals.	Rhombus: rhomboid with congruent sides.	Kite: quadrilateral with perpendicular diagonals and two pairs of adjacent congruent sides.
Rectangle: isosceles trapezium with congruent angles.	Square: rhombus with congruent angles.	Rhombus: kite with congruent sides.
Square: rectangle with congruent sides.		Square: rhombus with equal angles.

Table 1: Possible definitions and inclusive relationships between quadrilaterals

Analysis

For the analysis of the task answers, we considered three criteria:

- The economy of the definition, i.e., if the definition included the minimum and sufficient conditions or redundant conditions. In addition, we considered if the definitions derived from the perceptual references of prototypical examples (*prototypical judgement, type 1*), from the attributes of prototypical examples (*prototypical judgement, type 2*) or using attributes relevant to the concept (*analytical features, type 3 judgement*) (Hershkowitz, 1990).
- The *specialization* of the definitions, i.e., to what extent it was considered that adding attributes to the general class generate a subclass. For example, in task 1c, to define a rhombus, it is sufficient to add having equal sides to the definition of a kite (see Table 1).
- The *transitivity* of the definitions (A→B and B→C then A→C). For example, that the definitions given allow us to see that, if a square is a rhombus, and a rhombus is a rhomboid, then the square is a rhomboid, but not inversely (there are rhomboids that are not squares) (asymmetric relations between the inclusion relations between geometric figures).

Regarding of procedure followed, we analysed the answers following the above criteria (Figure 3) and then compared the answers with each other, identifying similarities and differences, generating patterns of answers. These patterns of answers were considered as profiles PPT's anticipatory competence of anticipating primary school students' answers to a task on figure classification as a part of the acquisition of the curricular noticing. One student out of 28 provided an answer that was inconsistent with the required task, so it was not considered in characterising the profiles.

Preservice teacher 17 (PPT17)'s answer	**Pre-analytical comments**
Rhomboid: Parallelogram with interior angles less than 180°, and with 4 sides parallel two by two. *Rhombus: Rhomboid with 4 equal sides and perpendicular diagonals* *Square: Rhombus with four right angles*	- *Uneconomical* definitions (non-minimal) (e.g., if you say the rhomboid is a parallelogram, you do not need to say 4 parallel sides two by two) - There is *transitivity* (the square is a rhombus; the rhombus is a rhomboid). - Use *specialization* by adding attributes to the more general class it belongs to.

Figure 3: Example of analysis of the PPT17's answer

Results

The analysis has generated three PPT's profiles.

Profile 1. No use of hierarchical definitions nor of the transitivity process (n=7)

Preservice primary teachers in this profile do not recognise the process of generating subclasses as they do not take into account any case the specialization of definitions, and they fail to consider the transitivity between inclusion relations. In this profile, PPT seem to base their definitions on prototypical images, generating *prototypical judgements of type 1*. For example, the preservice teacher PPT10 in task 1b, Rhomboid→Rhombus→Square, indicated,

- *Rhomboid: A geometric figure with four sides that do not form right angles, of which the opposite sides are equal, and the adjacent sides are unequal.*
- *Rhombus: Geometric figure with four equal sides that do not form right angles.*
- *Square: Figure having four equal sides that form four right angles.*

This PPT use partitive definitions and removes the possibility of generating subclasses from the more general classes by indicating that: the rhomboid has the *adjacent sides unequal*, excluding the rhombus as a subclass of the rhomboid; the rhombus *has no right angles*, eliminating the possibility of squares as a subclass of the rhombus; and the rhomboid *has no right angles*, excluding squares as a subclass and thus preventing *transitivity* between inclusion relations.

Profile 2. Partial use of hierarchical definitions, but not transitivity process (n=12)

PPT in this profile show a partial understanding of the specialization process. They define the figures considering only some of the inclusion conditions indicated in the task. Furthermore, they show a lack of understanding of the transitivity of inclusion relations by defining some figures using attributes of the prototypical figures, but not all the necessary attributes or using some definitions incorrectly. In this profile, PPT seem to use attributes of the prototypical examples as reference, generating *prototypical judgements of type 2*, as they use attributes of the identified figures. For example, the preservice teacher PPT2 in task 1a, Trapezium→Isosceles Trapezium→Rectangle→Square, indicated

> *As the student is at level 3, he can analyse the properties of geometric figures giving the sufficient and necessary indications, relating the properties to each other:*
>
> - *The trapezium is a quadrilateral with only two parallel sides.*
> - *The Isosceles trapezium is a quadrilateral with two equal angles two by two.*
> - *The rectangle is a quadrilateral with all right angles.*
> - *The square is a quadrilateral with all sides equal and all right angles.*

By defining trapezium as a quadrilateral with *only two parallel sides*, it excludes the possibility of considering the rest of the parallelograms (rectangle and square) as subclasses, and therefore does not take transitivity into account. By defining isosceles trapezium as a *quadrilateral with [two] equal angles two by two* it seems to be using some properties of the prototypical figure, but not all of them.

Finally, the definitions and the relation of inclusion between squares and rectangles are adequately indicated, reflecting judgements supported by the attributes of the prototypical figures and evidencing the characteristic of *specialization* between the rectangle and the square.

Profile 3. Use of hierarchical definitions involving the transitivity of inclusion relations (although with some non-economical definitions) (n=8)

In this profile, PPT use the specialization of definitions and the transitivity of inclusion relations. They define geometric figures considering, in all cases, the process of adding attributes to a class to generate a subclass (*specialization*), which entails the *transitivity* of inclusion relations. Furthermore, in most cases, they define geometric figures using the relevant attributes of the concepts (*analytical features*, *type 3 judgement*), although in some cases they provide redundant data. For example, the participant PPT17 in task 1b, Rhomboid→Rhombus→Square, explains,

- *Rhomboid: Parallelogram with interior angles less than 180º, and* <u>*with 4 sides parallel two by two*</u>.
- *Rhombus: rhomboid with four equal sides and* <u>*perpendicular diagonals*</u>.
- *Square: rhombus with four right angles.*

This PPT adds redundant information in the definition of rhomboid: *parallelogram with angles less than 180º and parallel sides two by two*. However, he explicitly uses the process of *specialization* of definitions (*a rhombus is a rhomboid that ...; a square is a rhombus that* ...), showing the *transitivity* of inclusion relations.

Discussion

The aim of this study was to characterise the PPT's curricular reasoning when anticipating correct answers in tasks involving inclusive relations between geometric figures as an aspect of *curricular noticing*. The results show three profiles of the PPT's curricular reasoning in the classification tasks, considering how they considered the *specialization* of definitions and the *transitivity* of inclusion relations. The three profiles described in the result section show two ideas. Firstly, about the preservice primary teachers' understanding of the relationship between defining and classifying, and secondly, about the characteristics of the PPT's curricular reasoning regarding the student's geometrical thinking in anticipating tasks.

Preservice primary teachers' understanding of the relationship between defining and classifying

The *specialization* of definitions and *transitivity* of inclusion relations are key to understanding the relationship between the processes of defining and classifying. On the one hand, the process of *specialization* allows us to understand how adding properties to figures generates a subclass of a class. Furthermore, considering the definitions of geometric figures to reflect two (or more) inclusion relations is a key aspect that must be understood by PPT when they think about the cognitive demand of tasks for the primary students. On the other hand, the *transitivity* of inclusion relations is linked to the process of *generalization* of definitions, which allows to include a figure (A) within a class (B) and, to include this second class, within another more general class (C) (Figure 1). Understanding these relationships between figures, dismissing misconceptions generated by prototypical examples (profile 3) and using *Analytical Features* (type 3) (Hershkowitz, 1990) seems to be a necessary condition in PPT's understanding of the potential of teaching activities in primary education. Thus, PPT using in their definitions some references to prototypical examples or attributes of figures to express the inclusive relationship between geometric figures (*prototypical judgement 1 and 2)* (Hershkowitz, 1990), seem have limited ability to anticipate primary school students' answers to the classification tasks of geometric figures.

Characteristics of preservice primary teachers' curricular reasoning in anticipating tasks

The PPT's curricular reasoning in anticipatory tasks indicates ways of thinking about teaching activities considering the students' geometrical thinking. In this study, the ideas of *specialization/generalization* of definitions and *transitivity* of inclusion relations have been considered as key aspects when PPT anticipate student answers in the teaching activities focused on the relationships between defining and classifying. In this way, anticipating student answers has shown features (profiles) of the PPT's curricular reasoning as part of *curricular noticing* (Amador et al., 2017). Furthermore, these results show that if a PPT uses correctly all the process to anticipate a hierarchical definition from a classification (specialization and transitivity), he/she has acquired one of the aspects of the curricular noticing.

For example, the inclusion relationship between the Rhomboid→Rhombus→Square has been the one in which the least students have applied the process of *specialization*, generating non-inclusive definitions and using the attributes of the prototypical figures, which tend to be exclusive. Possibly, using prototypical examples of the concept of quadrilateral makes it difficult for PPT to use the relevant attributes of different quadrilaterals to establish inclusive relationships between them. Moreover, in the definitions in which excluding attributes have been used the most, so that neither *specialization* of the definition nor *transitivity* of the inclusive relations can occur, it has been in the quadrilaterals in the first inclusion relation of the three graphs (e.g., in task 1b Rhomboid→Rhombus). We think that this may be due to the difficulty involved in the process of *generalization* (eliminating certain properties or substituting some for more general ones) when the concepts have already been acquired (*a posteriori classification*) (de Villiers, 1994) (Figure 3), to include the rest of the quadrilaterals that make up the inclusive relation within this more general class.

Acknowledgment

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Investigation the Geometric Thinking of Nine Graders´

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Keywords: van Hiele theory, geometric thinking, nine–grade pupils

Geometry is one of the most important topics of mathematics education. Even though most pupils just see geometry as the study of objects, points, shapes, geometry, is also applied in other subjects such as engineering, architecture or music. For these reasons, a very important teachers´ role is to arouse pupils´ interest in geometry and create a positive relationship with it. Sometimes it is a challenging mission because as we can observe in recent years, students in Slovakia achieve a relatively low level of geometric knowledge (Bočková, Pavlovičová, Čeretková, 2020). The pupils´ problems with geometry can be caused by the fact that pupils are at a very low level of geometric thinking or mathematics competencies in geometry. For this reason, it is very important to use van Hiele theory at all levels of education.

Dutch mathematicians Pierre van Hiele and Dina van Hiele – Geldof tried to understand and eliminate pupils' problems in geometry. Based on their observations, they described the model of geometric thinking in 1957. Van Hiele theory consists of five hierarchical and sequence levels (visualization, analysis, informal deduction, formal deduction, rigor) (Usinkin, 1982; Van Hiele, 1986). Today, van Hiele model of geometric thinking forms the basis of the content of education in various countries, such as the United States, Russia, Netherlands and Taiwan.

The main aim of this poster is to present the research about geometric thinking and mathematics competencies in the geometry of nine-grade pupils in Slovakia, who finished the lower middle school level of education. The research is focused on determining the level of geometric thinking of pupils The aim of the research is a quantitative evaluation of the solution of geometric problems, too. We also would like to find out the connection between the correct solutions of the geometric problem at different levels of mathematics competencies in geometry and the level of geometric thinking according to Van Hiele theory.

The levels of pupils´ geometric thinking were determined by using the van Hiele geometric test. The test was applied with the consent of Professor Zelman Usiskin. We suggested the valid and reliable Test of mathematical competencies in geometry, too. In the test, we used PISA division of mathematical competencies into three competency clusters: reproduction, connection and reflection. The test contains three tasks for each of the three levels of mathematical competencies. The research was realized in May – June 2021. The research sample consisted of 760 nine–grade pupils, who solved the van Hiele geometric test and Test of mathematical competencies in geometry.

According to Mc Anelly (2011), who has stated the separated levels of geometric thinking in the age category, pupils in the second stage of elementary schools should be at the level of analysis or informal deduction. Our research shows that 50.5 % of pupils achieved the required level of analysis or a higher level of geometric thinking (21.3 % - analysis level, 24.2 % - informal deduction level, 5 % - formal deduction level). The surprising finding is that 7.4 % of pupils did not even reach the level of visualization and 27.5 % of pupils achieved the visualization level. 14.6 % of pupils did not assign any level of geometric thinking. We compared same-age pupils from other countries with similar research, which was carried out by Usiskin (1982), Haviger, Vojkůvková (2013), Adelabu, Makgato, Ramaligela (2019) and Idris (2009). Slovak students have a similar or higher level of geometric thinking.

From the solution of geometric problems at different levels of mathematical competencies, it follows that pupils have a lot of misconceptions of basic knowledge in geometry. They cannot identify or use the formula for geometric shapes. They do not determine the length of the sides and the length of the altitude of geometric shapes from the picture, they have problems read from the image. Pupils also use the Pythagorean theorem in the general triangle and they have an incorrect conception of a straight angle.

We used statistical implicative analysis methods, specifically C.H.I.C. statistical software (Classification Hiérarchique Implicative et Cohésitive) to compare and reveal the connections between the level of mathematics competencies with the attained level of geometric thinking according to Van Hiele theory. An important finding is that we confirmed the interconnection between the solutions of individual tasks of both tests, with the achieved level of geometric thinking as well as with the correct solutions of tasks at a certain level of competence.

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Embodied abstraction in geometry learning in grades 5 to 6

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Keywords: *Embodied design, design research, geometry, abstraction*

To explain, interpret and distinguish the large diversity of visual information and spatial phenomena in today's world, children need to be able to grasp and understand the underlying mathematical and geometrical structures. Relating these real-life spatial phenomena with their underlying geometrical characteristics can be regarded as abstraction, one of the core aspects of mathematical thinking according to Drijvers (2015). How to foster the process of abstraction in geometry in primary school is the central question of my research. Practical and theoretical perspectives on geometry and abstraction, together with the theoretical lens of embodied cognition, led to design criteria for activities fostering abstraction in geometry. Accordingly, activities on mirroring for students from grade 5 to 6 have been designed, based on an embodied abstraction approach.

Theoretical background and research question

Geometry education in the Netherlands is characterized by a realistic approach, described by Freudenthal (1973) as grasping space, "the space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it" (p. 403). It encompasses problems, tasks and activities experienced in our direct surrounding, with the aim of learning about mathematical objects and concepts.

Based on the definition of abstraction by Dreyfus (2014), as the attempt to reach an understanding of the structure of a mathematical concept, and the notion of reflective abstraction by Piaget (1985) the process of abstraction in geometry can be regarded as the discovery and formalizing of new (to the learner) mathematic structures in spatial orientation through a process of reflection on experiences and explorations.

The theoretical lens of embodied cognition is used in this study for an embodied approach of abstraction in geometry. Embodied cognition theory emphasizes that all cognition is rooted in bodily experience; in the interaction of the body with the physical environment (Abrahamson & Lindgren, 2014). The definition of abstraction in geometry through the lens of embodied cognition that is used for the design reads: the process of describing, explaining, reflecting on, and structuring of action in the experienced world using mathematical artifacts. The design principles for the learning environment have their origin in action-based embodied design, one of the design genres introduced by Abrahamson et al. (2020). In action-based design: "participants … tackle motor-control problems: they are assigned the task of performing a technologically mediated manipulation of material or virtual objects, in an attempt to achieve a specified goal state." (p. 5). The research question we try to answer with this design is: how can embodied activities in geometry contribute to the process of abstraction by children in grades 5-6.

Methods

The study can be characterized as a design study (Bakker, 2018). During a series of design cycles, a learning environment on embodied tasks on mirroring is developed and tested. Criteria for the environment are outlined in a design table and based on the definition of the process of abstraction (see above) and elements of embodied design: enactment, expected verbalizing, degrees of freedom, and attentional anchors. The tasks focus on the geometrical properties of mirroring: perpendicularity, equidistance and angle of incidence begin equal to the angle of reflection. This is done using a see-through mirror, chess pieces and a laser pointer. The designs are tested during task-based interviews with 8 individual children from grade 5 and 6. The interviews, which took 30 minutes on average, are video-taped. Data are in the process of analysis in a bottom-up theory-guided coding process.

Results

During acting, all students discovered the correct position (equal distance to the mirror and on a straight line perpendicular to the mirror) of the chess pieces on either side of the see-through mirror. When explaining their solutions, they showed an understanding of equal distances of object and reflection image. Several students were able to write a clear instruction, including a drawing, of how to position the chess pieces, thus showing promising steps in the process of abstraction. Although they showed an understanding of the need of perpendicularity, they still struggled with the abstraction hereof.

Conclusion

This embodied design shows mixed, but also promising, results in fostering abstraction in the designed tasks on mirroring. More work is needed to further investigate embodied design in this and other areas of geometry.

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Developing prospective primary teachers' knowledge of mathematical reasoning processes in the context of a geometry task

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This paper aims to discuss the prospective primary teachers' knowledge of reasoning processes, namely the way they relate several reasoning processes, when solving a didactical task involving geometry. Data were collected by audio and video records of lessons, participant observation and the collection of written records of the prospective teachers. The results show how a group of prospective primary teachers may reach a high level of knowledge when involved in didactical tasks that are supported by relevant mathematical tasks and real classroom episodes, while working collaboratively. In particular, geometry tasks that involve spatial structuring favor the emergence of different reasoning processes and its relationships.

Keywords: Mathematical reasoning processes, Prospective primary teachers, Geometry, Spatial structuring

Introduction.

Teacher education should give special attention to mathematical reasoning, considering both the ability to reason, and knowledge about the reasoning processes (Stylianides & Stylianides, 2006). In particular, developing mathematical reasoning processes in the domain of geometry, in early years classrooms, implies specifically developing visualisation and spatial reasoning (Moss et al., 2015) since the central processes of generalising and justifying (Rodrigues et al., 2021), in this case, are founded on the geometric properties and on the objects´ structure.

This paper is part of the *Mathematical Reasoning and Teacher Education* (REASON) project, which aims to study the mathematical and didactical knowledge teachers need to carry out a practice that promotes pupils' mathematical reasoning and to study the ways to foster its development in prospective and practicing teachers of primary, middle and secondary school. In our communication we intend to discuss the knowledge of reasoning processes of a group of prospective primary teachers, when solving a task involving geometry, namely the way they relate several reasoning processes. This task was the fifth one implemented in a teacher education experiment carried out in 2019/20.

Conceptual framework

Reasoning in geometry

Reasoning geometrically about a spatial entity (object, diagram or concept) implies constituting an adequate mental model, that is, one that captures its relevant spatial structure and its geometric properties. Battista et al. (2018) state that "spatial and geometric structuring are types of spatial and geometric reasoning, respectively, that play vital roles in the construction of appropriate mental models for geometric reasoning" (p. 202). For spatial reasoning to adequately support geometric reasoning, these mental models must incorporate operational knowledge of relevant geometric properties and concepts, using mental models that integrate geometric properties into their structure and operation (Battista, 2007).

Lannin et al. (2011) distinguish two aspects in the generalisation: (i) identify common elements in different cases; (ii) extend reasoning beyond the domain for which common elements were initially identified, that is, thinking about a relationship, idea, representation, rule, pattern or other mathematical property considering it in a broader domain. For example, when, at the beginning of school, a student identifies squares as the figures that have four equal sides, he is making a generalisation that is false, but it is a generalisation. For these authors, the process of justifying consists of building a logical sequence of statements, each one relying on established knowledge in order to reach a conclusion. Thus, constructing a valid justification for a generalisation is not easy as it has to be verified that the generalisation is true for all cases in the domain, resorting to valid implicit relations. A valid justification must explain why by offering a view of the underlying relationships that exist in all cases.

Thus, we consider that the process of generalising is fundamental in Mathematics when we intend to "make general statements about properties, concepts or procedures" and that "justification is central to making it possible to mathematically validate" those statements (Mata-Pereira & Ponte, 2018, p. 783). These two processes interact with each other, as in many situations the language used in justification has to be general so that its applicability to the entire domain is clear; on the other hand, when some generalisations are established, it is because, at least implicitly, there are already justifications for them. For Jeannotte and Kieran (2017), exemplifying is an auxiliary process of generalising and justifying, which allows inferring data about a problem by generating elements that support those processes. In the process of generalising, it is essential to look for similarities and differences through the production of examples, in which case it is necessary to mobilize the process of comparing. In turn, in justifying the examples can be critical, for example when using counterexamples. For these authors, classifying consists, through the search for similarities and differences, identifying common and distinct points in different objects, joining them or separating them into a class of objects based on mathematical properties or definitions. This process involves comparing and, by stating that all elements of the class obey certain characteristics, it establishes a generalisation (Brunheira, 2019). For Mason (2001), “classification is not just about making distinctions and describing properties, but about justifying conjectures that all possible objects with those properties have been described or otherwise captured” (p.7). Mariotti and Fischbein (1997) state in the following way what means to classify in geometry:

> A classification task consists of stating an equivalence among similar but figurally different objects, towards a generalisation. That means overcoming the particular case and consider this particular case as an instance of a general class. In other terms, the process of classification consists of identifying pertinent common properties, which determine a category. (p. 244)

Thus, in addition to identifying the different reasoning processes, it is essential to have a deep understanding of the meaning of each one in order to establish relationships between them, thus reaching a high level of knowledge (Rodrigues et al., 2021).

Reasoning in preservice teacher education

Several studies (Lannin et al., 2011; Stylianides & Ball, 2008; Stylianides & Stylianides, 2009) indicate that prospective elementary school teachers must have opportunities to develop their mathematical reasoning if they are to work on it with their students, particularly in geometry.
In the field of geometry, Battista (2007) states that reasoning is strongly based on the spatial structuring of objects or situations, that is, on mental models that are activated to interpret and reason about these objects or situations. In the context of preservice teacher education in geometry, Brunheira (2019) suggests that processes such as classifying and justifying generalisations about geometric figures are influenced by the quality of spatial reasoning, but they are also promoters of its development. For Lehrer et al. (2013), geometric concepts such as shapes and relationships between them, for example congruence, constitute opportunities to build those relationships.
In addition to developing their own reasoning, Francisco and Maher (2011) refer to the need to create opportunities for teachers to learn about how to develop mathematical reasoning in students. In the same sense, Stylianides and Ball (2008) defend the need to develop in teachers the ability to plan and implement tasks that promote the development of reasoning in their students.

Methodology

The study reported here followed a qualitative-interpretative approach (Erickson, 1986) since it aims to understand the way prospective teachers relate several reasoning processes. Its context is a teacher education experiment developed in 2019/20 with 31 prospective primary teachers, attending a Master Degree certifying for teaching in primary schools (grade 1 to 4) and teaching Mathematics and Natural Sciences in grades 5 and 6. It took place during six lessons, one per week, each lasting two hours and 30 min and focused on mathematical reasoning, addressing specialised mathematics knowledge for teaching. All the tasks were initially explored autonomously by the prospective teachers, organised into eight groups, and were subsequently discussed by the class as a whole.

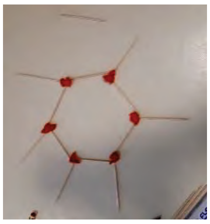
The data were collected through participant observation of the lessons, by the team Project, using audio and video recordings of the autonomous work carried out by two groups of prospective teachers (Groups 1 and 2) and the whole class discussions, and documents collection (all tasks resolutions). The data reported here are from Group 2. According to the ethical criterion of confidentiality, all the prospective teachers signed a free and informed consent form, in relation to the data collection methods, and are given fictitious names.

This paper refers to a didatical task about reasoning in geometry. Figure 1 presents an excerpt of the task.

Consider the task *Let's learn about pyramids*, proposed to 3rd grade students. In the previous year, the class had already come into contact with pyramids and prisms, in a first approach to their characteristics. So the teacher introduced the task by projecting the image below and asking *What is the intruder?*

After the initial discussion, the students started solving the task in pairs, using some models of pyramids in cardboard and wood, match sticks, toothpicks and plasticine balls.
3. Identify the reasoning processes involved.

4. Read the following dialog. The students had already analysed the possibility of building a pyramid with 13 edges using the material, as shown in the image. They were currently analysing the same issue for 15 edges.



Teacher — So, with 15 toothpicks, 15 edges, what happened?
Student — It would be missing 1.
Teacher — So and how many toothpicks did you put in the base?
Student — Eight.
Teacher — Eight. And now how many do you have to put on the side edges?
Student — Oh my God...
Teacher — OK, you can look at what you've done!
Student — Seven.
Teacher — So, can we build with 15?
Student — No… there was a toothpick missing… with odd numbers I couldn't do it.
Teacher — Ah! So tell me why it's not possible with odd numbers.
Student — Because one is missing or one is left.
Teacher — And why does this happen? What happens to the edges in the pyramids?
Student — Hmm... Here (points to the base) and here (points to the place where the side edges would be) must have the same number.

4.1. Discuss how the student's reasoning evolved, relating it to the interaction she established with the teacher.
4.2. Explain the role of the manipulative material in this situation and throughout the task.

Figure 1: An excerpt of the task

We used content analysis (Bardin, 2010) of the data using the framework we elaborated before (Rodrigues et al., 2021) concerned with knowledge of reasoning processes. The categories correspond to the reasoning processes worked in the teacher education experiment: generalising, justifying, exemplifying, comparing and classifying. Each of these categories were divided into subcategories corresponding to six levels of specialised mathematical knowledge of the content, presented in hierarchical form (Table 1).

Table 1: Framework for knowledge of mathematical reasoning processes

Category	Subcategories
Knowledge of the reasoning process	5. Knowledge of the process fits the definition presented, and includes its relationship with the other reasoning processes
	4. Knowledge of the process fits the definition presented, and is explicitly outlined by enunciating the properties of the process
	3. Knowledge of the process fits the definition presented, and is explicitly outlined through illustrative example(s)
	2. Recognising a reasoning process though considering only 'correct' processes
	1. Knowledge of the process takes on the meaning of the term in everyday language
	0. The process is confused with other processes

Results

Episode 1

In this episode, we analyse the reasoning processes that the group of prospective teachers identified to be used by the 3rd grade students during the solution of the intruder task (Figure 1). This lesson intended to support the introduction of the classification process which is, in our perspective, the main process in question. In fact, to find the intruder, students must identify pertinent common properties that determines the pyramids as a class (for example, they all have triangular faces that converge at a vertex) where the prism does not belong. For this, they should also ignore their particularities (for example, distinct bases) and consider them as representatives of a more general class (Mariotti & Fischbein, 1997).

Nuno:	OK. "Identify the reasoning processes involved". Here, I think that . . . the most obvious common is generalisation. They identify a property that fits all the pyramids.
Lara:	Yes.
Daniela:	Yes, Yes.
Nuno:	So one of them is to generalise. To compare...
Daniela:	Also to compare. You don't think so?
Nuno:	Between several...
Daniela:	Between figures, yes.
Lara:	To generalise they compare, don't they?
Nuno:	Yes, yes. They exemplify, here it does not...
Lara:	No.
Daniela:	No.
Daniela:	Compare between what?
Lara:	Among the different figures so that you can generalise.
Helena:	For example, here they made a comparison. When they had to select what it was [the intruder].
Nuno:	Yes, that's a fact. Between the ones that are and the ones that are not [pyramids].
Helena:	Exactly.

As expected, the prospective teachers did not report the classification process, but their analysis clearly identifies the generalising process that is strongly related to that process (Jeannotte & Kieran, 2017; Mariotti & Fischbein, 1997). Furthermore, Nuno explains the substantiation of this generalisation by saying that it corresponds to the property that “fits all pyramids”, an idea that gathers consensus. Associated with the process of generalising, the group also refers to the process of comparing as a support process because, as Lara says, it is necessary to compare the different figures “so that you can generalise”, which is also consistent with the literature.

The group agrees and is sure about the two identified processes, when one of the elements raises the hypothesis that the process of justifying may be also involved:

Lara:	To justify I don't know if it makes sense. Okay, you find properties that you can justify, but properties are generalisations.
Nuno:	Yes, yes. So, to generalise.
Helena:	I think it's enough to generalise and to compare.
Nuno:	Although, in order to generalise, they will also have to justify first, generalising is the most comprehensive of all. They will say first that the pyramid is a pyramid...
Helena:	Because so, so and so. Exactly.

Nuno:	It is a quadrangular pyramid, because the base is a square and because the faces are triangles, it has x vertices...
Daniela:	I think that, in order to reach the generalisation, they start with justification.

What started out as a tentative hypothesis from one of the prospective teachers turned out to be a meaningful possibility for everyone. The idea that the process of justifying may be involved derives from the perspective that, in this case, when we generalise we already have justification in mind or, to put it another way, we generalise because we know why. This idea is consistent with the suggestion by Mason (2001) when he states that the process of classifying also involves justifying conjectures that all possible objects with those properties have been described. Furthermore, in the particular case of geometry, as Brunheira (2019) states, the justification of generalisations concerning a class of geometric figures is based on a mental model of the class of objects, that is, its spatial structure that often presides over the formulation of generalisations.

In this way, we consider that the group's dialogues are quite relevant as they identify interactions between the processes of generalising and justifying, also showing understanding about all processes already dealt with, which corresponds to level 5.

Episode 2

Lara:	She started, she realized first that with 13 [toothpick] it would be left with one, right? That was the first thing she noticed. With 13 there would be one left, then with 15…
Daniela:	Yes, but here…
Lara:	One would be missing.
Lara:	She only realized the 15… she only gave the answer to 15 so quickly, because she had already done the one for 13.
Helena:	Because she had already done for 13.
Lara:	Because she even said there was one missing. Do you understand?
Daniela:	Yes…
Nuno:	Then we have to make a comparison with what the teacher was saying. Right here at the beginning, the teacher refers to another example, so she can...
Lara:	So, she can make a generalisation.
Nuno:	A generalisation, exactly.
Daniela:	So, the student began by understanding that with 13 toothpicks, one would be missing to complete the pyramid.
Nuno:	Then the teacher … encourages the student to go further…
Lara:	By giving another example.
Nuno:	…and it presents a new example, in this case with 15 toothpicks. Then the teacher encourages the student to go further by presenting a new example. She presents a new example, enabling the student to use the reasoning process, to generalise. She gave this example so that she could later conclude that it couldn't be an odd number.

In the written record with the answers to the task, the group summarizes the previous ideas and adds:

> In order to lead the student to a generalisation, the teacher guided her, leading her to understand that the number of toothpicks could not be odd. Also, the teacher asks why this happens, prompting justification. In a first moment, the student does not justify it, she only describes what happened. After the teacher's insistence, the student points to the material and justifies why her generalisation is valid. (Group’s record on question 4)

In this episode, the prospective teachers elect three processes that are mobilized: to generalise (that there are no pyramids with an odd number of edges), to exemplify (for 13 and 15 edges) and to justify (why an odd number of edges is impossible). Regarding the first process, the group correctly identifies that it is a generalisation when the student extends her conclusion (about 13 and 15) to the domain of odd numbers, corresponding to the definition of the generalisation process that was established. With regard to the process of justifying, it is noteworthy that the group is able to distinguish a simple description of an event (when the student says "Because one is missing or one is left") from a justification, relating this process with the investigation of the underlying reasons why it is true (Lannin et al., 2011). Finally, about the process of exemplifying, actually the examples used (with 13 and 15 edges) are suggested in the task. However, the group recognizes the support these examples provide for both the processes of generalising and justifying (Jeannotte & Kieran, 2017). In the first case, they consider that it is based on the attempt to build a pyramid with 13 edges that the student quickly concludes that it is impossible to construct a pyramid with 15 edges. In addition, they also realize that these two examples are fundamental to generalise and to justify, as they recognise the understanding of the situation they generate, allowing the student to understand why the number of edges cannot be odd.

Conclusion

The didactical task led the group of future teachers in a discussion about the reasoning processes involved in task on the properties of pyramids. In this discussion, the group was able to easily recognize, in context, the characteristics of the process of generalising, as well as its relationship with the process of comparing and exemplifying. However, the richness of the context involved–the establishment of the class of pyramids–enhances the emergence of various reasoning processes that occur in a non-linear way, generating a discussion about the distinction between generalising and justifying. Despite some hesitation, participants used the association between the process of justifying and the understanding of why a relationship works (Lannin et al., 2011) as a selection criterion for that process, which is found to be appropriate. Furthermore, they are also able to understand the supporting role that the process of exemplifying assumes in establishing a justification without confusing the role of empirical examples in establishing a statement, which is very common (Stylianides & Stylianides, 2009). On the contrary, the group seems to recognize that, in geometry, a justification must be associated with the spatial structure of objects (Brunheira, 2019).

This paper focused on a group of prospective teachers who shows a maximum level of knowledge about reasoning processes and its relationships, highlighting the importance and potential of didactical tasks that promote this knowledge, including the idea that different kinds of tasks can offer different kinds of opportunities for reasoning (Stylianides & Stylianides, 2006), which may be promoted using real classroom episodes. This research should be developed further highlighting the difficulties felt by other future teachers and ways to address them.

Acknowledgment

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Concept Formation of Geometrical Reflection in a Digital Learning Setting

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This paper gives insights into an ongoing study that investigates pairs of students working on geometrical tasks in a cooperative way. The goal of the study is to learn more about students' understanding of geometrical reflection. First results of the study show that «symmetry» and «congruence» are useful categories to describe the students' learning processes on reflections. This paper briefly summarises the theoretical background of the study. Subsequently, it discusses two examples of observed phenomena about these categories of the conceptual understanding.

Keywords: Geometrical reflection, conceptual learning, basic ideas.

Introduction

The formation of conceptual structures is a central aim in the teaching of mathematics, in particular in geometry. The students have to form mental images of objects and their mathematical properties mediating between their real-world experiences on the one hand and the abstract mathematical structures on the other hand. Developing these representations and concepts to capture the meaning of concepts requires a social process (Dörfler, 1995). This includes explanations, drawings, negotiations and discussions in interactive settings (Krummheuer, 1995; Voigt, 1995). During the pandemic these learning-processes had to take place in online learning environments. Schwob and Gudladt (2021) show that online learning environments can support successful interactions and enable the exchange of individual knowledge. Based on this research, the present project uses the dynamic geometry software «GeoGebra» within such online learning environments to gain insights into the students´ understanding of the congruence mapping reflection. This contribution discusses very first results of the ongoing project, focusing on the following question: Which conceptual aspects of reflection can be observed?

Theoretical Background

This paper focuses on the teaching and learning of mathematical concepts. More precisely speaking, the aim is on the students´ understanding of reflection. From our theoretical point of view, the matter of understanding a concept goes beyond the knowledge about the mathematical definition of concepts (Tall & Vinner, 1981). Roughly speaking, the students have to develop *basic ideas*[1] about what reflections mean within different demands. For example, in figure 1 below we can understand reflection in a statical way: If you see one quadrangle *ABCD* you can use the characteristics of reflection to check out whether *g* is a symmetry-axis of the figure. Hence, symmetry is understood as

[1] "basic ideas" is our translation of the term "*Grundvorstellungen*", which is used by vom Hofe and Blum (2016).

a static basic idea of reflection. If the students ask themselves whether both halves are equal or not while solving the tasks, the category «symmetry» may be prominent. On the other hand, you can also see the triangles ABC and ACD and ask yourself, whether these two triangles are congruent to each other. In this case, you have to activate a more dynamic view on reflections: Can we move one triangle on the other one by reflection at the axis *g*? In order to build a comprehensive concept it is necessary for the students to work empirically by operating with the objects and thus gaining insights in their properties (Dörfler, 1995). In terms of a comprehensive concept students would benefit from getting to know static and dynamic aspects of reflection.

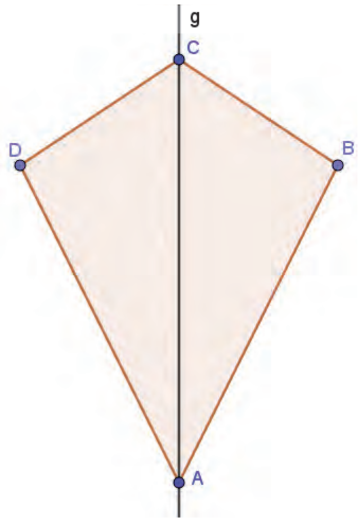

Figure 1: Basic ideas of reflection at quadrangle ABCD

Of course, for the mathematical experts, both meanings of reflection result from the same mathematical definition. But, from our point of view, they are different *basic ideas* that serve as a bridge between individual images and officially and formal definitions.

> So, *Grundvorstellungen* can be construed as mediating elements or as objects of transition between the world of mathematics and the individual conceptual world of the learner. GVs thus describe relationships between mathematical structures, individual-psychological processes, and subject-related contexts, or, in short: the relationships between mathematics, the individual, and reality. (vom Hofe & Blum, 2016, p. 231)

These basic ideas can be differentiated into two aspects: The *normative aspect* describes what concepts the students should have (vom Hofe & Blum, 2016). The *descriptive aspect* shows the occurring activated concepts of individual students (vom Hofe & Blum, 2016). The focus of the present project is on the descriptive aspects of basic ideas.

The Benefit of Using Dynamic Geometry Software[2]

Researchers like Ng and Sinclair (2015) combined analogue (paper-and-pencil work) and digital settings (dynamic computer-based environment) for reasoning about symmetry. The students worked

[2] Hereafter often abbreviated as DGS.

on the topic of symmetry for three lessons and increasingly used functional relationships to discuss the symmetry of objects e.g. by dragging them to explore the connection of pre-image and image. Starting from a static understanding of reflection, students seem to have extended their knowledge by operating with the respective figures in this manner. In addition, students developed new terms while communicating about symmetry (Ng & Sinclair, 2015). In other studies, the influence of the dynamic geometry software GeoGebra was investigated. Here it was shown that students who had to reflect drawn objects by using GeoGebra performed better in geometric transformations, especially in reflection (Pavethira & Kwan Eu, 2016).

Methodology and Methods

With the aim of gaining insights into descriptive aspects of the explored basic ideas we planned online interviews via the meeting tool «BigBlueButton» (https://bigbluebutton.org/) with pairs of students in secondary education (communication *through* technology) (Drijvers et al., 2016). While the interviews feature pairs of same-aged students, every student works on his or her own computer at home without enabling their webcams. One of them gets access to the tasks in «GeoGebra Classroom» (https://www.geogebra.org/) and has the authority to manipulate applets and share his screen. Based on the shared screen the students and the interviewer discuss the effects of manipulating reflections in the dynamic geometry software «GeoGebra» (communication *of* technology) (Drijvers et al., 2016). The interviews feature students from sixth to tenth grade. This age range was deliberately chosen because curricular standards on the topics symmetry and congruence are often not properly implemented in German schools. The students in the following episodes are in grade 9 (episode 1) and grade 6 (episode 2). The examples show that an adequate conceptual understanding of reflection doesn’t necessarily depend on the grade level. Both student groups have basic knowledge of the DGS «GeoGebra» and its functionality.

Working on the student´s understandings, we decided to organise our research as a *qualitative* study. The interviews are being transcripted[3] and analysed applying ethnomethodological methods (Krummheuer, 1995; Voigt, 1995) to work out different conceptual understandings of reflection. Furthermore, the transcripted episodes will be analysed by using comparative analysis to connect the theoretical background and empirical results via reconstructive categories (Krummheuer, 2018).

The Learning Environment

Altogether the learning environment includes eight tasks. Based on the work of Senftleben (1996) the tasks contain four phases: First, the researcher asks about the effects of a geometric operation and supports the students with a visualisation of the initial geometric construction of the object (Figure 2) below. Secondly, the students work on the question in a cooperative way. It is important that they do not use any representation within this phase – all operations are performed in their mind only. At the end of this phase, the students make a hypothesis about the effects of the discussed operation. Within the next phase, this hypothesis is validated by performing the operation within «GeoGebra». If the original hypothesis proves to be wrong, the students are then asked to find an explanation for

[3] All transcripts in the present paper have been translated to English by the authors.

their observation. While one of the students may manipulate objects on the screen at his free will, the other student is forced to verbalise his ideas and give instructions. Through this interaction both students must *discuss-the-screen* (Drijvers et al., 2010).

Example: Two Episodes

The students in the following episodes worked on six tasks before. The first three tasks encompass the reflection at one axis: The *Pentagon1* was reflected at the straight line g, so that the image *Pentagon2* originates. The students worked on the manipulation of *Pentagon1* (one vertex and position of *Pentagon1*) and the single axis of reflection g and their effects on *Pentagon2*. Afterwards the tasks focus on a double reflection at two parallel axes (figure 2): First the *Pentagon1* was reflected at the straight line *g*, so that the image *Pentagon2* originates. In addition, *Pentagon2* was reflected at the straight line *h*, the image is named *Pentagon3*. First, the interviewer explains this double reflection. In the following we provide an analysis of two episodes that focus on the following question:

"*Imagine you move the axis of reflection g towards pentagon1. What happens to the pentagon 2 and 3? Describe and explain.*"

In the following episodes the pairs of students work on this question. The students have already discussed their ideas within the first phase of the task, i.e. they built a hypothesis about the expected movements of the pentagons. The episodes start with the (very short) discussion of the accompanying hypothesis.

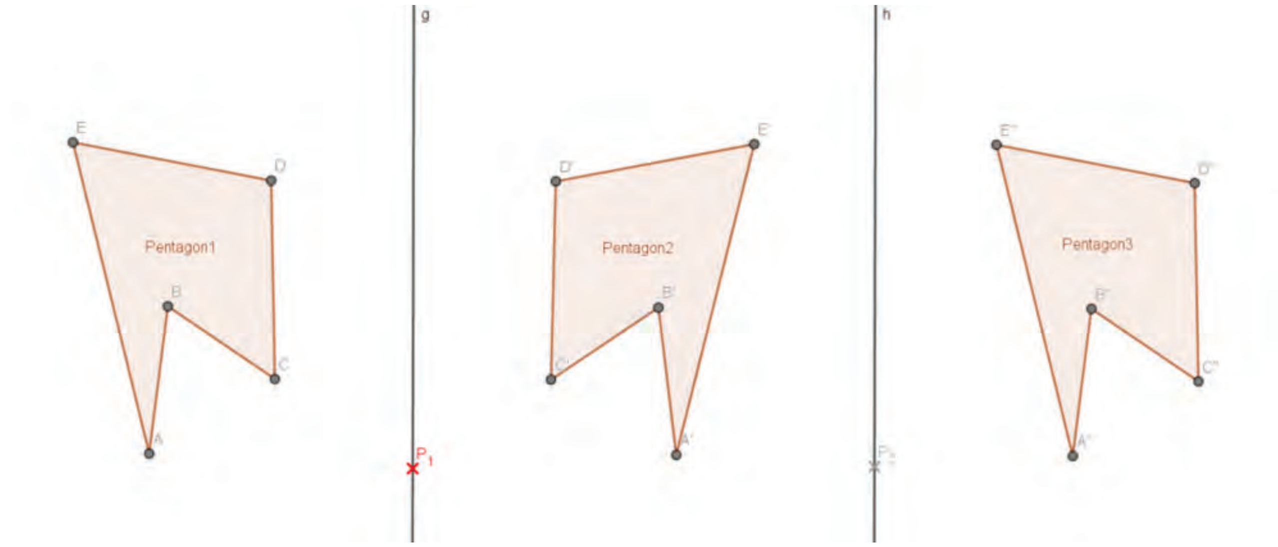

Figure 2: Construction of the task

Episode 1: Symmetry

Both students assumed what happens to the pentagons while moving the axis of reflection *g* . This episode begins with the verbalisation of the hypothesis and continues with the interviewer's question on what happens to the pentagons:

203	Michael:	this is exactly the same as the task before, that 2 stops and that the 1, err pentagon 1 moves to the left
204	Nicolas:	yes and 3 stops also, doesn't it?

During line 205 and 208 the students verify their hypothesis.

209	Interviewer:	what happened now?
210	Michael:	err pentagon 2 moves towards pentagon 1 and 3 moves to the side
211	Interviewer:	and why?
212	Michael:	so to the right (…) do you have an idea why?
213	Nicolas:	um move the axis a little bit to the left (8 sec.) yes because, yes because the first reflects err the second which is symmetrical again and then the second is mirrored again err to the third and the third must also have the same distance again as the second so they are symmetrical again. you know?
214	Michael:	oh yes.

Michael and Nicolas relate their hypothesis to the previous task, dealing with the same pentagons and reflections. But, within that task, the reflection axis *h* was moved. Hence, only pentagon 3 moved. Both assume that pentagons 2 and 3 wouldn't move (turn 204) because the axis of reflection *h* isn't shifted. They hypothesise, that only pentagon 1 would show an effect by "[moving] to the left" (turn 203). At first sight, this hypothesis sounds strange. But, having in mind a statical conceptional understanding of reflection, it seems quite plausible: Assume that the students understand pentagons 1 and 2 as a single figure and the line *g* as its symmetry-axis. Within this interpretation of the figure, it is very rational (besides alternative hypotheses) to expect that a movement of *g* would cause a movement of pentagon 1 in order to maintain the symmetry-axis of the figure.

When the students check their hypothesis by manipulating the objects in the «GeoGebra» applet (turn 210 and 212), they are confronted with the correct movements of the pentagons. The interviewer then asks them to find a reasoning for their observation (turn 211). Nicolas comes up with the explanation that pairs of subsequent pentagons are symmetrical (turn 213). One possible interpretation could be that he combines pentagons 1 and 2, which are described as symmetrical to each other, as well as pentagons 2 and 3. In this interpretation, pentagon 2 belongs to both combined symmetrical figures and represents one half of each figure. As Nicolas focuses on pentagons 2 and 3 and their respective distances to h, a static understanding of reflection can be assumed. The role that pentagon 2 has within this interpretation won't be further explained here. Within this episode, the research question can be answered as follows: The students Michael and Nicolas seem to have a statical basic idea of reflection, as they apparently combine two pentagons to one symmetrical figure. So, it seems that the category of «symmetry» has been activated in the students in this context.

Episode 2: Congruence

Both students assumed what happens to the figures while moving the axis of reflection *g*. At the beginning of this episode Niklas and Jonas present their hypothesis:

395	Niklas:	pentagon 2 pulled itself <u>towards</u> pentagon 1 and pentagon3 moves itself <u>away</u> from pentagon 2

During line 396 and 400 the students verify their hypothesis.

401	Interviewer:	very good. okay. now here’s the same question again. why is nothing happening with pentagon 1?
402	Niklas:	um (*moves the line g in the applet back and forth*)
403	Jonas:	maybe because pentagon 1 is the main figure (*hesitant*) (..)* that was drawn created?
404	Niklas:	* (*laughs*)
405	Interviewer:	what do you mean by main figure?
406	Jonas:	the one that has been drawn first and the others are just the reflections of it
407	Interviewer:	mhm (*affirmative*)
408	Niklas:	oh so this is the starting figure of the reflected pentagon 2 and 3 both of which are reflections. so they must be reflected err, towards pentagon 1 don’t they? so pentagon 3 to pentagon 2 of course.
409	Jonas:	yes.
410	Interviewer:	what is the difference between the main figure or the, the original figure and the pentagons 2 and 3?
411	Niklas:	um pentagon 2 and 3 (*mumbling*) (6 sec.) so they sort of need this exact reflection to continue being reflections, don’t they?
412	Jonas:	they even need pentagon 1 to exist because the other two are only reflections

Niklas presents a correct hypothesis about the movement of pentagon 2 and 3 when the axis of reflection *g* is shifted in the direction of pentagon 1. He assumes that they would move according to the respective changes in distance to the axes of reflection *g* and *h* (turn 395). As the students verify their hypothesis, they are asked to find a reason why pentagon 1 doesn’t move (turn 401). They describe pentagon 1 as the main figure (turn 403) that was drawn first (turn 406). In contrast to this main pentagon, the others are identified as only reflections (turn 408). Since they describe the three pentagons with different terms, one can assume that the students interpret them as distinct geometrical objects, which are congruent to each other. The interviewer then asks them to explain the difference between the main figure and the reflected figures (turn 410). At this point, Jonas stresses the necessity of the main figure (pentagon 1) for the existence of the reflected figures (pentagons 2 and 3; turn 412). Within this episode, the research question can be answered as follows: The students Niklas and Jonas seem to have a dynamic view of reflection, as they apparently compare the pentagon 1 as the main figure with the pentagons 2 and 3 as reflected figures. So, it seems that the category of «congruence» has been activated in the students in this context.

Discussion

A comparison of the two episodes shows very different views on the geometrical objects and thematised operations. These different views depend on the question, whether the students understand the pentagons as different objects or not. An interpretation of the pentagons as halves of a summarised figure, as in the first episode, leads to an understanding of the lines *g* and *h* as axes of symmetry. If, on the other hand, the students’ concept of reflection is different, as within the second episode, the lines are interpreted as axes of a congruence mapping that leads to a movement, i.e. the basic idea of congruence is activated. Both interpretations of reflection are important for students’ understanding of the concept of geometrical reflection. In some situations, the static basic idea «symmetry» may enable students to construct symmetrical figures or identify properties of figures etc. In contrast the dynamic basic idea «congruence» may be important in situations where more figures and the functional relationship between them are relevant. To be clear the tasks in the learning environment focus from a normative point of view on the dynamic basic idea «congruence». However, both basic ideas can be used to solve the tasks successfully.

The flexibility of switching through previous tasks in «GeoGebra Classroom» may support students in answering questions like "Did figure one move in previous tasks?" or “Which operations have been executed in these tasks?”. By dragging objects in the respective applets, students may realise, that figure one only ever moves if it is dragged directly. Based on this observation, technology could potentially support the understanding of the functional relationship between the pre-image and its images. Thus, conducting research on how a connection between image and pre-image is established, could lead to further insights. In this context it might be interesting to consider whether the categories «symmetry» and «congruence» affect the students’ understanding of this functional relationship.

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Classifying 3d figures by prospective primary teachers

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This research aims to characterize how prospective primary school teachers describe and classify solids. A professional task is designed and implemented with 148 future primary teachers. The task is structured in four sections in which construction, composition and decomposition, visualization, identification of attributes of 3D figures and their classification are promoted. The development of the task involves the use of manipulative material (half cubes). For the analysis, some tools of the model of Didactic-mathematical Knowledge and Competences of the mathematics teacher are used. It is observed that most of the future teachers recognize mainly perceptual elements of the figures, which makes them construct dichotomous classifications, and only a few interpret the classification utilizing global characteristics.

Keywords: Didactic-mathematical knowledge, future teachers, solids, classification, primary education

Introduction

It is widely accepted that the geometry of solids has been neglected in many curricula, and 3D geometry teaching receives limited or insufficient attention in primary school classrooms. Prospective teachers are suddenly engaged in plane representations of solids, but when doing classifications, they are proposed to analyse just general polyhedra observations and constructions. Many teachers who deal with geometric concepts have an approach in which they memorize the names of the figures and some of their characteristics are privileged (Copley, 2000). Furthermore, the representations used to show these figures are centred on a prototypical view of them. Authors such as Sinclair and Bruce (2015) point out that little research has focused on the geometric thinking of children (and prospective teachers) concerning these types of figures and the process of classification.

Therefore, our study aims to show an approximation to the mathematical knowledge evidenced by future primary school teachers when they tackle a task of construction, analysis and classification of 3D figures, based on the tools of the CCDM perspective (Godino, 2009; Pino-Fan & Godino, 2015; Godino et al., 2016).

Theoretical framework

In previous research, figures that students or future teachers must recognize are presented by the teacher, noting how they identify the properties that underlie these forms and if this helps the teacher classify them (Guillen & Figueras, 2005; Patkin, 2015). According to the approaches of Vinner (1991) and Fujita (2012) when students decide whether an object belongs to a certain category, they can give biased answers because in their decision prototypical or common forms are privileged. This suggests that figures presented in the same position and with little variety of examples allow deeper consideration and analysis of them. We assume that we should have greater knowledge regarding how future teachers face such classification problems of solids. Identifying this knowledge is a key aspect to adapt and improve training proposals to promote geometric reasoning in future teachers.

In this model CCDM, it is considered that two key competencies that the mathematics teacher should develop are mathematical competence and the competence of analysis and didactic intervention. It is suggested that the knowledge necessary for teaching mathematics implies a deep knowledge of mathematics and its teaching that is, a didactic-mathematical knowledge. However, mathematical knowledge alone is not sufficient for the adequate practice of the mathematics teacher (Pino-Fan et al., 2015). From the CCDM model, it is proposed that to achieve suitable teaching, a mathematics teacher must possess different types of knowledge. On the one hand, you have to know the school mathematics of the educational level at which you teach. Additionally, it is necessary to understand elements of later levels, what is referred to as the 'knowledge of the mathematical content per-se'. This knowledge is divided into two types: common mathematical knowledge and extended mathematical knowledge.

The first refers to the knowledge regarding the mathematical object that is necessary to put into play to solve problems and activities related to a specific mathematical topic at a particular educational level. It is generally associated with the level at which it is taught. The second refers to the fact that the teacher, in addition to knowing how to face problems and activities on a certain topic, must have advanced knowledge, which is part of higher levels.

Methodology

A professional task was designed and implemented with 148 students studying Primary Education at a university in Spain. The task introduces the process of constructing solids using one, two, three and four half cubes. The future teachers construct the figures by using plane development. They talk about the observed properties of the shapes and provide one or more classification ideas for the shapes. The key questions analysed in this paper are described in table 1. The task was solved collaboratively, which resulted in the configuration of 33 groups. Our data is the protocols written by each of the groups. From the analysis of the productions of future teachers, the types of constructions are identified; the criteria used to classify 3D figures, and the value given to the classification process from the didactic perspective.

As it is in our interest to analyze the types of knowledge that emerged in the development of the proposed professional task, we will base ourselves on the analysis of three fundamental questions that reveal fundamental elements for the discussion that we want to raise. Table 1 presents the selected questions, the type of knowledge that is valued and its intentionality.

Nº	Question	Type of knowledge	Intention
2.b-c	Build all the polyhedra that could be constructed with the four pieces (half cubes). Make a table where you present the graphic	Common	Obtain figures through composition. Identify a variety of shapes

	representation, the name and the characteristics of the obtained figures.		Know specific language to describe geometric figures
2.d	Define criteria and classify the figures that you have obtained in section b. Explain what you have considered to define the criteria	Extended	Generate categories from the characteristics of different figures.
2.e	Why do you consider that when we study various types of figures, a fundamental process to work with is Classification?	Extended and Meta-didactic	Promote an attitude of inquiry. Recognize the intentions and motivations of a classification task.

Table1. Questions in the professional task

We grouped similar responses belonging to the same category. From this categorization, information was obtained that allowed for describing difficulties, errors and justifications presented in the common knowledge of the content that future teachers have concerning 3D figures and their classification. To systematize the answers given to a question about classification, we show the process of recognizing emerging categories associated with the texts of future teachers.

Examples of texts found	Emergent category
They establish a table where it is grouped according to the number of vertices. [Identify most constructed figures that have 8 vertices. Most have between 6 and 12 vertices. None have 11 ...]	Perceptive appearance according to Battista (2012).
Figures 1,2 and 3 (cube, octahedron and triangular prism) can be classified within the same group since they are regular polyhedra, while figures 4,5, 6 are totally irregular, and therefore do not have all the equal faces. Therefore, we have two groups, regular and irregular.	Attempt to classify by grouping known features in 2D (regular, irregular) but with errors.
We have taken into account the type of polyhedron according to its bases, the regularity, the number of faces (bases and sides), the shape of its bases,	Correct recognition of 2D characteristics applied to 3D.
We have considered if it is concave or convex, the number of vertices and edges, and the symmetry.	Recognition of characteristics associated with the global observation of 3D figures.

Table 2. Emerging categories in ideas and classification criteria

To systematize the look at mathematical objects, we looked for the type of characteristics used from the classification. For this, the idea of the configuration of objects and processes of the Onto-semiotic perspective is used, looking at the definitions, arguments and propositions used.

Example of given argumentation	Emergent category
"… Since it allows children to group objects according to their similarities and differences, based on different criteria: shape, colour, size ... These relationships serve as the basis for the construction of logical-mathematical thinking. Piaget considers that these logical relations are the base of the classification, seriation, notion of the number and graphical representation".	It allows to establish logical relationships and discover new concepts
"The fact of classifying allows us to compare and see the differences of the other groups. Thus, it also allows us to look for regularities between shapes and their properties, to compare similarities and differences using the appropriate vocabulary, to understand the relationships between different three-dimensional figures, using the properties that define them, and to look carefully at the regularities and changes that occur they produce in a collection or a sequence".	It allows to compare, recognize common characteristics, through similarities and differences
"Classification is important because it allows ordering or organizing the figures into groups following common criteria, which facilitates the development of logic".	Allows an organization

Table 3. Argumentation categories given to the rank value

Results

We then recognized characteristics of the mathematical knowledge of the students in three sections: common knowledge about 3D figures; extended knowledge about the meaning given to the classification and the value of the solid classification itself.

Common knowledge of 3D figures. Each group built and recognized different polyhedra, the triangular prism being the one that the majority built (27.7%), followed by the trapezoidal prism (in some cases incorrectly named hexahedron) and the rectangular prism, both with a 23.1 %. Only four groups (6.2%) speak of a parallelepiped. Two groups (3.1%) mention an octahedron. When some groups were unable to assign a "known" mathematical name to the constructed figure, they proposed real names such as "chair", "bridge", "podium", among others (28.6%). In the descriptions they make of each figure, some teams state characteristics related to parallels, incidents, or perpendicularity, and express them correctly.

Students consider length as a differentiating element of solids, thus, for example, a triangular prism built with two half-cubes assumes it differently from a triangular prism built with three half-cubes. On the other hand, they identify regular and irregular bodies (which is used as an initial classification criterion in some cases). The use of informal language is evidenced to describe elements that are named in the geometry of solids; they name vertices, corners etc. Most students use the term equality to refer to situations of congruence, which in addition to being an error, triggers elements of greater concern, such as the mixture of terms from plane geometry with the geometry of space. It seems to us that it is due to the difference between equality and congruence, which is not sufficiently emphasized in school geometry.

Among the errors, we find eight groups that form a solid with the four half cubes that they identify with a cube, but which is not. Indeed, with four half cubes, we can never obtain a cube. This shows a difficulty associated with visualization because when joining the four half-cubes, a rectangular prism with two square faces and four rectangular faces is obtained as shown in Figure 2. Only in some cases do they show that this is not the case, saying that it is a prism with a square base, but with a height less than the side of the square. The other wrong example is to name one of the constructed figures an octahedron since by combining the four half cubes we can never get a face that is an equilateral triangle. In this case, there is only one group making this mistake.

Regarding the extended knowledge about classification. Most of the groups point to elements of a visual and descriptive nature to classify the solids. They mention, for example, the number of faces, vertices, sides and bases that the polyhedra they managed to build have. In these cases, they classify considering these attributes.

Although other groups allude to global characteristics proposing a classification of the figures as regular and irregular polyhedra, they do so in the incorrect way, since none of the constructed figures is a regular polyhedron. Other groups, when doing the classification, speak of figures that it is not possible to get with half cubes such as pyramids. Few groups speak of prisms as convex polyhedra, which is correct. Table 3 shows the percentages according to the elements that the future teachers considered for the classifications.

Reference to a classification	**Perceptive appearance**	**Characteristics Symmetry**	**Other classification**	**Global view**	**None**
Number of teams	18	1	10	3	1
Average	54,54 %	3,03 %	30,3%	9,09 %	3,03%

Table 4. Elements used by prospective teachers for classifying

Most of the classification criteria depend on a mere description of the object, that is, they do not address some more advanced theoretical element. Rather remain in a process of visual verification; only one group manages to rescue elements from the origin of the half cube as a section of the cube and the configuration of the 2D figures when locating to build a body in 3D. The classification process depends on the ability to identify similarities and differences between figures and explain why a certain figure is an example of such a class (Walcott et al., 2009). When describing elements of the classification, the examples and characteristics are usually shown but not so much the common and distinguishing criteria, although they are being valued as important. In the same way, we found that future teachers find it difficult to recognize how certain visualization elements are associated with representations that allow recognizing properties. For example, we expected that there would be classification proposals where the number of concavities or symmetries were considered.

Extended and meta-didactic knowledge about the value of the classification process. The importance given to the classification process by future teachers focuses on aspects such as establishing relationships; the identification of properties; the recognition of common and uncommon characteristics of the idea of classification as a grouping. As can be seen in Table 5, all groups focus on the importance of classification as it allows distinguishing attributes based on the comparison. Others simply refer to the classification as separation, posing simple dichotomous groupings, but without an adequate appropriation of the characteristics of the figures.

Explanation Categories	Nº of groups
It allows to establish logical relationships and discover new concepts	17
It allows to compare, recognize common characteristics, through similarities and differences	33
Allows an organization	4

Table 5. Elements valued as important in the classification process

After what has been observed, we can say that most of the groups identify the recognition of common characteristics as a great value of the classification, but at the time of classifying, they do not explicitly state the criteria, but rather emphasize the name given to a group of figures as evidence of the classification.

Final considerations

In relation to the common knowledge of mathematical content, this study found that the perceptual appearance dominates over certain conceptual aspects (Bernabeu et al., 2017; Gonzato et al., 2013). Future teachers identify visual arguments, similar to those used by children themselves in the early ages. An application of the assumed mathematical knowledge of the known classifications of figures in the plane is not observed in the case of prisms.

The language that is established in the group of students for teachers is mostly descriptive and suggests that its members are at the second level of van Hiele (Gutiérrez & Jaime, 1998). We can say that we interpret our results in the sense of showing the difficulty in relating the understanding of geometric figures to the coordination of two semiotic systems of representation. The discursive (oral or written) and the non-discursive (drawings, photos of the figures) (Duval, 2017). Indeed, despite having manipulative elements, students are not capable of formulating mathematically well-formulated explanations.

The analysis carried out has made it possible to identify a perceptual domain in the characterization of 3D figures and a low level of structuring from properties. Although the arguments given by future teachers about the value of the classification are at an initial level, it shows how they could deal with this process in the mathematics class in primary education in the future. Solving the question itself led future teachers to question the classification models that they had learned in their school years, but as they did not have a broad appropriation of the attributes of solids, they could not recognize generalizable properties that would allow them to make rankings richer.

The results of this study show that some future teachers like to define a mathematical object by describing its characteristics or properties. Future teachers manage to be surprised by the great variety of figures that can be built with the manipulative material worked on in the professional task. Moreover, it does not seem enough to recognize, for example, that different types of prisms can be obtained, and that this variety. It makes it possible to propose various types of classifications, starting with those in which one type of attribute is privileged (dichotomy), to then give way to more complex ones where there is a combination of attributes. No inclusive classifications are noticed.

The use of manipulative material as a mediator of the observations and responses allowed opportunities for reflection on the congruence of figures that can be observed in different positions. Moreover, common mathematical knowledge is not sufficient, because many groups are not able to systematize the process of the construction of figures, and they are satisfied with having found a few. Photography has made it possible to overcome the problems of drawing, which in turn has helped the classification process. Despite this, some groups were not able to identify the global pieces built, and they continue to look at the individual elements of shapes.

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Impact of a geometry course on pre-service teachers' images of geometry through the lens of fundamental ideas

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Keywords: Geometry, teacher education, fundamental ideas, mental image.

Introduction and theoretical framework

Geometry is an important part of mathematics instruction. It is the basis of other sub-areas of mathematics, different fields in science, technology, and arts (Wittmann, 1999). Furthermore, it is a tool for the acquisition of intellectual, cognitive, and practical life competencies (Wittmann, 1999). Despite the recognition of the role that geometry plays in school mathematics, it is often neglected compared to other areas of mathematics, such as arithmetic (Backe-Neuwald, 2000). These findings raise certain questions regarding current geometry education: What meanings do learners (i.e., students, pre-service teachers) assign to geometry? What geometrical concepts do they learn?

Geometry curriculum is nowadays structured around so-called fundamental ideas, which can be seen as "tools to organize the phenomena of the physical, social and mental world" (Freudenthal, 1973, p. 41). In the literature (e.g., Backe-Neuwald, 2000), different conceptualizations of fundamental ideas for structuring the geometry curricula are reported. Based on Wittmann's model (1999) of fundamental ideas of geometry, Kuzle and Glasnović Gracin (2020) proposed the following empirical model: geometric forms and their construction, operations with forms, coordinates, special relationships and reasoning, measurement, geometric patterns, geometric forms in the environment, and geometrization. They showed that primary grade students (Grades 3–6) had a rather narrow image of geometry regarding fundamental ideas, which primarily reflected the first fundamental idea. This was partially due to the mathematics curriculum itself (Kuzle & Glasnović Gracin, 2020), but may have also been due to the role that geometry plays in school mathematics as well as in-service teachers' insecurity both regarding the selection of central geometry contents and the criteria that determine their significance (Backe-Neuwald, 2000).

Research questions and method

Based on the theoretical considerations and empirical results, we sought to study pre-service teachers' images of geometry through the lens of fundamental ideas of geometry as they progressed through a two-semester geometry course. The term image "refers to mental representations of a cognitive structure associated with a particular concept (i.e., geometry), built up over the years through various experiences" (Kuzle & Glasnović Gracin, 2020, p. 13). Two research questions guided the study:

- What fundamental ideas of geometry can be seen in pre-service teachers' drawings?
- How do pre-service teachers' images of geometry change during the geometry course?

Within the exploratory qualitative study presented in this poster, a cohort of pre-service teachers in primary mathematics (Grades 1–6) that were enrolled in the course "Geometry and its didactics I and II" participated in the study. The course itself was structured around the fundamental ideas of geometry, which were continuously made explicit. The main source of data were two drawings, which the pre-service teachers were to submit at the beginning and the end of the course. Drawings as a data tool were chosen because they are understood as expressions of mental images and provide unique insights into individuals' minds (Kuzle & Glasnović Gracin; 2020, Luquet, 1927/2001). The data included only 43 sets of drawings due to the second semester being administered in a digital format. The first author analyzed the data using an inventory developed by Kuzle and Glasnović Gracin (2020). Here, every drawn object was categorized into one or more fundamental ideas according to its interpreted meaning. Afterward, the descriptive statistics were calculated.

Results and discussion

The results revealed that the first fundamental idea was illustrated by all pre-service teachers at both measuring times. All other fundamental ideas were drawn by more pre-service teachers at the end of the two-semester course. The largest increase showed the idea of geometric patterns (37% more second drawings included a pattern compared to the first drawings), followed by geometrization with 32%. The number of drawings including the idea of operation with forms increased by 26%, followed by the idea of coordinates, special relationships and reasoning (18%), and geometric forms in the environment (14%). The lowest increase had the idea of measurement (9%), which was already drawn by a relatively high number of pre-service teachers at the beginning of the course. Thus, the fundamental ideas that were already highly presented at the beginning of the course showed a slight increase, whereas the fundamental ideas that were not that strongly represented showed a stronger increase. The results showed that the pre-service teachers' image of geometry broadened and became more multifaceted during the two-semester geometry course. Most of the pre-service teachers thought of more different fundamental ideas than before. Concretely, the drawings revealed, which course topics, thoughts, and new insights shaped the pre-service teachers' images. Based on these findings, the geometry course can be further developed to enrich the pre-service teachers' ideas of geometry even more, and consequently counteract their insecurities regarding teaching geometry.

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The teaching of geometry in primary education: Is geometry still neglected in school mathematics?

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Primary school mathematics has undergone many changes in the last twenty years. Geometry in particular was re-evaluated due to a paradigm change and was attributed greater importance in curricula worldwide. Despite a clear consensus that geometry instruction is an indispensable component of mathematics even in primary education, the status of elementary school geometry is still considered unsatisfactory in many cases. This report aimed to find out if earlier findings that geometry is neglected in school mathematics instruction can be confirmed based on new empirical data using a questionnaire. In total, 120 primary grade teachers participated in the study. The study confirmed the results from previous research, but also shed light on new emerging factors contributing to the unsatisfactory state of current geometry teaching in primary education.

Keywords: Geometry, teaching geometry, primary education, in-service teachers.

Introduction

For several decades, researchers have emphasized the importance of teaching geometry (Franke & Reinhold, 2016; Sinclair & Bruce, 2015; Vigilante, 1967). Engagement with geometric content promotes basic cognitive skills, allows the development of specific mathematical ways of thinking, and contributes greatly to understanding the world in which we live (Bauersfeld, 1992; Franke & Reinhold, 2016). Furthermore, all mathematics is permeated by geometric ways of thinking, because whenever a piece of visual information is perceived, analyzed, and stored, geometric thinking is fundamental to this process. Despite the acknowledged importance of geometry, it seems to have lost its position in school mathematics developing the reputation of being the "problem child" of mathematics teaching as was reported by Backe-Neuwald (2000) more than two decades ago.

Mathematics teaching in elementary schools has been influenced by many changes that have taken place in the last 20 years. Firstly, to counteract the worldwide trend of the reduction of geometry in the school curriculum (Clements, 2003; Van de Walle & Lovin, 2006), the role of geometry concerning perspectives on the teaching of geometry for the 21st century has been reassessed (Mammana & Villani, 1998). Secondly, the teaching of geometry has been re-evaluated due to a paradigm shift in school education to a focus on process-oriented mathematical competencies, such as problem solving and argumentation. The fundamental question of whether geometry is being neglected remains open, arguably with good reason, as the status of elementary school geometry instruction is still considered unsatisfactory, especially compared to arithmetic (Clements, 2003; Franke & Reinhold, 2016). Similar results were reported in the study by Backe-Neuwald (2000) in which nearly 80% of primary grade teachers agreed with the statement that the teaching of geometry is neglected and mostly serves as entertainment rather than important mathematical content. Since these findings, however, the mathematics curriculum in Germany has been revised, making a reassessment of the empirical findings of Backe-Neuwald (2000) necessary. Furthermore, only a few studies (e.g., Sitter, 2019; Wiese, 2016) are based on recent data subsequent to the paradigm shift.

This paper takes into consideration a number of the recent discussions in the literature by focusing on the role of geometry instruction in elementary school mathematics. Specifically, utilizing two questionnaire items adapted from Backe-Neuwald (2000), I investigate whether there are any changes in the previously reported situation regarding the neglect of geometry in school mathematics.

Related literature

The study by Backe-Neuwald (2000) involving 108 in-service teachers represents a milestone in the contemporary empirical examination of geometry in primary schools and its role in the school structure, its results have been followed up in the form of a replication (Sitter, 2019) and an extension study (Wiese, 2016). The teachers' perceptions of school geometry were very ambivalent: when asked about associations with geometry instruction in elementary school, some teachers described it as "a welcome change" or "an exciting thing," whereas others considered it "secondary," "beside the point", or "not important" (Backe-Neuwald, 2000, p. 16–18.). As noted earlier, almost 80% of the in-service teachers surveyed believed that geometry was neglected in elementary school, not because of the curricular requirements of the syllabi, but due to issues concerning its actual implementation in the classroom (Backe-Neuwald, 2000). Only 15% of the teachers surveyed disagreed with the statement. In 89 cases (82.4%), the teachers stated that they have to utilize the available instructional time for more important mathematical content, namely arithmetic, which was identified as the main reason for the neglect of geometry in school mathematics. Moreover, many teachers (n = 65; 60.2%) shied away from the preparation intensity of geometry instruction (Backe-Neuwald, 2000) which was identified as the second reason for geometry being neglected in school mathematics. This was confirmed in the work of Sitter (2019). Along these lines, the third reason reported was that too little support in the form of additional materials is available for the preparation of geometry lessons.

Teachers themselves play a crucial role when it comes to explaining the deficit in the teaching of geometry. Teacher training has been strongly criticized in the past. Bauersfeld (1992) noted that elementary school teacher education is divided among different faculties into subject and pedagogical content knowledge which does not reflect school reality and, above all, limits development in the field of didactics. Such shortfalls are specifically attributed to geometry courses, which lead to subject-didactic uncertainties that either partially or completely prevent teachers from teaching geometry (Sitter, 2019). Further, Clements (2003) reported that a large proportion of prospective mathematics teachers only reached Level 2 of the van Hiele model of geometric reasoning (i.e., descriptive reasoning). Similar phenomena were reported in the study by Backe-Neuwald (2000). The teachers viewed their own previous education with regard to geometry topics and content as insufficient (n = 28; 25.9%) and they felt insecure teaching geometry (n = 21; 19.4%). The latter was most prominent in teachers with little teaching experience as well as out-of-field teachers (i.e., non-specialists). According to Backe-Neuwald (2000), this leads to a high degree of professional uncertainty in the conveyance of geometric content and, ultimately, to the deprioritization of geometry lessons. Textbooks were identified as the sixth main factor (n = 18; 16.7%) contributing to the neglect of geometry because textbook suggestions were perceived as being too trivial, easy and not challenging enough. Other reasons, such as time intensive methodological preparation, intensity of subject preparation, and difficulty of student assessment, were mentioned by less than 10% of teachers. Difficulty of student assessment was also reported by Sitter (2019).

Research process

For this study, a mixed-methods **research design** was chosen. Elementary schools were selected through existing contact with the resercher's university and random inquiries. The **sample** of 120 in-service primary teachers (Grades 1–6) consisted of 22 male teachers (18.3%) and 97 female teachers (80.8%). One participant did not provide gender information. A total of 90 teachers taught mathematics as subject specialists (75%) and 29 of them were non-subject specialists (24.2%). The data of one participant was not provided. In terms of professional experience, 16 teachers (13.3%) have been teaching mathematics for less than or up to two years, 18 (15%) for up to 5 years, 19 (15.8%) for up to 10 years, 16 (13.3%) for up to 20 years, and 51 (42.5%) for more than 20 years.

The **main source of data** was a questionnaire on the state of the teaching of geometry in primary grades that was based on an adaptation of the instrument of Backe-Neuwald (2000). Additionally, new items were developed based on literature published in the last 20 years (e.g., Clements, 2003; Franke & Reinhold, 2016). To cover a broad field of research, six sections were included. Each section consisted of several items with both open and closed questions. In addition to personal information, there were questions about the characteristics of geometry lessons, the use of materials, the goals and aspects of teaching geometry, possible reasons for the neglect of geometry, and the participants' personal attitude toward teaching geometry. The questionnaires were either distributed on site or were sent by e-mail. All questionnaires were returned to the author by mail.

In this report, I focus on the section of the questionnaire dealing with the possible neglect of geometry, which was measured with two items. Both items were **analyzed** in order to answer the research question. The first item was as follows: *"It is frequently claimed that geometry instruction is neglected in elementary education and only leads a 'Cinderella existence' there. What is your opinion?"* (Backe-Neuwald, 2000). Three options were given for the item where one response was to be marked and then justified in a free-form text response. In addition to *"I agree because..."* and *"I do not agree because…"*, *"Other"* was provided as a third option. It was assumed that there may be teachers who are not in a position to or are unwilling to take a position. The analysis of the item was based on qualitative content analysis according to Mayring (2010). For this report, the analysis-guiding categories emerged inductively from the hypothesis, in particular, the analysis variables of teaching experience, professional background, and teacher gender. In contrast to the first item, the second item was a standardized item which was slightly adapted from Backe-Neuwald (2000). It included statements on reasons leading to the neglect of geometry instruction. Here, 12 statements taken from Backe-Neuwald (2000) were supplemented with three additional statements (i.e., 5.2.1, 5.2.6, 5.2.7) which were identified in the recent literature (Clements, 2003; Franke & Reinhold, 2016), with an option of writing an additional statement (i.e., 5.2.16). The in-service teachers were asked to mark the aspects that were most significant to them, and to rank five responses accordingly (e.g., 1-most significant). The item was evaluated according to the frequencies of all response options as well as according to assigned rankings. The items complemented each other; by evaluating the open-ended item, teachers could express themselves freely without having been steered in one direction by predetermined response options. Through the additional evaluation of the standardized item, the spectrum of answers was expanded.

Results

Here, I present the results concerning the two items. The first item explicitly addressed the question of whether geometry instruction is neglected in the teachers' opinion and asked them to give reasons for their answer. In total, 45.8% of the participants agreed with the statement (n = 55), 35.8% of them disagreed (n = 43), and 14.2% did not take a clear position (n = 17). Five surveys were disregarded because either no information was provided or more than one answer was given. The affirmative responses were relatively evenly distributed across gender, professional background, and teaching experience in the groups with up to 5, 20, and more than 20 years of experience. However, affirmative responses were represented above average (60%–66%) in teachers with up to 2 years and 10 years of teaching experience. Conversely, 40% to 44% of those teachers with up to 5, 20, and more than 20 years of experience did not agree with the statement.

For the qualitative analysis of the item, the responses were categorized along the chosen focal variables of teaching experience, professional background, and teacher gender and presented as a selection, since there was often considerable overlap between the answers. Looking at the teachers' justifications for their respective statements, several commonalities stand out. Geometry instruction is <u>neglected</u> in elementary school [...] because

> "for almost all teachers arithmetic is in the foreground." (female, up to 20 years, specialist)
> "when there is time pressure, practicing 'arithmetic' is preferred." (female, over 20 years, specialist)
> "the weighting of work in the textbooks is also evident. Few textbooks deal adequately with geometry topics, especially in Grades 1–4. This means that the teacher has to provide a lot of additional material and it also gives the impression that it is of less importance." (female, up to 10 years, specialist)
> "the RLP [curriculum] is crammed with other elementary basic knowledge and a lot of time is lost to the sustainable learning of the basic arithmetic operations." (female, up to 10 years, specialist)
> "many consider the topic of 'Numbers and operations' to be the absolute No. 1 and since it is so packed, geometry goes under." (female, up to 10 years, specialist)

The predominant reasons given for the neglect of geometry were time pressure (n = 15), and the importance of arithmetic (n = 22). The latter was often mentioned in connection to the mathematics standards for primary education. Furthermore, the often very time-consuming and material-intensive preparation was also frequently mentioned as a factor contributing to the neglect of geometry (n = 7), which was also brought up in connection with textbooks. The reasons given mainly focused on practical requirements and perceived constraints (i.e., time pressure, the importance of arithmetic) or convenience (i.e., intensity of preparation). A disregard for the content of geometry was rarely communicated in the statements, but did occur occasionally:

> "it does not have the same value as imparting basic knowledge." (female, over 20 years, specialist)

Finally, in five cases the opinion was expressed, especially by those with more teaching experience, that non-specialist teachers in particular simply do not have the necessary qualifications to teach geometry confidently and appropriately:

> "many (colleagues) feel insecure or perceive it as a 'gimmick'." (male, up to 20 years, specialist)
> "there is a lack of trained professionals." (female, over 20 years, specialist)
> "lateral entry employees often lack the ability of methodical approach in teaching geometry." (female, over 20 years, specialist)

There are some comments stating that geometry is not neglected, but on closer inspection these display a misunderstanding of the question: they do not offer a general assessment of elementary school geometry as a whole (except in three cases), but rather refer primarily to the teacher's own practice based on school or state curricula (n = 33). Geometry instruction is <u>not neglected</u> in elementary school [...] because

> "I always meet the curriculum requirements." (female, over 20 years, specialist)
> "one lesson per week is a fixed geometry lesson." (female, up to 5 years, non-specialist)

The results from the standardized item of the questionnaire concerning the reasons for the neglect of geometry instruction in primary school is shown in Table 1. In addition to marking the factors influencing the implementation of geometry in school mathematics, the teachers were asked to rank the five most significant factors (i.e., statements). From this, a simple score was generated, which awarded the most important reason five points, the second most important four points, and so on. The score, as opposed to just looking at the absolute frequencies, allows a better assessment of the importance of the reasons for neglecting the teaching of geometry – even though both metrics show an almost equal ordering of the reasons. Out of 120 surveys, 117 teachers filled out this item. Additionally, not all participants ranked their answers (n = 24), so that this data was not taken into consideration for the Top 5 Score. The teachers' answers on this item predominantly confirmed the findings of the aforementioned open responses item. Accordingly, the importance of arithmetic was again the primary reason given for neglecting geometry (items 5.2.6 and 5.2.8). Item 5.2.8 was evenly distributed across the professional background with two-thirds of participants from both groups agreeing with this statement. The intensity of preparation for geometry continued to be significant, especially as evidenced by the comparatively high score (item 5.2.3). Similarly, the role of the teacher was addressed in the sense of uncertainty in the conveyance of geometry content (items 5.2.11 and 5.2.2). Item 5.2.11 was also relatively evenly distributed across professional background; 38.9% of subject specialists and 48.3% of non-specialists agreed with this statement.

Table 1: Reasons for neglect of the teaching of geometry

Item	Statement	Absolute frequency (max. 93)	Score Top 5
5.2.8	The arithmetic topics have to be worked through, leaving little time for geometry topics.	80	251
5.2.3	Geometry lessons require a lot of preparation in terms of the creation and provision of materials.	76	193
5.2.6	The arithmetic topics are more relevant for the school track than geometry topics.	54	137
5.2.11	Teachers feel insecure when teaching geometry.	49	113
5.2.5	Student performance is difficult to assess and time-consuming.	47	85
5.2.2	Geometry instruction requires a lot of subject preparation.	41	105

5.2.9	There are too few aids for the preparation of geometry lessons.	40	92
5.2.4	Geometry lessons require a lot of methodical preparation effort.	36	89
5.2.1	The learning objectives for teaching geometry do not seem as clear as for teaching arithmetic.	29	88
5.2.15	Many of the suggestions in textbooks for teaching geometry are trivial, too easy, and not challenging enough.	25	46
5.2.10	Teachers perceive their own previous education with regard to geometry topics and content as insufficient.	21	50
5.2.7	Geometry topics seem arbitrary compared to arithmetic instruction.	14	40
5.2.16	Other: __________________	14	23
5.2.13	The development of geometric skills is subject to an inner maturation process in children, which can neither be accelerated nor halted.	13	25
5.2.12	The goals of geometry instruction are not as important because ________.	5	6
5.2.14	Geometry instruction does not begin until secondary school.	2	0

Thus, the issue of neglect seems to originate predominantly from temporal-practical reasons, and not from a disregard for the importance of geometry. One of the most frequent reasons that the teachers gave on item 5.2.16 (i.e., "Other: ________") was the limited material availability in schools, including both analog and digital tools. This opens up a certain area of tension, since apparently teaching material in the sense of geometry as a visual instruction is quite appreciated, but it is more likely to be made available by the school rather than to be developed as a part of the teachers' own lesson preparation. Moreover, the teachers also reported that mathematics textbooks contributed to the negative trend for the following reasons: poor coverage of geometry topics, lack of usable tasks, and lack of integration of geometry topics into the "overall mathematics".

Discussion and conclusions

In the literature, it has been postulated – often on an insufficient empirical basis – that geometry is neglected in mathematics. This paper focuses on the question of whether there are any changes regarding the neglect of geometry teaching in the last two decades in relation to two items adapted from Backe-Neuwald (2000). The study results confirmed the hypothesis that despite the paradigm shift that occurred two decades ago, geometry is still insufficiently taught compared to other areas of mathematics, but positive changes were observable compared to findings reported two decades ago.

While about 80% of the participants in the study by Backe-Neuwald (2000) agreed with the thesis that geometry instruction is neglected in elementary school, only 45.8% of participants in this study agreed with that statement. In other words, 15% of participants from the Backe-Neuwald (2000) study and 35.8% of participants in this study disagreed with the thesis. Despite the positive trend over the last two decades, the perception that geometry is neglected still exists to a high degree that cannot be disregarded. The main reason for this seems to lie predominantly in the perceived importance of arithmetic (n = 80; 86%, item 5.2.8), especially in the early school years as has been reported previously in the work of Backe-Neuwald (2000) (n = 89; 82.4%). The fact that geometry can and must be the basis of arithmetic, both in terms of content and in the formation of mathematical teaching-learning processes (Bauersfeld, 1992), is not taken into consideration by the teachers at all. Similarly, the teachers from both studies identified the preparation intensity of geometry instruction in terms of material creation and provision (item 5.2.3) as the second reason for the neglect of geometry. Both groups also emphasized that standardized materials were only available to a limited extent. Geometry lessons are guided by the visualization of different concepts and the provision and use of materials in this respect is paramount, but the reality in schools still seems to be deplorable. If improvements in this situation would make it possible to upgrade elementary school geometry, such measures should certainly be taken, because the value of continuous material use for the learning success, especially in the area of spatial imagination, is undisputed (Franke & Reinhold, 2016). It is worth noting, however, that an improved availability of materials, as an essential requirement of making changes to the teaching of geometry, would not alleviate the perceived time pressure under the impression of the excessive weight of arithmetic; yet this is a particularly pressing problem of geometry instruction for teachers. Two reasons for neglecting geometry that played a minimal role in Backe-Neuwald (2000) with less than 10% of teachers identifying them, seem to play a greater role two decades on. Difficulties regarding performance assessment in geometry (item 5.2.5) was ranked fifth (n = 47; 50.5%), and intensity of subject preparation (item 5.2.2) was ranked sixth (n = 41; 44.1%). This may be due to curricular changes which placed extensive focus on process-oriented mathematical competencies such as problem solving and argumentation.

The present work has highlighted various aspects of geometry teaching that have direct implications for professional practice. As in the study by Backe-Neuwald (2000), teachers felt insecure when teaching geometry (item 5.2.11). Even though this was ranked as the fourth reason in this study and the fifth reason in the study by Backe-Neuwald (2000), the new results call for attention. The feeling of insecurity more than doubled in the last two decades, from 19.4% to 52.7%. Whereas in the study by Backe-Neuwald (2000) the perception of inadequate previous education (item 5.2.10) was the fourth reason given (25.9%), this was reported by 22.6% of teachers, and hence was ranked 11th. The open-response items in particular bring these issues even more to light. The finding that the qualifications of many teachers, especially those who are not fully trained, not only are insufficient to teach geometry competently but also raise insecurities in teaching geometry should give cause for concern, especially since this aspect has not yet appeared as a prominent reason in the literature. Comprehensive teacher training focusing on both content and pedagogical content knowledge is essential as well as supplementary training in order to dispel uncertainties when teaching geometry (Jones & Mooney, 2003). Also, the collaboration between researchers and practitioners (e.g., design research, action research) may provide teachers with the needed support. We need to understand how teachers may be better prepared to play the roles that have been emphasized in the literature as well as ongoing developments (Sinclair & Bruce, 2015). The importance of this aspect can hardly be overemphasized, especially given the increasing proportion of not fully trained teachers. It should not be forgotten that the participating teachers only represent in-service primary teachers to a limited extent. Despite this drawback, the study results not only provide an up-to-date insight into elementary school geometry teaching but also provide new evidence for questions that have not yet been examined in the literature which call for immediate attention that should be examined by educators.

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Escape From Plato's Cave: An Enactivist Argument for Learning 3D Geometry by Constructing Tangible Models

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Like Plato's allegorical cave-dwellers, students of three-dimensional geometry seldom get to handle the real thing, working instead with two-dimensional silhouettes. Such historical sensory deprivation may partially explain students' generally poor conceptual understanding of this core content and alienation from the field. Operating from a perspective of embodied learning, our design-based research study invited middle-school students to collaboratively construct and investigate voluminous objects. We present qualitative analyses of empirical results from implementing two experimental geometry activities. For both cases, we characterize students' critical insights as shifts in perceptuomotor attention leading to refinement of geometric argumentation. We implicate students' realization of an available 3D medium affordances catalyzing these shifts. The findings contribute to a socio-material elaboration of embodied learning for school geometry.

Keywords: Spatial geometry, Embodiment, Manipulatives, Visualization

Introduction

Historically, the mathematical discipline of geometry originated from mundane practice—situated, embodied know-how serving personal, social, and professional contexts, such as carpentry, agriculture, and navigation. However, while humans live and act in 3D space and engage voluminous objects as part of this naturalistic spatial comportment, geometry scholarship—the reification and scrutiny of these objects' structural properties—has historically depended on 2D material media and their consequent "flat" perspectives (Alsina, 2010). This dependence has a price. For example, a recent study by Fujita et al. (2020) documents elementary- and middle-school students' poor performance on spatial geometry problems, especially those requiring multiple reasoning steps.

Scholars from different disciplines have claimed that if students began studying geometry with their embodied sensibilities, we could preempt their poor engagement and low performance in the discipline (Freudenthal 1971; Thompson, 2013). Pedagogical advocacy to work with "the thing itself" harks back to Enlightenment (Rousseau, 1755/1979; Froebel, 1885/2005) and modernity (Montessori, 1949/1967), bolstered by Gaspard Monge and Felix Klein's approach to the development of intuition about complex structures through the construction of concrete models (Mueller, 2001). Still, after over 60 years of empirical research on the use of manipulatives in mathematical classrooms, their cognitive effects and optimal utilization have yet to be established (Bartolini Bussi et al., 2010).

We argue that realizing a material transformation in (spatial) geometry instruction, from 2D to 3D, demands a paradigmatic epistemological shift from idealistic to realistic views on geometry; from a representationalism model that separates perception, cognition, and action to an embodied model.

Cognitive argument: from a perception-cognition-action model to embodied cognition

Per Plato, she who seeks mathematical knowledge should strive to obtain mental representations as close as possible to the ideal (non-physical) forms. Millennia later, Platonic metaphysics persevere through cognitive-science theoretical models that box the mind inside the skull as an amodal ethereal switchboard between its earthly perception input and action output (Hurley, 2010). Over the recent decades, however, cumulative data from various fields (neurobiology, robotics, kinesiology) is casting doubt on this classical model (e.g., Willems & Francken, 2012).

The embodied turn in cognitive science rejects the hierarchical mind–body separation and stresses that perception and action are formatively constitutive of our thinking—cognition is modal and situated activity (e.g., Chemero, 2013). The mind's function is not to *represent* the environment precisely but to *engage* with it dynamically vis-à-vis socio–biological task demands and emergent contextual contingencies. The environment offers opportunities for potential action—*affordances* (Gibson, 1986)—that the agent interactively discerns and incorporates. When we engage the world with fellow humans, we coordinate with them perceptual orientations in relation to shared situations, from early development (Tomasello, 2019) and through to professional practice (Goodwin, 2018).

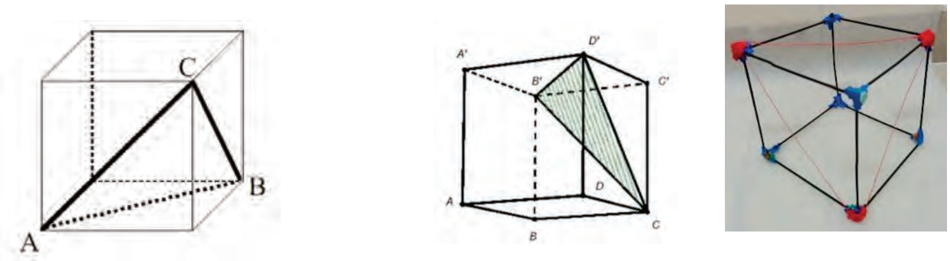
Imported to mathematics instruction, tenets from the embodied paradigm of the cognitive sciences suggest that learning new concepts begins with discovering new ways to act in the environment, using new instruments to perform tasks on discovered affordances (Abrahamson & Bakker, 2016). Working with the things themselves, students develop a capacity to act efficiently, describe the world mathematically to coordinate collaborative actions, iteratively encounter more complex problems, and ultimately modify the environments to solve emergent problems (Abrahamson et al., 2020).

Spatial geometry challenges and possible theoretical and practical solutions

Roth and collaborators demonstrated that geometry knowledge may emerge as enacted exploration of concrete instructional resources. Their examination of primary-school students' classification of geometric objects fuses material phenomenology and phenomenological sociology to view geometry as cultural–historical motivated sensuous labor (e.g. Bautista & Roth, 2012). Yet, whereas overtly embodied routines, such as gesturing, manipulating objects, and applying mechanical construction tools, are considered essential for young students' geometric reasoning and problem solving (Kaur & Sinclair, 2014), these resources almost disappear from older schoolchildren's instructional activities. Perhaps most acutely, mainstream spatial geometry education skips "the real thing," immediately requiring of students to visualize 3D objects given their 2D representations (Widder et al., 2019). Consequently, students struggle "to overcome the perceptual appearance (or 'look') of the given diagram" (Fujita et al., 2020, p. 235). For instance, for one of their survey items (see Figure 1, on the left), just 17 % of the 5th-grade students, 34% of 7th-grade students, and 52% of 9th-grade students marked the correct answer (percentages rounded). In light of their results, Fujita et al. (2020) call to revise primary and secondary school curriculum to provide students with more opportunities to develop both spatial skills and geometric knowledge for productive argumentation. However, the researchers do not indicate what types of tasks could possibly serve as context for realizing this call.

Investigating a 3D DGE (dynamic graphic environment), Mithalal and Balacheff (2019) explored conditions in which construction tasks stimulate students' transition from working with drawings and iconic visualizations to perceiving geometric properties of figures and non-iconic visual displays.

They claim that this transition depends on students' ability to perform certain figural operations (dimensional and instrumental deconstruction) in tasks designed to hone this ability.



In a cube, can you identify the shape ABC? Choose your answer from (a) – (e).

(a) Right-angled triangle
(b) Isosceles triangle.
(c) Right-angled isosceles triangle
(d) Equilateral triangle
(e) Scalene triangle

ABCDA'B'C'D' is a cube. Answer the true/ false questions and explain your reasoning.

1. CB'D' is a right-angled triangle. The right angle is ___?
2. B'D' is the shortest side of the triangle CB'D'.
3. Triangle CB'D' has an obtuse angle.
4. In triangle CB'D', all angles are equal.

Figure 1: Tasks involving 2D representations of 3D geometrical shapes

With Fujita et al. (2020) and Mithalal and Balacheff (2019), we acknowledge the key cognitive role of sensory perception in understanding spatial geometrical forms. Yet, *we conceptualize sensory perception as necessarily serving and emerging from goal-oriented action*. We thus seek to investigate how students ground geometry concepts in action-oriented perception. Our study accordingly evaluates a set of enactive tasks designed for high-school students to develop geometrical perceptions through multimodal action-based interactions with concrete material.

Constructing tangible models as embodied design for spatial geometry learning

Supported by the embodied perspective, we are looking to capture and theorize conceptually significant shifts in students' perceptuomotor attention towards voluminous geometrical solids. The two vignettes, both from video-recorded data gathered in Jerusalem, Israel[1], illustrate how each activity's unique sensorimotor affordances enabled students to engage in conceptually generative collaborative enactment and argumentation. The first vignette (V1) presents a task (Figure 1, right) designed to stimulate perceptual coordination of 2D diagrams and their 3D counterparts: students use a "3D pen" to construct a voluminous cube from its "flat" image, then they manipulate the model to investigate its properties per the problem instructions. The vignette exemplifies a "first step out of Plato's cave," where students tentatively realize what they can, may, and should do with a 3D model. In the second vignette (V2), students assemble a very large multi-unit geometrical form and then identify the "hidden" form that emerged in between the units. This vignette exemplifies students' zealous unshackling of any remaining "geometry-is-flat" predilections.

[1] Qualitative analysis of additional cases is currently underway to evaluate for generalizability of these case studies

V1. Stepping out of the cave: Learning a 3D model's affordances for spatial problem solving

The teaching experiment presented in this vignette is a part of a wider educational research project evaluating students' experiences with 3D pen sketching while solving spatial geometry problems (for details, see Rosenski & Palatnik, 2021). Three 10th grade students (T, G, & M) faced the task shown in Figure 1 (right) for approximately 20 minutes. Male student G drew the triangle inside the cube with a 3D pen. Next, over approximately 8 minutes, the students discussed the problem but made no significant progress. They were inclined to believe that CB'D' is a right-angled triangle.

T:	No, but you know that this is a cube, and this (cube face) is a square, and then, this is 90 degrees, then this is 90 degrees, and then it (diagonal) bisects the angle, so it is 45(degrees).
G:	The question is, if we rotate (two faces, where the edge is an axis) [gestures "rotation" with two palms as faces], would it be the same angle? Could it be?
M:	To them, it (the angle) in the picture also looks like that (90 degrees).
T:	In the picture, it is just from a different angle, if you turn it like this [adjusts the model], you can see that this [points] is the right angle if this is stretched [pulls up the slightly sagging plastic diagonal of the top face of the cube.]

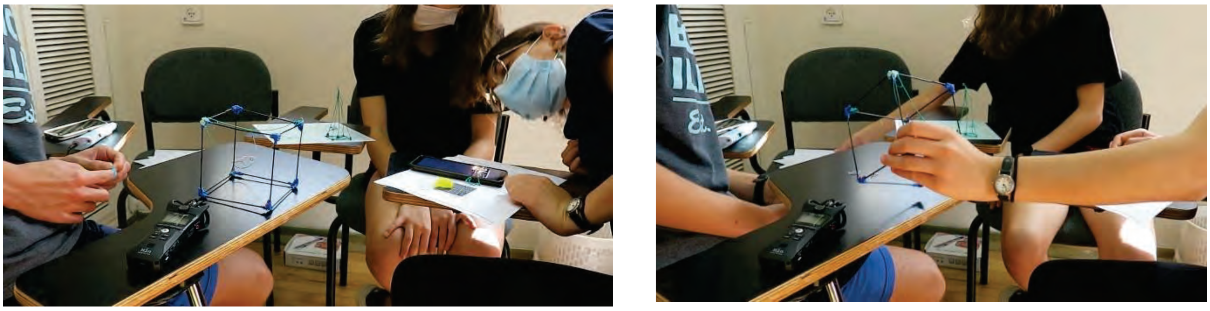


Figure 2: Investigating 2D representations of 3D geometrical shapes

During these eight minutes, students left the model standing between them on G's writing surface (see Figure 2). Remarkably, the students made inferences and conjectures about the task without taking the model in hand, only lightly touching it and making minor adjustments. Most of these adjustments reoriented the 3D model vis-à-vis the given 2D diagram.

In this phase of problem solving, the students were reluctant to make inferences based on the appearance of the 3D model. They approached the 3D model to correlate it with a cultural form that they are used to—a 2D diagram. In particular, the students sought to "flatten" the 3D model such that it would be seen as identical to the 2D diagram, where CB'D' presents an apparent right angle (see Figure 2-left, where T twists her body). However, a retinal image of a 3D object is not similar to a 2D projection. As Gibson (1986) has argued, we notice optical invariances of the object under the movement of the source of light, movement of the observer, movement of an observer's head, and manipulations and local transformations of the object itself. Naturalistic interaction could possibly untether the students from "paper math." Soon after, indeed, the students utilized these affordances of their 3D model, discerned it invariant features by handling it, and made correct inferences.

Several factors prepared the A-ha moment. For instance, G was unsatisfied with the claim that CB'D' is a right-angled triangle. He argued: "If these two angles look like the same shape, if they are both right angles and a triangle has a total of 180 degrees, then the triangle cannot exist”; “This is impossible. If you rotate it each way to make it (one of the angles) look like a right angle, then you can rotate it in a different way (to make other angle look like a right angle)”. Yet his teammates, still “deep in the cave,” had still to be convinced.

At the 14th minute of tackling the task, we witness a shift in students’ interaction with a model. T took the model in hand (Figure 2-right) for the first time, rotated it for five seconds, and 20 seconds later said, “Now I get it.” In the next minute, T tried to formulate her vision, accompanying her explanation with more than 5 complex model rotations (i.e., more than 2 axes involved). The core of her argument was in line with what G had previously argued: from different angles, the triangle appears isosceles, and yet its angle “can’t be 90 for all of them.” During this minute, G’s attention was on the rotating model, enabling him to see it under these transformations. In contrast, M focused only on the 2D diagram and tried several times to draw her peers’ attention to it: “Look at the picture!” It took another minute for G to formulate an answer and a valid argument.

T:	So you think all the angles are 60 degrees? Is it an equilateral triangle?
G:	Yeah, look, all the sides are of the same length [traces with finger three sides in succession] if you look at it. Is that true?
T:	Mmmm…I don’t know.
G:	Look [adjusts the model], all the sides (of the triangle) are exactly a diagonal [traces a diagonal of the upper face], the diagonal of a square. All the squares are the same. And that means, if all of the sides are equal [points on a different face], all of the angles are equal, and they’re all 60 degrees.
T:	Ok... [types into the response form] We think that all sides are equal; therefore, the angles are also equal... equal to 60. It’s an equilateral triangle.

To summarize, the task was difficult for the students, even with a 3D pen and a model. In line with Fujita et al. (2020), the task demanded that students harmonize their spatial reasoning skills with domain-specific knowledge of planar geometry (properties of squares and triangles). It took students time to utilize an available 3D medium and realize its affordances (a spectrum of perspectives on the equilateral triangle for the team members). M’s decision to stick to a 2D drawing may explain her low contribution to the final effort. In contrast, when T physically rotated the model, her actions apparently spurred and supported her spatial reasoning and were visible to G. He, in turn, observed these *physical* rotations, which allowed him to refine his previous arguments based on *mental* rotations and apply the corresponding geometric knowledge to the new 3D situation.

V2. Further steps: Constructing enactive argumentation—gesture, action, medium

In the second teaching experiment, part of “Geometry In… and Out” (Benally et al., 2021), four 7th grade students constructed voluminous solids (Figure 3) then worked on the following questions: “Comparing the volumes of the large and small tetrahedra that you built, how many times the volume of the large tetrahedron is greater? Explain your answer. Several small tetrahedra compose the large tetrahedron. Can you describe a three-dimensional shape between them? Can you construct it?”

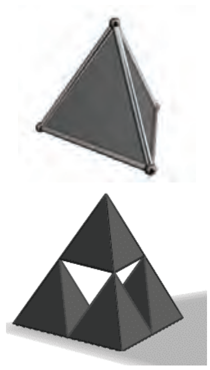
Once the group had constructed the first small pyramid, Yali placed it on his head (Figure 4a). Nami, using Yali as a stand, gestured on him that this polyhedron is called “arba-on” (“arba” is four in Hebrew). The palms of her hands present the polyhedron’s faces. Then, removing the model off Yali’s head, Nami gestured similarly, though with her forearms, to present the same four faces (Figure 4b).

It took the students approximately 11 minutes of collaborative work to construct a large model and begin answering the items. Their plan for estimating the large tetrahedron's volume was to decompose it into its component parts. They easily recognized four small tetrahedra: "three at the base, and one at the top." However, the students were not sure about the shape of a three-dimensional hollow between the tetrahedra (the octahedron outlined in red, for your convenience, in Figure 4c).

1) Your team has to construct a three-dimensional model of the following geometrical solid using a construction kit. The solid has the following properties:

- All the faces are congruent equilateral triangles.
- The same number of edges converge at each vertex.

2) The polyhedron you've constructed is called a tetrahedron. Construct a similar polyhedron whose edges are 2 times larger than the original one. You can use the image below for construction.



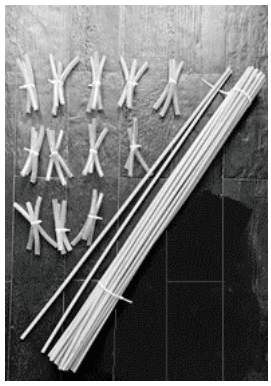


Figure 3: The tetrahedron construction task and materials.

a b c d

Figure 4: Different affordances of the available media as manifested in students' actions

Tami, Nami, and Gali suggested that the hollow is also shaped as a tetrahedron. Yali disagreed and offered to count the faces of the "empty space." He rotated the large model, hoping to render it more familiar, yet that action proved unhelpful. Tami remonstrated, "You just can't *see* this (tetrahedron)." To support her claim, she grabbed two sheets of paper lying on the desk and applied them successively as the polyhedron's faces, expecting these to total at four. Immediately, Yali appropriated Tami's strategy, just to disprove her. Summoning more paper sheets and distributing them over more group members, he marshaled an "octopus of hands" to simultaneously cover all the polyhedron's faces. The introduction of these auxiliary objects helped students to solidify the shape (Figure 4d), count the faces, and eventually write the following definition: "The polyhedron between four triangular (pyramids) has eight identical faces. Each face is an equilateral triangle."

This vignette illustrated the emergence of students' enactive argumentation through collaborative semiotic evolution of gestures into concrete media. Students' hands, semi-constructed models, and even repurposed found objects became instrumental in shaping a void—rendering a contested obscure object into an articulated, unequivocal, and publicly inspectable form. The hidden octahedron was born as a "prospective indexical" (Goodwin, 2018) then came forth through pointing, formative gestures, and construction media. Thus, the hollow solid inhabiting the larger structure was substantiated, reified. Eventually, once the form was delegated from imagination to media, the students could allocate cognitive resources to enumerate its facets and name the new geometric object.

Discussion and Conclusion

This article aimed to present theoretical foundations and empirical arguments for a set of embodied spatial-geometry curricular resources for middle school. We submitted that historical dependence of geometry education on 2D media is implicitly rooted in an idealistic Platonic tradition and the representational cognitivist model. We conjectured that tasks in which students construct 3D objects are more than “working with manipulatives”—they let students use their natural capacities of multimodal perception and collaborative action. We supported our argument through qualitative analysis of two vignettes exemplifying embodied design for spatial geometry learning.

We demonstrated how a group of middle-school students grounded geometric concepts of solids, their cross-sections, faces, and edges in goal-oriented, situated activity of constructing concrete 3D models. Our first vignette demonstrated that coordination of traditional and novel medium is not easy for students. They tentatively experiment with their new degrees of modal freedom—looking at objects, pointing at them, touching them, lifting and rotating them. New affordances catalyze shifts of students’ attention to relevant features supporting their geometric reasoning.

Mithalal and Balacheff (2019) considered the possibility of a continuous evolution from iconic to non-iconic visualization, where the figural operation of instrumental deconstruction would play a cohesive role. Our second vignette provides an empirical basis for this assumption. The transition from iconic visualization to non-iconic visualization was carried out by introducing tangible auxiliary elements (paper sheets in the form of polyhedron faces) into the 3D model. Thus, the students performed a naive instrumental deconstruction of shape, a mereological deconstruction of a 3-dimensional shape into four tetrahedrons and an unfamiliar shape, and finally a dimensional deconstruction of a 3-dimensional shape, which focused their attention on the 2-dimensional faces of the octahedron (see Palatnik & Sigler, 2021, for a theoretical discussion on the introduction of an auxiliary element as a shift in attention). This argumentation by action was later transformed into a normative formulation of properties—the formal definition of a geometric solid.

Constructing and manipulating tangible models creates opportunities for students to harmonize spatial skills and rigorous geometric argumentation as well as bridge iconic and non-iconic visualization. Yet, it is challenging to step out of Plato’s cave after a lifetime of unwitting incarceration. Middle school geometry should organize students’ engagement with 3D objects as one would any artifact—with untampered senses; with gross and fine motor actions; with all the tacit, evolutionarily endowed naturalistic sensibilities for orienting in the environment. Once students realize how to work with three-dimensional objects in mathematical activities, they can tap their know-how to build valid arguments grounded in enactive experience.

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Developing tools to analyse secondary teachers' mathematical knowledge for teaching basic geometry

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There has been extensive research on investigating teachers' mathematical knowledge in different topics and more recently at different school levels, with some work towards this in geometry (e.g., Ko & Herbst, 2020). In this paper we provide examples of research tasks used to describe and investigate teachers' knowledge of basic geometry. By basic geometry we mean concepts, theorems and properties of lines, angles and triangles in the junior secondary curriculum of South Africa. Since the choice of examples is a key element of any task, we offer a set of criteria for developing aspects of the example (i.e., figure in case of geometry) in tasks. The tasks, in turn, provoke teacher responses that enable us to describe aspects of teachers' mathematical knowledge for teaching geometry. We do this using a framework for producing tasks of varying difficulty level for teachers that could be used to illuminate their geometry knowledge for teaching.

Keywords: geometry, mathematics teacher knowledge, tasks.

Background of the study

Geometry is an important topic in school mathematics. Euclidean geometry is part of the school curriculum in many countries and the foundation for Euclidean geometry starts from primary grades by introducing shapes and their properties. However, geometry continues to be a topic that students struggle with (e.g., Luneta, 2014; Steele, 2013). One of the many reasons for students' low achievements, especially in geometry, is teachers' difficulty in teaching geometry (e.g., Bowie, 2013; Steele, 2013). Based on recent experience of working closely with different groups of secondary teachers in Gauteng, South Africa, the team of the Wits Maths Connect Secondary (WMCS) project recognized the need to support teachers to improve their knowledge of basic geometry.

The aim of our project is to understand the knowledge that teachers need to teach basic geometry in schools and then to support them through a professional development (PD) programme. Our definition of teachers' knowledge is based on Ma's (2010) conception of profound understanding of the content with breadth and depth of knowledge. We can also consider this knowledge as common content knowledge (CCK) (Ball et al., 2008) but it is specialised and needed for teaching. In order to understand this knowledge, we are in the process of developing a framework which focuses on developing teachers' mathematical knowledge *for* teaching, which is defined as including both *knowledge of mathematics* and *knowledge of mathematics teaching* (for details, see Adler, 2021).

We did not find any literature dealing with teacher knowledge when their mathematical knowledge is weak and little dealing with teaching basic geometry or teacher knowledge of basic geometry. By basic geometry we mean concepts, theorems and properties of lines, angles and triangles in junior secondary curriculum of South Africa. We are working with a relatively unique group of teachers who acknowledged the weaknesses in their geometry knowledge base. They openly discussed these and requested a course to help them to improve their geometry knowledge.

The PD provided us with a unique opportunity to understand what knowledge they bring, what is lacking and how we might help them to build their knowledge to the point where they can teach geometry with confidence and competence. So, from a research perspective, we asked ourselves how do we "measure" teachers' starting base and then "measure" gains over time? We do not focus in this paper on teachers' knowledge *per se*, nor the PD, nor do we look at change. None of these is possible without a framework that is geometry-specific for classifying tasks that would appropriately elicit teachers' knowledge. A key element of any task is the example/s used and in this paper we offer a set of criteria for developing examples within the tasks. In turn, these tasks provoke teacher responses and thus enable us to describe aspects of their mathematical knowledge for teaching geometry.

Literature Review

Several scholars since Shulman (1986) have tried to define specialised aspects of teachers' knowledge. Ball, Thames and Phelps (2008) categorised teachers' knowledge to specify different aspects which need to be focused upon. Following from Ball and colleagues' delineations of subject-matter knowledge (SMK) for teaching, Herbst and colleagues (e.g. Ko & Herbst, 2020) designed an instrument SMK-G to measure teachers' knowledge of geometry. They focused on tasks for teaching to capture differences in teachers' mathematical knowledge for teaching high school geometry. Through this work they offer distinctions between different SMK-specific tasks of teaching. For example, in one item teachers are provided with four algebraic expressions and asked to choose a set of measures for angles in a parallelogram that can be given to students as an exercise after they have learnt about interior angles of a parallelogram.

There are several other studies on designing assessment instruments to measure teachers' mathematical knowledge for teaching geometry. For example, in contrast to the assessment tools developed for assessing teachers' mathematical knowledge in a broader range of mathematical topics, Martinovic and Manizade (2018) proposed "that the measures should be designed as 'probes' around specific topics commonly taught by a targeted group of teachers" (p. 613). They developed a probe for a task on *Area of a Trapezoid* and developed a pedagogical content knowledge (PCK) instrument with rubrics used to "measure teachers' responses and … create PCK profiles" (p. 615). They propose that secondary mathematics teachers' PCK is a complex and multidimensional phenomenon with five dimensions: geometric knowledge; ability to provide geometric extensions; knowledge of applicable instructional strategies and tools; ability to ask diagnostic questions and knowledge of student challenges and conceptions. In another approach, Steele (2013) designed open-ended tasks and rubrics to understand teachers' thought processes in solving problems on geometry and measurement.

In the South African context, Bowie (2013) argued that both the research and curriculum documents acknowledge the difficulty that teachers face in enacting geometry in the classroom. In a study with 128 first year pre-service teachers, Luneta (2014) found that the majority of the participants had limited knowledge in basic geometry, falling within Level 1 of Van Hiele's theory. He argued that the participants "required not remedial but re-learning of basic these concepts" (p. 71). An earlier investigation in South Africa echoed this and showed that both pre-service and in-service teachers failed to reach the competence level expected of Grade 7 students (van der Sandt, 2007). It is therefore clear that the teachers we are working with in this study constitute a small sample of a much larger group of teachers in South Africa who have limited knowledge of basic geometry.

We draw on the existing literature to design tasks to investigate teachers' knowledge in geometry. However, to engage with pedagogical aspects successfully, teachers need strong content knowledge. In the context of our work, teachers needed support in understanding *basic* knowledge of geometry. In the existing literature, examples that are used in tasks are not problematised. The examples are a critical aspect of designing tasks for teachers, particularly those with a low knowledge base because it is essential that teachers are able to make a start on the tasks, even if they cannot complete all aspects of the task. In this paper, we focus on exemplification including tasks and examples for the purpose of describing teachers' knowledge. We now turn our attention to the theoretical framework which foregrounds examples in understanding teachers' knowledge.

Theoretical framework

We turned to Adler and Ronda (2015) for a theoretical framework on Mathematical Discourse in Instruction (MDI) where they focused on *object of learning*, *exemplification*, and *explanatory talk* (see Figure 1). Using a socio-cultural framework (Vygotsky, 1978), the authors (ibid) consider mathematics teaching as goal directed. In MDI, how an object of learning is exemplified is important. Another aspect, which is important in exemplification is the principle of variation. In their work, Adler and Pournara (2020) developed the use of exemplification for mathematics teacher education. They focused on algebra and functions. In MDI for algebra, examples and figures are different. By contrast, in geometry the example and the figure are the same (or at least inextricably linked). We therefore asked ourselves what does it mean to use exemplification in the topic of geometry? In the MDI framework (see Figure 1), exemplification is classified into examples, representations and tasks. This requires adaptation for geometry.

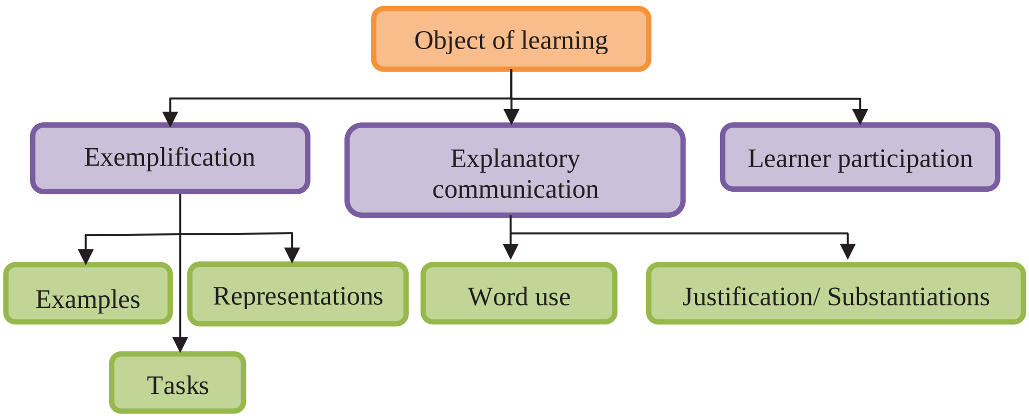


Figure 1: The MDI conceptual framework (Adler & Mosvold, forthcoming adapted from Adler & Ronda, 2015).

Developing the framework for geometry: Exemplification

The framework MDI focuses on the object of learning, exemplification, explanatory communication and learner participation. In developing the current framework for geometry, we borrowed these elements from MDI framework and modified them according to the needs of using it to examine the tasks used in the data collection tools pre-assessment, worksheets, and questions for the focused interview (see Takker et al., 2021). In this paper, we will focus on exemplification in geometry using examples as a means to access and ultimately to describe teachers' knowledge.

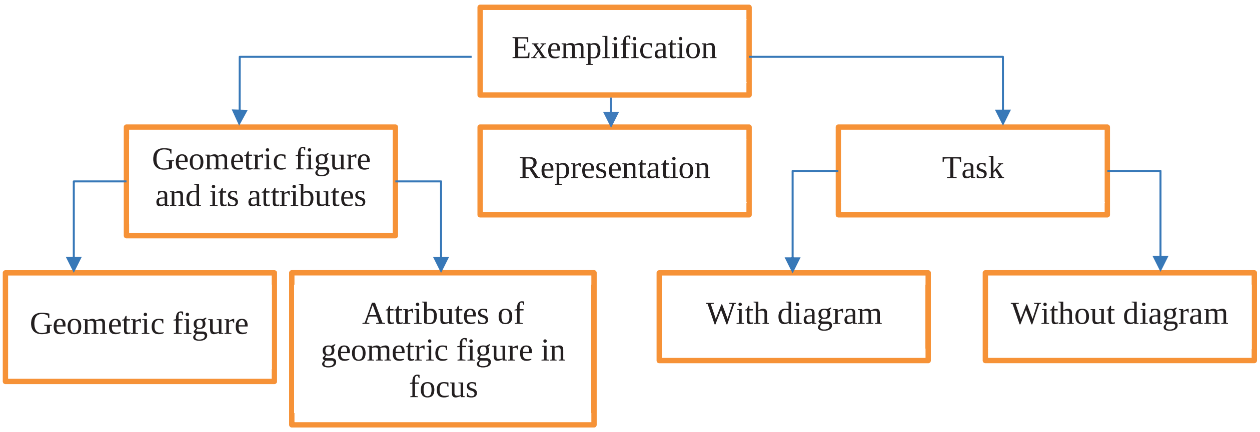


Figure 2: Exemplification aspect of the framework

The exemplification element is divided into three categories: (a) Geometric Figure (GF) and its attributes, (b) Representations, and (c) Tasks. These three categories are further divided into subcategories (see Figure 2). For instance, GF and its attributes is sub-divided into GF and attributes of GF in focus, whereas the task category is sub-divided into tasks with a diagram and tasks without a diagram. These subcategories are then examined under different levels. We needed a relatively fine-grained measure to describe and categorise teachers' knowledge, particularly as we worked in the largely unchartered waters of teachers' low knowledge base in geometry where we ultimately seek to investigate changes in their knowledge.

For us, categorising the tasks and examples is a start. And this is what is being offered in this paper. For example, GF is categorised into three levels with two sub levels based on the number of GFs in the given diagram, namely: one GF; two GFs (with same type of figures or different) and three or more GFs (with same type of figures or different). The *attributes of GF in focus* are categorised into three levels depending on axioms, definitions or theorems needed to solve the problem. In geometry it is important to identify the properties when the diagram is in a familiar orientation as well as in an unfamiliar orientation. Therefore, we categorised representation in standard orientation, in non-standard orientation or in multiple orientations. By *standard orientation* we mean the most familiar and common orientation. For example, having the "unequal side" of an isosceles triangle in horizontal position. And *non-standard orientation* refers to any other orientation than the familiar or most common orientation. In a complex figure, if one or more GF is in standard orientation and others in non-standard orientations then we consider the diagram is in multiple orientations. The subcategories for the third main category, i.e., task, are distinguished by whether the task comes with a diagram, or whether it is expected that a diagram (or a part of it is) be drawn as part of the solution. Six levels were set for a task which comes with a diagram whereas five levels were set for a question which does not come with a diagram. The levels for the task category were also based on the difficulty level of the task. For example, is it a numerical measure problem, or does it include algebra or a simple proof, or a complex proof without numerical measures, or perhaps algebra is also required? In rating the tasks of each data collection tool, the first two authors examined them individually and then discussed to ensure the validity. There were not many differences in each examination and agreement was quickly reached. We present three different tasks and their classifications using the framework.

EXEMPLIFICATION	
Geometric figure (GF) and its attributes	
GF (and diagram complexity)	*Attributes of GF in focus*
L1: One GF L2(a): Two GFs – same type of GFs L2(b): Two GFs – different types of GFs L3(a): Three or more GFs – same type of GFs L3(b): Three or more GFs – different types of GFs	L1: Use of axioms, definitions to identify instances of GF L2: Use of axioms, definitions, or properties in solving problems. L3: Use of axioms, definitions, properties or theorems in solving problems
Representation (diagram and symbols)	
L1: Standard orientation L2: Non-standard orientation	L3: Multiple orientations
Task	
With diagram	*Without diagram*
L1: Numerical measure problem L2: Algebraic measure problem L3: Numerical proof problem L4: Algebraic proof problem L5: Simple proof L6: Complex proofs and converse	L1: Draw a diagram based on given information for one GF L2: Extend a given figure L3: Draw a diagram based on given information for two or more GFs L4: Construct a simple figure with one GF L5: Construct a complex figure with two or more GFs

Figure 3: Levels in exemplification aspect of the framework

Examples of Tasks

We provide examples of three tasks and their analysis using the levels of the framework. The exemplification aspect of the framework guided us to design geometric problems with different combinations of levels. For instance, Example 1 (see Figure 4) has one GF in standard orientation and a numerical measure. The properties that can be used include sum of the interior angles of a triangle and angles opposite equal sides of an isosceles triangle. Therefore, this was categorised as L1, 2, 1, 1 which is explained in Figure 4.

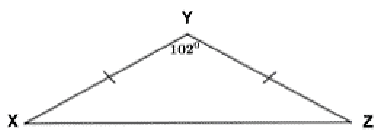
In ΔXYZ, $YX = YZ$ and $\hat{Y} = 102°$. Is the information given in the question sufficient to find the size of $\hat{Z}$. Give reasons. 	**GF -** L1: one GF **Attributes of GF -** L2: two geometric properties to solve: sum of the interior angles add up to 180^o and opposite angles to opposite equal sides are equal **Representation -** L1: Standard orientation **Task: with a diagram -** L1: a numerical measure problem

Figure 4: Analysis of Example 1

In the second example, (see Figure 5) there are two GFs in a non-standard orientation (because the parallel lines are not horizontal) and algebraic measure. The properties that can be used include angles formed on a pair of parallel lines cut by a transversal; vertically opposite angles and angles on a straight line. Therefore, this was categorised as L2(a), 2, 2, 2.

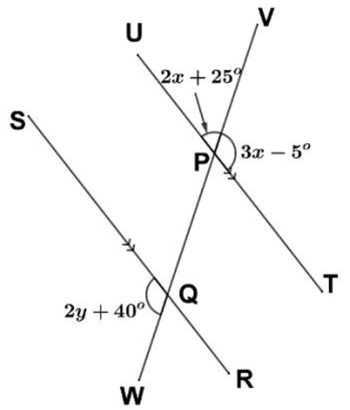
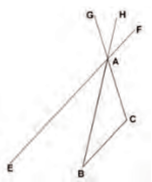
You will use algebra to answer this question. (a) Determine the value of x. Give reasons. (b) Determine the value of y. Give reasons. (c) Fill in the sizes of all 8 angles on the diagram. 	**GF** – L2(a): two GF **-** a pair of parallel lines with a transversal **Attributes of GF -** L2: geometric properties such as: angles formed on a pair of lines cut by a transversal; vertically opposite angles; and angles on a straight line **Representation -** L2: non-standard orientation **Task (with diagram)** – L2: Algebraic measure problem

Figure 5: Analysis of Example 2

The third example involves geometric proof. Example 3 (Figure 6) was categorised under L3(a), 2, 2, 6. The diagram given in the question includes three GFs - an isosceles triangle, a line parallel to one side and an angle bisector, and therefore coded as L3(a). The task was coded with L2 for *Attributes of GF in focus* since axioms, definitions and properties such as the properties of an isosceles triangle, definition of a bisector of an angle, properties of parallel lines, angles on a straight line, angles at a point need to be used to solve this problem. The diagram of Example 3 is not in standard orientation and that is the reason why it was classified as L2 for *Representation*.

While this is not a very complicated geometry task, it proved to be quite a challenging problem for the target group of teachers, as we saw in their difficulties with this and similar tasks. Therefore, we categorised Example 3 under L6 for the *task* aspect - a complex proof involving *if-then-because* reasoning.

In the given diagram, lines GC, HB and FE intersect at point A, BC = AC, and AH bisects GF. Prove that EF is parallel to BC. 	**GF -** L3(b): three GFs **-** isosceles triangle, a pair of parallel lines and an angle bisector **Attributes of GF -** L2: geometric properties such as: opposite angles to opposite equal sides are equal; sum of the interior angles add up to 180^o; bisection of an angle; angles formed on a pair of line cut by a transversal; vertically opposite angles need to be used to solve the task

	Representation - L2: non-standard orientation **Task (with diagram)** – L6: a proof of a non-numerical rider involving *if-then-because* reasoning

Figure 6: Analysis of Example 3

Discussion and Conclusion

The existing literature (e.g., Ko & Herbst, 2020), contains research on the use of measures to assess teacher knowledge in geometry. But there is no instrument to investigate teacher knowledge of *basic geometry* and hence a focus on content knowledge needed by teachers with a low knowledge base. We have presented part of the framework on exemplification used to develop tasks for investigating mathematical knowledge for teaching *basic geometry*. We showed how the framework is used to categorise the difficulty levels of the tasks. While using the exemplification aspect of the framework to design tasks for different levels of difficulty, we use the explanatory communication aspect to analyse the teacher responses to course tasks, including the pre-assessment, worksheets, and a focused interview. By analysing data, we hope to identify the mathematical knowledge that is needed to teach basic geometry. Although we are working with a small and special group of teachers, they represent a much larger group of teachers in South Africa, and possibly in other parts of the world, with low levels of geometry knowledge. This unique opportunity holds the potential to identify the structure of profound knowledge needed for teaching basic geometric ideas, relevant for teachers and teacher educators in planning tasks to support teacher learning.

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Construction of an equilateral triangle in the plane and space: an analysis from the theory of semiotic mediation

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Dynamic geometry environments provide users with several tools to construct geometric objects. In the use of these tools, convergence of personal to mathematical meanings has been recognized, because the tools provided by the environment embody theoretical relationships of Euclidean geometry. In this document we analyze the productions of two mathematically gifted students when solving a construction problem in a dual 2-dimensional and 3-dimensional setting. Supported by the theory of semiotic mediation, we show the nature of the signs exhibited by the students and some difficulties that can occur when they move from a 2-D representation to a 3-D one.

Keywords: Theory of semiotic mediation, 3-Dimensional geometry, Geometrical construction, Dynamic geometry environments.

Introduction

There is wide recognition of dynamic geometry environments (DGE), their characteristics and the challenges to incorporate this resource in ordinary teaching and learning of mathematics (Sinclair et al., 2016). These authors inform on efforts by mathematics education researchers made in this regard. Despite this, many research has been carried out focusing on DGE for 2D geometry, but it is much scarcer the research on the use of DGE for 3D geometry (Gutiérrez & Jaime, 2015).

The relationships established among mathematics and DGE is of an epistemological nature, since it happens through a process of mediation (Bartolini-Bussi & Mariotti, 2008). In this process, mathematical objects are shared and thought through their representations. Technology offers possibilities for such representations, as well as a space for communication between student and teacher in which a shared language circulates (Noss & Hoyles, 1996).

The construction on DGE of geometric objects and the justification of its validity can be conceived as a route towards the identification of geometric properties in the representations offered by the software, since the available construction tools allow establish a semiotic relationship between (i) personal meanings caused by the activities of construction and verification by dragging, and (ii) theoretical aspects of geometry that underlie these constructions through the tools used, which support the validity of the construction (Mariotti, 2012). To analyze this relationship, a semiotic perspective about the students' mathematical activity while solving geometric construction problems is pertinent. We have adopted the *theory of semiotic mediation* (TSM) (Mariotti, 2019) to study the production, nature, and evolution of signs present in students' activity in a DGE. In this paper we present solutions to a problem asking to make a construction in 2-D and next to make the same construction in 3-D and justify that the constructions are correct. The research objective is to analyze the students' solutions to that construction problem to identify the signs and signs' meanings exhibited by the students while solving the problem, and the influence of the 2-D construction on the 3-D construction.

The Theory of Semiotic Mediation

The TSM is supported by the Vygotskian notion of *semiotic mediation*, as well as the role of technology (Mariotti, 2019). This theory explains the process of learning mathematical contents when an artifact is used to solve a task (Drijvers et al., 2010). When experts (teachers) use an artifact to solve a problem, they recognize mathematical notions through its use. However, when novices (students) use the artifact, they do not recognize immediately the mathematical meanings that emerge, but students see them linked to the artifactual context in which the mathematical contents are used (Mariotti, 2019).

A *sign*, from Pierce's perspective, is "anything" that represents "something" to "someone" in some aspect or degree (Bartolini-Bussi et al., 2012). The TSM interprets students' production of signs and the evolution of such signs, whose meanings move from personal to mathematical, close to the teaching objective (Mariotti, 2019). In this process, the use of specific tools by the teacher, together with her actions, to promote students' learning is recognized and has relevance (Mariotti, 2012).

The signs can be characterized according to the nature of students' productions (verbalizations, writings, gestures…) and the ways the artifact is used. Some signs are close to the activity done with the artifact and others are close to the aimed mathematical meanings. When student's explanations move from references to the use of the artifact towards references to the mathematical context, different signs are used, which mirror the state of students' learning process. We consider three types of signs (Bartolini-Bussi & Mariotti, 2008): *Artifacts signs* point the use of the artifact or to actions linked to it. *Mathematics signs* refer to mathematical elements of the context; they are related to mathematical meanings and are expressed by propositions that satisfy standards of the mathematical community. They constitute the achievement of the semiotic mediation process orchestrated by the teacher. *Pivot signs* are present in actions, carried out with the artifact, where specific instrumented actions and natural language referring to mathematical contents are involved; their polysemic nature implies that these signs are used to advance from artifactual to mathematical context.

Construction problems and learning to prove within a DGE

DGE support the learning of proof (Mariotti, 2012). In our study, special emphasis is placed on learning of proof in the context of construction problems; we consider a proof as a mathematical argument, both empirical and deductive, aiming to convince of the validity of a mathematical statement, in our case, the validity of a construction in the DGE. Construction problems consists of (i) creating in the DGE a figure having some properties that remain constant under dragging and (ii) explaining the procedure used to construct the figure and validating it (Mariotti, 2019). DGE tools allow the construction of geometric objects on the screen, which provoke personal meanings by suggesting dependency relationships, that may be confirmed by dragging some objects. Furthermore, the tools are related to theoretical elements of Euclidean geometry that could help students to create a proof of the validity of constructions (Mariotti, 2012). Since DGE embody systems of theoretical relationships, solving construction problems leads students to accept the possibilities that the software offers them and the underlying logical system. Therefore, geometric constructions also have a purely theoretical nature, so the solutions may involve proving a theorem to validate them (Mariotti, 2019).

According to Mariotti (2019), in some DGE tools we can recognize presence of (i) graphical representations of geometric objects having some known properties, which are useful for the construction of the objects thanks to the verification by dragging; and (ii) geometric properties that can be evoked, useful to support constructions and that are framed in a mathematical theoretical domain. Therefore, solving geometrical construction problems in a DGE can make students evoke the theoretical meaning of the constructions embodied in the artifact (Mariotti, 2019).

Methodology

The content of this paper is part of a doctoral research, based on case study methodology. We analyze mathematically gifted students' learning of mathematical proof in 3-D geometry with GeoGebra mediating their activity. We designed and implemented a sequence of 18 construction problems in 60-minute sessions, conducted by the first author. Students initially solved the problem and then discussed their results with him, who conducted the dialogue to justify of the results. The problems involved objects and properties of equidistance in the plane and the space, in a way that each problem provided instrumental and conceptual elements useful to solve subsequent problems. The students had to construct the geometric objects with GeoGebra and justify that the constructions were correct.

Four Spanish mathematically gifted students (11 to 14 years old) in grades 1 to 4 of secondary school participated in the experiments. Besides the ordinary schooling, the students had participated in programs of attention to general giftedness (AVAST) and mathematical giftedness (ESTALMAT). As students were in different school grades and had different previous knowledge, the teaching sessions were organized as individual clinical interviews.

We present a fragment of the solutions to the 7th problem by two students, Juan (14 years old in grade 4) and Jorge (11 years old in grade 1); the names are pseudonymous. This problem asked to create an isosceles and an equilateral triangle in GeoGebra 2D and next in GeoGebra 3D. We analyze the two students' solutions to show the similarities and differences. In their approaches to the solution of the problem, we identified whether their expressions alluded exclusively to actions carried out with GeoGebra (artifact sign), if they also included mathematical properties close to those expected (pivot sign) or if they were completely in the domain of mathematics (mathematic sign).

An example: creating an equilateral triangle in the plane and the space

Students had to create an isosceles triangle and then an equilateral triangle, both having a given segment as a side. The triangles had to be created first in GeoGebra 2D and then in GeoGebra 3D, where the given segment was not contained in the XY plane. In previous problems the students had learned to create and use in GeoGebra spheres, circles, perpendicular bisectors, and bisector planes. Here we analyze the construction of the equilateral triangles.

Juan's solution

Juan had constructed 2D isosceles triangles by using the given segment, AB, its perpendicular bisector, and the circle with center in B and AB as radius (Figure 1a). The teacher asked Juan to delete the constructed triangles and find the position of the third vertex of the requested triangle. Juan pointed at point E, one of the intersection points of the circle and the perpendicular bisector, as a solution (Figure 1b). He explained that *the distances BE and BA will be the same… Then, AB and AE* *will be the same… Then, the three* [segments] *will be equal and it will be equilateral*. In this answer, Juan did not evoke actions with GeoGebra, but only geometrical properties used in the construction and other properties logically derived from them, which shows the emergence of a mathematical sign.

Figure 1: Juan's construction of an equilateral triangle in 2-D

As it was not clear how Juan had deduced the congruence of the three segments, the teacher asked him for a more detailed justification. Juan justified it in a different way: *By symmetry, if I draw the circle with center A and radius AB, the intersection would be… E… Then, AE would be equal to AB. AB would be equal to BE, all sides would be equal*. In this new explanation, Juan evoked an auxiliary construction and properties of it allowing him to prove that the triangle is equilateral. This answer reveals the (mental) use of GeoGebra and mathematical properties, so now Juan showed a pivot sign.

The last answer led the teacher to make Juan note that he had not used the perpendicular bisector of AB, previously used by Juan to create an isosceles triangle (Figure 1a). Juan changed again his justification: *If I draw the perpendicular bisector of AE, it passes through B. As BA is a radius, it is equal to BE. And since E is in the perpendicular bisector of AB, then all 3 sides are equal, and the triangle is equilateral*. This new answer had the potential to support the validity of the construction and, like the previous justification, has the characteristics of a pivot sign.

In GeoGebra 3D, Juan first constructed the bisector plane of AB and a point C on this plane, to create an isosceles triangle ABC (Figure 2a). Then, he built the sphere with center B and point A (Figure 2b). Initially, Juan wanted point C to be at the intersection of the sphere and the plane, although later he expressed that he wanted point C to be *on the perpendicular bisector of AB and on the bisector plane*, a property that he re-stated as C *being at the intersection ... between the plane and the sphere, but at the midpoint*. These last ideas were not clear, so Juan continued exploring the construction, and he created the circle intersection between the sphere and the bisector plane (Figure 2c).

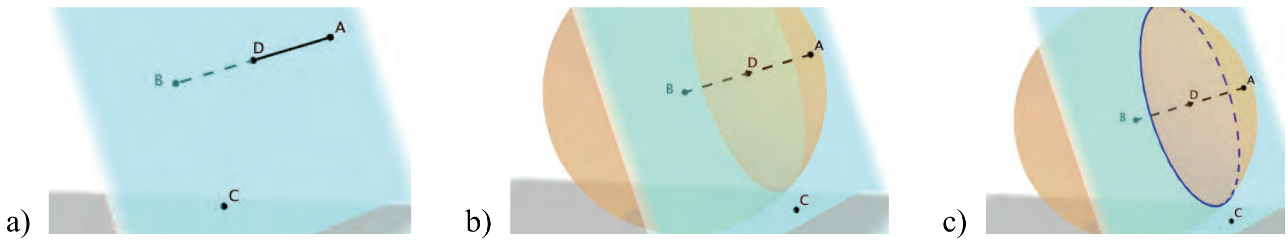

Figure 2: Juan's construction of an equilateral triangle in 3-D (first part)

Juan needed point C to belong to this circle, so he dragged C to the circle. Juan explained that point C *would have to be like here ... Or here* [while dragging C to two specific positions in the circle (Figure 3). Juan's specificity in his answer led the teacher to question these positions for point C. Juan considered it necessary because *there, C would be in the perpendicular bisector of A and B*. The teacher asked him about the possibility of placing C in another position, and Juan replied that *any point of the intersection serves*. To finish, Juan explained that C must be in the circle because *the distance from A to C will be the same as from C to B, because it* [C] *is in the bisector plane*. He completed this justification by saying that *since the sphere has the center in B, and radius BA and also BC, because C is at the intersection of the sphere and the bisector plane, then the three segments* [AB, AC, BC] *are equal*.

Figure 3: Juan's construction of an equilateral triangle in 3-D (second part)

By using the bisector plane of segment AB as reference and placing point C in the positions presented in Figure 3 indicate that, although using GeoGebra 3D, Juan used 3-D geometric objects as if they were in GeoGebra 2D. Juan's idea of using the intersection between the sphere and the plane suggests the recognition, albeit implicit, that the characteristic properties of the points of the 2-D circle used in the first part of the problem are also properties related to equidistance between the constructed points in the 3-D circle; hence this discourse shows a pivot sign, since it combines mathematical and artifactual elements. Additionally, the actions declared by Juan at the beginning of the second construction show that he knew the way to obtain the third vertex of the equilateral triangle given the 3-D configuration that he had on the screen. This leads to his actions serving a declared objective, which is clear from a mathematical point of view, as well as permeated by a deductive language. Therefore, this part of the solution evidences a mathematical sign.

Jorge's solution

Given segment GH as side, Jorge's construction of the 2-D equilateral triangle began with the construction of the circle with center in G and point H (Figure 4a). Jorge then created point I on the circle, segment GI and the circle with center at H and point G (Figure 4b). Jorge dragged point I to an intersection of the circles (Figure 4c), but he changed his mind: eliminated point I, constructed a new point I as one of the intersection points of the circles, and created the triangle GHI (Figure 4c).

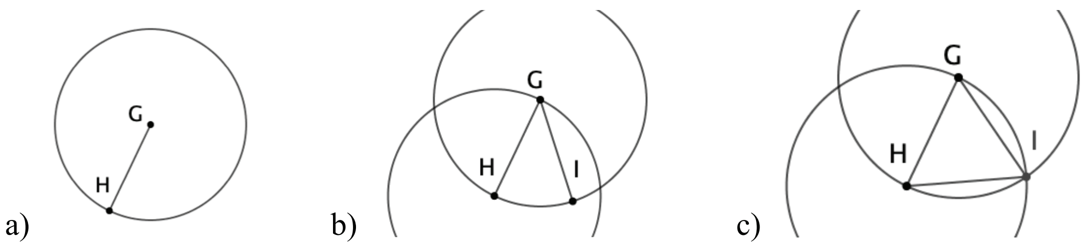

Figure 4: Jorge's construction of an equilateral triangle in 2-D

When the teacher asked Jorge about the validity of his construction, he replied: *the circles… were to copy segments, so I have used them so that all sides are the same length*. Jorge was asked about the choice of point I as the intersection of the circles, and he mentioned that any other point would be closer to one point (G or H) than the other.

To create the 2-D triangle, from the very beginning Jorge tried to use two circles having segment GH as their radii. So, the third vertex would then be one of the intersection points of these circles. Jorge first wanted to guarantee the congruence of segments GH and GI and then, with the help of the second circle, determine the place to which point I should be dragged so the triangle GHI would be equilateral. This strategy was modified, and the justification offered by Jorge was supported by the representation on the screen. His actions had been mobilized as reaction to what he observed on the GeoGebra 2D screen, for example the location of point I. Jorge did not use mathematical elements to justify the congruence of the segments, so these actions determinate a pivot sign.

In GeoGebra 3D, given the segment AB, Jorge created the sphere with center A and point B, and the sphere with center B and point A (Figure 5a). He constructed point C very close to the intersection of the two spheres and the triangle ABC (Figure 5b). Jorge believed that this triangle was equilateral because *each point is at the same distance from the two others... A from B and C, B from C and A, and C from A and B*. He continued explaining *we already have an intersection point. Well, we have two that are the same distance from A and B... we already have an equilateral triangle*. The teacher asked Jorge about the number of solutions, and he replied that he believed that there were more.

The construction that Jorge proposed, although correct, did not have any theoretical support. It is interesting to note that the solution presented by Jorge only considers two points at the intersection between the spheres as potential vertices of the equilateral triangle. This is evidence that Jorge had in mind the 2-D construction of the equilateral triangle he had just made, where the intersection of two congruent circles with the same radii leaded to this solution. We then see a potentially useful idea worked in GeoGebra but not fully developed, so presenting a pivot sign.

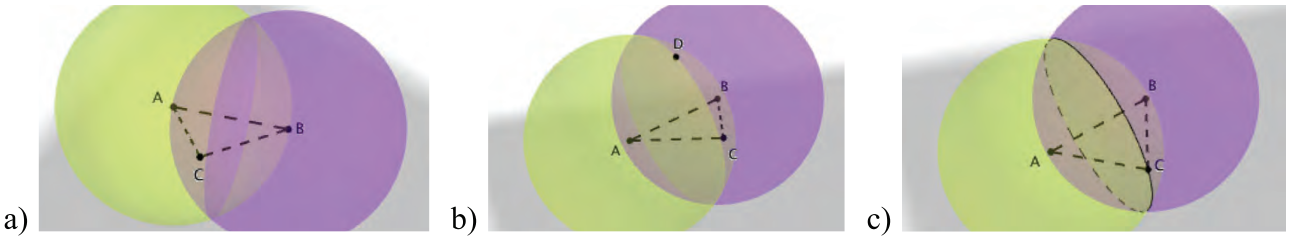

Figure 5: Jorge's construction of an equilateral triangle in 3-D

With some teacher's help, Jorge stated that the intersection of the spheres contained the solutions to the problem, so he created it on the screen (Figure 5c). Jorge tried to validate his answer by constructing some points in the intersection circle and calculating the distances from them to the centers of the spheres, A and B, but the teacher did not allow him to develop this idea and, instead, he asked Jorge to justify why the equidistance that he had stated was true. Jorge replied that *it is a circle, and all the points in a circle are at the same distance from the center*. Although the teacher showed Jorge that the centers of the spheres were not the center of the intersection circle, Jorge explained that *they are like a kind of center that has moved ... it is not in the center, but it is like a line that crosses the center*. Jorge tried to validate his construction based on some cases with the help of the artifact (an artifactual sign). He then made a progress by incorporating theoretical elements, such as the definition of circle, albeit in a wrong way, and equidistance to support the equidistance of each point of the intersection circle to the centers of the spheres. It seems that Jorge considered the centers of the spheres as points belonging to a line perpendicular to the circle and passing through its center; this constitutes a pivot sign that could have been used by the teacher to delve in his ideas. Even so, the language used is not precise and does not confirm this hypothesis.

The teacher did not develop Jorge's last idea but, instead, he asked him about the nature and properties of the circle in which the solutions to the problem were contained. Jorge mentioned that the circle was the intersection of the spheres, *which measure the same*, referring to the congruence of their radii. Additionally, the conversation with the teacher about a new point E on the circle led Jorge to mention that *they are at the same distance… Because E is at the intersection of A and B. So, like point C… it could be made equilateral triangles at all positions of the intersection*. This explanation was not entirely convincing, so the teacher insisted on asking why the points on the circle were equidistant from the centers of the spheres. Initially Jorge assured that this was true because *it is where… the spheres A and B intersect. So, it is like when the perpendicular bisector intersects the segment, they will always be at the same distance, any point on the circle*. The teacher modified his question and asked Jorge to explain why the distances AB and AE were the same. Jorge quickly replied: *Because they are in the… the spheres, any point is the same distance, right? from the center ... So, if there is a point where two spheres intersect, that point will be the same distance from the two centers*. We see that, in this conversation, Jorge managed, with the help of the 2-D construction, to connect the necessary elements of the 3-D construction to deductively justify the validity of his conjecture. This reveals at the end of Jorge's explanations a mathematical sign.

Discussion and conclusions

We have illustrated the mediating role of DGE and some semiotic signs produced by two mathematically gifted students while solving a construction problem, as well as differences and similarities between them. An interesting aspect which arises is the dynamic nature of the semiotic signs: in the excerpts presented, we can see that, at different moments in the solution of a problem, the students have exhibited signs of different types, not hierarchically sequenced; it happened that, even when mathematical signs were identified, setbacks with the presence of other types of signs were observed later. This could be because the problem selected was in the middle of the experimental sequence and students felt the need to produce deductive justifications, although they also showed traces of empirical, artifactual reasoning or parts of informal reasoning and language.

A second interesting aspect has to do with the content of students' outcomes when solving the problems in GeoGebra 3D. When constructing the 2-D equilateral triangle, differences could be observed in the students' performances, both in the geometric objects used and the degree of abstraction with which they justified their constructions. However, when working on the 3-D construction, both students agreed on the number of points that could serve as the vertices of the equilateral triangle. The solutions to this problem illustrate the ease with which the students modified the objects involved in the 2-D construction to adapt them to the 3-D construction. However, the reasoning that accompanied the constructions carried out was not adapted to the 3-D space; for instance, both students, guided by their 2-D experience, only identified two 3-D points as solutions.

Although this error was easily fixed, it is interesting to note the sustained influence of the representations and results got in GeoGebra 2D when moving to GeoGebra 3D.

We have presented only one short episode. More far-reaching studies are necessary to be able to advance in the characterization of the semiotic signs present in the justifications of solutions to geometric construction problems, the evolution in the meanings of the signs used, the relationship between the signs' meanings and the types of proofs produced, and the influence of the DGE.

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Secondary students’ experience using 3D pen in spatial geometry: affective states while problem solving

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One’s affective state can change our actions and thought process while learning. The interplay of student’s emotion and problem solving in spatial geometry has not been thoroughly studied. We present qualitative analysis of individual and collaborative problem solving of spatial geometry tasks by middle school students (8 students in a lab setting, and 21 students in group interviews in a classroom setting). We use the concept of embodied learning design, with the 3D printing pen as a medium, to make the process of converting 2D sketches to 3D models more explicit. Findings revealed that the students’ affective state significantly influenced the way they solve the problems in spatial geometry. 3D sketching environment allows students to build a bond (intimacy) with the material and use their emotions as signals for heuristic changes (integrity). The discrepancy between 2D and 3D visualization in spatial geometry tasks may lead to students’ emotional tension.

Keywords: Spatial geometry, secondary school mathematics, embodied learning, 3D sketching affective states, problem solving

Introduction and Theoretical Background

Spatial geometry

Grasping spatial geometry calls for command of five different types of mathematical thinking: spatial structuring, measurement, representing 3D objects, spatial ability, and conceptualizing mathematical properties (Pittalis & Christou, 2010). Students and teachers have historically relied on traditional 2D media and materials and their resulting “flat” perspectives (Alsina, 2010). Students struggle with spatial geometry tasks, especially those requiring several reasoning steps (Fujita et al., 2020). Applying a dynamic geometry environment (DGE) to spatial geometry fosters innovation in didactic and research methods and may stimulate geometrical reasoning (Mithalal & Balacheff, 2019). However, even 3D DGE are limited to the two-dimensional medium (screen) and thus may create perceptual problems (Dimmel & Bock, 2017). Therefore, we are seeking a teaching method that can provide students with a three dimensional and multimodal experience and facilitate the development of the five types of thinking mentioned above.

3D pen—a novel technology for freehand drawing of 3D models

The 3D pen is a relatively novel technology which application to spatial geometry instruction has not been significantly studied. The pen excretes hot plastic which allows one to draw in three dimensions free hand. The use of the 3D pen in mathematics education was presented by Ng, et al., (2020) in which elementary students were studied using the pen in comparison to dynamic geometry programs. It was found that long term retention of the group using the 3D pen was significantly better.

Our study focuses on cognitive and affective states of middle school students to find the potential of this novel medium in learning environment that promotes interaction between the object and the person (via gestures, movements and embodiment of the physical world) which is not possible when using a 2D medium (paper or screen). Assuming that using a 3D pen will make the 2D to 3D conversion visible and therefore easier and more natural for learners, we aim to characterize learners' emotions when solving non-standard spatial geometry problems.

Embodied learning of geometry

Embodied learning, an educational approach based on the role of the body — through movement, action, and gesture as a powerful tool for understanding and learning school subjects — is gaining ground in educational research (Abrahamson et al., 2020; Kim et al., 2011). Embodied learning is based on embodied cognition which suggests there is some level of interaction between the body's physical actions and movements in its environment and our cognition (Wilson, 2002).

The activities in which students solve spatial geometry questions by constructing tangible models may potentially bridge the gap between intuitive and disciplinary ways of mathematical reasoning (Palatnik & Abrahamson, 2021). Tasks in which students sketch with the 3D pen to visualize a problem can be considered as a sub-genre of embodied design activity—compelling students to use movements and gestures (c.f. Abrahamson & Lindgren, 2014).Research that combines the theory of embodied design with the effect on students' affective state is relatively rare (Sinclair & Heyd-Metzuyanim, 2014). This study focuses on students' emotions and meta-affective abilities while solving nonroutine for them problems (spatial geometry) using non-routine medium (3D sketching pen).

Theoretical framework for influence of affect on problem solving

While solving non-routine problems, many students are overcome by their emotions and unable to continue to solve the problem, however other students can maneuver their emotions, think of an alternative solution, and therefore solve the problem (McLeod, 1988). When students are aware of the feeling of frustration, for example, they can use it as a sign to give up and move on to the next problem, or as a sign to reevaluate their solution and find an alternative solution (Debellis & Goldin, 2006; McLeod 1988; Hannula, 2015).

Debellis and Goldin (2006) define mathematical intimacy as the students' emotional engagement with mathematics and mathematic integrity as their sense of "fundamental commitment to mathematical truth" and pursuit of understanding (Debellis & Goldin, 2006, p.132). The individual's ability to solve problems is influenced by their control of their emotions, their awareness, context and situation, mathematical education, attitudes, values, beliefs held by themselves and the environment and normative emotional expectation (Debellis & Goldin, 2006). Educators, curriculum developers, instructional coordinators, and textbook writers who are aware of students' emotions and the effects on their learning patterns, can use this to their advantage when searching for methods to create optimal learning environments.

McLeod (1988) characterizes these affective states based on magnitude (intensity), direction (positive or negative), level of awareness (of their meta-affective processing), duration, and level of control.

According to Debellis and Goldin (2006) mathematical intimacy refers to a specific, emotional engagement with mathematics. Mathematical integrity relates to one’s affective state when determining if a solution is satisfactory or should be praised. This leads to the pursuit of understanding and the students’ commitment (or lack thereof) to the mathematical truth. Students with high mathematical integrity recognize their insufficient knowledge or understanding and decide to change their solution or direction when problem solving.

Our research is situated on a nexus of spatial geometry content, embodied pedagogy, and the emotional component of students’ learning. This research report focuses on the following RQ:

How does the experience of 3D sketching influence the students’ affective state and their awareness of these states while solving spatial geometry problems?

Methods

The cases reported in this paper are part of a larger research project which explores the effect of the use of embodied learning design on spatial geometry problems (Palatnik & Abrahamson, 2021; Rosenski, 2021). We tested 9 students in individual interviews in a lab setting, and 25 students in group interviews in a classroom setting. Of the 9 students interviewed in the lab setting, 8 were included in the data analysis. The students were chosen based grade level (9^{th}-11^{th} grade), as they have all studied concepts in planar geometry necessary to solve the problems given. Of the 25 students interviewed in group settings, 21 students were included in the data analysis. The others were omitted for not completing the entire question set.

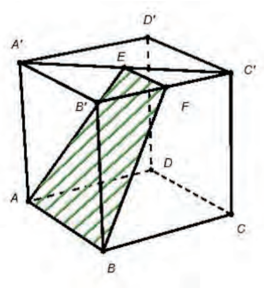
	Given	Section	Question	True / False	
	ABCDA’B’C’D’ represents a cube. E is the middle points of A’C. F is the middle point of B’C’.	1	B’ is a point on AE.	True	False
		2	ABFE is an isosceles trapezoid. The equal edges are _____.	True	False
		3	ABFE is a right-angled trapezoid. The right angles are_____.	True	False
		4	AE is the longest edge of quadrilateral ABFE.	True	False

Figure 1: Example of question #3

Before beginning the main activity, the students were asked to answer a computerized test of 30 questions that test spatial ability. The test is based on mental rotation and ability to rotate 3D objects projected on the screen. Subsequently, the students were given a 3D pen and were asked to sketch a pyramid with the 3D pen to practice drawing lines in the air. Most students were able to sketch freely with the pen within 10 minutes. The main activity of the study includes 3 questions in spatial geometry. Figure 1 shows the last question in the series of items. The students answered the questions by sketching a 3D object (shown to them on a 2D sketch) inside a plastic wire model of the cube and then using mathematical reasoning. Our decision to have the students sketch the drawings, is not only for the use of the 3D pen technology, but also to make the process of converting 2D sketches to 3D models more explicit and potentially overcome the difficulties associated with this process as described by Fujita, et al. (2020). We hypothesize that students struggle to use spatial visualization and property based spatial analytic reasoning, but the physical act of sketching in 3D allows the student to develop new ways to interact with the 3D model using embodied cognition to overcome this issue. After the students have sketched their 3D models and move on to the phase of problem solving. After solving each question, the students were asked to express how confident they were in their answers. This allowed us to see if students who answered incorrectly were sure or unsure of their solutions. The three questions increase in terms of difficulty and familiarize students with the 3D model affordances.

After completing the questions, the students participated in a task-based interview (Goldin, 2000) about the questions themselves along with a semi-structured interview about their overall experience and feelings. Some of the question in the interview were open ended while some questions asked students to 'rate their feelings' on a scale from 1-10. In open ended questions, many students do not elaborate, but in the direct 'rating' questions students felt comfortable answering. The one-on-one interviews allows participants to expand on their experience of answering questions. The entire session was audio and video recorded (the students' face off frame) in order to properly record and analysis the students' actions and arguments and answers.

Our analysis used triangulation of data from the videos, responses, and field notes of the researcher. The data was transferred to detailed transcripts which were coded freely. The student's internal affects are inferred from external factors such as observable behavior in individual children's mathematical problem-solving during task-based interviews (Debellis & Goldin, 2006). The affective states and emotions of the students were characterized as suggested by McLeod (1988) magnitude (intensity), direction (positive or negative), level of awareness (of their meta-affective processing), duration, and level of control. Throughout the coding process, the main themes surfaced, and we were able to categorize the various aspects of the students' experience.

Findings

Mathematical intimacy

Mathematical intimacy is the deep engagement of participants while solving the problems (Debellis, & Goldin, 2006). These feelings can be positive or negative and have varying magnitudes. Most of the participants of our study experienced mathematical intimacy while facing the tasks. Namely, the students were emotionally connected to the activity and showed (verbally and via behaviors) their deep care as the activity advanced from question to question. For instance, when answering the questions, a participant C was supposed to have a break for lunch. However, he did not agree to leave the room for the break until he finished the question he was working on. Note that C initially answered most of the questions incorrectly, but then saying aloud, "I'm not sure I sketched it correctly," felt he had made a mistake and corrected all his answers to the correct answer.

Each participant had a different range of emotions at all stages of the study, but there are several "common scenarios." Some participants were excited when they drew a 3D model but sad when they could not immediately answer the question, were satisfied when answering the question, and described strong positive emotions about the activity during the interview. The other participants were frustrated at their inability to find the correct explanation for their answer, making them give up and move on to another question. These students still got the correct answers and were able to solve the problem. Some participants were restrained when drawing with a 3D pen but expressed extreme excitement when asked about their experience during interviews.

The emotions in the students after finishing the activity were generally positive in fairly high magnitude. Most participants felt the activity was generally good, while others felt deep feelings of excitement and amusement, directly connecting a 3D pen experience with this feeling. For instance, participant O5 claimed it was “a thousand times more fun” than a “traditional math class.” When asked directly to rate their experience of 3D sketching while solving a problem on a scale of 1-10 in one of the emotionally charged characteristics, participants answered in the following way:

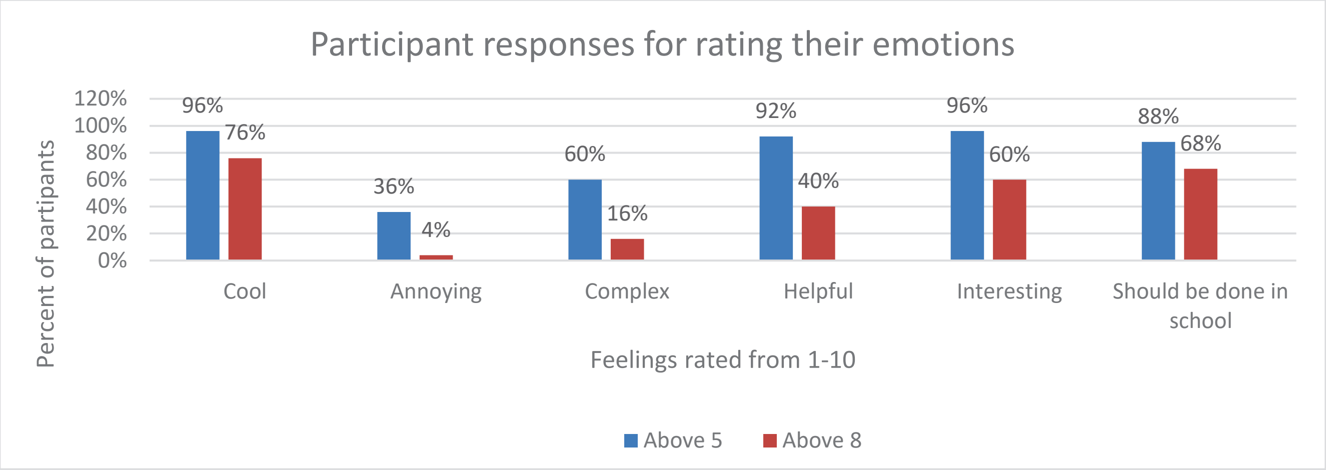


Figure 2: Participant responses for rating their emotions, post activity

As can be seen in Figure 2, the students felt that the activity and use of the 3D pen was cool, not very annoying, not very complex, quite helpful, interesting, and should be done later in the school year. While asking students to describe the magnitude of their emotions is not necessarily the most accurate measurement, these numbers relate to a general feeling among the students and is corroborated with the findings throughout the interview. When solving the problem, the participant A, after 3 minutes of searching for the solution, suddenly found the answer and screamed “Oh! I know! The diagonal and the side are never equal in a square.” His task-based interview confirmed his deep satisfaction in solving the problem: “It was really fun, there was a lot of hands-on work which is nice. It makes you think in a way you usually don’t”

Mathematical integrity.

Some of the participants of our study demonstrated mathematical integrity related to problem solving in general and spatial reasoning in particular. For instance, M, struggled to understand the characteristics of the 3D figure and when asked if the trapezoid she had drawn was isosceles (shown in Figure 1, question #3, section 2) she said “Based on how it looks I think no [not an isosceles trapezoid] I’m trying to think how I can prove why….um, wait I need to understand this” and later said “wow this is hard.” Her acknowledgement of lack of understanding, eventually led her to the correct answer. She used this seemingly negative emotion to alter her problem-solving plans and change her thinking. In this case, M's negative feelings were at a low magnitude, and they allowed her to change direction without being overcome by anxiety or despair.

We also observed interesting manifestations of mathematical integrity pertinent to the material and visual aspects of 3D sketching activities. One participant explained that the 3D pen's biggest disadvantage is not being "exact" enough, the lines drawn are "not straight" and it's too bad that they did not use a ruler. This may relate to the participants' values or previous beliefs about mathematics in general, but also shows his search for 'exact' and 'true' mathematical representations, rather than ones that are not exact. Another student, A, consistently looked to the given 2D sketch when solving the problem instead of using the 3D model (c.f. Palatnik & Abrahamson, 2021 for the similar phenomenon). He explained in the interview that "I used the sketch because it's more exact.". It seems that A feels that in order to align with his own 'mathematical truth' all the sketches must be exact. Throughout the problem-solving process, he decided to use calculations with Pythagorean theorem, despite not needing them to prove his answer. However, for him, it may have been the best way to solve the problem. When solving the trapezoid question (shown in Figure 1) section 4, he used complex calculations and an auxiliary line drawn with the 3D pen to explain his answer. When trying to thoroughly explain the calculations to the researcher he laughed and seemed to realize the excess calculations. He took this struggle with explanation 'lightly' and was able to solve the problem.

The participants of the study used their mathematical integrity to solve non-routine problems successfully without allowing their affective states of frustration or anxiety overcome their ability to make heuristic decisions and changes. This allowed them to describe their experience of using 3D pen and a tangible model to solve spatial geometry problems as positive. Students who can feel comfortable in a state of 'unknown' while searching for the right solution, will be able to solve non-routine problems (Debellis, & Goldin, 2006).

Discussion.

We conducted a study of middle-school students facing the spatial geometry tasks while sketching with a 3D pen. We sought to find out how does the experience of 3D sketching influence the students' affective state and their awareness of these states while geometry problem solving. The participants demonstrated a high level of mathematic intimacy—connection and strong emotions and mathematic integrity, the ability to use the intimacy to their favor. The students showed strong positive emotions such as excitement, satisfaction, and awe. The nature of the activity in the study gives students the ability to sketch 3D figures by themselves, rotate the figures, and change or add auxiliary lines. The two-dimensional diagrams on which the school's spatial geometry curriculum is based represent three-dimensional bodies in a distorted form (Widder, et al., 2019). This quality can create a sense of alienation in students. However, the affordances given by the 3D sketching develops the students' bond with a problem. This bond gave them general positive feelings about the activity, despite being a nonroutine and challenging problem to solve.

Students in the study showed a high level of mathematic integrity as they were able to recognize their own lack of spatial understanding or geometric knowledge which led them to reassess their problem-solving strategy. Most of the students navigated their emotions of dissatisfaction, lack of confidence, or frustration and were able to successfully solve the problems. In certain cases, we observed a type of mathematical integrity specific to 3D representation and sketching. Students are accustomed to using 2D sketches of 3D figures, and therefore some of them felt that the 2D sketch was more exact or closer to their own ‘mathematic truth.’ Educators and educational designers should be aware that the discrepancy between 2D and 3D visualization in spatial geometry tasks can lead to emotional tension for students. Some secondary students may have feelings of antagonism towards the novel medium of 3D sketches and require mediation before beginning the activity.

The participants' level of intimacy and integrity shows a high level of comfort and openness throughout the activity. Our study used a learning environment based on the principles of embodied learning. Abramson and Lindgren (2014) suggest that one may create a deep understanding by physical interaction with the environment. The 3D model of the cube allows students to “off load” at least some of the cognitive processes (mental rotations, imaginary auxiliary elements) needed to complete the task (Wilson, 2012). The use of the 3D pen affords students a physical visualization of the conversion process from a 2D sketch to a 3D model. In its "purely mental" form, this process proves to be very challenging for students of all ages (Fujita, et al., 2020). Throughout the intervention phases, the participants’ affective states can be seen as ‘pathways’ that may guide them to various directions and changes in the heuristic decisions (Debellis, & Goldin, 2006). The students in our study used their recognition of inadequate justifications as a signal to make a heuristic change (McLeod, 1988). In many cases this recognition was directly connected to the manipulation of 3D model and in particular 3D sketching.

In this study we found that the students’ affective state significantly influenced the way they solve problems. The awareness of the students’ emotions and its effect on how they solve problems can allow both students and teachers to mitigate the negative emotions and reroute them to an alternate solution. This topic demands further examination with research using a variety of geometry tasks and populations. The high school curriculum in Israel is focused on having students successfully pass matriculation exams and spatial geometry tasks are part of the exams on all the levels. Thus, drawing on our findings on secondary students’ experience while using 3D pen in spatial geometry, we suggest that teacher professional development programs consider recent advancements of embodied learning, novel medium (as 3D pen) ,and student’s affective states as well as the connection between these fields.

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TWG05: Probability and Statistics Education

Introduction to the work of TWG5: Probability and Statistics Education

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Introduction

The working group gathered 39 participants from 14 countries, from Asia to South America, with 26 papers and six posters. The participants were introduced to the three subthemes identified as emerging across all submissions. In particular, submissions focused on "Teacher education", "Reasoning about data" and "Statistical and Probabilistic Thinking and Reasoning". We prepared three guiding questions to support in the discussions on the three subthemes. The questions were: "*Considering these presentations, what contribution do they make to what we now know about teacher education/reading the data/statistical and probabilistic reasoning?*", "*Considering these presentations, what issues (regarding (mis-)conceptions, teaching approaches, resources, assessment etc.) do they highlight/raise about teacher education/reading the data/statistical and probabilistic reasoning*?" and "*What comments or thoughts do you have now with regard to teacher education/reading the data/statistical and probabilistic reasoning, from a research and practice point of view, to advance future research?*".

Teacher education

There were ten papers focusing on the subtheme of *Teacher Education.* They represented a broad gamut of research examining the disciplinary foci of statistics, probability and STEM. These research studies considered the experiences of students, preservice and practicing teachers across various educational settings, including early childhood, primary, secondary and college-level learning environments. They also explored a variety of learning opportunities to support the development of understanding including the use of innovative technologies, lesson study and a range of professional development opportunities. One of the themes that emerged from the research presentations was the need for researchers to engage in more effective communication with statistics and probability educators. The field is a fast-evolving landscape and new knowledge and understandings about pedagogies are constantly being developed. Our responsibility is to assist teachers in identifying these critical and desirable facets of statistics and probability education. This communication should focus on:

(1) The purpose and rationale for teaching statistics education. We need to remind teachers that our goal as educators is to engage students in statistical thinking and reasoning, develop statistical literacy and learn how to critically read the world and how we represent it statistically.

(2) Identifying the desirable feature of good statistics teaching. This includes teaching statistics through projects and investigations, open questions, and the need to address meaningful problems. We need to better communicate and provide a justification for how these desirable practices relate to the previously expressed purpose and rationale of statistics education

(3) Explaining the limitations of some common approaches to teaching statistics and probability. Examples are a focus on technical procedures, calculations of descriptive statistics in isolation from developing conceptual understanding and the use of closed questions that don't promote inquiry and a questioning stance.

(4) Providing insights into what it feels like to engage in these desirable pedagogical practices. It is important that teachers are provided with rich experiences of engaging in statistical investigations so that inert theoretical knowledge becomes meaningful. It is very valuable for teachers to have the opportunity to experience the practices of statisticians and experience the benefits for the learner of engaging in these practices.

A second theme that arose was recognition that statistics and probability education continue to evolve rapidly. Consequently, we need to continually remain open to and expand our conceptualization of the field. Moving forward the statistics and probability educational research community need to better leverage the use of contexts in our practices. This involves moving beyond meaningless and trite textbooks problems to a focus on addressing meaningful problem that may have personal relevance to learners or engage them in considering real life global problems and challenges. Many of these types of problems occur in rich interdisciplinary contexts such as STEM and embracing these new contexts provides numerous opportunities for the development and use of innovative pedagogies. Some of these opportunities include a move to new technologies and tools that allow us to explore large and open-source data sets that represent real problems and provide new lenses and ways of visualising data. Considering these responsibilities as researchers to embrace and manage change, we identified the emerging challenge for us as statistics and probability educators to balance context and at the same time ensure a focus on desirable learning outcomes. We communicated this concern and culminated our discussions in the form of a driving question for future consideration: How can we foreground statistics and probability when engaged in interdisciplinary collaborations and at the same time not lose sight of the important contexts that drive and underpin investigations and inquiry?

Statistical and Probabilistic Thinking and Reasoning

Eleven papers were presented under the subtheme of Statistical and Probabilistic Thinking and Reasoning. Five of those focused on probabilistic reasoning in relation to risk perception, decision making under uncertainty, random variation and sample space, covariation tasks in a Bayesian situation, and combinatorics. In these studies, the participants were primary, secondary, and university students. The other six papers investigated students' statistical conceptions with regard to frequency tables, statistical content in terms of statistical literacy and reading levels of statistical graphs in the mathematics textbook, the exploration of real and rich data with the use of technology tools and communication of results. Most of these studies involved secondary school students, but one study focused on primary school students. Textbook analysis research included primary and secondary grades.

From the discussions of these presented papers, three main themes emerged: 1) The emphasis on statistical reasoning in all phases of statistical investigation cycle and consideration of individual's dispositions like attitude towards statistics led to a broader view of statistical reasoning; 2) The research suggested the importance of rich learning experiences in support of critical thinking through the use of meaningful context and real data sets, the implementation of project-based learning in statistics education, and promoting probabilistic reasoning in data-based decision-making processes. 3) Fostering informal ideas with regard to statistical and probabilistic reasoning with younger learners is still relevant. In addition, some concerns have been expressed about the readiness of teachers and students for open-ended tasks suggested by research as well as the limited sources and time for implementing statistical projects. Critical discussions on the future of statistics and probability education have raised the questions of (1) relying on "better" textbooks versus "more" digital tools in school education, (2) communicating appropriate use of probability language, especially related to the everyday language (e.g., chance, luck, randomness), in the classroom, and (3) increasing the role of researchers in task design, textbook and curriculum development related to probability and statistics.

Reasoning about data

Within the five papers in the section "reasoning about data" four papers have had a specific focus on Data Science Education and to related fields like big data and machine learning. We discussed several core ideas and fundamental aspects to develop a competent reasoning about data in a sustainable way and across all age levels.

(1) At first we identified that it is important that there is a continuous development of data competence (from primary school to adult education) in the sense of a spiral curriculum.

(2) Second, the appropriate use of digital tools (like Gapminder, CODAP and TinkerPlots) can reduce the extraneous load in working processes and make learners able to explore large and multivariate data and to explore their data with regard to their specific inquiry questions. One crucial point in this respect is the choice of the digital tool. Educational software like TinkerPlots do not need a specific programming language (but are limited in some sense with regard to the data exploration capabilities). In contrast professional software like R or Python offer a broad range and landscape of statistical activities, but are more difficult to learn and learners may concentrate on programming and technical issues rather than on the content and the statistical exploration. So there is the danger that technical issues distract from the content issues.

(3) A third core idea to develop a competent reasoning about data, which was raised in the discussion, was the cooperation with other disciplines, e.g., the STEM disciplines.

(4) A fourth point is that the kind of data which is used for teaching and learning issues plays an important role: learners should be given real, meaningful data, which is authentic and offers

multivariate explorations. In addition to that the role of task design is central. In the discussion it was mentioned that teachers are often not comfortable with complex open problems which do not show clear steps to solve the task. To prepare teacher to consider using open tasks and problems and to make them familiar with these kinds of tasks was identified as a huge task for teacher education.

(5) A last, but fundamental issue with regard to reasoning about data which was mentioned was Data Science. Data Science was figured out to be an emerging field in statistics education and includes aspects like Big data, Open data, other data collection methods (e.g., Sensors, Webscraping). These new concepts, issues and ideas of Data Science led to the re-interpretation of fundamental ideas and concepts in statistics education (e.g., PPDAC cycle).

Looking ahead in the context of Data Science new approaches like Machine Learning in education are very new topics and one should use the opportunity to share all the different approaches arising and to include all the things we "know" by now (open projects, problem oriented learning, using new technology…). Given that four of five paper in the rubric "Reasoning about data" have tackled issues with regard to Data Science, we see that that this topic becomes more and more important in the statistics education landscape. Looking forward three big issues were identified which seem to be very relevant for a future perspective on reasoning about the data. Specifically more qualitative, design-based and quantitative research is needed in the following three fields: integration of Data Science into the classroom, connection of informal and formal concepts for reasoning about data and the connection of data, chance and context.

Organization of the TWG sessions

In the first session, we explained the organization of the Sessions. We were divided into two groups for some sessions: TWG5a with Caterina Primi, Sibel Kazak and Orlando Rafael Gonzalez and TWG5b with Aisling Leavy and Daniel Frischemeier. Each group used breakout rooms to create smaller groups. In this way, also with virtual modality, we tried to promote the famous three Cs of CERME Communication, Cooperation and Collaboration. To create a collaborative atmosphere that would support the discussions and feedback over the following days, we started with an ice-breaker activity "Speed dating" involving all the participants. Participants got to know their TG5 colleagues in a friendly context through this activity. We created groups of three randomly and for 3 minutes each participant introduced themself shortly before swapping to a new random group. At the end of each session for each group, a Padlet has been created to document and to write down the results of the discussion. There was a corresponding card in the Padlet for each of the guiding questions. In addition to that, there were two further cards for raising other issues, thoughts, etc. regarding the papers. All the contributions we collected were significant for a successful and substantial concluding discussion on the last day of CERME-12 when we summed up all our insights during the CERME-12 week. The last day we had a culminating session with all participants with the aim being to engage in an in-depth discussion on subthemes and to share the contributions discussed during each session.

Supporting teachers' confidence to teach statistics through a blended professional development approach

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The Teaching Statistics Through Data Investigations (TSDI) professional development course is a Massive Open Online Course (MOOC) designed to strengthen participants' ability to use a statistical investigation cycle to teach statistics and help students explore data using technology tools and make evidence-based claims. The case of interest for this research were 63 MOOC participants who also participated in one of nine professional learning teams (PLT) during 2016-2017. Quantitative data was analyzed using statistical tests to measure significance in growth of confidence to teach statistics, and qualitative data used to confirm or refute quantitative data-based claims. Results from the study indicate that a blended approach to professional development is an effective way to increase statistics educators' confidence to teach statistics.

Keywords: Statistics education, teacher confidence, MOOC, professional growth.

Background

In their seminal paper from a study of how statisticians think and work, Wild and Pfannkuch (1999) claimed that doing statistics involves a process of data investigations and specific ways of thinking about data. Teachers need to be knowledgeable about this method to enhance students' learning (Franklin et al., 2007). The GAISE K-12 report represented the work of Wild, Pfannkuch, and others to organize teaching statistics as phases in a statistical investigative cycle: posing a question, collecting data, analyzing data, and interpreting results.

In 2015, Hollylynne Lee and colleagues launched the Teaching Statistics through Data Investigations (TSDI) massive open online course (MOOC) to prepare middle school, high school, and postsecondary teachers in pedagogy for teaching statistics (see http://go.ncsu.edu/tsdi). The TSDI MOOC was part of a larger effort at the Friday Institute for Educational Innovation at North Carolina State University to create and implement MOOCs designed specifically for educators, to use design principles based on effective professional development and are specifically targeted at better preparing educators for the current challenges in K-12 education (Kleiman et al., 2013).

Lee and Tran (2015) proposed the Students' Approaches to Statistical Investigations (SASI) framework to help statistics teachers support students. The SASI framework builds on the GAISE K-12 framework (Franklin et al., 2007) and is grounded within four phases of statistical investigation (pose a question, collect data, analyze data, interpret results). The TSDI course introduced participants to this framework and built opportunities for participants to learn how to apply these ideas in task design, engage in data investigations themselves, and analyze students' work. It was designed to strengthen participants' skills and prepare them to use a statistical investigation cycle to teach statistics and help students explore data to make evidence based claims (see http://go.ncsu.edu/tsdi). There was a two-step registration process; first registration was for the platform that offers the course along with other MOOCs, and the second registration was specifically for the course. There were five units in the TSDI MOOC course. Unit 1 was titled *Considering the Possibilities of Teaching Statistics with Data* and focused on what statistics is and why it is taught in schools. Unit 2 was titled *Engaging in Statistics*, and it offered a careful look at what it means to engage in statistics. Unit 3 focused on *Introducing Levels of Statistical Sophistication* and presented a framework for supporting growth in students' statistical sophistication and digged deeper into statistical habits of mind. Unit 4 was titled *Delving Deeper into the Investigation Cycle*, and it provided teaching and learning materials to assist participants in understanding the different components of a statistical investigation, including several resources that can be used directly with students. Finally, Unit 5 engaged participants in *Putting It All Together* to consider how to change teaching practices that can really engage students in doing statistics with real data. Each unit was similarly structured by containing the following sections: *Hear from your instructor*; *Engage with essentials*; *Learn from our expert panel*; *Dive into data*; *Investigate: assessment items*; *Discuss with your colleagues*; *Extend your learning*; *Demonstrate your learning*; and *Unit feedback survey*. The course included instructor videos explaining the units, expert panel videos, videos of real students and teachers engaging in statistics work, animations depicting student work in statistics, brief readings, excerpted readings from articles or books, various data analysis tools (open source), and lesson plans, tasks, online apps, and videos that can be used in the classroom.

With a small grant funded by the American Statistical Association, Hollylynne Lee aimed to support local changes in statistics teaching and collaborative learning among teachers by supporting professional learning teams (PLT) among TSDI MOOC-Ed participants during fall 2016 and spring 2017. In those small teams, groups of participants were supposed to meet several times (online and if possible, face-to-face) and share their learning and experiences in the course, as well as their statistics teaching practices. The plan was to pursue a blended approach to professional development among teachers of statistics.

There is a common and strong issue with teachers' confidence to teach statistics (Estrada et al., 2011; Lovett, 2016; Stohl, 2005). Teachers mostly escape from teaching statistics for various reasons. One of those reasons was the lack of their knowledge about different strategies in teaching statistics (Hill et al., 2005). Thus, given the description of MOOC and PLT above, the research question guided this study is: In what ways does participation in a MOOC and in a PLT focused on teaching statistics impact teachers' confidence to teach statistics?

Theoretical framework

Clarke and Hollingsworth (2002) explains teachers' professional development and recommends key considerations for in-service and pre-service teacher training programs. The authors describe six perspectives about teacher change: (1) changes as training; (2) change as adaptation; (3) change as personal development; (4) change as local reform; (5) change as systemic restructuring; and (6) change as growth or learning. The authors state that those perspectives are not mutually exclusive, and they are interrelated. However, most professional developments align with the change as growth or learning perspective. In this perspective, change is identified with learning, and it is regarded as a natural and expected component of the professional activity of teachers and schools (Avineri, 2016) and, historically, "teacher change has been directly linked with planned professional development activities" (Clarke & Hollingsworth, 2002, p. 948). A model to explain the process of teacher change was developed by an international group of researchers (Teacher Professional Growth Consortium, 1994), as the Interconnected Model of Teacher Professional Growth. The Interconnected Model of Teacher Professional Growth suggests that professional growth (change) occurs through the mediating processes of reflection and enactment in four domains that encompass the teacher. These domains are (1) the personal domain (teachers' knowledge, beliefs, and attitudes); (2) the domain of process (teachers' professional experimentation); (3) the domain of consequence (salient outcomes of professional development); and, (4) the external domain (sources of information, stimulus, and support). The Interconnected Model of Professional Growth is updated to frame this study and used as the model for explaining teachers' change or professional growth in this study (Figure 1).

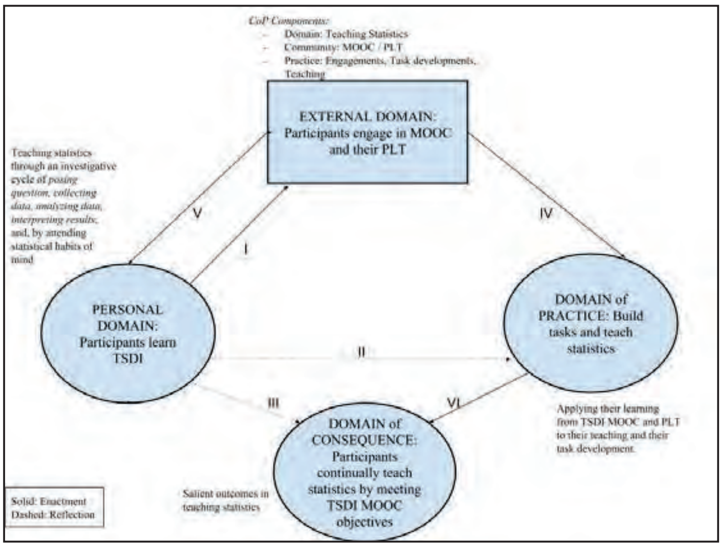

Figure 1: The interconnected growth model for blended PDs of MOOC and PLT

According to Clarke and Hollingsworth (2002), there are two different mechanisms (or two mediating processes) to account for change or professional growth effects: enactment and reflection. Enactment is the mechanism by which a teacher puts a new idea, belief, or a practice into action. Reflection, on the other hand, is defined as "active, persistent, and careful consideration" (directly quoted in [Clarke and Hollingsworth, 2002, p. 954], from [Dewey, 1910, p. 6]). In order to answer the research question of this study *(In what ways does participation in a MOOC and in a PLT focused on teaching statistics impact teachers' confidence to teach statistics?)* the following construct is provided as guiding hypotheses.

> <u>External domain >> Personal domain:</u> We hypothesize that participants' engagements in MOOC and PLT influences their knowledge of and confidence in teaching statistics. For example, a teacher explores a conceptual misconception discussed in a MOOC forum and realizes that he/she also has that misconception and tries to overcome it. (*Reflection)*

Methods

The largest group in this study is all TSDI MOOC participants (804 enrolled participants in total) for fall 2016 and spring 2017. The case of interest for this research are the 63 TSDI MOOC participants that also joined one of nine PLTs. There were four PLTs formed during fall 2016, and five PLTs formed during spring 2017. The study focuses on understanding the experiences of members of the nine PLTs situated within the larger MOOC community to find out about the change in their confidence about teaching statistics.

The Self-Efficacy for Teaching Statistics Survey (SETS; Harrell-Williams et al., 2014; Harrell-Williams et al., 2017) is designed to evaluate participants' confidence about teaching various aspects of statistics (e.g., estimate a population mean or proportion using data from survey; calculate the correlation coefficient between two variables, using technology; interpret measures of association). According to Lovett (2016), the SETS survey aligns with Bandura's (2006) construct of self-efficacy measuring expectations, and it measures teachers' efficacy for tasks and task levels that align with the GAISE framework. Those levels (Levels A, B, and C) are considered to have increasing statistical sophistication. The levels are aligned to some specific statistical content. Level A represents topics for novice statistics learners; Level B represents more complex content; and Level C represents more advanced content (Franklin et al., 2007). The SETS instrument consists of 44 items which are categorized as Level A, Level B, and Level C items. 11 items in the SETS instrument correspond to GAISE Level A (confidence to teach basic statistical content); 15 items correspond to GAISE Level B (confidence to teach more complex content); and 18 items correspond to GAISE Level C (confidence to teach more advanced statistical content). The SETS instrument has been validated as an appropriate instrument to measure confidence, on a scale from 1-6, for each of the 44 items. For each item in the SETS survey, participants rate their confidence in teaching the skills necessary to successfully complete the task on the following scale: 1-not at all confident; 2-only a little confident; 3-somewhat confident; 4-confident; 5-very confident; 6-completely confident. Content validity is established using experts' (college-level statistics educators, including teacher educators) judgement, focus groups (preservice and in-service teachers), and several pilot studies. Structural, substantive, and content validity evidence for the scores from the SETS instrument were outlined, and confirmatory factor analysis results provided evidence for treating the SETS as a two-dimensional (for SETS-Middle School, Harrell-Williams et al., 2014), and three-dimensional (for SETS-High School, Harrell-Williams et al., 2017) instrument aligned with GAISE Pre-K-12 framework. Within the TSDI MOOC, the SETS surveys were given to the participants twice. Participants could take the survey while they were in Unit 1 of the course (pre-SETS), and the survey was again available in Unit 5 (post-SETS). There was one open-ended question on the survey for participants to elaborate on factors that affect their confidence. In Unit 5, they were specifically asked to discuss any changes in their confidence in discussion forums. Thus, the SETS survey results can help describe the growth in participants' confidence about teaching statistics. The quantitative data for this research included SETS survey results. The statistical method for analyzing the SETS survey results is to examine scores for their total confidence pre and post, as well as gain scores for each level (A, B, C) of statistical sophistication of topics. The PLT participants who took the post-test were identified and pre-test takers were filtered to be in the same group. The SETS questionnaire has 44 Likert-scale prompts; these prompts are separated into three blocks, and each block represents different levels of sophistication (A, B, or C). The analysis is conducted for every sub-scaled score of these different levels and for the total. After matching pre- and post-surveys, 28 PLT participants took both the pre- and post-SETS. A dependent samples t-test was used to determine whether there was an increase in participants' gain scores in confidence to teach statistics after their participation in the course and PLT. The increase in participants' confidence to teach statistics is examined by analyzing SETS results and other qualitative data.

Results

Since a paired t-test is a parametric test with an assumption of normality, the distribution of gain scores for the total confidence score (postTotalSETS-preTotalSETS) should be checked for normality. Using Wessa's (2017) online tool, a Normal QQ Plot was created with the total SETS gain scores, which indicated the distribution was somewhat normal (See Figure 2). The assumption of normality was then assessed via Kolmogorov-Smirnov (*K-S*) test, and the results suggested that normality was a reasonable assumption for our data set. For example, the *K-S* test for total SETS gain scores had a test statistic value of .1546. Larger values of the *K-S* test statistic indicate the distribution does not follow a normal distribution. Since .1546 < .2499 (*K-S* test critical value for α=.05), the data is a reasonably good fit with the normal distribution, and a paired t-test can be used.

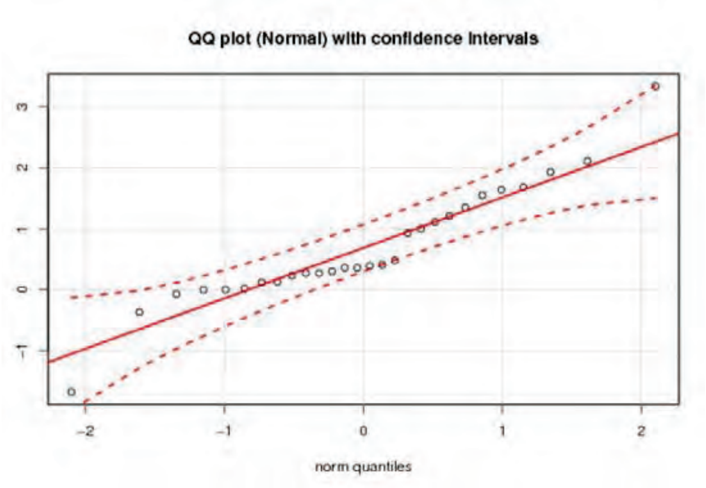

Figure 2: Title of the figure, no dot at the end

The question being asked for this analysis is: Is there sufficient evidence to suggest that PLT and MOOC participants' confidence in teaching statistics was increased after they took the TSDI MOOC and they participated in PLT? In other words, is there sufficient evidence to suggest that the mean score is greater for the post-SETS than the pre-SETS?

Ho: There is no difference between post and pre-SETS test scores. $Md = M2 - M1 = 0$

Ha: $M2 > M1$

Table 1 represents the SETS survey data for 28 PLT participants that completed both the pre- and post-SETS instrument. As seen in the table, in all three levels (A, B and C), it is observed that by participating in the MOOC and the PLT, participants SETS scores increased. Thus, we reject the null hypothesis and conclude that there is sufficient evidence to say that participants' confidence to teach statistics was increased by this phenomenon (participating in both MOOC and PLT). As a conclusion of looking to quantitative data of SETS results, we could claim that after participating in both MOOC and PLT PD projects, the PLT participants' confidence to teach statistics increased.

	Pre-SETS Mean	Post-SETS Mean	Difference Mean	Standard Deviation	*t*-test
LevelA Scaled[a]	4.19	4.82	.63	.90	3.71***
Level B Scaled[a]	3.87	4.63	.75	1.06	3.76***
Level C Scaled[a]	3.83	4.48	.65	1.05	3.30***
Total Scaled[a]	3.93	4.62	.68	.96	3.77*

$N = 28$.
[a] Scaled to 1 - 6.
* $p < .05$, ** $p < .01$, *** $p < .001$.

Table 1: PLT participants' SETS results

At the end of the pre-SETS survey, participants answered a similar question with minor differences. The open-ended question in pre-SETS was as follows: *“Consider some of the topics that you were most or least confident teaching. What may be some reasons that teachers might be more or less confident in teaching those topics?”*

In the end post-SETS survey, participants were expected to answer the following open-ended question: “*Consider some of the topics that you were most or least confident teaching. Did anything we did in this MOOC help you gain confidence in those areas? For the areas you are still less confident with, what may help build your confidence?”*

In the pre-SETS question, the goal was to make participants reflect on their own explanations of reasons for their least and most confident topics in teaching statistics; in the post-SETS question, we aimed to have them re-visit those topics and reflect on their own perspectives about how their participation affected their confidences to teach them.

Our desire was to figure out whether there was a change in their reflection about their confidence to teach statistics after participating in the MOOC and the PLT. Table 2 provides a sample of four PLT members’ responses to the open-ended questions.

Term/Years of Experience	**Pre-SETS**	**Post-SETS**
Fall / 9	I've only rarely taught about fitting data to models.	The concept of sample vs. population and the ideas around variability are two areas that I find are greatly improved using some of the real-time examples and simulations from this MOOC.
Fall / 1	I feel like I do not have enough practice or knowledge about these topics to be able to teach them to my students.	I am more confident in creating experiments and gathering data. I feel like I could have used more help in analyzing the data.
Spring / 5	Teachers might be more or less confident in teaching certain topics due to experience and education background.	Being able to talk with others and play with data helped me to gain confidence in those areas. The whole idea of exploration really helped me to gain confidence. I think having resources and activities with the activities I am less confident in would help me to gain confidence.
Spring / 27	As math teachers, we probably feel more confident with topics that involve calculations and less confident about topics that involve things that can limit the conclusions of those calculations, such as lurking variables and weaknesses in the data collection techniques.	I gained confidence in the technology end of things by reviewing some of the resources (that are bookmarked, but I do not recall their names off of the top of my head) presented in this course.

Table 2: SETS open-ended response examples

As seen in the table above, these four participants reflected on the change in their confidence in their own words. For example, a fall participant with one year of experience answered the question saying that he or she did not have enough experience or knowledge; on the other hand, the post-survey response of this participant included a detailed description of the impact stating that he or she had gained confidence in creating experiments and collecting data. The teacher directly reflects on the change in his or her confidence in conducting a statistics lesson using SASI framework.

The participant on the fourth row stated that math teachers used to feel more confident in more 'mathematical' subjects, and statistics is not one of those. In the post-survey, the participant said that she or he gained confidence in using technological tools. As a teacher with 27 years of experience, that participant gives a good sense about the strong connection between confidence to teach statistics and being able to use technology for teaching.

There were 15 participants (out of 28 who took both pre- and post-SETS survey) who gave us evidence of self-reflection on the positive change in their confidence to teach statistics. Looking at both participants' SETS quantitative results and open-ended responses, results showed that the phenomenon (participating in MOOC and PLT) helped them to gain confidence in teaching statistics.

A response given to a question in the end-of-course survey ("What was the most valuable aspect of this MOOC-Ed in supporting your personal or professional learning goals?") also showed evidence supporting increased confidence after participating in MOOC and PLT.

"This course gave me some great lesson plan ideas. It also made me more confident in teaching statistics where before I felt like I knew nothing about teaching statistics. After this course, I feel like I would teach it to my students." (PLT member, 8 years of experience)

Conclusion

Participating in the MOOC and PLT increased participants' confidence to teach statistics. Most of the teachers went from a "I know nothing about teaching statistics," or "I am clueless about statistics myself, how could I teach it?" level, to an "I can do it!" level. Attending the TSDI MOOC and PLT, the participants reflected that their self-confidence to teach statistics increased. This claim is supported by both SETS quantitative results and open-ended responses in SETS. The results of the study are valuable, because as stated before, there is a common and strong issue with teachers' confidence to teach statistics (Estrada et al., 2011; Lovett, 2016; Stohl, 2005).

The findings can be used to help statistics education researchers, as well as others in any educational discipline, to design more MOOC and PLT professional development projects to contribute to teachers' professional growth. The findings also can be used to create a common drive to change problematic aspects of statistics in curriculum.

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Covariational reasoning in primary school: A qualitative approach.

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Keywords: Covariational reasoning, gapminder's dollarstreet, education for sustainable development.

Introduction

Covariational reasoning includes recognizing patterns and interpreting the relationship in bivariate data (Garfield & Ben-Zvi, 2008). Although it is important to focus on enhancing young learners statistical thinking in various facets (Ben-Zvi, 2018), most curricula address this topic only in later years of statistics education. With this poster, we want to present the idea of using qualitative data such as photos from the Dollarstreet to address young learners covariational reasoning.

Theoretical framework

Many researchers recognize the value of a real context meaningful for students (e.g., Garfield & Ben-Zvi, 2008) and emphasize the importance of teaching and learning of statistics for democracy and citizen education (Nicholson et al., 2018). Recommendations to regard students as active learners when constructing their statistical knowledge often imply the use of technology and tools to explore real datasets (Garfield & Ben-Zvi, 2008). The databases and visualization tools provided by Gapminder (Rosling et al., 2005) including the photo database Dollarstreet (Rosling, 2015) give many opportunities to explore real datasets in statistics education at school level. In that direction, Andre et al. (2020) investigated how the context of sustainable development can enrich pupils' statistics education. Anyway, many authors also found difficulties in addressing young learners covariational reasoning, and recommend using alternatives to scatterplots (Konold, 2002).

Methods and Implementation

Following design-based research approaches (Bakker, 2018), in the summer term of 2021, we designed a seminar for six primary school teacher students focusing on important issues for Education for Sustainable Development and statistics education (Andre et al., 2021). The major task for the students in this seminar was to design and implement a learning trajectory for pupils of grade 1. Besides addressing other statistical ideas, it was mandatory for students to include two tasks as shown in figure 1 where pupils should relate the quality of sanitation facilities to various income levels and interpret a scatterplot in the same context. The implementation took place in a primary school class with 18 children. The students had to write a report and transcribed the interviews with the children when solving the mandatory tasks. We analyzed these interviews and the students' reports in several dimensions. This poster focusses on pupils covariational reasoning with qualitative data.

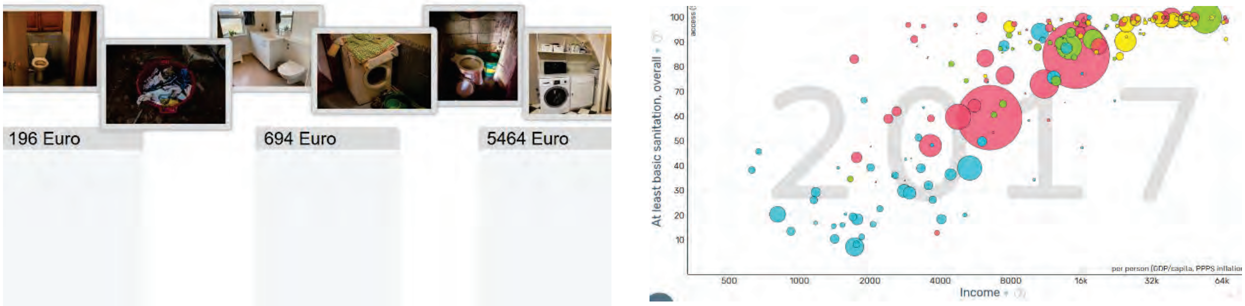

Figure 1: Left: Drag-and-drop task for qualitative covariational reasoning; Right: Gapminder's scatterplot showing income data and data of sanitation standards (Andre et al., 2021)

Results

While struggling to interpret the scatterplot, all children managed to relate qualitative data from photos to quantitative data, i.e., the income level. In our analysis we found several opportunities to further address supplementary aspects such as causality and variance (Andre et al., 2021).

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Reading levels in the activities related to statistical graphs in Costa Rica textbooks

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In this paper we present results of analysing the reading levels, according to the model proposed by Curcio and collaborators of statistical graphs in Costa Rica compulsory education textbooks. The activities linked to statistical graphs included in three complete textbooks series in each of the primary (grades1 to 6) and secondary education (grades 7 to 9) were analysed.

Keywords: Statistical graphs, reading levels, textbook analysis, compulsory education.

Introduction

Statistical graphs are essential for communicating information in many different areas, and appear frequently in the media, and for this reason a minimal statistical literacy is needed for the citizen to be able to interpret these graphs critically (Weiland, 2017). This need is still more urgent in the current pandemic situation, where citizens need to understand statistical information mainly distributed in graphs to collaborate with political and health authorities (Batanero et al., 2021). Moreover, graphical competence is part of statistical literacy defined by Gal (2002) as the ability to interpret and critically evaluate statistical information, data-related arguments, or stochastic phenomena, which they may encounter in diverse contexts, joined to the ability to discuss or communicate reactions to such statistical information, when required (pp. 2-3).

Consequently, graphical competence is a main goal in school curricula in many countries. This is the case of the compulsory education curricula in Costa Rica (MEP, 2012) that, for primary education recommend that children are requested to collect and record data to answer statistical questions about themselves and their environment, using bar and line graphs, tables and pictograms, as well as to read and interpret these representations since the 1st grade. Further study of graphs is accomplished in secondary education, where histograms, frequency polygons and pie charts are also introduced with the aim of improving students' graphical abilities. These goals can only be achieved with a correct teaching of the topic and the use of adequate textbooks.

Research analysing different content in textbooks is increasing today, since these books are a main didactical tool for teachers and students and constitute an intermediate stage between the official curricular guidelines and the teaching implemented in the classroom (Fan et al., 2013). Consequently, the aim of this research was to analyse the activities where statistical graphs are used in the Costa Rica textbooks and, more specifically the reading levels required from the students to complete these activities.

Below we describe the foundations, method and results of the research and we finish with some considerations to improve the teaching of statistical graphs in compulsory education.

Foundations

The work is supported, in the first place, by theoretical models on difficulty levels in reading graphs, and secondly on some research that have previously studied statistical graphs in primary school textbooks which is used to compare with our results.

Reading Levels of Graphs

A graph is a complex semiotic object, due to the different components that need to be comprehended in order to fully understand the information displayed (title, scales, types and amount of data and graphical elements used to represent the data). The graphic builder encodes the information represented by means of these elements and the reader of the graphic should carry out a series of interpretative processes for each of these components and for the graph as a whole as well as of the relationship of the graph with the information context (Arteaga & Batanero, 2011; Sharma, 2006; Spence, 1991).

Consequently, the interpretation of a graph is more or less complex, depending on the information that needs to be extracted from the graph and for this reason, several authors have defined levels in the reading of graphs. These levels reflect the fact that on the same statistical graph, we can raise questions of different levels of difficulty, which can refer to the title and scales, the variables and values being represented, the interpolation or extrapolation of values, and even to the detection of biases in the graph or in statements based on the graph. In this research, we relied on those defined by Curcio and his collaborators (Friel et al., 2001; Shaughnessy, et al. 1996; Curcio, 1989):

L1. Reading the data, where only the literal reading of a graph element is requested, for example, reading the graph title, the scale labels or the frequency for a given value.

L2. Reading between the data. Besides the literal reading of the graph, in this level the child has to compare various data represented in the graph or complete some arithmetical calculations with the data.

L3. Reading beyond the data. This level involves a generalization of the values in the graph, for example, interpolating or extrapolating the information displayed.

L4. Reading behind the data. A person attains this level when he or she is able to make a critical valuation of the graph, of the way it has been constructed or can discuss a statement related to the graph content.

Previous research

We also draw on other research that analyses statistical graphics in primary school textbooks, most of which only dealt with primary education textbooks. Díaz-Levicoy in various works analyses statistical graphs in primary school textbooks (primary education Chilean and Spanish textbooks in Díaz-Levicoy et al. (2016), grades 4 to 6, Argentinean textbooks in Díaz-Levicoy et al. (2017), Peruvian textbooks in Díaz-Levicoy et al. (2018)), with very similar conclusions. The authors study the type of graph proposed, the activity requested of the child, and the reading levels required in the activity. They conclude that bar graphs are most frequently used, with little weight given to other types of graphs in the curriculum. The most frequent activity in Spanish textbooks is reading graphs, while calculating was predominant in the Chilean books.

The most frequent reading levels in Curcio's (1989) classification were L1 followed by L2 in Chile and Spain; L2 being the most common level in Argentina. Level L3 appears from 4th grade in Spain and 3rd grade in Chile, and is not proposed in Argentina textbooks. As regards L4 only grades 5 and 6 in Spain and grades 4 and 6 in Argentina consider this level. The tendency was different in the Peruvian textbooks, were half activities in grade 1 correspond to level L1 and the remaining to L2, which predominates in all the other grades with very scarce activities in levels L3 or L4,

A direct precursor to this work is that by Jiménez-Castro et al. (2020), who analysed statistical graphs and reading levels in the activities related to them in two series of primary school textbooks (grades 1 to 6 in Costa Rica). Bar and line diagrams (single and multiple), pie charts, dot plots and pictograms were observed, with a large predominance of bar charts. Generally, the reading level required in the activities was L2 level.

All the above research deals with primary education textbooks and only that by Jiménez-Castro et al. (2020) was carried out in Costa Rica. This paper expands research by Jiménez Castro et al. (2020) with the analysis of a new editorial (6 new books) and expand the study with an investigation of three complete editorials for secondary education (grades 7 to 9, additional 18 textbooks) with particular focus on the reading levels requires in the activities proposed for statistical graphs in these textbooks. Below we describe the method, present the analysis categories and discuss the results. We conclude with some implications for the teaching of statistical graphs.

Method

This is a qualitative research, based on content analysis, which was carried out through the following phases (Porta & Silva, 2003):

- Fixing the object of analysis: In each textbook, the activities that incorporate statistical graphs were studied.
- Determining the categories or coding rules: The categories were the reading levels identified a priori based on previous research (Friel et al., 2001; Shaughnessy, et al., 1996; Curcio, 1989)
- Checking the reliability of the coding-categorisation system: the coding was reviewed several times by one author and discussed with other colleagues until an agreement on the coding was reached. These categories are presented in section called “Analysis categories”.

The information of the variable “Reading level” was coded, following an inductive and cyclical process; a detailed reading of each activity was carried out, the data obtained were interpreted and reviewed to make sense of the analysis, the data were related to the defined categories. This process was carried out of one of the authors of the paper as part as his doctoral thesis, consulted with the director of the doctoral thesis and with other colleagues, to give greater validity to the study.

Sample of textbooks and activities

The sample includes all the activities related to statistical graphs (activities where statistical graphs appear or activities that propose to build a statistical graph) in three books series of primary education (grades 1 to 6, 18 textbooks) and in another three series of textbooks directed to secondary education (grades 7 to 9, 18 additional textbooks). All the editorials were selected because they are most widely used in Costa Rica and are listed in the Appendix. In total. the number of activities analysed were 579 activities in primary education and 363 in secondary education.

Analysis categories

We used the reading level required, using the classification by Curcio (1987), and Friel et al. (2001). Below we include examples of activities classified in each of these levels:

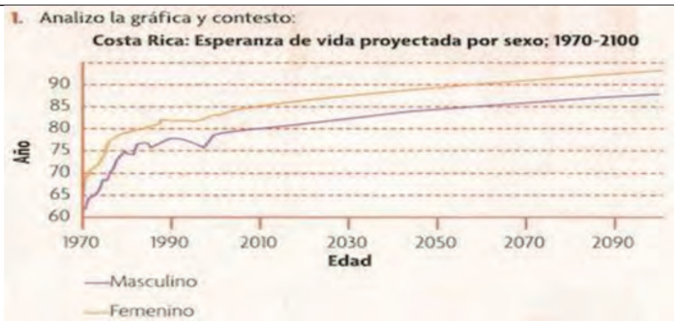
L1. Reading the data: At this level the activity concentrates on answering questions about the frequency of a value for a variable (or the value to which a frequency corresponds), without going so far as to make other connections or perform other calculations with the data. An example is shown in Figure 1, where children should complete a bar graph to represent the frequency of a series of objects (previously classified) shown in an image.

Figure 1. Example of reading level L1 "reading the data" (Source S1, p. 298)

L2. Reading between the data: this level applies when comparisons or operations must be made with the data obtained from a graph in order to solve the activity. It involves identifying relationships between the data, as in questions b of the activity presented in Figure 2, where, after reading the data, operations have to be performed on the data. Question f also corresponds to this reading level, because depending on the year of birth, the student must locate such data on the graph, and if the year is not a multiple of 10, he/she needs to locate it on the X-axis in order to identify the life expectancy corresponding to that year.

L3. Reading beyond the data: This reading level is achieved in activities where we ask to obtain data not explicitly represented within the graph, which includes interpolating or extrapolating a piece of data on the graph. We classify questions a and c in Figure 2 at this level as well, as the trend is not explicitly represented but must be inferred from reading the values on the graph and comparing along the time axis.

L4. Reading behind the data. The solution of activities qualified at this level require from the solver a critical appraisal of the method of data collection, validity, or reliability or to justify the conclusions made. Although this level of reading is not common in textbooks, we have encountered some examples in activities such as the one presented in question d in Figure 2, where the student is asked to think about the reasons explaining the increase in life expectancy along time.



a. Which tendency shows the life expectancy? Does it increase or decrease? Justify
b. How many years of difference there is between 1970 and currently?
c. Is the tendency similar for both genres? Explain
d. Suggest two reasons why the life expectancy follows this tendency
e. Comment two consequences of this tendency for our society
f. Taking into account the year you were born. Which is your life expectancy?

Figure 2: Task reproduced from AL5 p. 166

Results and discussion

In Table 1 we present the percentage of activities that were classified in each of the reading levels defined by Curcio and his collaborators (Friel et al., 2001; Shaughnessy, et al., 1996; Curcio, 1989) by school grade. We can observe the predominance of level L1 (reading the data), where only a literal reading of the data displayed in the graph is required from the students in the first three grades, in particular in the first grade, where the children are only starting to be introduced to some statistical ideas. This level diminishes in relevance since grade 4 in favour of upper reading levels.

Table 1: Percentage of tasks by reading level and school grade

	Primary education						Secondary education		
Reading levels	1st (n=50)	2nd (n=108)	3rd (n=97)	4th (n=109)	5th (n=93)	6th (n=122)	7th (n=176)	8th (n=82)	9th (n=105)
L1	68.0	42.6	44.3	24.8	22.6	32.8	21.0	30.5	29.5
L2	30.0	43.5	40.2	52.3	52.7	50.8	59.1	52.4	46.7
L3	2.0	7.4	6.2	16.5	16.1	7.4	13.6	15.9	10.5
L4		6.5	9.3	6.4	8.6	9.0	6.3	1.2	13.3

The reading level L2 (reading between the data, where students are also requested to perform some computations or comparisons with the data in the graph) was very frequent along all the school grades, including secondary education. It was the most frequent level in grades 4 to 9 because in these levels' graphs are also used to facilitate the computation of central tendency and spread measures and for this reason, the students are requested not only to read the graph but to compute mean, mode, range or other statistics with the data represented in the graph. Interpolation and extrapolation activities (reading level L3), as well as critical reading activities (Level L4) appear with low frequency in all the grades, although they are a bit more frequent in secondary education.

These results contrasts with what is reflected in the study described by Díaz-Levicoy et al. (2016), where Spanish and Chilean books remain only at reading level L1 in the first two grades and is still the most frequent level along all the primary education. In Costa Rica, reading levels 3 and 4 appear significantly from the second year onwards and increase in frequency until the sixth year. In the Spanish context, the maximum reading level only appeared in the fifth grade, whereas in the Chilean context, it is incorporated in a sustained manner from the fourth grade onwards, coinciding with the recommendations given in NCTM (2000) to promote the making of inferences and the obtaining of predictions from the data analysed. In Argentina, L3 is not taken into account and L4 only in grades 4 and 6. So, the distribution of reading levels in primary education in our research is more similar to that in the Peruvian textbooks, were half activities in grade 1 correspond to level L1 and the remaining to L2, which predominates in all the other grades with very scarce activities in levels L3 or L4.

In Table 2 we analyse the reading levels of the activities analysed by editorial, and we can observe some differences. As regards primary education, AL present more than half the activities in the L2 reading between data level, which suggest a stronger emphasis in computation in this publisher. EV is the editorial with less activities in this level, while it put more weight in L1 reading the data, in order to reinforce in the children, the learning of the different graphs before going further in more complicate activities. All the three editors include a small percentage of L3 and L4 activities which are those promoting statistical reasoning in dealing with predicting variables not in the graph and reinforcing the critical reading of the information.

Table 2: Activities with different reading levels in the various editorials

	Primary education			Secondary education		
Reading levels	AL (n=147)	EV (n=181)	SA (n=251)	PI (n=126)	PO (n=92)	SA (n=145)
L1	27.2	47.0	34.3	31.7	33.7	15.2
L2	54.4	40.3	46.2	58.7	41.3	57.9
L3	8.8	5.5	13.5	7.1	13.0	18.6
L4	9.5	7.2	6.0	2.4	12.0	8.3

As regards secondary education, SA is the editor with less L1 reading the data activities, which are less important at these teaching levels, given the age of students. PO is the publisher with less L2 activities, implying a smaller emphasis in computation. This is the editorial with a more balanced distribution of reading levels in the activities, since it includes more than a 10% of activities in each of the levels L3 and L4.

Conclusions

In this paper we present the analysis of the reading levels collaborators (according to the model by (Friel et al., 2001; Shaughnessy, et al., 1996; Curcio, 1989) required to complete the activities linked to statistical graphs contained in the most widely used textbooks in Costa Rica, at primary and secondary school levels.

The study reveals a predominance of the first two levels “reading the data” where only a direct reading of the information in the graph is required and “reading between the data”, where only comparisons or computation with that information is needed. Specifically, this second level is predominant in most grades even at secondary education level, which suggest an excessive emphasis in developing the computational abilities, instead of focusing in fostering their statistical literacy or reasoning.

We particularly point to the scarceness of level L4 “reading behind the data” activities, which will be particularly relevant in the education of secondary school students. These activities serve to develop in the student their ability to reason with statistical information, to discuss arguments based in statistics information and to assess the reliability of such information and in this way, help improving the students’ statistical literacy.

Also note that the L3 level in secondary school textbooks is not much considered, this type of level requires the student to be able to perform interpolations and extrapolations of the data presented in a graph, which is considered an important part of graph comprehension (Friel et al., 2001)

We suggest that especially in secondary education, it would be important to incorporate a greater number of L3 and L4 activities, which are closely related to the skills necessary to develop good statistical literacy levels (Gal, 2002)

As we pointed out in the previous sections, there are scarce research examining statistical graphs in secondary school education textbooks, and therefore this research should be continued with the analysis of other variables and textbooks in other countries. However, we hope our research make teachers conscious of the importance of taking into account all the possible reading levels when working with statistical graphs at school and help them organise teaching to take into account this didactical variable.

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Appendix: List of editorials used in the analysis

Primary education textbooks

Serie 1: *Asociación Libros para Todos (AL)*. Textbooks directed to grades 1 to 6 published in 2017.

Serie 2: *Editorial Santillana Serie Casa del Saber (SA)*. Textbooks directed to grades 1 to 6 published in 2016.

Serie 3: *Editorial Eduvisión. Visión Matemática (EV)*. Textbooks directed to grades 1 to 6 published in 2014.

Secondary education textbooks

Serie 1: *Editorial Publicaciones Innovadoras en Matemáticas (PI)*. Textbooks directed to grades 7 to 9 published in 2017.

Serie 2: *Editorial Publicaciones Porras y Gamboa (PO)*. Textbooks directed to grades 7 to 9 published in 2017.

Serie 3: *Editorial Santillana (SA).* Textbooks directed to grades 7 to 9 published in 2017.

Insights into the design of an introductory course for data science and machine learning for engineering students

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Due to their interdisciplinary nature, data science methods, such as machine learning, can be taught in many different ways. This paper presents an approach that takes advantage of the close content connection to statistics and of the mathematical structure of data science methods to develop an introductory course for engineering students. Following the research methodology of design research, we discuss the theoretical motivation and methodological implementation of the design principles for the course and show first insights into empirical results from the design cycles.

Keywords: Data Science, Machine Learning, Design Research, SRLE.

Introduction

A large proportion of the methods attributed to data science are (computer-aided) applications of statistics, making data science an essential topic in the statistics education research community (Engel, 2017; Gould, 2017). The possibility of applying data science methods in almost all areas of industry and research has created a need for subject-specific concepts for teaching the methods over the past years (Engel, 2017; Grillenberg & Romeike, 2018).

Due to the interdisciplinarity of data science between mathematics, statistics, computer science, ethics, and the respective reference science, there are many ways to approach the topic. We want to show insights into the design of an introductory course for data science (DS), particularly machine learning (ML), for early mechanical engineering bachelor students in a few lectures focusing on statistics, particularly mathematics. The development of the course follows the methodology of design research (Gravemeijer & Cobb, 2006).

In this paper, we motivate the design principles of the course and present their respective elaboration. For this reason, the central part of this paper is theoretical, followed by examples from the introductory course that illustrate the explicit implementation of the design principles. In the end, we switch to a first empirical evaluation and give a brief insight into the students' views on the developed introductory course.

Theoretical considerations for designing the introductory course

This section gives insights into the current state of research regarding teaching DS and in the methodology of design research, followed by the theoretically motivated design principles.

The current state of research in teaching data science with a focus on machine learning

In a DS study program, the versatility of what can be taught and in which ways it can be taught is wide (Grillenberg & Romeike, 2018). There are different approaches to the concretization of a DS curriculum for schools (Heinemann et al., 2018), undergraduate programs (De Veaux et al., 2017), and competence models for sub-aspects such as data literacy (Ridsdale et al., 2017) or data management (Grillenberg, 2019). One subfield of DS and part of many DS curricula is data analysis, especially ML (Grillenberg & Romeike, 2018). There are many open questions and few empirical studies about how learning ML occurs under different teaching methods (Steinbach et al., 2020).

Especially for students without a mathematical or computer science background, there are different approaches to how to deal with the more complex mathematical and programming details that seem to be a hurdle for students (Lavesson, 2010). Suppose one additionally considers the easy accessibility of methods nowadays, there is a danger: Using ML without theoretical expertise, for example, on fundamental mathematics and statistics, creates the risk of harmful socio-technical systems (Heuer et al., 2010). To date, there is little consideration of the role of ML in the context of statistical literacy and data literacy (Grant, 2017; Kadijevich & Stephens, 2020; Schüller, 2017). In this context, the distinction between the terms statistical and data literacy is still fluid, with broad similarities, and somewhat arbitrary (Gould, 2017; Schüller, 2017).

Theoretical considerations on design research

The research methodology of design research focuses on the close connection between the systematic design of teaching-learning material and the investigation of learning processes working with this material (Gravemeijer & Cobb, 2006). Especially in the case of little empirically tested teaching-learning material, design research can be used sensibly with the two following goals: To get empirically tested and cyclically improved teaching-learning material and to get research results on the learning processes of the target group when working with the material.

For this purpose, first, a prototype of the teaching-learning material is developed, considering the so-called design principles (see section *The design principles for the introductory course*). The development of the prototype also includes theoretical considerations about the students' learning processes, so-called intended learning trajectories. Subsequently, the prototype is tested with the target group in the so-called design experiments, for example to compare the students' individual learning paths with the intended learning trajectories. By analyzing the design experiments, a local (concerning the target group and the material) teaching-learning theory emerges, which contributes to the further development of the material. A cyclical continuation then provides improved teaching-learning material and a sharpened local teaching-learning theory (Gravemeijer & Cobb, 2006).

The design principles for the introductory course

In this section we motivate the design principles (DP) of the course and explain their methodical implementation. In the following section *Insights into the course,* two examples, *The unit square* and *Reflection tasks*, illustrate how the design principles are incorporated into the design of the course.

The first design principle is *Strong inclusion of statistics and mathematics to approach the DS/ML methods* (DP1). There are two main reasons for this design principle: One is the proximity between DS and statistics, respectively mathematics, in terms of content and the personal interest in this connection. The other is the fact that engineering students are, due to their curriculum, a target group with a comparatively strong mathematical background.

To implement the first design principle, we use the “four-level approach for specifying and structuring mathematical learning content” (Hußmann & Prediger, 2016). The four-level approach illustrates how to proceed methodically when the focus within a design research project is on analyzing the learning content. Using the approach, the prototype of the material emerges by answering a series of systematic questions on three theoretical levels in the sense of a “classic didactical analysis of subject matters” (Hußmann & Prediger, 2016). The first level, the formal level, addresses the logical structure and the formal representation of the (mathematical) learning content (objects and procedures). The second, the semantic level, addresses the sense and meaning of the objects and procedures under study; helpful representations and mental models are identified and linked to each other (see a concrete example in section *The unit square*). On the subsequent concrete level, the last theoretical level, learning situations, and examples for experiencing the concepts and procedures are developed (see a concrete example in section *Reflection tasks*). The fourth level then equals the implementation and evaluation of the design experiments.

The second design principle is *Embedding all methods in the overall context of data analysis* (DP2). It is an idea, which is used in several different contexts while learning methods to handle with data (Heinemann et al., 2018; Wild & Pfannkuch, 1999).

This design principle is implemented by using the “CRoss-Industry Standard Process for Data Mining” (CRISP-DM, Chapman et al., 2000) model to structure the course and some teaching activities. The CRISP-DM model is a process model that describes all essential steps of a DS process in an industrial context, starting with a question and ending with the implementation of the results. It has already been fruitfully used in other projects to structure teaching activities in the context of DS (Heinemann et al., 2018). The CRISP-DM also gives an overview of some core ideas of DS, according to DP3 (*core ideas of DS and ML,* see next paragraph), and we additionally use it in the sense of DP4 (*classroom activities,* see next paragraph) to design tasks that encourage students to discuss core ideas and own proceedings (see a concrete example in section *Reflection tasks*).

The further four design principles (DP3 to DP6) refer to the basic ideas of the “Statistical Reasoning Learning Environment” (SRLE, Garfield & Ben-Zvi, 2008). The SRLE is a well-structured and proven approach to create teaching-learning environments in the context of data. The origins of the SRLE go back to Cobb (1992) and were developed further by different statistics educators within the following decades. Because of the close proximity of DS and statistics in terms of content, it offers to use some ideas of the SRLE as design principles for the introductory course.

The following ideas from the SRLE (Garfield & Ben-Zvi, 2008) are adopted as design principles: *Focusing on the developing core ideas of DS and ML* (DP3, original: “Focus on developing central statistical ideas”), *Using classroom activities to support the development of students’ reasoning* (DP4), *Using realistic and motivating data sets* (DP5) and *Integration of appropriate technological tools* (DP6). DP4, DP5 and DP6 are adopted literally from the SRLE.

Insights into the course

To give an overview, we first present the content components of the course. Then we illustrate how the design principles shape the course by giving two examples.

The course components

The selection of the learning content is mainly based on the subjectively set goal of the course to convey the usefulness, practical relevance, and methodology of DS, especially ML, in the engineering sciences. Students shall be enabled to delve deeper into the topic of DS and ML. This goal results in three sessions of approximately 3 hours each:

Session 1 - fundamentals: A first overview of the possibilities to use DS methods in engineering is shown, and the CRISP-DM model is introduced. The handling of data within this setting is discussed and the basic concepts of ML up to classification are introduced.

Session 2 - k-nearest-neighbor classification (kNN): The basic concepts of ML are explored in depth by discussing the kNN as a possible method for classification.

Session 3 - model quality of classification models: Model properties (variance and bias), as well as different performance measures (accuracy, precision, recall), are discussed to be able to evaluate and compare classification models and to select the model parameters for a specific question.

The following example *The unit square* shows the use of the "four-level approach" on the first two levels, and thus gives insights into the implementation of DP1. The next example *Reflection task* shows the use of CRISP-DM as the elaboration of DP2 and some synergies with the design principles adopted from the SRLE (DP4 to DP6).

The unit square – An example for the analysis of the learning content on the first two levels

In Session 3, *model quality of classification models*, different model characteristics and performance measures are discussed with the students. When creating a classification model, the available data set consists of examples with the characteristics of the independent variables (called features) and a dependent variable (called a label). The total data set is first divided into training data and test data. The training data is used to build the model, and the test data is then used to check how well the model can predict the correct label. Concerning a binary classification model, the testing phase is usually represented using a confusion matrix as in Figure 1. Here, the number of correctly classified examples (true positive and true negative) separated by class is on the main diagonal, and the number of incorrectly classified examples (false positive and false negative) is on the opposite diagonal.

		Actual Class	
		Class 1	Class 2
Predicted Class	Class 1	true positive	false positive
	Class 2	false negative	true negative

Figure 1: Example of a confusion-matrix

All performance measures and performance criteria are derived from the values in the confusion matrix. A content analysis of the learning object, as it has been done on the formal level of the four-level approach, reveals that all performance measures can be represented by a probability space, which explains the relations of the values among each other:

Each example classified by the model[1] can be represented as $\omega^i = \left(\omega_1^i, \omega_2^i\right), i = 1, \dots n,$ where $\omega_1^i, \omega_2^i \in \{1,2\}$ with ω_1^i representing the actual class of the example and ω_2^i representing the new

[1] We continue to consider a binary classification problem here, a transfer to higher dimensionality does not pose a problem.

classified class. n is the number of examples. This gives $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$ with $\Omega = \{\omega^i : i = 1, \dots, n\}$ and $\mathbb{P}$ as the normalized count measure, a probability space that represents the testing phase.

Going on in the formal level, all addressed performance measures and their interrelationships among each other can be represented based on this probability space (for example, certain performance measures are equivalent to conditional probabilities). On the subsequent semantic level, the following questions arise concerning the prerequisites of the target group: How can the performance measures and their interrelationships be communicated without addressing this probability space and, for example, conditional probabilities?

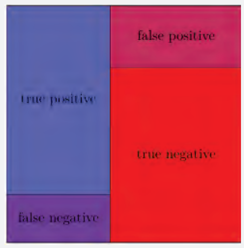

Figure 2: Example of a unit square

The unit square (see Figure 2) is considered to be a proven means to visualize proportions and probabilities up to conditional probability (Böcherer-Linder et al., 2017). The analysis in the two levels reveals that the unit square provides a way of representing the testing phase. The unit square in addition can visualize the performance measures to show them as formulas based on the values of the confusion matrix. Thus, by linking the content to mathematics, the unit square is included as a visual representation in the course.

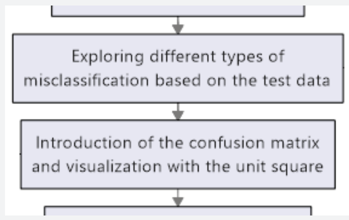

Figure 3: Excerpt of an intended learning trajectory

Besides the elaboration of different representatives of the performance measures (in this example, the unit square, the formula, and the values calculated by hand or with Python), the core ideas of Session 3 (such as the distinction between types of misclassification, which can also be visualized in the unit square) emerge from analyzing the learning content up to the semantic level. The analysis of core ideas, connections, and representatives results in a theoretical sequence in which they can be worked out: the intended learning trajectory (see Figure 3).

Reflection task – An example for considerations on the third level

The work areas of the CRISP-DM are learning objectives (see Session 1). The CRISP-DM is also used to structure the course (see section *The design principles for the introductory course*). The structure is implemented, among others, by giving students a task at the end of each session to reflect on the learnings in the framing of the CRISP-DM. For example, at the end of the third session, when students have to design a model and use the model for a decision afterwards (Bata et al., in press), this task reads:

Reflect on your decision in the Jupyter Notebook together in groups in the context of your notes from the last lecture[2] *and the CRISP-DM. If you find it useful, complete your answers again.*

This task demonstrates the incorporation and the interrelation of some design principles while formulating explicit learning opportunities on the third level of the “four-level approach”: Students work with a data set regarding the quality of steel (*realistic data sets*, see DP5), the Jupyter Notebooks are used as technical support throughout the task (*appropriate technological tools*, see DP6). The

[2] This refers to the notes of the reflection task of the past lecture (Session 2).

open assignment encourages students, who are working together in groups at this point, to discuss their results and to defend them argumentatively using the CRISP-DM (*overall context*, see DP2, *classroom activities,* see DP4).

First empirical insights

The design cycles and data collection

The introductory course has been conducted in two design cycles in different settings so far. In the first cycle, seven students participated in a laboratory setting (in groups of two and three, accompanied by the lecturer during the processing of the tasks). In the second cycle, 39 students participated in a course setting. The third session was additionally conducted with 4x2 students in a laboratory setting. All sessions took place via an online conference tool. Each session was video-recorded and transcribed. The group work was additionally documented using written products.

In addition, data were collected using a one-minute paper in each session. Students were asked five questions per session, each to be answered at one point after the session within a given time (usually one minute) and without looking into the learning material. The five questions were intended, among others, to help gather information about the learning environment. The evaluation gives a first insight into the students' views, from which we present first results.

Results

The question of the one-minute paper "How relevant do you think the content of the past lecture is to your studies and future career, and why?" was evaluated using points to characterize the students' answers: 0 points means no relevance, 1 point means medium relevance and 2 points means high relevance. In addition, the reasons were collected and grouped into content-related groups. Mean values between 1.81 and 1.92 across the cycles indicate that most students perceive the learning content as very relevant. However, the reasons for their ratings varied: Only about 10 percent of the students justify the relevance with concrete content like "validation of ML models"; instead, general facts are used as reasons. For example, students mention the presence and relevance of DS and ML in engineering or everyday life or Python as an essential competence for jobs and studies.

Two questions of each one-minute paper focused on the content goals of the particular session, for example: "For which data sets is the performance measure accuracy not recommended?" To evaluate the questions, 1 point (answered completely correctly), 3 points (answered partially correctly), or 5 points (answered incorrectly or not answered at all) were assigned to each response. The scores give an overview regarding the students' learning results concerning the questions. The questions were largely answered in a meaningful way in terms of content, the mean values of the answer points per question ranged from 1.95 to 2.63 across both cycles.

Discussion

This paper gives insights into the design principles and development of a short introductory course for DS and ML for engineering students. Especially the first design principle, implemented by the approach of specifying and structuring the learning content focusing on its statistical and mathematical aspects, opens a way to analyze ML methods, which have so far rarely been investigated from the perspective of the classic didactical analysis of subject matters. For example, the connection to the unit square has two potentials: On the one hand, learning methods with threshold parameters, which are discussed in every advanced ML course, can be transferred to the representation of the performance measures with an animated unit square. This visualization can show the direct influence of the threshold parameter on the performance measures. On the other hand, the very visual representation of the unit square can be used when students' backgrounds are not as mathematical as in the case of mechanical engineers.

The first analysis of the one-minute-paper questions shows a pleasing result, as the planned contents seem to reach the students and seem relevant to them. Nevertheless, the question arises about how the design principles, and the resulting developed or chosen representations, visualizations, and instructional activities contribute to the students' learning processes. The overall design study aims to explore students' individual learning paths through a qualitative analysis of the resulting video material to address this question. From this analysis, results are expected on whether and how the statistical and mathematical details are learned by students (which is unanswered by now) and used when applying the methods (first results see Bata et al., in press). Based on these findings, the role of statistics and mathematics in ML, specifically in the context of data and statistical literacy, can be addressed in greater depth.

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University students reflecting on a problem involving uncertainty: what if the coin is not fair?

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How do university master's students in mathematics reason about a simple problem involving coin tosses? We answer to this question by analyzing the participants' written reflections in terms of recency and equiprobability effects understood non-normatively. Overall, the participants display a strong tendency to deem the problem's coin to be fair despite indication that this should be the case and contrary to explicit evidence presented in the problem. In view of this, we further elaborate on the connection between the participants' answers and their education in mathematical probability.

Keywords: Probabilistic reasoning, university students, recency effects, equiprobability effect.

Introduction

The recency and equiprobability effects are well-known phenomena related to people's interpretations of problems involving uncertainty. The *positive recency effect* is the tendency to interpret the manifestation of some event as evidence that the same event is likely to happen again in the future. Conversely, the *negative recency effect* is the tendency to interpret the manifestation of some event as evidence that the same event is less likely to happen again in the future. Moreover, the *equiprobability effect* is the tendency to judge a set of events as all equally likely to happen.

These definitions differ from the ones usually employed in the literature (e.g., Gauvrit & Morsanyi, 2014; Morsanyi & Szucs, 2014; Chiesi & Primi, 2009) essentially in the fact that they are explicitly non-normative: i.e., they do not account for whether the aforementioned effects are against the assumed normatively-correct interpretation of the problem under consideration. Indeed, these effects have been often termed "biases" or "fallacies" or "misconceptions" as they have been mostly problematized with reference to cases in which the respondents' answers to word-problems were considered wrong (cf. Chernoff & Sriraman, 2020; Batanero, 2020).

For instance, an influential article by Fischbein and Schnarch (1997) contained the following question posed to, among others, 18 prospective teachers specializing in mathematics:

> When tossing a coin, there are two possible outcomes: either heads or tails. Ronni flipped a coin three times and in all cases heads came up. Ronni intends to flip the coin again. What is the chance of getting heads the fourth time? (p. 98)

The answer "equal to the chance of getting tails" was taken to be the correct answer (given by 17 students). The answer "smaller than the chance of getting tails" was taken to be evidence of a negative recency effect (given by no student), while the answer "greater than the chance of getting tails" was taken to be evidence of a positive recency effect (given by one student).

As to this student, was he or she simply incorrect or perhaps just interpreting the described fictional situation differently? After all, nothing in the statement of Fischbein and Schnarch's word problem as reported suggests that the hypothetical coin tossed by Ronni has to be considered a *fair* coin or that the way in which Ronni tosses the coin is not biased towards heads. As Gigerenzer (1991, 1996) has argued, probabilistic word-problems usually do not have only one correct answer over which there exists unquestioned consensus. It is true that often people's answers deviate problematically from the generally accepted norm. However, this discrepancy could be caused by the respondents' divergent interpretation of the situation presented to them (Chiesi & Primi, 2009, p. 152). How come that, we may in turn ask, the great majority of the students tested by Fischbein and Schnarch were keen to interpret the problem as involving a fair coin and a fair toss despite the information provided? According to the non-normative definition given above, we may also read these responses as manifestations of an equiprobability effect which we may also problematize. And what about the other responses? Unfortunately, Fischbein and Schnarch did not report on the participants' reflections and hence it is not possible to reconstruct their reasoning.

In this paper we address the following research question by means of a problem involving coin tosses analogous to the one employed Fischbein and Schnarch: *how do university students in mathematics reason about uncertainty?* We answer by means of an analysis of the students' written responses based on the definitions of recency and equiprobability effects given above. Furthermore, nuancing the students' reflections will give us the opportunity of discussing them in relation to their usage of acquired probabilistic concepts and ideas. Notice that our intention in this study is mainly descriptive: while we will discuss the problematic nature of many of the participants' responses, we will not attempt here to suggest any related pedagogical or curricular ameliorations.

Summary of relevant research

The study of Fischbein and Schnarch tested primary and secondary school students together with, as said, prospective teachers specializing in mathematics. Concerning the problem quoted above, the authors found that the negative recency effect decreases with age, while the positive recency effect is almost negligible. Normative equiprobability answers to this problem also increase with age. The authors hypothesized that their findings could be linked to the participants' education in probability. Rubel (2007) in turn tested various problems involving coin tosses on secondary students and analysed the justifications the participants gave for their responses (classified according to a belief framework). Overall, she showed that older children are not more subject to errors connected equiprobability and recency than younger children. However, she hypothesized that this could be due to the fact that the students she tested had only limited exposure to instruction in probability.

Chiesi and Primi (2009) tested primary school children and university students with problems involving drawing marbles with replacement from two bags. As to problems involving bags containing an equal number of marbles, they showed that the positive recency effect decreases with age while the negative recency effect increases and is found at remarkable rates in university students. The equiprobability effect linked to such questions also increases with age but is stable at a significant rate after Grade 5. When considering bags with different numbers of marbles, non-normative equiprobability answers were also remarkably chosen by university students. Furthermore, Morsanyi et al. (2009) enacted a cross-educational and cross-national study testing mainly university students in psychology with various problems involving uncertainty. They found that non-normative equiprobability answers are correlated to the participants' formal education in statistics. Overall, researchers have established an interesting link between the equiprobability effect (understood normatively as a bias) and education: the effect increases with age and is correlated to education in probability and statistics (Chiesi & Primi, 2009; Morsanyi et al., 2009; Saenen et al., 2015). In particular, Chiesi and Primi (2009) and Morsanyi et al. (2009) explicitly argued that the equiprobability effect could be a consequence or a "side-effect" of formal education.

These findings seem to echo insights from qualitative socio-cultural research focusing on word problems. These have suggested that many people approach word problems as activities reduced to the execution of some predetermined operations or algorithms without consideration of possible reality-constraints that might be implied by the problem itself and often neglecting their own everyday knowledge (cf. Verschaffel et al., 2020). Verschaffel and colleagues – surveying a large set of empirical studies mostly involving lower grades pupils – conclude that such conducts develop as a consequence of education in schools, which tacitly but systematically determines how students have to behave. Comparable conducts were reported by research on theorem proving testing both compulsory school pupils (e.g., Harel & Sowder, 1998; Paola & Robutti, 2001) as well as undergraduate students in mathematics (Stylianou et al., 2006): students seem to privilege justifications of mathematical theorems via ritualistic and/or authoritarian proof schemes, possibly as an effect of their years-long exposure to traditional teaching practices.

The present study

Participants

The participants are 84 students (34 males and 50 females of median age 24) of the course "Didactics of Mathematics 1" within the master's degree in mathematics at the University of Turin, Italy. This is a program focusing on advanced mathematics. Access to the program is conditional on holding a bachelor's degree in mathematics obtained with good proficiency from a recognized university. All the participants have passed at least one compulsory university course in probability and statistics (typically presented within an axiomatic framework). The following experiment was performed by testing two cohorts of students in two subsequent years with the same procedure involving a short computer-based questionnaire.

Procedure

The questionnaire was divided into two tasks requiring the students to work individually. The questionnaire was part of a larger written assignment which included a variety of mathematical questions and problems proposed to the students during the class as a test aiming to evaluate their general competence in solving mathematical problems. As to the questionnaire relevant for this study, each participant was presented with the following multiple-choice question:

Task 1

Sara tosses for ten times a coin and for ten times she obtains heads. Sara then asks to Piero to bet on the outcome of the next toss. Piero then bets five euros on the next toss resulting in heads again.
Do you agree with Piero's choice?

Yes No In part I am not sure

After the student submitted the answer, the computer immediately presented a related second task:

Task 2

Explain your reasoning.

The student could answer by submitting a text possibly containing mathematical symbols. The answers were automatically recorded by the computer.

Explanation of the choices

The possibilities provided in the multiple-choice question of Task 1 were selected as to allow the participants to nuance broadly their judgement concerning the fictional situation presented. We chose to proceed in this way in order to not necessarily force on the participant a yes/no answer, but rather to stimulate reflection over the problem in view of Task 2, the main focus of this paper. We leave a detailed analysis of the interrelations of the participants' answers to Task 1 and Task 2 to a subsequent paper. Task 2 in turn was formulated as an open question in order to offer to the participants opportunity for ample reflection in consideration of our research question. Overall, the problem was formulated as asking for a judgement towards a fictional decision-problem rather than as a problem of direct estimation of probability, likehood, or chance. This choice was made in order to induce in the participants a detached viewpoint towards the situation described and in order to avoid the difficulties of interpretation connected with most-likely/least-likely questions (cf. Rubel, 2007, pp. 533–534). Nonetheless, we hypothesized that the students would themselves refer to the possibility of estimating the outcome of a hypothetical 11th coin toss. Thus, we anticipated to be able to nuance the participants' reflections on the problem in relation to recency and equiprobability and in connection to acquired probabilistic concepts.

Method of analysis

For the present article we concentrate primarily on the analysis of the written responses to Task 2. Concerning the latter, we classified each answer according to a deductive coding procedure (cf. Braun & Clarke, 2006, pp. 83-84) based on the definitions of equiprobability and recency effects given in the introduction. Having found that the great majority of the participants articulated an equiprobability answer, we decided to further divide this category of answers into two groups (equiprobability with or without reservation). This choice was made in order to nuance whether the participant questioned equiprobability or else simply assumed it without problematizing it. More explicitly, we classified an answer as *Equiprobable tout court (Group A)* if the text argued without reservation that the outcomes of an 11th coin toss are equiprobable. On the other hand, we classified an answer as *Equiprobable with reservation (Group B)* if the text argued that the outcomes are equiprobable but explicitly expressed some reservation about it. We further classified an answer as *Heads is more likely (Group C)* or *Tails is more likely (Group D)* if the text argued that outcome of an 11th coin toss is more likely to be heads or tails respectively. Finally, we classified an answer as *Mixed (Group E)*, if the text did not conclusively favor any of the above options.

Results

Table 1 summarizes the participants' answers.

Table 1: the participants' answers to Task 1 and Task 2

	Group A Equiprobable tout court	Group B Equiprobable with reservation	Group C Heads more likely	Group D Tails more likely	Group E Mixed	Empty	Total
Yes	3	1	6	0	0	1	11
No	14	13	0	3	0	0	30
In part	15	15	5	0	5	0	40
Not sure	2	1	0	0	0	0	3
Total	34	30	11	3	5	1	84

As to Task 2, we present each group of answers individually together with exemplifying excerpts from the submitted texts which were translated from Italian into English as literally as possible. In presenting the results we pay particular attention to the structure of the arguments expressed and to how these connect to probabilistic concepts and ideas.

Group A

The majority of the participants state that heads and tails are equiprobable without explicit reservation (34 participants). Indeed, most students state that it does not matter what Piero chooses, nor what happened in the first 10 tosses.

> The events of heads and tails are equiprobable. The fact that heads was the outcome […] does not determine a greater probability in the following event.

All the texts categorized in this group are structurally very similar. These students typically state the fact that the events are equiprobable as an unquestioned starting point of their reasoning. This is usually formulated as a statement of the logical-geometric properties of sample-spaces linked to idealized fair coins as described in typical textbooks in probability theory. Other information concerning the previous tosses is then simply dismissed in view of the assumed hypothesis of equiprobability or sometimes not even discussed.

Group B

A consistent group of participants similarly argue that the events of heads and tails are equiprobable. However, these students are careful to explicitly indicate that this depends on the assumption that the coin or the toss is not biased (30 participants). The equiprobability assumption is then given as an explicit (but questionable) hypothesis from which their argument develops. The option of the coin or the game being biased is usually briefly considered as a possibility at the start of the text but dismissed as a result of a deliberate argumentative choice in line with the usual assumptions underlying the practice of problem-solving in probability courses.

I started from the assumption that the coin was not rigged. The probability […] is the same […] independently from the previous results.

Interestingly, 5 participants explicitly characterize assuming the equiprobability hypothesis as the more "mathematical" way of reasoning.

Mathematically, the next toss is independent from the previous tosses.

Group C

A smaller group of students argues that an 11th toss is more likely to result in heads (11 participants). This is argued by deeming implausible that a fair toss involving a fair coin could land ten times in a row on heads. Remarkably, 4 participants express this as a contradiction between mathematical thinking and everyday thinking.

Mathematically the probability for each side is one half. However, since the outcome was 10 times heads, I think that the coin is loaded.

Thus, according to these students, "mathematically" the probability of obtaining head or tails is the same. Nevertheless, if we disregard this, then we can conclude that the coin is loaded.

Group D

Very few students state that an 11th coin toss is more likely to result in tails (3 participants). Interestingly, all these students justify this by giving a mathematical (unsound) argument. For instance, one student argues that if he bets on heads then

[…] Piero has only $1/2^{11}$ probabilities to win.

Group E

Some texts contain considerations on different conflicting aspects of the fictional situation, which they leave unresolved (5 participants). Among these, 3 texts contain reference to unsound mathematical arguments similar to those given by students in Group D.

Discussion

In summary, only a small number of participants shows the positive recency effect (Group C), while a negligible amount manifests negative recency (Group D). A large majority instead inclines towards the equiprobability solution (Group A and B). These results contrast with the findings on university students of Chiesi and Primi (2009) and align better with the results of Fischbein and Schnarch (1997). This happens possibly because the curriculum of studies of our participants is more similar to the curriculum of the students involved in Fischbein and Schnarch's experiment.

As to the participants who manifest the equiprobability effect, their answers show to be connected either directly or indirectly to their education in mathematical probability. The answers of those who do not problematize the equiprobability hypothesis (Group A) may be seen as resulting from applying it as an unquestioned assumption associated with the usual presentation of fictional situations involving idealized games of chance within mathematical textbooks or courses in probability and statistics. As to the students who are keen to problematize the equiprobability hypothesis (Group B and C), they nonetheless state equiprobability as (even explicitly) the more mathematically-proper assumption, i.e., as the assumption which is more appropriate to adopt in a mathematical context. Some of the students even describe an openly-perceived conflict between mathematical and everyday reasoning (a phenomenon also discussed by Rubel, 2007). In particular, positive recency answers and mathematical-probabilistic assumptions were rather presented by the students as conflicting (Group C). On the contrary, the participants who manifested negative recency (Group D) did so when trying to frame the problem in terms of mathematical probability. Given that these were a negligible amount against the total of the participants, we do not attempt to infer more general implications from this particular finding. However, it could be the case that a similar phenomenon is also at work in groups of advanced students in other disciplines who manifest negative recency more consistently (e.g., Chiesi & Primi, 2009).

Thus, we observed an overwhelming tendency by these university students in mathematics to deem the problem's coin to be fair despite indication that this should be the case and contrary to explicit evidence presented in the problem. The equiprobability effect (understood non-normatively) appears to be related to the participants' education in probability, in a way which resembles analogous phenomena reported by socio-cultural qualitative research on word-problems (Verschaffel et al., 2020) and on mathematical proving (Stylianou et al., 2006). Further research testing participants with different mathematical backgrounds using the same procedure would be needed to substantiate this conclusion. Additional research would also be needed to understand if the same students would give substantially different responses when asked the same or an analogous question in a different setting (e.g., in a non-educational setting) or presented by means of a more realistic procedure (e.g., by observing an experiment featuring actual coin tosses). More theoretical elaboration as well as further empirical data would serve in turn to illuminate the relationship between the equiprobability effect and the equiprobability bias.

In conclusion, we have shown how university students in mathematics reason about a problem involving uncertainty. The discussed connection between the equiprobability effect and instruction in probability may be in itself not surprising. Indeed, the equiprobability effect is in general the result of a sound mathematical axiomatization of uncertainty (cf. Gauvrit & Morsanyi, 2014). Some could even argue that the outcome of a formal education in probability should prompt equiprobability answers in all relevant cases. However, the way in which our participants answered show that many of them applied equiprobability as an unquestioned assumption, possibly as a result to concepts and definitions narrowly presented by textbooks and courses in probability and statistics (cf. Batanero, 2020, p. 685). This fact in turn may be problematized as contrary to a full critical understanding of situations involving uncertainty. What if the coin is not fair?

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Communicating data exploration insights through posters – A preliminary analysis of primary students´ learning outcomes

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Since data and its representations have gained a significant role in our society, mathematics education has to adapt by finding ways of teaching students statistical literacy from primary school onwards. In a teaching unit that was designed as part of a bachelor´s thesis, students of a fourth grade (10 to 11 years old) were introduced to group comparisons with the help of TinkerPlots and asked to create posters on statistical mini-projects. This paper aims at providing first ideas for a framework for the analysis of such means of communication to understand what results the young students communicate and how they do so. The posters show that the primary school pupils already employ not only local perspectives but utilize more global views to interpret the data.

Keywords: Data exploration, group comparisons, poster presentations.

Introduction: The relevance of teaching statistics

Understanding statistics has become a highly relevant skill set in our times as data and its representations confront us daily. It is for that reason that we need to become literate with regards to what the data tells us and how to interpret it (English, 2014). This has led to the acceptance of statistics into school curricula, even as early as the primary level. In Germany, for instance, creating graphical representations of data and their interpretation are part of the curriculum (Hasemann & Mirwald, 2012). The United States, as well as Australia and New Zealand, have also introduced standards at the beginning of the century to ensure that these topics are covered obligatory at school (Ben-Zvi & Garfield, 2004). Hence, the need for teaching concepts and new insights into the understanding of statistics as part of a successful didactic approach to the field has become apparent. There is more and more research on statistical inquiries in the fields of mathematics education. Some fundamental ideas of statistics have been identified for instance by Burrill & Biehler (2011). Concerning reading and interpreting statistical displays, Friel et al. (2001) have developed a framework consisting of three different levels of understanding statistical representations of data and there is a variety of sets of recommendations to follow when designing a statistical teaching-learning environment (Garfield & Ben-Zvi, 2008). These conceptions are further supplemented with subject-unrelated approaches, for instance using design-based research as a method to develop theory and simultaneously successful teaching-learning sequences (Bakker & van Eerde, 2015). Another relevant aspect is the increasing interest in using digital tools when exploring data (Biehler et al., 2013) - they facilitate the creation of graphical representations in large and multivariate data, leaving more cognitive capacities to work on higher-level tasks such as interpreting the presented data (Garfield & Ben-Zvi, 2008). TinkerPlots is an adequate tool that also finds application in primary contexts and enables learners to realize also sophisticated statistical activities like group comparisons on a basic level (Konold, 2007). Therefore, a teaching unit has been developed that is grounded on the specific characteristics and requirements mentioned above (Schäfers, 2017). The goal of the teaching unit was that the young learners realize and conduct their own statistical (mini-)projects answering a statistical question that is derived from the students´ interests. This means that the young students experience all phases of the PPDAC cycle (Wild & Pfannkuch, 1999). Posters were created as a means of communicating results of the statistical project. In this paper, we take a closer look at the learning outcomes of primary school students and how they communicate their findings of data exploration in form of posters. The aim of this paper is twofold: it will (1) suggest first ideas for a framework to evaluate the learning outcomes which are communicated through the posters and (2) it will present first empirical insights on how primary school students´ communicate findings of their data explorations through posters. By working on these issues, we wish to contribute to the advancement of data exploration at a primary level and the communication of results and lay a possible groundwork for a deeper understanding of students´ learning processes in this particular field.

A teaching unit to develop reasoning about data in a primary school setting

The designed teaching unit aimed at promoting reasoning about data (in particular group comparisons) among fourth graders (10 to 11 years old) in a public German primary school and concluded in individually conducted statistical projects towards the end of the unit.

Methodology of the development of the teaching unit & design principles

The teaching unit was part of a design-based research project (Bakker & van Eerde, 2015) which creates theory on primary school students´ abilities to compare groups using TinkerPlots as a digital data analysis tool. In multiple cycles, the (re-)design of teaching elements, their theoretical foundation as well as their evaluation based on classroom experiments are combined to foster the interrelations between teaching practice and theory. The FUNKEN-model (Prediger & Zwetzschler, 2013) poses a similar approach focusing on the specifics of particular teaching contents. This teaching unit was conducted as the second cycle within a design experiment to develop local learning theories on group comparisons in primary schools and on supporting learning arrangements for this content. Building on the first cycle and its results, adaptations had been made on teaching materials concerning the group comparisons.

Research suggests certain design elements for developing statistical reasoning. Garfield and Ben-Zvi (2008) refer to them as *Statistical Reasoning Learning Environments* and argue in favor of, for instance, the use of real data, a focus on core competencies of statistics, utilizing digital tools, and an adapted form of assessment to develop and deepen students´ statistical reasoning. The teaching unit implemented these aspects - having students collecting their data on statistically relevant questions of interest to them, using TinkerPlots to explore real and multivariate data, and realizing statistical (mini-)projects with posters as learning outcomes that were used for assessment.

Data explorations and group comparisons as central topics in the teaching unit

Developing statistical reasoning is a central concern of teaching statistics. It is considered to be “the study of arguments that use statistics as evidence” (Burrill & Biehler, 2011, pp. 59f). This comprises understanding data and its possible representations, amongst other competencies (Ben-Zvi & Garfield, 2004). Graphs are graphical representations of data and therefore central to statistics. Friel et al. (2001) have differentiated between three levels of graph comprehension: *reading the data* as the ability to read the individual data (representations), *reading between the data* as the ability to understand and interpret relationships within or between different data sets and *reading beyond the data* as the ability to infer consequences and possible predictions from the represented data. It implies a shift from a rather local point of view towards a more global perspective which is based on interrelations and possible dependencies (Bakker & Gravemeijer, 2004). Group comparisons focus on more than one variable and investigate differences between groups defined by one of the variables (Konold, 2007). They are considered engaging, yet complex from an educational point of view. Group comparisons are closely connected to the idea of variation (Garfield & Ben-Zvi, 2008), which Burrill and Biehler (2011) defined as one fundamental statistical concept among others like representation. TinkerPlots can be used to support young learners to create graphical representations of data to facilitate the group comparison process. Even though it facilitates the drawing of the graph, it also stimulates the fundamental skills needed to create and understand the visual representations (Konold, 2007). Stacked dot plots and hat plots form the graph types that are employed by the tool and modal clumps are used as a possible means of comparison (Konold et al., 2002). Hat plots can be considered a more flexible form of a box plot as the crown can be regulated as to what percentage of the dataset it should enclose (Garfield & Ben-Zvi, 2008; Watson et al., 2008).

Poster as a means of communication in the teaching unit

Communicating (results) plays a central role in the mathematics classroom and is therefore regarded as a procedural competence within the German educational standards, also applying to the teaching of statistics (Sill & Kurtzmann, 2019). According to Wild and Pfannkuch (1999), communication is central to statistical tasks, and therefore included in their adapted PPDAC cycle within the phase of conclusions. For statistical classrooms, the aim is to foster communicational skills so that results can be summarized, more global coherences can be harnessed to interpret the data, and the context of data is considered (Peck, 2005). Posters are a possible way to communicate mathematical results and were used within the teaching unit as an assessment and a means to document the students´ learning outcomes. As posters are frequently used as means of communication, there are criteria to evaluate them by. For a contest, the ISLP has proposed a set of possible standards that focuses on the statistical contents of the posters (International Statistical Literacy Project, 2020). Selected important criteria include the interpretation of the collected data with regards to the statistical question, the integration of adequate graphical representations, as well as the portrayal of the collection of data. These standards will be included when evaluating the posters and proposing a possible framework for the analysis. Statistical representations such as graphs bear the opportunity to function as a means of communication themselves when being properly interpreted and understood (Friel et al., 2001). One challenge for young students is to verbalize their findings from data exploration, for instance in the context of group comparisons.

The contents of the teaching unit

The teaching unit on which this paper is based comprised ten lessons, each lasting 45 minutes. Students were introduced to the basics of statistics, the sub-processes involved in a statistical project, the use of the Software TinkerPlots as well as group comparisons (Schäfers, 2017). In the first part of the teaching unit (Lessons 1-3) the students are introduced to the first ideas of empirical and statistical work (getting to know about variables, learning about the PPDAC cycle as a means for structuring own statistical projects, collecting data, etc.). In the second part (Lessons 4-5) the students analyzed data about their class and learned, for instance, how to draw and read bar graphs. In Lessons 6-8, the students were introduced to the software TinkerPlots. Finally, in Lessons 9 and 10, the students worked on a statistical (mini-)project on their own, applying and testing said acquired competencies when working on individually developed statistical questions regarding group comparisons and presented their findings to their classmates via posters. More details on the teaching unit can be found in Frischemeier (2020). The posters were used as an opportunity to document the results of the data exploration and were to include a title, their statistical question as well as the results of their project answering their question.

An accompanying empirical study: Communicating results of a data exploration through posters

Research questions

Since the original research has suggested the validity of teaching group comparisons to primary school children (Frischemeier, 2020; Schäfers, 2017), the focus of this sub-project is to concentrate on how the learning outcomes of a data exploration are communicated through posters. It has led to the following research questions: What results of their own statistical (mini-)projects do fourth graders of a German primary school communicate through posters after 10 lessons on the foundations of group comparisons? In which way do they communicate their results of a data exploration through posters?

Data collection & participants

As the purpose of this article is to analyze the students´ learning outcomes and to present first ideas on a framework for further use within teaching arrangements like the one presented, we concentrate on the stage of conclusions within the PPDAC cycle (Wild & Pfannkuch, 1999). The students were asked to create posters in small groups to present their outcomes so that these posters were collected and will be used as the data to be analyzed. The posters consisted of a headline, a representation, and a box in which the representation was described and the statistical question answered. Twelve fourth graders of a German primary school (Ages 10 to 11) attended the lessons of the teaching unit. All of them had been introduced to data collection and presentation in grade three but there was no standardized test to pinpoint their prior knowledge on the acquired competencies or other skills that might help the statistical group comparison (Frischemeier, 2020).

Data analysis

To fulfill the purpose of this paper, a qualitative research approach was chosen for the sub-project. With the help of qualitative content analysis (Mayring, 2014), three dimensions were formed that serve as the basis for the proposed framework. They were derived from the current state of research on characteristics of group comparisons, graphical representations of data as well as pre-existing criteria on posters as a means of communication and the contents of the teaching unit. Therefore, the dimensions of special interest are (i) the statistical features included in the group comparisons (Frischemeier, 2019) and their level of abstraction, (ii) the levels of graph comprehension (Friel et al., 2001), as well as (iii) the components of a statistical project as recommended in the ISLP poster requirements (International Statistical Literacy Project, 2020). They function as the main points of interest to which segments of the texts and of the data material are sorted and then evaluated (Mayring, 2014). The coding manual as well as specific categories and sub-categories need to be further developed in a next step. Following the aim of this paper and the chosen qualitative approach, one poster will be analyzed exemplarily along the above-mentioned dimensions to give first insights into how results of a group comparison are communicated by fourth graders. The poster was chosen as a representative example of young students´ data exploration processes.

Results

A group of four students worked on the yellow poster which presents data on the shoe sizes of all fourth graders of their school. The goal was to find out whether the boys on average have larger feet than the girls. 77 pupils were questioned, 37 of them female and 40 males.

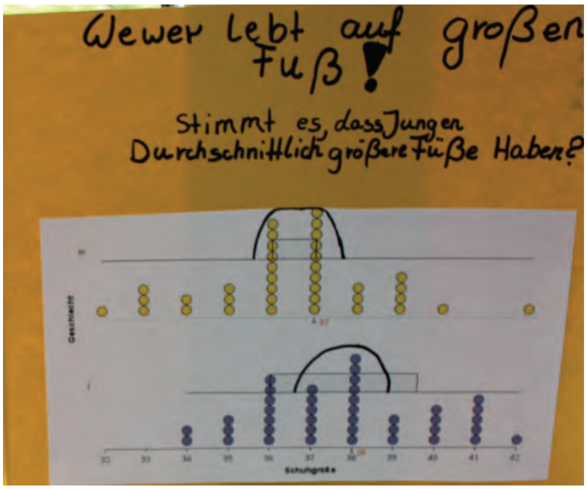

Figure 1: Headline and statistical display of the Data poster "Wewer lives on a large foot" [in German] from the group *Yellow*

Group Yellow: Wewer lives on a large foot! Is it true that boys on average have larger feet than girls? The fourth graders of the Almeschule in Wewer (Germany) were asked which shoe size they had. Among the girls, the smallest shoe size was 32 and the largest 42. Among the boys, the shoe sizes are closer together. Here, the smallest shoe size was 34 and not 32 like among the girls. The largest shoe size is 42, just like among the girls. Many shoe sizes of the boys accumulate at shoe sizes 37, 38. Among the girls, there is an accumulation at shoe sizes 36,37. Among the boys, the brim is slightly shorter than among the girls. But the crown is longer. Among the girls, it is the other way around. Here the brim is long and the crown is short. The median provides a center value of the distribution. The median of the girls is 37 and the median of the boys is 38. The center value of the girls is, therefore, smaller than the one of the boys. This means that the boys on average have larger feet. These results of the survey show that boys on average have larger feet.

To answer their statistical question, the students compared the maximum and the minimum of the distributions, argued with the help of the median as well as the components and positions of the hat plots. Furthermore, they considered modal clumps as accumulations of data with a certain value. Even though local characteristics such as the maximum as well as the minimum are considered, the most reliable source of information to the group is the median. It is misinterpreted as the arithmetic means of the distribution and articulated with the word "average" ("The median provides a centre value of the distribution […]. The centre value of the girls is, therefore, smaller than the one of the boys. This means that the boys on average have larger feet."). As crown and brim of hat plots are based on ideas of the spread of the middle half of the distribution, they are understood to be rather global. Since they are taken into account, it can be deduced that the group of students has succeeded in working towards a more global view on data even though some characteristics of the groups such as differing group sizes were ignored. Regarding the levels of graph comprehension, the students prove that their understanding exceeds *reading the data* as more global features such as the median are employed to answer the statistical question ("The median provides a centre value of the distribution. The median of the girls is 37 and the median of the boys is 38. The centre value of the girls is, therefore, smaller than the one of the boys. This means that the boys on average have larger feet."). In addition, as two datasets are compared based on relationships within each distribution, we can assume that our young students show the ability to *read between the data* ("Among the girls, there is an accumulation at shoe sizes 36, 37. Among the boys, the brim is slightly shorter than among the girls. But the crown is longer. Among the girls it is the other way around. Here the brim is long and the crown is short."). As no consequences or further deductions are proposed, the third level of *reading beyond the data* may not be attested at this time. The poster as a means of communication presents a title, the statistical question in form of a hypothesis, a graph that visualizes the distributions, the corresponding hat plots, and manually added modal clumps with labels for the axes (Figure 1). The text box gives information about the participants, which insights the group gained, and how they concluded that it is true that the boys have larger feet than the participating girls on average. Regarding the components of the poster (as suggested by the ISLP poster competition criteria), it only lacks in the sense that the method of data collection is not explicitly mentioned as well as a clear connection between the chosen title of the poster and the statistical question fails to appear. Although some minor mistakes were made mathematically, the group used appropriate means to compare the groups and to answer their question. In conclusion coherence between the interpretation of the data and the research interest can be attested.

Conclusion

Teaching statistical group comparisons in primary school settings with the help of digital tools such as TinkerPlots can be successful. Of special interest for this paper are the communicative means that the students employed. This paper has therefore shown first empirical insights into how primary students´ learning outcomes in form of posters can be analyzed based on fundamental concepts of statistics education as well as standards on statistical posters that might function as the grounds for a framework. One main concern is the shift from a rather local view to a more global perspective on the represented data (Bakker & Gravemeijer, 2004) which can already be seen on the posters. To communicate their findings, students use statistical properties of the data by expressing them verbally and also visually within the graph. Further research must be conducted on how students come to said learning outcomes, complementing the analysis of products with interviews to create an understanding of learning processes as well. In the next steps, the implied framework will be developed further and is supposed to be supported by a coding manual including exact definitions of categories with their sub-categories and examples to strengthen the comparability of the results. It should then also be applied to the other posters derived from the teaching unit as well as a generally bigger sample size to be able to advance the framework. Furthermore, the use of gestures and facial expressions as means of communication should be taken into consideration for a more in-depth analysis of the students´ presentations. As the teaching unit was part of the second cycle of a design experiment, further adaptations have to be made based on classroom experiences and student interviews to promote the development of local learning theories on group comparisons and reasoning about data as well as the advancement of supporting teaching-learning arrangements.

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Learning opportunities for pre-service teachers to develop pedagogical content knowledge for statistical inference

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Recently, researchers have encouraged the teaching of statistical inference to students at all levels. However, what constitutes pre-service teachers' pedagogical content knowledge for statistical inference has not yet been given specific attention in research. This paper presents a qualitative study of pre-service teachers participating in a collaborative learning setup in a mathematics course to be prepared for teaching statistics in primary school aged 6–10 years. The study reported here is the first cycle of a design research project. The findings show that pre-service teachers' learning opportunities regarding pedagogical content knowledge for statistical inference are insufficient. Based on the initial results, an initial conjecture map was constructed that guides the forthcoming design cycle.

Keywords: Content representation, design research, pedagogical content knowledge, pre-service teachers, statistical inference.

Introduction

Informal statistical inference (ISI) (Makar & Rubin, 2009) is often considered an essential ability of statistically literate citizens and the root of understanding formal inference (Biehler & Pratt, 2012). Much research focusing on ISI has been carried out with recommendations for powerful statistical ideas to introduce in primary school (Makar & Rubin, 2018). Despite prevalent international research findings, ISI has only made breakthroughs in the primary school curricula of a few countries. Therefore, researchers have called for ISI to be explicitly included in school curricula and teacher education (e.g., Langrall et al., 2017). However, in teacher education, limited time is spent on statistics. Thus, we need to learn to endorse ISI pedagogical knowledge for pre-service teachers through well-designed and effective arrangements (de Vetten et al., 2018). In line with Makar and Rubin (2018), we shift the focus from questions about what constitutes ISI to more knowledge of teaching statistical inference (SI) and design educational settings that provide prerequisites for pedagogical content knowledge (PCK) regarding SI.

Lehrer and English (2018) emphasised that children's informal reasoning about insecurity is formed both in and outside formal schooling. The same can be expected to apply to pre-service teachers, which makes it interesting to investigate the extent to which they pay attention to inference as a learning object in an existing learning environment. This paper presents the first cycle of a design research project (cf. Bakker, 2018), conducted in a mathematics course for pre-service teachers in Sweden. The design aims to provide knowledge to improve teacher education, from an educational setting focusing on statistical literacy to a setting supporting pre-service teachers' development of PCK for SI. We ask: How can a collaborative teaching strategy that mainly promotes statistical literacy skills support pre-service teachers' PCK for SI? Based on the findings, we propose an improved teaching strategy that is beneficial for developing pre-service teachers' PCK for SI.

Theoretical background

Statistical inference in statistics education

SI is usually associated with methods and concepts at the tertiary level, such as point interval estimates and hypothesis testing. By proposing a broader use of the term inference and making inferential reasoning available to all ages, Makar and Rubin (2009, p. 85) identified three key principles of ISI: "(1) *generalization*, including predictions, parameter estimates, and conclusions, that extend *beyond describing the given data*; (2) the use of data *as evidence* for those generalizations; and (3) employment of probabilistic language in describing the generalization, including informal reference to levels of certainty about the conclusions drawn". Acting on these principles can direct teachers' attention to children's abilities to communicate probabilistic predictions and generalisations from data. For example, Shaughnessy (2019) proposes a learning trajectory for the early years where children are initially encouraged to explore the use of statements such as 'likely' or 'unlikely', and thereafter propose and justify conclusions and predictions based on data, followed by designing studies for further investigation. Furthermore, Makar and Rubin (2018) acknowledge the *integration of contextual knowledge* and *consideration of the aggregate* as having key roles in making inferences. The proposed broad approach aligns with frameworks such as *data modelling* and the *statistical investigation cycle*, all of which incorporate the importance of teaching the overall picture of a statistical investigation rather than divided parts (Lehrer & English, 2018).

Several frameworks have been developed to describe a statistical investigative process. Findings from prior research (e.g., Blomberg, 2015) suggest that *data*, *distribution*, and *inference*, each associated with the earlier phases, constitute the core of statistics in education at all stages, including younger students. At the school level, the GAISE Pre-K–12 Report (Bargagliotti, 2020) proposes a model of four components: 1) Formulate Statistical Investigative Questions; 2. Collect/Consider the Data; 3) Analyse the Data; and 4) Interpret the Results. In line with Shaughnessy (2019), we consider distribution and inference to be the heart and soul of statistics. When composed into a somewhat simplified format that we believe can serve pre-service teachers who are to teach younger children, a statistical process follows three phases: data generation (e.g., problem context, statistical question, collect and organise data), analysis (e.g., use appropriate displays to represent data, distribution, variability, expected value), and communication (e.g., inference, generalisation, based on data, with uncertainty). To summarise, for students of all ages to have access to the aforementioned desirable inferential ideas, pre-service teachers not only need content knowledge of statistics, but they also need PCK for SI. What constitutes PCK and how to measure PCK for SI in teacher education are presented in the following section.

The quality of pre-service teachers' pedagogical content knowledge

Since the introduction of PCK (Shulman, 1986), the concept has been interpreted and operated in various ways in different research domains. The resulting theoretical diversity of PCK models motivated Carlson and Daehler (2019) to develop a refined consensus model (RCM). In addition to *content knowledge, pedagogical knowledge, knowledge of students, curricular knowledge,* and *assessment knowledge*, this model also considers *collective* (*cPCK*), *personal* (*pPCK*) and *enacted* (*ePCK*) as aspects of teachers' professional knowledge. To shed light on how to support pre-service teachers' development, the RCM of PCK is expected to contribute to situating the research and identifying elements to study. Based on the RCM of PCK, Chan et al. (2019) suggested a framework for measuring the quality of teachers' PCK. The framework consists of the following five key components: 1) selection and connection of big ideas, 2) selection of instructional strategies and representations, 3) recognition of variations in student understanding, 4) integration between PCK components and 5) pedagogical reasoning. These PCK components are not intended to constitute a fixed framework but are rather open to adaptation within different contexts. Thus, they are flexible enough to measure the quality of different variants of pre-service teachers' PCK, such as what a group of pre-service teachers knows, what a single pre-service teacher knows and does, and expressed pedagogical reasons for judgement and action.

Methods

Context and participants

The RCM model of PCK centres around the teacher's ePCK, which can be characterised by planning, carrying out that plan and reflecting on instructions and student outcomes. However, in this study, the pre-service teachers' reports did not include an enacted teaching context with students in schools. The empirical data occurred when groups of pre-service teachers planned and reflected upon a hypothetical teaching situation. Thus, this study falls within the realm of cPCK, which is considered a result of transformations of pre-service teachers' pPCK in a learning context in which pre-service teachers initially individually reflect and thereafter collectively reflect and plan. As the study was conducted during the COVID-19 pandemic, a remote teaching setup was used. Lectures and group work were conducted online in digital meetings.

The participants in the study came from one class in teacher education that focused on becoming a teacher for students aged 6–10 years. The participants studied a course in mathematics, including statistics, during the second of four years of study. The class consisted of 62 pre-service teachers, 33 of whom signed an informed consent form to participate in the study. The mean age of the participants was 29 years, with a standard deviation of 7.7 years. The research project has been ethically reviewed and approved by the Faculty of Health, Science and Technology at Karlstad University.

Design, data, and analysis

The teacher educators have been involved in planning and evaluating the study's design and have provided significant knowledge of the inherent possibilities in hypothetical learning trajectories and design conjectures (Bakker, 2018). Hence, a research model was developed that matched the purpose of the study and the teacher educators' planning. The teaching of statistics was carried out in six steps over two weeks. The learning objectives with these six occasions can be summarised as emphasising statistical literacy, developing statistical thinking and conceptual understanding rather than knowledge of the modelling process and procedural skills. Teaching statistical literacy emphasises statistical knowledge and skills in contextual settings and the need to critically evaluate studies and reports based on statistics and to communicate critical ideas (Sharma, 2017). Such an approach calls for developing pre-service teachers' ability to plan, enact and evaluate teaching-learning settings that focus on developing skills to evaluate statistical studies' quality and findings rather than practical skills that come with performing studies.

The six occasions are briefly depicted next: 1) A homework assignment that consists of reading and performing tasks in teaching material on probability and statistics for teachers. The learning content consists of basic concepts from the descriptive statistics such as mean, median, mode, range, boxplot, and how to compile and represent data with different types of diagrams. In addition to a short section that addresses the challenge teachers face in choosing an appropriate activity for teaching statistics, the teaching material includes no more than a short repetition of what the pre-service teachers have encountered in primary and secondary school. 2) A lecture, which highlights the purpose of statistics in teaching and in everyday life; descriptions of the subject and central concepts (as mentioned above, reliability and validity), drawing on practical examples. 3) A homework assignment involving individual work of self-selected statistical material and reflections based on support questions from the teacher educator. One question concerns inference: "In what way can conclusions be drawn (generalise, predict, etc.) and with what certainty do you think it can be done?" 4) Group work: The participants in the study were randomly divided into 11 groups, with 3–4 members in each group. Each group chooses one of the group members' self-selected statistical materials from the third occasion. Each group completed a description of the content in a hypothetical teaching situation covering three big ideas for teaching statistics for students aged 6–10 years, supported by a so-called *Content Representation* (CoRe) (e.g., Hume & Berry, 2011). A CoRe is a template for teachers to portray big ideas of concepts and skills related to a particular topic with answers to critical pedagogical challenges. 5) Each group presents their planned teaching situation by recording a video. Finally, 6) Group work: Each group compiles written feedback for three films.

This study focuses on data materials from occasion four, the pre-service teachers' CoRes, compiled in groups. Using CoRe as a reflective tool can transform teachers' tacit pPCK into explicit cPCK (Alonzo et al., 2019). Therefore, the knowledge expressed in a CoRe that was put together by a group of pre-service teachers represents their cPCK (Carlson & Daehler, 2019). A *content analysis* approach was used in this study (Robson & McCartan, 2016). The analysis used the five key PCK components (Chan et al., 2019) as operationalised categories. Within each of these components, the pre-service teachers' written outcomes related to statistical conceptual ideas were identified. Our focus was on identifying content related to inference. The first author of this paper is responsible for the analysis. To ensure validity, the findings have been discussed with both teacher educators and co-authors.

Results

The first key PCK component concerns the selection, connection and coherence of big ideas. In total, the 11 groups proposed 29 big ideas. To get an idea of the pre-service teachers' knowledge of big ideas in statistics, their stated outcomes of big ideas were interpreted and placed in one of the following five categories and their accompanying distribution and examples of the stated big idea followed by the expected learning outcome: 1. Data (10): "How to measure and compare data – To be able to formulate a question to be able to find out a specific matter", "With the help of statistics, you can find out various questions", "Statistics is a collection ... – Collection is an important process; it is important for students to get an understanding of who to ask or what to look for", 2. Analysis (9):

“Interpret – reading the diagram”, “The bars represent data – To be able to create bar charts after collected data”, “Analyse – comparison of the data presented in the diagrams”, 3. Communication (0), 4. Critical (1): “Statistics can be misused and misunderstood - All available statistics should not be trusted. It is important to be critical of sources”, and 5. Other topic (9): “Stress – We think that students should learn that stress can be both positive and negative depending on how you deal with it”, “Risks with an increased use of the Internet”, “Values – We want our students to gain knowledge about the Convention on the Rights of the Child”. Summing up, the analysis of the pre-service teachers’ CoRes showed that most of their stated big ideas connected to data generation, followed by analytical skills and other topic. One group stated a critical approach to statistics and no big idea was qualified in the category of communication.

The second key PCK component, teaching strategies, highlights the selection of instructional strategies and representations regarding the stated big ideas. Several groups suggest a modelling approach, meaning that students conduct their surveys and create diagrams. Furthermore, we can see traces of outcomes of PCK for teaching statistics, such as starting from everyday questions and showing examples of statistics that can be interesting, showing films about statistical investigation and practicing interpretations and comparisons of existing diagrams. We also note some outcomes of general pedagogical ideas, such as individual work, classroom discussions and a combination of these (individual-pairs-all), without relating to statistics.

The third key PCK component, students’ understanding, draws attention to a student-focused classroom climate based on the individuals’ knowledge and recognises the variation in students’ learning. In addition to activities that engage students’ interests, it is also desirable to uncover their thinking. Overall, pre-service teachers identify a wide range of student-focused factors that affect their teaching ideas at a general level. Raised challenges are, for example, different levels of students’ preconceptions, language difficulties, lack of interest and unproductive attitude. Pedagogical ideas to address these issues were, for example, conducting investigation, diagram creation, presenting these to each other, making comparisons and discussing each other’s presentations. When it comes to outcomes regarding inference, there are no traces of possible difficulties that students may have.

In the fourth key PCK component, integration between PCK components, attention is drawn to the teachers’ ability to plan the next teaching step and adapt this to how the students have received teaching. Since the study does not include a context in which pre-service teachers practice teaching, it is impossible to infer anything about their adjustment skills within this component. However, based on existing data, it is possible to identify pre-service teachers’ presumptive ideas about obtaining students’ learning big ideas and adjusting teaching practices. For example, some groups expressed ideas about controlling how students design, interpret, and analyse diagrams.

The fifth key PCK component, pedagogical reasoning, focuses on pedagogical issues as a teacher’s ability to justify decisions and actions within a teaching situation. Supporting questions in the CoRe evoked pedagogical reasoning and thus became applicable as an empirical basis for this fifth component. The arguments for the selected ideas are anchored in the importance of being a competent consumer of statistics. It was seen as important that the students understand the purpose of using diagrams and being critical of the statistics presented to them. Some stated the importance of students engaging with an investigation, compiling diagrams and interpreting and drawing conclusions based on the results. Both teaching methods and the arguments behind these choices vary. Some groups highlighted the importance of motivating students to learn and suggested using up-to-date social issues and topics that were close to the students' interests. Others mentioned teaching methods such as classroom discussion to introduce students to making interpretations and conclusions and equal collective understanding to reduce the variation in students' knowledge in the classroom. In addition to these outcomes of reasoning, most of the pedagogical reasonings was inadequate or missing.

Conclusion and discussion

Regarding the implications and limitations of the study, we would like to draw the reader's attention to the fact that the study should not be regarded as an isolated case study but as a first sub-study of design research consisting of several iterative studies (Bakker, 2018). The study was conducted in a specific context that limits the generalizability of the results to similar conditions. The study was also limited to one group of students who carried out all work during remote teaching due to the pandemic. The dropout rate was 47 %, most likely due to the pandemic.

Regarding the question of how the collaborative teaching strategy promotes pre-service teachers' PCK, we see, with the support of Carlson and Daehler (2019), that when students in groups plan to teach and then assess each other's work, they are offered the opportunity to exchange knowledge from pPCK to cPCK and back to pPCK on several occasions. In addition, students acquired professional knowledge bases such as content knowledge about statistics and curriculum knowledge via the internet. The results from the study depict a somewhat fragmented picture of the pre-service teachers' knowledge about big ideas in statistics education as emphasised by Shaughnessy (2019). The pre-service teachers' outcomes regarding inference in accordance with Makar and Rubin's (2018) ideas are almost non-existent. However, there is a clear emphasis on compiling and organising data, interpreting data, and being statistically literate. It is noteworthy that the setting triggered nearly half of the participating groups to choose topics from statistical contexts, separate from statistics, as big ideas. Indeed, *knowledge of the problem context* is a central conception in statistics education, and the stated big ideas, which were identified as statements about societal issues, are relevant to students. However, in this learning context, statistics was the subject of focus. Therefore, the results indicate that learning objectives for statistics are partly outperformed by learning objectives for other topics. Based on these results, we infer that a statistical literacy approach to teaching does not provide satisfactory outcomes in pre-service teachers' PCK for SI. Therefore, we suggest a draft of a conjecture map (Table 1) to guide the next cycle.

In accordance with Lehrer and English (2018), our suggestion for an improved strategy for high-level teacher educators with high-level conjecture is to emphasise a more explicit focus on practicing data modelling and highlight what characterises SI in education. We choose to complement the design conjectures of existing tools (e.g., CoRe) and participant structures (collaborative learning) within the embodiment of putting the conceptual framework of ISI modelling into practice. The design includes a lesson plan in which the development of content knowledge encompasses PCK. Our ideas of theoretical conjectures, meaning the produced desired outcomes from the mediating processes, include appropriately emphasising inference as a big idea, describing its key principles in accordance with Makar and Rubin (2009), and connecting inference to other core statistical conceptions such as data generating and distribution. Furthermore, according to Table 1, we suggest a designed mediating process that produces outcomes for how SI can be made teachable for the age group in focus, as well as pedagogical reasoning for selected teaching methods at both class and individual levels. The initial conjecture map presented in Table 1 will guide the forthcoming design cycle of our design research project. The plan is to carry out a second-phase study at the end of 2021. We intend to present preliminary results from the forthcoming study at CERME 12.

Table 1: Conjecture Map - Statistical Inference Modelling Teaching

High-level conjecture	Embodiment/design	Mediating processes	Pre-service teachers' outcomes
Conceptual framework of ISI modelling in education, and cPCK can contribute to pre-service teachers' ability to plan and reflect on teaching SI appropriate for a particular age.	• Putting ISI modelling into practice. • Completing a CoRe. • Planning for a hypothetical lesson or lesson sequence in statistics. • Using peer review. • The development of content knowledge encompasses PCK. • Collaborative learning as a participant structure.	• Experiencing ISI modelling cycle. • Identifying inference as a core concept and its connections to other big ideas in statistics education. • Expressing in speech and writing about planning for SI modelling teaching. • Reflecting in writing on other students' video presentations.	• Describing inference as a big idea together with other big ideas and concepts in statistics education. • Specifying how SI can be made teachable for the age group in focus. • Arguing for selected teaching methods at class and individual levels.

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(Co-)Variation of parameters in a Bayesian Situation – An analysis of students' reasoning about the effect of base rate changes

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Evaluating the risk of a hypothesis given some indicators for the hypothesis is a crucial example for conditional probability reasoning. Calculating the probability of a risk when a set of parameters (e. g. the so-called base rate, true- and false-positive rate) is given, is a task which is referred to as a Bayesian task, as it can be solved with the Bayes' formula. The conceptual understanding of a mathematical formula (and of Bayesian tasks more specifically) implies being able to reason about effects of changes in the given quantities. Based on the dimensions of the concept of functional thinking we propose to refer to this aspect as "Covariation" of Bayesian reasoning. However, hardly any studies have so far investigated Covariation in a Bayesian situation. In this paper we present a study in which participants are asked to reason about changes of the base rate and introduce a coding system with which their answers can be analyzed.

Keywords: Bayesian reasoning, covariation in Bayesian situations, statistics education.

Introduction

Evaluating Bayesian situations, i. e. the risk of a hypothesis (*H*) given some indicators (*I*) for the hypothesis, is important for experts from different fields (e. g. medicine and law) in everyday practice and a crucial part of conditional probability reasoning. The comprehension of Bayesian situations has so far almost exclusively been tested with the capability to calculate a probability for the positive predictive value (PPV), that is $P(H|I)$. This is a conditional probability indicating that a hypothesis H (e. g. a medical condition) is true, if an indicator I for the hypothesis (e. g. a positive test result in a medical test) is given. This probability can be calculated with the Bayes' formula when the base rate of the hypothesis $P(H)$, the true-positive rate $P(I|H)$ and the false-positive rate $P(I|\overline{H})$ of the indicator are given: $P(H|I) = \frac{P(H)\cdot P(I|H)}{P(H)\cdot P(I|H)+P(\overline{H})\cdot P(I|\overline{H})}$.

Summarizing the results of previous works (Binder et al., 2021; Böcherer-Linder et al., 2017; Böcherer-Linder & Eichler, 2017) of our ongoing research program, we argue that three aspects are necessary for a comprehensive understanding of a Bayesian situation: Being able to calculate one specific probability such as $P(H|I)$ in a Bayesian situation using Bayes' formula (we refer to this aspect as "Performance"), to assess the influence of changes of the given parameters of the situation (we refer to this aspect as "Covariation") and to discuss the result's impact (we refer to this aspect as "Communication"). While the aspect of Performance has been studied repeatedly, we are aware of hardly any studies for the remaining two aspects.

As part of a larger project we test and train all aspects of understanding a Bayesian situation (http://bayesianreasoning.de/en/bayes_en.html). In the training, participants learn to systematically combine two beneficial strategies, first representing the statistical information of the Bayesian situation in natural frequencies (McDowell & Jacobs, 2017) and second depicting the structure of the problem in a suitable visualization (e. g. Binder et al., 2020; Böcherer-Linder & Eichler, 2019; Khan et al., 2015). In this paper we present insights from a study which was conducted in preparation for designing these trainings. Thereby, we focus on the aspect of Covariation which we regard referring to the medical situation similar to the one shown in figure 1.

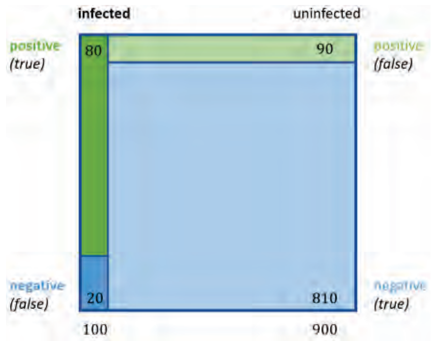
<table>
<tr><td colspan="2">Example for a medical diagnostic test: $P(H) = 0.1, P(I|H) = 0.8, P(I|\overline{H}) = 0.1$</td></tr>
<tr><td>Representation of the statistical information in form of natural frequencies:
100 out of 1000 people are infected.
80 out of these 100 infected people receive a positive test result.
90 out of the 900 uninfected people mistakenly receive a positive test result.
Calculation of the positive predictive value (PPV):
$$P(H|I) = \frac{80}{80+90} \approx 47\%$$</td><td>
</td></tr>
</table>

Figure 1: Example for the two beneficial strategies representing the statistical information with natural frequencies and depicting the situation in a visualization, here the unit square

To the best of our knowledge, Covariation as an aspect of Bayesian problems has only been tested once before (Böcherer-Linder et al., 2017). There, it was demonstrated that the effect of base rate change – that is, an increase or a decrease of $P(H)$ - is easier to understand with a unit square than with a frequency tree as a supporting visualization. Yet, this study does not shed light on the question why participants made mistakes and how they approached these questions.

Therefore, we want to provide an insight into first results of our study in which participants' reasonings for evaluating the change of the base rate in a Bayesian situation are analyzed. Our main question in this paper is: How can students' reasonings about Covariation (with a focus on a base rate change) in a Bayesian situation be categorized? In order to answer the question, we introduce a category system in this paper with which it is possible to analyze and cluster the level of different reasonings for evaluating the change of the base rate.

Theoretical Background

"It is believed that teaching students how to perceive formulas as covariational entities based on the provided context is essential. This skill can enable them to consider formulas as dynamic functions" (Sokolowski, 2021, p. 184). Even though this quote by Sokolowski has been formulated in the context of teaching physics, we argue that the covariational understanding of a function is equally important in conditional probability reasoning. We thereby refer to Borovcnik (2012, p. 21) who proposes "to investigate the influence of variations of input parameters on the result [i.e. $P(H|I)$]" aiming to strengthen a conceptual understanding instead of a more superficial numerical understanding of the concept of conditional probability and thereby of Bayesian situations.

Hence, we understand the Bayes' theorem not only as a formula but as a function that expresses the dependence of the posterior probability on three parameters. By appreciating Bayesian situations in the context of functions we combine two fields of mathematics education, i.e. statistical education with functional thinking. Consequently, we comprehend the central aspect of functional thinking, i.e. *Covariation* (Lichti & Roth, 2019), also as a part of Bayesian thinking. Covariation stresses the dependence of the independent variable on the dependent variable and the association between changes of both. A typical question for Covariation in the field of Bayesian reasoning is the following: "How does the positive predictive value $P(H|I)$ change when the base rate $P(H)$ increases/decreases?" In the following figure 2, it is illustrated in the unit square how the PPV is affected by an increase/decrease of the base rate.

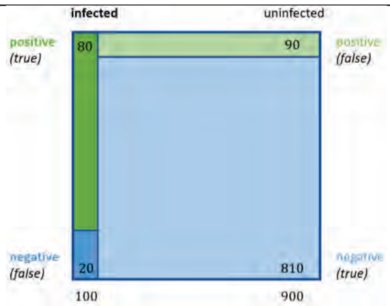
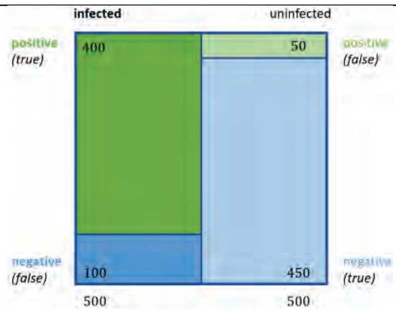
Base rate	100 out of 1.000 (10%)	500 out of 1.000 (50%)
Unit square		
PPV	$\frac{80}{80+90} \approx 47\%$	$\frac{400}{400+50} \approx 89\%$

Figure 2: Dependence of the PPV on the base rate illustrated in the unit square

Considering the effects of changes of the base rate is specifically important, as its influence often causes errors and misunderstandings when calculating the PPV (Kahneman & Tversky, 1982). For instance, in the situation of medical diagnostic tests (Fig. 1), very low base rates can cause a counterintuitive statistical phenomenon since in that case false-positive test results are more likely than true-positive test results and therefore the PPV is considerably low despite good test parameters (i.e. high true- and low false-positive rate). In such situations people tend to ignore the influence of the base rate which is called the "base rate neglect" (Kahneman & Tversky, 1982).

Analysing students reasonings about changes of the base rate can be significant in order to identify their conceptual understanding of the Bayesian situation. The SOLO-taxonomy of Biggs and Collies (1982) proposes a model with which it is possible to cluster the Structure of Observed Learning Outcomes (SOLO) into distinct levels. These levels differ with regard to the amount of (relevant) information which is used and linked to the cue in the students' arguments. Thereby, they distinguish five different levels of observed learning outcomes which can be applied to the teaching of various topics. We propose to apply the SOLO taxonomy by Biggs and Collis (1982) to the categorization of reasonings which are given about changes of the base rate in a Bayesian situation. In order to apply this model to tasks on Covariation, we identify what information is relevant for this task and thereby adapt the levels by Biggs and Collies to Covariation tasks in a Bayesian situation.

The PPV $P(H|I)$ in a Bayesian situation is calculated by $P(H|I) = \frac{P(I|H) \cdot P(H)}{P(I|H) \cdot P(H) + P(I|\bar{H}) \cdot P(\bar{H})}$ (Fig. 1). Thus, when reasoning about changes of the PPV it is necessary to consider alterations in both quantities representing the multiplied probabilities $P(I|H) \cdot P(H) = P(H \cap I)$ and $P(I|\overline{H}) \cdot P(\overline{H}) = P(\overline{H} \cap I)$ and then analyze their effect on the fraction (i.e. the PPV). Both quantities are dependent on the base rate. Illustrated in the unit squares in Fig. 2, one can see that the amount of infected people with a positive test result represented by $P(H \cap I)$ in the Bayesian formula increases with a higher base rate, whereas the amount of uninfected people with a positive test result represented by $P(\overline{H} \cap I)$ decreases when true- and false-positive rates stay the same. The relative increment of infected people with a positive test result (nominator) is higher than the relative increment of all people with a positive test result (denominator). Therefore, the PPV increases with an increase of the base rate. The different levels in the SOLO-model adapted to a Covariation task are described in figure 3.

Level	Description by Biggs & Collis	Covariation task in a Bayesian problem
Pre-structural	An irrelevant feature might be linked to the cue.	No explanation is given for the described relation between base rate and PPV or irrelevant consequences of the base rate change are described (e. g. on $P(\overline{H} \cap \overline{I})$).
Uni-structural	One relevant feature is linked to the cue.	The effect on only one of the relevant quantities ($P(H \cap I)$ or $P(\overline{H} \cap I)$) is considered or the effect of several quantities in the Bayesian situation is considered, but only one of them is relevant.
Multi-structural	Several relevant features are linked to the cue.	Both relevant quantities ($P(H \cap I)$ and $P(\overline{H} \cap I)$) are considered, but their relation to the PPV is not clearly spelled out.
Relational	All relevant data are considered and put into a conceptual scheme.	Both relevant quantities ($P(H \cap I)$ and $P(\overline{H} \cap I)$) are considered and their relation to the PPV is clearly spelled out.
Extended abstract	All relevant data are considered and subsumed in an abstract model.	A general model for the dependence of the PPV on changes of the base rate is generated.

Figure 3: SOLO model (Biggs & Collis, 1982) applied to Covariation tasks in Bayesian situations

In this study, we want to describe how we developed a coding system for the reasonings about Covariational tasks in Bayesian situations in order to classify them within these categories. Moreover, we will describe which further categories we inductively derived in order to code differences. Consequently, we will outline in the results section how the different levels of reasoning are distributed among preservice teachers, who were the subjects of our study.

Material and methods

Study design

Every participant answered five questions about a Bayesian situation such as the medical situation shown in Fig. 1. Thereby, we used two different Bayesian situations: one context about breathalyzers and another one about a mammography screening. The first task was always to a) calculate the PPV. The consecutive tasks were to determine how b) an increase of the true-positive rate, c) an increase of the false-positive rate, d) a decrease of the base rate and e) an equally large increase of the true- and the false-positive rate simultaneously affected the PPV. Answers were given in form of single choice questions with three options: the PPV i) decreases, ii) stays the same, iii) increases. Task a) was always the first one and task e) was always the last one. Tasks b) to d) were presented in a random order. Each participant was asked to give a reasoning for their choice of answer in one of the single-choice questions of b) to e). The answer to which a participant was asked to formulate a reasoning was chosen randomly. In the results section we only refer to the reasonings which participants have given for their decision of the effect of the base rate change.

Materials

Since visualizations and natural frequencies help to understand the influence of the base rate (Böcherer-Linder et al. 2017, also compare Fig. 2), we investigate how people reason about Covariation with the help of these two beneficial strategies (Fig. 1). As visualizations, we used a double-tree and a unit square which were more supportive in tasks of Performance than the simple tree diagram (Böcherer-Linder & Eichler, 2019) as we suppose that Performance is a prerequisite for Covariational tasks. Thus, the Bayesian situation was described with probabilities and also displayed in a visualization (unit square or double-tree) with frequencies.

Participants

230 pre-service teachers (181 females, 47 males, 2 unknown) participated in this study. They all study to become teachers in mathematics and another subject but for different age groups (e. g. some for primary school others for secondary school). They did not receive any prior training in stochastics. 60 out of the 230 participants were asked to give a reasoning for their answer to the single choice question about changes of the base rate.

Results

Developing categories for the data analysis

In this section we illustrate, how we coded the different reasonings and their belonging to the different levels of the SOLO taxonomy. Thereby, we refer to four exemplarily reasonings below. The initial question was: “Imagine: The probability that a driver is under the influence of alcohol is actually smaller than 10%. How does that affect the probability that a driver is actually under the influence of alcohol, if (s)he receives a positive test result in the breathalyzer?” After selecting if $P(H|I)$ increases, decreases or remains constant, the participants were asked to explain their choice:

Example 1: “The precision of the test is not changed by the description in the text.”

Example 2: “$0.1 \cdot 0.9 = 9\%$, *something smaller than* $0.1 \cdot 0.9 =$ *something smaller than* 9%”

Example 3: “As the number of people who are under the influence of alcohol decreases, the probability also decreases, that a positively tested person is also under the influence of alcohol. The nominator of the fraction decreases and the denominator stays the same. Thus, the result is smaller.“

Example 4: “Number of people under the influence of alcohol decreases, analogously 90% positive → less under the influence of alcohol and positive and number of false positively tested bigger. Denominator bigger and nominator smaller therefore result is smaller.”

First, we coded to which probabilities/quantities of the Bayesian situation the reasonings made a reference to. Example 1 refers to the true- and false-positive rate of the test (“precision of the test”), whereas examples 2 and 3 both refer to the quantity of true-positives (the numbers of the Bayesian context were chosen in a distinct way so we made this inference in example 2). In example 4 references to the quantity of true-positives and false-positives are drawn. However, in example 4 we can also observe that the conclusion for the denominator of the fraction for the PPV is wrong (i.e. it actually decreases as well). In total we observed six types of quantity-references (QR), some including further subtypes: no reference to any probability or quantity (e. g. "somehow seems logic", QR0), references to one or more probabilities of the test (e. g. true- or false-positive rate, QR11-QR14), references to one of the joint probabilities (e. g. true- or false-positives, QR21-QR22), references to two joint probabilities or joint events with only one of them being relevant (QR31-QR32) or both of them being relevant (QR33-QR34), description of a direct link between a decreased base rate and a decrease PPV without further explanation (QR40) and between a decreased base rate and an increased PPV (QR50). The types QR33 and QR34 differ only in their implications on the PPV: while both correctly describe the changes in the two joint probabilities, only QR34 draws the correct conclusions from it while QR33 doesn't. In figure 4 we display how we have assigned the different observed probability/quantity references to the levels in the SOLO taxonomy.

Level	Observed probability/quantity reference
No level	QR0 and QR50
Pre-structural	QR11-QR14 + QR40
Uni-structural	QR21-QR22 + QR31-QR32
Multi-structural	QR33
Relational	QR34
Extended abstract	No observations.

Figure 4: Observations of quantity/probability references in the different levels of the SOLO model

Apart from differences in quantity references we noted further differences and inductively generated additional categories to quantify these differences. First, we noticed a difference in the reference to the context of the Bayesian situation. While in example 2 there is no reference to the context at all, examples 3 and 4 refer to the specific context of the Bayesian situation. Example 1 refers to a more general context, as the "precision of the test" is just as suitable for the breathalyzer context as for the mammography context. Therefore, we differentiated between the connection of a rationale with a specific context with three codes: no context (C0), specific context (C1) and general context (C2). Second, we observed a difference in the representation of the described (changed or unchanged) quantities/probabilities. In example 2 only percentages are used, in example 4 percentages, references to a fraction and absolute frequencies are used. In example 3 probabilities, references to a fraction and absolute frequencies are used. For each reasoning we coded if each of the following types of representations was used (code 1) or not (code 0): probability (R1), percentage (R2), frequencies (R3), fraction (R4), proportion (R5), quota (R6) and size of an area in the unit square (R7).

Reporting the data within the categories

Two raters independently coded all 60 reasonings for effects of the base rate change with the coding system described above. We report inter-rater-reliability that was assessed for all categories described in a first attempt without further training. For the reference to the context Cohen's Kappa was 0.75 (95% CI: [0.59;0.9]) and therefore substantial. For each representation an interclass correlation coefficient (ICC) was calculated. The ICC for a quota as a representation was poor with an ICC of 0.381 (95% CI: [0.144;0.577]. Yet, the other representation types were coded with a good to excellent inter-rater-reliability with a range from 0.687 (95% CI: [0.528;0.8]) for frequency, to 0.92 (95% CI: [0.841;0.94] for proportions. For the quantity-representation the Cohen’s Kappa was 0.76 (95% CI: [0.63-0.88]). Overall, apart from the coding of the quotas as representation the inter-rater reliability seems satisfying. The results of the ratings of both raters are represented in figure 5.

	Context			**Representation**						
	no (C0)	specific (C1)	general (C2)	prob. (R1)	perc. (R2)	frequ. (R3)	frac. (R4)	prop. (R5)	quota (R6)	area (R7)
No level (25/20)	(12/11)	(11/9)	(2/0)	(13/8)	(6/5)	(6/1)	(4/2)	(2/0)	(3/1)	(2/0)
Prestructural (15/23)	(2/9)	(10/11)	(3/3)	(11/13)	(8/8)	(3/2)	(0/2)	(1/4)	(1/0)	(1/0)
Unistructural (18/15)	(1/1)	(15/14)	(2/0)	(10/9)	(10/4)	(9/11)	(3/1)	(2/0)	(0/0)	(1/1)
Multistructural (1/1)	(0/0)	(1/1)	(0/0)	(0/0)	(1/1)	(1/1)	(1/1)	(0/0)	(0/0)	(0/0)
Relational (1/1)	(0/0)	(1/1)	(0/0)	(1/1)	(0/0)	(1/1)	(0/0)	(1/1)	(0/0)	(0/0)

Figure 5: Number of Answers in each of the categories (rater 1/rater 2)

Discussion

Bayesian tasks have so far almost exclusively been studied as tasks of Performance. These tasks are known to be difficult and one of the assumed reasons for that is the base rate neglect, thus the cognitive error by which the influence of the base rate on the PPV is overlooked. We have introduced the concept of Covariation (which is an established dimension of functional thinking) to Bayesian tasks. With Covariation tasks in a Bayesian situation one can directly assess the participants’ ability to judge the influence of the base rate on the PPV. Asking for a reasoning of the given answer (as we did in this study) allows to qualitatively study the participants understanding of the Bayesian situation. The coding system which we have introduced in this paper can be of special help when teaching about Bayesian tasks, since arguments which belong to the different levels of the SOLO-model reveal different issues about the understanding of the Bayesian situation. For instance, students who reason according to the prestructural level clearly lack an understanding of the Bayesian situation itself as they are unaware of the relevant quantities which have to be considered in the particular Bayesian task. Therefore, they should revise the basics again (e. g. what sets and subsets are described in the situation and how can they be quantified and used to calculate the PPV). On the other hand, students whose reasoning belongs to the multistructural level are very well aware of the structure of a Bayesian situation. They might only need some support in how to argue about changes in a fraction. Thus, the coding system can help to tailor the support to the students’ needs. Moreover, it is evident, that the reasoning level in Bayesian situations is generally rather poor with 40 out of 60 students whose reasoning remains on the pre-structural level or cannot be assigned to either level. This observation confirms prior research about Bayesian reasoning in so far as Bayesian tasks have generally been shown to be considerably challenging and counter intuitive without any training.

A consecutive cluster analysis with the data derived from the coding system together with additional information from the study will reveal if and how the levels in the reasoning about Covariation coincide with other aspects (e. g. the capability to correctly calculate the PPV or the type of visualization which was used as a supportive tool). This will shed more light on how to successfully teach a conceptual understanding of conditional probabilities.

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Prospective primary school teachers' recognition of proportional reasoning in pupils' solution to probability comparison tasks

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In order to foster the learning of mathematics, the teacher must be able to interpret and analyse the students' mathematical activity. This cognitive analysis competence allows the teacher to understand the processes of mathematical learning, to foresee conflicts of meanings and to establish different possibilities for institutionalising the mathematical knowledge involved. In this paper, we are concerned with assessing the initial knowledge and competence of prospective primary school teachers in order to analyse responses of primary school pupils when solving urn probability comparison tasks. Specifically, we are interested in analysing what degrees of proportional reasoning the prospective teachers identify in the pupils' answers. The results reveal the prospective teachers' limitations for correctly identify proportional reasoning, specially to discriminate additive and multiplicative comparison.

Keywords: Probability, proportional reasoning, specialized probabilistic knowledge, teacher's education, urn tasks.

Introduction

A critical issue in mathematics education research is to clarify the type of didactic-mathematical knowledge that mathematics teachers should have in order to develop their teaching work in an appropriate manner (Chapman, 2014; Mason, 2016). The mathematics education research community accepts that teachers should have a certain level of mathematical competence, that is, they should be able to perform the mathematical practices necessary to solve the problems that the curriculum proposes and to articulate them with the subsequent mathematical contents. Teachers should also have a specialized knowledge of the content itself, of the transformations that have to be applied to it in teaching and learning processes, and of the psychological, sociological and pedagogical factors, among others, that condition these processes. Although the analysis of pupils' thinking is considered one of the main tasks of mathematics teaching, identifying the mathematical ideas inherent to the strategies that a pupil uses during mathematical problem solving could be difficult for the teacher (Fernández et al., 2013). In this sense, several researches indicate that both pre-service and in-service teachers have difficulties to interpret the responses of primary education students when solving mathematical tasks involving proportional reasoning, as well as making action decisions based on how pupils seems to understand proportionality (Buforn et al., 2020; Fernández et al., 2013). These investigations concluded that further research is needed on pre-service teachers' didactic and mathematical knowledge related to proportional reasoning.

The aim of this work is to assess the knowledge and skills of prospective primary school teachers to interpret pupils' responses to probability comparison tasks, identify incorrect strategies and recognise proportional reasoning in their mathematical activity. We address the following research questions:

What do prospective teachers understand by proportional reasoning in the context of probability comparison tasks? How do they identify and use it to justify their assessment of pupils' solutions?

Proportional and probabilistic reasoning are strongly linked; both involve quantitative and qualitative analysis, establishing relationships, making inferences and predicting outcomes. The results will allow us to design, implement and evaluate training actions to develop these didactic-mathematical knowledge and skills in prospective mathematics teachers.

Previous research

Proportional reasoning, understood as the ability to establish multiplicative relationships between two quantities and to extend this relationship to another pair of quantities (Lamon, 2007), is an objective present in the Primary Education curriculum, which integrates both the various interpretations of the rational number (ratio, operator, part-whole, measure and quotient) and the ways of reasoning with these meanings (up and down reasoning, relational thinking, covariance, etc.). Proportional reasoning is "a form of mathematical reasoning that involves a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information. Proportional reasoning is very much concerned with inference and prediction and involves both qualitative and quantitative methods of thought" (Lesh et al., 1988, p. 93).

Extensive research has focused on investigating pupils' strategies and levels of proportional reasoning in the context of probability, mainly in the urns probabilities comparison setting (Bryant & Nunes, 2012; Cañizares & Batanero, 1998; Langrall & Mooney, 2005; Van Dooren, 2014; Watson, 2005). According to Falk et al. (1980), probability can be thought of as being composed of two sub-constructs: chance and proportion. Various researches in mathematics education reveal that both students and teachers have difficulties in applying proportional reasoning in statistical and probabilistic contexts (Gal, 2002). Specifically, the lack of proportional reasoning to solve elementary probability comparison problems is found not only in students, but also in prospective primary school teachers (Contreras et al., 2011; Gómez et al., 2016; Vásquez & Alsina, 2015). Begolli et al. (2021) suggest that "prior knowledge of proportional reasoning reveals deeper insights into students' potential for learning about probabilistic reasoning, than prior knowledge of the concept of probability itself" (p. 462). For this reason, it is essential that prospective teachers be aware of the different components of proportional reasoning and how they appear in probability.

Didactic-mathematical knowledge and competence model

The study of the type of didactic and mathematical knowledge and competences that teachers should have in order to manage the pupils' learning process has generated several models that intend to characterize such teachers' knowledge and competences (Chapman, 2014; Hill et al., 2008). In this research we adopt the teacher's Didactic-Mathematical Knowledge and Competence (DMKC) model (Godino et al., 2017) developed within the Onto-Semiotic Approach (OSA). The DMKC model considers that the teacher should have a *common mathematical knowledge* regarding a certain educational level where he/she teaches, as well as an *expanded mathematical content knowledge* that allows him/her to articulate the content with higher educational levels. In addition, as some mathematical content is put at stake, the teacher should have a didactic-mathematical or *specialized knowledge* of the different facets involved in the educational process: *epistemic* (institutional content meanings), *ecological* (aligning tasks according to institutional mandatory curriculum), *cognitive* (understanding student's thinking), *affective* (reacting to anguish, indifference, anger, etc., manifested by students), *interactional* (identifying and answering to students' conflicts and interactions), and *mediational* (choosing the best suitable resources for teaching). In the epistemic facet, the specialized knowledge allows the teacher to recognize the diversity of meanings involved, be able to solve the task using different strategies and justify the accuracy of the procedures. In the cognitive facet, the specialized knowledge guarantees the teacher being able to understand the ways of reasoning, difficulties and personal meanings that students may present when working with the specific mathematical situation. It makes the teacher competent to identifying possible different solution strategies in a probability problem, assessing students' responses and recognizing the mathematical objects involved. In particular, analyzing the proportional reasoning put at stake in the mathematical practices involved in their resolutions. This will provide teachers appropriate responses to real classroom situations.

Method

We performed a content and descriptive analysis of the written solutions to the problem in order to classify the responses into different categories building on previous research and refining these categories through a cyclical and inductive process; this is typical of qualitative research. This research was conducted with 116 prospective primary school teachers (PPT in the following) at a Spanish university. During their undergraduate studies, these prospective teachers received specific preparation on the epistemic, cognitive, instructional and curricular aspects of teaching statistics and probability. Specifically, the intervention was carried out once the training process of the PPTs on the main contents of Data Processing, Chance and Probability had been completed. This deals with the fundamentals of the Didactics of Mathematics in terms of the main concepts, properties and procedures that form the primary school mathematics curriculum, mathematical learning, errors and difficulties and instructional aspects (tasks, materials and resources) related to this content. The task (see Figure 1) was proposed to the PPTs to be solved individually and voluntarily. The written answers of the PPTs to this task were analysed using content and descriptive analysis methods.

Below you find a problem and some solutions to it developed by primary school pupils.

> In box A two white balls and two black balls have been placed. In box B there are 4 white balls and 4 black balls. In which box is there a greater chance of getting a black ball?
>
> Alba: "In box B because it has 2 more black balls than box A".
> Daniel: "The same, because in box B there are 2 more white balls, but there are also two more black balls".
> Lucía: "The same, because in both the white balls are half as many as the black balls".
> Salva: "The same because in box B the number of white and black balls has been multiplied by 2 compared to A".

Justify whether each pupil's answer seems correct or incorrect and identify proportional reasoning in the pupils' answers.

Figure 1: Task proposed to PPTs to assess the analysis of pupils' solutions to a probability comparison problem

To assess the cognitive facet of the PPTs' didactical-mathematical knowledge and competence, we proposed to the participants to analyse the correctness degree of different pupils' solutions to a probability comparison problem in urns, identifying the proportional reasoning involved or not, as a relevant mathematical element of pupils' mathematical thinking when solving this type of task.

Notice that Alba's answer is incorrect, as she only compares the favourable cases, which leads her to decide that in box B the probability of drawing the black ball is higher. To Daniel, the probability is the same, "because in box B there are two more white balls, but there are also two more black balls". Daniel compares in an additive way the unfavourable cases (white balls) and the favourable cases (black balls) of both boxes: as the difference between the unfavourable cases of both boxes and the favourable cases of both boxes is the same, two, he concludes that the probability of drawing a black ball in both boxes is the same. This argument is not correct, since it only leads to a right answer in the case where the number of favourable cases matches the number of unfavourable cases. In Lucía's solution, it is true that "if the number of unfavourable cases is the number of favourable cases multiplied by a scalar (in our case 2) the probability remains constant". However, it is not true that "in both cases the white balls are half of the black balls". It seems that Lucía confuses "unfavourable cases" with "possible cases". We note that Lucía uses a correspondence strategy (proportional reasoning) but the multiplicative relationship that she establishes in one urn to be extended to the other is not correct. Finally, to justify that the probability of drawing a black ball in both boxes is the same, Salva uses proportional reasoning: the probability, as the ratio between the number of favourable and possible cases (favourable plus unfavourable) remains constant if both the number of favourable and unfavourable cases is multiplied by a scalar.

Results

Of the 116 participants, 100 (i.e., 86.21%) considered Daniel's answer to be correct, but only 73 (62.93%) provided some conclusive justification. Table 1 shows that more than half of PPTs (50.86%) considered Lucía's solution to be right, however, just 24 (20.69%) of them gave a clear description of why the argument used by this student is appropriate. Of the 84 (i.e. 72.41%) PPTs who explained why they considered the solution given by Salva to be correct or not, 78 (67.24%) considered it to be correct and only 6 (5.17%) considered his argument to be inadequate (they believe that it is only valid in this particular case or that he should have relied on the use of Laplace's rule). In addition, two PPTs considered all pupils' answers to be correct, without justifying their assessment. These two PPTs were the only ones who implicitly considered Alba's solution being right, for which 106 (91.38%) PPTs justified her error.

Table 1: Frequencies (percentages) in the assessment of the correctness degree of pupils' responses

	Alba	Lucía	Daniel	Salva
No answer/Not conclusive evaluation	10 (8.62)	35 (30.17)	34 (29.31)	32 (27.59)
Pupil's solution correct	0 (0)	24 (20.69)	73 (62.93)	78 (67.24)
Pupil's solution incorrect	106 (91.38)	57 (49.14)	19 (16.38)	6 (5.17)
Total	116 (100)	116 (100)	116 (100)	116 (100)

In view of the interest of this paper, of the arguments used by PPTs to justify their assessment of primary school pupils' answers as correct or incorrect, we focus our attention on those evaluations that refer to the presence or absence of proportional reasoning (see Table 1). This will give us useful information about what prospective teachers understand by proportional reasoning in the context of probability comparison problems and when and how they identify it in the pupils' responses.

As we see in Table 2, eleven PPTs consider that comparing only favourable cases in an additive way shows an absence of proportional reasoning in Alba's strategy, which they interpret in the context of probability as the ability to establish a proportion, or to establish a multiplicative relation/comparison. For example, PPT41's answer:

PPT41: I have been able to identify proportional reasoning in pupils who think that there is the same probability due to the fact that the number of balls in one box is proportional to the number of balls in the other box. Error that Alba may have made is assuming that because there are more black balls in one box than in the other, there is a much higher probability of getting a black ball, regardless of the number of white balls.

Table 2: Reference to proportional reasoning in the PPTs' assessment of pupils' solutions

Student	Reference to proportional reasoning	Frequency
Alba	Lack of proportional reasoning (multiplicative comparison, covariation) as a cause of error	11
	Evidence of proportional reasoning understood as a "more, more" type comparison	7
	Evidence of proportional reasoning understood as an additive comparison	3
Lucía	Evidence of proportional reasoning (multiplicative relationship) leading to the conclusion that there is the same probability	31
	Inadequate or incomplete proportional reasoning (multiplicative relationship) as a cause for her error	10
Daniel	Proportional reasoning (proportion, equivalence of fractions) as a guarantee of successful response	19
Salva	Proportional reasoning (proportion, equivalence of fractions) as a guarantee of successful response	13
	Proportional reasoning (multiplicative relation, variation) guarantee of successful answer	26

Furthermore, we note that seven PPTs of the 106 that identify and justify Alba's answer as incorrect, identify proportional reasoning like a "more, more" type comparison. See for example, PPT52's answer:

PPT52: […] In Alba's solution, a proportional reasoning can be identified, since the student observes more balls in box B and therefore thinks that there is a higher probability of drawing one in this box.

Three other PPTs also identify proportional reasoning in Alba's incorrect answer, which they interpret as an additive relationship. For example:

PPT110: In Alba's answer we can notice some proportional reasoning, she has taken into account the increase of two balls in the black colour [...] That would explain her mistake, to have thought proportionally in only one colour.

Likewise, other participants contemplate that Lucía has an error due to the use of proportional reasoning when it was not appropriate ("the main error refers to using proportional reasoning when it was not appropriate to do so", PPT24). They consider that the most suitable strategy is to stablish an additive comparison. In this sense, the analysis of the PPTs' reports shows that several participants show inadequate knowledge of proportional reasoning because they consider it in terms of additive comparisons. That is the case of PPT18:

PPT18: We can say that proportional reasoning is found in the following three answers [referred to Daniel, Lucía and Salva], in all three the same amount is added or taken equally in the two boxes. There is the same probability in box A as in box B.

Furthermore, 22 PPTs points out that pupils correctly apply proportional reasoning when they understand and use relevantly the equivalence of fractions and employ it to calculate and compare probabilities. They consider this leads to success for Daniel and Salva in their response. For instance, PPT38:

PPT38: Regarding proportional reasoning, I identify it in two cases, Daniel and Salva, since both have taken into account the equivalence of fractions. Thus, they have observed that in box A there are 2/4 white and 2/4 black, while in box B there are 4/8 white and 4/8 black, which means that the probability in both cases is the same.

To 50 (i.e., 43.10%) of the PPTs, proportional reasoning is involved in those pupils' answers in which a multiplicative comparison is established (see Table 2). This is observed in the evaluations of Lucía or Salva:

PPT11: Lucía has reasoned that in box A, there are half as many black balls and half as many white balls as in box B. She aimed to say, in some way, that the quantity in one box and the other changes proportionally.

PPT45: Salva uses proportional reasoning because he says that in box B both black and white balls have been multiplied by two with respect to box A.

As we can see, PPT11 and PPT45 consider the multiplicative relation between the favorable cases (and unfavorable cases) of both boxes. Other participants look at the correspondence within each of the boxes to ensure the same probability. For instance, PPT38 considers "there is the same number of both white (2) and black (2) balls in box A and, also the same number of white (4) and black (4) balls in box B, so the ratio is maintained and the probability is the same".

In summary, we have noticed that, a high-rise percentage of prospective teachers consider proportional reasoning to be based on "more …, more …" relationships (these descriptions appear in 23.91% of the occasions in which PPTs identify proportional reasoning). Besides, even when some prospective teachers discriminate between additive comparisons and multiplicative comparisons, they do not always properly describe the multiplicative relationship, or the magnitudes involved in the proportionality correspondence. That is, they do not always correctly establish a multiplicative comparison between the ratios of favorable and unfavorable cases (or of favorable and possible cases) within both boxes, or between the ratios of favorable cases between the boxes and of unfavorable or possible cases between the boxes. Furthermore, some participants, consider that the establishment of a multiplicative relationship only between favorable cases or (also solely) between possible cases is sufficient to identify proportional reasoning and respond successfully to the task.

Implications for teaching and research

"The identification of the relevant mathematical elements in a problem and the interpretation of how they are present in the students' answers allow prospective teachers to be in better conditions to make relevant instructional decisions and help students develop their proportional reasoning" (Llinares, 2013, p. 81). Hence to interpret different pupils' solution to a probability comparison task by recognising how proportional reasoning is involved in their answers, can help to enhance prospective teachers' didactic-mathematical knowledge and competences (especially in the cognitive facet) regarding this topic.

Our results show the need to strengthen teachers' education in relation to the connection between proportional and probabilistic reasoning. Prospective teachers find limitations in identifying and justify possible erroneous strategies behind pupils' incorrect answers. A biased or insufficient knowledge of proportional reasoning could explain why PPTs do not identify it in the incorrect answers and when they do, they show errors when interpreting the proportionality relationship and the properties that characterise it (Burgos & Godino, 2021).

We think that our results provide additional valuable information for the design of materials in teacher education programs that consider the characteristics of prospective teachers' understanding of proportional reasoning in probability tasks. First, to guarantee that prospective teachers are able to recognize and respond to students' errors, teacher education programs should develop a deep understanding of the conceptual, propositional and argumentative components of proportional reasoning involved in probability setting. On the one hand, proportional reasoning is an integral part of probabilistic reasoning. But on the other hand, probability is an enabling environment for future teachers to overcome a limited view of proportional reasoning linked to solving missing-value problems. "Balancing the amount of probability instruction with proportional reasoning instruction may be more successful than teaching only about probabilities" (Begolli et al., 2021, p 463). Therefore, specific actions should be designed in teacher training to reinforce the structural components of proportional reasoning by integrating it with probabilistic reasoning.

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Learning opportunities for statistical literacy in German middle school mathematics textbooks

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The development of statistical literacy is an important goal for middle schools, where statistics education mostly takes place within mathematics classrooms. Here, textbooks provide the most important tool for many teachers, guiding the content of their lessons. However, little is known about the statistical content of middle school mathematics textbooks. This study reports on a qualitative document analysis of three German Grade 6 textbooks. The results show that a large majority of tasks in textbooks revolve around technical constructions of diagrams and calculations of measures. Less space is allocated towards more conceptually demanding tasks like interpreting models or analysing and reflecting statistical arguments. This implies that teachers need to actively adapt their textbooks in order to unlock the potential for developing statistical literacy of these textbooks.

Keywords: Statistics education research, statistical literacy, textbooks, document analysis, middle school.

Introduction

In recent years, researchers in statistics education research have elaborated the importance of statistical literacy for aspects of digitalization such as big data and machine learning (François et al., 2020). It is becoming increasingly clear that statistical literacy cannot be reduced to a skill of specialists, but rather will be important for every citizen in the 21st century (Wild, 2017). Therefore, the development of statistical literacy becomes an important task for middle schools.This is a challenging task, as in many countries, statistics is considered only a small part of middle school mathematics instruction, and only very limited time can be allocated to statistics (Zieffler et al., 2018). Although researchers in statistics education research have begun to address this issue, insights into how statistical literacy can be developed in middle schools are limited yet (Büscher, in press). Thus, teachers have to contend themselves with the learning opportunities for developing statistical literacy that are provided by their mathematics textbooks.

This makes mathematics textbooks an important object of study. The content of textbooks largely defines the content of mathematics classrooms, as content that is not included in textbooks generally is not taught in class (Stein et al., 2007). Textbooks need to provide teachers with the suitable didactical instruments for developing statistical literacy. This study aims to provide insights into the learning opportunities for statistical literacy afforded by middle school mathematics textbooks.

Theoretical background

Statistical literacy as selective and imaginative readings of statistical information

Statistical literacy generally refers to the ability to understand and to critically evaluate statistical information presented in everyday media like newspapers, articles, or infographics (Gal, 2002). Whereas earlier conceptualizations of statistical literacy mostly related citizens to the role of “data consumers” (Gal, 2002), researchers recently have emphasized that statistical literacy also requires the development of skills more in line with data producers (Weiland, 2017). In order to integrate the two perspectives of data producer and data consumer, this study conceptualizes statistical literacy as the two processes of *selective and imaginative reading of statistical information* (Figure 1, Büscher, in press). *Selective reading* refers to the process producing concise statistical arguments. During this process, selective activities reduce the available information (illustrated by the progressively smaller boxes in Figure 1): A phenomenon is *encoded* into data by selecting only certain aspects that are then quantified. The data are then *abstracted* into a model by mathematizing certain relationships within the data. Finally, the model is *interpreted* by combining some of these relationships with a claim about the phenomenon under investigation, resulting in a statistical argument.

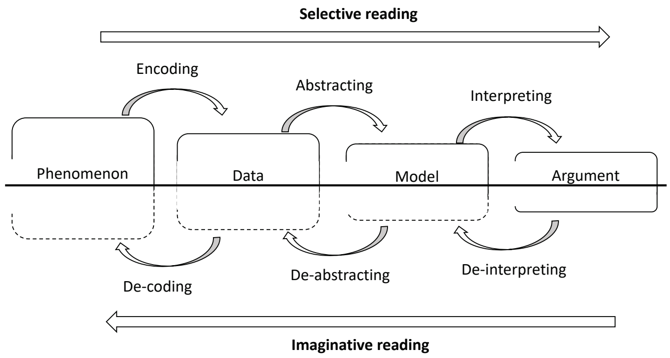

Figure 1: Statistical literacy encompasses activities of selective and imaginative reading

Crucially, a reader that is presented with a statistical argument in, for example, a social media post, likely does not have access to the underlying model or data. In order to critically evaluate the statistical argument, one has to revert the acts of selective reading through *imaginative reading* of what could have resulted in the argument (the dashed boxes in Figure 1). A statistical argument has to be *de-interpreted* to intuit the underlying model behind the argument, for example by guessing the type of measure of centre that an argument simply refers to as "average". Such a model only represents relationships in data, not the data themselves. The reader has to *de-abstract* from the model to imagine possible data behind the model, and what features of these data a median might or might not represent well. And finally, one has to recognize that the data only provide a quantified description of some aspects of the phenomenon that were obtained through certain methods of data collection. A *decoding* of the data might reveal important aspects that cannot be captured by the data. In this way, imaginative reading aims to discover possible causes and possible limitations of a statistical argument even if crucial information is missing.

This specification of the learning content of statistical literacy allows to decompose the larger construct into smaller activities that each can be the object of focused instruction. Instead of a holistic approach, teachers can create focused learning opportunities for each of the activities of selective and imaginative reading. This should not be taken as the claim that these activities should always be treated separately. Still, by identifying smaller activities, this conceptualization allows to identify the potential contributions to statistical literacy in many statistical tasks which are presented in textbooks.

Textbooks in statistics education research

Textbooks have a large impact on the enacted curriculum of schools, and Weiland (2019) proposes that this is especially true for statistics, where teachers have little prior experience. Thus, they might tend to adhere to the textbook more closely with statistics than with other subjects. In his study on United States high school textbooks, Weiland (2019) investigates what kinds of contexts are supplied in textbooks and how they are used. He finds that the contexts used "generally go no further than those typical of 'small talk', such as the weather, sports, personal preferences, or related to work or business" (Weiland, 2019, p. 32). He instead calls for textbooks to feature controversial socio-political issues to prepare students to be critical citizens. Tran and Tarr (2018) also investigate US high school textbooks, focusing on the complexity of the investigation of bivariate data in textbook tasks. They find that students are not required to formulate their own statistical questions, but are always given a fixed question in the tasks. Most of the time, students are provided with the data, which generally consists of fewer than 20 values and show no "messy" features like missing values. Thus, the textbooks provide little learning opportunities for organizing real, unstructured data.

Apart from these studies, not much research could be found that investigate the statistics content of textbooks. Under the statistical literacy perspective employed in this study, the existing studies suggest that the statistical arguments about contexts in the textbooks are uncontroversial, and thus might not motivate a deeper investigation of the sources of possible controversies through imaginative reading. Where selective readings are elicited, they are performed in a very fixed way, possibly emphasizing activities of abstracting over the more open activities of encoding and interpreting.

Research questions

A statistically literate citizen needs to be able to engage in activities of selective and imaginative reading. Textbooks need to provide teachers with suitable instruments to create learning opportunities for these activities. The little empirical knowledge available about textbooks suggests that textbooks might not be well equipped for this task, but further insights are needed. This study aims to provide a contribution by investigating the following research question:

(RQ 1) Which learning opportunities for activities of selective and imaginative reading are provided by German middle school textbooks?

(RQ 2) Which differences in learning opportunities exist between German middle school textbooks?

Method

Selection of textbooks

This study took the form of qualitative document analysis (Bowen, 2009). As a first step, relevant textbook series to be used in the analysis had to be selected. This proved a difficult task: In Germany, educational policy is a matter of the 16 federal states, which leads to variations in the mathematics curriculum and to state-specific textbooks. Additionally, textbook publishers generally do not disclose the market shares of their textbooks, so that little objective criteria exist for selecting textbook series for study. In the end, a theoretical sampling resulted in the selection of three textbook series. Two of these series, *Lambacher Schweizer* ("LS", Jörgens, 2009) and *Elemente der Mathematik* ("EdM", Griesel et al., 2014), are textbook series used in German middle schools tracked for academic education. According to the publishers' description of their teaching conception, both textbook series provide a clearly structured learning progression with possibilities for differentiation and an emphasis on exercises. These series were selected to allow the identification of possible differences in learning opportunities for similar teaching conceptions. In contrast, *mathe live* ("ml", Glöckel et al., 2014) is a textbook series for integrated middle schools that introduce academic tracking only in later school years. According to the publisher, the teaching conception focuses on exploring mathematics in real-life situations and on individual approaches to mathematics.

From each textbook series, only the textbook for Grade 6 was included in the analysis. This decision was made because the mathematics curriculum for Grade 6 includes a relatively large part of statistics in relation to other grades. Content includes the construction and critical evaluation of various diagrams as well as measures of centre, which are important models for a statistically literate citizen. Only the chapters focusing on statistics were included in the analysis, and chapters focusing on probability were not included. This resulted in a data corpus of 371 tasks.

Data analysis

For data analysis, codes were assigned to each task according to the activities of selective and imaginative reading elicited by the tasks. For this, a coding scheme had to be developed in a multi-step approach consisting of deductive and inductive analytic phases. The assigned codes were compared and contrasted to identify possible incongruences in assigning the codes and to find the central categorial cuts between the codes. In the end, a coding scheme emerged that identified codes based on the *source* and the *goal* types of statistical information (phenomenon, data, model, argument). The source refers to the type of statistical information given in the task; the goal refers to the type of statistical information required as a solution to the task. The identification of the type of statistical information considered the language employed for giving the information: (a) statistical information on a *phenomenon* is characterized by rich descriptions of contextual knowledge without exact quantification. (b) Statistical information on *data* is characterized by atomic quantifications of certain aspects of the phenomenon. This includes categorical data as well as frequency data. (c) Statistical information on *models* refer to relationships within the data that are not reported by the data itself, but by additional models. Such models can be measures of centre as well as diagrams like pie charts, which can illustrate the proportional relationships between frequency data. Finally, (d) statistical information on *arguments* comprises justifiable claims about the phenomenon that are based on a model. Mere verbal descriptions of models are not considered statistical arguments; instead, an interpretative step has to be performed that situates the model in the larger phenomenon by incorporating additional context knowledge or by generalizing from the model.

Table 1 gives illustrates the final coding scheme. This scheme was applied in a final deductive step of analysis by identifying source and goal of the statistical information and assigning codes according to the coding manual in Table 1. Throughout the whole process, the assigned codes were discussed in the research team of the author and two colleagues. Not every task fit neatly into the coding scheme. These cases were discussed with the research team to provide a consensual validation of the coding.

In some cases, the team concluded that it was not possible to assign a code to a task. Such tasks often consisted of purely mathematical questions or were too unspecific to be clearly assigned to any activity. Although a skilled teacher might still use these tasks to develop statistical literacy, they could not be included in the analysis. Due to the complexity of the coding and the need for inductive development of the coding scheme with a research team, a test of inter-rater reliability was not applicable. Thus, the results of the analysis should be interpreted as exploratory findings on statistical literacy content in textbooks.

Table 1: The coding scheme for identifying activities of selective and imaginative reading

Activity	**Source**	**Goal**	**Example task**
Encoding	Phenomenon	Data	- *Conduct a survey about leisure activities in your class*
Abstracting	Data	Model	- *Given is a table with frequency data on leisure activities in class, Draw a pie chart.* - *Given is a table with daily sleep time in the class. Find the average.*
Interpreting	Model	Argument	- *Given is a chart on students' long jumps. Who should win the competition?*
De-Interpreting	Argument	Model	- *Given is a newspaper article. What does the author mean by "average"?* - *Tobias claims that boys are keener on sports than girls, Aylin disagrees. Find a justification for the claims in the data.*
De-Abstracting	Model	Data	- *The median daily sleep time is 8h. Which data could have produced the median?*
De-Coding	Data	Phenomenon	- *Given is data on youths' internet activities. Does this fit your own experiences?* - *Given is data on students' commutes to school by bus. Find a possible reason why some values are very high.*

Results

Table 2 provides an overview about the codes given in the analysis. The data is also visualized in Figure 2. Percentages do not add up to 100%, as some tasks were assigned multiple codes, and few tasks did not fit the coding scheme and were not assigned any codes.

Table 2: Codes assigned to the tasks in three textbooks

Textbook	**Tasks**	**Encoding**	**Abstracting**	**Interpreting**	**De-Interpreting**	**De-Abstracting**	**De-Coding**	**n/A**
EdM	130	14 *(11%)*	77 (59%)	9 (7%)	14 (11%)	13 (10%)	5 (4%)	19 (14%)
LS	146	8 (5%)	88 (60%)	3 (2%)	6 (4%)	35 (24%)	2 (1%)	15 (10%)
ml	95	3 (3%)	57 (60%)	11 (12%)	7 (7%)	11 (12%)	4 (4%)	6 (6%)

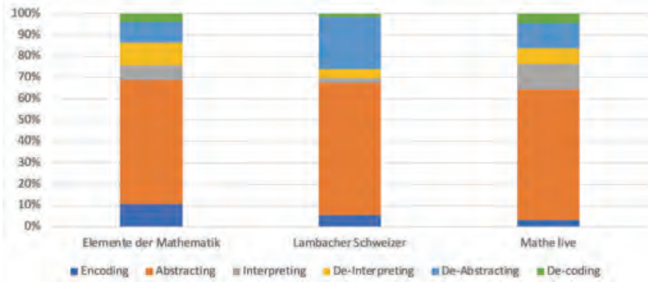

Figure 2: Textbooks heavily emphasize abstracting, but other activities vary

The data show that across all textbooks, the large majority of tasks (about 60%) elicit the selective activity of abstracting. The imaginative activity of de-coding shows the least occurrence in all of the textbooks, with only 1% to 4% of tasks eliciting this activity. Apart from these similarities, the other activities are featured very differently in the textbooks. *EdM* has the largest occurrence of encoding (11%) and de-interpreting (11%), *LS* by far has the largest amount de-abstracting (24%), and *ml* places a stronger emphasis on interpreting (12%) compared to the other textbooks.

Dimensions of differences between the textbooks

A comparison between the textbooks shows the very different learning opportunities for statistical literacy provided by the textbooks. A first dimension of the differences concerns the *technical nature of the tasks*. The activities of abstracting and de-abstracting in the textbooks are often elicited in tasks where a diagram has to be drawn based on given data (abstracting), or where possible data values have to be reconstructed given a measure of centre (de-abstracting). Both activities require no contextual knowledge, and can be solved in a very technical way. In *LS,* these two activities combined make up for 83% of the tasks. This technical nature could be the result of the exercise-driven approach of the textbook – although *EdM* also emphasizes exercises, and seems to provide more learning opportunities for more conceptually demanding tasks. Another dimension of the differences could be provided by the different treatment of the activities of interpreting and de-interpreting. *EdM* and *ml* show a very different emphasis on these activities, with *EdM* favoring de-interpreting (11%) over interpreting (7%), and *ml* favouring interpreting (12%) over de-interpreting (7%). This could be explained by the different approaches of the textbooks. *ml* claims to follow an exploratory approach. This resonates with the activity of interpreting, in which conclusions about a phenomenon have to be drawn actively based on a model. In contrast, the activity of de-interpreting consists of searching for a fitting model for a given conclusion. As such, the differences between these two textbooks could be considered on a dimension of *exploratory vs. confirmatory data analysis*.

Summary

For all textbooks, the activity of abstracting is by far the activity with the most learning opportunities, as more than half of all tasks elicit this activity. Generally, all textbooks provide more learning opportunities for selective reading than for imaginative reading. This does not mean that the textbooks adequately support the development of selective reading, and do not support the development of imaginative reading. The overemphasis on abstracting also leads to only few learning opportunities for the other activities of selective reading. The results suggest that the textbooks themselves are less suited for developing the whole processes of selective or imaginative reading. The activity of decoding is almost absent from textbooks. However, learning opportunities for other selective or imaginative activities do exist in the textbooks. Where the textbooks do not mostly consist of technical tasks, they provide different learning opportunities for (de-)interpreting depending on an exploratory of confirmatory approach to data analysis.

Conclusion

Statistical literacy is an important skill for all students (Wild, 2017). As statistics is commonly taught within mathematics, statistical literacy needs to be developed in middle school mathematics classrooms. Here, textbooks play an important factor for the enacted curriculum. However, not much is known about the statistical content of mathematics textbooks in middle schools (Weiland, 2019). To uncover the statistical literacy potential in mathematics textbooks, this study provided a specification of the learning content of statistical literacy through different activities concerning the selective and imaginative reading of statistical information (Büscher, in press). Selective reading refers to the process during which a phenomenon gets encoded into data, from which a model is abstracted, which is then interpreted in a statistical argument. During each of these steps, information gets lost. Imaginative reading refers to the process of intuiting this lost information in activities here denominated as de-interpreting, de-abstracting, and de-coding. Three German Grade 6 textbooks were then investigated in a qualitative document analysis study (Bowen, 2009). The results show that all textbooks overemphasize the selective reading activity of abstracting. Learning opportunities for other more conceptually demanding activities of statistical literacy do exist, but risk to be overshadowed by the possibly very technical activities of abstracting and de-abstracting.

These results can highlight the importance of the teacher for developing statistical literacy. Teachers need to use textbooks as didactical instruments to provide and adapt them for their own goals. One the one hand, a skilled teacher could add a small question that elicits interpreting or de-interpreting to a task which itself only requires abstracting. The specification of the activities of statistical literacy might provide such a teacher with a conceptual framework for analysing and adapting their teaching. On the other hand, inexperienced teachers might very well overlook the potentials for developing statistical literacy, and studies have shown that teachers tend to adapt tasks mostly to make them "manageable", instead as to adapt the content of the tasks (van Steenbrugge & Ryve, 2018).

Limitations

The textbooks analysed here provide only a limited window into the development of statistical literacy in middle schools. Even in Germany, there are many regional differences in textbooks, and the selection of textbooks cannot be considered representative even for German schools. It also might well be the case that the potential for statistical literacy is different in the textbooks for grades other than Grade 6. Finally, as outlined above, there exist a difference between the textbook content and the curriculum enacted by teachers. Instead of providing empirical insights into the actual state of teaching, this study shows that learning opportunities for statistical literacy can vary wildly between different textbooks, and that textbooks run the danger of overemphasizing parts of statistical literacy. This study's specification of activities of selective and imaginative reading can provide a framework for researchers and teachers to evaluate learning opportunities for statistical literacy as well as an orientation for adapting their own teaching or didactic materials for fostering statistical literacy.

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How Does Online Professional Development Nurture Teachers' Noticing Skills of Students' Thinking?: The Case of Data Analysis

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This paper presenting one aspect of a design-based research project examines the development of a teacher's noticing skills in the context of data analysis and interpretation during her involvement in an online professional development program (OPD). For this purpose, the noticing skills of the teacher for the strategies used in the given students' responses were identified according to the modified version of the Jacobs et al.'s (2010) framework. By examining the teachers' discussions in the OPD, how the teacher's noticing expertise evolved throughout this process was investigated. Findings show that the teacher had difficulties in each interrelated skill of noticing at first. However, enhancement in her noticing of students' thinking in the context of data analysis was observed throughout her participation in the OPD.

Keywords: Collaboration, data analysis and interpretation, online professional development, teacher noticing

Teacher noticing, which is a key part of teaching expertise (van Es, & Sherin, 2021) and has been a prominent construct in the mathematics education literature for the last 20 years, is generally defined as all the processes that teachers engage to deal with the ongoing information in the classroom (Sherin, Jacobs, & Philipp, 2011). Teacher noticing is essential for efficient instructional practice (Blömeke et al., 2015) and has a direct influence on students' learning (van Es & Sherin, 2021). Although different conceptualizations exist about teacher noticing, researchers generally regard it as including two main processes: paying attention to and making sense of the notable situations in the classroom (Sherin et al., 2011). Jacobs and her colleagues (2010) add one more process to the above description, which is decision-making. Furthermore, they focus particularly on noticing students' mathematical thinking, which is an essential practice to promote students' learning (National Council of Teachers of Mathematics [NCTM], 2014).

Despite the value of teacher noticing, the relevant studies suggest that noticing is not an innate ability and the teaching experience alone does not provide sufficient improvement (Jacobs et al., 2010). Therefore, researchers attempted to develop teachers' noticing skills through variety of ways. For instance, Fernandez, Llinares, and Rojas (2020) showed evidence of enhancement in prospective teachers' noticing skills through sharing narratives about their own practices in an online forum and collaborating with their partners and tutor. Moreover, Fernandez et al. (2020) and several other researchers (e.g. Klein, Fukawa-Connelly, & Silverman, 2017) agree that online systems "slow down the process of learning" (Clay, Silverman, & Fischer, 2012, p. 762) by giving teachers extra time to reflect on the situations, which allows them from different contexts to collaborate around instructional situations. The necessity for such environments has been felt even more during the COVID-19 pandemic. Taking into account the advantages of online environments for teachers' professional development and its particular need during the pandemic, in this paper, we report one aspect of a larger research project which aims to support middle school mathematics teachers' noticing skills of students' mathematical thinking through their involvement in an Online Professional Development (OPD) program.

Although we focus on various mathematical contents in the project, the specific domain of students' mathematical thinking considered in this paper is about analyzing and interpreting data which is the core process of the statistical reasoning (Jones et al., 2004). More specifically, this process includes "recognizing patterns and trends in the data and making inferences and predictions from data." (Jones et al. 2004, p.103). In this research paper, our aim is to examine to what extent a middle school mathematics teacher attends to and interpret students' inferences from a given data and how the teacher base their decisions on students' understandings. Moreover, we are interested in how the teacher's participation in the OPD could support the development of her noticing skills. In other words, the following research question guided the research study: How do a middle school mathematics teacher's noticing of students' thinking in the context of analyzing and interpreting data develop throughout her involvement in an online professional development program?

Alternative to the studies that use online systems for developing teachers' noticing skills (Fernandez, Lilinares, & Valls, 2012; Fernandez et al., 2020), synchronous modes of communication were also used in the present study. Furthermore, in this research, we work with in-service teachers who have less than 15 years of experience. By this way, we have a chance to investigate novice teachers' development of noticing skills regarding students' understanding through collaboration with other colleagues from different schools across Turkey. Lastly, we particularly focus on the content of data analysis and interpretation, which gets relatively less attention in the available literature. Hence, this study may provide valuable information to the teacher noticing literature about the nature and development of middle school mathematics teachers' noticing skills in the context of data analysis.

METHOD

This study is part of a large research project, which adopt design-based research as its methodology (Bakker, 2018). However, in the current study, we limited ourselves to a teacher, Aslı, as our case and examined how her noticing skills developed during her participation in the OPD.

Participants

The teacher Aslı was selected among the project participants who are 35 middle school mathematics teachers whose professional experience did not exceed 15 years and working in public schools in different cities of Turkey. We selected Aslı, who has 8 years of teaching experience, for this study because she was one of the very active teachers in the OPD. She tried to enhance collaborative discussion environment by challenging the other teachers through her questions and making comments to their opinions.

Data collection

The data of the study was collected through the basketball problem, shown in Figure 1 below. This item was adapted by Gökce (2019) from the work of McGatha, Cobb and McClain (2002). In the problem, the scores of two basketball players were given during the last 10 games and the students were asked to decide which player the coach should choose. To decide, students should analyze the given data by using appropriate measures of central tendency and variability, and interpret the results in the given context. Research suggests two strategies that students generally use while answering such comparison questions. Therefore, student responses including those strategies are provided under the problem to examine how teachers notice them. As shown in Figure 1, Student 1 (S1) chose Barış by focusing on the mean score of each player. Her reasoning was partially correct since she ignored the variability of the given data. On the other hand, Student 2 (S2) also chose Barış by only focusing on the scores above a certain value, 15. Since he ignored the data less than 15, reasoning of S2 was not correct.

The coach will select one player from the school's basketball team to play in the all-star tournament. Below is a listing of points scored by the two candidates, which the coach thinks choosing, for the last ten games of the season.

Players	**Points Scored in Games**									
Arda	8	14	11	13	15	12	10	10	9	13
Barış	4	15	8	17	18	9	19	6	18	5

Which player would you recommend the coach to choose? Explain why.

Your students, Student 1 and Student 2, responded as follows:

Student 1 (S1)'s response:

He should choose Barış because while Arda's average is 11.5, Barış's average is 11.9.

Student 2 (S2)'s response:

He should choose Barış because he scored 15 or more in half of the games that he played.

Figure 1. The basketball problem

To determine the teachers' noticing skills before the discussions held in the OPD, teachers were asked to answer three questions suggested by Jacobs et al. (2010) based on the given cases. Subsequently, the basketball problem was discussed in the OPD with all teachers and research team during 3 weeks period. To summarize the discussion in the OPD, a synchronous session was conducted by the facilitator, a member of the design team and researcher who has many research studies on statistical thinking. In that session, teachers were expected to ask any questions they have in their mind regarding the problem and state their opinions regarding the online asynchronous discussion held among the teachers. Moreover, the facilitator mentioned some big ideas, such as the importance of the context in interpreting the given data, that did not get enough attention in the discussion. Lastly, semi-structured interviews were conducted to get in-depth information regarding the change in teacher's noticing skills after asynchronous and synchronous discussions. Teacher Aslı's written answers about two students' strategies, participating teachers' comments in the asynchronous discussion held in the OPD and summary session, and Aslı's interview transcripts were constituted the data for this study.

Data Analysis

For the analysis of the data, the theoretical framework by Jacobs et al. (2010) was modified by taking into consideration the data collected and the Teacher Recognition Skills framework developed by Kilic and Dogan (2021). Accordingly, while teachers' attending skills was examined at four levels, the skills of interpreting students' understanding and deciding how to respond were examined at five levels. In particular, teachers' attending skills, the ability of teachers to identify the mathematical details in students' strategies, were categorized as No Attempt, Lack, Limited and Robust Evidence as different from Jacobs et al.'s (2010) framework. For interpreting skill, identification of students' understanding based on their strategies, one more category, Substantial Evidence was added to the categories of attention. In relation to the last skill, the teachers' ability to base their decisions on students' understandings, were not evaluated as lack, limited and robust. Instead, categories named as No Attempt, Ignorance, Questioning, Challenging and lastly Responding to child and Incorporating were used based on the data. Based on this framework, we firstly determined the noticing skills of teacher Aslı. Then, by examining the teachers' discussions in the OPD and the interview of Aslı, we tried to find evidence in the development of her noticing skills and how those developments occurred.

Findings

In this paper we aimed to examine the development of a teacher's noticing skills during her involvement in the OPD program. Below, teacher's responses and the changes in her ideas during and after the discussions in the OPD will be provided for each noticing skill, respectively.

The teacher's evolving expertise in attending and interpreting skill

The below response presents teacher's attention to the given students' strategies before the discussion in the OPD:

Aslı: For S1, mean is the best way while comparing two groups. For S2, if the range is too much among the given data, it is correct to use median and the data which are above the median. We can say that statistical reasoning of S2 was correct if we just evaluated the player Barış. However, if the range is not too much while comparing two groups, as in the two groups in this question, comparison using mean will be a better approach just like S1 did. (Pre-test)

In this response, teacher attended that S1 used mean while comparing two groups, but she could not notice that S1 did not consider the variability while interpreting the given data. In other words, since she could not attend all the details in S1's strategy, she evaluated her reasoning inaccurately. The teacher managed to attend S2's reasoning as not correct; however, she focused on a concept, median, which was not used by the student. For these reasons, the level of teacher attention to the given student strategies was coded as lack.

Aslı began the discussion in the OPD with the same arguments in the pre-test . Following her comments in system, the below discussion held among some teachers.

T1: …I believe that S2 chose Barış by considering the mode of the given data. Even though the answer of S2 is correct, I do not think that his reasoning is true.

Aslı: If S2 used the mode of the given data, should not he consider 18 [the mode of the data for Barış]? If S2 were to use mode, did not he say that 18 repeated much in the given situation; thus, I choose Barış? I thought that since the sixth data is 15 when all of them are ordered, S2 focused on the median of the given data.

T1: I stated that S2 used the mode by grouping all the data, which are above 15. I do not think that we can take 15 as median since there are 10 games in total.

T3: Differently from above discussion, I think that neither S1 nor S2 reasoned correctly about this question. S1's reasoning is missing since she does not know that it is not enough to just look at one central tendency when the averages are so close to each other. In here, she should look at the ranges of the given data additionally. I could not understand which statistical concept S2 used with his statement of '15 and more points'. I do not think that he used the mode or median as mentioned in above comments.

Since teachers could not agree on evaluating S2's strategy, facilitator posed some questions to let teachers think and interpret the meaning of mode and median for the given data sets. After a long discussion in the system, teachers agreed that S2 did not use the concepts of mode or median, but simpler reasoning. In other words, they agreed that he only concentrated on the highest points in the given data. The following comment of Aslı in the discussion indicated that teacher's expertise in attending S2's strategy improved after this fruitful discussion.

Aslı: I believe that to organize data by grouping (She means to group the data with respect to highest and lowest scores) is correct to understand the nature/tendency of the data. However, it is missing. This is also valid for S2. While thinking the scores which are above 15, the data which are below 15 were lost. (Discussion)

When the data was analyzed in terms of interpreting skills, again her response indicated the lack of evidence in interpreting the students' mathematical understanding. The following response was received before the discussion in the OPD when Aslı was asked to interpret the students' mathematical understanding by considering the given students' strategies:

Aslı: For S1, the mean; that is, all values in the data group, is important because it is affected by the change of each data. For S2, on the other hand, if the difference among the data is high, to look at the median and the data above the median is a more accurate way to interpret the data. Because of the skewed distribution of the data , looking at the median is the most useful method for S2. (Pre-test)

Although teacher thought that S1considered the importance of all data, S1's solution is not enough to make this claim. For S2's strategy, the teacher again focused on the concept of median as she attended in the first question. She argued that S2 knew to use the median when the data set is skewed; which indeed cannot be concluded from the student's strategy. Furthermore, the teacher did not mention variability or representativeness while interpreting the students' mathematical understanding.

When teachers were asked to interpret students' mathematical understanding in the given student responses in the OPD, Aslı's comment was as follows:

Aslı: S1 used only the mean when comparing the data groups, but did not include the concept of range, she should interpret these two concepts together. In S2's interpretation, grouping (She means to group the data with respect to highest and lowest scores) can be used to organize the data, but it is insufficient on its own for data analysis. It is seen that S2 has no knowledge of the concepts of mean, median, and range when looking at the solution. I think that the starting points of both students' reasoning are correct, but they are insufficient. To interpret the data with a single concept will not be sufficient in data analysis. (Discussion)

Most of the other teachers agreed with Aslı's comment in the following parts of the discussion. It was observed that Aslı mentioned the concepts of mean and range, which are necessary to interpret the mathematical understanding of the S1's strategy. For S2, teacher gave up her idea of median. Since teachers did not focus on the meaning that S1 attributed to the mean, facilitator let teachers discuss this through her questions. After this discussion, she interpreted S1's strategy by referring to the concept of representativeness during the interview while she did not mention it in her first comment in the OPD. Lastly, Aslı's following expression regarding her own development gave us one more evidence that discussions nurtured her interpreting skills of students' understanding. "The discussions in the OPD enabled us to brainstorm. It made me realize some new ideas. In particular, I became aware of the concept of range for this question".

The teacher's expertise in deciding skill

Before the discussion in the OPD, Aslı responded her next instructional moves as in the following:

Aslı: To show that both students' reasoning can differ with respect to the different situations, I give two data sets whose ranges are high and low and I ask which student is more successful. For example,

Case 1: Ayşe's savings during five months: 10, 50, 90, 90, 100 (Mean=68)
Case 2: Ali's savings during five months: 70, 80, 85, 85, 100 (Mean=84)

If we look at the mean, we interpret correctly and say that Ali is more successful. I can relate this question with the given problem situation by stating that it is not correct to say that Ayşe is more successful since she has 3 data which are above 90. (Pre-test)

During the interview, Aslı stated that she would use the above example for S2. Since Aslı believed that the use of mean is better for this problem, she asked such a question to S2 so that student can realize that he should use mean while answering this question. In other words, her aim of asking this question is to lead S2 to the teacher's correct answer. In addition, Aslı added some questions to understand both students' strategies better such as "Why did you use 15 in your answer?" or "Why did you use the mean?". For these reasons, the level of her expertise in deciding how to respond based on the given students' strategies were determined as questioning.

On the other hand, Aslı began the asynchronous discussion held in the OPD with the argument below:

Aslı: S1 should feel the need to look at the range while interpreting the data. So, for this purpose, I would ask which players she would choose if the mean of them were equal. Similarly, I would ask S2 which players he would choose if the number of data, which are above 15, were equal. My aim in this part is S1 and S2 should notice that their reasoning is missing and only one method is not enough while analyzing data. (Discussion)

Different from her first comment in the pre-test, Aslı asked some probing questions for S1 to let her notice that she should consider the variability of the given data. In the same way, the questions for S2 were to guide him to consider the data which are below 15. In other words, the teacher started challenging students, a higher level in the framework, to make them realize that their reasoning is underdeveloped. Although Aslı did not provide any explanation regarding how to guide S2 to the use of average in her first comment in the OPD, she offered some ways for this aim after the facilitator's question. During the discussion in the OPD, most of the teachers agreed with the Aslı's first comment. On the other hand, during the interview, when she was asked about her next problems to the students after the basketball problem, she stated that "I again present a problem that students can provide different comments…Indeed, I would change the context. However, the characteristics of the data would be the same so that to calculate just the mean would not be sufficient." With this description, we can say that the teacher thought providing new contexts for students. However, her aim seems to make students practice the ideas discussed in the given problem, not to emphasize the importance of the context, which was discussed by the teachers and facilitator during the synchronous session.

Discussion

The case of Aslı was presented to illustrate that the OPD can help teachers in their development of noticing skills. At the beginning, the teacher had difficulties in each interrelated skill of noticing. However, the findings showed enhancement in her noticing of students' thinking in the context of data analysis throughout her participation in the OPD. During the discussion, teachers had a chance of observing different ideas, posing questions, and making suggestions to each other. Also, the facilitator let teachers focus on the important concepts through her prompts. All these interactions might direct the teacher's focus on students' mathematical thinking deeply as consistent with the study of Fernandez et al. (2020). In other words, it can be inferred that the collaboration among the teachers and the facilitator throughout the discussion might allow the teacher to attend the details of the given strategies and interpret students' understanding by covering all the essential concepts such as variability and representativeness. Moreover, the teachers were provided with two students' strategies and the solution of S2 was particularly challenging and unfamiliar for the teachers. This could have led the discussion environment to be more productive, which in turn might have increased the collaboration among teachers.

Another factor contributing to the development in the teacher's noticing skills could be the asynchronous nature of the OPD. Clay et al. (2012) argues this mode of communication slows down the process of learning, which could enable teachers spent more time to consider and revise their ideas (Fernandez et al., 2012; Klein et al., 2017). Indeed, this was shown by Aslı's one of the statements during the interview: "Discussions in the OPD provided us to think more on the students' answers." Thus, asynchronous modes of communication provided via the OPD might play an effective role in developing the teacher's attention to the given task; leading fruitful communication among teachers which in turn enhance her noticing skills.

Although there was clear improvement in attending and interpreting skills of the teacher, deciding how to respond based on students' understandings was more difficult for her. Preceding the discussion, the teacher's questions were like directing students to the correct answer which was also observed by Klein et al. (2017). Even though the teacher suggested asking probing questions to challenge students throughout the discussion, she had difficulty in extending students' understanding to a further point after the discussions. Although some new ideas were presented by the facilitator during the synchronous session such as the use of graphical representations of the data presented in the basketball problem and the importance of context for interpreting data, these ideas were not reflected in the teacher's responses during the interview. Therefore, in future research, we suggest carrying such specific ideas into the OPD to enable teachers create more productive discussion environment for the deciding skill. Lastly, it would be good to examine how the teachers' evolving expertise is reflected into her instructional practices in further research studies.

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Teaching statistics in Hungarian schools: situation analysis and development opportunities

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In Hungary, teaching of statistics has been part of the curriculum for only 20 years. The introduction has been gradual: while the earlier curricula aimed at a low level of statistical literacy, the recent curricular changes have moved to a higher level. Despite initial difficulties, the subject is now popular with students and teachers alike. To maintain this, it is important that teachers are given adequate support to teach statistics at a higher level. In this article, we describe the development and current situation of teaching statistics in Hungary, and the research and development activities that have been taking place in this field. We examine whether teachers' attitudes, skills and interest in statistics (education) are congruent with the development of the statistics curriculum in Hungarian schools.

Keywords: Statistic education, curriculum development, teacher attitudes.

Introduction, theoretical framework

There is a wide range of international literature on the development and teaching statistical literacy. Developing statistical literacy is an important task for schools in our data-driven everyday lives (Watson, 2003a). Secondary schools play an important role and offer many opportunities in this process (Watson, 2003b). Over the last 20 years, there has been a growing demand for education to focus more on statistical literacy, reasoning and thinking (Ben-Zvi & Garfield, 2004). However, it is not unique that statistics education is nevertheless often simply a matter of learning and practising definitions and new procedures, without the need for a deeper understanding of the underlying concept (Batanero & Borovcnik, 2016). The term statistical literacy increasingly carries the connotation that an individual can be goal-oriented in a specific field, rather than simply acquiring basic knowledge (Gal, 2002). In recent years, the critical handling and interpretation of data has also become emphasised in terms of growing up and taking responsibility (Schiller & Engel, 2018). Another important task related to this topic is designing curricula around statistics and the targeted organisation of classroom procedures (Ridgeway, 2015). Examining teachers' skills and attitudes towards teaching statistics in primary level (Chick & Pierce, 2008) and secondary level is a crucial phenomenon for development.

In Hungary, statistics was introduced as a separate subject in secondary education in the 1950s, but only in vocational schools (Tóth, 2006). In secondary grammar schools, the subject of statistics did not appear in the final exam requirements before 2000. However, in the last 20 years, there have been many changes in this field in Hungary. In this essay we present a brief history of the topic, the current situation and possible directions for further development. First, we would like to demonstrate the changes in the curriculum, then we analyse the requirements of school-leaving examination, and finally introduce the results of a teacher survey. In this way, we aim to provide a complex picture of the circumstances of statistics education in Hungary today.

As for the possible way forward, several doctoral research projects have currently been dealing with the teaching of statistics in Hungary, and this is one of the main elements of a four-year research project launched by the Hungarian Academy of Sciences in September 2021. As an essential element of the thorough investigation of sources conducted by the authors of this article, a system for defining levels of statistical literacy has also been developed. The system defines 3 levels, grouped according to statistical content (measures of central tendency and variability, charts). The first level is characterised by the application of basic conceptual knowledge and the ability to perform simple calculations. The second level is characterised by the skill to solve more complex, multi-step tasks, and to compare and contrast them. The third level is characterised by reasoning, argumentation, generalisation and abstraction. Considering that the incorporation of elements of statistical literacy is still at an early stage, 3 levels are sufficient to describe the characteristics currently want to be observed.

Development of teaching statistics in Hungary since 2000

In order to understand the changes in teaching statistics, we will examine different segments of education: the official curricula, the requirements and tasks of the final examination (which have a greater impact on the content of education than the curriculum itself).

Curricula

In Hungarian public education, teaching statistics was not included in the requirements at all in the 20th century. The first time when statistical knowledge appeared within mathematics was in the framework curricula of the National Curriculum, which were prepared in 1995 and came into force in 2000[1]. Since then, the National Curriculum and the framework curricula have undergone several major and minor changes (one major change in 2012, the other in 2020). The 2000 (Sajtóiroda, 2004) and 2012 framework curricula (Oktatási Hivatal, n.d.) had very similar requirements within the subject of statistics. In grades 1–4, the requirements included collecting data, making tables and graphs, reading and interpreting them; interpreting the arithmetic mean of some numbers, introducing the concept of "average", using it to characterise sets of data. In grades 5–8, the pie chart, the mode, the median and the analysis of data sets were also introduced. In addition, in grades 9–12, the ability to measure the standard deviation of data, to know the sampling with or without replacement and to solve problems related to them.

The 2020 framework (Oktatási Hivatal, n.d.) has introduced significant changes in several aspects, mainly in the requirements for secondary schools. In grades 9–10, the interpretation and evaluation

[1] In Hungary, since 2000, the content of teaching and learning in schools has been determined by a three-tier curriculum regulation, as prescribed by law. The top level of regulation is the National Core Curriculum. The second level of content regulation is the framework curricula, which provide both curricular and methodological support. The third, local, level of regulation is the pedagogical programme of the schools and the local curriculum drawn up by the teachers in the schools.

of data, the drawing of simple statistical conclusions, the choice of the appropriate type of graph to represent a specific set of data and to answer a statistical question, the creation and reversal of a pie chart into a bar chart, the recognition and correction of graphical manipulations in graphs are new items. Also new materials in grades 11–12: visualisation of the concept of a representative sample; classification and characterisation of statistical data using quartiles, means and scatter; construction and use of box-plot diagrams; interpretation and evaluation of data, statistical inference; recognition of statistical manipulations in graphics and text.

In addition to the changes in content, another important difference is that the 2020 framework proposes significantly more time (more than 50% more than in the 2012 curriculum) for teaching probability and statistics.

Requirements and tasks of the final exam

One of the characteristics of Hungarian mathematics education is that classroom practice is influenced more by the requirements and tasks of the final exams than by the curriculum itself. In Hungary, a new final exam[2] system was introduced in 2005 (Csapodi, 2016). The requirements for the intermediate level exam were determined by the content of the National Core Curriculum and the basic curriculum, while the requirements for the advanced level exam were more in line with the needs of higher education. Changes in the curriculum are of course always followed by changes in the requirements for the final examination. Knowing that the secondary school curriculum devotes 9% of the available time to probability and statistics, it is interesting to see that the examination requirements of these topics in the examination papers for the baccalaureate are 15%, which is much closer to the 12% timeframe in the new curriculum.

The current requirements for the exam are generally quite simple, falling into level 1 mentioned in the introduction: the ability to draw pie charts and bar graphs and to read data from them; to calculate mean, mode, median and standard deviation. The only requirement that goes perhaps a little beyond these is to be able to compare data sets using the statistical indicators learned.

Due to the changes in the curriculum in 2020 and the increase in the time available for teaching statistics, the requirements for the baccalaureate will also change from 2024. In addition to the previous requirements, the concept of quartiles and box-plot diagrams will be introduced, but more importantly, the requirements include the ability to choose the appropriate diagram for the situation and to argue for the choice, and at advanced level the ability to choose a mean value that well characterises the data set, to argue for the choice, and to evaluate and analyse the statistical indicators obtained and to draw statistical conclusions. These are a very big changes from the more traditional closed-ended tasks in Hungary.

Looking at the sets of tasks over the last 15 years, we see that the tasks on statistics do not go beyond level 1 of statistical literacy (the level system mentioned in point 1), as required: making simple diagrams, reading data from diagrams, calculating means and dispersion coefficients. The evaluation

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[2] By *final exam* we mean an exam that is to be performed by those who leave secondary school at the age of 16 or 18, before entering higher education. This type of exam is also called *matura*, *baccalauréat* or *graduation exam*.

of the results obtained, and the analysis of the data have not been included in the questions so far. In the 50 or so sets of intermediate level questions that have been used since the introduction of the new examination system, there are few that require more complex thinking.

Research on the detailed analysis of final exams (Csapodi, 2016; Csapodi & Koncz, 2016) show that students at both levels are successful in solving statistics problems and, where there is an opportunity, they are willing to choose the corresponding problems[3]. Knowing that statistics is a relatively new subject in schools, there are two possible explanations for this. It is possible that the statistical literacy of teachers (see the results of the teacher survey later) and the available teaching materials have improved as well. On the other hand, we have also seen that, out of understandable caution, the requirements for these subjects only ask candidates to demonstrate basic knowledge. One of the consequences of this is that the examination requirements for these subjects from 2024 onwards will require a greater amount of knowledge and more complex competences from candidates than in the past, albeit to a lesser extent.

RQ 1. Whether or not teachers' attitudes, skills, and interest in statistics (education) are congruent with the development in the curriculum of statistics at Hungarian schools?

RQ 2. What is the attitude of Hungarian math teachers to further expansion of the statistical curriculum?

Methodology

Designing the teacher questionnaire survey

In connection with the current changes, we have conducted a survey for research purposes, involving secondary school teachers. As a research method, a questionnaire survey was chosen due to the uncertainties of the epidemic situation. This quantitative research method is easy to conduct online, and the results can also be evaluated in this form. The aim of the survey was to obtain information on the studies, attitudes, and classroom habits of teachers in the field of teaching descriptive statistics. Taking these details into consideration, further education and development opportunities should become more targeted, both in terms of teacher education and teacher training.

The questionnaire consisted of four main parts: background data on the respondents; rating of statements on a 5-point Likert scale in 2 groups of questions: (1) How much do they agree with the given statements about the teaching of statistics? (2) To what extent do they feel that the given statements about the teaching of statistics are true for them? The fourth part of the questionnaire measured the popularity of teaching particular mathematics areas. A total of 28 statements were assessed in parts 2 and 3. The main statements: 7 + 6 + 7 statements were included for basic attitudes, assessment of students' knowledge, and framework knowledge. The sub-statements: 3 + 3 + 2 statements are listed in order of opinion on the working methods used, about the assessment and the current requirements. The questionnaire was tested on a small sample, using convenience sampling.

[3] It is not necessary to solve all the problems in the exam to get the maximum score, both at intermediate and at advanced level students can skip one of the exercises.

Results of a teacher survey

The questionnaire for an online survey was completed by 181 teachers. Although the survey is not representative, we can say in terms of background data, that the composition of the respondents is similar in terms of both settlement type and school type to the proportions measurable in the society of Hungarian teachers. As far as professional background is concerned, more than 80% of respondents have been teaching at secondary grammar schools for more than 15 years. Nearly 30% indicated their own secondary school studies as the source of their knowledge needed to teach statistics, 56% chose their own university studies, 76% chose 'their own teaching experience', and almost 60% mentioned their 'own research'. The following table summarizes the proportion of groups with different professional experience in details (measured in years) that indicated their own secondary education as a source, their university studies and their teaching experience gained during their work. (Several sources could be marked, not just their main one.)

Table 1: Distribution of teachers by teaching experience and source of statistical knowledge

Years in public education	Number of teachers	Secondary school studies	University studies	Teaching experience
More than 25 years	65	8 (12%)	38 (59%)	58 (89%)
15–24 years	78	12 (15%)	38 (49%)	61 (78%)
5–14 years	24	13 (54%)	15 (63%)	18 (75%)
Less than 5 years	14	14 (100%)	11 (79%)	8 (57%)

Given the detailed data, just a small proportion of teachers who have been teaching for more than 15 years have high school statistical studies and also in these age groups there is the lowest indication of university statistical qualification. In the case of those, who have been teaching less than 15 years, the presence of high school statistics education can be well traced. The teaching experience is also expected to decrease as the number of teaching years decreases.

During the examination of the sources used to teach statistics, we found that, that 78% of respondents use the available textbooks and 91% use available exercise books and 92% work with statistical tasks in the graduation task series of previous years. Thus, we can say that the need for purposeful preparation for output requirements is very strong, and that it is substantial to examine the available textbooks and exercise books when inspecting the teaching of statistics thoroughly.

Based on the evaluation of the 12 statements of the second part, we can say that the majority of teachers (60%) fully or largely agree that the students' knowledge meets the current requirements, their fulfilment does not mean any difficulties. The statistical questions of the final leaving exam are not considered difficult, however, they do not support raising the level of questions, as well as raising the number of lessons devoted to the topic and expanding the curriculum in this field. On the other hand, there is a great deal of interest and willingness to participate in in-service teacher training on statistics (74.6%).

The 16 statements in the third part included, for example, statements about the framework curriculum. It was found that only 50% of the respondents stated that they were aware of the changes in the framework curriculum, on the other hand, as a result of all this, there is a very high degree of indifference – more than 57% – to the changes in the framework curriculum. The feedback on classroom habits is more varied. The new elements – reasoning, data evaluation (S2, S3), analysis (S13, S14), statistical conclusions (S4) – were not welcomed, although, according to the answers, the graph shows that the new methodological expectations have become part of the teachers' daily work.

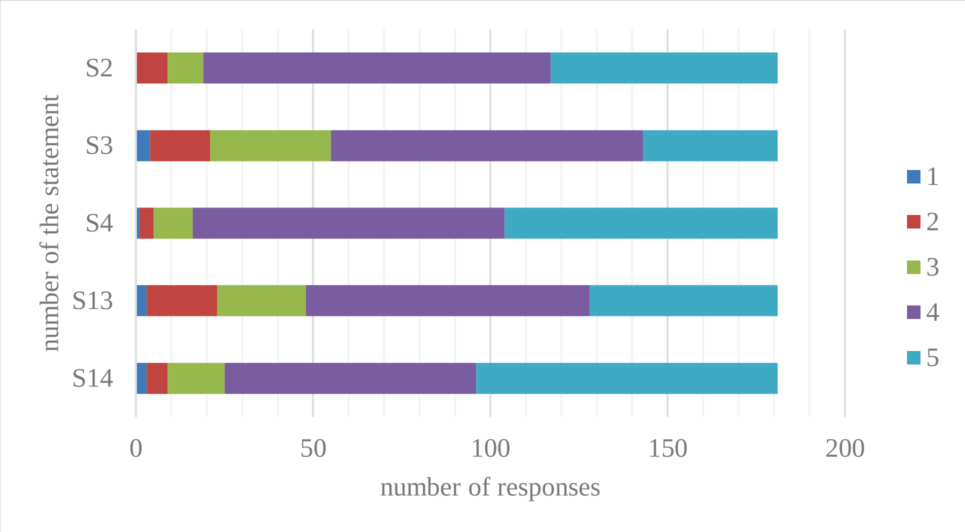

Figure 1: Grading statements about the appearance of interpretation and evaluation in teachers' own classroom work

In written school exercises and tests, open items that require justification or reasoning are not used and are not welcomed.

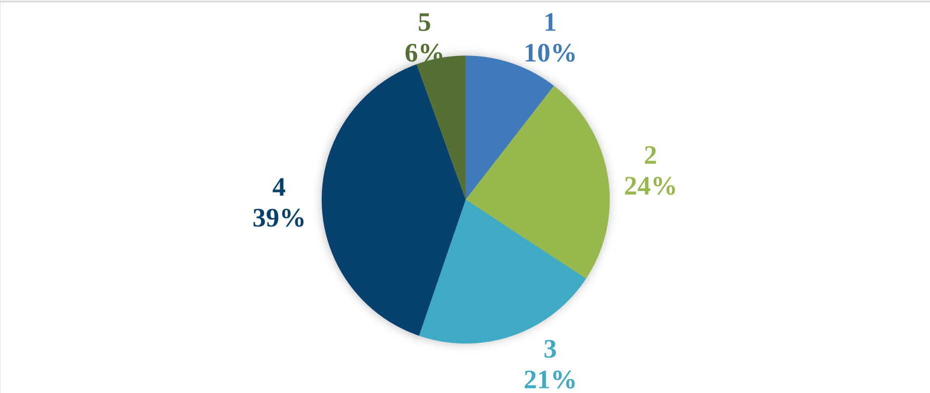

Figure 2: Integrating analysis and evaluation into tests with statistical content

In the fourth, concluding part of the questionnaire, the respondents rated how willing they were to teach certain areas of mathematics. (1 = very much dislikes, 2 = dislikes, 3 = indifferent, 4 = likes, 5 = very much likes). According to this ranking, it seems that the classical topics with a long tradition (equations, algebra, sets, functions, etc.) are the most popular whereas, for example, logic and probability only achieved lower scores, statistics and graphs were in the lower midfield.

Conclusion

Based on the responses to the 28 statements in the questionnaire, we can therefore say that teachers are generally satisfied with the current statistical requirements, the number of the proposed lessons and the achievement of their students (RQ1). However, the direction of stepping forward is less popular, both in terms of increasing the number of lessons in the field, expanding the content, and changes in final exam requirements (RQ2). Preparation for the exam requirements is still highly emphasised. To change the basic approach, it is necessary to incorporate new expectations and methodological issues in both teacher education and teacher training in statistics education; and to support colleagues' work by creating a novel, free-to-use task collection.

Summary and outlook

Regarding to the situation of descriptive statistics education in Hungary, based on the above, we can state that many positive changes have taken place in this field since 2000: basic knowledge of statistics has been included in the curriculum and among the final exam requirements; based on the research, statistics have become a popular topic for students, they successfully solve these types of tasks, for example on the final exam; the proportion of this topic has increased during the latest curriculum developments; there has been a shift in the curriculum and in the graduation requirements from the simple return of conceptual knowledge towards application, interpretation and analysis; basically teachers like teaching descriptive statistics. At the same time, it is evident in addition to positive changes, that there are areas where further development may be substantial: it is worthwhile (in line with international practice) to add additional elements to the statistical part of the mathematics curriculum, for example, the possibility of regression and correlation calculation, as well as the possibility of hypothesis testing in the curriculum should be examined; the statistical part of textbooks and exercise books should be expanded and enriched, the authors of this article play an active role in this work; to deepen the knowledge of teachers, to help their methodological preparation, to consider important to expand statistical knowledge, to prefer to deal with the teaching of statistics.

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How does noise affect our health? Analysing a project-based activity in Statistics at secondary level

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Teaching statistics through projects brings to life in the classroom aspects of working with data that are not usually part of traditional paper-and-pencil activities. There exists a generalized agreement about the positive effects of this kind of teaching proposals and number of international and local initiatives to foster them at secondary and tertiary level. Despite all this, their sustainability as normalized classroom activities is very fragile, given in particular the extra amount of work required to teachers willing to implement them. Research in didactics can help better understand the conditions that enable the introduction and management of statistics projects in the classroom, as well as the – sometimes invisible – constraints that hinder their development. We illustrate it with a case study based on the implementation of a statistics project in a classroom of Grade 9 students, using recent developments of the anthropological theory of the didactic.

Keywords: Statistics education, project-based teaching, noise pollution, study and research paths, anthropological theory of the didactic.

Introduction: Statistics project competitions in Catalonia and Spain

Despite the influence of recent didactics research in the Spanish intended curricula at secondary level, one can still find in textbooks “some inertias, inherited from previous curricula and from traditional uses of statistics, with many algorithmic procedures, leaving few spaces for critical thinking, decision making, and analysing solutions within a context. Additionally, the official intended curriculum still has some outdated standards regarding the use of tables, and an excessive emphasis on mathematization of statistical procedures” (Rodríguez-Muñiz et al., 2018, p. 419). Furthermore, the topics of statistics tend to remain the last to be taught at the end of the academic year, and thus the first to suffer reduction due to rescheduling necessities (Angulo et al., 2013).

Since 2006 in Spain and 2009 in Catalonia, university departments of statistics supported by scientific societies launch annual competitions for projects based on surveys and experiments carried out by secondary school students. In the case of Catalonia, about 70 projects are presented every year, with 200 students, 30 teachers and 15 schools (1.5% of the total). The contest includes three categories. These competitions appear as good opportunities to involve teachers and schools in the implementation of inquiry- or project-based teaching proposals, creating an external motivation for students and increasing the visibility of their work. What teachers and students do and how they organise their inquiries is certainly very diverse. According to Markulin et al. (2021), project-based learning in statistics is a broad term that refer to different teaching proposals, ranging from open individual inquiries to strongly guided collective activities.

The teaching proposal we are analysing emerged in this context of statistics competitions. It was implemented by the first author, a mathematics teacher with 16 years of experience. It has evolved during the past five years as the teacher, together with the students and schools, have gained more experience. We focus on the last project implemented in the academic year 2020/2021 with Grade 9 students, which won both the Catalan and the Spanish awards in their category. After describing the experience, we will introduce some analytical models from the anthropological theory of the didactic, especially the notion of "study and research paths" (Chevallard, 2015) to present a critical analysis of the didactic process. The results obtained will help us draw some hypotheses about the difficulties found for the implementation of such proposals and possible ways to overcome them.

A Statistics project in Grade 9 about noise pollution

Five years developing projects to teach statistics

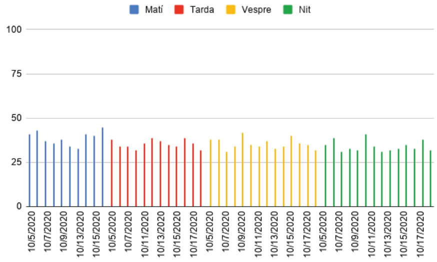
The project we are considering was implemented in a secondary school in Rubí, an industrial town in the outskirts of Barcelona with around 90.000 inhabitants. The idea of teaching statistics through projects in secondary education first appeared six years ago when the teacher had to deal with a group of unmotivated students with low skills in mathematics. To raise interest in the class she designed a survey with a topic of interest for the students, they gathered the information and then perform a simple descriptive statistical study: tables of frequency, graphics and centralization and deviation measures. The following years, the implementation of the projects went on, and the way of teaching statistics was improved through them. One fact that fostered this improvement was the contest "Incubator/Seedbed of surveys and experiments". In addition, the attendance to the awards ceremony, where winner projects are presented, also helped widen the teacher's perspectives. This led to last year's project "The noise and how it affects our health", which was run during 24 weekly sessions of 50 minutes with two groups of 30 students and consisted of two phases or parts.

First part: How noisy is our town?

The first phase of the project took place in the first 12 sessions. It consisted of an investigation to know how noisy Rubí is. We can distinguish different steps in the project development. For the first two sessions, the teacher searched previous studies and information about the topic and prepared a presentation about what the noise is, an official noise map of the city published in the Town Hall website, a Google Maps for each student to locate a pin in their home address. Once the class saw that students' neighbourhoods could "cover" a vast enough space, the teacher proposed a survey with questions about the characteristics of the streets that can affect the noise level. She also presented DecibelX, a sound level meter app. Students answered the survey and carried out a descriptive statistical analysis. With all the information, students made their own hypotheses.

The data gathering was done in two steps. With the help of the mobile application DecibelX, students measured the amount of noise outside their houses. Measurements were carried out for two weeks, starting on the 5th of October 2020 and finishing on the 18th of October 2020, and on certain times: from 7am to 8am (morning), from 5pm to 9pm (afternoon), from 9pm to 11pm (evening) and from 11pm to 7 am (night). With every measurement, students filled out a Google Forms with the email address, the number of decibels, the time and a picture of the measurement from the app.

For the statistical analysis, all the information in the Google Form was downloaded to an Excel file. The students, in groups of 4, calculated the mean, standard deviation and coefficient of variation of the measures per each time slot and made graphics (Figure 1).



	Mean	Std Dev	CV
Morning	39,04	3,67	9,41%
Afternoon	35,56	2,41	6,78%
Evening	35,63	2,95	8,29%
Night	34,27	2,97	8,67%

Figure 1: Results of the statistical analysis of one of Rubí's areas

With the analysis, students could contrast the hypotheses previously raised. They discovered, among others, that:

1. Two main parts of the city could be distinguished: the city centre (called "the red zone") and the outskirts (called "the green zone"), where the difference of decibels was notorious.
2. Some items not considered in the hypotheses could be identified; for instance, the proximity to children parks, which affected the amount of noise in the afternoon.
3. The Noise Map provided by the Town Hall had higher levels of noise in some parts of the city centre than those measured by students. As a possible reason it was pointed up that some streets had become pedestrian since the map was published in 2012.

Finally, to show the results of the study, the teacher selected the places where students had gathered enough noise measures (at least 70% of the number agreed in advance) and uploaded the results in a digital map provided by the Cartographic Institute of Catalonia, called Instamaps (Figure 2).

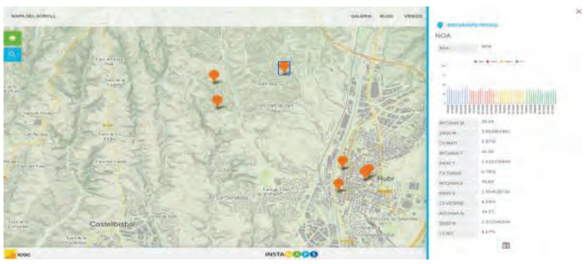


Figure 2: Instamaps of Rubí (www.instamaps.cat)[1]

Second part: How does noise affect our health?

The second part of the project followed the same structure as the first one and took place during the next 12 sessions. However, in this case, the previous study was carried out entirely by the students. In groups, they looked for information about how the noise affects our health and summarised it in an infographic made with Canva, which they shared with the rest of the students making an oral exposition in class. They then made their own hypotheses considering the previous research and gathered data in two steps. First, students, always in groups, proposed the type of questions they

[1] https://www.instamaps.cat/visor.html?businessid=b099e4b7093f76d5bf574d1e26dc4893&3D=false#14/41.5007/2.0225

should ask to the inhabitants of the city to study if the noise affected their health. Questions were shared in a padlet, commented in class to select the most suitable ones. Students administrated the survey in the two different parts of the city identified in the previous phase of the project: the city centre or red zone and the outskirts or the green zone. A total of 435 answers were collected.

Again, for the statistical analysis, all the information in the Google Form was downloaded to an Excel file and the students, in groups of 4, made a descriptive statistical analysis (Figure 3).

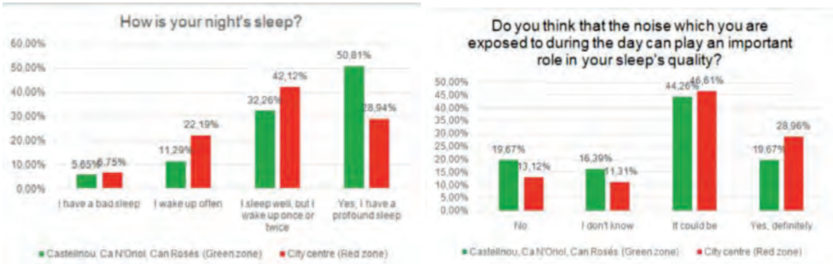


Figure 3. Examples of figures summarising two answers from the survey

As final results, students found that: most of the people were aware that the exposition to high amounts of noise for a long period of time could cause health problems; there was a difference between the people in the two zones in terms of their sleeping quality; and people suffered more from anxiety and stress than from physical health problems. At the end of the project, students made some general proposals to improve the noise pollution in the town, like installing porous paving which absorbs a high quantity of the acoustic wave, increasing the amount of pedestrian zones, organising workshops at schools, and using the local media (radio, magazines, newspapers) to make people aware of noise pollution and give advice on how to reduce the noise.

Analysing a teaching experience from a research perspective

Study and research paths as descriptive and analytical models

Researchers working in the Anthropological Theory of the Didactic (ATD, Chevallard, 2015) have been developing a methodology of analysis for teaching experiences that can be ranged into the category of inquiry-, problem- or project-based learning and are conceptualised as *study and research paths* (SRPs). These instructional practices correspond to the pedagogical *paradigm of questioning the world*, where one studies open questions and develops knowledge – with other kind of tools – to provide answers to them. Some difficulties found in implementing projects can be explained by the prevalence, at school, of the *paradigm of visiting works*, in which curricula are first proposed in terms of organisations of knowledge to study (or "visit") and problems, projects or investigations tend to be subordinate to them: one approaches a question – or carries out a project – to encounter some specific knowledge organisation.

As shown by Bosch (2018), SRPs are not only instructional proposals designed and implemented within the ATD, but they can also be used as *models* in the scientific meaning of the term: not examples to follow, but conceptual constructions to better analyse empirical experiences. The main research question motivating these analyses is to better know the conditions that can foster the implementation of project-based teaching and the barriers or constraints that hinder it. The set of conditions and constraints of all kinds (curricula and pedagogical resources, classroom management, school organisation, society determinants, etc.) correspond to what we call the *ecology* of teaching and learning processes.

The study we present here is an example of the use of SRPs as a descriptive and analytical tool applied to an empirical teaching process organised in terms of project-based learning that was implemented by a teacher without any connection to didactics research, let alone the ATD. In fact, the teacher decided to teach statistics through projects without relying on any specific PBL perspective. Our purpose is to show some key elements of the teaching process – now interpreted as an SRP – that help to better understand its ecology. In the next sections, we introduce at the same time the notion of SRP and the analysis of the teaching process only for the first part of the project.

The Herbartian schema

In an SRP, the generating question is the main purpose of the inquiry. Students, guided by the teacher, address them by proposing new derived questions, searching data and pieces of information potentially useful, and studying them. The results of the study are then contrasted and validated with the data and old information available, which usually produces new interrogations calling for new data and information, etc. At the end, what matters is to be able to provide an acceptable answer to the initial question and disseminate it. The Herbartian schema proposed by Chevallard (2011) identifies some key elements of an SRP. The reduced form of the schema $S(X; Y; Q) \hookrightarrow R^{\heartsuit}$ indicates a didactic system S where a group of students X with the help of a group of teachers Y address a question Q to provide their own answer $R^{\heartsuit}$. The developed form of the schema $[S(X; Y; Q) \rightarrowtail M] \hookrightarrow R^{\heartsuit}$ includes a *milieu* M with all the resources used by $S(X, Y, Q)$ during the inquiry: questions Q_i derived from Q, external answers or works $A_j^{\lozenge}$ elaborated by others that seem useful to address Q, empirical data D_k and other pieces of knowledge, virtual and material objects O_m :

$$[S(X; Y; Q) \rightarrowtail \{Q_i, A_j^{\lozenge}, D_k, O_m\}] \hookrightarrow R^{\heartsuit}.$$

Some commonalities have been identified in the teaching experience:

- The starting point is an initial question, called the "generating question", the teacher raises in a global way, to the whole group of students.
- The project is a collective work, that is, the answer to the question is a joint report from the whole class, which will be submitted to the contest. Despite some of the work was organized in small teams, students share their results and the new questions or directions to follow.
- The work is guided through partial questions, with some given or requested answers.
- Several supports and tools were used: interactive maps, mobile applications, digital campus, etc.

Using the Herbartian schema, we have a single teacher $Y=\{y\}$, a whole class X consisting of students x_i sometimes organised in teams X_j. The generating question Q was divided by the teacher into two sub-questions: how noisy the city is ($Q1$) and how noise affects our health ($Q2$). Along the experience other questions, proposed by X or y, appeared and they were related to institutional answers $A_j^{\lozenge}$, (like the definition and measure of noise, or the statistical notions to determine a sample, organise data, summarise and visualise it, etc.). Material and digital tools were mobilised: Excel, DecibelX, Instamaps, Canva, GoogleForms, etc., together with the students' familiarity and knowledge about the town and what produces noise. Finally, an important element of the project is the data collected D_k, linked to new knowledge work as its quality, reliability, representativity, etc.

However, an SRP is a dynamic entity where questions generate the need for answers with, in turn, raise new questions in a kind of selfsustained process (Bosch & Winslow, 2015). Next some examples are given, to show the dialectic between questions and answers and its crucial role in the dynamics of the process.

The questions-answers dialectic of the first part of the project

In an SRP many questions are raised, some more explicitly than others. Pieces of answers are provided to these questions, by the teacher or the students, using available resources or producing them by themselves. Space limitation only permits us to present a short account of the questions and answers that appeared during the first part of the project, those more implicit are indicated into brackets. We will next use this summary to analyse the teacher's and students' role in the inquiry.

Q1 Is Rubí very noisy?

Q1.1 What is noise and how to measure it?

→ A1.1 Teacher's presentation about noise A1.1.2 Introduction of the app DecibelX

Q1.2 [What do we know about Rubí's noise?]

→ A1.2 Official noise map of Rubí in 2012

Q1.3 Is the map still valid today? Can we reproduce it partially?

  Q1.3.1 Where do we live in Rubí?

  → A1.3.1. We are distributed in different zones of the town

  Q1.3.2 What are the characteristics of the surroundings of where we live?

  →A1.3.2 Google Forms answering some questions

  Q1.3.3 [How to collect data?]

  → A1.3.3 Using DecibelX in each student's zone in different hours, with a survey

  Q1.3.4 How many data are necessary?

  → A1.3.4 Formula and determination of the sample size

  Q1.3.5 How are we going to analyse the data?

  →A1.3.5 With the mean, standard deviation, coefficient of variation and graphics

  Q1.3.6 How many data do we have? How are they? Are there errors?

  →A1.3.6 Data collection check

  Q1.3.7 What do the collected data say?

  → A1.3.7 Data cleansing, numerical and graphical summaries, interpretation

→ A1.3 In some zones, Rubí is less noisy than in 2012 […]

Figure 4 shows the map of questions and answers only for the first phase, using the above notation.

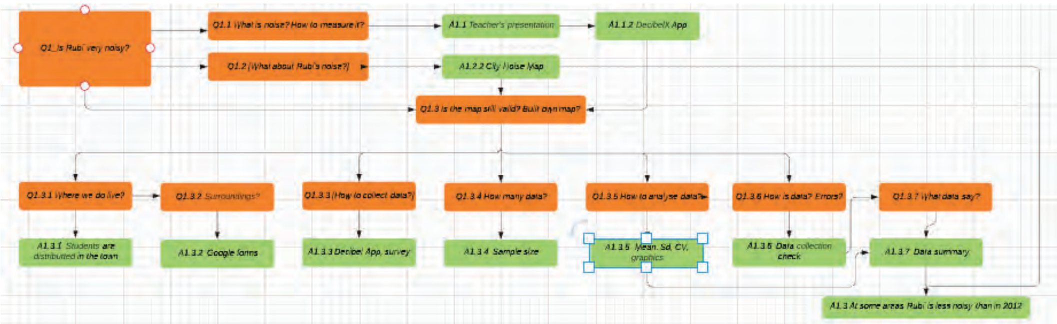


Figure 4: Map of questions and answers of the first part of the project

Some of the questions were proposed directly by the teacher, such as $Q1.1$ and $Q1.2$; few of them by the students. Some questions were answered using established works introduced by the teacher (the study dimension of SRP), while others need ad-hoc elaborations based on students' hypotheses, data compilation and exploitation (research dimension). The sharing of responsibilities between teacher and students was analysed. Table 1 summarises the involved elements (what), the class organisation (how) and the persons assuming the main active role (who) for the first 12 sessions. Remember that y represents the teacher, x_i individual students and X_j a students' team.

Table 1: Elements, methodology and main agent for the first 12 sessions of the project

#	What (elements)	How	Who	#	What	How	Who
1	Q1, Q1.1, A1.1, Q1.2, A1.2	Class	y	7	A1.3.5	Class	y, x_i
2	Q1.3, Q1.3.1, A1.3.1, Q1.3.2	Class + Indiv	y, x_i	8	Q1.3.6, A1.3.6	Class	y, x_i
3	Q1.3.3, A1.3.3, Q1.3.4, A1.3.4	Class	y	9	Q1.3.7, A1.3.7	Indiv	x_i
4	A1.3.2	Class + Teams	y, X_j	10	A1.3.7	Teams	X_j
5	A1.3.2	Teams + Class	X_j	11	A1.3.7	Teams + Class	X_j, y
6	Q1.3.5. A1.3.5	Class	y, x_i	12	A1.3	Teams + Class	X_j, y

The table shows that the teacher acted as main agent with a lot of active participation in most of sessions. However, an important characteristic of this project that is crucial in the paradigm of questioning the world is that all the new information, tools and pieces of knowledge introduced by the teachers were studied and activated by the students because they needed them to answer the question. The new knowledge was at the service of the project, not the other way round.

Concluding remarks

To analyse the constraints hindering the development of this teaching experience, we will consider two specific aspects. On the one hand, the experience was developed in response to a generic proposal, such as the announcement of a contest. The contest invites statistical work in the context of secondary education, as it is a topic included in the curriculum. The initial motivation of the contest is to influence the value given to statistics, especially its usefulness, to motivate students and get them interested in statistics. The generic recommendations and previous examples offered by the organisation of the contest include the case of formulating a question, a hypothesis or to do an experiment from which questions can be formulated. However, no detailed guidelines are given nor a specific PBL approach suggested; it is an open setting, passing the responsibility to the participating teachers, which leaves them alone and requires much more dedication. It also gives them more flexibility and allows more creativity.

On the other hand, although it may be influenced by instructional formats that are in vogue, defined independently of the content, the teacher did not have guidelines to follow. The implementation of projects in class was not a requirement of the school. It appeared as the teacher's response to the demotivation detected in a group of students, with the aim to show the usefulness of statistics. The teacher worked alone to prepare the project questions and answers, provide the tools to be used and guide the students' inquiry. The result was clearly an excessive burden of time that puts sustainability at risk. The analysis of the project through the ATD approach shows some characteristics of the project that could have been organised differently, for instance in the management of the questions-answer dialectic and the sharing of responsibilities between teacher and students: what was explicitly stated, what remained implicit; what was done by the teacher, what by the students and why, etc. Collaborative work between teachers and researchers can help identify the constraints that hinder the long-term sustainability of project-based teaching in statistics and, the most important, implement new conditions to overcome them.

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Infodemiology and COVID-19: Using big data analytics to enhance secondary students' statistical thinking in an artificial intelligence era

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The goal of this article is to illustrate how two Grade 12 Thai students, working together on an activity using Google Trends and COVID-19 open data from the Department of Disease Control of Thailand, engaged in the six phases of the framework for statistical thinking for the era of big data and artificial intelligence developed by González, Isoda and Araya (2020). The obtained results provide empirical evidence that senior high school students can successfully engage in infodemiology, a multidisciplinary field through which it is possible to observe the health-seeking behaviors of people involved in social media and their attitudes toward health and illness. This study documents the feasibility of enhancing secondary students' statistical thinking and literacy through the engagement in big data analytics to monitor new infectious diseases such as the COVID-19.

Keywords: Statistical education, COVID-19 data, statistical thinking, big data transnumeration.

Introduction

The interest in big data has been growing in many areas of today's society, including education. Big data (i.e., data that are too big for standard database software to process; Manyika et al., 2011), represents a paradigm shift in the ways that we understand and study our world, and at the very least, it is seen as a way to better use and creatively analyze information for public and private benefit. In order to make sense of big data, it is necessary to explore large, complex and seemingly unrelated sets of raw data, looking for significant correlations and new and unanticipated connections among them, using artificial intelligence (AI)-driven data analytics platforms, powerful enough to handle data sets with sizes in the order of terabytes, petabytes and zettabytes (Claverie-Berge, 2012; Glaser, 2018). These platforms are IT environments in which raw big data sets can be transnumerated under user request (i.e., converted into a new type of representation, within the platform restrictions, standards, and degrees of access) by AI (Ferguson, 2012). For example, search engines like Google and Google Trends use AI algorithms to analyze vast amounts of text online and determine, as quickly as possible, the most appropriate result for a particular search.

The concept of big data "refers to datasets whose size is beyond the ability of typical database software tools to capture, store, manage, and analyze" (Manyika et al., 2011, p.1). Additionally, big data is often associated with key characteristics that go beyond the question of size, namely the 5 Vs: volume, velocity, variety, veracity and value (Storey & Song, 2017). Big data is dispersed among various platforms that operate with different standards, providers and degrees of access (Ferguson, 2012). For example, a lot of work in big data focuses on Twitter, the blogosphere, and search engine queries. All of these activities are not undertaken equally by the whole population, which raises concerning issues around the question of whose data traces will be analyzed using big data.

Big data is opportunistic data (i.e., data already collected by others, not through personally-planned processes conducted by the user, and hosted somewhere). One way in which big data can be converted into a new type of knowledge is through machine learning, which is an automated process that extracts patterns from data using AI algorithms (Kelleher et al., 2015). This approach to data analysis, the big data analytics, is an iterative and exploratory process of using advanced technologies and analytical techniques on big data to reveal critical information, such as hidden and/or meaningful data patterns, trends and associations, which are data-first answers, and then users will work backward to find the questions that should have been asked (Claverie-Berge, 2012; Cumming et al., 2017; Glaser, 2018).

The big data analytics approach to data analysis is rarely, if ever, dealt with at school level (Wild et al., 2018). Instead, statistical thinking frameworks using a question-then-answer approach (e.g., such the PPDAC cycle), are commonly used when teaching statistics in the school. These question-then-answer approaches focus on data gathered systematically for a purpose, using planned processes, and chosen on statistical grounds (e.g., random sampling) to justify certain types of inferences and conclusions. This is one of the most relevant criticisms to current statistical thinking frameworks, because analyzing opportunistic data requires statistical thinking processes that do not necessarily follow the line of thought sketched by models such as the PPDAC cycle (Wild et al., 2018).

A framework for statistical thinking for the era of big data and AI

Addressing the aforementioned weakness of statistical thinking frameworks using a question-then-answer approach was one of the aims of the "Inclusive Mathematics for Sustainability in a Digital Economy" (InMside) project, supported by the APEC Secretariat[1]. One of the outcomes of the InMside project was a six-phase framework for statistical thinking for the era of big data and AI, describing how a big data user engages in statistical thinking with the support of an AI-driven platform (González et al., 2020). This framework understands statistical thinking as a cognitive process comprised of the following six phases[2].

- Big data quality assessment: Before engaging in creative discovery with big data, users must assess the quality of the big data at hand, in order to establish, among other things, from where the data came, how they were collected, and what instrument or questions were used to collect such data. In other words, this is a step in which the users assess big data's veracity (i.e., the assurance of quality or credibility of the collected data for the intended use).
- Patterns and relationships from data: Iteratively look for trends within big data sets, such as patterns and linear or nonlinear relationships between variables, with the support of an AI-driven data analytics platform (e.g., Google Trends), based on a particular interest.
- Questions: Pose critical and worry questions, in order to find plausible explanations to the patterns and relationships found. These questions are not digging into data searching for specific metrics, as in the traditional data handling approach.
- Objectives: Set objectives related to the posed questions, in order to analyze the data.
- Data mining: In order to achieve the objectives previously set, the user re-examines the data in the light of the objectives, explore old and new data sources, or introduce new variables for

[1] For more information, please refer to https://aimp2.apec.org/sites/PDB/Lists/Proposals/ DispForm.aspx?ID=2247

[2] For detailed information on this framework, the interested reader may consult the article by González et al. (2020).

consideration, using AI, machine learning and statistics.

- Designing: Provide ideas for new searches for trends within big data sets, based on the understanding of the past and present, and design plans and strategies for the future, based on the results from the data mining.

Using big data analytics to engage in infodemiology

Big data analytics allows data users and scientists to engage in a new research field known as infodemiology, which is the study of big data generated during a pandemic or other significant event that may impact public health, using web-based resources, in order to repurpose such data to inform public health and health policymaking (Kurian et al., 2020). During these times of COVID-19, using health-related data in a statistical investigation allow students to put into practice statistical thinking and big data analytics in the context of a real-life problem, as well as to fully experience statistical processes such as acknowledging, modelling of and reasoning about variability, data representation, data-driven argumentation, and informal inference (González et al., 2020; Watson et al., 2018).

Research methodology overview

Participants and data gathering procedure

For this study, a task named “Use Google Trends to tell a data story” was developed by the author, with the support of a Grade 12 mathematics teacher working in a large public high school in Bangkok, who administered the task to her 24 students (9 boys, 15 girls) via Zoom in real time on September 13, 2021, in two 50-minute-long consecutive sessions. This task required students to look for stories told by COVID-19 data using Google Trends (https://trends.google.com). All the participants were already familiar with using Google Trends and Excel, stating the “what” and the “why” while making charts with the application, using the six phases of the framework for statistical thinking for the era of big data and AI (González et al., 2020), and using and interpreting the COVID-19 Dashboard data from the Department of Disease Control of Thailand (https://ddc.moph.go.th/ covid19-dashboard). The students, arranged in pairs, were placed in breakout rooms and engaged in solving the task after being instructed by the teacher. Students were not assisted in any way as to influence their responses. When the task was completed, each group submitted their stories in a Word document to the teacher, including captures of Google Trends charts and any additional graphical representation or calculation made for the activity. A qualitative interpretive micro-analysis was used to analyze the data.

Summary of results: Yai and Nan’s answer

In order to illustrate the potential of this activity to enhance secondary students’ statistical thinking and literacy through the implementation of the six-phase framework for statistical thinking for the era of big data and AI (González et al., 2020), this paper presents, as a case study, a description of the answer given by one pair of students (two girls nicknamed Yai and Nan). This was due to the rich answers that occurred in this pair and the fact that these answers were deemed to be typical of the ones that occurred in the other pairs. The analysis of the collected data will be presented in terms of the six phases of the aforementioned framework for statistical thinking (González et al., 2020).

Big data quality assessment

The veracity of the big data to be used in this task was assessed by the researcher and the classroom teacher, who provided students with data from a trustful source (i.e., the Department of Disease Control of Thailand). Also, students were previously instructed to approach to big data analytics posing worry questions such as “Who created the source?” “What methodology did they follow in collecting the data?” “Did the data creators, or anyone else, summarize, edited or modified the data? Answers to these questions are necessary to assess and determine the veracity and quality of the data.

Patterns and relationships from data

Browsing news and media reports on the Internet, Yai and Nan found the date of the first confirmed case of COVID-19 in Thailand (January 12, 2020) and China (December 8, 2019). Then, they decided to set January 12, 2020 as the “Day Zero”, and created a data set from December 8, 2019, until April 4, 2020, the date in which all commercial international flights were suspended in Thailand, and lockdown measures were implemented in varying degrees throughout the country. When asked why they named January 12 as “Day Zero”, they answered that they learned from the movies that the first person infected by a disease is usually called “Patient Zero”.

Using Google Trends, this group chose three keywords to observe the online health-seeking behaviors of Thai people involved in social media, in relation to COVID-19, over the selected time period. The keywords were the following: flu, fever, coronavirus. Figure 1 shows the evolution of the online search trends for these three keywords in Thailand during the chosen timeframe.

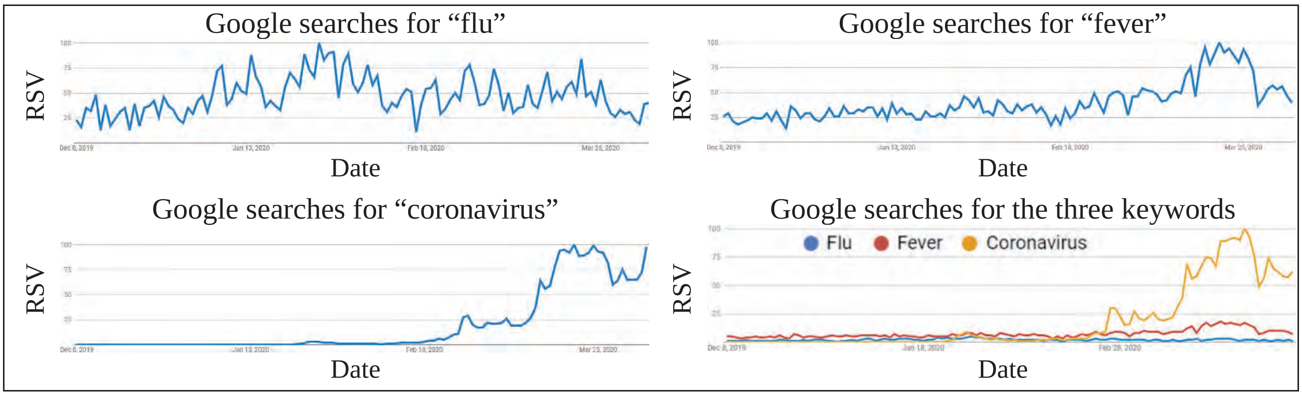

Figure 1: Relative search volumes (RSVs) of the Google queries “flu”, “fever”, “coronavirus”, and their simultaneous comparison, during December 8, 2019 to April 4, 2020 in Thailand

Making a naked eye analysis of the graphs depicted in Figure 1, Yai and Nan mentioned that the general trends of the Google searches for each of the keywords of interest changed after the “Day Zero”. In relation to this idea, they wrote the following:

> Yai and Nan: General trends on each graph show that interest in Thailand for searching “flu” and “fever” in Google spiked in “Day Zero”, but not for “coronavirus”. The Google Trends graphs for “flu”, “fever” and “coronavirus” were nearly parallel until January 26, 2020, when the searches for “coronavirus” started to increase, and more clearly from February 22, 2020, when the Google searches for “fever” and “coronavirus” increased in comparison to the searches for “flu”.

So, in order to write their data story, Yai and Nan explored sets of raw big data using Google Trends looking for patterns and connections, instead of digging into data searching for specific metrics. This approach provided them with data-first answers to questions they had not yet thought to ask.

Questions

Some of the questions posed by Yai and Nan, based on the patterns and relationships they found from their interpretation of Google Trends graphs depicted in Figure 1, are shown below.

- Why did the interest in Thailand for Google searching “coronavirus” started late in January 2020, almost two weeks after “Day Zero”?
- Why did the Google searches for “fever” and “coronavirus” start to show a noticeable increase after February 22, 2020, in comparison with the Google searches for “flu”?
- What happened in Thailand from March 13 to 25, 2020, that there was a common increase in the Google searches for “flu”, “fever” and “coronavirus”?
- Is there a relationship between the volume of Google searches for “flu”, “fever” and “coronavirus” and the number of daily new COVID-19 cases reported in Thailand during that time period?

Then, from the data-first answers generated in the previous phase, Yai and Nan worked backward to find the questions that should have been asked. These questions can lead to the creation of more data representations in order to find explanations to the patterns and relationships previously found.

Objectives

After posing questions to arrive at insights about the patterns and relationships previously found, Yai and Nan, were able to set clear objectives to continue their data analysis, such as the following:

- To identify the reasons why the interest in Thailand for Google searching “coronavirus” started late in January 2020, almost two weeks after “Day Zero”.
- To identify the reasons why Google searches for “fever” and “coronavirus” started to show a noticeable increase after February 22, 2020, in comparison with the Google searches for “flu”.
- To determine what happened in Thailand from March 13 to 25, 2020, that there was a common increase in the Google searches for “flu”, “fever” and “coronavirus”.
- To determine whether there is a relationship between the volume of Google searches for “flu”, “fever” and “coronavirus” and the number of daily new COVID-19 cases reported in Thailand during that time period.

Data mining

In order to address the objectives previously set, Yai and Nan engaged in data mining. Data mining is the process of finding and extracting relevant data, previously unknown but potentially useful, in a heterogeneous data repository, using AI, machine learning and statistics (Cumming et al., 2017). In order to exemplify this phase in Yai and Nan’s work, let us briefly address the objectives stated above.

Any verbal statement can be considered to be a hypothesis. Therefore, in the context of data mining, a hypothesis is an explanation of some phenomenon (e.g., a cause-effect relationship or a trend in data) that can be argued by the means of data analysis. Then, Yai and Nan hypothesized, in relation to the first objective, that the term “coronavirus” was not on most Thai people’s health radars by that time. Yai and Nan confirmed this hypothesis when they found, through Google searches, that the terms “novel coronavirus” and “2019-nCoV” were mentioned by the World Health Organization (WHO) in their first situation report, published in January 21, 2020.

In relation to the second objective, Yai and Nan surfed the Internet to find plausible answers. They concluded the following:

Yai and Nan: It is very likely that the noticeable increase in Google searches for “fever” and “coronavirus” that started in Thailand after February 22, 2020, was due to the news of the 10 towns locked down in Italy, which was released on February 23, 2020, after as the number of reported cases in Italy grew from fewer than five to more than 150. Also, the Ministry of Public Health and the Department of Disease Control of Thailand had a daily broadcast to instruct population about COVID-19, and by that time thermometers were placed in most public spaces, so people were watching their temperature, since people knew that “fever” was one of the most common symptoms of COVID-19.

In relation to the third objective, Yai and Nan concluded, after surfing the Internet, the following:

Yai and Nan: We think that the noticeable spike in Google searches for “flu”, “fever” and “coronavirus” that happened in Thailand from March 13 to 25, 2020, was mainly because of the state of emergency to control COVID-19 declared by the Thai Prime Minister and Cabinet on March 17, to become effective on and from March 26 to April 30, 2020. This measure prohibited entering into Thailand, entering in crowded areas and places at risk of COVID-19 infection, gatherings of many people, and only certain businesses were allowed to open. Also, on March 22, Thailand reported 188 new COVID-19 cases, the largest single-day rise until then, and the first time that more than 100 new cases were reported. Another possible reason could be the announcement of the death by COVID-19 of four Thai nationals on March 24. Before that, only one person had died of the virus in Thailand.

In relation to the last objective, Yai and Nan used the data from the Department of Disease Control of Thailand to obtain the number of daily new COVID-19 cases reported in Thailand from “Day Zero” to April 4, 2020, and plotted them in Excel against the RSVs for each of their Google queries, which they downloaded from Google Trends. The results are depicted in Figure 2.

Pearson correlation coefficients were calculated between each Google query’ RSV and the daily new COVID-19 cases in Thailand. On this matter, Yai and Nan wrote the following:

Yai and Nan: The Excel scatterplots revealed a moderate to strong relationship between the volume of Google searches in Thailand for “fever” and the daily new COVID-19 cases reported from “Day Zero” to April 4 ($r = \sqrt{0.363} = 0.602$). A strong relationship was found between the volume of Google searches in Thailand for “coronavirus” and the daily new COVID-19 cases reported during the chosen period ($r = \sqrt{0.6756} = 0.822$). We interpret these results as evidence that, as the number of COVID-19 cases increased, Thai people’s interest in searching information on “fever” and “coronavirus” also increased. However, the relationship between the volume of Google searches in Thailand for “flu” and the daily new COVID-19 cases reported during the chosen period was negative, because of the decreasing linear trend, and moderate ($r = -\sqrt{0.094} = -0.307$). We think that when more was known about the nature of COVID-19, so agencies and Thai people got to know that COVID-19 wasn’t a flu variety, then the volume of Google searches in Thailand for “flu” wasn’t as large as the volume of Google searches for “coronavirus” or “fever”, one of the most common symptoms of COVID-19.

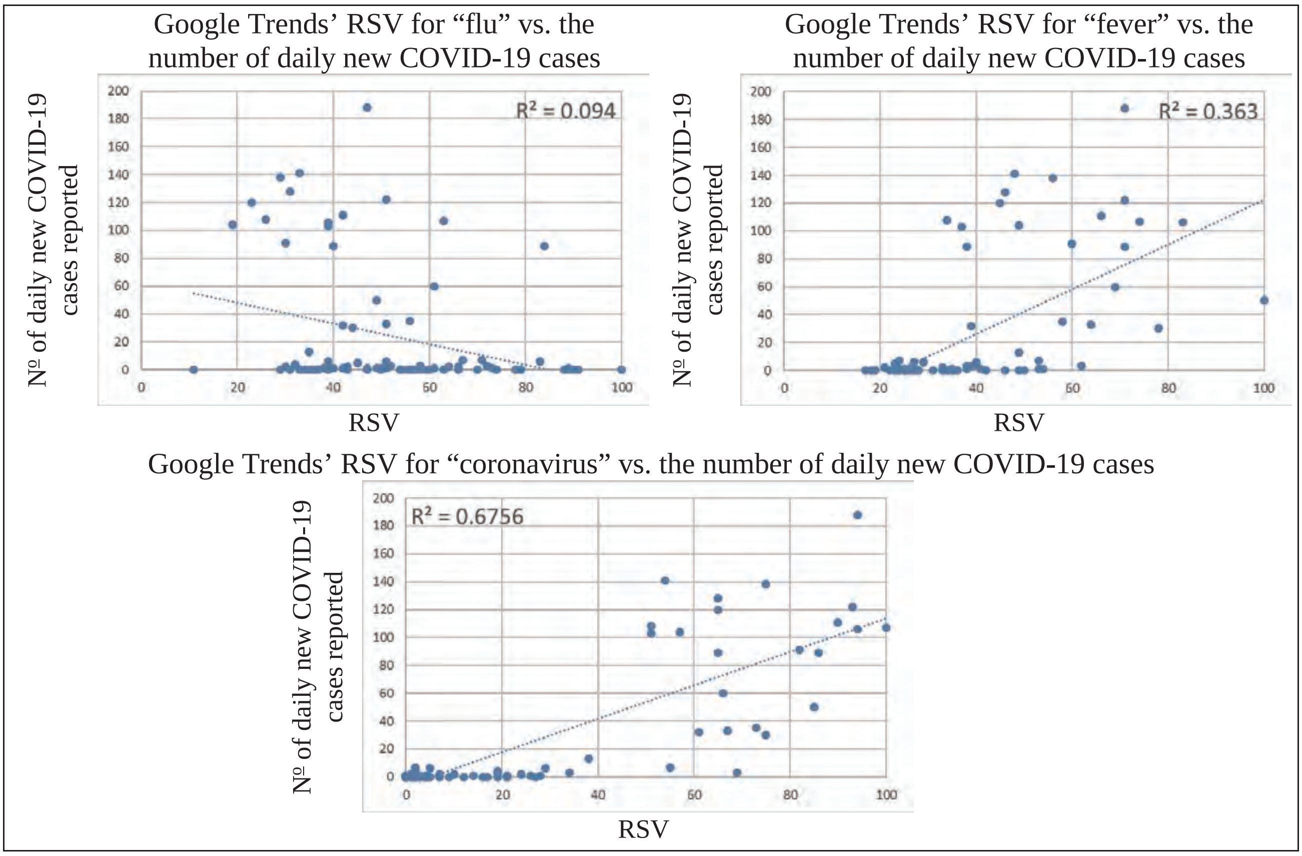

Figure 2: Scatterplots of the daily new COVID-19 cases reported in Thailand from January 12 to April 4, 2020, as a function of the RSVs of the Google queries "flu", "fever", and "coronavirus"

Designing

From all the insight and valuable knowledge Yai and Nan gained through engaging in the data mining phase (e.g., Thai people learning that COVID-19 was not a flu variety), they were able to design plans and strategies for the future, to generate value of some sort. For example, they suggested the creation of health education campaigns about COVID-19 using music and animated cartoons, so Thai people avoid considering COVID-19 as a flu variety, self-medication, and underestimating its symptoms.

Summary and implications

Google Trends is a proven AI-driven tool for evaluating people's information-seeking activities. If Google Trends is used to describe interest on health issues or health-seeking behaviors of people involved in social media, then we are engaging in infodemiology, which is the area of scientific research that focuses on scanning the Internet, publicly available data (e.g., public health and government databases) and other sources for user-contributed health-related data. This study reveals the benefits of engaging secondary students in infodemiology, through big data analytics using Google Trends, for their development of statistical thinking in this era of big data and AI. In fact, while engaged in data mining, Yai and Nan provided answers similar to those by Kurian et al. (2020).

Although this short description is far from exhaustive, the findings reported in this paper have practical implications for the teaching and learning of statistics, suggesting how an activity designed to engage students in infodemiology and big data analytics is particularly suited to support the development of statistical thinking for the era of big data and artificial intelligence through the implementation of the six phases of the big data analytics approach identified by González et al. (2020). Even though the traditional question-then-answer approach can still be used to find answers when we have specific needs, when it comes to finding the most impactful ways to reveal stories from big data, people that let the data provide the questions first, as in the approach described by González et al. (2020), Claverie-Berge (2012) and Glaser (2018), will get the best and more insightful results.

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Teaching data science in school: Digital learning material on predictive text systems

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Data science and especially machine learning issues are currently the subject of lively discussions in society. Many research areas now use machine learning methods, which, especially in combination with increased computer power, has led to major advances in recent years. One example is natural language processing. A large number of technologies and applications that we use every day are based on methods from this area. For example, students encounter these technologies in everyday life through the use of Siri and Alexa but also when chatting with friends they are supported by assistance systems such as predictive text systems that give suggestions for the next word. This proximity to everyday life is used to give students a motivating approach to data science concepts. In this paper we will show how mathematical modeling of data science problems can be addressed with students from tenth grade or higher using digital learning material on predictive text systems.

Keywords: mathematics education, Jupyter Notebooks, natural language processing, mathematical modeling, data science.

Motivation and classification of the research area

The use of mobile devices has increased enormously in recent years. This results in a high demand for fast and reliable input methods. One of these assistance systems for typing is word prediction. It makes suggestions for the next word based on expressions already written to save the time of the user when typing. But how does this assistance system know what the user wants to write next? How can word suggestions be generated in such a way that they suggest the desired word with high probability? Answering these questions is the goal of the interactive workshop.

Word predictions are predictive text systems, which are developed within the scientific field of natural language processing. The main goal of natural language processing is to study, how humans understand and use language in order to develop software programs, that allow computers to simulate human behavior (Chowdhury, 2005, p. 51). Predictive text systems also attempt to mimic the language of the user. Specifically, they involve the prediction of letters, words, or even entire word sequences. However, predictive text systems are not only used in typing, but also in speech recognition, spell correction, machine translation, and handwriting recognition (Ghayoomi & Momtazi, 2009, p. 5233). Predictive text systems are thus “one of the important tasks in most natural language processing applications” (Ghayoomi & Momtazi, 2009, p. 5233) and are encountered by students in their everyday life. This makes it an authentic and relevant topic.

In order to give predictions for a text, predictive text systems work with large text data to extract knowledge from it. The N-gram concept, which is a simple decision tree model, is used in the workshop to bild suitable word suggestions out of the text data (Bahl et al., 1998, p. 1002). This makes it a typical problem in the field of machine learning and data science.

Since data now play a central role in all areas of our lives, it is necessary to give learners an understanding of data science issues and to train them in the critical use of data (Gould, 2021; Opel et al., 2019). Therefore, based on the mathematical and technical knowledge in the modeling of data science questions, digital learning material for high school students should be developed and tested on real problems. The topic of predictive text systems was seen as a particularly suitable example to convey machine learning and data science concepts in an easily accessible way in school lessons. In a design-based research approach, learning material on this topic should be designed, developed and improved in cycles of implementation and redesign with systematic feedback from students.

Background information on the learning material

In the workshop, high school students work out how occurrence frequencies of word sequences can be estimated using large data sets and how they can be used to generate word suggestions. On several worksheets, they collect first ideas for developing a prediction model and then develop different basic models for generating word suggestions. Finally, they apply these models to a large training data set.

The learning material is suitable for students from tenth grade and higher. The workshop assumes prior knowledge of relative and absolute frequencies, as well as prior knowledge of the concept of functions. In addition, students should understand and be able to calculate probabilities and perform and evaluate simple multistage random experiments. Programming skills are not required. The learning material can be used in a compact one- or two-day workshop or divided into several lessons of a teaching unit, both in presence but also in online classes.

A problem-oriented workshop

The developed learning material uses the example of word prediction to show how real-world problems solved using big data can be prepared for students. It approaches the topic in a problem-oriented manner and introduces mathematical content when it is needed to find and understand the solution of the problem. Thus, the problem is the focus and mathematics is experienced as a tool that helps to understand the world, so that the students do not only learn mathematical methods but can recognize the usefulness of mathematics for everyday life. This strategy is part of all workshops that are created within the CAMMP (Computational and mathematical modeling program) project of the Karlsruhe Institute of Technology[1] and the RWTH Aachen University[2]. The overall goal of the project is to promote competencies in mathematical modeling among learners through a variety of learning opportunities and to highlight the importance of mathematical modeling and simulation science for society. CAMMP aims to achieve this goal by providing education and training for teachers in mathematical modeling and by continuously developing and testing new learning material. All workshops created in the project are implemented as digital learning material.

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[1] http://www.scc.kit.edu/forschung/CAMMP

[2] https://blog.rwth-aachen.de/cammp/

Digital learning material

Digital learning material in school context is already a much discussed topic due to the advancing technological development of society and is currently gaining even more importance for schools teaching due to the Covid-19 crisis and the distance learning associated with it. For mathematical modeling, digital learning material has a special significance, as it enables the solution of real-world problems with large amounts of data in school context and can be a useful support for learners, especially in complex reality-based problems (Geefrath & Siller, 2018, p. 9-10).

This learning material is therefore implemented in the form of digital worksheets, so-called Jupyter Notebooks (see Figure 1), which can be accessed via a cloud platform hosted by the Karlsruhe Institute of Technology. The material can thus be edited directly in the web browser. The login process to the platform is described at www.cammp.online/english/214.php. The digital working material contains many different building blocks clearly arranged in a single file. Instructions, formulas, illustrations, but also code fields can stand directly next to each other and facilitate the learners work in the workshop. Digital differentiation material, such as staged help as fold-out text or in the form of a link to a separate file, as well as consolidation tasks, make the learning material very suitable for heterogeneous learning groups. Additional differentiation is provided by information sheets that are linked to the worksheets and can be called up by the learners if required. For example, learners who have no experience in programming are supported by information about for-loops or if-statements. The subdivision of the problem into smaller tasks as well as adaptive, automated feedback on the solutions enable the learners to work through the material very independently. A more detailed description of the form of the learning material can be found in Gerhard et al. (in press).

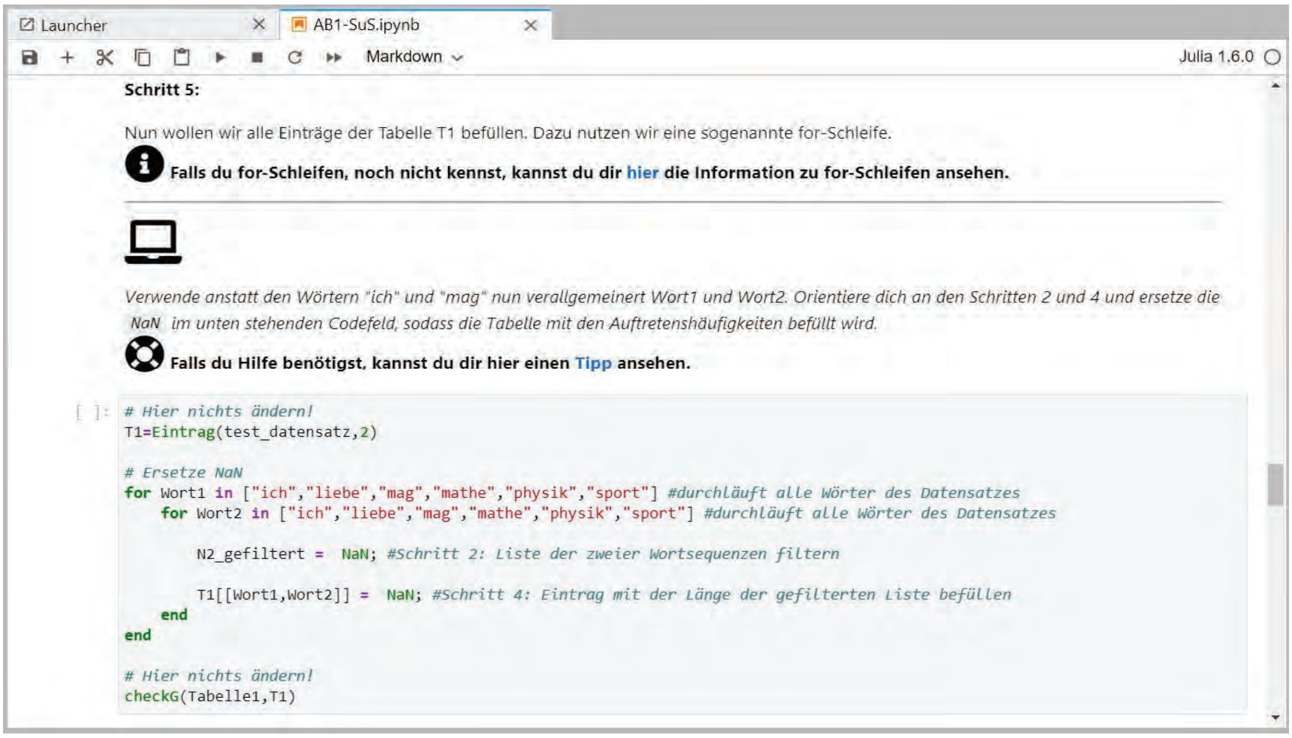


Figure 1: Screenshot of a digital worksheet from the learning material on word predictions

Didactic preliminary considerations

In the development of the learning material, particular attention was paid to the independent implementation of the prediction model by learners. At the end, the solution of the problem should not only be understood theoretically, but also worked out practically. Only minor technical details, which are less relevant for the basic mathematical understanding, are executed in the background.

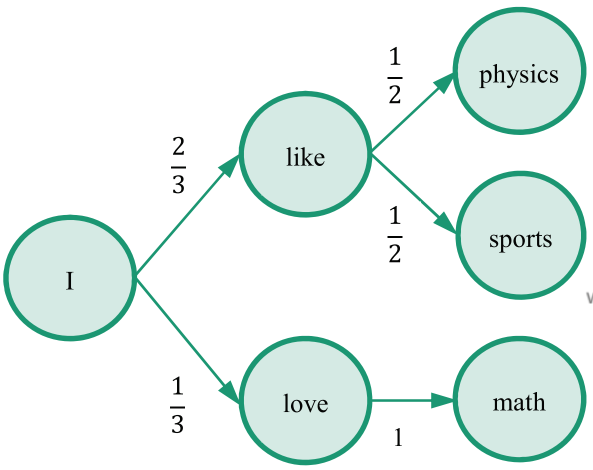
Furthermore, care was taken to avoid unnecessary technical terms, like “Markov chain” or “matrix”. Merely the nth-order Markov assumption is relevant for the workshop. Students will only learn about the applied assumption to the word prediction example. Here it means that the frequency of occurrence of words depends only on the n previous words. In the workshop the assumption thus simplifies the estimation of the occurrence frequency. Also, the term matrix, since it is not known by some learners depending on their previous schooling, is circumvented by paraphrasing it with tables. Since the matrix is only used as a storage form, this can be done without problems in the workshop.

Detailed description of the material

Worksheet 1: A first prediction model

To understand how word predictions are generated, the workshop will first look at a small example. As data basis, which will be used later for comparison, the three sentences "I like physics. I like sports. I love math." will be used. This represents the so-called training data of the prediction model and, in contrast to the training data sets used in practice, which comprise several thousand words, is chosen to be very small in order to be able to clearly illustrate the principle of word prediction. With the help of this data set, the word that the user would most likely write next should be suggested, if possible. It is first assumed that the user has already typed the word "I". Ideas and thoughts about possible suggestions for the next word and how to build them based on the data set, are first collected in a short brainstorming session in plenary. So far, the students always discovered independently that one possible strategy for generating the word suggestions is to compare the already typed word sequence, in our case the word "I", with the data set. In this way, we can determine which word has already been typed in the past after this word sequence and use this information for our prediction.

In our small training data set, the word "I" is found in three places. Now the words which follow can be identified. These are the words "like" and "love". One can therefore assume that these are popular



word2

word1	I	love	like	math	physics	sports
I	0	$\frac{1}{3}$	$\frac{2}{3}$	0	0	0
love	0	0	0	1	0	0
like	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
math	0	0	0	0	0	0
physics	0	0	0	0	0	0
sports	0	0	0	0	0	0

Figure 2: Transition graph (left) and transition table (right) of the bi-gram model

subsequent words after the word "I" and thus also represent meaningful suggestions for the next word. The word "like" follows the word "I" more often than the word "love". The probability of the word "like" occurring after the word "I" should thus be estimated as higher according to our training data. Quantitively, this so-called transition probability of the word "like" occurring after the word "I" $\mathrm{P}(\mathrm{I} \rightarrow \text{like})$ can be estimated via the occurrence frequency of the two-word sequence (I, like) as well as the occurrence frequency of the two-word sequence (I, any word) and results in

$$\mathrm{P}(\mathrm{I} \rightarrow \text{like}) = \frac{\mathrm{N}(\mathrm{I}, \text{like})}{\mathrm{N}(\mathrm{I}, \mathrm{I}) + \mathrm{N}(\mathrm{I}, \text{like}) + \mathrm{N}(\mathrm{I}, \text{love}) + \cdots} = \frac{2}{0 + 2 + 1 + 0 + 0 + 0} = \frac{2}{3}.$$

These two-word sequences are also named bi-grams, which explains, why the corresponding model is called the bi-gram model. In order to display and store the different transition probabilities in a suitable form, the students learn two different possibilities. The visual representation as a transition graph (see Figure 2, left side) serves mainly as a visual support for the learners, while the transition table (see Figure 2, right side) is especially suitable for a larger data set as a storage location for the transition probabilities. The row of the transition table represents the already typed word (word1), the so-called word history, and the column represents the following word (word2). If a word is now typed, the next word with the highest probability can be found in the corresponding row and indicated as a suggestion.

Based on this, learners develop a general strategy for estimating transition probabilities from a training data set using the occurrence frequencies of two-word sequences and automate the process for all possible transitions so that the calculations no longer need to be done by hand. Now the model can be tested on a larger data set. A larger data set is important so that meaningful suggestions for the next word can be made for as many already typed words as possible. As soon as the already typed word does not occur in the training data set, no suggestion can be made using the bi-gram model. The training data set used for this purpose consists of the German-language texts of the corpus "What's up, Switzerland" (Stark et al., 2014-2020) and the texts of the category "Belletristik" of the corpus "LIMAS" (Research group LIMAS, 1970-1971). The training data set contains more than 300,000 words. From this, part of the data is retained for later testing of the model.

The model is first trained with the help of sample texts, the so-called training data set. Subsequently, the correlations detected from these can be used for prediction on an unknown data set. Bi-Gram models therefore use typical machine learning strategies.

Worksheet 2: Uni-gram and tri-gram model

When testing the bi-gram model with different word histories, learners are tasked with identifying various problems of the model and coming up with possible model improvements. Among other weaknesses of the model, learners recognized that the prediction model uses only the last word for word prediction and that the word history needs to be extended to more than one word for more context-based suggestions. To do this, students use the learning material to develop a tri-gram model that takes the last two written words into account to build a suggestion.

Another problem, which learners will identify through examples in the workshop, occurs when the word already typed does not appear in the data set. In this case, there is an option to suggest the words that occur most frequently overall in the training data set. This model is called a uni-gram model, because the occurrence frequencies of one-word sequences are determined instead of two-word sequences, as in the original model. The transition probability is calculated by dividing the occurrence frequency of the single word by the total number of words at

$$\mathrm{P}(\text{word 2}) = \frac{\mathrm{N}(\text{word 2})}{\mathrm{N}(\text{every word})}.$$

In the workshop, learners now collect advantages and disadvantages of the three n-gram models, to finally realize that all models have different strengths and weaknesses and that they need to be combined for the best possible prediction of the next word.

Worksheet 3 and 4: Combined models and model evaluation

The combination of the n-gram models makes it possible to realize the probability estimation more reliably and at the same time context-based. For the combination of the models, the back-off procedure or the interpolation are mainly used in natural language processing (Wendemuth et al., 2004, p. 30-31). The back-off procedure is a fall-back strategy. In the case of an unseen bi-gram as a word history, the tri-gram model does not give any suggestions for the next word. Therefore, the bi-gram model is used. If the last word in the word history also does not appear in the training data set, the uni-gram model is used and the words that appear most frequently in the data set are suggested. The students implement the back-off procedure with the help of a simple if statement. As a result, the students learn different combinations of possible n-gram models, while they are improving their basic programming skills.

However, with the back-off procedure, the model may count on the tri-gram probability, which is often not as reliable. This is because there are much more tri-grams than bi-grams or uni-grams. Therefore, many tri-grams do not appear in the training data set, or appear only very rarely. This means that the tri-gram likelihood is often based on very little data compared to the bi- or uni-gram model. It is therefore always best to use all transition probabilities of the individual models and combine them with a weighted sum to produce an overall transition probability. The new estimator of the transition probability of word2 with word history (word0, word1) is thus given by

$$\widetilde{\mathrm{P}}(\text{word 0}, \text{word 1} \rightarrow \text{word 2}) = \mathrm{g}_1 \cdot \mathrm{P}(\text{word 2}) + \mathrm{g}_2 \cdot \mathrm{P}(\text{word 1} \rightarrow \text{word 2})$$
$$+\mathrm{g}_3 \cdot \mathrm{P}(\text{word 0}, \text{word 1} \rightarrow \text{word 2}).$$

The interpolation weights g_1, g_2 and g_3 are initially determined by the learners in a range selected by logical considerations. Later, the weights can be optimized by minimizing an error measure. In this context, the distinction between training data, which are used to estimate the transition probabilities, and test data, which are used to determine the goodness of the language model, but also the weights of the interpolation, is important. Here, the students have the opportunity to independently develop an optimization procedure for minimizing the error measure as a function of the weights or, guided by another worksheet, to learn about one possible procedure.

At the end of the workshop, the initially qualitative considerations regarding the advantages of combining the n-gram models compared to a single n-gram model can also be confirmed quantitatively by calculating an error measure.

Additional tasks

Another optional task for learners is to use the prediction model, they have developed, to generate whole texts. Learners can choose different prediction models (a combined model or a single smoothed n-gram model) and observe differences in text generation. The higher the n-gram length is chosen, the better the generated text will be in both grammatical and contextual terms. At higher orders, the generation is comparable to copying individual sentence strings from the training data set.

Furthermore, the workshop is followed by some social questions about the benefits and dangers of these assistance systems, which can be debated with the learners in an open exchange. Questions such as "Do assistance systems, like word prediction, influence writing behavior?" or "How do biased word suggestions arise and what are their effects?" are particularly suitable for critical discussion. On the one hand, dealing with these questions serves to dive deeper into the topic. On the other hand, it shows a second perspective on the problem from a socio-scientific point of view and, by critically illuminating the assistance systems in plenary, it contributes to empowering the students to form independent critical judgments. The learning material is therefore suitable for application in interdisciplinary lessons, for example in a seminar or a project week on natural and social sciences.

In addition, another worksheet is in progress, which will give students the opportunity to use different data sets as training data for the prediction model. In this way, the influence of the training data on the language model can be highlighted and thus attention can be drawn to the importance of a suitable training data set.

Experience and summary

Data science in school is desirable and possible. This is demonstrated by the learning material described above, but also by other projects such as ProDaBi (Opel et al., 2019), in which a data science curriculum was developed and tested in high school, or the experience of Narges Norouzi et al. (2020) with learning material in machine learning and natural language of the COSMOS Summer School for high school students.

In this workshop, students learn how knowledge can be generated from data and used to support decisions when typing. In doing so, they use typical machine learning strategies. The learning material has already been conducted in an online workshop with 28 learners and will be tested more frequently and continuously improved in the future. The conduction took place as a two-day event during the summer vacations with students from grades ten to thirteen. Additionally, the workshop was conducted and examined from a didactic point of view with 29 teacher trainees of the RWTH Aachen University in one day as part of a didactic seminar. Especially the independent work of the high school students on the open tasks to optimize the interpolation weights and to generate their own text using the prediction model on the second workshop day shows how quickly the students were able to familiarize themselves with a topic that was new to them. After conducting the online workshop with students from grades 10 to 13, the students were asked to fill out an online questionnaire in order to use the results and past experiences to iteratively improve the workshop. The students named machine learning, mathematical modeling, as well as programming basics, among others, as areas in which they could see an increase in learning after the workshop. The workshop was rated with an average grade of two (good).

During the conduct, the learners contributed interesting ideas to the mathematical model, but also to the socio-critical discussion. The high oral participation of the learners shows how interested they are in understanding and solving the problem and the positive feedback from the learners also speaks in favor of the project: “I found the insight into generating word suggestions very interesting and it was a very good example of how mathematical modeling is applied in everyday life.” (Participant’s answer in the evaluation of an online workshop, personal communication, August 27, 2021).

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Task design to foster informal statistical inference in upper secondary school: focusing on correlation with multivariate data

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Keywords: Informal statistical inference, correlation, multivariate data.

Introduction

Recent statistical education research has focused on informal statistical inference (ISI), without using formal inferential statistical procedures to understand the concept of uncertainty (Makar & Rubin, 2018). Kazak et al. (2021) focused on the relationship between multivariate data and ISI, and identified students' difficulties about the relationship between correlation and uncertainty. On the other hand, there is a few studies related to the task of improving students' understanding with correlation and uncertainty, and promoting ISI. Accordingly, this study discusses the instructional guidance needed to resolve them and the task promoted ISI with multivariate data. This study develops a task to promote ISI, by focusing on correlation with multivariate data.

Method

In this study, the principles of the task designed from a theoretical perspective are based on collaboration with researchers and teachers. Makar and Rubin (2009) defined ISI as comprising three elements — "generalization beyond the data at hand," "use of data as evidence of generalization," and "representation of uncertainty with probabilistic language." Further, Engel and Sedlmeier (2011) pointed out the importance of understanding the concept of correlation as distinct from that of regression, and the need to understand data as a construct of signal and noise, whilst also considering the role of variability. The principles of the task design for this study, based on previous literature, are presented in Table 1. Subsequently, a tentative task, the "Decathlon Data Project," was developed based on the principle of the presented task design.

Principle	Description of the principle
Principle 1	Researchers and teachers should develop problems that enable students to use correlations to identify the relationship between variables through real multivariate data and to, subsequently, make generalized predictions.
Principle 2	Teachers should plan their instructional guidance to enable students for technological modeling and making generalizations based on evidence on the relationship between various variables.
Principle 3	Teachers should encourage students to reflect on the process and conclusions of data analysis, as well as account for the explanations (Signal) and unexplained variations (Noise) in the data.

Table1: The principles of task design in this study

Results and Conclusion

The developed task pertains to the Olympic Decathlon (Figure 1).

The "King of Athletes" is a decathlon in athletics, in which the participants' ranking is determined by their total score from ten different events. In the 2021 Tokyo Olympics, two Japanese athletes, who competed in the previous Rio Olympics (Keisuke Ushiro and Akihiko Nakamura) could not participate because they were unable to achieve the participation standard score of 8350 points. Thus, based on the score data of the 23 athletes who participated in the Tokyo Olympics and the two Japanese athletes, what events would you suggest to the Japanese athletes to practice and participate in, in order to win a medal in the next Olympics?

Figure 1: The task of the "Decathlon Data Project"

Principle 1: This task aims to predict the events that the Japanese athletes should focus on for the next Olympic Games, by considering the relationship between the total score and each event and between the events based on real data of the decathlon. *Principle 2*: This task was designed to enable students to predict the events that the athletes should focus on by determining the correlation coefficients between the total score and the points or records of each competition, using CODAP (Finzer, 2017) and Excel, and by creating scatter plots. *Principle 3*: In this task, the teacher encouraged students, through reflection, to recognize the need to further analyze data about world competition, since the proposed events are only based on the characteristics and trends of the Tokyo Olympics. It is expected to help students understand the relationship between correlation and uncertainty, and promote ISI. After this study, the author and the teacher will collaborate to validate the effectiveness of the task from a practical perspective and elaborate the principles of task design.

Acknowledgment

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Allgemeinbildung and Statistical Reasoning with Digital Technologies

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Keywords: Allgemeinbildung, statistical reasoning, digital technologies, distribution.

The historical development of the Danish school is deeply influenced by the German *Bildung*-tradition. Through all periods, the curricular aims were formulated both in terms of *Allgemeinbildung* and education (Børne- og Undervisningsministeriet, 2022). *Allgemeinbildung* addresses the need for a democratic society to have enlightened and empowered citizens. Mathematics teaching must, as well as all other school subjects, contribute to students *Allgemeinbildung* (Niss, 2000). Heinrich Winter (1995) offers a specification of how mathematics education should contribute to this aim. He determines three basis experiences, which mathematics teaching should allow students to create.

In today's data-rich society, all citizens should be familiar with statistical ideas. Statistical reasoning is a way of moving back and forth between the particular and the general and the purpose of statistical investigations are to learn new about the context (Zieffler et al., 2018). Because of the embedded connection to the context, statistics teaching seems to comprise potentials particularly for Winter's first basis experience: "to perceive and understand the phenomena of the world around us in nature, society and culture in a specific way" (translated in Biehler, 2019, p. 153).

The inclusion of digital technologies (DT) in statistics teaching comprises some potentials, for example by liberating students from routine calculations and drawing of graphs and making instant shifts between different representations and simulations possible (Biehler et al., 2013). The question is, however, if statistical reasoning with the use of DT comprises new potentials for *Allgemeinbildung*. The research question for this project is: How can statistics teaching in lower secondary school be designed so that students' statistical reasoning with digital technologies can serve as contributions to their *Allgemeinbildung*?

Distribution is a central statistical idea and is a way of moving from seeing data as individual values to interpreting distribution as a conceptual entity. It is difficult for middle-grade students to liberate their view on data from the measurement value of an object to see data in terms of distributions. A conceptual understanding of distribution is, at the same time, a prerequisite for developing the ability to choose appropriate statistical measures. The inclusion of DT can support the development of an informal understanding of distribution (Bakker & Gravemeijer, 2004). The development of students' conceptual understanding might give rise to experiencing the world around them in new ways. Their interaction with DT while reasoning about data in terms of distribution might be essential in creating the opportunity for such experiences. To conceptualize students' interaction with digital technology and investigate how the inclusion of DT affects the ability to articulate their reasoning, the Theory of Instrumental Genesis (TIG) is included in the project. TIG conceptualizes the role of an artefact when humans carry out a task. Drijvers et al. (2013) describe TIG in three dualities. The artefact-instrument-duality describes the process where a person builds an instrument out of an artefact. The instrumentation-instrumentalisation-duality is bilateral and addresses how a student knowledge forms the building of an instrument, and, at the same time, the artefact forms the students thinking. The scheme-technique duality addresses the duality between the students thinking and gesture. In this project, the focus will be on the process where students build an instrument for reasoning about distribution from educational software for statistics, but also on how the software forms their conceptual understanding about distribution.

This PhD project is expected to connect *Allgemeinbildung* and the use of digital technologies and develop teaching principles for statistics teaching in lower secondary school. The methodological approach is Design-Based Research (DBR), which is anchored in authentic learning situations through iterative processes. Each iteration comprises preparation, experiment and retrospective analysis (Bakker & van Erde, 2015).

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Preservice mathematics teachers' experiences of acquiring and organizing image-based data

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There is a growing interest in engagement with non-traditional data generated from a variety of sources, such as sensory devices, smartphones/watches, and social media tools. This paper reports on experiences of a group of pre-service mathematics teachers in acquiring and organizing image-based data to analyze categorical data as part of a course assignment. To identify the actions pre-service teachers took to transform observations of butterflies' photos to cases to answer statistical questions related to categorical data, the data from retrospective interviews and artifacts generated during the data exploration and organization were analyzed. The findings suggest that data handling and structuring became an important component of the statistical investigation before formulating statistical questions. While the use of hierarchical tables in structuring data by hand appeared to be intuitive, it could be challenging to use such a structure already built in a data organizing tool.

Keywords: Statistical inquiry, data acquiring, data organizing, image-based data, categorical data.

Introduction

Understanding and reasoning with data become increasingly essential part of our everyday lives as we need to handle information related to global issues on health, environment and so on, such as the COVID19 pandemic and wild fires due to extreme weather conditions. The nature of data also evolves with the advanced digital technologies. Large quantities and different forms of data are generated from a variety of sources, such as sensory devices, smartphones/watches, social media tools etc., every day. To analyze and make predictions from these data require new skills. Therefore, an important goal of statistics education is to develop necessary data skills starting from early school years.

Statistics is typically taught as part of school mathematics curriculum and there has been a shift from simply computing numerical and graphical representations of data to an inquiry-based approach (Watson, Jones, & Pratt, 2013). In this approach to the teaching of statistics, the focus is on the statistical problem-solving process (Franklin, Kader, Mewborn, Moreno, et al., 2007; Bargagliotti, Franklin, Arnold, Gould, et al., 2020). According to *the Pre-K–12 Guidelines for Assessment and Instruction in Statistics Education II (GAISE II): A Framework for Statistics and Data Science Education*, the purpose of this process is "to collect and analyze data to answer statistical investigative questions" (Bargagliotti et al., 2020, p. 13). As seen in Figure 1, it has four components that are interlinked: (1) Formulate statistical investigative questions, (2) Collect/consider the data, (3) Analyze the data, and (4) Interpret the results.

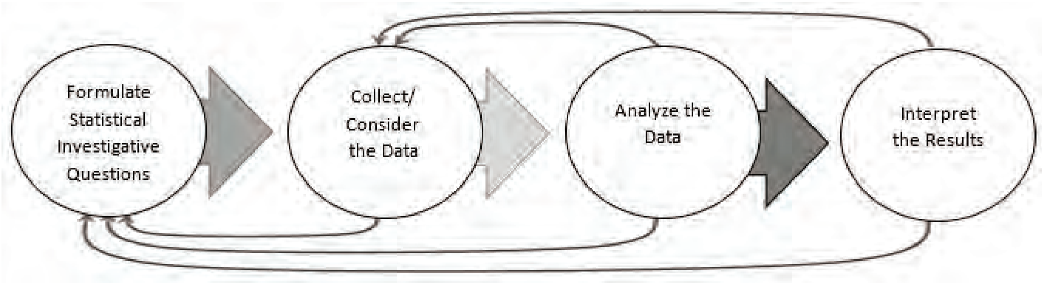

Figure 1: Statistical problem-solving process (reproduced from Bargagliotti et al., 2020)

Traditional statistical education focuses on data from random samples and making inferences from a sample to an unknown population. In today's digital age, however, data come in various forms (big, messy, unstructured data, repurposed data, image/text/sound-based data etc.) and do not necessarily comply with the structure required for statistical inference in the traditional sense (Gould, Bargagliotti, & Johnson, 2017). Along with the emerging field of data science, statistics education tends to focus more on the engagement with such non-traditional data sets.

Image-based data, i.e., the use of photographs as data, becomes increasingly common in daily encounters. The GAISE II report (Bargagliotti et al., 2020) acknowledges the use of this kind of non-traditional data as part of the statistical problem-solving process and helping students to make sense of these non-traditional data. As an example, Bargagliotti et al. use data from Dollar Street website as part of the Gapminder Foundation project (www.gapminder.org/dollar-street). The Dollar Street website displays the world as a street ordered by monthly income per person in the family from different countries and continents, and currently uses 43685 photos of 422 families in 66 different countries. Using these image-based data, students are expected to investigate "How are people's concepts of family and living spaces similar or different across the world?" (Bargagliotti et al., 2020, p. 63). The guidelines in the GAISE II report offers some instructional ideas with regard to the use of these image-based data through each four components of the statistical problem-solving process mentioned above. Bargagliotti et al. (2020) suggest that this kind of non-traditional, multivariate data sets requires a great amount of data exploration time to make observations and wonderings to analyze data. Engagement with other types of non-traditional (large and complex) data from secondary data sources, such as participatory sensing data (Gould et al., 2017) and public data sets (Wilkerson, Lanouette, & Shareff, 2021), also appears to require more emphasis on considering data and data preparation phases in statistical investigation process. As seen in these studies, such existing multivariate data sets are constructed and made publicly available by others with a particular purpose and investigators often repurpose them to explore new questions. To do so, data handling and structuring becomes an important part of statistical investigations.

The Collect/Consider the Data component of the statistical problem-solving process involves recording/acquiring, measuring and organizing the data to answer statistical questions with the acknowledgment of variability in data (Bargagliotti et al., 2020). Konold, Finzer, and Kreetong (2017) call attention to this important process of transforming observations into data and focus on table format as a typical means of data recording and structuring either by hand or using software. Two common table formats in statistics are case-data table and summary table as called by Konold et al. (2017). While case-data tables are mainly used for collecting and storing raw data, summary tables usually include only some of the information gathered and are organized in a way to compare groups or detect trends in the data. The data analysis software, such as TinkerPlots (Konold & Miller, 2011) and CODAP (https://codap.concord.org/), uses attribute-based structure for recording data where each column includes a variable and each row holds one observation, called a 'case', as described by Konold et al. (2017). However, entering the information collected to organize the data in such a format can be challenging for the learners. Konold et al. point out the need for thinking of data based on attributes which requires considering properties of the observations along with their values case by case. Their study revealed that students and adults tended to create two types of case tables, flat and hierarchical, when they were asked to construct data sheets to record and organize the data by hand from the pictures showing traffic along two road segments (including information about the vehicle type, speed, direction etc.) in a given time and date. Compared to the flat tables (attributes as columns and cases as rows) as seen in most data analysis software, the hierarchical tables include the cases at more than one level created by nested structure. This type of hierarchical structure of observations is available in CODAP and allows exploration of multilevel data sets.

Given the scarce of research on how students/adults come to record and organize data, especially with non-traditional image-based data, as part of a statistical problem-solving process, the aim of this paper is to present a case of three pre-service mathematics teachers who chose to use photos of butterflies as data to analyze categorical data. More specifically, the following question will be investigated in this exploratory study: What actions did pre-service teachers take to transform observations of butterflies' photos to cases to answer statistical questions related to categorical data?

Method

As part of 3rd year course, Teaching of Statistics and Probability, in the mathematics education program, the pre-service teachers were given a group assignment to design a statistical investigation activity that is aligned with the learning outcomes in the middle school mathematics curriculum. Two of the groups were expected to focus on analyzing categorical data and comparing two or three groups using existing data sets available in publicly available internet sources. One of these groups, including Arya, Yelda and Hale (pseudonyms), chose image-based data using "Kelebek-Türk" (Butterflies-Turk) group's website (https://www.kelebek-turk.com/) while the other group designed their data investigation activity involving data collected through survey questions on readings books.

Due to the uniqueness of transforming image-based data into a statistical analysis of categorical data with group comparisons, an interview was conducted with these three pre-service teachers to gain insights into their approaches when acquiring and organizing information from photographs of butterflies. Other artifacts, such as the documents created during the data exploration and organization, were also collected for analysis to answer the research question. Recordings of group interview and the student artifacts were analyzed using the perspectives on collect data/consider data suggested by Bargagliotti et al. (2020) and Konold et al. (2017).

Findings

Preservice teachers began with considering possible data appropriate for analyzing categorical data with comparing groups, which was given in the course assignment. Due to the difficulty in finding at least three categorical variables in data sets using publicly available data sources, such as Turkish

Statistical Institution website, they stated that some of the data explored during the course, such as the ladybug data (Bargagliotti et al., 2020, p. 32), inspired them to consider similar data about other animals. Yelda came up with the idea of searching butterfly data on the internet and found the group of Kelebek-Türk that provides photographs and observational data provided by their members from different age and occupation groups. After reading the "About us" information on the website, the pre-service teachers got their initial information about how the data were collected. Then they decided to use some of these photographs to construct their own observations involving categorical data.

According to the Kelebek-Türk website, there are 416 types of butterflies under nine different families in Turkey and their members have photographed 393 of them. As seen in Figure 2, there are nine butterfly families with different number of types, such as the Family Hesperiidae has 43 types, observed in Turkey and Figure 3 shows a sample of photos of different butterfly types belonging to the Family Hesperiidae.

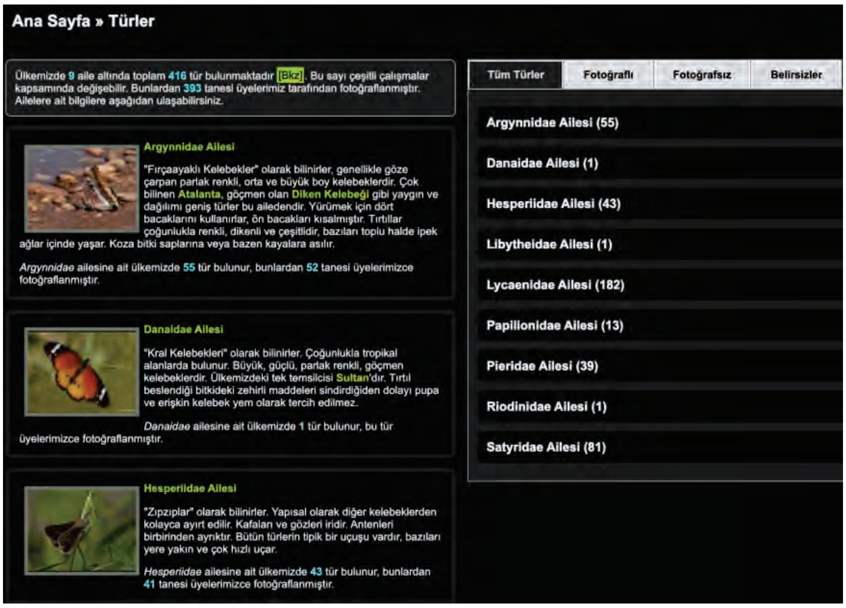

Figure 2: Screenshot of the website displaying the information about all nine different butterfly families

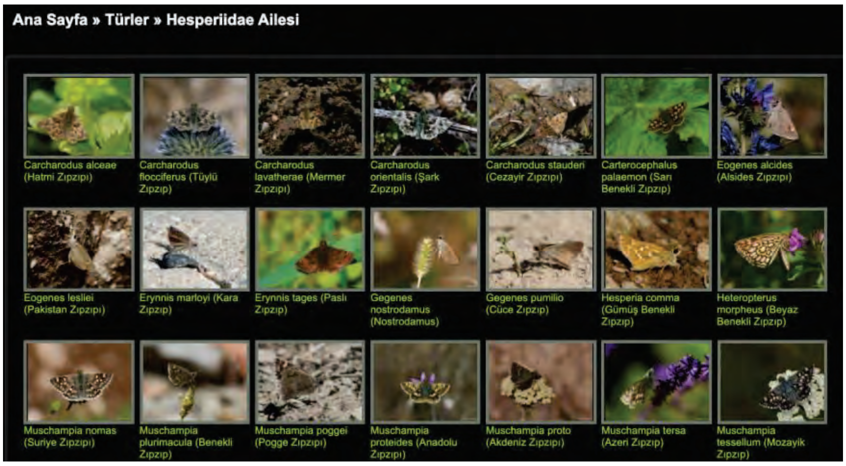

Figure 3: Screenshot of the website displaying the photographs of the butterfly types belonging to the Family Hesperiidae

Following the exploration of the information given in these pages, the pre-service teachers decided to focus on two families with a similar number of butterfly types (the Family Hesperiidae with 43 types and the Family Pieridae with 39 types) for comparison purposes given in the assignment. To collect information about these two families, they then began to click on each photo as seen in Figure 3. For example, Figure 4 shows various information about Muschampia poggei from the Family Hesperiidae, including numerical (frequencies of photographs by cities), geographic (distributions of butterflies across Turkey), photographic (captured images of butterflies with/without identified gender), tabular (months they were seen) and graphical (distribution of altitudes seen by months) representations. From this information, they selected to focus on the image of butterfly with unidentified gender (since not all types had gender information) to generate attributes, such as wing color and wing appearance which were considered as distinguishing characters of butterflies, and the seasons the butterflies seen, which involved grouping of months data given on the website.

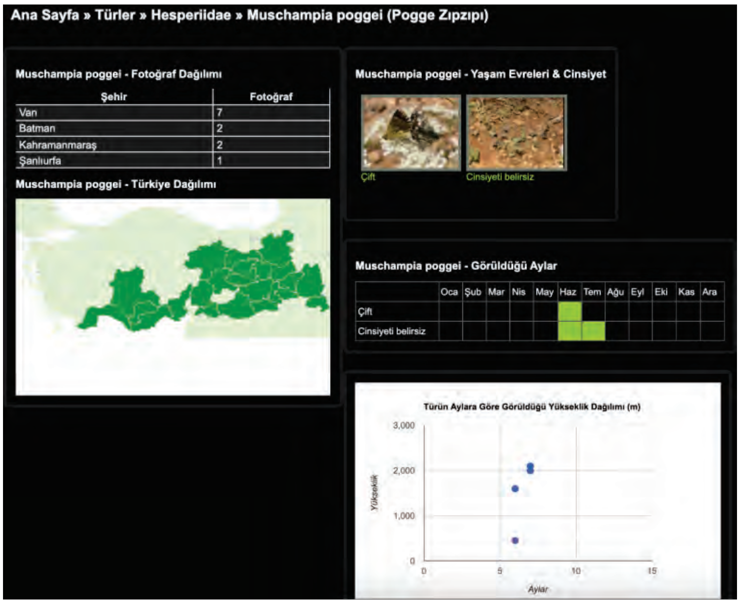

Figure 4: Screenshot of the website displaying various information about Muschampia poggei from the Family Hesperiidae

After this extensive exploration on the website, the pre-service teachers began to make an ordered list for each type of butterflies in both families, including the name of the butterfly, its photo, and the season it is seen, on a Word document. As part of the course assignment, they needed to use CODAP to analyze the data to answer questions considered in the Formulate statistical investigative questions phase of the statistical problem-solving process in Figure 1. Three questions emerged during this exploration: 1) What colors are the wings of the Family Hesperiidae and the Family Pieridae? 2) How do the appearances of the wings of the Family Hesperiidae and the Family Pieridae look like? 3) What seasons do the Family Hesperiidae and the Family Pieridae tend to be seen? Then they constructed a table, as seen in Figure 5, displaying the family name, type name, wing color, wing appearance, and season seen in columns respectively and case values in rows.

Kelebek Aileleri	Kelebek türleri	Kanat Rengi	Kanat görünümü	Yaşadıkları Mevsimler
Hesperidae (Zıpzıplar ailesi)	Pyrgus cinarae (Güzel Zıpzıp)	Siyah	Desenli	Yaz
	Pyrgus serratulae(Zeytuni zıpzıp)	Kahve	Desenli	İlkbahar-Yaz
	Eogenes alcides (Alsides Zıpzıpı)	Beyaz	Düz	Yaz
	Pyrgus sidae (Sarıbandlı Zıpzıp)	Siyah	Desenli	İlkbahar-yaz
	Pyrgus jupei (Kafkasya Zıpzıpı)	Kahve	Desenli	Yaz

Figure 5: Part of the table created for organizing the data

Organizing the data as shown in Figure 5 made entering them into CODAP table relatively easier due to its case-based structure. One difficulty the pre-service teachers encountered in this process was having more than one value for season variable for some cases, such as Pyrgus serratulae seen in both spring and summer. Therefore, they chose to create two different data sets, one including 70 cases of butterflies in two families with family names, wing color and wing appearance variables (Figure 6, table on the left) and the other including 120 cases with season variable for two families (Figure 6, table on the right). Due to the repeated cases for each season a butterfly seen, the number of cases was increased in the second data set. The pre-service teachers decided to separate multiple seasons since they wanted to answer their third question: What seasons do the Family Hesperiidae and the Family Pieridae tend to be seen? In either of the data sets, they also did not include type name which was displayed in the table in Figure 5 as they considered that having the type names would not allow to see two categories (the families) which were compared in the data analysis.

Yeni Veri Seti

Durumlar (70 cases)

index	Kelebek Aileleri	Kelebeklerin Kanat Renkleri	Kelebeklerin Kanat Görünümü
1	Hesperidae (Zıpzıplar ailesi)	Siyah	Desenli
2	Hesperidae (Zıpzıplar ailesi)	Kahve	Desenli
3	Hesperidae (Zıpzıplar ailesi)	Beyaz	Düz
4	Hesperidae (Zıpzıplar ailesi)	Siyah	Desenli
5	Hesperidae (Zıpzıplar ailesi)	Kahve	Desenli
6	Hesperidae (Zıpzıplar ailesi)	Kahve	Desenli
7	Hesperidae (Zıpzıplar ailesi)	Boz	Desenli
8	Hesperidae (Zıpzıplar ailesi)	Turuncu	Desenli
9	Hesperidae (Zıpzıplar ailesi)	Kahve	Desenli
10	Hesperidae (Zıpzıplar ailesi)	Boz	Desenli
11	Hesperidae (Zıpzıplar ailesi)	Kahve	Desenli
12	Hesperidae (Zıpzıplar ailesi)	Boz	Desenli
13	Hesperidae (Zıpzıplar ailesi)	Siyah	Düz
14	Hesperidae (Zıpzıplar ailesi)	Kahve	Desenli
15	Hesperidae (Zıpzıplar ailesi)	Beyaz	Desenli
16	Hesperidae (Zıpzıplar ailesi)	Boz	Desenli
17	Hesperidae (Zıpzıplar ailesi)	Boz	Düz
18	Hesperidae (Zıpzıplar ailesi)	Siyah	Desenli
19	Hesperidae (Zıpzıplar ailesi)	Kahve	Desenli
20	Hesperidae (Zıpzıplar ailesi)	Boz	Düz

Yeni Veri Seti

Durumlar (120 cases)

index	Kelebek Aileleri	Kelebeklerin Uçtuğu Mevsimler
1	Hesperidae (Zıpzıplar ailesi)	İlkbahar
2	Hesperidae (Zıpzıplar ailesi)	İlkbahar
3	Hesperidae (Zıpzıplar ailesi)	İlkbahar
4	Hesperidae (Zıpzıplar ailesi)	İlkbahar
5	Hesperidae (Zıpzıplar ailesi)	İlkbahar
6	Hesperidae (Zıpzıplar ailesi)	İlkbahar
7	Hesperidae (Zıpzıplar ailesi)	İlkbahar
8	Hesperidae (Zıpzıplar ailesi)	İlkbahar
9	Hesperidae (Zıpzıplar ailesi)	İlkbahar
10	Hesperidae (Zıpzıplar ailesi)	İlkbahar
11	Hesperidae (Zıpzıplar ailesi)	İlkbahar
12	Hesperidae (Zıpzıplar ailesi)	İlkbahar
13	Hesperidae (Zıpzıplar ailesi)	İlkbahar
14	Hesperidae (Zıpzıplar ailesi)	İlkbahar
15	Hesperidae (Zıpzıplar ailesi)	İlkbahar
16	Hesperidae (Zıpzıplar ailesi)	İlkbahar
17	Hesperidae (Zıpzıplar ailesi)	İlkbahar
18	Hesperidae (Zıpzıplar ailesi)	Yaz
19	Hesperidae (Zıpzıplar ailesi)	Yaz
20	Hesperidae (Zıpzıplar ailesi)	Yaz

Figure 6: The data set tables created in CODAP

Concluding remarks

This paper presented insights from a group of pre-service teachers’ experience of acquiring and organizing image-based data to analyze categorical data by comparing two groups as part of a course assignment. As pointed out by Bargagliotti et al. (2020), the pre-service teachers spent a great deal of time to explore the non-traditional data involving photographs and other types of data collected by others for specific purposes as mentioned in the Kelebek-Türk website. In accord with the findings of Gould et al. (2017) and Wilkerson et al. (2021), data handling and structuring became an important component of the statistical investigation before formulating statistical questions. They needed to consider attributes/variables and criteria for identifying categoric values for them, such as assigning values, plain or pattern, for wing appearance based on the dots or lines seen in the photos of butterflies. The pre-service teachers did not seem to have any issue with preparing the data for the analysis in CODAP as they were able to consider the data as attribute-based. This could be due to their prior experiences with using CODAP for data analysis during the Statistics course taken in the previous semester. However, transforming the initial table created by hand (Figure 5) into a single data set table in CODAP was challenging for them. Since their initial table included subsets of data within the season variable, the flat table did not work. Instead, they needed to create hierarchical data sets, which is possible in CODAP, but this feature was not used in the course before. It was also evident in the work of Konold et al. (2017) that the use of hierarchical tables in structuring data by hand was more intuitive, but it could be challenging to use such a structure that is already built in CODAP environment.

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Secondary school students' strategies in solving combination problems

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This work is part of a wider investigation conducted in Italy, which aims to explore the effects of instruction in combinatorial reasoning of secondary school students. Two groups of students with and without instruction were given a questionnaire adapted from Navarro-Pelayo's research in order to analyze the students' performances and strategies used in the solutions, as well as the effect of instruction. In this work we present the results found in three combination problems. The students did not generally present particular difficulties in solving problems of combinations even though students who received instruction showed an improvement in their performances. In the no instruction group enumeration was the main strategy while the group of instruction used a more heterogeneous distribution of strategies.

Keywords: Combination problems, strategies, secondary students' performance.

Introduction

The study of combinatorics is an important component of mathematics, especially within discrete mathematics and in the school curriculum, where often plays a key role in the teaching of probability. For instance, combinatorial reasoning is needed to form the sample space in probability problems, as well as to understand some discrete distributions (e.g., the binomial distribution). However, not much attention has been given to the teaching of this topic, especially in the Italian school system. Moreover, some previous research, such as that by Navarro-Pelayo (1994) have described the students' difficulties in solving combinatorial problems, and this justifies an interest toward this field of research.

Piaget and Inhelder (1951) considered that combinatorial reasoning is a pre-requisite for a full understanding of the notions of randomness and probability. In addition, combinatorics is an area where students can both develop and train problem solving strategies, as well as mathematical processes such as generalization and recursive thinking (Kapur, 1970). Although there has been previous research on students' strategies in solving combinatorial problems, most of this research have been carried out with small children (e.g., English 1991; 2005) or university students (Godino et al., 2005) and none of this research considered Italian students.

The aim of this research was exploring the Italian students' combinatorial capacity and its change with instruction, as well as describing strategies developed by secondary school students before and after teaching. In this paper we present a part of the results of the global project; more specifically we analyze both performances and strategies used by a sample of secondary school students in solving three combination problems, also comparing the results obtained by the students with and without instruction.

Background

We based our research on the work by Fischbein and Gazit (1988), who analyzed the combinatorial capacity of children since 10th year of age and proved that they can learn to solve combinatorial problems when they receive a specific instruction. They suggested as main variables affecting the difficulty of the tasks the type of combinatorial operations (suggesting that permutations are the hardest), the type of element to be combined (numbers and letters are easier than objects or people) and the total number of combinations to be formed (a variable that we will describe as the dimension of the problem). Concerning secondary school students, Batanero et al. (1997) showed on a sample of 720 14-15-years-old students the effect on the difficulty of combinatorial problems of the implicit combinatorial model (together with the type of elements to be combined, the type of combinatorial operations and instruction). This variable was firstly described by Dubois (1984) who stated that every elementary combinatorial problem (that is a problem that could be solved through the application of a single combinatorial operation) could be classified through one of three possible schemes, implicitly present within the problem: selection, distribution or partition, depending on the combinatorial operation suggested in the text. Henceforth, we will refer to a problem by saying that it belongs to a particular model depending on its combinatorial scheme.

Regarding combinatorial strategies, English (2005) described enumeration used by small children in combinatorial problems and considered both a-systematic (consisting in a random or incomplete selection of elements) and systematic enumeration. Systematic enumeration involves repeating the selection, fixing an element (for example the first element in a permutation) and combining it with all the other elements, and then repeating the procedure with the remaining elements until all the configurations are listed. Lockwood and Gibson (2016) worked with 42 undergraduate students concluding that even creating partial lists of the set of outcomes led to significant improvements in the students' performance in solving combinatorial problems; the authors also suggested that instruction should facilitate systematic enumeration processes in the students. In a case study with university students, Godino et al. (2005) found that some students still used enumeration at this educational level and described other strategies such as use of a formula, use of the sum or product rule, posing new problems related to the original or breaking down the problem into simpler subproblems. In our investigation we will analyze all the strategies used by the students in the sample and its effectiveness to get a correct solution, as well as the differences between students with and without instruction.

Method

The sample consisted in 115 secondary school Italian students from different school grades and specialties, 64 of which had not received instruction on combinatorics and 51 students who received instruction on the subject. Although the sample of students was not random, because we depended on the availability of schools and teachers, we included classes of different schools and with different teachers, in order to obtain a sample of students as heterogeneous as possible. The group of students who did not received instruction was formed by classes of grade 10, 11 and 12 (i.e., second, third and fourth year of secondary school with students aged between 15 and 18). On the other hand, most of the students who received instruction on combinatorics belonged to grade 12 (fourth year of secondary school with age 17-18). There were also three students belonging to grade 10 (second year of secondary school) who received instruction on combinatorics in extracurricular courses and who were, then, included in this group. The instruction was mostly formal, based on the use of formulas to solve combinatorial problems, with the exception of one class of 12 students who approached the topic in a constructivist way, mostly through exercises and examples, constructing combinatorial rules before being taught a formula. The students were given a questionnaire of 13 open-ended problems (in an Italian translation) adapted from the one used in Navarro-Pelayo (1994) and Batanero et al. (1997) and proposed in two different versions presenting the same problems with an inverted order, aiming to cancel a possible effect of time on the data collection and obtaining a sufficient amount of data on every item of the questionnaire. Depending on the availability of the school, students had 60 or 90 minutes to complete the questionnaire. We remark, though, that time did not represent an issue, considering that most of the students were able to complete the questionnaire widely before the smaller time limit.

In this work we will focus on three items taken from the whole questionnaire and corresponding with combination problems:

> **Item 1**. Supposing we have three identical letters, we want to place them into four different colored envelopes: yellow, blue, red and green. It is only possible to introduce one letter in each different envelope. How many ways can the three identical letters be placed into the four different envelopes? For example, we could introduce a letter into the yellow envelope, another into the blue envelope and the last one into the green envelope.
>
> **Item 2**. School students must elect their representatives. Five students are the candidates: Elisabeth, Ferdinand, George, Lucy and Mary. In how many different ways can three of the five candidates be chosen? For example, Elisabeth, Mary and George could be elected.
>
> **Item 3**. Mary and Cindy buy four lipsticks of different shades, numbered from 1 to 4. They decide to share out the lipsticks, two for each of them. In how many ways can they share out the lipsticks? For example, Mary could hold lipsticks with shades 1 and 2 and Cindy those with shades 3 and 4.

All the three items are small dimensional, meaning that the solution requires less than 10 combinatorial configurations. Item 1 is a problem belonging to the distribution model (Dubois, 1984) in which students are requested to distribute some objects (the letters) in three containers (the envelopes), and whose solution is $C_{4,3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!} = 4$. Item 2 belongs to the selection model and students are asked to select 3 people from a group of five and the solution is $C_{5,3} = \frac{5!}{3!(5-3)!} = 10$. Finally, item 3 belongs to the partition model and students are asked to form two subsets from a set of numbers and the solution is $C_{4,2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = \frac{4!}{4} = 6$.

Once the written responses of each student were collected, a content analysis (Neuendorf, 2017) was performed. Every solution has been codified through numerical vectors including all the information related to the problem (e.g., ID of the student, group, class, correctness of the resolution, strategy of resolution). We also included qualitative information on the strategy used; for example, when a student employed a formula but not enumeration in a problem the respective entries of the vector include a number identifying how the formula has been used (e.g., correct formula, wrong formula, correct formula with arithmetical mistake, correct formula introduced with a wrong name, etc.) and a zero in correspondence of enumeration. After the codification of the students' responses to the questionnaires, we proceeded with a quantitative analysis focusing mainly on two variables: on one hand the correctness (or not) of the problem solution and on the other hand, the strategy used by the student (in this work we focused only on the bare use of a strategy and not on how a particular strategy is used). The solution to a problem was considered correct if the student either identified the total number of combinations by using a formula or correct arithmetic operations or else if he or she provided a complete list of all the possible configurations. Regarding the strategies, we observed all the procedures of the students considering all the strategy appearing in a solution, even when one strategy was used as a support for another one.

Results

In Table 1 we present the percentages of correct solutions to each item in both groups. We note that in both groups item 3 (partition of a set of numbers into two disjointed subsets) was significantly easier than the other two problems and that, generally, there was an improvement in students' performances after the instruction, even though some students in the instruction group were still not able to solve the problems.

Table 1: Percentages of students in each group correctly solving the items (above) and comparison with results in Navarro-Pelayo (below)

Item 1 $C_{4,3}$ Distribution	Item 2 $C_{5,3}$ Selection	Item 3 $C_{4,2}$ Partition	Item 1 $C_{4,3}$ Distribution	Item 2 $C_{5,3}$ Selection	Item 3 $C_{4,2}$ Partition
No instruction (n=64)			Instruction (n=51)		
39,1	12,5	51,6	51,0	54,9	66,7
No instruction (n=348) [Navarro-Pelayo, 1994]			Instruction (n=352) [Navarro-Pelayo, 1994]		
26,9	22,5	31,0	26,7	46,0	37,2

We observe that despite the small dimension (only 10 different configurations), item 2 was difficult for students with no instruction. We suppose that this is due to the fact that the students were not used to deal with the selection of people (while they were more familiar to work with numbers, that are used in item 3, and letters, appearing in item 1). We can observe that mistakes mostly derive from either not considering that the ordering of elements does not have an influence in the problem or considering as different elements that are identical instead; for example, in item 3 many students provided a solution that considered different the permutations of a same solving configuration or in item 1 students assumed that different letters should be distributed in the envelopes. This error was described both in Batanero et al. (1997) and Fischbein and Gazit (1988) as linked to combination problems. Comparing our results with those in Navarro-Pelayo (1994) – see Table 1 – we can observe a similar trend of improvement for Spanish students. While, on one hand, the percentages are different from those obtained in our sample, probably because of the different sample size (115 students in our research and 720 in Navarro-Pelayo's), on the other hand we observe a similar pattern of results: students globally improve after the instruction and they perform better on item 1 and item 3 (with the latter as the best performing item), but it is in item 2 where can be noted the best improvement after instruction.

Students' strategies

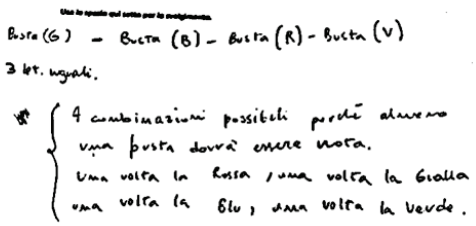
In analyzing the solutions, the following strategies were considered: a) *Enumeration*, found in previous research (Batanero et al., 1997; Godino et al., 2005; English, 2005; Lockwood and Gibson, 2016) with students of different ages and consisting in the explicit listing of all the possible combinations to be formed, according to the problem statement. b) *Tree- diagram*: the student builds a tree diagram as a help in producing all the configurations; according to Fischbein and Gazit (1988) the resolution of the problem is facilitated with the involvement of this tool. We also considered the following strategies, as described in Godino et al. (2005): c) *Formula*: the student recognizes the combinatorial operation of combination (or another operation, when the students proceed changing the combinatorial model or refers to another problem that he or she found equivalent) as a solution to the problem and remembers its formula. d) *Reference to other problem*: the student transforms the problem in another equivalent which is used to obtain the solution. e) *Sub-problem decomposition*: the original problem is divided into several combination problems of smaller dimension and the resolutions of which are combined to get the solution to the initial problem. f) *Sum, product or quotient rules*: the student does not remember the formula of combinations and tries to solve the problem using the elementary arithmetical rules of sum, product or quotient. g) *Other strategies*, generally giving a wrong answer with no justification.

In Table 2 we present a summary of the results obtained from the analysis of strategies, with the percentage of use of every strategy considered together with the percentage of students getting to a correct solution. We observe how students without instruction mostly relied on an enumerative strategy in every item, even though this did not always lead to a correct resolution, especially in item 2. The majority of enumerations were systematic, probably due to the reduced dimension of the problem, that allowed students to better control their listing procedures. We remark how students proceeding through enumeration failed in reaching the correct answer due to different reason: on one side, in item 1 and item 3 their mistakes were mostly connected to a misunderstanding of the problem (i.e., considering objects different/identical or taking/not taking into account the importance of order) rather than something imputable to the chosen procedure. However, in these items students tended to overestimate the number of configurations.

A different situation is observed in item 2 where students mostly produced an incomplete list of configurations, underestimating the solution of the problem, providing a partial resolution to the problem (a solution which does not contain procedural mistakes but that fails to reach a correct solution). In fact, we observed how most of the enumerations end with a list of 9 out of 10 configurations.

Table 2: Percentages of global and correct use of each strategy in the no instruction group (n=64) and in the instruction group (n=51, percentages in brackets)

	Item 1		Item 2		Item 3	
Strategy	Global	Correct	Global	Correct	Global	Correct
Systematic enumeration	45,5 (21,3)	30,9 (19,1)	75,4 (17,0)	12,3 (8,5)	75,9 (39,6)	53,4 (33,3)
A-systematic enumeration	5,5 (10,6)	(6,4)	1,8 (4,3)	(2,1)	(4,2)	(4,2)
Tree diagram	16,4 (2,1)	5,5	1,8 (2,1)		8,6 (2,1)	3,4
Formula	(29,8)	(19,1)	(57,4)	(46,8)	1,7 (31,3)	(25,0)
Reference to other problem	(6,4)	(6,4)	(6,4)	(4,3)	1,7 (6,3)	(4,2)
Sub-problems decomposition	7,3		10,5 (2,1)		6,9 (4,2)	3,4 (2,1)
Sum rule	10,9 (8,5)	9,1 (8,5)	5,3		(4,2)	(2,1)
Product rule	25,5 (29,8)		21,1 (14,9)	1,8 (2,1)	20,7 (16,7)	3,4 (2,1)
Quotient rule	(6,4)	(6,4)	(10,6)	(6,4)	(12,5)	(8,3)
Other	3,6	1,8	(2,1)		(4,2)	(4,2)



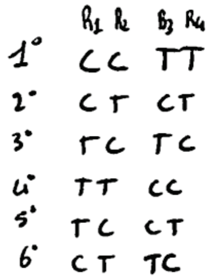


Figure 1: (a, left) Resolution of item 1 through the implicit use of the sum rule

(b, right) Resolution of item 3 through enumeration

Some of the students also developed different solving strategies but there were very few cases in which they were able to get to a correct solution. Most of these were reached through the use of the sum rule, by changing the model of the problem from distribution to selection: an example of this resolution is presented in Figure 1a, where the student explains that, instead of distributing letters he can choose the envelope to be left empty and so there are four possibilities (basically, every sub-problem is represented by a single configuration of the envelope that is discarded so that the global solution is reached by summing up the results obtained in the four subproblems). A change of model can be also observed in Figure 1b, where the student distributed people instead of partitioning lipsticks and proceeds by listing all the possible configurations. We also notice how in the group of students with instruction – see Table 2, percentages in brackets – the choice of strategy shifted from a majority of enumerations to a more heterogeneous use of strategies. In fact, most of the considered strategies appeared in the solutions of the whole group with the exception of the sub-problems decomposition and tree diagram, that were scarcely used. The most frequent strategies were formula, enumeration and product rule, even though the latter was mostly connected to a non-resolution of the problem. Students of the instruction group who applied a formula mostly got to a correct solution, even though we can observe some students using a wrong formula due to, we suppose, a misunderstanding of the statement of the problem.

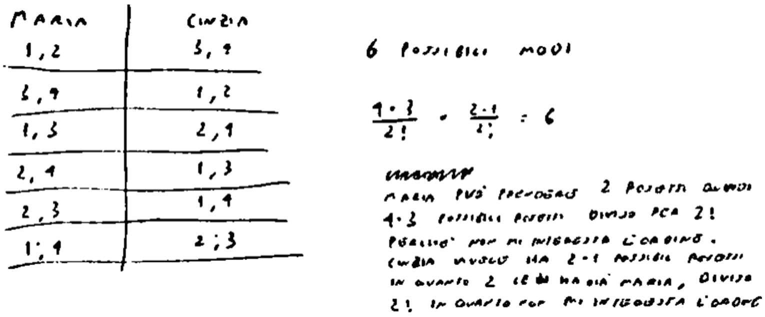


Figure 2: Resolution of item 3 through the double use of enumeration and quotient rule

We observe how part of the students used enumeration; we suppose, also considering results of the whole questionnaire, that this is due to the fact that the problem is a small-dimensional, so students easily produce a complete list of configurations. This strategy led to a correct solution nearly in most of the cases for item 1 and item 3 while, similarly to what was observed for the group of no instruction, students were not always able to get to the complete list of enumerations in item 2, for which resulted more effective the use of the formula. We suppose that the reason is that the formulation of item 2 is very similar to usual combination problems one can find in textbooks. Finally, we notice that some of the students who received instruction solved the problem using multiple strategies within the same resolution; for example, we can observe in Figure 2 a resolution of a student involving both enumeration and quotient rule. On one hand, on the left of the figure, the student properly produces the complete list of solving configurations and then provides an argument involving the arithmetic rule of quotient explaining how the result of the problem is 6. We suppose that students that provided a double solution proceeded in this way in order to justify their calculations, in case they were not sure about the formula or the rule they were using. In conclusion, instruction was found to be useful to increase the competence of the tested students to solve combinatorial problems, in agreement to what stated by Fischbein and Gazit (1988). Students' performances (in terms of correctly solved problems) increased significatively after the instruction, even though some difficulties remained, and the students were not always able to get to the correct solution. In spite of a (mostly) formal instruction based on the learning of formulas it is worth observing that even after acquiring new procedures some of the students of the group with instruction continued to rely on enumeration, even though sometimes they used it together with formulas and arithmetical rules, providing a double solution to a problem. Comparing our results with the ones presented in Godino et al. (2005) for university students, we notice that the distribution of strategies for secondary school students is different, shifting toward a higher use of enumeration. On the other side, focusing only on the results of the group with instruction, we can observe that the distribution of strategy was more similar to what observed in university students with higher mathematical preparation that, consequently, does not seem to play a key role in the development of solving strategies.

Conclusions

Despite the small dimension of the tested sample, this explorative research provides new insights on the solving strategies students activate when solving combination problems. Most of the correct answers come from the use of either enumeration or formulas even though students also developed other different procedures. We underline the fact that knowing the strategies that the students activate would allow the teacher to better implement a teaching activity, for example introducing different techniques. However, further and deeper steps would be needed in order to proceed with this research; for example, widening the dimension of the sample and including more focused qualitative analysis, aiming to better understand the mechanisms that lay behind the choice of a solving strategy.

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Digital learning environments for mathematics education – A workshop on stochastic paradoxes

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Keywords: Educational Media, Education innovation, Pre-service teacher training.

Preconditions and motivation

Digital technologies can support and benefit mathematical learning processes, especially the development of students' statistical reasoning, but also have their limitations (Thurm, 2020, Biehler et al., 2013). In order for student teachers to reflectively use digital technologies at an early stage of their teacher training, the seminar "Digitalbasierte Lernkontexte des Mathematikunterrichts" (Eng. Digital learning environments for mathematics educations) was developed. There, student teachers develop, test, evaluate, and reflect on digital learning environments for relevant, authentic applications of mathematics in a Design-Based research approach (Bakker & Eerde, 2015). Below, we present a workshop developed by students who pursued the initial research question: How can we profitably employ digital technologies in teaching stochastic paradoxes?

Storyline

The 90-minute workshop starts with a joint introduction to the situation via radio recording. The state "Tralien" is struggling with the serious disease "Frost" as well as with substantial financial problems. A pharmaceutical company presents two drugs to cure the disease. However, due to a lack of financial resources, a decision has to be made in favour of the production of one of the drugs. Furthermore, when aid money is put up, it is discovered that "Tralien" is missing large amounts of tax money. This causes the financial problems and leads to a second task: tax evaders must be exposed. Upon this introduction, students form two expert groups who work on the problems almost independently guided by GeoGebra books and additional material in Excel and R. The first group learns about the Simpson paradox and how it relates to the choice of one of the two drugs. The second group deals with Benford's law and applies it to expose tax evaders. Finally, each group presents the findings.

Didactic considerations and digital tools

A statement is judged as paradoxical, if it conflicts with another self-evident statement. Hence, a reason for working with stochastic paradoxes is that students conflict with their intuition and have to correct misconceptions. In doing so, existing conceptions must be recognized as incorrect, underlying errors must be understood and then a correct conception can be formed. This process of understanding errors and building new conceptions is common in many learning areas (Winter, 1992). Simpson's paradox (Meyer, 2018) and Benford's law (Humenberger, 2018) offer many starting points here. Another reason to choose stochastic paradoxes is their relation to students' everyday life. Students are exposed to statistics of large data sets and their interpretation. Thus, they should internalize the relevance of proper handling of statistical results as well as how to evaluate and interpret data sets.

To further emphasize the relevance of the content, the workshop is based on real, publicly available data and students are introduced to software used in data analysis (R, Excel). These tools allow the demonstration of the processing of large amounts of data as well as its visualization, while reducing schematic operations. Currently, however, students do not program themselves. Instead, the focus is on the discussion and interpretation of the finished code and relevant diagrams. In further revisions, we plan to enable students to complete programming tasks too, using Jupyter Notebooks. For now, GeoGebra books guide through the workshop. The many different tools in GeoGebra enable a diverse design and a clear structure of learning tasks, as well as visualisation and change of representation to promote comprehension. GeoGebra serves as an information and presentation tool, so students can work on the materials almost independently and are not bound by space or time. In addition, GeoGebra Classroom provides a virtual classroom, where teachers can view students' answers and identify potential problems and misunderstanding, allowing for individualized learning guidance.

Acknowledgment

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Supporting teachers' early statistical development and PCK through lesson study: an informal inferential reasoning experience with kindergarten through fourth grade teachers

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Developing statistical thinking at the early school levels is a challenge that in-service primary school teachers must face. One of the actions that would achieve this goal is aimed at generating teaching proposals from the teachers themselves in their schools, from contexts that give meaning to the analysis of data in situations of uncertainty from an informal perspective. This report presents how collaboratively, a group of preschool and primary school teachers and researchers participated in a Lesson Study Group on the Informal Statistical Inference (ISI) approach for nine weeks. This allowed them to move in a cycle of goal setting, planning, implementation and evaluation of a lesson plan, acquiring knowledge of the content and pedagogical knowledge of ISI reflecting on their professional learning and the teaching of ISI.

Keywords: Early statistics, primary education, informal statistical inference, lesson study, pedagogical content knowledge.

Introduction

Statistical thinking makes it possible to more accurately analyze the world and overcome the fallibility of intuition. The use of statistics allows decisions to be made about phenomena in which uncertainty is present and facilitates the establishment of inferences in situations that we try to foresee according to the behavior of the data. It is necessary to incorporate at the school level experiences that allow students to face critically various information sources that provide data, such as those related to epidemiological phenomena, scientific findings, electoral results, weather forecasts, economic models, among others. In turn, they are required to be able to recognize variation and understand the randomness present in many daily activities. Given this scenario, teachers could use informal statistical inference (ISI) as a theoretical and pedagogical approach that contributes to developing in their students' ways of reasoning in situations of uncertainty, understanding ISI as "a generalized conclusion expressed with uncertainty and evidenced by, yet extending beyond, available data" (Ben-Zvi, et al., 2015, p. 293).

On the other hand, preservice teachers and in-service teachers, require a pedagogical content knowledge of the informal inferential reasoning (IIR) around central concepts of ISI, so that they understand key ideas in statistics, anticipate the difficulties and errors of students and can build tasks that provide opportunities for the development of inferential reasoning (Leavy, 2010). In this sense, the present study addresses early statistical education with preschool and primary educators and investigates their pedagogical content knowledge (PCK) by planning a learning sequence on ISI in early childhood education designed collaboratively within a lesson study group (LSG), and by researching their lessons in the classroom.

Lesson study for the transformation of teaching

To initiate a Lesson Study cycle, a team of 3 or more teachers is needed to form a LSG, which meets periodically to establish a lesson plan. Through Lesson Study cycles, understood as a process of instructional improvement, teachers can transform traditional ways of teaching mathematics into an instance of teacher professional development. Research has shown that it is a way for teachers to develop their work collaboratively and understand how their students learn. Lesson Study (LS) focuses attention on teamwork and shared responsibility around a lesson plan, its implementation, and its improvement. Thus, one or more teachers prepare the lesson, selecting the materials needed to achieve the objective stipulated in the lesson plan. Subsequently, one of the teachers involved in the planning implements the lesson, which in some cases is observed by other teachers or researchers. Once the lesson is completed, the teachers and observers meet in a session in which the implemented lesson is reviewed and analyzed, which will allow for the improvement of the lesson plan and its subsequent dissemination. According to Isoda et al. (2021), LS allows the development of teachers' Content Knowledge and Pedagogical Knowledge, promotes their ability to observe students' productions in class, motivates the improvement of their teaching proposals and allows the establishment of collaborative relationships among colleagues.

Teachers' knowledge and Informal Statistical Inference

Since the 1980s, Shulman (1987) stated that teaching begins when the teacher reflects on what needs to be learned and how students will be taught. In these reflective processes, beliefs, implicit theories, and other forms of thinking interact with context variables to shape actions that take place in the classroom. The PCK construct seeks answers that contribute to specifying the professional knowledge needed in teaching to intensify student learning (Shulman, 1987).

The framework presented in Table 1 (Chick et al., 2006), accounts for the components of PCK that are evident in teaching and the way pedagogical and content knowledge are intertwined. This framework considers three components: Clearly PCK (involves aspects that are more of a mixture of content and pedagogy); Content knowledge in a pedagogical context (involves aspects drawn more directly from the content); and Pedagogical knowledge in a content context (includes knowledge drawn more directly from pedagogy).

Table 1: Framework for analyzing Pedagogical Content Knowledge (Chick et al., 2006)

PCK Category	
Clearly PCK	Teaching Strategies
	Student Thinking
	Student Thinking - Misconceptions
	Cognitive Demands of Task
	Appropriate and Detailed Representations of Concepts
	Explanations
	Knowledge of Examples
	Knowledge of Resources
	Curriculum Knowledge
	Purpose of Content Knowledge
Content knowledge in a pedagogical context	Profound Understanding of Fundamental Mathematics (PUFM)
	Deconstructing Content to Key Components
	Mathematical Structure and Connections

PCK Category	
	Procedural Knowledge
	Methods of Solution
Pedagogical knowledge in a content context	Goals for Learning
	Getting and Maintaining Student Focus
	Classroom Techniques

The research in which this report is framed, involves characterizing the progression of the IIR from kindergarten through fourth grade attending, in turn, to the development of the PCK of the teachers when designing and implements a lesson plan in an LSG. In this sense, it is considered that the PCK accounts for the knowledge of teachers to understand the concepts, problems or emerging difficulties adapted according to the interests and skills of the students (Estrella et al., 2015).

This framework is combined with the informal statistical inference components characterized by Makar and Rubin (2009) for the generation of teaching proposals that seek to promote IIR in students at the school level. Various frameworks have been proposed when characterizing the ISI and the reasoning that it supports, whose common components correspond to using data as evidence requiring to assess and consider the available data to establish arguments associated with a question or problem, privileging certain evidence that they provide over personal experiences or opinions; generalizing beyond the data accounts for the ability to communicate conclusions derived from particular data, generating inferences that apply to a broader set; and expressing uncertainty implies manifesting the uncertain in generalization, being aware that statements are not certainties, but consider a margin of error.

In this way, one of the purposes of this study focuses on how the work in a LSG that seeks to promote ISI can contribute to the development of the PCK of the teachers linked to the experience. Although the work with the teachers included stages of planning, execution and improvement of a lesson plan, this report focuses on the following question: How does a teacher's PCK manifest itself when planning a lesson plan on ISI at the preschool to fourth grade levels?

Methodology

To answer this question, -and based upon consensus among the authors-, a qualitative, descriptive-interpretative approach is adopted, analyzing the dialogues transcribed from nine video recordings: seven working sessions within the LSG and two lessons of implementation of the lesson plans of each teacher, according to the categories adapted from the PCK components of Chick and collaborators (2006), integrating the key components of the IIR.

Participants

The LSG consisted of six teachers (three primary school teachers, one preschool educator, two special educators) and three researchers with expertise in statistics education, two of them with LS research experience. Regarding their years of professional experience, three of the teachers in this school had more than 15 years of teaching experience, two of them had between 10 and 15 years of experience and one had less than 10 years of teaching experience. The teachers work in a municipal educational establishment located in the Metropolitan Region of Chile and carried out the execution of the lesson plans with a total of 70 students distributed in grades K-4.

The LS cycle for generating a lesson plan that promotes IIR.

The LSG met weekly for 2 hours during 9 weeks. The activities of the LS cycle in which the teachers had to go through were delineated by the team of researchers based on four work axes: A. Authentic experiences of informal inferential reasoning; B. Planning a learning sequence; C. Implementation, revision and adjustment of the lesson plan that promotes IIR; and D. Analysis and reflection of the LS process.

In Table 2, although the activities are presented sequentially, some actions are carried out cyclically throughout the process, for example implementation of the lesson plan, review and adjustment of the lesson, identify student needs, reformulate objectives, propose new possible teaching interventions, among others.

Table 2: Overview of the activities according to the LSG work

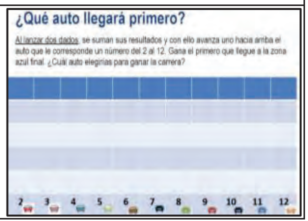
Week	Activity
1–2	**Authentic experiences of informal inferential reasoning: Exploring content knowledge in a pedagogical context** •Task 1. How will Tom get to school tomorrow? •Task 2. Which car would he choose to win the race? What elements characterize IIR: Comparing responses of LSG teachers and some other teachers in the country, in light of the components: Using data as evidence, expressing uncertainty, and generalizing beyond data.
3–4	**What do we expect from the IIR of K-4 students? - Planning a learning sequence** Identify student needs, formulate objectives, set a task to promote IIR, anticipate student difficulties and responses, outline possible teaching interventions, and determine aspects to assess.
5	**Implementation of the lesson plan** The planned lesson plan is implemented, some teachers and researchers observe and record, in videos and/or field notes, what happens.
6	**Review and adjustment of the lesson plan according to post-session analysis** The experiences of the teachers in the implementation are shared, the LSG reviews and adjusts the lesson plan considering some suggestions and recommendations that arise in the post-session analysis.
7	**Implementation of the adjusted version of the lesson plan** The lesson plan is implemented in light of the proposed adjustments and the LSG's recommendations.
8 - 9	**Analysis and reflection of the LS process** Video records and field notes are reviewed to identify strengths and aspects of the intended lesson plan, and group reflection on all activities of the process describing the LS experience, professional learning and challenges, socioemotional aspects and some discoveries.

ISI tasks proposed to the teachers in the first LS stage

In the first LS stage, the teachers had two experiences that triggered their RII (see Table 3). The first situation is from the inferential domain and is based on one of the items reported by Watson and Callingham (2003). The second situation belongs to the probability domain and is related to a randomized experiment with dice. After its realization, the in-service teachers were able to know the elements that have been characterized at the research and pedagogical level of the ISI approach, with the purpose of integrating these elements in the lesson plan that would be formulated, implemented and evaluated in the following stages.

Table 3: ISI tasks proposed to the educators and teachers in the first stage of the LS

Situation	Concepts	ISI Ability	Resources

Reading a data representation with a missing data to make a prediction in a scenario of uncertainty.	Variable, categories of the variable; data; majority; prediction.	Predicting the variable category in which the missing data correspond beyond the data provided in the representation.	
Conducting an experiment of throwing two dice simultaneously, advancing on game board according to the sum of the results of the dice faces.	Random variable; sample; population; random experiment; variability; sample size; prediction.	Predicting the greatest chance of winning by considering various samples.	

The lesson plan and the core task proposed within the LSG

The lesson plan was jointly designed based on previous LSG discussions on IIR and was intended to be implemented transversally with K-4 students. This plan considers a central statistical task of a playful type involving the randomized coin-tossing experiment, called "the frog race".

The game, whose instructions are illustrated in Figure 1, favors the manipulation of the material and the recording of data on a game board. Before starting the game, each player must choose the frog that he/she thinks can reach the goal before the other two. The game ends when one of the frogs has reached the goal. The game can be repeated as many times as desired, although three or more completed games are recommended.

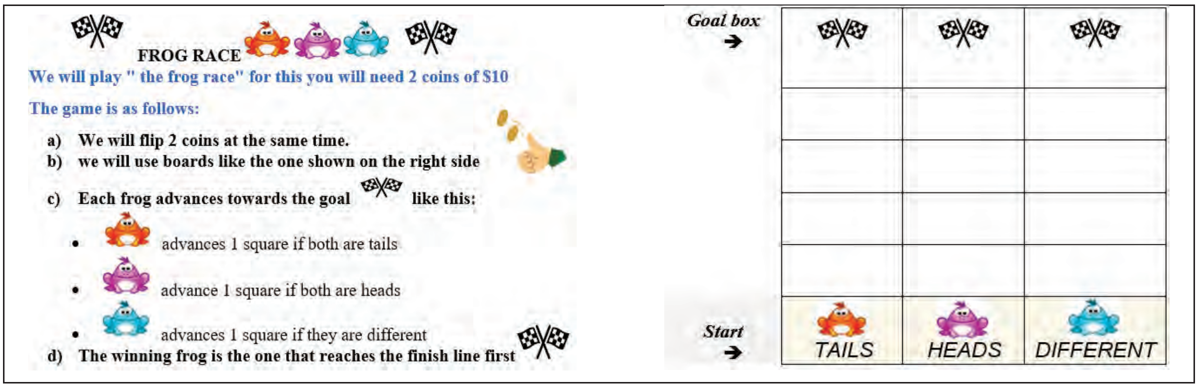


Figure 1: Board and game instructions “Frog race” (Estrella et al., 2022)

After completing several games with the data recorded on the boards, students are encouraged to establish generalizations beyond the data, using arguments based on the samples obtained (their own and those of their classmates) and to express uncertainty about which of the frogs is more likely to win; the guiding question was "If you had to give someone advice, which frog would you advise them to choose? Why?". Each teacher gave two lessons on ISI in two groups of the same course.

Results

To account for teachers' PCK when generating, implementing, and improving a lesson plan that promotes IIR, some illustrative episodes were selected from the categories proposed by Chick et al. (2006). Given the limitation of this paper, we chose to show only the lesson planning stage and the categories used to analyze the teachers' interventions (each teacher is labeled with T1 to T5, and the researchers as R1 and R2).

Conformation of a lesson plan

As an instance prior to the planning of the lesson plan, the teachers faced IIR eliciting tasks, as an authentic experience that not only allowed them to activate their content knowledge on ISI, but also enabled them in a subsequent exercise, to analyze their own and other colleagues' answers, considering the IIR components brought into play. At the same time, they learned in parallel some aspects of the ISI approach as an alternative that makes possible the teaching of contents associated with uncertainty prior to the use of formal inference techniques. These actions were key in the formulation of a central task of the lesson plan.

Some manifestations of teachers' PCK according to the categories (1) Clearly PCK, (2) Content knowledge in a pedagogical context and (3) Pedagogical knowledge in a content context, when participating in an LSG are illustrated in Table 4 below.

Table 4: PCK manifestations of two teachers according to PCK categories

Stage of the LS cycle	**PCK Category**		
Planning	1	2	3
T1: The central problem of the class... considering K to 4th grade, What scenario... evidence... express uncertainty...	X	X	X
T2: It would have to be something like the car race thinking that something super meaningful and known to the kids...			X
T1: You can think of the generalization in the fact that if they see that a car is winning... When they are doing the game ...If a partner asks them	X	X	
T1: A visual question to see who is winning... visually they realize who is winning... and when the children realize that they are winning... they will immediately say "yes", I experience this daily with my students. They want to get to the winning situation, yes or yes!	X		X

In this episode, T1 points out in his intervention key aspects to promote in his students, identifying components of the IIR that are fundamental for students to understand and establish ISI, in particular the use of data as evidence and the establishment of expressions with uncertainty. T2 proposes to the LSG to involve playfulness as a fundamental feature of the task, so that students can become emotionally attached to its development. Then T1 points out to his colleagues that employing a visual representation could allow students to reason through identifying patterns in the data, and analyze the results more favorably. Finally, T1 discusses strategies for engaging students.

Table 5: PCK manifestations of five teachers according to PCK categories

Stage of the LS cycle	**PCK Category**		
Planning	1	2	3
R1: Regarding the car race, in kindergarten I think there would be difficulty with the sum of the dice, ... there would be too much prominence of the teacher, 2 dice are going to be slower (e.g. 6 and 6).	X		
T1: Maybe the children could roll [roll the dice] and one [teacher] could count.			X
T2: But T1, that has to be done [live and direct with the students] how do you plan to do it with Kindergarten in this [Pandemic] scenario?			X
T1: A of course .... I was thinking about that... how do you apply it? It has to be something we can apply now [Pandemic] right?			X
T3: With R2 we thought of a race, but with colored frogs, and that, instead of rolling dice, you would roll a coin 2 times or 2 coins at the same time. The frogs would advance, for example, the orange one if the two coins go "tails", the pink one if the two coins are "heads"., and the blue one if different [head and tail].		X	

R1: Given the level of the children it could be 2 coins. I want to know which frog you would vote for.	X		
T4: I'm leaning...I have no prior training ...but I think there's a better chance of 1 or 1 [heads and seal]. R1: and why? T4: I don't know... Thinking it's harder for [both] 2 heads and 2 tails to fall out. R2: Do you have 2 coins? let's toss them 10 times!		X	
T5: I already did it, I got 2-times [both] head, 1-time tails [both], and 7-times tail and head T4: I won!		X	
T1: Playing the game I feel it's super practical, because the parents could pull or the children, and helping them to advance. We [teachers] wouldn't have to do it, my kids could do it.			X
T4: It could be playing as a family and completing a register...	X		

In the episode (see Table 5), R1 invites to anticipate certain difficulties of the students in the operation of a randomized experiment of throwing two dice simultaneously given the limited numerical scope of the kindergarten through fourth grade students. Some LSG members propose alternatives to reduce its complexity, in this case using two coins (three non-equiprobable events) instead of dice (six equiprobable events; or 36 non-equiprobable events). On the other hand, teachers discuss some pedagogical strategies that can be adjusted regarding their management in an online learning environment given the pandemic. In turn, R1's intervention provokes teachers to bring into play their content knowledge in the pedagogical context of task planning vis-à-vis possible predictions, the operation of the randomized experiment, and possible outcomes.

Discussion

It is recognized that teachers face challenges in teaching statistics, as many lack experiences in school or in their initial teacher training, particularly on aspects specific to early statistics (Estrella et al., 2020). Although research regarding teachers' PCK about ISI and ways of preparing to teach ISI is scarce (de Vetten et al., 2017), it is also recognized that teacher professional development experiences conducive to promoting PCK on ISI and associated reasoning can be generated. In this perspective, the manifestation of K-4 teachers' PCK was analyzed when planning, executing and improving the lesson plan on ISI.

Authentic informal inferential reasoning experiences (Table 3) helped to foster in teachers, an awareness of the uncertainty inherent in the outcomes presented in the situations, which led to proposing a central task for K-4 students to develop their IIR early; in turn, allowed teachers to analyze their own IIR, and to delimit future actions regarding the planning, management and evaluation of a learning sequence that could develop the IIR of their students. The planned learning sequence considered the materials, motivation, online education context, and students' prior knowledge and experiences, delineating possible teaching interventions, in the core task promoting IIR. During the implementation, review, and adjustment of the lesson plan, teachers were able to reflect on the achievement of the lesson objectives, providing feedback leading to the fine-tuning of the lesson. Finally, the instance of reflection of the LS process allowed teachers to make explicit the professional challenges, the perceived socioemotional aspects and some discoveries.

Although this report shows part of the comprehensive analysis of teachers' PCK in only one of the stages of the LSG cycle, the categories employed can be considered relevant to analyze in detail the manifestation of aspects of PCK in relation to IIR. The generation of statistical learning sequences created with the expertise of the teachers themselves together with LSG collaborating researchers in an LS that lasted several weeks, hold promise for the promotion of statistical thinking at the early school levels. It is projected that the spirit of improvement of the LS methodology will permeate both the learning sequence constructed and the experience and teaching of this group of teachers.

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Game-variation to support probabilistic reasoning

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This study investigates game-variation as a means for supporting reasoning on the bi-directional relationship between underlying probability distribution and data. A teaching experiment where students in Grade 5 (11-12 years old students) are playing three games of the Color-run constitutes the context of investigation. The study shows how variations in the sample space between the games can pull students into probabilistic reasoning about chance, random variation and sample space. The study also shows how probabilistic reasoning can face challenges from deterministic reasoning.

Keywords: Probabilistic reasoning, deterministic reasoning, random variation, sample space, inferentialism.

Introduction

This study intends to contribute to the call for increased attention on how learning environments can be designed to support students' statistical and probabilistic reasoning (Ben-Zvi et al., 2018; Langrall et al., 2017). This is approached by the research question:

> How can variation in sample space between games be used to support reasoning on the bi-directional relationship between underlying probability distribution and data?

A teaching experiment where Swedish grade 5 students are playing three games of the Color-run (Nilsson, 2020) constitutes thew context of investigation.

Theoretical background

The present study takes an inferentialist perspective on reasoning. This means that reasoning is understood to be a constructive process, taking place as students participate in and contribute to the practice of giving and asking for reasons (GoGAR) (Brandom, 2000). Think of the concept "probability", in the claim, "The probability of an outcome of six is 1/6 when rolling the die." Being an act of reasoning in GoGAR, the claim implies the assumption of a fair die, that the relative frequency of sixes stabilizes around 1/6 as we increase the number of trials, and that the probability of not having a six is 5/6. This example involves many reasons related to the concept of probability, of which only a few have been made explicit here. The main point, however, is to show that reasoning consists of knowing what follows from a claim and what it follows from, what would be evidence for it and what is incompatible with it (Brandom, 2002). For the learning of probability, this implies teaching where students are invited to make claims (take a stand) and are challenged to ask and give reasons for claims, in explorative practices of experimenting with data, making connections, explaining, inferring and generalizing (Nilsson, 2019).

Understanding the bi-directional relationship between underlying probability distribution and data implies understanding how the sample space is mapped in data, taking into consideration the effect of chance (Nilsson, 2014). On this issue, research shows some contradicting results.

Lecoutre (1992) for instance, shows that students often fail to use information of sample space when an element of chance is involved. Students argued that outcomes of a random experiment are equiprobable, no matter of the composition of the sample space, because the experiment is just a matter of chance. Noll and Shaughnessy (2012) and English and Watson (2016), on the other hand, show that students tend to stress proportional reasoning between sample space and frequency outcomes. Several studies also show that students tend to appreciate contextual, material and idiosyncratic explanations (Nilsson et al., 2018; Watson et al., 2007) over probabilistic and data-centered reasons when asked to predict or explain the distribution of data.

Method

The teaching experiment

This study is part of a small-scaled teaching experiment over three lessons in a Swedish grade 5 class (11–12 years old) with the purpose of exploring informal hypothesis testing in a probability context. The present study concentrates on the first of three lessons, with the aim of introducing the students to reasoning on random variation and sample space in the context of the Color-run (Nilsson, 2020). Three bottles with three different sample spaces were design. After shaking the bottle, the color of the marble that appears in the neck of the bottle is recorded. The color first recorded seven times wins the game.

Figure 1 shows the bottle used in the first game and the results of the three games. The entire lesson was in the format of whole-class teaching. One camera was placed at the back of the classroom to capture the whole-class discussions. In the two first games there were 15 students in the class. Another student joined in during Game 2.

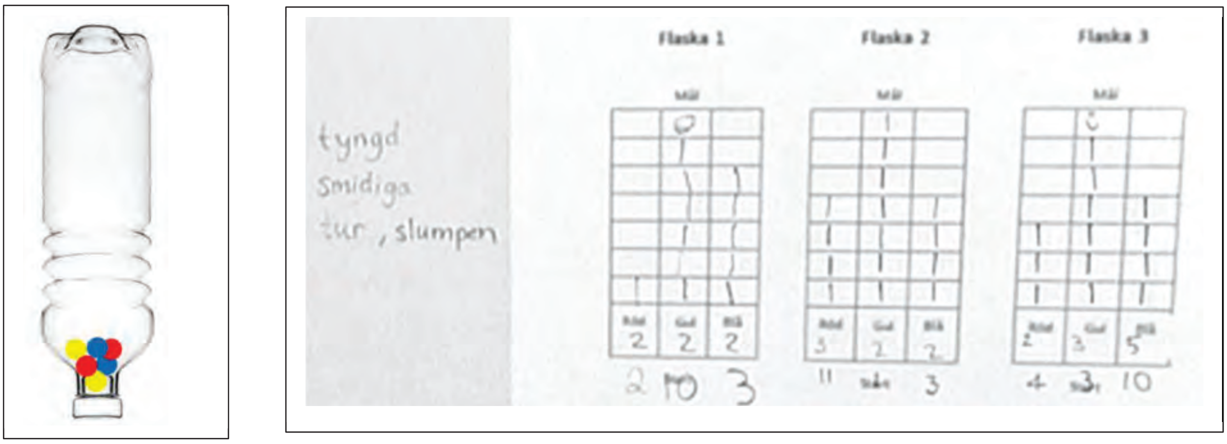

Figure 1: The bottle and the three games. The order of the three games moves from left to right. A list of reasons is to the left of Game 1. At the top of the list is 'weight', then 'agilent' and at the bottom 'luck, chance'. The row at the bottom of the playing-boards tells the colors – red to the left, yellow in the middle and blue to the right - and the number of each color in the bottle. At the very bottom are the students' votes on each game.

The rationale of the design of the game

The teaching experiment was designed by me, the researcher and author of the paper. Sara[1], the class-teacher, run the lessons and was supposed to act as she was used to. She was used to elicit and build discussion on students' ideas. If no student made connections between the win of a game and the sample space, Sara should take some careful initiatives to support such connections.

Each game was structured in three phases: predicting-playing-reflecting. First, the students were asked to predict which color they think will win and to provide reasons for their predictions. Second, the game was played. Third, the students were asked to reflect on the outcomes of the game, connecting back to their predictions. It was expected that a game both would ask for reasons and give reasons. It would ask for reasons if a game outcome was different from students' expectations. Experiences from previous games would help in giving reasons for predicting a subsequent game.

A principal idea of the variation theory is that we can only discern an aspect of a learning object if we experience a variation in that aspect (Runesson, 2006). In the present study the bi-directional relationship between sample space and data was an aspect students were supposed to discern. On this account, the sample space was changed between the three games. The game should also incite reflections on random variation. This raised questions on the length of a game. A long game would stress the regularity between sample space and data but mask random variation. A short game would stress random variation but mask the relationship between sample space and data. In preparing the lesson, I played the game of different length and decided to use a seven-step playing board (Figure, 1).

Method of analysis

In the analysis I looked for signs of taking a stance and of asking and giving reasons for a stance taken. Taking a stance (expressing a claim) on an outcome, was initiated by the teacher asking, "Who do you think will win?". Taking a stance was also made explicit in voting on a color. Signs of asking for reasons are 'why-questions" or questions like, 'how are you thinking?'. Signs of giving reasons come with signal words like, 'because' and 'since'. The results are structured according to the phases of predicting-playing-reflecting in each game.

Analysis and Results

Game 1

The sample space in Game 1 was, 2R, 2Y and 2B (Figure, 1).

Predicting on Game 1

After introducing the game and making all aware of that it is two marbles of each color in the bottle, Sara asked, "Which color do you think wins?". The votes from the students were two on

[1] Sara is a pseudonym for the teacher's real name.

red, ten on yellow and three on blue. The students gave both deterministic and probabilistic reasons for their votes.

Of the deterministic reasons were an idea of distance the most emphasized. When Sara asked the students to vote, she held the bottle horizontally and a yellow marble was closest to the bottle top. The students noted that the yellow was “At the front”. Hence, being at the front, the yellow marble has the shortest distance to the bottle top and so, most likely be first to the top when turning the bottle around. Sara did not shake the bottle to challenge the idea.

Probabilistic reasons are reasons that take into consideration sample space and random variation. For instance, when Sara asked Marcus why he voted for blue he said, “I don’t know. One should pick one and it doesn’t matter which one”. We can interpret Marcus as if he considers the win of the game to be just a matter of chance, since each color has as many favorable outcomes.

Playing Game 1

Sara turned the bottle each time. One of her students helped her at the whiteboard to mark the outcomes in the race. Sara moved around in the classroom to engage all students in reporting on outcomes. The first game ended with one on red, seven on yellow and five on blue.

Reflecting on Game 1

Sara asked, “Do you have any ideas why yellow won?” Students still argued for the distance to the bottle top. Now, Sara challenged this, “But I shook the bottle. I thought I was shaking a lot” [Sara shakes the bottle]. No student responded, probably because the shortcomings of their idea became obvious to them. Another deterministic reason that came up was, “Maybe because it is heavier? [giggling] They [the yellow] look bigger”. Sara looked close to the bottle, and responded with a long yes, which can be interpreted as if she wanted to tell that all marbles are exactly the same, without dismissing the student’s contribution. In general, Sara struggled between an attitude that no answer is wrong and trying to turn the students away from deterministic reasons towards probabilistic reasons.

Albin explained the win of yellow by, ”Yellow has luck”. After Albin’s suggestion Sara made a list on the whiteboard of the reasons students came up with (Figure, 1). Noting ‘weight’ and ‘at the front’, she asked if she had missed anything. One student repeated luck. Sara revoiced luck but with hesitation in the tone. She also expressed, “Lucky color maybe”. Sara added ‘Luck’ to the list. It seems as if Sara considered luck not a characteristic of randomness but a material property of the yellow color. Why Sara made this interpretation might be because Albin said, “Yellow has luck” and not “Yellow had luck”. The former signals that yellow possesses luck as a property. Regardless of if Sara made this interpretation or not, the situation shows how important it is to make inferences explicit in teaching, so the participants know they are talking about the same thing.

Game 2

The sample space in game 2 was, 3R, 2Y and 2B (Figure, 1).

Predicting on Game 2.

Sara made explicit the number of colors in the bottle. Eleven students voted on red, one on yellow and three on blue (Figure, 1) and Sara asked, "Yellow was a color of luck, it had the weight for winning, but now we do not believe in yellow anymore, it is only one that believes in yellow? Why do we believe most in red this time?". Daniel articulated, "It is most of them". Several students agreed with Daniel. Mario, who claimed that yellow won Game 1 because they looked bigger (heavier) now claimed that red looks bigger. However, when Sara claimed the marbles are exactly the same he moved to probabilistic reasoning:

Mario:	But, it can be because, it is more, it is greater, like, 65% percent chance that red wins as they are one more than the other".
Sara:	Where did you get the 65% chance from?
Mario:	Sorry, I say 60% chance. 60% chance and the other have 20% chance, because, yes red, because, last time it was 50, 50 50.

Mario seems to confuse two ways of quantifying the probabilities. He ended with, "last time it was 50, 50 50", which implies that he connects equally likely outcomes to a 50-50 situation. When he articulated 65% chance on red, we do not know if he changes the chance also of the other two colors or if he ends up in the triplet 65-50-50. However, Mario's change to 60% can be understood as if he discovers that the total probability needs to be 100%. Nevertheless, extending on Daniel's sample space reasoning, Mario tries to give reasons for the probabilities of the three outcomes by making connections to the underlying sample space. Being able to compare between different games helped him in making this connection.

Sara held the bottle horizontally also when the students were predicting the second game. However, this time, no student expressed they vote on a color because it is at the front.

Playing Game 2

After five observations in Game 2, there were three yellow, one red and one blue. When the third yellow appeared in the fifth observation one student voiced, "What?". Sara picked up on that, asking, "Anyone who has any thoughts now when yellow has run away with three?". Following his previous line of reasoning, Albin responded, "They (yellow) are lucky!"

Sara stopped the game again after nine observations. Now there were six on yellow, two on red and one on blue. Sara asked, "How can this be?". Reasons given were that it is something with yellow; that yellow is closes to the bottle top most times and that yellow won last time.

Next come five observations with no yellow. One student claimed yellow got tired and another that yellow wants to create a thrill. Since these answers are accompanied with a laugh, I interpret them more as jokes than as real reasons. No student referred to luck or chance. The next observation is yellow, and the game ended with four on red, seven on yellow and four on blue.

Reflecting on Game 2

Sara initiated reflections on Game 2 by connecting the game-variations to the list of reasons posted on the whiteboard. She asked, "Do we still think it is about these things [moving her hand over the list] or is it something new, something we should delete, because we are going to play another game with another bottle, so we need to have some thoughts before". The students negotiated on deleting the social reason 'Most votes' and the distance reason 'At the front'.

They kept weight and added that yellow is agilent and bouncy. However, most students raised their hand for “luck” as a reason for why yellow won. There is reason to believe that this was stimulated by how Sara was pausing the game, highlighting aspects of random variation.

No student referred to the role of sample space, which was probably because the situation was about explaining why yellow won. The sample space did not answer to why yellow won.

Game 3

The sample space in Game 3 was, 2R, 3Y and 5B (Figure, 1).

Predicting on Game 3

In Game 3 it was evident that the sample space is a strong reason for many students. In Game 3 there are 16 students but, there are 17 votes, distributed as, four on red, three on yellow and nine on blue. From the video it is hard to figure out where the extra vote comes from. However, most likely it is an extra vote on yellow, because it appeared some confusion when Sara counted the votes on yellow. Most students voted on blue, because there are most blue in the bottle. So, even if the students did not receive empirical evidence in Game 2 for the role of the sample space, most students put forward properties of the sample space as the strongest reason for why it is highest chance that blue wins Game 3.

Charles took a specific turn on the sample space. He voted for red because there are two red marbles in the bottle. He shaped this reason from the two first games, “It is always two marbles that win”. Sara responded, “Okay, it is always those [colors] with two marbles, yes it was two yellow [looking at the playing-board of Game 2] and two yellow also there [looking at the bottle of Game 1]. Okay, so when it is two of the same sorts, it wins!”. Another student, Albin, challenged this by noting that there were two of each color in Game 1 and that there were also two blue in Game 2. Sara’s first reaction to Albin was to silence him but, almost immediately she acknowledged his objection:

Sara: “But now it's Charles who ..., here you're [talking to Sven] talking about there are two. Yes, here it was two of each [pointing to Game 1]. Here then [looking at Game 2]? Yes, there were two blues as you said. Yes, hmm [nodding her head from side to side].

Even if Sara did not move further on this contradiction, the episode shows how game-variation can invite for reasoning on sample space in relation to exploring patterns between games.

Playing Game 3

Again, the game provided students opportunities to experience random variation in short series. After seven observations Sara made a short break. She did not ask for any reflections or thoughts. She just let the student take in that there is one observation on red, two on yellow and four on blue. No more blue appeared in Game 3. Yellow won again. In the last seven observations it appeared six yellow and two red. The game ended by, three on red, seven on yellow and four on blue.

Reflecting on Game 3

Several students thought that the yellow marbles are heavier and therefore drop down to the bottle top faster than the other. It is probably unlikely to many students that yellow wins over blue only because of chance, since there were so many more blue than yellow marbles in the bottle. In terms of hypothesis testing, we can say they consider the result of seven yellow out of a total of 14 observations so extreme that they reject the null hypothesis P(yellow)=3/10. The result on yellow is rather unlikely, but not significant on the level of 0,05 (two-sided test). However, we need to take into account that yellow won all three games. This experience most likely further support that chance does not tell the whole story for the results on yellow, compared to the other colors, within and between the games.

Chance was also highlighted in reflecting on Game 3. When Sara asked for other ideas on why yellow won, Albin again answered “luck”. Charles then added “the chance”, which Sara added to the list of reasons, after luck (Figure, 1). The meaning of luck or chance was not given any further consideration. The class negotiated on deleting bouncy and keeping agilent in the list of reasons.

Concluding discussion

How can variation in sample space between games be used to support reasoning on the bi-directional relationship between underlying probability distribution and data? The present study answers to this research question by reporting on a teaching experiment where students in Grade 5 (11-12 years old students) were playing three games of the Color-run (Nilsson, 2020).

In line with previous research (e.g., Nilsson et al., 2018; Watson et al., 2007), this study shows how probabilistic reasoning can face challenges from deterministic reasoning. Deterministic reasoning was manifested as distance reasoning and reasoning on material properties of the random generator. Distance reasoning relates to the perception that random experiments can be controlled (Pratt, 2000): you can control the outcomes by ordering them in a certain way. Distance reasoning neglects the random process, which, in the present study, took place in shaking the bottle. Reasoning on properties took place in reflecting of material difference between the colors in the bottles. Material differences were manifested as differences in weight, bounce, and agility. Reasoning on material differences does not need to neglect the random process, where students predict too narrow ranges of variation (Noll & Shaughnessy, 2012). After Game 3, for instance, material differences were given increased attention. The result of yellow in Game 3 was almost significant for rejecting the parameter value, according to sample space. Hence, it was reasonable to reflect on if it could be something more to the situation, beyond sample space and chance, to explain the strongly deviating result of Game 3.

In contrast to Lecoutre (1992), the present study shows that students can express strong commitments to the role of the sample space. In Game 3, for instance, even though the number of favorable outcomes was not decisive for the outcome of Game 2, most students used the number of favorable outcomes to give reason for why blue would win in Game 3.

Several students referred to luck, to explain the outcome of the games. What students mean by luck, is not always easy to tell. Does it refer to a property of an outcome or is it synonymous to chance? However, it would not be inferentially possible to substitute “Yellow has luck” with

“Yellow has chance”. Hence, the present study suggests further research on the meaning of luck. Taking an inferentialist perspective (Brandom, 2000), the study particularly suggests investigating situations in which students are supported to develop GoGARs of many inferential relationships that include the concept of luck.

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Enhancing students' data literacy. When learners evaluate their school's sustainable development.

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Keywords: Education for sustainable development, data collection, processing data.

Introduction

Important decisions in politics, economics, health, and society are based on statistics. In order to participate in society as a responsible citizen, data management is an essential skill (e.g., Nichelson et al., 2018). For students growing up in a data-driven world, it is crucial to not only know how to analyse data, but also to think about how data are generated and what data can or cannot tell us (Wild et al., 2018). The present study follows the work of Andre et al. (2020), aiming to re-design their learning trajectory towards a student-centred approach on data collection and processing of data.

Theoretical background

Wild et al. (2018) list key statistical ideas important to understand at a deeper level including data collection, statistical modeling, covariance, and others. Student-centred methods lead to deeper understanding of concepts and the use of technology when exploring real datasets is crucial therefore (e.g., Wild et al., 2018). The PPDAC-cycle (Wild et al., 2018) describes such statistical investigation processes. Moreover, many studies emphasize that a context meaningful for students contributes to their development of competences (e.g., Makar und Ben-Zvi, 2011). Andre et al. (2020) take sustainability issues as such a context for students' statistical investigations. With the doughnut economy, Raworth (2017) presents an economic model for describing sustainable development (see figure 1), where social thresholds should be fulfilled while ecological boundaries should be obeyed.

Methods and Implementation

Following design-based research approaches (Bakker, 2018), a learning trajectory was generated to guide students of the 'Secondary School Centre' of Vipiteno in the district of South Tyrol through their own statistical research on sustainable development of their school's community. The aim of the study was to identify opportunities and challenges that arise during their statistical investigations.

31 students of two grade 12 classes participated in this study. They completed 12 lessons of 45 minutes each. In the first part, students were asked to identify factors contributing to a good and equitable life in the school community. Based on these results, basic ideas of the doughnut model (Raworth, 2017) were introduced, and a *school doughnut* was assembled containing students' ideas of their schools' sustainable development including issues such as energy consumption, CO_2-emissions on their way to school, political voice, mental health and many more. Subsequently, students were guided in their working processes of collecting and analysing data on these issues. Currently, results of the assignments and video recordings of the lessons are analysed qualitatively.

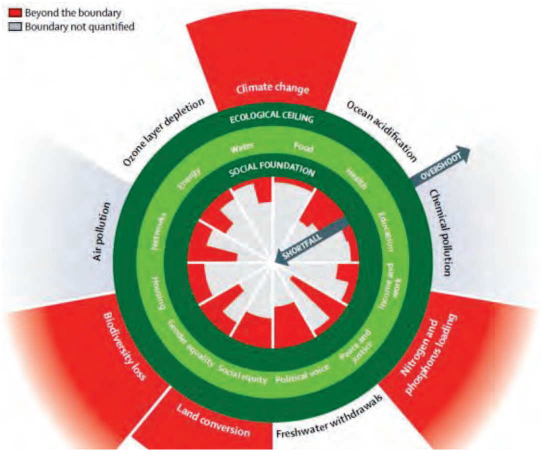

Figure 1: Doughnut Model (Raworth, 2017, p. e48)

First Results

Following the PPDAC-cycle, students managed to generate a doughnut model of their schools' sustainable development integrating several crucial topics. Thus, students could develop a deeper understanding of how to describe a phenomenon with statistical procedures. Two main categories of our data analysis are processes of data collection and generating statistical questions where we found major difficulties. Moreover, a third main category were students' struggles when defining specific boundaries and thresholds to classify the results of their investigations. These results are used to re-implement an improved learning trajectory in order to support students to overcome these difficulties.

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How secondary school students build frequency tables?

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The ability to build and read statistical tables is part of statistical literacy and its teaching is included in the Spanish curriculum. We present results of an evaluation study carried out with 128 students in the 3rd grade of Compulsory Secondary Education (14-16 years old) who were asked to construct a distribution table with absolute and relative frequencies from a data list. About half of the sample performs a correct construction and calculation of absolute frequencies and 15% of relative frequencies. Based on the ontosemiotic approach, semiotic conflicts such as confusion of relative, percentage and absolute frequencies and erroneous procedures in the calculation of these frequencies are identified.

Keywords: Statistical education, statistical tables, student evaluation.

Introduction

Statistical tables are an important tool for summarizing and communicating information both in the media (Estrella, 2014) and in the scientific field (Feinberg & Wainer, 2011). Consequently, the ability to read, interpret and build statistical tables should be a component of statistical literacy for all citizens (Gal, 2019). Taking into account this goal, different curricular guidelines, including the Spanish documents (MECD, 2014) propose working with statistical tables throughout primary education (6 to 11 year-olds) to record and classify qualitative and quantitative data, as well as to build absolute and relative frequencies tables. Although statistical tables are considered easy by teachers, due to their wide use (Koschat, 2005) and their high presence in textbooks (Pallauta et al., 2021), research shows the need to reinforce its teaching, given that tables require specific cognitive skills for their understanding and construction (Martí, 2009). The aim of this study is analysing the construction of one-variable distribution table and identifying the semiotic conflicts that may arise in the process of organizing and summarizing a data list in a group of Spanish secondary education 3rd grade students.

Fundamentals

Theoretical framework

We base on the Onto Semiotic Approach to mathematical knowledge and instruction (OSA; Godino et al., 2007; 2019), which assumes that mathematical objects emerge from the mathematical practices (actions or operations) put into play to solve problem-situations. In this theoretical framework, the mathematical object can be considered from an institutional (e.g., a teaching institution) or personal (a person) perspective. The following types of mathematical objects are distinguished: problem-situation, language, concepts, propositions, procedures, and arguments. In the OSA the understanding of a mathematical object involves a synchrony between the institutional and personal meaning attributed to that object, which should be built progressively during the teaching-learning process (Godino, 2002). Furthermore, when there is a discrepancy between both meanings, a *semiotic conflict* is produced and may be classified according to the type of mathematical object involved. A conflict is conceptual if it refers to a misunderstanding of concepts or properties, and procedural when the students confuse a procedure.

Background

Research analysing the students' performance in the construction of statistical tables is scarce, despite this type of representation is widely used in the classroom (Estrella & Estrella, 2020). Such construction, according to Chick (2004), among other processes, requires the ordering and classification of data, their grouping and calculation of frequencies, which implies high cognitive demand, especially for elementary school students (Nisbet et al., 2003). Pfannkuch and Rubick (2002) point out that students are unaware of the need to summarize information, and it is complex for them to establish classification criteria when they analyse a large list of data. In this sense, Marti et al. (2011) suggest that the construction of tables requires segmentation processes along with deciding the variable(s) to be represented, the frequencies to be computed and adjusting such information in the spatial structure that characterizes the table. Research with primary school children confirms these difficulties: it is not easy for them to translate graphs to frequency tables (Díaz-Levicoy et al., 2018), or find criteria to organize data in tables (Estrella & Estrella, 2020). As regards adolescents, Álvarez et al. (2020) conducted an investigation with 100 Colombian students aged 15 to 18 years to analyse their difficulties in constructing frequency tables. The authors observed errors in the construction of tables with grouped data; confusion of different types of frequencies with the variable values and arithmetic errors in compute different types of frequencies. We note that most of research developed with secondary school students has focused on the two-way tables, so our work intends to contribute knowledge of these students' difficulties when constructing a one-dimensional frequency table.

Method

The study involved 128 Spanish students just beginning the 3rd grade of Compulsory Secondary Education from two schools in the same region, which split in a total of 70 males and 58 females ranging in age from 14 to 16 years. The participants had worked with statistical tables throughout primary education and their knowledge had been reinforced in 1st and 2nd grades of secondary education in accordance with a common standard for learning of the Andalusian curricular guidelines: "Organises data, [...] of qualitative or quantitative variables in tables, calculates their absolute and relative frequencies". (Consejería de Educación y Deporte, 2021, p. 785, p. 788). To these students we proposed a situation to be solved individually during the mathematics lesson, with paper and pencil (Figure 1), which is part of a wider questionnaire aiming to analyse the secondary students' knowledge on statistical table and to identify the semiotic conflicts that may arise in their answers. The task was adapted from a textbook (Kheong et al., 2017), and selected by suggestion of a panel of experts among three possible items. This type of table is frequently used in secondary school in Spain. Although the structure of the statistical table is provided in the task, no example was provided to complete it so, firstly, the student must identify the different modalities of the variable; no order of the categories is required because the variable is qualitative. Secondly, he/she had to find the absolute and relative frequencies associated with each modality, and finally calculate the corresponding totals. A similar activity was used by Díaz-Levicoy et al. (2020) with Chilean primary school students in the 3rd grade (8 years old), who had to translate a data list to a table, where a column was provided to record the count and absolute frequency of each modality and no relative frequencies were requested.

Claudia is very fond of insects and one morning she made a list of those she could observe in her garden:

ant - butterfly - ant - bee - ant - ant - mosquito - mosquito - bee - mosquito - butterfly - ant - ant - mosquito - ant - ant

Complete the table with the data collected by Claudia:

Insect	Absolute frequency	Relative frequency
Total		

Figure 1: Task proposed to the students

A content analysis (Bisquerra, 2019) was carried out to analyse the students' productions, in which the authors systematically reviewed the categories established in the previous analysis, and in the discordant cases a consensus was reached to ensure the reliability of the coding. Thus, the evaluation of the productions around the construction of the table were discussed in two rounds and were classified into three categories: correct, partially correct and incorrect; and additionally various semiotic conflicts that arose in this process were identified according to our knowledge and results of previous research in three rounds.

Results

This section presents firstly the results of the evaluation of the tables constructed by the students and then the different cognitive semiotic conflicts identified along with their distribution.

Statistical table construction

The criteria for classifying the construction of the table were as follows:

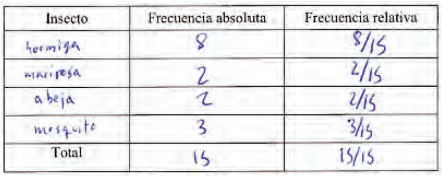
Correct table. The student adequately completes the columns corresponding to the absolute and relative frequency of each modality, using the information given (Figure 2), where the relative frequency can be represented through decimal numbers or fractions (simplified or not).

Insecto	Frecuencia absoluta	Frecuencia relativa
Hormiga	8	$\frac{8}{16}$
Mariposa	2	$\frac{2}{16}$
Abeja	2	$\frac{2}{16}$
Mosquito	4	$\frac{4}{16}$
Total	16	$\frac{16}{16}$

Figure 2: Correct construction of the statistical table (student E16)

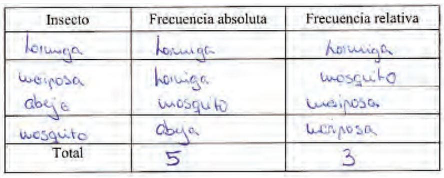
Partially correct table. In this category we classified the answers in which most of the cells were filled in with correct values, but there were some errors or blank cells. An example, with a counting error in one absolute frequency, which affects the calculation of the total, is presented in Figure 3 (left hand side).

Incorrect table. The table records values that make no sense, showing a lack of knowledge of the concepts of absolute and relative frequency, such as the example shown in Figure 3 (right hand side).



Insecto	Frecuencia absoluta	Frecuencia relativa
hormiga	8	8/15
mariposa	2	2/15
abeja	2	2/15
mosquito	3	3/15
Total	15	15/15

Partially correct construction of the statistical table with conflict in data classification (student E9)



Insecto	Frecuencia absoluta	Frecuencia relativa
hormiga	hormiga	hormiga
mariposa	hormiga	mosquito
abeja	mosquito	mariposa
mosquito	abeja	mariposa
Total	5	3

Incorrect construction of the statistical table (student E5)

Figure 3: Partially correct and incorrect construction of the statistical tables

The results on the building of the statistical table according to its correctness are presented in Table 1. A high percentage of students incorrectly computed the frequencies (absolute: 23.4%; relative: 62.5%) or did not compute them (absolute or relative: 18.8%). This result indicates that the task of classifying the values of the variable and forming the distribution was not simple. In the study developed by Díaz-Levicoy et al. (2020) with primary education 3rd grade students, most participants computed absolute frequencies correctly (68.4%); however, they were given a column to complete the counting, and the modalities were previously given, which facilitated task. The authors did not ask to complete the total. In our study, a low percentage computed the relative frequencies successfully (14.8%), while results were better for absolute frequencies.

Table 1: Frequency (percentage) of tables according to its correctness and frequencies computed

Table	Frequencies computed	
	Absolute	Relative
Correct	63(49.2)	19(14.8)
Partially correct	11(8.6)	5(3.9)
Incorrect	30(23.4)	80(62.5)
Not calculate	24(18.8)	24(18.8)
Total	128	128

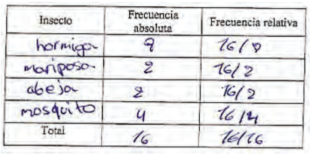
Semiotic conflicts in the construction of the table

A second analysis of the responses containing partially correct and incorrect construction of the statistical tables led to the identification of some semiotic conflicts, which are described below according to whether a concept o procedure is misunderstood. Some students showed more than one of these conflicts, and consequently the percentages in Table 3 do not add up to 100%. The conflicts were classified as procedural or conceptual.

Procedural semiotic conflicts

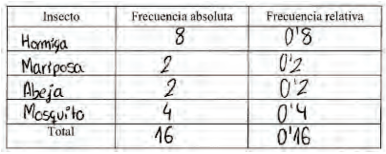
- P1. *Confused or incorrect classification of values*. Some data are classified incorrectly, leading to an error in the calculation of frequencies, as in the example shown in the left-hand side of Figure 3, which alters the total.
- P2. *Conflict in the computing relative frequencies*, which were classified in three different types:
  - P2.1. *Inverting the numerator and denominator* in the calculation of the relative frequency (Figure 4), which evidences a confusion in the definition of this concept, as well as misunderstanding the part-whole relationship (Álvarez et al., 2020).
  - P2.2. *Multiplying or dividing by a power of 10 to calculate the relative frequency,* instead of

using the total number of insects in the sample (Figure 5). An explanation is that the student tries to incorrectly obtain the relative frequencies from the percentage.



Insecto	Frecuencia absoluta	Frecuencia relativa
hormiga	9	16/9
mariposa	2	16/2
abeja	2	16/2
mosquito	4	16/4
Total	16	16/16

Figure 4: Construction of the statistical table with conflict P2.1 (student E249)



Insecto	Frecuencia absoluta	Frecuencia relativa
Hormiga	8	0'8
Mariposa	2	0'2
Abeja	2	0'2
Mosquito	4	0'4
Total	16	0'16

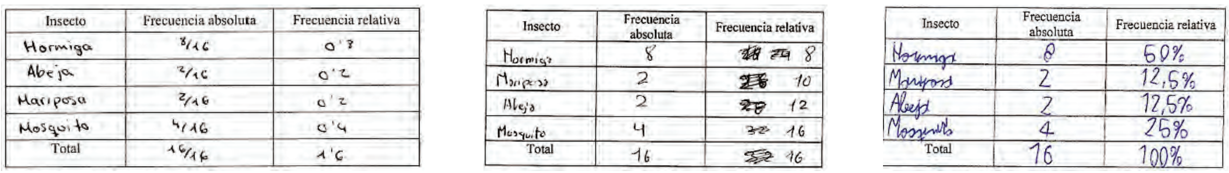
Figure 5: Construction of the statistical table with conflict P2.2 (student E58)

- P2.3. *Not computing the relative frequencies.* When the student leaves the column provided to record the relative frequencies incomplete.

- P3. *Conflict in computing totals (of absolute or relative frequencies).* These are the answers in which the total of the frequencies is not calculated, or show confusion in the addition algorithm.

Conceptual semiotic conflicts

- C1. *Confusion of different types of frequencies*, which found in previous research (Álvarez et al., 2020; Batanero & Godino, 2001; Fernandes et al., 2019) and can be classified into several types:
  - C1.1. *Confusing absolute and relative frequencies.* An example is given in Figure 6 (left hand side), where E23 completed the absolute frequencies using the formula for relative frequencies; this conflict was also presented inversely, that is, some students record absolute frequencies both in the absolute and relative frequencies columns.
  - C1.2. *Confusing relative and cumulative frequencies.* Similar to the conflict described above, some students provided cumulative frequencies in the column of relative frequencies (Álvarez et al., 2020) as can be seen in Figure 6 (centre, answer of E130).
  - C1.3. *Exchanging relative frequencies and percentages.* This response was less common, and includes the students recording percentages instead of relative frequencies for each modality of the variable as can be seen in the response of E248 in Figure 6 (right hand side).



Insecto	Frecuencia absoluta	Frecuencia relativa
Hormiga	8/16	0'8
Abeja	2/16	0'2
Mariposa	2/16	0'2
Mosquito	4/16	0'4
Total	16/16	1'6

Conflict C1.1 (student E23)

Insecto	Frecuencia absoluta	Frecuencia relativa
Hormiga	8	8
Mariposa	2	10
Abeja	2	12
Mosquito	4	16
Total	16	16

Conflict C1.2 (student E130)

Insecto	Frecuencia absoluta	Frecuencia relativa
Hormiga	8	50%
Mariposa	2	12,5%
Abeja	2	12,5%
Mosquito	4	25%
Total	16	100%

Conflict C1.3 (student E248)

Figure 6: Students' construction of the statistical table with conflict C1

- C2. *Not understanding the table content and meaning of the concepts involved in its structure.* This type of answer is uncommon and is shown in Figure 3 (right hand side), where the student records in the cells combinations of three insects, instead of computing absolute and relative frequencies. The concepts of variable, modality and frequency are not understood.

The semiotic conflicts presented in the construction of the statistical tables according to the content analysis made and whether they refer to absolute or relative frequencies are showed in Table 2. It should be noted that some answers showed more than one conflict (total percentages exceed 100%).

Table 2: Frequency (percentage) of conflicts in the tables according to the frequencies computed

Type of semiotic conflict	Abs. Freq.	Relative Freq.	Total (n=128)
P1. Misclassification of values	25(19.5)		25(19.5)
P2. Miscomputation of relative frequencies			
P2.1. Inverting numerator and denominator		8(6.3)	8(6.3)
P2.2. Dividing or multiplying by 10		11(8.6)	11(8.6)
P2.3. Not computing the relative frequencies		46(35.9)	46(35.9)
P3. Incorrect or not computing the total	39(30.5)	84(65.6)	123(96.1)
C1. Exchanging different frequencies			
C1.1. Absolute and relative	4(3.1)	6(4.7)	10(7.8)
C1.2. Relative and cumulative		5(3.9)	5(3.9)
C1.3. Relative and percentages		1(0.8)	1(0.8)
C2. Not understanding the table meaning and structure	7(5.5)	6(4.7)	13(10.2)

The most frequent semiotic conflict was P3 (96.1%) consisting of errors or absence of totals, particularly in the relative frequency (65.6%) that may be due to the lack of knowledge in the properties associated with the table (e.g., sum of total relative frequency is 1). It was followed by P2.3 (35.9%) linked to not computing the relative frequencies, giving evidence of the lack of proportional reasoning; then P1 (19.5%) corresponding to misclassification of values; and C2 (10.2%) lack of knowledge in the table structure and the concepts involved in its construction. Infrequent conflicts were P2.2 (8.6%) in which the relative frequency is obtained by dividing or multiplying the absolute frequency by powers of base 10, or C1.1, consisting in exchanging absolute and relative frequencies (7.8%). In absolute frequencies there were few conflicts (P3: 30.5%, P1: 19.5% and C2: 5.5%).

Discussion

The analysis of students' productions suggests that the process of building a statistical table is not simple, which coincides with other studies (Estrella & Estrella, 2020; Marti et al., 2011). It is striking that given the age of the students and their educational career, some of them lack minimal knowledge of the process of building a one-dimensional frequency table. Thus, a wide variety of semiotic conflicts in this process was detected, the most recurrent being the absence or incorrect calculations of the total, followed by the absence of calculation of relative frequency. Other students who tried to complete it, confused frequencies or showed a confusion of the concept manifested in procedural conflicts. These results confirm the need to make the teaching statistical table construction explicit (Martí, 2009), as well as further research within different educational levels and various situations to that described in our study such as showing the structure of the table completely, partially or not at all (e.g., Gea et al., 2020), and the translation between different representations (graphs, statistical measure, etc.), among others.

Acknowledgment

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An example of rich, real and multivariate survey data for use in school

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Data exploration and getting new insights from data is more important today than ever before. Exploring data is an important part of data science and can be a fruitful topic in middle school. New insights can be gained from rich data, but this requires a good data basis. Therefore, we present multivariate and rich data of over 1200 young people who answered an online survey on more than 150 questions. Several examples of possible data explorations of this data are presented. A short glimpse is given to a corresponding teaching unit for grade 8-10. With a suitable data science tool, students can handle and analyze such rich data. Students' presentations at the end of the teaching unit show the insights students get from the data but also, show challenges when exploring multivariate data.

Keywords: Data science, multivariate data, statistics education, CODAP.

Introduction

Statistics and data exploration have a long tradition and is fortunately also getting more attention in the classroom. With the rise of the new field of data science in recent years, its importance is once again increased (Ridgway, 2016) and gets new perspectives from computer science and special domain knowledge. In a data science project (Rubin & Mokros, 2018), various skills can be acquired and used that are helpful for good data exploration. On the one hand, this requires having rich, real data for students to draw real insights and conclusions that are important and motivating to them (Garfield & Ben-Zvi, 2008). On the other hand, a tool is needed that provides easy access to data analysis (Biehler, Ben-Zvi, Bakker, & Makar, 2013). In this paper, we present an example of such data which were collected as part of the ProDaBi project (https://www.prodabi.de/en/) and use the CODAP tool for exemplary explorations.

Background

The Project Data Science and Big Data in Schools (ProDaBi) aims at investigating in which way and with what topics data science can be implemented in the school curriculum. The project was initiated by Deutsche Telekom Stiftung and is conducted by an interdisciplinary team with members from statistics and computer science education. In this context, we collaborated with a media education research association[1] that conducts representative surveys among young people. We received an elaborated questionnaire for telephone interviews about media use of young people (so called "JIM-study"; JIM=Youth, Information, Media) and adapted this as an online survey. Since 2019, many young people participated in this online survey, so that we got rich and real data on young people's media use, which we call *JIM-PB*, based on the official JIM study and our regional reference

[1] https://www.mpfs.de

(PB=Paderborn). Please note, in contrast to the official data from the JIM-study, our data does not claim to be representative.

The data is used in different teaching settings. At first, it is used as an introduction in a project course on data science in grade 12 (Frischemeier, Biehler, Podworny, & Budde, 2021). This project course consists of three modules: (1) basics of data analysis and statistical thinking, (2) algorithmic thinking and machine learning, and (3) application in a comprehensive project.

Second, adapted from these modules 1 and 2, we have developed a teaching unit for grades 8-10, which also introduces statistical reasoning and machine learning. Here, students work entirely with the CODAP tool, both for data exploration and for creating decision trees based on data. This entire unit makes use of the JIM-PB data. Enhancing statistical thinking as proposed by Wild and Pfannkuch (1999) and introducing predictive modelling (Ridgway, Ridgway, & Nicholson, 2018) with the use of decision trees are the two main purposes of the teaching unit.

Rich data that allow multivariate explorations consist of many variables of different types to motivate creative investigations. How the JIM-PB data fulfils this is shown in the next section.

The data “JIM-PB”

The JIM-PB data is based on an online survey that contains various thematic blocks. The first block contains questions on age, sex, grade, type of school. The following blocks contain question in the style “How often do you…”

- … do leisure activities (e.g. meeting friends, playing an instrument, etc.),
- … use classical and digital media (e.g. newspapers, radio, etc.),
- … use media devices (e.g. tablet, gaming console, PC, etc.),
- … use social media (e.g. Facebook, Twitter, WhatsApp, etc.),
- … watch different YouTube videos (e.g. on product tests, letsplays, sports, etc.),
- … play different electronic games (e.g. car racing, shooter, adventure, etc.),
- … use specific apps (e.g. for news, public transport, school, etc.).

Values for all the corresponding questions on the frequency range from “daily”, over “several times a week”, “once a week”, “once in a fortnight”, “once a month”, “less often” to “never”.

Some questions are posed concerning the weekly time in minutes e.g. on watching TV, playing electronic games or using a PC for school at home. Additionally, questions are asked about availability or owning devices for media use (e.g. PC, Laptop, Tablet, WiFi, etc.). Questions are like “Is the device available at your home?” with values “available at home” or “not available at home” and “Do you own such a device?” with values “I own it myself” and “I do not own it”.

This leads to 161 different questions, resp. variables. 1287 young people fully answered the survey, so the data contains as many cases from September 2020 until June 2021. A list of variables lists the original questions and the variable’s names in the data. Data handling is done in advance. Variables are defined and little data cleaning like correction of German umlauts is necessary.

This results in rich and multivariate data with numerical and many categorical variables. In particular, the large number of variables makes it possible to develop and investigate very creative questions.

The Jim-PB data in a teaching unit for middle school

The idea of using the Jim-PB data in school is mainly to introduce reasoning about data in the frame of a data project. The PPDAC-cycle (Wild & Pfannkuch, 1999) frames the teaching unit and

> [w]orking with SP [statistical projects] thus represents a strategy that can enrich curricula because each phase involved in developing a project entails the use of various statistical concepts and processes that go beyond the topics normally included in curricula (Gómez-Blancarte & Ortega, 2018, p. 5)

For use in school, we created two versions of the data. The "standard" version is didactically reduced and contains 50 variables. A selection was made for this based on content criteria. For example, all questions about "owning" a device and playing different games were abandoned. We recommend using this version in grade 8-10. The "large" version contains all 161 variables.

The selection of a tool is crucial for doing data science in middle school. We selected and applied our JIM-PB data to the online data exploration platform CODAP (https://codap.concord.org), which is a free and online software that allows an easy and quick start of data exploration for beginners. For a detailed description of CODAP see for example Haldar, Wong, Heller, and Konold (2018).

For the teaching unit in middle school, we have chosen personalized advertising as the context. This has a motivation and an everyday relevance for students. The students are asked to explore the data with regard to four topics: user groups of TikTok, online newspapers, Youtube Letsplay videos and use of game consoles. Eight lessons with 45 minutes each can be used to enhance students' statistical skills when exploring Jim-PB data with CODAP in the first part. Additional eight lessons then build on findings from the first part to create and understand the method of decision trees for predictive modelling within CODAP. For the first part, three lessons can be dedicated to introducing basic terms, the data and tool, and concepts like row, column and cell percentages. Posing statistical questions (Arnold, 2013) and preparing group work can happen in lesson 4. The following two lessons can be used for students' own data exploration and preparation of a presentation. That results in a following presentation lesson. Reflections on the data, the PPDAC cycle and the conclusions can happen in the eighth lesson of the first part.

Explorations of the Jim-PB data with CODAP

What's in the Jim-PB data? At the beginning of an analysis, it is worth taking a brief look at one-dimensional distributions to characterize the sample of respondents.

Figure 1: Distribution of sex and age in the Jim-PB data

A little more female (56%) than male (44%) students participated in the survey, aged 10-20 with peak at age 15 (Figure 1). For example, 78% have a game console available at home and a tablet is used by slightly less than half for gaming (Figure 2).

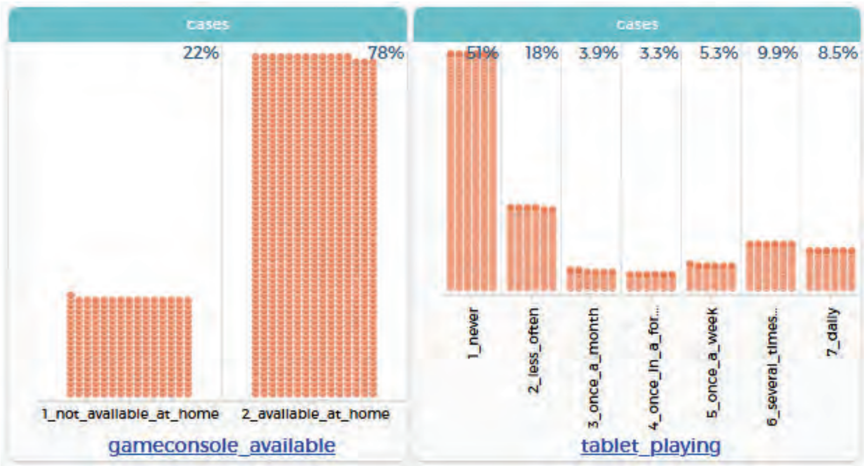

Figure 2: Distribution of game console available at home (left) and playing on a tablet (right)

Exemplarily, we look at a typical distribution of how often the participants use a tablet or Instagram (Figure 3). Such an "u" shape often occurs in these data, which means that the students can be split in two major subgroups ("rarely" and "frequently").

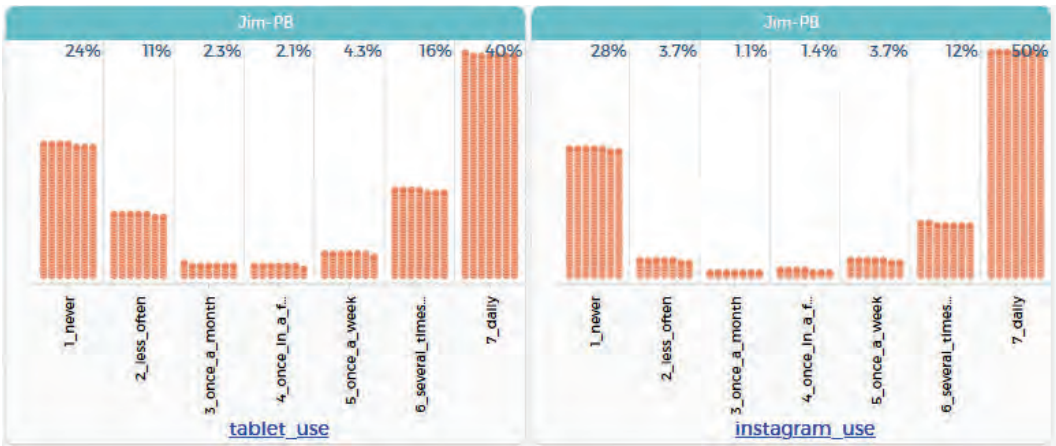

Figure 3: Distribution of tablet use (left) and Instagram use (right) in the Jim-PB data

Who belongs to those who daily use a tablet? Are those, who use daily a tablet those who use Instagram daily? For answering questions like this, two-dimensional diagrams are necessary like in Figure 4.

The challenge of interpreting complex diagrams like the one in Figure 4 is to use the correct percentages. This is often difficult for students (Watson & Callingham, 2014) and must therefore be well addressed in class. In Figure 4, row percentages are shown. A correct interpretation of the value 53 % at the top right corner is: Of those who use a tablet daily, 53 % use Instagram daily. But this is the same for those who never use a tablet (lowest right value in Figure 4)! Now we look at those who use a tablet daily.

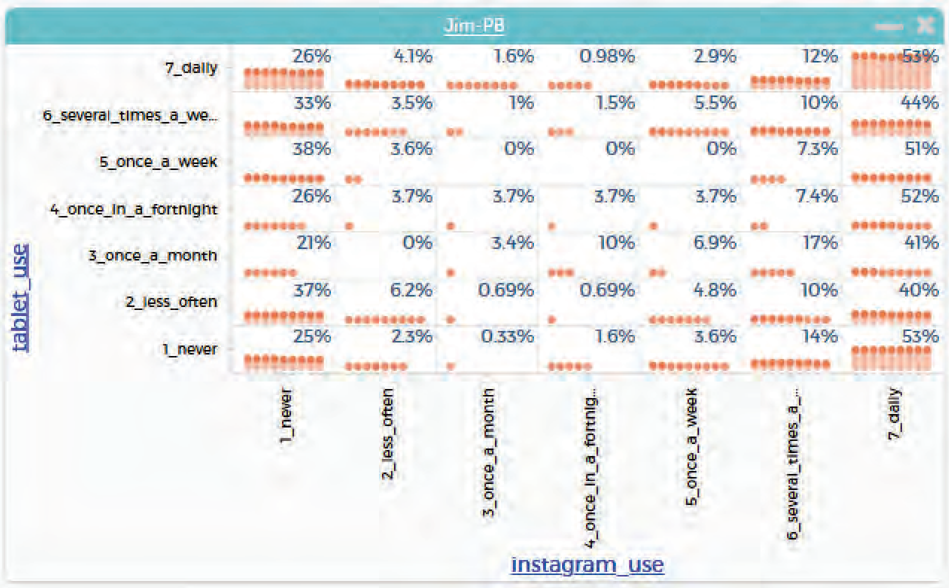

Figure 4: Do those who daily use a tablet those who daily use Instagram? (with row percentages)

7x7 tables like in Figure 4 are quite complex and hard to interpret not only for middle school students. By using data moves (Erickson, Wilkerson, Finzer, & Reichsman, 2019) like filtering with the "eye" function in CODAP, one can concentrate on a subgroup of the participants by selecting the corresponding cases and hiding all other cases (Figure 5).

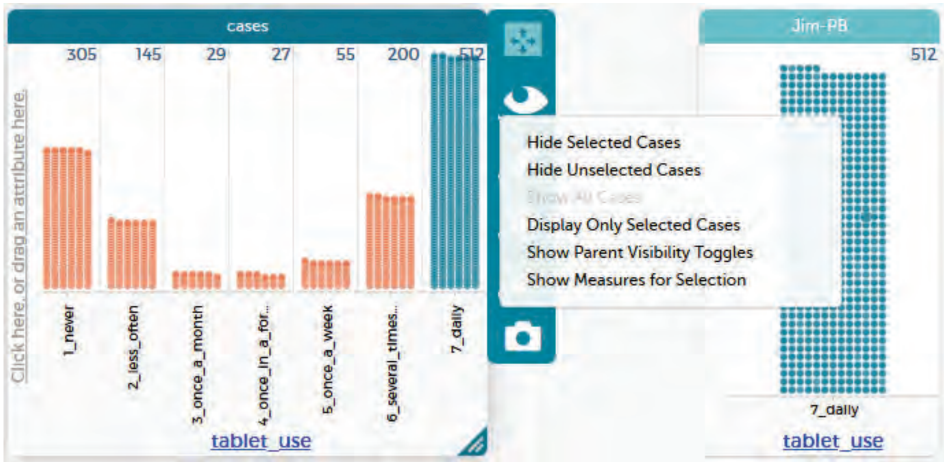

Figure 5: Filtering the data by selecting a subgroup (left) and the subgroup only (right)

Now the analysis can be focused on the subgroup of participants of those who use a tablet daily. For example, in this subgroup two third are female (Figure 6 left) in contrast to all participants (56%, Figure 1); distributed over all grades (Figure 6 middle) like in the whole group. 38% play often with the tablet (Figure 6 right, values for 'several times a week' plus 'daily') in contrast to the all participants, of whom less than 20% play tablet often (Figure 2 right). As a story, daily-tablet-user are more often female than male and use a tablet for playing more often than the whole group.

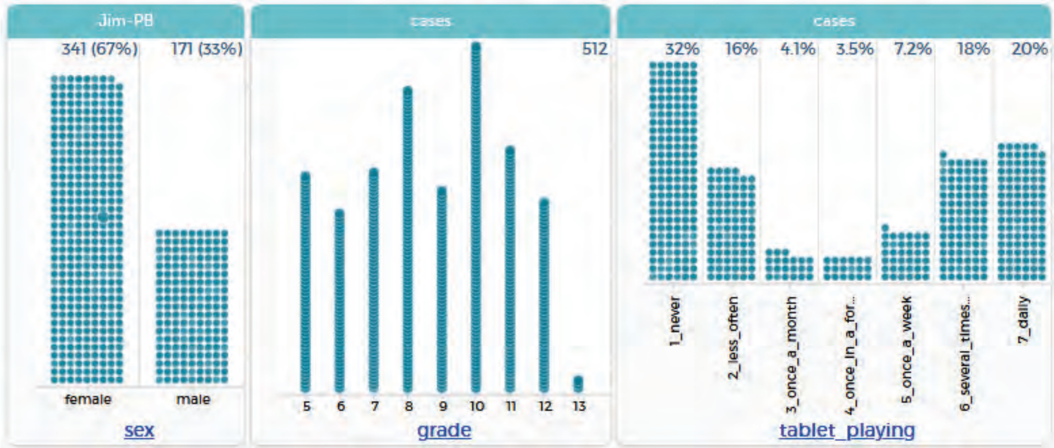

Figure 6: Subgroup displays of those who use a tablet daily

This is a small impression of an analysis of the Jim-PB data that can be done with CODAP.

Students' explorations of the JIM-PB data: some impressions

The first part of the teaching unit was implemented in four different classes in different schools in 2020, once in grade 9 and 10 and twice in grade 11. This includes a total of 77 students aged 13 (grade 9) to 17 (grade 11). All students had nearly no previous knowledge in statistics and data analysis. We collected their final presentations from lesson 7 and analyzed their diagrams and written interpretations. Student groups had between two and six members, we collected n=15 presentations because one group did not hand in their presentation.

Here we give a short overview of the 15 students' presentations, the diagrams used and written interpretations. Major differences between the three grades could not be identified, but the groups differed greatly in their work, independently of the grade. There were great and poor presentations in every grade. So, we analyzed on all presentations without taking the grade into account.

The number of slides and diagrams varied much between the presentations. The number of slides varied between two and 37, with an average of 13. Many slides of a presentation had only little or no text, except one presentation that was like a report. Every group stated their topic at the beginning.

The number of diagrams per presentation varied between two and 16 with an average of six. One group did not include any diagram, they only reported several percentages (that must have been figured out with diagrams in CODAP). The presentations often had no written explanations in the slides but were explained orally during the presentation lesson. Three groups used column percentages, all other used row percentages which are the default setting in CODAP. For these groups it is not clear whether they also considered using column percentages to answer their underlying question. Of course, a rearrangement of the variables would be equivalent to changing row and column percentages. Using and interpreting percentages was often challenging for students. The use of row percentages did not always go along with the interpretations, several times column or cell percentages would have been appropriate for the written statements. This is similar to results found by Watson and Callingham (2014). Two groups used the filtering function to focus on subgroups like in Figure 5.

A nice example is from a group from grade eleven that looked at how the frequency of reading online newspapers is changing with age. They computed the average age of readers in the different groups as shown in Figure 7.

Figure 7: Students' exploration of age and online newspaper reading

As an interpretation they wrote "For each 'reading frequency' you can see the average age indicated by the blue line. Newspapers are read online from the age of about 15, while the average age of non-online newspaper readers is 13. Therefore, it can be said that the target group of online newspapers has the age of more than 15 years, while the largest and most active part of the readers is represented by people aged 16 years and older."

Students' investigations were rich and showed interesting relationships in the data. Diagrams created and used for presentation by the students offered great opportunity to tell many stories about the data. Nevertheless, only few detailed descriptions like the one for Figure 7 occurred and these were not yet completely satisfactory due to inaccuracies in the explanations. Cultivating interpretations turned out to be a challenge.

Conclusion

The JIM-PB data provides rich exploration opportunities and is suitable for students aged 13 and up. For students, the version with 50 variables is sufficient to make interesting discoveries and looking for many relationships in the data. CODAP has proven as an appropriate tool for an easy entrance to explorations.

Such rich data not only brings advantages, but also challenges. The type of the data means that variables with seven values have to be related to each other. This results in 7x7 matrices with a total of 49 entries. In class, it became apparent that the interpretation of 7x7 matrices was challenging for students and patterns were rarely well described and interpreted. As an implication, we added a lesson on data preparation in the teaching unit, where the seven values can easily be merged so that only the values "frequently" and "rarely" remain by using the "hierarchize" function in the CODAP's table. Then only four-field tables occur, which are much easier to interpret than the 7x7 matrices.

Additionally, modelling aspects of such a data preparation with relation to possible interpretations can be discussed in class. As another implication, data moves (Erickson et al., 2019) like filtering (easily done in CODAP with the "eye" function) are a useful concept that can enrich students' data science projects in middle school.

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The role of probabilistic reasoning in risk perception and intentional behaviors during the COVID-19 Pandemic

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During the last year, governments around the world used statistics data to keep people informed about Covid-19, to stress the importance of managing this disease and to encourage to adopt preventive behaviors. In this study, we investigated a mechanism underlying intentional non-precautionary behaviors during the COVID-19 lockdown. We hypothesized that the comprehension of COVID-19 risk statistics information had a central role in mediating the relationship between probabilistic reasoning ability and perceived statistics value (the antecedents) and the intention to act non-precautionary behaviors. Participants were 141 university students enrolled in an online study. Results revealed that probabilistic reasoning ability and perception of statistics value had a role in reducing the likelihood of engaging in transgressive behaviors through their impact on the ability to adequately reason with statistics data referred to the COVID-19 epidemic.

Keywords: Probabilistic reasoning, COVID-19 risk statistics information, statistic value, statistical literacy.

Introduction

The 2019 coronavirus disease (COVID-19) outbreak, caused by the novel coronavirus SARS-CoV-2, has become a global health threat, which has prompted the scientific community to question how to deal with it and mitigate its impact. It has been reported for the first time in Wuhan, Hubei Province (China) in late December 2019 and it has rapidly spread around all the five continents. As of 30 April 2020, the number of totals confirmed cases worldwide has exceeded 3 million (World Health Organization, 2020). On 20 February 2020, Italy identified its first case of local transmission and since March 20, Italy had surpassed China as the country with the highest amount of people who died from COVID-19 in the world (World Health Organization, WHO, 2020a). The number of deaths in Italy remained the highest worldwide until April 13, when the USA has become the nation with the highest number of deaths (WHO, 2020b). Even though governments around the world adopted different response strategies to tackle the pandemic, at some stage most countries either enforced or encouraged policies targeting preventive behaviors such as social distancing. However, these measures of containment have requested citizens a limitation of their activities and a modulation of their behaviors.

In medical decision-making, it has been widely demonstrated that adherence to health behaviors is influenced by risk comprehension related to the medical issue/problem. For example, risk comprehension has been found to influence risky decisions related to sexual behavior (Patel et al., 2007), substance use (Lundborg & Lindgren, 2002) and tobacco consumption (Lundborg & Lindgren, 2004). In a systematic review about poor risk comprehension's health outcomes, it has been found that a lower comprehension of health information was associated with more hospitalizations, less mammography screening and influenza vaccination (Berkan et al., 2011).

Several studies about risk comprehension were then carried out to investigate what improves or hinders it. In this regard, as much of health-related information is expressed numerically (Reyna et al., 2009), a growing body of literature has sought to determine how health numeracy skills, the ability to understand and make use of health-related statistics (Låg et al., 2014), can improve people's risk comprehension. Health numeracy, intended as the ability related to probabilities, proportions, and percentages in the health domain, is low in the general population (Lipkus et al., 2001; Peters, 2012). However, it seems to have important impact on risk comprehension (Låg et al., 2014; Rolison et al., 2020).

Anyway, numeracy is a complex concept, encompassing several functional elements. Numeracy skills can be defined along a continuum that goes from elementary arithmetic skills to higher levels that encompass the ability to master probabilistic information and risk estimates (Reyna et al., 2009). Among these numerical skills, a growing interest has been recently posed on *probabilistic reasoning abilities* (Donati et al., 2014; Donovan et al., 2017; Hertwig et al., 2008; Primi et al., 2017). Probabilistic reasoning can be defined as the ability to think statistically about uncertain outcomes, and to make decisions based on probabilistic information. Probabilistic reasoning ability represents an important skill to correctly understand information related to risks. Interestingly, it has been highly documented that the majority of people have an inadequate comprehension of probabilities (Gigerenzer et al., 2005), even if they are highly educated (Lipkus et al., 2001). In the health domain, risk comprehension demands people to face uncertain outcomes and, therefore, with probabilities. For example, it has been demonstrated that the inadequate comprehension of risk and probabilities is critical in many areas, as the understanding of diagnostic tests (Gigerenzer et al., 2007) and drugs' side effects (Gigerenzer & Galesic, 2012). Specifically referring to COVID-19 related risks, it has been shown that when introductory statistic course students had higher probabilistic reasoning ability, they were more proficient in understanding COVID-19 risks (Primi et al., 2021). Additionally, probability reasoning ability, reflective ability and statistics interest had a role in reducing the likelihood of engaging in transgressive behaviors through their profitable impact on the ability to adequately reason with statistics data referred to the COVID-19 pandemic.

In line with this premise, we were interested in investigating the relation between probability reasoning ability and Covid Statistic Risk Comprehension in graduate students with more experience in statistics. We hypothesized that people who properly understand probability would be better in understanding and evaluating Covid Statistic Risk information. Additionally, as mass media communication about COVID-19 has been based on statistical concepts, such as "frequency", "shape of the curve", "flattening the curve", "positive rates", we hypothesized that a fundamental prerequisite to personally engage in data understanding would be the perception of the statistics value. It represents the usefulness, relevance, and worth of statistics in personal and professional life (Schau et al., 2003). Finally, as individuals who perceived risk related to COVID-19 as higher declared are more likely to implement protective behaviors (de Bruin & Bennett, 2020, we included intentional non-precautionary behaviors during the COVID-19 lockdown as dependent variable.

In sum, our research question was to investigate the role of the probabilistic reasoning ability and the perceived value of statistics (positively related to each other) as antecedents of risk comprehension of COVID-19 statistics and this variable was hypothesized as the intermediary variable (mediator) between its antecedents and the dependent variable, that was conceptualized as intentional non-precautionary behaviors during the COVID-19 lockdown.

Method

Participants

Participants were 141 students attending graduate programs (60% female; mean age = 23.4; *SD* = 4.83) at the University of Florence (Italy). Introductory stats courses are compulsory in all the programs. For the pandemic all the programs were online. All students participated on a voluntary basis after they were given information about the general aim of the investigation. Participants completed an online survey on April 2020, during the first Italian lockdown.

Measures and Procedure

The *Probabilistic Reasoning Scale* (*PRS-B*; Primi et al., 2019) consists of 9 multiple-choice questions. The items include questions about simple, conditional and conjunct rule in probability, and the numerical data are presented in frequencies or percentages. A single composite score, based on the sum of correct responses, was calculated.

The *Value* subscale of the *Survey of Attitude towards Statistics* (*SATS-36*; Schau, 2003) is one of the subscales of the instrument measuring the six components of attitude toward statistics. The specific subscale consists of 9 Likert-type items using a 7-point scale ranging from *strongly disagree* to *strongly agree*. A single composite score was computed based on the sum of the responses, with higher ratings representing a higher perception of the Statistic Value.

The *Statistics Risk Comprehension Scale-Covid 19* (*SRCS-Covid 19*) was developed for the purpose of this study. In detail, we constructed a scale aimed at investigating people's understanding of the statistics about the epidemiological situation regarding the COVID-19 epidemic that was spreading in Italy in that specific historical period. Eight multiple-choice items with three response options (among which only one was the correct one) were created with the aim of covering the most debated issues in the Italian mass media concerning the COVID-19 at that time (e.g., cases of infections, dead cases out of infections, prevalence rates of positive COVID-19 tests). A single composite score, based on the sum of correct responses, was calculated. An example of item is: "*On 19 March 2020, in Italy there are about 40,000 infections and about 33,000 people who are still positive for the virus. This means that: a) As of that date, there are about 73,000 cases of infections; b) As of that date, there are about 7,000 cases between deceased and recovered; c) As of that date, there are about 7,000 cases of healed*".

In order to investigate the intention to act not precautionary behaviors put in place during the lockdown period, we developed a brief questionnaire through which participants were asked to indicate whether they have intention to act the listed behaviors. Behaviors were defined as not precautionary on the basis of the restrictions imposed by Italian government through the Decree of President of the Council of Ministers (DPCM) issued on March 9, 2020 (https://www.gazzettaufficiale.it/eli/id/2020/03/09/20A01558/sg). An example of not precautionary behavior was "*Take a walk with some friends*". For each of the ten listed behaviors, participants had to respond "*no*" (scored as 0) if they had not the intention to engage in that behavior, or "*yes*" (scored as 1) if they had the intention to do that behavior, by specifically referring to the lockdown period. Among the listed behaviors, four were classified as not precautionary. In order to obtain a measure of intentional not precautionary behaviors, a total score was computed by summing responses given to the items investigating those kinds of behaviors.

After giving the informed consent, each scale was briefly introduced, and instructions for completion were given. All participants completed the *PRS-B*, *Value* subscale of the *SATS-36*, *SRCS-Covid 19* and the scale investigating intentional precautionary behaviors during the Covid 19 lockdown period. Time administration was about 30 min.

Results

To analyze the relationships between COVID19-related statistics risk comprehension and the scores relative to probabilistic reasoning ability, perception of statistics value, and intention to engage in not precautionary behaviors, correlations among the variables were calculated (Table 1).

	1.	2.	3.	4.
1. COVID19-related statistics risk comprehension	-			
2. Probabilistic reasoning ability	.46***	-		
3. Perception of statistics value	.33***	.36***	-	
4. Number of intentional not precautionary behaviors during the Covid 19 lockdown	-.22**	-.13	-.22*	-
M (*SD*)	5.09(1.52)	6.43(1.72)	46.71(8.36)	1.94(1.71)

*p<.05, **p<.01, ***p<.001

Table 1: Summary of intercorrelations, means, and standard deviations for scores of COVID19-related statistics risk comprehension, probabilistic reasoning ability, perception of statistics value, and intention to engage in not precautionary behaviors during the Covid 19 lockdown

As expected, the SRCS-Covid 19 score significantly and positively correlated with probabilistic reasoning ability and perception of statistics value. It was also significantly and negatively correlated with the number of not precautionary behaviors that are likely to take be placed. Moreover, probabilistic reasoning ability and perception of statistics value were positively inter-related, and the number of intentional not precautionary behaviors were significantly and negatively correlated with both probabilistic reasoning ability and perception of statistics value.

In order to investigate our hypothesis on the mechanisms underlying the relationships among these variables, we conducted a path analysis employing the maximum likelihood (ML) method using AMOS 16 software (Arbuckle, 2007). The model included probabilistic reasoning ability and perception of statistics value as COVID-19 statistics risk comprehension's antecedents (positively related to each other). In turn, statistics risk comprehension was hypothesized as the intermediary variable (mediator) between the antecedents and the dependent variable, that was conceptualized as the number of intentional not precautionary behaviors during the Covid 19 lockdown.

The presence of the mediated effect was investigated through the test of indirect effects (Cheung & Lau, 2008). In AMOS the Bootstrap confidence interval method is used to define the confidence intervals for indirect effects (MacKinnon et al., 2004). In mediation analysis, bootstrapping is used to generate an empirically derived representation of the sampling distribution of the indirect effect, and this empirical representation is used for the construction of a confidence interval for the indirect effect. The 90% bias-corrected confidence interval percentile method was implemented, using 2,000 bootstrap samples. Confidence intervals for the indirect effects which do not contain 0 are considered as indicative of significant indirect effects, thus meaning the presence of a mediated effect. Several goodness-of-fit indices were used to test the adequacy of the model: Comparative Fit Index (CFI) (Bentler, 1990), the Tucker-Lewis Index (TLI) (Tucker & Lewis, 1973) and the Root Mean Square Error of Approximation (RMSEA) (Steiger & Ling, 1980). CFI and TLI values equal to .90 or greater and RMSEA values of .08 or below are considered as indices of adequate fit. The model showed a good fit to the data (CFI = .976, TLI = .927, RMSEA = .074). All coefficients were statistically significant in the expected directions. Specifically, results revealed that probabilistic reasoning ability and perception of statistics value – positively inter-correlated – had a significant direct and positive effect on COVID-19 statistics risk comprehension. In turn, COVID-19 statistics risk comprehension was directly and negatively related to the number of intentional not precautionary behaviors during the Covid 19 lockdown. Results also showed significant and negative indirect effects from the independent variables on the number of intentional not precautionary behaviors during the Covid 19 lockdown, indicating that probabilistic reasoning ability and perception of statistics value had a role in reducing the likelihood of engaging in transgressive behaviors through their profitable impact on the ability to adequately reason with statistics data referred to the COVID-19 epidemic (Figure 1).

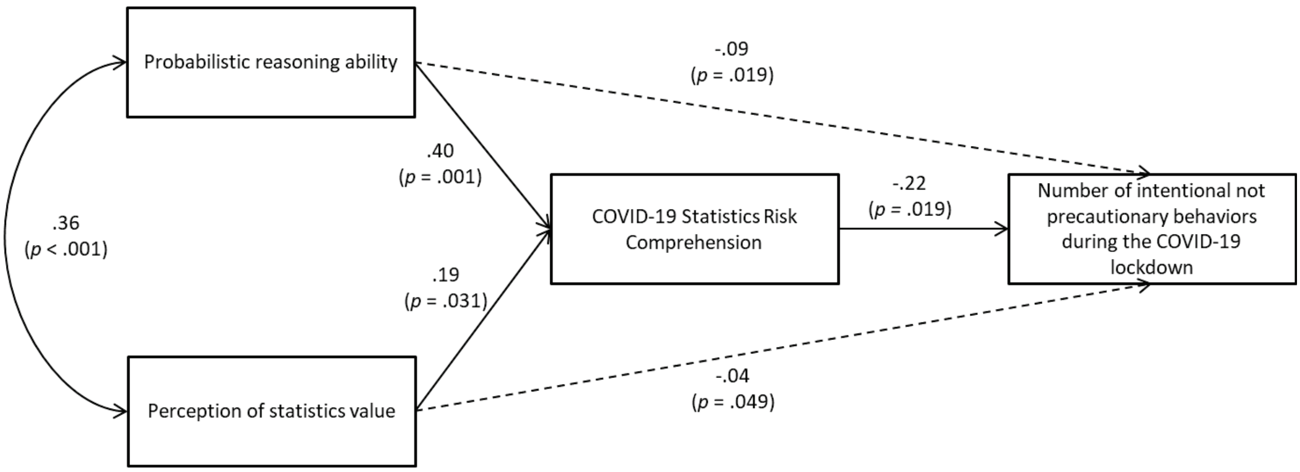

Figure 1: Path model among the variables. Straight lines indicate direct effects. Dotted lines indicate indirect effects. All the coefficients are standardized

Conclusion

The emergency caused by coronavirus disease 2019 (COVID-19) has led to a surge in interest in trustworthy statistics and a greater number of people accessing statistical information about their own communities. Results showed the role of mediator of the statistics risk comprehension confirming the need to have a deep understanding of statistics, especially in the context of a global pandemic such as Covid-19. We found that probabilistic reasoning ability and perception of value in statistic had indirect effects on the intention to act non-precautionary behaviors through statistic risk comprehension. This result is in line with other studies concerning diverse health contexts, that showed that numeracy (in its different components) has consistently been related to risk perception, more accurate understanding of risks, and better decisions (Garcia-Retamero et al., 2019), and with explanation models related to behavioral conducted in the pandemic (Primi et al., 2021). More generally, our findings show the importance of statistical literacy as an essential skill for all citizens, especially in the context of a global pandemic such as Covid-19**.**

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Onto-semiotic competence in the statistics content of early childhood education of recently graduated teachers

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This paper presents a training action with ten recently graduated teachers who are taking an advanced training Master's Degree in Spain. The objective of the action is to develop their didactic-mathematical competences and, in particular, the onto-semiotic competence in the field of statistics in early childhood education. To achieve this competence, the teachers identified the emergent objects in a real videotaped mathematical practice. The results obtained from a qualitative analysis indicate that the participants have difficulties in recognizing the properties and the arguments, as well as some statistical concepts, whereas they identify the procedures quite well. It is concluded that the competence development requires increasing the temporal suitability of the action and also dedicating more hours to the initial training of teachers in statistical content.

Keywords: Teacher education, instructional design, onto-semiotic competence, statistics, early childhood education.

Introduction

The development of statistical literacy should begin in early childhood education (NCTM, 2000). Carrying out small statistical studies with a qualitative variable is suitable for this. However, in Spain, there is still some reluctance to do so. One of the causes of the statistics teaching and learning processes implementation slowdown in the early ages is that statistical content is not well integrated into the Spanish educational curriculum for early childhood education. But this is not the only reason.

In the last decade, there has been a warning that, in the initial training of teachers at this stage, not enough hours are dedicated to statistics (Alsina & Vásquez, 2016; Franklin et al., 2015; among others). Along these lines, Alsina (2020) reviews the study plans of the Degree in Early Childhood Education of seventeen Spanish universities and concludes that most of them dedicate only 6 credits to the whole didactic-mathematical training and that only three of these universities, allocates between 6 and 9 credits to strictly mathematical or statistical content. Consequently, there is an immediate need to improve the didactic-mathematical skills and knowledge in statistics of kindergarten teachers. In this sense, Batanero (2019) suggests that research on the teacher's statistical knowledge focus on training actions that favor the development of their specialized knowledge and professional competence.

This work is part of a larger investigation whose aim is to explore and develop the didactic-mathematical skills and knowledge about statistics in early childhood education of a group of recently graduated teachers after a training action designed for this purpose. This paper addresses the onto-semiotic competence and it is motivated by the question of which mathematical objects identify the participants in a statistical practice of early childhood education.

It is dedicated the following section to expose the theoretical framework and describe the methodological approach, as well as the training action, in the third point. In the fourth section, the main results regarding the onto-semiotic competence are collected and, finally, the conclusions of the work and some final reflections are presented.

Framework

In recent decades, the field of mathematical education research has focused on determining the knowledge of the mathematics teacher. In this sense, the Schulman model (1987) and the Mathematical Knowledge for Teaching model (MKT) (Hill et al., 2008) stand out internationally. Different approaches have also been interested in teacher skills, such as Lesson Study (Fernandez & Yoshida, 2004) or Noticing (Mason, 2002).

This research is developed within the Teacher's Didactic-Mathematical Knowledge and Competence (DMKC) model (Godino et al., 2017) of Onto-semiotic Approach (OSA) (Godino et al., 2007) theoretical framework, because it offers detailed categories of knowledge and skills.

This model considers that the mathematics teacher must have three types of knowledge. The mathematical knowledge includes the common knowledge of the content of MKT; the didactic-mathematical knowledge comprises the specialized knowledge of MKT and also cognitive, affective, interactional, mediational and ecological aspects and, finally, the third is the meta didactic-mathematical, which includes knowledge about the norms that condition a teaching and learning process, as well as on the didactic suitability assessment criteria (Pino-Fan & Godino, 2015). Likewise, the DMKC determines that the mathematics teacher's main competence is the general competence of didactic analysis and intervention. This competence is made up of five different competences: global meaning analysis competence, onto-semiotic analysis competence of mathematical practices, didactic configurations analysis and management competence, normative analysis competence and didactic suitability analysis and assessment competence (Godino et al., 2017).

Specifically, in this work, we focus the attention on one of them, on the onto-semiotic analysis of mathematical practices competence. This competence consists in carrying out an onto-semiotic analysis, that is, in identifying the objects (situation-problem, concepts, linguistic elements, procedures, properties and arguments) and processes involved in a mathematical practice (Godino et al., 2017). To achieve the development of this competence, current research (Burgos et al., 2018; Godino et al., 2018; among others) is committed to specific training actions in which it is necessary, first, to produce an evolution of the personal meanings; then, introduce the tools of the OSA and put them into practice and, finally, institutionalize the acquired knowledge through discussions.

Method

Methodological approach

This research is framed within the exploratory qualitative approach and the design research paradigm (Kelly et al., 2008), which pursues the development of knowledge about an educational reality that is desired to be improved through the design of an innovative intervention in it, while researching on the said design (Cobb et al., 2003).

Context and participants

The training action was carried out with the ten students of the Master's Degree in Advanced Training of Early Childhood and Primary Education Teachers of a Spanish university. All of the participants are recently graduated teachers with no work experience. In particular, two of them have dedicated 6 ECTS of their initial training to didactic-statistical content for the teaching and learning of statistics in primary education and, the other eight, 0.5 ECTS to statistics in early childhood education. The teachers worked in pairs throughout the training action.

Video material and implementation

In the formative experience of this research, video is the main material used. Its use of scaffolding coincides with that indicated by various authors (Coles, 2014; Gaudin & Chaliès, 2015) in teacher training, since it is used to develop analysis and reflection skills, such as competence in onto-semiotic analysis. Specifically, two real teaching and learning processes videotaped are used in this action.

The main video shows the mathematical practices of an early childhood education class (5-6 year-old) when carrying out a statistical study. Children want to discover "What is the game that the class likes the most?". First, they collect the data. Each child has a paper with five pictures that represent a class' game (building game, cars, doctors, hairdressers and jigsaw puzzles) and has to cut the one he likes the most. Then, children all together build a single pictogram with the pictures they have cut. Finally, they compare frequencies and identify the mode. In the following sessions, the transnumeration is worked, since they build a graphic experientially and another, with Lego pieces.

The other video is about geometry in 5th grade (10-11 year-old) and it is used as a preliminary practice with the OSA.

Implementation

The action designed and implemented consists, principally, of two content blocks (Figure 1). In the first block, of provocation and exploration of personal meanings, the participants assessed, according to their criteria, the statistical teaching-learning process of the main video. Then, they shared their impressions and ideas. This block is completed with an oral game about the personal meanings of the training teachers about the mathematical processes defined by NCTM (2000). The second block focuses on onto-semiotic competence. First, the lecturer introduced some theoretical notions of OSA, such as the kinds of mathematical objects and onto-semiotic analysis. Then, in Task 1, the teachers searched for mosaics and then identified the emergent objects in them. In Task 2, they watched the geometry video and recognized the mathematical processes underlying in the video. Afterwards, they shared their work. Finally, in task 3, they had to identify contextualized examples of each type of mathematical object and process that emerged in the statistical teaching-learning process videotaped.

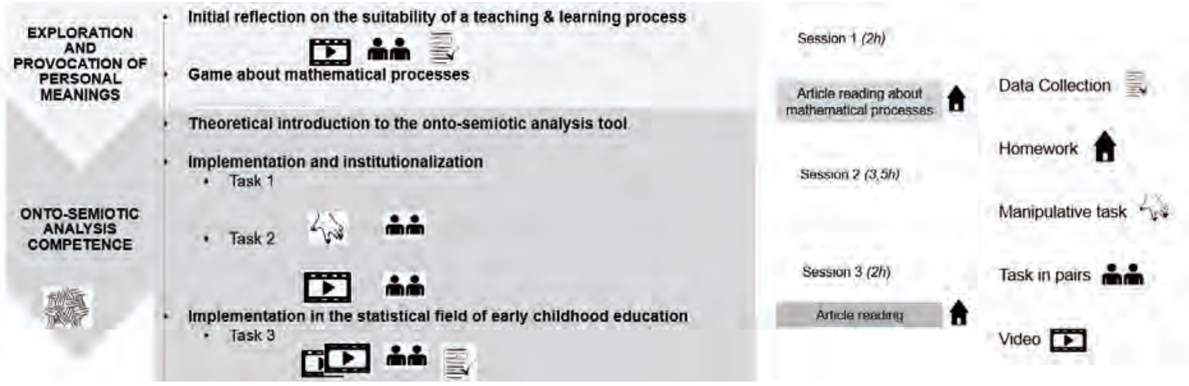

Figure 1: Characteristics of designed and implemented formative intervention

Data collection and analysis

In this paper, it is analyzed the information collected from the teacher's pairs written responses corresponding to task 3 applying the onto-semiotic analysis techniques. Specifically, first, an expert onto-semiotic analysis of the emergent objects in the videotaped was carried out and then, based on the content analysis, the onto-semiotic analysis of each pair was compared with the expert.

Results and discussion

As far as concepts are concerned, certain difficulties have been detected. Some of them have also been observed in other training experiences with primary and secondary school teachers and with other mathematical content. For example, not all the elements that the participants present as a concept are, really, a concept. Likewise, not all the mathematical concepts identified are actually mathematical concepts that emerge in the teaching and learning process, since some of them do not emerge in the videotaped process or, simply, as Beltrán-Pellicer et al. (2020) suggest, do not respond to an operational need, such as "statistics" or "reasoning". Another aspect observed is that some couples confuse the curricular contents with the concepts. For example, one of the pairs presents as a concept "the identification of the numerical symbols". Similarly, another couple does the same with "writing the number symbols." As highlighted by Beltrán-Pellicer et al. (2020), the teachers in training include in mathematical concepts, concepts that, from the didactic-mathematical perspective, are rather linguistic elements, such as "tie" to refer that two categories of the variable have the same absolute frequency or "drawing", to symbolize the iconic figure of a category. Another remarkable characteristic is that teachers have difficulty seeing concepts that emerge from a routine situation in the early ages classroom. In the videotaped process, and specifically, for data collection, a child has to distribute scissors to each partner, so that they make a correspondence. Only two couples have been able to detect this correspondence.

In the process, mainly numerical and statistical concepts emerge. It should be noted that the teachers in training recognize the first quite well. However, the results are not good in terms of statistical concepts. Table 1 shows the statistical concepts that emerge and the number of pairs that identifies each of them. As can be seen, the concepts that they mostly recognize are mode and pictogram, although it is surprising that, since mode is the main concept, it is not explicitly recognized by all the groups. Likewise, concepts so elementary as variable or absolute frequency have not been identified by the teachers in training. This result is in line with that obtained by Gea (2014), in which the percentage of identification of the measures of central tendency by prospective secondary school teachers is higher than that of the statistical variable.

Table 1: Number of pairs that identifies each statistical concept

Statistical concepts	Number of pairs
Absolute frequency	0
Categories	0
Graph	2
Mode	3
Pictogram	3
Population	0
Qualitative statistical variable	0
Sample	0

With regard to linguistic elements, the mathematical videotaped practice brings into play four types of language: verbal expressions, numerical, graphic and iconic language. Now, while four pairs identify verbal and graphic language, only one of these four also recognizes numerical and iconic language. One of the pairs only identifies graphic language.

It should be noted that in the videotaped, the children represent the data in three ways. However, only one of the couples is able to refer to the experiential graph and the Legos graph. This result is worrying given the importance of experiential and manipulative mathematics in the early ages.

Regarding the procedures, most teachers recognize the data collection and the graphs construction, but, on the contrary, do not identify the data reading. An important fact observed in the analysis of this kind of object is that, in some cases, the teachers' personal meaning of the mathematical procedure coincides with that of the mathematical process. In fact, Beltrán-Pellicer et al. (2020) point to certain coincidences between both constructs that make it difficult to differentiate them. This could explain why teachers do not view level 2 of data reading as a procedure, but as a reasoning process. This level requires interpretation of the data from the comparison or ordering, typical entities of the reasoning. Finally, with regard to the statements and arguments, the level of competence shown by the teachers in training has been very low.

In the videotape, eleven emerged properties or propositions are considered informal and, most of them, are linked to the data reading. Three couples identified only the following: "The game with the most votes was hairdressers." The remaining, none of them. As for the only formal property that is put into play, "the mode represents a group, so it provides information on the entire set and not on specific elements", it should be noted that no one recognized this property. We believe that there are two reasons for the low number of correctly identified properties and propositions. The first lies in a lack of understanding of what is a mathematical proposition, property or statement (Giacomone et al., 2018). The second is that most of the properties of the videotaped process are justified from informal deductive arguments based on a graph; consequently, the properties and propositions do not emerge accompanied by their arguments.

The result is not better in the case of arguments. Four of the five couples have not been able to correctly identify any argument. In fact, it has been observed that training teachers confuse arguments and properties and propositions, that is, they can present an argument as a property and a property as an argument. Or, they, simply, do not separate the proposition to be justified from the argument itself, that is, they present a justifying statement (Beltrán-Pellicer et al., 2020) with a proposition-conjunction-argument structure.

Conclusions

The competence development of the teachers has not been uniform for all the objects. The level of recognition of linguistic elements and procedures is sufficient, but that of properties, propositions and arguments is insufficient. As far as concepts are concerned, the level is beginner, but not sufficient, since they recognize numerical concepts well, but not statistical ones.

In our opinion, in addition to the aspects indicated in the previous section that hinder the development of competences, we point out the early childhood context. Specifically, we refer to the fact that common verbal expressions abound in early age mathematical practices and that the mathematical name of the concepts in them is not institutionalized. This requires a higher cognitive effort on the part of the teacher to identify mathematical objects, because there is a "camouflage" effect. Therefore, it is necessary to educate the professional noticing of kindergarten teachers in this regard.

We emphasize that the level of recognition of the procedures shows that the mathematics with which the participants have been trained has been basically algorithmic, rather than constructive. Moreover, they identify most of the procedures involved, but do not distinguish the reading and interpretation of the data, to which it seems that they are not so used.

Finally, it is necessary to add that the didactic suitability of the formative action is quite high, but for possible replications it is suggested to leave more time for the discussions and institutionalizations of the mathematical objects, even for the assimilation of their meanings. However, it is difficult to increase the degree of cognitive suitability of the action if the teachers have not previously received an initial training that favors, at least, their common content knowledge.

It is evident, now more than ever in the reality that we live as a consequence of the pandemic, that statistics are necessary for society and science to advance. Therefore, it is a good time to push for a change in university curricula.

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Subjective probability in use: reasoning about a chess game

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Subjective probability is often left aside in elementary school curricula, despite it is often used in daily-life. In this paper, we expose a didactic experiment with 5th grade students (10-11 years old) in which they were asked to handle subjective probability. Findings show how students were able to make probability judgements even without a numerical support, they changed their initial judgements when more quality information was provided, and they refrained from state a founded judgement when they considered available information was not enough. Results endorse the importance of introducing this type of probabilistic reasoning in elementary school.

Keywords: Elementary school, Probabilistic reasoning, Subjective probability, Verbal communication.

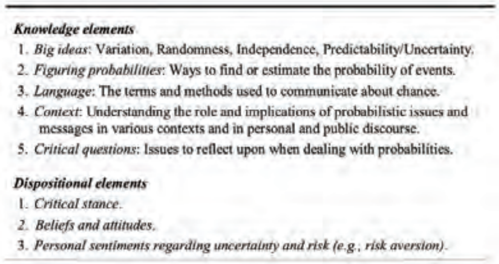
Introduction

The notion of chance is polysemic and, often, controversially interpreted in common contexts, including scientific ones. This affects how probability is understood in our society, and how it is perceived in school contexts. Despite the axiomatic approach solved the formal mathematics obstacles, “there are still controversies over the interpretation of basic concepts and about their impact on the practice of statistics” (Batanero et al., 2005, pp. 15–16). Classical and frequentist (or experimental) approaches are the most common in elementary school when teaching probability, being the subjective (or Bayesian) approach often barely mentioned, or totally ignored (Carranza & Kuzniak, 2008; Gómez-Torres et al., 2014). However, subjective probability is largely present in daily-life, where events cannot be simplified into counting possible outcomes of the random experiment, or repeating it under the same conditions, as when predicting sport results or assessing the risk of being infected by a virus (Muñiz-Rodríguez et al., 2020).

At early school ages, the subjective meaning of probability is strongly related to the use of verbal reasoning and, particularly, chance language (Kazak & Leavy, 2018). This conceptualization is often described as the intuitive meaning of probability (Batanero, 2005), in which the use of linguistic quantifiers and terms about chance helps children making qualitative probabilistic judgements. In this paper, the results of a didactic experiment developed with 5th grade students (10-11 years old) in a Spanish school, consisting of probabilistic reasoning about a chess game, are described.

Theoretical framework

The intimate relationship between probabilistic reasoning and the notion of chance produces a greater space for intuition (and for counterintuition) when learning and dealing with probability. Handling this intuition and making it coherent with the mathematical development of probability is a challenge in probabilistic education at the elementary school, as Fischbein (1975) already pointed out. Even when subjective probability has not been a trending topic on probabilistic education research literature, the didactic experiment described here can be framed within the general paradigm of probability literacy (Gal, 2005), due to its link to the basic probabilistic instruction at elementary school. Gal's framework includes five knowledge and three dispositional elements (see Figure 1). This research study focuses on figuring probabilities (estimates), language (of chance), and context (a chess game), and mainly dealing with beliefs and attitudes.



Knowledge elements

1. *Big ideas*: Variation, Randomness, Independence, Predictability/Uncertainty.
2. *Figuring probabilities*: Ways to find or estimate the probability of events.
3. *Language*: The terms and methods used to communicate about chance.
4. *Context*: Understanding the role and implications of probabilistic issues and messages in various contexts and in personal and public discourse.
5. *Critical questions*: Issues to reflect upon when dealing with probabilities.

Dispositional elements

1. *Critical stance.*
2. *Beliefs and attitudes.*
3. *Personal sentiments regarding uncertainty and risk (e.g., risk aversion).*

Figure 1: Elements of probability literacy (Gal, 2005, p. 51)

De Finetti's claim (1974) "probability does not exist" illustrates the subjective meaning of probability, neglecting the objective existence of "one" probability, and assuming it as the degree of belief about the occurrence of a given event. Such a personal judgement about the outcome of a random experiment is necessarily based on the information about the experiment or similar experiments that the involved person has. Thus, estimating the probability is dependent not only on the quantity of information but also on its quality. Different people have different type of information, and process it into very different ways, depending on his/her knowledge of the problem/context, experiences, and beliefs. And when new information is provided, people revise their initial probability assignment (the formalization of this revision process should be made in terms of Bayes' formula, which led to alternatively use Bayesian for this subjective approach). The subjective theoretical approach differs from the classical one, based on Laplace's rule, which is only applicable under equiprobability of the elementary events in a finite sample space, and assumes the existence of a real/certain value of probability for a given event. Nevertheless, subjective does not mean ungrounded: the probability elicitation depends on the person, but in obvious situations, as simple random experiments with games (coins, cards, dice, etc.), the subjective probability assignment of a minimally informed person should converge with the classical one, unless evidence from the running experiment is opposed (for instance, an unfair dice). Hence, Pratt and Kazak (2018, p. 206) discussed "the dual notion of probability" as a degree of belief and as the stable frequency in a long run.

Going into the elementary school framework, probability is introduced in different countries by means of the language of probability (Vásquez Ortiz & Alsina, 2019). Some experiences can be found in the research literature to promote a proper probabilistic reasoning by linguistically labelling, interpreting, and ranking the probability of certain events (e.g., Alsina et al., 2020; Kazak & Leavy, 2018). But, after this linguistic approach, most national curricular guidelines go directly to the formal definition of probability based either on Laplace's rule or on the experimental probability (Batanero et al, 2005). These guidelines rarely move towards the subjective definition, despite many daily-life situations are not suitable to apply Laplace's rule, neither experimentation is possible (either similar conditions are not reached, or repetition is impossible).

This paper addresses Pratt and Kazak's claim (2018, p. 222): "there is still a scarcity of research […] in the area of subjective probability at the school level". By means of a didactic experiment based on a chess game, we aim at analyzing 5th grade students' probabilistic reasoning in a context of uncertainty without conditions for applying Laplace's rule nor for experimenting by repetition. Previous research (Helmerich, 2015; Huber & Huber, 1987; Kazak, 2015; Pratt & Kazak, 2018; Vásquez Ortiz & Alsina, 2019) showed that students can deal with the language of probability and produce qualitative probability estimates, and that, under certain conditions, they mobilize their personal knowledge and beliefs to confront with experimental information. However, these studies considered situations which could be analyzed in combinatorial terms, that is, counting cases, even when they were not undertaken by using that strategy. The task that we present here is not suitable for determining probabilities by counting cases and, therefore, the cognitive demand is greater. Thus, we raise the following research question: How do 5th-graders integrate contextual information to deal with probability estimates not based on counting cases?

Methodology

Context

The didactic experiment was carried out in April 2021, at an elementary school in Oviedo (Spain). Thirty-one students (20 boys and 11 girls) in 5th grade (10-11 years old) participated, 4 of which played chess (3 boys and 1 girl). Curricular guidelines in 5th include to identify random situations and to make estimates on some game results (coins, dice, cards). In previous years, students worked with probability language (impossible, likely, certain), whereas the definition of probability based on Laplace's rule is to be studied in 6th grade. The experiment was conducted by one of the researchers.

Experiment

The didactic experiment consisted of a chess-based situation about a game between two elite players, organized into three steps:

1st step: In Grenke's chess tournament Anish Giri (white) will face the current world champion, Magnus Carlsen, in black. Of all the games that Anish Giri has played (908) almost half (456) were drawn. By contrast, Magnus Carlsen lost just three games in the last two and a half years. What do you think the result will be? Will Anish Giri win the world champion with white? Justify your answer. Would Giri be less likely to win if he played black?

2nd step: Due to the COVID-19 pandemic, both players spent several months without playing slow games. The three games that Carlsen lost in the last two and a half years were after pandemic lockdown. Meanwhile, Anish Giri did not lose any games after the lockdown. Do you think confinement was more beneficial for one than the other? After knowing these data, would you change the answer in the previous section?

3rd step: If that game had happened before the pandemic, when Carlsen had 120 games without losing, what do you think would be Giri's probability of winning?

A sheet with the wording of the situation was given to each student. First, each student individually wrote down his/her answer to each question on the sheet. Then, students orally shared their answers in a group discussion (due to COVID-19 restrictions personal contact was limited).

Obviously, in this situation the probability cannot be determined by using Laplace's rule, as there are too many variables involved (pieces color, opening preparation, level of concentration, etc.).

Information from previous matches can help to make an estimate but, clearly, not the repetition of the same random experiment under similar conditions. Moreover, the data provided for both players refer to different periods. Students must mobilize their knowledge about the context and handle their beliefs about games. There was not a right or wrong answer, but a better or worse reasoning based on the way students integrated and aggregated information and beliefs, and how they handled possible biases in probabilistic reasoning. Depending on the amount and quality of information, students could critically analyze the situation and produce an argued answer. During the didactic experiment, some explanations about chess had to be provided regarding the supposed advantage that the player with white pieces has. Also, a brief profile of Giri and Carlsen was required by students, explaining why they are considered world top players. Even being a complex argumentation, this task can be framed within the aforementioned Gal's three knowledge elements in probabilistic literacy, combined with beliefs and attitudes as a dispositional element.

Data collection and analysis

Students' written answers were considered as the primary unit of analysis. One of the researchers acted as teacher during the didactic experiment. She gave some instructions at the beginning about how to tackle the task. During the group discussion, she asked few questions to prompt students to justify their answers and took observation notes. Data was analyzed using a mixed technique, with a frequentist analysis and also a qualitative content analysis (Krippendorff, 2018) to find evidence of the use of probability language, the estimation of probabilities from a subjective point of view, and changes in the initial estimates when new information is provided. Considering that students' answers were rather brief and not too elaborated, data was first examined using a grounded theory approach. Each of the emerging categories was later analyzed in relation to the aforementioned Gal's knowledge and dispositional elements. For instance, students' answers to the 1st step questions were classified into blank or unclear justification supporting that Giri wins, justifications based on Giri's trajectory, justifications based on Carlsen's trajectory, and justifications supporting a draw between both players. Later, each category was reviewed to understand how students build their justifications using probability language, estimates, and information from the context (knowledge elements) together with their beliefs and attitudes (dispositional element).

Results

Table 1 shows the answers to 1st step questions. Justifications about Giri's win with white were based on players' trajectories: "Anish Giri would win because he has not lost a game, while Carlsen lost three", "Carlsen is the champion", or "Carlsen only lost 3 games in one year and a half". There were unclear justifications ("Giri is going to be lucky"). Those who supported the draw argued that both players are excellent. When asked if Giri would be less likely to win if he played black, 6 students agreed, and the rest (25) said there would be no difference. One of the arguments supporting the less likelihood was expressed with probability language ("The piece color matters because there are higher probabilities of winning when playing white"), while the rest were mainly based on students' beliefs ("Black pieces give worse luck"). Most students supporting there was no less probability argued that the color did not matter. Between the first and the second step, the teacher conducted a discussion with the students about their answers. Many of the students who did not know too much about chess waited to hear the argumentation provided by those who played chess, looking for endorsement or extra information. Key points of such discussion are stressed in the next section.

Table 1: Answers to "Will Giri win with white? Would Giri be less likely to win with black?"

	With white				With black	
Type of answers and justifications	Giri wins, blank or unclear	Giri wins, based on his trajectory	Carlsen wins, based on his trajectory	Draw	Less likely	No difference
Frequency	14	5	9	3	6	25

The 2nd step of the experiment was about reasoning if the confinement was more beneficial for Carlsen or Giri and checking how students integrated the related information. Table 2 shows the answers. Most who agreed that the confinement was more beneficial for Giri ("He did not lose any game after it") kept their initial answer about Carlsen's favoritism, based on his role as champion ("I trust in Carlsen, he is the champion"), but there were three answers providing an intuitive probabilistic reasoning (for instance, "Maybe Carlsen just had a bad run"). There was one student supporting the confinement was not beneficial for any of the players, who noted that the provided information was about the number of losses, but nothing was said about the number of draws (where Giri had much more than Carlsen), so it justified keeping Carlsen as the favorite player to win. We did not find solid argumentations among those changing their initial view.

Table 2: Answers to "Do you think confinement was more beneficial for one than the other? After knowing these data, would you change the answer in the previous section?"

Type of answers	For Giri, not changing	For Giri, changing	To any of them, not changing
Frequency	18	4	9

In the 3rd step of the experiment, students were asked about Giri's probability of winning if the game had been held before the pandemic. Table 3 shows the types of arguments used by students. Some recognized the difficulty of estimating a probability ("I do not know how to measure probabilities, but I think it would be around 45%"), others simply provided a value ("From 1 to 10, an 8"). Some other reasonings were based on Carlsen's apparent superiority, eliciting a low estimate for the probability ("Very low, 2 out of 10, 20%, because Carlsen was undefeated", "About 33% or very low, because Carlsen was undefeated during 120 games"). Two answers explicitly based the argument on the previous results: "Giri would draw, because he usually draws", and "Giri is going to win because he was never the champion and Carlsen started losing games".

Table 3: Answers to "If that game had happened before the pandemic, […] what do you think would be Giri's probability of winning?"

Type of arguments	Blank	No reason	Verbal	Numerical	Verbal and numerical
Frequency	3	2	11	14	1

Discussion and conclusions

Results revealed different examples of probabilistic heuristics, as well as some biases in students' reasoning. In the first step, students used their previous knowledge about the context (a knowledge element in Gal, 2005) to guess the possible result of the game, but their beliefs (a dispositional element in Gal, 2005) were also involved. Since previous knowledge differed from one student to another, we found answers based on a proper integration of previous information and knowledge about chess ("Anish Giri would win because he did not lose a game, while Carlsen lost three") and also other answers built on maximum uncertainty ("There will be a draw because both players are excellent"). There were also some mistakes regarding information processing or reading comprehension, getting confused between Carlsen having lost three games and having won all but three (that is, not considering draw as a possible result). Thus, this illustrates how students differently integrate and process the contextual information in terms of deciding what is the most probable event.

In the second step, extra information was added, so the students had to decide whether they keep or change their initial decision, again within a meaningful context (Gal, 2005). It was striking how most of the students integrated the information about the confinement and the players but decided to keep their initial decision, because most answers were supported by the personal belief on Carlsen's quality as player, and also a poorly explained favoritism to him (Batanero, 2005; Vásquez Ortiz & Alsina, 2019). There was also an interesting answer trying to find a rationale between Carlsen's category as world champion and the three games he lost after confinement ("Maybe Carlsen just had a bad run"). Also remarkable was how students used personal experience regarding chess (for instance, the one arguing that whites have greater chances to win was one of the students playing chess and he used his records to support this idea). In line with Kazak and Leavy (2018), we confirmed how the meaning of subjectivity is used when dealing with previous and added information and personal experiences and beliefs to estimate a probability, stressing in this experience the importance of beliefs as an attitudinal element (Gal, 2005).

In the third step, the difficulties to estimate a probability without a counting or experimental situation were revealed. Nevertheless, students were able to find a rationale supporting their linguistic or numerical assignments (knowledge elements in Gal, 2005). Rather curious was how students expressed probability in many different scales (percentage, 1-10 scale, linguistic quantifiers, etc.), and how they were able to compare even without a numerical assignment. Besides numerical language, half of the students used verbal language to express such probability using different terms ("impossible", "very low", "none"). In some cases, a wrong understanding of the probabilistic language was reflected when considering an improbable or unlikely event as impossible. Students also used different approaches to integrate the information from previous games: some used it to find a trend in terms of experimental probability ("He usually draws"), others reasoned by using a version of gambler's fallacy ("Giri is going to win because he was never the champion"). The latter shows the outcome approach described by Konold (1989): when asked about a probability there are students answering about the outcome of the next trial.

Regarding research question, the development of the discussion between the first and second steps became particularly crucial. We witnessed how students who were not familiar with the context (chess game) started from a total uncertainty (even ignorance) about the answer, but when they heard their colleagues who regularly play chess reasoning about the situation, they started either totally following them or modifying their opinions in terms of what was expressed by experts. That is, they acknowledged the quality of the information provided by others and modified their initial beliefs about the probability in an informal Bayesian reasoning. As stated by Kazak and Leavy (2018), such reformulations seem rather intuitive for children of this age.

The implications on this didactic experiment are rather promising for future research and practice. Despite the difficulties encountered on probabilistic reasoning, students are able to quite naturally handle probability in informal and even numerical terms. Evidence pointed to the use of Bayesian procedures to integrate added information about the experiment. Choosing a chess-based situation instead of a more popular sport (as football or basket) made students felt too much apart from the context and impeded arguing about their reasonings. Thus, to promote students' probability literacy, particularly probabilistic reasoning, it seems necessary to widely implement similar contextualized situations that are relevant to students and help them to understand the different meanings of probability. Similar experiences, with an increasing complexity in the situations and the cognitive demand, could be also designed for secondary education, where the use of the subjective meaning of probability is also usually omitted (Blanco-Fernández et al., 2016; Rodríguez-Muñiz et al., 2019). More precisely, the presence of didactic experiments about subjective probability should be increased by teachers making use of daily-life situations instead of archetypical examples, such as urns, coins, dice, cards, or lottery.

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Causal inference for beginners: designing a massive open online course

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Keywords: Causal inference, data literacy, curriculum development, instructional materials.

Data literacy and causal inference

It takes much more than the ability to insert numbers into formulas to be data literate. According to Schüller (2020), data literacy includes behaviors and attitudes that allow us to create value or make decisions with the help of data. In the age of "big data", the basis for decision making is often observational in nature and contains many variables, which increases the necessity to "think clearly about correlation and causation" (Rohrer, 2018). As causal conclusions are ubiquitous, we should provide a framework to discuss the necessary assumptions (Rohrer et al., 2021). Several authors advocate for the inclusion of causal inference in the curriculum, at least in post-secondary education (Cummiskey et al.; 2020, Lübke et al., 2020). The presented course therefor aims at introducing basic concepts of causal inference to improve data literacy and foster understanding of sources of bias. Ignoring these has led to many incorrect conclusions in, e.g., Covid-19 research (Herbert et al., 2020).

Massive open online course

The course "Introduction to causal inference" is designed as a self-paced online course and will be hosted by https://ki-campus.org/, a project funded by the German Federal Ministry of Education and Research (BMBF) focused on developing the prototype for a digital learning platform specifically geared towards AI.

Interactive modules and videos provide learning opportunities on the potential outcomes framework and causal effects. Correlations based on confounding as well as collider bias are presented and explained with the help of basic elements of causal diagrams (chain, fork, inverted-fork). With these diagrams direct, total and indirect effects are introduced as well as the consequences of (not) adjusting for covariables when estimating causal effects. The mathematical foundations provided include conditional probabilities, conditional and unconditional independence as well as Bayes theorem. Within this framework, design principles like random sampling and allocation are explained and motivated. Probability and statistics are used to explain everyday phenomena as well as "paradoxes" like the ones attributed to Simpson and Berkson. The course is published with the CC BY-SA 4.0 license and can be integrated in existing curricula, as questions for summative assessment are provided.

Technology

Computer-based tools are used and integrated to support learning. The R package learnr is used to provide course participants a web-based opportunity to explore data with R or to run simulations to investigate how different assumptions about the data generating process affect conclusions. Furthermore, assignments and quizzes with automatic feedback are provided. Embedded videos give additional motivation and illustration by e.g. interviews of researchers and practitioners in the field.

Thanks to R/exams, it is possible to generate randomized tests which can be integrated in different learning management systems. The flexible modules of this course allow for assessment of up to 1 ECTS.

Evaluation plan

A major concern with massive open online courses is that most participants never finish. Who does and who does not finish furthermore depends on individual characteristics (such as motivation, background knowledge), raising the possibility that a large number of people who could, in principle, profit from a course are left behind. To improve our understanding of how well different materials work for participants with different backgrounds, future participants of the course are encouraged to take part in a longitudinal evaluation alongside the course. This evaluation contains quantitative but also qualitative assessments of the provided learning materials as well as obstacles encountered by participants. It will be realized with the help of the survey framework formr.

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Data Science and Machine Learning in mathematics education: High-school students working on the Netflix Prize

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One goal of contemporary mathematical modeling classes in schools should be to include up-to-date problems or interesting, new technologies from the everyday life of students - especially if these allow the didactical reduction to elementary (school-)mathematical knowledge and thus have the potential to enrich mathematics education. Data Science and Machine Learning is applied in numerous areas of science and technology and used in many applications in our everyday life. Using movie recommender systems and the so-called Netflix Prize as an example, this paper discusses how mathematics education can be enriched by modeling real-world, student-centered problems from the field of Machine Learning in school. For this purpose, we describe tested digital learning material from guided modeling projects and share our experience with giving the problem of developing a recommender system as a completely open problem to upper secondary students.

Keywords: Data Science, Machine Learning, mathematical modeling, recommender system, digital learning material.

Motivation

Many technologies and applications that students use in their everyday life are based on methods from the fields of Data Science (DS) and Machine Learning (ML). An important tool for answering questions from these areas is (mathematical) modeling. Therefore, DS problems offer a great opportunity to design mathematical modeling activities on student-oriented, up-to-date problems. In this way, not only modeling competencies but also the handling of data can and should be trained. Especially in today's world, it is essential to deal with data in an understanding, responsible and critical way. Some profitable approaches to implement problems from the field of DS and ML in the classroom are advanced by the Paderborn project ProDaBi (Opel et al., 2019). In addition, Sube (2019) and Schönbrodt et al. (2021) already worked out how mathematical modeling lessons on DS questions can be designed using real-world problems.

We developed digital learning material in which upper secondary students can acquire an understanding of essential steps in solving data-heavy problems based on methods from DS and ML by actively working on an authentic, and relevant problem (Schönbrodt & Frank, 2021). The learning material presented in the following can be implemented within a mathematical modeling day or be distributed over several school lessons.

The problem – Authentic and relevant

Netflix, Amazon, and many other e-commerce companies that the students encounter every day mainly rely on one thing for customer loyalty: personalized recommendations for new products, movies, etc. For this purpose, recommender systems are developed, which should predict what the respective user might like. The developed learning material is based on the Netflix Prize which

Netflix launched in 2006 to further improve their own recommendation system: The team, that could predict at least 10% more accurately than Netflix's system, which movies a user would like had a chance to win the grand prize of one million USD (Feuerverger et al., 2012, p. 203).

Mathematical modeling days – Guided digital learning

The key questions of the developed learning material are: "How can we model user preferences by exploiting given user ratings?" and "How can we predict unknown user ratings in the best possible way to then use the predicted ratings to suggest relevant new movies to users?" (Schönbrodt et al., 2021).

In the learning material the students work with the original dataset of the Netflix Prize. On digital worksheets they explore the dataset and collect their own ideas for the development of a recommender system. Afterwards, they develop a mathematical model and apply it to the Netflix dataset. The implementation of the data exploration and the modeling process is discussed in more detail below. The digital tool used to develop the digital worksheets is Jupyter Notebook[1]. Jupyter Notebooks are widely used in numerous branches of industry, research, and economy. Thus, not only the learning content but also the digital tool provides an authentic insight into current problem-solving strategies and technical implementations in applied mathematics or more generally in STEM fields. Put in simple terms, Jupyter Notebooks are digital documents which can be used to write text, execute code (to compute mathematical terms), and can be edited interactively by the students. Also, Biehler & Fleischer (2021) and Opel et al. (2019) highlight the possibilities of Jupyter Notebooks for designing learning material on ML methods. In these papers the focus is stronger on statistics and computer science teaching whereas we focus more on the process of mathematical modeling and related competencies.

The developed learning material exemplifies how data-heavy problems from the students' everyday life can be prepared and carried out in the context of modeling projects, both in distance and face-to-face learning. The material is available on a cloud-platform of the Karlsruhe Institute of Technology for direct use in class and can both be accessed and edited in a web browser (see www.cammp.online/english/214.php). The material can be used in heterogeneous learning groups by means of digital differentiation material (scaffolded tips, optional tasks for more advanced students) and individual, automated feedback on the students' solutions. The dataset and the main modeling steps of the learning material are briefly described below.

For the processing of the learning material, it is assumed that the students are familiar with mean values and standard deviations. In addition, an understanding of functional relationships is also central. Knowledge of vectors and the scalar product is preferable. Matrices, however, are not a prerequisite for the comprehensible processing of the learning material. They are rather introduced in a problem-oriented manner as explained below. Furthermore, no programming knowledge is required on the part of the students. Instead, a fill in the gap approach is implemented. The students simply need to replace a placeholder within the code (specifically *NaN*, which stands for Not a Number) with

[1] For further information see https://jupyter.org, last accessed: 05 August 2021

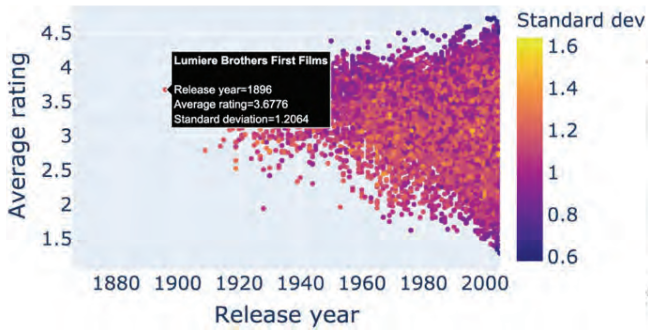
the relevant solution to the task (a number, formula, or function). They receive task-specific feedback on their input and, if necessary, a tip on how to correct their solution.

Unit 1: Understanding and analyzing the data

The starting point of the learning material is the Netflix dataset. This consists of 17,700 movies, 480,189 (anonymized) users, and 100,480,507 ratings from one (worst) to five (best) that users gave to the movies (Feuerverger et al., 2012, p. 204). The title and release year of the movies are known.

Figure 1: Rating matrix / table containing users' ratings for the movies

The students examine the dataset using various forms of representation such as the rating matrix / table (see Figure 1) or interactive scatter diagrams and tables (see Figure 2). In doing so, they answer questions such as "For which movie do the ratings differ the least from the mean rating?" or "Which conclusions can you make regarding the distribution of the data over time?". The students can modify the interactive plots shown on the worksheets to answer the questions. For example, they can zoom into the scatterplot or sort the table in Figure 2 by any column. They decide independently which representation of the data is suitable for answering which questions.



Titles	Number of ratings	Average rating	Standard deviation	Release year
Miss Congeniality	232944	3.3613	1.1117	2000
Independence Day	216596	3.7242	1.0391	1996
The Patriot	200832	3.7839	1.097	2000
The Day After Tomorrow	196397	3.4422	1.0993	2004
Pirates of the Caribbean: The Curse of the Bla...	193941	4.1539	0.9071	2003

Figure 2: Visualizations of the Netflix dataset on the digital worksheets

When introducing the tabular representation of the rating data (see Figure 1), the concept of a matrix is avoided at first, as this should not be a prerequisite for the processing of the learning material.

Unit 2: Developing a mathematical model

After exploring the data, a brainstorming takes place. Whilst discussing with their fellow students the students collect their own ideas on how to use the known ratings in the rating table to predict the unknown ones. During the lessons conducted with secondary students so far, the students mentioned diverse approaches. Among them:

- Find similar users and then find the movies the similar users liked.

- Identify users' areas of interest, such as a specific genre.
- Determine the genre of the movies a user liked. Then suggest movies from that genre.

Next, using small rating tables, the students start with the development of a selected mathematical model: the decomposition of the rating table into a user-feature table (short user table) and a movie-feature table (short movie table, see Figure 3). The core of the described model is a matrix factorization. This was also the basis of the model used by the winning team of the Netflix Prize (Koren et al., 2009, p. 32). Thus, not only the problem but also the mathematical methods used are authentic (Vos, 2011). The user table indicates how much a user likes certain properties, such as the action or comedy genres. The movie table specifies the extent to which a film contains these features (see Figure 3). The students independently develop a formula to calculate the known ratings from the rows of the user table and the columns of the movie table and transfer this to the calculation of unknown ratings. This is basically achieved by the scalar product of the row vector of the user table and the column vector of the movie table.

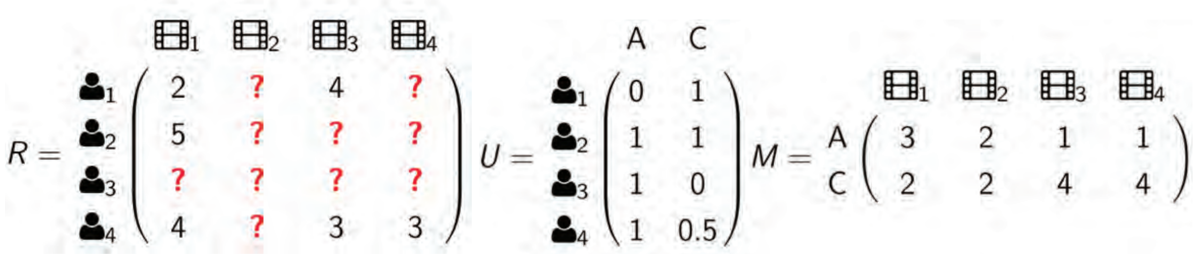


Figure 3: Exemplary rating table R and corresponding user table U and movie table M. The user and movie tables reflect the interests of the users and the characteristics of the movies regarding the features "action" (A) and "comedy" (C)

Motivated by the fact that the Netflix dataset does not provide any information on the features of the movies and thus both the user and the movie table must be calculated appropriately, the students should first determine a suitable decomposition for a 2x2 rating table themselves using pen and paper.

Note: Another modeling approach for which learning material for guided modeling projects was developed and tested is the modeling of similarities - either between users or between movies. Students can be creative in their choice of similarity measures, but also learn about common similarity measures such as (adjusted) cosine similarity and Pearson correlation (Sarwar et al., 2001, p. 287). This approach falls in the class of so-called neighborhood methods.

Unit 3: Error measure and optimization

Calculating a decomposition "by hand" is not feasible for large rating matrices. Therefore, the goal is to leave the calculation to the computer. For this purpose, a step-by-step optimization procedure is applied, which provides a sufficiently good decomposition. The students first define an error measure by which they (and later the computer) can evaluate whether a found decomposition is already sufficiently good. The error measure is defined by looking at a small example of a given rating table R and predicted ratings given in the table P (see Figure 4).

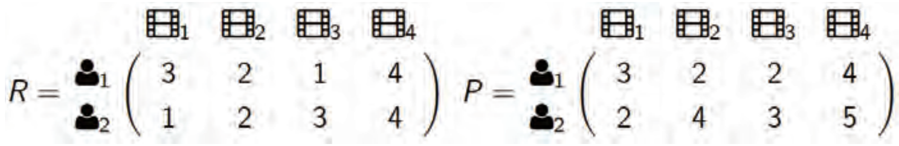


Figure 4: Exemplary rating table R and table P containing the predicted ratings

The self-defined error measures are then compared with the mean absolute error and the mean squared error. Then an alternating optimization algorithm is used to compute a decomposition that leads to the smallest possible error on the known data. The algorithm does not have to be developed by the students themselves. Instead, an optimization package of the used programming language, in our case Julia (Python would also be possible), is applied as a black box.

Unit 4: Application to the Netflix dataset and critical discussion

So far, we have only evaluated how well a found decomposition is suited to represent known ratings, but unknown ratings are to be predicted. We apply an essential strategy of supervised ML: Dividing the known rating data into those used to compute a decomposition (training data) and those used to afterwards validate how well the prediction works on "unknown" data (test data). Students apply this strategy to the Netflix dataset. Finally, they conduct a parameter study (varying the number of features considered) and try to improve the results on the test data.

In a final discussion, limitations of the developed model are discussed, such as: "How do we deal with new users who have not yet submitted ratings?". A critical discussion on the possibilities of manipulating recommender systems, the problems that could arise from de-anonymizing[2] users, and the extent to which so-called filter bubbles could be problematic takes place. The opinions and the students' experiences with recommender systems are particularly included in this discussion.

Experiences with students

The digital learning material was already implemented with more than 90 students from Grade 10 to Grade 13 (this corresponds to an age range of approx. 15 to 19 years) within school lessons (4-5 lessons 90 minutes each) or in the framework of a mathematical modeling day (approx. 5 hours) in Germany. During the implementations, the diverse ideas of the students for the development of a recommender system but also the numerous arguments in the critical reflection of such systems stood out. The very lively discussions also showed that the students were highly interested in solving the problem.

While working with the learning material numerous school mathematical contents are used; among them mean values, standard deviation, vectors, scalar product, and functions. Therefore, the problem setting is not only extremely student-centered but especially real and authentic and thus provides an answer to the question "What's the point of math?".

[2] In 2008, two researchers at the University of Texas showed that de-anonymization of the dataset was partially possible by combining it with another publicly available film dataset (Narayanan & Shmatikov, 2006).

Open modeling within a mathematical modeling week

The Netflix Challenge was given to a group of six upper secondary students within a so-called modeling week. The task of the students was to develop a movie recommender system for a smaller subset of the Netflix dataset from scratch. The students developed the recommender system and thus the mathematical model all by themselves. In contrast to the more guided modeling days described above within the modeling week the students did not receive any learning material with pre-structured subtasks. They only received a description of the datasets and the problem as well as a short introduction in using Jupyter Notebook. The students implemented code themselves and filled the Notebooks with content on their own. They worked on the problem for four full days and were supported by a scientific advisor. The advisor only helped with programming related questions but did not intervene in the modeling process of the students at all. The students not only developed a recommender system but also documented their modeling process in a report and gave a presentation on their results at the end of the modeling week. The core idea of the model the students developed was to measure similarities between users by computing the mean absolute deviation of the ratings of different users. Once they determined the similarity measure and computed the similarities between any pair of users the students predicted the rating of a specific user by taking a weighted sum of the ratings of the most similar users to the selected one. The model developed by the students without any guidance, can be linked to the class of neighborhood methods mentioned above. These are methods that are used by experts in this field (Sarwar, 2001, p. 285).

During the modeling process, the students discovered a relevant trade-off in the field of ML:

“Furthermore, we concluded that the more data we use in the algorithm, the more accurate our result will be. One problem that arose was the long, very rapidly increasing, computation time that the program needs to get results with the large amount of data” (quote from the students’ report).

Summary and Outlook

Through the learning material, secondary students collect experience in dealing with large amounts of data and gain an active insight into essential strategies of mathematical modeling and ML. We already implemented the learning material on the Netflix Challenge with numerous learning groups as part of stronger guided modeling projects (modeling days). Jupyter Notebooks were used as a digital tool to structure and guide the modeling process through smaller subtasks. The feedback from the students on the modeling activities, which were so far only carried out virtually, showed that there is a great interest in this type of problem.

On top, the Netflix Challenge was posed as a problem in the context of an open modeling project (modeling week) once. The results of the students at the end of the modeling week showed that they were indeed able to work independently in a small team and to develop and validate a recommender system on a real-world dataset. To make a more general statement about the extent to which this problem is suitable for open modeling projects, further testing with a more heterogeneous student group would be necessary. Here, the participating students were interested in mathematics and some of them already had programming skills.

The tool Jupyter Notebook provides different design possibilities, variability of the programming language and the neat combination of code and text in one document. Nevertheless, it would also be feasible to provide the data (or a part of the data) in the form of CSV files and to have the neighborhood methods described above developed with the help of a table calculation program.

In addition to the problem presented here, various other applications based on ML methods offer a good opportunity to design modeling activities on real and up-to-date problems. The following are examples of possible ML methods which can be reduced to school-mathematical concepts: The so-called support vector machine (SVM), which is widely used for classification problems. The SVM is based on minimizing distances between points and hyperplanes. Vectors, straight lines, and planes as well as the scalar product can find an authentic application (Schönbrodt et al., 2021). Starting from a classification problem with two- or three-dimensional data, the method could first be developed with students in the visual case using rich visualizations implemented in Jupyter Notebooks. After that the method can be abstracted and applied to higher dimensional problems (e. g., for the classification of images with faces). Another example is represented by so-called n-gram models, which are used to predict words when typing a message on a smartphone. In this case conditional probabilities and absolute and relative frequencies play a central role. Finally, Neural Networks, which are the basis for countless applications in our everyday life (Schlichtig et al., 2019), rely on various school mathematics contents such as the chain rule, vectors, and the scalar product.

It seems important to prepare a selection of these methods to give meaning to (school) mathematical content and to offer students an insight into methods and problem-solving strategies that increasingly influence technologies and our society now and in the future. Therefore, the development of a curriculum that precisely addresses these kinds of methods and meaningfully combines content from mathematics and computer science education is necessary. Important steps in this direction are already accomplished by the projects ProDaBi and the International Data Science in Schools Project (Biehler et al., 2018; IDSSP, 2019). These projects as well as the learning material on the Netflix Prize presented in this paper underline that addressing problems from DS and ML with high-school students is feasible and necessary.

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Learning to teach statistics through study and research paths

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In Brazil, as in many other countries, there is a gap between the type of statistics preservice teachers learn in graduate programmes and what they will teach at compulsory education. This paper presents the design of a study and research path for teacher education (SRP-TE) as a tool to reflect, develop, analyse, and experience teaching proposals in statistics for the final grades of the lower secondary school. The investigation is based on some principles of the Anthropological Theory of the Didactic and its proposals to locate teacher education within the paradigm of questioning the world. The proposed SRP-TE starts from a school activity about the water distribution in Brazil and its extension to incorporate dimensions of the statistical work that tend to be absent from secondary education, such as the search, collection, cleansing, and representation of data.

Keywords: Anthropological Theory of the Didactic, study and research path for teacher education, statistics teaching, secondary school, working with data.

Introduction – Research context and aim

In this paper, we present a PhD investigation developed at the *Universidade Federal de Mato Grosso do Sul* (Brazil) in partnership with the *Universitat de Barcelona* (Spain). The question motivating the research project came from a perceived limitation in the preservice teacher education process followed by the first author during her degree in mathematics education. In the case of statistics, it seems that the double discontinuity described by Klein (Isaev & Eichler, 2017) appeared there with even greater prominence. The subject *Probability and Statistics* did not provide future mathematics teachers with a practical vision of the area. It is then common to hear reports from new mathematics teachers expressing difficulty working with this theme in Brazilian compulsory education[1] when they take up a classroom, as happened with the first author of this article. A significant gap is perceived between the education received and the teaching practice, and the situation is not limited to Brazil. According to Martignon (Batanero et al., 2011):

> […] the gap between disciplinary statistics and school statistics has to be taken seriously into account when preparing future school teachers of statistics who have to be aware that they will be providing future citizens with "statistical literacy". In other words, future citizens need to be endowed with tools for interpreting statistical information in the media, for dealing with relative and absolute risks, and for understanding the effect of base rates on the predictive accuracy of medical tests" (Batanero et al., 2011, p. 34)

[1] In Brazil, the compulsory education is composed by preschool (for children between four and five years old); primary and lower secondary education (for children between six and fourteen years old), that is also known as elementary education; and upper secondary education or high school (for teenagers between fifteen and seventeen years old).

In her master's thesis, Verbisck (2019) focused on the case of probability and the teaching proposals of different textbook collections from grade 1 to grade 12. In all cases, only a small part is devoted to probability and statistics, with a tendency to locate it in the last chapters of each book. More than ten years ago, Lopes and Ferreira (2004, p.12, our translation) stated that "until the implementation of [...] PCNs[2] (MEC, 1998), the teaching of statistics at elementary and high school grades was very restricted and marginal. The topics covered were included in the mathematics discipline, in the most advanced grades and, generally, it was one of the last topics in the textbook [...]". However, this persists today, at least in more than half of the textbooks approved by the Brazilian ministry of education in 2017.

There is an agreement in the international research community on the importance of adopting a broad view of statistics that includes aspects such as searching for and collecting data, cleaning and organising data, visualising data, using specific software, preparing reports, all within a perspective of solving open questions and studying variability. These perspectives are described in terms of: "statistical reasoning" (Garfield & Ben-Zvi, 2008), "statistical literacy" (Watson, 2006), "statistical thinking" (Wild & Pfannkuch, 1999), "informal inferential reasoning" (IIR) (Leavy, 2010) and "informal statistical inference (ISI)" (de Vetten et al., 2019).

How to transpose this perspective in teacher education is still an open problem. Statistical investigations are often taken as a basis for enlarging teachers content knowledge while providing them with new pedagogical resources (Makar & Fielding, 2011, Pfannkuchm & Ben-Zvi, 2011, Santos & da Ponte, 2014). Our PhD research proposal follows this line of research, focusing on lower secondary school teacher education. We aim to investigate the possibilities and contributions of implementing a teacher education proposal to focus their reflections on current statistical practice and literacy. The proposal should also provide teachers with experience in carrying out statistical inquiries. Thus, the broad research question framing the development of the thesis is: *What educational proposal is it possible to implement with a group of preservice mathematics teachers in Brazil and how this proposal contributes to providing future teachers with tools to design, analyse and implement new didactic processes for the teaching of statistics in lower secondary school?* This question is approached within the Anthropological Theory of the Didactic (ATD) and, more particularly, a specific teacher education proposal called "study and research paths for teacher education" (SRP-TE). After presenting the theoretical approach, we will describe the design of an SRP-TE that will be implemented during this academic year 2022, with a pilot version during the last term of 2021. The SRP-TE starts from an open question taken from a Brazilian textbook. We will illustrate in this paper how the inquiry promoted by an SRP-TE can highlight the different dimensions of the statistical work and also be used as a tool to analyse curriculum and textbook proposals.

Theoretical framework: Study and research paths for teacher education

Chevallard (2015) states that, in our societies, teaching mathematics—and teaching in general—participates from what he calls the *paradigm of visiting works*. In this paradigm, the role of students is to study ready and finished knowledge organisations built in topics, areas, domains, and disciplines. It is up to the students to "look and admire" these works without necessarily questioning their validity or value. So, in this paradigm, topics and subjects are like monuments: students cannot change them; they do not need to know their *raison d'être*; they just have to study them. At most, questions about knowledge value and validity are posed by the teacher, characterised as "the one who knows".

[2] *Parâmetros Curriculares Nacionais* (National Curriculum Parameters), official documents used in Brazil as a curricular tool for educators and educational institutions. These curriculum documents were published in 1997 and 1998 and have recently been replaced by the document: BNCC, *Base Nacional Comum Curricular* (Common National Curriculum Base). The BNCC was approved by the Brazilian Education Minister on December 20, 2017. With this document, "school systems and public and private educational institutions now have a mandatory national reference for the preparation or adaptation of their curricula and pedagogical proposals" (Brasil, 2017, p. 5, our translation).

In contrast to this first paradigm, Chevallard describes the *paradigm of questioning the world* in which knowledge organisations change into *questions*. Teaching and learning processes become inquiry processes aiming at answering the questions. In other words, students are the inquirers of the generating questions *Q* proposed by the teachers (or by the students themselves). There may be times when visiting works is required in the search for answers to a question, but with a specific *raison d'être* (answering *Q*). They also have to search for answers in medias (as the internet, books, experts, etc.), and they have to validate these answers and check their utility to answer *Q*. This paradigm creates significant changes in the teachers' and students' roles. The first ones are no longer the "holders of knowledge", and the students raise questions, investigate, search or elaborate answers and even validate them. To study the conditions needed to transition to this second paradigm, Chevallard (2015) proposes a general inquiry format called *study and research paths* (SRP). The interplay between questions and answers plays a crucial role in the dynamics of SRPs. Students, helped by teachers, address an initial question *Q*, display *Q* into derived questions Q_i, search or elaborate answers A_i, find new questions during the process which, in turn, call for new answers, etc. Bosch and Winsløw (2015) point at the importance of such a dialectic between questions and answers to ensure the dynamics of SRPs, which is usually represented using *questions-answers maps* (Figure 1).

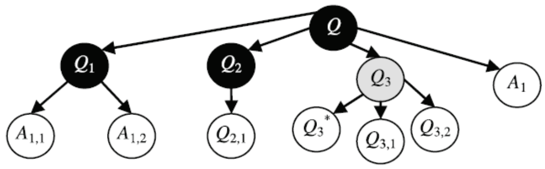


Figure 1: Example of Questions-Answers maps (Q-A) (Winsløw et al., 2013, p. 271)

Barquero, Bosch and Romo (2015) argue that teacher education proposals also need to be conceived within the new paradigm of q*uestioning the world*. Thus, they consider implementing *study and research paths for teacher education* (SRP-TE) "as a way to provide teachers with pertinent (theoretical and practical) tools to nourish and sustain their professional development" (Barquero et al., 2015, p. 810). An SRP-TE consists of five modules, which will be described below.

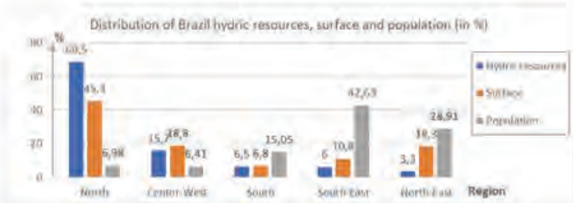
Module 0 is to propose the open question (generating question $Q_{0\text{-TE}}$) related to the subject to be discussed. In our case, our $Q_{0\text{-TE}}$ is: "How to teach statistics in the final grades of lower secondary school?". Other derived questions will appear, such as: What statistics should we teach at these levels according to the curriculum? What kind of activities are proposed by official textbooks? What other proposals can exist? Partial answers to $Q_{0\text{-TE}}$ will appear during the entire SRP-TE (and hopefully also afterwards), making it a transversal module of the educational process. *Module 1* consists of proposing the group of teachers an SRP with a relevant question that could be approached in a real classroom as students. With this, the group of teachers will be introduced to an inquiry process, a strategy also suggested by Makar and Fielding (2011). *Module 2* is the moment for teachers to analyse the experienced SRP using epistemological and didactic tools spontaneously accessed by the students or provided by the educators. In *Module 3*, teachers design and implement (if possible) an SRP for the schooling level under discussion. The new design is carried out based on the analyses made in the previous module, adapting, adding, or excluding (if necessary) didactic tools that seem necessary – or at least useful – for the implementation of this new SRP. Finally, *Module 4* consists of the teachers sharing their experiences through an a posteriori analysis. Again, the mathematical and didactic tools used in modules 2 and 3 are present here and have great importance, "not only to provide some provisional answers to the question that was at the origin of the whole process (e.g., "How to teach …?"), but also as a means to analyse other possible alternative answers" (Barquero at al., 2015, pp. 810-811), as those answers found in Module 0.

These modules structure the SRP-TE. We present below an exercise identified in a Brazilian textbook of the eighth grade of secondary school that gave us the initial idea of a generating question to be studied in the SRP when developing module 1 with the group of teachers.

Designing a teacher education proposal

The proposed SRP for the teachers starts from a school activity about “the water distribution in Brazil”. Looking at the approach to statistics in some textbooks from the eighth and ninth grades of secondary school, we find the exercise below:

Brazil has about 13.7% of the total fresh water in the world, being considered a territory rich in water terms. However, the country is experiencing serious problems related to both the degradation of water quality, especially in the urban areas' proximity, and the lack of control of excess and lack of water, which affect several Brazilian locations. Not just the floods affect Brazilian cities: water scarcity also imposes serious restrictions and high costs on economic and social development of large cities. Looking at the chart below, answer in your notebook:



Information obtained in: Ministério do Meio Ambiente <http://wwww.mma.gov.br/estruturas/ sedr_proecotur/_publicaco/140_publicacao09062009025910.pdf>. Accessed: 1st July 2018.

a) What kind of chart is this?
b) Indicate the Brazilian region:
- with the largest surface;
- with more water resources;
- with the second lowest population concentration.

c) Which region has the lowest percentage rate of water resources in our country?
d) In which region is there the greatest concentration of population?
e) Can it be said that the larger the surface of the region, the greater the number of inhabitants? Justify your answer.
f) How many percent of the world's fresh water is in the Southeast region of Brazil? Explain how you elaborate your answer.
g) Can it be said that the region with the most water resources is the one with the largest population?

Figure 2: Brazilian exercise (Giovanni Júnior & Castrucci, 2018, p. 26, our translation)

This type of exercise shown in Figure 2, and which is relatively common in the textbooks, is clearly located in the paradigm of visit works. Although it presents a relevant topic to be discussed (and even includes a link to the source of the data), it does not pose any open question, nor does it encourage students to search the data, organise and summarise them, and investigate other issues related to the topic. The questions raised in the textbook only require students to look at the graph, identify the largest or smallest bar, and perform some simple percentage calculations. If we look at the official curriculum guidelines, we can observe that this is not the kind of activity that encourages students to develop the skills assigned to this level of education, which are:

- Assess the suitability of different types of charts to represent a survey dataset.
- Classify the frequencies of a continuous variable of a survey into classes so that they summarise the data in a way that is suitable for decision making.
- Obtain the values of measures of central tendency of a statistical survey (mean, mode and median) with an understanding of their meanings and relate them to the data dispersion, indicated by the range.
- Select reasons of different natures (physical, ethical, or economic) that justify conducting sample and non-census surveys and recognise that sample selection can be done in different ways (simple random, systematic, and stratified sampling).

- Plan and execute a sample survey, selecting an appropriate sampling technique, and write a report that contains the appropriate graphs to represent the data sets, highlighting aspects such as measures of central tendency, range, and conclusions. (Brasil, 2017, p. 315, our translation)

Despite the limitations of the exercise proposed, the topic presents many interesting questions that could generate a potential SRP within and SRP-TE. For this, we need to move from the paradigm of visiting works to the paradigm of questioning the world and *question* the piece of information given. At the same time, we will see how the inquiry process it generates can incorporate dimensions of the statistical work that tend to be absent from secondary education, like the search, collection, cleansing and representation of data, together with a critical reading of quantitative information. Thus, for the SRP to be developed with the group of preservice teachers in Module 1, we propose to start with Q0 and initiate a questioning that can lead to the following derived questions (starting from the same text and graphical information as before):

Q0: How can we explain the contradiction between the abundance of water resources and the water problems (scarcity, quality degradation, lack of control, etc.) of Brazilian locations? How are hydric resources distributed in Brazil compared to the population and surface?

Q1_Water: What do we know about water distribution in Brazil?

Q1.1. Are there studies about the water problem in Brazil? Where can we find them?

Q1.2. What disciplines are involved in the studies: geography, politics, geology?

…

Q2_Graph: What information can we draw from the graph?

Q2.1. What variables appear in the graph? What others could it be interesting to consider?

Q2.2: What are hydric resources? How are they measured?

Q2.3: Why are the variables in percentages? How are these percentages calculated?

Q2.4: How will be the graph if we use units instead of percentages?

Q2.5: Can we improve the graph using another type or adding/omitting information?

…

Q3_Data: What data is used to make the graph?

Q3.1 Is it available? Where? [The presented link does not work.]

Q3.2. Is the data available also reliable? How is it obtained?

Q3.3. What are the units of the different variables in the data source?

Q3.4. Can we use the available data to reproduce or update the graph?

Q3.5. Are there other interesting variables with available data to consider?

...

Q4_Working with data: How to download the data to start working with it?

Q4.1. How to clean the table of data to make it ready to use?

Q4.2. What tools are available for data processing, and which ones can we use?

Q4.3. What types of numerical and graphical summaries are appropriate?

…

We started from the main subject of the text presented in the textbook exercise and elaborated a relevant open question (Q0) that could be worked on in a classroom. The exercise presented a graph with the distribution of water in Brazil and the population and surface area by region. This graph will also serve as a possible answer for a derived question since its source is the Ministry of the Environment. In this initial work to list our generating question and possible derived questions, we identified two large groups of questions: those related to the topic “water” (Q1, Q1.1, Q1.2, ...) and those related to the topic “graph” (Q2, Q2.1, Q2.2, ...). Still, within the topic “graph”, another large group of questions appears, which is the topic related to “data” (Q3, Q3.1, Q3.2, ...). By raising the questions related to “data”, we entered the topic “working with data” (Q4), which also made us think of derived questions (Q4.1, Q4.2, ...). With this, we obtain an a priori questions-answers map (Figure 3) that is part of SRP-TE.

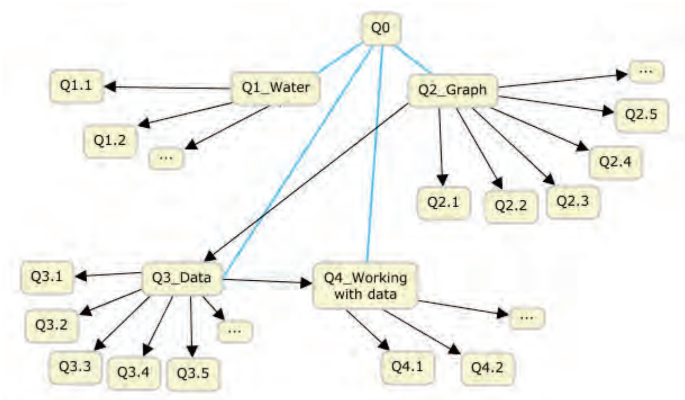


Figure 3: Questions-answers map about water distribution in Brazil

We started from an exercise proposed in a Brazilian textbook and reworked it to display the initial and derived questions that may arise during the inquiry process. This questions-answers map serves as a possible SRP to be developed with the group of preservice mathematics teachers. The answers are still being developed since we have not yet worked directly with the group. It is an a priori study about the proposed theme: distribution of water in Brazil and, with this SRP in development, we believe it will be possible to work with dimensions of the statistical work that tend to be absent from the final grades of the lower secondary education, as the search, collection, cleansing and representation of data. The inclusion of provisional answers will likely enrich the map with more questions and potential maps to follow. It will also introduce more connections between the derived questions and the different branches they form.

Conclusions

The a priori questions-answers map shows the productivity of changing attitude from *visiting works* to *questioning the world*. It first gives visibility to many interrogations that are usually hidden in the way school tends to present (quantitative and qualitative) information, giving no room for doubts or questioning. It also shows how these interrogations go beyond the strict statistical reading and analysis of data to merge with other disciplines or areas concerned by the question addressed. This is not specific to statistics, even if it plays a crucial role there. Teaching modelling also suffers from the same phenomenon of disciplinary confinement. It makes many connections between different topics of statistics that tend to be presented separately even if they nourish each other (like the definition and measurement of variables, the reliability of data, the ambiguity of percentages, the pertinence of the type of graphs chosen, etc.). Finally, it provides future teachers with a broader and more realistic vision of statistics that can help them detach from the narrow perspective proposed by textbook exercises and facilitate the introduction of engaging statistical activities in the classrooms. We hypothesise that the design and implementation of an SRP-TE in statistics will provide students with tools to reflect, develop, analyse, and implement new statistics teaching proposals when working in lower secondary school and develop a critical posture when acting in their profession. The change of paradigms proposed by Chevallard (2015) seems particularly appropriate in the case of statistical inquiries for the specific descriptive tools it provides (Markulin et al., 2021). It also makes us, researchers, adopt a critical stance towards the social, educational, political and epistemological dimensions that involve the themes of teacher education and statistics teaching.

Acknowledgments

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Exploring the statistical practices in classroom: STEM teachers' experiences using dynamic data analysis technology.

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In the 21st century, schools have been increasingly expected to raise students prepared for a world of quantities and uncertainty. As statistical practices become an important skill for every individual, researchers and educators work on ways to translate the methodologies of professional statisticians in the teaching and learning environments. Technological tools are helpful in translating statistical investigations into the classroom. The purpose of our research is to understand teachers' experiences about integrating statistical practices into their STEM lesson plans by using LabStar, a dynamic data collection and data analysis tool. Findings were investigated under four categories: personal development, in-class implementation, views about the mobile application and the tool, and statistical investigation environment. Future research was recommended to investigate teachers' professional development opportunities to integrate meaningful statistical practices in their lesson plans.

Keywords: Statistics education, statistical investigation, dynamic data analysis technology.

Introduction

In the Big Data era being competent in making data driven decisions in all aspects of daily life is inevitable. Many decisions in politics, economics and social sciences are based on thorough analysis of data. The omnipresence of data requires individuals with strong analytical and reasoning skills who can base their decisions on their inferences beyond the data in the way that professional statisticians do (Frischemeier, 2020). However, international reports point out the lack of individuals, who demonstrate necessary skills to cope with Big Data even at the tertiary level (Manyika et. al., 2011; Puang-Ngern, Bilgin, Kyng, 2017). From this point, practice of statistics became a crucial element to integrate in primary and secondary school curricula.

In 21st century, schools have been increasingly expected to raise students prepared for a world of quantities and uncertainty. With the availability of data from variety of sources and in a variety of formats students are required to manage the complexities to reach data driven conclusions (Makar & Rubin, 2009). Statistics, therefore, has become an essential element of school mathematics curricula of many countries (e.g., Australian Curriculum, Assessment and Reporting Authority, 2015; Common Core State Standards Initiative, 2010; Ministry of National Education, 2018).

As statistical practices become an important skill for every individual, researchers and educators work on ways to translate the methodologies of professional statisticians in the teaching and learning environments. Statistics education is becoming an agenda across many levels of schooling. Debates about how to help students gain an understanding about statistics requires us to think about the nature of statistics. Statistics is about carrying out statistical investigations on a specific statistical question to understand a phenomenon. Statistical investigations are based on exploring the variation, by engaging data collection, data representation, data reduction, level of certainty and informal inference practices (Watson et al., 2018).

Statistical investigation is recommended to be the focus of the statistics teaching-learning activities in classroom contexts. The parts of the investigation process have been also included in mathematics curricula starting from elementary grades. As the practices of statistical investigation become a part in mathematics curriculum, the role of the teachers comes forth. Teachers are expected to design authentic contexts where students can engage in data collection, analysis, and informal inference processes. Research also underlines the importance of teacher knowledge on statistics and statistical thinking to design effective learning experiences for their students (Rodrigues & de Ponte, 2020).

Technological tools are helpful in translating statistical investigations into classroom. Dynamic technologies that allow students to experiment with data collection and analysis techniques were started to be researched in terms of the opportunities they provide for statistical practices (Meletiou-Mavrotheris et al., 2008).

COVID-19 caused schools to get shut off, which made STEM and statistics education to be conducted on online portals. This probably led to changes in the teaching method and the environment. The aim of creating statistics literate students that use statistics tools, software, and can interpret the data became the priority. In order to fulfill this aim, collaboration, communication and multidisciplinary studies that involves data science and education must be addressed.

Based on this rationale the purpose of our research is to understand teachers' experiences about integrating statistical practices into their STEM lesson plans by using LabStar, a dynamic data collection and data analysis tool. Teachers were provided with educational videos about integration of statistical investigations into STEM lesson plans and the use of LabStar tool. After watching all the videos teachers implemented the STEM lesson plans, designed with authentic statistical investigation tasks. Our aim is to explore the lived experiences of teachers about the implementation of these lesson plans. Our research questions are:

- How STEM teachers contextualize statistics and mathematics into daily life problems using STEM integration?
- How STEM teachers perceive their roles in terms of integration statistical investigation in STEM education?
- How STEM teachers evaluate the educational videos in terms of designing and implementing statistical investigations in STEM lessons?

Methodology

The research used a phenomenological method among qualitative research designs. The study group is four STEM teachers who used statistical investigation practices in their classrooms during 2020-2021 Spring Semester. T1 was a middle grades mathematics teacher, T2 and T3 middle grades science teachers, and T4 was a high school physics teacher. They implemented STEM lesson plans integrated with statistical investigation in various grades and with different disciplines in the center of the lesson plan. They used LabStar as a dynamic data analysis technology. LabStar is a dynamic data collection and analysis tool. It works with the physical tool and the mobile application. The mobile application can be used also separately from the tool by analyzing open resource data. All teachers and students had the LabStar mobile application, one of them (T4) had both the physical data collection tool and the mobile application. Teachers used data from open resources to analyze in their in-class implementations. Before the implementation they were provided with training videos regarding ways of using dynamic data analysis technology in their STEM lesson plans.

For data collection we conducted face-to-face interviews with the teachers. To ensure a prolonged relationship, interviews lasted at least 60 min. Audit groups were formed within the research group and member check processes were planned to ensure trustworthiness. Researchers are also using peer-debriefing during data collection and analysis phases. STEM lesson plans and field notes of the teachers were used as data sources to ensure triangulation in data collection. For data analysis, interviews were transcribed, and thematic analysis was conducted by using constant comparative method. Because multiple coders will analyze the transcripts, the methods for ensuring intercoder reliability were used. Documents of teachers (lesson plans and notes) will be analyzed by content analysis. Data analysis will be finished in November 2021.

Findings

Analysis of the interviews and the documents revealed myriad findings about the experiences they had during integration of STEM education into their classes. Findings were investigated under four categories: personal development, in-class implementation, views about the mobile application and the tool, and statistical investigation environment.

Personal Development

This theme emerged as the participants mentioned about the training they have received and research they have done before and/or during applying a STEM activity in the class. All teachers expressed that the educational videos that they have watched before attending to the classes, helped them to learn and grasp the nature of the application before introducing the software to the students. One of the teachers, T1, commented that "With the help of the videos, especially the fourth one, we were able to lead the students in the right direction". T2 added that: "The videos were sufficient, I received STEM education last year, I knew nothing. In a year I learnt about STEM education and some of the questions were given to us that are in the videos."

Furthermore, some of the teachers had to get help from various resources because they felt that they did not have sufficient knowledge.

> I am a science teacher, when I was studying on it, I got help from my husband who is a mathematician, (I asked about) upper quartile, lower quartile etc. I taught students with the information I got from him too. I also got tremendous help from the educational videos. (T3)

This view also showed that teachers also had an opportunity to discover knowledge related to other disciplines. A teacher had sufficient technological knowledge and even integrated STEM education to work with a software simultaneously.

> I thought about what I could do, I set up a meeting with the students from 9th and 10th grades. A student said that we should test its (LabStar's) accuracy. We tested it and it was accurate. We conducted two experiments, firstly

> we created sound waves and measured its decibel. We have a software called Scratch; we created the sound waves then integrated it with LabStar. The data were almost equal. We used it to compare the sound waves, their wavelengths. I found it usable for advancing in physics. It is a nice software (T4).

The teachers started their lessons with introductory activities to motivate students. T1 stated “We asked the students about number of cases of covid-19. Then we led them to the relevant website.”. T2 added “I asked the properties of microscopic life forms. With the feedback I have received, I mentioned about the bacteria then gave the table”. Although T2 did not ask anything before the lesson, she let the students explore by themselves. This allowed the students learn by discovery.

> I did not ask anything, I let them do experiments because I was experimenting too. When I asked what do you see, they told me that the wave is increasing, I asked how a wave can increase? Is it on the direction x or y? Then we got data, but now they tell numbers instead. Are they increasing or decreasing? We call them data. As the waves increase, the numbers increase, what do we call it? Wavelength. (T4)

Some teachers worked as a group to put out the lesson plan.

> We worked as a group. I usually build interdisciplinary relations with mathematics, but I always get help from the biology. We do gas pressure, sound waves, they are all about biology. There were machines in STEM laboratory, I used the distance sensor there. (T4)

The teachers explained that they plan to apply their knowledge into their other lessons too.

> Maybe it is not possible to apply it to all topics, since physics, chemistry and biology are different. But for velocity, race, order of the velocity, density of a matter, there can be data and it can be commented on. I think that it could be integrated depending on the area. (T2)

T4 also asserted that, “I will use it to explain illumination density. I have the topic movement on 10th grade, location, distance, translocation, I will use graphs. It is fun to make students active in 9th and 10th grades.”

In-class implementation

This theme has emerged as the teachers expressed their experiences about the application of a STEM lesson in a class. A problem occurred because of the mismatch between the current grade’s curricula and the statistical terms in the tool. T1 claimed that “Some terms were troublesome since they did not fit to my class”. T2 added that “In my opinion it was not suitable for 5th graders, the terms mode, median in the graphs”. T1 further explained that: “The students didn’t know the terms and their meanings since the terms do not fit to fifth grade’s curriculum. They did not understand it because they did not learn it, and I did not pre-teach the terms.”

Some teachers had problems because of the students’ technological skills being insufficient. T1 claimed “Some students found it hard to transfer data to an Excel sheet”, She further said “At first they found it hard to create a table. There were students who did not know the shortcuts”.

The teachers explained that they were not able to hold the attention of the whole class nor make all students participate in the activity. T2 claimed that “If you have 50 students, only 10 students give positive feedback. There were students who did not download the mobile application or participate in the lessons”.

The teachers who had the mobile application but not the physical tool claimed that they could have reached to more students if they had the physical tool to collect real time data. Some teachers observed that some students were interested in the lesson:

> There were many students who were interested in the lesson. There was data related to basketball, it drew the attention of male students. If the topics were different there could be different things. But I received positive reactions, some students asked if they could collect and analyze another data. (T2)

Views about the mobile application and the tool

This theme has emerged as the teachers stated their views about the mobile application and the tool that they have used to carry out their STEM lesson plans.

Another challenge was born because of the language of the application being English. This was a common problem amongst all teachers. T1 stated "The only problem was the application being in English, it could be in Turkish. If they don't know English, they need to use translation". Some schools did not receive the physical tool, but they only had the mobile application. The teachers believe that they need the tool itself.

> It would be better if we had the tool. For instance, in a 7th grade, I have conducted a lesson about darker colors absorbing light thus being colder. I used heat and light sensor simultaneously to observe the graph. It was a brilliant lesson. I did not have such fun in any lesson before. So, it would be better if we had the data collection tool. (T3)

T2 claimed that. "It is vague for 5th graders, for a fifth grader, the tool being here and the process of collecting the data is necessary. It needs to be concrete". There were some problems that originated from the application and technology use. Overall, the teachers were satisfied with the experience they had with LabStar. T2 claimed "The teachers can use LabStar easily and create STEM lesson plans.", T4 added "LabStar can analyze data, it is accurate and fun".

Statistical Investigation Environment

The teachers have different expectations about the statistical terms that the students must know and express in a statistical investigation environment.

> It is crucial to be able to read the graphs, where is the peak, what is the increment this month, being able to comment, creating a cause-and-effect relationship is important for science. Mathematical interpretation, peak point, outliers. I believe that LabStar application is related to mathematics at most. (T2)

> The students do not know how to read the graphs, they cannot even read tables. They have learnt how to do it; I believe that that is the significant contribution. (T3)

The teachers claimed that STEM integration made observing statistical terms easy.

> In the topic waves, there are fundamental notions. Frequency, amplitude, and wavelength. You can create a sinus wave, you can observe the wavelength, then we can measure it. We talked about range and modes, we created 2-3 waves, x wave and y wave then we compared them. At first, they talked about their image, then we compared them as data. (T4)

Some students were not interested in terms related to statistics. When teachers were asked if the students asked about statistical terms, T2 said “No, they did not ask. In 6th and 7th grades the activities would be more efficient. However, since the total number of lessons are less in 5th grades, it limits us.” T1 added that, in online environment, only the students who are passionate about mathematics asked, but we did not have a whole-class discussion. Since the number of curious students were low, they contacted us through private messages. (T1)

Discussion

The teachers reported that by seeing the graphs live, students grasp the concepts of mathematics and statistics in a fun and interactive way. This may be because in most cases formal statistics teaching leaves students in a confused state as most of the time they are not able to apply the theoretical knowledge into real life (Bakker et al., 2017), whereas the fun factor comes into play with the activities. The teachers worked together with other teachers to set up a multidisciplinary context, this is because developing mathematics, and integrating it with the other disciplines in STEM is the main aim in STEM education (Li & Schoenfeld, 2019). Even though context integration should be used to teach STEM (Kertil & Gurel, 2016), the teachers most of the time failed at integration of STEM with other disciplines, they mostly worked on their own disciplines. Therefore, creating statistical investigation environments was evaluated to be essential (Anderson & Li, 2020).

Teachers used real data from various resources to analyze with the mobile application and believed that real life situations are shown with excellence by using the tool and mobile application. it is probable that go prepare students to real world and its requisites, they feel that real life problems should be the main scope of the curriculum and the lessons in the schools to provide problem solving skills in which they need to provide a solution, a model to achieve the goals clearly (English & Watters, 2005). The teachers preferred experimenting with the students mostly because In the 21st century, using computers and being proficient is necessary and in STEM education, for the most of the educators experimentation is the preferred way rather than the computation (Li et al., 2020). The teachers dealt with problems that arise from the statistical terms, but they were caused from the curriculum not including the related terms, and the teacher’s lack of self-awareness to do a pre-teach session. Future research was recommended to investigate teachers’ professional development opportunities to integrate meaningful statistical practices in their lesson plans.

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TWG06: Applications and modelling

Introduction to the papers of TWG6: Applications and modelling

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Introduction

The thematic working group TWG6 takes on the importance of exploring the role of mathematics when it is used and applied in the study of real-world phenomena in educational contexts. TWG6 considers the interplay between empirical research results and theoretical approaches to the teaching and learning of applications and mathematical modelling, in mathematics and other subjects, at primary, secondary and tertiary school levels, as well as in teacher education. It started at the fourth ERME Congress (CERME4) in 2005 and has since then continued to be an active thematic working group in the following nine congresses. In total, more than 200 papers and posters have been presented in the TWG6. In CERME12 the TWG6 received 36 submissions, resulting in a total of 25 papers and 5 posters presented. See Figure 1 for a summary of the TWG6 contributions in the different CERMEs. At CERME12, we have discussed theoretical, methodological, and empirical research contributions aiming to address a variety of topics, such as application, modelling and simulations in connection to other subjects, the use and impact of technology and other resources to support modelling, teacher education for applications and modelling, and assessment practices for mathematical modelling, among others. The contributions discussed at the congress revealed a strong and fruitful diversity in the research questions considered, the school levels addressed, the theoretical approaches chosen and their methodological development.

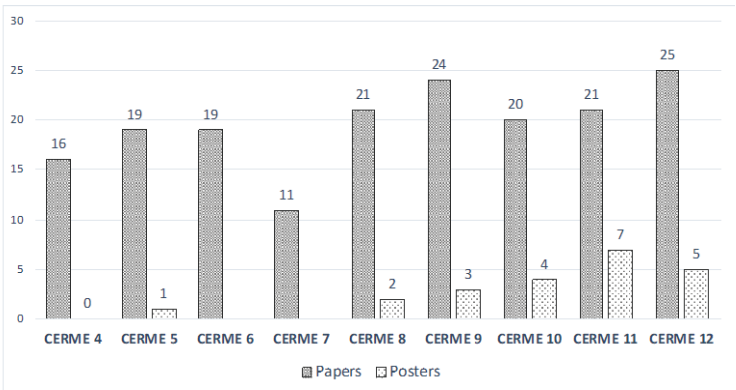

Figure 1: Evolution of the papers and posters presented in TWG6

A total of 45 participants participated in the online CERME12, representing 14 countries –most of them from Europe, but also from South and Central America, Israel, South Africa and Japan. The papers and posters were divided into six leading themes (see Table 1), so that in each session papers and posters were discussed on one of the themes.

Table 1: Leading themes (LT) defined and number of contributions in the proceedings

Leading themes	Papers	Posters
LT1. Theoretical and methodological developments in modelling	3	
LT2. Modelling in STEM topics and extra-mathematical topics	3	2
LT3. Design and analysis of modelling tasks and processes – Learning modelling at primary or secondary school education	6	2
LT4. Design and analysis of modelling tasks and processes – Teaching and learning of modelling at tertiary education	6	
LT5. Teacher education for mathematical modelling and application	4	1
LT6. Technology and other resources in mathematical modelling	3	

In the first leading theme, the reader can find papers focusing on theoretical and methodological developments of tools for the analysis of the modelling tasks and practices. The second theme focuses on the interplay and connections between mathematical modelling and other subjects, referring to the STEM topics and/or engineering context. The third and fourth themes address different strategies to support the design and analysis of modelling tasks and of the strategies to foster students' work on modelling tasks. These contributions are organised into two sub-topics according to the educational level the research contributed to: primary or secondary school education (LT3), or university education (LT4), the latter also including university students who are being prepared to become teachers. This last sub-topic links coherently with the fifth one, which covered teacher education for modelling and its applications, presenting several instructional proposals for preservice and in-service teacher education for application and modelling. Finally, the sixth theme focuses on the use of technology and other resources in mathematical modelling, mainly about how to combine different resources with technology in concept development by means of real word contexts. Table 2 summarizes the papers and posters included in the proceedings of CERME12, which we elaborate on in this introductory report discussing the leading themes.

Leading themes and overarching questions

LT1: Theoretical and methodological developments in mathematical modelling

This first theme focuses on proposing research tools and methodologies to analyse modelling practices beyond the cognitive aspects and to discuss the rationale and specificities of mathematical modelling in comparison to other domains or disciplinary approach to modelling. Firstly, *Vos* and

Frejd present a suggestion for extending the modelling cycle by the dimensions of metacognitive strategies, tool use and social norms. Secondly, the paper of *Ärleback* and *Frejd* considers a dual integrated modelling approach to the teaching and learning of mathematics and the teaching and learning of mathematical modelling at the same time. Thirdly, *Kawakami* and *Ärleback* present a review on the characterization and comparison among the rationales of statistical modelling and mathematical modelling. In all three contributions it became clear that the teaching of modelling and the analysis of modelling processes should not be limited to modelling competencies but should also include other aspects such as mathematical concepts, metacognition, affect, social norms, group work and tool use. Some questions for discussion emerge for future developments on this topic:

- What might a modelling cycle as an analysis tool look like, that includes aspects such as metacognition, affect, tools, social norms or group work, in addition to the analysis of sub-competencies?
- What would a concept for teaching look like, in which modelling competencies and mathematical concepts or knowledge about the context can be taught at the same time?
- To what extent is it necessary to distinguish between mathematical modelling and, for example, statistical modelling? Does statistical modelling represent a sub-form of mathematical modelling?

LT2: Modelling in STEM topics and extra-mathematical topics

The LT2-papers study the role of mathematical modelling in extra-mathematical, engineering, and STEM contexts. The paper by *Kacerja* and colleagues presents an analysis of teachers' discussions on the modelling involved in the body mass index (BMI). *Vásquez* and colleagues discuss a study and research path (SRP) for the teaching of modelling in secondary school in relation to the evolution of COVID-19, which connects to the poster from *Donner* and *Bauer* on a modelling project about the pandemic for grades 9 and 10. The poster by *Fleischmann* and colleagues presents the design of an SRP about modelling climate change. Finally, *Pablo-Díaz* and *Romo-Vazquez* present the design of a didactic activity for engineering education based on the Hazen-Williams model. The main areas and driving questions of the discussion with respect to LT2 can be summarized and highlighted as:

- The complexity of realistic and real-world (ill-defined) problems. How to deal with and conceptualize the extra-mathematical domain? How to transpose complex real-world modelling problems into different grade classrooms without losing authenticity, relevance and richness? How to support or motivate students to validate their results with the real context?
- Critical perspective, ethical considerations, and sensitivity towards students. How to support a critical perspective on the use of models in society (e.g. when addressing issues related to Covid-19, BMI)? How to consider in classrooms the potential implications of using the models for decision-making?
- Implications of using real data in modelling activities. How to foster a critical stance towards, and thinking about, data in students? Starting from the given data: what questions can be answered? Starting from questions: what data is needed in order to provide an answer?
- The role and importance of various aspects of posing questions. What is the role of problem posing in the context of applications and mathematical modelling? What can students learn

from posing questions with respect to modelling and extra-mathematical questions? How to support students in posing productive and answerable extra-mathematical questions?

- The nature of the models used and developed: To what extent are STEM contexts and activities providing new and different types of models to explore?

STEM and other extra-mathematical contexts provide rich sources for modelling problems but introduce new types of challenges and demands for students, teachers and researchers. We need to conceptualize and theorize about how to design activities and learning environments (including technology) that fundamentally connect to, and use, knowledge from other disciplines and extra-mathematical domains.

LT3: Design and analysis of modelling tasks and processes – Learning modelling at primary or secondary school education

The papers related to this topic refer to different strategies to support students when solving modelling problems, teachers when guiding their implementation and strategies for the design and analysis of modelling activities. As one of the themes that are grouping more contributions, we can distinguish different focuses and issues raised.

Firstly, *Elias* and colleagues focus on examining the notion of the rate of change in modelling problems. The authors examine which aspects of the notion of rate of change are prone to subjective reasoning by learners, due to their structure or due to missing information, and which aspects are objective. Secondly, some contributions focus more on some particular steps of the modelling process and analyse, for instance, the influence of students' beliefs about modelling or the interest in the context of modelling tasks. *Geisler* focuses on the steps of validation and investigates how students validate their models within modelling tasks with experiments. Additionally, *Kanefke* and *Schukajlow* focus on the difficulties of students noticing when data are missing from some modelling problems. They aim to analyze the extent to which students noticed missing data while processing modelling problems with missing data. Furthermore, the poster from *Kaemmerer* presents a comparison between students' work on modelling tasks when they have an interest (or do not) in the real-world context where the task is proposed. The poster from *Surel* discusses strategies for a better comprehension of modelling tasks related to students' engagement. Some common questions for discussion linked to these contributions are about:

- What impact do students' beliefs about the nature of mathematics (e.g., exactness of mathematics) have on students' approaches to working with (missing) data? How can we influence students' beliefs, so they are able to be more flexible with their models (adaptations/reformulation/validation)?
- Are students' affective reactions dependent more on the type of modelling task or the phase in the modelling process? Are there important differences in students' affective reactions?

Thirdly, other papers are more focused on the proposal of analytic tools for the ideal, individual, or collective modelling routes. The paper from *Schneider* and colleagues investigates whether and to what extent knowledge about ideal-typical modelling processes have an influence on phase transitions in individual modelling. *Göksen-Zayim* and colleagues present an observation instrument to study collaborative modelling. The authors use three main components: collaborative learning, mathematical modelling and the language that students use while working together. Moreover, *Bassi* and *Brunetto* are interested in the affective factors, cognitive and motivational, that can be associated with the different modelling activities. Some common questions about these contributions refer to:

- To what extent does group work condition the modelling process? How to take into consideration the aspects related to group work conditions?
- Which tools are more useful for the systematic analysis of the influence of interactions on the resolution process?

LT4: Design and analysis of modelling tasks and processes – Teaching and learning of modelling at university

This fourth leading theme, closely related to the previous and the next ones, is now focused on the design and analysis of mathematical modelling at university level. Most of papers on this topic choose the university students who are in fact preservice teachers.

Zhou and *Hansen* investigate how the introduction of the pedagogical method "mathematics in three acts" to preservice teachers influenced their mathematical modelling. *Hartmann* and colleagues investigate the modelling activities that take place when posing problems that are based on given real-world situations and the extent to which modelling activities occur with different problem-posing activities. *Segura* and *Ferrando* aim to categorize the different types of errors when university students solve some modelling tasks involving estimations and to inquire into the efficient use and learning from errors to improve initial teacher training. *Sevinc* and *Ferrando* analyze the commonalities and divergences in the resolution of a group of Turkish and Spanish preservice mathematics teachers' ways of modeling approaching a Fermi-based modeling problem. The contribution from *Andresen* aims to inquire about learning trajectories in the context of modelling with differential equations. Textual analysis of the reports from teachers in a masters' program implied the marking of notions and terms related to progressive, horizontal and vertical mathematising,

We can divide the main topics into three areas. They are school-university transfer, modelling processes and modelling tasks. In the *school-university transfer*, the importance of the transfer of studies from school to university, was discussed. Here, the question arises, how to compare the conditions under which the implementation runs and the results. Another question discussed is for example: How to take into account the university context and the institutions to which some modelling tasks are to be transferred?

In the *context of modelling processes*, the question of understanding reality and mathematics in the modelling process as "part of the world" was discussed. Modelling is seen, especially at university, as a way to learn new advanced mathematics rather than just using familiar university-level content. The connection between mathematical modelling and problem posing and the importance of their relationship was also pointed out here.

The discussion of *modelling tasks* for university students includes criteria for evaluating the "quality of the problem" (for example openness and complexity). Fermi tasks also have great potential as modelling tasks at the university. They provide an opportunity to analyse possible errors in more detail. Modelling tasks examples at university, if used in teacher education, should also be considered in terms of their pedagogical significance. Looking to the future with respect to this fourth topic, questions arise about:

- What differences between engineering students and preservice teachers are relevant to the construction of modelling opportunities?
- What are the main differences between mathematical modelling in university and mathematical modelling in the school?

LT5: Teacher education for mathematical modelling and application (professional content knowledge and competencies)

This fifth theme is focused on teacher education, based on the acknowledged need for preparing preservice and practicing teachers for the teaching of applications and modelling. Papers on this leading theme are closely related to the previous ones. As mentioned before, papers in LT4 present the analysis of preservice teacher activity (considered as university students) and, in the current one, the focus is on professional content knowledge for mathematical modelling, pedagogical knowledge for modelling and simulation, and/or on teachers modelling skills.

On the one hand, *Greefrath* and colleagues consider the professional content knowledge of mathematical modelling as a competence facet of teachers to present and discuss the development and use of a test instrument. In this same direction, but referring more to pedagogical knowledge, *Gerber* and colleagues propose a theory-based model and subsequently present items of an associated test instrument to measure the preservice mathematics teachers' professional knowledge for teaching simulations and mathematical modelling with digital tools. On the other hand, *Montejo-Gámez* and colleagues present the characterization and the analysis of the kinds of assumptions made by a group of preservice teachers on the models considered and the impact on modelling outcomes and teachers' skills. The poster presented by *Ulbrich* and colleagues focuses on developing mathematical modelling skills for mathematics teachers through 3D Modelling and 3D Printing. Some main issues and questions related to this leading theme are about:

Content knowledge (CK) and Pedagogical content knowledge for mathematical modelling (PCK): Considering the specificities of the knowledge domains (mathematics, stochastics, grade level, ...), how can we describe mathematical, technological and modelling knowledge involved in PCK in order to design assessment tools for measuring PCK?

Noticing mathematical modelling processes: How is teachers' noticing competencies and knowledge (regarding metacognition) in modelling and its development related to one another?

Validity and replicability of research in teacher education: How do we deal with the question about validity and replicability in our field of research (in particular, in teacher education for mathematical modelling)? What might be the impact of teacher education for modelling on teachers' practice? How can we replicate studies in light of the diversity of conditions (and limitations) under which teachers can act? In the future development of our field of research, questions arise about what knowledge on technologies (which tools, when to use tools, black boxing, instrumental genesis, ...) might also be involved in the discussion of teacher knowledge for mathematical modelling and of teacher education.

LT6: Technology and other resources in mathematical modelling

This theme focused on the use of technology and digital resources in mathematical modelling. *Touma* and *Olsher* explore the design principles of computer-based modelling activities. From design-based research, the results show that technology plays several relevant roles, namely simulation, investigating scenarios, and simplifying procedures. In her study, *Jessen* aims at characterising the roles of digital resources in mathematical modelling, by using the Anthropological Theory of the Didactic in terms of media-milieu dialectics. The digital tools are seen both as media and milieu. *Lieban* and *Bueno* examine students' use of 2D and 3D resources to find out how ideas from the Three Worlds of Mathematics come into play. They conclude that connections result from experimentation, exploration, understanding, and manipulation, in physical or digital environments. Some of the matters largely discussed were:

Contributions of digital tools to the modelling activity: What is the relationship between simulation and modelling? In considering the different roles that digital tools can play in the modelling process, to what extent does technology define the modelling activity? How does technology trigger students' creativity in the modelling process?

Design of technology-based tasks and environments: How can we find a balance between the teacher's and the students' intervention in a modelling task? How to design good tasks that foster the students' modelling competencies using digital tools? How to develop modelling tasks considering the diversity of students' knowledge of technology in a class? How to promote student's self-confidence on the use of technology for modelling?

Teachers' knowledge on mathematical modelling with digital tools and STEM practices: Is there a specialized PCK concerning the implementation of mathematical modelling with technology? How can technology be used as a communication tool and a collaboration tool in modelling activities? Is it possible to establish a relationship between a modelling task and the technology that is more effective to carry out some or all the steps of the modelling process?

In the future development of our field of research, we need more studies on technology-based modelling activities. New conceptualizations and theoretical approaches concerning mathematical modelling with technology are also necessary. The design of modelling activities with digital tools is important in feeding the research on this theme.

Concluding remarks and perspectives

The leading themes addressed by the TWG6 show again the variety of research approaches and questions the papers dealt with (Carreira et al., 2019; Kaiser & Sriraman, 2006). Furthermore, the educational levels spanned from primary to tertiary education, also covering preservice and in-service teacher education. With the overview presented in the previous section, we now focus on each of the six leading themes central in CERME12 and summarize the main driving questions of the discussion, as well as some remarks and questions, looking ahead for future research with respect to each leading theme.

The first leading theme (LT1) discusses research tools and methodologies to analyse modelling practices beyond the cognitive aspects, suggesting extending the modelling cycle by considering metacognitive strategies, social norms, affect, group work and tool use. Moreover, this discussion is complemented by the comparison between mathematical modelling with modelling in other domains, such as statistical modelling, to discuss their possible meeting point in terms, for instance, of their rationale or conceptualisation. The second theme (LT2) argues the role of mathematical modelling in extra-mathematical, engineering and STEM contexts (or more in general, interdisciplinary contexts). There is a long tradition in TGW6 of discussing examples under this theme. The discussions highlight the need to conceptualize and theorize about how to design activities and learning environments that connect disciplinary knowledge, in particular mathematical and mathematical modelling knowledge, to other disciplines and extra-mathematical domains.

The third and fourth themes are more associated with the design and analysis of modelling tasks and processes in primary and secondary education (LT3) and tertiary education (LT4). These two themes group about half of the contributions. LT3 groups the papers that refer to different strategies to support students when solving mathematical modelling problems and strategies to design and analyse modelling activities at primary and secondary school levels. Closely related to the previous one, LT4 is focused on the design and analysis of mathematical modelling at tertiary education. Most of the papers choose the university training of preservice teachers to analyse what future teachers do when they are being trained (as students) at university.

The fifth theme (LT5) focuses on teacher education, based on the need of preparing preservice and in-service teachers for the teaching of applications and modelling. LT5, with the previously mentioned LT4, included many papers, continuing with the dynamics initiated in the previous CERME11. Teachers and their initial and continuous professional development are crucial for the integration of mathematical modelling into mathematics education at all school levels. Some questions about the linkages and replicability of teacher education and its impact on teachers' practice remain open for future research. Finally, the sixth theme (LT6) is focused on the use of technology and digital resources in the teaching and learning of modelling. Questions about the need for theoretically-based designs of modelling activities with digital tools or the need for extending research on teacher education for the use of simulations and modelling remain highlighted for future development of our field of research.

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We want to thank all the participants of TWG6 at CERME 12 for their presentations and participation in the discussion and for working together in the online intensive sessions.

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Learning trajectory as a complex cluster of activity

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This paper presents a study of students' learning mathematics by modelling. In the study learning was conceptualized in terms of emergent modelling following the four-layer model by Gravemeijer et al. (Gravemeijer, 2020). Aim of the study was to inquire learning trajectories in the context of modelling with differential equations in 21 students' reports. Textual analysis of the reports implied the marking of notions and terms related to progressive, horizontal and vertical mathematising, followed by the linking of them with the four layers. The paper focuses on one case of students' modelling in one report. It demonstrates how the students' learning trajectory was identified and determined, for interaction between populations as an emergent sub-model of a differential equations model. Thereby it contributes to the discussion of i) conceptualizations of modelling and its relations to learning, and ii) methods for research in students' learning.

Keywords: Mathematical models, learning processes, active learning.

Introduction.

The issue of this paper is to interpret students' learning mathematics from modelling with differential equation (DE) systems in terms of emergent models, by vertical and horizontal mathematising (Gravemeijer & Stephan 2002). The study's students do not build up the model from scratch. The study's conceptualization of learning from modelling is in contrast with cases of 'learning by modelling' that refer to learning about/developing competence in modelling (see Blomhøj & Ärlebäck, 2018). The present study was part of a more comprehensive study of data, consisting of the 21 reports described below. In the presentation of the larger study by Andresen (2021) the methods used in the present study were described and discussed. The larger study also encompassed another case study, going into details about the interplay between the DE model, the problem modelled, and the tool used for modelling, presented in Andresen (2020).

Theoretical framework

In this section the basic idea of emergent models is presented and operationalized into a tool for textual analysis, following the larger study by Andresen (2021). Whereas the presentation in (Andresen, 2021) concentrated on conceptualizations of learning in terms of emergent modelling, the aim of this paper's study was to trace and identify the students' learning trajectory for cyclic interaction between populations in the predator – prey DE model as an example of learning about a mathematical concept by modelling.

Emergent mental models of mathematical concepts and relations

Gravemeijer and Stephan (2002) expressed modelling in terms of emergent models. The notions of emergent models and modelling originated in Realistic Mathematics Education (RME) which sees mathematics as a human activity (see Freudenthal, 1991). One of the basic principles of RME was students’ horizontal and vertical mathematising, described by Gravemeijer and Stephan (2002) as the passing of four levels of activity where a new mathematical reality is created at each level. The levels of activity were:

1. situational level with descriptions in natural language and own wording,

2. referential level where a ‘model of’ is created and inquired. A ‘model of’ is identified by the students’ use of situation related terms and half-way formalized explanations, for example, that ‘the number of sick persons will grow exponentially over time’,

3. general level with creation and handling of a ‘model for’. A ‘model for’ is identified by the students’ use of general expressions and terms with no visible relation to the situation, for example, that ‘We find that the graph of I(t) hits the maximum value if the parameter has a value of 0.259’,

4. formal level with general reasoning and considerations. For example, considerations and reflections about modelling with DE.

Horizontal mathematising is described as the passing from first to second level, vertical mathematising as passing further up. This four-layer model was the basis for the design heuristics of emergent models that aim to support the students’ processes of emergent modelling (Gravemeijer & Stephan, 2002). Gravemeijer (2020) described emergent modelling as an incremental process in which models and mathematical conceptions co-evolve. Central to the emergent modelling design heuristic is the use of a series of sub-models. Together they substantiate an overarching model which develops from a model of informal mathematical activity to a model for more formal mathematical reasoning. The overarching model is mental, with Gravemeijer (2020) mentioning, for example, the concepts of ‘distribution’ and ‘function’. In this view, modelling is not separated from mathematics nor from ‘reality’. The two goals, modelling for the sake of mathematics and mathematics for the sake of modelling, mentioned by Niss and Blum (2020), are here intertwined.

According to Cobb (2002, p. 193) the four-layer model might ‘facilitate (…) the analysis of mathematical learning in instructional situations (…). The explication of a mapping between a situation and a model might then be viewed as a description of the way that the situation became structured during modelling activity’. Based hereon, the four-layer model was in this study operationalized into a tool for textual analysis of reports to interpret the students’ mathematical concept formation. Signs, displayed by the wording in reports, of students’ activity were stratified regarding the levels. The progressive mathematization, then, was detected as progressive variation between the levels and interpreted as steps of the students’ concept formation, in the form of sub-models evolving into an emergent model. Accordingly, the mathematical learning outcome was conceptualized as emergent models of essential mathematical concepts, following Cobb (2002). This paper’s case study of modelling with DE was focused on sub models in the form of details of the interaction between populations. The overarching (mental) model was the mathematical idea to model interaction between two populations, namely foxes and small rodents, by the predator – prey differential equations model (Figure 1).

Students’ mathematical modelling

The modeling took place in a group of Norwegian teachers in a masters’ program that requires 60 ECTS in mathematics and two years of professional practice as a mathematics teacher. The teachers, in the following called students, attended the course ‘Modelling in and for mathematics teaching and learning’ which is a 15 ECTS course in a masters’ program in mathematics education.

The DE part of the course was based on Blanchard et al. (2002) which progressively build up the DE models and examples, and balance between qualitative, quantitative, and numerical methods. Hereby inspired, groups of students (2-3 persons) under sparse supervision were asked to formulate, complete, and present a project that encompassed a simple DE model, and to report the project. Each group choose what DE model they wanted to study. The course’s individual examination included an interview about this project (10 minutes). The students were expected stepwise to establish an overarching DE-systems model in a process, contrasting with bare application of a model, i.e., contrasting with picking out a ready-made model and fit its parameters with data. However, the students would not be able to establish a model stepwise from scratch, purely based on progressive mathematising and formation of sub-models. Therefore, they were free to involve and build on ready-made models like the predator-prey model and its modifications.

$$\frac{dx}{dt} = \alpha x - \beta xy,$$
$$\frac{dy}{dt} = \delta xy - \gamma y,$$

Where x is the number of prey, y is the number of some predator,

dx/dt and dy/dt represent the instantaneous growth rates of the two populations, t represents time, and α, β, ɤ, and δ are positive real parameters describing the interaction of the two species

Figure 1: The predator prey DE model

Methods

The research question was formulated in terms of a goal, i.e., the aim of the study was to identify the students’ learning trajectory for cyclic interaction between populations in the predator – prey DE model. The research question, hence, was: *What appearance can a learning trajectory have in a modelling process?* The study’s case was picked out of 21 reports prepared in 2014–2020 and evaluated by the author. The students were not familiar with De-models nor modelling in advance. For this presentation the cyclic interaction between populations was used as the example of an emergent sub-model. In the case, the students’ collective learning processes were documented by their own descriptions of the mathematical modelling activities, and by the reflections reproduced in the reports. The descriptions and reflections were reported in a convincing way: they used a first-person perspective in their writings in the report which is mentioned as a sign of being an active learner by Ju and Kwon (2007). This impression of validity was supported by the interviews under the individual examinations. The reliability of the analysis rested on the condensation of meaning from units of the convincing texts in accordance with the qualitative methodology described in Kvale (2001). The meaning of each unit was interpreted in accordance with the emergent modelling framework.

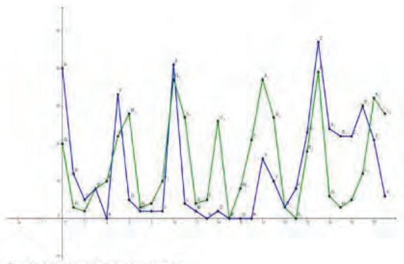
The textual analysis, presupposing that the students’ mental activities were reflected by the wording in their reports, served to stratify the mathematical learning process in terms of passing through the four levels, situational, referential, general, and formal. Each appearance of a mathematical concept or its related notions in the text was assigned to a level of activity in the four-layer model (Gravemeijer & Stephan 2002). Thereby, the marking served to stratify the signs of mental activity concerning the emergence of sub-models into the four levels of progressive mathematising. These signs constitute the study's documentation of the students' learning path or learning trajectory in the example.

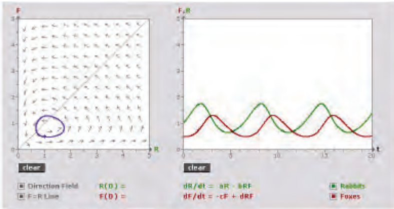
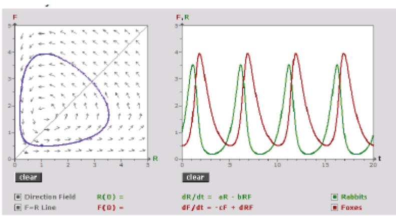
Data

During the textual analysis all elements were coded and subsequently sorted by their (related) mathematical content. This case considers emergence of the sub-model cyclic interaction. The result, hence, of the textual analysis was a cluster of signs of activity at different levels concerning interaction between the populations of foxes and small rodents in the predator – prey DE model. Table 1 displays, chronologically, some of the excerpts in the cluster concerning (cyclic) interaction. The numbers in Table 1's most left column refer to the complete cluster. The level of activity refers to the four-layer model by Gravemeijer and Stephan (2002).

Table 1

Number	Excerpt, translated from Norwegian by the author	Level of activity
1	In the work with our model, we start from a simple model, and on the basis of the changed model, we study what happens to the prey population and the predator population. We will then expand the model and look at factors that are significant for prey-predator interaction. (…).	*Situational -> referential*; Horizontal mathematising
3	(…). It turns out that the number of mountain foxes varies in step with the number of small rodents (…). At population peaks of lemmings and other small rodents, we see that there is a large increase in mountain foxes. Correspondingly, there is a significant decline in the population of mountain foxes when the population of small rodents declines. In other words, reproduction increases for the arctic fox when the supply of prey increases. (…).	*Referential level*; number of foxes and number of small rodents are half-way formalized but still linked with the situation. The same goes for the peaks of lemmings and other small rodents, and the large increase in foxes.

5	 Figur 2: Tallmaterialet presentert i Geogebra. Figure 2	*General level*; Except for the figure text the graphs have no connection with the situation
9	Cyclic oscillations It is often the case that predator populations and prey are interdependent and that populations fluctuate depending on the number of the other species. If there is one species that eats another species, then we have two sizes that vary with time. In such a model, we will thus have two dependent variables, both of which are a function of time.	*Situational -> referential level* (horizontal mathematising)
12	Two of the factors are important to clarify the sign of: • $-\boldsymbol{\beta BR}$ will always be negative, except if $\boldsymbol{B}$ and / or $\boldsymbol{R}$ is equal to zero, because $\boldsymbol{\beta}> \mathbf{0}$ • $+\ \boldsymbol{\delta BR}$ will always be positive, except if B and / or R are equal to zero, because $\boldsymbol{\delta}> \mathbf{0}$ This means that if the prey population increases, then the growth rate of the predators will increase. A solution of this first-order system will be two functions, B (t) and R (t). • $\boldsymbol{B}\ (\boldsymbol{t})$ describes the prey population as a function of time. • $\boldsymbol{R}\ (\boldsymbol{t})$ describes the predator population as a function of time.	*General -> referential level* The first two bullets are at general level (no links with the situation), the text and the last two bullets interpret / link them with the situation
24-26	We perform the calculations using spreadsheets and draw the curve in Geogebra:  Figur 5: Løsningskurve for byttedyr-rovdyrsystem med initialbetingelser $(B_0, R_0) = (1, 0.5)$ We see that this curve has a characteristic oval shape that reflects that the number of individuals in the populations changes cyclically.	*General level*, *General -> referential level* Interpretation of the shape of the curve in the referential level (number of individuals changes cyclically)
36	(…) Phase diagram for different values of α: Initial conditions: (B0, R0) = (1, 0.5)	*General level.* The phase diagrams, though, are interpreted

	(…) What happens when a = 1?  *Figur 13: Fasediagram for a=1* See that there will be nice, smooth cyclical fluctuations. Predators (red curve) depend on prey (green curve). Can imagine that there are limited external factors that affect. What happens when a = 2?  *Figur 13: Fasediagram for a=2* See that there are smooth, cyclical oscillations. The peaks of the curves are higher than the peaks of the curves when a = 1. These curves are more similar to our starting point. (…)	and linked with the situation: The activity at general level is followed by experiments at *referential level* with different values of α (the model's parameter) in graphic representation, and then followed by interpretation into the situation of the results of the experiments

Comments to Table 1

Ad 1: Prey population and predator population and their interaction are at the situational level. The simple model is seen by the students as a means for study of the interaction between them on the one hand, and the study is intended to enrich and develop the model on the other hand.

Ad 3: Here the sub-model emerges; Peaks of small rodents give large increase in foxes and decline of small rodents give decline in foxes. Reproduction increases for the fox when the supply of rodents increases.

Ad 5: The emergent sub-model in a new graphic representation, re-produced by the students using GeoGebra.

Ad 9: The mentioning of two dependent variables is (cyclic) interaction at an early stage

Ad 12: Increase in prey population gives increase of growth rate of predators; this is initial characterization of interaction.

Ad 24-26 Solution curve, reproduced by the students. The number of individuals in the populations changes cyclically in accordance with the sub-model.

Ad 36: The cyclic interaction is identified in the graphs.

Results

The aim of the case study was to identify the students' learning trajectory for cyclic interaction between populations in the predator – prey DE model. As illustrated by Table 1, the report displayed signs of activity at situational, referential and general level and, in total in the report, shifts in both directions between these levels. Besides, the students' demonstrated representational literacy by shifting between plain language and wordings, formal language and formulars, and graphic representations in the report. In contrast with an (ideal?) smoother passing up through the four levels from situational to referential and further up to general, their learning trajectory can be identified as the complex cluster of their activities.

Learning trajectory for cyclic interaction as a cluster of activities

A brief sketch of the activities starts with the introduction in plain language (3) based on observations of population curves from a Norwegian official web site. Next, the students reproduce population curves in GeoGebra (5) and compare these. Then they introduce models for growth of populations in general terms. This introduction is detached from the graphs and from the plain language description. After the presentation of the general predator – prey model the students mention cyclic interaction for the first time (9) before they proceed to discuss qualitative and quantitative methods. They use Euler's formula and GeoGebra to prepare a curve that illustrate the cyclic interaction (24-26). Finally, they prepare a series of graphic representations (36) to compare with their initial data to control the cyclic interaction.

Perspectives

This cluster of activities can be seen as the result of the rich environment for the students' modelling. They had the general predator – prey model, series of authentic data, tools in the form of GeoGebra and others, and their own prior mathematical experiences as the resources to include. The students' initial task was open, and they had little guidance during their project and preparation of the report. At a first glance, the process and the report might appear messy with no clear goals and direction. In a classroom context, such unstructured student activities could easily trigger the teacher to act and make an intervention like, for example, asking the students' leading questions and guide their process. Taking the students' evident learning outcome into account in the present study, as it was evaluated in informal conversations and in the interviews at the oral examination, though, the process in which the students to a high degree worked without guidance from the lecturer showed to be fruitful. There seems to be a paradoxical discrepancy between the messy and unstructured process documented in the report on the one hand, and the learning outcome on the other hand.

Conceptualization of learning in terms of emergent modelling and models seems to be one promising way to explain this apparent paradox. In particular, the idea of learning trajectories as clusters of activity in these terms, may contribute to widen and nuance the discussion of the relations between mathematics modelling and the learning of mathematics. Metaphorically speaking, the non-linear learning fits well with the non-linear modelling represented not only in the four-layer model but also in the traditional modelling circle by Niss and Blum (2020).

Besides, the study presented in this paper can give inspiration to the development of research methods for inquiry of students' learning: the tool for textual analysis presented in this paper may be useful for studies based on data in the form of students' written reports and other materials.

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Possible strategies to enhance students' learning and achievement in mathematical modelling teaching experiences

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In the present work preliminary results about strategies to enhance students' learning and achievement in mathematical modelling teaching experiences are presented. The results confirm the importance of employing real world problems which are perceived as important by students. Moreover, they also suggest that the design of teaching experiences in which the initial steps of the modelling cycle can be carried out with a few or even any mathematical language could enhance students' perception of self-efficacy, thus providing a motivational push also for the subsequent steps, where specific disciplinary language must be employed. Finally, also the use of ICT and mathematical tools which are perceived by the students as belonging to the university and employment world can increase students' perception of value.

Keywords: Real-world problem, enjoyment, self-efficacy, modelling cycle.

Introduction and theoretical background

In the last decades mathematical modelling (MM) has been recognized as relevant both by scholars and educational policies for teaching mathematics (Barquero et al., 2019). However, bringing MM within the mathematical curriculum is demanding both for teachers and students, so that in everyday mathematics teaching practice, there is still relatively few genuine modelling (Blum, 2015). Indeed, MM requires mathematical and extra-mathematical knowledge, as well as appropriate beliefs and attitudes, especially for more complex modelling activities (Blum, 2015; Niss, 2003). We agree with the scholars that claim the relevance of affect factors for teaching and learning mathematics exploiting MM. In particular, we agree with the fact that using MM to deal with real world situations increases students' enjoyment and interest, motivates them and makes them retain longer what they learned. The research *problematique* of the present paper frames within two of the five leading themes emerged during the discussion of the thematic working group TWG6 at CERME 11 (Barquero et al., 2019), namely *T1. Analysis of modelling process when solving modeling problems* and *T3. Strategies to support design and implementation of modelling.* To that end, in this work we present the findings of a preliminary study about MM in high school.

Mathematical modelling and modelling cycle

The term *mathematical modelling* indicates the process of translating between the real world and mathematics in both directions (Blum & Borromeo-Ferri, 2009). More precisely, *a mathematical model* is a "deliberately simplified and formalized image of some part of the real world" (Blum, 2015, p. 77), such as nature, society, everyday life and other disciplines. The purpose of building and making use of a model is to understand how to tackle problems related to the real world (Niss et al., 2007). For our purpose, we focus on the modelling cycle (Blum, 2015; Greefrath, 2011), which is composed of seven steps within and between reality and mathematics (see Figure 1).

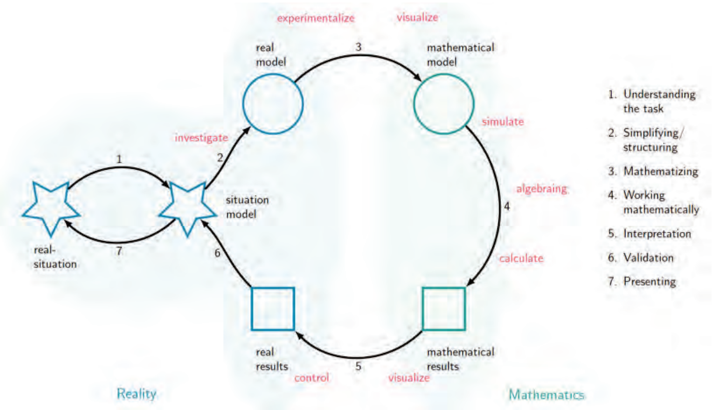

Figure 1: Blum's modelling cycle with added influence of digital tools (red terms)

The first two steps are within the reality and allow to move from the real situation to situation model (understanding) and to the real model (simplifying); the third step (mathematizing) bring to the mathematical model within the mathematics domain, where it is possible to work mathematically (step 4) reaching a mathematical result. The further two steps allow to go back to reality interpreting (step 5) and validating (step 6) the results. In the last step, the results (if any) are presented in the real situation. According to Greefrath (2011), we share the importance of digital tools along such a cycle. For instance, a digital tool may support the learner to calculate in step 4. Teaching mathematics using MM means to carry out all steps of the modelling cycle and drive students through them. To achieve that, certain competencies or sub-competencies are required (Niss, 2003), for instance understanding a given real world situation or interpreting mathematical results in relation to a situation (Blum, 2015). Modelling competency means the ability to construct, to use/apply mathematical models by carrying out appropriate steps as well as to analyze or to compare given models (Blum, 2015). Moreover, non-mathematical and extra-mathematical knowledge are required for MM. As a result this activity is highly demanding: several studies (see Blum (2015) and the references within) have shown that each step can be a cognitive barrier for students, even the first one "understanding the situation and constructing a situation model".

Affect framework

The role of affect factors, such as "emotions", "beliefs" and "attitudes", is pivotal in learning mathematics, also when students deal with problems (Schukajlow et al., 2012; Hannula, 2012). The characterization of affective factors is challenging for the researchers, because affective variables tend to form a cluster (Liljedhal, 2018). But in one of the possible classifications, Hannula (2012) identifies three types of affects: i) *cognitive* (e.g., beliefs), ii) *motivational* (e.g., values), and iii) *emotional* (e.g., feelings). *Cognitive* type refers to the affect variables that concern the beliefs towards the learning process and the achievement. We dwell on the self-efficacy belief (Bandura, 1977), which is the engine for mathematical thinking and doing (Andrà et al., 2020). More precisely, self-efficacy is a major factor in whether students will attempt a given task, how much effort they will put on it, and how resilient they will be when difficulties arise. Self-efficacy beliefs can be represented with the variable "I can", that range continuously from zero (lack of) to a maximum (strong) (Andrà et al., 2020). *Motivational* type concerns value and interest. The value is the perceived importance attributed to something (Eccles & Wigfield, 2002), while the interest concerns a specific relationship between a person and something in one's "life-space" (Krapp, 2000, p.111). Value and interest are nuances of motivational factors, for instance one can recognize that mathematical skills are valuable for obtaining a specific job but has not any interest in developing mathematical skills because that job is not part of one's "life-space". Finally, *emotional* types are all those feelings which do not belong to the previous ones. Emotions prepare our actions, accompany these actions, and influence reflection about their outcomes (Schukajlow et al., 2012). Among them, we distinguish positive and negative emotions (Pekrun et al., 2002), that can be task- and self-related or social related. For instance, *hope*, *boredom* and *anxiety* are task- and self-related emotions, while *sympathy* and *antipathy* are social related. In this paper we strongly focus on the emotion *enjoyment*, since it has been proven that there is a positive correlation between students' enjoyment and academic achievement (Pekrun et al., 2002). The levels of enjoyment can be seen as nuances of the variable "like" (Di Martino & Zan, 2015), that can range continuously from zero (boredom) to a maximum.

Research questions

We recall that the rationale behind the present study is the problematique of finding possible strategies which can be employed in the design of mathematical modelling teaching experiences in order to enhance students' learning and achievement. In particular, due to the strong connection between learning process and affective factors, we formulate the following specific questions: RQ1. Which activity carried out during the project the students enjoy the most? And RQ2. Which affective factors, cognitive and motivational, linked to the emotion enjoyment, can be associated to the different activities of the project?

Methodology

The research context

The present study, which is part of a wider MM project, involved two researchers (the authors), 18 students (grade-11) and their teacher. The data analyzed in this work has a peculiarity because the didactical situation stems from a real-world problem posed by a stakeholder. The alderman to the culture of a small town wanted to know how young people use and what they would like to find in the municipal library. The stakeholder involved the local high school for answering her question. The project was carried out through eight meetings of two hours each. The first three meetings, delivered as frontal lessons by statistics experts, aim at providing some fundamental statistical instruments. The remaining five meetings, delivered by the authors of the paper, had a student-centered approach such as group work and classroom discussion and focused on the mathematical modelling process. Such methodologies have been used since they have proven to be particularly suited for modelling activity: they are able to activate students both cognitively and metacognitively (Blum, 2015) and they have a positive impact in terms of enjoyment, interest and self-efficacy (Schukajlow et al., 2012). Since the question posed by the stakeholder is open-ended, as highlighted in (Blomhøj et al., 2003), this could generate in the students a feeling of "perplexity due to too many roads to take and no compass given". For this reason, the students have been guided through the modelling process by means of the activities described in the following subsection.

The mathematical modelling activities

The question posed by the stakeholder (how young people use and what they would like to find in the municipal library) has been investigated by means of a survey which has been built and successively analyzed by the students using statistical and ICT instruments. In the last part of the experience the students wrote a scientific report which has been delivered to the stakeholder. The activities were designed to allow students to pass through all the steps of the Blum's modelling cycle, but the ones considered in this work are related to steps 2, 3, 4 and 5. More precisely, during activity A1 (*Proposing the more relevant questions*) students were asked to work in groups and propose questions for the survey, suggesting them to take inspiration from surveys to which they answered in the past and from the internet. Then, they were asked to identify which dimensions (e.g., frequency of attendance, reason for attendance) of the phenomenon "library" their questions aim at exploring. Finally, the questions were classified according to such a criterium. Therefore, students discarded the questions which were not relevant with respect to the stakeholder's request. Thus, A1 is related to the "step 2. simplifying/structuring" of the Blum's modelling cycle. While activities A2 (*Formulating the questions for the survey*) and A3 (*Formulating the answer options for the survey*) are linked to "step 3, mathematizing". Indeed, during such activities, the students were asked to work in groups on the formulation of questions and answers to make them as clear as possible for the participants to the survey, to reduce possible ambiguities in their answers and to answer effectively to the initial stakeholder's question. The main part of such activities concerns the characterization of variables beyond the identified dimensions. Roughly speaking, the students decided which form to give to a particular question (e.g., Likert scale, open question, multiple choice). Even if it isn't written in mathematical language, the survey produced by the students is the mathematical model of the initial problem posed by the stakeholder. Activities A4 (*Using the statistics software R*) and A5 (A*nalyzing the results of the survey*) are strongly intertwined. In activity A4, a lesson-tutorial about the software R has been delivered to the students and the students were asked to practice with the software, with given datasets and with some preliminary data coming from the first draft of the survey. While during activity A5, the students were asked to work in groups with the software R to analyze and to interpret the results of the survey. We can then relate both activities A4 and A5 to step 4 ("working mathematically") and 5 ("interpreting"). In particular, the software R support the students in "calculate" and "visualize" (Greefrath, 2011).

Data gathering and method of analysis

The participants of this study are 18 high school students (grade-11), 4 male and 14 female. The average grade of the class in maths is good (6.9 over 10), according to their last report card. To address the above research questions, one survey has been developed and administered online by google form the day after the last meeting. The survey is composed of a set of Likert (5 levels) questions and a set of open questions, according to the narrative analysis (Di Martino & Zan, 2015). Question Q1 and Q2 investigate the "like" and "can" variables, respectively. They ask students to rank each of five activities according to their level of "like" and "can", more precisely: *Q1. Consider the following activities you took during the project and indicate how much you liked them from 1 (I didn't like it at all) to 5 (I liked it very much). Q2. Consider the following activities you took during the project and indicate how much you perceived it as easy from 1 (Not easy at all) to 5 (Very easy).*

For questions Q1 and Q2 and for each activity the mean of the answers has been computed. Cognitive, motivational and emotional factors related to the emotion enjoyment have been investigated more in detail by means of the following two open questions: *Q3. Choose the activity that you liked MOST among the options of the previous question and describe what you liked of that activity and why. Q4. Choose the activity that you liked LESS among the options of the previous question and describe what you DIDN'T like of that activity and why.*

The answers to both questions have been qualitatively analyzed based on the affective types which can be retrieved in them. These types have been identified in students' answers by means of keywords and sentences. For example, the answer *"Formulating the questions for the survey because we had interesting discussions and it was also easy"* is labelled as both motivational type (interest) and cognitive type (self-efficacy). Finally for each activity, the frequency of a particular type has been computed.

Data analysis and findings

Figure 2 (left) summarizes the analysis of the responses to question Q1 and Q2. For each activity, we employed the mean of the score assigned for the level "like" and "can". In this way each activity can be seen as a point in the cartesian plane. We can notice that the activities which have been enjoyed the most were activities A2 (Formulating the questions for the survey) and A3 (Formulating the answer options for the survey), closely followed by activity A1 (Proposing the most relevant questions). We then find at an intermediate level activity A5 (Analyzing the results of the survey), while activity A4 (Using the software R) has been considered as the less enjoyable one. Figure 2 is also a first indication about the fact that, as expected, high levels of enjoyment (I like) correspond to high levels of perceived control and self-efficacy (I can), even if for all the activities the "I like" level is greater than the corresponding "I can" level.

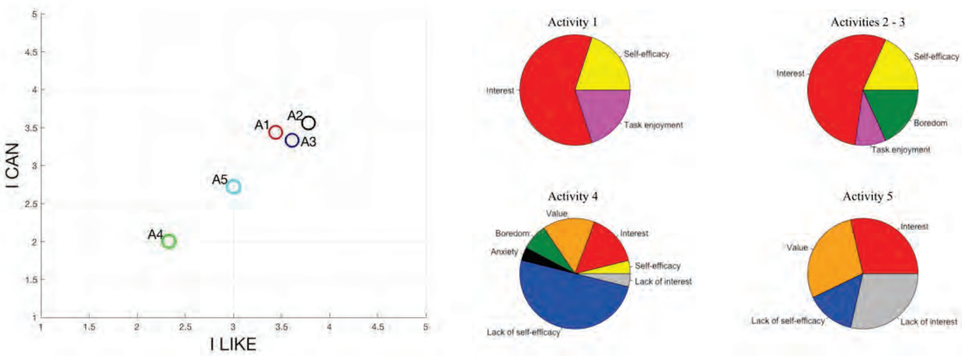

Figure 2: Left: Graphical representation of the dimensions I can – I like. Right: Frequencies of emotional, motivational and cognitive types for the different activities

In order to identify which cognitive, motivational and emotional factors, related to the emotion enjoyment, are reported by the students when carrying out different activities (RQ3), we analyze the answers to the open questions Q3 and Q4. Figure 2 (right) shows the overview of such an analysis, and in the following we report detailed examples of the students' responses.

Considering activity A1, 5 students have indicated it as the preferred one. Analyzing their answers, we notice that the motivational factor (interest) is the more frequent justification for their enjoyment, as can be seen from the following answer: *"thinking about the questions because they are tightly bound to reality"*. In other answers we can retrieve the emotion of task related enjoyment and the self-efficacy: *"because in general I like sharing my point of view with other people and discussing points of view different from mine [...]"* and *"We have been challenged in finding questions inherent to the project which had to be carried out"*. Activities A2 and A3 have been indicated as the most enjoyable ones by 6 out of 18 students. Also for these activities interest was the most popular justification. Moreover, confirming the trend highlighted in Figure 2, also the self-efficacy and perception of control emerge: *"[...] because we had interesting discussions and it was also easy"*. As we can notice in Figure 2, even if activities A1, A2 and A3 have been appreciated by the majority of students, their score in terms of "I like" is not the maximum possible. Analyzing the open question Q4, we can infer that these activities have sometimes triggered negative emotions, like boredom: *"[...] because, for what I am, I didn't find this activity exciting and sometimes boring"*. We now consider activity A4, using the statistics software R, which has been chosen as the less enjoyable activity by 15 out of 18 students. Confirming the trend reported in Figure 2, we can notice that the lack of control and self-efficacy is the most frequent reason for their choice, as can be seen from the following answer: *"I wasn't able to use the software as I would have to"*. In other cases, the activity also triggered negative emotions, as anxiety and boredom: *"Using R needs precise codes and high attention in every little detail of them otherwise an error is computed"* and *"The initial approach was difficult and, since we spent many hours using R, the activity seemed quite repetitive to me. Overall, it was however interesting learning this new language"*. Notice that in the last few lines of the previous answer also the interest factor comes out. Even if activity A4 was the less preferred activity for the majority of students, 3 students out of 18 have indicated this activity as the preferred one. Also in their answers we can recognize the types of interest and value: *"Using the statistics software R is the activity I enjoyed the most because it was a new experience and mostly because it can be useful for my future studies"*. Activity A5 was preferred by only 3 out of 18 students, due to value and interest: *"To me it seemed the most useful and interesting thing among the different activities"*. Summing up, for activities A1, A2 and A3, which have been considered as the most enjoyable, enjoyment is mainly associated to positive types such as interest and, to a lesser extent, task related enjoyment and self-efficacy, even if also the negative emotion of boredom emerges. Activity A4, which has been evaluated as the less enjoyable, triggers mainly negative types. The most frequent one is the low self-efficacy. However also positive motivational types, such interest and value, come out. Finally, in activity A5, which has received an intermediate score with respect to enjoyment, positive and negative types are more balanced: we find in similar proportions value, interest and lack of self-efficacy and lack of interest.

Discussion and conclusions

This study focuses on the relation between the emotion enjoyment and mathematical modelling activities. More precisely, we wonder which activity the students enjoy the most and which cognitive and motivational types can be associated to the different activities. Briefly, we were able to identify self-efficacy and lack of self-efficacy, concerning the cognitive factor; value, interest and lack of interest for the motivational factor; task enjoyment, boredom and anxiety for the emotional factor. Back to the MM, in Section 4 we argue the correspondence between activities and steps of the Blum's modelling cycle. Focusing on "Simplifying" step (activity A1), Blum (2015) argues that this step is usually challenging for the students since they are not used to making assumptions by themselves. However, in our case students show a high sense of self-efficacy. An explanation could be the strong link with reality of activity A1, which could have enhanced students' interest. This is a confirmation of the fact that the employment of real-world problems felt as important by students is fundamental in order to make students interested. Also for the "Mathematising" step, a potential cognitive barrier, students' answers show high levels of self-efficacy. In this case an aspect which could have impacted is the fact that this step has been realized without employing explicitly specific mathematical language. In fact, in activities A2 and A3 only common language was employed. The design of teaching experiences in which a few specific mathematical language is employed in the "Mathematising" step, could increase students' perception of self-efficacy, thus providing them a motivational push also for the following steps of the modelling cycle. On the contrary, in activities A4 and A5, associated to the Blum's step of "Working mathematically" and "Interpreting", the students' difficulty in dealing with mathematical language and problems linked to instrumental genesis strongly come out: the lack of self-efficacy is the cognitive factor more frequently reported in their answers. However, also for these steps, which have been perceived as difficult, the students reported positive motivational types such as interest and value. In particular, value was instead absent in the students' answers referring to the "Simplifying" and "Mathematising" steps. An explanation for this could be that students perceived the statistical and ICT tools as close to the university and employment world, thus contributing to raising the perceived value of the "Working mathematically" and "Interpreting" steps. The use of such tools could be important to enhance the value students attribute to the modelling experience. In order to verify the hypotheses just exposed we are planning to extend the present study by involving more students and by designing additional modelling teaching experiences based on different real-world scenarios.

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Towards a dual integrated modeling approach to the teaching and learning of mathematics: Challenges and tensions in implementation

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This paper elaborates on the theoretical conceptualization of a so-called dual integrated modeling approach to the teaching and learning of modeling and the teaching and learning of mathematics through modeling. Departing from a sociocultural perspective on teaching, the models and modeling perspective, and using the previously developed theoretical constructs of the Teaching Triad and the Expanded Mediated Triangle, we discuss modeling in terms of (i) micro modeling in which modeling is used as a vehicle to teach and learning other curricula content; and (ii) macro modeling in which modeling is a self-standing mathematics curricula goal. We illustrate this dual integrated approach by analyzing and discussing a lower secondary teachers' implementation of a modeling activity focusing on a simple statistical investigation and measures of central tendency in statistics.

Keywords: Dual integrated modeling approach, expanded mediational triangle, measures of central tendency, models and modeling perspective, teaching triad.

Introduction

There are different goals and rationales for implementing modeling tasks in the teaching and learning of mathematics (Blum & Niss, 1991; Kaiser & Sriraman, 2006). One way to broadly characterize these is to draw on Julie and Mudaly's (2007) discussion of modeling as either *a self-standing mathematical content* or as *a vehicle for learning other more specific (mathematics) ((curricula)) objectives* (see also Niss & Blum (2020)). By enabling the option to put 'mathematics' and 'curricula' in brackets in the last sentence, "*(mathematics)*" and "*((curricula))*", the latter category then includes socio-critical and discursive perspectives on modeling (cf. Barbosa, 2006) as well as ethnomodeling (Orey & Rosa, 2021). There are plenty of examples of research studying modeling as understood more or less exclusively in one or the other of these two categories. In the case of modeling as a self-standing mathematical content much research focusing on the development of students' modeling competencies qualifies, see Cevikbas et al. (2021) for examples. Regarding modeling as a vehicle for learning other more specific (mathematics) ((curricula)) content, Barbosa (2006) and Orey and Rosa (2021) illustrate this strand of research, as do much of the research carried out based on the realistic mathematics education (RME) programme (cf. Gravemeijer, 1999). However, theoretical and empirical research on integrated modeling approaches, combining both rationales and goals outlined above and given close to similar emphasis on both, is to our knowledge spares. Even in the so-called *educational modeling perspectives* (cf. Kaiser & Sriraman, 2006), which potentially have pedagogical as well as subject-related goals, are commonly divided into two strands focusing either on *didactical modeling* or *conceptual modeling*.

In this theoretical paper, we seek to start building and elaborating on a theoretical foundation for a dual integrated modeling approach based on the models and modeling perspective (MMP) (Lesh & Doerr, 2003). We use the MMP to at the macro level of teaching focus on modeling as a self-standing mathematical content in its own right (*macro modeling*), as well as at the micro level of teaching using modeling as a vehicle for teaching more specific mathematical content matter (*micro modeling*). In terms of Blum and Niss' (1991) six basic approaches to include applications and modeling in mathematics teaching and learning, our conceptualization of this dual integrated modeling approach is akin to what they term "*[T]he interdisciplinary integrated approach*" (p. 61, italics in original).

As part of our initial work and ongoing thinking, we in this paper outline our current theoretical conceptualization of this dual integrated modeling approach. We also report on a first attempt to apply this framework as an analytical lens, to discern its adequacy to capture aspects and challenges of teaching that arises when the dual approach is implemented in the classroom. The question guiding our theoretical exploration in this paper is: *What teaching challenges come to the fore in the initial phase of when a teacher tries to adapt a dual integrated modeling approach in his/her teaching?*

To further theorize, we use a sociocultural perspective on teaching to try and capture and describe teaching challenges on both the micro- and macro level of the teaching and learning of, and through, modeling, when a lower secondary teacher implements a modeling activity involving a simple statistics investigation focusing on measures of central tendency in statistics.

Theoretical considerations

We now elaborate on the notion of the dual integrated modeling approach adapted in the paper, and how we conceptualize modeling at both a micro- and macro level (*micro*- and *macro modeling*) using the Teaching Triad (Jaworski, 1994) and the Expanded Mediational Triangle (Engeström, 1998).

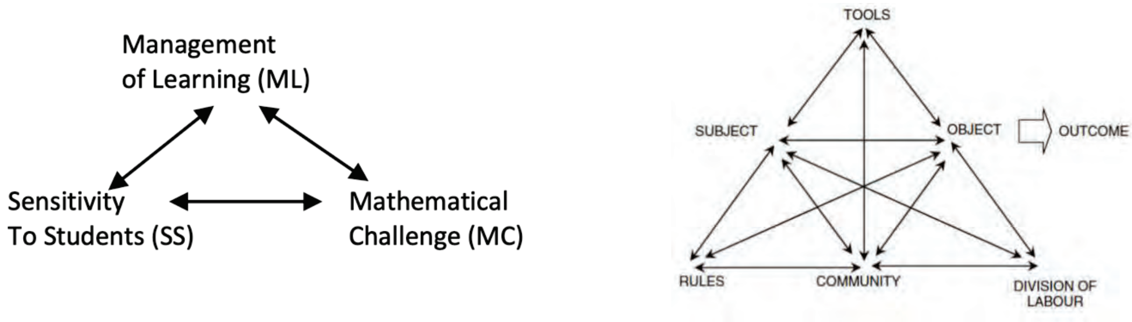
The models and modeling perspective

The perspective on modeling adapted in this paper is the *models and modeling perspective* (MMP), in which a model is defined as a general system consisting of elements, relationships, rules and operations that can be used to describe, predict, make sense of, or explain some other system. A mathematical model focuses on the structural characteristics of the system at hand (Lesh & Doerr, 2003). Learning from a MMP is understood as developing useful and generalized models consisting of a set of concepts and procedures. The concepts are used to describe or explain the mathematical objects and aspects in the context relevant to the phenomenon studied using or re-using the procedures to engage in or create goal-directed constructions, manipulations, or predictions (Lesh & Harel, 2003). In the MMP three different types of structurally related activities organized in so-called *model development sequences* can be used to purposefully support students' learning towards a given learning goal: *model eliciting activities* (MEAs) which aim to elicit the students' ideas they bring to the activity; *model exploration activities* (MXAs) that focus on the underling mathematical structure elicited by students; and *model application activities* (MAAs) were students apply their models in similar or new contexts (Lesh et al., 2003). In all three types of activities students iteratively engage in expressing, testing, revising, and developing their models (Lesh & Doerr, 2003; Lesh et al., 2003).

The Teaching Triad and the Expanded Mediational Triangle

Following Jaworski and Potari (2009) and Jaworski et al. (2017) we in addition stress the sociocultural aspects of teaching as a mediating process connecting the content of mathematics, students, and teachers. In particular, we elaborate on the notions of *micro*- and *macro modeling* from both an institutional perspective and the nature of more local interactions between teachers and students engaged in modeling activities.

The Teaching Triad (TT) captures three interdependent and interlinked dimensions of teaching (see Figure 1a): *Management of Learning (ML)* aims at describing the organization of the learning environment by the teacher such as the planning and orchestrating of tasks and activities, use of resources, and forms of working in the classroom; *Sensitivity to Students (SS)* focuses on students' affective-, cognitive- and social needs and the ways in which these are considered by the teacher in interactions; and *Mathematical Challenge (MC)* on what mathematical content, thinking and activities are offered to students in the learning environment including for example posed questions, sets and sequences of tasks, and metacognitive demands and processing.



(a) (b)

Figure 1: (a) The Teaching Triad (Jaworski, 1994); and (b) The Expanded Mediational Triangle (Engeström, 1998)

The Expanded Mediational Triangle (EMT; see Figure 1b) by (Engeström, 1998) models the structure of human activity built around Leont'ev's three levels of human activity where the concepts *activity*, *action* and *operation* are central and dialectical theoretical constructs. An *activity* is described as a complex evolving structure of mediated and collective human agency connected to specific *motive* that both distinguishes activities from each other and gives the activity directionality. Activities are constituted by *actions* directed toward specific and conscious *goals* that realize and sustain the activity. Actions in turns are constituted of *operations* that realize the actions in the activity and are carried out under the premises determined by the *conditions* of the activity and the environment in which the activity takes place (Leont'ev, 1979). In the EMT, the *subject* transforms the *object* into an *outcome* using *tools* and artifacts. However, the mediation facilitating this transformation is both supported and conditioned by the culture and *community* carrying out the activity. Here, the *rules* account for norms, conventions and regulations within a community or activity, whereas the *division of labour* highlights both the actual division of labour and responsibility for achieving the goal/motive of the actions/activity and status- and power relations within the community (Engeström, 1998).

The dual integrated modeling approach: micro- and macro modeling in terms of TT and EMT

In terms of EMT, we consider the activity in the dual integrated modeling approach to be the teacher teaching mathematics with the overall motive of the activity for the students to learn mathematics within the boundaries of the teacher's and students' school setting. The actions that constitute the activity are the actions of the teacher as s/he based on her/his set goals interact and engage with the students to support their learning. One of the goals in the dual integrated modeling approach is for the teacher to draw on and implement the ideas and principles of the models and modeling perspective, and indeed to for the students to learn modeling, meaning for the teacher to make the students cognizant about the modeling processes that inherently comes with the way of working based on the models and modeling perspective and with model development sequences (macro modeling). In realizing the actions, the teacher carries out various operations conditioned by the conditions manifested in the activity and the institutional and cultural setting in which the activity is carried out. In other words, and analogous to Jaworski et al. (2017), we conceptualize the teacher as the *subject* in the activity with the *objective* to teach and support students to learn and make sense of mathematical content (such as modeling as a self-standing mathematical content – macro modeling). The teacher does this using various *tools* in mediating the mathematics – including pedagogical and didactical strategies and theories – and in particular the elements of the models and modeling perspective and micro modeling (modeling as a vehicle). However, the mediation of mathematics is in addition supported as well as conditioned by the cultural norms of the schooling (*community*), the school's norms and regulations (*rules*), and status- and power relations between actors in the classroom (*division of labour*).

In the dual integrated modeling approach this framing using the EMT is complemented by the TT framework to really focus in on the acts of teaching and how the teachers' actions and managing of the learning are related to the mathematical content and the students' learning – and in particular the role and function of all aspects related to micro modeling (the use of modeling to teach other content) and how these relate and support developing macro modeling as a mathematical content to be learnt.

An example of the use of the dual integrated modeling approach

The example we will now discuss and theorize on comes from a grade 7 classroom of 22 students of which the majority was girls (86%). The class was considered to be somewhat loud and challenging to keep on track in order to not lose focus on the topic at hand and the work needed to be done (providing clues about governing behavioral- and interactional norms in the class (community) as well as the power- and status relations in the classroom (division of labour)). The teacher had taught the class for almost a year at the time when the modeling activity, The Paper Helicopter Activity, was implemented.

The teacher participated in a project aimed at developing the teaching of statistics using the MMP, and he was familiar with, and had some prior experiences with, the fundamental philosophy and ideas of this perspective. In the context of this paper, the adoption of the MMP as the framework guiding and supporting the design of the learning opportunities for the students, induced principles for how to implement micro modeling in terms of model development sequences. With respect to the content matter of statistics being in focus in the project, simple statistical investigations and measures of central tendency in statistics, macro modeling in the context of this paper is (somewhat artificially) equated with the whole statistical (model) investigating process.

In addition, one of the goals the teacher expressed for participating in the project was to be able to work more independently from the textbook used in the school (breaking an established rule) and for his students to be comfortable in not relying on the textbook all the time as the sole source for mathematical knowledge and truth, and hence wishing to change the epistemic status of the textbook (a community norm) as well as its authoritative status (shifting the division of labour).

Within the project focusing on the teaching of statistics, the teacher and the two authors together developed a sequence spanning 12 lessons of on average 60 minutes covering the statistical content prescribed in the curricula; mainly descriptive statistics, simple statistical investigations, and measures of central tendency (mean, median and mode). The allocated amount of time and predefined mathematical subject-matter content establish natural conditions on what actions and operations were feasible to implement and enact to support the students mathematical learning. To document the teaching, the classroom was videotaped using two cameras: one following the teacher's movement and actions in the classroom which in addition picked up all his interactions and conversations (using a small portable microphone); and a fixed camera in the back of the classroom facing front to capture the dynamics of the classroom. The teacher also recorder short pre- and post-lesson audio-memos to document the teachers thinking regarding the goals and plans made before the lesson, and then the teacher's reflections on what happened in the classroom in relation to these goals and plans during the lesson. All audio-memos were collected, as were all the written work done by the students. In the account provided here we only draw on the video and audio data. Next, we use data from this project in a first attempt to apply and evaluate the outlined dual integrated modeling approach as a lens to identify and teaching challenges that surfaced during an implementation of a modeling activity involving a simple statistical investigation and focusing on measures of central tendency in statistics.

Identifying of teaching challenges applying the dual integrated modeling approach as a lens

As part of the sequence of the 12 lessons on statistics the teacher introduced an adapted version of the paper helicopter activity originating from the work by Box (1992) who used it to teach experimental design to engineering students. The activity has also recently been used as a modeling activity involving conjecturing, experimenting, and evaluating 10-11-year-old students' ideas about statistical distributions (Kawakami, 2017). In the activity, the students were presented with a scenario to evaluate three competing designs of airdrop devices by conducting a small statical investigation of miniature replicas of the designs in terms of paper helicopters (see Figure 2). The aim for the students were to decide which helicopter (a) had the longest flying-time; and (b) which was the more accurate in terms of coming closets to the airdrop target – and also considering if difference in the loaded weight might influence the decisions. The class worked on the activity on and off during four consecutive lessons (lessons 6-9 in the sequence), whereby we will focus on the first two lessons.

Figure 2. Helicopter

In the first lesson, the teachers used 20 minutes to introduce the activity using a by the researchers pre-prepared PowerPoint presentation (a new tool) aimed at raising the students' interest and provide a meaningful as well as motivating framing for the activity. This had the effect that the class fell dead silent and focused all their attention on the teacher's introduction. This behavior was atypical for the class in question, indicating that the use of this type of presentations might be an effective strategy in managing the learning (ML) to tackle this class' students' attitudes toward mathematics and what it means to do mathematics in a, for the students, intriguing and inspiring way (SS), and hence make them better prepared to engage in the mathematical challenges (MC) to come. The introduction ended with the teacher leading a MEA-inspired whole class discussion around the questions *What features of the paper helicopter affects its (a) precision (how far from the intended target it lands)? (b) travel time (time spent in the air after being released?* and *How do you think these different features affect the precision and travel time of the helicopter?* The features suggested by the students to affect the precision were wind, distance to fall, air pressure and gravitational force. Features suggested to affect the travel time were the size of the rotor blades, the weight of the helicopter, the body-shape of the helicopter, and the falling velocity. Note that all the features related to precision are about external factors that might affect the behavior of the helicopter rather than actual features *of* the helicopter.

The second lesson (70 minutes) began with the teacher recapitulating and discussing the students' suggested ideas of what factors might influence the paper helicopter. The teacher explained that the class were to evaluate three competing paper helicopter designs by collecting and analyzing data to determine which design is the best. In addition, the teacher also qualified the explicit goal of the activity to in addition investigate (i) if heavier load decreases the travel time? (ii) if heavier load increases the precision? (iii) if larger rotor blades increase the travel time? and (iv) if larger rotor blades increase the precision? After having established these goals, the teacher turned to illustrate an airdrop with a prepared cut out paper helicopter, explicitly showing how to fold the rotors of the helicopter to form a suitable angle relative to its body, how to hold it to minimize interference on its trajectory, and from what height to drop the helicopter (the edges of the ceiling lamps). In doing this, the teacher also multiple times explicitly stressed the need for the students to be consistent in their experiment procedure and data collection. Initially, this way of managing the learning (ML) was interpreted as being over-sensitive to the students' needs (SS) just to get them to understand the mathematical challenge at hand (MC). However, this later turned out to be an important clarification act by the teacher to eliminate misconceptions, as the students' following discussions revealed that some of the student had the impression that the task dealt with real maneuvering helicopters.

Most of the time in the second lesson were spent on collecting and analyzing the data in groups. The students took measurement series of 10 values using either 0, 1 or 2 paperclips as helicopter wight. At this point the students were not totally free to explore the data and think about how their data set could be used to answer the posed questions. Rather, they were implicitly prompted by the semester overall planning listing the content for the week as "mean, median and mode" as well as explicitly by the teacher's statements like "I wonder what the mean, median and mode of that data will tell us about which helicopter is best?!". Here, a tension between two conflicting goals came to the fore: the teacher's goal for the students to engage in realistic statistical investigation and to freely explore the data set (macro modeling) on the one hand, and to use the activity as a context for the students to learn more about (and apply) mean, median and mode (micro modeling). This tension was then accentuated when the nature of the students' collected data not in all cases was suitable for determining the mode of the data.

In summary, the first attempt to apply the dual integrated approach as a lens on a teaching sequence provided information on teaching challenges related to establishing student's autonomy for learning both micro and macro modeling simultaneously. The teachers' role, highlighted by our theoretical analysis, of directing students on what to learn and at the same time leave enough space and time for students' own experience in both micro and macro modeling is at the core of the teaching challenges.

Discussion and conclusions

This paper provides the first steps of theorizing toward a theoretical conceptualization of a so-called *dual integrated modeling approach* to the teaching and learning of (i) modeling; and (ii) mathematics through modeling. In this initial work, a potentially fruitful interplay between what is discerned from the TT- and the EMT-related framing of the ongoing teaching has come to the fore. On the one hand, the EMT provides a background helping to contextualize and deeper understand the relationships and dynamics revealed in the TT-based analysis of students' learning as mediated by the teacher's actions in managing of the mathematical challenges (MC) while being sensitive to students' various needs (SS) given the organized and planned management of learning (ML). An example of how the EMT elevates the effect of the teacher's actions, is the effect the use of the introductory PowerPoint had on focusing the students and making them more susceptible and ready to engage with the mathematical content. On the other hand, challenges for the teacher in obtaining the learning goal revealed by the TT-based analysis highlight conflicts and tensions between the micro level of teaching as constrained by institutional and cultural aspects manifested at the macro level provided by the EMT. As such, the TT-based analysis might be helpful in pointing at more systemic oriented changes needed in the EMT to improve the teachers' possible actions to increase the students' opportunities to learn mathematics in the classroom. This could for example be introducing a new tool such as a pedagogic strategy or assessment tool that facilitate the teacher in transforming his/her objectives for the students to learn mathematics toward the desired outcome, or to pin-point where measures need to be taken, such as replace dysfunctional power relationships (division of labour) or classroom behavioral norms (rules).

The development of the dual integrated modeling approach is just in its infancy, and all notions and concepts still need to be further concretized and elaborated. However, our analysis shows that the dual integrated approach as a lens provides some useful information for developing teaching practice, and we hope that what we have presented in this paper can spark an interesting discussion and result in productive suggestions for how to continue develop this line of thinking and theorizing.

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A topic-specific design research approach for modelling Covid-19 in grade 9 and 10

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Keywords: design research, iteration, epidemiology.

Method and Description of the research topic

In the framework of topic-specific design research, see e.g. Gravemeijer and Prediger (2019) and the literature cited therein, we consider dynamical systems both from content and design perspective. We report on the introductory two steps of the design research cycle according to the FUNKEN model by specifying and structuring the content and discussing initial ideas for design development. As a result, we outline a first teaching-learning arrangement dealing with discrete epidemic models. Design experiments are to follow in a next step.

Specifying and structuring the content, design principles

On a formal level (Gravemeijer and Prediger, 2019, p.47), Epidemiology is both an interdisciplinary field and highly relevant for understanding the global issues in treating the Covid-19 pandemic and therefore chosen as underlying topic. The professionally recognised basic model is the SIR model according to Kermack and McKendrick (1927), which is used both in a discrete and in its continuous variant. Our design goal is a teaching-learning arrangement for grades 9 and 10, so for curricular reasons only discrete models can be considered. On a semantic level the proposed arrangement is intended to tie in with exponential growth, the idea of iteration and simultaneously to enable a wide range of modelling activities. As a design principle, we use the four-step model "Mathematise → Mat. Work → Interpret → Validate" (Blum, 2015, p. 82). Our second design principle is visualisation to support structuring. Students are asked to work with the aid of flow-charts throughout the intended tasks, which is an excellent tool for clarifying complex iterative processes by providing a stringent pictographic representation (Sommer and Venke, 2020). For an example of a possible flow-chart see Figure 1. The teacher should focus on supporting students by translating the flow-charts into formal difference equations. The use of a spreadsheet program (e.g. Excel) and the graphical representation of the results scaffolds mathematical working.

Sketch of the developed design

We organise the modelling activities according to runs of the modelling cycle. In each step, a difference equation is set up (mathematisation), implemented with help in a spreadsheet programme (inner-mathematical work), the result is displayed graphically, interpreted and validated.

Exponential model: $I_{n+1} = I_n + \alpha I_n$. Focus in this step lies on the *effective contact rate* α giving the average rate of contacts of an infected person leading to a new infection. Problem: Exponential growth. After a while there is nobody left, who could be infected.

SI-model: In addition to the class of the infected I, the class of susceptibles S is introduced. The probability that the contact person of an infected is susceptible is modelled by the Lagrange probability $\frac{S}{N}$ where N is the size of the whole population $N = S + I$ leading to the difference equations $S_{n+1} = S_n - \alpha \cdot I_n \cdot \frac{S_n}{N}$ and $I_{n+1} = I_n + \alpha \cdot I_n \cdot \frac{S_n}{N}$. Compared to the formulation of the infection term as aSI (Ableitinger, 2011), the parameter α retains its meaning here in the transition from the exponential model to the SI-model. Problem: In the end all people are infected. There is no recovery and immunisation.

SIR-like model: The group I is split into several groups, depending on the number of days a person is already infected. After a few days, the exact number has to be specified – here we assume 8 days –, the infected recover. The final model is depicted in Figure 1.

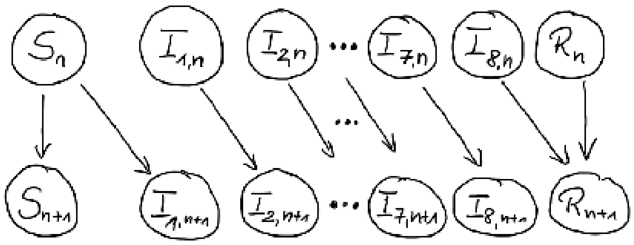

Figure 1: Flow-chart of a discrete SIR model with age structure

In subsequent modelling runs, contact restriction measures, testing and quarantine strategies, vaccinations and different virus variants can also be taken into account. Initial experiments show that students can arrive at sensible models here, at least at the pictographic level.

As a next step, this proposal is going to be empirically studied and further specified and improved. Our first questions are: To what extent 1) do students form a semantic understanding of the terms and parameters involved, 2) are students enabled to translate flow-charts in difference equations, and 3) are students enabled to break down complex real-life situations in epidemiological contexts into easier sub-problems through this teaching-learning arrangement?

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Objective and subjective aspects of mathematics and context - the case of rate of change

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The concept of Rate of Change (RoC) is often presented in an Extra-Mathematical Context (EMC) which evokes subjective judgments due to interpretations of the described real-life situation and of the missing information in the problem. In this study, we investigated different learners' interpretations of several EMC problems involving RoC, with the aim to examine which aspects of the notion of RoC are prone to subjective reasoning, due to their structure or due to missing information, and which aspects are objective. We found that while the problems raised subjective thoughts for different learners, analysis of both the EMC and the mathematical concepts can help predict which aspects of the mathematics are prone to be subjective, and which are not.

Keywords: Rate of change, context, calculus, missing information.

Introduction

Mathematics is often regarded as a domain in which students do not have much room to express their opinion, but rather one in which they need to find the correct answer. Problems that are based on Extra-Mathematical Context (EMC), offer an opportunity in this sense: the interpretation of the described real-life situation and of the missing information in the problem may allow for personal interpretation, and as a result, the problem may have several correct answers. Among other rationales for using EMC, Rubel and McCloskey (2021) claim that it motivates students to learn mathematics, supports learning mathematics, as well as how to solve everyday problems.

EMC based problems assume students have some factual knowledge and experience with the context of the problem. This, together with the need for reading comprehension, may lead to missing information situations. These situations require interpretation, which is necessarily subjective. Furthermore, the content of EMC problems may bring into play students' subjective thoughts and experience about the given context. These characteristics give EMC based mathematical problems a subjective shade that must be taken into consideration.

Personal thoughts, subjective by nature, may be based on objective or subjective considerations. Objective considerations include agreed paradigms, whereas subjective considerations include personal life experience. It seems that mathematics students have learned that only objective considerations are acceptable by the system, but in EMC problems, students react differently than in "pure mathematics" problems.

The concept of rate of change (RoC) is a central concept in calculus. A central issue for mathematics educators is how to make fundamental ideas of calculus, such as RoC, meaningful for students. From a mathematical point of view, in the concept of RoC (and in many other mathematical concepts) there are aspects that are prone to subjective judgments. The study presented here aims to examine which aspects of the notion of RoC are prone to subjective reasoning, due to their structure or due to missing information, and which aspects are objective.

Missing information and subjectivity in word problems

Word problems found in textbooks are typically well-defined with all the necessary information given. This is because such problems are explicitly designed to provide a way for students to practice the mathematical procedures they have just learned. De Lange (1995) introduced categories of context, one of which is referred to as “camouflage context”. These "artificial" word problems are situations where the context is used only to “dress-up” the mathematical problem. The frequent use of "dressed-up" word problems has caused students to develop restricted beliefs about word problems. Specifically, such beliefs are that all of the relevant information is given and that every problem has a single precise numerical answer (Reusser & Stebler, 1997). Thus, in these problems, students usually do not apply subjective ways of thinking.

Problems with missing information are an important part of mathematics education as well as real life (Blum, 2015). Solving problems with missing information requires skills and solution strategies that are different from the ones required to solve well-defined problems (Jonassen, 2000). For example, Fermi problems are what Ärlebäck (2009) defines them as "open, non-standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations" (p. 331). For example: How long would it take to count to a million? Or: How many cups of water are there in a bathtub? Ross and Ross (1986) recommend teachers present such problems, because it gives the students a more nuanced picture of mathematics, showing that doing mathematics is not always about executing well-defined procedures.

Krawitz et al. (2018) compared students' performance in solving word problems that are problematic from a realistic perspective versus quantitative problems with no numerical information, such as Fermi problems. They found that students did not notice the missing data in the word problems and as a result developed unrealistic solutions. In Fermi problems, when no numerical data are given but a numerical answer is required, the missing information is obvious. On the other hand, in questions in which students fail to notice the missing information, this may prevent them from arriving at a realistic response. An example for a question that is problematic from a realistic point of view: "A man wants to have a rope long enough to stretch between two poles 12 m apart, but he has only pieces of rope 1.5 m long. How many of these pieces would he need to tie together to stretch between the poles?". In this question, students have to notice that some length of rope will be used to tie knots, and this has to be taken into account. How much rope exactly is needed for a knot, is not given in the problem, although other numbers are given. Thus, with problems that have missing information, a student needs to first notice the missing information and then deal appropriately with the situation.

Missing information in a problem brings into play different subjective ways of thinking that students use to fill in what they don't know. These subjective thoughts are important to understand when teaching different mathematical concepts with EMC.

Rate of change

The concept of RoC, a central concept in calculus, describes how one quantity changes in relation to another quantity and is expressed as a ratio between a change in one quantity relative to a corresponding change in the other. Herbert & Pierce (2011) state that while rate is an important mathematical concept with many everyday applications, it is commonly misunderstood. Thompson (2013) made the point that the notion of RoC "entails a complex coordination of understandings of quantity, variation, relative change, accumulation, and proportionality" (p. 60).

From a mathematical-epistemological analysis of the concept of RoC, the following aspects seem vital for the existence of RoC in an extra-mathematical situation: (1) RoC always involves the measures of two quantities; (2) The quantities involved are varying ; (3) The change is continuous (or at least intuitively perceived by the student as continuous); (4) The nature of the relationship between the variables is relevant - The notion of RoC is closely related to that of function. A function describes a relationship between two quantities, while the RoC describes how one quantity changes with respect to the other. Not any relation between two quantities can be described by a function. There needs to be some specific sort of relationship between two variables for their connection to be fit to be described by a function, and thus potentially have a RoC of one with respect to the other. The definition of that relationship is relatively simple when we talk about pure mathematics but is not straightforward when the real world comes into play.

Some of these aspects seem to be objective (for example: in a situation in which one variable is described, it can be concluded without using personal judgment, that there aren't two variables in the situation) and other aspects seem to be subjective (for example: the nature of the connection between the variables). Others maybe objective at times, and subjective at other times, depending on the familiarity with the described situation (for example: whether the variable is discrete or continuous).

Methodology

The aim of our research study was to investigate which aspects of RoC tend to elicit subjective reasoning and which aspects tend to elicit objective reasoning, in an EMC based discussion. In the instrument used in this study, various situations were selected according to the aim mentioned. These situations differ in the sort of information which is given, the sort of information which is missing, and the characteristics of the situation: the number of quantities involved in the situation, the kind of quantities involved, whether the quantities vary and how they vary (continuous vs. discrete), and whether they co-vary or not (in our opinion). While these are not classic Fermi problems, since they require no numerical estimations, they are of the same character: open problems requiring the students to make assumptions about the problem situation. Based on the analysis of pilot trials, the situations as well as the formulation of the questions were repeatedly modified.

The final situations and questions have been used as base for semi-structured task-based interviews (Goldin, 2000). The interviewees were students and prospective teachers that were asked whether, in their opinion, it made sense to talk about RoC in these situations. The interviews were recorded and transcribed. When analyzing an interview, we identified which aspects of RoC were judged as relevant for each situation by the interviewee. We needed to establish which utterances are indicative of an interviewee's subjective considerations. To do this, we identified several relevant criteria, which helped us tag utterances as subjective considerations: (1) Statement of opinion – utterances explicitly qualified by the interviewee as their own belief, opinion or interpretation; (2) Non institutionalized (mathematically) argumentation – the utterance contains non-general arguments that are determined by personal life experiment or/and non-formal personal interpretations; (3) Adjustment completion – the utterance contains interviewee's use of concepts and ideas that were not a part of the given situation, which make the situation more logical and complete for the interviewee. The interviews were analyzed by the authors using content analysis methods (Smith, 2000) using the above mentioned criteria of subjectivity.

Findings and their discussion

Here, we present the preface to the final version of the interview, and three of the situations, as they were presented to the students. For each situation, a few representative student answers are given. The students were interviewed one by one, but the answers are presented here together. The quotes in this chapter symbol that the text is presented as was presented to the interviewees.

Preface

"Students learned in class about rates of change of quantities in different situations. Two situations were discussed in class: (1) A car that drives on a road. In this situation, they agreed that the distance driven by the car changes with time and it makes sense to talk about the RoC of the distance with respect to time. This RoC is the speed of the car. (2) Water flows into a tank. In this situation, they agreed that the volume of water in the tank changes with time and it makes sense to talk about the RoC of the volume with respect to time. After class, the students continued discussing this topic for other everyday situations."

Boris' situation

"Boris said: I am thinking of the rate of the dollar to the shekel and the temperature of the Mediterranean Sea. Does it make sense to talk about rate of change in Boris' situation? If it does – what is that rate of change? If it doesn't – why doesn't it make sense?"

Boris describes two variables, with a questionable connection between them. In this situation there is missing information regarding the nature of the connection between the two variables, meaning that the aspect of RoC which may be considered subjective, the type of connection between the two variables, is at the core of the situation. We ignore potential continuity issues of the exchange rate. The following are responses to Boris' situation (translated from Hebrew):

Tina	There is no connection between these two variables. One does not influence the other. It isn't possible to treat one as a function of the other.
Rob	The temperature of the Mediterranean Sea doesn't influence the rate of the dollar, so there's no rate of change.
Oliver	They're asking me how the rate (of the dollar to the shekel) influences the temperature? […] In this case it doesn't make sense to talk about rate, because it's difficult to find something that connects them. Maybe there's an oil company that works in the Mediterranean, and if the exchange rate goes up, the company will work harder, I don't know. […] Maybe in the summer the rate of the dollar

goes up and in winter it goes down? But no, I don't think so. [After discussing different situations, Oliver returned to Boris' situation] You can talk about a rate of anything. The question is if it makes sense… If there is some benefit in it for me. […] If there's no connection, then one doesn't influence the other; then the rate of change will be zero, because it isn't having an influence.

Typical for this type of open-ended questions, the respondents gave answers at different levels of complexity. Tina and Rob both stated that one variable has no influence on the other, but each of them drew a somewhat different conclusion. Rob drew the immediate conclusion that if there is no influence then there is no RoC. Tina answered indirectly, and considered one not being a function of the other a satisfactory answer to the question she was asked, about the existence of RoC. Oliver, on the other hand, gave the impression that he was fearing that he lacked some knowledge regarding the connection between the two variables. Tina and Rob made an assumption regarding the connection between the variables, maybe without being aware that this is an assumption, since obviously they may lack some unknown knowledge regarding the situation. Oliver tried to "find the answer", although it is an impossible mission. Later on, Oliver reached two surprising conclusions. The first is that for a situation to have a RoC, there needs to be some benefit to the discussion of RoC in the given situation. The conclusion regarding the benefit of the discussion, is the effect of basing a mathematical problem on EMC. The second surprising conclusion is that no connection means the RoC equals zero. This is a confusion. Zero RoC means no change, rather than no connection.

All students characterized the connection between the variables (or rather the lack of such a connection) as the criterion for the inability to talk about RoC in this situation. Oliver's response was labeled as subjective due to the 'non-institutionalized (mathematically) argumentation' expressed by the issues raised (oil company, summer-winter).

Anat's situation

"Anat said: While I am driving to the Dead Sea, I am thinking of my height above sea level and my distance from the Dead Sea. Does it make sense to talk about rate of change in Anat's situation? If it does – what is that rate of change? If it doesn't – why doesn't it make sense?"

Anat describes two continuously changing quantities which are closely related. This is a situation in which RoC is objectively relevant (or at least, it may be assumed that all the vital aspects for RoC are present), if the student is familiar with it. The following are two responses to Anat's situation:

Tina	Yes. You can talk of the rate of change of the distance in relation to the height.
Tanya	If the speed is constant then the distance and the height will change accordingly. Speed multiplied by time equals distance. So if I want to know the rate of change then I will divide the distance by the time. The distance changes here, and so does the time [variable]. The magnitudes here are changing all the time but I don't know if you can estimate the rate of change. The question is how you define rate of change. No, I can't estimate the rate of change. As time goes by, the distance changes and so does the height above sea level. […] You can talk about the speed in respect to distance and height. Maybe the height changes with respect to the distance I've traveled. I think that there is a time parameter here, a distance parameter and a height parameter.

While Tina immediately stated her answer, Tanya had trouble talking about the RoC of distance with respect to height (or height with respect to distance). In this situation, “time” isn’t explicitly involved and Tanya imposed it, presumably because she found quantities that “change over time” to be easier to discuss, than two non-temporal quantities changing one with respect to the other. This is what Jones (2017) called “invoking time”. This is a tendency for some students to insert time into what are otherwise time-less contexts. “For some students, invoking time seemed to be beneficial since it helped them to organize the covariation between the two quantities and to reach those higher covariational reasoning levels in which changing rates of change are considered.” (p. 107). Due to the 'adjustment completion' used to invoke time, which was not part of the described situation, Tanya's reply was labeled as subjective.

Hadas' situation

"Hadas said: I am thinking about the fact that nurses at baby-care centers weigh the babies, and each baby has its own weight. Does it make sense to talk about rate of change in Hadas' situation? If it does – what is that rate of change? If it doesn't – why doesn't it make sense?"

Hadas describes one magnitude only, which is, objectively, a situation in which RoC is not relevant. The following are responses to Hadas' situation:

Rob	Every baby has his or her own weight. In rate of change we talk about rate between two things, right? […] I don't understand – the weight in respect to what?
Oliver	I see only one variable here. […] You can talk about rate of change but it would be… There's no connection between the babies. It [the RoC] would be difficult to measure and it wouldn't give me anything. Would I be numbering the babies? […] You can talk about rate of change but it would be degenerated. You can talk about change, but not about rate of change because it's not continuous. […] If you weigh 10 babies every hour then you can force the rate in here.

While both students understood that this is a situation with only one variable, Rob was quick to determine that one variable is not enough to talk about RoC. Oliver, on the other hand, expressed some confusion, although he stated clearly that he only sees one variable in the situation. We witness him trying to find a second variable (numbering the babies) and reasoning why this wouldn't solve the problem. Of the aspects which are vital for the existence of a RoC of one variable with respect to another, the one which is relevant in this situation is that two quantities need to be involved. This aspect was considered objective by the researchers and witnessed in practice as objective. Although subjective thoughts have been raised (for example "numbering the babies" which uses 'adjustment completion') the criterion remains central and valid.

It is interesting that Oliver talks about the number of babies that are weighed per hour, since it has to do with the influence of language. In Hebrew, the words 'rate', 'rhythm' and 'pace', are the same word. Oliver talks about the pace of work that the nurses manage as an option to insert rate into the situation. This has nothing to do with the mathematical concept of RoC which is not relevant in this situation.

In general, our findings in this study include notions which are considered necessary for the notion of RoC. Some of these were considered objective, which have one "correct" judgment (having 2 quantities involved in the situation or having quantities that “change” – related to as quantities that takes on different values). The findings demonstrate how, in objective situations, students still bring in their subjective ways of thinking (such as invoking time or completing a missing variable by numbering a single variable).

Other notions are subjective, and different interviewees had different opinions, none of which were incorrect. The first is "A connective relationship" – Students argued that in order to talk about RoC, two quantities need to be connected, meaning that one influences the other: “There’s no RoC because there’s no connection between the two variables”. This is a subjective judgment, and different interviewees found different levels of relationships in the same given scenario. The second subjective notion is "Having a benefit" – Students stated that in real-life scenarios, some benefit must come out of talking about RoC: “you can talk about RoC, but the question is what you get from it”.

Conclusion

It seems that, in the case of the concept of RoC, EMC has a considerable influence on students’ interpretation of the concept and on their decision-making process. Interviewees based their decisions on their own life experience and their personal ways of thinking. Regarding the different aspects of RoC, even aspects that the researchers thought to be objective proved to elicit subjective thoughts when the design of the task was based on real-life situations. Missing information played a significant role when the two described variables had a vague connection. While some students assumed (without stating the assumption) that no connection exists between the variables, others tried to fill in the missing information with imaginary connections.

Viewing mathematics as a basis for many engineering and scientific fields makes working with EMC vital for a proper mathematical education. But working with EMC elicits subjective ways of thinking, which are difficult to foresee and not easy to analyze. Understanding which knowledge elements are more prone to be objective, and which inherently propose a situation in which there is missing information, may enable mathematics educators to combine EMC in a more constructive manner. As a benefit, students learn that assumptions must sometimes be made, in order to solve a mathematical problem, sometimes there is information which is missing in the problem, and sometimes there is more than one correct answer - doing mathematics is not always about getting exact answers by means of well-defined procedures.

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Modelling climate change through Study and Research Paths

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Keywords: Herbartian schema, mathematical modelling, paradigm of questioning the world, Study and Research Paths, the Anthropological Theory of the Didactic.

Research topic

The upcoming research accounted for in this poster is related to CiviMatics, an Erasmus+ funded project involving Austria, Germany, Norway (NTNU), and Romania (see https://www.civimatics.eu). CiviMatics aims to combine approaches of mathematics education and political education by offering educational tools to enhance the competences of citizens, especially regarding the interaction with global issues facing our societies. A societal issue dealt with in this project is climate change, currently one of the most urgent problems requiring an initiative. At NTNU, the course MA3001 (see https://bit.ly/3DDy0df) has been designed for the sake of CiviMatics to conduct experiments with the modelling of climate change through so-called *Study and Research Paths* (SRPs; Chevallard, 2022). MA3001 will be taught in the spring 2022 as part of a master's programme for Grades 8–13, "Natural Science with Teacher Education". The accompanying research will be centred on the following questions: *What are the main constraints related to implementing SRPs? What is the student teachers' learning outcome? What are the successful points associated with conducting SRPs?*

Theoretical framework and methodology

The didactic design to be employed and the research to be conducted in MA3001 are rooted in the *Anthropological Theory of the Didactic* (ATD; Chevallard, 2022). Our conceptualization of modelling is based on the notions of *system* and *model* as explained by Strømskag and Chevallard (2021): A system $\mathcal{S}$ is any entity subject to laws of its own. Given a system $\mathcal{S}$, a system $\mathcal{S}'$ is said to be a model of $\mathcal{S}$ if, by studying $\mathcal{S}'$, one can answer certain questions Q about $\mathcal{S}$. In practice, given a question Q relating to $\mathcal{S}$ which one wants to answer, one tries to build up a model $\mathcal{S}'$ of $\mathcal{S}$ (or choose an existing one) whose study with respect to Q is easier, safer, quicker than by a "direct" study of $\mathcal{S}$.

The modelling activities to be conducted in MA3001 are based on the new didactic paradigm proposed by ATD, that of *questioning the world* (Chevallard, 2022). In this paradigm, the point is to create a didactic system $S(X; Y; Q)$ that inquiries into a question (or a set of questions) Q. A praxeological function that studying a question Q assumes is to lead to studying all sorts of works (including derived questions Q_i). How can we describe what happens in a didactic system S when a class X studies a question Q under the supervision of a teacher/teachers Y? The model provided by the ATD is the *reduced Herbartian schema* (Chevallard, 2022): $S(X; Y; Q) \Rightarrow A^{\heartsuit}$. The heart hints at the fact that the answer A to Q will be "at the heart" of the didactic system, at least for some time.

The next step in building up a model of the inquiry is the introduction of the didactic *milieu*, M, which is the set of tools that the class gathers in order to carry out their inquiry into question Q. This results in the *semi-developed* Herbartian schema (Chevallard, 2022): $[S(X; Y; Q) \Rightarrow M] \Rightarrow A^{\heartsuit}$. Here, the didactic system is seen to *create* (denoted by ➡) the milieu M and to *generate* (denoted by ➥) the answer $A^{\heartsuit}$ by drawing upon the milieu M. In the quest for an answer A to the question Q, three main components stick out: (1) The search in the literature (including Internet queries) for *existing answers* $A^{\lozenge}$ offered by other persons or institutions. (2) To draw upon the answers $A_i^{\lozenge}$, the didactic system has recourse to *works* W_j of various kinds, like theories, experiments, essays, etc. (3) To use these works, the student needs to study them. This involves studying a number of *questions* Q_w about the work under study. Hence, the milieu M takes on the following appearance: $M = \{A_1^{\lozenge}, A_2^{\lozenge}, \ldots, A_m^{\lozenge}, W_1, W_2, \ldots, W_n, Q_1, Q_2, \ldots, Q_p\}$. We will use the Herbartian schema as a tool to regulate and analyse inquiries.

An SRP in MA3001 will be organised as follows: The generating question Q of the SRP will be presented by the teacher who is supervising the SRP. Student teachers (hereafter 'students') work in teams to study and answer the question Q. The teams and the teacher meet regularly to review the teams' work on Q and to confirm, adjust or extend the SRP for a specific period. In addition, seminars will be arranged where the teams present preliminary progress reports and receive feedback from each other and from the teacher on the condition of the answer under construction.

This is the generating question Q_1 (with subquestions $Q_{1.1}$ and $Q_{1.2}$) that the students will be given:

Q_1. How is future climate change caused by long- and short-term pollutants modelled in the scientific literature?

$Q_{1.1}$. What assertions and simplifications are made in these models?

$Q_{1.2}$. What knowledge seems indispensable to understand these models (focusing on mathematics, but also considering civic education, chemistry, biology, physics)?

At present, the focus is on an a priori analysis of SRPs which involve inquiring into Q_1 and its subquestions. We will draw on the many SRPs reported on by the ATD research community (e.g., Barquero et al, 2018; Bartolomé et al., 2018). Data to answer the research questions for the follow-up research presented in the first section, will include observations of students' team work, students' reports from the SRPs, and questionnaires about the functioning of the didactic systems at work.

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Students’ approaches concerning model validation and model improvement when solving modelling tasks with experiments

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Mathematical modelling is important yet challenging. Especially the validation step is a hurdle for many students. One approach, discussed in the literature, for fostering students’ validation competence is to use modelling tasks that use data gathered in experiments. Analyzing the validation of 71 students in two modelling tasks with experiments, we found that many students struggle with validation. In particular, most students seem to put more trust in their model than in their experimental data and therefore suggest to improve experiments instead of working on an alternative model. Stimulated recall interviews show that students’ lack of mathematical knowledge as well as a lack of understanding of the relation between data and model can be the reasons for these findings.

Keywords: Experiments, mathematical modelling, model-validation.

Introduction

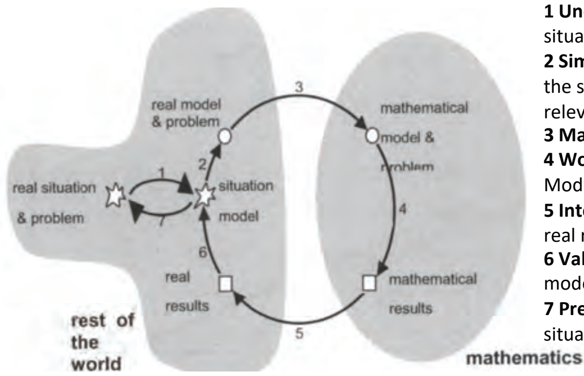
Mathematical modelling is a central mathematical competence, which is reflected in several curricular documents around the world (e.g., National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010). Nevertheless, modelling remains complex and challenging for students. Previous studies have shown that especially the validation of one’s models is a hurdle in the modelling process for many students (e.g. Blum & Leiß, 2007). Therefore, supporting students when validating their models is necessary. According to Engel (2010) real data is needed for authentic modelling and validation, since real data usually does not fit perfectly to an intended model. In contrast, using already smoothened or unrealistic data that fits well to an intended model, leads to a modelling process in which the relevance of validation is not stressed (Engel, 2010). One approach to integrate real data into the modelling process is to use modelling tasks with experiments in which students first perform experiments and gather data that can be used for the subsequent modelling (cf. Zell & Beckmann, 2009). This approach is frequently discussed in practical teaching literature (e.g. Ludwig & Oldenburg, 2007) but little empirical evidence for the anticipated positive effects on students’ validation competence exists.

The project *Mathematical Modelling with Experiments* (MaMEx) follows the objectives to design and evaluate modelling tasks with experiments and to gather insights into students’ validation processes. In a pilot study, Geisler (2021) used a modelling task with experiment concerning the decay of beer-froth and analyzed students’ validation approaches. Students in this pilot study still struggled with validation. Moreover, there was confusion among students, whether the mathematical model should be fitted to the data or vice versa in the sense that the measurement of the data should be improved in order to fit better to the mathematical model. In this contribution, we present an extension to the aforementioned pilot study to get more insights into students’ validation approaches.

Theoretical Background

Mathematical Modelling

Mathematical modelling is usually described as a circular process (e.g. Galbraith et al., 2010). In this contribution, we follow the modelling cycle proposed by Blum and Leiß (2007) describing seven steps in the modelling process (Figure 1). According to Niss (1994), the validation step is the most important step in the modelling-process. In this step students evaluate whether the real results obtained in the modelling process and the used mathematical model itself are adequate with respect to the real situation. As a consequence of this step, the model itself could be revised if it is not found suitable. Unfortunately, validation is very challenging for many students and some students do not see validation as a necessary step (Hankeln, 2020). When students work on modelling tasks, the validation is often absent (Blum & Leiß, 2007) and those students who validate their models often rely on a rather intuitive feeling that their model might be suitable or not (Borromeo Ferri, 2006). In accordance with these results, Borromeo Ferri et al. (2013) report that students scored weakest concerning their validation competence compared to the other modelling sub-competences. Furthermore, within their intervention-study they showed that fostering students' validation-competence is possible even with short interventions like modelling project days.



1 Understanding/Constructing (understanding the real situation and setting up a (mental) situation model)
2 Simplifying/Structuring (simplifying and structuring the situation model into a real model focusing on the relevant information)
3 Mathematising (setting up a mathematical model)
4 Working mathematically (working with the math. Model to obtain mathematical results)
5 Interpreting (interpreting the math. results to obtain real results)
6 Validating (validating the real results and the whole modelling-process with respect to the situation model)
7 Present (present real results with respect to the real situation)

Figure 1: Modelling-Cycle following Blum & Leiß (2007)

Fostering validation with modelling tasks with experiments

In the literature, there exist several concrete teaching ideas for combining experiments and modelling tasks (e.g., Ludwig & Oldenburg, 2007). Halverscheid (2008, p. 226) sees a natural link between experiments and modelling as experiments "represent the *rest of the world* for which mathematical models are built." Ludwig and Oldenburg (2007) emphasize benefits of experiments for the whole modelling process and especially the validation step, because students can validate their models based on real data instead of theoretical considerations. According to Zell and Beckmann (2009, p. 2216) experiments offer good occasions for validation: "Because of measurement errors the formula is never correct. So it is natural to talk about the correctness and the limitations of the model and its results."

Even if these considerations propose a high potential of modelling tasks with experiments for validation, empirical research in this field is scarce. So far only a few qualitative studies with small samples have dealt with validation in the context of modelling tasks with experiments. Zell and Beckmann (2009) used modelling tasks in combination with physics experiments and interviewed secondary students after working on these tasks. Students struggled with accepting the imperfection of their models due to measurement errors at first but eventually were able to validate their models. In contrast, Maull and Berry (2001) observed undergraduate students who worked on a modelling task concerning the cooling off of tea and found that many students were not able to validate their models without help of the lecturer. It seems that using experiments can even hinder the validation process: Carrejo and Marshall (2007) report about an undergraduate course in which students experimented with motion. When talking about the validation of their models, students attributed even systematic shortcomings to measurement errors instead of unfavorable model specifications.

In a pilot study of the MaMEx-project, Geisler (2021) implemented a modelling task concerning the decay of beer-froth in two upper secondary classes. Students performed an experiment and used their gathered data to model the decay by exponential functions. Analyzing students written solutions, Geisler (2021) found that students' acceptance or rejection of a model was very subjective, in the sense that models identified as adequate by the students as well as models that students judged as inadequate showed the same systematic shortcomings (e.g. systematically over- or underestimating the data). These shortcomings have often been justified via measurement errors by the students which is in line with Carrejo and Marshall's (2007) findings. Furthermore, being asked how to improve the fit between data and function, most students stated that they would improve their experimental procedures instead of searching for an alternative model. It seems that students put more trust into their mathematical model than in their data and therefore try to improve data in order to fit the data to the model instead of improving the model with respect to the data. However, the analyzed sample ($N = 19$) in this pilot study was rather small and all students came from the same school.

The current study

Objectives and Research Questions

In this paper, we present an extension to the aforementioned study. We implemented two modelling tasks in four secondary classes to determine whether results are comparable within a larger as well as more diverse sample and with different modelling tasks. Furthermore, we wanted to understand students' ideas behind their approaches to improve their models. Our guiding research question was:

How do students validate their models within modelling tasks with experiments with respect to their gathered data and which approaches to improve models do they propose?

In the following, we give an overview of the used modelling tasks before describing the methods of data collection and analysis.

Design Principles of the Modelling Tasks

All modelling tasks with experiments used in the MaMEx-project follow four design-principles (cf. Galbraith et al., 2010): P1) The contexts of the task should be realistic and use only those physical quantities that are already known by the students. P2) The mathematics should be foregrounded in the whole process. Therefore, experiments should be easy to conduct and take only a few minutes. P3) It should take not too much time to set up a mathematical model but the task should offer relevant occasions for validation. P4) A validation prompt should be implemented since many students do not validate their models spontaneously.

The task “Cold Coffee” (in the following named “CC”) is adapted from Ludwig and Oldenburg (2007). The following served as an introduction to the task:

> After brewing, coffee needs some time to cool off in order to be conveniently drinkable. The desired drinking temperature differs from person to person. Model the temperature decrease and evaluate at which time the coffee can be delightfully drunken.

The context is familiar to students. Students are asked to formulate a hypothesis concerning the decrease of temperature before performing an experiment. Students measure the temperature of freshly brewed coffee every minute for 10 minutes. Only well-known quantities (time and temperature) are used in the experiment (P1) and it can be done with everyday material and takes only 10 minutes to perform (P2). After performing the experiment, students were asked to model the cooling using a function. No hint was given which kind of function is suitable. However, setting up a mathematical model is not to complicated (P3) since the temperature can be modelled using an exponential function of the form $f(x) = b \cdot a^x + c, b > 0, c > 0, 0 < a < 1$ with x as the time in minutes after brewing the coffee. The parameters can be calculated by using the data from the experiment: c can be estimated as the room temperature and b can be calculated as the difference between the temperature at the beginning of the experiment and the room temperature. One possibility to estimate a is to use two consecutive measured values, e.g. $a = \frac{\text{temp (1 min)} - b}{\text{temp (0 min)} - b}$. Students could use GeoGebra to plot graphs of functions and compare them with their data. There are several occasions for validation, since the decrease is not perfectly exponential. Furthermore, students sometimes forget to consider that the coffee cannot cool down under the room temperature, so that their first attempt to model the decrease is often not adequate. As a validation prompt (P4) serves the following subtask:

> Compare your function with your measurement-data. Does your function describe the data accurate enough? How could your model be improved?

Besides the CC task, a second modelling task “Stale Beer” (“SB” in the following) was used. In this task students were asked to model the decay of beer-froth and to evaluate the beer quality based on their results (Geisler, 2021). The SB task had the same structure as the CC task, in the sense that the same subtasks (e.g., the same validation prompt) were used. However, the SB task can be considered to be a little easier. The decay of beer-froth can be modelled using an exponential function of the type $f(x) = b \cdot a^x$ and therefore less parameters have to be computed. A detailed description of this task and the related experiment can be found in Geisler (2021).

Methods

Both modelling tasks were implemented in four upper secondary classes (all in year 10) from three different German schools. 71 students (age between 15 and 17 years, 54 % female) voluntarily participated in the study. Based on their last mathematics grades, the students represent a large variety of achievement levels reaching from grade 1 (very good) to grade 6 (not sufficient) (mean: 3.1 – sufficient). All classes had covered exponential functions as well as the calculation of the function’s parameters based on given values some month prior to the implementation of the tasks. Students were not informed that exponential functions could be used to solve the tasks. In every class, half of the students worked on the CC task whereas the other students solved the SB task during a 90 minutes lesson. Students performed the experiments in pairs of two. Students' answers to the validation prompt served as data in order to answer the research question. Since not all students answered all subtasks, data is available for 64 students. Furthermore, we conducted stimulated recall interviews with six students after working on the task to better understand students' validation ideas and ideas for model improvement. We chose students for the interviews based on their answers to the validation prompt, to achieve a certain variety of students' approaches. Accordingly, we used their answers to the validation prompt as a stimulus in the interviews. Interviews have been recorded and transcripted.

Students' answers have been analyzed using qualitative content analysis (Mayring, 2010). We used an inductive coding guide. All answers were first coded regarding students' judgement whether their model was adequate with respect to their data. Second, students' ideas to improve their models were coded. All answers have been coded independently by two raters and Cohen's kappa has been calculated indicating a very good interrater-reliability of $\kappa = 0.9$.

Results

Since both tasks are very similar in structure, we do not discuss the results divided by tasks. However, numbers concerning the different approaches related to the tasks are provided in brackets. Except of two (SB: 1, CC: 1), all students were able to set up an exponential function as a model for the decay of beer-froth respectively the decrease of the temperature of the coffee. However, most students working on the CC task did not consider the room temperature and set up functions of the form $f(x) = b \cdot a^x$ resulting in models that mostly only describe adequately the temperature during the first minutes of the experiment, as can be seen in the solutions in Figure 2.

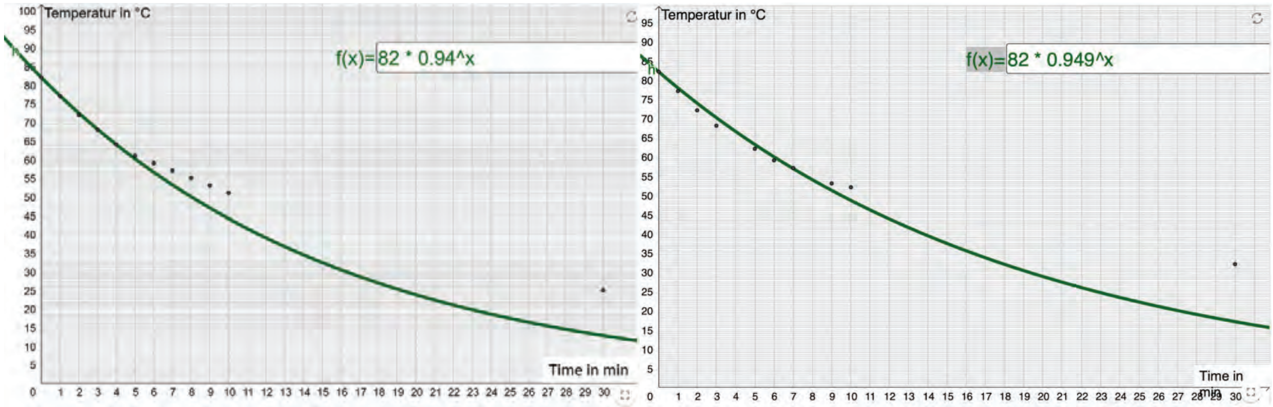


Figure 2: Screenshots of students' GeoGebra files with measured data and related functions – CC task

Model Validation

28 students (SB: 16, CC: 12) stated that they see no or minimal deviations between their model and data, indicating that they consider their model to be adequate (e.g., "*The function describes our measured values quite exact*", CC). In contrast, 25 students (SB: 13, CC: 12) see substantial deviations between their data and their model and therefore judge their model as not adequate (e.g., "*No, the function is decreasing more steep than our measured values*", SB). Two of these students wrote that they are satisfied with their model but that they consider their measured values to be "*very imprecise*". It seems that these students attribute deviations between model and data to their data instead of the specifications of the model. 11 students (SB: 6, CC: 5) gave a more sophisticated

evaluation, stating in which timespans the model is useful to describe their data (e.g., "*The measured values deviate a little bit from the function especially at the beginning. At the end the values match with the function*", SB). In these answers, first ideas concerning the limitations of models become apparent. These judgements are nearly equally distributed among the two tasks. Furthermore, students' judgements are very subjective, as can be seen in the solutions in Figure 2. While both models show similar shortcomings (good fit during first minutes but clear deviations later), the left model was considered to be not adequate by students and the right model was considered sufficient.

Approaches for Model Improvement

Five students did not answer the question concerning the improvement of their models. Moreover, ten students (SB: 4, CC: 6) explicitly stated that they have no concrete idea for improving the fit between model and data. The majority of the students (n = 39, SB: 22, CC: 17) only gave ideas to improve the experimental procedure instead of working on the model itself. Most of these ideas contain only simple improvements like taking more measurement values or measuring the values more precisely. However, two students argued that one should repeat the experiment several times with partly different materials ("*Take more values for a longer timespan for a more precise model, repeated experiments with different bottles of the same beer*", SB). Only ten students (SB: 4, CC: 6) wrote ideas for an alternative model for their data, like calculating the parameter "a" in another way (e.g., "*It would be helpful to calculate a for all measured values and to take the mean for it*", CC).

It seems that many students have no concrete idea how an alternative model based on their already measured values can be set up. Furthermore, some students seem to put more trust in their model than in their data. This impression is supported by students' answers in the stimulated recall-interviews. Out of the six students interviewed, only two (we call them student *A* and student *B*, both worked on the CC task) offered ideas how to improve the function. Student *A* offered the idea to use more than one function (in the sense of a composite function) in order to adequately model different parts of the data: "We recognized that beginning with a certain value it [the coffee temperature] remained constant. […] Perhaps one can split it into two functions: one for the first part and one for the constant part". Student *B* followed a different idea that involved using all measured values to set up a function:

Interviewer: You have thought about how one can improve your function and you wrote "one could calculate *a* for every single value and take the mean then." What do you mean by every single value?

Student *B*: We have values for every timepoint and we could calculate an *a* for every timepoint and calculate the mean of these *a*s.

Interviewer: And what would happen if you do so?

Student *B*: Then one value would lie exactly on the function and it would be consistent what lies over and under the graph.

Student *B* seems to be aware that not all measured values will lie exactly on a function and that it is desirable that a similar number of values lies over and under the function, that is the function does not systematically over- or underestimate the measured data.

All other students said that they have no idea how to further work on their functions. Two students argued that the model depends heavily on the experimental data and that the experiment should be improved accordingly. These students are aware that measuring new data will result in a slightly different function, as becomes apparent in the dialogue with student *C*, who worked on the SB task:

Interviewer: Do you have ideas how to improve your model if you already have the measured values? […]

Student *C*: So I think the analysis is really dependent on the experiment and so it is more relevant what values we gather from the experiment to have the right analysis.

Interviewer: Ok, so you say it depends on the data. What would you say would happen to your function if you gather other data, if you measure more often or more precisely?

Student *C*: I think nothing. It would be the same. It would just be more detailed and one could better see how the function…

Interviewer: So if you would have measured more precisely it would be the same function but everything would fit better?

Student *C*: So maybe there would be a different number [for the parameter] but basically it would be the same.

The last two interviewed students, one is student *D* who worked on the SB task, stated that improving the experiments will lead to the same function they already have:

Interviewer: You wrote that one could use a better measuring cylinder to better measure the change [of the beer-froth]. These are ideas that set in during experimentation. Do you have also ideas how to improve your function if you already have the values and you cannot change them? […]

Student *D*: I have no idea!

Interviewer: And if you would have a better measuring cylinder and you measured the values more precisely, do you think at the end you would have a function that fits better to the measured values?

Student *D*: I think we would have the same function then, but eventually the measured values would fit better to it.

Discussion

All in all, the results of the study at hand are similar to those of the pilot study of the MaMEx project (Geisler, 2021) and support the findings of Carrejo and Marschall (2007): Most students seem to attribute deviations between their model and the experimental data to measurement errors and primarily suggest to improve the experimental procedure instead of considering alternative models. This is even the case, if systematic shortcomings of the models exist. These results show that integrating experiments in lessons does not guarantee successful validation but experiments could be used as a starting point for fruitful discussions on validation in the classroom. Regarding students' statements in the interviews, two main problems can be identified: 1) Students seem to have not enough knowledge about exponential functions to reconsider their model and 2) there seems to be a lack of understanding concerning the relation of the reality (data) and the model (function) which has to be addressed in mathematics lessons. The latter problem might be due to the perception that the model is solely determined by the data and that gathering good data is the most important step in the whole process. This is remarkable since ones' model depends on both: the data used and the decisions made (e.g., type of function) in order to set up the model. However, in the answers of student *D* a real misconception concerning the relation between data and model comes apparent. Student D seems to see his function as the "right" model and rather independent from the actual data. It remains uncertain where this trust in his model comes from. One possibility could be that many students believe that mathematics is always precise and mathematics tasks have only one right answer. These beliefs are quite common among students and have been proven to influence their approaches to mathematics tasks (Schoenfeld, 1992). Measuring students' beliefs concerning the nature of mathematics prior to their work on modelling tasks with experiments might shed light on this hypothesis.

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Pre-service teachers' pedagogical content knowledge for teaching simulations and mathematical modelling with digital tools

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Dealing with simulation and modelling tasks with digital tools in mathematics lessons puts high demands on teachers when it comes to the preparation and implementation of lessons. These must be met in teacher education. To measure the pre-service mathematics teachers' professional knowledge for teaching simulations and mathematical modelling with digital tools, we propose a theory-based model and subsequently present items of an associated test instrument. Using a one-parameter Rasch model, we show that the underlying model can be confirmed empirically and discuss the potentials and limitations of the results.

Keywords: Mathematical modelling, simulations, digital tools, pedagogical content knowledge, measurement.

Introduction

Large-scale studies – such as Kunter et al. (2013) and Blömeke et al. (2014) – have shown how the pedagogical content knowledge of mathematics teachers can be described theoretically and empirically. Wess et al. (2021b) were able to use these conceptualizations and other research results from recent years to examine the pedagogical content knowledge of pre-service mathematics teachers specifically for mathematical modelling. Due to the increasing importance of digitalization in teaching and learning, it now seems sensible to integrate digital tools into this existing concept and to reinterpret the construct of Wess et al. (2021b). Based on this, the first results of the development of a test instrument that focuses on the professional knowledge for teaching simulations and mathematical modelling with digital tools will be presented in the following. The main question is: *To what extent can the pre-service teachers' pedagogical content knowledge for teaching simulations and mathematical modelling with digital tools be empirically captured as a construct?*

Theoretical Background

The term **mathematical modelling** describes the investigation of extra-mathematical processes and relations with mathematical tools. This includes the structuring of the extra-mathematical situation, the well-justified construction of a model to describe the reality, translation processes between the extra-mathematical and intra-mathematical world (in both directions), mathematical considerations, and the interpretation and validation of the results obtained (Niss et al., 2007). These modelling processes can sometimes be carried out several times as well.

If a mathematical model, which can be used for experimentation, of a reality-related situation is already available, **simulations** can also contribute to the exploration of reality. Simulations then enable dynamic, experiment-like processes that provide insights into the real system represented in the model (Greefrath & Siller, 2018). Simulations can also help to validate and optimise the mathematical model (Greefrath & Vorhölter, 2016).

Digital tools that are used in mathematics education represent a subcategory of digital media. While digital media are used, among other things, to communicate and document information, digital mathematics tools specifically support mathematical learning processes and the investigation of mathematical relations (Drijvers et al., 2016; Hillmayr et al., 2020). For example, they can generate and process larger amounts of data, visualize interrelationships dynamically, take over calculation processes, reduce complex function terms and offer new possibilities for information research (Greefrath et al., 2018). Thus, on the one hand their use therefore enables – especially in simulation and modelling processes – the treatment of previously inaccessible content. On the other hand, other focal points in mathematical considerations are now made possible (Greefrath & Siller, 2018). Examples of digital mathematics tools (we will speak of "digital tools" in the following) are Computer Algebra Systems, Dynamic Geometry Software, spreadsheets, and function plotters.

Several authors, such as Molina-Toro et al. (2019), investigated the integration of **digital tools in modelling processes**. They showed that digital tools can be used to support various processing phases and sub-competencies of mathematical modelling. Communication with the digital tool is essential here: on the one hand, mathematical descriptions must be translated into the language that the digital tool can understand and process and on the other hand, the results of the digital tool have to be translated back into the mathematical terms and operations. Greefrath et al. (2018) therefore extend the modelling cycle of Blum and Leiss (2007) by a technological world that takes into account the translation processes with the digital tool (Figure 1).

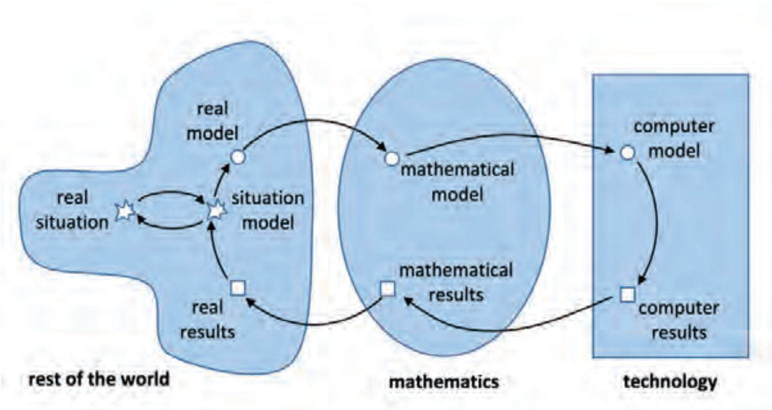


Figure 1: Extended modelling cycle (cf. Greefrath et al., 2018, p. 235)

Concrete functions of the digital tool can be implemented in different modelling phases: investigate, experimentalize, visualize, simulate, calculate, control (Greefrath et al., 2018). As mentioned above, mathematical simulation fits into reality-related contexts as an experiment-like process with the already existing mathematical model.

To investigate **professional knowledge for teaching mathematical modelling**, Wess et al. (2021b) developed a structural model of professional competence for teaching mathematical modelling. It serves as the initial basis of our test instrument and uses the model of Kunter et al. (2013) and research by Borromeo Ferri (2018). Therefore, the pedagogical content knowledge for teaching mathematical modelling includes a *theoretical dimension* (e.g. knowledge about modelling cycles as well as aims, perspectives and criteria for the use of modelling tasks), a *task dimension* (e.g. knowledge about solution processes, analyses and development of modelling tasks), a *diagnostic dimension* (e.g. recognition of modelling phases and difficulties in the modelling process) and an *instruction dimension* (e.g. knowledge about interventions during students' modelling processes) (cf. Borromeo Ferri, 2018; Wess et al., 2021b).

For the current test development to **measure the pedagogical content knowledge for teaching simulations and mathematical modelling with digital tools**, mathematical simulation is included in this structural model at the above-mentioned intersection with mathematical modelling. Additionally, the four teaching competencies are focused on the use of digital tools (cf. Figure 2).

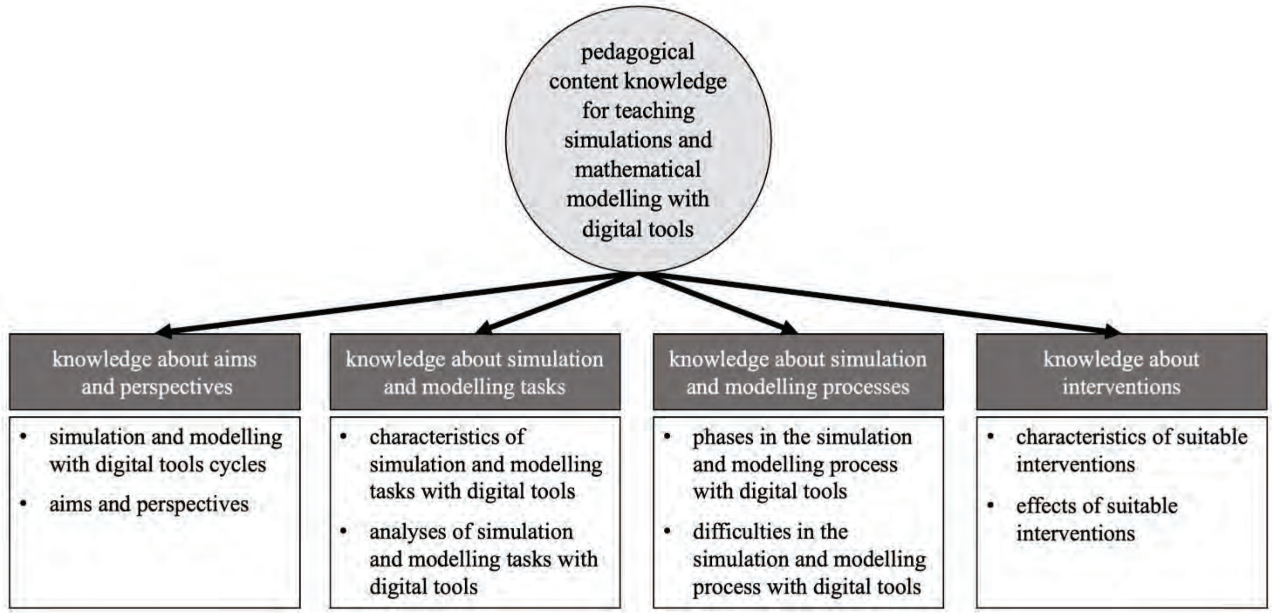


Figure 2: Pedagogical content knowledge for teaching simulations and mathematical modelling with digital tools (following Wess et al., 2021a)

Test construction

Based on the model shown in Figure 2, we developed 84 items in a deductive test construction to empirically (quantitatively) capture the construct described above in four dimensions. The preliminary test design was first qualitatively pre-piloted with experts on simulation, modelling, and digital tools ($N = 11$). Based on their feedback and edits, the content of the test draft was revised. We then presented the test draft to ten pre-service mathematics teachers at the University of Muenster and the University of Wuerzburg for further qualitative pre-piloting. Using think-aloud and verbal probing methods, items that were difficult to understand were identified and then revised or eliminated. Finally, a test draft with 79 closed items in the four theoretically derived dimensions *aims and perspectives* (13 items), *tasks* (10 items), *processes* (28 items) and *interventions* (28 items) was developed. As an example, we would first like to present one item each from the dimensions *aims and perspectives* (Figure 3) and *tasks* (Figure 4):

5.3	The use of digital tools …	
	… requires a standardized approach to mathematical modelling.	☐
	… in mathematical modelling is only possible in calculation.	☐
	… makes it possible to work on mathematical models with complex function terms.	☐
	… is not helpful in understanding the factual context.	☐

Figure 3: Example item of the dimension *aims and perspectives*

6.3	Modelling tasks with digital tools in pre-built configurations (e.g., dynamic worksheets) ...	true	false
	… prevent different ways of solving the problem.	☐	☐
	… enable a targeted reduction of task complexity.	☐	☐
	… enable students to use the digital tool independently right from the start.	☐	☐

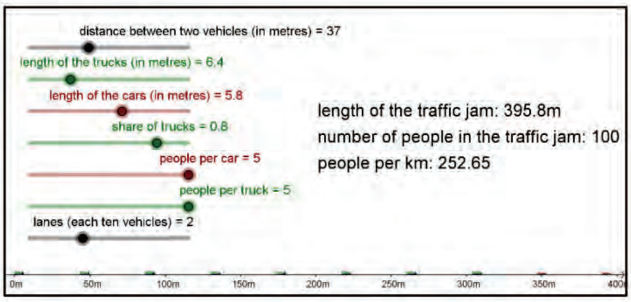
Figure 4: Example item of the dimension *tasks*

Following Wess et al. (2021b), the scales consist of multiple-choice and combined-true-false formats, which are to be evaluated dichotomously. Both items (as well as the following) were initially constructed in German and then translated into English for this paper.

The dimensions *processes* and *interventions* are captured with case-based text vignettes. The text vignettes each contain a simulation and/or modelling task and associated conversations between students in a concrete processing phase of the task with digital tools. The text vignette "Traffic Jam" serves as an example here (cf. Figure 5, task, and Figure 6, conversation).

7.4 Traffic Jam (9th grade)

At the beginning of the summer vacations, traffic jams often occur. Christina is stuck in a 20 km traffic jam for six hours. She thinks about how many people are in the traffic jam with her.
Estimate the number of people in the traffic jam.
Also use the GeoGebra applet on the right.



[The simulation shows a traffic jam with ten vehicles per lane. The students can make different assumptions for the traffic jam using the sliders. The simulation then calculates the length of the traffic jam, the number of people in the traffic jam and the number of people per kilometer in this traffic jam.]

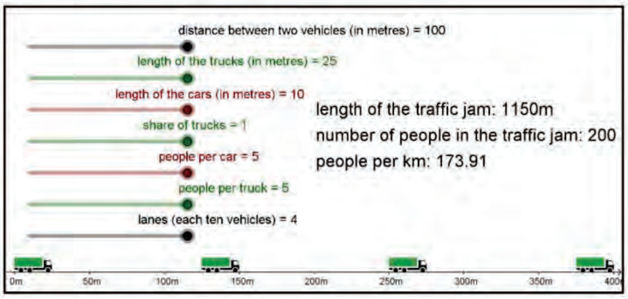
Figure 5: Task "Traffic Jam" (following Maaß & Gurlitt, 2011; Wess et al., 2021b)

STUDENT 1: First, we need to know how many cars and how many people are even in the traffic jam. Or rather in an average traffic jam. Otherwise we can't estimate.

STUDENT 2: Exactly. In the simulation, by the way, you can also adjust the trucks. There are fewer people in them, or at least they are longer than a normal car.

STUDENT 3: I'll set everything so that the traffic jam is as big as possible.

STUDENT 3: [Sets the maximum distance and the maximum vehicle length, among other things, on the sliders, see right].



STUDENT 3: Huh? I don't even get to 20 km. The traffic jam is always shorter in the simulation. We can't even solve the problem with that.

STUDENT 2: I have no idea how to do that.

Figure 6: Conversation of the students while solving the task "Traffic Jam"

Based on the task and conversation, the participants should subsequently diagnose the students' problem in the solving process (cf. Figure 7) and derive suitable interventions (cf. Figure 8) in these situations by answering the corresponding items.

7.4.2	Which function of the digital tools do the students **mainly** use in this situation? Please place <u>**one**</u> mark.	
	investigate	☐
	simulate	☐
	visualize	☐
	control	☐

Figure 7: Example item of the dimension *processes*

	Please mark whether each of the following interventions is suitable for the autonomy-oriented promotion of modelling or simulation competencies in this situation. Please place <u>**one**</u> mark for <u>**each**</u> intervention.	suitable	unsuitable	do not know
7.4.5	"Check that the simulation allows you to make statements about at least part of the problem."	☐	☐	☐

Figure 8: Example item of the dimension *interventions* (following Wess et al., 2021b)

Evaluation methods and results

To make the quantitative evaluation of the test design in the context of ongoing item revision and selection possible, the test was presented to a suitable sample under standardized conditions. The sample consisted of $N = 128$ pre-service mathematics teachers from the University of Muenster, the University of Wuerzburg and the University of Erlangen-Nuremberg.

The four dimensions were each scaled with a one-parameter Rasch model (cf. e.g., Rost, 2004). For the calculations, the software R with the packages TAM (Robitzsch et al., 2021) and eRm (Mair & Hatzinger, 2007) was used. Following PISA (OECD, 2012), items with a discrimination index under 0.2 were removed from the test. In line with Bond and Fox (2007), we only left items with adequate mean square fit (MNSQ) statistics in the test. Thus, items whose infit and outfit values were not between 0.8 and 1.2 were gradually eliminated. We made an exception for two items from the dimension *interventions*. The two items show an overfit that, however, is not significant at a level of five per cent. Since the two items are of great importance from a didactic point of view, they remain in the test nonetheless and only their phrasing is revised. In future evaluations, they are to be critically examined again.

After selection and revision, the test contains 54 items in the four dimensions. The one-dimensionality of the scales was tested globally with the help of Andersen tests (cf. e.g., Rost, 2004). According to Lienert and Raatz (1998), the EAP reliabilities of the individual dimensions are sufficient for group comparisons. The results of the tests are summarized in Table 1.

Table 1: Results of the analyses

<table>
<tr><th>Scale</th><th>Number of items</th><th>EAP reliability</th><th>Andersen test</th><th>MNSQ</th><th>Pt.-bis. corr.</th></tr>
<tr><td>Aims and persp.</td><td>9</td><td>.57</td><td>1</td><td rowspan="4">0.82* to 1.15
* Exception: Two items are closely below 0.8 (overfit).</td><td rowspan="4">> 0.22</td></tr>
<tr><td>Tasks</td><td>9</td><td>.56</td><td>.95</td></tr>
<tr><td>Processes</td><td>18</td><td>.62</td><td>.26</td></tr>
<tr><td>Interventions</td><td>18</td><td>.78</td><td>.16</td></tr>
</table>

Summary and Outlook

This article focused the extent to which the pre-service teachers' pedagogical content knowledge for teaching simulations and mathematical modelling with digital tools can be empirically captured as a construct. Based on the structural model of professional competence for teaching mathematical modelling (Wess et al., 2021b), items were constructed using a deductive test theory. It was found that – in the studied group – pedagogical content knowledge for teaching simulations and mathematical modelling with digital tools can be adequately captured as a construct using the developed test instrument. The data collected confirms the four scales *tasks*, *aims and perspectives*, *processes* and *interventions*. Nevertheless, the scales *tasks* and *aims and perspectives* need to be focused on separately in the further course of the investigations due to the comparatively poorer EAP reliabilities. It needs to be checked whether a multidimensional approach, which takes into account correlations between the latent traits, increases the EAP reliabilities.

Combined with the results obtained from the Andersen test and in the Mean Square Fit (MNSQ) statistics for the two scales, the developed test in its current form seems to enable the measuring of pedagogical content knowledge for teaching simulations and mathematical modelling with digital tools. At the same time, the results are to be confirmed again in cross-validation.

The promising results must be viewed – analogously to Wess et al. (2021a) – against the background that the dichotomous item construction has to allow for definitive true or false answers. Particularly in the field of reality-based tasks, this leads to an additional narrowing of an already very narrow construct, so that many items and text vignettes had to be excluded at the outset. In addition, the scalability and meaningfulness of the current test instrument have so far only been demonstrated for the participating universities. Although we have taken into account the representativeness of the sample according to objective parameters (e.g., study progress, subject combination, previous achievements if applicable) in our evaluation, differences in teacher education in the area of reality-based tasks cannot be ruled out. The question of generalizing the present results therefore remains open for the time being.

In addition to the previous results, pedagogical content knowledge, beliefs and self-efficacy will now also be evaluated and presented in a structural equation model. The complete test instrument will then be used in the coming semesters in courses at the University of Muenster and the University of Wuerzburg in a pretest-posttest control group design.

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An observation instrument to analyze the collaboration between students while solving mathematical modelling problems

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Collaborative learning is a well-established approach to elicit reasoning. The ability to solve mathematical modelling problems depends much on the ability to reason, because mathematical modelling problems are usually presented to students through texts. Decisions have to be made at all stages of the mathematical modelling cycle, for instance on assumptions, simplifications and feasibility of the model. In this paper we present an observation instrument to study collaborative modelling. There are three core components: collaborative learning, mathematical modelling and the language that students use while working together.

Keywords: Mathematical modelling, collaborative learning, observation, language proficiency.

Introduction

In the Netherlands, mathematical modelling has become a small but obligatory part of the upper secondary curriculum. However, it is missing in the greater part of lower secondary education. To improve the vertical coherence in the curriculum, our research focuses on modelling assignments in lower secondary education.

The purpose of our larger research is to determine whether the open-ended group assignments we developed promote collaborative mathematical modelling. In this study, we aim to construct an observation instrument. The research question we address is: How do we examine the quality of students' collaborative learning in tasks that aim at mathematical modelling?

Theoretical framework

Many studies have shown that collaborative learning has a positive effect on students' mathematics learning (e.g. Dekker & Elshout-Mohr, 1998, 2004; Pijls, 2007; Webb, 2009; Webb et al., 2020; Yackel et al.,1991). Discussions in small groups can also promote the development of modelling competencies (Maaβ, 2006; Galbraith & Clatworthy, 1990). We discuss three features of the interaction between students, that - based upon the literature - can be considered relevant when aiming at mathematical modelling with group tasks: (1) collaborative learning, (2) mathematical modelling and (3) the language that students use while working collaboratively on the modelling task.

Collaborative learning: Various researchers have shown that collaborative learning in heterogeneous groups raises the level of abstract reasoning as suggested by van Hiele and Freudenthal when the task enhances students to verbalize their understanding, to explain their thinking to one another and to criticize one another's way of thinking (Calor et al., 2019; Dekker & Elshout-Mohr, 1998; Palha, 2013; Pijls, 2007; Yackel et al., 1991).

Dekker and Elshout-Mohr (1998), for example, defined four key activities in their Process Model for structuring small group discussions of students when working at a mathematical task. Initially the students are working individually on the same mathematical task. Their works will be different. After some time, they compare the findings: they show their own work, they explain their own work, they justify their own work and eventually reconstruct their own work.

Collaborative learning activities that have been identified as key activities for mathematical level raising, such as explaining and justifying, have also be mentioned by scholars who focus on features of productive collaborative learning (Mercer, 2006; van Boxtel et al., 2000; van der Linden et al., 2000). According to Mercer (2006) three types of talk can be discerned: disputational talk, cumulative talk and exploratory talk. Disputational talk is characterized by disagreement and individualized decision making, cumulative talk takes place when students share knowledge but contributions are uncritically accepted by the group. In exploratory talk students engage critically and constructively with each other's ideas.

Mathematical modelling: In the field of mathematical modelling various definitions of the modelling cycle are found. The modelling cycle starts with a situation or problem in the real world. Thus is it important to understand the situation. Blum and Leiβ (2005) define this understanding of the original situation as the situation model. Then, the given situation has to be simplified, structured and made more precise, which results in a real model (Blum & Leiβ, 2005). In the first step of the modelling cycle it is essential to become aware of the meaning of the problem. Furthermore, modelling encompasses various activities that are carried out consecutively, including mathematising, working mathematically, interpreting, and validating (Blum & Leiβ, 2005). Maaβ (2006) identifies specific sub-competencies related to the mathematical modelling process and argues that modelling competencies include more than the steps of a modelling process. Assumption making is one of these sub-competencies to understand the real problem and to set up a model based on reality.

The use of language: The language used at school often forms an obstacle to learning mathematics (Van Eerde & Hajer, 2009). Plath and Leiβ (2018) point out that a low language proficiency may result in comprehension problems and a low performance on modelling tasks.

Mathematical modelling problems are usually presented to students through texts in context-rich assignments. In the context of modelling tasks, it is important to examine what language students use while modelling. Cummins (2000) distinguishes between Basic Interpersonal Communicative Skills (BICS) for communication every day and Cognitive Academic Language Proficiency (CALP) for communication in education. During collaboration, students may explain difficult words to each other, and use mathematical terminology during their collaborative work. Webb (1991) classifies this as the explainer: the learner who translates unknown vocabulary into language that is known to other students. Especially for modelling tasks is it important to understand the situation of the given problem (Blum & Leiβ, 2005).

Method

In this study, a series of five different modelling tasks for the domains algebra and geometry have been developed for grade eight students (age 13-14). In the Netherlands, grade eight is part of lower secondary education. The modelling tasks were developed according to design principles that we derived from Galbraith (2006) and Geiger et al. (2021) and literature on collaborative learning.

For collaborative learning, a task should be complex and rely on multiple skills (van Boxtel et al., 2000; Cohen, 1994). Furthermore, the designed modelling tasks are open-ended problems that require making the necessary assumptions, are linked to the real world and motivate students (Galbraith, 2006; Geiger et al., 2021). The tasks were improved using feedback from educational experts, mathematics teachers and a pilot with grade eight students.

In total, five secondary schools with ten classes and seven mathematics teachers, located in an urban environment participated in this research. The collaborative learning groups consisted of three students. Each group worked on one task during one lesson. Three or four randomly selected groups in each class were video-taped and audio-recorded, and the written group products of all groups were collected. In each class, the tasks were introduced with the same introduction and brief information about mathematical modelling.

To analyze students' verbal interaction while collaborating on mathematical modelling tasks, we need a valid and reliable observation instrument. This observation instrument was designed in three steps: (I) development of the instrument based on theory about collaborative learning and mathematical modelling and recordings of three randomly selected groups; (II) a pilot study in which we used six randomly selected video-recordings of group conversations to further operationalize the categories in the instrument and to investigate the reliability of the instrument; (III) adjustment of the instrument.

Results

First design of the observation instrument: Recordings from three randomly chosen groups were selected to design a first version of the observation instrument. A variety of two tasks was included. We defined three main categories of student interaction while working collaboratively on mathematical modelling tasks, that can be considered relevant: (1) collaborative learning, (2) mathematical modelling and (3) the language that students use while working collaboratively on the modelling task. In addition, to improve the designed open-ended group assignments, it is important to examine questions students ask the teacher about the modelling task. This additional category is named questioning. Furthermore, for each subcategory, we added space to note comments.

Collaborative learning: It is important to discern whether students work primarily individually or together at our tasks. Therefore, the first subcategory in the main category collaborative learning is collaboration. The second subcategory is critical considerations focused on the solution strategies (critical considerations and strategies). The next four subcategories are the four key activities of Dekker and Elshout-Mohr's (1998) Process Model. The last sub-category is the type of discussion. According to Mercer (2006) three types of talk can be discerned: disputation talk, cumulative talk and exploratory talk.

Mathematical modelling: Mathematical modelling encompasses various activities that are carried out consecutively. For these activities we make use of the modelling cycle of Blum and Leiβ (2005). The steps in the modelling cycle are sequentially added as subcategories to the main category mathematical modelling of the observation instrument. In the first step of the modelling cycle it is essential to get aware of the meaning of the problem. Therefore, we first focus on whether students have understood the problem by observing whether students clarify the purpose of the modelling task (clarifying). In the second subcategory we observe if students discuss how to tackle the problem (addressing). In the third subcategory, we observe whether students understand the problem and if they transform the problem for reasoning (transforming). The fourth subcategory, we observe whether students simplify or structure the problem (simplifying). In the next subcategories we observe the assumption making, mathematizing, working mathematically, the way students are solving the problem (solving strategies), interpretation of the solution (interpreting) and whether students validate (in between or at the end) throughout the modelling process (validating).

The use of language: In the first subcategory we observe the use of everyday language (BICS). To find out whether the designed tasks contain difficult words for the students, we observe whether students explain certain words to each other while collaborating on the modelling task. In the second subcategory we observe the extent to which students use mathematical terms (CALP). To support coding, a list has been added with the most common mathematical concepts for the designed modelling tasks. In the third subcategory we observe whether students explain the mathematical concepts to each other (CALP to BICS). The last subcategory focuses on students' use of modelling language. We investigate whether students formulate their solution in the context of the modelling task (formulating) and whether they use specific modelling language in the solution process.

Questions about the task: The last category is added to examine if the students are asking the teacher a question and to write down the question(s) asked (questioning).

Results of the pilot study: Six randomly chosen video-recordings were coded by two researchers (the first two authors). The percentage of agreement between the researchers are shown in Diagram 1. For the main category collaborative learning the subcategories with the least agreements are: explaining their own work, justifying their own work and reconstructing their own work. For the main category mathematical modelling the subcategories with the least agreements are: clarification of the purpose, assumption making, mathematization and validation. For the main category use of language the agreement is acceptable. This is also for the additional category: questioning.

Revision: As a result of these findings, the categories with a low percentage of agreement, were revised. Clarifications have been added to the key activities. For example, what is meant with to show, to explain, to justify and to reconstruct. The fourth category, with subcategory to explain, of the observation instrument is shown in Table 1.

Table 1: Fourth subcategory of the observation instrument for main category collaborative learning

Collaborative learning	☒	Notes
4. To Explain Explaining or clarifying the way of working or thinking		
A. No student explains his way of working or thinking to the rest of the group		
B. One student explains his way of working or thinking to the rest of the group		
C. Several students explain their way of working or thinking to the rest of the group		

An explanation has also been added to the main category mathematical modelling, subcategories making assumptions and validating. In the main category collaborative learning, critical considerations focused on solution strategies have been adapted into two subcategories. Critical considerations students make with each other while collaborating is still a subcategory of collaborative learning. Solution strategies have been moved to the category mathematical modelling, because it fits more in the processes for solving modelling problems. With this category we examine whether students apply one or more strategies to solve the modelling problem.

We also adjusted the sequence of the subcategories in the main category mathematical modelling. In the revised instrument, we started with the subcategory clarification of the purpose of the task, followed by addressing the problem, understanding and simplifying the problem, and assumptions making. After that, the findings of a mathematical model or mathematization were noted, followed by mathematical solution, interpretation and validation. As discussed above, another subcategory had to be added, namely the solution strategies. As a result, the revised observation instrument included ten subcategories for mathematical modelling. By choosing an order that best suits how students work through the task, it becomes easier to observe.

No adjustments were made for the components: language proficiency and question to teacher.

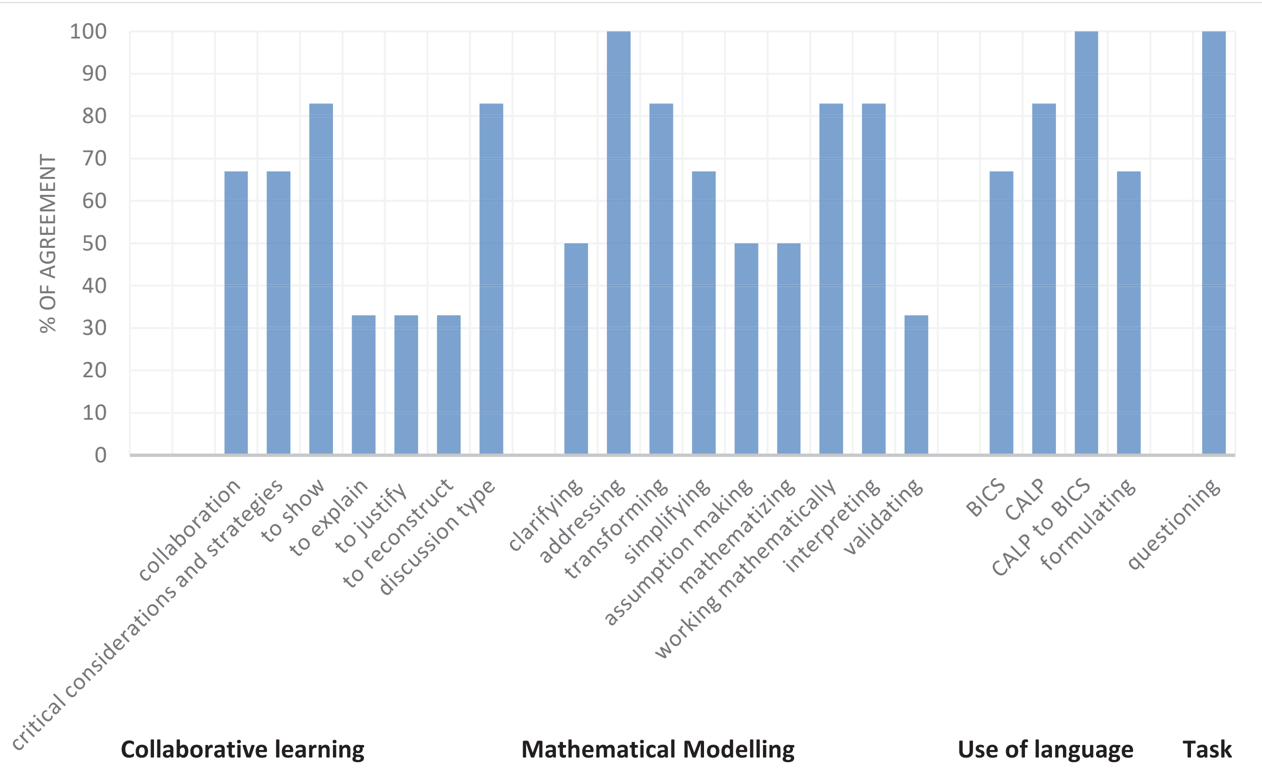

Diagram 1: Percentages of agreement in each subcategory for the pilot with the observation instrument (six collaborative learning dialogues)

Conclusion and discussion

In order to evaluate the quality of students' conversation, we have developed an observation instrument with which we can analyze the students collaborative learning while solving mathematical modelling problems. We observe the data on basis of three main categories. The first category is collaborative learning of students. We are using the Process Model (Dekker & Elshout-Mohr, 1998) to observe how students work together to increase their mathematical understanding. The second category is mathematical modelling, using the phases of the modelling cycle (Blum & Leiβ, 2005). The last category is the language that students use while working collaboratively on the modelling task. We observe the way students explain difficult words to each other, the use of language related to mathematics and modelling.

For the main category collaborative learning, the least agreement has been reached for the key activities of Dekker and Elshout-Mohr's (1998) Process Model. One reason for this could be that these key activities are applied by Dekker and Elshout-Mohr to students who initially work individually on the same task and then compare it with each other. While the students in our study work together on the tasks throughout the modelling process. For the main category mathematical modelling, subcategories have also been identified in which a low agreement has been reached. These subcategories, have been supplemented with an explanation that has been added or adjusted. Therefore, the revised observation instrument should be retested.

Successful solving of a mathematical modelling problems requires modelling competencies (Maaβ, 2006), but also metacognitive strategies in solving complex modelling problems in groups (Vorhölter, 2019). In the instrument we focus on the students' multiple solution strategies, but we do not observe whether students are (collaboratively) engaged in task orientation or planning. It is difficult to include that, because we want to use the observation instrument to know whether the developed modelling tasks are suitable. If there is a lot of regulation during collaboration, it can mean that the task is too open or complex, but it can also mean that students discuss this well. In addition, the developed assignments were performed in groups without teacher guidance and this can lead to confusion of students' thinking (Goos et al., 2002). A more qualitative analysis would then be necessary to gain more insight into this. Follow-up research could focus on how these metacognitive activities and teacher guidance could be included in the observation instrument.

In our next study, we will apply this observation instrument to investigate the quality of student interaction in order to evaluate the suitability of the group tasks we developed in order to develop students' mathematical modelling competencies.

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Professional content knowledge of mathematical modelling of pre-service teachers: First considerations for a new test instrument

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Modelling competence is a central concept in research on teaching and learning mathematical modelling. Numerous studies support the theoretical foundation of modelling competence as well as the existence of tested and already modified test instruments for different age levels. In addition to other aspects of professional competence, such as pedagogical content knowledge, pre-service teachers must also possess modelling competence themselves. Even though test instruments for modelling competence already exist, the aspect of professional content knowledge of mathematical modelling as a competence facet of teachers has received less attention thus far. In this theoretically oriented paper, first theoretical considerations for the development of such a test instrument are presented on the one hand, and on the other hand, ideas for further development are discussed based on sample items.

Keywords: Professional competences, content knowledge, test instrument, mathematical modelling.

Introduction

The importance of pedagogical content knowledge in mathematical modelling teacher education has already been highlighted as significant, but only more systematically in recent research. Therefore, a focus will now also be placed on the content knowledge of modelling. In this paper, we mainly present theoretical considerations regarding the content knowledge of modelling in the overall context of teacher education on modelling. In addition, we present examples of concrete items. Accordingly, we would like to address the challenge of developing a test instrument, which will differ from already existing tests in the still pending development process.

Therefore, we first review the following theoretical foundations, which form a theoretical framework for us as a basis for the systematic construction of items. This concerns professional competence in the models of Kunter et al. (2013) and Shulman (1986). Especially regarding Krauss et al. (2013), we derive our considerations of levels of mathematical expertise, which we concretise with respect to calculus, with which we start as a mathematical subject area with item construction. Relevant aspects of modelling sub-competencies follow, as well as a discussion of existing test instruments. Finally, the item examples illustrate first considerations and show the challenge of construction. The main goal of this paper is to demonstrate we need a new test instrument for the content knowledge of mathematical modelling in teacher education, including which aspects of test construction need to be considered.

Professional competence

The professional competencies of mathematics teachers and pre-service mathematics teachers have been extensively studied (e.g. Blömeke & Delaney, 2012; Kunter et al., 2013). The concepts of professional competence developed in these studies share key commonalities. In particular, professional knowledge is seen as consisting of different knowledge domains: content knowledge, pedagogical content knowledge and pedagogical–psychological knowledge (Shulman, 1986). In addition to these cognitively oriented knowledge dimensions, affective aspects are also considered. Further conceptions of pedagogical content knowledge for mathematics teachers, some of which also include content knowledge and pedagogical–psychological knowledge, are relevant (Depaepe et al., 2013). The importance of teachers' content knowledge is highlighted in various studies (Blömeke & Delaney, 2012). The high correlation of content knowledge and pedagogical content knowledge is well known (Krauss et al., 2008; Yang et al., 2021), highlighting the importance of the content knowledge of pre-service teachers in particular.

Levels of mathematical content knowledge

Mathematical content knowledge can be described using different models. Based on Shulman (1986), Krauss et al. (2013) proposed four levels to describe mathematical content knowledge. They distinguish everyday mathematical knowledge, the mastery of school-level mathematical knowledge, a deep understanding of the content of the secondary school mathematics curriculum and university-level knowledge (Krauss et al., 2013, p. 155). This classification begins with a first level of mathematical content knowledge that all adults must possess. The second level is school knowledge. At this level, we consider competence to use mathematics in the context of the knowledge usually taught at school. This *school mathematical competence* goes beyond everyday mathematical competence and includes, for example the competences required to complete a task in the Abitur examination (KMK, 2012). A third level describes the mathematical content knowledge required for a deeper understanding of the subject content at the secondary level. We refer to this level as *in-depth school mathematical competence*. This includes elementary mathematics from a higher standpoint (Klein, 2016) as taught at university. The fourth level can be called *university competence.* This includes mathematical knowledge taught at university with virtually no connection to the school curriculum, for example algebraic number theory (Krauss et al., 2013). Depending on the course of study, *in-depth school mathematics competency* or university mathematics competency are achieved at university in a teaching degree with mathematics as a subject. Content knowledge, as conceptualised in COACTIV, lies between the second and the third level, as shown in the published example item 'Is $2^{1024} - 1$ a prime number?' (Krauss et al., 2013, p. 152).

Modelling cycle and sub-competencies of modelling

Mathematical modelling processes can be illustrated by a modelling cycle that depicts the sub-processes or sub-competencies of simplifying, mathematising, interpreting and validating. Former research describes many cycles that vary mainly in the number of individual sub-processes (Borromeo Ferri, 2006). The modelling cycles from applied mathematics (Pollak, 1977) describe the processes deterministically in three or four phases. In didactically oriented discussion, there are four-phase, but also six- and seven-phase cycles (Blum & Leiß, 2007) or extended cycles concerning the use of technologies (Greefrath et al., 2018). Regardless of the number of sub-processes, modelling competence is required. In this context, modelling competence is understood as the ability to identify a real-world problem in a given situation, translate it into mathematics and interpret and validate the solution to the corresponding mathematical problem about the given situation (Niss et al., 2007). *Global modelling competence* refers to the entire modelling process and to general competences, such as a structured and goal-oriented approach to tasks, reasoned argumentation and independent reflection on the modelling process (Kaiser & Brand, 2015). The *sub-competencies of mathematical modelling*, however, refer to the sub-processes in the modelling cycle mentioned above or identified in the various cycles (Maaß, 2006).

The assessment of students' modelling competencies using tests has already been empirically demonstrated in many studies for different age groups (Haines et al., 2001; Hankeln et al., 2019; Kaiser & Brand, 2015). Two basic principles can be distinguished. On the one hand, there are items that focus on the sub-processes of modelling (atomistic approach) and, on the other hand, items that require the complete pass through of a modelling cycle (i.e. a holistic approach; Blomhøj & Jensen, 2003). In our test construction, we prefer an atomistic approach and limit ourselves to the sub-competencies simplify, mathematise, interpret and validate.

Mathematical modelling and analysis

In his characterisation of applied mathematics and modelling, Pollak (1977) attributed early special importance to analysis. To work on a modelling task, a knowledge of mathematical content is required in addition to process-related competences, which are described within the framework of the German educational standards for the Abitur examination. The central subject here is calculus (KMK, 2012).

Textbooks often contain examples from calculus as modelling tasks, for example for modelling growth processes. These tasks are particularly suitable for students because of their accessibility through a reference to real life and use of a well-known mathematical topic. With the help of differential calculus, many real or application situations can be modelled, such as topics in the context of traffic (Siller, 2013). Simultaneously, calculus is a central subject area in the German Abitur examination and in mathematics teacher education; it is receiving special attention in research (Rach & Heinze, 2017).

Development of a test instrument

Previous test instruments on university students' modelling-specific mathematical content knowledge use either rather clear Level 2 mathematical content knowledge for modelling, that is the school level (Yang et al., 2021), or Level 3 to Level 4, that is university knowledge or knowledge clearly beyond the secondary level (Czocher et al., 2021; Haines et al., 2001). While Level 2 may seem too low for pre-service teachers in higher semesters, Levels 3 to 4 may overwhelm some pre-service teachers. Sample items from the test of content knowledge from COACTIV could be used as a guide for items on modelling specific content knowledge between Levels 2 and 3.

Based on the previous considerations, we design an *atomistic* test instrument for the modelling-specific content knowledge of pre-service teachers with subject-specific content at *Levels 2 to 3* from the subject area of *calculus* and the focus on one of the four *selected sub-competencies*: simplify, mathematise, interpret and validate. Four example items will illustrate this direction in the following.

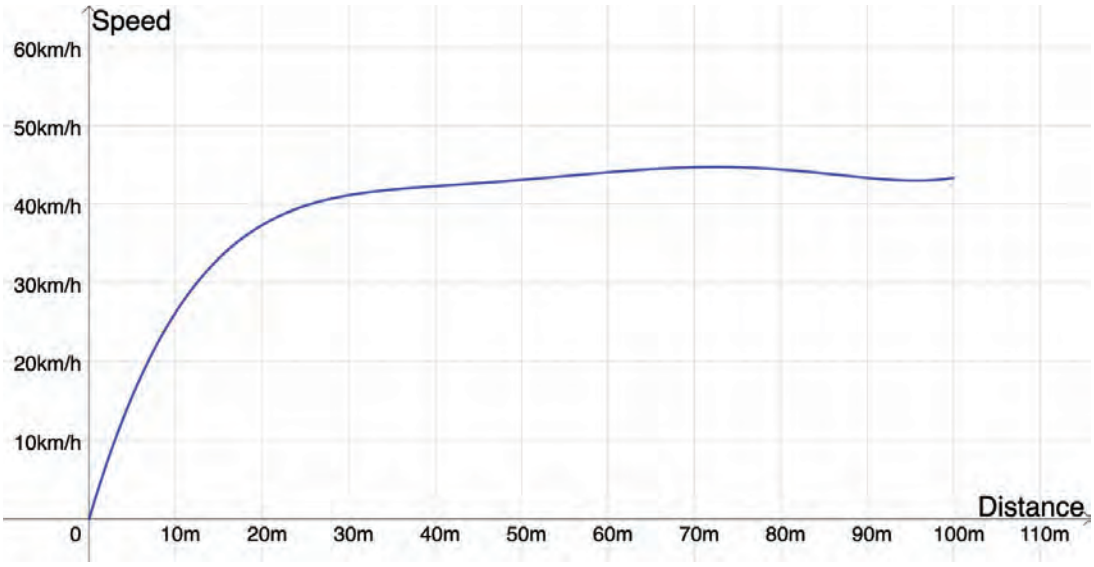
Example item 1 (mathematising)

Spruces represent an important timber species in Germany. The temporal development of the thickness of spruces is modelled by a function d. The planted seedling has a diameter of 0.04 m. After 160 years, a spruce has typically reached a diameter of 0.96 m. Which of the following mathematical models best fits the problem described?

$d(t) = \frac{1}{1+e^{-0,04\,(t-80)}}$	$d(t) = 0,00575\,t + 0.04$	$d(t) = 0,04\,e^{\frac{1}{50}t}$	$d(t) = \frac{1}{e^{-0,04\,(t-80)}-1} + \frac{t}{80}$
●	●	●	●

Example item 2 (interpreting)

In 2009, Usain Bolt set a new world record over 100m running with a time of 9.58s. In the figure, the course of his speed during the world record race is approximated.



Describe the progression of Bolt's speed during the race, considering the real-world context of the record run.

Example item 3 (interpreting)

Tim and his friends are standing on a federal highway in the slow-moving traffic that has seemingly formed 'out of nowhere'. Caro is annoyed: 'These damn 70 zones! If they would lift the speed limit here, then everyone could drive faster, there wouldn't be so many vehicles piling up, and we'd get there sooner!'

Tim has come up with the following 'half-speed rule': $d = \frac{v}{2}$:

v = speed
l = vehicle length
d = distance between two vehicles
S = route of the convoy
D = vehicle throughput per hour

$$D = \frac{S}{l+d} = \frac{1000 \cdot v}{l+d} = \frac{1000 \cdot v}{6+d} = \frac{1000 \cdot v}{6+\frac{v}{2}}$$

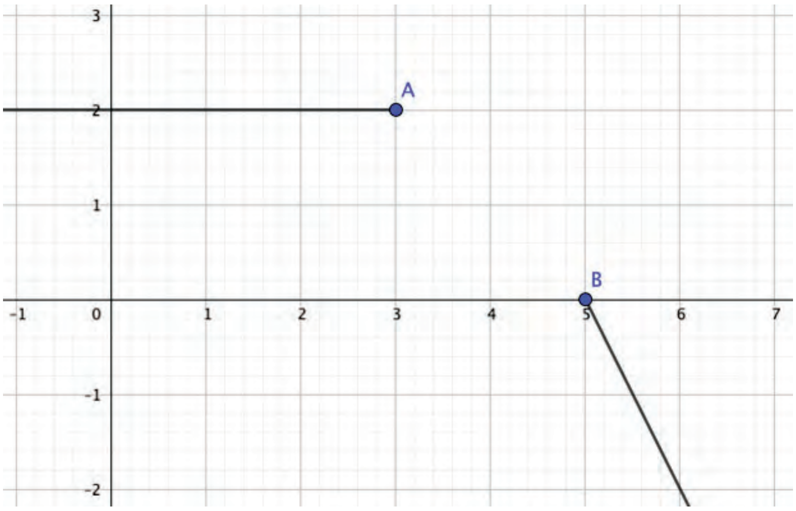
Tim achieves an approximate vehicle throughput of 1,600 vehicles per hour.

What did Tim calculate with his result?

a) The throughput grows infinitely with increasing speed.
b) The risk of rear-end collisions increases due to a guideline speed of 50km/h.
c) The throughput grows limited with increasing speed.
d) The maximum vehicle throughput is only achieved at a higher speed.

Example item 4 (validating)

The shape of a road between two existing roads should be modelled mathematically. The roads can be presented in the coordinate system as follows.



The missing piece of the road between A and B should be modelled mathematically. The model chosen was the appropriate arc of the function $f(x) = \sqrt{4-(x-3)^2}$. Explain why this result would not be used by engineers in reality.

Discussion and Conclusion

Previous research shows that there is a need for a special test instrument on professional content knowledge in mathematical modelling, which is, on the one hand, more demanding in content than average modelling tasks for school students and, on the other hand, below the level of university students who study mathematics as a major (cf. tests by Czocher et al., 2021; Haines et al., 2001). Furthermore, it is desirable that sub-competencies of modelling are considered atomistical, so a separate consideration of sub-competencies becomes possible. In this case, there is already experience with modelling tests for students (e.g. Hankeln et al., 2019; Kaiser & Brand, 2015), whereas in Haines et al. (2001), there is a focus on the first steps of the modelling cycle and no aim at a distinction between sub-processes. Another aspect concerns the common mathematical content of teacher education. Here, several items from the well-established test instrument of Haines et al. (2001) seem less suitable, as they mainly target linear optimisation. Therefore, we focused on calculus as a well-known subject area from upper secondary education and developed items for (pre-service) teachers that go beyond the content of lower secondary education, such as in Yang et al. (2021).

Example item 1 refers to finding a suitable mathematical model in the context of exponential functions. Hence, it is an important sub-area of calculus that is also covered in school. Not all the models given (e.g. logistic growth) are typically school subjects, but go beyond this. Therefore, this test item is to be placed between Levels 2 and 3 of mathematical knowledge. Basic experience with (plant) growth is sufficient to identify the appropriate mathematical model. Based on the problem, the participants can consider which type of growth is appropriate for this problem. The possible answers include linear growth, exponential growth, and logistic growth. A backward validation by inserting values is not necessarily goal-oriented because the terms available for selection correctly represent the boundary points in each case. The mathematical properties of the function must therefore be recognised from the properties of the real problem. The item, then, targets mathematising. For an objective evaluation, a multiple-choice task is recommended here. The task format is like well-known tasks by Haines (2001), but the level of difficulty is adapted to the target group.

Example item 2 describes a world record race using a graph of a function. This graph represents speed and needs to be interpreted. This belongs to the subject area of calculus and can be assigned to secondary level II. Like example 3, it belongs to interpreting, because the results of a mathematical model need to relate to the respective situation. In example item 3, the mathematical content goes beyond the school material. Example item 4 addresses validation. Students must critically examine the mathematical model used (Item 4) in the context of the real-world situation. Item 4 addresses a familiar high school context with a functional equation not usually used in that context. Further additional items for simplification can be taken from the test by Haines et al. (2001), if necessary. A broad coverage of sub-competencies seems important to us to obtain a valid measurement instrument that considers all steps of the modelling cycle for pre-service teachers.

All items, therefore, can be assigned to the subject area of calculus and use school material from upper secondary school or surpass it. The mathematical requirements, then, are implicit. It should be discussed whether explicit items on working mathematically should also be included. In various test instruments for students (Hankeln et al., 2019), this has not been done. The reason in this case was that the processes of modelling were the focus of the study and the test length should remain within certain limits. The question of the test length is also to be discussed here. Other test instruments (Kaiser & Brand, 2015), however, consider holistic tasks and implicitly include mathematical work. However, we also consider it attractive for teacher educators to use items precisely for certain sub-competencies, so that a very focused diagnosis is possible.

Some of the items developed thus far can be objectively coded using multiple choice items, while others are evaluated using criterion-guided coding manuals in the partial credit model. In addition to the items presented here, further items for a complete test of professional content knowledge for modelling need to be developed and empirically tested. A prerequisite is a discussion of the competencies to be tested (atomistic or holistic), level of difficulty, item format and mathematic subject area.

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Modelling while problem posing – A case study of preservice teachers

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Because problem posing might enhance activities that are necessary for solving real-world problems, it has the potential to foster modelling. However, systematic research on the connection between problem posing and modelling is largely missing. Therefore, in the present study, we investigated (1) the modelling activities that take place when posing problems that are based on given real-world situations and (2) the extent to which modelling activities occur with different problem-posing activities. To address these questions, we asked seven preservice teachers to first pose a problem based on a given real-world situation and then to solve their self-generated problem. A qualitative content analysis revealed that modelling activities that are close to the real-world situation (e.g., understanding, simplifying, and structuring the given pieces of information) are involved in problem posing. This result indicates that problem posing has the potential to foster mathematical modelling.

Keywords: Problem posing, modelling, teaching approach, preservice teachers.

Introduction

Modelling is one of the key competencies of mathematical learning as it enables students to understand their environment with the help of mathematics (Niss & Blum, 2020). However, modelling is a demanding process for both students and teachers alike (Schukajlow et al., 2018). Therefore, beneficial approaches for fostering mathematical modelling are needed (Schukajlow et al., 2018). Problem solving research has indicated that problem posing has a positive influence on problem solving because problem posing has been claimed to trigger important problem solving processes, for example, analyzing the given situation in an in-depth manner (Cai & Leikin, 2020). As modelling can be characterized as real-world problem solving and begins with a given real-world situation that has to be understood, simplified, and structured, it is possible that problem posing based on given real-world situations (i.e., modelling-related problem posing) is beneficial for fostering mathematical modelling. Surprisingly, there is only a little research on modelling through problem posing. To investigate the potential of problem posing for modelling, we aimed to analyze the connection between problem posing and modelling from a cognitive perspective in this study (Schukajlow et al., accepted).

Theoretical Background

Mathematical Modelling

Mathematical modelling can be characterized by a demanding process of translating information between the real world and the mathematical world with the goal of solving a real-world problem with the help of mathematics (Niss & Blum, 2020). The modelling process can be described as idealized in a circular theoretical model consisting of various activities (Blum & Leiß, 2007). The process begins with activities that are located in the real world. The given real-world situation, which is often presented as a textual description, first has to be *understood* by reading the text and supplementing the information with experience, thus ending up in the construction of an individual situation model. In the next step, the situation model has to be transformed into the real model by *simplifying and structuring* the given situation. Through *mathematization*, the translation from the real world into the mathematical world begins. The real model is translated into a mathematical model or problem. By *working mathematically*, a mathematical result can be calculated, and then it has to be *interpreted* with respect how it applies to the real world so that a real result is achieved. Finally, the real result must be *validated* with respect to whether the existing models and results are appropriate for describing the given situation.

Each of the activities described above can be demanding for students and may represent a potential barrier in the solution process (Blum & Leiß, 2007; Schukajlow et al., 2018). Especially the activities located in the real world (understanding, simplifying, and structuring) are challenging as real-world aspects are often neglected (Krawitz et al., 2018). Therefore, there is a need for approaches that can help overcome these barriers.

Problem Posing

In recent years, problem posing has become an important topic in mathematics education as it has great potential for both the teaching and learning of mathematics (Cai & Leikin, 2020). Problem posing includes a variety of processes. In addition to the generation of new problems, it also comprises the reformulation of given problems that can take place before, during, or after problem solving (Silver, 1994). Further, different stimuli can initiate problem posing. In addition to the categorization of these stimuli on the basis of the structure of the given situation (Stoyanova, 1997), the stimuli can also be differentiated on the basis of their connection to reality. With regard to the differentiation of real-world problems (Blum & Niss, 1991), problem posing can be initiated by situations with and without a connection to reality. In our study, we define problem posing as the generation of new problems on the basis of given real-world situations before problem solving and refer to this type of problem in the following as modelling-related problem posing. An example of a given real-world situation is depictured in Figure 1. An exemplary question that can be posed based on the "Cable Car" situation is a question about the length of cable needed for the new cable car.

Cable Car

For more than 90 years, the *Nebelhorn* cable car has taken numerous guests up into the heights. Now it can go into well-earned retirement. Beginning in the summer of 2021, a new cable car will transport enthusiastic outdoor fans to the Nebelhorn mountain. The aim of the project is to avoid long waiting times, to provide seated transportation with an optimal view from every seat, and to increase carrying capacity.

Technical data of the old cable car:

Model:	Großkabinen-Pendelbahn
Weight empty cabin:	1,600 kg
Weight full cabin:	3,900 kg
Height valley station:	1,933 m
Height top station:	2,214.20 m
Horizontal difference:	905.77 m
Speed:	8 m/s
Carrying capacity:	500 person/h
Power unit:	120 PS

Figure 1: Real-world "Cable Car" situation

Mathematical Modelling in Problem Posing

The connection between mathematical modelling and problem posing can be regarded from two perspectives. On the one hand, questions can be raised during modelling (e.g., Barquero et al., 2019). On the other hand, modelling activities can be already involved in posing a problem. In the following we want to focus the latter connection. In out-of-school modelling scenarios, even though discovering or generating a problem typically takes place before the problem is solved, not much research has investigated the relationship between problem posing and modelling. On the basis of theoretical considerations, generating one's own problems can have a positive influence on the subsequent modelling process. To pose problems that are based on given real-world situations, the given situation has to be *understood* and *explored* with regard to possibilities for posing a problem by distinguishing relevant from irrelevant information and establishing relationships between the relevant information (Bonotto & Santo, 2015). Based on these relationships, possible mathematical problems can be *generated* and *evaluated* with respect to whether or not the given information is coherent and sufficient for solving the problem (Bonotto & Santo, 2015). The associated in-depth analysis of the situation might therefore already involve the modelling activities that are needed to construct an adequate real model and have benefits for the solution process (Hartmann et al., 2021). Empirical research on modelling-related problem posing has supported this assumption. Bonotto and Santo (2015) showed that students included real-world aspects in their solutions when solving self-generated problems using real-world artefacts as problem-posing stimuli (e.g., supermarket bills, restaurant menus). Further, empirical results indicate that the sequence of posed problems is guided by the problem-solving strategies that are typically employed (Cai & Hwang, 2002). Therefore, it is possible that problem posers are involved in *problem solving* while posing a problem that is based on given real-world situations by planning possible solution strategies. As part of planning a possible solution strategy, students might therefore already be engaged in the activities (e.g., structuring and mathematizing) that are needed to solve the self-generated problem. However, an open question is which modelling activities are already taking place during problem posing.

Research Questions and Method

The goal of the study was to analyze the connection between problem posing and modelling by investigating the occurrence of modelling activities in problem posing process. Therefore, we focused on the following research questions:

1. Which modelling activities take place when posing problems based on given real-world situations?
2. With which problem posing activities do the modelling activities co-occur?

Sample

To find an answer to these research questions, we conducted a study with seven preservice mathematics teachers from a large university in Germany (3 men, 4 women) between the ages of 20 and 26 years old (M = 22.86, SD = 1.95). Five of them participated in a program for a higher track secondary school teachers' degree and two of them for a middle track secondary school teachers' degree. All of them already had experience in solving modelling problems and six of them in problem posing. We used heterogeneity sampling to select preservice teachers with different mathematics performance levels, with different levels of experience in problem posing and solving, and who were participating in different university programs.

Procedure and Instruments

To collect the data, we used a qualitative design that included thinking aloud and stimulated recall in order to get deep insights in cognitive processes. The preservice teachers were given three real-world situations and were asked to first pose a problem that was based on the given situation and after posing it to solve their self-generated problem while thinking aloud. We recorded both processes. To supplement the thinking aloud data, we conducted a stimulated recall for every posing and solving process. We used the recorded videos of their posing and solving processes, including their writing, speaking, gestures, and facial expressions, to stimulate the processes. As stimuli for problem posing, we used real-world situations as they are described in modelling problems and extended them by adding further authentic information to allow them to pose a variety of problems. An example of a real-world situation is displayed in Figure 1.

Data Analysis

For data analysis, we first transcribed the videos that had been recorded of the problem posing and stimulated recall and paraphrased the transcripts into sequences, each describing an activity in the process. Then, we analyzed the transcripts by using Mayring's (2015) content analysis. The coding scheme is based on the problem posing and modelling activities described in the literature, and the problem-posing activities were extended on the basis of the given material. The coding schema for the problem-posing activities and the modelling activities are presented in Figure 2. The activity of understanding is included in both coding schema and conceptualized in the same way.

Problem-Posing Activities		Modelling Activities	
Understanding	Comprehending and understanding the given situation and information using the description that is given about the situation.	Understanding	Comprehending and understanding the given situation and information using the description that is given about the situation.
Exploring	Discovering/ gathering relevant information to develop possible questions and organizing the information.	Simplifying/ Structuring	Simplifying and structuring the given real-world situation by differentiating between relevant and irrelevant information, identifying missing information, making assumptions on the basis of this information, and identifying possible solution steps.
Generating	Raising and formulating possible questions and defining a question.	Mathematizing	Translating the selected information into a mathematical model (e.g., table, term, diagram).
Problem Solving	Planning a more or less concrete way to solve the generated problem.	Working Mathematically	Performing the mathematical operations to generate a mathematical result.
Evaluating	Evaluating possible questions on the basis of individual criteria (solvable, meaningful, complete, appropriate formulation, difficulty, suitable for a particular target group).	Interpreting	Interpreting back the mathematical result with respect to the real-world situation and question.
		Validating	Checking models/results for plausibility and appropriateness with respect to the situation.

Figure 2: Coding Schemes for Problem Posing and Modelling Activities

All data were coded by the first author, and over 50% of the data were coded by a second well-trained rater. Interrater reliability was at least moderate for problem posing (Cohen's Kappa between κ =.81 and κ =.95) and for modelling activities (Cohen's Kappa between κ = .76 and κ = .92). To gain an overall picture to which extent the modelling activities are already involved in the problem posing process, we analyzed the transcripts regarding the duration (time of sequences) and the frequency (number of sequences) of the sequences assigned to individual modelling activities.

Results

To assess the occurrence of modelling activities in problem posing, we analyzed the frequency and duration of the sequences in which modelling activities took place while participants posed a problem (see Figure 3). The figure shows that all modelling activities except validation took place during the problem-posing process. However, there were strong differences in the frequencies of the individual activities. Simplifying and structuring could be identified most frequently in problem posing. Second most, understanding could be identified during problem posing, whereas mathematizing, working mathematically, and interpreting could be identified only rarely in the problem-posing process.

Regarding the duration of the individual activities, the overall picture was similar, but it was noticeable that the duration of understanding was significantly higher when compared with the number of sequences. Accordingly, the sequences to which understanding was assigned included a long duration. Overall, participants spent most of their time understanding, simplifying, and structuring, whereas they addressed the activities of mathematization, working mathematically, and interpreting for only very short periods of time.

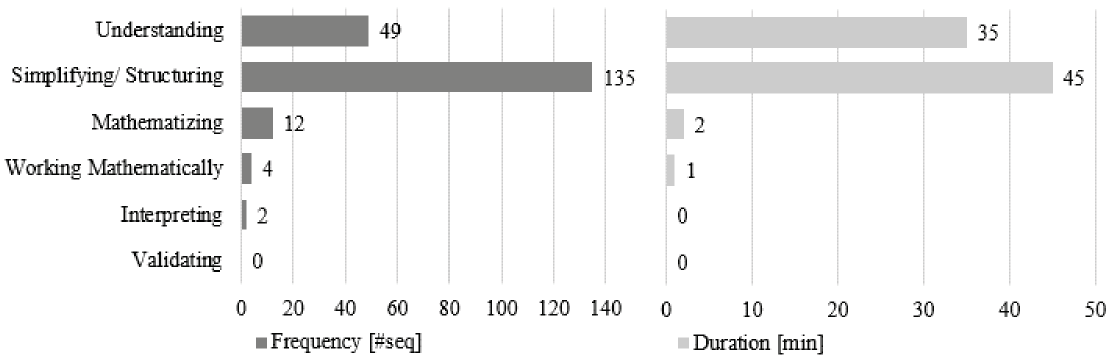

Figure 3: Frequency (left) and duration (right) of modelling activities during problem posing

To understand the connection between mathematical modelling and problem posing, it is important to analyze the co-occurrence of modelling and problem-posing activities. Table 1 presents an overview of the co-occurrence of these activities.

Table 1: Co-occurrence of problem posing and modelling activities

Problem Posing **Modelling**	**Understanding**	**Exploring**	**Generating**	**Problem Solving**	**Evaluating**	**Σ**
Understanding	49	0	0	0	0	49
Simplifying/ Structuring	0	94	6	14	21	135
Mathematizing	0	0	0	12	0	12

Working math.	0	2	0	2	0	4
Interpreting	0	1	0	1	0	2
Validating	0	0	0	0	0	0

Overall, the results revealed that the modelling activity of understanding exclusively occurred with the problem posing activity of understanding. Simplifying and structuring took place most often when participants were exploring the given situation. In the following excerpt, Max is exploring the given real-world situation by filtering relevant from irrelevant data.

Max: Let's take another look at the data for the old Nebelhornbahn. Um, I find the large-cabin aerial tramway rather irrelevant now. Weight empty cabin 1,600 kg and full cabin 3,900 kg.

However, simplifying and structuring also occurred during generation, problem solving, and evaluation. In the context of generation, simplifying and structuring occurred while participants were making assumptions or mentioning relevant information for solving the problem in the formulation of the self-generated problem. For example, in the following excerpt, Nina supplemented her self-generated problem (*What is the best way to shorten the waiting time for the new cable car?*) with the information that the number of people and the speed should be taken into account when solving the problem.

Nina: *(Supplements problem)* Consider the number of people and speed. Something like this.

During problem solving, simplifying and structuring occurred when participants were planning the solution steps they would follow to solve the self-generated problem. In the following excerpt, Max is planning a possible solution for his self-generated problem.

Max: But then we also have the travel speed of 8 m/s. This means that one could theoretically also determine the travel time if we have the length of the route. How long the cable car needs from one station to the next. That would be the next solution step so to speak.

In the context of evaluation, the possibility of solving the self-generated problem is evaluated by checking whether all the information is given. For example, in the following excerpt, Lea evaluated her posed problem by identifying the information that was relevant for checking whether all the information needed to solve the problem was given in the situation.

Lea: Because we know how fast it is, we know where it starts, we know how it's going, and we can say that it's just going straight, so it's kind of going up as a linear function; Then you could/This is a nice question.

Mathematizing came up exclusively for problem solving and working mathematically, and interpreting occurred during exploration and problem solving.

Discussion and Conclusion

In the present study, we investigated the connection between problem posing and mathematical modelling and in particular the occurrence of modelling activities during modelling-related problem posing. The analysis of the problem-posing processes of seven preservice teachers revealed the involvement of nearly all modelling activities. We found that especially the modelling activities of understanding, simplifying, and structuring were already involved in posing a problem that was based on given real-world situations. Understanding was involved in problem posing because it is an essential part of both conceptualizations (problem posing and modelling). Simplifying and structuring are similar to the problem-posing activity of exploring as they are aimed at analyzing the given situation in an in-depth manner, and they typically co-occurred with this activity. Hence, while posing a problem, a situation and real-world model might already be developed. Mathematizing is less involved in the problem-posing process, but it typically co-occurred with the problem-posing activity of problem solving. Problem-solving activities while posing a problem may help problem posers to plan possible solution steps by creating a partial mathematical model. The other modelling activities (i.e., working mathematically, interpreting, and validating) occurred only rarely or not at all, and therefore, these activities might not be triggered by posing a problem.

Our results indicate that especially the modelling activities located in the real world occur while posing a problem. Consequently, problem posing might stimulate an in-depth analysis of the context, something that is important to do for modelling. This result contributes to a theoretical model of the relationship between problem posing and modelling, and it needs to be examined in future studies. As the modelling activities in the real world represent a major cognitive barrier (Krawitz et al., 2018), posing a problem with respect to a real-world situation might help problem solvers overcome these cognitive barriers. Some indication about the importance of problem posing for modelling activities in the real world comes from results in a study by Bonotto and Santo (2015) who found that after problem posing, students often considered real-world aspects of the problem. Further research should investigate how self-generated problems are solved and whether problem posing can affect modelling performance. As a practical implication, our results suggest that modelling-related problem posing could be an innovative and fruitful approach for teaching modelling activities that are located in the real world.

Our study has some limitations that we want to acknowledge. We used a qualitative research approach with a small sample to describe the connection between problem posing and modelling in an in-depth and detailed manner. Due to the design, we could make only hypothetical generalizations, which must be verified in future studies. Further limitations result from the real-world situations we used. These limitations should be kept in mind when interpreting the results of the study. Despite the listed limitations, the study allowed us to contribute to research on modelling by qualitatively exploring the connection between problem posing and modelling. We conclude that there is a close relationship between problem posing and modelling and that, therefore, modelling-related problem posing has great potential for fostering modelling.

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Characterising the roles of digital resources in mathematical modelling

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Over the last decades digital resources of various kinds have entered mathematics education including mathematical modelling. In this paper we present the a priori analysis of a modelling activity based on Study and Research Path using the notions of media-milieu dialectic and Herbartian schema from the Anthropological Theory of the Didactic. We use these notions to differ between resources: digital tools, textbooks, data and to differ between the pragmatic and epistemic value of the resources. Finally, we discuss how task design can benefit from addressing the potential roles resources can play in modelling processes.

Keywords: Mathematical modelling, media-milieu dialectics, Herbartian schema, digital tools, study and research paths.

Introduction

Through computers, with numbers of digital tools, and smartphones students have access to numerous resources potentially supporting their mathematical work including modelling. Initially, emphasis has been put on Dynamic Geometry Software (DGS) and Computer Algebra Systems (CAS), but other resources emerge continuously. The presence of the resources affects the objects, techniques and practices, students are engaging with in mathematics. Artigue claims that “mathematical objects are not absolute objects but are entities which arise from the practices of given institutions” (2002, p. 248). Thus, when institutions allow the use of more resources it changes the practices for example within mathematical modelling and during the last CERME, the following questions were raised:

> ”How can digital and physical tools change ways of approaching modelling problems? How do they change the meaning of “working mathematically” today? Do we need to rethink the definition and conceptualizations of the steps of the modelling processes when using technology and simulations?” (Barquero et al., 2020, p. 1108)

Though still open, the questions are not new. Previously, Siller and Greefrath (2010) have argued how the merger of modelling and technology might serve both pedagogical, psychological, cultural and pragmatic aims for the teaching of modelling where more advanced calculations and realistic problems can be addressed. They have described the modelling process with an addition (an extra cycle) to the modelling cycle representing the role of computational media (Greefrath et al., 2011). Doerr, Ärlebäck and Misfeldt challenge this representation of the merger arguing that this approach does not capture the potentials of the new resources: “computational media both empower the mathematical processes involved in modelling activities by providing new “worlds” to explore and potentially shape the world we try to model.” (Doerr et al., 2017, p. 79). In other words, emphasis must be put on epistemic values just as much as pragmatic values. Though more realistic problems, requiring more advanced calculations, and larger amounts of data might favour the pragmatic value of computational media. Hence, the role of computational media should be central in our studies.

Greefrath et al. examined this in a study where performances by students who employed GeoGebra compared to those using pen and paper, did not differ. Though students' attitudes towards GeoGebra and self-efficacy related to the program, were predictors for their modelling competence. They conclude that technological tools "cannot simply be seen as a facilitator of learning mathematical modelling, at least not if the tasks are not changed." (Greefrath et al., 2018, p. 243). Thus, the need for analysing and understanding how we can design learning environments providing high-quality modelling education, which draw on "experimental materials and technology in modelling" (Carreira et al., 2019, p. 48) is still an open question. Schukajlow et al. (2018, p. 11) recommend authors in the future to "[...] take into account how technologies can be used for modelling or more generally what interaction between humans and media are meaningful […]". This aligns with the claim of Artigue (2002) and the notion of instrumental genesis, which she argues works in two directions: instrumentalization and instrumentation. Trouche formulates the processes as: "the instrumentation process is the tracer of the artifact on the subject's activity, while the instrumentalization process is the tracer of the subject's activity on the artifact" (Trouche, 2020, p. 410), where the artefact can be DGS, CAS or others. In other words, when tools are used for pragmatic purposes, the students operate the tools to gain certain answers, which can be considered instrumentalization. When we identify instrumentation, it is often linked to more epistemic purposes, where students' thinking is shaped by the tool and the explorations done within the tool. Thus, to capture the needs formulated by Schukajlow et al. (2018) and Carreira et al. (2019) we might need to link instrumental genesis with mathematical modelling, as done by Geiger (2017). He discusses the role of instrumental genesis and teachers' orchestration through a technology rich modelling task capturing traits of authenticity, open-endedness and connectivity as characteristics of high-quality modelling education. He concludes that "by improvising and revising his approach to orchestrating students' learning the teacher promoted changes in students' schemas of instrumented action related to both the digital tool and also the task" (Geiger, 2017, p. 299). He continues by stating the need for further research on requirements for designing modelling problems, where technology acts as enabler for other design principles, implementation, and the learning of mathematical modelling.

To address the challenge of designing and implementing modelling activities, where technology acts as enabler, we propose to explore the potentials of the Anthropological Theory of the Didactic (ATD) in terms of media-milieu dialectics and the Herbartian schema when designing Study and Research Paths (SRP).

The media-milieu dialectics and Herbartian schema

Below we present a generating question, Q, initiating a SRP modelling total income for persons with different levels of education. The question invites students to study of the extra-mathematical and mathematical domain further. Thus, we consider the modelling activity a driver for learning mathematics as seen in (Jessen, 2017). This paper presents the a priori analysis of the design. This is one of the four main components of didactical engineering from the perspective of ATD (Barquero & Bosch, 2015). The a priori analysis explore the potential role of digital resources for modelling and afterwards we discuss how this might inform design principles. From an ATD perspective, the modelling activity unfolds as a dialectic between study and research processes initiated by Q. Students pose derived questions, which leads them to study some media. The new knowledge gained from the media is reworked into partial answers to the derived, and hereby the generating question. This is the research process or reconstruction of knowledge, which takes place in the milieu (Jessen, 2017). The quality of the reconstructed knowledge is validated against the milieu. In ATD we use the Herbartian schema (Chevallard, 2008) to explain the media-milieu dialectic and identify factors affecting the dialectic. The schema describes the didactic system S working together for the development of an answer to the modelling problem Q:

$$S(X; Y; Q) \Rrightarrow A^{\heartsuit}$$

X represents the group of students (which can be a singleton), Y represents the group of teachers supporting X's study of the question Q. Y can be a group, a single teacher, y, or $\emptyset$ in case of self-study. The ➥ indicates that the system of $X; Y$ and Q bring into being a personal answer $A^{\heartsuit}$ to the modelling question. $A^{\heartsuit}$ is different from answers found in any resource, as it is the result of the personal or joined modelling process of a class. The modelling process in ATD is characterised as X and Y's interaction with the milieu M, which is "a fuzzy and changing set of didactic "tools" of different kinds that X, acting under the supervision of Y, has to bring together (➡)" (Chevallard, 2008, p. 2). The entire process can be depicted as:

$$[S(X; Y; Q) \Rightarrow \mathrm{M}] \Rrightarrow A^{\heartsuit}$$

$$M = \{A_1^{\Diamond}, A_2^{\Diamond} \dots, A_l^{\Diamond}, W_1, W_2, \dots, W_m, D_1, D_2 \dots, D_n\}$$

The elements of the milieu are media which "designate here any representation system of a part of the natural or social world addressed to a certain audience" (Chevallard, 2007, p. 1, our translation). The $A_i^{\Diamond}$ denote existing answers being students' previously learned knowledge. This includes instrumented techniques and ability to draw on different digital resources. The W_j cover resources to be studied during the modelling process. This can be a page of a textbook, web searches of different nature (Khan Academy, encyclopaedia etc.). The D_k denotes data in various forms. Chevallard (2019) argues that data can be of quantitative or qualitative nature. Students might generate data (from experiments or simulations), they can be found in databases, or they can be part of the formulation of the modelling problem. Thus, the milieu of the Herbartian Schema is the media brought together with the purpose of nurturing the study of the modelling problem Q. Note that the milieu can have an element of a-didactic potential as known from the work of Brousseau (1997), where "a system of objects acting as a fragment of "nature" for Q, able to produce objective feedback about its possible answers" (Kidron et al, 2014, s. 158). Thus, for modelling problems, their real-world contexts become objects to be studied by students when developing and validating an answer $A^{\heartsuit}$.

We consider the notions of media-milieu and Herbartian schema tools for designing the rich environments for high-quality modelling education as suggested by Carreira et al. (2019). Below we draw on elements of didactical engineering from the perspective of ATD when we address the research question of this paper: How can media-milieu dialectic and Herbartian schema support the characterisation of the potential roles played by digital resources in the SRP on modelling?

A modelling problem on total income

The generating question for the SRP has been designed to introduce the notion of piecewise linear functions in Danish upper secondary mathematics in grade 10 building on the notion of linear function from lower secondary mathematics. In Denmark the students are allowed to use CAS-tools for their written exit examination and to visit webpages previously used in class (Danish Ministry of Education, 2017). Therefore, the students are used to search all sorts of information online and familiar with basic uses of CAS-tools. The problem is formulated as:

Three friends completing lower secondary school in Denmark discuss their plans for further education. One is planning to get a job without pursuing further education. The two others plan to become a nurse (bachelor degree) and upper secondary teacher (master degree) respectively. They discuss when each of the different strategies have given the larges total income? Below you find a table of average income in Danish kroner (DKR), as result of years of completed education.

Table 1: The table of average income corresponding to years of education

Years of education	9	12	14	16	18	21
Average Income/year	210.000 DKR	310.000 DKR	365.000 DKR	370.000 DKR	490.000 DKR	520.000 DKR

The entries correspond to completing lower secondary, upper secondary, short further education, bachelor's degree, master's degree, and the PhD degree. The idea behind the problem is to create the need for piecewise defined functions and the specific notation linked to this. The teacher can enlarge the milieu by the end of the modelling process, introducing a work, W_j, presenting the notation. The idea of enlarging the milieu with intentions of the a-didactical situations has been reported on by Grønbæk & Winsløw (2015). Striving to promote self-study element on complex numbers for first year students, they designed media and milieus for students to explore. They argue for their design choice by stating "when students study and encounter an inference they do not follow, they are supposed to consider the text as a milieu that resources and constraints their efforts to fill the gab" (Grønbæk & Winsløw, 2015, p. 2132). The same idea has been explored by Jessen (2017), who explicitly designed a repository offered to the student to consult, when and if they encountered shortcomings of their existing knowledge and techniques. We will return to this idea later.

An a priori analysis of the SRP

As Grønbæk & Winsløw (2015) we here present the potential paths of students working with this SRP, not having a repository. We present the potential strategies, which serve as an argument for generating power of Q. Not all strategies require the use of digital resources. Basic pen and paper methods allow students to add income for each person. This requires knowledge about educational length for nurse and upper secondary teacher. This can be found online through webpages or educational guides. This represents data, D_i, brought into the milieu by the students. The strategy can be eased using spreadsheets as indicated in figure 1, where blue fields mark the years where the nurse's total income is largest, the orange is when the teacher exceeds both. This strategy draws on existing answers, $A_i^\lozenge$, in terms of arithmetic, and instrumented techniques if using spreadsheets.

Some students might consider the spreadsheets tables, they can create scatter plots from. Some might experiment with linear regression using Excel, GeoGebra, etc. Both instrumented techniques and existing answer, $A_i^\lozenge \ldots$The tables can be considered self-generated data, D_l. When the digital tool plays the role of media it provides the students with answers. When exploring what function fits data, we consider the digital tool the milieu for exploration. Thus, the scatter plot might be yet another answer or experiments with different types of function. It depends on the context and the students discourse regarding the problem of fitting data. If new knowledge is reconstructed, we consider it research and the digital tool is the milieu supporting this.

Other students might recall the notion of linear functions, $A_j^\lozenge$, identify the average income as the slope of a linear function. The x-axis represents time measured in years. When the linear function is expressed as $f(x) = ax + b$, where f is total income in thousands of DKR, they need to determine the constant b. For the person without further education, we have $b = 0$. In this scenario the students need to determine b for the nurse, where $f(6.5) = 0$, since it takes 6.5 years to complete nursing school. The slope of the function is 370, and students should solve the equation $0 = 370 \cdot 6.5 + b$. Which means $b = -2405$. This reasoning requires several notions and formulas (known answers, $A_1^\lozenge, A_2^\lozenge, \ldots,$) put together in new ways. The students' interaction with the milieu provides the students with new answers and further practiced for the teacher.

Besvarelse 2

År	Arbejder	Sygeplejerske	Gymnasielærer
9	0	0	0
9,5	105000	0	0
10	210000	0	0
10,5	315000	0	0
11	420000	0	0
11,5	525000	0	0
12	630000	0	0
12,5	735000	0	0
13	840000	155000	0
13,5	945000	310000	0
14	1050000	465000	0
14,5	1155000	620000	182500
15	1260000	775000	365000
15,5	1365000	930000	547500
16	1470000	1085000	730000
16,5	1575000	1240000	912500
17	1680000	1395000	1095000
17,5	1785000	1550000	1277500
18	1890000	1705000	1460000
18,5	1995000	1860000	1642500
19,5	2100000	2015000	1825000
20,5	2205000	2170000	2007500
21,5	2310000	2325000	2190000
22,5	2415000	2480000	2372500
23,5	2520000	2635000	2555000
24,5	2625000	2790000	2737500
25,5	2730000	2945000	2920000
26,5	2835000	3100000	3102500

Figure 1: spread sheet solution

Some students might choose to use CAS-tools and the 'solve' command to do the same. Depending on their instrumental genesis, this might affect their conception of the models developed. The pragmatic use of the CAS-tool provides them with an answer to be studied, W_i, and incorporated with the models they are building. From here the intersection points can be found solving the equation where two functions are equal or using apps in the CAS-tool. This leads to the construction of different mathematical knowledge. The simple models of total income are presented at the left side of figure 2. When shared with classmates, the models become new answers brought into the milieu by students for their classmates to study. The different approaches will provide them with broader perspective on choices made during the modelling process.

For some students the real world will drive them to elaborate their models. Most students know that Danish government provide financial support to students after turning 18. The amount depends on whether you live at home or away from home, and the parents' income. These data, $D_i \ldots$, are to be found online and to be use in the models describing the nurse and the teachers' total income. Other income from student jobs might also be included providing new pieces in each model.

This can be modelled by adding numbers by hand or in spreadsheets as an elaboration of the strategy shown in figure 1. Again, students might choose to experiment with scatter plots and linear regression for certain domains. Here the digital tools have epistemic value, when students' partial answer becomes objects to be explored further using notions, works and existing answers. Alternatively, they might try to use the strategy of being able to find the slope of each part of the function using the potential amounts of income (financial support, student jobs etc.).

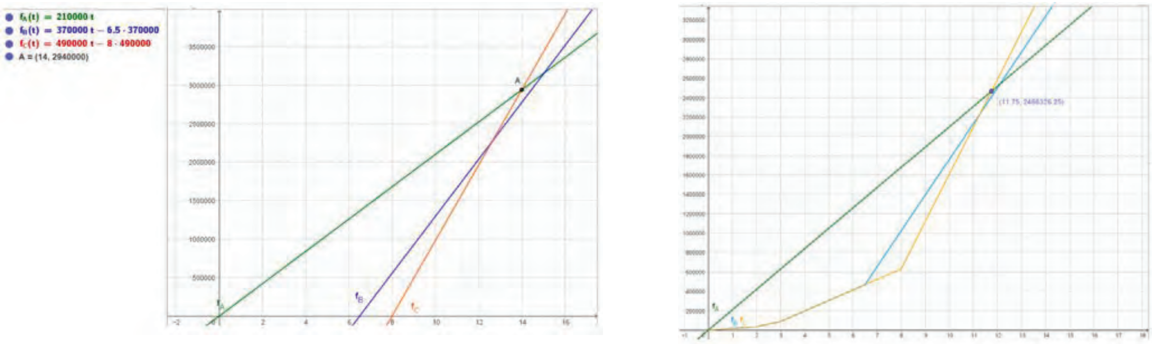

Figure 2: Two different solutions with linear and piecewise linear functions respectively

Then the strategies will be like those used to produce the models of the left side of figure 2 taking each domain into consideration. For this strategy the digital tools will mainly have pragmatic value. Though the tools allow the development of still more elaborate models to be done faster, using CAS-tools. This is valuable, when engaging students in continuously improving their models against the extra-mathematical context. Finally, when the teacher elaborates the milieu by introducing the notation for piecewise linear function and syntax for this using CAS-tools, the teacher provides the students with the a-didactical potential of formalising their previous work. Simple examples of models based on this new knowledge is presented at the right side of figure 2.

Not all students will be able to use this notation, why it is important to orchestrate validation of the different models by asking the students to compare and contrast their answers, helping them to realise which piece of the scatter plot is representing the blue fields from the spreadsheet and how is this represented in the formal notation of piecewise linear functions?

Discussion and concluding remarks

From the a priori analysis we can argue that media-milieu dialectic and Herbartian schema allow us to identify the roles played by digital resources in the SRP on modelling as being the provider of data, and new answers or work to be studied. In some cases, the tools might function as reminders of notions or tools taught previously. Also, the digital resources represent the perceived reality against which hypotheses, notions, newly constructed answers, and still more elaborated models are explored and validated. Moreover, SRP and the media-milieu dialectic seem to provide the students with the autonomy to enlarge the milieu or to choose known methods and answers as starting point for their modelling process. We can argue that SRP based modelling activities where the media-milieu dialectics are explicitly considered in the design process offers a richness in terms of number of strategies, students might pursue depending on previous learning. It allows students to experiment with data from various resources, and different uses of technology. We consider this the outset for high-quality modelling education as requested by Carreira et al., 2019. The notion of Herbartian schema naming resources as answers, works or data, which all can shift between being media to be studied or milieu employed in research processes can highlight changing roles that enrich modelling and enable us to harvest the pedagogical, psychological, cultural, and pragmatic aims of merging modelling and digital tools as Siller and Greefrath (2010) argued for. Jessen (2017) draw on media-milieu dialectics and Herbartian schema for the a posteriori analysis. If we do the same for technology rich modelling activities, we might be able to describe the potential meaningful interaction between humans and media as asked for by Schukajlow et al. (2018). This resonates with the instrumental genesis approach adopted by Geiger (2017), though his framework does not capture episodes where the digital tools provide new knowledge in terms of search for data or works to be studied. The study process might be the most novel contribution to modelling activities: to explore the potential of small elements of direct instructions, brought into the milieu by the students. Therefore, when Doerr et al. (2017); Geiger (2017) and Greefrath et al. (2018) argue that the presence of technologies in mathematics education calls for a change of tasks designed for modelling activities to capture new potentials. We propose to consider the digital tools and technologies both as media and milieu facilitating students' exploration of modelling as a dialectic study and research processes. Thus, from a task design perspective, we need to experiment with ways to nurture both study processes and research processes. We need to explore how technology and different tools can nurture those processes. What is gained, what is lost in the shaping of mathematical modelling? And if the modelling functions as driver for learning mathematics, how does this approach shape the mathematical object constructed through the modelling process?

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Prescriptive modelling – mathematics teachers' discussions of the BMI

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The study presented in this paper focuses on primary school teachers engaging with the Body Mass Index (BMI) as part of a university course. The BMI is seen as an example of prescriptive modelling where mathematics is used to keep track of the obesity phenomenon. Four categories are developed to characterize the teachers' discussions: the mathematical aspects of the BMI formula, meta-validation, the consequences of the use of the BMI and other indices in society, as well as on their teachability in the classroom. The results can contribute to developing an understanding of prescriptive modelling processes from a critical perspective.

Keywords: Prescriptive modelling, teachers' discussions, critical perspective.

Introduction

In this paper, we investigate the processes some in-service teachers go through when they discuss a task about a mathematical model like the BMI and its inclusion in mathematics education. Blum (2015) highlighted four purposes for using mathematical modelling in education: helping students to use mathematical knowledge to make sense of extra mathematical situations (pragmatic); developing argumentation and modelling competencies (formative); exploring the relationship of mathematics to real life and the mathematics' role in shaping society (cultural); and affective issues such as students' interest in mathematics (psychological). The study presented here is particularly situated in the cultural perspective, with a specific focus on a critical view of mathematics' role in society.

In mathematics curricula around the world, such as in the Department of Basic Education (2011) in South Africa, in the Common core state standards initiative (2010) in the USA, and in the Australian curriculum, assessment and reporting authority (2015), modelling and applications play a key role. Similarly, in the mathematics curriculum in Norway, *modelling & applications* is one of six core elements and concerns students' insight on how mathematical models are used to describe everyday life, work-life, and society, as well as students' competence to solve problems from reality by using mathematics (Ministry of Education and Research, 2020). *Democracy & citizenship* is an interdisciplinary topic in mathematics, aiming at students' awareness of the prerequisites and premises of the mathematical models used in society by giving them opportunities to work with real data sets from different fields. Modelling and models are seen as a possibility for students to understand the role of mathematics in society, which is the focus of socio-critical modelling and connected to the cultural arguments for using mathematical modelling in education (Blum, 2015). Barbosa (2006) defined this perspective as modelling as critic, where students' ability to criticize the mathematical models is achieved through the learning of mathematical concepts and modelling competencies.

Niss (2015) distinguished between two types of modelling. In descriptive modelling, the aim is to find a model that can be the answer to a problem from extra-mathematical domains. In prescriptive modelling, the aim is "to pave the way for taking action based on decisions resulting from a certain kind of mathematical considerations, in other words 'to change the world' rather than just 'to understand the world" (p. 69). The differences between the two types of modelling centre around the process of modelling. While examples of research in mathematics education focus mainly on descriptive modelling, Niss (2015) called for research on prescriptive modelling because of the impact such modelling has on society. Indices, like the BMI, are examples of prescriptive use of models in our society. We consider the term *index* as a measure of one or several chosen variables connected to a relatively large sample taken from a population or set. BMI is computed as the ratio between the weight (m) of a person in kilogram, and the squared height (h) in meters: m/h^2. The BMI values should be between 18.5 to 24.9 kg/m^2 for an adult to be considered of normal weight. The model is used extensively e.g. in medical contexts, in keeping track of obesity in individuals and populations, even though it has known limitations (e.g. Hall & Barwell, 2015; Kacerja et al., 2017).

Given the emphasis on modelling in the curricula, mathematics teachers across the world have the task to make mathematical modelling an integral part of students' learning of mathematics and connect it to students' development of critical competence and critical citizenship. Following Niss' (2015) call to focus on prescriptive modelling, we have used the BMI with in-service primary school teachers (grades 1.–7.) to facilitate discussions about the mathematical models' impact in society. The novelty in our study consists in the use of prescriptive modelling examples in teacher education settings and in collecting empirical data from teachers' discussions about indices. According to Niss, little is known about the processes when people engage with prescriptive modelling. In this paper, we address this gap by exploring the research question: what characterizes teachers' discussions about indices and how they, the BMI in particular, can be used in mathematics education? We include a critical mathematics perspective to provide insights into teachers' understanding of indices' roles in education and society.

Theoretical considerations

Several researchers have described processes involved in mathematical modelling. Niss (2012) defined the mathematical modelling cycle as the process that starts with "some extra-mathematical domain, moving into some mathematical realm so as to obtain mathematical conclusions and translating these back to the extra-mathematical domain" (p. 50). A model of the modelling cycle which is often referred to is the one by Blum and Leis (2007) with two connected worlds: the mathematics and the real world. Even though there are different modelling cycles presented for descriptive modelling, there are some common elements we find in all of them. An extra-mathematical problem is the starting point. Then through discussions, the problem is translated into a mathematical problem, and mathematical concepts and processes are used to find one or several models as solutions to the problem. An important phase of the modelling process is the de-mathematization of the solution, choosing the best solution (model) while interpreting it in relation to the original problem. If the solution does not make sense for the problem at hand, then a new modelling process must start.

In the socio-critical perspective in modelling, Rosa and Orey (2015) emphasized that "students are expected to understand, reflect, comprehend, analyze, and take action to solve problems taken from their own reality" (p. 390). They presented a social-critical mathematical modelling cycle, where the real problems are environmental, political, social etc. This focus is in line with the purpose that the socio-critical perspective applies for mathematical modelling, where students should understand the role of mathematics in society, and develop tools for their social-critical efficacy, which they can further apply in other cases as well. In the modelling process, the emphasis is on the individual modeller, and it includes action since the aim is for students to be able to act upon reality.

Doerr, Ärlebäck and Misfeldt (2017) underlined the necessity to have several representations of mathematical modelling to capture the multiplicity of perspectives in mathematics education. The modelling cycles such as the ones by Blum and Leis (2007) and Rosa and Orey (2015) are different and capture different aspects of modelling. They are however not enough to describe students' working processes when involved with prescriptive modelling (Niss, 2015). By exploring three examples, one of which is the BMI index, Niss argued for some of the limitations of the existing modelling cycles that "become very rudimentary when applied to the BMI model" (p. 71). In our study, while exploring an existing model such as the BMI index, the processes of idealizing the extra-mathematical situation and mathematizing the question posed, become trivial in the sense that the index already exists. As Niss (2015) discussed, the mathematical treatment reduces into replacing the weight and the height of a person into the BMI formula and the de-mathematization process reduces into finding the interval in which the person can be placed based on the number obtained (p. 70–71). Niss (2015) argued that two aspects of the modelling cycle that need to be more developed in existing models to adapt it to prescriptive modelling are meta-validation and critique of the model. Meta-validation requires looking critically at three points: how the modelling results influence the discourse around the problem that was modelled; how the obtained model is compared to other potentially relevant alternatives; and how a change in the requirements influences the modelling and its outcomes. An important contribution from our study is the attention towards teachers' reflections upon the possible uses of BMI and other indices in their classroom teaching. In this paper, we analyze the teachers' discussions to characterize how they talk about including the BMI in their teaching and which of these three, and other processes, they go through. This can be seen as a first step in developing an understanding of prescriptive modelling processes and their use in school settings.

Method

The participants in our study were twelve in-service primary school teachers who attended a course on Numeracy across the curriculum. After a teaching session in which one teacher educator presented some uses and misuses of mathematics and the idea of an index, the teachers were divided into two groups and given 60 minutes to work with the BMI task. The task had three sets of questions: the first included questions about what BMI is, the formula and purpose of the BMI, what it measures, and how it could look differently; the second concerned the use of the BMI in different contexts in society and the meanings of the use; and the third focused on the teachers' thoughts on using indices in their teaching of mathematics, possible reasons for including or excluding such topics in schools, as well as thoughts about similar index-related examples they have used. A picture of a muscular rugby player with a high BMI value was included. The purpose of the BMI task was to structure and guide the discussions towards the mathematics in the indices, the role that indices have in our society, and how teachers see possibilities and challenges in using indices to promote critical thinking with their students. There were two teacher educators present, one in each group, to observe the discussions and provide a better understanding of what the teachers said, and make sure that the teachers addressed all three sets of questions of the BMI task. After a few minutes of discussion, the groups were given the BMI formula and the cut-off points for six weight categories. The teachers did not follow the structure of the question sheet, they jumped back and forth, but in the end, they had covered all the questions. The discussions were audiotaped and transcribed.

All the authors of this paper worked together through several cycles to analyze the data applying a thematic analysis approach (Braun & Clarke, 2006). The three sets of questions from the question sheet were used as initial, overarching codes, combined with the recommendations by Niss (2015) on meta-validation and critique of the model. The teachers' utterances were analyzed within the framing of the question sheet. We went systematically through some initial parts of the data as a whole group and analyzed the teachers' utterances according to the different codes. Then we continued the coding process in smaller groups to complete the first coding before we as a group compared and refined the coding to generate the final categories. The initial analysis was based on the question sheet, but the succeeding code and retrieve process was based on what the teachers said. The analytical process was therefore twofold in which the main part of the categories and subcategories were generated inductively from the data. We also used premade categories deductively, but also these categories were refined based on what the teachers said.

Results

In Table 1 below, we present the results – the categories developed from the analysis of the teachers' discussions:

Table 1: The categories (underlined categories are generated from the data, the others are premade)

A Investigating the index (BMI) concept and formula A1 What is an index (BMI) A2 What does it measure? <u>A3 Variables</u> A4 Reflections about previous knowledge	**B Evaluating alternatives (meta-validation)** B1 Formula - how could it look differently <u>B2 Challenges (neglected variables, measurement inaccuracy etc.)</u> <u>B3 Point out existing alternatives, compare with alternatives</u>, adapt the index
C Influence and use in society <u>C1 Pros of using BMI</u> <u>C2 Cons of using BMI</u> <u>C3 Seeking/giving information</u> <u>C4 Critical</u> <u>C5 Examples</u>	**D Teachability** <u>D1 Reflections, appropriateness mathematically and thematically</u> <u>D2 Reflections, appropriateness ethically</u> <u>D3 Reflections on own knowledge about indices to use them in teaching</u>

Categories A (investigating the index) and B (evaluating alternatives) include the teachers' utterances when addressing mainly the first set of questions from the question sheet. Here the meta-validation questions by Niss (2015) are integrated as part of category B to characterize the discussions when teachers look at alternative formulas and point out challenges of the existing formula. Category C (influence and use in society), including critique, stems mainly from the teachers' discussions of the second set of questions. Here the notion of critique by Niss (2015) is found on discussions where the inappropriate use of the model is criticized (cons) and the results of such use on the discourses around the obesity problem are brought forward. Category D (teachability) is connected to the third set of questions from the question sheet.

Investigating the index (BMI) concept and formula (category A)

The following discussion takes place at the beginning of the discussion in group 1, where the teachers have not yet seen the formula of the BMI (the teachers are anonymized and numbered like this: T1, T2 …). One of the groups starts the discussion with T1 reading aloud the first question: "What is BMI?" T2 answers "it has something to do with the body", focusing on what the index measures (an A2 category utterance). T3 includes variables, "it has to do with height and weight" (A3) and adds "it is a ratio" (A2). The teachers search for answers together by saying what they think BMI is and what they seem to remember concerning body, height, weight, and ratio. They have not yet seen the formula, but they are closing in as the ratio is between the weight and the square of height.

Towards the end of the group discussions, when talking about how to teach about something like indices, the teachers go back to their initial reflections trying to make sense not only of the BMI but also of what an index is or can be. The question by T4, "When we measure temperature and rainfall, do we work with indices?", is such an example, where the teachers are trying to find out what qualifies to be an index. Similarly, T5 asks: "if you some days in advance get a considerable increase in the air pressure, you quite often see an improvement of the weather […]. Is that an index?" In this category, the teachers focus on what they know about BMI: what an index is and BMI in particular (A1); discussions about what it actually measures, usually related to different uses of BMI they know about (A2); the variables used to measure it, such as the weight and the height (A3); and in addition, they talk about their previous knowledge (A4) about BMI. The categories from A1-A3 are nuances of teachers' investigations of the BMI as they try to make sense of it. It is difficult to distinguish between the three subcategories as the answers are often intertwined, but they are valuable for being able to nuance the discussions.

Evaluating alternatives, meta-validation (category B)

Another question in the first part of the question sheet asks if the formula could look differently. This question is connected to the meta-validation process as introduced by Niss (2015). To answer the question, the teachers present examples of the different uses of the BMI they know about. One such example is the picture of a rugby player on the question sheet with a muscular body, but with a BMI of 35.98 kg/m2 is placed in the obese class II according to the cut-offs provided by the BMI model. T4 compares the rugby player with a person "who does not train, that has eaten too much, right. It does not say they are in the same shape; it just says they have the same weight". There is a discussion of muscles weighing more than fat, and how this is not taken into consideration in the BMI formula. This aspect is categorized as a challenge (B2) in terms of neglected variables in the formula.

Other examples, such as the use of weight and height graphs for small children, which even though they are not direct examples of the use of BMI, are referred to in the discussions. These graphs monitor children's development to ensure they grow as they should by comparing a child's measures to the curves of the average children at the same age. In these examples, the teachers focus more on how the BMI or other indices are used, sometimes without considering geographical factors. Other challenges of the BMI are taken into the discussions, such as measurement inaccuracies (B2) and the effect these can have on the results of the formula. The teachers point out existing alternatives such as waist circumference as a better measure that in some respects takes into account the fat vs muscles issue. They often express the need to combine those two measures to get a better picture of someone's health. These discussions are categorized as B3, as adaptions or alternatives to the BMI.

Use and influence in society (category C)

When the teachers in group 2 discuss the question "What do you think about BMI's role and use in society?", they use examples to illustrate their answers (C5). Examples vary from personal ones about themselves or their family members, to examples of extreme cases where the formula does not fit. A representative example is T6 saying "I am worried about who shall decide what is right about weight". T6 gives an example about a 14-year-old girl who was told her weight was a little high, but T6 did not agree with this at all. The example is personal (C5), and T6 is critical (C4) towards the uncritical use of BMI, without considering other factors besides the number from the BMI formula. The critique is also directed to the ones who use the formula and have decision power. The example can also be seen as being against the use of BMI (C2). Among the examples that support the use of the BMI (C1), we find: "I think, from a society perspective and when used sensible, that this is a good tool. What else shall health nurses or I use … if we don't have standards?" The reasons for accepting the use of the BMI are often connected to the need of having standard tools. In these discussions, the teachers often elaborate on the examples by arguing for why the formula is necessary (C1), or on the contrary, giving reasons for why the formula should not be used (C2). They often ask questions seeking for information or giving information (C3), and it is usually when taking into consideration the different examples that they are critical towards the use of the index (C4).

Teachability (category D)

The last section on the question sheet is connected to the participants' work as teachers and their thoughts about the possible use of indices in school teaching. In the following example from group 2, the participants are trying to make sense of the BMI formula, and T7 says, "I think it is very difficult to think that one also measures area", and T8 adds, "yes, surface area". At the same time, the teachers are thinking about their students, and T7 says: "Talking with the students about this and then you take kilos and then you divide it by the area of the body". T7 is thinking aloud about how to present the topic so that students can make sense of it from a mathematical point of view, which is an example of discussions of the mathematical appropriateness of the BMI (D1). T7 adds immediately after "hm ... there is something wrong, isn't it?" showing uncertainty on how to present it since the teachers themselves are having problems with figuring out how to talk about this with the students (D3).

In the teachers' answers, we identified several reflections about the BMI and other indices' appropriateness to be used in teaching. They discuss both thematic appropriateness in terms of the mathematical level (D2) and ethical appropriateness in terms of BMI representing obesity that can be a sensitive topic for their students (D1). All of these aspects came in addition to their discussions of teachers' knowledge about indices to include them in their teaching (D3). At this point, they often ask themselves the question: what is an index?

Discussion and concluding comments

In this paper, the focus has been on in-service primary school teachers' discussions of the BMI and the use of indices in education. We have looked at the discussions from a critical perspective where the aim is to facilitate an understanding of and criticize the role of mathematics in shaping society. As we found through the categories in the study, the teachers engaged in discussions about the mathematical aspects of the BMI formula and their knowledge of it (A); about alternatives to the BMI and its limitations, or meta-validation processes (B); about the BMI's use and influence in society (C) as well as about teachability of BMI and indices in general (D). These categories were also nuanced with subcategories that capture different aspects of teachers' discussions.

Like in several of the mathematical modelling cycles (e.g. Blum & Leis, 2007; Rosa & Orey, 2015), the starting point of the discussions in our data is an extra-mathematical situation, the BMI and its use in society. However, the discussion of the mathematics in prescriptive modelling is different from the aforementioned descriptive modelling cycles, as also Niss (2015) pointed out. In our data, teachers discuss an existing mathematical formula (categories A and B), by comparing it to alternative models (B3) and pointing out weaknesses such as missing variables (B2). Similarly to the socio-critical modelling cycle (Rosa & Orey, 2015), the teachers discuss the role of the BMI model in society (category C). At this point, our data allowed us to nuance the way teachers did this by weighing pros and cons for using BMI (C1 and C2), by being critical (C4), by seeking further information (C3) and by providing examples (C5). The categories and subcategories made it possible to further develop the aspects of meta-validation (B) and critique of the model (B and C) as processes of the prescriptive modelling cycle that Niss (2015) called for. Our study adds the teachability aspect (D) to prescriptive modelling. Since we work with teachers and their competence to engage students with examples of the uses of mathematics in society, it is important for us as teacher educators to know the teachers' challenges and possibilities for working this way. This adds another perspective for understanding ways of implementing these examples in teacher education and school mathematics.

The categories and subcategories are a step towards finding ways to represent working processes in prescriptive modelling. The importance of such representations was emphasized by Doerr, Ärlebäck and Misfeldt (2017) and Niss (2015). The representations can be used in further research about prescriptive modelling and socio-critical perspectives in modelling, but also for teaching about the mathematics' role in society. The categories show that the teachers were given the possibility to engage in critical discussions of BMI and indices in general, from different angles. Given this possibility, indices can be a starting point to develop a critical perspective in mathematics. As Niss (2015) and Hall and Barwell (2015) also recommended, focusing on such models allows for developing insights both into the mathematical and the societal aspects of the models and their consequences, which lies at the heart of the critical mathematics perspective.

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Comparison of the working on modelling tasks when students have or do not have personal interest in the real-world context of the task

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Keywords: Modelling task, real-world context, personal interest.

Motivation

Modelling means solving a real-world problem with the help of mathematics. A modelling process can be analyzed using an ideal modelling cycle with the steps: understanding, simplifying, mathematising, working mathematically, interpreting, validating and exposing (Greefrath & Vorhölter, 2016). Consequently, dealing with modelling tasks not only entails dealing with a mathematical problem, but also with the real-world situation described in the task. In a classroom there are children with a variety of personal interests. Thus, the question arises to what extent these personal interests influence the working on a modelling task. This study therefore compares the use of the real-world context by individual students when working on two modelling tasks in which students have much/little personal interest in the real-world context.

Theoretical framework

The person-object conception of interest differentiates between two types of interest: 1) Situational interest which is aroused by the conditions of a situation. 2) Personal interest which is a longer lasting characteristic of a person that goes along with a high subjective esteem, positive emotions for the content (Krapp, 2000) and a lot of knowledge about it (Renninger et al., 2002).

Ainley et al. (2002) identified a connection between interest in the topic of a reading task and persistence and learning outcome. When working on mathematical word problems in a context of high or low developed personal interest, Renninger et al. (2002) found out, that the influence of the context can be different depending on the interest in mathematics itself and the mathematical ability. The way of dealing with the real-world context of a modelling task can be very individual and variable. However, Busse (2011) was able to distinguish between “four different ideal types of dealing with the real-world context: reality bound, integrating, mathematics bound, ambivalent” (p. 38). Concerning mathematical application tasks Stillman (2000) showed that prior knowledge of an application task context can help students to engage with the task or check results for reasonableness. Nonetheless, the prior knowledge might still have negative or neutral effects as well. Krawitz and Schukajlow (2018) examined the impact of prior mathematical knowledge on the solution of modelling tasks. They discovered that it depends on the appropriateness of the activated knowledge, whether the prior knowledge promotes or interferes with the solution process. Since personal interest is often accompanied by a lot of knowledge, it could be interesting to investigate, whether the effects of prior knowledge are applicable in a similar way for personal interest.

Research questions and Methods

As there are hints to both positive and negative effects of interest on the working on a mathematical or reading task, the present study aims to investigate the impact of personal interest in the context of a modelling task on the working progress. The following research questions are examined:

- At which points in the working process do students refer to the real-world context of the modelling task when they have much or little interest in the context?
- At which points is personal interest mentioned by students when working on a modelling task in which they have a personal interest in the real-world context?
- To what extent does the personal interest in the real-world context of a modelling task have an impact on the working on the task?

The study consists of two parts: First, a class of 7th grade is presented an interest questionnaire. Based on the results students are selected for participation in the next step. Second, 10-15 students work individually on three modelling tasks using the “think-aloud” method. One task with a high personal interest in the context and one with low interest, respectively. In between there is a buffer task as a distraction. The tasks are followed by a short interview about the working process.

Outlook

Data collection is currently being carried out. Using a qualitative content analysis, the working process will be divided in the steps of the modelling cycle. For each student, a comparison is made of whether the points, at which the context is considered in the work, differ depending on whether there is much or little interest in the context. Also, the points, where interest or prior knowledge is used, are coded, so one can see if there are differences for tasks with much/little interest in the context.

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Students' processing of modelling problems with missing data

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The problems that are found in the real world, mathematics, and science are usually ill-defined problems. By contrast, the problems given in classrooms tend to be well-defined. It can be useful for students to solve modelling problems so they can learn how to deal with ill-defined problems. In a qualitative study of ninth to eleventh graders (N = 12) in different secondary school tracks in Germany, we investigated how students process modelling problems with missing data. We found that students have difficulties noticing when data are missing from some problems. When students notice the missing data, they notice it at the very beginning of their processing, after understanding the problem, or while validating the mathematical results. After noticing, either the students were able to make a realistic assumption about the missing data using various strategies, or they made no assumption at all. The theoretical and practical implications of the study are discussed.

Keywords: Mathematical modelling, ill-defined problems, missing data.

Introduction

Having students solve problems is an integral part of mathematics teaching all over the world. The problems that are typically given in school have a clear structure, include all relevant data, do not include any superfluous data, and have exactly one solution. This type of problem is known as a well-defined problem. By contrast, the problems encountered in everyday life and at work are often ill-defined and have multiple solutions. Consequently, their solution processes differ from the processes needed to solve well-defined problems. In mathematics education, ill-defined problems are defined as problems situated in a specific context, where one or more aspects of the problem are not well specified, the problem description is not clear, or all the data needed to solve the problem is not provided in the description (Jonassen, 2000). One important type of ill-defined problem is a problem with a connection to reality (e.g., a modelling problem). Modelling problems are characterized by a demanding process of transferring information between the real world and mathematics. Modelling problems can be ill-defined in different ways, for example, when the initial state of the problem is unclear and the initial data are missing, so-called modelling problems with missing data or open (-ended) modelling problems. Modelling problems as ill-defined problems have rarely been the focus of research yet. In this study, we aimed to analyze how students process modelling problems with missing data and how they overcome the difficulties that occur while solving these problems. The theoretical foundations of this research are theories about the processes involved in solving modelling problems (Blum & Leiss, 2007) and problems with missing data (Krawitz et al., 2018).

Theoretical background and research questions

Modelling problems

Mathematical modelling is an important competency that is part of mathematical literacy and is included in many national curricula and in mathematics teacher education. Modelling problems have been found to be difficult for students to solve due to the cognitive complexity of such problems. In order to describe which activities students need to engage in to solve modelling problems, several approaches and numerous theoretical models of the solution process have been developed in previous research (e.g., Blum & Leiss, 2007; Galbraith & Stillman, 2006). An idealized process for finding a solution to a modelling problem proposed by Blum and Leiss (2007) is the following: The first phase involves understanding the problem and constructing an individual situation model. Second, the students have to construct a real model by simplifying and structuring the situation model. Afterwards, the real model is transformed into a mathematical model. The mathematical model allows the student to apply mathematical procedures to compute a mathematical result. Then the mathematical result is interpreted with regard to reality in order to obtain a real result that should be validated with respect to the real situation. The validation can lead to the need to revise the solution and the constructed models by applying the modelling cycle again. There are various types of modelling problems with different characteristics (Maaß, 2010). An important characteristic of modelling problems that is particularly relevant to this study is that they often do not contain all the data needed to find an accurate solution (i.e., the initial state of the problem is unclear). Following Maaß (2010), such modelling problems will be titled modelling problems with missing data. An example of a modelling problem with missing data is the "Fire Brigade" modelling problem (see Figure 1). Missing data for this modelling problem include the position in which the fire engine is parked or the height of the fire engine where the ladder is attached.

Fire Brigade

There is great happiness at the fire brigade! The authorities have approved a request for a new turntable ladder vehicle. Now, the fire brigade can rescue people from great heights using a basket attached to the end of the ladder. The vehicle has an engine with over 250 hp and has space for 3 firefighters. The length of the turntable ladder is 30 meters. In addition, the firefighters were informed that some safety rules must be observed when using the new vehicle. The vehicle must maintain a safe distance of 7 meters when firefighting a burning house. In addition, care must be taken to ensure that the operation is not obstructed by power lines. Furthermore, in the event of a storm, the vehicle should not be operated as a matter of principle when wind speeds reach 100 km/h.

From what maximum height will the fire department be able to rescue people from danger with this new vehicle?

Figure 1: The "Fire Brigade" modelling problem, modified from Schukajlow and Krug (2014)

Characteristics of modelling problems with missing data

In ill-defined problems, the data, goals, and operators are not clearly specified (Jonassen, 2000). Modelling problems with missing data share these important characteristics with ill-defined problems, including the opportunity to develop different solutions to the same problem. From a normative point of view, the ability to solve modelling problems with missing data can be useful for students' actual and future lives because most of the problems that occur in their daily lives and at work are ill-defined. Schukajlow and Krug (2014) identified benefits of prompting students to find multiple solutions to modelling problems with missing data on students' interest, experiences of competence, and autonomy. In another study, a feeling of autonomy that came from solving modelling problems with missing data was found to be one of the important sources of students' interest in these problems (Schulze Elfringhoff & Schukajlow, 2021). In previous research, many studies have used modelling problems with missing data without emphasizing that data were missing, which is an important feature of the problem. A typical class of modelling problems with missing data are Fermi problems, where a large amount of information is missing, and reasonable assumptions and a series of estimates are necessary for finding a solution (Ärlebäck & Bergsten, 2013), much more than in the problem presented in Figure 1. Therefore, the special type of problem used in this study is characterized as unique because it is missing relatively little data, but the data that are missing are crucial for the solution with respect to reality.

In order to solve modelling problems with missing data, such as the one in Figure 1, three steps seem to be crucial from a theoretical point of view (Krawitz et al., 2018): (a) Students must identify the problem as an ill-defined problem by noticing that the numerical data given in the problem is not sufficient to solve the problem adequately; (b) they must identify which quantities have to be estimated; and (c) after noticing which data are missing, they have to make assumptions about how to deal with the missing data. In order to make assumptions, students need to conceptualize the real-world situation, which requires realistic considerations and extramathematical knowledge. Further, estimation skills and strategies, such as the reference point strategy, are necessary. For the "Tree Track Adventure Park" modelling problem (Figure 2), students must notice during their processing that Mr. Meier needs enough rope so that the rope that will connect the two trees can be securely attached to each tree. Thus, students must estimate how many meters of additional rope will be needed to securely attach the rope to each tree by wrapping the rope around the trees. To make an assumption, students can either just estimate how much rope will be needed, or they can compute the amount of rope that will be needed after estimating how thick the trees are and how many times the rope will be wrapped around them.

Tree Track Adventure Park

Mr. Meier is the owner of a tree track adventure park. He opened this attraction for young and old guests 12 years ago. Now he has to replace the steel ropes to which the safety lines are clipped. These steel ropes are very expensive. To replace them, Mr. Meier needed more of the steel cable than originally planned. He has only 22 m of steel rope left for the last zip line between two trees. The two trees are 17 m apart. So that visitors can then slide from one tree to the other, the steel rope is attached to the starting tree at a height of 10 m and to the ending tree at a height of 8.5 m.

Does Mr. Meier have to buy more steel rope or is the existing steel rope sufficient?

Figure 2: The "Tree Track Adventure Park" modelling problem, modified from Schukajlow and Krug (2014)

If the students neglect the real-world aspect of attaching the rope, they can still calculate how many meters of rope are needed between the trees and give this as an answer, thus concluding that Mr. Meier has enough rope, but this is in fact an unrealistic answer. The rope that is available is not enough because Mr. Meier needs more rope to attach the rope to the trees, so he needs to buy more rope.

No or inappropriate assumptions in a modelling problem can also result in an inadequate situation model and an inadequate mathematical model for the problem situation (Chang et al., 2020). Thus, it is important for students to master the steps needed to solve modelling problems with missing data. Previous studies have pointed out that students have trouble solving modelling problems with missing data because they seem to separate their knowledge of the real world from their mathematical knowledge and tend to ignore the context of the problem (Galbraith & Stillman, 2001). However, in-depth research on students' individual work and thinking with respect to the demands of such problems is rare.

Research questions

In this study, we addressed students' processing of modelling problems with missing data. Thereby, we first analyzed the extent to which students noticed missing data while processing modelling problems with missing data. Second, we investigated whether students made assumptions about the relevant missing data in the modelling problems used in this study. Thus, we posed the following research questions:

1. How do students process modelling problems with missing data?
    a. To what extent do students notice that data are missing?
    b. Did students make assumptions about the missing data?

Method

Participants

Participants were 12 students (seven girls and five boys) from different secondary school tracks in Germany. Participants came from high-, middle-, and low-track classes. They were between the ages of 14 and 16 and were in Grades 9 to 11. All students participated voluntarily in the study with their parents' permission. To select the participants, we followed the principle of maximum variation. Thus, we chose students who varied in gender, age, and mathematical performance. To assess students' mathematical performance, we considered students' math grades, the types of classes they were taking, and the types of schools they were attending. Students' grades ranged from very good to deficient. It was ensured that all participating students had already covered the unit on the Pythagorean theorem, which was required to solve the problems in the study, in previous mathematics lessons.

Procedure

In order to address the research questions, we conducted a three-step procedure that combined the methodological approaches of thinking aloud, stimulated recall, and interview. The procedure began with instructions for the think-aloud method, where students watched a video with a demonstration of the think-aloud method and practiced the method to solve a nonmathematical task. After the instructions, first, all students worked individually on the same given modelling problems with missing data using the think-aloud method. Students were videotaped as they processed the modelling problems. Second, after solving all the given problems, the stimulated recall was conducted. Video recordings of students working on all the problems administered in the study were shown to the students immediately after they finished working on the problems. Students were asked to pause the video recordings whenever they wanted to add something that was going through their minds about the situation that was shown on the screen. If the interviewer thought that the situation shown in the video recording was relevant to the research question, and the students did not pause the video recording, the researcher could pause it too. By doing this, the insights from the first step could be intensified. In the third step, the students were administered a semi-structured interview with in-depth questions about their work on the three problems. In particular, questions were asked about how students handled the missing data from the problems.

Modelling Problems

In this study, we used two modelling problems with missing data. Both problems could be solved by applying the Pythagorean theorem, which is part of national and international curricula. The two problems—"Fire Brigade" (Figure 1) and "Tree Track Adventure Park" (Figure 2)—were modified from prior studies (Schukajlow & Krug, 2014). For our study, the relevant data that was missing was always numerical. Thus, students needed to make assumptions about the missing quantities in order to solve the problems correctly. All problems were developed so that students were also able to perform calculations without noticing that important data were missing. For the "Fire Brigade" modelling problem, the height of the fire engine, and for the "Tree Track Adventure Park" problem, the additional rope needed to attach the rope to the trees were considered the relevant pieces of missing data.

Data Analysis

The transcripts of students' responses were analyzed using a qualitative content analysis (Mayring, 2014). The deductively developed coding scheme was used to code the sequences with regard to modelling activities from the modelling cycle by Blum and Leiss (2007). Another coding scheme was used to code how the students dealt with the characteristic demands of modelling problems with missing data (Krawitz et al., 2018). More specifically, we coded whether students had noticed the relevant missing data from the modelling problem and had made a realistic numerical assumption.

Results

We analyzed whether the students noticed that relevant data was missing from the problems (Research Question 1a). In the solutions given by 7 of the 12 students for the "Fire Brigade" problem, the students noticed that the height of the fire engine was missing (Figure 3). Among other objects, Anton's mathematical drawing included the fire engine with its height (Figure 3, left).

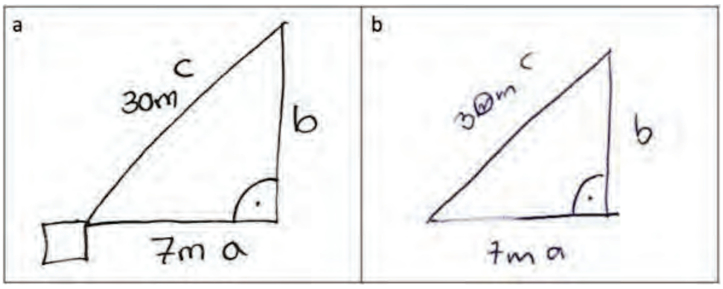

Figure 3: Excerpts from students' solutions to the "Fire Brigade" modelling problem

In the sequence presented below from Anton's processing while thinking aloud, he commented:

Anton:	Okay. (…) Here is a house wall, I don't know how big it is yet. (...) The safe distance must be seven meters. Seven. (…) And the height of the fire/the turntable ladder is thirty meters. (...) The turntable ladder is not completely on the ground.

By contrast, in the drawing made by Paula (Figure 3, right), data about the height of the fire engine was not considered. The sequence presented below illustrates that she was not thinking about the missing data while making her drawing.

Paula:	The turntable ladder is thirty meters and uhm (…) the safe distance of seven meters. So that's seven meters. And then (…) the ladder would have to be angled so thirty (…) thirty meters. And that is then (…) the height.

For the "Tree Track Adventure Park," students were less likely to notice the relevant missing data (only 2 out of 12 students noticed). For the "Fire Brigade" problem, we investigated how students' noticing of the missing data interacted with their modelling process by analyzing when (i.e., between which sequences of modelling activities) the students noticed that the height of the fire engine was missing. Figure 4 illustrates Anton's modelling process with its modelling activities (Blum & Leiss, 2007), and the arrow indicates when he noticed the missing data. It took Anton four minutes and 17 seconds to process the "Fire Brigade" modelling task.

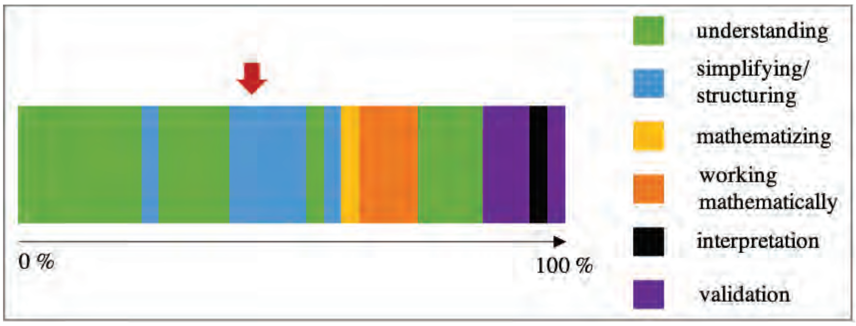

Figure 4: Anton's modelling process

Anton noticed the height of the engine between sequences in which he was simplifying/structuring. Students usually noticed the missing data in one of two different phases in the modelling process. First, the relevant missing data were noticed at the beginning of the modelling process after an initial understanding and after or during simplifying/structuring, but usually before working mathematically. An example of this case is Anton. Second, some students noticed the missing data after they had completed the modelling cycle activities (i.e., after mathematizing and working mathematically and usually between the validation sequences). These students noticed the missing data after they had already obtained an initial mathematical result. In the validation phase, students noticed that the answer they had calculated was not appropriate for the problem situation. In this case, some students began to correct their previous solution by considering the missing data, whereas other students stuck with their inadequate solution without making any changes. Further, noticing the missing data was not related to the activities of mathematizing or working mathematically. After students noticed that the height of the fire engine was relevant data, they should make a numerical assumption about the missing data and consider this assumption in the next steps of their modelling process. However, only five of the seven students who noticed the missing data also made a numerical assumption. The other two of the seven students did not make any assumptions. An illustration of this can be seen in a sequence of Julius' solution while thinking aloud.

Julius: Although, (…) the vehicle height, but I think it doesn't matter.

Julius does not see the height as being particularly important for solving the problem. In order to answer Research Question 1b, we analyzed the extent to which the students made appropriate reality-based numerical assumptions. Comparing the students' numerical assumptions with the real data, we found that all the students' assumptions were within an acceptable range around the real data. Different strategies for making a numerical assumption were found. For example, how Berta made her numerical assumption is described in the following sequence from the stimulated recall.

Interviewer: How did you do that?

Berta: I (…) when you look outside, you can see the trees, and then I thought to myself, or in the climbing forest, there are sometimes trees like that, and then I thought to myself that (…) yes, if you just put an arm around it, I don't know, then that's (…) usually a meter or two, if you, if those are thicker trees.

Summary and Discussion

We investigated students' processing of modelling problems with missing data (i.e., real-world problems where the initial state of the problem is ill-defined). In this study, we analyzed whether students noticed relevant missing data and whether they made realistic assumptions about the missing data. An important finding of the study was that students seem to have trouble noticing the relevant missing data in the modelling problems. One possible explanation for this finding could be that students fail to think about the real-world situation of the modelling problem and just take the quantities given in the problem description and then work mathematically with these quantities. This idea would contribute to previous research on mathematical modelling by pointing out that students do not appropriately consider the real-world context of a problem when they process problems with missing data. The modelling problems used in this study were designed to analyze how students deal with missing data and to make it possible for students to perform calculations without noticing the missing data. However, ignoring the missing data in these tasks leads to unrealistic answers. As the necessity of noticing missing data can easily be overlooked, an intensive consideration of the real-world context was essential for solving these problems. As the number of students who noticed the missing data varied considerably between the modelling problems, this finding suggests that whether students notice missing data depends on the specific characteristics of the task. These characteristics might be the context of the problem or the use and nature of the missing data. With respect to the modelling cycle (Blum & Leiss, 2007), we found that some students noticed the missing data directly after reading the problem description, indicating that these students try to develop a more in-depth understanding of the given situation before continuing to process by mathematizing and working mathematically. In the other cases in which students noticed the missing data, the in-depth understanding of the real-world context took part at the end of the processing, after mathematization and working mathematically. Another important finding of our study is that students who noticed the missing data did not always make a numerical assumption, but when students did make an assumption, the assumptions were realistic. This indicates that measurement skills or estimation strategies are not the key obstacle to making assumptions. One possible explanation for the results regarding difficulties in including assumptions in processing might be that problems with missing data and the ability to make assumptions might not be a part of students' mathematics classes. Another possible explanation may be students' conceptions and beliefs about mathematics.

A practical contribution of our study is that students need practice processing modelling problems with missing data in school. Therefore, teaching methods should address students' ability to notice missing data and make assumptions. As the results of our study show, close attention should be paid to the phases of simplifying, structuring, and validation that were particularly relevant for the noticing of missing data. Teaching methods need to stimulate these phases intensively.

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Characterising rationales for using statistical modelling in education research from a mathematical modelling perspective: A review

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This study developed and discussed a framework for characterising rationales for using statistical modelling from a mathematical modelling perspective based on a systematic literature review. We used this framework to provide an overview of the distributions of these rationales in the analysed studies, focusing on statistical modelling. The results identified three categories of rationales for using statistical modelling, namely, competency, content, and socially oriented types. This work discussed how the conceptualised rationales for using statistical modelling in the teaching and learning of statistics can guide the initial step in task design and curriculum development related to statistical modelling. These rationales may also serve as a common point of departure for discussions and collaboration between the mathematics and statistics education communities on the use of models and modelling.

Keywords: Statistical modelling, mathematical modelling, rationale, systematic review.

Introduction

Research on the teaching and learning of statistical modelling (SM) has increasingly gained attention in the statistics education community (e.g. Langrall et al., 2017; Pfannkuch et al., 2018) and mathematics education communities (e.g. Ärlebäck & Frejd, 2021; Frejd & Ärlebäck, 2021; Kawakami & Mineno, 2021). This trend owes its rise to the advent of data science and the need for models to address uncertainty, and underpinned by the worldwide growth of mathematical modelling (MM) in education research (e.g. Kaiser, 2017; Niss et al., 2007) and the rapid technological development.

The present study is part of an overall research goal to elaborate on the connections, boundaries, and boundary-crossing approaches between the teaching and learning of MM and statistics or SM. In this study, MM refers to the generic modelling process involving back-and-forth transitioning between the extra-mathematical world and mathematics (Niss et al., 2007). SM refers to 'any one of a number of practices: the development of a distribution (empirical or descriptive model) from data; the process of creating a theoretical (probability) model from an empirical model; and the practice of sampling from a theoretical model (simulation)' (Langrall et al., 2017, p. 502). MM and SM both include the generation, use, evaluation, and revision of models, and both emphasize the real-world context. However, a characteristic of SM that is not always found in MM is that of making uncertainty and variability central aspects of modelling (Langrall et al., 2017).

Pfannkuch et al. (2018) highlighted interpretations of SM and frameworks for describing students' reasoning with SM in statistics education literature. However, the design guidelines for SM tasks and curriculum components related to SM have not been identified and elaborated. As with MM in mathematics teaching and learning (Niss et al., 2007), such guidelines are essential for including SM as a significant component of statistics teaching and learning at all levels of education. This study examined the rationales for using SM, given the particular significance of rationales in designing SM tasks and curricula related to SM. Studies on teaching and learning SM have employed various rationales for using SM (Kawakami, 2019; Pfannkuch et al., 2018), but these rationales have yet to be systematically organised and structured. To facilitate the work of researchers, teachers, and curriculum developers in systematically designing purposeful SM tasks and to position SM practices appropriately within the statistics and mathematics curricula, this study aimed to (i) develop and discuss a framework for characterising the rationales for using SM from an MM perspective based on a systematic literature review; and (ii) provide an overview of the distributions of these rationales in studies focusing on SM.

Theoretical framework

Commonalities between rationales, goals, theories, and practices in research focusing on MM and SM have been noted and broadly pointed out (Frejd & Ärlebäck, 2021; Langrall et al., 2017). To investigate some aspects of these commonalities in more detail, we used the rationales for using MM in mathematics education to clarify the characteristics of SM from an MM perspective.

Rationales for using MM in mathematics education

The reasons for including the teaching and learning of MM in mathematics education have been discussed from at least two perspectives. Niss and colleagues (2007) introduced dual rationales for using MM: *modelling as a means for developing competency in applying mathematics and building mathematical models* and *modelling as a means for learning mathematics*. The former uses MM to develop a general MM competency (analysing and constructing mathematical models of extra-mathematical contexts and situations) by focusing on the use of mathematics in real-world contexts and problem solving. The present work referred to this rationale as *competency oriented*. The latter, modelling as a means for learning mathematics, uses MM to support the learning of mathematical contents through modelling activities 'by offering motivation for its study as well as interpretation, meaning, proper understanding and sustainable retention of its concepts, results, methods and theories' (Niss & Blum, 2020, p. 28). Our work referred to this rationale as *content oriented.* Niss and Blum (2020) stressed that these two rationales are not dichotomous but can be pursued simultaneously. However, the priority of either can change the aims and designs of mathematics lessons. In addition to these two rationales, Barbosa (2006) proposed a third rationale from a critical mathematics education perspective, namely, *modelling as a means for reflecting on the nature and role of mathematical models in society*. This perspective uses MM to 'emphasize critical thinking about the role of mathematics in society, the role and nature of mathematical models, and the function of mathematical modeling in society' (Kaiser, 2017, p. 274). We referred to this rationale as *socially oriented*.

Three potential rationales for using SM in the teaching and learning of statistics

Pfannkuch et al. (2018) noted and listed four distinct educational purposes for using SM in the teaching and learning of statistics: (P1) enculturating students engaged in the discipline with an approximation of professional statistical practice and reasoning, thinking, points of view, and beliefs; (P2) instructing students in key statistics knowledge and concepts; (P3) developing students' notions of the power, nature, role, purpose, and utility of SM; and (P4) enabling students to gain insights on particular situations. However, these four objectives have not been elaborated upon, nor are their relations discussed systematically. In this study, we sought to elaborate on the rationales for using SM by relating these four purposes to rationales for MM.

P1 employs SM as a means for developing students' statistical competency. In statistics education, such a statistical competency is often framed and discussed in terms of statistical literacy, statistical reasoning, or statistical thinking (Garfield & Ben-Zvi, 2008). According to Garfield and Ben-Zvi (2008), statistical literacy is the ability to interpret, evaluate, and communicate statistical information and messages; statistical reasoning is the ability to connect statistical concepts and explain statistical processes and results; statistical thinking is the ability to use statistical models, methods, and applications in advancing statistical investigation, such a PPDAC cycle (Problem–Plan–Data–Analysis–Conclusion) (Wild & Pfannkuch, 1999). In terms of the rationales for using MM, P1 can be described as *competency-oriented SM*. P2 and P3 employ SM for promoting the learning of statistical contents, such as statistical knowledge and concepts (e.g. variability, distribution, sample, and sampling) as well as knowledge and concepts related to statistical models and modelling. Employing the rationales for using MM, P2 and P3 can be described as *content-oriented SM*. Lastly, P4 employs SM for decision making in the real-world, social, and societal contexts, in which data are embedded and, in some cases, for developing a critical understanding of the use and role of statistics, statistical models, and modelling in these contexts. In terms of the rationales for using MM, P4 can be described as *socially oriented SM*.

In summary, we have elicited three potential rationales for using SM in the teaching and learning of statistics: (R1) *competency-oriented SM*, (R2) *content-oriented SM*, (R3) *socially oriented SM*. These three rationales for using SM, as well as the rationales for using MM, are not in opposition to one another. Indeed, multiple ones may merge within a single practice. As the priority of any of these rationales can influence the aims and design of statistics lessons, we make a distinction among these three rationales. The present study addressed two questions: (i) How can rationales R1–R3 be further elaborated and understood based on the use of SM as also done in empirical research? (ii) What is the use distribution of rationales R1–R3 in empirical research on SM?

Methodology

We conducted a systematic literature review of peer-reviewed research on SM in mathematics and statistics education. Research papers from the following influential mathematics education journals were identified: *Educational Studies in Mathematics* (ESM) (May 1968–July 2021), *ZDM: Mathematics Education* (ZDM) (1997–July 2021), *Mathematical Thinking and Learning* (MTL) (1999–July 2021), *Journal for Research in Mathematics Education* (JRME) (1970–July 2021), and *Journal of Mathematical Behavior* (JMB) (1995–July 2021). Research papers from the following internationally recognised journals in statistics education were also included: *Statistics Education Research Journal* (SERJ) (2002–July 2021) and *Journal of Statistics Education* (JSE) (1993–July

2021). The paper selection process is illustrated in Figure 1. As part of the systematic review process, we read all identified 63 papers, and then coded the rationales for using SM in the papers by assigning them into the three categories R1, R2, and R3. The coding was determined based on the explicit description(s) of the rationale(s) for using or investigating SM or modelling in the papers by focusing on the (1) purpose and position of the study; (2) intentions and purposes of the used teaching materials, curriculum, and teaching practices; as well as (3) research questions in the papers. Examples of such descriptions and their coding are shown in Table 1. The first author conducted the first analysis, which included classification, and the second author independently checked the assigned papers and the first analysis. Where discrepancies occurred, the authors discussed and resolved these issues.

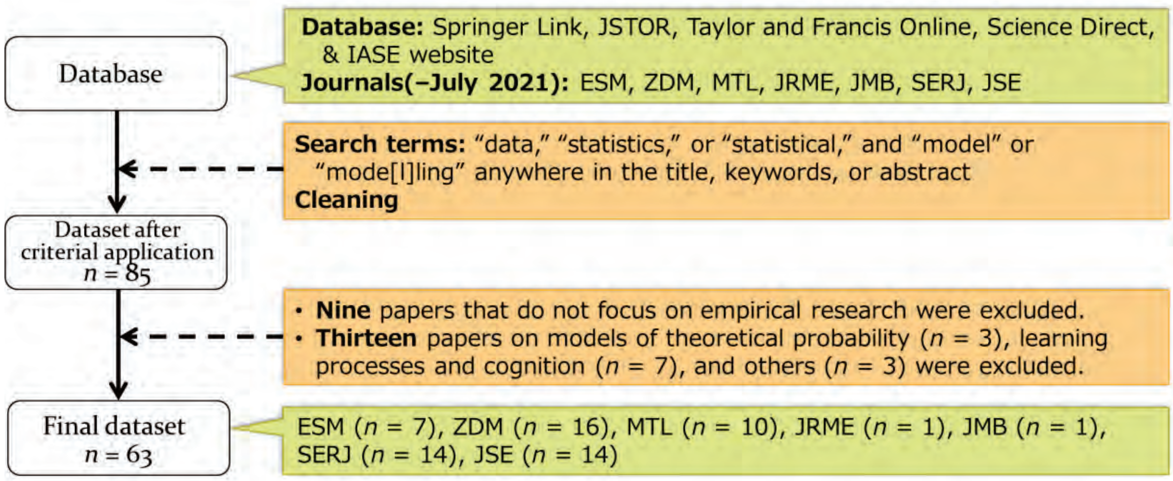

Figure 1: Paper selection process

Table 1: Examples of the descriptions from which categories R1–R3 were determined

Category	Examples of the description in the papers
R1	In this article, we have examined one approach to developing primary school students' statistical literacy, namely, through modelling with data. (English & Watson, 2018, p. 113)
R2	The study relied on classroom video and student artefacts to analyse aspects of the students' modelling experiences which exposed them to powerful statistical ideas, such as key repeatable structures and dispositions in statistics. (Makar & Allmond, 2018, p. 1139)
R3	Statistical modelling needs to become a tool for critical democracy. (Zapata-Cardona, 2018, p. 1220)

Results

Table 2 presents the identified rationales for using SM in the analysed literature with the frequency and examples within each category.

Table 2: SM rationales in the literature in terms of categories R1, R2, and R3 (*n* = 63)

Category	Freq. (%)	Examples (listed by author only given space constraints)
R1	19 (30)	Doerr & English (2003)$_{\text{JRME}}$, Biehler et al. (2017)$_{\text{SERJ}}$, Doerr et al. (2017)$_{\text{SERJ}}$, Noll & Kirin (2017)$_{\text{SERJ}}$, Dvir & Ben-Zvi (2018)$_{\text{ZDM}}$, Leavy & Hourigan (2018)$_{\text{ESM}}$
R2	13 (21)	Prodromou & Pratt (2006)$_{\text{SERJ}}$, Lesh et al. (2008)$_{\text{ESM}}$, Ainley & Pratt (2017)$_{\text{SERJ}}$, Büscher & Schnell (2017)$_{\text{SERJ}}$

R1 & R2	21 (33)	English (2012)$_{ESM}$, Manor & Ben-Zvi (2017)$_{SERJ}$, English & Watson (2018)$_{ZDM}$, Kazak et al. (2018)$_{ZDM}$, Makar & Allmond (2018)$_{ZDM}$, Patel & Pfannkuch (2018)$_{ZDM}$, van Dijke-Droogers et al. (2021)$_{ESM}$
R1 & R3	5 (8)	Simonoff (1997)$_{JSE}$, Biehler et al. (2018)$_{ZDM}$, Wilkerson & Laina (2018)$_{ZDM}$
R2 & R3	2 (3)	Zapata-Cardona (2018)$_{ZDM}$, Zapata-Cardona & Martinez-Castro (2021)$_{MTL}$
R1, R2, & R3	3 (5)	Jacobson et al. (2009)$_{JSE}$, Garfield et al. (2012)$_{ZDM}$, Kazak et al. (2021)$_{MTL}$

Only R1 rationale: Competency-oriented SM

Papers in which R1 was the only rationale used comprised the second largest category of papers, accounting for about 30% (*n* = 19). These studies typically used SM to develop statistical competencies (e.g. statistical literacy, reasoning, and thinking) as well as statistical processes (e.g. *statistical inquiry* in the sense of Wild and Pfannkuch [1999] and *informal statistical inference* in the sense of Makar and Rubin [2009]). These studies emphasised the role of statistical models and modelling in developing statistical competencies; for example, '[m]odels are important concepts in statistics and key components of learning to think statistically' (Noll & Kirin, 2017, p. 213). Exclusively, R1-coded papers often described SM as applicable for real-world problem solving and stressed the applied nature of statistics, emphasising the use of actual and authentic data in the educational setting.

Only R2 rationale: Content-oriented SM

Papers in which R2 was the only rationale used comprised the third largest category of papers, accounting for about 21% (*n* = 13). These papers used SM to elicit, develop, and deepen the understanding of a wide range of statistical contents, including an aggregate view of data, measures of distribution, signal and noise, variation, population, sample and sampling, theoretical distributions, statistical models, SM process, causality, and statistical inference. These papers tended to reframe statistical concepts and knowledge as models or modelling constructs in support of the notion of *emergent modelling* (Gravemeijer, 1999) and *model-eliciting activities* (Lesh & Doerr, 2003); for example, 'measures are understood as models, which can either be used to make sense of a given situation or to reason about the statistical measures themselves' (Büscher & Schnell, 2017, p. 144). Exclusively, R2-coded papers often expressed SM as an epistemic practice of statistics and a pedagogical tool.

Combined R1 and R2 rationales: Competency- and content-oriented SM

The largest category of papers employed both R1 and R2 (approximately 33%, *n* = 21). These papers used SM to combine and integrate statistical competencies (R1), such as statistical reasoning and informal inference, and statistical contents (R2). In these papers, the emphasis was not only on real-world problem solving, transiting between the real (data) and model (data) world, but also on conceptual developments within the model world. Some papers considered statistical literacy and reasoning as competency to carry out SM, comparable to MM competency. For example, Patel and Pfannkuch (2018) framed *SM reasoning* as the ability to transit between the physical and model worlds involving the following activities: starting with understanding of the real-world problem, applying structure to it toward a transition to the model world, refining the model and analysing simulated data, and interpreting the results back into understanding regarding the original real-world problem. R1-and-R2-coded papers often used SM as an integrator between statistical methods and statistical content.

R3 rationale: Socially oriented SM

R3 was the least commonly discerned rationale in literature (approximately 16%, n = 10). None of the papers used it as the sole rationale, only using it in tandem with R1 and/or R2. These studies used SM to enhance critical thinking on real-life, social, and societal contexts and to examine the power and limitations of statistical models and modelling. In some cases, the discipline of statistics embedded in these contexts were examined using the notions of authenticity, critical citizenship, ethics, and publicity. These papers focused on the role of statistical models and modelling in thinking critically about life and society. For example, Zapata-Cardona (2018, p. 1220) observed that '[s]tatistical modelling needs to become a tool for critical democracy' and '[m]odelling activity should focus on the functions of the applications in society'. R3-coded papers often described SM as a means and object of social criticism and decision-making based on data. They also emphasised the use of social issues and contexts in education.

Discussion and conclusion

Our analysis showed that all three rationales for using SM in research on the teaching and learning of statistics were used to various extents in empirical studies. In other words, SM is seen as a multifaceted means of achieving applied, epistemic, and social critical-related goals in the teaching and learning of statistics. However, based on the results of the systematic review in terms of R1 to R3, SM can also be understood more holistically: (i) R1 can be seen as stressing *SM as a component of statistical competencies*, i.e. the use of statistical models/modelling as a key element for promoting real-world problem solving and statistical inquiry (e.g. Noll & Kirin, 2017); (ii) R2 can be taken to promote *SM as integrating aspects of statistical contents*, meaning that the structure of statistical content is rooted and reflected in statistical models/modelling (e.g. Büscher & Schnell, 2017); and (iii) R3 portrays *SM as shaping and influencing real-life, social, and societal decision making*, indicating that statistical models/modelling form the basis for data-driven decision making (e.g. Zapata-Cardona, 2018). Figure 2 summarises these three conceptualised rationales for using SM in the teaching and learning of statistics. In each rationale in Figure 2, the goal–means relation is depicted with arrows. Inclusionary relations between the goal and means are shown by the ellipses.

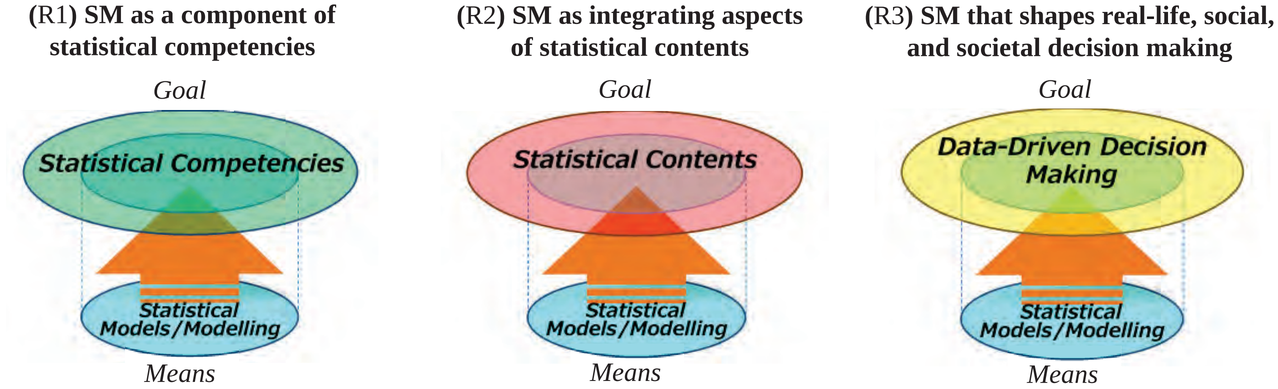

Figure 2: Three conceptualised rationales for using SM in the teaching and learning of statistics

The review results also revealed that these three rationales can be used together in one lesson, unit, curriculum, or project. Indeed, the exclusive use of R3 was not found in any of the 63 analysed papers. This is in contrast to the research on MM, where the lone use of the socially oriented rationale is common (cf. ethnomodelling). The combination of R1 and R2 was the most common category, with three papers using all three rationales (R1–R3). Hence, combining rationales is common in the context of SM, in contrast to much of the research on MM. These results may be related to the nature of statistics, which pertains to statistical methods, statistical content, and data with context (Wild & Pfannkuch, 1999). However, a more detailed review of MM literature is needed to establish further the viability of these differences between SM and MM.

The goal–means relations in Figure 2 support the hypothesis of the learning trajectory of SM. For example, learners can advance competency-oriented SM (R1) using contents constructed through content-oriented SM (R2). They can then perform socially oriented SM (R3) by making full use of competency and contents acquired through the other types. The inclusive relations in Figure 2 also suggest that different rationales have different domains and assumptions in which statistical models and modelling are placed and that, in practice, the functions/roles of statistical models and modelling can be changed dynamically. Therefore, it is also necessary to review the meaning of statistical models, modelling, and the description of SM specified in each study.

The conceptualised rationales for using SM in Figure 2 can guide the initial step in task design and curriculum development related to SM for researchers and teachers. It can also serve as a common lens for shared discussion and collaboration between the mathematics and statistics education communities on issues related to the teaching and learning of models and modelling (e.g. English & Watson, 2018; Langrall et al., 2017). However, to provide more concrete design guidelines, additional research is needed to identify and elaborate on more aspects and characteristics of SMs. The review results specifically demonstrated the applicability of the findings of MM education research to SM education research in terms of goals and rationales. A natural next step is to clarify the similarities and differences in teaching and learning SM and MM. It is also necessary to clarify to what extent we need to distinguish between MM and SM, or whether SM is a sub-form of MM. These, in turn, will shed light on the nature and role of models and modelling, and the underlying assumptions and hypothesis in, and for, MM research.

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An exploratory modelling task by slicing 3D shapes: the Three Worlds of Mathematics through an historical and technological approach

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In this work, high school students were assigned to investigate a problem which involves the transition from 2D to 3D interpretation using the method of exhaustion, one of the cornerstones ideas of Calculus. For this approach, the authors sought to develop the study through an exploratory modelling task, under the lens of the Three Worlds of Mathematics theory. In this scenario, the students were supported by an initial discussion based on Calculus early ideas and with the use of current technologies, such as GeoGebra and laser cutting machines. The data collected in this study came from a video recording of this initial discussion and from the portfolios of three students who explore the volume of the solid generated by the rotation of a cycloid, comparing it with classical results in two dimensions.

Keywords: Three worlds of mathematics, method of exhaustion, cycloid, GeoGebra.

Introduction

In this work, high school students were assigned to investigate a question which involves the transition from 2D to 3D interpretation using some of the cornerstone ideas of Calculus. For this approach, the authors sought to develop the study through an exploratory modelling task, under the lens of the Three Worlds of Mathematics theory (Tall, 2013). In this scenario, students were supported by an initial discussion based on Calculus early ideas and with the use of current technologies, such as GeoGebra and laser cutting machines. In this regard, dynamic simulations were digitally represented based on a physical model to initialize the discussion and enhance students' mathematical ideas. Observing the interactions during an online meeting and students' portfolios, the authors of this paper aimed to identify (or not) the benefits coming from either physical and digital prototypes and how they can support one another. Based on these ideas, the following research question emerged: *how do students understand and combine ideas from the Three Worlds of Mathematics while using 2D and 3D resources and prototypes to build basic ideas of Integral Calculus?*

Motivation

The nature of slicing a 3D shape to develop mathematical concepts goes back to Archimides, who resorted to the method of exhaustion in order to have a more rigorous proof of his theorems, particularly following his goal to determine the volume of a sphere (Bueno, 2020). Nevertheless, it is not hard to identify such a technique in day-by-day contexts and this can be a natural reason to connect 2D and 3D representations in geometric studies and working with modeling. These practices take part of the Conceptual Embodied world (Tall, 2004), which is developed by students' perceptions and actions in manipulating such shapes and its slices. The examples in Figure 1 illustrate some possibilities in this sense. While the tomato and sheets of paper easily connect to daily life contexts, the artcraft lighting shows an example developed by students in a Calculus undergraduate course.

The central picture is a physical model developed through the integration of GeoGebra and laser cutting techniques and it was used to spark the discussion with the students in this study.

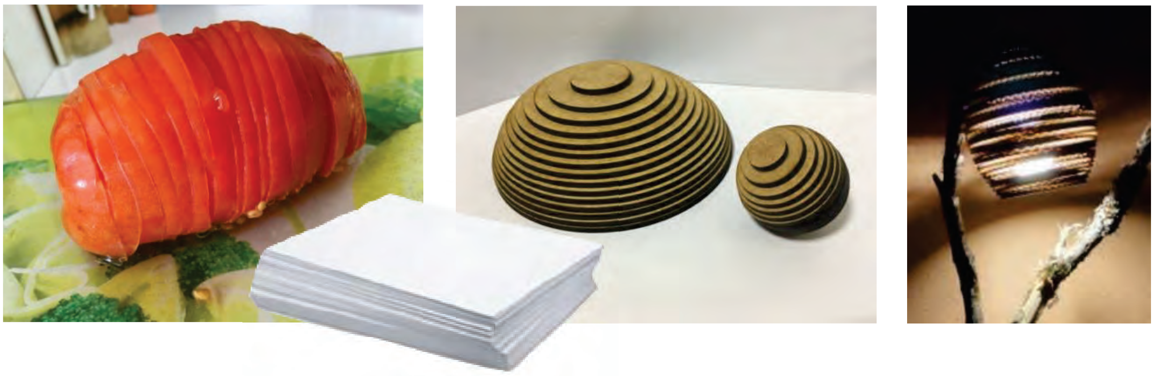


Figure 1: Conceptual Embodied world through shapes sub-divided by slices

Once the math behind the strategy to obtain the right volume of solids of revolution might rely on Integral Calculus, the problem posed to high school students was to approximate such volume based on geometrical concepts and supported by GeoGebra. Particularly, the task was to estimate the volume of the solid generated by the revolution of a Cycloid when it is turned around its symmetrical axes. The challenge is to investigate whether there is, or not, a straight connection with the circle that generates the Cycloid, since there is a known result in the 2D version of this problem, regarding their areas (Toffolo et al., 2020). To support the students' ideas, and in addition to the physical model that was shown, GeoGebra applets[1] were shared and discussed with student participation during an online meeting, when the problem was presented.

Theoretical Framework

David Tall, aiming to understand how mathematical thinking is built, developed the Three Worlds of Mathematics theory, which involves early perceptions about counting and arithmetic, reaches algebraic symbolism and goes on until mathematical border research. By creating the Conceptual Embodied, Operational Symbolic and Formal Axiomatic worlds, Tall (2013) brought ideas capable of supporting important thoughts about mathematics teaching and learning.

In this regard, Tall (2013) highlights that mathematical thinking development is more complex than the simple sum of new ideas to a fixed structure of pre-existing knowledge. Mathematical thinking is built, in fact, by a continuous rearrangement of mental conexions that evolve to reach, in time, knowledge structures more and more sophisticated.

Geometry study, for instance, starts with real objects manipulation. These objects have their physical particularities described by students in an informal way using the common spoken language. With time and development of news ideas, these particularities start being defined in a more accurate way and evolve until reaching more formal ways, like Euclidean Geometry. Some students who go to college to study exact sciences can go further and learn even more complex ideias, like Differential Geometry, for instance.

[1] https://www.geogebra.org/m/hacg6ex6 and https://www.geogebra.org/m/tykauqhp.

Arithmetics knowledge grows in a different way, because it is related to actions done with objects and not just with its embodied particularities, like shape or size. Therefore, the focus of arithmetics is counting, grouping and ordination, for example. These activities occur, inicialy, on the perceptual world and evolve until the use and manipulation of mathematical symbols that arise from operations and go further until its manipulation without great conscient cognitive effort.

Algebra arises from the generalization of the perceptions built with arithmetics. New rules are understood and fill out the blanks of previous studies until the moment that equations and functions start being used as cognitive resources to solve problems. With this gradual mathematical thinking sophistication, functions can be understood, for instance, by visualizations of the cartesian plane. In the long term, some algebraic structures can be formulated with axiomatic systems, generating fields, rings or groups.

According to Tall (2004) theory, ideas of the Conceptual Embodied world are developed by perceptions and actions that happen in the real world and evolve until the creation of mental images becomes more and more sophisticated. Conceptions that don't belong to this real world are built in an abstract way, like a perfect straight line.

The idea of embodiment, as stated by Lima (2007), is related to observation, action and thought about situations with physical objects, but that evolve until the development of mental experiences. In this case, with abstract thoughts, objects (reals or imaginary) start being manipulated and observed by using only thoughts, generating new ways to perceive these actions.

A relevant alternative for this embodiment complexification can be the use of digital resources to manipulate objects. Therefore, with tools like GeoGebra it is possible to develop actions and, in a blink of an eye, verify its implications on the screen. Beyond that, with 3D printers and laser cutting machines, many mental constructions can be brought to life in the real and physical world and help teachers in their class, illustrating important ideas and supporting individual and collective mathematical perceptions.

Tall (2003) understands that educators should use new technologies to help them to build more meanings in Mathematics studies. With GeoGebra, 3D printers, laser cutting machines and other resources it is possible to change ways to teach and learn mathematics, going beyond algebraic manipulation and static graphics and reaching discussions and learnings from embodyments made possible by digital technologies.

The Operational Symbolic world, as highlighted by Tall (2013), arises from symbols used in arithmetics, algebra or even Calculus. It begins with actions that, with some time and thoughts, can evolve and until it is perceived as mathematical processes or even as mathematical concepts. In this regard, some people reach only the process idea while others go further and understand symbols as processes to do and also as concepts to think about.

Discussing this duality of mathematical symbols, Gray and Tall (1991) brought the idea of "procept", understood as the amalgamation of process and concept. According to Lima (2019), if some individual understands symbols as "procepts" he/she will end up developing "proceptual" thinking.

The Formal Axiomatic world, as suggested by Tall (2004), can be characterized by axiomatic developments that generate definitions, theorems and corollaries (among others) that have its ideas deduced from mathematical demonstrations. In this mathematical environment thoughts are not restricted to embodiment objects and thoughts are expanded to reach mental experiences even more sophisticated that end up with constructs that provide mathematical abstract structures such as an abelian group. Even knowing that this world begins to be studied only in college it is important to perceive that "in any mathematical activity formal mathematics ideas can be found, because a student doesn't understand new mathematical concepts without some presence of its formal characteristics" (Lima, 2019, p. 8).

According to Tall and Mejía-Ramos (2004), each one of the Three Worlds of Mathematics has its ways to warrant the truth. In the Conceptual Embodied world the truth is built from human intuition to, on a more advanced level, be grounded in mental experiments. In other words, something is proved right if it seems to be true or if some experiments end up occurring in the expected way.

In the Operational Symbolic world a formula is proved right because it can be developed by a sequence of algebraic manipulations. In the Formal Axiomatic world a theorem is proved right if it can be built by formal logical developments based on axions and mathematical definitions.

To an individual to trill his/her journey through the Three Worlds of Mathematics and develop knowledge, some previous ideas are necessary. Some of this previous knowledge leads to increasing progress while others lead to alternative paths that can, in some cases, lead to erroneous conclusions.

Tall (2013) calls this previous knowledge as *met-before* and uses this term to describe how new situations can be understood from the perspective of previous experiences. *Met-before*, although, can't be restricted only to experiences because it is linked also to any memories generated by those previous experiences and held on into the brain and that can affect new cognitive constructs.

Methodology and Methods

The explorative nature of the study demanded an interpretative approach comprising qualitative methods such as observation, data collection, and analysis (Cohen et al., 2011). Our experiments were carried out with high school students through an online course in Brazil and lasted a period of three months with meetings in every second week on average, to follow the students' progress. At the phase of the course when the problem was proposed, all participants had already been introduced to GeoGebra and it is important to highlight that students were free to choose which problem they would work with, among three alternatives. Beyond the cycloid problem, the other two were related to combinatorial studies (applied in a logical game) and fractals. From 19 students who achieved this investigative and exploratory task, 3 of them chose the cycloid problem (12 chose the game task and 4, the fractal task). Nevertheless all problems were introduced and initially discussed with the whole group.

The data collected in this research came from two different sources: i) a video recording from the initial session (~30min) when the problem was introduced and discussed with the whole group, and ii) the portfolios of the three students who dealt with the cycloid problem, in which they extended the discussion started at the whole group session and developed their ideas, research and calculation. During the online session with students, the mediator (one of the authors of this paper) emphasized that despite this problem could be tackled using Calculus techniques, the main goal would be to explore the basis of these techniques, it means, the math ideas behind the Calculus (or supporting it).

From this point, the mediator asked if the participants had seen anything about cycloid before and one of the students mentioned some animations with a circle generating the curve that she had already seen online (in a math blog and also in a GeoGebra applet). These animations as well as some experiments that give the cycloid its known alternative names – brachistochrone and tautochrone – were then shared and mentioned some reasons why this curve has an important role in the field of sciences. Some short math videos (from the Portuguese Mathematics show “That is Mathematics”[2]) was also recommended. Finally, the problem was presented, with an important remark that it had been motivated by the genuine curiosity of a former student a few months earlier.

Data Analysis and Results

The starting point for this problem, as mentioned in the previous sessions, was the initial online meeting, when the physical model composed by the slices of the solid obtained by the cycloid rotated was shared. Therefore, the discussion was introduced from ideas of the Conceptual Embodied world. In the same class, the mediator shared with the students two digital applets.

Before sharing such physical (through the cam, only) and digital representations, nevertheless, an introduction about the relevance of the Cycloid and some of its characteristics were explained and this step was crucial to attract students’ attention and contributions. Despite the fact that online communication has been challenging in general, some students did not hesitate to express their guesses when they were asked about the ratio between the solid generated by the rotation of the cycloid and the correspondent sphere. At this moment, they already had seen that the area under the cycloid was three times the area of the correspondent circle through the first shared applet. After discussing some free guesses, they looked suspicious and uncertain (some guesses suggested the ratio would be kept as the planar situation). Then, the second applet was shared, with a 3D interactive representation, so they could rotate the cycloid and visualize the final shape of the generated solid. This second applet allowed users to approximate the solid by a set of slices, starting with a cone, through the interaction with two sliders and facilitate the comparison between the original shape and its approximation by the set of truncated cones. Such interaction supported students’ interpretation regarding the strategy to approximate the unknown volume (adding up slices as thin as desirable, they will tend to infinity), but their task was still open.

From this point and based on the group discussion, three students developed their digital representations, shown on Figure 2, to approximate the volume, but the analysis of their portfolio shows they had slightly contrasting approaches.

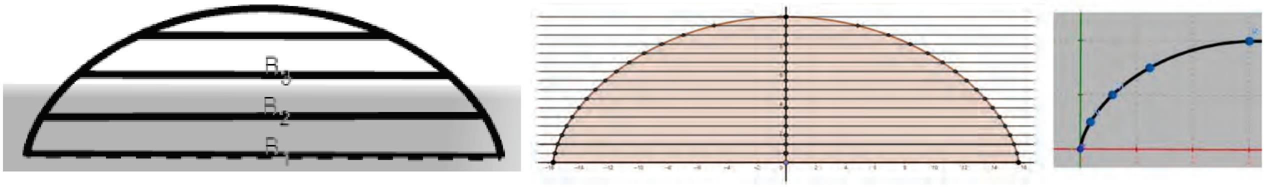
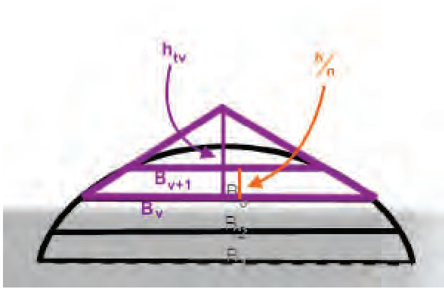


Figure 2: Shapes sub-divided by slices leading to ideas from the Conceptual Embodied wor

[2] https://www.youtube.com/istoematematica

Despite all three strategies arose from the Conceptual Embodied world, their goals were not exactly the same. Whereas one of them (student 1) sought to generalize an expression based on an arbitrary number of slices, the other two concerned to calculate the volume based on a particular example. In these last two cases, one of them (student 2) sought to determine the unknown ratio as accurately as possible, while the other one (student 3) tried to convince herself whether such a ratio was three, or not, since this was the ratio in the planar case, regarding the areas of the figures (cycloid and circle). These students' approaches indicate how the participants were able to recognize 3D shapes' properties and compare 3D shapes (in this case, solids of revolution), as well as understanding and visualizing the internal structure of the solid to calculate the volume of solids in the sense of 3D geometry abilities described by Pittalis et al. (2009).

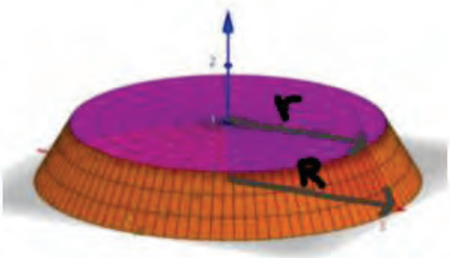
Considering the case of student 1, who tried to generalize the formula of the volume, it seems that he reached Operational Symbolic world ideas once he used digital algebraic descriptions to understand how developed ideas could be extended to more and more cases. Even though he did not obtain a final formula, he was able to generalize in some steps. In Figure 3 (right side), it is shown how he started defining the *height related to the basis B_1*, to obtain the height (named h_{tv}) related to an arbitrary basis B_v using similarity of triangles, understood by the authors as a met-before. The colorful remarks on the left side were added by the authors to illustrate the idea described in digital symbolic language by the student.



"h_t1 = Altura do triângulo de base B_1
*h_t1 = (-B_1*h)/n(B_2-B_1)*
*h_t1 = (-B_1*2r)/n(B_2-B_1)*
*h_tv = (-B_v*2r)/n(B_(v+1)-B_v)"*

Figure 3: Similarity of triangles to generalize a formula in the Operational Symbolic step

The other two students, as already mentioned, focused on calculating the volume of the truncated cones, also considering the radii from consecutives slices as their bases. In this case, news ideas arrived from this met-before (truncated cones). Particularly, student 2 used an online calculator to obtain the volume of the truncated cones and organized all data in a spreadsheet, as shown below, in Figure 4 (right side). The representation on the left side (Figure 4) is from one of the applets shared, but the notes (r and R) are from student 3.



Raio maior	Raio menor	Altura	Volume
15.7	15.6	0.62	477
15.6	15.4	0.62	468
15.4	15.1	0.62	453
15.1	14.8	0.62	435.3
14.8	14.4	0.62	415.2
14.4	14	0.62	392.8
14	13.4	0.62	365.6
13.4	12.8	0.62	334.3
12.8	12.2	0.62	304.4
12.2	11.4	0.62	271.3
11.4	10.5	0.62	233.7
10.5	9.5	0.62	195
9.5	8.4	0.62	156.2
8.4	7	0.62	115.8
7	5	0.62	70.8
5	0	0.62	16.2
	VOLUME TOTAL:		4704.6

Figure 4: Calculating the volume of the truncated cones

Despite both pictures above came from different students, they reveal the same strategy, with different number of slices. The ratio obtained by student 2 and student 3 were, respectively, 8,99 (with 16 slices) and 7,71 (with 4 slices), which are rather close to the real ratio, 9,10.

Conclusions

Physical and digital resources were combined in this research to support students in an exploratory modelling task, which the aim was to investigate the volume of the solid generated by the rotation of a cycloid around its symmetry axis. As motivation for the task developed in this paper the authors considered the interpretation from daily contexts of slicing and historical aspects that retrieve Archimedes' studies and the method of exhaustion, an important and cornerstone topic of Calculus. Using the Three Worlds of Mathematics as theoretical framework, the study intended to observe how the connections with such resources that use current technologies (GeoGebra and laser cutting machines) might help students on their journey through the Three Worlds of Mathematics – Conceptual Embodied, Operational Symbolic and Formal Axiomatic. Based on students' interaction during an online meeting and the portfolio of three students, the researchers found evidences that connect the conceptual embodied and operational symbolic in this task, specially connecting ideas from classical geometry, understood as met-before, to introduce seminal ideas of calculus.

In addition, this study is in line with Carreira (2019), who consider that modelling problems require a certain kind of mathematical thinking that could be called modelling thinking. For her, such thought aims to establish connections, analogies, similarities and relationships between different systems, among which mathematics is obviously included. These connections, according to the author, as well suggested by the evidences in this work, arise as a result of experimentation, exploration, understanding, conceptualization and manipulation of reality (in different environments, whether physical or digital). Although the problem in this study is sparked by the exploration of a mathematical object – the cycloid curve – the strategy applied might be naturally adapted for (and motivated by) other solids of revolution coming from the real world, such as illustrated in Figure 1. Thus, as highlighted by Carreira, the processes of mathematization, prototyping, experimentation and simulation take place in a cycle that involves the construction of a model (based on mathematics) that becomes multidimensional as the relationships among its variants (real model, physical model, mathematical model, computational model) are strengthened and consolidated.

The outlined examples are part of a set of experiments and activities developed in our practices. In future papers, we hope to report additional insights into connecting physical and digital modeling approaches, seeking to align with related and pertinent theories and pedagogies.

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On the information that prospective teachers assume when modelling and its impact on the modelling outcomes

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This paper characterises the mathematical models that prospective elementary teachers develop to solve a modelling task, as well as its dependence on those information and assumptions that students use without justification when developing the model. For this purpose, a sample of 74 prospective elementary teachers (45 women) was chosen to estimate the distance a ship is from shore when it sees the lighthouse on the horizon. The use of an analysis tool specific to characterize models and the analysis of those statements that individuals use in their models without justification showed a wealth of models. The relationship between individuals' awareness of certain information and the validity of such information was also suggested. Several assumptions on the curvature of the earth's surface or on the height of the ship resulted to had an impact on the models analysed.

Keywords: Mathematical models, educational research, prospective elementary teachers.

Introduction

Nowadays, mathematical educators in general and researchers in mathematics education in particular acknowledge the benefits of modelling at all levels of education. In the scope of teacher education, particularly, several authors point out the potential of the modelling activity to provide prospective teachers with challenging experiences allowing to explore mathematics content, serve as examples of teaching and help to manage students' beliefs (see Fernández-Ahumada & Montejo-Gámez, 2019 and references therein for a discussion on the benefits of modelling in teacher education). This makes modelling a field of great interest to researchers in mathematics education and teacher educators. One of the most relevant and differentiating skills of mathematical modelling is linked to taking actions that facilitate the application of mathematical tools to obtain knowledge about real systems, which are usually complex. In other words, modelling implies simplifying or organising a real system, in order to bring it closer to a mathematic formulation. Within Ärlebäck and Albarracín (2019) activity-based approach, for instance, those actions devoted to facilitate the application of mathematics are encompassed in the "Modelling" sub-activity. Under a modelling-cycle approach, on the other hand, they are included in the Simplifying/Structuring transition.

The role of assumptions in the transition of Simplifying/Structuring

Previous research has emphasised that difficulties in modelling are especially obvious when real-world situation has to be simplified (Dede, 2016; Kaiser et al., 2010), and there are studies describing the actions involved in the simplifying/structuring process of a real system (see e. g. Czocher, 2016; Maas, 2006; Montejo-Gámez & Fernández-Plaza, 2021). These include drawing/sketch of such real system, identifying and naming variables, operationalising relationships or patterns, introducing outside knowledge and estimating data, carrying out systematic experiments and using and formulating assumptions. In particular, there is consensus among the different authors on the importance of the management of assumptions within the modelling activity. However, some studies suggest that much of the difficulty in the simplification process arises when making assumptions (Dede, 2016; Hidiroğlu et al., 2014 cited by Dede, 2016), and the focus of teaching is hardly focused on the formulation, use or discussion of the assumptions of a model. On the other hand, Wozniak (2012) highlighted the importance of prospective teachers being able to explicitly express the actions carried out during the modelling activity. Consequently, it is appropriate to help students to become aware of the information they apply when modelling.

Research on the impact of this information on the modelling activity is not very extensive. Fernández-Ahumada and Montejo-Gámez (2019) introduced the term premises to refer to mathematical statements used in a model without prior justification. The analysis of this type of statements allowed the detection of difficulties in the modelling activity. Other studies made emphasis on the formulation of the assumptions. At this respect, Wozniak (2012) found 'mute praxeologies' when analysing prospective elementary teachers' activity: they used hypotheses, but generally did not formulate them. These studies suggest the usefulness of making visible the information that prospective teachers may assume when solving a modelling task and measuring the influence that this may have on the modelling outcomes. This leads to the following research question: what information do prospective elementary teachers assume as valid when approaching a modelling task and what is its impact on the models developed? To give an answer, this study aims to analyse written productions in order to access the assumptions made during the transition simplifying/structuring. The methods used are explained below.

Methods

A qualitative methodology was used for the development of the research. The sample used was made up of N = 74 students of the degree in Primary Education at the University of Granada (45 women and 29 men). The participants were organised according to the usual working groups in class for them: 1 group of two people, 12 groups of three people and 9 groups of four. The 22 groups worked for an hour and a half, without interactions between groups, to solve the following task, adapted from Kaiser (2014): *The Cabo Mayor lighthouse is situated at the North of Santander. The focus of this lighthouse is situated 91 m above sea level; thus, it is useful to warn ships that they were approaching the coast. How far, approximately, is a ship from the coast when it sees the lighthouse for the first time?* The work of the groups generated 22 written productions that reflected the models proposed by the students.

Analysis procedure

The research question was approached on the basis of a three-stage analysis. (1) The first one was to characterise and categorise the models developed. For this, the instrument and procedure presented by Montejo-Gámez et al. (2021) were used, which are schematised in Figure 1. The procedure starts by deciding what is to be considered mathematical content. Next, per each written production, the representations used in such production were identified, categorised and then analysed: On one hand, the questions of the system and the mathematical questions of the model were identified (depending on whether or not they contained mathematical content). On the other hand, the statements involved in the representations drove to identifying the relations of the system (those without mathematical content) and mathematical results (those with mathematical content) of the model. Then, the entities referred to in the relations were identified as the objects of the system, whereas the relevant quantities involved in the results were identified as the variables of the mathematisation. Finally, the underlying mathematical properties from the results were abstracted, and hence, the concepts involved. In this way, the system and the mathematisation that made up the mathematical model in the written production were obtained. The synthesis of the differentiating elements provided a summary characterisation which gives an overall idea of the model developed and highlights its main attributes (see Table 1 below). Once this was done with all the written productions, models were organised into emergent categories, according to such attributes and the validity of the models (to provide a reasonable answer to the task).

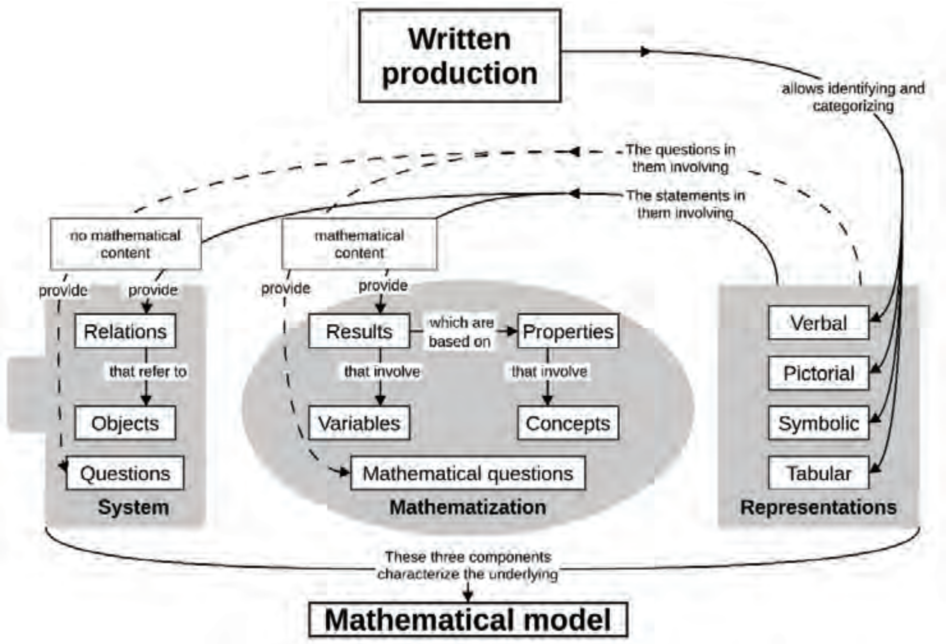

Figure 1: Flux diagram of the procedure used to characterise the models from the written productions in the first stage (Montejo-Gámez et al., 2021, p. 6)

(2) The second stage of the analysis consisted of identifying the information that students assume to be valid in their models, which was based on Fernández-Ahumada and Montejo-Gámez's (2019) notion of premise. In this way, the relations and the results and properties of the models analysed in the first stage were retrieved, and those statements that were used without prior justification were selected. Note that such statements were obtained from the representations of the models. For example, the first pictorial representations in Figure 2 below (from a to e) indicate that students assume as valid the flatness of the earth's surface. Once the statements assumed as valid were recovered, they were organised into emerging categories with a common meaning, and each category was assigned a synthetic formulation that summarises this meaning. These categories were obtained by two of the researchers independently and then discussed and reviewed by the third researcher. In order to find out the incidence of each category, the following variables were quantified: frequency of appearance in the productions, contextuality (percentage of statements in that category that were represented without any reference to mathematical content), awareness (percentage of statements in

that category that the authors recognise as assumptions in an obvious way) and validity (percentage of statements that have been used in an acceptable way in the context in which they were used). For example, the 82.4% validity shown in row S6 in Table 2 indicates that 82.4% of the times students used the Pythagorean theorem it was to apply it to right-angled triangles (on the remaining occasions they used it in situations where it does not apply). (3) Finally, in the third stage, the distribution of the statements that students considered valid according to each type of model was obtained. These data made it possible to visualise the impact of the information students assumed to be valid on the models produced.

Results

The characterisation and categorisation of the models developed by the students (first stage of the analysis) started from the choice of what was to be considered mathematical content: arithmetic operations, units of measurement, geometric properties of the circle and the Pythagorean theorem, content that belongs to the syllabus of the university course in which these students were enrolled. This first stage resulted in six categories, which are summarised in Table 1 and illustrated in Figure 2. The most repeated models (11 of the 22 models analysed) were those based on a right-angled triangle with the sea surface (flat) and the lighthouse. Nine of these constitute the M1 category and the remaining two make up the M2 category. M1 consists of the models that correctly identified the height of the lighthouse. None of them were valid for solving the task, as four of them assumed ad hoc data on the angles (Figure 2, a and b) and four others assumed values for the sides of the triangle (Figure 2, c and d). A final group gave no answer after considering the right-angled triangle.

Table 1: Summary characterisation and validity of the models developed by students

Model	*Frequency*	*Summary characterisation*	*Valid?*
M1	9	A right-angled triangle whose base is the Earth's surface (considered to be flat). The lighthouse height datum is correctly identified (Figure 2a-d).	No
M2	2	A right-angled triangle whose base is the Earth's surface (considered to be flat). The lighthouse height datum is incorrectly identified (Figure 2e).	No
M3	1	Circle of radius 91 m (Figure 2f).	No
M4	5	Right-angled triangle whose legs are the radius of the Earth and the lighthouse-horizon line of sight. The height of the ship is neglected (Figure 2g).	Yes
M5	3	Triangle whose legs are the radius of the Earth plus the height of the ship and the lighthouse-ship line of sight. The triangle is assumed to be right-angled (Figure 2h).	No
M6	2	Two right-angled triangles with common leg the radius of the Earth, and uncommon legs the ship-horizon and lighthouse-horizon lines of sight, respectively (Figure 2i).	Yes

M2 is made up of the models based on a right-angled triangle that misidentified the 91 m height datum (Figure 2e). The remaining categories use the circumference to address the task. In particular, category M3 has been assigned to a single model that interpreted the range of the lighthouse as 91 m and described the situation using a circle, which could not lead to an answer to the question posed (Figure 2f). In turn, category M4 encompasses the 5 models that used the curvature of the Earth's surface to model the situation and disregarded the height of the ship, which allowed them to obtain close-to-reality solutions (Figure 2g). Category M5 includes models that did not disregard the height of the ship but considered right-angled triangles that are not really right-angled, which was the case for two of the groups (Figure 2h). Finally, M6 comprises models that considered the ship's height and used two right-angled triangles to calculate the ship-to-horizon and horizon-to-lighthouse distances (Figure 2i).

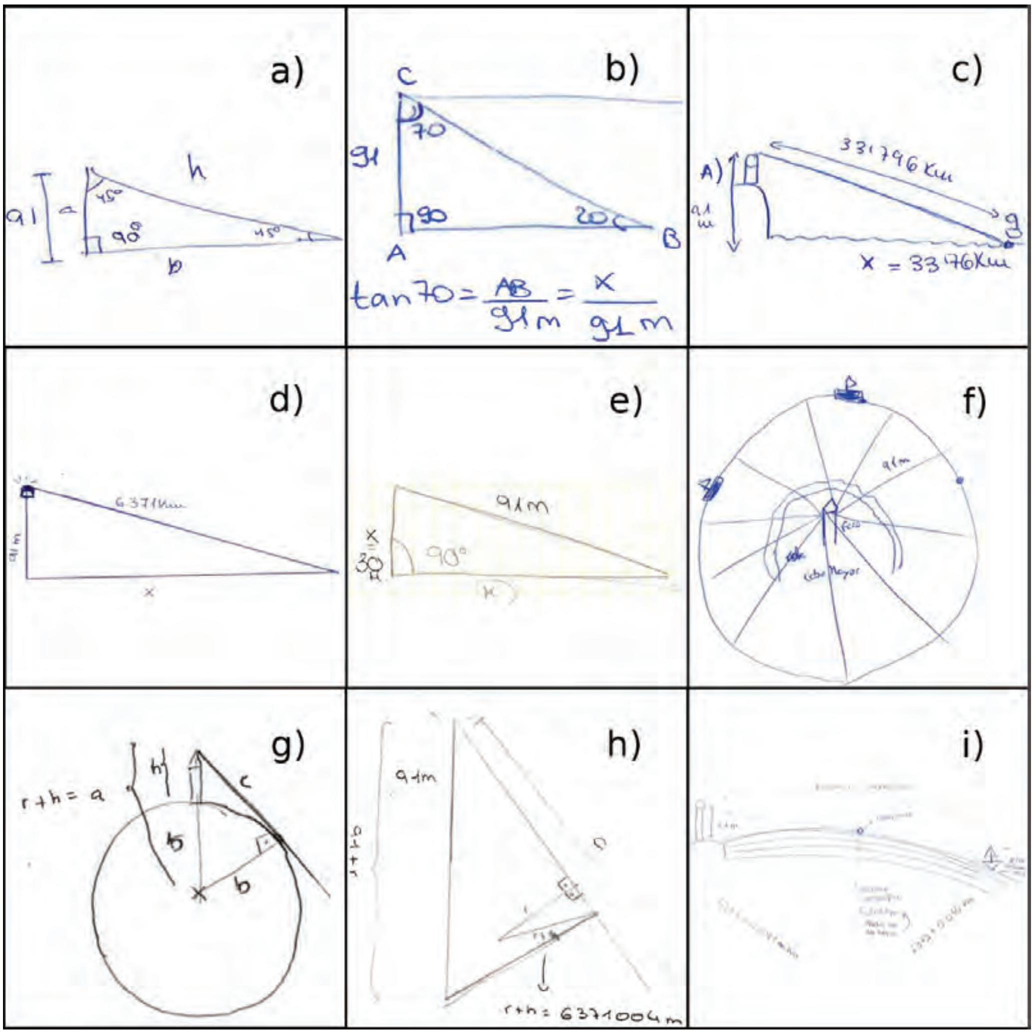

Figure 2: Pictorial and symbolic representations corresponding to the models found

Table 2, regarding the second stage of the analysis, summarises the information that students considered valid for the development of the models without the need for justification. There were 116 such statements, which were organised into seven categories. Four of them included statements expressed with hardly any mathematical content (rows S1-S4 in Table 2), and the remaining three were mostly expressed in mathematical language. In the case of S1-S4, we found hypotheses that students used without showing evidence that they were making assumptions. Of particular note were the assumptions of the flatness of the Earth's surface, the assumption that the height of the ship can be neglected, and different approximations to useful numerical parameters (angles in the M1 models, triangle side lengths in the M1, M2 and M3 models, and the radius of the Earth in M4, M5 and M6). In contrast, those models that used the curvature of the Earth or the height of the ship showed a higher degree of awareness. Furthermore, data indicate that statements with a higher degree of awareness have higher validity within the corresponding model. With regard to the last three categories (rows S5-S7 in Table 2), which describe applied mathematical knowledge, it is observed that students generally use valid mathematical content (the Pythagorean Theorem in most cases) without justifying it, and that they assume its applicability as a matter of course. However, about a quarter of the statements in S5 expose that the students show explicit awareness of having identified a variable whose numerical value provides the answer to the task.

Table 2: Categories of statements that students considered valid in their models

Synthetic formulation		*Frequency*		*Contextuality*		*Awareness*		*Validity*	
S1a. The Earth's surface is flat	S1b. The Earth's surface is curved	12	10	100%	100%	8.3%	80%	0%	100%
S2a. The height of the ship is negligible	S2b. The height of the ship is not negligible	18	4	100%	100%	0%	50%	100%	100%
S3. A variable has a certain numerical value		21		90.4%		28.6%		57.1%	
S4. There are no clouds or haze		2		100%		100%		100%	
S5. Calculating certain information answers the problem (identification of the unknown).		22		4.5%		22.7%		81.8%	
S6a. Pythagorean Th. is applicable to solve the problem	S6b. Trigonometry is applicable to solve the problem	17	2	0%	0%	0%	0%	82.4%	100%
S7. When the ship starts to sight the lighthouse, the ship-lighthouse line is tangent to the Earth.		8		0%		12.5%		100%	

Regarding the impact of the statements that students used without justification, Table 3 shows the distribution of these statements in the different models. In particular, row S1 shows that the use of the curvature of the Earth was decisive in the development of the models, so that M1, M2 and M3 are based on the flatness of the Earth's surface, while the remaining three types of models are based on taking advantage of its curvature. Among the latter are also concentrated the claims related to the tangency of the ship-lighthouse line of sight (line S7 in Table 3), although it seems reasonable that these claims are a consequence of using the curvature of the Earth, rather than a cause of the development of the models. In contrast, the use of ship height does discriminate between M4, which disregards it, and M5 and M6, which introduce it as a model parameter (row S2 of Table 3). As for the rest of the statements, the applicability of certain mathematical contents such as trigonometry could be a differentiating feature of the models (especially with regard to mathematics), but it has not been considered as such due to its residual nature in the context of pre-service education. Finally, it is observed that statements related to the adoption of values for the parameters and the identification of the unknown are evenly distributed across the different models, so that they did not exert any relevant influence on the students when constructing their models either.

Table 3: Distribution of the statements that students considered valid according to each type of model

Synthetic formulation		*M1 (9)*		*M2 (2)*		*M3 (1)*		*M4 (5)*		*M5 (3)*		*M6 (2)*	
S1a. The Earth's surface is flat.	S1b. The Earth's surface is curved.	9	0	2	0	1	0	0	5	0	3	0	2
S2a. The height of the ship is negligible	S2b. The height of the ship is not negligible	9	0	2	0	1	0	5	0	1	2	0	2

S3. A variable has a certain numerical value	7	3	1	5	3	2
S4. There are no clouds or haze	0	0	0	1	1	0
S5. Calculating certain information answers the problem (identification of the unknown).	9	2	1	5	3	2
S6a. Pythagorean Th. is applicable to solve the problem	6	1	0	5	3	2
S6b. Trigonometry is applicable to solve the problem	2	0	0	0	0	0
S7. When the ship starts to sight the lighthouse, the ship-lighthouse line is tangent to the Earth.	0	0	0	4	2	2

Note: The number in brackets next to M*i* indicates the number of models that fell into the *i-th* model category

Discussion and conclusion

This paper characterises the mathematical models developed by prospective elementary teachers when dealing with a task in which a property of the reference system, which is not explicit in the task but known to all participants, is key to finding a valid answer. The aim of this study was to analyse the information that prospective elementary teachers put into play without justifying it and its impact on the models developed. The main novelty of this paper is the analysis strategy used for this purpose and the first results found about this impact.

Analysis of the statements that prospective elementary teachers assumed to be true without justification revealed a large volume of information that was used in the models without obvious justification, with a balance between mathematical and extra-mathematical information. These results are aligned with those of Wozniak (2012), who found a large amount of mute praxeologies during modelling activity of prospective elementary teachers. In addition, it was observed that statements that students consciously impose are more valid for the model than those that are assumed without obvious awareness. As for the impact of these types of statements on the models developed, it was found that the use of the flat or curved character of the earth's surface was decisive. Indeed, all three types of models that assumed the flatness of the Earth led to non-valid answers, while two of the three types of models based on the curvature of the Earth did provide reasonable answers. The use of ship height was not decisive for the validity of the models, but it did differentiate them. On the other hand, data or information on the applicability of certain mathematical content had less impact on the models developed.

In brief, it has been found that the information that students use without justification when developing a model (especially in the simplification and structuring of the initial contextualised situation) can determine the modelling outcomes. Therefore, knowing this information is useful for understanding their models, finding difficulties (Fernández-Ahumada and Montejo-Gámez, 2019) and accounting for the contextual mathematical knowledge of prospective elementary teachers. The present study highlights the usefulness of written productions when studying this information that is used without justification and provides an analysis strategy for this purpose. The instrument developed by Montejo-Gámez et al. (2021) has been used, although tools for analysing written productions such as that of Ferrando et al. (2017) could also be used for this purpose. The didactic implication that emerges from this study is the relevance of expressing the mathematical knowledge that is activated during the modelling activity, which is especially necessary in prospective elementary teachers. Tools such as the Study and Research Paths (Barquero et al., 2019) or the implementation of modelling tasks focused on the specific activities of using, formulation or detection of hypotheses could be useful in this sense. Exploration of these possibilities will be developed as future work.

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Design of a didactic activity based on the Hazen-Williams model for engineering education

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The research presented is focused on engineering education, framed in the Anthropological Theory of Didactics (ATD), whose objective is to design a didactic activity of mathematical modelling. For this purpose, didactic engineering is considered. The starting point is the analysis of a civil engineers' workplace. A praxeology based on the Hazen-Williams model to design pipelines is identified using notions of hydraulics, topography, and mathematics. A didactic transposition on this praxeology is performed. A mathematical modelling activity is designed and implemented with students in the fluid mechanics' course. The central task is to determine the diameter of the pipe and ensure a water flow rate of 50lts/s. The handling of the mathematical model requires a qualitative analysis of the variables at stake and to relate knowledge of various kinds.

Keywords: Mathematical models, higher education, engineering education.

Introduction

Linking the mathematics that engineers use at work with the mathematics they study at university is a social demand, highlighted by Pollak (1988). Indeed, some research has been conducted in the workplace to identify the mathematics being used (e.g., Frejd & Bergsten, 2016; Gainsburg, 2007). Their results show the fundamental role of mathematical models, the management of which requires other knowledge, such as computational, practical, and engineering, and particularly, from experience, which is fundamental in decision-making, what Gainsburg (2007) calls the engineer's judgement. These types of mathematics can occur in training through didactic proposals that are inspired or based on the analysis of mathematics in the workplace, as suggested by Frejd and Bergsten (2016). In this line, this research was carried out within the framework of the Anthropological Theory of Didactics (ATD), in proposing to relate the workplace and specialised training as a first step for designing didactic proposals for the mathematics classroom. We mainly raise two research questions: What kind of activities of engineer's workplace can be transposed to engineering courses through mathematical modelling teaching proposals? Furthermore, what institutional conditions make it possible to integrate these proposals into engineering courses?

Some elements from ATD

The ATD proposed by Chevallard (1999; 2019) defines a model for analysing human activity in its institutional dimension. The praxeology $[T, \tau, \theta, \Theta]$ is a minimal unit of analysis of human activity. Its four components are the task type (T), the technique (τ), the technology (θ), and the theory (Θ). The 'task' refers to what is to be done; the 'technique' is how it is to be done; the 'technology' is a discourse that produces, justifies, and explains the 'technique'; while the 'theory' produces, justifies, and explains the 'technology'. Doing mathematics is a human activity closest to mathematical modelling because doing mathematics in this frame consists of and acting (produce, teach, uses) on mathematical models, as stated in Barquero et al. (2019). Institutions offer resources and conditions that allow their subjects to develop specific activities and establish restrictions. There are different types of institutions, and a subject may belong to several institutions, occupying various positions (e.g., teacher, student, parent, citizen). According to Chevallard (1999), praxeologies can circulate between institutions, undergoing, in effect, transpositive processes, i.e., transformations. To analyse this phenomenon in the case of the training of future engineers, Romo-Vázquez (2009) classified institutions according to their relationship with knowledge into three types: production, teaching and use. Production (or research) institutions are those that produce praxeologies, such as disciplines (e.g., mathematics, hydraulics); teaching institutions are in charge of transmitting praxeologies (e.g., school mathematics, school hydraulics) and using institutions are those in which praxeologies are used (e.g., industry, workplace). This classification is made considering the primary vocation of each institution. Still, it does not mean that praxeologies are not created or taught in the workplace or that no praxeologies are taught in the disciplines. However, when a disciplinary praxeology is taught, it undergoes a didactic transposition. It means that it is transformed to become an object of teaching. Thus, mathematical praxeologies become school mathematical praxeologies (Chevallard, 1991). The didactic transposition process is illustrated by Bosch & Gascón (2006, p. 56) as follows (figure 1):

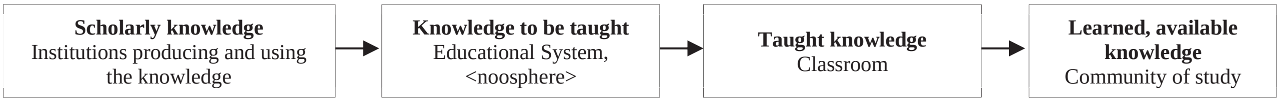

Figure 1: The didactic transposition process

In the case of training of future engineers, specific didactic transpositions can be performed. For example, transposing mathematical modelling praxeologies from the workplace to mathematics or engineering education, as suggested by some research (e.g., Frejd & Bergsten, 2016; Romo et al., 2017), as represented in figure 2.

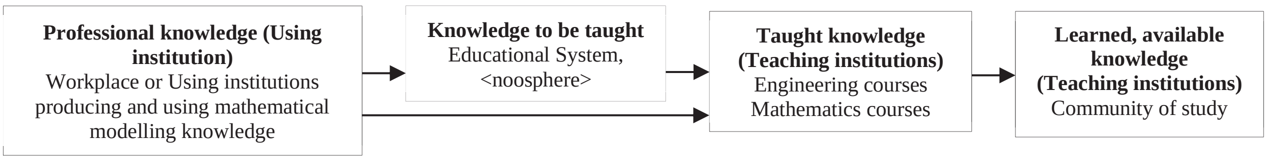

Figure 2: Transposing mathematical praxeologies from the workplace to teaching institutions

Performing this kind of transposition from Workplace to Engineering education demands identifying a local[1] mathematical modelling praxeology in a specific workplace of engineers W: $[T^e, \tau^{em}, \theta^{em}, \Theta^e] \leftarrow Pw$. Here, T^e is an engineering type task and the technique τ^{em} to perform this type of task has mathematical and engineering elements. The technology θ^{em} that justifies the technique is a mathematical model used in engineering, whereas the theory is from engineering. Praxeology Pw is then transformed into a school mathematical modelling praxeology Ps that can be constructed in engineering teaching (ET) as illustrated in figure 3.

[1] Praxeology with the same logos block for different practical blocks. It corresponds metaphorically to a modelling topic composed of various sub-topics.

$$\begin{bmatrix} T^{e} & \tau^{em} & \theta^{em} & \Theta^{e} \\ T^{se} & \tau^{sem} & \theta^{sem} & \Theta^{sem} \end{bmatrix} \begin{matrix} \leftarrow W \\ \leftarrow ET \end{matrix}$$

Figure 3: Scheme of transposition workplace praxeology to teaching praxeology

The process of transposition demands considering the institutional conditions: workplace and teaching, and the way of the original praxeology can be transformed to live in the teaching institutions. Likewise, *Pw* is conceived like an *epistemological model reference* in the sense proposed by Barquero, Bosch and Gascón (2019). Thereby, *Pw* must be a local modelling praxeology.

Methodology: Didactic engineering

Didactic engineering constitutes a solid research methodology (Artigue, 2020), which allows the design of tasks. Its four phases are preliminary analysis, activity design and a priori analysis, experimentation, and a posteriori analysis. Based on this, we design and analyse a didactic activity of mathematical modelling for engineering education, as illustrated below.

Phase 1. Preliminary analysis: Civil engineers' mathematical modelling praxeology of hydraulics. This analysis focused on characterising a mathematical modelling praxeology of hydraulics used by civil engineers to design pipelines in their workplaces. It was carried out jointly by a mathematics education researcher and a civil engineer with 29 years of experience, who is also a university professor (first author) of mathematics and engineering courses and a PhD student in mathematics education. Analysing his professional practice is a difficult task because some knowledge is no longer recognised, then a researcher in mathematics education questioned and asked for explanations. Civil engineers' mathematical modelling praxeology of hydraulics is represented as $[T^{h}, \tau^{hm}, \theta^{hm}, \Theta^{h}] \leftarrow Pw$, where *h* represents hydraulics and *m* mathematics. Thus, task type is T^{h}: Design a pipeline to transport water between two points taking advantage of the effect of gravity. Technique τ^{hm} and technology θ^{hm}: Step 1. Recognise the project's requirements and the minimum initial data to calculate the difference in level between the supply and distribution reservoirs and the total length of the pipeline path. Step 2. Determine the whole length of the pipeline path (L) using an established reference system (see figure 4). Step 3. Determine the hydraulic head of the system (ΔH), calculating the difference between supply and distribution reservoirs. ΔH is always greater than zero since the level of the distribution reservoir is lower than that of the supply reservoir, ensuring that the effect of gravity generates the flow of the fluid (figure 4).

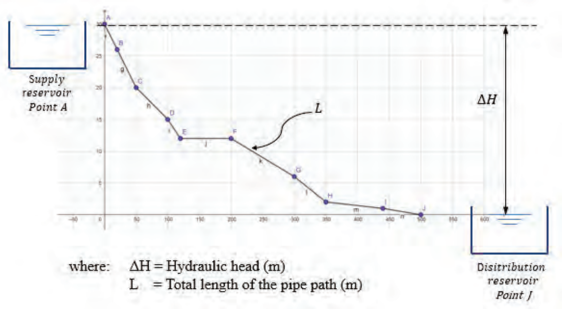

Figure 4: Schematic illustration of the pipeline path

Step 4. Select the material of manufacture of the pipe, considering the roughness of materials employing the Hazen-Williams coefficients (C)[2]. A material with high roughness will have a lower coefficient and a lower flow rate for the same pipe diameter. In contrast, the lower roughness of the material will have a higher coefficient and a higher

[2] See for example Alegret & Martínez (2019, p. 45).

flow rate. Step 5. Determine the flow rate (Q) using the Hazen-Williams mathematical model: $Q = 0.2785C\ D^{2.63}\ S^{0.54}$ where: Q = Flow rate at the pipe (m^3/sec); C = Hazen-Williams coefficient (dimensionless); D = Diameter of the pipe in meters (m) and S = hydraulic head loss per length of pipe (dimensionless factor) obtained with the following formula: $S = \Delta H/L$ where: ΔH = hydraulic head, expressed in meters, L = Total length of the pipe path interconnecting both reservoirs. The applicable limits of this formula are debatable; in the third edition of Hydraulic tables by Williams & Hazen (1933), the pipe diameters to be used were limited to 0.05 m (2 in.) < D < 1.85 m. (6 ft.). Furthermore, it is recommended to be used for flow velocities values below 10 ft./sec (3.05 m/s) and is valid only for water flowing at ordinary temperatures (5 °C - 25 °C) (Alegret & Martínez, 2019). It is worth mentioning that, in the case of not complying with the described parameters, the mathematical model could give results different from reality. Step 6. Verify that the flow rate (Q) obtained is the closest (approaching from a higher value) to the project's required flow rate (Q_p); Step 7. Check the water velocity (v) at the pipeline, which should be less than 10 ft/sec (3.048 m/sec). Step 8. Report the pipeline design results. To perform these eight steps requires that engineers properly use the Hazen – Williams model, recognising the mathematical relationship of the variables involved and how they can be satisfied from experimental conditions. Sometimes it is necessary to adjust data and, above all, to contrast the results obtained by the model with the existing pipe diameters on the market. The theory Θ^e is hydraulic engineering.

Phase 2. Didactic transposition of the civil engineers' mathematical modelling Pw, design of the didactic activity and a priori analysis. A didactic transposition was performed on the Pw, obtaining a school praxeology Ps. The task type is the same; the technique is organised in Ps through three stages using subtasks *Tn.m*. The technology in both praxeologies is the Hazen-Williams mathematical model, and the theory is hydraulic. However, in school praxeology, the fluid mechanic's course is also involved, see figure 5.

$[T^h, \tau^{hm}, \theta^{hm}, \Theta^h] \leftarrow Pw$		$[T^h, \tau^{shm}, \theta^{shm}, \Theta^{sh}] \leftarrow Ps$
T^h Design a pipeline	→	T^h Design a pipeline
τ^{hm}	→	τ^{shm}
Step 1		Stage 1 (3 subtasks): T1.1, T1.2., T1.3
Step 2 -7		Stage 2 (5 subtasks): T2.1, T2.2, T2.3, T2.4, T2.5, T2.6
Step 8		Stage 3 (1 subtask): T3.1
θ^{hm} Hazen-Williams mathematical model	→	θ^{shm} Hazen-Williams mathematical model
Θ^{shm} Hydraulic	→	Θ^{shm} Hydraulic– Fluid mechanics course

Figure 5: Scheme of didactical transposition from *Pw* to *Ps*

Thus, the school praxeology is described in this way. T^h: Design a pipeline to transport water between two points taking advantage of the effect of gravity. The technique has three stages. In stage 1, the requirements and initial data of the project are recognised through three subtasks. Firstly, T1.1. Identify the required flow rate (Q_p) at the distribution reservoir. This is the project's main datum, for example, 50lts/sec; T1.2. Analyse the topographic data (see Table 1), draw up a scheme and visualise the general conditions of the system to be designed. This is a first approach to the establishment of the reference system: location of the supply point as an initial point of the system, establishment of scales to use and labels to identify the implicit elements in the system. Finally, T1.3. Drawing up the topographic profile for recognising the natural terrain and identifying key points for the design of the pipeline. Civil engineers usually use the software like AutoCAD, favouring parameter manipulation for better visualisation and detailed analysis. Students can use GeoGebra, which provides an elevation view and elements necessary in a simplified two-dimension drawing to apply the Hazen-Williams model (see figure 6). Stage 2: designing a pipeline to transport water between two points by taking advantage of the effect of gravity involves five subtasks. T2.1. Determination of the pipeline's total length is important for calculating the hydraulic head loss per length of the pipe. The lengths of the pipe sections are calculated from point to point where the changes of direction in the path occur and add together. Students can perform these calculations in GeoGebra. T2.2. Calculation of the hydraulic head, a dominant factor for the natural movement of the fluid through the pipe. This is the difference in level between the point of supply and the point of distribution calculated in meters. T2.3. Selection of the type of pipe considering three main conditions: 1) installation needs, 2) cost of the pipe, and 3) the related Hazen-Williams coefficient, depending on the material of manufacture. The students do not have information on conditions 1 and 2, so they should focus on the efficiency of the material for fluid conduction, evaluating the Hazen-Williams coefficient. T2.4. Determine the flow rate. *This is the central task* and consists of the application of the Hazen-Williams mathematical model. Although their use is not identified in professional praxeology, it is considered that students could propose an initial value of the pipe Diameter using the continuity equation $A = Q/v$, where v is the limiting velocity of the water in the pipe, Q, the flow rate for which the design is made and A, the cross-sectional area of the pipe. Considering the formula of the area of a circle, it is feasible to obtain an initial datum for the Diameter. For the students to perform this same procedure, avoiding an initial random proposal of Diameter, they will be provided with a table that favours the qualitative analysis of relationships between variables. T2.5. and T2.6. Verification of compliance with limitations or restrictions of the method, guaranteeing the correct operation of the mathematical model to obtain results following reality. 1) Pipe Diameter limited to 0.05 m (2 inches) < D < 1.85 m. (6 ft.); 2) Design flow rate greater than or equal to that required in the project [$Q \geq Q_P$]; 3) Fluid velocity in the system must be less than 10 ft. per second (3.05 m/s). To verify it, the formula $v = Q/A$ is used again. Suppose any of the restrictions are not met. In that case, students are expected to propose another diameter, develop the procedure, checking whether all the restrictions are met again, i.e. an iterative process is generated until the pipe Diameter that meets the restrictions of the method is found. As a means of verification (immediate feedback) and approaching the professional reality, students can use "Epanet", specific software for the design of piping systems in which the Hazen-Williams mathematical model is encapsulated or implicit. Stage 3) Report the pipeline designed results. A T3.1 subtask is proposed to elaborate a report of the pipeline designed results.

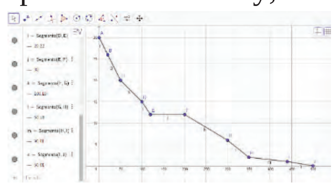

Figure 6: Topographic profile in GeoGebra

The objective of the didactic activity is to allow students to construct or reconstruct the school praxeology *Ps* in the classroom. For this purpose, a situation similar to those faced in workplace practice is proposed: "A team of engineers has to connect two reservoirs for the supply of water for domestic use in some town, whose flow rate average required is 50 lts/sec, according to the results of previous studies carried out by specialists. The topographic study carried out on the path of the pipeline that will connect these reservoirs yields the following data on the obligatory points (Table 1), to carry out the least amount of excavation possible."

Table 1: Topographic data of pipeline

Punto	Coordenadas		Punto	Coordenadas	
	X	Y		X	Y
A (supply)	0	30	F	200	12
B	20	26	G	300	6
C	50	20	H	350	2
D	100	15	I	440	1
E	120	12	J(distribution)	500	0

Phase 3. Experimentation. The study was developed at the Universidad Cristóbal Colón, Veracruz, Mexico, in engineering education. The didactical activity was implemented in a fluid mechanics course. The participants were 37 students of the fourth semester, from majors of Industrial Engineering and Petroleum Engineering.

Phase 4. A posteriori analysis. The analysis was performed on the basis of the students' worksheets and the teacher's notes on this activity. Three teams were chosen to analyse their praxeologies and, mainly, how they used the mathematical model, chose the Diameter and justified their solution to the project. The criteria for selecting the teams were the clarity and coherence of their reports and the students' commitment to the development of the activity (according to the teacher's notes).

Some results

This section presents a first analysis of the praxeologies developed by three teams of students, particularly in three subtasks: the determination of the Diameter and flow rate (T2.4) and the verification of compliance with limitations or restrictions of the method (T2.5 and T2.6), which were vital in the development of the didactic activity. When the students were faced with subtask T2.4, they had calculated the value of S and identified the material for their pipe and the Hazen-Williams coefficient C. So, they still had to determine the value of the Diameter D and Q (flow rate) by using the Hazen-Williams model: $Q = 0.2785CD^{2.63}\,S^{0.54}$. The students had no previous experience selecting pipe diameters; then the teacher provided them with a table with different diameter values. So that they could evaluate it in the mathematical model and, depending on the results of the flow calculation, continue with the proposal of diameters until they found the one that would provide the closest flow to that required by the project. Thus, we tried to avoid using the mathematical model as a formula and favour qualitative analysis and the relationship between the variables at stake. The students determined the Diameter and verified that it was as close as possible (approaching from a higher value) to the flow rate required by the project (Qp = 50Lts/s). (See Tables 2a, 2b, and 2c, corresponding to samples of work achieved by the teams 1, 2, and 3, in that order).[3]

Table 2a: Work by team 1

Diameter (in)	Diameter (mm)	Diameter (m)	Flow rate (m^3/sec)	Flow rate (Lt/sec)
5.7	144.78	0.14478	0.049066	49.0661151
5.8	147.32	0.14732	0.051363	51.3625321
5.9	149.86	0.14986	0.053724	53.7244023

Table 2b: Work by team 2

Diameter (in)	Diameter (mm)	Diameter (m)	Flow rate (m^3/sec)	Flow rate (Lt/sec)
3 3/4	95.25	0.09525	0.0543	54.3
3 1/2	88.9	0.0889	0.0453	45.3
4	101.6	0.1016	0.0644	64.4

Table 2c: Work by team 3

Diameter (in)	Diameter (mm)	Diameter (m)	Flow rate (m^3/sec)	Flow rate (Lt/sec)
5.75	146.05	0.1463	0.050431	50.4331434
5.744	145.8976	0.1459	0.050033	50.0334106
7	177.8	0.1778	0.5053	50.53

[3] These tables were made by the authors, considering some rows of the original tables. The green highlighted was also made by the authors.

Team 1 selected a Diameter of 5.8 inches; it is observed that for this selection, they started evaluating diameters from the table they were provided (2, 3, 5) until they found values close to the required project's flow rate, then by 0.1-inch diameter approximations (5.7, 5.8, 5.9). They found the closest value, approaching from a higher value to the project flow rate (see table 2a). *Team 2* selected a diameter of 3¾ inches. This team presents errors in the operation of the mathematical model, it obtains higher flow rates using smaller diameters; the process of selecting the pipe diameter is very similar to that carried out by team 1, with the difference that they use approximations of 1/4 of an inch. (see table 2b). *Team 3* selected a diameter of 5.744 inches. It can be seen that they used the same procedure for diameter selection as teams 1 and 2; in this case, the approximation of the diameter proposals is 0.001 inches, trying to obtain the value of the project flow exactly equal to that required (see table 2c). The mathematical model worked as a technology that allowed them to control the technique, not considering whether the Diameter obtained corresponds to existing pipes in the market. This knowledge is constructed in the engineer's workplace. Moreover, we could ask students to verify the pipe's existence in a new redesign of the modelling activity. Thus, they would have another element to validate the Diameter obtained with the Hazen-Williams model. Concerning the verification of the obtained flow rate to be the most adjusted (approaching from higher values), it was observed that in all cases the students selected the Diameter that provided the closest magnitude to the required project flow rate, always considering values higher than this one. T2.5 Regarding the review of the water velocity at the pipeline using the continuity equation: $v = Q/A$ (where A is the cross-sectional area of the pipe). It was observed that some teams did not perform it. Where this was done, they only indicated that the parameter was not met. However, they did not make any proposal to establish a design that meets this requirement.

As an extra activity, and as a possibility for future research, it was proposed to the students to work with more specialised software –used in the workplace–, which allows them to check their results and receive immediate feedback, favouring reflection on the work carried out, proposals for improvement and connection with reality. Modelling the system in Epanet allows inserting geometric data of the system (obtained from the topographic profile in GeoGebra, figure 6), characteristics of the pipe (diameter, length, material of manufacture), accessories and equipment (e.g., valves, pumps) with the corresponding characteristics obtained from the manufacturer. The software also allows an immediate validation of the results by consulting the design parameters; these are flow rate and fluid velocity. This can be seen on the screen by moving the pointer to the desired element (figures 7 and 8).

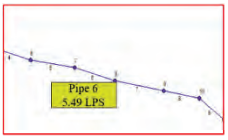

Figure 7: Epanet fluid flow rate

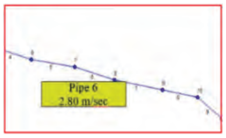

Figure 8: Epanet fluid

Conclusion

This research shows an avenue to design mathematical modelling activities for the training of engineers that connect using and teaching institutions. Didactic engineering allows performing a didactical transposition on the workplace's mathematical modelling praxeology. In this case, the analysis of the Civil engineers' mathematical modelling praxeology of hydraulics is developed by a subject of a using institution and a teaching institution. He analysed his workplace activity, identified a mathematical modelling praxeology, transposed it, generated a didactic activity and implemented it. His professional experience as an engineer and as a teacher was fundamental in all stages of didactic engineering. The students have no difficulties carrying out this activity, but their lack of experience limits the proposed solutions: expensive materials, non-existing pipe diameters, lack of practical verifications. The use of GeoGebra allows an approach to mathematics courses in the first semesters. In contrast, the possibility of using Epanet's software allows an approach to professional practice, guiding the student in the consideration and analysis of parameters and the relationship between them for the generation of the solution with elements very close to reality. From the teacher's perspective, the software used is complementary since GeoGebra provides relevant geometric data for modelling the project in Epanet. It is considered that this didactic activity can also be adapted to be implemented in a mathematics course, and even more, it can be proposed to be developed by students from different semesters with different backgrounds and experiences.

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How knowledge about ideal-typical modelling processes affects phase transitions in individual modelling routes

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Working on mathematical modelling tasks is usually challenging for students: several studies have shown, that students do not knowingly and consciously apply solution strategies when working on modelling tasks. In an empirical study, we investigate whether and to what extent knowledge about ideal-typical modelling processes has an influence on phase transitions in individual modelling routes. The individual acquires this knowledge in the form of an instruction that contains information about the modelling process, e.g. in form of a modelling cycle and a solution plan. In this article, the phase transitions of students who received an instruction about modelling processes are compared to those of students without such an instruction. The data for the study were collected, presented, and analysed using the Modelling-Activity-Interaction-Tool (MAI-Tool), which is based on quantitative methods to capture and analyse structures of modelling processes.

Keywords: Individual modelling routes, knowledge about ideal-typical modelling processes, phase transitions, MAI-Tool.

Introduction

Studies have already shown that knowledge about mathematical modelling has a positive influence on the modelling process (Stillman & Galbraith, 1998). This knowledge is often provided to students in the form of a solution plan (Beckschulte, 2019). Schneider et al. (2021) found out that individuals who received knowledge about the modelling process engaged intensely with mathematics and rarely deviated from the solution process.

In an empirical study, we compare individuals who have received knowledge about ideal-typical modelling processes (referred to as *the instructed*) with those who have not (*the non-instructed*). Knowledge about ideal-typical modelling processes is provided to students as an instruction in the form of the five-step modelling cycle of Kaiser and Stender (2013) (cf. Figure 1) and the corresponding solution plan of Beckschulte (2019).

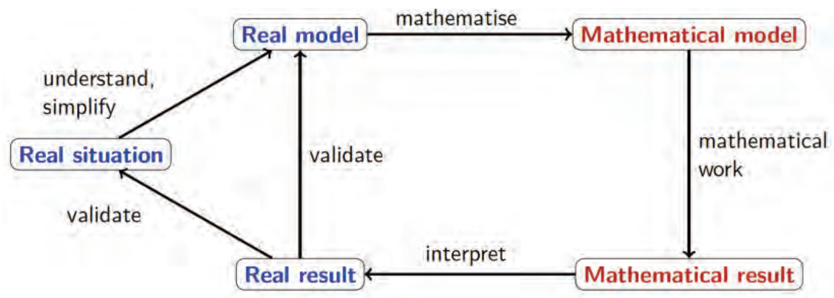

Figure 1: The modelling cycle of Kaiser and Stender (2013)

In addition, the solution process of a modelling problem is discussed using the problem of colouring the map of Germany (Leuders, 2007). The instruction is based on the fundamentals of Vorhölter and Kaiser (2016) and took about 20 minutes.

The aim of our research is to analyse how knowledge about ideal-typical modelling processes affects the structure of individual solution processes. Not only the phases, but also the phase transitions provide information about the influence of an instruction on the modelling process: especially with regard to the phases that are outside the modelling cycle (*summary solution process* and *miscellaneous*), the phase transitions can be used to show which phases occur before or after these phases. This particular view of individual modelling routes requires a tool that can represent structures of modelling processes and algorithmically evaluate them as numerical data: the MAI-Tool. In this article, we examine individual modelling routes for differences in the relative frequency of phase transitions, i.e., we investigate whether the relative number of the two cohorts differ significantly from each other.

Theoretical framework

Analysis of individual modelling routes

Methods for the representation and analysis of individual modelling processes already exist and have been applied in empirical studies. The modelling cycle is not passed linearly and there is a switching back and forth between the phases. This is what Borromeo Ferri (2007) calls an individual modelling route: Successive phases are connected by numerical arrows in the modelling cycle. Thus, the modelling cycle is not only suitable for representation, but also serves as an analysis tool for individual modelling processes.

The Modelling-Activity-Diagram (MAD) is another concept for representing and analyzing modelling processes (Ärlebäck and Albarracin, 2019): in a linear representation, activities of an individual are shown over time. The activities describe what is done in the phases of the modelling cycle.

The MAI-Tool is a newly developed tool for capturing, representing, and evaluating modelling processes (Ruzika & Schneider, 2019; Schneider et al., 2021): the tool is based on observable interactions within the group as they work on a modelling task. Following qualitative methodology, interaction units are entered into the MAI-Tool with the following information: interacting person(s) and content of the interaction. The content is assigned to a phase of the modelling cycle. A timestamp is automatically assigned to each interaction unit so that the duration is also stored. Individual modelling routes are automatically extracted from the group process. The evaluation is quantitative, as modelling processes are described with numerical data. Since the data are available digitally, they are evaluated by algorithms, so that an objective evaluation is given. This results in the advantage that the numerical data can be applied in a statistical test. The individual modelling routes are displayed graphically in the tool – depending on the focus of the evaluation.

Knowledge of mathematical modelling

Solving mathematical modelling problems is challenging for students (Blum, 2015): empirical studies have found that students rarely consciously follow a strategy and do not know how to proceed when they encounter difficulties (Kaiser et al., 2015). Therefore, it is important to provide students with strategies that can assist them in the solving process. In the form of a solution plan that includes the modelling cycle, the phases as well as their transitions are described in detail using activities. A solution plan belongs to the general strategy aids, as no specific help is given for the task (Borromeo Ferri, 2006). By describing the entire solution process in the solution plan, knowledge about the ideal-typical course of a modelling process can be acquired: "[the] solution plan is not meant as a schema that has to be used by the students' but as an aid for difficulties that may occur in the course of the solution process" (Blum & Borromeo Ferri, 2009, p. 55).

Schneider et al. (2021) showed in an empirical study that knowledge about ideal-typical modelling processes affects the structure of individual modelling routes: individual modelling routes of the instructed were compared with those of the non-instructed for differences in the relative number and relative duration of phases in the modelling process. In particular, the phases of the *mathematical world* (*mathematical model* and *mathematical solution*) occur more frequently and for longer for the instructed compared to the non-instructed students. In addition, two phenomena were discovered that have not been considered before: non-instructed students summarise their solution process more often and longer. Moreover, they digress more often from solving the modelling task. On the basis of these analysis, they concluded that the non-instructed students lack knowledge about ideal-typical modelling processes: since they do not know how to solve a modelling task in a structured way, they are not interested in an improved solution and are satisfied with their first solution found.

Classification of phase transitions in the modelling process

To investigate differences in individual modelling routes between the instructed and the non-instructed students with respect to phase transitions, we assign the possible phase transitions to different categories.

In a first step, we divide the phases into two phase types. Phases within the ideal-typical modelling cycle (INPHASE) are *real problem*, *real model*, *mathematical model*, *mathematical solution*, *real solution*, and *validating*. Phases outside the ideal-typical modelling cycle (OUTPHASE) are not part of the modelling cycle: *summary solution process* and *miscellaneous*.

Phase transitions are characterized by their start and end phases. We distinguish between typical and atypical transitions. Typical transitions include adjacent phase transitions (APT). An APT corresponds to an ideal-typical transition in the modelling cycle with INPHASES. There is a total of three atypical transitions that do not correspond to an ideal-typical transition in the modelling cycle: The adjacent backwards phase transition (ABT) is a phase transition in which the start and end phases are reversed compared to APT. ABT are not typical phase transitions because they do not follow one another in an ideal-typical manner as in the modelling cycle. The ABT as well as at the jumps (JUMPS) consist of INPHASES: JUMPS are neither APT nor ABT. The third atypical phase transition includes at least one OUTPHASE and is called phase transition outside the modelling cycle (PTOUT). Start and/or end phase of the phase transition is an OUTPHASE. An illustration of the phases can be found in Figure 2.

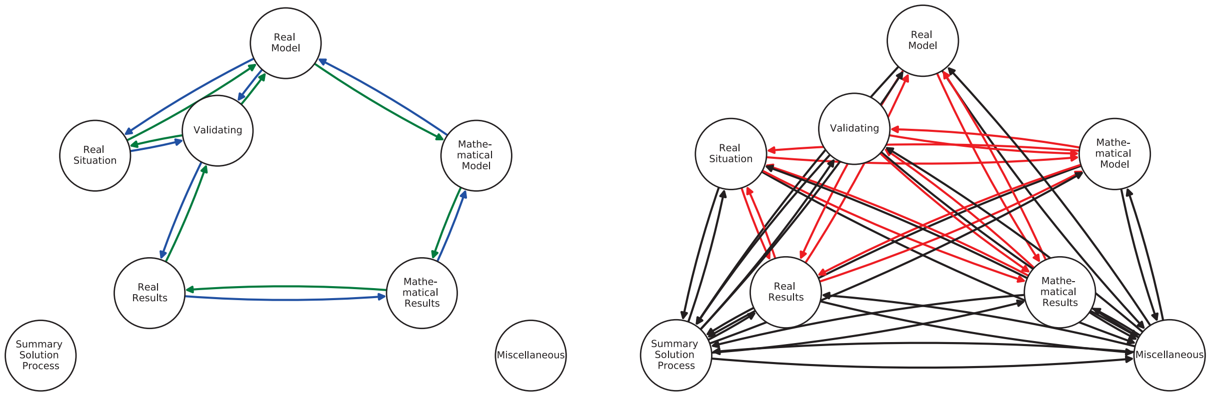

Figure 2: An overview of the phase transitions: APT (green), ABT (blue), JUMPS (red) and PTOUT (black) – illustrated on the modelling cycle

Research question and hypotheses

Based on the theoretical framework as well as on the classification of phase transitions in modelling processes, we formulate the central research question in this article:

To what extent does knowledge about ideal-typical modelling processes affect phase transitions in individual modelling routes, in particular: for which phase transitions does the relative number of individuals with instruction significantly differ from those without instruction?

Following the results of Schneider et al. (2021), we formulate hypotheses that are tested for significant differences in the phase transitions between the instructed and the non-instructed students. The instructed students engage with the mathematical world more frequently and more often during the solution process compared to the non-instructed students. From this, we derive the following hypothesis:

(H1) The instructed students have a significantly higher relative number of the APT *mathematical model* to *mathematical solution* (MM → MS) than the non-instructed students.

In particular, Schneider et al. (2021) highlighted the long duration and frequent occurrence of OUTPHASES for the non-instructed students. We hypothesize that this also affects phase transitions, especially PTOUT. Since the non-instructed students have not engaged as intensively with the *mathematical world*, it can be conjectured that they often switch from the *real model* to OUTPHASES. The following hypotheses emerge:

For the instructed students, the relative number

(H2) of the PTOUT,

(H3) of the PTOUT *summary solution process* to *miscellaneous* (SSP → MISC),

(H4) of the PTOUT *miscellaneous* to *summary solution process* (MISC → SSP),

(H5) of the PTOUT *real model* to *miscellaneous* (RM → MISC) and

(H6) of the PTOUT *real model* to *summary solution process* (RM → SSP)

is significantly lower than for the non-instructed students.

Methodological framework

The study uses a mixed-methods design, with data collection being qualitative and data analysis being quantitative. Tenth grade students from different Gymnasien (German Grammar School) were selected for the study. Only those who had indicated in a questionnaire that they had no previous experience in modelling and had not acquired knowledge about modelling were included in the study. Thus, we excluded the influence of previous experience in modelling. Working in groups of five, they complete the "filling-up" task in 30 minutes (Blum & Leiß, 2006). The sample includes 40 students (20 instructed, 20 non-instructed). The group work was videographed.

Interactions were recorded using the MAI-Tool: Grounded Theory (Strauss & Corbin, 1996) was used to qualitatively code the interaction units. Each interaction unit contained information on who interacted (with whom), which phase of the modelling process could be assigned an when the interaction began and ended. To guarantee reliability in the coding, two people coded the data. The interrater reliability was calculated using Cohen's Kappa (Cohen, 1960) and is $\kappa= [.72, .90]$ for each individual. Phase transitions are evaluated by absolute and relative frequencies in the MAI-Tool. The relative frequencies of both cohorts are analysed for significant differences using the Mann-Whitney U test. In contrast to t-tests, this non-parametric test requires fewer prerequisites. The significance level is set at $\alpha=0.05$. The effect size δ is calculated according to Fritz et al. (2012) and interpreted according to Cohen (1988). The Mann-Whitney U test and the calculation of the effect size was done using the software R. Because multiple hypotheses are tested from one data set, analyses are corrected using the Benjamini-Hochberg method (Benjamini & Hochberg, 1995).

Results of the study

The hypotheses are tested using the Mann-Whitney-U-test. The results are shown in Table 1. All hypotheses are accepted because the *p*-values for the respective variables are always below the significance level of $\alpha=0.05$. Thus, significant differences between the two cohorts can be measured for each variable. The effect size is at least weak for each hypothesis; this underlines that an effect is measured in each case.

Table 1: Results of the Mann-Whitney-U-test for group 0 (non-instructed) and group 1 (instructed) with the mean values (mv), standard deviation (sd), *p*-values and effect size δ (rounded to 3 decimal positions)

variable (corresponding hypothesis)	mv cohort 1	sd cohort 1	mv cohort 0	sd cohort 0	*p*-value	effect size δ
relative number MM → MS (H1)	0.024	0.027	0.006	0.014	0.006	0.351
relative number PTOUT (H2)	0.213	0.166	0.411	0.262	0.006	0.349
relative number	0	0	0.042	0.051	<0.001	0.53

SSP → MISC (H3)						
relative number MISC → SSP (H4)	0.001	0.004	0.025	0.045	0.032	0.259
relative number RM → MISC (H5)	0.027	0.036	0.08	0.089	0.007	0.343
relative number RM → SSP (H6)	0.01	0.012	0.002	0.006	0.007	0.343

Interpretation of the results

An overview of the phase transitions in the solution process of both cohorts can be found in Figure 3: the thicker an edge, the higher the relative number of the respective phase transition. In particular, the significant differences shown by the statistical test are evident in this figure.

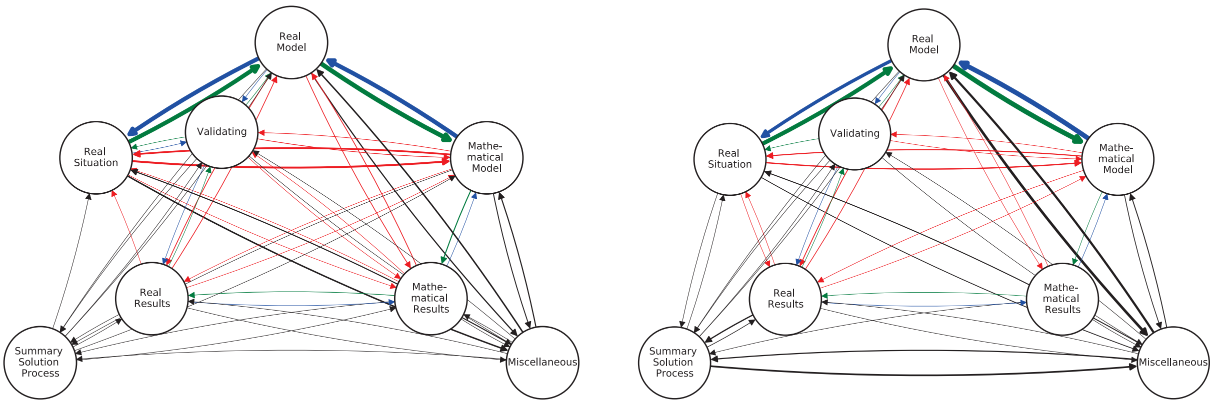

Figure 3: Overview of the phase transitions of the instructed (left) and the non-instructed (right)

The APT *mathematical model* to *mathematical solution* occurs often for the instructed in the solving process compared to the non-instructed students: when creating (and improving) the *mathematical model*, an attempt is made to translate reality into mathematics in the best possible way. From the instruction, students know that a *mathematical model* as well as its solution with mathematical concepts is relevant to solve the modelling task. In contrast, the non-instructed students have problems working mathematically: besides forming the *mathematical model*, there are also difficulties in applying mathematical concepts to solve the problem. Above all, solving the problem is a challenge for them, which is why the switch from *mathematical model* to *mathematical solution* rarely occurs.

Especially in the OUTPHASES the influence of the instruction becomes clear: the non-instructed students often switch between the OUTPHASES, which is reflected in the high relative number of PTOUT. Thus, they often stay outside the modelling cycle, i.e. also outside the processing of the modelling task. The instructed students proceed in the solution process more goal-oriented and follow the phases of the modelling cycle, which they learned in the instruction. Therefore, they deviate less often from the solution process when working on the modelling task. Since the non-instructed students lack knowledge about modelling processes, they move away from the problem (OUTPHASES) and/or summarize their previous solution. By switching between the two OUTPHASES, they again gather new ideas on how to proceed in the solution process. The frequent switches from *real model* to an OUTPHASE and vice versa show that the non-instructed students have problems forming the *mathematical model*. Either they switch to topics that have nothing to do with the processing of the task or they recapitulate their solution process. This can be related to the phase transition *mathematical model* to *mathematical solution*: the influence of an instruction mainly affects those phase transitions that are related to the *mathematical world*. Without knowledge about the formation of the *mathematical model*, individual modelling routes are less structured and often outside the ideal-typical modelling cycle.

Further research

We have shown that individual modelling routes of the instructed and non-instructed students differ with respect to the relative number of certain phase transitions. In the next step, we build on these results: using the MAI-Tool, we will examine individual modelling routes with respect to the sequence of phases. For this we will develop a concept to describe them more precisely on the basis of patterns.

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Comparing pre-service teachers' errors in individual and group resolutions of modelling tasks involving estimations

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Real-context estimation problems are useful for introducing modelling in primary school classrooms, although teachers have difficulties associated with the modelling process and also with measurement and estimation senses that may hinder their implementation. Based on a specific categorization, this work analyses the errors of N= 224 pre-service teachers when they draw up individual resolution plans for a sequence of modelling tasks involving estimations, and when they subsequently solve the same sequence in groups and on-site. Individual and group resolutions are compared with two objectives: the first is to determine whether working at the problem site and in groups helps to reduce the number of errors; the second is to find out whether the types of errors that are made are different. This would help to efficiently use learning from errors to improve initial teacher training in solving modelling problems.

Keywords: Fermi problems, errors, measurement, estimation, pre-service primary teachers

Introduction

Real-context estimation problems, also known as Fermi problems, are modelling tasks accessible to primary school students (Ärlebäck, 2009). Hagena (2015) argues that measurement sense and estimation are necessary to successfully solve many modelling tasks. Different studies have found that prospective teachers have difficulties and make many errors when solving modelling problems (Wess et al., 2021; Moreno et al., 2021). Furthermore, deficiencies have been found in the estimation skills of prospective teachers (Castillo-Mateo et al., 2012), especially when they have to reason about estimation and measurement in complex situations (Baturo & Nason, 1996). A previous work (Segura & Ferrando, 2021) developed a specific error categorization for one type of Fermi problem: those that require estimating a large number of elements on a delimited surface. This paper analyses errors in the productions of N = 224 pre-service primary school teachers when they are confronted with a sequence of this type of Fermi problems, whose statement is contextualized around the Faculty of Education building. The experience consists in two phases: firstly, individually and in-classroom, they have to make a resolution plan for each problem, secondly, they have to solve the sequence again in groups (N = 62) and perform measurements and estimations at the problems' site.

A comparative analysis allows us to answer the following research questions: Q1. Does group and on-site work help to reduce the number of errors made by prospective teachers when solving Fermi problems? Q2. Do error types evolve from individual resolution plan to group resolution at the problem site?

Theoretical framework

Lesh and Harel (2003) define a *mathematical model* as a system consisting of mathematical concepts, symbolic representations of reality, relationships, regularities or patterns, as well as the procedures, mathematical or otherwise, associated with their use. The development of a mathematical model is made up of phases, and these phases form a cycle (Borromeo Ferri, 2018) that starts from a real and open situation that has to be *simplified* and *structured*. This process leads to the *real model*, a representation that prefigures the mathematical model. In order to build the *mathematical model*, the real model must be *mathematized*. The next phase requires *working mathematically* within the model to find a solution to the problem in mathematical terms. Finally, the mathematical result must be *interpreted/validated* in the real situation.

Estimation tasks can be used as a means of initiation into mathematical modelling (Borromeo-Ferri, 2018). A specific case of estimation tasks are Fermi problems: these problems present a situation where little concrete information is known, forcing students to make assumptions and estimates to obtain a solution to the initial question (Ärlebäck, 2009, p. 331). In this study we will use a subset of Fermi problems, those that consist of estimating a large number of elements in a delimited surface. Ferrando et al. (2020) categorised the productions of prospective teachers when solving these problems into four types of resolution (counting, linearization, base unit and density). Productions that did not reflect a clear process for arriving at the estimate were classified as *incomplete*.

Studying errors made by pre-service teachers can be useful to understand with what competence and how they solve modelling tasks. Wess et al. (2021) provide a comprehensive categorization that provides an overview of the difficulties at each stage of the modelling cycle. Moreno et al. (2021) established a categorisation of errors (simplification error, mathematization error, resolution error, interpretation error) that has been one of the bases of the categorisation we have developed for Fermi problems. The other basis for establishing a specific error categorisation for Fermi problems consisting of estimating a large number of elements on a surface is the work of Castillo-Moreno et al. (2012). Their research analyses errors of measurement and estimation of length and area, which helps them to establish a categorisation of these errors, both conceptual (errors in perception or meaning of the magnitude; errors in the internalisation of the referents) and procedural (errors in the conversion of units of measurement or incorrect calculations). Basic understanding of quantity management poses difficulties that hinder reasoning about estimation and measurement in complex situations (Baturo and Nason, 1996).

Experimental design

The starting point of this work is a sequence of four problems. The four problems are contextualised in rectangular enclosures located in areas close to the Faculty of Education:

P1-People. How many students can stand on the faculty porch when it rains?

P2-Tiles. How many tiles are there between the education faculty building and the gym?

P3-Grass. How many blades of grass are there in this space?

P4-Cars. How many cars can fit in the faculty parking?

The sample is formed by N = 224 prospective teachers. The experience includes two parts: in-class individual work, on-site group work.

In-class individual work

We provided each prospective teacher the written statements of the four problems of the sequence and a little image for each problem. They worked individually for 90 minutes on solving the sequence. The following aspects were emphasized: in each problem they should raise a possible solution plan indicating the measures they would need to obtain the estimation; the work should be done individually; they should explain their procedures in written form and may use drawings or diagrams; and, finally, they were not expected to obtain a numerical estimate but rather to explain how to get the requested estimate. We have called this schematic solution a *resolution plan*.

On-site group work

Pre-service teachers were again confronted with the same sequence of four estimation tasks one week later, but they were randomly grouped into N = 62 groups of between 3 and 5 people, and they were taken to the real spaces in the Faculty of Education's environment where the problems are located. They were also provided with measuring instruments to take data and carry out the necessary calculations to obtain a numerical estimate. The session lasted 90 minutes.

Errors categorization

The following categorisation of errors (Table 1), specific to the Fermi problems which require estimating a large number of elements in a delimited surface, is based on a previous analysis of the resolution plans of prospective teachers (Segura & Ferrando, 2021) and on the background review. It is organized following the phases of the modelling cycle and also considers essential processes of estimation and measurement sense.

Table 1. Error categories and associated error types for modelling problems involving area and length estimates

➢ *Simplification error:*
E1. Incomplete real model associated with the lack of consideration of elements of the real situation.
E2. Incorrect real model due to error of perception of the magnitude.
E3. Incorrect real model due to inadequate internalisation of referents of the magnitude to be estimated.
E4. Does not build a real model.
➢ *Mathematization error:*
E5. Mathematical model incoherent with the real model due to an error in the meaning of the terms of the magnitude.
E6. Mathematical model incoherent with the real model due to inadequate internalisation of units of measurement of the S.I. of the magnitude to be estimated.
E7. Mathematical model incoherent with the real model due to the use of unsuitable units of measurement.
E8. Mathematical model is not constructed or is incomplete because elements of the real model are not quantified.
➢ *Mathematical working error:*
E9. Use of incorrect calculation procedures or calculation errors.
E10. Error in conversion of measurement units.

E11. Incomplete resolution procedure.
➢ *Interpretation error:*
E12. Absence of measurement units in the results.
E13. The estimate is clearly incompatible with the real situation.

Results and discussion

The analysis of the N = 224 resolution plans and the N = 62 group resolutions combines descriptive and qualitative analysis of categories and types of errors. Using the categorisation presented in Table 1, Table 2 show the overall results of the analysis of the N = 224 individual resolution plans and the N = 62 group resolutions.

Table 2. Frequency comparison of each type of error in individual resolution plans and group resolutions

Error type/ Category	Frequency in the N = 224 individual resolution plans	Frequency in the N = 62 group resolutions
E1	27 (5,86%)	5 (4,17%)
E2	100 (21,69%)	7 (5,83%)
E3	4 (0,87%)	5 (4,17%)
E4	41 (8,89%)	0
Simplification error	**172 (37,31%)**	**17 (14,17%)**
E5	66 (14,32%)	7 (5,83%)
E6	7 (1,52%)	25 (20,83%)
E7	24 (5,21%)	5 (4,17%)
E8	87 (18,87%)	3 (2,50%)
Mathematization error	**184 (39,91%)**	**40 (33,33%)**
E9	44 (9,54%)	7 (5,83%)
E10	1 (0,22%)	13 (10,83%)
E11	49 (10,63%)	1 (0,83%)
Mathematical working error	**94 (20,39%)**	**21 (17,49%)**
E12	2 (0,43%)	3 (2,50%)
E13	9 (1,95%)	39 (32,50%)
Interpretation error	**11 (2,39%)**	**41 (35%)**
Total	*461*	*120*

Among the 224 prospective teachers who participated in the in-class individual experience, 166 made at least one error, representing 74.11% of the total. The results show that pre-service teachers made a large number of errors (461) in their individual resolution plans, giving an average of 2.05 errors per solver. In particular, 166 pre-service teachers who made errors had an average of 2.78 errors per solver. On the other hand, when these prospective teachers were confronted with the same problems in groups and on-site, they made a total of 120 errors, an average of 1.9 errors per group. 51 of the 62 groups made errors, or 82.26% of the total. The 51 groups that made an error had, therefore, an average of 2.35 errors per group. Thus, a higher proportion of groups made errors when they had to find a numerical estimate for each problem, but they made fewer errors than when, individually, each group member had to come up with a resolution scheme.

It is observed that the proportion of errors for each category and for each associated error type is different for individual resolution plans and for group resolutions. In the individual in-class experience 37.31% of the errors belong to the category of *simplification error*, when the real model has to be constructed from the simplification and structuring of the real situation. Specifically, error *E2. Incorrect real model due to error of perception of the magnitude* is the most frequent (21.69%). Qualitative analysis of the individual resolution plans shows that most of these errors are due to confusing length and area. It appears especially in problem P2-Tiles, where the surface area between the gymnasium and the Faculty of Education is confused with the distance between the two. It also appears in P1-People or P3-Grass; in which the width of a person or width of a blade of grass is confused with the surface area they occupy. However, simplification errors are drastically reduced in the on-site group resolutions, accounting for only 14.17%, the category with the lowest error frequency. It is relevant that there is no group resolution that has not developed a real model (E4, leading to incomplete productions) while in the individual resolution plans the number was high (41).

Regarding the category *mathematization error*, number of errors is very high both in the individual in-class experience (39.91% of the total number of errors) and in the group and on-site experience (33.33%). However, the nature and severity of these errors are very different. The most frequent errors in the individual resolution plans (E5 and E8) are the least frequent in the group resolutions. Indeed, *E5. Mathematical model incoherent with the real model due to an error in the meaning of the terms of the magnitude* is a type of error related to the inappropriate use of different quantities in a procedure, mixing them without respecting dimensional homogeneity. A qualitative analysis shows that many errors of this type appear in individual resolution plans which, in order to obtain the number of elements, divide the magnitude measurements of different dimensions. For example, in P1- People, dividing the total area of the porch by the width of a person. We also find other examples of E5, such as the confusion of the concept of area and the concept of perimeter. On the other hand, *E8. Mathematical model is not constructed or is incomplete because elements of the real model are not quantified* relates to resolutions that do not develop the mathematical model sufficiently to know which strategy would give the estimate. There are numerous examples of this error in the individual resolution plans: for example, in P4- Cars it is mentioned that "the car is measured" and that "the parking is measured" but it is not stated precisely what magnitude is plan to be measured are, nor is it written with what mathematical procedures the number of cars would be estimated from these "measurements". There is a lack of definition of relevant variables for the resolution of the problem and the dependencies between these variables, which indicates difficulties in understanding the mathematical concepts involved in the real model of the situation. Errors E8 and E5, numerous in individual resolution plans (18.87% and 14.32%, respectively) are the most serious in this category. They denote important conceptual shortcomings, especially E8 is often linked to resolutions categorised as incomplete. In contrast, in on-site and group resolutions, the most frequent type of *mathematization error* (20.83% of the total number of errors) is *E6. Mathematical model incoherent with the real model due to inadequate internalisation of units of measurement of the S.I. of the magnitude to be estimated*. This type of error is caused by assigning a value to an estimation reference in conventional units of measurement that does not correspond to reality. We find many examples in group resolutions: for example, in P2-Tiles, those who consider that one foot is equivalent to one metre. Another example: in P3-Grass, a solver estimates that the dimensions of a blade of grass are 1 metre wide and 3 metres long. It is clear from these examples that the conventional measures for the length of a foot or the area of a blade of grass are not well internalised by the solver. These are errors that denote shortcomings in the measurement sense, but do not hinder the development of a mathematical model to obtain an estimate. In group resolutions only 2.50% presented incomplete mathematical models (E8).

Regarding the category *mathematical working error*, number of errors is similar for individual resolution plans (20.39%) and group resolutions (17.49%). However, the associated error types also change drastically. In the individual resolution plans the most numerous are E9 and E11. For *E9. Use of incorrect calculation procedures or calculation errors*, a qualitative analysis shows that, in individual resolution plans, this error is due to the use of the inverse algorithm (Ivars and Fernandez, 2016), that is, make a multiplication in a situation with a measure division structure. For example, the procedure $surface\ area \div element\ area = number\ of\ elements$ is inverted to $surface\ area \cdot element\ area = number\ of\ elements$. Errors of type *E11. Incomplete resolution procedure* are due to the fact that, in many individual resolution plans, the solver has not explicitly written that, in order to find the number of elements of a surface using the area of the surface and the area occupied by the element, he/she must divide these data. Solvers do not write down the process because, perhaps, it is taken for granted. In contrast, in the on-site group experience the most numerous error in this category is E10, which only appears once in the individual plans. E9 and E11 hardly appear at all. Thus, in group resolutions, numerous errors of type *E10. Error in conversion of measurement units* are made, especially when converting units of area (for example, from cm^2 to m^2 by multiplying by 100).

Finally, the most important change is observed in the *interpretation error* category: from representing only 2.35% of the total errors in individual resolution plans, it becomes the most frequent in group resolutions (35% of the total). This is not because working in a group and on-site makes it difficult to interpret and validate the results, but because the schematic nature of the resolution plan, which did not require a numerical estimate, meant that there was no solution to interpret, as most plans only point out how they would get there. In group resolutions, the most frequent error is *E13. The estimate is clearly incompatible with the real situation*, representing 32.50% of the total errors. In fact, it is the most frequent error made in all phases of group resolutions. A qualitative analysis shows that most of these errors occur because the numerical nature of the result is incompatible with reality. For example, a decimal number is given as an estimate of the number of cars that fit in a car park, or people that fit on the porch. This error is because the solvers have not checked the mathematical result and have presented it as a solution without first interpreting it in the real situation.

Conclusions

The discussion of the results above allows us to answer the two research questions:

Q1: *Does group and on-site work help to reduce the number of errors made by prospective teachers when solving Fermi problems?*

Working in groups and on-site does not reduce the proportion of participants who make errors, but increases slightly. This may be because the group resolution is more comprehensive than the individual resolution plan, as it requires measurements and numerical estimations. A larger number of processes may allow more errors to be made. However, working in groups and on-site does slightly reduce the number of errors made and, above all, their seriousness. Errors that prevent the development of a real model (simplification errors, especially E4) and those that prevent the development of a mathematical model (especially E8) are drastically reduced. This explains, from the point of view of errors, a previous work (Segura et al., 2021) which shows that, for this same sample, the percentage of incomplete productions drops from 15.6% in the individual experience to 2% in the group productions. Working in groups and on-site, therefore, is a scaffolding tool that, although it does not reduce the number of errors committed, it does drastically reduce their relevance. Errors made in groups do not impede the development of a model and the achievement of a reasoned estimate, even if it may be wrong.

Q2: *Do error types evolve from individual resolution plan to group resolution at the problem site?*

The nature of the errors indeed differs from individual resolution plans to group and on-site resolutions. The evolution at each stage of the modelling process is as follows: in group resolutions there is a sharp reduction of error rates of the simplification phase compared to individual resolution plans. Working in the place of the problem improves the perception of space and, therefore, of the magnitudes involved in the real model and which will be quantified in the mathematical model. This allows for a low number of incomplete or incoherent real models, when in the resolution plans this number was high.

In the mathematization phase overall proportion of errors does not change much, but their nature does. The types of errors associated with this phase in individual resolution plans were mostly serious conceptual errors that often impede the development of a mathematical model and result in incomplete productions. However, in group resolutions most errors in this phase denote a lack of skills in the measurement sense, but do not prevent the development of a mathematical model to reach an estimate. Again, working in groups and on-site helps in the modelling process, although the detailed measurement work brings out errors in the sense of measurement and estimation.

The proportion of errors in the mathematical working phase is similar in the individual in-class experience and in the on-site group experience. But the types of error are different in each experience. In the individual resolution plans there is a majority presence of errors of a structural type in the estimation calculation procedures (use of the wrong algorithm, or lack of calculations). However, in the group resolutions, most of the errors in this phase are due to failures in the conversion between units of measurement, which implies a poor understanding of the concept. This difference is due to the schematic character of the individual resolution plan: it is not required to execute the numerical operations, only to explain them, which may lead to errors in the choice of algorithms that would not be made if actually executed, but avoids making numerical errors (such as in conversion).

The change in the proportion of errors of the interpretation phase is drastic: these errors increase considerably in group resolutions. The explanation is obvious: the schematic nature of the individual resolution plan makes it unnecessary to interpret the result, as it only explains how to get an estimate, and very few solvers provide a numerical estimate. However, in the group resolution they have to reach a numerical estimate, and many groups do not reflect on the nature of the result obtained: they use decimal numbers to express the number of cars or people, or give clearly implausible estimates (e.g. estimating that there are 30 blades of grass in a garden).

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Turkish and Spanish Prospective Mathematics Teachers' Solutions to a Fermi-Based Modeling Problem

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This study presents a part of a larger project investigating Turkish and Spanish prospective mathematics teachers' ways of modeling in Fermi-Based Modeling Problems. In this particular study, we focused on 10 Turkish and 10 Spanish prospective teachers' models to solve Toilet Paper Rolls Problem. We collected their written responses to the Fermi-based modeling problems. The data analysis process involved series of coding and categorizing the characteristics of the solutions. We presented the models of prospective teachers from two countries separately and articulated the commonalities and divergences in their ways of approaching a Fermi-based modeling problem. Although most models indicated a numerical approach, we observed that some models were pedagogically more powerful than others.

Keywords: Fermi-based modeling problems, Prospective mathematics teachers, real context.

Introduction

In this study, we focused on open-ended and ill-defined Fermi-based modeling problems. Fermi-problems involve situations that people may face in their daily lives and engage in making strategic estimations to identify quantities (Ärlebäck, 2009). Therefore, researchers found Fermi problems useful to integrate into a modeling perspective and investigated students' ways of thinking from the modeling perspective (e.g., Albarracín et al., 2019; Albarracín & Gorgorió, 2014, 2019; Gallart et al., 2017). Considering Fermi problems as a base for modeling is a very recent approach, and much needs to be known. For instance, there is still little known about the pre-service or in-service teachers' modeling process in Fermi-Problems (e.g., Ferrando et al., 2021). To address this gap, we designed a project to investigate prospective mathematics teachers' ways of modeling in Fermi-Based Modeling Problems that incorporated a global real context, Covid Pandemic. We used such context because it was experienced somehow similarly all over the world. Researchers investigating the models in Fermi problems argued that the familiarity of the context played a role in problem-solvers' models (e.g., Ferrando et al., 2021). We therefore conjectured that the context of the problem would not intervene the models that was produced by the prospective teachers in both countries and so we could attend prospective teachers' ways of mathematical thinking that were provoked by but not depend on the Covid Pandemic context. In the larger project, we aimed to articulate the commonalities and divergences in Turkish and Spanish prospective teachers' ways of approaching Fermi-based modeling problems. This study, however, presents our initial exploration of Turkish and Spanish prospective teachers' approaches to one Fermi-based modeling problem through an in-depth content analysis of their work. Through this exploration, we expect to see a range of solution approaches that the prospective teachers might develop in a Fermi-based modeling problem situation.

Theoretical Framework

There are various modeling perspectives addressing different epistemological assumptions (Kaiser & Sriraman, 2006). Still, they share a common understanding that modeling involves a transition between the real-world and mathematical world, and this transition embraces a series of modeling cycles (Sevinc & Brady, 2019). Hence, the modeling process involves a series of express-test-revise cycles where the problem-solvers express the real-life situation mathematically, develop and test mathematical solutions, and revise the solutions until they satisfy the real-life problem (Lesh & Doerr, 2003). Since the nature of Fermi problems are “directly related to the daily environment” and therefore “offer more pedagogical possibilities” (Sriraman & Lesh, 2006, p.248), they have been considered as effective situations to initiate a modeling process.

Some examples of Fermi problems ask for the time needed to get into the top of the Empire State Building using the stairs (Ärlebäck, 2009), estimate the number of elements in a specified area (see Albarracín & Gorgorió, 2014 or Ferrando et al., 2021). As seen, these problems are realistic, open-ended, and do not often involve a numerical value and therefore ill-defined. Due to these characteristics of the Fermi problems, researchers considered that they had the potential to initiate and develop a model in response to the given problematic situation. Models refer to

> (a) a conceptual system for describing or explaining the relevant mathematical objects, relations, actions, patterns, and regularities that are attributed to the problem-solving situation; and (b) accompanying procedures for generating useful constructions, manipulations, or predictions for achieving clearly recognized goals. (Lesh & Harel, 2003, p. 159)

According to this definition, models present one’s ways of thinking about a situation. However, those ways of thinking needed to be useful for a particular situation and therefore required a goal-directed conceptual reorganization.

In this study, we focused on a Fermi-based modeling problem that was given a realistic context and not bounded by cultural issues (see Figure 1). In other words, our Fermi-based modeling problem addressed a global situation that people experienced all over the world. Modeling this realistic problem required one’s analysis of the situation, making relevant and realistic assumptions, producing a solution that will be useful in that situation. Although the problem did not explicitly state, the ill-defined nature of the problem requires problem-solvers to produce a general model (e.g., a formula or a list of steps and/or procedures) that will be applicable in similar situations. In this respect, we expect prospective teachers to engage in a series of conceptual reorganizations and produce a model that is shareable and re-usable, one of the identifying characteristics of a model (Lesh & Doerr, 2006).

Methods

This study presents a small portion of a larger project involving a set of Fermi-based modeling problems and 30 Turkish and 34 Spanish prospective mathematics teachers as participants. The participants in both countries were senior college students. Particularly, Turkish prospective teachers were in their last semester of a four year-long teacher education program. They have been trained to teach mathematics in middle schools (i.e., grade 5-8). Similarly, Spanish participants were in the second term of the compulsory professional master's degree for secondary school mathematics teachers. Since we are still at the initial phase of data analysis, here, we focused only on 10 Turkish and 10 Spanish prospective teachers' solutions that were randomly selected from the entire data set.

We collected prospective teachers' written responses to the problems and, in this study, analyzed their responses to one of the Fermi-based modeling problems called the Toilet Paper Rolls Problem (see Figure 1).

Toilet Paper Rolls

It has recently been one year since the declaration of the global pandemic by COVID19. In the first months after the start of the pandemic, most governments decided to place citizens in strict home confinement for a couple of weeks. In most of our homes, products were stockpiled to avoid unnecessary trips to the supermarket, and one of the most stockpiled products in our homes (all over the world) was toilet paper rolls.

Based on the picture, how many rolls of toilet paper do you think can be accumulated in this wardrobe?

Figure 1: The Toilet Paper Rolls Problem

The data analysis process involved series of reading the responses, coding the characteristics of the solutions, and categorizing the solutions and characteristics. Data-driven coding process characterizes our data analysis process: we did not start with a code list but constructed the one as codes emerge from the data. Two authors also met to articulate and refine our interpretations of teachers' models.

Findings

In this section, we present Turkish and Spanish prospective mathematics teachers' models separately and, in the following section, discuss the commonalities and divergences in their way of approaching a Fermi-based modeling problem. As we communicate about their ways of solutions, we use T# (e.g., T1, T2, …) for Turkish participants and S# (e.g., S1, S2, …) for Spanish participants.

Turkish Prospective Mathematics Teachers' Models

The participants first identified what information was needed to solve the Toilet Paper Rolls Problem. Depending on the approach they took, they either identified the needed information with literal symbols or numerical values, which were not provided in the problem. For instance, T4 made a list (Figure 2a) with estimated numerical values. Not only T4 but five other prospective teachers listed them and assigned estimated values to conjecture a solution for the problem. Prospective teachers who approached the problem algebraically also listed the same items (Figure 2b), but they listed their labels with literal symbols instead of the numerical values.

Numerical Approach	Algebraic Approach
Width of the cabinet: 32 cm Depth of the cabinet: 30 cm Height of the cabinet: 40 cm } Only for one shelf of the cabinet Height of the toilet paper roll: 12 cm Radius of the toilet paper roll: 5 cm	*To formulate a general formula, I label the height of the cabinet (a), width of the base of the cabinet (b), length of the base (c), diameter of the toilet paper roll (R), and height of the toilet paper roll (h).*

Figure 2a and 2b: The list of information made by T4 (left) and T1 (right) to solve the problem

We observed that five of the ten prospective teachers preferred estimating numerical values and those estimations were mostly blind guessing. Only one of them stated that she relied on the picture provided in the problem and showed her marking on the given picture. There was one prospective teacher who showed both an algebraic solution and a numerical solution and one who provided a description of a solution but did not produce an algebraic expression or a numerical value.

Those seven of the ten prospective teachers considered a regular arrangement of toilet paper rolls. The regular arrangement that they described was involving four steps: finding the number of rolls that would fit along with (i) the length, (ii) the width, and (iii) the height of one shelf the cabinet, and (iv) multiplying that result by 2 to find the total number of rolls that would fit into two shelves of the cabinet. Not only the prospective teachers who approached numerically but also the ones who approached algebraically considered this regular arrangement. Figure 3 below presents both cases.

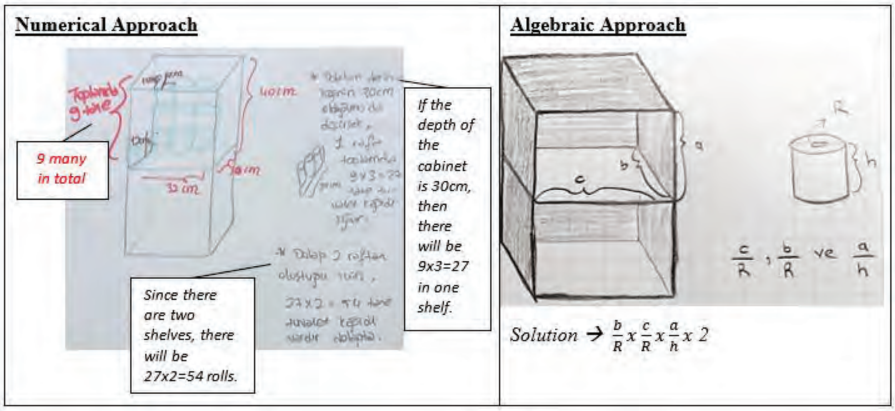

Figure 3a and 3b: Numerical and algebraic solutions based on a regular arrangement

Another algebraic solution was produced by T9 without referring to the arrangement of the rolls. Her solution involved a ratio of the volume of the cabinet to the volume of the toilet paper rolls.

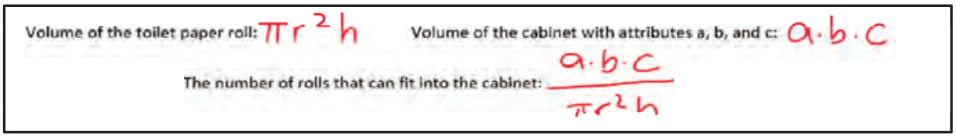

Figure 4: Algebraic solutions without a particular arrangement consideration

Although the prospective teachers expressed that the problem was different from stereotypical textbook problems, these solutions indicated they followed a procedural prescription of their curricular knowledge. There was only one prospective teacher, T6, who considered realistic and contextual assumptions and a variety of (regular and irregular) arrangements. In this sense, his model was more detailed than other prospective teachers' models (see Figure 5).

T6 stated that, for the arrangement in Solution 2, he was inspired by his mother's arrangement of cans in a cellar, and this solution resulted in one more row to put toilet paper rolls. For Solution 3, he stated that he needed to consider a realistic need – the need for storing as many rolls as possible during the pandemic lockdown period – and therefore, he would not care about the shapes of the rolls in such a case. So, he decided to distort the shapes of some rolls to fit more into the cabinet.

Spanish Prospective Mathematics Teachers' Models

Similar to Turkish prospective teachers, we observed that in all the productions the variables necessary to address the problem were identified and, in most cases, these were initially presented literally (except in the case of S7, which, as shown in Figure 6a, directly gives values to the variables). For example, S2 states: "To solve this problem, I would need to know the dimensions (width, height, and depth) of each of the two shelves of the cabinet. In addition, I would also need to know the diameter and height of each paper roll", while S6, as shown in Figure 6b, with the help of a graphical representation, indicated the quantities he needed to know.

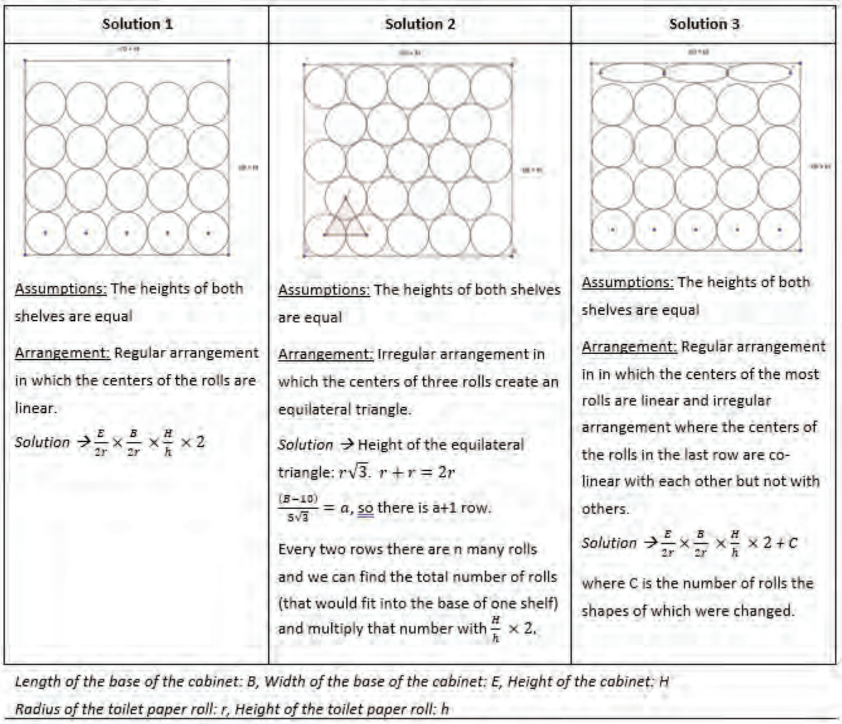

Figure 5: Three algebraic solutions with various arrangement considerations

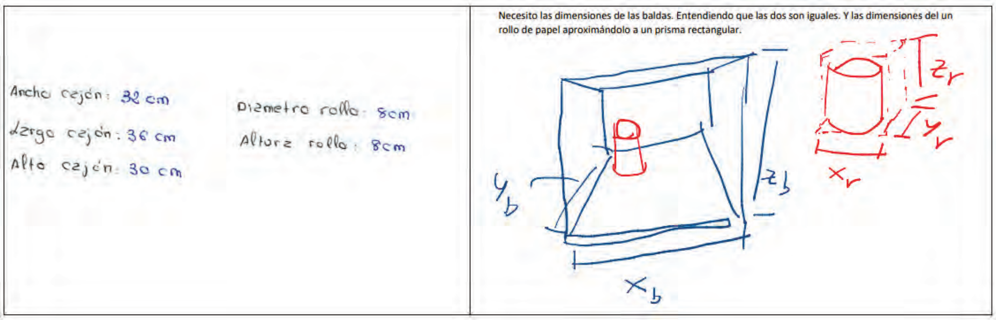

Figure 6a and 6b: The list of information made by S7 (left) and S6 (right) to solve the problem

Most of the productions did not include indications on how to obtain values of the variables; as we will see later on, these productions presented, in almost all cases, algebraic resolutions. However, some participants introduced estimated values to the magnitudes involved, sometimes in an unjustified way (such as S7 in Figure 6a). In other cases, relying on the image accompanying the statement, S3 noted that "From what can be seen in the image, the maximum height per shelf seems to be four rolls. We can make an estimate of the maximum depth, also assuming that there are four rolls. Finally, we can also get the length of the wardrobe, which seems to be three rolls".

The statement of the problem suggests that the rolls are organized in the cabinet, and this is how it has been interpreted in practically all the resolutions that propose one or more models for the arrangement of the rolls.

Only S4 avoided indicating a specific arrangement of the rolls; however, he explained the following: “I first thought that we could reason directly from the volumes of the paper roll and the cabinet, but I discard this option because we would not have enough information to know how many could be correctly arranged.” As we have already seen in the analysis of Turkish prospective teachers’ productions, in some cases, they proposed more than one possible arrangement, both in the way the rolls were placed and in the way they were oriented on the shelf (see Figure 7).

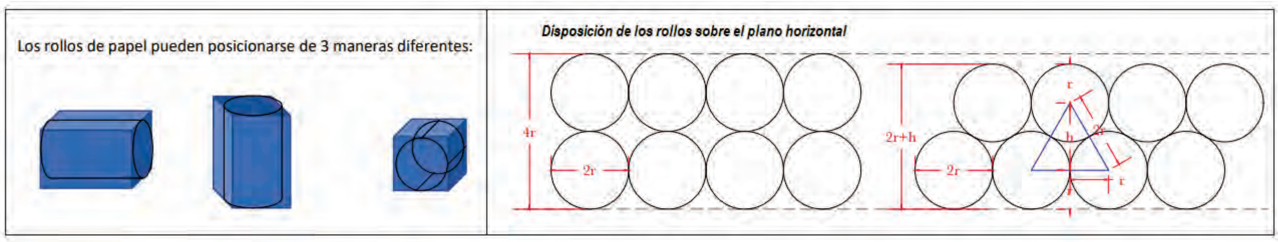

Figure 7a and 7b: Different orientation of rolls on the plane (S1) and different arrangements (S5)

Once the variables were fixed (and, in some cases, their values estimated) and the arrangement(s) defined, there were two possible procedures to arrive at the solution. Starting from the estimated number of rolls that would fit in rows, columns, and shelves, a final estimate could be obtained. One of the prospective teachers defined this as “finding the volume by taking the rolls as the unit of measurement.” Another less direct way of arriving at the solution was to reason from the dimensions of the elements (the rolls and the cabinet) according to a fixed arrangement.

Finally, the proposed solutions were categorized according to the language used, following the same criteria as in the analysis of the previously described Turkish productions. However, in the case of Spain, there was no production that presented the solution both in numerical and algebraic language. In relation to the introduction of elements of reality in the resolutions, we have identified two prospective teachers who considered particular characteristics of toilet rolls in their resolutions: both S4 and S9 commented that it was necessary to determine whether or not the roll could be deformed.

Conclusion

As we presented in the previous section, the prospective teachers approached the Toilet Paper Rolls Problem in different ways, but they all did make sense of the problem situation. Below, we summarized the characteristics of Turkish and Spanish prospective mathematics teachers’ models in the Toilet Paper Rolls Problem. As our analysis of the Fermi-based modeling problem indicated, there were similarities in prospective teachers’ ways of thinking (see, for instance, Table 1).

Actually, a considerable number of Turkish and Spanish participants approached the problem numerically. We interpreted that those who operated with estimated numerical values did not feel comfortable with indeterminacy in the problem and transform the problem into a one that is manageable. Although those prospective teachers estimated numerical values to conjecture a solution, their solution process involved systematic steps of finding the number of rolls on each attribute and multiplying them.

Table 1: Characteristics of prospective mathematics teachers' models

<table>
<tr><td rowspan="2">Step 1. Identification of variables</td><td colspan="3">Numerical</td><td>Literal</td></tr>
<tr><td colspan="3">T2, T4, T5, T7, T8, T10
S7</td><td>T1, T3, T5, T6, T9
S1, S2, S3, S4, S5, S6, S8, S9, S10</td></tr>
<tr><td rowspan="2">Step 2. Estimating numerical values to use in the conjectured operations</td><td>Only estimation</td><td>Making a search</td><td>Relying on the visual image</td><td>NO</td></tr>
<tr><td>T4, T7, T8, T10
S7</td><td>T6</td><td>T2
S3, S9</td><td>T1, T5, T3, T9
S1, S2, S4, S5, S6, S8, S9</td></tr>
<tr><td rowspan="2">Step 3. Regular arrangement of elements</td><td>YES, ONE</td><td>YES, MORE THAN ONE</td><td colspan="2">NO</td></tr>
<tr><td>T1, T2, T3, T4, T5, T7, T8, T10
S2, S3, S6, S7, S8, S10</td><td>T6
S1, S5, S9</td><td colspan="2">T9
S4</td></tr>
<tr><td rowspan="2">Step 4. Procedure</td><td colspan="2">Direct from the number of rolls in each dimension</td><td>From dimensions of rolls and cabinet</td><td>Not specified</td></tr>
<tr><td colspan="2">T2, T5, T7, T8,
S3, S8, S10</td><td>T1, T3, T4, T6, T9, T10
S1, S4, S5, S6, S7, S9</td><td>S2</td></tr>
<tr><td rowspan="2">Step 5. Structure of solution</td><td>Numerical</td><td>Both numerical and algebraic</td><td>Algebraic</td><td>Verbal</td></tr>
<tr><td>T2, T4, T7, T8, T10
S7, S8, S10</td><td>T5</td><td>T1, T6, T9
S2, S5, S6, S9</td><td>T3
S1, S3, S4</td></tr>
</table>

In this sense, the models relying on the numerical approach – and the algebraic ones – indicated ways of reasoning that were useful in the situation and are re-usable in similar situations, aligned with Lesh and Doerr's remark of model's being sharable and re-usable (2006). In addition, we observed a few prospective teachers considered realistic and contextual assumptions that influenced their solutions (e.g., T6 considering the different arrangements and S1 considering the different orientation of rolls), consistent with the research found that students tended not to consider realistic assumptions while solving real-life problems (Verschaffel, de Corte & Lasure, 1994; Verschaffel et al., 2000). However, even though they were a few, we found these models more sophisticated than others because their assumptions and approaches resulted in more than one solution. We believe that such models are pedagogically more powerful to develop solid mathematical reasoning for both teachers and students because they allow problem-solvers to think different aspects of the situation, which resulted in a different solution. In our presentation, we will further address these pedagogical implications. Our analysis involved only written responses and therefore hindered us to observe prospective teachers' modeling process. Although it is not in the scope of this particular study, in the data analysis of the larger project, we seek for the ways of which final models might signal possible shifts in problem-solvers' mind.

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Written engagement with modelling task situations as a strategy to enhance the comprehension process

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Keywords: Modelling task, situation model, comprehension process.

Motivation

When it comes to mathematical modelling tasks, students often have difficulties when working on them (Blum & Leiss, 2007). A high proportion of the occurring errors are errors in the comprehension process (Wijaya et al., 2014). These errors lead to further errors in subsequent modelling steps and to incorrect solutions. Although understanding the context of a modelling task and forming a suitable situation model is so important, it is often not focused on in mathematics lessons (Prediger & Krägeloh, 2015). In order to help students to solve modelling tasks correctly, it would be advisable to focus on the comprehension process. One way to integrate this into regular mathematics lessons could be to guide students to engage with the situation of a modelling task. In more detail one option could be to have students retell the given task situation in writing before solving the task.

Theoretical framework

According to the modelling cycle by Blum and Leiss (2007), the first step in solving a modelling task is to understand the task text and the situation described in it. In this step, a mental model of the given situation, the so-called situation model, is formed. A situation model is a mental representation of the given situation that is built in the mind while understanding a task and is influenced by variables of process-, person- and task-related characteristics (Leiss et al., 2019). Since the context of modelling tasks is not interchangeable, modelling tasks cannot be solved without considering the context (Leiss et al., 2010). The comprehension of the task and the associated formation of a suitable situation model is crucial for the further processing of the task. Nevertheless, when working on modelling tasks in mathematics lessons, the focus quickly shifts to mathematizing or to directly working mathematically (Prediger & Krägeloh, 2015). This makes it difficult for students to form a suitable situation model.

To counteract this, it would be advisable to focus more intensively on the given situation when working on modelling tasks. One possibility would be to encourage students to do this by retelling the given task situation in detail. In order for all students to have the opportunity to engage with the task situation individually by retelling it, this could be done in writing. The argument in favor of a written engagement is that the thinking process is slowed down during the writing process, which allows students to deal with the situation more intensively (Kuntze & Prediger, 2005). Therefore, this strategy could help students to form a suitable situation model. However, text production requires certain writing skills, what might bring new hurdles that need to be overcome (Bossé & Faulconer, 2008). It is therefore unclear to what extend an engagement with the task situation of a modelling task in the form of written retelling is beneficial for the processing of a modelling task.

Research questions and method

With the theoretical background, the following research questions arise:

1) To what extend does an engagement with the situation of a modelling task influence the success of solving a modelling task?
2) How does the processing of a modelling task differ when students retell the task situation in writing before solving the task?

To investigate the research questions, a quantitative study is planned. Seventh graders are given two modelling tasks to solve. The first task is to be solved by each student without further instructions. For the second task, the students receive further instructions in which they are asked to retell the given task situation in writing. Afterwards, they are asked to solve the task as well. The order of the tasks will be permuted over all students.

The students' written solution paths are then analyzed. For the evaluation of the results, first the number of correct solutions will be compared for the processing with and without applying the presented strategy. Subsequently, some details, such as the number and type of errors made, can be focused on in more detail. For example, it could be analyzed whether certain errors are made less frequently when retelling the task situation before solving the task instead of directly solving the task. These analyses should help to determine whether written retelling of the task situation as a strategy to engage with the task situation is beneficial for solving a modelling task. Data collection will begin in the spring of 2022.

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Designing computer-based modeling activities in mathematics

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In the present study, we asked: what are the design principles of computer-based modeling activities in mathematics? Using a design-based research approach, we formulated our essential components for computer-based modeling activities according to the literature. Next, based on enactment of the activities, we redesigned it and formulated five design principles. We describe a four-phases study demonstrating the "Head-to-head dice competition" activity.

Keywords: Task design, mathematical modeling, technology, simulation.

Theoretical background

A main challenge in learning mathematics is the students' difficulty in making connections between reality and mathematics, and in realizing that math is important and useful for understanding and solving reality-related problems. For this reason, one of the goals of the mathematics curriculum worldwide is to provide opportunities for developing an understanding of mathematical models, structures, and simulations applicable to many disciplines (NCTM, 1989), which should infuse students' mathematical interest with meaning.

This paper conceptualizes a model as a system of elements and the relations between them, which can be used to describe, explain, or predict the behavior of some other familiar system (Doerr & English, 2003). This definition is consistent with Shternberg and Yerushalmy's (2003) definition of a didactic model as "a collection of objects that are familiar to the learner, and operations defined on them which support the construction of an understanding of more formal reasoning about mathematical concepts and about their essential properties". The model serves as an accessory to thinking and a testbed for alternate assumptions, considering that the predictions of the model depend on the accuracy of its input data (Wilensky & Rand, 2015). Schwartz (2013) claimed that mathematics education is about teaching people how to build and use effective models of phenomena, therefore, education should also focus on formulating the relations between the elements of the models, which are referred to as measures.

Modeling problems are authentic, complex, and reality-related situations, in which students connect their extra-mathematical knowledge (or knowledge of the context) with the mathematical ideas to which they have access (Ferri, 2018). To deal with these reality-related problems, many researchers agree that students need to go through a modeling cycle which starts with simplifying and understanding the situation of a given problem and ends with interpreting the mathematical results in the real world as real results. This process includes getting through constructing a real model of the situation, transforming the real model into a mathematical model by working mathematically, getting mathematical results, and validating it (Ferri, 2018; Greefrath, 2011).

Schwartz (2013) stressed that the formulation of measures is an important first step in the act of modeling, and that in order to have a model students must be able to predict the values of a measure they are interested in. Modeling problems often involves two steps: selecting an already formulated model that describes the phenomena given in the question, then selecting the “correct” relation between the relevant aspects of the situation and evaluating it quantitatively (Schwartz, 2013).

In recent years, technology has been a great assistance for teachers and learners alike, by offering dynamic forms of mathematical representation such as models and simulations, which facilitate access to previously unfathomable concepts (Chance et al., 2007) and assist in building students’ understanding, facilitating their thinking about the domain by promoting a visualization approach to learning (Konold & Lehrer, 2008). Indeed, some researchers claim that technology can play a pivotal role in supporting and promoting mathematical modeling (Greefrath, 2011), especially that it illustrates the relationships between the different components of the given model, enabling students to investigate different contexts and to submit more than one correct solution for various input data (Wilensky & Rand, 2015). Greefrath (2011) claims that technology can be utilized at several steps in the modelling cycle such as for investigating, experimenting, visualizing, simulating, and calculating (see Figure 1). In the present study, we are interested in exploring design principles of computer-based modeling activities in mathematics that includes models and simulations.

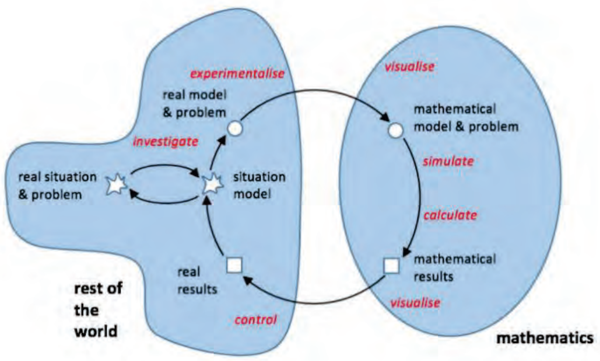

Figure 1: Modelling cycle with added influence of digital tools (Greefrath, 2011)

Methodology

The present study is part of a larger research project on the design principles of modeling activities in a digital environment, in which we ask which design principles of computer-based modeling activities support students’ modeling processes. The project is carried out in Israel. In the present study, we asked: what are the design principles of computer-based modeling activities in mathematics that includes models and simulations? We focused on principles related to activity structure, given tools, and teacher role.

Research settings

The design of new educational materials is a crucial part of design-based research in education, and it is interwoven with the development and testing of theory through iterative cycles (Bakker, 2018). To explore the design principles of computer-based modeling activities, we conducted design-based research, focusing on the “Head-to-head dice competition” activity. The research was conducted in four phases. In the first phase, we explored the design principles of modeling activities in the literature, and formulated our essential components for modeling activities. According to the design principles we articulated in the first phase, we designed new modeling activities. One of the designed activities, “Head-to-head dice competition”, is demonstrated in the research tools section. In the second phase, we presented, implemented, and consulted about the designed activities with a team of mathematics education research and development professionals, and two teacher educators and mathematics teachers. During the second phase, we further clarified and consolidated the design principles. In the third phase, we implemented the activity in mathematics lessons. As a result of this phase, 5 design principles were formulated. In the fourth phase, we examined and redesigned the activities according to the outcomes of the third phase. In this way, we integrated the design and the enactment insights derived from the research into the activities.

Population

To clarify and consolidate the design principles, in the second phase, we presented the designed activities and consulted with three groups: (a) our research team, which included graduate students, post-doctoral fellows, content developers, and researchers; (b) two teacher educators with expertise in accelerated mathematics teaching; and (c) 30 secondary mathematics teachers who participated in a professional development program and were furnished with pedagogical tools for using modeling activities in their mathematics lessons. For the third phase, we used a population of accelerated 9th grade students who were taught by two teachers in two schools, and who participated in a preparatory session to familiarize themselves with the digital environment. Both classes performed the Head-to-head dice competition activity.

Research tools

We used questionnaires, class observations, lessons’ recording, and modeling activity designed in a digital environment. The Head-to-head dice competition activity includes six tasks, and it deals with probability. This activity focuses on the students' strategies for assigning numbers to two fair six-sided dice, an orange and a yellow one, to create situations of equal and different probabilities of winning the game for each die. This activity deals with head-to-head competition in which two players play against each other. After asking students to write randomly the numbers 1 to 12 on the sides of the dice (Figure 2a), the first player selects randomly a die. Next, the players roll their own die. The player who gets the higher number wins a point. The winner of the game is the player with the most points. The activity includes a simulation that enables students to model how many times to roll the dice, and presents a histogram showing the relative frequency with which each number appeared, and the total points for each die (Figure 2b). In addition, tasks 2-6 include two-dimensional matrix model in which the first yellow row represents the six sides of the yellow die and the left orange column represents the six sides of the orange die (Figure 2c). When writing numbers on the sides of the dice, the squares in the matrix are colored according to the winner of the outcome of the roll. For example if the orange die shows 1 and the yellow die shows 5 then the square will be colored yellow - which means that the player with the yellow die is the one who wins in this case. This model allows students to look at the probabilities through an interactive area model.

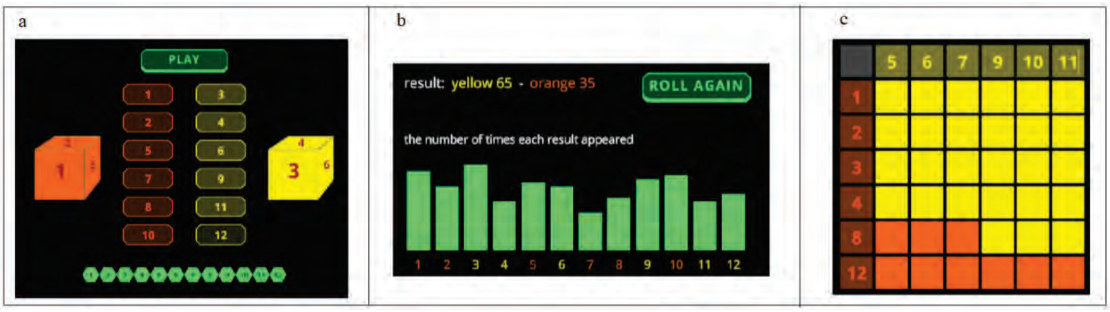

Figure 2: Head-to-head dice competition activity (a) writing numbers on the dice model (b) histogram of the frequency (c) the two-dimensional matrix model

This activity is considered a modeling activity as it deals with playing a simple game, a head-to-head competition, which helps students to understand the difference between probability and statistics by presenting probability quantitatively through the two-dimensional matrix model, while the statistics are inferred after simulating rolling the dice and interpreting the data obtained from the simulation.

Data sources

Data for this report consist of students' responses (from questionnaires, submissions and observation), comments by our research team, reviews of the two teacher educators and the mathematics teachers, and field notes of the first author taken during the various phases. Because of the COVID-19 epidemic, learning and meetings were conducted virtually. The lessons were recorded: the screen was video-recorded and the students' voices were audio-recorded. In the course of the meetings, all of the participants worked on the activities as students, asked questions, and discussed the various aspects of each activity with the designers. The authors took notes and documented the suggestions.

In the third phase, we implemented the activity in mathematics lessons: each class was asked to solve the dice activity during a mathematics lesson. Students worked individually and submissions were collected automatically by the digital environment. The students' notes and conversations were audio-recorded and correlated with the collected data. During the class observation of the virtual classes, we focused on the students' comments as they were interacting with the modeling activities, and later followed up their interaction with the teacher in the class discussion.

Data analysis

To answer our research question, in the first phase, we explored the design principles of modeling activities reported in the literature (Ferri, 2018; Schwartz, 2013; Greefrath, 2011). To design our computer-based modeling activities, we began by formulating our own design principles. We defined our designed activities as literacy activities that describe authentic and complex situations related to reality, which require students to understand the context, collect data, make assumptions, and evaluate their answers. Similar to Schwartz's (2013) approach, our activities involved formulating measures using the given data of the phenomenon we want to describe, arranging the data and describing how it is related. According to the design principles we articulated in the first phase, we designed new modeling activities. In the second phase, we gathered notes and comments of professionals, and then we coded and categorized it according to its relation to the activity structure, the tools for students and teachers, and the teacher role while students interact with the activity. In the third phase, we gathered the questions that students asked while they were working on the activity, their comments during the class discussion, and their responses to a questionnaire concerning the activity. We summarized all these data, coded it and categorized it according to how common is each phenomenon. Then we analyzed submissions to elicit evidence for common phenomena. In the fourth phase, we classified the design principles of modeling activities according to the categories we formulated in the second phase, after incorporating the insights derived from the third phase.

Results

The present study focuses on the design principles of computer-based modeling activities. In this section, we describe the four design phases. For each phase, we explain and demonstrate the design principles that were developed.

First phase: First attempts at formulating design principles

After exploring the design principles of modeling activities reported in the literature (Ferri, 2018; Schwartz, 2013; Greefrath, 2011), we formulated our essential components for our modeling activities. Unlike some other approaches (e.g., Ferri, 2018), our modeling activities already included models that represent the given situations, which enabled students to work mathematically to transform the real model into a mathematical one, then interpret the results in the real world based on the mathematical results they obtained.

The main components of our modeling activities were: (a) Simulation: each task of the designed modeling activities includes a simulation and digital tools that allow interaction with the various components of the model, offloading the student's mathematical work to these tools; (b) Example eliciting: the designed modeling activities consist of example-eliciting tasks, asking students to submit more than one correct solution for various input data. Example eliciting is a vital element in the modeling process, especially given that examples generated by students reflect their understanding of particular mathematical concepts, their difficulties, and possible inadequacies in their perceptions; (c) Modularity: the modeling activities were designed in a modular way, so that the first tasks serve to create familiarity with the context of the activity, whereas the advanced tasks require a deeper level of thinking and aim to stimulate students to generalize and consider the wider aspects of their solutions.

According to the design principles we articulated in the first phase, we designed a new modeling activity: “Head-to-head dice competition” activity.

Second phase: Clarifying and consolidating the design principles using professional advice

In the second phase, we further examined and clarified the design principles with a group of professionals. Their experience with the activity resulted in four main suggestions, described below in the context in which they emerged.

1. Introductory task. During the professional development meetings, teachers expressed a need for an introductory task that helps students familiarize themselves with the context and the various tools of

the simulation. Specifically, for the Head-to-head competition activity, teachers suggested adding a task after the first task to facilitate the transition between the given models (the dice and the table):

> I feel that there is a missing task that asks for a 50-50 probability using the table, which will help students understand the transition from task 1 to task 2. The task will help students using the example they suggested of a 50-50 probability, to change some of the numbers so they get a greater probability for the orange die to win (Orna, a teacher, 4th meeting at the professional development program, summer 2020)

2. 5-10-minute videos. Two types of videos were suggested; a technical one that guides students in using the model and the simulation in each activity, as suggested by one of the teachers: “You should add a short video that explains how to use the tools and the simulation, and the teacher should be present in case there are any questions” (Tami, a teacher educator, 2nd meeting, 2020). In addition, another video was suggested that summarizes classroom performance of the activity and the class discussion: “I would be happy to hear from other teachers who performed the activities in class, what are the main points of the discussion they conducted and what are their recommendations to other teachers” (Marta, a teacher, 5th meeting at the professional development program, summer 2021)

3. Teacher’s guide. Many teachers agreed with the teacher that suggested to have a written guide for each activity intended for teachers: “As a teacher, to make it easier and more convenient for me, I would like to have a written guide for each task for whom it is suitable, for which classes, and what its mathematical topic is” (Toni, a teacher educator, summer 2021)

4. Flexibility of design. Enabling teachers to edit the activity (add or to remove tasks or change other attributes) to adjust it to their classroom. As one teacher pointed out, teachers can use this feature for various aims, e.g., to adapt activities to a particular lesson or to change the formulation of the tasks: “Due to system constraints, I do not have enough time to complete all tasks. I prefer to have the ability to select some of the tasks from the activity” (Sara, a teacher, 5th meeting at the professional development program, summer 2021)

Third phase: Enactment of the activity

In the third phase, we implemented the activity in mathematics lessons and examined the students’ interaction with the modeling activity and with the teacher in the class discussion. We report the results of the students’ work in four parts, according to the various components of the modeling process that appear in each activity: 1. Simplifying and understanding the situation of the problem. Because of the difficulty in understanding the given problem, as observed in the class and reported by the students, we found that it was necessary for the teacher to solve and discuss the first introductory task together with the students, and to use short videos that explain how to use the technological model. One student reported: “I could not always understand the situation on my own. I needed the explanation of the teacher, to introduce the activity and then I could work on the other tasks on my own” (Hana, a 9th grade student); 2. Transforming the real model into a mathematical model by working mathematically. The students’ submissions and self-reports indicated that they managed the transformation from visual representation (two-dimensional matrix) to the mathematical model in the dice activity. However, there is a need to conduct a discussion after each task during which students raise mathematical ideas and compare the various solutions; 3. Obtaining mathematical results that are interpreted in the real world as real results. During the class observation, students succeeded in interpreting the mathematical results as real results. Most students submitted correct answers for the fourth task (Figure 2c), in which they were asked to assign numbers to the sides of the dice in such a way that the orange die has precisely a 1/4 chance of winning. Figure 2c shows an answer that includes 9 orange squares (¼×36=9), as required in the task; 4. Validating the results. 83% of students reported that they used the simulation to evaluate their simulation results, and that it saved them many long calculations. The class discussion following the activity validated their answers.

Fourth phase: Evaluation and redesign

Based on the results of the consultation phase and the enactment of the activity, we redesigned it according to the following principles: (1) Principles related to the activity structure: (a) In addition to the modularity of the tasks, each activity will include an introductory task aimed at assisting students in becoming familiar with the technological environment and the various tools of the simulation. The task is intended to be mediated by a teacher. (b) The teachers will be able to edit the activity (to add, to remove, and to edit tasks) according to their needs. (2) Principles related to the given tools for students and for teachers: (a) Short videos will be prepared for teachers and students on each activity. (b) A written guide for teachers that will include teachers' comments and reviews of the activity. (3) Principle related to the teacher role: The transition between the tasks will include a discussion with the students to illustrate the aim of each task. This will be noted in the written guide for teachers.

Note that all principles mentioned above will be applied also to the new activities that will be developed as part of the larger research project.

Discussion

In this paper, we described research leading to design principles of mathematical modeling activities, giving examples based on "Head-to-head dice competition" activity. As researchers stress, the modeling process starts with simplifying and understanding the situation of a given problem (Ferri, 2018; Greefrath, 2011). In the present study, the results indicate that many students had difficulty in understanding the dice problem, which means they needed help in starting the modeling process. This points out to the important role of the teacher in introducing the activity, mediating the first task and discussing students' answers after each task. Furthermore, we found that technology played an important role as students were solving the tasks: the simulation assisted in checking the results of the students' assumptions and in simplifying difficult procedures. The technology illustrated the relationships between the different components of the given model, enabling students to investigate different contexts and to submit more than one correct solution for various input data (Wilensky & Rand, 2015). Our findings confirm the claim of some researchers that technology can play a pivotal role in supporting and promoting mathematical modeling (Greefrath, 2011).

The process of consolidating design principles of modeling activities attests to the value and the need for communication tools (e.g., short videos and a written guide), both during the development process (Bakker, 2018), to connect between developers and teachers, and during the use of the activities by the teachers. This conclusion will be taken into account in the consolidating process of the design principles in the continued research.

Acknowledgments

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Developing mathematical modelling skills for mathematics teachers through 3D modelling and 3D printing (3DMP)

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Keywords: Teacher education programs, problem solving, educational technology.

Description of Research Topic

Mathematical modelling (MM) skills are considered highly valuable for Science, Technology, Arts, Engineering, and Mathematics (STEAM)-based education. MM is also part of geometry as is used for example in 3D modelling and 3D printing (3DMP). This overlapping can contain new forms of training teachers in an emerging technology, their MM skills, and providing them with tools to teach MM based exercises. MM aims at connecting real world problems with mathematics as means to solve these problems with a feedback improvement loop (Blum & Leiss, 2005). A mathematisation and de-mathematisation process has to be performed as a link from the mathematical model to the real-world solution and after the modelling step, a validation step is performed to investigate whether adaptation is necessary (Jankvist & Niss, 2020). This process has similarities to 3DMP where real world problems are also tackled by mathematising the problem, modelling it, de-matematising it by the following 3D print, and validating the result by iteratively checking whether improving the model is necessary. For example, to receive 8 cookies from a given dough size, a 3DMP cookie cutter could be created where the solution can be tested after printing. This suggests that at least for some areas such as geometry, 3DMP can be used as an instance of a MM process. Approaches to use 3D printing can improve MM skills as research indicates in the training of 50 mathematics teachers in 3D printing that reported perceived improvement of their MM skills as was also visible from their 3D prints (Asempapa & Love, 2021). This trains modelling skills, by transforming real-world information into mathematical concepts and skills, referring to the model of Blum and Leiss (2005), are highly needed for teachers to instruct their students in mathematics education. As both MM and adopting new technologies can be challenging for teachers, ways to improve technological problem-solving skills and exercises teachers can use should gain more attention (Drijvers et al. 2013). In this poster, formatted by A0 size and containing links to GeoGebra resources, we will present data driven elements for a pedagogical framework of a 3DMP course that aims at supporting Austrian pre-service-mathematics-teachers (PSMT). Our question is: what are attributes of a course framework that trains 3DMP and MM skills? We want to find hints for insight into needed ingredients that help them master this technology and provide MM exercises the PSMT's can use in their later lessons.

Steps Towards a Course Framework and Preliminary Results of the first attempt

The basis of our first course attempt was experiences of 3DMP workshops in 10 schools in Montenegro which were conducted in a design-based research manner introducing the technology and possible exercises. A task for the teachers was to create a cookie cutter with specific constraints such as size- and thickness limitations of the 3D printed version. A theoretical part contained examples of applications of the technology in mathematics lessons and personalized aids for students, theory about 3D models and modelling as well as the theory about the 3D printing process. A practical part aimed at getting familiar modelling first models, printing and revising them after adaptations to the model iteratively. After a year, we interviewed 37 of them to learn about their view and usage of the technology. During the analysis of the transcripts we found that these teachers reported that the course and working with 3D modelling in particular improved, amongst others, their MM skills. So, the concept of first course modules was based on the structure and experiences of these workshops.

We created three 3DMP modules focusing on 3D modelling for 60 Austrian PSMTs with a content structure similar to the workshops in Montenegro that accompanied a geometry course for second level education. The PSMTs had to create their own projects documented in GeoGebra books. The projects should explain a problem and the models to solve it had to fulfil the constraints of a certain printing time. They had to report their ideas, give feedback to each other's projects and the models had to be revised accordingly. This led to new and improved 3D models similar to the process described by Blum and Leiss (2005). We read through the documentation and found hints that the 3D modelling part was especially useful to train the mathematization step. We took these learnings to design a course structure concept to support 3DMP and MM skill improvement.

Discussion and Presentation: A Course Framework Concept for Teachers

The poster contains a description of the setup of the course, a description of the evaluation of the first course and a framework for upcoming courses. Based on this first course, we will develop a course dedicated only to 3DMP and check the PSMTs MM skills during the course. In the course, 3DMP and MM will be introduced. We give the participants a geometrical modelling task by letting them create mathematical 3D mazes in GeoGebra, then will conduct the course and give them a task requiring similar skills that can also be solved in GeoGebra. This task has to be solved in individual projects. A questionnaire and looking at the documentation in the GeoGebra books about their experiences and possible changes will provides more insight if and which aspects of the course changed their skills in mathematical modelling in geometry. This will help to identify aspects of how a 3DMP course should be shaped that also helps improve MM skills. We hope to create a new possibility of raising the sensitivity of future mathematics teachers for MM by connecting it to 3DMP.

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The role of models and modelling in the pandemics' evolution: transposing an 'study and research path' to secondary school

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This paper discusses the design of a teacher education proposal aiming to provide tools to secondary school teachers to deal with the teaching of modelling in interdisciplinarity contexts. We start by briefly presenting the case of a study and research path for teacher education (SRP-TE) about disseminating the fundamental role of models and modelling when interacting in the modelling process of the pandemics. We then focus more specifically on the last modules of the SRP-TE, when some of the interdisciplinary research projects are adapted and transposed to secondary school. We present an experience during the academic years 2020/21 with an open teaching project to inquire into the impact of the COVID-19 led by a team of teachers from different disciplines intervene. We analyze the conditions that facilitate the development of this project and the limitations that hinder its progress as a richer modelling activity.

Keywords: Modelling, interdisciplinarity, study and research paths, secondary school, pandemics.

Introduction

Research in mathematics education has recognised the importance of including applications and mathematical modelling in mathematics teaching and learning (Blum, 2015). Pollak (1969, p. 401) introduces the importance of problem posing stating that "the student has as much right to participate in the derivation for the mathematical model and in checking the degree of its validity as he has to repeat any experiment to satisfy himself of its validity". In this sense, including mathematical modelling applications involve finding genuine problems for students to pose good questions to study.

Besides all the progress made in research and the support of educational policies and curriculum reforms, implementing a well-established activity on modelling must confront big constraints for its long-term and large-scale dissemination. In this respect, we have several examples of major issues in our society that require a collective scientific effort working across the boundaries of the scientific disciplines, where mathematics and mathematical modelling can be seen and act as service subjects.

The COVID-19 pandemic has shown more than ever that students and, more in general, citizens need to understand how mathematical and scientific advances contribute to understanding societal phenomena. In addition, "the pandemic illustrates how the operation of science changes when questions of urgency, stakes, values and uncertainty collide (Saltelli et al., 2020). People feel the need to understand what mathematical models can provide, how we may interpret the predictions and, more generally, how they help understand complex systems such as the pandemics' evolution.

There is no doubt about the critical constraints that hinder the long-term "survival" of modelling activities in interdisciplinarity contexts in schools. They can be interpreted as a consequence of important didactic phenomena that exist in school institutions, such as the isolation of disciplines and the prevalence of monodisciplinary curricula (Michelsen, 2006, p. 269), the dominant way to organise the teaching and learning of school disciplines (based on the logic of concepts rather than the logic of problems), and the inexistence of epistemological and didactic tools to approach modelling in the interaction among disciplines.

When discussing interdisciplinary education, it is clearly related to the importance of STEM education (Maass et al., 2019) in three different approaches: twenty-first-century skills, mathematical modelling, and education for responsible citizenship. English (2016, p. 362) makes the role of modelling in STEM education clear “modelling is a powerful vehicle for bringing features of 21st century problems into the mathematics classroom”.

Within the framework of the anthropological theory of the didactic (ATD), a change of school paradigm (Chevallard, 2015) is proposed to overcome some of the main didactic phenomena linked to the “monumentalisation” of the taught knowledge. This change has been described in terms of a paradigm shift, from the paradigm of visiting works to the paradigm of questioning the world. Chevallard characterises the transformation in mathematics education not only at the pedagogical level (“how to teach?”) but also includes the changes the paradigm shift may have on the didactic level, dealing with the question about “what and how to teach?” In the paradigm of questioning the world, the knowledge to be taught is associated with the inquiry of relevant questions. Approaching these questions includes moments of study (searching for available answers in the media) and moments of inquiry (deconstruction and reconstruction of knowledge to generate one’s answer). Implementing question-led study processes helps the knowledge to be taught to become dynamic, provisional, and collective (compared to the traditional notion of knowledge in school institutions). In the ATD, the so-called *study and research paths* have been introduced to facilitate the inclusion of mathematical modelling in educational systems and, more importantly, to explicitly situate mathematical modelling problems at the centre of teaching and learning practices (see Bosch, 2018). More recently, our research team have been working with the proposal of *study and research paths for teacher education* (SRP-TE) (Barquero et al., 2018), an inquiry-based process combining practical and theoretical questioning of outside and inside school scientific activities.

Given the fundamental role of models and modelling in the understanding and social dissemination of the pandemics, we present our experience with the design of an SRP-TE about decoding the evolution of the pandemics. Its design and implementation, developed in the framework of the European project IDENTITIES, wanted to work with teachers to question the necessary tools for the analysis and design of interdisciplinary projects in secondary school. We then focus on the last modules of the SRP-TE when some of the interdisciplinary research projects are adapted and transposed into secondary school. We focus on the experience during the academic years 2020/21 with an open teaching project inquiring into the impact of the evolution of the COVID-19 where a team of teachers from different disciplines intervened. In this paper, we continue the study to answer the research questions: *What are the conditions facilitating the development of this project and the limitations hindering its progress towards a more prosperous interdisciplinary modelling activity?*

First contact with the experience with an SRP-TE about the COVID-19 evolution

In the context of the IDENTITIES project (https://identitiesproject.eu), it has been proposed the design of an instructional proposal for preservice secondary school teachers based on an adaptation of the structure of the SRP-TE (Barquero et al., 2018). One of the four proposals has focused on the role of models and modelling to understand the COVID-19 evolution. It was developed in a local implementation in the University of Barcelona and in an international Summer School, where the first two authors were involved, as a participant and as an educator, respectively. The adaptation of the general structure of the SRP-TE consisted of four modules. Participants had to assume different roles to facilitate questioning together (teachers and educators) the way to describe, analyse and design possible modelling activities that could be transposed to secondary schools to address live societal questions that emerged during the scientific approach of the pandemics.

In *module 1*, participants might act as "explorers" to analyse a set of news and research dissemination papers that the researchers-designers had selected to see the evolution of the problems addressed by the scientific community and analyse the role assigned to the disciplines, in particular, to mathematics. From this first analysis, participants with educators delimited some possible lines of inquiry that involved models and modelling and the interaction among different disciplines. The topic addressed in each line were: (1) The complexity of delimiting the system to model: analysing data, (2) The role of the equation-based models: what can we consider a 'good' model? what are models for?; and (3) Agent-based models and simulations: Simulating scenarios to help make decisions about societal restrictions. *Module 2* asked participants to experience an SRP, previously designed by the researcher-educators, about the lines mentioned above of inquiry. Participants had to assume the role of "student". The main goal of this module is to make participants carry out an unfamiliar activity that could, to a certain extent, exist in an ordinary secondary school classroom. *Module 3* refers to the collective analysis of the SRP that they come to be experienced as students, but now adopting a role of "analyst". Some specific tools were here introduced to help teachers carry out the analysis of the activity carried out. For instance, one of the provided tools takes the form of a questions-answers map (Winsløw et al., 2013) which help to make explicit the kind of disciplinary and interdisciplinary questions and answers that emerged in their experience with the SRP. In *module 4,* teachers in training worked in the design of an adaptation of the experienced SRP. And, in case they had the chance, they implemented them in real secondary school classrooms. The work developed with one of the participants (researcher in didactics and the paper's first author) is the case study we focus on in the following sections.

Design of an SRP about modelling the COVID-19 evolution for secondary school

Institutional context and conditions for the implementation

The SRP about modelling the COVID-19 evolution has been implemented twice, in April-June 2020, with the beginning of the pandemic, and in February-March 2021. Due to the exceptional conditions of the first implementation, this paper focuses on the second implementation as its design was improved and the conditions for implementation were more stable (at least, than during the confinement). The implementation was carried out at Col·legi Natzaret, in Esplugues de Llobregat, a town near Barcelona (Spain), with 60 students of grade 10 (15-17 years old) distributed in two parallel groups. It was developed as an interdisciplinary project involving the subjects of mathematics, biology, and oral and written expression. Students were organised in working teams of six members, with heterogeneity in relation to their academic performance. The SRP run over 17 sessions of 1 hour during the official hours of mathematics, biology and plurilingual expression. It ran under relatively regular conditions, although the limitations due to the pandemics: the parallel groups could not interact, and each teacher was assigned to only one of the groups. Four teachers participated in the implementation: two mathematics teachers (one being the paper's first author), a biology teacher, and an English teacher (both teachers of optional subjects, who each had half of the students). In collaboration with her research team, the first author developed the *a priori* design of the SRP. The rest of the teachers had no direct involvement in the design. Still, they got actively engaged in deciding how to present the project to students and in the *in vivo* analysis during its implementation. Some special sessions were organised with all the teachers to agree on how to introduce the project, the timing, the way to distribute the students and the strategy to manage the SRP. Then, during the implementation, the teachers shared a journal where they daily reported their work with the class and the teaching materials (their presentations, students' reports, evaluation criteria, among other aspects).

Students worked collaboratively with online and digital tools. The teachers used a shared google presentation to report the progress of each working team. In addition, after each session, the working teams worked with the same template to document the advances of their inquiry. They had to report on the questions they had addressed, the temporary answers found, the tasks developed individually and in groups, and the new questions to follow with. Besides these shared documents, students had access to a presentation with some common instructions, indicating what was expected from their work and the steps to follow. From the start, the students were informed that they were responsible for defining the questions to address and the hypothesis they had about the pandemic evolution. They had to update their question-answer map regularly and, in the end, prepare an informative video presenting the results from their research to be distributed to the school community. The SRP teachers evaluated the students' presentations, with some invited teachers from other subjects.

The openness of the generating question of the SRP and its devolution

One crucial difference with other previous implemented SRP is that this one did not start from the same common generating question. On the contrary, students were asked to confront a more general extra-mathematical problem of particular social relevance with an excess of news related to the pandemic. As the older students in the compulsory secondary school level, they were asked to run an awareness campaign for the school about the pandemic and its impact on society. They were responsible for providing contrasted and scientifically founded information and defining what they wanted to address.

In the beginning, the working teams were told to approach their research from three complementary points of view: from the available data (accessible through the Spanish government website) and the mathematical models they could use (*Which data may be selected to understand the evolution of the pandemic? How can mathematics and mathematical models help us to understand the pandemics evolution?*); from the biological knowledge of the disease (*How is the virus behaving?*); and from the societal impact of the pandemic (*What impact and effects are the pandemics having on our society?*). The working teams were asked to delimit their focus by always keeping in mind these three complementary general questions. Students started by gathering the concerns of the educational community, starting with their own and surveying their classmates and families. This helped them define the questions they wanted to address and plan the first steps of their particular SRP. At the end of these first steps, each team had to present the general topic and identify three interrelated "researchable" questions concerning the mathematical, biological and societal aspects. Some examples of the researchable questions they posed are: *How long does the COVID-19 survive on a surface? What are the characteristics of the virus that make it so deadly? What age groups are the ones more affected? What are the physical sequelae of the disease?* About the societal questions, examples of the ones proposed by the students were: *How has the pandemic affected tourism in Barcelona? How has confinement affected people's daily lives? What restrictions were implemented in Madrid during the three waves in comparison to Barcelona?* Concerning the mathematics questions, those with a descriptive nature were more frequent: *Which autonomic communities in Spain have been more affected? How can we measure if the first wave was worse than the rest? Are there important differences between the evolution of the case numbers (infected, death, recovered) among the two consecutive years?* Moreover, there were also some groups that included questions about the evolution of the data: *How has COVID evolved in Catalonia? How has it evolved in the different counties?* As it can be read in the project presentation (available at https://bit.ly/3tRQpPz), the whole implementation followed three main phases. The first phase with the (a) generation of researchable questions, (b) exploration of databases, and (c) presentation of specific questions and hypotheses to address. A second one is where they focus more on (a) looking for and organising the most relevant data for their inquiry, and (b) analysing data and proposing models to fit data and/or predict the evolution of the pandemics. A third and last step, where students had to work on the informative video. In the following section, we focus on the students' work in the first two phases, paying special attention to the researchable questions with a mathematics intervention.

Results of the experienced SRP about the COVID-19 evolution

During the sessions guided by the mathematics teachers, the different working groups addressed their researchable questions. To facilitate their work, the mathematics teachers were asked to follow a structured report. On the one hand, each working team had to make explicit the main questions they wanted to address, their hypothesis or preliminary answers, and the data they worked with. On the other hand, they had to fill out their *map of questions and answers* to describe the particular study and research trajectory they were following. This device, which was used during the whole implementation, took a crucial role for several reasons. First, it allowed students to have a common instrument for all the sessions and make explicit the evolution of the inquiry. Second, it facilitated that the teachers from the different disciplines could follow the work of the working teams. Moreover, students used this organisation to address the questions of each discipline with the corresponding teachers. Additionally, at the end of the implementation, the assessment of these maps considers the completeness and classification of all the elements, the relevance of the questions, their creativity and accuracy.

Five sessions were devoted to the first phase of the project. During these sessions, students were provided with a database from the Spanish ministry that regularly updated the data about the evolution of the pandemic since its beginning. Students found different spreadsheets with accumulated data on cases, deaths, ICU admissions. These worksheets also included information by sex, age groups, provinces, and communities. This large amount of data created important limitations. On the one hand, they had to be very careful in defining what they were interested in looking at, that is, to delimit and construct the *system*, as well as the particular questions they wanted to address. That is why the teachers were especially attentive to help them on delimiting the system by selecting the variables to consider, formulating the initial hypothesis to contrast, etc. On the other hand, they needed to learn some techniques to work with Excel to manipulate big spreadsheets easily. They had some experience with Excel but as beginners' users. Then, the mathematics teachers had to dedicate some common sessions to respond to these necessities. For instance, students were asked how to sort a list of data by value, how to filter by defining some criteria (e.g., provinces or age groups), among other utilities.

In the particular case of *Team A*, they were first interested in this initial question: "$Q_{\text{0-Team A:}}$ *Which has been the "worst" wave of the pandemics in Spain? Has the second wave been worse than the first one* (as said in the media)*?*". In these first steps, they started to define what they wanted to address (length, number of infections, hospitalise and death, in global and by different groupings):

$Q_{\text{1\_dates}}$: How long did each wave take?

$Q_{1.1}$: When do we start counting the beginning of a wave and its ending?

$Q_{\text{2\_infections}}$: How many infections have there been in each wave?

$Q_{2.1}$: How many deaths by sex were there? In total? By sex? By age group?

$Q_{2.2}$: Which was the age group more infected?

$Q_{2.2.1}$: How many are infected between 0-9 years old? Between 10-19? Between 20-29?...

$Q_{\text{3\_hospitalizations}}$: How many people were hospitalised during the first wave?

$Q_{3.1}$: How many deaths by sex were there? In total? By sex? By age group?

$Q_{\text{4\_deathss:}}$ What is the number of deaths in each wave?

$Q_{4.1:}$ How many deaths by sex were there? In total? By sex? By age group?

$Q_{4.1.1:}$ What can explain that men seem more likely to die?

$Q_{4.1.2:}$ Do we have the same tendency of deaths growth by each group age?

These questions mainly correspond to delimiting the system and representing the data numerically and graphically. The same happened with the rest of the groups, who mostly worked on the graphical representation of the data (once selected and manipulated). For groups who pose some questions about the pandemic evolution, the most common was the graphical representation of data concerning time. For instance, we present the questions made by *Team B*:

$Q_{\text{0-Team B}}$: Which wave has affected Madrid the most?

Q_1: Which wave shows the highest speed in the increase in the number of cases? And less?

$Q_{1.1}$: What is the date with the most cases registered? $Q_{1.2}$: How long has each wave lasted? $Q_{1.3}$: Why are there fewer registered cases on the weekends? $Q_{1.4}$: How do restrictions affect the registration of cases?

$Q_{2:}$ Which wave has the highest number of cases, regardless of its duration?

$Q_{3:}$ Which wave presents the highest number of cases, considering its duration?

This team wanted to analyse the variation in the number of cases. They defined and calculated the *speed* of the number of cases using variation taxes. However, none of them started to propose equation-based models to fit the data and forecast what can happen in the future. All the groups posed questions to describe the data or compare them by age or place, but none of them tried to go further.

In the last session, the teams made presentations to their peers. They discussed questions and suggestions. Interesting ideas emerged: which is the correct way of comparing the data from two or more provinces? From this discussion, they concluded that it was necessary to introduce *relative variables*, dividing the cases by the population, to compare them with accuracy.

Conclusions and discussion for future implementations

In this paper. we want to focus on the ecology of the SRP. More concretely, about what conditions and limitations the SRP reveals. About the conditions that facilitate the implementation of the SRP, we can detect three different aspects: (1) the *a priori* design of the SRP, (2) the SRP's management tools, and (3) the questions and answers maps.

When preparing the design of the SRP, the choice of the initial question is essential, but so can be the development of the path that will lead to the potential mathematical activity. For this, teachers must develop the pathway in advance: mastering, at a professional level, the activity of mathematical modelling, so that they can identify the student's processes and know-how to redirect them without forcing them or giving them the feeling that they are being deprived of the ability to decide the process of the research. But also, to identify when students open different lines of inquiry and distinguish whether these will lead to a rich mathematical activity. About the SRP management tools, we note the shared teacher diary as a tool that helped teachers keep an overview of the process, even when they did not share it with all the students. On the other hand, the scoring system for student contributions, where teachers had a daily record of which students had made a positive contribution, helped generate class discussions where students shared their progress and commented on their opinions of other students' work. Finally, the *questions and answer map* was found by the students as a rich tool. These students were used to working with this kind of tool, so they had no problems or difficulties understanding what was being asked of them. Also, as it was recommended to build since the beginning of their research, they used this tool to keep track of their daily progress and to present the final product.

Related to the constraints that hindered the progress of the SRP, we can identify (1) the lack of coordination time available, (2) the awareness of the interdisciplinary concept, and (3) the time slots available to do the project. We can highlight the lack of coordination time available to teachers, who rarely have time to share and reflect on what happened in class and have to use moments between corridors or before entering the classroom. Interdisciplinary work needs quality time for the teachers to share opinions and make decisions. The responsibility of this aspect belongs to the school, which has to provide teachers with resources to facilitate the success of the teaching activity. Besides, teachers also have some responsibility for the success of the SRP. They need to be aware of the difference between carrying out a multidisciplinary project or an interdisciplinary one. In this study case, teachers had not been explicitly given this difference. In the next experimentations, we will try to introduce this difference to see if teachers become more involved in disciplines other than their own. Finally, the last constraint to be considered was the time organisation of the project. All the students were taking mathematics, but others were not taking biology or multilingual expression.

They were taking technology, which was not involved in the project. Therefore, some students dedicated more hours than others to the development of the project. In the same way, the mathematics teachers spent 4 hours per week and the teachers of the optional subjects only 3 hours per week. This was an important constraint for both teachers and students, who were unequally involved. This causes hierarchies in the project production and sometimes results in students not being able to participate as they feel disconnected from their team's progress.

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The modelling cycle as analytic research tool and how it can be enriched beyond the cognitive dimension

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The modelling cycle is a theoretical construct frequently applied in research studies on mathematical modelling. On the one hand, the modelling cycle highlights essential aspects of modelling, which makes it a tool for conceptualizing. On the other hand, the modeling cycle can be used as a research tool for analysis of students' work. In the latter case, it has the limitation of primarily yielding results of a cognitive nature. We sought ways to include other aspects to analyze, such as metacognitive strategies, tool use, and social norms. These aspects support and change the cognitive activities involved in mathematical modelling practice. Rather than the standard modelling cycle, we propose an enriched modelling cycle with overarching layers for analysis of results. The enriched modelling cycle is a wider theoretical framework with interacting dimensions that affect the phases in the modelling cycle. We discuss potentials and challenges of this framework for new research studies.

Keywords: cognition, mathematical modelling, modelling cycle, social norms, theoretical frame.

Introduction

In this theoretical paper, we focus on theoretical constructs applied in many research studies on the teaching, learning, and assessing of mathematical modelling. Review studies on research on mathematical modelling education have been published by Cevikbas et al. (2021), Geiger and Frejd (2015), Kaiser and Brand (2015), Schukajlow et al. (2018) and Stillman (2019). These reviews show that a considerable number of studies apply a modelling cycle (MC) as theoretical framework for analysis of data in their research. With this paper, we aim at opening a discussion on benefits and limitations of MCs as a tool for analysis. This leads us to present possibilities to enrich the MC as theoretical framework, so it can assist researchers in further analyzing and theorizing mathematical modelling education[1].

This paper arose from a discussion on the importance of collaboration in mathematical modelling, as also noted by Blum (2002): how can aspects of groupwork, such as *agency* and *accountability* be analyzed in research applying an MC? Also, we discussed that research using a MC-based framework observed students having *blockages* in the modelling process, and that these were of a *cognitive* nature, that is, in students' minds (e.g., Galbraith & Stillman, 2006). However, students' problems could also be *blockages* caused by the norms of the didactical contract (Brousseau, 2002), where students are taught, for example, to avoid using extra-mathematical knowledge in mathematics task. When analyzed as blockages caused by the environment, the research results shed light on underlying mechanisms that hinder change; blockages in the educational environments may be more persistent

[1] This paper extends a short paragraph in Frejd and Vos (2021), where we presented the enriched modelling cycle. In the present paper we have more room for backgrounds and elaborations.

and harder to remove than cognitive blockages in students. So, why does research applying a MC-based framework yield cognitive blockages and not *socio-cultural* blockages?

Our discussion led to studying the essence of MC-based frameworks, regarding their benefits and limitations in analyzing aspects in modelling activities. This inspired us to develop an enrichment perspective on MC-based frameworks, so these can zoom out and yield analytic results that could be, for example, *social* in nature. This enrichment perspective should assist researchers of modelling education to create new angles in their data analysis, and thus reach new research results.

The modelling cycle as tool for conceptualizing and analyzing

Much research on mathematical modelling describes mathematical modelling through a *modelling cycle* (Niss & Blum, 2020; Geiger & Frejd, 2015). A MC is a schematic diagram showing mathematical modelling as a cyclic process, which consists of subsequent phases. See Figure 1 for an often-used example from Blum (2015), which shows seven phases in the modelling process; other MCs may have fewer or more phases and other wordings (Perrenet & Zwaneveld, 2012).

The MC in Figure 1 builds on an earlier version by Blum and Leiβ (2007), in which the 1st phase was named *understanding*, to indicate that the modeling process starts from a problem situation that needs to be understood. In the new version of this MC (Blum, 2015), it is written *constructing* to indicate that a modeller needs to create a mental model of the problem and the task ahead. After this start, the modeller goes through different phases by structuring and simplifying the problem context (e.g., making a rough drawing of the problem situation), which is mathematizable (e.g., by creating algebraic formulas), and which can be worked on mathematically (e.g., by manipulating the algebraic formulas). The mathematical results thereof can be interpreted and validated considering the original problem. In case the results are considered inadequate for the real situation, the entire modelling process is repeated. If the modeller is ‘ready’, the results can be *exposed*, that is: presented to others.

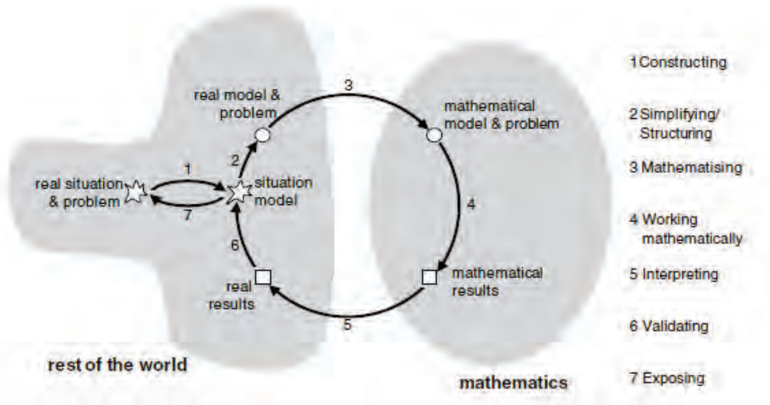

Figure 1: Modelling cycle from Blum (2015)

When students are given a modelling task, students follow other routes than what is described in a MC, ‘jumping’ back-and-forth between phases (Borromeo Ferri, 2006; Ärlebäck, 2009). However, most phases are somewhere observed in students’ activities. Thus, a MC does neither show what a modeller does step-by-step, nor is it a recipe to be strictly followed. Niss and Blum (2020) explain that a MC “should be understood as an *analytic* (ideal-type) *reconstruction* of the steps of modelling necessarily present, explicitly or implicitly, as an instrument for capturing and understanding the principal processes in mathematical modelling" (p. 14, italics by the authors). Thus, a MC is a *tool* for researchers and teachers to apprehend, comprehend, recognize, explain, and analyze important aspects in modelling, independent of whether it is done by an expert or a novice. Thus, a MC does not offer a definition, that is, it does not offer an explicit statement clarifying what mathematical modelling is. Also, a MC does not characterize mathematical modelling; that is, it does not offer qualities of modelling. Rather, a MC *conceptualizes* mathematical modelling; that is, it offers an abstract and structured idea of essential aspects, which is simplified so it is practical for use in teacher education, in educational-political discussions, and in research. In other words, a MC is a *model*.

The advantages of conceptualizing mathematical modelling through a MC are manifold. For instance, MCs show that modelling is complex, and that each phase affects others dynamically. Also, MCs show that modelling starts from real life and returns to it, and that mathematics is a useful toolbox in the solution process. Also, MCs show that modelling is not a purely mathematical activity, yet that mathematical activities play a central role. Also, MCs show that modelling differs from 'applying mathematics', which starts from a mathematical object, concept or algorithm that subsequently is used in a non-mathematical context, regardless of whether then a problem will be solved.

Apart from using MCs as conceptualization tool, researchers use MCs as an *analytic tool* to analyze their data in light of the different phases that a MC distinguishes. For example, we see that MCs are used to analyze students' activities regarding when they are in which phase (e.g., Ärlebäck, 2009), to analyze students' modelling competences regarding whether students are able to 'pass' a certain phase (e.g., Haines, Crouch, & Davis, 2000), to analyze mathematics tasks for certain emphases of modelling (Frejd, 2011), or to analyze classroom culture for an emphasis on certain modelling phases (Brady & Jung, 2021). The use of MCs as analytic tool yields a rich body of knowledge.

The modelling cycle with other dimensions than the cognitive dimension

When MCs are used as analytic tool in research of mathematical modelling, the results will be framed by it. The standard MC describes *cognitive* activities, which are activities that a researcher can observe in, or deduct from, a modeller's speech, gestures, writings, reactions and other explicit or implicit expressions. More generally, cognitive activities involve mental efforts to use and make sense of information. Activities such as speaking, listening, reading, remembering, non-routine problem solving, decision making, and sense making are mentioned as examples of cognitive activities. Cognitive activities can be learnt through experience or by being taught.

When an analytic framework has a cognitive focus, the research results will accordingly be primarily of a cognitive nature. This means that these have an individual's or a group's mental activities as unit of analysis. With an emphasis on cognitive aspects, the research may not capture other aspects that also play a role in mathematical modelling. Below, we give a few aspects that are not immediately captured by a theoretical framework based on the cognitive activities in a MC.

A dimension for metacognitive strategies

Successful mathematical modelling involves *metacognitive strategies* (e.g., Maaβ, 2006; Stillman, 1998, Vorhölter, 2018). These are needed for regulating and coordinating the many processes in modelling, both individual and group processes. During the modelling work, aims and outcomes need to be coordinated and regulated considering (1) goals in the task, (2) resources present, (3) the didactical contract from the teacher, and so forth. Different metacognitive strategies can be linked to each of the different phases in a MC, see Table 1. For instance, when starting, students need to ‘read’ the intentions into a task description and anticipate what they can do to reach a satisfying answer. In each of the phases in the MC, they can expect unexpected situations and may reflectively change the initial plans. They need to anticipate, reflect, plan, monitor, etc. From a research point of view, to analyze metacognitive strategies, one needs a different theoretical framework than for cognitive activities. Yet, metacognitive strategies and cognitive activities are intertwined. So, one can perceive the metacognitive strategies as an overarching layer over the standard MC, whereby the metacognitive strategies and the cognitive activities are two dimensions in one theoretical framework.

A dimension for tool use

Another aspect in mathematical modelling not captured in the standard MC is the use of *tools*. Therefore, Greefrath (2011) drew an alternative MC describing functions of digital tools in each phase of the MC. We want to extend this idea, building on Vygotskian theory (Williams & Goos, 2013), which explains that any cognitive activity is always *mediated* by tools, such as pens, blackboards, or digital tools. Mediation entails that the tool changes both the results of the activity (e.g., a mathematical answer becomes more precise), but also changes the cognitive activities (e.g., writing down intermediate steps off-loads memory demands). When starting on a modelling task, a modeller can try to understand the problem by using *Wikipedia* as inquiry tool. Another tool at the start of a modelling process is the *task sheet*, which offers students the information to be used and the guidelines to follow. Important tools in modelling are paper and pencil for making notes and sketches. At the very end of the modelling process, a modeller will present the results of the activity, possibly in written form or in an oral presentation to an audience. Thus, tool-use can be another analytic dimension that can be an overarching layer over the standard MC, see Table 1 and Figure 2.

A dimension for social norms

Another analytic dimension for research on mathematical modelling can be *social norms*. These are socially shared, implicit or explicit standards of acceptable behavior. As Blum’s (2015) MC shows, modelling takes place in two worlds: the ‘mathematical world’ and the ‘rest of the world’, in which there are different social norms. For instance, in the ‘rest of the world’, number answers can be estimations and, hence, not so mathematically precise. Yet, when presenting the final answer of the problem to the *client*, a modeller will abide to presentation norms (e.g., correct spelling, attractive lay-out). Regarding norms in the mathematical world, Yackel and Cobb (1996) described *socio-mathematical norms*, such as the use of preferred symbols (e.g., x and y) rather than creative inventions (e.g., ȝ and ͼ), and the specific way to justify claims (by giving a proof rather than a few examples). Also, there are classroom norms, also known as the didactical contract (Brousseau, 2002). Also, in groupwork, there may be competing norms, with some students making the effort because they consider the activity relevant, whereas others do it to pass the exam (Hernandez-Martinez & Vos, 2018). Thus, social norms will impact any modelling activity in many ways, and these may differ between the phases. We put some norms indicatively in Table 1 without claim of completeness, since research on this theme is still scarce and recent (e.g., Bonotto, 2020; Dede, 2019). Table 1 shows the phases in the MC with analytic dimensions for metacognition, tool use, and social norms that all differently interact and modify the cognitive modelling activities.

Table 1: Phases in the modelling process with indicative dimensions for cognitive activities, metacognitive strategies, tool use and social norms

	Cognitive activities	Metacognitive strategies	Tool use	Social norms
1	Con-structing	strategies to understand and reformulate the problem, to use additional information	Interpret task sheet, investigate resources (e.g. Wikipedia)	Norms within the team, in the classroom, norms of the *owner/client* of the problem
2	Simpli-fying/ structuring	strategies to select and organize information, develop plans, anticipate later actions, to monitor progress	Experiment with pen-and-paper (p&p), sketch & drawing tools, spreadsheets, etc.	Norms within the team, in the classroom, norms about what aspects to choose and the extent to which creativity is permitted
3	Mathema-tizing	strategies to organize infor-mation, develop & implement plans, to monitor progress	Visualize and organize with p&p, spreadsheet, plotter	Norms within the team, in the classroom, norms on using standard methods and being creative
4	Working mathema-tically	strategies to implement plans, to monitor progress	Calculate & simulate with pen-and-paper, Geogebra, CAS, etc,	Norms within the team, socio-mathematical norms on rigor, accuracy, use of common sense
5	Inter-preting	strategies to interpret results, to face unplanned outcomes	Visualize with p&p, presentation tools	Norms within the team, in the classroom and with the client in the process of interpreting
6	Validating	strategies to verify results, to invite critique, to evaluate the process and products	Control using p&p, information resources	Norms within the team, in the classroom and with the client on what can be regarded as 'validating'
7	Exposing	strategies to present results, to communicate and convince	Present using p&p or digital tools	Norms in the team, classroom, with the *client*, focusing on convincing

Conclusion

In this paper, we have looked at MC-based frameworks for analyzing aspects of modelling education and observed that such research has yielded a rich body of results, but that these are primarily of a cognitive nature and may obscure other aspects that play a role in modelling. Therefore, we suggest enriching the MC with overarching dimensions, such as metacognition, tool use or social norms, see Figure 2. The overarching dimensions can be supplemented or replaced by other dimensions that also affect the MC phases in different ways. Examples of alternative dimensions are *creativity* (Lu &

Kaiser, 2021), *flexibility* (Andresen, 2007) or *language* (Vorhölter et al., 2013). We suggest that also students’ *attitudes* may differ between the phases; so far, research connecting modelling to attitudes has focused on how modelling activities relate to students’ attitudes in mathematics in general (Chamberlin & Sriraman, 2019), and not on affect in different phases of modelling

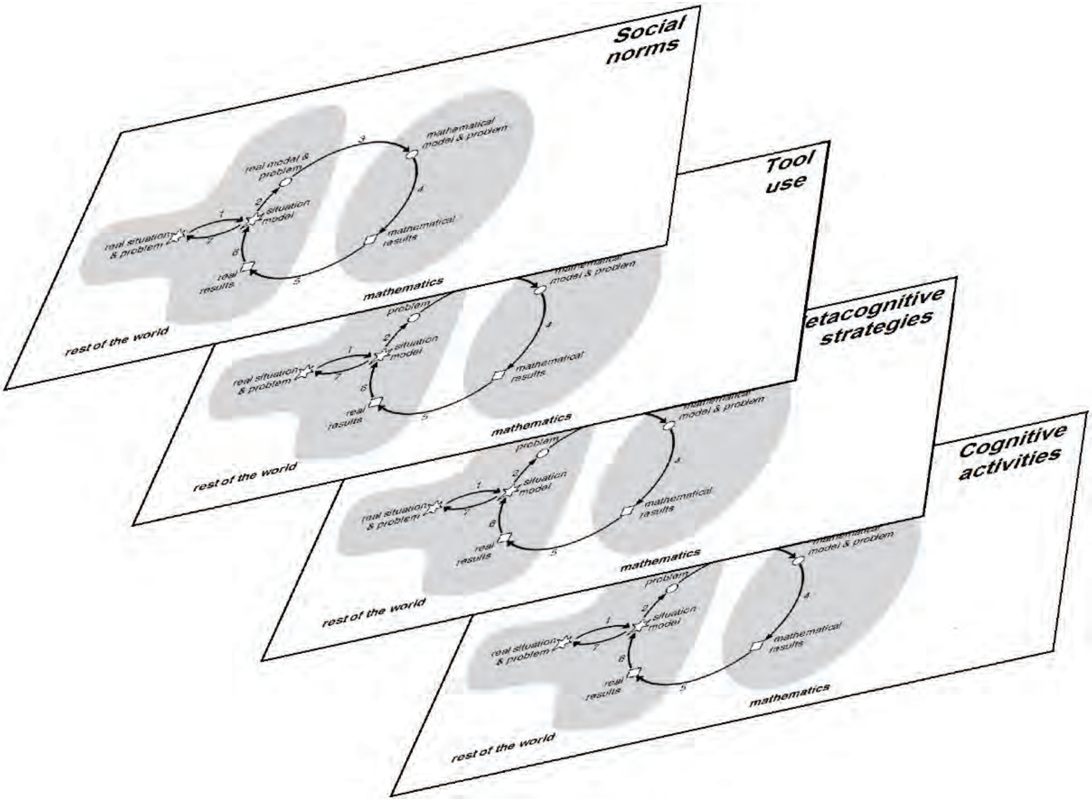

Figure 2: The enriched modelling cycle with four dimensions for analyzing modelling activities

Mathematical modelling is a complex and dynamic activity, and because of that it deserves to be studied from different perspectives. An enriched MC with a variety of overarching dimensions over the standard MC may enable modelling researchers to extend and deepen their research. This should give theoretical insights into how different dimensions interact and may reveal how students’ cognitive modelling competencies are affected by various aspects that haven been obscured so far.

Of course, any theoretical frame has its limitations. The enriched MC is an analytic tool for students’ modelling activities, and not a tool to design modelling tasks or a modelling curriculum. Also, it may be overwhelming to novices, and therefore, in teacher education or in educational-political discussions, the standard MC is more practical to keep a focus on aspects of mathematical modelling. However, the enriched MC should enable researchers to zoom out beyond the cognitive dimension and to further theorize the learning, teaching and assessing of mathematical modelling.

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When mathematics in three acts meets mathematical modelling

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In this article, we investigate how the introduction of the pedagogical method, "mathematics in three acts" to preservice teachers influenced their mathematical modelling, while on practicum. We analyzed documentation, of groups of preservice teachers, on their experience of teaching modelling lessons. One finding was that the groups were able to use "mathematics in three acts" to produce modelling problems that fulfilled certain criteria in our selected framework, but they had difficulty finding modelling problems for the lower grades (grades 1-3). We also found that the preservice teachers appeared to emphasize the subprocess of working mathematically and devalue the subprocesses simplifying/structuring, mathematising, and validating, when guiding the pupils through a modelling process in a "mathematics in three acts" lesson.

Keywords: Mathematical modelling, "mathematics in three acts", modelling problem, modelling process.

Introduction and literature review

Internationally, mathematical modelling was traditionally reserved for secondary schools. Only in the last two decades did researchers begin to see value in modelling for primary education (e.g., English & Watters, 2005). In Norway, modelling was introduced to primary school (grades 1-7) in 2020, when a new and revised version of the national curriculum included modelling and applications as one of several core elements. At our university, mathematical modelling was not part of the teacher education for preservice teachers for grades 1-7 until 2018. In 2018, a research project, LATACME[1], began, which focused on mathematics teacher education for grades 1-7, including a focus on modelling. LATACME uses an educational design research approach, that uses an analysis of experiences in design cycles, to refine or change teacher education practices in forthcoming cycles with the overall aim of improving educational practices and developing theory about those practices. According to Borromeo Ferri (2018), educational modelling courses should keep a balance between theory and practice. Consequently, at the beginning of the project, the LATACME research-practitioner team recommended teacher educators to introduce theories about mathematical modelling in their work with preservice teachers (PSTs), and to ask the PSTs to prepare and implement a modelling lesson when on practicum.

Research by Paolucci and Wessels (2017) had shown that while PSTs were relatively proficient in identifying and presenting relevant real-world problem contexts to young children, they had difficulty with formulating problems satisfying certain design principles which characterized "good" modelling problems. Reflecting on data from the first-year design cycle (2018-19) researchers in LATACME

[1] Learning about teaching argumentation and critical mathematics education in multilingual classrooms (LATACME, https://prosjekt.hvl.np/latacme/) funded by the Research Council of Norway. The project period is 2018–2022.

observed that the modelling problems suggested by their PSTs had been based on real-world situations that were familiar to the pupils, but the modelling processes were seldom fully developed. More detailed investigations showed that the PSTs had difficulties with balancing pupils' independence when guiding the different work subprocesses (Hansen, 2021). In the next design cycle (2019-20), the research-practitioner team recommended the teacher educators introduced the PSTs to the didactical method "mathematics in three acts" (MITA) to investigate whether this method could assist PSTs in introducing modelling problems which could support pupils with an appropriate amount of guidance to work through a complete modelling process. MITA was designed by Meyer (2011) to encourage learners to pose and work on mathematical problems and it was further developed in Lomax et al (2017). The basic features of the method are illustrated in Figure 1.

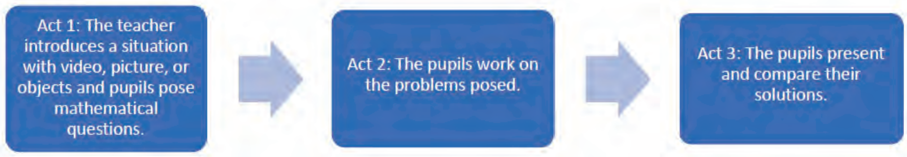

Figure 1: Mathematics in Three Acts (inspired by Meyer, 2011; Lomax et al., 2007)

However, it was by no means certain that MITA would achieve the desired effect. Dogan (2020) had found that a group of PSTs were successful in creating problems that were based on real-world contexts, but only a part of the problems could be classified as being model eliciting. To study how effective MITA was in assisting the PSTs in introducing problems that could be classified as "good" modelling problems in Act 1, we decided to evaluate the problems the PSTs adopted in their modelling lessons via MITA in terms of criteria to Dogan (2020), who proposed four criteria: reality, openness, complexity, and model eliciting. This would form the second cycle of the education design research approach for this aspect of LATACME.

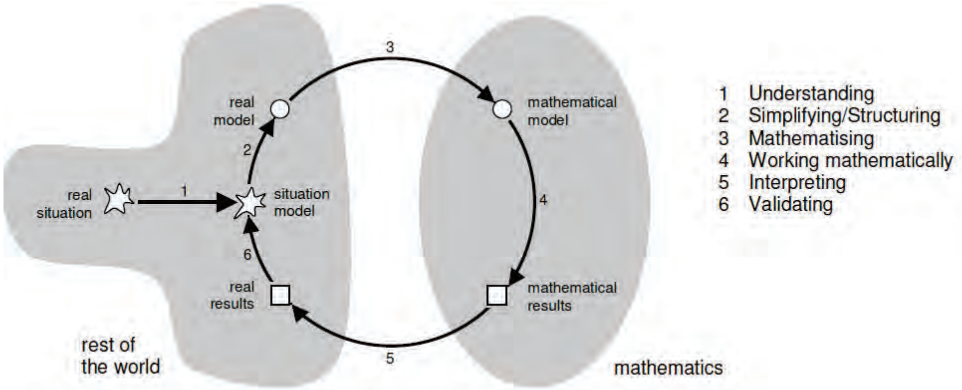

Figure 2: Modelling Cycle (Blum & Leiß, 2006)

After a modelling problem is chosen, solving it involves several subprocesses, which are not usually carried out linearly. The modelling process has been illustrated in various ways. As shown in Figure 2, Blum and Leiß (2006) emphasizes the cyclical nature of a mathematical modelling process and describes the common subprocesses, that in return require different competencies involved in solving a modelling problem (Maaβ, 2007).

While MITA suggests a linear working process (see Figure 1), the modelling cycle (see Figure 2) indicates the constant movement between the sub-processes. This suggests that some of the subprocesses, such as validating, in typical modelling cyclical processes, may easier to be overlooked when engaged in MITA. The second aim of our study was therefore to investigate to what extent subprocesses of the modelling cycle (Blum & Leiß, 2006) were present in lessons based on MITA.

Conceptual framework and research questions

Borromeo Ferri stated that in modelling lessons, "the selection and the quality of tasks for lessons are essential for mathematical understanding, for promoting students' mathematical practices and competencies" (2018, p.41). Synthesizing earlier research, Dogan (2020) proposed four criteria to evaluate modelling problems posed by PSTs: reality, openness, complexity, and model eliciting. The reality criterion requires that the modelling problem comes from a real-world situation and aligns with the reality of the pupils (Lesh & Doerr, 2003; Maaβ, 2007; Dogan, 2020). A modelling problem satisfies the reality criterion if it allows pupils to interpret the problem based on their experience and their mathematical knowledge. The openness criterion requires a modelling problem to be interpretable in multiple ways, open-ended and to allow for different solution paths (Maaβ, 207; Dogan, 2020). A complex modelling problem requires the pupils to understand the context and search for relevant data, and to be cognitively demanding to solve (Dogan, 2020; Borromeo Ferri, 2018). The model eliciting property ensures that a modelling problem should promote the modelling process, and requires the students to use mathematics to construct, describe or explain situations (Dogan, 2020; Lesh & Doerr, 2003).

Based on our twin aims for this paper, our research questions are about the MITA lessons described by groups of PSTs when reflecting on their practicum:

RQ1: Which of the four criteria (reality, openness, complexity and model eliciting), were most commonly found to be fulfilled by the modelling problems in the lessons?

RQ2: Which of the subprocesses of the modelling cycle were present in the PSTs' descriptions of the pupils' work processes?

Research Method

The investigation was conducted by using document analysis (Bowen, 2009). The units of analysis were 16 written assignments from groups of PSTs for 1-7 grades. The assignments were classified according to the four criteria in Dogan (2020) and the subprocesses of the modelling cycle by Blum and Leiß (2006) depicted in Figure 2.

Research Context and Participants

These PSTs were in the first semester of their second year of their teacher education. General theories on mathematical modelling and MITA were introduced to them through 3 three-hour lectures combined with literature reading. The PST-groups were then asked to plan and carry out a modelling lesson while on practicum. After practicum, the PST-groups were asked to describe and reflect on their modelling lessons. A total of 16 PST-groups with 3-5 PSTs in each group, gave permission for us to analyze their assignments. Of these groups, 15 had used the MITA structure for their modelling lessons. One of these groups reported to have performed two different modelling lessons. Therefore, we analyzed 15 assignments which contained 16 modelling lessons.

Data analysis

To evaluate modelling problems used by the PST-groups in their modelling lessons via MITA (RQ1), we used the four criteria reality, openness, complexity, and model eliciting (Dogan, 2020; Lesh & Doerr, 2003). After having identified the modelling problems the PST-groups described in their assignments, we categorized them according to the four criteria according to the questions in Table 1 which was based on the work of Dogan (2020). In the next section, we provide an example from one of the PSTs' assignments and a detailed description of how the analysis was carried out.

Table 1: Criteria for evaluating modelling problems via MITA

Criteria	Guiding questions for each criterion
Reality	1.Whether the problem sprang from real life of the pupils. 2. Whether the problem was suitable for the pupils' academic level.
Openness	Whether the problem was open for different interpretations or solving methods.
Complexity	1. Whether the problem was cognitively demanding for the pupils to interpret the problem. 2. Whether the problem could make the pupils to see the need of mathematics.
Model eliciting	Whether the problem required the pupils to generate a model.

To identify the modelling subprocesses the PSTs described the pupils going through, we analyzed the documents using the characteristics of the subprocesses described by Blum and Leiß (2006) (see Figure 2).

Results

The assignments contained MITA lessons. Some included dialogues from Act 1, while others described the dialogues implicitly. We chose here to present an extract from an assignment of one of the PST-groups, where the dialogues from Act 1 were described explicitly. This was typical of the assignment data and we use it to explain how our data analysis in more detail.

Extract from an assignment

This PST group described that they had implemented a modelling session over three lessons of 45 minutes each, in a grade 4 class with 22 pupils, using the MITA structure.

The PSTs described that in Act 1 they had presented a video about global warming. The PSTs and the pupils talked about this video and tried to understand some graphs about climate change. The PSTs described that the pupil had raised many concerns, among which the PSTs had identified two interesting questions, "*Is it possible to find out* how *much warmer it will be when I grow up*?" and "*How many years will it take before the sea rises over the dock (Bryggen) in Bergen*?" The PSTs reported that the last question had been chosen for this modelling session.

The document then reported that in Act 2 the PST-group had started the lesson with a discussion about what would happen if the sea level continued to rise. They had afterwards asked the pupils to think about what they needed to know to answer this question. The PSTs together with the pupils had concluded that one must know the present height and how much the sea level rises per year. The PSTs then wrote *"we found that the dock height above the sea level is 720 mm"*, and *"… that the sea level rises 3.4 mm per year. Since the pupils have not learned decimal numbers, we decided to round down to 3 mm per year"*. Afterwards, they had provided the pupils with a table with two columns, where one column was a list of the years from 2016 to 2022 plus the year 2032, and the other column were the sea levels with the first three years' sea levels 43 mm, 46 mm and 49 mm. In groups, the pupils had been supposed to fill in the table, with the PSTs being available for the pupils' questions. The PSTs also mentioned that some pupils were critical about the table, by for example saying that "*It is not certain the water rises all the time*".

In Act 3, the PST group had divided the pupils into groups of four. Having their table and calculations at hand, the pupils had been asked to answer the question "how many years will it take before the sea rises over the dock (Bryggen) in Bergen?". After about 30 minutes of group work, the PSTs had a summary, with one of the PSTs showing a solution method with centicubes.

Analysis of the example

The problem "How many years will it take before the sea rises over the dock (Bryggen) in Bergen?" was chosen after Act 1. It is likely that Bryggen would be familiar to most pupils in the class and as such most pupils would want to find out what could happen in the future, using this as a benchmark to understand the implications of climate change. From the PSTs' description, the mathematics involved seemed to be suitable for grade 4 pupils. The problem, therefore, fulfilled the realistic criterion. The problem also fulfilled the openness criterion as it allowed several different interpretations (e.g., what does "over the dock" mean?) and different solving strategies. It also required solvers to orient themselves, simplify, find needed information, and to use mathematics to find the solution(s). Consequently, we interpreted it as being complex. As the problem also seemed to invite pupils to generate a model, the problem was also classified as model eliciting. Thus, we considered the problem to be realistic, open, complex and model eliciting.

In regard to the second research question, we identified the modelling sub-processes that were evident in what the PSTs described the pupils as doing in Act 2 and Act 3 of the modelling sequence. For the first subprocess, "understanding", the PSTs indicated that the pupils had the opportunity to interpret the modelling problem in classroom discussions about the consequences of sea-level rise. They also had to identify and understand the information they needed to answer the problem. However, from what they described, it seemed that the PSTs took over the next subprocess "simplifying/structuring" themselves, since they provided the height of the dock and rounded down the height that the average sea level rises per year. In addition, they presented a table to represent the situation, an activity that we interpreted as part of the "mathematising" sub-process, so that the pupils just had to complete the table. By completing the table, the pupils concentrated on the subprocess "working mathematically" and implicitly the subprocess "interpreting" in the context of the real model. The subprocess of validating was not promoted by the PSTs.

General results

There were 11 problems in total (see Table 2) that satisfied all the quality criteria. Of those which did not fulfill all the criteria, most of the problems were for pupils in the first few grades of school. One modelling problem for grade 1 failed the reality criterion, and 3 modelling problems for grade 2 and 1 modelling problem for grade 3 failed the openness, complexity or/and modelling eliciting criteria.

Table 2: Number of modelling problems that fulfil each criterion

	Reality	Openness	Complexity	Model-eliciting
Number of modelling problems out of 16	15	13	12	12

Table 3: Analysis of the modelling subprocesses that were present

	Understanding	Simplifying	Mathematising	Working mathematically	Interpreting	Validating
No.	11	1	6	11	11	5

We found that where the PSTs' modelling problems did not fulfil all four criteria in our framework, there were also lack of the modelling subprocesses in the descriptions. For four of the modelling problems this was because that the complexity of the problems was too low for the grades (complexity criterion was not fulfilled). Therefore, we restricted identification of modelling subprocesses to the 11 modelling lessons in which the modelling problems satisfied all four criteria for good problems. Table 3 presents the analysis of the modelling process for these 11 modelling lessons. It shows that the sub-processes simplifying, mathematising, and validating were often absent in these lessons. In particular, the subprocess simplifying only appeared in one of the modelling lessons. In the one modelling lesson that involved all subprocesses failed to repeat subprocesses for one or more cycles, suggesting a linear rather than a cyclical approach to modelling.

Discussion

When it comes to the first research question, our result showed that 11 out of 16 lessons adopted modelling problems that satisfied all four criteria, and 15 of the problems satisfied the reality criterion (Table 2). We can compare this with the results by Dogan (2020), who found that PSTs had difficulties constructing modelling questions that fulfilled all four criteria (5 out 17 in his case fulfilled all criteria, and 12 of 17 fulfilled the reality criterion). In our analysis, we interpreted that through MITA the PSTs were able to encourage the pupils to come up with accessible modelling problems from familiar contexts. In some of the lessons, the pupils asked a wide range of questions through Act 1, which gave the PSTs some freedom to choose open, model eliciting problems with adequate complexity. This can be illustrated by an example from a PST group reported having presented the pupils with a short video of Usain Bolt running a competition. Some questions that the pupils asked were: how fast did Usain Bolt run in that competition? How many centimeters did he run? How much does he earn as a runner? How old will Usain Bolt become?

The fact that the 5 cases that failed one or more criteria in Table 1 were for lower grades (1-3 grades) is interesting. Paolucci and Wessels (2017) raised also the concern about the PSTs' capacity to create modelling problems for lower grades, especially for grade 1. One of the PST groups stated, "since there is not much theory [mathematical knowledge] available for 2nd graders, we choose to look at the mathematical curricular goals". This indicates that the PSTs thought the difficulty lay in the fact that pupils in lower grades did not have sufficient knowledge in mathematics, so that there are a limited number of mathematical topics to work with. This suggests that PSTs need more guidance from teacher educators in designing modelling problems for lower grades.

In response to the second research question, we found that the main modelling activities took place in Act 2 and Act 3. In these acts, the PSTs guided the pupils through the modelling process to solve the problems posed in Act 1. Our analysis showed that the subprocess simplifying/constructing was missing for 10 of the 11 modelling processes. In most cases the PSTs took over the modelling problems and simplified and structured them for the pupils. Only in 6 of the 11 modelling processes did the pupils need to transfer the real model to a mathematical model. Ng (2018) and Hansen (2021) pointed out similar tendencies among experienced teachers and PSTs respectively that were new beginners in teaching modelling, that is, that the teachers and PSTs tended to provide scaffolding because they perceived mathematical modelling to be challenging for their pupils.

The PSTs also did not often include the validating subprocess in the modelling cycle. Only 5 out of 11 PST groups used this sub-process. Our result showed that in the descriptions in the assignments the cyclic nature of the modelling process seemed to be absent in the lessons, whereas the subprocess "working mathematically" was always part of the process. We suspect MITA could have affected both the cyclicity of the modelling process and the focus on working mathematically. Act 3 in MITA was designed to let the pupils to compare and reflect over solution methods, and it did not suggest that pupils should go back to re-solve the problem or refine the solution to the problem. Therefore, the cyclicity was not part of the process suggested by MITA.

Conclusion

There are two main conclusions from this study. The first is that we found through MITA the PSTs were able to arrive at modelling problems that included the pupils' perspective, but that they had difficulty to find modelling problems for the lower grades (grades 1-3). The second conclusion is that even if the PSTs arrived at appropriate modelling problems together with the pupils, the corresponding modelling process through MITA did not necessarily contain all the subprocesses of the modelling cycle. In particular, the subprocesses simplifying/structuring, mathematising, and validating were often missing. This is interesting, and it asks for more attention towards how to instruct PSTs to include these subprocesses in their modelling practice in teacher education.

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TWG07: Adult Mathematics Education

Introduction to TWG07 Adult Mathematics Education

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This paper is a summary paper of the Thematic Working Group (TWG) on Adult Mathematics Education (AME). As the only thematic working group that focuses on adults' lived experiences of mathematics, the research makes an important contribution to the field of Mathematics Education. The main themes in this group identify that adult numerical behaviour goes beyond the mathematics skills, knowledge, and procedures taught in formal education It is multifaceted, requiring the use of higher order skills of analysis and judgement, applied within a broad array of life's contexts, experienced through a range of emotions. The research in this group points to the need to raise the profile of research that shows the benefits to adults of learning mathematics but also the long term economic disbenefits in the neglect of teaching and teacher training for this group.

Keywords: Adult Mathematics Education, Numeracy, PIAAC,

Introduction

One thing the pandemic has taught us is that understanding mathematics and its role and influence on peoples' lives really matters. Every day we were shown graphs and charts indicating trends that were used to make predictions about the world's health and economic situations with significant implications on our individual behaviour. We were shown countless exponential graphs, numbers in the billions and given numerical risk analyses of actions, which increased the demands on our mathematical interpreting, understanding, and reasoning to make sensible decisions on our behaviour with huge impact on our lives. Yet the mathematics being learned in schools seemed to be reduced in content and unequal in access (Hodgen, Taylor, Jacques, Tereshchenko, Kwok, & Cockerill, 2020), distant from what was needed to help understand the situation.

Hoogland and Díez- Palomar (2022, this publication) argue that society is becoming increasingly mathematised, and the demands on adults to make numerate decisions that impact on their lives is considerable, yet interest in research into the world of adults learning mathematics and life-long learning lacks attention. This is already a long-lasting problem. Coben (1994) argued interest in research into adult numeracy only started to be recognised for its importance when she highlighted the differences in cultures between academic research and those involved in adult numeracy and mathematics education. Reinforcing the idea that adults learning mathematics was different from school based mathematics, Withnall (1994) further argued rather than it being a set of mechanistic, isolated mathematical skills which can be acquired free of context and then applied to real life, numeracy is better understood in relation to the demands on adults' lives in their broader contexts, where communication skills are key to interpreting the variety of 'mathematical codes' encountered and on which we are asked to exercise judgement to ensure a useful outcome.

Since then, with nearly 30 years of research plus the establishment of a global forum Adults Learning Mathematics (ALM), practitioners and researchers have been exploring their understanding of adults' ability to learn and utilise mathematical information. In response to economic, social, and technological changes, our understanding of the field of adults learning mathematics has developed, yet still many citizens lack the numeracy competencies to enable them to fully participate (Gal, Grotlüschen, Tout & Kaiser, 2020; Hoogland & Díez-Palomar, 2022, this publication; Yasukawa, Rogers, Jackson, & Street, 2018) and few trust numbers enough to deal with, let alone challenge, a world in which fake-news can flourish. These factors have profound implications for the way in which numerical skills are taught and learnt in adult education programmes.

Emerging themes in Adult Mathematics Education

The content of this working group requires us to consider the notion of numerate behaviour in its widest sense, exploring a range of ideas, contexts and applications, probing further the relationship between numeracy and mathematics.

Contributions focus on:

- Reflecting on the field of adult numeracy and its relationship with mathematics.
- Numeracy as a lived experience of mathematical calculations and procedures in social economic and political contexts.
- Developing numeracy and mathematical skills with adults, including vocational education.

These themes reinforce the need to recognise that adults' experiences with mathematics inform judgements affecting their lives far beyond the classroom. The breadth of papers in this working group reflects the wide scope of the field of Adult Numeracy. The single working group focused on adults' mathematical experiences in this CERME 12 conference points to an important contribution that the understanding of adults' lived experiences of mathematics, both inside and outside formal state sponsored education systems, adds to our understanding of Mathematics Education. Adults and their numeracy skills are intertwined with the political, social and economic systems, are shaped by, but also shape those lived experiences (Lerman, 2000). In the next paragraphs we introduce the papers outlining the contributions to the three areas of focus.

Reflecting on the field of adult numeracy and its relationship with mathematics

Kaye, past chair of ALM, discusses the historical development of the concept of numeracy. Taking a philosophical view, using Thomas Kuhn's notion of lexicon and incommensurability, Kaye likens the development of the field of adult numeracy to a paradigm shift. Emerging from the field of mathematics education research, almost in a Darwinian sense (Kuhn, Conant, & Haugeland, 2000, p 98-101), responding to historical, environmental, and social changes. Kaye seeks to research the notion of adult numeracy using Kuhn's paradigm shift. He argues for the need to find agreement in the concepts of what numeracy is, so that it can be utilised to critique mathematics research. Although he concedes that many languages do not have a word for 'numeracy', he maintains many countries and contexts have similar adult learners in similar situations with similar numerate behaviours.

Diez- Palomar and Hoogland (2022, this publication) undertook a literature review into the field of adult numeracy over the last 20 years, to identify the topics investigated and where gaps have appeared that could point to new lines of enquiry. The main findings suggest research most often cited focused on the contexts of health and social care, followed by articles on numeracy in everyday life. Most higher order skills cited were processing information, problem-solving and critical thinking. Most of the cited articles were based on quantities and numbers, perhaps not surprisingly given the time span, few studies focused on apps or digital skills. When considering dispositions most focused on self-confidence.

It is apparent that while many within the field of adult numeracy research are aware of important contributions, the work is less well known outside this field and so work needs to be done to raise the visibility of our research in other domains and with policy makers. Increased visibility is important to inform policies on adult mathematics education, with as main focal point the idea that numeracy is a multi-faceted concept.

Adult numeracy is more than mathematical skills, knowledge, and procedures taught in schools. It is complex encompassing higher order skills, applied to a broad range of contexts, experienced through a range of emotions. The Common European Numeracy Framework (CENF) by Hoogland and Diaz-Palomar (2022, this publication) brings up to date our understanding of what society requires from adults. They identify four aspects to numerate behaviour. Suggesting adults need to be able to recognise and apply appropriate mathematical concepts but within many different everyday contexts; to use higher order skills to process, reason and analyse the numerical information that influences decisions affecting their daily lives. All of this while recognising adults have many beliefs and feelings related to mathematics itself, developed through lived experiences, which will affect their judgement when dealing with numbers. Spiegelhalter (2017) also writes about this in relation to the notion of 'trust' in numbers when surrounded by fake-news. The CENF illustrates a way to research aspects of the field of adult numeracy encompassing the lived experience of mathematics education. Hoogland and Diez- Palomar further point to the increased 'mathematisation of society' through technological developments, where people must make numeracy-based decisions all day long. While arguing that equipping people with the necessary skills will be a big challenge, they posit this will require us moving away from 'mastering the execution of calculations with pen and paper to recognising numeracy as a multifaceted concept needed for the 21st century'. Work related to the CENF framework will, in the future be linked with PIAAC as well as the development of teaching resources and professional development modules, to help disseminate the ideas futher.

Numeracy as a lived experience of mathematical calculations and procedures, in political, economic, and social contexts

Kelly, (2022, this publication) chair of ALM, explores the financial literacy skills and knowledge needed to survive in complex financial systems. She uses the notion of financial vulnerability (Gal et al, 2020) to research the extent of global economic need, the impact of western economic systems, community influences and individual risk factors. She identifies financial vulnerability as a global issue, affecting women more than men, arguing that this has a lot to do with societies' gendered expectations of female and male roles. Other vulnerable groups include those who are disabled, single parents and people living in marginalised communities, and the COVID pandemic has made life more difficult for those already financially squeezed. Mathematical calculations, such as interest, are key to understanding many financial concepts. But making sound financial judgements in real life also requires the ability to analyse, and reason what those interest rates mean, for example, when taking out loans. She found most current national child-focused financial education resources in the UK focus on dispositions towards money considering psychological and emotional relationships with money and mathematics. While adult numeracy skills and knowledge are essential to making sound judgements in financial decision-making, so are literacy and digital capability skills as well as understanding emotional attachments. Kelly argues that financial literacy education can be seen as one way to combat financial vulnerability but to be effective needs to focus on learners' priorities.

Byrne and Harrison (2022, this publication) give an insight into numeracy education in prisons, a particular but very important context within society. Mathematics education in prison is a basic life skill (Council of Europe, 1990) yet Byrne and Harrison found it varies considerably across countries and within national systems. Despite this variation the Mathematics in Prisons (MiP) group was established by two practitioners, one in Ireland and one in the USA, whose research focus is centred upon those learning mathematics within the 'secure estate'. Prisons over the centuries have been seen as agents of the state to reform, to penalize, to encourage desistance from crime, and more recently to encourage lifelong learning and personal transformation. The MiP group shares the later goals and explores similar experiences in the maths classrooms. All tutors work with students who may not have chosen to study mathematics were they not in prison. Yet Byrne and Harrison found this experience can encourage reflection and become a turning point in an individual's life, as mathematics is a gateway to further and higher education. The MiP group plans to support those working across international boundaries, overcoming the challenges of variation in language and culture, exploring the value of learning mathematics for different groups and aim to use this CERME platform to further raise awareness of the group and to recruit members.

Dulam and Hoogland (2022, this puplication) utilise the large database provided through PIAAC to explore numbers in an economic context examining the consequences of a mismatch between workforce skills and employment on the economy. At individual (micro) level, where skills-mismatch can lead to lower job satisfaction and wages. At company (meso) level where mismatch leads to a higher staff turnover and inefficiencies and at a country (macro) level, leading to unemployment, lower productivity, and lower economic growth mainly due to wasting human capital (OECD, 2013).

They find skill levels vary over time in adults' lives and with employment opportunities. But they also question how reliable the actual measurements of over and under skill levels are. In their research they found being over-skilled for a job is more likely for men, younger age-groups, those with higher education, and for people who use their numeracy skills often at work. Also, the likelihood of being over-skilled increases as the frequency of using numeracy skills at work increases. Further consideration needs to be made to fully appreciate the implications of some of the findings. For example, employing over skilled men might be due to an inbuilt gender bias within certain job roles and industries where males are more likely to be employed over females, which would affect a skills mismatch. Also, some countries' employment policies are attached to certain financial penalties which are imposed if people do not accept employment offered, consequently this can 'push' over-skilled people into lower skilled jobs.

Exploring context as a broader notion of democracy Lindenskov (2022, this publication), considers the skills and knowledge essential to understanding how mathematical thinking underpins the very essence of our political and social systems. She argues that mathematics underpins our lives both inside and outside education, in both formal and informal ways. For example, using the same numbers but comparing election outcomes in countries that use ‘first past the post’ algorithms with those using a proportional representation system. Lindenskov argues that such exercises give a better understanding of how mathematics shapes our parliaments, our laws and our lives. She further posits democracy can influence the classroom through the teaching approaches used, as well as the topics covered. This links back to the role that numerate behaviour has in adults’ lives going beyond the classroom, and how research into the field of numeracy helps us better understand how mathematics can inform, influence, and empower adults’ daily lives.

Developing numeracy and mathematical skills with adults, including vocational education.

Investigating mathematics education with adults and different aspects of the mathematical skills and knowledge of teachers and their students is a rich source of research into adults learning mathematics. This section points to several gaps in teacher training and continuing professional development courses for those involved with mathematics education for adults, especially those asked to teach numeracy to adults but without the specialist knowledge to support their endeavours. Is it any wonder that results from PIAAC and OECD surveys show that in all but one participating country, at least 10% of the adults are proficient below level 1 of the 6-point scale in literacy or numeracy (Hoogland, Kelly & Díez-Palomar, 2019, p.1294)?

Bradtke and Ferri (2022, this publication) explore the mathematical competence of vocational teachers whose main subject is economics, comparing prospective teachers who have mathematics as a subject in their curriculum with those who have not. In their research they found that most vocational teachers did not receive any mathematical training in their studies, although they use, among other things, the percentage calculation in business lessons. In a pilot study, an instrument was validated to record whether the mathematical model of compound interest is recognised and successfully applied in different situations, by the two groups of teachers. The results showed that even student teachers with mathematics as a subject have difficulties with its application to the economic problems. This is interesting when reflecting on the importance of understanding and applying percentage calculations at a global level to economic growth figures, but also to more individual financial decision-making when applied to making interest payments on savings and credit. It points to an important deficiency in education systems that can have serious widespread repercussions, where teachers’ lack of mathematical understanding is passed on to the students, known in economic terms as the multiplier effect.

Responding to a national push for problem-solving to be introduced into the mathematics taught in schools and colleges in order that it can be utilised more effectively in people’s jobs and lives, Faulkner, Breen, Prendergast and Carr (2022, this publication) compare the problem-solving and procedural skills of adults in mathematics education in Ireland. They find that adults in mathematics education in Ireland have significantly weaker problem-solving skills in mathematics when compared with procedural skills. This lack of problem-solving skills aligns with findings amongst third level students in schools, however the same investment/interest in improving the provision of mathematics education for adult learners is not present. They further argue that these findings are misaligned with one of the key recommendations of the ‘Adult numeracy: Assessment and development’ policy to “invest in the development of national capacities to measure and improve adult numeracy” (Gal & UNESCO, 2020, p.3).

This lack of investment in appropriate teacher training for adult numeracy educators is exacerbated for those who are asked to take on the responsibility of teaching adults’ numeracy without the requisite skills and knowledge, as research shows by Prendergast, O’Meara, O’Sullivan, and Faulkner (2022, this publication). Their research highlights an unmet demand for professional development in adult numeracy education, with many numeracy practitioners looking for opportunities to develop their practice. In response to this need this research group aims to establish a series of online ‘Numeracy-Meets’ for adult numeracy practitioners. They outline a new model of support that focuses on topics such as Family Numeracy and Financial Literacy rather than teaching traditional topics such as fractions in isolation. Through the establishment of an informal community of practice they aim to meet the professional development needs of practitioners. It is early days for this project which, if successful, hopes to expand its reach both nationally and internationally.

Forster, Faulkner and Prendergast (2022, this publication) have also undertaken further study into the relationship between psychosocial factors, demographics, the level of mathematics studied and progression for Access students. Their findings show that students studying foundation mathematics had significantly higher scores for amotivation and neuroticism. While those who chose to study advanced mathematics, students had ranked more highly for general self-efficacy (GSE), belief about mathematics ability (BMA) and intrinsic motivation to know. Additionally, female students were significantly less likely to study advanced mathematics than males. Non-Irish nationals studying advanced mathematics were significantly less likely to progress to higher education than their peers. All these findings need to be further explored in relation to motivation and reasons for study in relation to the notion of success. The research recommends teachers engaging students in enactive self-mastery but also the importance of role-modelling and verbal persuasion in encouraging progression to higher education and higher-level mathematics studies.

Stacey (2022, this publication) in her doctoral research seeks to explore possible changing perceptions in relation to adults’ motivations, mathematics anxieties around mathematics and confidence while learning mathematics. Working with adult numeracy learners (19+) returning to the further education sector in the UK to study GCSE Mathematics. Although still in the pilot phase, her paper also raises several key issues in relation to continuing professional development for numeracy teachers. She highlights the very under-researched group of adult learners (+19), as most research on learners’ perceptions of mathematics has been conducted with school aged, 16–18-year-olds in further and higher education students. She also raises the issue of variation in professional development for teachers on offer, highlighting a lack of understanding of how different countries and universities organise both full and part time doctoral education as part of their professional development programmes.

Forwarding the field

The papers in this workshop show research into the field of adult numeracy has a huge amount to offer the teaching of mathematics in schools, as it reflects the lived experience of numbers recognising its complexities. Yet as Diez-Palomar and Hoogland (2022, this publication) found, numeracy has very little visibility in the research world which means it is out of the reach of policy makers. This is undoubtedly a result of Coben's (1994) ideas of differences in cultures in the research world on school mathematics education and the practitioners' experience of teaching adults. But it can also be seen as developing from a historical deficit of political interest and funding in adult education outside university level. For example, in the UK until PIAAC identified national skills discrepancies there was little interest or investment in developing the numeracy and literacy skills of adults. Researchers in the field of adult numeracy need to explore ways to promote the wealth of research that exsits in this area, but also the significant gaps. The pandemic and the greater use of technology also offer more opportunities for practitioners and researchers to join, network, and promote the benefits of adult numeracy.

The papers outline a strong case for interest and investment in adults' numeracy education, when considering the impact on people's lives and the potential empowering and transforming nature of mathematics. However, the research into the resultant effects of an over skilled or under skilled workforce on global economic inefficiencies, job dissatisfaction, high turnover costs and loss of productivity are clearly made. Yet papers in this working group provide clear evidence of underinvestment in adult education with the serious consequences it has for learners at all levels.

Conclusion

Employers, politicians, and educators want people to have a higher quality of numerate and mathematical behaviour to handle numeracy situations effectively in their daily lives, but typically use standardized school mathematics tests to assess these qualities. Unfortunately, most of these tests are not designed to capture the multifaceted nature of numerate behaviour and therefore provide an image that is too narrow to reflect such behaviour. The research field of adult numeracy and mathematics education recognises the complexity of numerate and mathematical behaviour and offers insights into possible approaches that can be used within mathematics education, casting a broader light on the intricacies of mathematical and numerate behaviour in the context of everyday life.

The notion that technology is leading to a society becoming increasingly 'mathematised' should be considered more seriously in research on mathematics education, for instance by using global research databases, such as PIAAC (OECD, 2019), that enables us to compare and analyse how different societies and countries deal with this issue in their educational policies and practices.

Research in this working group shows the importance of adult numeracy research in relation to life opportunities and inclusive societies, yet point to a lack of visibility in academic research with the consequent lack of effect in learning from work undertaken into adult numeracy research and practice. Understandably policy makers are currently focused on the long-term effects of the Covid -19 pandemic on school children and their learning. Yet these few texts indicate a real need for resources to be invested beyond schools to respond to the demands on adults' numeracy requirements to engage fully in life chances relating to health, finances, employability and through civic society.

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The mathematical competence of vocational teachers with a focus on economics - an empirical comparison between prospective teachers with and without mathematics as a subject

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Keywords: Grundvorstellungen, teacher education, professional mathematical knowledge, percentage calculation.

Theoretical background

Percentage calculation is one of the most important mathematical topics for everyday life and work (Parker & Leinhardt 1995). Vocational teachers focusing on economics use it in their teaching, but little is known about their own knowledge of it. It is likely that they may draw on their mathematical knowledge from school and their training does not usually include any mathematics education content on percentage calculation. Initial studies indicate that their understanding is incomplete (Bradtke 2018). Therefore, we are investigating the research question: Do vocational teachers focusing on economics, who do not specialise in mathematics, have a professional mathematical knowledge of the content percentage calculation?

In German research on mathematics education, the existence of an adequate understanding of the concept is examined by checking whether the normative Grundvorstellungen (desired ideas about a mathematical content) are present and can also be applied situationally. To test this, vocational teachers focusing on economics were asked to solve tasks involving percentage calculation. It is argued that if they can solve the tasks, it is likely that they have the appropriate Grundvorstellungen, otherwise, deficits are to be expected. Furthermore, the Grundvorstellungen of a mathematical concept are considered sustainable if they can be activated in different situations (Padberg & Wartha 2017). Compound interest also plays a special role in commercial calculations but only with solid/broad Grundvorstellungen is one competent in dealing with the mathematical content. Difficulties with this operation are also increasingly reported in Anglo-American countries (Parker & Leinhardt 1995).

If we look at the essential mathematical activities of vocational teachers focusing on economics in commercial education, then the teacher should be able to solve the tasks him/herself as well as recognise errors in the processing of tasks and describe the error. Along these three contexts, we have set twelve tasks involving compound interest in order to determine how viable the conceptions of this term are.

Methodology

In a pilot study, 77 secondary school student teachers' mathematics was surveyed (semester: mean 3.44; standard deviation 1.93). The pilot study took place in the winter semester 2021/2022 at the University of Kassel. No vocational teachers with a focus on economics were surveyed, as they were held back for the main study due to their low numbers. Participants in the pilot study were secondary school student teachers with mathematics as a subject. It is argued if they have difficulties with the concept of compound interest, then it is very likely that vocational student teachers without mathematics as a subject do as well. The study was designed as an online achievement test. The invitation link was sent to prospective students on our course on mathematics education in the first secondary school via Moodle. Participation in the study was linked to compensation opportunities for academic performance.

Results

Compound interest was tested through nine tasks with varying demand situations. On average, 7.4 tasks could be solved with a standard deviation of 2.54. The scale formed an alpha coefficient of α = .755. For three items, the item discriminatory power is slightly below .3. These are to be revised in the next pilot study. The distribution of scores is as shown in the histogram (see Figure 1).

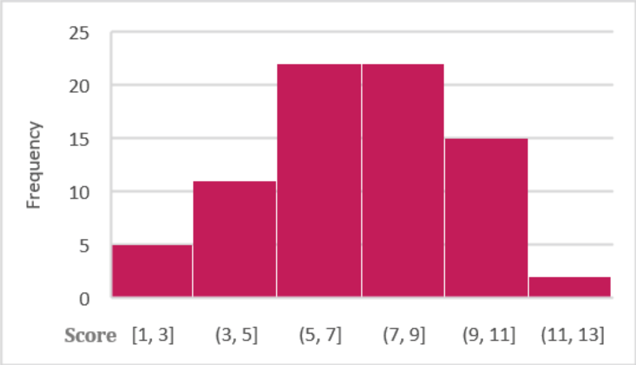

Figure 1: Results on the viability of the Grundvorstellungen for multiple changes of the original size

Discussion

The secondary school student teachers with mathematics as a subject do not seem to be able to solve compound interest problems in all contexts. This suggests that they do not recognise the problem type of compound interest in some contexts and therefore cannot apply the Grundvorstellungen associated with this problem type. If student teachers with mathematics already have this difficulty, then it can be assumed that this is even more serious for vocational student teachers without mathematics as a subject. Whether vocational teacher have difficulties in this regard will be investigated in a further study.

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ALM Mathematics in Prison (MiP)

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Mathematics in Prison (MiP) topic group is a new initiative developed by Adults Learning Mathematics - International Research Forum (ALM). Our mission is to link practitioners and researchers interested in the field of Mathematics education in prisons and correctional education contexts. The topic group was developed in response to the ALM virtual seminar series, which featured a variety of speakers on various topics related to adults and Mathematics during 2021 and 2022. Discussions among participants at some events expressed interest in the field, and ALM responded by facilitating the development of a dedicated Mathematics in Prison group. Members joined from different parts of the world, all with an interest in Mathematics education in the secure estate. The group focuses on the challenges and opportunities of learning mathematics in this unique and under researched context.

Keywords: Adults mathematics education, teaching resources, correctional education, desistance.

Introduction

The MiP topic group first presented on the work of their work in a workshop at ALM 28, a virtual conference from the University of Hamburg. Members of the group presented an informal session where each taught a class on ratio, as this is a topic many of the group find is a challenge for their adult students. Participants were encouraged to share their perspectives and experiences in this field.

The two authors of this paper are practitioners in the field who are also researching for a PhD and wish to share their teaching experiences. Their research and to connect with practitioners. These goals were extended were shared within the topic group, to offers a forum to share good practice, discuss methodologies and explore the challenges practitioners and researchers face working in the secure estate. The group encourages informal conversations between practitioners and researchers to identify common themes and investigate ways that the MiP topic group might support mathematics education in this context in the future. Practitioners study experiences, literature, and best practices (Ginsburg 2019), and investigate topics such as teaching fractions, assessment and technology enhanced learning.

The focus of the paper is to open this topic for discussion and invite others from the field to connect with us, from any background, and together we can develop the vision activities of the group. We hope in future that we can identify some common themes emerging in the field of mathematics education in prisons. As the group has been in existence for a very short time, this paper reflects how we opened this discussion rather than describing findings. We would like in future to gather relevant literature and give practitioners the opportunity to reflect on the literature as it relates to their practice, possibly in the form of a professional book club.

The Prison Education Context

In our meetings we found that Mathematics education in different countries varies in language, culture, curriculum, philosophy, delivery, and statutory basis. For example, there are different terms within the group's members for students, for staff, for the education service, and even for the subject of Mathematics itself. Our first task was to discuss the language of our work with the others in the group so we could communicate in a nonjudgmental way with each other.

There are many differences in the education services in prisons across the world, and this is reflected in the language used. Prisons over the centuries been seen as agents of the state to reform, to penalize, to encourage desistance from crime, and to encourage lifelong learning and personal transformation. In spite of this, in the MIP group, we felt that we share many similar experiences in our Maths classrooms.

We all work with students who may not have chosen to study Mathematics except that they found themselves in prison. This experience can encourage reflection and become a turning point in an individual's' life, as mathematics is a gateway to further and higher education. Many have spent years away from formal education and may never have considered returning to education if they were not in this context. The process of capturing the stories and experiences of students learning Mathematics in informal contexts such as prisons, and workplace education settings (Kelly 2016), may help develop insights and understanding for future researchers, teachers, and teacher trainers.

Overview of the Mathematics in Prison Topic Group

As stated, the MiP group grew out of the discussion prompted by the fourth virtual ALM seminar "Adults Mathematics in Prison Education". This was led by Linda Ahl who presented her research on the teaching of Adults mathematics in prison education". After the discussion, ALM trustees decided to advertise the first topic group meeting through the ALM mailing list. The first gathering included participants from Sweden, UK, Canada, USA, Ireland, Northern Ireland. While some could not attend, there was wide interest who wish to keep in touch with developments of the group. We have also had queries from Asia.

The MiP group has permeable boundaries with few barriers to relations with outsiders. We have been contacted by approximately twenty people who expressed interest in the group. Not all have attended all meetings but remain in touch with the group. Some of the members of the MiP group are currently teachers of mathematics in prisons, while others work in universities, and others are researchers. People who contacted the group are mostly mathematics teachers and many are teaching at different levels from basic to advanced. Generally, education in prison follows the national curriculum of the country at many different levels, but this is an area we have not investigated in detail as this group is still at an exploratory stage.

Communication was going to be a challenge for the group, due to the different time zones and the digital divide (Hopkins et al. 2015) between those working in prisons and those in the community. The impact of this was that the groups was not all able to access video calls in work hours, so the decision was made to rotate the times of the meetings, communicate outside our work settings, to communicate online and send content to each other using technology platforms in advance of the meetings.

We reflected on the best way to communicate with those in the group before and during the meeting, and we decided to use Google Jamboard. This interactive smartboard enables teachers and students to collaborate on a virtual whiteboard, to allow to brainstorm ideas and create sketches (Virto et al 2020). We set up a link to Google Jamboard with questions composed by the group leaders ahead of the meeting. At the meeting, we invited participants to answer the questions and contribute to discussions suggested on the Jamboard.

Table 1: Questions on MiP questionnaire

Question	**Purpose**
What is your name?	To connect
What is your workplace?	To find out type of institutions, for example, secure or open.
Where s it located?	To find geographical context and possibly form local clusters.
What is your specific interest in Mathematics teaching?	To develop topic clusters.
What is your students' profile?	To see age and nationality profile
What are your students' goals?	To understand motivations.
What are your goals for the future of MiP?	To help plan.
What are your optimal times to meet?	To plan a calendar and rotate times for different time zones.
What dilemmas and challenges do you meet in your work?	To understand ethical and moral issues relevant to the work.
What words would you include in our community glossary?	Suggestion: include terms for students, staff, Mathematics.

We invited the group participants to answer on Jamboard on topics including their name, their workplace, their location, their student profile (general comments on age and nationalities), the goals of our students and their Mathematics learning objectives, the best times to meet, dilemmas and challenges we experience at work, our specific interests in mathematics teaching, our goals for the future of the MiP group, setting up a community glossary relating to our work. We are keen that the group remains open to future questions and observations from the existing members and future ones, so we plan to be flexible in organizing meetings. We are considering future formats and may look at other platforms, such as padlet.

The disadvantage of Jamboard is that anyone can remove and add material. As participants are working in prisons, there is awareness of the need for privacy and security. The decision was made to safeguard the group's discussion by sharing the link during the meeting so people could add ideas and then we closed the link so no one else could change or share it. Finally, we shared a PDF of the meeting and brainstorming with the participants. A summary of the discussion was shared with the ALM trustees.

The content gathered indicated that educators were working with a wide range of age groups, in different departmental systems and with varied resources. The use of IT within the secure systems was diverse and this is one area that could be an important source sharing ideas and approaches to developing mathematics.

One of the challenges in teaching mathematics is this sector is how to be authentic and remain true to the needs of our students. Ahl (2020) comments that discussions on prisons can create curiosity in those who are not familiar with the field. As Szifris (2018) comments, this curiosity may be fed by the lack of educational research. Other professions have researched education in prison, from their own professional lenses, whether through the perspective of criminology and the impact ion desistance, or from a psychological point of view. The MiP group are clear in our mission, which is to support and investigate Mathematics education in prisons, while respecting the dignity and privacy of staff and students.

Rationale and Statutory basis for teaching and researching mathematics in prisons

Education has been a part of everyday life in prisons for centuries, yet it is still generally under theorized (Szifris 2018). While adult Mathematics is a more limited field of research than mathematics education in mainstream areas (Safford-Ramus et al 2016) yet have skills and knowledge and adult Mathematics education in prison is even more limited. Ahl (2020) has recently added to the field in a study based on mathematics education in Swedish prisons, and who argues for more research into this specialist field. Anecdotal evidence from practitioners in the MiP meetings and from research (Creese 2016) suggests that prisoners and people in detention have unmet needs in mathematics and numeracy.

Mathematics education in prison is a basic life skill (Council of Europe 1990). Yet it varies considerably across countries and within national systems. Levels of mathematics education have been investigated in prisons in the UK (Creese 2016) and the USA (Rampey et al 2016). In Ireland priority is given to those in prison with basic educational needs, including numeracy and literacy (Irish Prison Service 2019). In the UK, Coates (2016) advocates for development of basic skills in mathematics, as well as English, and Information and Communications Technology (ICT).

Future ideas for activities in the MiP Topic Group

We look forward to presenting at CERME12 as it will help us to plan for our activities in the future, regarding the approach the group will take. We plan to gather more reflections on experiences from within mathematics classrooms in this context. This may include instructional strategies, teachers' professional development needs and experiences. We plan to collaborate on the instructional strategies and materials and their impact in classrooms that teachers share with us. Assessment in adult education is a core issue and mathematics assessment in prisons is an important concern as mathematics is a gateway subject to further education. We plan to connect with other groups related to Mathematics and education in the secure estate and correctional settings.

We plan to keep the gatherings in our groups as open as possible, some sources for discussion in this group include an educator developing a curriculum or resources for a specific group of adult learners. Another option may be to pose a problem for the group on a specific problem or project the group could reflect on why the students struggle with this topic and offer strategies on how to deal with it. Another idea could be to reflect on the impact and relevance for the practitioner of an article or book, like a professional book club. This could lead to activities and experiments in-class in the future which may confirm or challenge our original findings.

Recently two members of the MiP group presented a workshop at the Australian Correctional Education Association conference, held online. This opened the topic to a new set of researchers, practitioners and policy makers. The topic provoked discussions and interest and we expect to have new members in the group in the future, and develop links with related Australasian groups.

Conclusion

We anticipate that this topic group will provide a forum, within ALM, where Maths in prison practitioners and researchers can connect, share resources, and support each other in their work. The group is at an early stage, and we look forward to the developments in the future.

We hope that this paper and conference will help to publicize the topic group and add to the limited research in this field. The meetings have provoked lively discussion and have illustrated that the field of mathematics education in prisons has more features in common than it has differences.

We look forward to developing a dialogue on continuing professional development within this sector across national boundaries. This could take the form of pieces about practice, or it may include action research from individual practitioners' or teaching experiments. It may involve learners, their communities, and teachers, in line with all appropriate ethical guidelines. We do not yet have an overview of the different policies regarding curriculum and accreditation of Maths education in the prison context, so we hope to learn more about the role of governments in education policy on different countries.

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Research contributions to adults learning mathematics in the field of numeracy in the last twenty years

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This paper explores the contributions of research to the field of adults learning mathematics (ALM) in the last twenty years. The results of the review of the literature on ALM show that the most cited studies that have been published in the last twenty years tend to focus on the field of numeracy to understand health data (such as understanding how to dose a medicine in a medical treatment). However, we know little about key aspects of how adults learn mathematics, what obstacles they encounter, and how they overcome them. This paper identifies the main gaps that ALM research faces in the coming years.

Keywords: Numeracy, Adult Education.

The concept of numeracy

In its origins, the word numeracy was closely related to the concept of quantitative literacy (Tout, 2020). Thinking quantitatively has been the focus in educational research focused on mathematics since the last century. As Sowder (1989) says, most of the research on how we learn mathematics over several decades focused on mathematical intuition, higher-order thinking, mental schemes for understanding mathematics, and the development of quantitative thinking. In the last forty years, it should be added that research in the field of mathematics didactics has diversified into other areas beyond the strictly cognitive.

However, almost all the research that has increased our understanding of how we learn mathematics and how we can develop those innate mathematical skills that we all have, for the mere fact of being human beings, focuses on early childhood, adolescence and up to the age of university studies (Carpenter, Dossey, & Koehler, 2004). Instead, in "adulthood", there are few investigations, and of those that do exist almost all related to literacy, functional literacy, dispositions towards mathematics, and mathematics embedded in different contexts (workplace, etc.). There are studies on what mathematics we use in adult life, measuring the ‘mathematics’ we know, the challenges that adults should know to be ‘mathematically competent’ in today's world and explorations into the vulnerability of certain people and certain people social groups who lack certain knowledge and skills. Nevertheless, there is a big gap (comparatively speaking) in research into how adults learn mathematics, what research has contributed to this regard, and the gaps that need to be filled in the coming years or decades.

The objective of this paper is to review what research is being carried out in the last twenty years in the field of ALM, to identify what are the topics that are being investigated, and where there are gaps that would open new lines of research in the future on how adults learn mathematics.

Background

The first piece of evidence we have about ALM is that adults already know maths. Mathematical knowledge (the ability to think and reason in mathematical terms) is an innate skill. In the same way, that speech is a human skill that manifests itself early when children begin to speak. Mathematical thinking is also part of our ability to represent the world and solve problems. Therefore, when we talk about learning mathematics as adults we talk about academic, formal and ‘pencil and paper’ mathematics, similar to what we refer to when we talk about literacy, for example where we speak about reading and writing in the printed letter.

Unlike literacy, where researchers claim that it refers to the knowledge and proficiency in the use of texts, in the case of numeracy, the definitions we usually use go beyond defining numeracy as the understanding and use of the ‘printed number’. As Tout (2020) says numeracy not only refers to mathematical content but is a ‘way of thinking, of reasoning, of acting’. It has more to do with a set of skills related to ‘numerated’ behavior that goes beyond the simple use of mathematical objects and their representations. That is why it is difficult to establish the boundaries of the concept and, therefore, establish the bases of didactic research on how to ‘learn’ numeracy. Numeracy seems to be an innate human trait. However, it is also learned and, from the point of view of contents, procedures and ways of thinking and reasoning mathematically, it is a complex set of knowledge.

Another problem is the invisibility that some authors, such as Wedege (2010), have pointed out around mathematics and numeracy. That is, there are forms of thought and reasoning that are typical of mathematics. However, as they are part of our way of representing and understanding the world around us, they are not recognized as ‘mathematics’ (nor as numeracy). Examples that help to understand this paradox (well-studied in the field of ALM) are, for example, making estimates, identifying quantities, and comparing them with each other, considering risks associated with the subjective perception of probability, etc. Many people play the lottery because they are hoping to win the prize. They also decide to go out for the weekend after watching the weather forecast on TV. However, it is more difficult to find people who, before buying the lottery number, use Laplace's rule to calculate the probability they have of winning the prize, we do not make a count using the laws of combinatorics of how many different chances there are of obtaining the six winning numbers of the weekly lottery (among the 36 possible), nor do we use the density distribution of the event ‘being sunny’ or ‘raining’ when we prepare to spend the weekend away from home.

This apparent ‘over-definition’ of the concept of numeracy poses difficulties, as evidenced by the fact that it has been a concept that has been evolving since it first began to be used in the mid-twentieth century (Hoogland et al., 2019). During all these years, the concept of numeracy has gone from being defined as the knowledge of basic arithmetic objects and procedures (knowing how to add, subtract, multiply and divide, and some of the rules and properties of basic operations) to being defined as a social practice that involves the use of numbers (and mathematics by extension) to solve problems of everyday life, make decisions, value information and act on the world around us (multifaceted concept).

Teaching numeracy to adults

Different studies show that when children enter kindergarten at 3 or 4 years old, they already have developed what we would call ‘number sense’ (Westwood, 2021). Children know how to make comparisons between numbers, they know how to discriminate between quantities, and they know how to count accurately. So, when they get to school, what do they learn? According to research in our field, it seems clear that what they learn is the ‘academic’ language of mathematics, including the written representation of numbers and their characteristics and properties. They learn to reason with numbers. They are presented with ‘problems’ and asked to solve them, training them to develop that innate capacity that we all have, which is to solve problems.

Furthermore, that is called ‘teaching math’. According to the type of mathematical object being taught, we will discuss arithmetic, algebra, geometry, etc. Mathematical objects, as we know, are part of conceptual structures, the components of which are very precisely related. For some, mathematics is the ‘language of precision’, such as Ernest (2003) reflects upon when he analyzes the different philosophies of mathematics.

We have some indications of explanatory models of how learning works: for example, Dreyfus and Dreyfus (2005) argue that teaching first goes through verbal descriptions and rules, then the second phase of association, and a third of automation; they affirm that learning is a more global process (holistic), in which holistic patterns are recognized because of interaction with the environment in multiple different situations. Hatano (1988) confirms this, claiming that learning goes beyond rule-based knowledge.

Nevertheless, what happens when we talk about teaching and learning mathematics in adults? There is no accumulated knowledge base on how adults learn mathematics comparable to the work that has been done with children and young people.

Therefore, in this paper, we have conducted a systematic review of the literature, looking for everything that has been written about adults and numeracy in the last twenty years to see to what extent the previous statement is true (or not).

Methods

To answer the research question posed here, we have conducted a systematic review of studies into numeracy and adults published in journals included in the Web of Science database from January 1, 2000, to June 1, 2021. To perform the query, the words *numeracy* AND *adults* were used.

The search generated a first database of a total of 843 articles published in that period. To select a sample of them, the following procedure was followed:

(1) We sorted all articles by the total number of citations, from the one with the most citations (1,836 citations) to those with no citations.

(2) The articles were grouped by quartiles, generating four groups ordered according to citations: A, B, C and D, as shown in the attached Table.

Table 1: Grouping of articles according to quartile by the level of citations

	Range of citations		Range of citations
A (25%)	19- 1,836	C (75%)	2-6
B (50%)	7-18	D (100%)	0-1

(3) From each quartile, (the first) ten articles that met the following criteria were chosen:

- Articles with a defined research question, a clearly explained methodology (that meets the criteria of replicability, reliability, and validity), a section of results, and conclusions (answer to the research question). We excluded all articles of a conceptual, reflective nature and those which are not based on empirical research.
- Articles that deal with the learning of mathematics of adults.
- Articles that comply with the dimensions and components of empirical studies from an adaptation by Taylor et al. (2021), as shown in Table 2.

Table 2. Criteria characterizing the components used to select the empirical studies on numeracy by dimension (adapted from Taylor et al. 2021)

Criteria	Dimension	Component
Clear identified research question(s)	Research Design	Rigor
Appropriate unit of study (school, classroom, adult learner,...)	Research Design	Rigor
Clear description of the research design	Research Design	Rigor
Target population defined	Research Design	Rigor
Appropriate statistical analysis	Research Design	Rigor
Appropriate qualitative analysis	Research Design	Rigor
Clear defined unit of analysis	Research Design	Rigor
Consent forms	Research Design	Rigor
Valid and reliable measurement	Research Design	Measurement validity
Size and significance effects are reported	Research Design	Reliability
Limitations are reported	Research Design	Reliability
Use of randomness as criteria for selecting the sample	Research Design	Reliability
Triangulation	Research Design	Reliability

Evidence of transferability to other contexts	Research Design	Generability
Adequate description of the sample /intended population	Research Design	Generability
Description of the research activities conducted	Research Design	Replicability
Clear information about the financial sources (grants, etc.)	Research Design	Independence of the researcher
Consistent, statistically significant positive effect on an outcome	Effectiveness	Overall effectiveness
Findings are reported across different contexts	Effectiveness	Consistency of effects
Contributions persist over time	Effectiveness	Sustainability
Clear criteria to identify the target population	Impact	Inclusion criteria
Detailed documentation of all components of the study (how, to whom, and by whom is intended to be conducted)	Impact	Implementation feasibility
A clear description of the intended effects in the consent forms	Impact	Implementation feasibility

The total sample of selected articles has been 40 articles, ten from each of the quartiles.

Results

Below are the results obtained from the analysis of the sample of forty selected articles.

First quartile

Almost all the articles in the group that generate the most citations are articles in the field of health, published in journals of medicine, nursing, or health psychology. The most recurrent theme that appears in these articles is the study of the extent to which adults who participate in them can read and interpret the instructions of the medicines to apply the doses correctly. Many studies investigate the skills of nurses to apply the doses of medicines to patients correctly. There are also some studies on the understanding of risk and probability in decision-making. From the point of view of higher-order skills, most of the selected articles focus on the processing information skill or the problem-solving skill. In half of them, we have identified the skill of critical thinking' none refer to managing situations. Regarding the contents that appear in the selected articles, most focus on quantity and number'. The two other contents that appear are 'pattern, relationships and change' and data and chance. There is no article that talks about the use of the calculator, the use of apps or spreadsheets, or digital skills.

Second quartile

In the second quartile, we find the same trend as in the first quartile: most of the articles that appear are published in health journals (medical research, neurological sciences, neuropsychiatry), although we also find a journal of gerontology and another on literacy and numeracy studies. Finally, there are two articles published in a generic journal as Plos One. Regarding the research topics, the topics are more diverse than in the case of the investigations of the first quartile. In the previous group, we also found some studies on misunderstandings in adults with cognitive impairment when reading medication instructions. However, in addition to that topic, we find other research topics: the understanding of risk estimates using an online risk calculator when outcomes are expressed using integers, the impact of age on non-optimal decision-making, the management of domestic aspects using the Internet (shopping and banking skills), the financial literacy of adults, among others. On higher-order skills, unlike the studies reported in the first group, here we find that most refer to critical thinking or processing information. We also find somewhere that the problem-solving skills of adults are studied. From a content standpoint, most focus on quantity and number. We also find other topics more punctually, such as data and chance or the patterns, relationships, and change. It highlights that several of them include the use of digital applications.

Third quartile

In this third group of articles, some study of the field of medicine continues to appear. Nevertheless, unlike previous groups, these types of studies are the minority. The studies in this group are published in journals from various disciplines, such as education and development, policies, and assessment. As you can see, most are journals in the field of education. The topics studied are numeracy skills and labour market outcomes among indigenous populations, comparison between groups of adults from different socio-economic classes based on PIAAC results, assessment of objective and subjective health about numerical skills, perception of the incidence of specific disease and the survival rate, interactions between adults and children in contexts of learning informal mathematics (grocery store), etc. From the point of view of higher-order skills, in this group, the studies focus on skills such as processing information, problem-solving or critical thinking in the vast majority. We found no case of mathematizing or managing situations. Regarding the analyzed mathematical contents, those related to the dimension of quantity and number appear above all. In some cases, also what refers to data (using and reading statistics) and chance (making predictions, assessing the opportunity/risk of making a particular decision).

Fourth quartile

In this fourth group, the articles that are most cited are published in education or lifelong learning journals, specifically aimed at adult education. There are also some articles in specialized journals of mathematics education. In a minority way, journals from the field of health appear. From the point of view of the topics studied in the articles of this group, there are studies on how adults solve problems in the context of everyday life (evidence problems), the result of tests to measure the level of numeracy measured with tests (both standardized, such as PIAAC, and tests created *ad hoc*). There are also several studies related to motivation (emotional dimension), anxiety about mathematics, etc. Finally, we also find studies on the type of mathematics and mathematical skills needed to develop certain occupations in the labour market, mainly of an ethnographic (observational) nature. From the point of view of higher-order skills, in these studies, it is customary to study the skills of analyzing situations, processing information, and to a lesser extent, problem-solving and critical thinking. As far as mathematical content is concerned, they usually focus on aspects of quantity and number, or data and chance.

Discussion and conclusions

The first result we have is that the vast majority of empirical studies are in the field of health and refer to aspects such as literacy and numeracy to understand the instructions regarding the dosage of medicines, or the understanding and application of medical treatments, or related to considering the pros and cons to make a particular health decision. In all these cases, the most outstanding aspect that appears in the studies is how adults can understand quantitative data, read the representations of that data (graphs, tables; but also, data of absolute frequencies, or percentages), and compare one set of data with another, to make decisions based on that understanding. In addition, study aspects include the understanding and management of risks and the making of decisions in environments of uncertainty with limited information.

A second result that we have found is that there is an excess of articles which consists of reflections, narratives, conceptualizations, etc., with as main content considerations, theoretical points of view, which are illustrated with concrete examples, discussions on research already published (meta-research), and specific cases of the knowledge of the author(s).

Third, the empirical studies on ALM suggest that adults learn from making connections: new mathematical objects are incorporated into our previous cognitive schemes. Several studies suggest that adult math learning is contextual: we use our experience knowledge (in the workplace, at home, in the supermarket, etc.) to incorporate new mathematical objects into our mathematical knowledge. To do this, we usually use skills such as processing information, problem-solving, and critical thinking.

Fourth, this literature review shows that one of the lines of research in ALM is on emotions. There are several studies related to the emotional response that adults give to mathematics and its learning.

Finally, fifth, the most remarkable result we have found is that the actual process of learning mathematics by adults is hardly investigated. This is reflected in the topics that appear in scientific databases such as WoS (Web of Science) which focuses mainly on what topics adults should learn, such as functions, logarithms, derivatives, geometric theorems, spatial thinking, probabilistic reasoning and Markov chains. We didn’t find much research on what actually happens when an adult is learning a mathematical concept and how s/he solves mathematical problems (Contextually? Drawing on everyday experience? Through interactions?). We didn’t find evidence if what we know from studies from different perspectives, such as the semantic fields, the cognitivist approach, socio-cultural studies, also applies to the case of adults.

Our main conclusion is that much more research is needed in the field of ALM, at least to contrast with what we know about research on learning mathematics with children and young people. Does it apply in the same way to adults? Or does adult learning works differently and, therefore, do we have to teach/support it in other ways (impact on the curriculum), and evaluate it also in other ways (impact on assessment, such as PIAAC, for example)?

Limitations

The scientific literature review study presented here is not exhaustive. A systematic literature review process has been followed, using WoS (Web of Science) to identify scientific articles on numeracy in the field of adults learning mathematics. However, we are aware that despite this effort to systematize, it is impossible to have reached the total number of studies carried out in this area and on this subject throughout the world.

The sample has a noticeable bias towards certain topics, because the different fields of research have different traditions from the point of view of citations. Thus, for example, in medicine, there is a greater tradition of citation than in other areas, which explains the over-representation in our sample of articles published in this type of journal. For further studies, it is planned to correct for this effect. But on the other hand, we must also consider that the level of citation is also an indicator of the visibility of the work.

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Measuring numeracy skills mismatch with PIAAC data

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We assess the incidence of numeracy skills mismatch in five countries: Belgium, Chile, Italy, Netherlands, and the United States of America. To do this, we make use of a new approach (Brun-Schamme & Rey, 2021), namely by identifying someone as being mismatched if the score for numeracy skills is outside the interval [median – SD , median + SD]. *We make use of the PIAAC dataset, collected by the OECD, a survey that measures adults' proficiency in numeracy among other type of skills. We find that 14% of the workers are over-skilled, whereas 16% are under-skilled. Being over-skilled is more likely for men, younger age-groups, having a high level of education, using numeracy skills often at work, and having studied science, mathematics, and engineering.*

Keywords: Numeracy, skills mismatch, over-skilled, under-skilled, occupations.

Introduction

The world has seen major developments in technological progress, human capital formation, and labour demand. Numeracy has increasingly become one of the crucial basic skills for adults to cope with the digitalised and technologized 21st-century society. Having an adequate numeracy level determines the success of individuals' participation in their roles as citizens and professionals. Hence there is a need to measure the numeracy proficiency and whether there is a good match between the possessed skills and required skills in numeracy.

Skills mismatch, defined as possessing qualifications or skills that does not adequately meet the qualifications or skills necessary for the doing one's job, has negative effects at all levels of the economy: at individual (micro) level, skills mismatch is leads to lower job satisfaction and wages. At company (meso) level mismatch leads to a higher staff turnover, and inefficiencies. At country (macro) level, to unemployment, lower productivity[1] and lower economic growth mainly due to wasting human capital (OECD, 2013). The aim of this study is to inform national policymakers on lifelong learning especially regarding numeracy and the mismatch of skills.

The Programme for International Assessment of Adult Competencies (PIAAC), a major survey conducted by the OECD in over 40 countries, provides the opportunity to measure skills proficiency (in literacy, numeracy, and problem solving in a technological rich environment) and the degree to which people are well-matched in a harmonized way. This paper focusses on numeracy skills, because 1) these skills are the most comparable throughout different countries (Perry et al., 2016) and 2) these skills have a mathematical foundation.

[1] Being under-skilled can lead to lower productivity because the worker is performing below the required skills.

The OECD (2013) defines numeracy skills as “the ability to access, use, interpret and communicate mathematical information and ideas in order to engage in and manage the mathematical demands of a range of situations in adult life.” To this end, numeracy involves “managing a situation or solving a problem in a real context, by responding to mathematical content/information/ideas represented in multiple ways.”

Theoretical background

Human capital is formed by the skills and education an individual gains over time (Wiederhold & Ackermann-Piek, 2014). Human capital positively affects an individual’s success, and productivity. To put skills to effective use, it is important that they are aligned with the required skills at work. Wiederhold and Ackermann-Piek (2014) discuss the reasons why workers may be over-skilled or under-skilled. Factors that may play a role are shifts or changes in the economy, the occupation type, the timing in the professional career (experience), discrimination in the labour market, and family responsibilities.

Pellizari and Fichen (2013) developed a theoretical framework to define and measure skills mismatch with PIAAC data. In this framework jobs are defined as production functions and skill use, which is treated as an endogenous choice of the worker, is considered as the only input. The model furthermore assumes that there are fixed costs to carry out the job and that the marginal product of used skills is locally constant and that it declines above a certain threshold (it is equal to zero). These assumptions lead two critical values in the definition of skills mismatch, namely that workers with a skill proficiency below the lower critical value are under-skilled and workers above the upper critical value are over-skilled. Furthermore, the model assumes that production technologies of firms do not change and that the skills mismatch is measured in the short run.

Several studies (OECD, 2013; Perry et al., 2016, McGuinness et al., 2018, Flisi et al., 2017, Allen et al., 2011) have used PIAAC data to measure skills-mismatch in various ways. This paper applies the latest approach as developed by Brun-Schammé and Rey (2021) to measure numeracy mismatch. Flisi et al. (2017) provide 20 indicators for occupational mismatch for 17 European countries, whereas Perry et al. (2016) evaluates six measures for mismatch. The preferred method by the OECD, developed by Pellizari and Fichen (2013), is the method where self-reported[2] mismatch is identified as an objective measure (whether the score for numeracy skills exceeds 95th percentile of the distribution within the same occupation or whether it is lower than the 5th percentile of the distribution). The main argument against this method is firstly, the bias raised due to overconfidence or misinterpretation, and secondly, that the mismatch is measured within one-digit occupational code leaving little room for the heterogeneity within the 1-digit occupation. Brun-Schammé and Rey’s approach therefore make use of the two-digit occupation code to take some more heterogeneity into account (ending up with 40 occupational categories instead of 10). It assesses the skills mismatch for France and categorizes someone as being mismatch if the score for numeracy skills is outside the mean and one standard deviation. Using this approach, we assess the prevalence of skills mismatch

[2] By asking workers whether they can do a more demanding job or whether they need extra training to their job.

in five countries, and look for associations between mismatch and socio-demographic and job-related characteristics such as job satisfaction, wages, and skills use.

Data and methods

The PIAAC dataset is based on an international comparable survey conducted by the OECD in over 40 countries in three rounds: the first one in 2011-12, the second in 2014-15, and third in 2017. The data we use are for Belgium, Netherlands, and Italy from the first round, Chile the second, and the USA from the third round. Around 5000 non-institutionalized people per country were surveyed. To obtain representative results, the sample was chosen through a multistage clustered design. The proficiency scores in the original dataset were based on the Item Response Theory scaling methodology resulting in 10 plausible values for each type of skill proficiency in the dataset (Yamamoto et al., 2013).

We perform a quantitative analysis, by measuring the incidence of numeracy skills mismatch conforming to Brun-Schammé and Rey (2021) as follows. Firstly, we calculate the median and standard deviation of the numeracy skills score[3] for each two-digit ISCO occupation. Secondly, we qualify a worker as being over-skilled if the score for numeracy proficiency score is above the median plus one standard deviation and as being under-skilled if the numeracy proficiency score is below the median minus one standard deviation.

We furthermore perform binary logistic regression to study the association between mismatch and socio-demographic and job-related variables. Our sample size is 12,166 in total.

Table 1 below provides the statistics. We see that on average 14% of the workers are over-skilled and 16% under-skilled. A critique from Pellizari and Fichen (2013) is that having a mismatch percentage of 30% can be attributed to the normal distribution of numeracy proficiency skills score. In a normal distribution, for instance, 32% will score below or above one standard deviation from the median. Nevertheless, it can be interesting to find out which variables are associated with being over-skilled and under-skilled respectively.

Table 1: Descriptive statistics

Variable (in percent)	Total	Belgium	Chile	Italy	Netherlands	USA
Over-skilled	14.13	13.56	15.49	14.29	13.18	14.6
Under-skilled	16.37	17.35	16.43	16.66	16.24	14.76
Gender (% of women)	50.64	49.76	50.84	48.69	50.6	53.87
Education level						
Lower secondary or less	19.86	11.42	24.38	27.53	25.68	7.63

[3] Based on the 10 plausible values of the numeracy proficiency and taking the corrected standard error into account by using the Repest command (Keslair, 2020). Occupations with less than 25 observations were eliminated.

Upper secondary	42.13	41.64	44.8	49.78	40.31	34.02
Post-secondary, non-tertiary	2.16	3.87		1.29		
Tertiary – professional degree	12.32	26.29	17.77	0.25	4.34	7.25
Tertiary – bachelor degree	14.93	1.84	11.5	18.69	20.42	11.9
Tertiary – master/research degree	8.61	14.95	1.55	2.47	9.25	24.77
Area of study						
General programmes	12.23	13.1	21.4	8.35	9.31	7.48
Teacher training and education science	8.96	9.67	10.15	4.64	7.71	13.27
Humanities, languages and arts	8.76	7	11.41	16.63	3.39	8.87
Social sciences, business and law	20.23	16.62	10.48	21.54	29.47	22.22
Science, mathematics and computing	12.01	10.94	11.84	20.08	6.44	14.97
Engineering, manufacturing and construction	17.44	24.94	16.27	13.19	16.89	11.42
Agriculture and veterinary	2.3	1.95	2.13	2.32	3.34	1.23
Health and welfare	12.82	12.26	7.81	6.76	19.48	15.9
Services	5.25	3.52	8.52	6.49	3.98	4.63
Age group						
24 or less	12.77	9.99	15.61	5.44	16.77	14.12
25-34	22.78	24.11	28.29	20.66	18.63	23.04
35-44	24.93	25.65	20.79	33.71	22.81	23.42
45-54	24.97	29.06	22.05	27.63	25.08	19.85
55 plus	14.55	11.19	13.25	12.56	16.71	19.58
Occupation						
Armed forces	6.99	7.85	2	1.09	11.08	11.84
Legislators, senior officials and managers	19.46	23.21	11.86	15.22	22.44	23.69
Professionals	17.54	16.15	13.29	21.5	17.62	20.71
Technicians and associate professionals	11.73	13.48	11.5	13.64	12.78	5.62

Clerks	18.12	13.59	22.38	17.1	17.81	20.66
Service workers and shop and market sales workers	0.89	0.3	2.65	0.69	0.66	
Skilled agricultural and fishery workers	.62	10.78	10.64	10.43	6.8	4
Craft and related trades workers	6.02	6.57	7.17	10.03	2.52	5.3
Plant and machine operators and assemblers	10.62	8.07	18.51	10.28	8.31	8.17
Elementary occupations						
Immigrant (born abroad)	8.65	8.15	2.61	9.74	7.9	17.47
Working part-time (< 30 h/ week)	23.05	18.83	17.53	18.34	38.67	15.04
Firmsize						
1-10 people	26.18	19.56	37.75	37.12	20.23	18.6
11-50 people	29.16	27.79	28.09	27.09	32.47	29.15
51-250 people	24.54	29.67	19.53	19.77	26.68	25.37
251-1000 people	11.53	14.53	8.68	8.3	11.93	13.85
More than 1000 people	8.58	8.45	5.95	7.71	8.68	13.03
Numeracy use at work						
All zero response	26.62	27.22	27.03	36.73	25.93	15.31
Lowest to 20%	15.85	18.02	15.65	13.3	17.43	13.03
More than 20% to 40%	13.77	14.12	13.7	13.84	13.72	13.36
More than 40% to 60%	14.75	14.98	14.76	10.73	15.58	17.36
More than 60% to 80%	14.05	12.09	14.51	12.41	13.47	19.04
More than 80%	14.98	13.56	14.35	13	13.88	21.9

Results

Figures 1-4 below are based on performing a binary logistic regression of being over-skilled on gender, age-group, education level (alternated by area of study), migrant status, occupation, working part-time or not, firm size, and numeracy use at work. The country was entered as a control variable. In the figures we see the probability of being over-skilled by each of these variables.

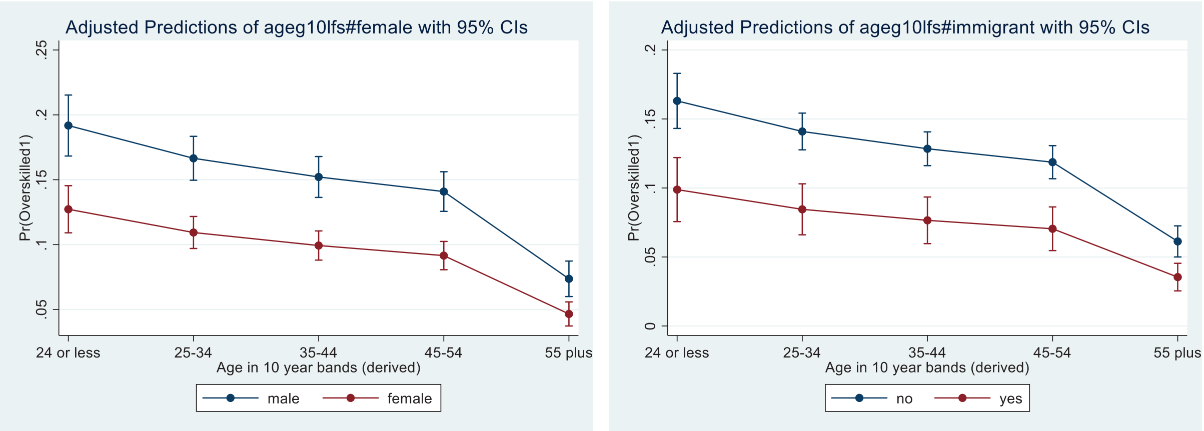


Figure 1: The probability of being over-skilled by gender, age-group, and migrant status

Men are significantly more likely to be over-skilled than women, controlling for other factors. The probability is 18 percent for men, compared to 14% for women. The probability of being over-skilled declines over years.

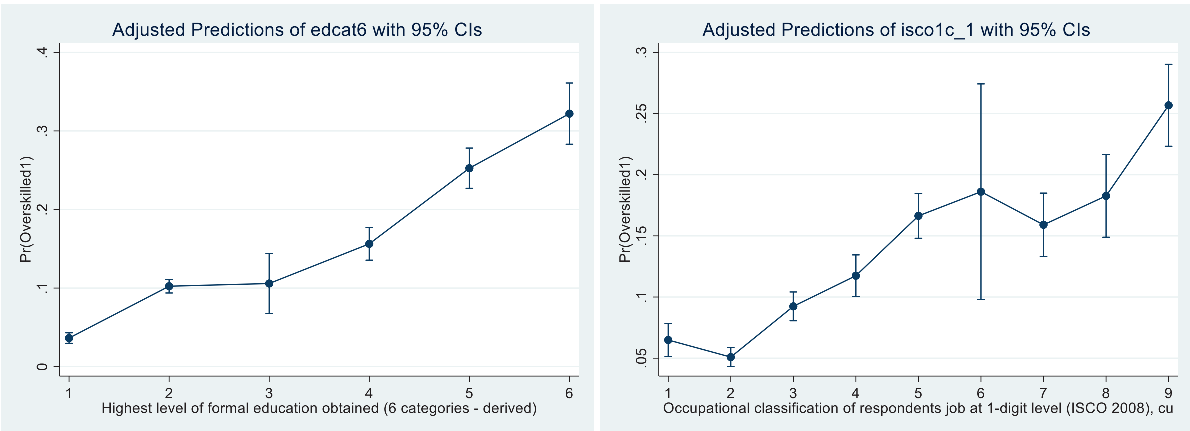
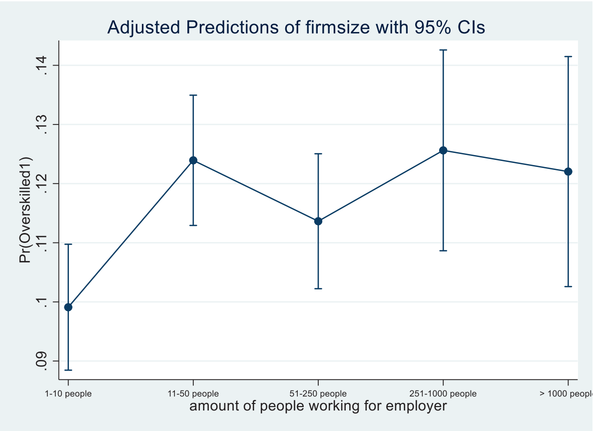


Figure 2: The probability of being over-skilled by education level[4] and occupation[5]

Being higher educated has a significant positive association with being over-skilled, controlling for other factors. People in elementary occupations are more likely to be over-skilled than people in other occupations.

[4] Legenda: 1 = Lower secondary or less (ISCED 1,2, 3C short or less), 2= Upper secondary (ISCED 3A-B, C long), 3 = Post-secondary, non-tertiary (ISCED 4A-B-C), 4 = Tertiary – professional degree (ISCED 5B), 5 = Tertiary – bachelor degree (ISCED 5A), 6= Tertiary – master/research degree (ISCED 5A/6)

[5] Legenda: 0 = Armed forces, 1 = Legislators, senior officials and managers, 2 = Professionals, 3 = Technicians and associate professionals, 4 = Clerks, 5 = Service workers and shop and market sales workers, 6 = Skilled agricultural and fishery workers, 7 = Craft and related trades workers, 8 = Plant and machine operators and assemblers, 9 = Elementary occupations



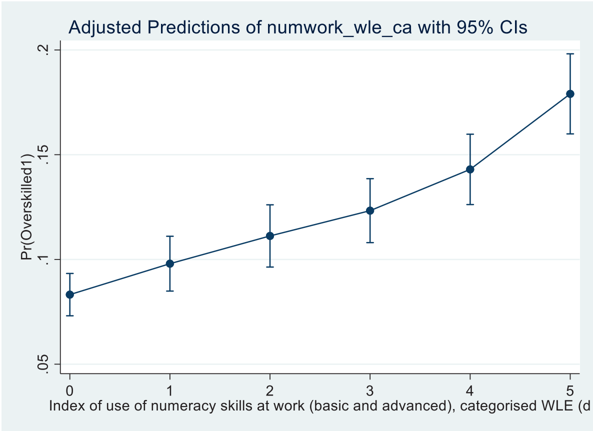


Figure 3: The probability of being over-skilled by firm-size and use of numeracy skills[6]

There is no significant difference in the probability of being over-skilled across firm of various sizes. Furthermore, we see that the likelihood of being over-skilled increases as the frequency of using numeracy skills at work increases.

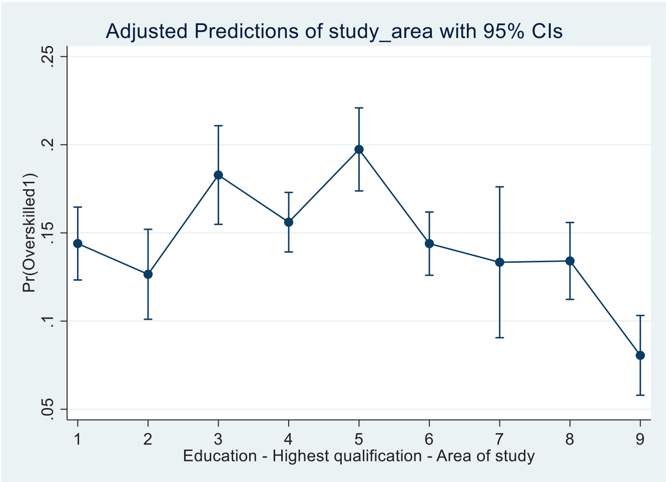


Figure 4: The probability of being over-skilled by area of study7

Being over-skilled is significantly more likely for people who studied science, mathematics and computing and significantly less likely for people who studied services.

Conclusion

Being over-skilled is more likely for men, younger age-groups, higher education, and for people who use their numeracy skills often at work. Also, people who studied science, mathematics and computing are significantly more likely to be over-skilled. Our results are largely in line with what earlier studies showed, although we used a different measure and other sample set. Further studies

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[6] Legenda: 0 = All zero response, 1= Lowest to 20%., 2 = More than 20% to 40%, 3 = More than 40% to 60%, 4 = More than 60% to 80%, and 5 =More than 80%

[7] 1 = General programmes, 2 = Teacher training and education science, 3 = Humanities, languages and arts, 4 = Social sciences, business and law, 5 = Science, mathematics and computing, 6 = Engineering, manufacturing and construction, 7 = Agriculture and veterinary, 8 = Health and welfare, 9 = Services

should focus on the reasons why certain study areas are significantly associated with the probability of being over-skilled and on improving the measure for mismatch.

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A comparison of the problem solving and procedural skills of those in adult mathematics education in Ireland

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A desire to have students and citizens who are numerate and effective problem solvers is a common goal internationally (Gal & UNESCO, 2020). This desire is no different in an Irish context across all areas of education including adult mathematics education (SOLAS, 2020). This research used diagnostic testing to determine and compare the mathematical problem solving and procedural skills of a cohort of adult students in an Irish University. The diagnostic test, which was developed by the authors using the second level mathematics curriculum in Ireland as a guide, revealed that all adult students had statistically significantly lower mathematical problem solving skills when compared to procedural skills. These findings raise concern about the problem solving skills of adult learners. Discussion around the implications of such findings for best practice in adult mathematics education are outlined.

Keywords: Adult mathematics education, undergraduate mathematics education, diagnostic testing, procedural skills, problem solving skills.

Introduction

The desire of having numerate citizens and students who can problem solve effectively is often driven by a nations need to stay economically competitive (Perkins & Clerkin, 2020). Adult mathematical numeracy is essential to ensuring job markets, economies and societies prosper in addition to the having a key role in improving the lives of citizens (Gal & UNESCO, 2020). Much literature exists on the importance of the development of problem solving skills in mathematics (IBEC, 2015) and a subsequent need to use effective pedagogical practices to support this (Foshay & Kirkley, 2003). Despite these uncontested and commonly reported findings adult mathematics education is often overlooked when it comes to providing appropriate provision in terms of its development and improvement (Gal & UNESCO, 2020). The aforementioned body of research has resulted in an overhaul of the second level mathematics curriculum in Ireland in addition to heavily funded government programmes to upskill second level teachers. The same opportunities or investment have not been afforded in adult mathematics education. In an era where equality, diversity and inclusion are at the fore front of all higher education policies it seems remiss that those less advantaged, i.e., adult students on an [1]Access Foundation Programme, are not offered at least the same supports and opportunities as the more advantaged in society. Some details of the overhaul to the second level mathematics curriculum in Ireland in 2010 as well as some literature surrounding problem solving in general and in the context of the new mathematics curriculum in second level education is also discussed in order to provide context for the development of the diagnostic test used within this research.

Literature Review

The second level mathematics curriculum in Ireland

In 2010 a new mathematics curriculum was introduced in second level education in Ireland entitled ‘Project Maths’ (PM) (Prendergast et al., 2017). The rationale for the introduction of a new curriculum was to change from the traditional ‘chalk and talk’ teaching methodologies predominately being employed so that students would have improved understanding of mathematical concepts and improved engagement with the subject (Hourigan & O’Donoghue, 2007). The change from more traditional approaches to teaching mathematics to more student-centred approaches which focus on teaching for understanding was partially due to Irish students' comparative poor performance on an international platform (Humphreys, 2015). Another motivator for the change was the emerging literature relating to the constant and steady decline in students’ basic mathematics skills on entry to third level education over the past two decades (Faulkner et al., 2010).

Like previous studies in the area Treacy and Faulkner (2015) found that students’ basic mathematical skills on entry to third level education, between 2003 and 2013, had declined. This study examined a time period which included students pre and post PM. What seemed positive about this study's findings however was the suggestion that students' problem solving skills may have improved over this time period. An examination of lecturers' perceptions of the change in mathematical performances, if any, after the introduction of PM found that lecturers felt that students’ procedural skills were still on the decline but that possibly their openness to working with mathematics problems and problems that were not familiar to them when compared to previous cohorts of students had improved (Prendergast et al., 2017)

Problem Solving in Mathematics

The need to document and improve students' problem solving skills through new and innovative pedagogical practices is omnipresent in education literature however it is not a straightforward practice to implement or even a concept that is very well understood (Kilpatrick, 1969). Despite the challenge of defining problem solving in mathematics and understanding the best approaches to teaching it effectively it remains to be a skill that is very much desired by both employers and those in higher education institutions (Vordermann et al., 2011). The desire for this problem solving skill set often stems from it reportedly not being at the disposal of many school leavers (Jones et al., 2014).

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[1]The Access Foundation Programme is a one year preparatory programme for students wishing to peruse an undergraduate programme at certificate or degree level in Ireland. The programme provides a root to education for students from communities which lack a strong tradition of participation in third level education.

Additionally, research in this area emphasises the long length of time that students spend in formal education learning mathematical skills which they then find very challenging to transfer to further education and/or a working environment (Treilibs, 1979). Such realisations have resulted in many countries, including Ireland, trying to change their school curricula in response to the issues students are displaying with applying problem solving skills in mathematics upon leaving school (Soh, 2008). This "hiatus" between formal mathematics education and the workplace has also been acknowledged in research by FitzSimons and Boistrup (2017) who offer some suggestions as to how to overcome such issues by recontextualising the mathematics in different types of work and how the recontextualization could be incorporated into formal education settings so that the mathematics can be identified, and a context can be offered. The approach to improving problem solving employed in second level education in Ireland is considered next.

Problem Solving and Mathematics Education in Irish Second Level Schools

Project Maths and Problem Solving

The second level Project Maths curriculum in Ireland defines problem solving as the following:

"Problem Solving means engaging in a task for which the solution is not immediately obvious. Problem solving is integral to mathematical learning. In day-to-day life and in the workplace the ability to problem solve is a highly advantageous skill. In the mathematics classroom problem solving should not be met in isolation but should permeate all aspects of the teaching and learning experience. Problems may concern purely mathematical matters or some applied context."

The description of problem solving outlined in the PM curriculum can be seen to be in line with Polya's (1945) interpretation of problem solving which involves engaging with real problems by guessing, discovering and trying to make sense of mathematics. It has also been likened to the Programme for International Student Assessment (PISA) which consists of theoretical underpinnings focus on everyday problems that often occur when interacting for example with a device that is unfamiliar for the first time (OECD, 2014). This could be seen to be in contrast with the previous focus of the Irish second level mathematics curriculum in which problem solving was more commonly described as 'an ideal of problem solving' (McClure, 2013) where the focus is on solving a set of 'problems' using prescribed and practiced techniques (Faulkner et al., 2021). Details of the diagnostic test that was developed, and informed by PM, are outlined in the methodology section which follows.

Methodology

Measuring Problem Solving: The Diagnostic Test

A diagnostic test was developed to determine students' problem solving and procedural skills in mathematics on entry to adult and undergraduate education.

Diagnostic Test Design

Four mathematics educators from two higher education institutions in Ireland developed the paper based diagnostic test. The diagnostic test was designed with a focus on examining the basic mathematical skills of students beginning their studies who hoped to pursue degree programmes with an engineering/technology focus. Several controls were put in place to ensure that the test measured what it had set out to. Such controls included using the second level mathematics curriculum and its interpretation of what constitutes problem solving skills as a guide. The initial draft of the test was examined critically by five mathematics education specialists from different institutions around Ireland and the feedback was incorporated to improve the test prior to distribution.

Contents of the Diagnostic Test

The diagnostic test is broken down into two sections: Procedural questions (Section A) and problem solving questions (Section B). Both sections consist of 11 questions in total. Each question in section A of the test is paired with a question in section B i.e., the paired questions require the same procedural skill(s) to successfully complete them with the section B questions also involving some real-world context. All of the section B questions were sourced from state examination papers that second level students take when completing their second level schooling in Ireland. It was thought that taking questions from the state examination papers allowed for the diagnostic test to directly mirror the second level curriculum's interpretation of what constitutes problem solving (Faulkner et al., 2021).

Data Analysis

The Statistical Package for Social Sciences (Version 22.0) was used to analyse and interpret the diagnostic test data. Independent samples t-tests were used to test for statistically significant differences between the mean performances of participants with different demographic backgrounds. Chi-squared tests were used to test for statistically significant associations between the qualitative variables. A 5% level of significance was used for all tests and no adjustment were made for multiple testing. It should be noted that the aim of the research was purely quantitative in nature, i.e. to determine and compare students procedural and problem solving skills in mathematics, rather than a more in depth investigation into students approaches to completing the diagnostic test questions.

Sample

Within this study 87 students undertook the test. Of these students 34.5% (30) were adult students from the Access Foundation Programme. These adult students were enrolled on an advanced mathematics module with the intention of pursuing a degree programme which has a technology/engineering focus. The advanced mathematics module contains content that mirrors that completed by second level students. 65.5% (57) were first year undergraduate students enrolled in an engineering programme. These two groups of students were chosen for comparison in this research as both were currently/intending to pursue a similar undergraduate discipline (engineering/technical focus) and it was of interest to the authors to determine and compare their mathematical skills and preparedness. It was also of interest to see if the traditional students coming from second level mathematics education were demonstrating equally as good problem solving skills as procedural skills in mathematics or did the adult students perform better than the traditional students and could this possibly be linked to skills previously learned in the workplace. It should be noted that all the Access Foundation Students with the except of 1 were male.

Results

Adult and engineering students were examined as one group initially and the results showed that students performed statistically significantly better in the section A ($\bar{x}$= 57.2) than they did in section B of the test ($\bar{x}$= 32.3) (p=0.00).

The data was then examined by analysing how the adult students' performance compared to the engineering students' performance. This analysis found that the adult students' performed to a statistically significantly lower standard when compared to the engineering students in both section A and section B of the diagnostic test (p=0.00).

Further analysis of the adult students found that they performed statistically significantly worse in section B (=18.7) of the diagnostic test when compared to section A (=43.1) ($p<0.001$). The same analysis was carried out for engineering students and found that they performed statistically significantly worse in section B ($\bar{x}$=39.5) of the test when compared to section A ($\bar{x}$= 64.6) ($p <0.001$).

Discussion

All students (adult and engineering) performed statistically significantly better in the procedural section of the diagnostic test (i.e., Section A) when compared to the problem solving section of the test (i.e., Section B). Therefore, the strengths and weaknesses of each group of students was found to be the same despite the adult students performing to a lower standard overall. The demographic of students that was most likely to perform better in all aspects of the diagnostic test were those who were enrolled on the engineering programme and were male and Irish. Despite this however this characterisation of student still performed statistically significantly worse in section B of the diagnostic test compared to section A. These findings highlight the apparent challenge that all student cohorts within this research have with applying what could be considered basic mathematical concepts to applied scenarios. The concepts required to answer the paired questions are the same. For example, Q5 in section A asks students to calculate the difference in area between a rectangle and a circle and the paired question in section B asks students to determine how much of a pool area (rectangular in shape) is taken up by a jacuzzi (circular in shape)

It is however acknowledged that there is a possibility of some intrinsic difference in difficulty level between the questions in each section of the test due to the problem solving questions requiring more processing to successfully complete the questions. This is certainly likely to be the case when the adult mathematics students are considered as they have not been exposed to a second level school system for a long period of time and if/when they did the teaching approach would have been very much focussed on chalk and talk teaching methodologies with little focus on context or problem solving. The finding is therefore of more note when the engineering students are considered as the teaching approaches that they have been exposed to in second level education in recent years (i.e., problem solving focussed) might be expected to negate or at least reduce the additional challenge that the problem solving questions may present (Faulkner et al., 2021).

When we consider the example of Q5 previously given in the context of the engineering students it raises questions about how traditional students are adapting to learning mathematics in second level education which is not solely based on procedural skills and involves some level of real world thinking and contextualisation. Research has indicated that engineers are graduating with good knowledge of fundamental engineering science and computer literacy but little ability to apply this in practice (Mills and Treagust, 2003). The suggested challenges faced by traditional students in undergraduate mathematics who have been exposed to problem solving teaching methodologies in second level education should be recognised by those in adult mathematics education wishing to employ such teaching methodologies.

As previously alluded to perhaps it is not as surprising that the adult learners do not perform as well as the traditional students predominately entering higher education directly from second level education. It is possible that literacy issues also play a key role in the challenge that the adult learners face. This has been found to be the case in related research into adult mathematics education (Prendergast et al., 2017). Although the anticipated challenge of literacy was considered and acknowledged in the development stages of Project Maths it was deemed a necessary challenge for those wishing to be successful in problem solving across all disciplines (Foshay & Kirkley, 2003; Faulkner et al., 2021).

So, what can we learn in terms of adult mathematics education? We know that those in adult mathematics education display similar patterns of challenges in terms of their procedural and problem solving skills as compared with traditional students however to a larger degree. We also know that adult learners despite this common initial underperformance and categorisation of being 'at risk' of failing mathematics often improve to a large degree due largely to engagement with mathematical support services (Faulkner et al., 2021). The focus however for those in adult mathematics education is so often on remediation rather than on proactive measures to ensure best practice is in place for them for example via engaging in mathematics education with appropriately trained mathematics educationalists. A focus in adult mathematics education on problem solving skills in mathematics is needed for the same reasons that it is advocated and provided for in second level education. Adult mathematics education deserves at least the same level of teaching qualifications that those in primary and secondary education in Ireland have. There is a new phenomenon in educational research currently being examined relating to what have become known as out-of-field teachers of mathematics. These are teachers in second level education who are teaching mathematics but not suitably qualified to do so. Continuous professional development programmes are now being put in place to upskill such teachers (Hobbs & Törner, 2019). The norm is adult mathematics education is to have out-of-field teachers and no such measures have been put in place; this needs to become part of the national and international agenda.

Conclusion and Recommendations

Although initial studies into PM have indicated that students' problem solving skills may have improved in recent years (Treacy & Faulkner, 2015) and that lecturers perceive students as being more open to engaging with unseen mathematics problems (Prendergast et al., 2017), this research highlights the comparatively weaker skills that those in adult mathematics education demonstrate in problem solving in mathematics compared to procedural skills in mathematics. This pattern of performance is mirrored amongst those in beginning undergraduate engineering programmes despite them coming directly from a second level mathematics education which has come to focus more on problem solving and teaching for understanding. Acknowledgement should be given to future work in this area taking into account the greater conceptual challenge faced by students when completing the problem solving questions and a more detailed marking scheme than right/wrong should be therefore be considered when analysing students' performance in this area.

The rationale behind the changes made to second level mathematics education is research based and changes/improvements in students' performances on entry to higher education may take time to become embedded. Attention must be given however to the relatively similar challenges evidently faced by adult mathematics students in problem solving and the lack of provision and investment in their mathematics curriculum and teacher training. Such disparity has been raised in this research paper and should be investigated further to provide a platform for improved future provision of adult mathematics education. The 'Adult numeracy: Assessment and development' policy outlines the significant numeracy challenges faced internationally and makes 3 recommendations one of which involves "investing in the development of national capacities to measure and improve adult numeracy" (Gal & UNESCO, 2020, p.3). This international and recent recommendation very clearly aligns with the research being outlined.

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Psychosocial factors, level of mathematics and progression in an access programme

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This study examined the relationship between Access students' psychosocial characteristics, the level of mathematics module (advanced, intermediate or fundamental) they chose and their progression to higher education. A quantitative approach was adopted for this portion of the study, which took place over three academic years, 2017 - 2020. Questionnaires were completed by 184 students in the Access programme at Technological University Dublin. Results revealed that students with higher belief in their mathematics ability were more likely to study advanced mathematics and more likely to progress to higher education. Male students were more likely to study advanced mathematics than females and non-Irish nationals who studied advanced mathematics had higher belief in their mathematics abilities than Irish nationals but were less likely to progress than their peers.

Keywords: Access programme, psychosocial, mathematics level, self-belief, progression.

Introduction

Higher education has many societal and personal benefits but there are low participation rates for students from some sections of society, including students who are socioeconomically disadvantaged (Archer et al., 2005) and adults aged 25 – 64 years old (Eurostat, 2019). Access programmes have been established to address these inequalities by tackling the social, educational, and financial barriers that some students experience in accessing higher education (O'Reilly, 2008). To date, there has been little research on Access student progression, particularly in relation to mathematics. The goal of this study is to examine whether the psychosocial factors of motivation, personality traits, general self-efficacy (GSE) and belief about mathematics abilities (BMA) affect the level of mathematics module Access students choose and their progression to higher education.

Psychosocial Factors Affecting Progression in Higher Education

Personality traits, including extraversion, agreeableness, conscientiousness, neuroticism, and openness to experience (McCrae & John, 1992), play a role in determining a student's educational attainment (Lenton, 2014). The personality factors of conscientiousness and openness have been found to affect students' mathematics grades (Furnham et al., 2009). Lipnevich et al. (2016) contend that conscientiousness may be beneficial for mathematics performance because it results in persistent and thorough learning, while openness has been linked to deep learning. Personality traits affect a student's likelihood to progress in higher education (Altman, 2017).

Self-efficacy affects individuals' perceptions about their abilities related to a given task (Hutchison et al., 2006) and their ability to learn (Schulze & Schulze, 2003). Schöber et al. (2018) found that self-efficacy affected mathematics achievement. According to Hall and Ponton (2005), positive experiences with mathematics increase students' self-efficacy. Overall, research indicates that students with higher self-efficacy are more likely to progress (Erb & Drysfales, 2017).

Intrinsic and extrinsic motivation have been widely studied. Intrinsically motivated individuals do something for the inherent satisfaction they get from a behaviour, extrinsically motivated individuals engage in a behaviour for the reward they gain through external control or self-regulation (Ryan & Deci, 2000). According to Ryan and Deci, amotivated individuals are not motivated to engage in a behaviour and feel they have no control over that behaviour. Some researchers contend that lower intrinsic motivation negatively affects student performance (Augustyniak et al., 2016) and their progression in education (Vallerand et al., 1997). There is a positive relationship between mathematics self-efficacy and students' intrinsic motivation and progression (Skaalvik et al., 2015). This study examined the relationship between psychosocial factors, Access students' mathematical experiences and their progression to undergraduate studies.

Method

Technological University Dublin (TU Dublin) offers a one-year Access programme, which provides an alternative route to higher education for mature students (students aged 23 years and older) and for young adults (students aged 22 years and under) who are socio-economically disadvantaged (Technological University Dublin, 2020). Participating students choose one mathematics module each semester at fundamental, intermediate or advanced level.

The main study, which took place over three academic years, 2017 - 2020, adopted an explanatory, sequential mixed methods approach. The ethics committee at TU Dublin provided ethical approval for the study. During the quantitative phase of the research, Access students completed a 29-item questionnaire at the start of the academic year. The questionnaire included the 28-item Academic Motivation Scale (Vallerand et al., 1992). It also included John and Srivastava's (1999) 44-item Big Five Inventory, which organizes personality traits in terms of the five dimensions of extraversion, agreeableness, conscientiousness, neuroticism, and openness to experiences. Schwarzer & Jerusalem's (1995)10-item General Self-Efficacy Scale was included to assess students' ability to deal with unusual or difficult situations. Additionally, students rated their BMA using a five-point Likert scale, where 1 represented 'excellent' and 5 represented 'poor'.

Progression was measured based on whether students were offered a place at a higher education institution or not. The data was analysed using SPSS. Mann-Whitney U tests (U) were conducted to compare the mean ranks of data where one variable was dichotomous, and the other variable was ordinal. Chi-square tests ($\chi 2$) were employed when both variables were dichotomous. Independence of observations was observed for all Mann-Whitney and chi-square tests.

Results

One hundred and eighty-four Access students completed questionnaires over the three years of the study. Forty-nine percent were female, 51 percent were male, 43 percent were young adults and 57 percent were mature students. Overall, 25 percent of Access students were enrolled in fundamental mathematics, 64 percent in intermediate mathematics and 11 percent in advanced mathematics. A Mann-Whitney test revealed that students who studied advanced mathematics had significantly higher mean ranks for intrinsic motivation to know than their peers (U = 945.5, p = .071).

Additionally, students who studied intermediate mathematics had higher mean ranks for extrinsic motivation external than their peers (U = 2639.5, p = .033). Students who studied fundamental mathematics had significantly higher mean ranks for amotivation than students who studied advanced or intermediate mathematics (U = 1732.5, p = .002). Young adult students had a significantly higher mean rank for extrinsic motivation than mature students (U = 2197, p = .038).

There was no significant difference in mean ranks for most personality traits based on the level of mathematics students studied. However, students who studied fundamental mathematics had a higher mean rank for neuroticism than their peers (U = 1613, p = .020). Additionally, young adults had a significantly higher mean rank for extroversion than mature students (U = 1910, p = .001).

Fundamental mathematics students had a significantly lower mean rank for general self-efficacy (GSE) than students who studied intermediate or advanced mathematics (U = 3090, p = .038). Moreover, students who studied intermediate mathematics had a significantly higher mean rank for self-efficacy than their peers (U = 2732, p = .017). Although male and female students did not differ in their mean ranks for GSE (U = 3409.5, p = .826), males were significantly more likely to study advanced mathematics than females ($\chi2$ = 4.93, df = 1, p = .026). Additionally, non-Irish nationals had significantly higher mean ranks for GSE than Irish nationals (U = 3999, p = .007) and were more likely to study advanced mathematics or intermediate mathematics than Irish nationals ($\chi2$ = 3.58, df = 1, p = .059).

Students who studied fundamental mathematics had a significantly lower mean rank for belief about mathematics their abilities (BMA) than those studying intermediate or advanced mathematics (U = 3562, p = .014), and students studying advanced mathematics had a higher mean rank for BMA than their peers (U = 97.6, p < .001). Additionally, non-Irish nationals had a significantly higher mean rank for BMA than their Irish peers (U = 4916.5, p < .001). The findings related to psychosocial factors are outlined in Table 1.

Table 1: A Comparison of Mann Whitney Mean Ranks for Psychosocial Factors by Mathematics Level, Age and Nationality

	Mann Whitney Mean Ranks				
	Mathematics Level			Age	Nationality
	Fundamental	Intermediate	Advanced	Young Adult	Non-Irish National
Motivation					
Intrinsic motivation to Know			Higher		
Extrinsic Motivation External		Higher			
Extrinsic Motivation Total				Higher	
Amotivation	Higher				

Personality					
Neuroticism	Higher				
Extroversion				Higher	
Self-Efficacy	Lower	Higher			Higher
BMA	Lower				Higher

Mathematics and Progression

Overall, 85% of students who studied fundamental mathematics, 84% of those who studied intermediate mathematics and 89% of students who studied advanced mathematics progressed to higher education. Statistically, students studying intermediate mathematics were more likely to progress (χ^2 = 7.57, df = 1, *p* = .006) than those studying fundamental or advanced mathematics.

Moreover, mature students studying intermediate mathematics were more likely to progress than their young adult peers ($\chi2$ = 8.39, df = 1, *p* = .004), but non-Irish nationals who studied advanced mathematics had lower progression rates than Irish nationals ($\chi2$ = 2.92, df = 1, *p* = .087).

There was no statistically significant difference in progression based on students' mean ranks for intrinsic motivation total (*U* = 2647.5, *p* = .552) or extrinsic motivation total (*U* = 2363, *p* = .969). Moreover, students' mean ranks for amotivation were not significantly different depending on whether they progressed or not (*U* = 3076.5, *p* = .329).

Overall, there was no significant difference in mean ranks for personality traits based on whether Access students progressed or not – extraversion (*U* = 2521, *p* = .909), agreeableness (*U* = 2245, *p* = .872), conscientiousness (*U* = 1978, *p* = .140), neuroticism (*U* = 2681, *p* = .856) or openness (*U* = 2149, *p* = .114).

Additionally, there was no statistically significant difference in progression based on Access students' mean rank for GSE (*U* = 2815.5, *p* = .425). However, students who progressed had significantly higher mean ranks for BMA than those who did not progress (*U* = 2807.5, *p* = .022).

Discussion

This study aimed to determine whether there was a relationship between the psychosocial factors of motivation, personality traits, GSE and BMA and the level of mathematics Access students study as well as their progression to higher education. The findings revealed a relationship between all four psychosocial factors and the level of mathematics module Access students studied.

Access students who studied fundamental mathematics had significantly higher mean ranks for amotivation and neuroticism and significantly lower mean ranks for GSE and BMA than their peers. Prior performance has been found to be a predictor of students' self-efficacy in mathematics (Lopez and Lent, 1992), while students with higher self-belief in their ability to succeed in higher education mathematics classes have better mathematical skills (Hall and Panton, 2005). Access students with lower GSE and BMA may have weaker mathematics skills or their past performance in mathematics may have affected their BMA as indicated by Lopez and Lent (1992). Additionally, research indicates that neuroticism creates negative emotions, results in failure to progress and results in a negative reaction to the fear of failure (Barthelemy & Lounsbury, 2009). Fundamental mathematics students' lower mean ranks for GSE and BMA, in conjunction with their higher neuroticism scores, may have resulted in amotivation, as individuals who are not motivated to engage in a behaviour, may feel they have no control over that behaviour (Ryan & Deci, 2000).

Alternatively, advanced mathematics students had higher mean ranks for GSE, BMA and intrinsic motivation to know, which is regulated by the pleasure of learning something. Mueller et al. (2011) contended that intrinsic motivation increases self-efficacy and results in the development of more favourable tendencies towards learning mathematics, which may have influenced students' decision to choose the advanced mathematics module in the Access programme.

Although male and female student had similar mean ranks for GSE and BMA, females were less likely to study advanced mathematics. This may be because mathematics is seen as masculine (Mendick, 2005) and because males are more likely to choose mathematics intensive careers (Law, 2018). Access students choose the level of their mathematics module based on the higher education course they wish to pursue, and male Access students were more likely to aspire to study mathematics intensive fields such as engineering, physics and computer science than female Access students.

Access students with higher BMA scores had higher progression rates than their peers. Students generally have a good awareness of their academic abilities (Mattern & Shaw, 2010; Reason, 2003). Mattern and Shaw (2010) also found that students with higher BMA had higher GPAs and were more likely to progress from first to second year of higher education.

Although there were no significant differences in progression in relation to personality traits, motivation or GSE, mature students studying intermediate mathematics had higher progression rates than young adults. Young adult students had higher mean rank scores for extrinsic motivation than mature students. Research indicates that students thrive in an educational setting where they are more intrinsically motivated (Ryan & Deci, 2000), which may explain the lower progression rates for young adults studying intermediate mathematics. Moreover, young adult Access students had significantly higher extroversion scores than mature students, and extroversion is negatively related to educational attainment (van Eijck & DeGraaf, 2004).

Non-Irish nationals studying advanced mathematics were significantly less likely to progress than their peers although they had higher mean ranks for BMA and GSE. Non-Irish nationals, who were non-native English speakers, may have failed to progress because they experienced difficulties in modules that required advanced English language skills. Higher education students' academic achievement can be affected by English competency (Harris & Ní Chonaill, 2016).

Limitations and Recommendations

The sample size was relatively small, although it represented 67 percent of participants in the Access programme over the three years of the study. Additionally, the General Self-efficacy Scale was employed in the questionnaire, but a college self-efficacy scale may have been more pertinent. Asking participants to complete the AMS and the GSE at the end of the Access programme as well as at the start would have indicated whether Access students' self-efficacy, BMA or intrinsic motivation increased during their studies.

Future research should examine more closely why Access students choose the level of mathematics modules that they do, their previous mathematics performance and the reasons why non-Irish nationals studying advanced mathematics are less likely to progress than their Irish peers. This data would help to determine whether the relationships identified in the current study are causal relationships or whether they are affected by other confounding factors.

Conclusion

Although there is no difference in progression, overall, depending on the psychosocial factors of motivation, personality and GSE, these factors may affect the level of mathematics modules that students choose to study. This in turn may affect the higher education courses students aspire to, as mathematics intensive courses may require advanced mathematics. Therefore, it is important that students are made aware of the opportunities that advanced mathematics can afford them.

Given that students with higher BMA had higher progression rates, overall, Access students should be encouraged to improve their BMA and their GSE by following Heslin and Kelhe's (2006) recommendations of engaging students in enactive self-mastery, role-modelling and verbal persuasion.

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The Mathematisation of society: rethinking basic skills for adults

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Technology, data use, and digitisation are based on mathematical structures, and this permeates many aspects of our daily lives: apps, online activities, and all kinds of communication. Equipping people to deal with this mathematisation of society is a big challenge. Which competences are needed, which skills must be mastered? Which dispositions are helpful? These are the questions that matter in the development of adult education. The concept of numeracy is mentioned already for many years as a possible useful approach to equip adults with the necessary skills. In this paper we will argue that is only true when numeracy is defined as a multifaceted concept which combines knowledges, skills, higher order skills, context and dispositions.

Keywords: Numeracy, basic skills, adult education.

Introduction

World-wide, too many citizens lack the necessary numeracy competencies to participate autonomously and effectively in our technologized and number-drenched societies and consequently many citizens are overlooked for certain jobs and have problems in their daily life, dealing with the fast-growing abundance of number-related situations. In literature, these numeracy competencies are mentioned specifically in studies on 21st century skills, global competences, and skills for the 4th industrial revolution (OECD, 2016; Schwab, 2016; Voogt & Roblin, 2012).

Considering numeracy as a social practice seems to be the most promising way forward to battle low numeracy and empower adults with a broad and effective repertoire of numerate behaviour to cope with situations in work and daily life. We cite (Oughton, 2013): “A social practice view of numeracy not only takes into account the different contexts in which numeracy is practised, such as school, college, work and home, but also how people’s life and histories, goals, values and attitudes will influence the way they carry out numeracy (p16). An even richer collection of ideas on this approach can be found in Numeracy as a Social Practice (Yasukawa et al., 2018).

Background

For the past three years, we have been working in an Erasmus+ project on a Common European Numeracy Framework, which uses a multifaceted definition of numerate behaviour, incorporating knowledge and skills, use of tools, higher order skills, and dispositions. The combination of these aspects is seen as increasingly crucial in the quality of numerate behaviour to act autonomously, reflect critically, and make sound decisions in our number-drenched society. In a sense, numeracy is the new literacy in today’s society. This gives rise to reconsider fundamentally the idea of “basic skills” in mathematics and numeracy for primary and secondary education. Numeracy is about how people deal with the quantitative and multidimensional phenomena in the world around us, both in daily-life situations and professional contexts. In the latest and most state-of-the art definitions of numeracy, it is described as a broad and multifaceted concept and as a social practice. It manifests itself in a plethora of observed numerate practices of people, showing that numerate behaviour is affected by cultural, social, personal, emotional traits, and societal power relations. In short, numeracy takes the person and his/her relationship with the world as a starting point.

A historical timeline

The picture in Figure 1 is by some researchers considered to be one of the he first examples of human numerate practices in "writing".

Figure 1: Schoyen collection: ms-1717-beer-inanna-uruk

It is a Sumerian pictographic script from the 31st century BC on a tablet, which is part of the Schøyen collection in Oslo, Norway (The Schoeyen Collection, 2021). The tablet is presumed to be giving information on the production of beer 5000 years ago. Reading from right to left, we see the barley delivered, then the brick building - presumably the brewery -, and the barley in the jar resulting in the beer. It is the earliest representation in history of an industrial process. Looking at this tablet from a mathematical perspective, we see that the use of quantitative notations is meaningful and purposeful. For the documentation of the brewing process an external tool is used in this tablet. And a mix of cognitive tools is used: pictorials, quantities and order of process. So, using schematic, symbolic, quantitative and visual notations to describe the real world and to communicate with others, which is mathematics, is as old as the written language of the first civilized societies. It is good to remember that we as mankind have used symbols and other visual representations to describe ánd control the world for millennia, basically as long as we have had civilized societies.

Before the art of printing, the use of writing material and the use of a mathematical repertoire was only for expert and privileged citizen. So was the use of mathematical knowledge and skills.

Nevertheless, like people have always used language to express themselves, they also have always used logic reasoning, problem solving, estimating, planning and many more cognitive processes which we now categorize under mathematical cognitive processes or meta-cognitive processes.

The wide-spread use of pen-and-paper has given people the opportunity to use this "new technology" to do actual calculations, which was one of the driving forces of the industrial revolution.

The pen-and-paper techniques were introduced in the mass education which roughly started at the beginning of the 20th century and has since then been dominant in steering the goals of primary mathematics education, likewise the pen-and-paper algebraic manipulation which brought acting mathematically to a much broader mass was steering the content of secondary maths education.

If we look a bit closer to the last 75 years, we see that the pen and paper calculations with numbers or variables permeated so deep into the educational system, that it has become a goal in itself, instead of a means to provide answers to typical kind of problems. It permeated so deep that it now even has survived its relevancy and is more a driving force for wide-spread math anxiety than that it delivers students a useful repertoire.

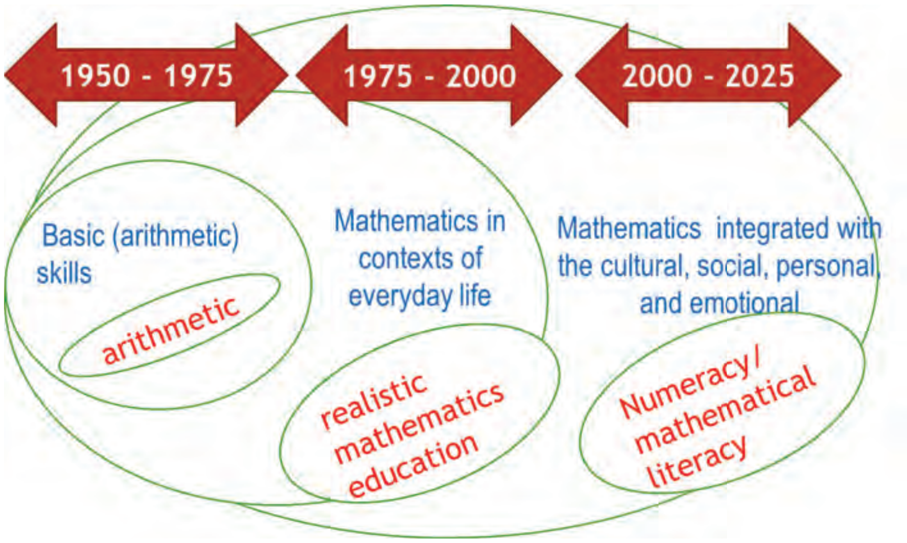

Figure 2: Relevant mathematics for use in daily life

After WWII, in the third quarter of last century (1950-1975), there was a broad consensus that the best way to prepare students to the needs of society should consist of practicing operations on bare numbers according to fixed procedures by hand and on paper: adding, subtracting, multiplying and dividing. Long division is the iconic image of this perspective.

Was that a relevant activity? Yes, very relevant. Before 1975 there were simply no electronic calculators available to the common user and almost everything had to be calculated by hand and with pen and paper. Every factory built in the reconstruction of society after WWII, every rocket shot to the moon, every conveyor belt production process was created by engineers who carried out complex calculations with pen and paper. But also, almost all retail transactions were based on manual calculations. However, the quantitative side of the world today looks very different and performing large calculations with only pen and paper has almost completely lost its relevance.

In the fourth quarter of last century (1975 – 2000) all kinds of mechanical and electronical tools made an appearance which made manual work for a large part superfluous: calculators, electronic cash registers, spreadsheets, et cetera became part and parcel of daily life activities. In education, the desire arose not only to learn the pen-and-paper skills, but also to teach where and how these skills could be used to solve problems from daily life. Situations from reality became part of education in many forms: applications, simulations, contexts, projects, et cetera. The emergence of Realistic Mathematics Education (Gravemeijer, 1994; van den Heuvel-Panhuizen, 2001) is an example of this perspective.

In the first quarter of this century (2000-2025), a third perspective on numeracy is rapidly gaining popularity worldwide. From this perspective students have to be "numerate" in order to deal with the quantitative side of the world. Numeracy takes the person and his/her relationship with the world as a starting point. The quantitative side of the world is so rich, so varied and sometimes so complex, that people need a very extensive repertoire to cope with it. From this perspective, numeracy is an inseparable part of personal development. Immediately after birth, the first interactions of the young born with numbers, patterns and structures in time and space, are discernible. Our brains are said to be hardwired to deal with numbers, structures and patterns (Butterworth, 1999; Dehaene, 2011; Devlin, 1996) The body supports numeracy development by moving and relating to the three-dimensional physical environment. And manifestation of mathematical structures are visible in all cultures (Bishop, 1991).

Interest in the psychological side of mathematics learning is also increasingly the subject of study. Many difficulties student have with mathematics and numeracy can be associated with psychological problems caused by educational settings of exclusion and selection and strict right/wrong regimes (see for example the literature on math anxiety (Dowker et al., 2016; Maloney et al., 2013)). It is good to realise that math anxiety is not a student characteristic, but rather an educational characteristic.

Part of numeracy is also how to deal with the avalanche of quantitative data that today's society produces and uses for economic traffic, the political process and daily life. It is mainly about drawing conclusions from numerical information. Interpreting, analysing, organizing, estimating, structuring, selecting and critically considering quantitative information are skills associated with numeracy. Appropriate education in this area is developing worldwide.

However, the development of a functional and modern conceptualization of numeracy is slow. In politics and the (social) media, the perspective from 1950-1975 is often still dominant.

What are then the aspects of the numerate world in the 21st century? In our search for the aspects of numerate behaviour in today's world, we came quick-and-dirty to the list of cognitions and manifestations in figure 3.

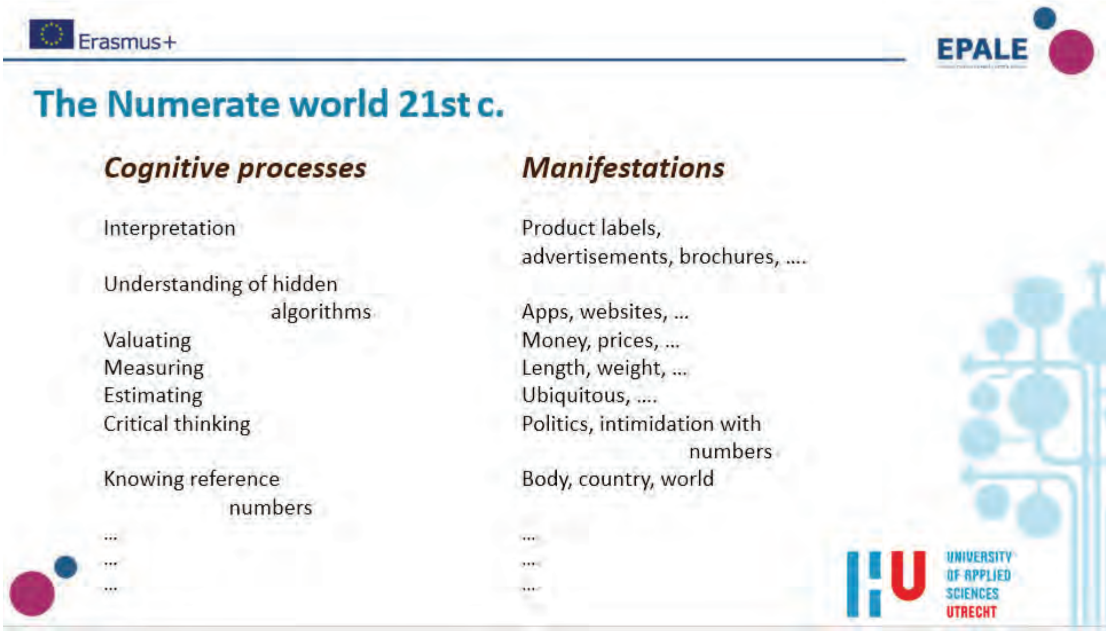

Figure 3: Cognitions and manifestations

A multifaceted framework

What are then these new basic skills needed to cope with living in contemporary society? Among researchers and practitioners there is a growing consensus that a numeracy framework which describes numerate behaviour and numeracy practices should contain much more than only content descriptions. As important are dispositions, attitudes, higher order skills, and aspects of agency, and self-efficacy.

In 2019, funded by the European Union, an Erasmus+ project started under the name Common European Numeracy Framework (CENF) to create an overview of the relevant aspects which matter in the quality of numerate behaviour of a citizen. This was based on a literature review on emergent themes in numeracy, a wide-scale European Numeracy Survey, and expert consultations. The main categories of aspects which were discerned, are: Content knowledge and skills, Context, Cognitive processes (especially higher order skills), and Dispositions. See figure 4.

In the CENF, for each category and subcategory (e.g., Quantity and Number, Self-efficacy, Mathematising) descriptions of observable numerate behaviour were developed.

The aspects which are categorised under cognitive processes and dispositions are used also quite often in educational research and literature in a generic way. They also appear in other kind of frameworks. There is, for instance, a substantial overlap of the category ‘Higher order skills’ with frameworks for 21st century skills (Csapó & Funke, 2017; Geiger et al., 2015; Hoogland et al., 2018; Voogt & Roblin, 2012)The aspects under disposition can also be found in literature on motivation in education and even more general literature on personal traits (Eccles, 1996). In this numeracy framework we have described these aspects in terms of numerate behaviour, or otherwise stated, in terms of how these aspects play out in numeracy situations.

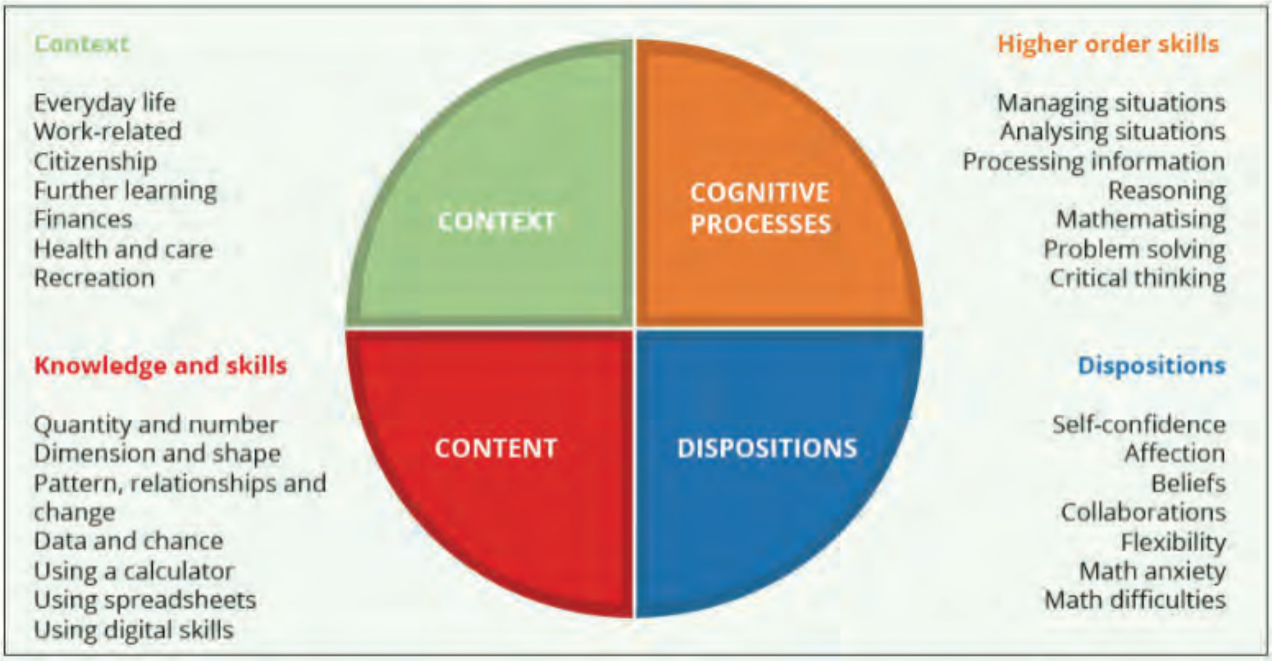

Figure 4: What matters in the quality of numerate behaviour?

Furthermore, the descriptors are formulated as a rubric with six levels, so that they may give indications for possible learning trajectories. By this, teachers and learners in adult education can establish together which (combinations of) aspects of numerate behaviour can be addressed to improve the quality of the learner's numerate behaviour.

The six levels are labelled X1, X2, Y1, Y2, Z1, and Z2. Roughly, the levels X refer to use of numeracy in daily-life house-hold situations, the levels Y refer to the numeracy activities of a citizen actively participating in nowadays societies, including being critical to numerate communication in news and social media, and the levels Z refer to the use of numeracy in professional settings, by users and producers of numerate communication. Clearly, there is an overlap in all these levels and activities, but the distinction is made to make it easier to have a focuses discussion on numeracy, necessary numeracy, developing numeracy skills, and improving the quality of numerate behaviour.

A few aspects highlighted

In the quadrant of content knowledge and skills, we see a reference to the well-known content categories as we can find in subsequent assessment frameworks of PISA and PIAAC (OECD, 2012, 2013, 2021a, 2021b; PIAAC Numeracy Expert Group, 2009). But added to that is the use of computational tools. The relevant and adequate use of computational tools is becoming more and more a basic skill, than the calculation the tool performs. There is a wide-spread narrative that pen-and-paper skills must proceed the development of tool use, but there is now evidence for the causality of that. Working together on using tools, understanding numbers and calculations, and establishing which basis facts should be part of the mental repertoire seems to be more an obvious choice for designing educational trajectories.

In the quadrant on dispositions, we want to highlight the aspect of math anxiety. We like to stress the fact that math anxiety is a typical school product, produced by endless feedback on right and wrong answers with corresponding feelings of stress and fear causing in many individuals a long-lasting urge to deflect any mathematical connotations, and sometimes any connotations to numbers. In adult mathematics education coping with math anxiety is one of the biggest challenges. Without school experiences there exists no math anxiety. Young children don't have anxieties in using numbers, quantities, measurements et cetera. It comes naturally to them. The brain and the body are hardwired for it. Maths is arguably originating from abstracted actions and gestures in concrete situations (Butterworth, 1999).

The rubrics are all first versions developed in recent years. The aim is to discuss them with experts, policy makers and practitioners to see whether the descriptions can be clarified, refined, and linked to educational numerate practice.

Discussion and conclusions

For adults to deal with the quantitative aspects of real life, a new set of basic skills is necessary. Limiting basic skills to only calculations and manipulation with variables and algebraic forms are clearly not sufficient anymore to deal with the abundance of mathematical and mathematical based manifestations. As long as education keeps focusing on a 19^{th} and 20^{th} century perspective of mastering the execution of calculations with pen and paper, adults will not gain the necessary competences for full participation in contemporary society.

The Common European Numeracy Framework as it is developed in an Erasmus+ project during the years 2018-2021 in close collaboration between European countries can contribute to a highly necessary shift in the definition of necessary basic skills for adults to cope with the problems in daily life and professions. The framework advocates a more holistic approach which intertwines knowledge and skills, context, higher order skills and dispositions.

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Adult numeracy – a paradigm shift?

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I present the ground-breaking theories about scientific revolutions by Thomas Kuhn and his development of those theories, particularly the concepts of incommensurability and lexical change. I review the long-standing theoretical discussion about adult numeracy and the extensive attempts to reach some agreement upon which to build a research domain alongside mathematics education research. I propose that Kuhn's conceptual analysis can be applied to the emergence of adult numeracy out of the field of mathematics education research to form a new paradigm of education research.

Keywords: Numeracy, paradigm, incommensurability.

Introduction

I will summarise the development of Thomas Kuhn's theory in the 30 years after the publication of *The Structure of Scientific Revolutions*, (1962) particularly in the works published as *The road since structure* (Kuhn, et al., 2002) with reference to fluid lexicons and incommensurability. I will use these philosophical concepts to critique the repeated attempts to explain and define adult numeracy, and the many contexts in which mathematics is found wanting in describing an adults' numerate accomplishments.

My main proposal is to apply Kuhn's analysis of a scientific revolution to the emergence of adult numeracy in the field of mathematics education research. Subsequently can I use the resultant research process to establish adult numeracy as a discipline in its own right and initiate further research along similar lines? The hope is that such research can produce a body of knowledge that will transform how the majority of the adult population views numeracy education and practice.

Kuhn, scientific revolution, paradigm change

The concept of the paradigm and paradigm change is central to Kuhn's argument. Unfortunately, the term 'paradigm' has been much over-used (see for example Rogers, 2002, p. 487) and so it is worth looking at what Kuhn said.

> The transition from a paradigm in crisis to a new one from which a new tradition of normal science can emerge is far from a cumulative process, one achieved by an articulation or extension of the old paradigm. Rather it is a reconstruction of the field from new fundamentals, a reconstruction that changes some of the field's most elementary theoretical generalizations as well as many of its paradigm methods and applications. (Kuhn, 1962, p. 84)

This makes us realize that that nature of the change involved is deep, significant and conceptually challenging. In proposing the theory Kuhn sees a very significant role for the growing dominance of a new paradigm making changes to the direction of future research. He says "But paradigm debates are not really about relative problem-solving ability . . . Instead, the issue is which paradigm should in the future guide research on problems" (Kuhn et al., 2002, p. 156).

Revisit Structure –lexicon – incommensurability

In his later work, revisiting the theory of scientific revolutions and responding to the very many criticisms that were made of ‘Structure’ Kuhn puts greater emphasis on the use of language and the significance of a research community sharing ‘a core’. The introduction of two additional concepts, lexical change and incommensurability, make the theory even more applicable to the emergence of adult numeracy as a new research field, as I will show.

> People who share a core, like those who share a lexical structure, can understand each other, communicate about their differences, and so on. If on the other hand, core or lexical structures differ, then what appears to be disagreement about fact (which kind does a particular term belong to?) proves to be incomprehension (the two are using the same name for different kinds). The would-be communicants have encountered incommensurability, and communication breaks down in an especially frustrating way. . .. What the participants in communication fail to share is not so much belief as a common culture. (Kuhn et al., 2002, p. 239-240)

Kuhn later clarifies what he means by a lexicon.

> A lexicon or lexical structure is the long-term product of tribal experience in the natural and social worlds, but its logical status, like that of word meanings in general, is that of convention. Each lexicon makes possible a corresponding form of life within which the truth or falsity of propositions may be both claimed and rationally justified, but the justification of lexicons or of lexical change can only be pragmatic. (Kuhn et al., 2002, p. 244)

This theoretical approach begins to give us a vocabulary and philosophical approach to explore and explain the gulf that has opened up between adult numeracy education researchers and practitioners and others in the mathematics education world.

Revisiting numeracy – again

I have been involved in examining how numeracy has been used and defined in the particular context of Adults Learning Mathematics for twenty-five years and give for reference two conference proceedings. (Kaye, 2003 and Kaye, 2015)

This is continuing a discussion that takes place in every conference about teaching adults' numeracy (and mathematics). Making a strong connection with the tradition we are making in CERME here is a view of the situation by Kees Hoogland in his introduction to the papers presented at CERME 11. This both gives an overview and identifies some of the problems.

> Although there is a broad acknowledgment that an array of psychological and sociological factors are important in (the results of) adult education, there is not yet a well-researched set of examples how in practice this can be taken into account in a more systematic and effective way. The practice of adult numeracy education is still a plethora of different content descriptions and goals that vary from very back-to-the-basics to very sophisticated higher-order skills. . .. The twin goals of trying to establish a firm definition of numeracy which acknowledges the multifaceted nature of adult’s mathematical practices while identifying consistent teaching approaches should be of concern to the global AME community. (Hoogland et al., 2019, pp. 1298/9)

Two recent summaries published in 2015 and 2020 in the ZDM mathematical journal review recent research into numeracy. In the introductory paper by Geiger et al (2015) various statements about what numeracy is (or is not) are presented.

> Although what is meant by numeracy varies between countries, it is now broadly accepted that being numerate extends beyond the mastery of basic arithmetic skills to how to connect the mathematics learnt in formal situations, such as school classrooms, to real world problems. (p. 531)

> While there is an increasing focus on numeracy internationally, there is not yet a widely accepted definition for this construct or of the fundamental characteristics that describe this idea. Thus, the meaning of numeracy still varies widely across international borders . . . (p. 532)

This is followed by a section headed ‘Facets of numeracy’

> The intrinsic usefulness of mathematics means that it provides a way of reasoning about, and functioning within, different societies with diverse social norms, values, cultures, and traditions. Unsurprisingly, this gives rise to related but distinct ways of conceptualising and identifying numeracy practices. (p. 535)

We see that numeracy is described as both an ‘idea’ and a ‘construct’; we see repeatedly stated that there is no agreement on what ‘numeracy’ means; and we find that ‘mathematics’ creeps into the descriptions, without being defined.

Another useful summary is the survey paper by Gal, Grotlüschen, Tout and Kaiser (2020). The title makes clear it has a focus on adults ‘Numeracy, adult education, and vulnerable adults: a critical view of a neglected field’ and provides its own definition of numeracy.

> Adult numeracy is a construct related to the ways people cope with the many mathematical, quantitative, and statistical demands of adult life. Some definitions of *numeracy* emphasize basic computational skills or focus on emergent numeracy at a young age. In this paper, however, numeracy is used broadly to encompass a set of diverse skills, knowledge-bases, dispositions and affect, communication abilities, and practices and behaviors, that range from simple to very advanced, relate to mathematics and statistics, and that individuals need or use in order to engage and manage diverse life situations and tasks in the adult world. (Gal et al., 2020, p. 378)

Many similar definitions have been given over the years. Other researchers have looked at mathematics in other contexts and chosen alternatives to ‘adult numeracy’ such as ethnomathematics and indigenous mathematics These fields have seen the debates around ‘numeracy’ as irrelevant (or a distraction) and have sought in other ways, and for other purposes, to redefine ‘mathematics’ and ‘mathematics education’. The use of these terms is part of the lexical change I am drawing attention to.

Ethnomathematics is summarized in this introduction to the topic group on ethnomathematics at ICME 13 (2016)

> The application of ethnomathematical approaches allows us the opportunity to examine local knowledge systems and give insight into forms of mathematics used in diverse contexts and cultural groups. The pedagogical approach that connects this diversity of mathematics is best

represented by a process of translation and elaboration of the problems and questions taken from daily phenomena.

In addition, I also wish to reference the works of Gelsa Knjnick (2021) with the Landless Movement in Brazil and that of Marta Civil (2021) with parents from mainly Mexican origin in the USA.

These studies and investigations are often left out of the discussion around adult numeracy. They focus on presenting and defending the mathematics within particular cultures, which along with all other social, political, and cultural traditions and values have been suppressed. These need to be included as they too are trying to redefine mathematics education.

The mathematics of indigenous people needs to be noted too. One particularly powerful article is 'Mathematix: 'Towards a way of being' by Rochelle Gutierrez (2019) which presents the case of the mathematics (mathematx) of the indigenous peoples of North America.

> In this paper, I seek to analyze some of the specific ways that Western mathematics in US schools operates as a form of dispossession and how mathematx addresses the need for Indigenous people to continue to remake themselves while interacting with non-Indigenous people. (p. 67)

Gutierrez later gives an example in which the relationship to, and naming of, geometric shapes is very different to the Western tradition.

> For example, the word for square literally means "it squares itself" which is an action performed rather than an enduring property of the object. Ardoch Algonquin First Nation member and Anishinaabemowin teacher Marjolaine LaPointe recounts, I was thinking of this word, kakadeyaa, which means a square, but it's not a noun. It's a verb which means it's squaring itself. And kakadeyaa has this k sound, which is a cutting, a grouping, a separating, and when we break the language down into those sound-based parts, we start thinking in three dimensions as opposed to binaries of "this is this and this is that" and we see relationships between many things. (pp. 73-74)

There are numerous other examples drawing on indigenous numeracy (or 'mathematics') making a case to have a much wider understanding of what 'mathematics' might be. For example, the work of Linda Furuto (2018) on the traditions of Hawai'I and the Pacific Also the work of Marcia Ascher (2005), which deals with similar indigenous concepts outside of the education sector. And not forgetting the trail-blazing publication by Claudia Zaslavsky's 'Africa Counts' (1979) originally published in 1973.

What about mathematics?

As I have shown in this paper the concept, principle, practice and justification for 'numeracy' has been repeated many, many times. However, whilst 'numeracy' is explained in some way as doing something with mathematical concepts (or skills) 'mathematics' is very rarely defined. As Coben (2006, p. 21) has said "The nature of the relationship between numeracy and mathematics is elusive". Coben (2006, p. 21) noted that "'Mathematics' is taken to mean mathematics learned and taught at any level, including the most basic". Anecdotally, I re-call having a discussion about this with a group of trainee adult numeracy teachers and one contribution was (as I remember it) "Everyone knows what mathematics is – there is a degree in it". I have been thinking about this ever since and have previously given examples of statements about mathematics, alongside those of numeracy, in a guide to teachers of adult numeracy (Kaye, 2013).

Numeracy, incommensurability and lexical change

I now return to the ideas of incommensurability and lexical change to argue that these need to be applied to using numeracy in the context of both adult education and people's lives. As has been shown there is a very strong research tradition that is comfortable with recognizing the importance of numeracy in adult education and many of those studies have arisen because of concerns about 'poor numeracy' in the adult population (O'Donoghue, 2000, p. 103). In fact, these concerns go back for about 60 years. All reviews of using numeracy speak of the Cockroft report published in 1982, which in turn refers back to the Crowther Report published in 1959. Within paragraphs 35 to 39 of the Cockcroft report there is a discussion about the scope and range of 'numeracy'. As is the case in many official and research documents (on mathematics education) the target group is school children. However, as I have argued elsewhere (Kaye, 2018, pp. 11-37) once societal, cultural and political aspects of life are under consideration this is far more relevant to adult or lifelong learning, than school mathematics. What is important is that the argument is still the same between a 'small numeracy' or a 'big numeracy'. In the words of the Cockroft report (1982) (emphasis in the original report):

> *We would wish the word 'numerate' to imply the possession of two attributes*. The first of these is an 'at-homeness with numbers and an ability to make use of mathematical skills . . . The second is an ability to have some appreciation and understanding of information which is presented in mathematical terms, for instance in graphs, charts or tables or by reference to percentage increase or decrease. . . . *Our concern is that those who set out to make their pupils 'numerate' should pay attention to the wider aspects of numeracy and not be content merely to develop the skill of computation. (*Cockcroft, 1982, para. 39 p. 11)

The same discussion has been taking place ever since.

If the (adult) numeracy research community has been trying to fit numeracy practices into mathematics education and cannot find a 'good fit' then I think we have a conceptual problem of incommensurability. With a broad view of adult numeracy encompassing people's lives, including their work and cultural experiences, coming from every educational, class and ethnic background there is no common lexicon with a mathematics education built on the foundations of pure mathematics finding truth through logic and proof.

A new paradigm

There are indications that some researchers over the last twenty years have become aware of the need to question the current research domains and look for developments that establish what I consider is a new research paradigm.

In Jablonka's (2003, p. 87) section on 'Mathematical literacy for environmental awareness', a discussion about the fundamental nature of mathematical knowledge is introduced. It concludes "Consequently it is argued that mathematical literacy involves an attempt at changing the perception of mathematics towards a more human view in the hope that this may eventually even lead to the development of new forms of mathematics".

Coben (2006) published the article ‘What is specific about research in adult numeracy and mathematics education? She introduced a ‘professional knowledge domain’ and expressed her view of the current state of research.

> Research in adult numeracy and mathematics teaching and learning is still in the exploratory phase of development. There is much conceptual uncertainty, manifested in shifting relationships between numeracy and mathematics and between numeracy and literacy, and reflecting similarly shifting relationships between these areas in educational practice. (Coben, 2006, p. 29)

In a later work on numeracy (and mathematical literacy) curricula Jablonka (2015) expressed a great deal of concern about the nature of assessment items used to measure adults’ levels of numeracy. Jablonka critiques how mathematics is contextualisesd in the numeracy/mathematical literacy assessments.

> The incorporation of N/ML activities into mathematics curricula that specify mathematical knowledge in terms of generalisation and specialisation of *mathematical* meanings entails a recontextualisation that brings about a shift of criteria for what counts as an accomplishment of a task. This needs to be recognised in any attempt of inserting N/ ML activities into a *mathematics* curriculum (Jablonka, 2015, p. 606/7)

The reference to N/ML is to Numeracy/Mathematical Literacy and the italics is in the original. This again shows that trying to talk about adult numeracy in terms of ‘mathematical meaning’ brings about considerable confusion, most of which arises from ‘context’ in its broadest sense. In some ways it is not surprising as everything we have looked at so far emphasises that adult numeracy and numeracy activities and practices contain or are defined by ‘context’. Mathematics and traditional ‘mathematics education’ is considered context free. They are inconmenesurable as previously discussed.

Conclusion

This report presents a theoretical approach to research into adult numeracy education and practice. It uses the philosophical approach of Thomas Kuhn of the paradigm change, first described in the Structure of the Scientific Revolution (published in 1962) and subsequently developed over the next 30 years. The introduction of concepts like incommensurability and lexicon fluidity allowed this powerful analytical tool to be applied to the ongoing debate about adult numeracy. Combining this theoretical approach with a well-informed account of defining adult numeracy I have made the claim that adult numeracy research represents a paradigm change in the context of mathematical education research. There have been repeated attempts to explain and explore adult numeracy with the incommensurable lexicon of pure mathematics and its related educational processes, which have failed.

As Kuhn (2002, pp. 97-99) himself says this can only become really apparent if future research recognises the lexical divergence and with more and more researchers (and practitioners) using the new taxonomy, a new branch of education research evolves as does a new species through Darwinian evolution. A new habitat is defined where adult numeracy can grow freely.

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Financial literacy and numeracy: exploring the skills and knowledge needed to survive in complex financial systems

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Research shows that even before the pandemic many people were financially vulnerable, in the sense that they were struggling to 'make ends meet' relying on loans to cover even day-to-day costs. This paper outlines literature research undertaken to support a European funded project, called Money Matters, that aims to develop resources to help improve the financial literacy skills and knowledge of families defined as disadvantaged or financially vulnerable. Developing financial literacy skills, knowledge and behaviours is identified as one way to support families who feel insecure about their financial situations. The paper examines what is meant by financial literacy, the knowledge, skills, and pedagogies, and identifies a considerable range of factors related to financial vulnerability and points to the challenges of developing the wide range of necessary skills and knowledge, including numeracy, that families need to survive in an economy with complex financial tools on limited resources.

Key words: Financial literacy, financial vulnerability, family learning, money, financial numeracy.

Introduction

This research supports a project funded through Erasmus + called Money Matters, that seeks to help disadvantaged families develop their financial literacy skills and knowledge. Building on a variety of skills from 6 partner organisations across Europe it aims to develop the financial literacy skills of a network of educators trained to use a pedagogic approach that includes the whole family utilising both online games and apps as well as comic book resources to help achieve greater financial wellbeing.

Developing financial literacy skills, knowledge and behaviours are promoted by many international organisations to support people in overcoming financial disadvantage. Producing resources to support families identified as vulnerable in the development of their financial literacy skills requires the combination of many different sets of knowledge, skills and related pedagogies including financial literacy, numeracy and literacy skills, family learning approaches and digital technologies developments. This paper limits its focus to exploring the notion of financial literacy, financial vulnerability as well as approaches taken in current financial education in the UK aimed at families. The research explores the role of numeracy and how it contributes to the development of financial literacy skills and knowledge. It moves on to consider the challenge of developing the wide range of necessary skills and knowledge, including numeracy, needed by those experiencing financial vulnerability and challenges in their everyday lives.

Financial skills, knowledge, and behaviours

Many countries and international organisations promote the development of people's financial skills and knowledge as important for inclusion in society and as a way of countering financial vulnerability, thus encapsulating the twin aims of empowering citizens but also contributing to the overall stability of current financial systems (OECD/INEF, 2020). When identifying essential financial skills, the Organisation for Economic Development International Network for Financial Education (OECD/INEF) and the European Union (EU), tend to use the term financial literacy, while the United Kingdom (UK) and the World Bank (2013) use the term financial capability, although often these terms seem to be used interchangeably.

Essentially both financial capability and financial literacy are similar in their range of characteristics and look at the skills and knowledge needed by individuals to function within Western economies. By 2018 both the OECD/INEF, the EU and the UK include wellbeing as a key aspect of financial literacy and consider how to develop the resilience to achieve this. Arguing improving financial literacy is pertinent during times of economic volatility, suggesting financial resilience consists of developing

> '…a financial cushion, coping with a financial shortfall and stress, and behavioural traits promoting long-term planning and saving, keeping control over money, taking care with expenditure, and avoiding financial fraud' (OECD/INEF, 2020, p. 7)

However, Storchi and Johnson (2016) argue that developing financial wellbeing is not simply about improving how people access financial services and support or identifying 'optimal behaviours'. Rather they posit people in different social, cultural, and economic environments may have different financial and wellbeing goals. Hence people may access the same resources but use them in different ways, reflecting their own notions of wellbeing. Given that nearly half of the population in the UK are benefit dependent or on low incomes (Money Advice Service, 2016) their notions of wellbeing may be very different to someone who has the resources to consider long term investments.

Storchi and Johnson (2016) also argue that a simple lack of money may affect people's decisions more than lack of financial knowledge.

What is financial vulnerability?

Research shows that significant numbers of people were already struggling financially before the COVID-19 pandemic and now, given the impact on the world economy, are under even more financial pressure (Lusardi, Hasler, & Yakoboski, 2020; OECD/INEF, 2020; Social Metrics Commission, 2020). There are many ways to describe people in difficult financial situations who are 'struggling to make ends meet' and experiencing feelings of insecurity relating to debt and savings. In this article we use the term financial vulnerability termed by Gal, Grotlüschen, Tout and Kaiser (2020) to describe people who might be identified as financially fragile, financially stressed, squeezed, or struggling.

When families who are already financially vulnerable are caught up in unexpected single negative events, such as the loss of employment or being furloughed, they can be left in difficult positions with a limited number of options available to them to help overcome their problems. Certainly, the COVID-19 pandemic could be identified as one such significant negative event with long term implications for many families who have little or no savings or 'back up money' (Kelly, Smith & Kaye, 2022) to support them during this time. While debt consolidation and restructuring may be one option, more often other more precarious choices are used including short-term ‘pay day’ loans with excessive interest rates or the overuse of credit cards with minimum repayments (Gathergood, 2012).

Some countries do have financial organisations that are committed to working with families in debt, to help plan a way out of distressful situations (Lusardi, Hasler & Yakoboski, 2020; Financial Capability Lab, 2018). However, research also points out that financially vulnerable families have little awareness, or confidence, in accessing financial organisations and support that is available to them. Hence families with low levels of financial literacy, who are experiencing debt, struggle to get out of the cycle of debt (Financial Capability Lab, May 2018), reinforcing the notion of a financial environment that appears to provide limited effective support that might help overcome vulnerability.

Factors influencing financial vulnerability

Psychological research tends to focus financial literacy more on individual personality types, linking those with problems of self-control to the use of quick access but high-cost solutions such as ‘pay day’ loans (Gathergood, 2012). Exploring the wider notion of vulnerability Gal, Grotlüschen, Tout and Kaiser (2020) identify that financial vulnerability is linked to debt and identify numeracy skills as key to understanding complex financial decision making.

The OECD (2013) found that in many countries, women displayed lower financial knowledge than men and identified as being less confident in their financial knowledge. The report argues that barriers to women developing financial literacy are often linked to their socio-economic position in society and the different social norms around finance expected between men and women. However, women tend to live longer than men but on average earn less which suggests deeper structural barriers within societies that women experience. Hence Gal, Grotlüschen, Tout and Kaiser (2020) point to gender as important issue when researching financial literacy and vulnerability. Further research identifies other factors linking lower levels of financial literacy to financial vulnerability, including being ‘young, a single parent, in poor health, unemployed and with low income’ (West & Worthington, 2018, p 331), having a disability (Lusardi, Hasler &Yakoboski, 2020), as well as age and limited digital capabilities (OECD/INEF, 2020). The number of factors identified in relation to financial vulnerability points to the need for a variety of approaches to develop relevant and useful financial skills and knowledge.

Financial vulnerability is a global issue. Research by the OECD/INEF (2020) found that 42% of individuals responding to their survey, across 26 countries from Asia, Europe, and Latin America, felt ‘financially stressed’, in that they worry about meeting their everyday living expenses. Lusardi, Hasler and Yakoboski (2020) point to 27% of the US population being ‘financially fragile’, in that they would have difficulty ‘coming up with $2000 in an emergency, and this was in January 2020 before the pandemic hit. The Money Advice Service (2016) report indicates that well before the 2020 COVID crisis, nearly half of the population of the UK were either on very low incomes, benefit dependant or heavily reliant on credit. This ‘squeezed’ group are often working nonetheless have ‘significant financial commitments but little provision for coping with sudden changes such as unexpected bills, being made redundant, unexpected medical costs.

Baron’s (2015) research with marginalised communities in Canada reinforces the notion of increased financial vulnerability amongst the most disadvantaged in society. She argues living in communities where there is a low median income, where unemployment is common, income is low and there is little surplus available for long-term savings or investments as contributing to financial vulnerability. A financial environment where temporary contracts and low levels of pay are common, such as in the UK, also contributes to in-work poverty, meaning many families struggle to make ends meet relying on benefits and food banks to get by (Beatty, Bennet & Hawkins, 2021). When considering financial education for disadvantaged groups the notion of putting aside money for savings and long-term investments may be relevant but less of a day-to-day priority.

Financial education-developing financial skills and knowledge

Still the OECD/INEF argues that the main factor hindering possible resolutions of family debt is the lack of key financial literacy skills. Focusing on individuals' behaviours Lusardi (2019) argues financial education is needed to help people counteract ineffective spending and expensive borrowing by learning about financial planning and debt management. Lusardi, Hasler and Yakoboski (2020) recognise that people who are financially vulnerable are hardest hit during economic crises yet they still argue that financial education is important for the well–being of individuals as well as the whole of society. Hence it should be provided in schools, workplaces and in the community, targeting different age groups related to different financial imperatives. However, Willis (2008) posits the issue of financial vulnerability is more systemic and not easily solved through financial education courses. He identifies the current economic environment as made up of complex financial products and services that are constantly changing, making financial vulnerability a consequence of such financial systems rather than an individual problem of skills deficit and behaviour.

While Willis (2008) continues to see little value in financial education in schools, others argue the notion of financial literacy development in later life is less effective pointing to research that shows most people's financial behaviours and habits are developed while very young and become fixed (Whitebread and Bingham, 2013). However, Mundy (2011) posits that 'financial teachable moments' happen at particular times throughout life when big changes happen, such as when a baby is expected, going to college, getting married or divorced. Kelly, Moulton, and Stone (2014) identified one such 'teachable moment' when a significant change occurred in the UK benefit system during the introduction of Universal Credit. Adults experienced a dramatic change in their benefit payments from weekly cash payments to monthly online payments, requiring claimants to develop digital as well as financial skills to access and manage their money.

Financial literacy, numeracy skills and family learning

Researchers (Baron, 2015; Skagerlund, Lind, Strömbäck, Tinghög and Västfjäll, 2018) argue that mathematics is the driving force behind becoming financially literate and has a dual effect on the population's skill, raising both mathematical skills and knowledge and financial literacy levels.

Research also suggests the focus and reasons for developing financial and numeracy skills change over time. In families with younger children, Skwarchuk, Sowinski, and LeFevre (2014) identify both formal and non-formal activities in the home as important because they develop different aspects of literacy and numeracy skills. In numeracy, formal activities tended to develop symbolic skills such as number skills, whereas informal activities such as games and cooking tended to influence attitudes towards mathematics activities. When developing mathematical skills and attitudes in younger children Ramani and Siegler (2015) highlight the importance of the social context of home learning and the contribution informal activities, such as cooking and shopping, make to acquiring new skills, such as helping to develop a sense of size, weight, volume and even ratio. Whitebread and Bingham (2013) also posit that young children respond to social and emotional interactions with parents, grandparents and carers and enjoy joining in with adult financial practices such as shopping, where they can be given responsibilities for spending and budgeting. They see learning within the family as key in the development of younger children's financial 'habits of mind' arguing they develop 'executive functions' of mind that influence more specific financial understandings which come later. They posit that 2- to 3-year-olds can be encouraged to start counting, comparing sizes and quantities helping them develop language. By 7 years old they start to have a sense of 'value' and equivalences so, for example, they can understand that a small 5 pence piece is equivalent to 5 big pennies, so value is not indicated by the size of the coin. With older children it might be useful to link discussions around income, expenditure, and affordability to the impact of purchases on wider society and the environment. Where price can relate to where the item is made, working conditions of those producing the items and the cost to the environment in the making and transportation of the item. Gal, Grotlüschen, Tout and Kaiser (2020) point to the need for good numeracy skills to support financial skills later in life in order to understand credit, compare utility bill offers and undertake more complex longer term financial planning needed for retirement or taxation. Indeed, they argue that the term financial numeracy should be used rather than financial literacy to emphasise the importance of those underpinning skills.

But the Money Advice Service (2016) points to a wider range of skills that enable or inhibit financial well-being, these include financial confidence, digital engagement, and the ability to seek advice and guidance. Two current large scale financial education programmes aimed at supporting families in the UK focus on the psychological and emotional aspects of finances highlighting the importance of talking about money. They also reason those early financial habits learned from parents influence children in their attitudes to money later in life. (Campaign for Learning, 2021; Made of Money, 2021).

Marta Civil (2016) points to the importance of ensuring learning is centred on families' lives and their lived experiences. In her work with excluded people in US society, she argues it is important to view the parents not just as adult learners but also as 'funds of knowledge'. Using this approach Civil posits teaching and learning should focus 'our attention on the culturally situated everyday practices of the learners. It should be grounded in a dynamic view of culture as lived experiences' (2016, p 46) hence financial education needs to start from what the learners view as important in their daily lives, which may differ from that of the educator or trainer. This awareness is important when developing resources with diverse families who may be identified as financially disadvantaged or vulnerable.

Conclusions

The notion of financial vulnerability can vary greatly in different contexts but encompasses individuals and families who are struggling to make ends meet financially, and who have little 'back-up money' or no safety net when confronted with a loss of income or a new necessary expense. Research into financial vulnerability identifies a wide range of influential factors including psychological dispositions such as lack of self-confidence or self-control, while others point to vulnerability linked to gender, age, income, and disability. Further research considers the financial context as a significant contributing factor such as belonging to local marginalised communities while others point to the complexity of the wider social and economic systems that people find increasingly difficult to navigate. More research into what is most important to different groups identified as disadvantaged is essential.

The US and the UK, two so-called developed economies, have at least a quarter of their populations in financial stress. The dominant thinking across many western countries is that improving a family's financial literacy skills and knowledge will enable them to reduce debt and increase their long-term financial security. While helping people understand the choice and methods of calculations of financial transactions in an economic system can be helpful, the research points to the difficulties of following orthodoxies of saving money for future negative events if families find themselves on low incomes and unable or unwilling to seek financial advice available.

Research also points to istrong links between financial literacy and numeracy skills, however current online teaching programmes in the UK tend to focus on psychological and emotional relationships with money. Research into marginalised communities highlights the need to develop resources in the Money Matters project that are relevant to people's lives, starting from the learners' input and experiences, which may or may not necessarily focus on all the financial literacy advice currently promoted through dominant orthodoxies. This requires the teaching and learning approaches used in the resources to be flexible and respond to learners' priorities while still enabling participants to develop a better understanding of their current financial situation and possibilities for the future.

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Democracy and mathematics instruction for adults - three philosophical perspectives

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Some might view mathematics education and democracy as two distinct arenas without mutual relations. Contrary to such an understanding, this paper presents part of a larger study on potential relations between the two. More specifically, I explore relations between democracy and mathematics teaching or instruction for adults. The paper seeks to identify philosophical underpinnings and inspirations for describing and understanding core relations between democracy and mathematics education. I present snapshots from the works of three prominent scholars: theDane N.F.S. Grundtvig's philosophy of lifelong learning; the German Jürgen Habermas' theory ofcommunicative action; and the German Hartmut Rosa's concepts of resonance and world relation. Highlighting how these theories might offer new perspectives of how mathematics education can support democracy? The perspectives differ in their depth and breadth but share certain similarities.

Keywords: Mathematics instruction, democracy, educational environment, adult education.

Setting the scene

In my view, Nordic understandings of mathematics education for democracy that emerged around 1990 (Mellin-Olsen 1987, Niss 1994, Skovsmose 1990) remain relevant today. I believe it is time that these issues be placed high on the agenda when reflecting upon contemporary classrooms for adults and to revisit and more deeply examine the relationship between democracy and mathematics education, when exploring both contemporary research and current instructional practices in mathematics classrooms.

First, I will revisit and reiterate a double-sided view of what constitutes democracy. I argue that democracy understood in its formal sense and as a steering mechanism must be supplemented by lived democracy as comprised by people's participation and communicative interactions in and across the many levels and spheres of society. Back at ALM8, Paola Valero and I termed these two forms of democracy Democracy 1 and Democracy 2 (Lindenskov & Valero, 2002). Second, I wish to incorporate democratic reflections on the questions of what should be taught, who should be taught, and how it should be taught within mathematics education for adults?

I have combined these questions in a two-dimensional matrix model (see Figure 1) that I first presented at ALM28 (Lindenskov, 2021). The first dimension concerns formal representative democracy as a form of rule contra informal democracy in the form of everyday participation and communication. The second dimension highlights the questions of what mathematics to teach, to whom, and how. I developed the model as part of a collective effort with colleagues at the Department of Educational Theory and Curriculum Studies at the Danish School of Education. At our annual working seminar in 2017, we decided to write an anthology on democracy and 'subject didactics' [German: Fachdidaktik] (Lindenskov, 2020).

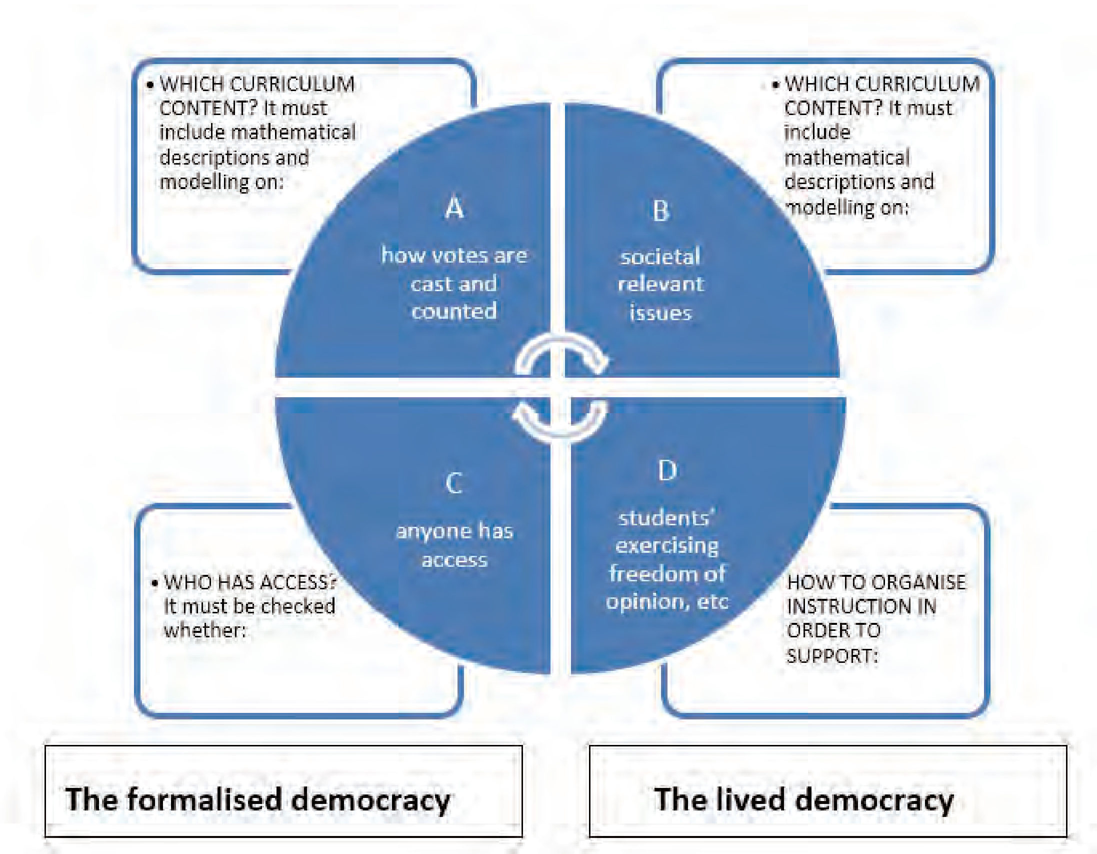

Figure 1: Matrix model for mathematics/numeracy instruction and democracy

Besides revisiting the previously mentioned Nordic understandings of mathematics education for democracy from around 1990, I re-examined my own recent analyses of Danish mathematics curricula and related public debate (Lindenskov 2018, 2019). I became convinced that the previous focus on democracy within critical studies of mathematics education has currently been replaced by other themes. This change of focus is clear in Yasukawa et al. (2018), where the thirteen contributions on numeracy as social practice make little mention of democracy. I noted that democracy is only addressed as a theme in two of the contributions – one from South Africa and one from India – but is otherwise absent. Nevertheless, the publication distils several issues contributing to what I term 'lived democracy'. In addition, democracy is a theme in the 'social injustice' framework developed by Berry III et al. (2020), exemplified by activities concerning the use of algorithms in formal voting systems.

The bottom left-hand corner of Figure 1 includes reflections on who has the right to receive high-quality mathematical education. It is essential in democracy that anyone has rights and access to acquire appropriate mathematic expertise and authority. However, history – and the present day – provides many examples of unequal access to education and unequal learning opportunities. On the right-hand side, the model presents democracy as a 'way of life' rather than a form of rule (Jakobsen, 2010). At the top right-hand corner, you see how democracy 'is done' daily in the mass media (newspapers, Radio, TV), via social media or direct person-to-person dialogues (written or oral). Much of the background information supporting political decisions at the local, national, and international levels includes numerical measures and the results of mathematical problem-solving and modelling.

The bottom right-hand corner of Figure 1 focuses on the questions of whether and how mathematics lessons constitute lived democracy. What characterizes the educational environment for mathematics and numeracy learning? Are mathematics classrooms a site for lived democracy, where students and teachers exercise freedom of opinion, discuss openly, demonstrate critical faculties, and adhere to principles of tolerance, trust, and justice? And do all students have the right to expect to bevalued equally and to experience mathematics as meaningful?

One example of a mathematical activity relating to formalised democracy illustrates part A in Figure 1 and how votes are cast and counted in different democratic voting systems. Figure 2 and 3 allow learners to compare the effects of different electoral algorithms. While figure 2 shows the representation of the various political parties in the Danish Parliament applying the algorithm used in Danish parliamentary elections, figure 3 illustrates how things would look applying a first-past-the-post algorithm such as those found in Great Britain and the USA.

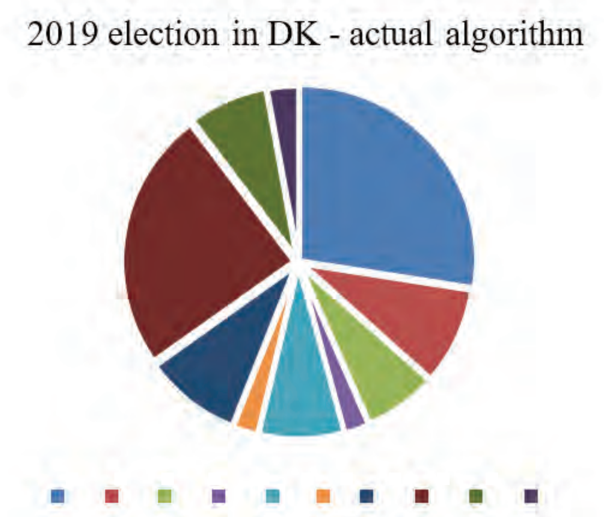

Figure 2. Actual party representatives in the Danish Parliament following the 2019 election

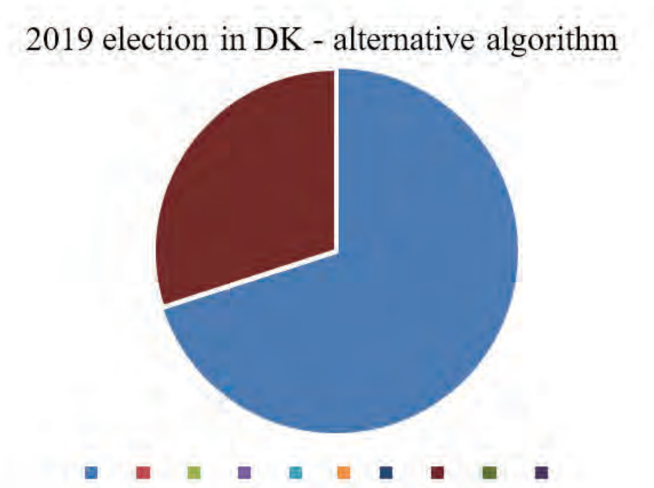

Figure 3: Hypothetical partyrepresentatives in the Danish Parliament following the 2019 election if applying a first-past-the-post algorithm

In the following, I explore possible philosophical underpinnings to help clarify the model from Figure 1 and develop tools for analysing curriculum materials, classroom activities and classroom life in relation to democracy.

N.F.S. Grundtvig – Lifelong learning

Focusing on adult education implies a concern for lifelong learning. For me as a Dane, the scholar N.F.S. Grundtvig (1783-1872) is unavoidable when considering such matters as one of the founders of adult education institutions in Denmark, termed ‘folk high schools’ (Broadbridge et al, 2011). Grundtvig is also considered an important figure in other countries. Many American scholars acknowledge the inspiration that Malcolm Knowles(1913-1997), who is widely considered the father of adult learning theory in the US, drew from Grundtvig (Warren, 1989). Furthermore, a European Union programme 2000 – 2013 to support adult learning was former named after Grundtvig, forming a branch of the Socrates programmes.

In addition, a branch of the Danish Lutheran Church is named after Grundtvig, standing in opposition to Pietistic Lutheranism. As a priest, Grundtvig wrote more than1500 hymns. For this presentation on education and democracy, the most important thing to note is, firstly, Grundtvig’s educational philosophy with its goal of making education accessible to all adults. Secondly, it is notable that Grundtvig influenced the shift from autocracy to formal democracy in Denmark, with the first democratic constitution of 1849 giving approximately 15% of the population the right to vote (Korsgaard, 2014).

All legislation governing adult education in Denmark have included the aim of empowering adults and encouraging them to participate in democracy and of offering instruction that supports participants’ active involvement. Nevertheless, as the Danish scholar Gunhild Nissen has pointed out, Grundtvig viewed mathematics with scepticism. Grundtvig used wordings as mathematics ‘neither cared for life nor for death’. The ‘folk high schools’, he argued, should be ‘schools for life’ and should offer courses that would empower ordinary people. Grundtvig refused to give mathematics teaching a place in his folk high schools. Grundtvig likewise advocated to deny giving mathematics teaching a place in primary education, convinced that mathematics with his wording would give children ‘ink in their veins instead of blood’ (Nissen, 1991).Taking a more optimistic view, Nissen drew on Habermas' concepts of the system and lifeworld to support her hope that,in the future, mathematics would not only be a tool of the system and part of its structural foundation but could be integrated within the communicative contexts that build and expand the lifeworld (Nissen, 1992). To summarize, Grundtvig espoused everyone’s right to education (part C in Figure 1) and the importance of making lived democracy part of education (parts B and D in Figure 1). Even today, his scepticism as to whether mathematics can contribute positively to the lives of ordinary peopleshould be taken seriously as a topic for discussion and reflection.

Jürgen Habermas – Communicative actions

Jürgen Habermas, born 1929, is a German philosopher and sociologist who has played a highly influential role in the development of critical theory. In the book *The Theory of Communicative Action* (1984, 1987), he outlines the elements constituting what he called ‘communicative actions’ as opposed to ‘strategic actions’. While strategic action can be considered successful when those involved achieve their individual goals, successful communicative action is when a consensus is freely reached on reasonable goals that can be achieved through cooperation. Communicative action involves a shared effort to reach a rationally motivated consensus through the medium of language. But in which ways can such consensus be achieved? Habermas has analysed the conditions for successful speech acts leading to mutual understanding. Sometimes, the listeners may immediately be able to sense whether they consider the speech act acceptable and their reasons for accepting or rejecting the proposed goal.

In other cases, there may be a need for explicit reasoning supported by empirical claims, appeals to a sense of moral or ethical good, assertions of authenticity, personal sincerity, or aesthetic value.

I find it interesting that Habermas' communicative actions are used by Jenny Cramer and Christine Knipping in their analyses of student-teacher interactions in determining whether particular mathematical results are correct (Cramer & Klipping, 2018): How do communicative actions and strategic actions between teacher and students and among students play out? Which communicative actions for reaching a consensus includes that students express and interpret mathematical claims and other types of claims? Which strategic actions are convincing other students? Cramer and Knipping focused on discourse ethics, and language, included in students' claims and discussions in mathematics classrooms. Especially Habermas' rules of discourse ethics were found to offer a range of explanations for obstacles that may hinder student participation in discussion. To summarize, Habermas' specific focus on different kinds of communicative acts provides us with analytical tools for acknowledging distinct democratic facets when organizing mathematics instruction (as in part D of Figure1). In my view, Cramer and Knipping's use of Habermas' concepts has rich potential for studying andsupporting democratic learning processes in mathematics instruction for adults.

Hartmut Rosa – Resonance and being in the world

Hartmut Rosa, born 1965, is a German sociologist who contributed to the further development of critical theory by adding a concept of resonance (Rosa, 2019). Rosa sees resonance as a way of encountering the world – people, things, matter, history, nature, and life as such. He highlights four crucial qualities of resonance. The first of these qualities is affection, which is a feeling of being truly touched or moved by someone or something. A second quality is emotion and an experience of self-efficacy. Third, by being touched and affected and by reacting and answering, transformations occur. The fourth quality characterizing resonance is that you never can be sure whether you will experience resonance.This elusiveness makes it impossible to predict or control what happens. Nevertheless, Rosa sees such attempts to get in touch with the world in the sense of resonance as inherent to human nature.

Rosa's ideas are grounded in Habermas' theory of communicative action, but Rosa goes a step further, arguing that Habermas gave too much respect to rationalism. In addition, he puts more emphasis on education, studying what resonance and being in the world might mean in an educational context.He has developed what he refers to as the triangle of resonance. This triangle consists of teacher, students and material, and he adds what he terms the axis of resonance between the students. Rosa gives an example of the role played by materials – also in mathematics, while also underlining the importance of the teacher (2019, p. 244):

> Education as a process of disclosing the world begins with the excitement of the teacher, who in a way functions as the first tuning fork, preparing her students for resonance so that in the resonance event between them the material (whether classical drama, a mathematical formula, the rules of grammar of a foreign language, or a political party platform) comes alive and begins to speak. Thought it might sound overly poetic expressed this way, it is basically an almost everyday occurrence, playing out hundreds of times in hundreds of different ways in every school on every school day.

Rosa points to four preconditions for establishing a triangle of resonance between teacher, students and material and for establishing an axis of resonance between the students. The first concerns students' freedom from fear and anxiety, expectations of self-efficacy, and mutual trust between teacher and students and among students. The second is a teacher who believes that she has something to say about the material, i.e., the material appeals to her and is important to her. The third is that students must approach the material openly, with a willingnessto engage with and be moved by it, and a belief that they can 'make it speak.' Finally, it is a precondition for establishing an axis of resonance between the students that there is an atmosphere in the classroom that facilitates students' openness to resonance–where students do not risk mockery, disparagement, malicious remarks, bullying, etc.

I fully agree with Rosa's claim that recent debates in the field of education and pedagogy assign too reductive arole to the teacher. Regarding teachers solely as professional moderators and mediators, helping students and sharing their know-how, underestimates the real importance and potentials of teachers. Rosa refers to Hattie's findings of the teachers' influence on students' learning (Hattie, 2012), and Rosa suggests that it might be teachers' influence of what he terms the web of resonance in the classroom that is the most important for students' learning. Rosa also suggests that this web of resonance might be one of the core mechanisms behind school's reproduction of sociostructural differences. He refers to Maaz et al. (2014), who state that these differences are reproduced and even exacerbated within German education (Rosa, 2019, p.246). To summarize, Rosa extends the perspective offered by Habermas' concept of communicative acts between people (as in part D in Figure 1) to include a strong focus on the relationship between materials, teachers and students in educational contexts and on peoples' relations to the world (as in part A and part B in Figure 1).

Concluding remarks

The three scholars whose ideas are at the centre of this paper share an interest in ensuring everyone has access to education and that this education supports and strengthens democratic processes within society. Meanwhile, Grundtvig's scepticism regarding whether mathematics can contribute positively to the lives of ordinary people is a warning we ought to seriously discuss and reflect upon – for instance, by using the model presented in Figure 1 to continuously ask how high-quality mathematics instruction for adults can contribute to formal and lived democracy. It is my assertion that the three scholars provide several possible underpinnings that can help clarify the model in Figure 1.

Grundtvig shows that part C in Figure 1 is underpinned by everyone's right to education and parts B and D by letting lived democracy be part of education. Habermas shows that part D is underpinned by different kinds of communicative acts that offer analytical tools for acknowledging distinct democratic facets when organizing mathematics instruction. As mentioned, Cramer and Knipping's use of Habermas' concepts has rich potential for studying and supporting democratic learning processes in mathematics instruction for adults. Rosa shows that part D is underpinned by his extended perspective on communicative acts between people, while parts A and B are underpinned by his focus on the material and the teacher and on people's relations to the world.

However, none of these scholars show any great interest in numeracy and mathematics. As such, it is up to those of us conducting research on and developing mathematics education to build on and concretize their insights. The model in Figure 1 is my modest contribution.

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Numeracy-Meets: professional development and networking for adult numeracy practitioners

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Despite the clear and obvious need for adults to be proficient in numeracy, international studies suggest that many adults continue to struggle in this area. One of the main challenges continues to be the availability and quality of adult numeracy education. Research has highlighted an unmet demand for professional development in this area with many numeracy practitioners looking for opportunities to further develop their practice. Moreover, practitioners have expressed a desire for networking opportunities with colleagues to discuss and share their experiences. Thus, this research aims to establish a series of online 'Numeracy-Meets' for adult numeracy practitioners. These 'Meets' will be an organised but informal community of practice for practitioners to share pedagogy, practical innovations, and personal insights into teaching adult numeracy.

Keywords: Adult numeracy practitioners, needs analysis, professional development model.

Background to the Research

The development of a numerate society is an international and national priority in education. Governments, policymakers, and educators around the world have stated that numeracy is a capability that everyone needs to possess to meet the demands of everyday life (United Nations Sustainable Development Goals [UNSDG], https://sdgs.un.org/2030agenda). Numeracy skills are critically important for the adult population to allow individuals to meaningfully engage in society; to earn a good wage; and to protect their physical and mental wellbeing (Carpentieri, Litster, & Frumkin, 2010; Parsons & Bynner, 2005). Research shows that adults with higher competency in literacy, numeracy and problem solving in today's world tend to have better outcomes in attaining a job than their less-proficient peers (OECD, 2019). On the other hand, low numeracy levels amongst adults can contribute to intergenerational cycles of inequality and disadvantage in families (Carpentieri et al., 2013). For example, research has shown that adults who struggle with numeracy are more likely than others to have lower incomes, have trouble finding employment, and suffer from poorer physical and mental health (Carpentieri et al., 2010; Parsons & Bynner, 2005).

Thus, despite the clear and obvious need for adults to be proficient in numeracy, international studies suggest that many adults struggle in this area. In the UK, a study conducted by National Numeracy (2019) found that 56% of adults displayed numeracy skills which were the equivalent of that expected of a primary school child, while only a quarter of the adult population displayed levels of proficiency in the area of numeracy at or above the level of that expected of a 16-year-old. In addition to reports such as this, the Programme for the International Assessment of Adult Competencies [PIAAC] has also been used regularly by Governments and policy makers worldwide to determine adults' level of proficiency in the area of numeracy. This international assessment measures adults' skills and competencies in a number of different areas, including numeracy, and categorises their skills into one of six proficiency levels. When Ireland took part in PIAAC (2012), it revealed that over one quarter (25.3%) of adults in Ireland scored at or below Level 1 on the numeracy scale (OECD, 2013). This score ranked Ireland 19th out of 24 participating countries and suggested that 754,000 Irish people struggle with everyday maths and may be unable to do a simple maths calculation such as subtraction (NALA, 2017). In 2015, NALA undertook another analysis of the PIAAC data, this time focusing on the themes of skills in the workplace and social wellbeing. When it came to numeracy, the main findings showed that:

- Over half (60%) of the sample with numeracy at level 1 or lower were women.
- The average age of the respondents with level 1 or lower numeracy was 43 years.
- Almost half (49%) of the sample with level 1 or lower numeracy had lower secondary education or less.
- Almost one third (31%) of the sample with level 1 or lower numeracy had had no paid work in the last five years.

(NALA, 2017, pp. 15-16)

The most recent PIAAC study shows that, on average, across all 28 OECD counties surveyed, 22.7% of adults are performing at or below Level 1 (OECD, 2016). In essence, these adults are not capable of going beyond one-step processes in the area of numeracy nor are they capable of dealing with problem scenarios where the numeracy component is not wholly explicit.

In addition to the aforementioned low levels of proficiency in the area of numeracy, research also indicates that many adults also hold negative attitudes towards the domain. According to the work of Breen (2003) and Southwood (2011) fear is the emotion often reported by adults when confronted with numeracy tasks and it has a negative impact on their willingness to engage with numeracy and on their performance in the domain. Mathematics anxiety has been defined by Richardson and Suinn (1972) as "feelings of tension …that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" (p. 571). While Martinez and Martinez (1996) determine it to be a construct with multiple causes, many link its origins to negative classroom experiences from the past. Such experiences may include the use of traditional teaching methodologies, where mathematics involves the memorisation of formulas, and the following of rules and procedures (Idris, 2006; Prendergast et al., 2014). While there have been many changes to the teaching and learning of mainstream mathematics in recent years, Carpentieri et al. (2010) determine that in numeracy teaching it is sometimes easier to quantify "bad practice" than to define "good practice". According to Swain (2005), "bad" practice involves the teacher using a series of procedures, where the students learn by rote and without understanding. No connections are made to other areas of numeracy. With this in mind, and in line with the recommendations of Goos et al. (2021), there is much work to be done in relation to facilitating effective continuing professional development (CPD) for adult numeracy practitioners to help improve their practice. Effective and sustained provision is of huge importance, particularly for those practitioners who may be working in isolation in small centres and also those practitioners who may not have a background in the area.

CPD is also essential for bringing about the pedagogical transformation and educational culture change which is required for the effective implementation of innovative approaches (Bray & Tangney, 2017).

The Irish Context

In Ireland, adult numeracy provision is provided by the Education and Training Boards (ETBs) through their local adult literacy service. Each literacy service is organised by an adult literacy organiser (ALO), and numeracy tutors provide tuition on a one-to-one or group basis.

In addition to numeracy, the literacy service offers a range of programmes that include family learning, English to Speakers of Other Languages (ESOL), and workplace basic education. Goos et al. (2021) found that many adults are reluctant to admit their numeracy learning needs, and instead approach an ETB looking for courses in areas other than numeracy. However, once these adult participants develop confidence and comfort in the learning environment of the ETB, tutors and ETB staff are alert to opportunities for identifying their numeracy learning needs and directing them into integrated courses with a numeracy focus. The vast majority of adult learners engage in 'mainstream' tuition which consists of either one-to-one or group tuition, typically for two to four hours a week during the academic year.

Adult literacy services across the country design and deliver a wide range of programmes to meet the needs of adult learners. Some are accredited at levels 1 – 5 on the National Framework of Qualifications; others are non-accredited. The National Framework of Qualifications is a 10-level framework of standards for accreditation purposes. Levels 1 – 4 are of most relevance to those with basic skills needs and provide an opportunity for certification – often for the first time. Tuition is also available in a number of other education settings including community education, Youthreach, Community Training Centres, National Learning Network, probation projects, disability services and special schools

Research carried out by NALA (2013) found that over 60% of adult numeracy tutors reported that they did not have enough training in teaching numeracy to adults, and 15% reported that they had no training at all. More recently Goos et al. (2021) reported that adult numeracy provision in Ireland is predominantly dependent on part-time numeracy tutors. Only three ETBs had full-time staff members involved in adult numeracy. NALA (2015) developed a framework for meeting the professional development needs of tutors of adult numeracy in the Irish Further Education and Training sector. This framework recognises that tutors are also adult learners in the context of professional development. It recommends that professional development should be underpinned by a broad and dynamic view of numeracy that is internationally recognised. The framework also identifies important qualities and knowledge that adult numeracy tutors need to possess, including excellent understanding of elementary mathematics; digital literacy skills; and a view of mathematics as part of everyday life. Goos et al. (2021) recommended that ETBs should consider ways of supporting adult numeracy tutors to develop these qualities, and of making such opportunities accessible to tutors while avoiding costs to tutors in terms of time and financial commitment

The Study

It is imperative that if we are to meet the literacy and numeracy targets set out by UNSDG (https://sdgs.un.org/2030agenda), we must address the teaching of numeracy in adult education. This proposal seeks to address Goos et al.'s (2021) recommendation and explore ways of supporting adult numeracy practitioners. In doing so, it aims to work with the National Adult Literacy Agency (NALA) to help practitioners around Ireland in developing their skills in the teaching of numeracy to adult learners. In addition to this, another aim of this project is to create a network of expertise between partners in academia and NALA to support practitioners in fostering a positive and engaging environment for teaching numeracy within the adult education spectrum. NALA is an independent charity committed to making sure people with literacy and numeracy difficulties in Ireland can fully take part in society and have access to learning opportunities that meet their needs. They have a long track record in the provision of professional development opportunities for adult numeracy practitioners in Ireland and the researchers felt that the organisation's knowledge and networks in the area would be a very welcome addition to the project team.

Thus, this study aims to bring together expertise in the field of numeracy, teacher education, adult numeracy education and those delivering courses to adults enrolled in numeracy development programmes.

There are two main objectives to the research, namely to:

- Investigate the professional development and resource needs of adult numeracy practitioners in Ireland.
- Design, implement, and evaluate a professional development and networking model that addresses some of these needs.

Theoretical Framework – The Numeracy-Meet Model

We have termed the proposed professional development and networking model as a series of Numeracy-Meets. These are based on the TeachMeet model which was developed in Scotland in 2006 and mainly involved primary and secondary school teachers focusing on teaching strategies and classroom practices. According to Amond et al. (2018) a TeachMeet is "an event held after-hours between teachers to share practice and ideas, making short presentations and hosting conversations in a convivial and playful atmosphere." Bennett (2012) notes that TeachMeets usually last a couple of hours and are focused on teachers sharing ideas with one another based the things that they've used and found effective in their teaching. While the Meets take an informal structure, they do require some sort of facilitation to encourage participation and arrange the running order.

Although there is currently a dearth of research in this area, TeachMeets fulfil needs for CPD and communities of practice (CoP) (Amond et al., 2018; Amond et al., 2020). From an adult numeracy practitioner perspective, the TeachMeet model can meet many of the components of NALA's (2015) framework for developing the CPD needs of adult numeracy tutors. For example, the use of the TeachMeet model is accessible in terms to time and cost and can focus on understanding of elementary mathematics and viewing mathematics as part of everyday life. Thus, this study will adapt the TeachMeet model to suit the needs of adult numeracy practitioners in Ireland. It is proposed that each Numeracy-Meet will be structured around a theme, thereby allowing attendees to focus on one aspect of the broader concerns. In line with Charles (2021, p.3), our Numeracy-Meet model will be developed with the following goals:

- Create a means of communication around pedagogy
- Provide a forum for sharing of expertise and insights
- Create a pool of instructional resources and strategies
- Gain input in steering instruction and assessment
- Foster teamwork and enable networking

Proposed Methodology

There will be four stages to the study:

- Stage 1: Preplanning which includes the design of a research instrument for an online scoping survey in which practitioners can outline the types of professional development and resources necessary to improve their own teaching of numeracy. The design of this instrument is s currently underway, and it is anticipated that there will be three sections:
    - Section A will focus on practitioners understanding of numeracy and issues they identify around the teaching and learning of adult numeracy in Ireland.
    - Section B will involve a needs analysis in which practitioners can outline the types of professional development and resources necessary to improve their own teaching of numeracy.
    - Section C will seek to determine practitioners' perspectives on CoP and to ascertain any prior experience they have of engaging in such CoP and other CPD opportunities.

  The survey will be piloted with a group of five numeracy practitioners who will be invited to participate on the basis of the expertise they could bring to the research and the contemporary experiences they have in similar peer groups to the research participants.

- Stage 2: Needs Analysis to Guide the Design and Implementation of the Numeracy-Meet Model (January - February 2022). The research instrument will be circulated online through social media and existing networks (including a NALA mailing list of adult numeracy practitioners around Ireland who have signed up to be kept informed of their CPD offerings) to a sample of numeracy practitioners using a snowball sampling method. It is difficult to quantify the number of numeracy practitioners nationally and so a response rate will be difficult to quantify. The data gathered will provide an evidence base around the specific needs of adult numeracy practitioners and will guide the design and implementation of the Numeracy-Meet model.

- Stage 3: Implementation of six Numeracy-Meets (February - May 2022)
  It is anticipated that here will be six Numeracy-Meets hosted between January and June 2022 (one Numeracy-Meet per month). Potential areas of focus for each Numeracy-Meet will be guided by existing research and by the data from Stage 2 but a sample programme may include:
    1. Introduction to numeracy
    2. Family numeracy

3. Financial numeracy
4. Numeracy for health
5. Numeracy in a digital world
6. Overcoming mathematics anxiety

Each Numeracy-Meet will have a similar structure, with the exception of the Introductory Meet. It is envisaged that each Meet will be facilitated by an expert in mathematics/numeracy education who will open the Meet and share some ideas and resources about teaching a particular aspect of numeracy. Following this, the Meet will then involve active participation from adult numeracy practitioners who are in attendance. These practitioners will have volunteered to share their expertise and insights in advance of the Meet. Their involvement may see them sharing a resource that they found worked well or a teaching strategy that they found beneficial. Any lesson plans or resources discussed in the Meet will be collated and packaged by the project team and shared with all participants. As such, a folder of resources and plans would accompany each Numeracy-Meet. The Meet will conclude with participants completing out a brief online evaluation.

- Stage 4: Evaluation of Numeracy-Meets (February - June 2022)
  In parallel with Stage 3, Stage 4 will focus on the evaluation of the Numeracy-Meets. This evaluation will adopt a mixed methods research approach and as such will yield both qualitative and quantitative data. As mentioned, on completion of each of the six Numeracy Meets, participating practitioners will be asked to complete a brief online evaluation. This evaluation will allow practitioners to respond to the material shared during the Meet, outline what they feel they gained from the Meet and offer suggestions for future Meets. All evaluations will be anonymous to encourage participants to be honest in their responses and the project team will use the feedback to shape future Meets and to inform the overall evaluation of the project. Furthermore, once all six Numeracy-Meets have taken place, numeracy practitioners who attended two or more of these events will be invited to participate in a focus group which will be facilitated by the project team. The focus group will be used to ascertain practitioners' insights into their perceived effectiveness of the project and the future sustainability of such a community of practice among numeracy practitioners.

Conclusion

As recommended by Goos et al. (2021), professional development for adult numeracy practitioners needs to be widely promoted and accessible and involve practitioners in sharing their practice as well as learning new teaching approaches. It should be coordinated with the aim of establishing CoP of adult numeracy tutors while raising the profile of adult numeracy provision. The Numeracy-Meet model proposed in this study will specifically address this recommendation and offer a structure to CPD activities which are informal and led by adult numeracy practitioners. It is also anticipated that it will create a network of expertise between partners in academia and NALA to support practitioners in fostering a positive and engaging environment for teaching numeracy within the adult education spectrum in Ireland.

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Changing perceptions among adult learners (19+) in further education studying GCSE mathematics:

Methodology and data analysis - the importance of the pilot

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A Doctorate in Education in the UK generally involves four modules prior to the thesis phase. Two of the modules are focussed on the pilot study, the first on the methodological rationale, and the last on the analysis of the data collected. Although this process can be seen by post graduate researchers as a delay to the actual data collection, it can be highly informative and make a substantial contribution to the thesis. In this example, the efficacy of a questionnaire, developed from the Abbreviated Maths Anxiety Scale (AMAS) (Hopko, Mahadevan, Bare, & Hunt, 2019), was tested with adults studying GCSE mathematics, whilst developing an understanding of action research, insider research and thematic analysis.

Keywords: GCSE mathematics, further education, pilot study, perceptions, math anxiety.

Introduction

Adult learners studying GCSE mathematics in a Further Education (FE) college, whilst being highly motivated to re-engage with the classes, often due to extrinsic motivations such as career changes, can present with a degree of anxiety about either mathematics, or examinations, or both. It is unknown whether for these learners' mathematics and exam anxiety are linked, if the levels change during the courses, and if confidence, or lack of it, impacts on exam results.

The prevalence of mathematics and exam anxiety is becoming increasingly well documented. Recent research funded by the Nuffield Foundation (Nuffield Foundation, 2018) includes two studies, one on GCSE resits mathematics for 16- to 18-year-olds (Hough, Solomon, Dickinson, & Gough, 2018), and one on mathematics anxiety in pupils at Primary and Secondary schools (Szucs, McLellan, & Dowker, 2017). Research also exists which has analysed the experiences of traditional pathway students in Higher Education (Evans J., 2000). However, few studies exist which consider the effects of previous experiences in mathematics classrooms on non-traditional adult learners returning to education after a break or map the attitudes of those adults towards mathematics or examinations.

The pilot study mapped confidence levels for both mathematics and examinations against exam results for a small group of adults (n=18), to evaluate for a correlation between confidence and success. It might seem that it is likely that more anxious learners are less likely to succeed, but not all research supports this view (Mandler & Sarason, 1952; Tavani & Losh, 2019). There also seems to be a lack of clarification around whether mathematics or exam confidence, or both, are linked to exam success.

The research design for the pilot study was a mix of quantitative and qualitative methods, to gain a broad understanding of the experiences of adult learners (Clough & Nutbrown, 2007). A modified 'Abbreviated Mathematics Anxiety Survey' (AMAS) (Hopko, Mahadevan, Bare, & Hunt, 2019), a tried and tested survey in the public domain, was revised to change the labelling of the scale to include confidence, rather than just anxiety, and to gather comments from learners. It was re-named it MECS (Mathematics and Examinations Confidence Scale) to differentiate it from the original. Learners were surveyed in April, after curriculum input and as learners move into the revision phase, using the questionnaire.

Questionnaire responses were matched to examination results, to evaluate for a correlation between confidence levels of both mathematics and examinations against outcomes. This analysis is presented on scatter graphs, and expanded into gender, nationality, and age comparisons.

The doctoral thesis of which this pilot study forms a part aspires to make an original contribution to the understanding of factors in mathematics delivery for adults in the UK, which may differ from their experience in previous educational settings.

Summary of research questions

The main research questions are:

- Do courses in FE colleges change adult learners' perceptions of GCSE mathematics and/or examinations and, if so, how, and why?
- Is a relatively high confidence level necessary for exam success?
- Are more anxious students less likely to pass?
- The research questions for the pilot study are:
- Does the MECS questionnaire give sufficient information to formulate answers to these questions?
- Did participants find it easy to understand and use?
- Is an analysis by themes helpful for understanding the data collected?

Methodology and methods of the pilot study

Action Research

Action research is the phrase used to describe research, which is conducted by practitioners in the field, linking research to practice (Burgess, Sieminski, & Arthur, 2006; Munn-Giddings, 2017). It can often be linked to an intervention, which is designed to improve practice for teachers, or outcomes for students, or both (Wilson, 2017; Munn-Giddings, 2017; McNiff, 2016). The project aspires to both describe the experiences of adults returning to mathematics classes, and work towards interventions that could improve outcomes for learners, either at a micro or macro level, i.e., either for a learner, a particular classroom, adult learners in FE colleges, or even to the wider mathematics educators' community (Creswell, 2012).

In addition to describing this pilot study as action research, it can also be described as participatory action research, as participants are involved in the research process. This is because, in addition to collecting data, some participants have been asked to evaluate the questionnaire (Munn-Giddings, 2017, p. 72; Cohen, Manion, & Morrison, 2018).

Mixed Methods

A mixed method approach was chosen to allow for some 'triangulation' of data, as it can be cross referenced to show reliability of the findings (Maguire, 2019; Cohen, Manion, & Morrison, 2000). Cross referencing refers to using more than one method to collect data (Mason, 2018; McNiff, 2016), or by using more than one set of people (McNiff, 2016). In the pilot study both quantitative and qualitative data have been collected in the questionnaire.

There is an issue with a mixed method study, that it may be seen as pseudo-scientific, blurring the demarcation line between positivist and interpretivist approaches, and making the research sound more scientific than may be justified (Cohen, Manion, & Morrison, 2018; Mason, 2018).

The Survey Method: A questionnaire

The method for data collection tested in the pilot was a questionnaire, because in the mathematics education field the Abbreviated Mathematics Anxiety Scale is often mentioned in research, and has been adapted in many ways, including for use with younger children (Hopko, Mahadevan, Bare, & Hunt, 2019; Psychological Toolkit, 2018). Adapting an existing survey both adds academic weight to the main study, as it is tried and tested by other researchers, and avoids the major challenge of constructing and testing a survey from scratch (Wilson, 2017; McNiff, 2016).

Questionnaires can generate a lot of data quickly, as they are generally easy to complete, can include both quantitative and qualitative questions, and responses can be easily anonymised (Clough & Nutbrown, 2007; Cohen, Manion, & Morrison, 2018). The pilot study evaluated the questionnaire to see whether the participants could respond positively, neutrally, or negatively to the questions posed, and whether patterns emerged, even though there was a small sample size (Taber, 2017; Cohen, Manion, & Morrison, 2018).

The disadvantages of a questionnaire include that the information they generate might be quite superficial, as respondents will not necessarily answer thoughtfully and accurately, and that there is little or no opportunity for the researcher to follow up on comments, unless permission for contact has been included in the questionnaire (Cohen, Manion, & Morrison, 2000; Clough & Nutbrown, 2007)

One of the significant changes made to the AMAS questionnaire was to alter the Likert Scale (Hopko, Mahadevan, Bare, & Hunt, 2019). The original uses the word 'anxiety' throughout, as students are asked to grade themselves in the following way:

1 Low anxiety, 2 Some anxiety, 3 Moderate anxiety, 4 Quite a bit of anxiety, and 5 High anxiety.

This was altered to:

1 Very confident, 2 Confident, 3 Neutral- neither confident nor anxious, 4 Anxious, and 5 Very anxious

This was altered to moderate the potential for a presumption for anxiety. The original survey could be interpreted as leading the students to an expectation of an answer, an outcome that should be avoided, as it can lack objectivity (Burgess, Sieminski, & Arthur, 2006; Sfard, 2013; Wilson, 2017).

In extreme examples the questions "can influence respondents and alert them to ideas they had not thought about before" (McNiff, 2016, p. 184).

The second change was to include comment lines, to allow learners to add to their numerical response, which changed the questionnaire into a mix of quantitative and qualitative data and canvassed opinions on the statements. An 'any other comments' section at the end of the questionnaire was also included, to enable participants to share other issues or thoughts.

Other minor adaptations introduced were to change the American language to a more UK version of English, and to add one additional question, to ask how confident learners would feel taking any other, non-mathematics exam. This was to try to evaluate whether it is all examinations that students feel anxious about, or specifically exams in mathematics.

Sampling

Opinions on sample sizes for pilot studies can be very divided, with some sources saying that a few respondents would be enough (Burgess, Sieminski, & Arthur, 2006), but other sources specifying a minimum of 30 (Cohen, Manion, & Morrison, 2000). In this pilot study the number of participants was eighteen (n=18).

The sample for the pilot study was drawn from one FE college, so may be unrepresentative of the total population of adults studying in colleges and could also be seen to be biased (Burgess, Sieminski, & Arthur, 2006; Oates, 2006), as the learners may have modified their answers knowing that their 'teacher as researcher' was going to see them. This is described as the halo effect in Cohen et al (2018), but as the pilot study tested the format of the questionnaire, rather than used the data, it has still given a useful insight on whether the questionnaire is acceptable to participants.

There was also no attempt to gain a representative sample from the classes, such as a mix of those who might be confident or anxious, or a mix of gender, ages, or nationalities (Oates, 2006). Participation by learners was voluntary and on an opt in basis.

Data Analysis: chosen method and analysis

Thematic Analysis

In this pilot study a mix of quantitative and qualitative data has been collected in a mixed method study. Thematic analysis seems to be a suitable theoretical framework for data analysis and interpretation, as it is a flexible approach which can encompass both types of data (Nowell, Norris, White, & Moules, 2017; Braun & Clarke, 2013). It can also reflect the complexity of the data, as use of several different perspectives or data sources can be shown in the codes, and these can be used to support and illustrate the themes, and to triangulate the data (Creswell, 2012; Cohen, Manion, & Morrison, 2018).

The pilot study data analysis looked for patterns in the data around themes, such as course content and examinations, and learner characteristics such as age, gender, and nationality, to see if this way of analysing yielded useful insights and added to understanding, if only to the complexity of educational research.

Research Findings

The total score for each participant from the questionnaire was used in a number of scatter graphs to compare confidence levels with exam results; the decision to use these in the analysis was based on the need to test whether they would be a useful way to display the two forms of primary data in a single format (Burgess, Sieminski, & Arthur, 2006; Spiegelhalter, 2019). Scatter graphs are a statistical representation designed to show comparisons between two variables, which could have an appeal to wider audiences (Creswell, 2012; Thomas, 2013). In scatter graphs it is possible to see if patterns emerge, and if correlations exist between data sets (Braun & Clarke, 2013; Evans M., 2017). The numbers involved in the pilot study means that it was a small data set, and no conclusions can be drawn on the content, but the use of scatter graphs has been explored in this pilot study.

Four scatter graphs were constructed. The first two illustrated the very broad themes of confidence and anxiety compared to exam performance but contrasted the use of grades and scores. In these two graphs the slope of the trend line was virtually identical, which relieves a practical challenge for the research, as scores will be more difficult to capture than grades, as learners only receive grades. There is a further benefit as colleges that agree to participate in the main study are likely to use different exam boards with unique marking structures for grades, so scores would not be a viable method of comparison.

There was a slight correlation between the level of confidence and the outcome of the exams. This could indicate that there was a causal relationship between anxiety and success, and more anxious learners were less likely to pass (Spiegelhalter, 2019; Creswell, 2012). However, the slope of the trend line, (also known as a line of best fit, or regression line), and even its aspect, varied depending on which part of the group is investigated, which illustrates the importance of examining data by different themes, to enhance its validity (Cohen, Manion, & Morrison, 2018; Creswell, 2012).

Further scatter graphs illustrated analytical differences in gender and age, two themes identified as present in existing literature on the subject. When males are selected from the participants, there is a stronger relationship between levels of confidence and success in the exam, which means for females the correlation is weaker. However, if females are examined as a group, there are some clear examples of participants who are very anxious and high achievers, who could be seen as outliers. In this small sample size, it is not possible to draw any conclusions from this, but it does confirm the important of collecting the gender of participants for investigation and confirms the use of gender as a theme.

When participants under the average age were selected, the aspect of the trend line changed so that now more anxious students were more likely to pass; this could reflect the need for a certain amount of anxiety to maximise performance (Yerkes & Dodson, 1908), but it was unexpected, and shows the advantage interrogating data thoroughly using thematic analysis to ensure its validity (Nowell, Norris, White, & Moules, 2017). It also probably shows the danger of drawing inferences from small samples (Cohen, Manion, & Morrison, 2018; Creswell, 2012).

In addition to organising this data into themes, other calculations have taken place, such as enumeration of the range used on the Likert scale. One concern was that participants might be unlikely to use the full range of a Likert scale, as most would use 2, 3 or 4, if the scale is 1 to 5, and avoid the extremities (Cohen, Manion, & Morrison, 2018). This was tested in the pilot study and participants had no problem with using the full range of the scale, indicating that they had no problem with deciding how strongly or otherwise they felt about the content. This may reflect the strength of feeling there is about mathematics generally, either positively or negatively. The use of a Likert scale allowed for the data to be analysed in several different ways, such as by ranking the questions from most confident to most anxious, which could yield different insights and understanding.

Over half of the twenty respondents used the comments sections, and on average each person commented on approximately half of the questions.

Learners involved in the review confirmed that they felt the AMAS document was leading, in terms of anxiety, and that they preferred the MECS questionnaire. When asked about the use of questionnaires in general, they felt that the intervention had been beneficial, as it implied that it was 'okay to be anxious', that it was normal for the whole class and thus acceptable. The survey took around 5 to 10 minutes to complete, which was not too demanding on peoples' time and energy. The small group review of the AMAS vs MECS questionnaires took approximately 20 minutes.

Conclusions

To repeat, the research questions for the pilot study are: Does the MECS questionnaire give sufficient information to formulate answers to these questions? Did the learners find it easy to understand and use? Is thematic analysis a useful approach for data analysis?

The current MECS questionnaire seems to have worked well, as it covered assessment in various forms, classroom dynamics, and course content such as number work and algebra, and the language used in it seems to have been understood by the participants.

There were no negative comments from learners about the language used, or the time taken. The participants completed the questionnaire in different ways, some using the numbered scale, and some the words attached to the numbers, which demonstrated a useful flexibility.

Thematic analysis as a framework to organise, interrogate and interpret the information seems to be a useful method, both for the insights it could yield, and for the accessibility of the data in subsequent presentations.

The contribution of the pilot study to the final project and subsequent thesis is clearly substantial. The interrogation of the content and process of implementation of the questionnaire used in the pilot study, and the development of themes and codes, have all made significant contributions towards an understanding of the research issues, which will be highly beneficial for the main study. It could also lead to an expansion of the data, "to formulate new questions and levels of interpretation" (Coffey & Atkinson, 1996, p. 30; Gibbs, 2017), leading to the generation of new theories to explain phenomena in mathematics education for adults.

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TWG08: Affect and the teaching and learning of mathematics

Introduction to the work of TWG08: Affect and the teaching and learning of mathematics

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Introduction

The work of the Thematic Working Group 8 (TWG08) "Affect and the teaching and learning of mathematics" started with a revision of the call for papers in 2019. The call for papers included theoretical, methodological and empirical fields of research on affective constructs for students and teachers. Because of the variety of affective constructs, we could point out in the call only few examples of affective constructs in the area of beliefs, attitudes, emotions and motivation. Spreading of the COVID pandemic all over the world affected our preparation for the CERME 12 conference, which was moved from February 2021 to February 2022. In order to make the communication on affect possible, we followed the call of the program committee to offer a short virtual meeting on affect in the framework of the virtual CERME in 2021. In this virtual meeting, we presented an overview of research on affect and discussed urgent questions for mathematics education (e.g. What is the impact of COVID pandemic on research in affect?) and questions of importance for research on affect (e.g. Do emotions and motivation affect performance in mathematics?) in small groups and in the whole group. We continued with the preparation of the group's work for CERME 12, which was eventually held as a virtual conference. In this chapter, we would like to introduce our work in TWG08, discussions on affect in teaching and learning of mathematics and new developments in the field.

The sound number of submissions and participants confirmed the long-term interest of researchers in affect-related research in mathematics education. In total, 26 papers and 4 posters were submitted to our group and 24 papers and 3 posters were accepted for presentation. Of the presented studies, 23 papers and 3 posters were accepted for publication in the conference proceedings. In addition to the different European countries, our group included presentations from Australia, Canada, Israel and the United States of America. Researchers from 13 countries participated in the work of our group. Many newcomers joined the group, indicating that interest in research in the area of affect is constantly increasing.

We started our work with an ice-breaking activity. Each participant talked about his country and topic of research interest to the audience. After the ice-breaking activity, a reflection of the prior work done in CERME conferences was presented in the whole group and open questions in research on affect were discussed in breakout rooms. Following a tradition of the TWG08 group regarding importance of small group discussions for scientific activities, we scheduled a considerable amount of time for this type of work in following sessions. The contributions in each session were assigned to the topics beliefs, attitudes, motivation, emotions or other affective constructs. After a short presentation (3 – 5 slides, 5 min) and clarifying questions (3 min) for each paper in a session, we assigned participants to breakout rooms with 4 – 6 participants. In order to structure the small group work and to support the content of the discussions, at least one co-leader and one of the authors joined each group. Each small group could decide which of the presented papers or posters they would like to discuss with the author(s) and other group members (25 – 40 min). In the last 10 minutes of each session, we shared our thoughts about the topics of discussions with the whole group. In the last session, we presented a synopsis of the small and whole group discussion, which were again discussed in breakout rooms. At the end, we reminded participants on the importance of revisions of contributions in the final phase of the revision process.

Reflection on prior research in affect

Inés Gómez-Chacón presented an overview of the work that has been done in the past conferences in TWG08. An essential part of the overview was presenting the changes in research on affective concepts over the past 15 years. A part of the overview was included in the final report about the work of TWG 8 that was videotaped by Çiğdem Haser and Stanislaw Schukajlow.

A review paper by McLeod (1992) serves, usually, as the starting point for systematic research on affect in mathematics education. According to this taxonomy, research on affect can be assigned to one of the three major categories: beliefs, attitudes or emotions. In McLeod's framework, beliefs are the most stable in time, less intense and most cognitive. On the opposite side of the range, McLeod posed emotions, which are considered as less stable in time, most affective and less cognitive. Attitudes are situated between beliefs and emotions regarding temporal stability, affect and cognition.

In the last decades, researchers elaborated repeatedly on the taxonomy of affect proposed by McLeod. One influential model was proposed by Hannula (2011) in CERME 7. He analysed research in mathematics education and suggested distinguishing between cognitive, motivational and affective dimensions, unstable states and stable traits as well as social, psychological or physiological dimensions of affect.

Another possibility to think about affect is to distinguish between three characteristics: object of the affect (e.g., mathematics, problem solving or strategy use), subject (e.g., teacher, students or policy maker) or valence (positive, negative or neutral; Schukajlow, Rakoczy, & Pekrun, 2017). Further, the theoretical approach, such as acquisitionist or participationist, can be considered of importance for research on affect. A strong interdependence of affective constructs is another essential characteristic of affect. In research on beliefs, these interdependencies are reflected in the emphasis of the so-called beliefs system. A strong interdependence of affective constructs is another essential characteristic of affect. An affective system includes different affect components that are closely related to each other. Changes in one component of affect system result in changes in other components and vice versa. For example, increasing interest in mathematics may positively affect enjoyment and negatively boredom. A holistic view on the affect system might be important for getting a comprehensive view on affect and its development in students and teachers. This research perspective calls for using multidimensional approach to affect and applying different methods (e.g. qualitative, quantitative and mixed methods).

In the past, we observed growing attention to the clarification of concepts that were used in research on affect. As many affective constructs are defined as complex phenomena, different affective constructs revealed a conceptual overlapping in some parts. For example, emotions include a part of physiological, affective and expressive components, and also cognitive and motivational parts. Therefore, an emotion can overlap with motivational constructs, such as interest or self-efficacy beliefs, on the theoretical level.

A relationship between different affective constructs and cognitive outcomes, such as achievement related choices and performance, was another topic of high interest in the past. Prevalent questions were: (a) What affective reactions can be observed during learning of mathematics? and (b) Whether changes in students' emotions, motivation, beliefs or attitudes are related to students' learning and learning outcomes. The underlying theme of research was often an idea that improving students' affect will increase motivation, positive emotions, growing mindset or positive attitude.

Contributions in the TWG08

Now we would like to present the papers and posters that are published in the proceeding. Andrà et al. found out that mathematical views of undergraduate mathematics students are related to achievement and preferences for different teachers' lesson types. Haser analysed students' beliefs about problem posing and their place in mathematics-related belief systems. The poster by Brunetto and colleagues presented a methodology for clustering students' mathematical views. Vankúš at al. reviewed in their poster the methodology used in the research in affective domain.

Schukajlow and Rellensmann analysed a relationship between motivational components (self-efficacy, value, and cost) and gender. Krawitz and Hartman investigated preservice teachers' interest and self-efficacy while posing problems to descriptions of real world situations. Developing and validating survey instruments for assessing beliefs and motivation in mathematics were presented by Pedersen and Haavold. Capone and Lepore analysed engagement, motivation and participation of undergraduate students during Covid pandemic. Pan et al. were interested in indications of students' wellbeing in terms of primary students' values.

Sumpter et al. investigated gendered self-evaluation in mathematics. Herset and Ghami presented an experimental study on difficulty level marking in mathematics tasks. Biton at al. analysed students' perceptions of mathematics learning environment in a virtual communication messenger. Stereotypes in a polarized world and their relation to mathematical identity were the focus of the study by Kaspersen and Gjøvik.

Vasilopoulou and Triantafillou presented students' perspectives on inclusion and peer-collaboration. Zakariya et al. analysed the relations between attitudes, prior knowledge, self-efficacy and grades in mathematics. Kourti and Potari identified and analysed pivotal teaching moments of emerging emotions during decision making. Weber et al. investigated a development of mathematical anxiety of prospective elementary teachers. Holm explored mathematical stories of future elementary teachers and shifts in affect in mathematics. Viola and Gambini presented an analysis of cooperative group work in game theory activities. Analysis of attitudes of women and racially/ethnically minoritized students was presented by Uysal and Clark. Viitala carried out a case study on teacher identity and long-term professional development.

Báró studied changes in affect due to problem posing in Hungary and Romania. Pierri explored engagement and affect in the context of storytelling. Liljedahl analysed social persuasion through a teacher’s actions in a mathematics classroom. Courtney et al. investigated teacher’s understanding of mathematics in a remedial course and the role of affect in the process of understanding.

Evolution of the TWG

This year many passionate young researchers joined the group. The number of contributions addressed similar topics as in the past conferences (e.g., self-efficacy expectations). Some new topics, such as mathematical well-being and stereotypes, and a new group of research subjects, such as marginalized students, were addressed in CERME 12. More studies focused on the relationships between affect and cognition. New theoretical approaches emerged in the group. Some examples of these approaches are dual process theory, value-fulfillment theory, activity theory, positioning theory or model of problem posing. New measurement instruments were developed and analysed by using sophisticated qualitative and quantitative methods. In several contributions, researchers addressed gender studies and derived implications for research on affect. Different objects of affect in varied contexts, of very different sorts, were presented in the group. Some examples of these objects are drawing strategy, level marking, social media, pivotal teaching moments, intelligent tutoring systems, and digital interactive storytelling.

During our final discussion, we summarized directions for future research. One example was expanding the view on affective variables and see them (a) as independent constructs that can be manipulated in order to increase performance, (b) as dependent variables that can be affected by, for instance, different teaching methods, (c) as mediation variables that can transmit the effects of teaching methods on cognition, and (d) as moderating variables that affect the effects of teaching methods on cognition. Furthermore, we call for more research on linking affect to cognition, more attention to theorising and embodiment, more focus on equity, more longitudinal and comparative studies and stronger focus on cross-domain research.

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Mathematical views: a preliminary analysis on undergraduate mathematics students

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We investigate the mathematical views of a sample of undergraduate mathematics students in their third year of studies, namely during the semester that precedes the Bachelor degree. We collected data about the students' mathematical views, their personal information and academic achievement, analysing the relationships among them. We also study mathematical views' influence on the appreciation of different aspects of teaching. Finally, we investigate the extent to which undergraduate studies can change a student's view of mathematics. Data are collected by means of multiple-choice demographic questions and Likert-scale affective questions. Results reveal that different views have an impact on third-year undergraduate mathematics students' achievement, as well as on the preference for more transmissive formats of the lesson, or more participative ones, based on different mathematical views.

Keywords: Beliefs, cluster analysis, gender, mathematical views, postsecondary education.

Introduction

There is an increasing number of studies on the crucial role of demographic and affective variables in tertiary mathematics learning. Gender is an example for demographic variables, especially for the *clichè* for female students of being more diligent, as this has an impact on their learning strategies and examination success (Griese, 2018). The students' views of mathematics is an example for affective variables. Roesken, Hannula and Pehkonen (2011) maintain that mathematical views comprise beliefs, wants and feelings. The authors also argue about the key role of different school backgrounds, different math curricula and different views and expectations towards mathematics in undergraduate students. To this regard, Daskalogianni and Simpson (2001) note that some beliefs, developed during school days, are carried forward in university, and this may cause difficulties. Liebendoerfer and Schukajlow (2017) consider undergraduate students' mathematics-related beliefs with respect to the nature of mathematics during the first year of undergraduate studies, and find out that different views of mathematics, hold by the students at the beginning of their studies, predict different interest towards mathematics at the end of the first and the second terms.

Many researches, like the aforementioned ones, focus on the first year of undergraduate studies and highlight difficulties in the secondary-tertiary transition. We, instead, focus on the effects of mathematical view at the end of undergraduate studies, to examine possible changes with respect to the first days at university. We firstly recall the theoretical background of our research.

Theoretical framework

Beliefs are propositions about a certain topic that are regarded as true (Philipp, 2007), and tend to form clusters as they “always come in sets or groups, never in complete independence of one another” (Green, 1971, p. 41). For this reason, according to Grigutsch, Raatz & Törner (1998), beliefs can be seen as “world views”, outlining four different views: a *process-oriented* one that represents mathematics as a creative activity consisting of solving problems using different and individual ways; an *application-oriented* view that represents the utility of mathematics for real world problems as the main aspect of the nature of mathematics; a *formalist* view that represents mathematics as characterized by a strongly logical and formal structure; a *schema-oriented* view that represents mathematics as a set of calculation rules and procedures to apply for routine tasks.

Di Martino and Gregorio (2019) underline that mathematical views have an impact on an undergraduate student’s choices and can prevent one to enroll in an undergraduate course only because there is some mathematics in it. Di Martino and Gregorio (2019) observe that also undergraduate *mathematics* students may face difficulties and they may develop different mathematical views that, likewise any student, may prompt them to make certain choices in place of other ones. On these premises, our research aims at answering the following four research questions. RQ1: Is there a relationship between undergraduate mathematics students’ demographic variables, such as gender or school type, and their mathematical views? RQ2: Is there an effect of university studies on mathematical views? RQ3: Are there differences in students’ academic achievement, such as the exam grade, with respect to their mathematical views? RQ4: Does the students’ appreciation of how they are taught mathematics at university differ based on their mathematical views?

Answering to RQ1 would allow us to shed some light on the origin of different mathematical views, while answering to RQ2 would allow us to understand if and how three years of undergraduate studies in mathematics can change one’s view. Answering to RQ3 and RQ4, indeed, would allow us to examine whether low- and high-achieving students tend to hold specific, yet different, world views, and the relations between such views and the aspects of teaching they mostly appreciate.

Methodology

The aim of the study is to examine the effects of undergraduate studies on students’ mathematical views, in particular the achievement at the end of their studies. Hence, the participants are 93 students enrolled *in the third (and last) year* of undergraduate studies in mathematics at the University of Torino. This sample represents 37.8% of the entire population of 246 students enrolled in the third year. They participated on a voluntary basis and have been contacted via email, proposing to participate in an anonymous online survey, in May 2021. The analyzed sample is composed of 86 responses, because 7 are incomplete data.

The investigation of the mathematical views played a central role. The main idea is to figure out whether there is any correlation between affective and cognitive variables, expecially between beliefs and the students’ ways of relating, in practice, to mathematics (i.e., passing exams, achieving good results, being satisfied with one's own choice of the curriculum, etcetera). To collect data on students’ mathematical views, we translate into Italian a 5-points Likert-scale questionnaire developed by Erens and Eichler (2019), who design a set of 24 statements concerning the nature of mathematics based on the model proposed by Grigutsch et al. (1998). Each statement is assigned to one mathematical view. Examples of statements are: “the ideas of mathematics are of general and fundamental use to society” (A); “mathematics is a logically coherent edifice free of contradiction consisting of precisely defined terms and statements which can unequivocally be proven” (F) “there is usually more than one way to solve a task or problem in mathematics” (P), and “Mathematics consists of memorising, recalling and applying procedures” (S).

Students have been clustered with respect to mathematical views through a network analysis technique that is called *community detection* (Brunetto, Tassone and Cravero, submitted). Briefly, a network with students as nodes (i.e., with 86 nodes) has been created, and a link has been established among two nodes (i.e., two students) if both agree to the same statement concerning a mathematical view. We, thus, identified the clusters of students who are, as nodes, the most connected to each other if compared to the entire network. Four clusters have been identified and each cluster has been further characterized based on the student information collected in the survey (such as the gender, so that a cluster can have a majority of males and another one a majority of females, for example).

In order to answer RQ1, the participants were asked to declare their gender (with the possibility of choosing to not declare it), and to indicate their high school type. To this respect, in the Italian high school system there is a school type that has a strong scientific and mathematical curriculum, and usually the huge majority of undergraduate students in mathematics attended it. However, there are significant differences in the extent and depth of mathematics learning that depend on the individual teacher in each school (Lombardo, 2015), hence it makes sense to investigate the views developed by individual students even if they attended the same school type. We computed the percentages of males and females in each group defined by a specific mathematical view. Similarly, we computed the percentages of students coming from the school type with a strong mathematical background.

In order to answer RQ2, we created four 5-points Likert-scale questions and asked the participants to rate their agreement about: (a) during the undergraduate studies, one has discovered -in a positive sense- aspects of mathematics that she did not imagined before; (b) during the undergraduate studies, one has been disappointed by the way mathematics has been taught; (c) one has significantly changed her beliefs about mathematics; (d) the undergraduate studies have confirmed the view of mathematics one had when she was a freshman. We compared the averages of answers (from 1 to 5) to each question in each cluster of views of mathematics, considering an average of 3 as being neutral, an average lower (higher) than 3 as discordance (accordance) for the cluster.

In order to answer RQ3, the participants were asked to self-declare their average score at exams, because the questionnaire was anonymous, and it was not possible to recollect student data from the university official database. We computed for each group of mathematical views the percentages of students with average scores falling into one of the intervals considered “low”, “medium”, “medium high” and “excellent” in the Italian university system.

In order to answer RQ4, we borrowed the questions to be asked from a National survey on undergraduate students’ appreciation of various aspects of teaching and assessment. The questionnaire is made of four 5-points Likert-scale. We also created a series of Likert-scale questions on the distance teaching experienced during the pandemic (March 2020 - June 2021). For each mathematical view, we compared the average of answers at each question, considering an average of 3 as the group being neutral with respect to a particular aspect, an average lower than 3 as disaccordance (and we distinguished different degrees of discordance), and an average higher than 3 as accordance (at various degrees).

Results

We refer to Brunetto et al. (submitted) for more details about how the students have been associated with a mathematical view. In short, the community detection algorithm makes out 4 clusters:

- 28 students have a predominately schema-oriented view (labeled S in the following).
- 21 ones have a primarily process-oriented and a secondarily application-oriented (P-a) view.
- 20 ones hold a primarily formalist and a secondarily application-oriented (F-a) view.
- 17 students show an application-oriented (A) view.

From these data, we can see that students sorted themselves almost equally in four groups with respect to mathematical views. Having identified the clusters, the presentation of results is organized in four parts, each of one dedicated to answer one of the aforementioned research questions.

RQ1: *Is there a relationship between undergraduate mathematics students' demographic variables and their mathematical views?* Within the sample of 86 respondents, 3 students selected "other/I prefer not to reveal my gender". Table 1 shows the distribution of gender in the overall sample and within each group of students identified with the same mathematical view.

Table 1: Distribution of gender in the sample of respondents

	Overall sample	**S view**	**F-a view**	**P-a view**	**A view**
Male	35%	26%	32%	38%	53%
Female	65%	74%	68%	62%	47%

Looking at Figure 1, we infer that 29 (35%) respondents are male and 54 (65%) are female. We also notice that in cluster S there is a higher percentage (74%) of female students compared to the percentage of females (65%) in the entire sample of interviewees. Conversely, there is a significantly higher percentage (53%) of males in cluster A, compared to the distribution of gender in the entire sample of valid responses. Percentages of males and females in clusters P-a and F-a mirror the entire sample's ones with respect to gender.

As regards the secondary school type, we notice that over 75% of students come from the type of school that has a strong mathematical curriculum. Furthermore, this subsample shows to divide itself almost equally in the four groups identified for the mathematical views, but a slightly higher percentage belongs to the group of those who hold a P-a one. In the group S,, 50% of students come from secondary school types where the level of mathematics is lower, while in the other 3 groups the percentage of this kind of students resembles the general one (i.e., 25%).

RQ2: *Is there an effect of university studies on mathematical views?* Looking at Table 2, we can see that students in cluster F-a discovered positive aspects of mathematics even more than the entire sample, for which the average is high, anyway (4.5 points over 5). Only S students are less in agreement with this statement. P-a students are the most in disagreement with the statement that it was disappointing to be taught in the way it has been during the undergraduate courses. All clusters agree that beliefs about mathematics change during undergraduate studies, and all students except P-a cluster disagree that "the undergraduate studies have confirmed the view of mathematics they had when they were freshmen".

Table 2: Averages of agreement to each item in the entire sample and in each cluster. Only in case of significant difference with respect to the entire sample, the cluster average is reported

Statement	**Entire sample**	**Cluster A**	**Cluster F-a**	**Cluster P-a**	**Cluster S**
During the undergraduate studies, aspects of mathematics, which have not been imagined before, have been discovered in a positive sense	4.5	same of entire sample	**4.75**	same of entire sample	**3.9**
During the undergraduate studies, the way mathematics has been taught was disappointing	2.5	same of entire sample	same of entire sample	**2**	same of entire sample
My beliefs about mathematics has greatly changed during the undergraduate studies	4	same of entire sample	same of entire sample	same of entire sample	same of entire sample
The undergraduate studies have confirmed the view of mathematics they had when they were freshmen	2.7	**2.5**	**2.5**	**3**	**2**

RQ3: *Are there differences in the undergraduate mathematics students' achievement with respect to their mathematical views?* In the Italian university system, exam pass scores range from 18 (sufficient) to 30 (excellent). Scores between 18 and 20 are considered low achieving, the range 21-24 is considered medium, 25-28 medium-high and 29-30 excellent. The sample average score is summarised in Table 3. From it, we can see that the percentage of excellent students in the A group (30%) is higher than the one computed in the general sample (18%), and that the majority of excellent students tend to hold a process-oriented view. In this group, also the percentage of medium-average students (23%) is lower if compared to the general sample (40%). No excellent student belongs to the S group and the percentage of those who have a medium score average (60%) is higher if compared with the general sample (40%). F group's percentages mirror the general sample's one. These observations lead us to conclude that there are differences in the undergraduate mathematics students' scores at exams with respect to mathematical views.

Table 3: Average scores at exams and mathematical views. The percentages in brackets refer to each total reported in the last row of the table

	General sample	Application-oriented (A)	Formalist+application-oriented (F-a)	Process+application-oriented (P-a)	Schema-oriented (S)
21-24	35 (40%)	6 (35%)	7 (37%)	5 (23%)	17 (60%)
25-28	36 (42%)	6 (35%)	10 (53%)	9 (41%)	11 (40%)
29-30	15 (18%)	5 (30%)	2 (10%)	8 (36%)	0
	86	17	19	22	28

RQ4: *Does the students' appreciation of how they are taught mathematics at university differ based on their mathematical views?* Table 4 summarises the students' average appreciation of various aspects. P-a students tend to like the least step-by-step explanations and many exercises during the lessons, but to like the most being prompted to reflect. S students like the least to be provided justifications for definitions and methods, but to like the most many exercises.

Table 4: Averages of appreciation to each item in the entire sample and in each cluster. Significant differences with respect to the entire sample are in bold

Statement	Entire sample	Cluster A	Cluster F-a	Cluster P-a	Cluster S
Exhaustive, step by step explanations in frontal lessons	3.1	3.2	3	2.8	3.2
Justification of the definitions and methods	2	2.3	3	2.2	1.5
A good number of exercises	2.9	3	2.75	2	3
Stimulating reflections on a student's side	2	2	1.5	3	2

We also asked questions about the at-distance teaching that affected these students because of the pandemic, but no significant difference can be seen with respect to the different views. The sole exception is a question about the lack of opportunities to talk and interact directly with one's mates during the lockdown, to which A, F-a and P-a students show to be in accordance (rank above 3), while S ones are in discordance (rank below 3).

Discussion and preliminary conclusions

The research presented in this paper concerns a sample of undergraduate mathematics students and aims at examining the relations among their mathematical views and their academic achievement. Having considered only the third year can be seen as a limitation, but in our study this was quite unavoidable, since we needed that the students lived at least some semesters at university and we wanted that they had experience of this in pre-pandemic times.

Our results show that a schema-oriented view attracts more females, while an application-oriented view attracts more males. Among the students with a strong mathematical curriculum at secondary school, a process-oriented view is slightly predominant. This leads us to conclude that, with respect to gender and school type, mathematical views are different and, thus, it is possible that the origin of one's view depends also on their gender and/or school type. Our findings connect both to Griese's (2018) ones about females being more diligent, as the S view that seems to be predominant for females might be seen as mirroring this schema-oriented approach to learning, and to Roesken et al.'s (2011) one that students from different school types hold different views.

Furthermore, all the students in the sample strongly agree that, during the undergraduate studies, they were positively surprised by discovering aspects of mathematics they never imagined before, but for the students with a schema-oriented view such an agreement is more modest. Furthermore, for these students, it does not hold true that the undergraduate studies have confirmed the view of mathematics they had when they were freshmen, hence it seems that, for the students in this group, the mathematics that has been faced at university was different, but not always it was pleasant to discover such novelties. We know that professors who teach in the undergraduate mathematics course tend to promote a formalist, or an application-oriented, or a process-oriented view, hence we interpret S students' more modest appreciation of such a change with respect to school mathematics as an indicator of impermeability, of these students, to the novelties embodied in new views of mathematics. Liebendörfer and Schukajlow (2017) also remark that application-oriented views are positively correlated to interest in the first term, and process-oriented, schema-oriented and formalist views do not predict interest in the first or in the second terms. For the schema-oriented cluster, Libendoerfer and Schukajlow's findings seem to be confirmed also for students at the end of the third year. If we widen our focus to consider not only students' mathematical views, but also their professors' ones, we have to admit that, to our knowledge, no research study investigated the effect of different views, held by mathematics professors, on students' achievement and views in undergraduate studies. This could be a prompt for future research in this area.

With respect to the exam scores, we firstly notice that no low achieving student answered our questionnaire. This represents a limitation and, in a follow-up study, we will investigate why these students did not show up. Secondly, it emerged that no excellent student has a schema-oriented view. Conversely, most excellent students have a process- or an application-oriented view. Most students with a formalist view have a medium-high average of exam scores, whilst the majority of students with a schema-oriented view have a medium one. This leads to the conclusion that to develop a formalist, or an application- or a process-oriented view, can be considered a proxy for higher achievement at undergraduate studies. We can also notice that the students with a schema-oriented view are those who more strongly disagree with respect to exhaustive justifications of definitions and procedures during a lecture, and those who missed less the interaction with their peers during the pandemic. Somehow, these students show a sort of passive acceptance of mathematical algorithms and models and an individualistic approach to learning. We know that this kind of attitude towards mathematical learning are proxies for general weakness in mathematical achievement, as it is also confirmed in our data, being these students the ones who achieve the lowest grades.

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Positive changes in affective variables: Two-round action research in Hungary and Romania

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The author of this paper is a mathematics teacher in Romania, and she is interested in students' motivation and attitude towards mathematics. Teachers face challenges in the online learning environment, such as fewer direct social interactions, one of the effects of which is a reduced possibility of motivation. In collaboration with two academic experts and another teacher from Hungary, the research group sought answers to these challenges. In two-round action research, the impact of problem-posing on motivation was investigated. Informed by the experience of the other teacher in the first round, the author redesigned the second round, with more emphasis on emotional factors. The author argues that problem-posing can be successfully adapted during online learning, and by incorporating interpersonal interactions into the problem-posing process, the reduced motivational effect observed in online learning can be compensated.

Keywords: Action research, mathematics activities, problem posing.

Introduction

The global outbreak of the COVID-19 pandemic affected almost every country and territory in the world. Due to the pandemic, lockdown and rules of social distancing have led to closures of schools in Hungary and Romania, like in most countries, decreasing social interactions in the learning process. When teachers were asked to select their top three concerns about distance learning on students, the most common answers were: students' social isolation, a decrease in student well-being, and potential learning loss. Surprisingly, educators ranked students' social needs above learning loss (Flack et al., 2020). Hattie (2008) states that social interaction within classrooms is positively associated with learning outcomes.

The pandemic is also affecting teaching methods. However, teachers cannot give up specific methods such as problem-posing and its positive effects on learning. Problem-posing positively impacts motivation, while motivation and interest have a close relationship with context personalization of students' tasks (Walkington et al., 2013). Moreover, personalization is an energizing factor, which is significant for the student's motivation (Suriakumaran et al., 2017). Our research team, two teacher-researchers (including the author) and two experts, implemented problem-posing in an online environment. The principle was to use info-communication tools that the learners and teachers were comfortable using. In this article, the author focuses on a single aspect of the problem-posing, namely its positive impact on motivation (Silver, 1994). The online learning environment reduces interpersonal relationships, so the positive effect of problem-posing on motivation in the online environment may decrease. Moreover, decreased interpersonal relationships lead to decreased motivation since interpersonal relationships in students' lives contribute to their motivation (Martin & Dowson, 2009). As a result, the importance of emotional factors through interpersonal relationships in online education should be growing, which implies our research question: *Does the increased role of emotional factors impact the context personalization and, through this, students' motivation?* The author inferred motivation using content analysis that unfolds context-personalization characteristics. An online questionnaire followed the analysis with questions on motivation.

Theoretical background

Motivation can be described as the student's willingness or desire to participate and succeed in the learning process. Weiner (1992) defines motivation as an individual's desire to act in specific, personal ways. Walter and Hart (2009) describe sources of motivation as task interest, social environment, opportunity to discover, knowing why, using objects, and helping others. Some researchers emphasize the importance of context personalization that helps to target students' out-of-school interests and experiences (Cordova & Lepper, 1996). Walkington et al. (2013) describe context personalization as an approach to learning in school. Personalized problems may make mathematics more accessible to students, they may help bring the "real world" problem solving closer to "school mathematics," and they may attract students' attention and interests to impact motivation (Boaler, 1994).

In this paper, the author uses the concept of problem posing in the following sense.

> By problem posing in mathematics education, we refer to several related types of activity that entail or support teachers and students formulating (or reformulating) and expressing a problem or task based on a particular context (which we refer to as the problem context or problem situation). (Cai & Hwang, 2020, p. 2)

Several aspects of problem-posing (PP) are thought to have meaningful relationships to student disposition toward mathematics. For example, posing offers a means of connecting mathematics to students' interests. Within a classroom community, students could be encouraged to pose problems that others in the class might find exciting or novel (Silver, 1994).

Ellerton (2013) proposes the Active Learning Framework (ALF) for PP in mathematics classes, defining four steps: 1. The teacher models an example (processing the new content) 2. Students solve problems based on model 3. Students pose problems with the same structure as model 4. Finally, the class discusses and solves problems posed by students as "My classmate's problem." This framework considers PP in classrooms an essential activity that allows students to consolidate their knowledge and think critically about it.

The affective domain is defined in many ways in educational and psychological literature. Often it is used as a broad umbrella concept that covers attitudes, beliefs, motivation, emotions, and all other noncognitive aspects of the human mind (Hannula, 2020). This paper focuses on self-concept, anxiety, motivation, perceived usefulness, and enjoyment of mathematics. Mathematical self-concept refers to one's ability to learn and perform mathematical tasks, how confident one is in learning new mathematical topics, and one's interest in mathematics (Reyes, 1984, p. 560). Mathematics anxiety is defined as a sensation of stress and apprehension that interferes with mathematics performance ability, number manipulation, and problem-solving (Richardson & Suinn, 1972). Perceived usefulness refers to how students can relate school mathematics to real-life (Reyes, 1984). Finally, the usefulness of mathematics includes liking mathematical terms, symbols, or routine counting, but liking mathematics problems as well (Aiken, 1974).

The circumstances of the action research

The challenge of action research is how to adapt the ALF model to the online environment while retaining the motivational potential of problem-posing. This question touches on all components of ALF, i.e., how to present the model problem, how to practice, how to do the problem creation, and how to process the classmate's task.

Participants

The participants of the research are four 6th grade classes from Hungary and Romania. The teachers of the two classes in Hungary are the same person. Similarly, the two classes in Romania are taught by the same mathematics teacher. 125 students participated in the experiment: 65 from School 1 (Hungary) and 60 from School 2 (Romania). The language of instruction is Hungarian in both schools, being Hungarian the students' mother language.

Method

Curricula and syllabi are different in the two countries, which led us to divide them into rounds of our action research. The first round took place in School 1 in the spring semester of 2020, while the second one in School 2 in the autumn semester of 2020. The researchers implemented the ALF method for the following curricular lessons in both schools. 1. Proportional division (two PP activities), 2. Straight and inverse proportionality (two PP activities), 3. Percentage calculation, calculation of the percentage base (one PP activity). Both rounds were organized under online teaching conditions ordered due to the viral situation. This circumstance allowed us to focus on the challenges and experiences gained from the first round and transform them into opportunities. Table 1 contains similarities and differences between the rounds in each phase of the ALF.

Table 1: Differences between the two rounds of ALF

ALF step	Round 1	Round 2
Model problem	Slideshow that the student works up at home.	Slideshow that students work on with the teacher in an online lesson.
Practice	Self-regulated learning through a presentation.	Teacher-regulated learning through a presentation in an online lesson.
Problem-posing	Homework to be sent to the teacher.	Homework to be sent to the teacher.
Classmate's problem	Based on the teacher's selection.	Based on the teacher's selection.
		Who is the problem poser?
	Homework.	Online classroom: individual work.
Evaluating the solution	By the teacher.	By the problem poser.

Round 1: The teacher prepared a slideshow with narration and timing, which included the stages of introduction and practice. Students had the opportunity to play the slideshow several times before they started the problem-posing activity. After collecting the posed problem, the teacher selects one (or two) as “My classmate’s problem” and sets as homework that are also collected.

Round 2: The teacher prepared the same slideshow, but instead of sending it out, she presented it via Google Meet lessons, where students can comment, ask, talk. Problem posing activity is homework as in Round 1, but “My classmate’s problem” is also discussed live and solved during online lessons. Solving “My classmate’s problem” was preceded by a discussion with the topic: Who is the problem poser? Why do you think that? As a new step, after revealing the problem poser’s name and solving the problem, they added the story that inspired them in the process of the PP. Solving the classmates’ problem was an individual task followed by a discussion of whether the problem poser accepts or declines the solution of the class. Through these conversations, emotional factors were given a more significant role in Round 2, letting us know each other’s stories. Essentially, it is about bringing the task to be solved emotionally close to the students, thus making them more motivated to solve it.

Collecting Data

Parts of the research material are students’ work, students’ answers to the questionnaire, video recordings of Round 2 lessons, and teachers’ reflections. All the student works were coded, taking into account the coding frame for evaluating out-of-school interest (sports, video games, pets, sports, info-communication tools, social media, food) based on Walkington et al. (2013). It was also identified two other ways (besides out-of-school-interest) that students use to express their presence in the posed problems that the author calls personal traits:

a. Student’s direct presence in the posed problem by using the first-person singular. The personalization is given by the conjugation and pronoun “Me/I” and not the activity. For example, “My favorite T-shirt was finally on discount.” In the second case, the name of the “hero” in the story coincides with the student’s name who posed the problem: “*Zsófi* does 3 km in 50 mins.”

b. Actual or happened situations: The written context refers to an event that happened lately. For example, the following problem refers to the actual Black Friday: “Peter buys a new laptop, with a 30% discount it is 1500 Ron on Black Friday. What is the laptop’s original price?”

Results

We observed two developmental aspects of affect in Round 2: the role of different emotions, values in PP, and the role of affective factors in interpersonal relationships between students, teachers.

Role of different emotions, values in problem posing

Having different emotions, values, hobbies, and interests encourages students to remark their presence (directly or indirectly) in the posed problems. These interests were discovered in many of the students’ work. They include their personal interests in their problems; this is how they express themselves and their bonding to the subject. We managed to bring the “real world” problem solving closer to “school mathematics” problem solving, which may attract students’ attention; they could be saying, “this happened to me/might have happened to me”.

The personal trait appears in 9% of all problems (Table 2), and there is no significant difference between the two schools (21% in School 1 and 14.8% in School 2; the Fischer test result is 0.12>0.05). Thus, this form of personalization follows from the PP activity itself, and the research suggests that emotional factors have little influence.

Table 2: Number of problems with the sign of a personal trait

Appearance of personalization	Problems with a sign of a personal trait	Problems with no sign of a personal trait	Sum
School 1	46	173	219
School 2	28	160	188
Sum	74	333	407

In Round 2, we observed that more personalized context appeared in the posed problems (Table 3). For example, code “out-of-school-interest” was identified in 67 cases in School 1 (30.6% of all problems), while in School 2, this number is 76 (40.4%).

Table 3: Number of problems with the sign of out-of-school interest

Appearance of personalization	Problems with the sign of out-of-school interest	Problems with no sign of out-of-school interest	Sum
School 1	67	152	219
School 2	76	112	188
Sum	143	264	407

Personal interest was more prevalent in the second round, and the difference is significant. Fischer’s exact test result is $p = 0.047$; the result is significant at a significance level of 0.05. We explain the difference by the increased role of emotional factors because they felt motivated to use their out-of-school interest in the posed problems.

Role of affective factors in interpersonal relationships between students, teachers

During Round 2, students were asked to guess the author of “My classmate’s problem”. The following discussion was transcribed from the second online lesson.

Teacher: [sharing her screen with the posed problem] Firstly, I would like you to guess who could have written this problem?
Student 66: Mrs. Teacher, it is obviously G. (Student 82).
Student 62: This is G.
Many: Yes, G. wrote it.
Teacher: Why is it obvious that it is G? She has hens with lice?
Student 69: She has stories like this.
Student 62: She has lived on a farm.
Many: [laughing]

Teacher:	G, got your hens lice, or what?
Student 82:	[laughing] No, but my grandma used to have hens. That's where the idea came from.
Teacher:	I love this problem.

As students talked about the possible problem posers, the teacher could join the conversation and observe their relationship. The teacher found out someone's pet's name, for example, and who had known that before. These conversations offer a social interaction between students, including the teacher, improving Walter and Hart's (2009) motivational factors such as task interest, social environment, and helping others. The teacher observed that developing the online social environment had been contributing to a more inclusive classroom atmosphere. These problems were very memorable; even a few months after the PP activities, some students fondly remembered the lessons, suggesting that they liked solving problems about G's hens.

Round 1's ALF included affective factors such as self-concept and motivation via personalization of the context. Telling and listening to classmates' problems added additional factors. Figure 2 shows that listening to a story that inspired a classmate arouses other students' interest, leading to the enjoyment of mathematics, decreasing anxiety. By telling the story that inspired the student in the process of PP, he or she would pose a more realistic problem, which can change the student's view of the usefulness of mathematics.

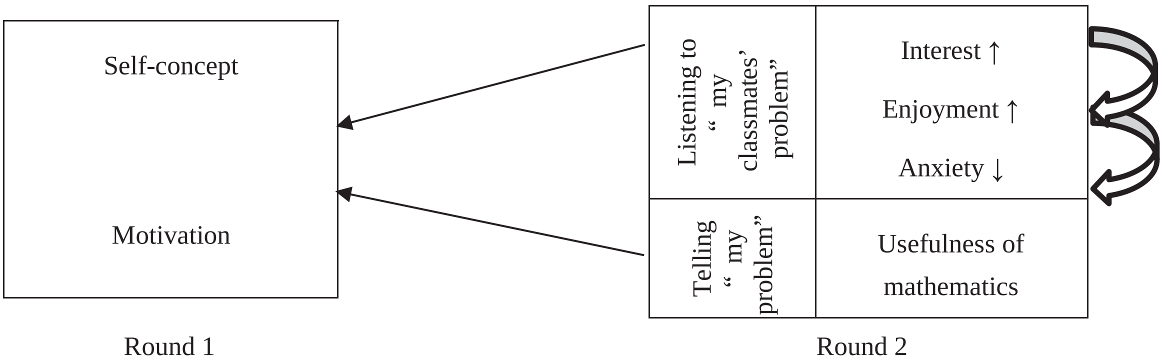

Figure 1: Affective factors added in Round 2

Knowing the background story while solving each other's problems seemed interesting to students, such as realizing that their story could have happened, which was revealed from the classroom discussions. Analysis of the answers to the questionnaire also supports this observation. At the end of the teaching unit, students were asked to answer 16 questions leaving their opinions about this new teaching-learning method. Three of them are highlighted here, I6, I8, and I9 (Table 4). We used Likert scale items with three grades according to the children's age specifics: Agree/Cannot decide/Disagree. Only definitive answers were evaluated (agree, disagree). Three items were assigned to motivation, belief, and emotion. To analyze the data statistically, contingency tables were created. The distribution of the answers is shown in Table 4.

Students enjoyed solving the classmate's problem in both schools, but the ratio of those who agreed with I9 was greater in the second round (95.7% versus 84.8%). However, the difference is not significant ($p = 0.12$). Table 4 also shows that almost 75% of students are happy to pose their problems. The author considers this fact as a sign that the adaption of the ALF paradigm was successful in both rounds. The positive answers to I8 suggest the idea that students are aware of the fact that their problems are realistic and support our finding that problem-posing positively influences students' perceptions of the relationship between school mathematics and real life. There was no significant difference between the two rounds in this respect.

Table 4: Students' responses to the three questionnaire items

	I9: I liked it when the classmate's problem was the homework.		I6: I liked to come up with my own task for math lessons.		I8: I have created tasks that can occur in real life.	
	Agree	Disagree	Agree	Disagree	Agree	Disagree
Round 1	28	5	22	8	24	4
Round 2	45	2	32	11	39	3

Conclusion

Although the participants of the two rounds were children from different countries, that could be a limitation of the study regarding the statistics, but also an advantage, which suggests a new plan for the future. The author desires to try out this method with the same students on different curricula and the same curriculum with other students; if the pandemic allows, in situ in the classroom. The author found that PP has a motivating effect in the online environment as well, and even this positive effect can be increased by including emotional factors. The author also claims that problem-posing opens up space for the expression of personality. It is an opportunity for the teacher, even in the online environment, to learn what is currently occupying the student, their hobby/circle of friends, even to conclude relationships between them. The author finds that students pose more context personalized problems by increasing emotional variables and are happier to solve their classmates' problems than with fewer affective factors. By and large, the adaption for the online learning environment went successfully. Based on the literature (Martin & Dowson, 2009) and the experiment, the author claims that more social interactions promote context-personalization, which triggers motivational factors.

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Students' perceptions of learning mathematics in the WhatsApp environment through the "Bagroup" project

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This paper is part of a larger quantitative and qualitative study that focuses on how teachers and students perceive learning of mathematics via WhatsApp. The main objective of the program is to allow for both teacher-student and student-student contributions of knowledge and to boost students' sense of self-confidence and authorship in the discipline. The research participants of this study comprise 152 Grade 11 and Grade 12 students who responded to a questionnaire distributed during the 2018 program. We present a quantitative analysis of the factors that students perceive to be pertinent in the WhatsApp Bagroup learning environment.

Keywords: Social media, communication, WhatsApp in mathematics education.

Introduction

The fact that technological social platforms can be useful for the purpose of schooling and education is increasingly recognized and studied (Greenhow & Lewin, 2016). Online social networking platforms have become highly popular virtual meeting places for youth and adults (Boyd, 2010; 2014; Greenhow & Askari, 2017).

It is not surprising then that, recently, there has been an influx in research focusing on the pedagogical benefits of using social media as productive sites for promoting learning. Having said that, an examination of recent studies shows that focus has been put on post-secondary settings (e.g., Dyson et al., 2015) or teacher education (e.g., Sendurur et al., 2015). Research on the use of social media for subject-specific learning purposes, such as mathematics, among K-12 remains understudied. Given the prevalence of the use of social media in everyday life especially among teenagers, we conjecture that the use of these platforms can potentially increase student-teacher and student-student interactions. These interactions around mathematical ideas may allow almost instantaneous response and feedback from others and surface mathematical misconceptions and misperceptions that can be directly addressed in follow up posts (Biton & Segal, 2021; Freeman et al., 2016).

Literature review

Recent studies have explored the use of social media platforms such as Twitter (Carpenter & Krutka, 2014), Pinterest (Hertel & Wessman-Enzinger, 2017), and Facebook (Biton et al., 2015). These studies have suggested that interaction over social media contributes to socializing (Madge et al., 2009) and increased self-esteem and social capital (Steinfield et al., 2008). Such affordances need to be further investigated within mathematics education. With respect to learning and instruction, Asterhan et al. (2013), and Asterhan and Bouton (2017) for example, discussed how social media can be used to develop innovative, collaborative forms of online learning that extend beyond the traditional classroom. Students perceive learning through social media to be very intensive and highly collaborative in nature (Meishar-Tal et al., 2012). From the students' standpoint, social media platforms have been found to also allow for the manifestation of learners' emerging conceptual understanding in mathematics (Baya'a & Daher, 2013; Freiman, 2008).

Relatively little research focused on students' perspectives on learning mathematics through WhatsApp. A case in point is a recent study by Rosenberg and Asterhan (2018). The researchers looked into teacher-student interactions; students' evaluation of the efficacy of contact with their teachers; the social dynamics in these groups compared with that in in-person classrooms. It was found that teachers used WhatsApp not only to manage housekeeping-related matters but also for instructional and learning purposes that were initiated either by them or by students in the group. Some of the drawbacks identified revealed issues regarding information overload, social pressure due to the public nature of the communication, and accessibility to WhatsApp. Drawing on this research, it still remains to be surfaced how students perceive the use of WhatsApp as a means to learning school mathematics, which is where our research aims to contribute to knowledge.

Methods

The study presented herein is part of a larger research project that focuses on exploring how students and teachers perceive learning over a social network—specifically WhatsApp—and identifying learning opportunities to eventually inform pedagogical approaches and instructional strategies. This paper particularly aims at identifying and characterizing students' perceptions of the WhatsApp "Bagroup" program learning environment. We formulated five overarching research questions for our larger research study. In this paper, we focus on the following: What factors do students perceive to be important when learning in the WhatsApp "Bagroup" program environment?

The "Bagroup" initiative.

The "Bagroup" is a state-wide program that was set up for the Ministry of Education by the Center for Educational Technology. The purpose for this initiative was to offer the use of the WhatsApp application to high school students throughout the country and across grade levels in preparation for their mathematics matriculation exams (*Bagrut* in Hebrew). Approximately 4,000 high school students joined the mathematics WhatsApp groups 2018. (This was an increase of more than 100% with respect to the two preceding years, which had less than 2,000 registrants.) Forty groups were set up with 100 students in each. An experienced teacher of mathematics was assigned for each group. Group members are generally not acquainted with each other at the outset. Learning is divided into various topics each of which are studied over a set time period and led by a teacher. Once every few days, the teacher introduces a topic or answers questions raised by the students and uses various digital means. The lessons are accompanied by a repository of materials in the accompanying site that are available for use by the students. Learning in the group takes place in a variety of ways: text messages, voice messages, photos and video-clips, question repositories, presentations, and more.

The groups ran between two to four months up to the exam date during which hundreds of thousands of messages were sent that included mathematical content, questions, and explanations to solutions. The Bagroup project offers the following to its students: [1] Support: Students can get an (almost) immediate response from a teacher or another student from the group. [2] Equal access: Using the freely available WhatsApp technology made it possible for anyone to join at no cost. [3] Use of mobile technology: Students’ digital devices supplemented classroom instruction and extended learning to anywhere and anytime. [4] Capacity building: The program is aimed at enhancing understanding of mathematical concepts and, by extension, affective dispositions towards mathematics. [5] Variety: Students shared multiple approaches to solving mathematical problems.

The program was launched in 2015 on Facebook for mathematics only and then in 2016, was moved to the WhatsApp platform. The program saw a growth from 614 students in 2015 to 8,500 students in 2019 (unpublished figures). Also, the 2019 program added the subjects of English and Hebrew.

Participants and data collection

Participants comprised 152 students in Grades 11 and 12 who answered a questionnaire that was distributed to the students in the 2018 program. Nonprobability sampling was used based on participants’ availability and willingness to reply to the questionnaire. The questionnaire was put together by the first and third author and revised for accuracy and consistency. The student questionnaire included nine open-ended questions and twelve Likert-type questions. The open-ended questions invited students to share a memorable experience or an episode they had in any of the WhatsApp interactions. They were also invited to share the most obvious advantages and disadvantages of using WhatsApp for the purpose of learning high school mathematics and to reflect on specific characteristics of WhatsApp that make the mathematical content accessible to the students. The open-ended questions also requested possible explanations regarding students who left the group. The other part of the questionnaire provided a 6-point Likert-type scale for 12 statements concerning the experience of learning mathematics via WhatsApp with 1 = Strongly Disagree and 6 = Strongly Agree.

Data analysis

Students’ responses were analyzed and coded for themes by the first two authors through content analysis of the responses to the open-ended questions. A Varimax rotation analysis was used of the data, the goal being to detect how the WhatsApp learning environment was perceived by the students by demonstrating the items are inter-correlated for internal consistency.

In this paper, we present an analysis of the quantitative data gleaned from the students’ questionnaires.

Findings

Analysis of the twelve Likert-type scales found high internal validity of three categories that explained more than half of the variance in the students’ answers (58.06%). The three categories were the following: [1] Factors in the WhatsApp environment that support students’ emotional needs. This category accounted for about a quarter of the variance among the students’ answers (five statements). [2] Factors in the WhatsApp environment that promote learning. This category accounted for approximately another fifth of the variance of the students (four statements). [3] Factors that hinder learning in the WhatsApp environment. This category accounted for an additional 8% of the variance in students’ answers (two statements).

Table 1 presents the mean scores and standard deviations of all the statements, divided into the three categories. The results indicate that students' perceptions about learning in the WhatsApp environment was varied.

Table 1: Means and standard deviations of the categories and statements regarding the students' experience learning mathematics via WhatsApp, N = 152

Statement	Mean	SD
Category 1: Contribution of the WhatsApp environment made to students' emotional needs	3.51	1.15
I manage to follow the lesson during the WhatsApp discourse	3.46	1.65
I managed to understand the material in the WhatsApp environment at the same level as in a regular class	3.84	1.30
WhatsApp allows participation without fear of making errors	3.73	1.68
The written discourse in WhatsApp is preferable over the verbal discourse in a regular class	2.80	1.53
Learning through WhatsApp allowed me to meet my specific needs as a learner more than in a regular classroom	3.72	1.61
Category 2: Factors that promote learning in the WhatsApp environment	3.81	1.13
Learning via WhatsApp allows me to learn from others more than in a regular class	3.99	1.61
The value of a lesson via WhatsApp is different from the value of a regular lesson	4.65	1.40
Learning via WhatsApp improved my mastery of available technologies for learning mathematics	3.27	1.68
I invest more time and effort in mathematics in the WhatsApp project compared to students in a regular class	3.04	1.68
In the future, WhatsApp will become an important tool for students for learning math	4.09	1.67
Category 3: Factors that hinder learning in the WhatsApp environment	3.09	1.29
There is some mathematical content that cannot be explained via WhatsApp	3.53	1.73
I can't manage to cooperate with other students the way I do in a regular class	2.66	1.47

To facilitate analysis, the levels of agreement were reduced to three: disagree, somewhat agree, and agree. Figure 1 graphically illustrates the percentage of students that agreed, somewhat agreed, or disagreed with each of the twelve statements.

In the first category Contribution of the WhatsApp environment to learners’ emotional needs, the

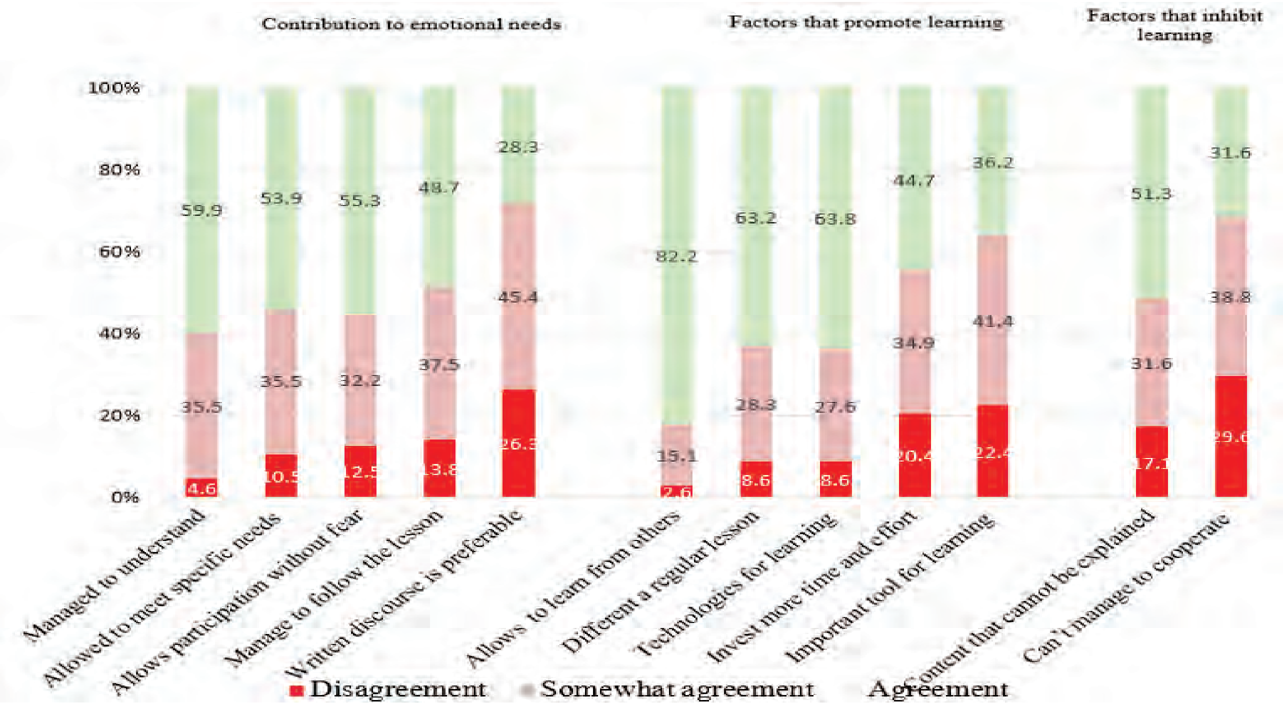

Figure 1: Proportion of students who agreed, somewhat agreed, or disagreed with the statements

highest agreement was with the statement: “I managed to understand the material in the WhatsApp environment at the same level as in a regular class.” Looking at Figure 1, we can see that 95.4% of the participating students agreed (59.9%) or somewhat agreed (35.5%) with this statement. The next highest statement in the category of emotional needs: “WhatsApp allows participation without fear of making errors,” with 55.3% and 32.2.% agreeing or somewhat agreeing with this statement, respectively. “Learning through WhatsApp allowed me to meet my specific needs as a learner more than in a regular classroom” scored third. Here, again, the great majority of students agreed or somewhat agreed with this statement (53.9% and 35.5%, respectively). However, the statement: “The written discourse in WhatsApp is preferable over the verbal discourse in a regular class” received the lowest mean score in this category, but still maintaining a high level of agreement 28.3% and 45.4%). Under the second category “Factors that promote learning in the WhatsApp environment,” there is a high degree of agreement with “The value of a lesson via WhatsApp is different from the value of a regular lesson.” Collectively, 91.5% agreed (63.2%) or somewhat agreed (28.3%) with this statement. Within this category, a high degree of agreement was also found with the statement: “In the future, WhatsApp will become an important tool for students for learning math” demonstrating that 77.6% of the students either agreed (36.2%) or somewhat agreed (41.4%) with this statement.

As well, the statement: “I invest more time and effort in mathematics in the WhatsApp project compared to students in a regular class” scored the lowest mean points (3.04%). However, overall, the great majority of the students still agree 44.7% or somewhat disagree 34.9% with the statement and the remaining 20.4% disagreeing with it. Notably, the WhatsApp environment, through students’ perspectives, allows to learn from others with 82.2% agreeing with this statement and 15.1% somewhat agreeing with this statement.

Discussion

The study examined the popular phenomenon of using WhatsApp, particularly its use to support learning through the national "Bagroup" program initiative whose objectives are to prepare students for the mathematics matriculation exams and contribute to students' self-confidence in mathematics. This paper reports some quantitative results on how students perceived learning mathematics using WhatsApp. The findings demonstrate that, for the most part, students felt that learning through WhatsApp contributed to their emotional needs. They were able to understand the material, found their peers' contributions helpful, felt confident to take an active part in the group without fear of making mistakes, and perceived the WhatsApp environment as one that was conducive to their learning from their peers more efficiently than in a regular classroom. Nevertheless, almost a fifth of the participants felt that this environment was not suitable for some mathematics topics. This requires further investigation as to the specific content they refer to and the pedagogical and instructional approaches to be adjusted. In addition, almost a third believed that the platform did not always allow the type of collaboration possible in a normal classroom. Here too, further investigation will shed more light on patterns of collaboration and learning mathematical content.

The findings also demonstrate that students believe that learning in the WhatsApp environment promoted their learning of mathematics (second category). with the most notable findings suggesting that this environment allows students to learn from each other and persevere more in solving mathematical problems (i.e., invest more time and effort in mathematics). We wonder whether opportunities to learn from other students (82.2% agreeing with the statement and 15.1% somewhat agreeing with the statement) may be associated with other factors such as not fearing making mistakes, or with the anonymity that the environment provides where classroom-related social roles become blurred and the threat of losing face (Lin & Yamaguchi, 2011) is mitigated.

The analysis of students' responses to the open-ended questions may shed some more light on the differences in students' experiences. Specifically, the relationship between students' perception of mathematical content that is not as easily managed on WhatsApp (Category 3) and their managing to understand the mathematical concepts that were taught via WhatsApp (Category 1). As well, looking into students' perception of not being able to cooperate with others (two thirds of the respondents agreed (31.6%) or somewhat disagreed (38.8%)) while noting their perception of WhatsApp allowing them to learn from others in the group (82.2.% agreeing and 15.1% somewhat agreeing).

The findings of this study suggest that using the WhatsApp environment in learning mathematics may provide positive experiences where learners receive opportunities to take ownership over mathematical ideas thus developing self-confidence and authorial identity in mathematics. This ties with Fellus' (2019) broad conceptualization of learners' mathematical identity that offers a unified framework to understanding students' experiences in learning mathematics. Specifically, how students talk to and about themselves as learners of mathematics, what opportunities they get to take ownership over mathematical content, and the socioculturally available metanarratives about who can and who cannot do mathematics. As we further investigate learners' experiences with the WhatsApp environment, we will be looking more deeply into the relationship between the affordances the platform offers and one's opportunities to take ownership over mathematical ideas and to develop tolerance towards errors as ways to deepen conceptual understanding and to foster perseverance.

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Community detection for undergraduate mathematical views

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Keywords: Academic aptitude, network analysis, postsecondary education, cluster analysis.

Beliefs are propositions about a certain topic that are regarded as true (Philipp, 2007), they are individually held (Erens & Eichler, 2019) and tend to form clusters as they "always come in sets or groups, never in complete independence of one another" (Green, 1971, p. 41). A belief is seldom isolated, but it is connected to other beliefs, which are part of the same cluster of beliefs. According to Green (1971), belief clusters are coherent families of beliefs across multiple contexts. For this reason, beliefs can be understood in terms of views of mathematics (Grigutsch, Raatz & Törner, 1998), namely as epistemological beliefs about mathematics, its teaching and learning. According to Grigutsch et al. (1998), it is possible to outline four different views: a process-oriented (P) view that represents mathematics as a creative activity consisting of solving problems using different and individual ways; an application-oriented (A) view that represents the utility of mathematics for real world problems as the main aspect of the nature of mathematics; a formalist view (F) that represents mathematics as characterised by a strongly logical and formal structure; a schema-oriented (S) view that represents mathematics as a set of calculation rules and procedures to apply for routine tasks. Erens and Eichler (2019) operationalised these four views into a Likert-scale questionnaire made of 24 statements, each of which is assigned to a specific view. Examples of statements are: "mathematics helps to solve tasks and problems that originate from daily life" (A), "logical strictness and precision are very essential aspects in mathematics" (F), "in order to comprehend and understand mathematics, one needs to create or (re-)discover new ideas" (P), "doing mathematics demands a lot of practice in adherence and applying to calculation rules and routines" (S).

In this poster, we aim at showing a methodology for clustering these statements and checking whether the four categories defined for the aforementioned four views are a posteriori confirmed by a survey administered to 93 students enrolled in the third year of undergraduate studies in mathematics at the University of Torino (more details can be found in Andrà, Magnano, Brunetto & Tassone, submitted). On one hand, we claim that the need for an a posteriori analysis resides in the importance to verify the reliability of the measurement tool. On the other hand, the methodology shown here represents a novelty, as it is based on network analysis.

A network is a set of nodes connected to each other through an edge. Indeed, a network N is a pair (V, L) where V is the set of nodes and L is a subset of the Cartesian product $V \times V$. There are two mathematical tools that allow one to analyze a network: (i) the graphical representation of the network, and (i) the adjacency matrix, which describes the connections between nodes as its component $a_{ij}>0$ if the nodes i and j are connected by a link, otherwise is it null.

One of the potentialities of network analysis is the possibility to use clustering tools that do not require particular metric definitions. These techniques are called community detection as the goal is to identify subnets of nodes characterized by relatively large internal connectivity, namely nodes that tend to connect much more with other nodes in the group than the rest of the network. To this end, we used the so-called "Louvain method" which is based on the optimization of a quantity called modularity (Q). Given a partition into k sub-graphs $\{C_1, C_2, \ldots, C_P\}$ of the network, the modularity (Q) is the normalized difference between the total weight of the links inside the sub-graphs C_k and the expected value of total weight in the randomized "null network model" (Newman, 2010).

It is also important to recall that a network can be built as a projection of a bipartite network, namely one made up of two distinct classes of nodes V_n and V_m, and links can only connect nodes of different classes. This is the case of the data analysed in our study, since the network has been built with nodes identified by both the 93 students and the 24 items. A student-node is connected to an item-node with an edge weighted 1, 2, 3, 4, or 5 depending on the rank assigned to it on a 5-point Likert scale. Student-nodes are not directly connected to each other, as well as item-nodes. Another typical example of a network of this sort is the Amazon network, where consumers are linked to each product if they have purchased it at least once. However, it is possible to define a link between consumers if they have purchased the same product, or a link between products if they were purchased from at least one same consumer. This method allows one to create a similarity metric between participants or between questions in the context of questionnaires (Brunetto, 2017).

In our poster, we confirm an almost perfect correspondence between a priori classification of Erens and Eichler's (2019) work and our investigation on a sample of undergraduate mathematics students a posteriori, and we extend clustering to other aspects of mathematics teaching and learning.

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Fuzzy cognitive analysis in undergraduate mathematics class on engagement, motivation, and participation during covid-pandemic

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This work focuses on Distance Learning during the COVID-19 pandemic to improve undergraduate students' motivation, participation, and engagement. Experimentation in a Mathematics STEM class evaluated the impact of Distance Learning on students' motivation, participation, and engagement levels, computed through a Fuzzy Cognitive Map. It was performed on some affective and interaction parameters derived from using an adaptive e-learning platform and from the answers of a semi-structured questionnaire. The results have been analyzed through Technological Pedagogical Content Knowledge and Instrumental Genesis theories.

Keywords: Virtual environment, fuzzy cognitive map, motivation, participation, engagement.

Introduction

Distance Learning (DL), at the time of COVID-19, was a didactic of "being there", as defined by the philosopher Heidegger, in the etymological sense of "being-out" (ex-sisto), in which “being” is meant as the projection to the future, an opening to possibilistic horizons. DL was characterized by using methodologies and technological tools to keep students' participation, engagement, and motivation high. It was not just a question of using technologies in teaching but an actual adaptive process to using technologies as the only way to learn. This study is framed by the theory of instrumental genesis (Verillon and Rabardel, 1995), integrated into the Technological Pedagogical Content Knowledge Framework (TPCK), showing how the didactic contents delivered in the presence have taken on a new aspect with distance learning (Mishra and Koehler, 2007). Some theories on motivation (Skinner, 1935; Fredricks et al., 2004) were used to analyze the students' motivation to follow lessons at a distance, to be virtually present in the classes (Shonfeld et al., 2020), and to use the teaching tools made available by the teachers. It is not rare to find many students who leave the online learning course shortly after the beginning; such a phenomenon, called dropout, is always more frequent among the students who are not sufficiently engaged and motivated with the learning experience. The root causes of students dropping out are the lack of motivation, engagement, and participation (J. Keller and K. Suzuki, 2004). Our contribution, in this paper, is to model teaching in an adaptive way using adequate technological tools to cope with panic and frustration in which students suddenly found themselves trying to stem a possible wide phenomenon of dropout (Capone, 2022). The motivation considers the level of interest in the course, the engagement represents the level of involvement in the learning experience (Kuh, G. D., 2003), whereas participation (Dominguez, R. G., 2012) refers to the action of taking part in activities and projects, the act of sharing in the activities of a group. For these reasons, a modern course cannot be limited to the simple learning content delivery task. Still, it should support the learners in their whole learning experience, leading them to reach their learning objectives successfully. A case study was conducted to evaluate whether our didactic models adapted to this emergency have helped to maintain a sufficient level of engagement, motivation, and participation. The study was carried out with first-year engineering students of 2019-2020 in mathematics class. The case study has nomothetic and ideographic intentions by collecting quantitative and qualitative data. The quantitative data include the students' interactions on the used e-learning platform, the results of the tests, and a semi-structured questionnaire according to the Likert scale. In addition, the questionnaire anonymously proposed to students also included open-ended questions from which to infer, through qualitative analysis, the motivational state of the student. Finally, from the qualitative and quantitative data that emerged, a Fuzzy Cognitive Map (FCM) was built to derive the levels of the students' engagement, motivation, and participation parameters. Based on completely remote teaching, the obtained results are compared with those obtained in a previous experiment (Capone and Lepore, 2021). The data seem to show that, thanks to the use of an adaptive e-learning platform and restructuration of the educational content through technologies, the students have maintained an adequate level of engagement and motivation parameters comparable to those of the previous years. On the other hand, participation was lower than the previous experience due to a drop in concentration and frustration of being at the computer the whole day to follow the online lectures. The dropout was instead contained within acceptable values, and in terms of students’ achievement of competencies, the results are comparable.

Conceptual Framework

The framework of the Technological Pedagogical Content Knowledge (TPCK), the idea of instrumental genesis proposed by Vérillon and Rabardel (1995), and the theory of Fuzzy Cognitive Map are tools to describe and model complex systems/environments.

Technological Pedagogical Content Knowledge Framework

The Technological Pedagogical Content Knowledge (TPCK) framework conceived by Shulman in 1986, defines which elements can characterize teaching supported by technologies without neglecting the pedagogical aspects and the specific teaching contents of the discipline. We follow Mishra and Koehler (2006) that make further clarifications by describing the meaning of the intersections between TK and CK, between PK and CK and between TK and PK. They specify the following: Pedagogical content knowledge (PCK) is concerned with the structure, organization, management, and teaching strategies for how the particular subject matter is taught; Technological content knowledge (TCK) is related to how one specific subject matter is represented in technology-rich environments. Teaching with technology requires knowing the subject and how subject matter can be changed with the application of technology, and this knowledge is called TCK; Technological pedagogical knowledge (TPK) is concerned with how teaching and learning change due to integrating technology into instruction and how a teacher should choose a particular tool for a specific task considering its affordances and limitations. Technological Pedagogical Content Knowledge (TPCK) "is an emergent form of knowledge that goes beyond all three components" (p. 1028). According to the transformative model, TPCK is different from "knowledge of a disciplinary or technology expert and from the general pedagogical knowledge shared by teachers across disciplines" (p. 1029). This model helps to interpret the attitude of teachers in the face of innovation in their didactic.

The Instrumental Genesis

The idea of instrumental genesis (Verillon and Rabardel, 1995) seems suitable to describe the use of Distance Learning in mathematics education. The instrumental genesis distinguishes an artifact (an artificial object/instrument) from an instrument (a psychological construct) by defining the instrument as a mixed entity composed of both components related to the characteristics of the artifact and subjective components (patterns of use) that come out from the situated instrumented activity or the activity involving a subject (as a user), an artifact (as an instrument) and an object (as epistemic transformation, for example, the knowledge of functions in two variables). This hybrid entity considers the thing and describes its practical use for the subject. A scheme is a systematic procedure for using a given instrument to achieve a given purpose. The elaboration and evolution of instruments is a long and complex process that Rabardel calls instrumental genesis. It is articulated in two processes: instrumentalization, related to the appearance and evolution of the different components of the artifact, for example, the progressive recognition of its potentials and limits; instrumentation, related to the appearance and development of patterns of use. In our case, the didactic activities carried out with AR involved students (as users and protagonists of the didactic-educational path), technological objects such as 3D glasses, tablets, and PCs (as instruments), and an object, intended in an educational sense as a mathematical item to be recognized, internalized and contextualized (the real functions of two real variables). The teacher thus makes use of technological artifacts that undergo a triple process of instrumental genesis: from the didactic point of view, because their use is aimed at generating knowledge and enhancing skills; from the pedagogical point of view, because their use is subordinated to suitable teaching methods activated by the teacher and aimed at the construction of mathematical meanings; from a technological point of view, because the use of technologies is not an end but implements an effective mobilization of strategies aimed at learning.

Fuzzy Cognitive Map

In this work, parameters related to Interaction, Participation, and Motivation, and their causal relationships, have been identified and analysed through a Fuzzy Cognitive Map to describe the student's state during the course. Kosko defines an FCM as a graph structure for representing casual relationships. It can symbolically describe complex systems/environments, highlighting events, processes, and states. In an FCM, a node of the graph is called a concept, and an edge is called weight. The edge allows for implementing a causal relationship between two concepts. The weight represents the strength of the influence of the relationship, described with a fuzzy linguistic term (e.g., low, high, very high, etc.).

An FCM can be formalized through a 4-tuple (N, E, B, f), where:

1. $N = \{N_1, N_2, \ldots, N_n\}$ is the set of n concepts that are represented by the nodes of the graph.
2. $E: (N_i, N_j) \rightarrow e_{i,j}$ is a function $(NxN \rightarrow [-1,1])$ which associates the weight $e_{i,j}$ to the edge between the pair of concepts (N_i, N_j);
3. $B: N_i \rightarrow B_i$ is the activation function which associates to each concept N_i a sequence of activation values, one for each time instant t: $\forall t, B_i(t) \in [0,1]$ is the activation value of the

concept N_i at time *t*. $B(0) \in [0,1]^n$ is the initial activation vector containing the initial values of all the concepts; $C(t) \in [0,1]^n$ is the state vector at a specific time instant *t.*

4. *f*: R → *[0,1]* is a transformation function with a recursive relation for $t \geq 0$ between $B(t + 1)$ and $B(t)$:

$$\forall i \in \{1, \dots, n\}, B_i(t+1) = f\left(\sum_{\substack{i=1 \\ j \neq i}}^{n} e_{ji} B_j(t)\right) \text{ (Eq. 1)}$$

FCM is developed by integrating existing experience and knowledge related to a system. As *f (x)*, different functions can be used. FCM can be used to make a what-if inference, starting from a given initial activation vector *B(0)*, to understand what will happen next to the modelled system/environment. Our FCM has been defined by a team of four experts. Each expert, starting from the status model we have defined, has proposed his FCM to identify the causal relationships and weights between the available concepts. The weights are represented by seven linguistic terms: *no impact 0.00, very low = 0.165, low = 0.335, medium = 0.50, almost high = 0.665, high = 0.835, very high = 1.00*. Then, we aggregate the different maps proposed by the experts to obtain one FCM. When some differences arise between the relationships and weights proposed by the experts, we asked them to discuss these differences and try to find an agreement, until they achieve a sufficient degree of consensus. This allowed us to obtain the FCM widely discussed in D’Aniello et al, 2020.

Methodology

The methodology is based on a single case study to identify intervention strategies on the specific situation of didactic hardship following the pandemic. Direct observation, understood as student-teacher interaction through information technologies collected empirical data on teaching effectiveness. The approach was both of an ideographic nature based on qualitative methods through a questionnaire to students (fully reported in appendix A) and of a nomothetic nature based on quantitative data that emerge from the results of the tests, from the responses to the questionnaire based on the Likert scale, and from students' interactions with the e-learning platform. Further qualitative information was collected from the protocols from the dialogues between students and between students and teachers on the e-learning system used. A specially created FCM summarizes both qualitative and quantitative data in terms of participation, motivation, and engagement, comparing them with the data of the previous cohort. As for the analysis of the disciplinary skills, a comparison between the results of the first and second mid-tests of two years was carried out.

The Case Study

The didactic experimentation involved the students of the first year of Mechanical Engineering and Management Engineering at the University of Salerno, the course teacher, two experts of the discipline for the exercises, three students attending the master's degree in engineering with the function of the tutor. The course was carried out during the second semester of the first year after students had attended and/or taken a Calculus 1 exam. Because of the COVID-19 pandemic, after the first two face-to-face lessons, the teaching activities of the Calculus II course were conducted through Distance-Learning. They were articulated into Asynchronous Online Learning and Synchronous Online Learning. The course has been restructured following the constructive alignment suggested by Biggs and Tang (2010). The course included 90 hours of lessons (six hours per week), divided into 54 hours of theory lessons and 36 hours of training; also 24 hours of exercises with the tutors, dividing the students into two sub-groups, 12 additional hours of activities for students who have reported an insufficient evaluation at the first test. The course has been designed considering the indications of the Teaching Council and the Lisbon descriptors. The following online resources have been used: custom adaptive e-learning platform (D'Aniello et al, 2020), Microsoft Teams, Doceri, Edmodo, the teacher's website, the teacher's YouTube channel, Geogebra AR. The use of these resources follows the framework proposed by Bray and Tagney (2016). This approach focuses on creating activities within the Transformation space: transformative uses of technology allow significant task redesign (modification) or permit the creation of tasks that would not be possible without the digital tools (redefinition). These aspects have contributed to implementing this course's activities, especially regarding engagement. The results obtained from the data available for completely remote teaching were compared with the data collected in a previous experiment. The course was conducted in blended mode, i.e., in the 2018/19 academic year. The Engagement parameters have been made available by the systems used through the collection of student interactions. The parameters relating to Motivation were determined through the analysis of questionnaires administered during the course. The parameters related to Participation or Emotions and social activities are established through sentimental analysis on the video streaming of the webcams filming the students and through the answers to the questions in the questionnaire. Dropout was calculated as the ratio of the number of students who attended the whole course to those who were present at the beginning of the course. The data provided by the system for the different parameters of interest are analysed through FCM execution to create the student's status; by comparing the parameters of the participants it will be possible to understand if the use of completely remote teaching did not negatively affect the status of students. The two classes are made up of 131 and 112 students respectively. For the experimentation, 60 samples from the first group and 60 samples from the second group were randomly selected. Cochran's formula was used to calculate the sample size: $n_0 = \frac{Z^2pq}{e^2}$ (Eq. 2), where: e is the desired level of precision (i.e. the margin of error); p is the (estimated) proportion of the population which has the attribute in question; q is $1 - p$; the z-value is found in a Z table. It is s the abscissa of the normal curve that cuts off an area α at the tails ($1 - \alpha$ equals the desired confidence level, e.g., 95%); n_0 is the sample size.

In our experimentation the chosen parameters were: $Z = 99\%; p = 0.90; e = 0.10$.

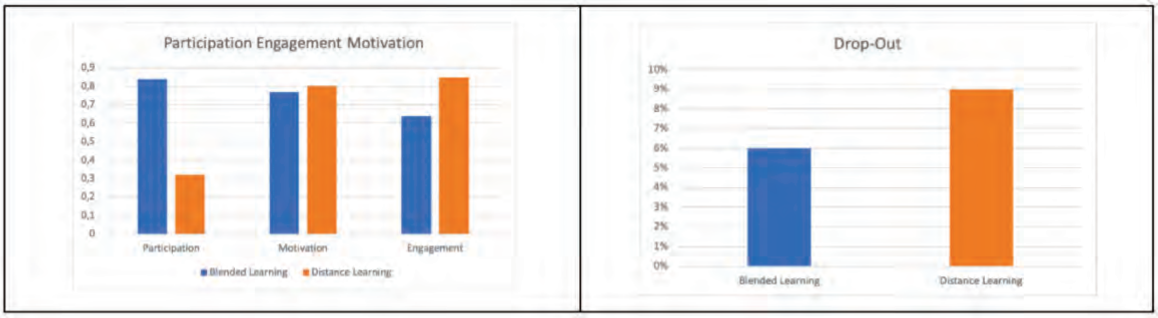

Fig. 1: a) Participation, Engagement and Motivation results. b) Dropout results

The average levels of engagement, motivation, and participation calculated through the execution of the FCM are shown in Fig.1a. The parameters used to obtain the motivation level were determined by analysing questionnaires administered during the course. The parameters related to participation were established through sentimental analysis on the video streaming of the webcams filming the students and through the answers to the questions in the questionnaire. At the same time, the parameters related to engagement were obtained, analysing mainly the data coming from the e-learning platform. The decline in student participation in the year 2019/20 compared to the previous year seems to be due to the negative impact of the emergency for the pandemic on the emotional state of students. In addition, the sense of alienation and frustration has led to a loss of concentration and poor social activity. The level of motivation is comparable for students of the year 2018/19, it is determined by intrinsic and social motivation for students of the year 2019/20 by an external tax. The engagement is slightly greater for students of the academic year 2019/20 who have massively used the e-learning platform as the only tool to follow the course and access the content made available by the teachers. Finally, the dropout graph shown in figure 1b shows how the two values are comparable (6% in the year 2028/19 and 9% in 2019/20) and how they remained below an acceptable threshold. The use of an adaptive e-learning platform and the restructuring of educational content using technologies, which led students to have a high level of engagement for the course duration, seem to have contributed to containing the phenomenon of dropout feared at the beginning of the course through distance learning methods. In addition, from the hierarchical questions of the questionnaire, it emerges that in the year 2019/20, 69.77% of students answered 4 or 5 on the Likert scale to the question of how frequently they confronted each other on the teaching activities of the course; while in 2018/19 40%. 92% of the students declared that they had interacted on the e-learning platform forum either assiduously or frequently (4 or 5 on the Likert scale) in 2019/20. As they declared, the forum allowed them to recreate the study room environment, although virtual, in which to discuss the solution of the exercises proposed in class. In 2018/19, however, only 25%. As a further element of qualitative analysis, it was decided to compare the data that emerged from the tests during the two academic years, shown in Fig.2.

Mark	First mid-term test 2018/2019	First mid-term test 2019/2020	Secondo mid-term test 2018/2019	Second mid-term test 2019/2020
A	4%	6%	15%	10%
B	12%	17%	18%	15%
C	31%	30%	23%	31%
D	22%	19%	12%	15%
Fail	32%	25%	32%	28%

Fig. 2: Results of the tests done by students during the two academic years

A less critical situation seems to emerge from the data in Fig. 2 than from the qualitative data. There is no statistically significant difference between the two data; indeed, it is possible to note that the percentage of those who failed the tests in 2019/20 is slightly lower than that of 2018/19. A possible interpretation of these data is suggested by having completely restructured the didactic activities. TCK, PCK and TPK have been intersected to convey the contents of the discipline using alternative teaching methodologies supported by the use of technologies. So, the main findings are:

Distance Learning is excellent as an additional and support methodology but highlights the ineffectiveness of completely remote teaching; the emergency alters motivation, participation, and engagement, and the distance learning used. The motivation went from intrinsic to extrinsic given the external constraint to take the online course as the only way to obtain attendance. Participation plummeted because the emotional state and the desire for socializing were adversely affected. Engagement increased as a result of massive use of adaptive learning platform and all the technological tools as the only means to follow the course and feel part of a community; by integrating TCK, PCK and TPK to use an adaptive e-learning platform, students have reached adequate levels of competence.

Conclusions

Distance learning, already widespread in recent years, has taken on a new meaning in this pandemic time. It has taken on the role of proximity education, as openness to the world and a willingness to enter into a relationship with things and others. It was indispensable for strengthening the web of relationships between teachers and students and between teachers. Distance Learning could lead to unexpected connections that are not just accessing a server from a client through protocols or software but could instead favour synaptic connections that produce cognition and generate positive emotions that do not make us become victims of an emotional abduction and transported by fear, by anguish in a non-adaptive way. This research focused on distance learning and the virtual environment during the Covid-19 pandemic, highlighting how motivation, participation, and engagement are affected. The purpose of the experiment was to verify how the use of completely remote teaching influences the student's status in terms of participation, interaction, engagement. A student with a high level of these indices proved to be more motivated to study the discipline, positively influencing skills improvement. It was possible to understand that completely remote teaching did not negatively influence the student's status, assessed through FCM, and allowed satisfactory results to be achieved in terms of acquired skills. The dropout was contained within acceptable values. The strategies implemented by the course lecturers aimed at integrating TK, PK, and CK following the TPCK framework. Finally, the processes of instrumentation and instrumentalization have caused an instrumental genesis of technological artifacts. The findings of this case study provide a fresh perspective and set the stage for future related research. The authors hope to undoubtedly return to face-to-face teaching, where human values and relationships take on a concrete and visible form while trying to exploit the potential of distance learning, integrating the strengths of the two methodologies. Distance learning offered the opportunity to rethink educational action from the point of view of contents, methodologies, and student-teacher interactions alike. At the end of the pandemic, as the next step, we think that some positive aspects of distance learning can be integrated with traditional teaching and enhance a blended modality of didactic action called Integrated Digital Teaching.

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Exploring Teachers' Affective Schemes through a Networking of Theories Approach

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In this report, we network the documentational approach to didactics and a theory of meanings to explore and develop models of one tertiary teacher's (Evelyn's) understandings of developmental mathematics, its students, and the resources she utilizes to support instruction. In addition, we discuss the affective aspects of the schemes of meanings that came to Evelyn's mind as a result of these understandings. Finally, we discuss future research to explore the design of interventions (e.g., professional learning experiences) that provoke changes in tertiary teachers' schemes of meanings to more productive structures.

Keywords: Affect, caring communities, sense of belonging, theory of meanings.

Introduction

In the United States, college and university policies typically require new students to be assessed in mathematics through some form of placement test. These tests are commonly designed and evaluated by a third-party (e.g., ALEKS, COMPASS). Students assessed as needing mathematics remediation are required to complete remedial (or developmental) math courses prior to taking a college-level math course. The need for remediation in math is not uncommon for students at both 2- and 4-year public institutions. Chen's (2016) analysis of beginning tertiary students' course taking between 2003 and 2009 found 59.3% of students who began their tertiary education at a 2-year public institution and 32.6% of students who began their tertiary education at a 4-year public intuition, took one or more remedial mathematics course. Unfortunately, a large percentage of students enrolled in developmental courses fail to pass them. According to Chen (2016), 20% of remedial math course takers beginning at both public 2- and 4-year institutions did not complete any of the remedial math courses they attempted. There exists substantial research documenting the characteristics (e.g., academic preparedness, age, ethnicity, socioeconomic status) of higher education developmental mathematics students that relate to retention and success (e.g., Benken et al., 2015). In addition, extensive research on efforts to reform developmental mathematics in higher education have focused on restructuring coursework, reforming developmental placement, redesigning curricula, and enhancing student support (e.g., Bickerstaff et al., 2019). Missing from this research are investigations into those at the forefront of attempts to implement such changes—the instructors of developmental mathematics courses. In this report, we focus on teaching and learning developmental mathematics at the tertiary level and addresses the following research questions: 1) How do tertiary mathematics teachers understand developmental mathematics? 2) How do tertiary mathematics teachers understand the students enrolled in developmental mathematics? 3) How do tertiary mathematics teachers understand the resources they utilize to support developmental mathematics instruction?

Theoretical Framework

The study utilizes a networking of theories approach (Bikner-Ahsbahs & Prediger, 2010) to connect frameworks and explore the research questions; specifically, we "network" the documentational approach to didactics (Trouche et al., 2020) and a theory of meanings (Thompson et al., 2014).

The Documentational Approach to Didactics

The documentational approach to didactics (DAD) analyses "teachers' work through the lens of 'resources' for and in teaching: what they prepare for supporting their classroom practices, and what is continuously renewed by/in these practices" (Trouche et al., 2020, pp. 237–238). In DAD, resource is defined as anything "developed and used by teachers and pupils in their interaction with mathematics in/for teaching and learning, inside and outside the classroom" (Pepin & Gueudet, 2020, pp. 172–173). Such resources include text (e.g., textbooks, worksheets, tests) and other material resources (e.g., calculators); digital-/ICT-based resources (e.g., online textbooks, GeoGebra); discussions between teachers, orally or online; students' written work; and teachers' discussions with mathematics teacher educators (Pepin & Gueudet, 2020). Integral to DAD is documentational genesis, which comprises two interrelated processes (Trouche et al., 2020): instrumentalisation, where a teacher's knowledge guides the choices she makes among various resources and the way these resources are appropriated; and instrumentation, where the features of the resource(s) impact the teacher's practice. Documentational genesis "gives birth" to a mixed entity (i.e., a document) linking resources and a utilization scheme for these resources, represented by the equation: Document = Resource(s) + Utilization Scheme. The concept of scheme, which can be viewed as a structure organizing a subject's activity with a resource or set of resources for a given goal, is central in DAD.

Theory of Meanings

Thompson and Harel's theory of meanings (Thompson et al., 2014; Table 1), is based on Piaget's notion of assimilation to a scheme, and focuses on teachers' (schemes of) meanings, where a scheme is defined as "an organization of actions, operations, images, or schemes—which can have many entry points that trigger action—and anticipations of outcomes of the organization's activity" (Thompson et al., 2014, p. 11).

Table 1: Definitions of Understanding, Meaning, and Ways of Thinking (Thompson et al., 2014)

Construct	Definition
Understanding (in-the-moment)	Cognitive state resulting from an assimilation
Meaning (in-the-moment)	The space of implications existing at the moment of understanding
Understanding (stable)	Cognitive state resulting from an assimilation to a scheme
Meaning (stable)	The space of implications that results from having assimilated to a scheme. The scheme is the meaning.
Way of Thinking	Habitual anticipation of specific meanings or ways of thinking in reasoning

As characterized by Thompson and Harel (Table 1), an understanding is a cognitive state of equilibrium, which may occur from assimilation to a scheme (i.e., stable understanding). According to Thompson et al. (2014), "A scheme, being stable, then constitutes the space of implications resulting from the person's assimilation of anything to it. The scheme is the meaning of the understanding that the person constructs in the moment" (p. 13). Alternatively, a cognitive state of equilibrium might be a state the "person has struggled to attain at that moment through functional accommodations to existing schemes . . . and is easily lost once the person's attention moves on" (i.e., in-the-moment understanding; Thompson et al., 2014, p. 13).

Methods

Since the study attempted to reveal teachers' understandings and explore how these conceptions impacted their teaching of developmental mathematics, it was necessary to make models of teachers' conceptions. The case study described in this report expresses our second- order models of teacher's understandings at various points throughout the study; that is, throughout the activities (i.e., interventions) designed to make teachers' meanings explicit.

Study Participants

In this report, we focus on one of two study participants (i.e., Evelyn). Evelyn had 16 years of tertiary teaching experience and was teaching a fully remote Beginning Algebra community college course during an 8-week summer course with 13 non-traditional college students. The course met twice each week for 90-minutes. Although Evelyn's summer course encompassed 8 weeks, the study itself was 12 weeks in length and included weekly group meetings before, during, and after the conclusion of her summer course. Eleven of the 13 students (84.6%) passed the course, a rate higher than results from Chen (2016), indicating Evelyn may serve as a model for effective developmental mathematics practices.

Data Collection

The study employed a modified version of the reflective investigation methodology (Trouche et al., 2020) for data collection, a methodology naturally associated with case studies and grounded by five main principles: (1) broad collection of resources; (2) long-term follow up; (3) in- and out-of-class follow-up; (4) reflective follow-up; and (5) confronting a teacher's views on her documentation work. Due to the length of the summer course and limits precluding in-person interactions, some principles could not be strictly adhered to, resulting in a modified version of the methodology. The data corpus consisted of Evelyn's semi-weekly self-recorded videos discussing her lesson plans, with accompanying documents and internet links to all materials she utilized during instruction and assessment; her reflections of lesson implementations; and video recordings and field notes from weekly online discussions designed to probe Evelyn's understandings in more detail. Data also included two tools specific to reflective investigation methodology: reflective mappings of Evelyn's resource system (RMRS) and inferred mappings of Evelyn's resource system (IMRS). According to Rezat et al. (2019) an RMRS is a methodological tool created by a teacher where the teacher is asked to draw a map (based on her own reflections) to present her resources in a structured way; whereas an IMRS is a methodological tool created by "the researcher based on the observations of and interviews with the teachers about their resource work" (Rezat et al., 2019, pp. 357–358).

Results

Making sense of Evelyn's understandings of developmental mathematics and her students required us to develop models of Evelyn's ways of operating—models that represented our interpretations of

Evelyn's interactions with study activities and digital resources. Using data generated from study activities and reflective investigation, these models were tested, modified, and refined through ongoing and retrospective conceptual analysis of the data corpus. Evelyn utilized a student survey at the beginning of her course to obtain information about her students' perceptions of mathematics. The survey was posted as an online assignment and asked students a variety of questions, including: "How do you feel about taking a math course synchronously/online?"; "What are your strengths and weaknesses in mathematics?"; and questions regarding students' experiences with the institution's videoconferencing application (i.e., Webex) and learning management system (i.e., Blackboard). As part of the Week 6 meeting, Evelyn described her initial RMRS. During the Week 7 meeting, the first author created an inferred mapping of Evelyn's resource system (IMRS), based on her RMRS and the Week 6 discussion. During the Week 9 meeting, Evelyn was confronted with this IMRS and her original RMRS. The Week 9 meeting resulted in the refined IMRS shown in Figure 1, which illustrates Evelyn's resources partitioned into three groups: resources specific to mathematics content, tools for communication and dissemination, and tools to deliver instruction.

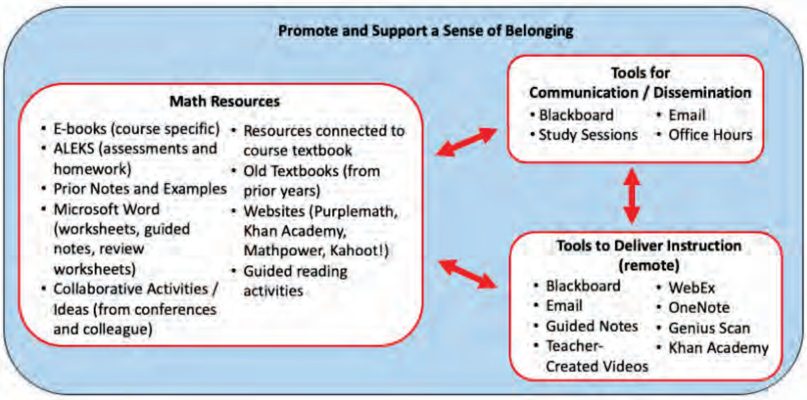

Figure 1: Refined Inferred Mapping of Evelyn's Resource System (IMRS)

Evelyn's RMRS did not include the arrows. Comments made during group meetings emphasized Evelyn's integration of these three types of resources. For example, whenever Evelyn spoke about "Math Resources" she used, she invariably included ideas about and a rationale for utilizing "Tools for Communication / Dissemination" and "Tools to Deliver Instruction." During the Week 11 meeting, Evelyn indicated her use of guided notes, Khan Academy, Microsoft Word, and ideas from her colleagues were "dependent on her students' needs . . . [and] were provided to help develop, promote, and support a caring community in the classroom." Throughout the data corpus, Evelyn indicated her "students' needs" referred to each student's individual circumstances and challenges, which included (Goldin, 2002; Hannula et al., 2019): math anxiety – "my students are so intimidated and scared of mathematics because of bad prior experiences" (Instructor 2, Week 2 meeting); test anxiety – "testing is always a traumatic experience for these students" (Instructor 2, Week 9 meeting); low math self-efficacy – "part of my job is to get these students to believe they can actually do the math" (Instructor 2, Week 11 meeting); and frustration – "many of my students with families and work get fed up trying to navigate school" (Instructor 2, Week 6 meeting). Therefore, Evelyn enacted her intent to "support a caring community in the classroom" through listening to and observing her students; demonstrating respect for her students; thinking and reflecting on ways to support her students; being approachable, available, and responsive to her students; and creating an environment where caring relations can flourish.

Throughout the group meetings, Evelyn's comments also emphasized an image that resources were in the foreground of her practice, whereas promoting and supporting a sense of belonging in the mathematics classroom was always in the background. Comments and assertions that illustrate this image include: mathematical identity – "I want to make sure each student believes they can be . . . they are a contributor to the mathematics classroom . . . to view themselves as mathematicians" (Instructor 2, Week 6 meeting); and values – "since most of my classes have both traditional and nontraditional students, I have to make certain to provide activities that encourage each group to value mathematics" (Instructor 2, Week 9 meeting) (e.g., Goldin, 2002; Hannula et al., 2019). According to Thompson and Harel (Table 1), the meaning of an understanding is the actions or schemes the current understanding implies. Therefore, the meaning of Evelyn's understandings "to develop, promote, and support a caring community in the classroom, dependent on my students' needs" and "to promote and support a sense of belonging in my students, as learners of and contributors to mathematics"—what comes to Evelyn's mind in situations related to developmental mathematics and its students—are comprised of affective aspects of her students (e.g., anxiety, frustration, identity, self-efficacy, values). Goldin (2002) defines beliefs as "internal representations to which the holder attributes truth, validity, or applicability, usually stable and highly cognitive, may be highly structured" (p. 61). For Philipp (2007) beliefs are "the lenses through which one looks when interpreting the world" (p. 258). Therefore, Evelyn's stable understandings related to developmental mathematics and its students are indicative of her beliefs about developmental mathematics and its students.

Teachers' Understanding of Remedial Mathematics

Evelyn exhibited meanings for development mathematics that involved more than just a series of pre-tertiary math topics (e.g., operations of real numbers, order of operations); rather, as described by Evelyn, "Developmental math involves content that is foundational to my students' future mathematics learning" (Week 10 meeting). Evelyn also indicated a desire to prepare students for subsequent math courses by helping her students "build or strengthen their mathematics foundation for their next course, including study strategies" (Week 3 meeting). Therefore, as characterized by Thompson and Harel (Table 1), Evelyn's stable understanding of developmental math is "*content foundational to students' subsequent math courses and dependent on each student's needs*." Although one might question how this stable understanding can be defined as a "cognitive" state resulting from an assimilation to a scheme, it is important to note that when "Piaget spoke of schemes, he had in mind organizations of mental and affective activity whose contents could be highly nuanced and could contain several layers of structure" (Thompson, 2016, p. 432). For Piaget (1973), the "affective and cognitive mechanisms always remain interrelated though distinct" (p. 261). Similarly, for Vergnaud (2013), "[c]ognition is affective, or it is not, and affectivity is cognitive, or it is not. A scheme conveys both characteristics" (p. 52). Therefore, we interpret a "cognitive state" as being inclusive of all cognitive, affective, and social activity. This interpretation aligns with A. G. Thompson (1992), who referred to teachers' understandings "as a more general mental structure, encompassing beliefs, meanings, concepts, propositions, rules, mental images, preferences, and the like" (p. 130). Finally, since meaning is implicative, Evelyn's meaning of developmental mathematics has implications for her further action. Specifically, Evelyn's actions focus on highlighting the "foundational to subsequent mathematics courses" aspect of development mathematics and affective aspects of "each student's needs" (e.g., anxiety, frustration, identity, self-efficacy, values).

Teachers' Understanding of the Students Enrolled in Remedial Mathematics

In addition to the initial survey, Evelyn also asked her students how they were thinking about the course (e.g., what they struggled with, what was useful) throughout the course. Evelyn developed models of her students based on her interpretations of students' responses to these queries. For example, one of Evelyn's students who had not taken a math class for over seven years asserted, "I had a lot of college credits, but just [gave] up on myself . . . [and] want to prove to myself and my children that it's never too late." As such, Evelyn made a point to "check-in" with this student on a regular basis to make certain she was feeling comfortable and achieving her goals. Since Evelyn was teaching remotely, she provided a live tutorial for her students, where she went through the Blackboard and Webex environments, making certain students had a sense for where to find the calendar, weekly and daily folders, assignments, assessments, video links, and chat box. Evelyn also provided her students with an unlimited amount of time to complete online quizzes, since she wanted to "make certain [they] feel like they have enough time to do really good work, to be thorough, and to show all of their work" (Week 10 meeting). Evelyn indicated this action helped reduce some math anxiety because students were not watching the clock. Furthermore, Evelyn promoted a collaborative and supportive environment by encouraging students to use "clapping" and "thumbs up" emojis to show appreciation for student or group virtual demonstration work. Finally, Evelyn utilized her understanding of her students' future courses to emphasize concepts and behaviors important to students' continued success. Therefore, Evelyn understood her (developmental) students as "*individuals with varying mathematics and school experiences in need of a caring environment and a sense of belonging as learners of and contributors to mathematics*"—indicative of Evelyn's beliefs about her students. Since meanings are implicative, Evelyn's actions focused on affective aspects of her students (e.g., anxiety, frustration, identity, self-efficacy, values).

Teachers' Understanding of the Resources They Utilize to Support Instruction

Evelyn selected resources that provided her students with opportunities to engage in mathematics in a supportive environment. This way of thinking is demonstrated by examining those resources Evelyn utilized to find and generate mathematics content and environments (e.g., Khan Academy, Microsoft Word; see Figure 1). Evelyn indicated she used Khan Academy because most of her students were familiar with it, either through a prior high school math course or their own children's use of the resource. Regarding use of her prior notes, Evelyn indicated her students had found guided notes to be quite beneficial during asynchronous classes over the past year. As such, she decided to utilize these same notes in the remote setting. Finally, Evelyn provided self-made video tutorials corresponding to her guided notes, again prepared over the past year, as an additional resource to support her students. Evelyn indicated her Word documents could easily be changed to pdf format and uploaded to Blackboard Ally, a tool that generates alternative formats for students to download, provides accessibility scores, and gives instructor feedback on how to improve their accessibility score. Finally, Evelyn indicated making digital course content more accessible to all her students supported her attempts to support student engagement and sense of belonging. Therefore, Evelyn understood these resources as "*tools to support each student, based on their individual needs, as they engage with mathematics and interact with their classmates and herself*"—indicative of Evelyn's beliefs about developmental mathematics and her students. Finally, since meanings are implicative—and as a result of the instrumentalisation process—Evelyn's actions with respect to these tools focused on her students' "individual needs" (e.g., anxiety, frustration, identity, self-efficacy, values).

Discussion and Conclusion

As illustrated here, Evelyn's schemes of meanings, indicative of her beliefs, focused on affective aspects of her students' varied needs and experiences. Aligned with Hackenberg (2005), we posit that Evelyn's assimilatory structures involve a scheme of decentering (Piaget, 1962), where decentering is the attempt to imagine one's experience from another perspective. Given the case study only included one participant, our findings are not generalizable. The study's small sample size and lack of range within teaching styles are a few additional notable limitations. Future research should explore the understandings of a larger sample of tertiary teachers of developmental mathematics to determine the meanings by which teachers operate; specifically, the ubiquity of schemes of decentering. Such research should also include examination of the types of professional learning experiences most propitious to fostering schemes of decentering in tertiary mathematics teachers. Furthermore, the "networked" approach presented here provides researchers and mathematics teacher educators with a framework focused on teachers' documentation work that supports identification of the schemes of meanings with which teachers teach—including teachers' belief structures—and has the potential to support the design of productive professional learning experiences for both pre-service and in-service teachers. Finally, such research should explore the role the instrumentation process might play in transforming teachers schemes of meanings, where the features and teacher's understanding of a resource or set of resources impact that teacher's schemes as a result of their interactions with the resource(s).

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Beliefs about problem posing

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Beliefs about problem posing are rarely addressed in mathematics education. The study explored 43 senior preservice middle grades mathematics teachers' (PTs) beliefs about problem posing and their possible place in their mathematics-related belief system with two open-ended surveys and one problem posing task. In line with their mathematics-related beliefs, PTs believed that problem posing was important because it supported students' conceptual understanding. However, they believed that problems for the low-level students should require applying simple procedures and these students would not be able to pose problems. Problems including challenging tasks could be posed for high-level students and they had skills to pose such problems. PTs' problem posing beliefs may be linked more to their beliefs about students' learning than those about nature of mathematics and problems.

Keywords: Problem posing, preservice mathematics teachers, beliefs, belief system.

Introduction

Problem posing is (a) producing new problems by rearranging an existing situation to a problem or (b) revising existing difficult problems to formulate more solvable problems (Silver, 1994). It is an important, challenging, and cognitively demanding mathematical activity as it enhances students' conceptual understanding, reasoning and dispositions towards mathematics (Singer et al., 2013), and enables teachers explore students' mathematical understandings and experiences (Silver, 1994). Hence, there is a demand for teachers to encourage students for posing problems (Crespo & Sinclair, 2008; Rosli et al., 2015). Inservice and preservice teachers' mathematics-related beliefs influence their response to the demands for meaningful mathematical experiences and their conceptualization of teaching and learning mathematics (Philipp, 2007). Despite the importance of problem posing in mathematics education and of inservice and preservice teachers' beliefs, beliefs about problem posing have received less attention (Li et al., 2020) within the studies about mathematics-related beliefs.

Middle school mathematics in Turkey aims to enhance students' conceptual understanding of mathematics by guiding them to construct new mathematics concepts on existing ones, focusing on conceptual understanding, emphasizing connections to other content area and real-life, and utilizing multiple representations (Milli Eğitim Bakanlığı (MEB), 2018). This approach is supported by mathematics teacher education programs that train teachers specifically to teach middle school mathematics. Problem posing is an effective tool to enhance students' and preservice and inservice teachers' conceptual understanding of mathematics, yet it is addressed less in teacher education programs compared to problem solving (Crespo & Sinclair, 2008). Teachers' beliefs about problem posing may influence their tendencies to utilize problem posing in their teaching (Li et al., 2020). Therefore, this study investigated preservice middle school mathematics teachers' (PTs) beliefs about problem posing in the beginning of a short problem posing module to draw conclusions for how these beliefs might be related to each other within the web of PTs' beliefs and inform the teacher education programs with the following research question: How are PTs' beliefs about problem posing related to their mathematics-related beliefs and beliefs about problem solving?

Belief Systems

It is often accepted that beliefs are held in *belief systems* including the following characteristics as identified by Green (1971): (i) Beliefs are located in *clusters* in which we hold non-conflicting beliefs; (ii) some beliefs are constructed in relation to others in our belief system, which makes them *quasi-logical*, where we have *primary* beliefs and their *derivatives*; (iii) strong and important beliefs are located at the *center/core* of the clusters and those that are less strong at the *peripheral*; and (iv) we hold some beliefs *evidentially* (based on evidence/reason and can be changed by rational criticism) and some *non-evidentially* (not evidence-based and not changed easily). He suggested that teaching should aim for developing evidential beliefs and a quasi-logical belief system as logical as possible.

Beliefs in a cluster may be considered as a web of primary and derivative beliefs (Beswick, 2018). If we have experiences that conflict with our existing beliefs, we (our belief system) tend to adapt to these new beliefs or try to explain our existing beliefs in different ways as an attempt to limit the conflicts. This adaptation takes place when we reflect on our experiences and reorganize our understandings from our observations (Philipp, 2007). In the end, we have new beliefs and a new (sub-)web of beliefs linked to the existing web, which illustrate adaptation of our belief system when there are new experiences and knowledge (Beswick, 2018).

Beliefs about Problem Posing

Previous studies revealed that PTs considered having problem posing skills as important and beneficial for teachers and students especially to increase students' interest in mathematics (Hošpesová & Tichá, 2015). They also believed that problem posing increased students' mathematical thinking, made students the owners of their mathematical knowledge, and served as an effective process to assess their learning (Hošpesová & Tichá, 2015). PTs tended to believe that problem posing could help students develop a better understanding of mathematical concepts, think beyond solving and posing problems, become more creative, and develop positive attitudes when they worked on the problems posed by students (Grundmeier, 2015). PTs mostly believed that the connection between real-life and mathematics would be useful for students while posing problems, but they did not focus on the benefits on students' learning (Lee, 2012).

When PTs posed problems, they paid attention to the difficulty of the problems for the age and grade level of students, authenticity of the problems, mathematical concepts in the problems, and realistic and understandable nature of the problems (Rosli et al. 2015). Yet, analysis of their problems showed that the problems they posed were not in high quality (Hošpesová & Tichá, 2015), did not focus on conceptual understanding but promoted memorization (Crespo & Sinclair, 2008), and their solution required one or two arithmetic operations (Lee, 2012). PTs did not increase the reasoning and thinking skills required to solve the problems; rather, they added another operation for solution to increase the complexity of the problems (Leavy & Hourigan, 2020). It is possible that since PTs tended to pose problems similar to the ones they solved earlier in teacher-assigned tasks and textbooks because they were generally were not given opportunities to pose problems (Crespo & Sinclair, 2008).

Studies showed that training on problem posing provided teachers and PTs with new perspectives on problem posing (Barlow & Cates, 2006; Leavy & Hourigan, 2020). Teachers developed favorable beliefs about employing problem posing in the classroom and tended to do so in their lessons because they observed that problem posing helped their students construct mathematics knowledge and increased their interest in mathematics and discussions in the lessons (Barlow & Cates, 2006). However, teachers and PTs believed that problem posing might be confusing for students in the beginning, consume time, result in ineffective mathematics lessons, and students might be more inclined to pose simple questions but not problems that require reasoning (Hošpesová & Tichá, 2015).

PTs can develop beliefs to support their practices for effective learning, such as beliefs about problem posing that will enable them to integrate problem posing effectively in their future teaching, when teacher education programs address their existing beliefs and aim for new ones (Philipp, 2007). However, this process requires knowing how PTs' existing mathematics-related beliefs are organized, where their (probably derivative) problem posing beliefs might be placed within their belief system in relation to the other beliefs, and what possible belief connections should be targeted while designing problem posing experiences for them. Therefore, the study focused on PTs' beliefs about problem posing in the beginning of a short module aiming to provide initial ideas on problem posing.

Method

A basic qualitative research design was employed to explore PTs' beliefs (Merriam, 2009) by open-ended surveys and problem posing tasks, which PTs provided written responses. The problems posed by PTs during the implementations were not the foci of the study and therefore, not reported.

Context and participants

The study was conducted in a four-year middle-grades (5 to 8) mathematics teacher education program in Turkey with 43 PTs (6 males and 42 females) who were taking the Practice Teaching course and consented for the study. Practice Teaching is offered in the last semester of the program and required 60 hours of observation of and teaching middle school mathematics lessons, and two-hour University component every week to discuss PTs' experiences and observations at schools.

Instruments and procedures

PTs participated in a four-hour module (2 two-hour meetings) on problem posing that took place at the University component of the Practice Teaching course. The study focused on the first two-hour meeting where PTs completed two open-ended surveys and one individual task. Surveys targeted PTs' thoughts, experiences, and observations about teaching and learning mathematics, problems, problem solving and posing. The reason for targeting a broad area of beliefs was to understand PTs' larger web of mathematics-related beliefs and possible derivative beliefs about problem posing within (Beswick, 2018). The task asked PTs to pose problems for students with different levels of knowledge and skills, and reflect on the process. Information about the data collection procedure is in Table 1.

Data analysis

Data analysis started by extracting belief statements after repeated careful reading of all responses (Beswick, 2018) where PTs expressed their understandings and reasons (Philipp, 2007) for the question content. I developed a code list based on the studies about PTs' mathematics-related beliefs in these programs (e.g., Haser & Doğan, 2012) and based on my repeated reading of the data with room for emerging codes especially for beliefs about problem posing. While I kept a list of possible codes from the literature (above) for problem posing, my focus was more on codes extracted from the data because I tried to see the connections among the beliefs within this specific data set. After determining the initial code list, I coded the data with some emerging codes, revised my coded data for consistency, and listed examples of belief statements for the codes. After coding was completed, I determined the themes and revised the examples of belief statements and data to ensure an adequate representation of PTs' beliefs by the themes. I summarized the themes narratively to see a clearer picture of connections which revealed a web of PT's beliefs grounded in data and their conflicting beliefs. I revised the conflicting beliefs one more time in detail for a better interpretation.

Table 1: Examples of questions, (*) notes, and the number of PTs responded

Survey 1: What are the behaviors of a middle school student with an in-depth understanding of mathematics concepts? / Based on your observations at schools, what can you tell about what teachers pay attention to in their teaching? (43 PTs, 12 questions)
Survey 2: Based on your observations at schools, how do the middle school teachers address problem solving in their teaching? / Can middle school students pose mathematics problems? Why? (43 PTs, 12 questions)
Task: Pose a problem which requires the multiplication of two simple fractions (and if necessary, other operations and thinking) and which low-/mid-/high-level students can solve by spending some effort. Please indicate the grade level. ((*)Adapted from Singer et al. (2011). Targeting a 6th grade objective (MEB, 2018), 39 PTs, 3 questions)

Findings

Mathematics-related beliefs

PTs believed that mathematical knowledge was about understanding the reasons and connections between the mathematical concepts by asking questions and thorough thinking. School mathematics had certain "definitions", "important terms, [and] proofs" of key importance as they were connected to other concepts and daily life. Although PTs expressed the importance of these connections for mathematical knowledge, they did not mention abstract concepts and relationships. Rules, formulas, the reasoning behind them and their connections were also part of the mathematical knowledge.

Teachers should "conduct activities that would increase students' interest", ask "questions that would make students question," and teach accordingly to ensure students' learning. PTs believed that students should not memorize rules and formulas; therefore, teachers "should reach the rule by proving with the students." Teachers should monitor students' learning by questions "that require making connections with other concepts" and by asking them to try different ways, generalize, build patterns, give counter examples, and use multiple representations. PTs criticized teachers' practice of solving several questions in the lessons. They would like to ask questions "that students will inquiry the most [and] will build cause and effect relationship" after "teach[ing] the rationale behind the concepts". It is better if students solved questions after the lesson because "they may forget easily and not be able to see their missing [knowledge] if they do not solve questions."

PTs believed that knowing fundamental concepts and building relationship between them were important for students. These required reasoning, questioning, and deeply thinking about the reasons and meanings. They stressed the use of formulas, but were against memorizing them. PTs believed that students, who paid attention to their teachers with utmost care and spent effort on learning during the lesson, would learn mathematics effectively.

Beliefs about problems, problem solving, and problem posing

Mathematical problems were real-life situations for PTs which required “understanding and thinking” of mathematical knowledge and “building relationships and logical inferences”. Problems could be solved in different ways with multiple results, but not immediately by a procedure or a formula. Middle school level problems were the same except for the level of the mathematical knowledge. Problem solving at schools, as observed by PTs, was the activity where teachers first asked a question and answered it, and then asked a similar question to the students to be solved in a similar way. “Unfortunately” they observed that some teachers assigned closed-ended drill-and-practice questions, which did not match PTs’ definition of problems, and students were not asked about their reasoning.

PTs would like to employ problem solving in the future with problems that would target students’ mathematical thinking or understanding of mathematical concepts and more difficult than what students were used to solve. PTs’ problems would require attention to and discussion of the problem situation, and the daily life contexts to be solved. Students were the problem solvers, not the teachers.

> I would like to consider a situation that actually requires a solution. I would like students to follow a solution [strategy] by considering what kinds of external factors would influence the results they would reach had they been in such a situation, instead of [asking a problem] that they would reach a single result directly by multiplication [and] division.

On the other hand, PTs did not have much experience in problem posing in their courses and only some of them were able to design and ask problems in real classroom settings, which received positive reaction from students and collaborating teachers. They gained initial ideas for their problems from course books or internet resources. Some focused on connections between the mathematical concepts and real life problems, and they employed real data.

The problems PTs posed in the study provided data on PTs’ thinking about problems for students with different levels of knowledge and skills, and the factors they would consider while posing problems. PTs posed problems requiring more/difficult operations for high-level students and those requiring “implementation of a known procedure” or “a memorized operation” for low-level students, contrary to their definition of problems and statements disaproving memorization. Although they indicated in the survey that problems did not have immediate solutions, some PTs posed problems that “would not lead [students] to difficulty and enable them to understand the concept and the question.” They did not take into account the properties of problems that they identified when they posed problems especially for low-level students. For example, one PT wrote a mathematical expression of multiplication of two simple fractions without a problem statement and stated that “[Students] will only implement what they have learned. Knowing the rule would be sufficient.”

PTs stated that they prioritized students’ knowledge, possible difficulties, the national curriculum, own experiences from the courses, and daily life connections and links to other mathematical concepts while posing problems, and targeted students’ thinking. The problems they posed for low-level middle grades (5 to 8) students included fundamental concepts most of which were covered in the elementary grades (1 to 4), such as the meanings of fractions and fraction multiplication, part-whole relationship, fair sharing, and part-of-a-part. Their problems for high-level students were about connections among the mathematics concepts and the meaning of other fraction operations. Some problems required more/difficult operations as students' level increased: "I identified simple, medium [and] difficult level operations and I identified the context of the question for these."

Beliefs about students' problem posing

Many PTs believed that students had the capability to see the mathematics in daily life because they had to deal with several problems in their daily lives, which illustrated the contexts for students' problems. They indicated that students were creative, better and quick learners, and their prerequisite knowledge was sufficient to pose problems.

> They have reasoning, they are involved in real life. They gain experiences [and] deal with troubles. Don't these show that they have a natural tendency to solve problems? [So] they can pose problems about some of the obstacles/questions that are in front of them while they improve their lives.

Yet, several PTs believed that middle school students could pose problems only "if [teachers] enable them to understand and conceptualize the topics [effectively]." Some believed that students' understanding of problems were limited and "they cannot examine what should be thought [and] whether it can be solved in multiple ways or not while posing the problem" because "posing a problem is more difficult than solving that problem" and students were "not used to posing problems". Therefore, "[problems] should be posed only by the people who know the concepts well."

Discussions and Conclusions

PTs' problem posing beliefs and mathematics-related beliefs seemed to be in line. They believed that problem posing provided students with deep thinking of mathematics concepts in context, and their connections to each other and daily life. These beliefs were probably derivative of PTs' primary beliefs because (a) PTs' experiences were not sufficient yet to build primary problem posing beliefs and (b) they had some observations and experiences of teaching mathematics in classrooms on which they were able to elaborate to develop new understandings of problem posing.

Even though PTs identified certain characteristics for problems, their related problems mostly lacked these characteristics. Their problems for high-level students were different from those for low-level students in terms of the number of operations involved in the solution, but not the thinking level, as can be seen in other studies (Leavy & Hourigan, 2020). PTs believed that problems should point to the links among the mathematical concepts, but they illustrated this link in their problems only for high-level students to some extent, not for low-level students. These differences showed that some of PTs' problem posing beliefs might be connected to their beliefs about students' learning more than to their beliefs about the nature of mathematical knowledge. They might develop contradictory beliefs within the same belief system in different clusters (Green, 1971) with an effort to reach stability by adapting to the new problem posing experience (Beswick, 2018).

PTs' beliefs about problems to be posed to high-level students (who are good at reasoning and dealing with challenge) seemed consistent with their beliefs about problems (challenging situations that required thorough thinking) and teaching and learning mathematics (mathematical knowledge is not for memorization). These beliefs, however, conflicted with their beliefs about problems to be posed to low-level students (simple problems as they could handle only certain and memorized mathematical procedures, but not challenge). PTs possibly observed certain practices at schools where teachers posed several questions solved by routine procedures and believed that this practice was better for low-level students, even though they criticized it. They might also tend to replicate the problems they have seen in mathematics lessons and resources (Crespo & Sinclair, 2008) due to their insufficient problem posing experiences. PTs might believe that certain less risky practices would be better in classrooms when they lacked problem posing experiences; therefore, tended to pose easier problems for low-level students. Although PTs prioritized approaches that would lead students to conceptual understanding of mathematical concepts, they might keep a consistent belief system by thinking about low-level students as a special case that required a different understanding, and accordingly holding beliefs about low-level students in different clusters (Green, 1971).

Longer and intensive problem posing experiences would provide PTs with opportunities to develop more consistent web of beliefs. The beliefs documented here revealed that when PTs did not have sufficient problem posing experiences, they tended to develop derivative beliefs, which should be explored and addressed carefully when providing PTs with problem posing training. It is advisable that such training for PTs should include more and varied experiences of posing problems and require reflection on experiences and rationale. Further studies should explore PTs' initial beliefs about problem posing in-depth before they are trained for it and how their problem posing beliefs are linked to their beliefs about students, problem solving, and teaching and learning mathematics while preparing for and teaching through problem posing.

It should be kept in mind that the instrument in the study revealed certain beliefs of this specific group of PTs, but not the others, limiting the conclusions of the study. The analysis presented here was an initial exploration of PTs' beliefs about problem posing with a more general perspective. Their actual belief system could be revealed when a further analysis for each PT is conducted to map their belief systems. The present data set would allow only for a limited view of this map. A more comprehensive and detailed illustration of belief system can be possible with additional data set including task-based interviews with PTs and observation of their problem posing practices in the classroom.

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The effect of level marking mathematical tasks on students' time spent on it: An experimental study

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The purpose of this pilot study was to investigate the effect of level-marking mathematical tasks on students' time spent on such tasks and correct solutions. The study was conducted among students in lower secondary schools in Norway. The effect was measured by comparing the control group with the experimental group. An independent samples t-test suggested that even if the students were given the same mathematical task, the time spent on the task depended on the level-marking. The results suggest that the level-marking of tasks has a significant effect on students' time spent on solving the task. In addition, the preliminary results show a negative effect of level-marking on the correct solutions of the given task.

Keywords: Ability grouping, textbook, self-efficacy, mathematics education, level of difficulty.

Introduction

In Norway, it is common to use mathematics textbooks in which tasks are grouped according to their difficulty. Several textbooks use different symbols (e.g., geometric figures or colours) to identify the level-marking of the tasks and indicate whether the task is easy, medium, or hard. This is called ability grouping and is used to adjust teaching to the students' ability. In this way of grouping abilities, a student with low skills in mathematics can choose to work with easy tasks by focusing on the tasks marked with a specific colour. The purpose of differentiation is to achieve an optimal learning effect and to improve self-efficacy (Dale & Wærness, 2003, pp. 90). The results from TIMMS (Trends in International Mathematics and Science Study) show that Norwegian students in 8th grade score lower in algebra compared to statistics, geometry, and numbers (Bergem, Kaarstein & Nilsen, 2016). Most textbooks in Norway use level-marking on the tasks; for example, in the textbook *Faktor* 61% of the tasks in algebra are grouped according to difficulty level, in *Maximum* 76% are grouped accordingly, and *Grunntall* has the highest proportion of level-marking, where 98% of the tasks have level markings and almost 1/3 of the marked tasks have the level-marking 'hard'. Therefore, it is important to study how this can affect mathematics education. As in Norway, Sweden also uses textbooks with ability grouping in mathematics, and Brändström (2005) concluded that tasks have a low difficulty level for all students regardless of their mathematical abilities. Earlier studies have shown extensive usage of textbooks in mathematics (Dolonen et al., 2016), but although there is a lot of research on textbooks in mathematics (Brändström, 2005; Howson, 1995; Jablonka & Johansson, 2010; Stein et al., 2007; Tesfamicael & Lundeby, 2019) there is still a lack of research on how the textbooks affect student's achievement in mathematics (Fan et al., 2013). The aim of this paper is therefore to study how level-marking of tasks in textbooks can affect students' time spent on given mathematics tasks. The idea behind this study was drawn from previous research that claims the amount of time spent on homework positively affects achievement (Keith, 1982; Cheema & Sheridan, 2015).

Theoretical framework and research question

In Norway, the Education Act § 1-3 states: “Education must be adapted to the abilities and aptitudes of the individual pupil, apprentice, candidate for a certificate of practice and training candidate” (Opplæringsloven, 1998). Adapted training is characterized by variation in working methods, use of teaching materials, varied teaching aids, the learning environment, curricula, assessments, and variations in the intensity and organization of the training (Utdanningsdirektoratet, 2020). The focus is on pedagogical differentiation, which is implemented by differentiating the teaching in an overall class (Imsen, 1997). The teacher can adapt the teaching by varying the degree of difficulty, called level differentiation as defined in (Imsen, 1997). This form of differentiation is widely used in mathematics teaching, where the teaching is adapted so that the students work on tasks that are adapted to their skills. To differentiate the teaching in an overall class, it is therefore suitable to use textbooks that contain mathematics tasks with different levels of difficulty so that all students can be in the same classroom even if their abilities are different. Diverse studies claim that mathematics textbooks are the major resource for planning and executing the teaching and thus have a strong position in mathematics teaching (Stein et al., 2007; Jablonka & Johansson, 2010; Howson, 1995). Robitaille and Travers (1992) argued that a textbook is ‘‘perhaps more characteristic of the teaching of mathematics than of any other subject’’ (p. 706). Howson (1995) also argued for the importance of the textbook and claimed that a textbook is one step nearer to the classroom reality than a national curriculum.

As discussed before several studies have investigated the importance of spending time on tasks (Cheema & Sheridan, 2015; Keith, 1982), but the hypothesis that more time spent on a given task by the student should result in better learning outcomes has not been confirmed. Karrie et al. (2021) showed a weak relationship between time and learning, and it has also been shown that the time spent on a task is affected by self-efficacy (Multon et al., 1991). Self-efficacy can be explained based on the social-cognitive theory of learning, and it is defined as “beliefs in one's capabilities to organize and execute the courses of action required to produce a given attainment” (Bandura, 1997, p. 3). Self-efficacy has significant effects on cognitive, affective, selective, and motivational processes, and it influences task choice, effort, persistence, resilience, and achievement (Bandura, 1997; Pajares & Miller, 1995; Zimmerman & Martines-Pons, 1990).

There is a limited number of empirical studies on the effect of level-marking a mathematical task on students' time spent on the task. However, there are several studies on how mathematical problems with *different* difficulty levels affect effort and persistence. Chen (2003) showed that the average effort judgment decreases linearly as the problems become more difficult. Montague and Applegate (1993) concluded that the difficulty of the task has a direct influence on persistence. The concept of *time spent on a task* and *effort* are related to each other, but they are still different because students' time is not necessarily used to do their best to solve the assignment. Persistence is about the person's ability to work with relatively high intensity over a long period and can be related to *time spent on a task* if the task is difficult. In the present study, we investigated persistence by measuring the time taken to solve a task with a high level of difficulty. Measurements of persistence in terms of students’ actual behaviour have been conducted previously by Shen et al. (2016). The authors measured persistence where students were solving challenging mathematics problems, and they claimed that it would be problematic to measure persistence based on easy problems because “more competent individuals may find solutions more quickly than less competent individuals” (Shen et al., 2016, p. 42). Based on the previous theoretical and empirical literature, our research questions were formulated as:

- *Can the level-marking of the mathematics task affect students’ time spent on such tasks?*
- *Can the level-marking of the mathematics task affect whether the students get the correct or incorrect answer to the task?*

Method

This paper is based on the collection of data through an empirical survey among lower secondary schools in Norway by distributing an online questionnaire and exploring students’ perceptions of level-marking tasks in algebra. A total of 28 female (54.9%) and 23 male (45.1%) participants took part in the pilot study. Participants were students in grade eight (80.4%) and grade nine (19.6%). The current study compares the questionnaire responses of students concerning students' time spent on solving the task. The survey was distributed randomly to the participants, and all students were given the same algebraic task (see Figure 1 and Table 1). The experimental groups got Task 1 with level marking “Difficult”, while the control group got the same task without level marking. To measure time, we chose a task with a high degree of difficulty because the students would take a long time to try to solve it. The high degree of difficulty of the task is an important criterion for choosing the task in order to provide information about the student's perseverance rather than how fast they solve a task (Shen et al., 2016). Given our illustrative focus, a total of 51 students who received Task 1 marked as difficult (Hard) (N = 21) and who received Task 1 without marking (N = 30) responded to the questionnaire in 2021. The analysis draws on two items from the student’s questionnaire.

Table 1: Task and description of the groups

<table>
<tr><th>Participant</th><th>Description</th><th>Task 1 (Authors’ translation)</th></tr>
<tr><td>Control group</td><td>Students who got the task with no level-marking</td><td rowspan="2">The father is retired. If you change the position of the two digits of the number of his age, you get the son's age. One year ago, the age of the father was twice the son's age. How old are father and son now?</td></tr>
<tr><td>Experimental group</td><td>Students who got the task with difficult level-marking</td></tr>
</table>

The computer measured the time each student spent on solving the task. The students got access to the electronic survey, and the computer registered the time from the students opening the questionnaire until they pressed the “send” button (see Figure 1). The time spent on the page included the time used to read the task and to solve it. The survey did not allow students to navigate between pages.


Løs oppgaven

Oppgave 1 - Høy vanskelighetsgrad

Min far er for gammel til å jobbe. Hvis du bytter om tallene i hans alder får du min alder. For ett år siden var min far dobbelt så gammel som meg. Hvor gammel er far og sønn nå?

Send

Figure 1: The questionnaire for the experimental group (for English version of the task, see Table 1)

The independent variables in this analysis were the control group and experimental group (see Table 1). Independent samples t-test was used to analyse the data.

Results

In the following, we will investigate and discuss the effect that level-marking of mathematics tasks has on students' time spent on the tasks. To do this, we used an independent samples t-test. Levene's test for equality of variances was significant ($p = 0.004 < 0.01$), which means that the variances in the two groups were assumed to be unequal. The questionnaire responses showed significant differences in the time spent when level-marking was used compared to the time spent when level-marking was not used. These differences are presented in Table 2, which indicates that students in the control group reported that they spent more time on the task ($M = 223.4$ s, $SD = 239.3$ s) than students in the experimental group ($M = 109.7$ s, $SD = 105.2$ s; $p = 0.027 < 0.05$). Note that the standard deviation in the control group was larger than the experimental group, meaning that students who received the task marked as difficult had less variation in time spent compared with students who did the task without level marking.

Table 2: Students' time spent on the task and correct solutions

	N	Mean (M)	Std. Deviation (SD)	Correct solutions
Control group	30	223.4*	239.3	17%
Experimental group	21	109.7*	105.2	14%

Note. Time: Seconds; M: mean; SD, standard deviation. Levene's test for equality of variances was significant ($p = 0.004$). * The mean difference is significant at the 0.05 level ($p < 0.05$).

The findings show that the time spent by the experimental group was significantly lower than time spent by the control group. The percentage of the students in the experimental group (14% ($N = 3$)) who correctly solved the task was lower compared to the control group (17% ($N = 5$)) (see Table 2.

Discussion

Due to the strong position of mathematics textbooks in mathematics teaching (Stein et al., 2007; Jablonka & Johansson, 2010), it is important to investigate whether level-marking can affect other aspects in mathematics education. However, previous research states that there is a lack of research on the relationship between textbooks and students' learning outcomes (Fan et al., 2013). In this study, we investigated whether level-marking on a mathematical task can affect the student's time spent on the task. The results from the experiment show that the control group spent more time with the mathematics task compared to the experimental group. It is therefore reasonable to hypothesize that there is a significant relationship between level-marking and time spent on the task. This result is important since previous research claims that time spent on a task positively affects achievement (Cheema & Sheridan, 2015; Keith, 1982). The preliminary results from our study show a negative effect of the level-marking on the correct resolution of the given task. The percentage of the students in the experimental group (14%) who correctly solved the task was lower compared to the control group (17%), which means a reduction of almost 18% of the percentage of students who solved the challenging task without level-marking. As mentioned before, a difficult task was chosen in order to measure time spent on the task. This could explain the lower percentage of correct solutions of the task in both groups and the slight differences between the two different groups. However, this shows the potential of a new research direction for investigating other mathematics tasks. The findings suggest that tasks without level-marking will lead students to spend more time on each assignment, which can positively affect the correct solutions (Cheema & Sheridan, 2015; Keith, 1982; Multon et al., 1991). On the other hand, this result suggests that tasks marked as difficult will lead the student to spend less time on the same task. As discussed in the method section, the grade of the chosen task was difficult in order to examine the time they spent *trying* to solve the task rather than how fast they solved it. It should also be noted that the interpretations of the findings in this paper have limitations related to the voluntary nature of participation in the survey and the sample size. The present study investigated only 51 students from nine different schools in Norway, and future research should take these limitations into account.

Conclusion

The results from this pilot study demonstrate that students' time spent on a task is predicted by the level-marking of the task's difficulty. Students who got the task with no level-marking spent significantly more time on the problem compared to students who got the same task marked 'hard'. Spending enough time on a given task can positively affect performance (Cheema & Sheridan, 2015; Keith, 1982). The preliminary results of this study show a negative effect of level-marking on the correct solutions of the given task and suggest that we should be careful in marking tasks as difficult. It is important to let the student assess the task and spend enough time thinking about the possibility of solving the problem. The results show that the standard deviation in the control group was larger than in the experimental group. The tasks might have a low difficulty level for all students regardless of their mathematical abilities even though they are marked difficult (Brändström, 2005). However, more research is needed to fully understand the relationship between level-marking and the time spent on the task by collecting larger amounts of data and investigating the relationship with other variables such as grades, gender, and self-efficacy. In addition, it is important to investigate other tasks with different degrees of difficulty.

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Exploring mathematical stories of future elementary teachers: an analysis of shifts in affect in mathematics

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This research focuses on examining shifts in emotions related to mathematics learning that impacts the attitudes that future teachers hold about mathematics. Initial analysis showed that almost half of the sample did not hold consistent emotions related to mathematics. In examining the catalyst events related to shifts in emotions, many of the shifts to a positive emotional reaction were related to internal structures, and those that shifted to a negative emotional reaction were external motivators. Understanding the events that lead to changes in emotion is important for supporting future teachers in considering how to support future students.

Keywords: Learning and perceptions, mathematics education, affect, emotions.

Introduction

The focus of this paper is understanding the timing and events that influence a change in future teachers' attitudes and emotions towards mathematics. Past research has shown a potential relationship between attitude and achievement (Ma & Kishor, 1997), as well as interconnections between attitudes and beliefs about teaching mathematics (e.g., Philipp, 2007). In Ontario, the teacher education program is a two-year after degree program, so the time with future teachers in a mathematics course is limited. If attitudes have an impact on beliefs and beliefs have an impact on choices made in the classroom (Wilkins, 2008), then understanding where changes in affect occur, is important. Initial research around emotions related to mathematics shows that there is not an exact classification of what makes a student enjoy or dislike mathematics with some things that individuals enjoy about mathematics (e.g., a focus on right and wrong) are the same reasons that others hate it (Holm, 2021). Philipp (2007) notes that in a review of literature around affect, the negative attitudes towards mathematics tend to be rooted in their experiences as learners, so this study seeks to explore further how previous experiences in mathematics impacts shifts in emotional response.

Literature Review and Framework

In Liljedahl and Andrà's (2020) work with future teachers, they note that "emotions are not to be avoided or controlled in mathematical activities, but they have to be provoked, made visible and talked about" and strive to include activities in preservice programs to call attention to emotions (p. 9). In the research study discussed in this paper, the attempt was made to have participants call attention to the emotions that have been provoked during their experiences with mathematics through recalling and writing about their mathematical histories. Hannula (2002) examines attitude as a connection between emotion and cognition in order to understand the influences with attitude, yet notes that although emotion and cognition are separate constructs, they are highly connected and cannot be fully understood separately. My intention in having participants write their stories was for them to understand how their feelings about mathematics were connected to their experiences, their attitudes, and their beliefs through reflection. In this research paper, the focus is on how their emotions were connected to their experiences in order to understand more about the factors influencing emotion and consequently their attitudes related to mathematics.

According to Zan et al. (2006) and Antilla et al. (2016), the area of emotions needs further research to add to the relatively low number of studies currently, and this research project attempts to continue filling this gap in the literature by adding to the discussion. Schukajlow et al. (2019) note that "emotions are defined as phenomena that included cognitive, affective, motivational, physiological and expressive parts. Affect and motivation are used in this definition thus for characterization and grounding of emotions" (p. 3). Emotions are a complex portion of the work in mathematics education research, so an understanding of the types of events that can trigger strong emotions (both positive and negative) can add to the body of work to support future teachers in moving forward.

To ground this research within its focus on emotions, the definition of emotions from DeBellis and Goldin (2006) is utilized: "Emotions describe rapidly-changing states of feeling experienced consciously or occurring preconsciously or unconsciously during mathematical (or other) activity. Emotional feelings range from mild to intense, and are local and contextually-embedded" (p. 134). Since emotions are tied to particular events within mathematics, participants in my research have been requested to write about their emotions connected to mathematical situations, in order to understand what events trigger a change in their connection with mathematics. Ekman (1992) notes that the basic emotions are anger, fear, sadness, enjoyment, disgust, and surprise, with contempt, shame, guilt, embarrassment, and awe being considered basic but share commonalities with the others.

Future teachers will be bringing their emotions about mathematics into their future classrooms, so it is important to understand the root causes of these emotions to help future teachers understand the impact on their own students. Although it is unrealistic to assume that all students will maintain a positive attitude towards mathematics due to the complexities of beliefs, emotions, and affect described in the research cited, an understanding of what causes shifts in attitudes can illuminate some areas of concern in teaching.

The study in this paper uses a portion of the framework that Hannula (2002) has suggested for understanding and analyzing attitude. Although considering all four categories of the framework would have added to the discussion, this paper focuses on two specific areas that could be observed in the data set "the emotions the student experiences during mathematics related activities" and "the emotions that the student automatically associates with the concept 'mathematics'" (p. 26). By focusing specifically on the emotional aspects of the attitude framework, this report looked at what emotionally charged events impacted their attitudes towards mathematics. The basic emotions in the work of Ekman (1992) was used to categorize the emotions in the stories. The basic emotions were categorized as "positive" if they could conceivably lead to a positive outcome in a classroom if presented by the teacher (enjoyment, surprise, and awe), or "negative" if they could conceivably lead to a negative outcome if presented by the teacher (disgust, fear, sadness, contempt, shame, and guilt). The goal of the study was to understand more about how and when future elementary teachers encountered a shift in their emotions related to mathematics. This information will inform how teacher education programs can both support a potential change during a limited time with future teachers, and raise awareness of future teachers on the impact of events within their own classrooms.

Methods

The data for this research paper focuses on three years of written stories collected from future teachers at an Ontario Faculty of Education. The participants would all be in the Bachelor of Education program, either in the Primary/Junior program (K to grade 6) or the Junior/Intermediate program (grade 4 to 10). The Bachelor of Education is a two-year, after degree program, so the participants completed another degree in any subject, including mathematics, although having a mathematics degree is rare in our Bachelor of Education program. The majority of participants will be teaching in K to 8 schools following the completion of their degree.

The stories were part of an assignment that all future teachers would complete in their first-year of the program as part of their initial mathematics methods course. The description of the portion of the assignment relative to this research is as follows: "Create a discussion about your own experiences as a mathematics student. Be sure to include a description/picture of your experiences and how you felt as a mathematics student." Participants would complete the initial assignment draft in the first two weeks of class (November), and then would have the opportunity to edit or add details to submit at the end of the first class (February) to allow for the opportunity for reflection. The research questions in this reported study were as follows: When in school do shifts in affect occur in mathematics? What do participants indicate causes the shift in affect in mathematics?

The initial analysis focused on emotions presented by the future teachers by highlighting events or portions of the stories related to mathematics activities or emotions immediately associated with mathematics (Hannula, 2002), providing a narrower scope for determining when or if a catalyst event occurred. Then the highlighted portions were coded based on the emotions in Eckman's (1992) work. The initial data analysis of the stories resulted in sorting them into different categories based on how they described their emotions about events in mathematics: neutral (no feeling included), only positive (enjoyment, surprise, and awe), only negative (disgust, fear, sadness, contempt, shame, and guilt), and ones that showed a shift from one emotion category to another. Initial descriptive statistics are presented to gain a picture of the total cohort over the three years. The category of neutral was not meant to convey future teachers who did not feel strongly in a single direction but instead a place to sort those who did not mention something related to the basic emotions at all in their responses related to mathematics. Liljedahl and Andrà (2020) note a concern over using a binary construct for considering attitudes and emotions, so this construct was only used as a means to show the range of story types, focusing on the ones where participants noted shifts in how they felt about mathematics. Approximately ten percent of the total number (18) were randomly selected to be independently coded by a critical colleague to determine initial reliability of the sorting of the papers. The independent coder sorted the papers as only positive, only negative, neutral, or showing a shift (including if positive to negative or negative to positive). Coding was compared to the author's codes showing an initial agreement of 83% (15/18) was achieved and after a conversation, the three disagreements were agreed upon between the reviewers further strengthening the lens for coding the remainder of the papers.

Further analysis was conducted on only the stories that described a shift in emotions for the purposes of this paper. These papers were chosen in order to investigate what causes a shift from either positive to negative or vice versa. With the Ontario program only being two years and only having two, ten-week mathematics education courses, time is limited, so it is important to understand how changes occur. The stories were grouped into subcategories to explore: those that shifted from positive to negative, those that shifted from negative to positive, and those who experienced multiple shifts over the course of their experiences. A fourth category was later created for those who did not have a "shift" in emotion but did not convey the same emotion throughout the story. These individuals related to Hannula's (2002) category of automatic response as opposed to reactions to mathematics. A thematic analysis (Braun & Clarke, 2006) was conducted on the stories in each of the sub-categories to understand what caused the shift. Sections of stories where the change was described were initially coded with a descriptive term. Descriptive terms were then gathered and collapsed to determine overarching themes to describe what caused a change in affect in mathematics. The collapsed codes were reviewed by and then discussed with the critical colleague to ensure that all initial codes were accounted for in the final analysis.

Results

In total 182 different stories were considered for the data analysis. Table 1 shows the stories by classification to begin the discussion. Words like "struggle" and "hard" were coded as neutral since they were not coupled with an emotion related to the event. Some individuals did attach an emotion, but some noted enjoyment because of the struggle and some noted dislike because of struggle.

Table 1: All stories categorized by emotion

Type of Emotion	Number	Percentage
Positive	38	20.9%
Negative	50	27.5%
Neutral	16	8.8%
Shift	78	42.9%

Table 2: All stories grouped by the type of shift

Type of Emotion	Number	Percentage
Positive to Negative	37	47.4%
Negative to Positive	17	21.8%
Multiple shifts beginning with positive	13	16.7%
Multiple shifts beginning with negative	6	7.7%
Multiple emotions throughout	5	6.4%

Based on the initial data analysis, only 78 were considered for the second analysis to determine what impacted a shift in the emotions of the future teachers when discussing their mathematical histories. Table 2 shows the breakdown of the new subcategories for this group, and each of the results from the subcategories will be discussed in detail below.

Positive to Negative Stories

In examining the stories for when a single shift occurred from positive to negative, only 35 of the total 37 stories included an approximate time frame. Table 3 shows a breakdown of the time bands.

Table 3: When the shift from positive to negative occurred

Type of Emotion	Number	Percentage
Grades 3-5	4	11.4%
Grades 6-8 (middle school)	3	8.6%
Secondary school (Grades 9-12)	21	60%
University	7	20%

The codes in this group of stories could be sorted into five overarching themes: external motivating factors, societal pressure, internal motivating factors, perception of purpose, and content. External motivating factors and societal pressure were the most commonly noted themes in this portion of the data set. External motivating factors included unhelpful or unapproachable teachers, classroom management styles, grades, and teaching pedagogy. Examples of societal pressure included teacher humiliation, peer pressure, family pressure, and a fear of looking “dumb” in front of peers. Internal motivating factors included frustration, a lack of understanding, fear of failure, and the work being harder that previous grades or units. Perception of purpose related to participants finding the mathematics unconnected to real life or any real-world application for being studied. Content included timed drills, functions, calculus, data management, and algebra.

Negative to Positive Stories

All 17 stories of a single shift from negative emotions to positive emotions related to mathematics contained a time frame for when the shift occurred. Table 4 notes the breakdown of the times that were included. As a note, only grades 11 and 12 were specifically mentioned, other participants noted “high school” in general as the time frame for when the shift occurred.

Table 4: When the shift from negative to positive occurred

Type of Emotion	Number	Percentage
Grades 6-8 (middle school)	4	23.5%
Secondary school (Grades 9-12)	7	41.2%
University	3	17.6%
Adult (after school years)	3	17.6%

The codes in this category could be broken into four overarching themes: internal motivating factors, external motivating factors, perception of purpose, and content. In this subcategory of stories, internal motivating factors was credited the most times as being why the shift occurred. Internal motivating factors included a feeling of confidence, skills development, success, and a feeling that the mathematics made sense. External motivating factors that caused a shift to a positive emotion related to mathematics were all related to specific traits of teachers: engaging teachers, feeling understood, appropriate pedagogy, and caring. Perception of purpose related to finding a connection with real life and application of the mathematics or being able to support their own children in mathematics. The only content mentioned in a positive shift was data management.

Multiple Shifts

When considering the stories that had multiple shifts, these were grouped together since their themes mirrored most of the categories already presented. In looking at those that started positive and had multiple shifts, as well as those that started negative and had multiple shifts, the most common threads were the importance of the teacher and the reliance on grades or success. The participants often pointed to having a singular moment with a teacher who caused a positive emotion that was often shifted by a singular teacher who caused a negative emotion. The feelings of confidence based on success in grades, as well as the feelings of frustration based on a lack of understanding were also often cited as reasons for going back and forth on the like/dislike of mathematics.

Multiple Emotions

Although a much smaller group of the sample, these five participants provided an interesting subgroup to consider. These participants did not have an actual "shift" in feeling around mathematics, but they included multiple emotions throughout the story that ranged from neutral to positive to negative. What was interesting about this group was the reasoning for the changes. Again, the teacher played a role in the feelings reported, where the participants would admit they liked mathematics but hated the class due to the teacher (or vice versa). Oftentimes it was different content areas or aspects of the pedagogy, like collaboration, that were enjoyed but the overall feeling towards mathematics was negative. One participant also changed opinions of the subject based on their feelings around success or failure in relation to the topic.

Conclusion/Discussion

The data from this study pointed to some interesting information of when students shifted from one emotion to another, which showed no discernable pattern, although the majority of the participants saw a shift in later years. Philipp's (2007) review of the literature pointed to affect being determined before college, but this research does not support this finding. A number of individuals found their opinions of mathematics shifted to a more positive outlook in either university or once they were part of the workforce and needing mathematics in real-life. University mathematics was also credited by multiple participants as causing a shift to a negative emotion towards mathematics. Not reporting on shifts in earlier years of school could also be a result of the recency of the events to the time of the reflection instead of attributed to the school or mathematics classes in the early years. One limitation of this study was that the stories were entirely self-reported, so there could potentially be missing details or memories that are not entirely accurate based on the time since the events had occurred. Not all stories started from the same point, so some individuals only reported on what they felt was most important to tell.

In comparing the motivating factors that caused a shift from negative to positive and positive to negative, it was interesting to note that internal motivation most often was credited for the shift to positive, and external or societal pressures were most often credited as a shift towards a negative conception of mathematics. Emotions related to motivational elements (Antilla et al., 2016) suggests that attitudes towards mathematics could continue to be affected with negative emotions that would lead to a decrease in motivation which could continue a cycle of negativity, raising concerns over the impact of emotions with motivation. This finding is particularly interesting for considering preservice teachers so that they understand the power they could have over the emotions of their own students.

The findings showed that some participants credited being challenged in mathematics as being the reason that they felt more positively about mathematics, while others noted that perceiving the mathematics as harder was the reason they now felt negatively about it. This research links to how mindset may affect the entire perception of the mathematics classroom; therefore, the need for a growth mindset (Boaler, 2016) as it may shift the entire perception of mathematics overall. The findings in this study also link to the caution by DeBellis and Golden (2006) that in their research they advocate for developing meta-affect that would allow them to use the negative feelings in a way that would lead to productive learning. Developing this meta-affect with future teachers could prove to be important, but also to make them aware of how to structure their classroom environment to encourage this outlook in their future students.

The smaller group of participants who reported a range of automatic emotions throughout their stories showed that the most prevalent portion of the Hannula (2002) framework in the study was emotions connected to mathematics events and not just automatic responses. This highlights the importance of considering the emotions related to specific events in teaching. This smaller group showed that their responses were caused by mainly external factors in that their feelings towards mathematics were entirely influenced by grades, success, or teacher impacts. As soon as grades dropped, success was less, or as the teacher was perceived as being unhelpful, mathematics was now seen with negative emotions. When the opposite was true, the mathematics was then seen in a more positive light. This research finding also aligns with what Philipp (2007) noted that there is a connection between perceived proficiency and affect.

One smaller theme that came out of this data was around English speakers taking mathematics in French while in French Immersion programs in elementary and secondary school. Although only a small number reported the category, there was a theme around a dislike of math when in the French Immersion program that was resolved when returning to an English program. The participants often cited they felt like the language was a barrier, so this would be a worthwhile area to study further to investigate how this may impact the emotions that a future teacher holds around mathematics.

Even though Philipp's (2007) review found no research linking affect to classroom decisions but notes that there is support for teachers in developing a positive affect. This research sought to begin the conversation about how to support future teachers in developing a more positive relationship with mathematics through understanding what causes shifts in their emotions related to mathematics. Future research directions could consider the intersections with beliefs and practices in mathematics.

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Stereotypes in a polarised world and how they relate to mathematical identity

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Previous studies have indicated that men are stereotyped as more mathematical than women. Since these stereotype images have shown psychological effects—particularly for negatively stereotyped groups—it is relevant to ask how variables other than gender relate to mathematics. In this study, we examined how stereotype characteristics of gender, occupation, political views, and personality relate to mathematical identity. From statistical analyses of Comparative Judgement data, we show significant associations between mathematical identity and extreme stereotypes (related to occupation, political views, and personality). No significant association was found between mathematical identity and gender.

Keywords: Stereotypes, mathematical identity, gender, occupation, political views, personality.

Introduction

Men are often stereotyped as more mathematical than women (e.g., Picker & Berry, 2000). Regardless of how accurately such stereotype images reflect reality; they affect people (e.g., Danielsson et al., 2019; Herzig, 2004; Jugović et al., 2012). An experimental study showed, for example, that women underperform relative to equally qualified men when they, incorrectly, are told that mathematics achievement tests are gender-biased. When they are not informed about this made-up gender bias, men and women perform equally (Spencer et al., 1999). Studies like these exemplify that stereotype images in mathematics are both real and significant.

If there exist stereotype images about gender and mathematics, and if these images affect people, then similar phenomena might exist for other variables. We argue, therefore, that stereotype research in mathematics education should examine: (1) how stereotype images other than gender relate to mathematics, and (2) how such images—insofar as they exist—affect people (in particular, how they affect those who are stereotyped negatively).

In the study on which we report in this paper, we addressed the first of these issues. That is, in addition to gender, we examined how people associate mathematics to three variables: occupation (income, education, and practical/theoretical work), political views (regarding the environment, taxes, state regulations, the EU, and immigration), and five facets of personality traits, known as “the Big Five” (Goldberg, 1993). We chose these variables based on a conjecture that they represent a tendency in Europe and elsewhere, namely that, besides gender, people identify themselves (and others) in extreme opposites; people regard themselves and others, not only as males and females, but also as left-wingers and right-wingers, practical and theoretical, extroverts and introverts, and so forth. How such variables relate to mathematics is unclear.

Using a Comparative Judgement (CJ) design (which we describe later), 34 Norwegian university students were first introduced to the term mathematical identity (MI), which we define in the next section. Subsequently, the students were shown multiple pairs of extreme characteristics (e.g., one person who is against eco-friendly policies; another person who has a high education), and they were asked to judge which of these characteristics that best reflect a person with a strong MI. The results on which we report in this paper answer the following research question: *How do stereotype characteristics of gender, occupation, political views, and personality relate to MI?*

Theoretical framework

Mathematical identity

Similar to Deaux (1993), we view in this paper MI as a relationship between *social MI* and *personal MI* (e.g., Kaspersen et al., 2017). When we refer to social MI, we mean a set of (relatively) agreed-upon characteristics of what it means to be mathematical within a specific context. For example, “making sense of proofs” is a member of social MI, but only insofar (1) most persons within an observed context agree that making sense of proofs is a characteristic of MI and (2) that they interpret in similar ways what proofs are and what it means to make sense of them. Limits for what counts as “most persons” and “in similar ways” depend on the context of the study.

When we refer to personal MI, we mean the extent to which individuals identify with the set of characteristics that constitute social MI within the activity in which they participate. For instance, if a person usually tries to make sense of proofs when she sees one, this characteristic is a part of her personal MI, but only if she participates in an activity where the characteristic is also a member of the social MI.

These definitions are motivated to allow sentences that include measures (e.g., “person A has a stronger MI than person B”). As described in Kaspersen (2018), these sentences are nonsense unless there exists a body of reference, and this is the role of social MI: It is the body of reference to which personal MIs can be measured. When people respond to a MI instrument, a rough interpretation is this: the items (and their psychometric properties) represent the social MI; how person A responds to the items represents person A’s personal MI.

Twenty characteristics of social MI have proven to have relatively robust psychometric properties within (but not necessarily between) a wide range of contexts (Kaspersen, 2018). The characteristics include: “liking to discuss mathematics”, “trying to understand formulas/algorithms”, “struggling with putting mathematics problems aside”, “taking time to understand why some methods in some cases do not work”, and “taking the initiative to learn more mathematics than what is required”. That the characteristics have robust psychometric properties means that they can be applied in instruments (e.g., questionnaires) that measure MI.

In this study, the expression “individuals with low MI” means persons who select the lower categories (e.g., “never”) when they respond to an MI instrument that contains characteristics of social MI. By contrast, “individuals with high MI” means persons who select the higher categories (e.g., “always”) when they respond to the same instrument.

Stereotypes

Stereotypes have been studied for more than a century (e.g., Brauer et al., 2001; Brigham, 1971; Spencer et al., 1999). Nonetheless, researchers disagree on how the term should be defined. In a review of stereotype definitions, Kanahara (2006) documented that some researchers include truth-values in their definitions. Some define stereotypes as relatively true beliefs; others define them to be mostly false. In this paper, we take Kanaharas (2006) position, namely, that the truth-value of a stereotype is a question of empirical matters: In some cases, stereotypes are relatively accurate; in other cases—for example, when mathematics is portrayed as a male domain—they represent the matters of facts fallaciously.

In his synthesis, Kanahara (2006) claimed that definitions of stereotype share two characteristics: (1) they describe stereotype as related to beliefs (e.g., Allport, 1958); and (2) they describe stereotype as a group concept (Giddens, 2001; Krech et al., 1962). Accordingly, Kanahara (2006) defined stereotype as "a belief of a group of individuals", and this is the definition we use in this paper. Specifically, we report in this paper on how students compare beliefs about several groups of individuals—males, females, left-winger, right-wingers, etc. —and how these stereotypical images associate with MI.

The four variables used in the study

We have studied how students compare stereotype images of persons with different genders, occupations, political views, and personality traits. Although none of these variables is dichotomous, the students in this study were asked to consider only extreme opposites. For *gender*, the students stereotyped males and females only. For *occupation*, the students stereotyped persons with six characteristics: low income, high income, low degree of education, high degree of education, practical work, and theoretical work. For *political views*, the students stereotyped persons with ten values: positiveness towards environment-friendly policies, low taxes, state regulations, the EU, and immigration, and negativeness towards these issues. For *personality*, the students stereotyped persons with high or low measures of the facets of the Big Five (Goldberg, 1993), that is, persons with high or low measures of conscientiousness, openness, extroversion, agreeable to experiences, and neuroticism. These characteristics and measures of how much the students associated them with MI are represented in Table 1.

Methods

Participants and data collection

The participants of the study were a convenient sample of 34 student teachers at a Norwegian university. In the first phase, each person responded to an instrument for measuring MI (Kaspersen et al., 2017), which contained 19 statements of MI (one item in the original instrument is reversely coded and was removed in this study). When they had finished, the participants were told that the instrument measures MI and that persons who respond in the lower categories have lower MI than persons who respond in the higher categories. Moreover, the participants were told that the purpose of this first phase was only to familiarise them with the concept of MI.

In the second phase, each participant responded to items of a CJ instrument. A detailed description of CJ methodology and how it can be applied in mathematics education was described by Jones and Inglis (2015). Here, we present a rough outline only. Briefly, CJ is based on a psychological principle: humans are poor at estimating measures but good at comparing them. For instance, it is easier for teachers and other experts to compare the qualities of two mathematical arguments, two mathematical texts, two forms of instructions, and so forth than for them to give measures to each of these things.

The data collection of CJ follows this principle: multiple individuals compare multiple pairs of items, usually using a digital platform. In this study, NoMoreMarking (nomoremarking.com) was used for data collection. For each comparison, the participants were asked: "Which of these persons do you think have the higher MI?" They could then choose between two stereotype descriptions, each holding one of the characteristics listed in Table 1.

For the personality traits, the respondents were not given category labels (e.g., "highly conscientious"), but instead a brief description of those categories (e.g., "a person that is performance-oriented, orderly, self-disciplined, and thorough"). For simplicity, however, we use the category labels when we report the results in this paper.

A sample task is presented in Figure 1. For the respondents, the pairs of items that appeared on the screen seemed to be randomly selected, although they were adaptively chosen by the software to increase the statistical information. Each person made 28 comparisons; thus, the data comprised 952 comparisons in total. To increase the statistical information further, we allowed comparisons of non-opposite characteristics (as the case in Figure 1 illustrates).

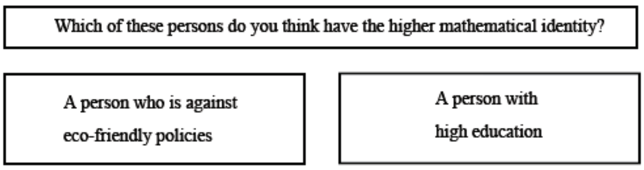

Figure 1: Sample task

Analyses of CJ data is, in principle, similar to Rasch analysis: A maximum likelihood estimation is conducted to estimate the most likely measures of each item (in this study: stereotype characteristics) given the observed data. Subsequently, analyses are conducted to assess dimensionality, judge agreement (i.e., the extent to which the results depend on individual judges), and reliability.

Regarding dimensionality, a crucial question is this: does it make sense to compare the items in the instrument? For instance, does it make sense to make the comparison in Figure 1? In the CJ paradigm, an answer is this: If the judges (here: the students) respond in ways predicted by the Rasch model, the items are sufficiently unidimensional; by contrast, if the judges respond unpredictably, the items are too multidimensional for comparisons to make sense. Infit Mnsq—an information-weighted squared difference between modelled and empirical data—is one indicator of uni-dimensionality. Roughly, when the Infit Mnsq of one item is substantially greater than 1 (in this study, we used 1.3 as a threshold), it indicates that this item is so different from the other items (i.e., it belongs to a different dimension) that it does not make sense to include it in the comparisons.

Regarding judge agreement, a crucial question is whether different judges make different comparisons for similar pairs of items. For instance, will different judges make different judgements on the task in Figure 1? In the CJ paradigm, each judge is associated with an Infit Mnsq value. The interpretation of judge Infit Mnsq is about the same as for item Infit Mnsq: If a judge Infit Mnsq is substantially greater than 1 (in this study, we used 1.3 as a threshold), it indicates that the judge makes judgements that differ substantially from the rest of the judges.

Overall, the analyses showed good psychometric qualities. The reliability was .91, and the largest item Infit Mnsq was 1.3 (“a person who has a theoretical work”). Two persons had Infit Mnsq greater than 1.3 (1.4 and 1.5 respectively), which means that they in some cases made unpredictable answers relative to the rest of the group. However, removing these persons from the analyses had no statistical implication. Thus, the results on which we report in the next section represent a shared agreement amongst the students who participated in the study.

When measures for the items had been estimated, we conducted classical t-tests for each pair of contrasting items using a Bonferroni adjusted alpha level of 0.004. For instance, a t-test was conducted to assess whether the students associated individuals with high income and those with low income differently to MI. To more directly compare the null hypothesis with the alternative, Bayesian t-tests were conducted. Under the null hypothesis, we expected an effect size of 0, and the alternative hypothesis was two-sided. Before observing the data, we assumed that δ followed a Cauchy distribution with scale $r = \frac{1}{\sqrt{2}}$. All analyses were conducted in R (R Core Team) and JASP (JASP Team, 2020), and all measures are reported in logit units.

Results

Gender

In contrast to previous studies, the results in this study indicated no significant differences ($p = .583$) in how students related gender to MI. The Bayes factor was $BF_{01} = 4.7$, which means that the observed data was almost five times as likely under the null hypothesis (i.e., that males and females are associated equally with MI) than under the alternative (i.e., that males and females are associated differently with MI).

Occupation

Three characteristics of occupation—income, education, and work—showed significant ($p < .001$) associations with MI. In effect, the students stereotyped persons with low income, low education, and practical work as having lower MI than persons with high income, high education, and theoretical work, who were stereotyped as having higher MI. In all cases, the Bayes factors ($BF_{10} = 6.9e + 10$ being the least) showed decisive evidence for the alternative hypothesis, namely that occupation characteristics associate differently with MI.

Political preferences

Associations between MI and five contrasting political preferences were assessed. The overall image is that students stereotyped persons positive towards immigration and environment-friendly policies as having a higher MI than persons hostile towards these issues ($BF_{10} = 18.1$ and $BF_{10} = 6760.2$ respectively). There was no significant difference between left-wing and right-wing values for the remaining political issues: attitudes towards taxes, state regulations, and the EU.

Personality

There were also significant differences in how the students connected personality traits and MI. Roughly, the students stereotyped conscientiousness persons as having a higher MI; persons with high measures on the remaining personality traits—extraversion, agreeableness, openness to experiences, and neuroticism—were stereotyped as having a lower MI. The most significant differences were found in conscientiousness and neuroticism. That is, persons that are laid-back, messy, and careless (low conscientiousness) were stereotyped as having a lower MI as compared to individuals that are performance-oriented, orderly, self-disciplined, and thorough (high conscientiousness). Moreover, persons that are sensitive, worried, cheerless, and have mood-changes (high neuroticism) were stereotyped as having a lower MI as compared to individuals that are emotionally robust, handling stress, and balanced (low neuroticism).

Table 1: Characteristics of stereotyped MI

	Stereotyped as lower MI	Measure	Stereotyped as higher MI	Measure	*p*
Gender					
	Male	0.23	Female	0.44	.583
Occupation					
	Low income	−2.43	High income	1.31	< .001
	Low education	−3.26	High education	4.09	< .001
	Practical work	0.14	Theoretical work	4.02	< .001
Political views					
	Non-environmental	−1.03	Environmental	0.22	.002
	Against low taxes	−0.49	Pro high taxes	0.22	.069
	Against state regul.	−0.91	Pro state regul.	−0.23	.077
	Against EU	−1.06	Pro EU	−0.81	.506
	Against immigration	−2.51	Pro immigration	−0.51	< .001
Personality					
	Conscientiousness (low)	−2.47	Conscientiousness (high)	4.28	< .001
	Openness (high)	−0.91	Openness (low)	1.88	< .001
	Extroversion (high)	−0.60	Extroversion (low)	1.42	< .001
	Agreeable to exp. (high)	−1.58	Agreeable to exp. (low)	−0.30	.001
	Neuroticism (high)	−1.28	Neuroticism (low)	2.18	< .001

Discussion

We have maintained that research in mathematics education should examine: (1) how stereotype images other than gender relate to mathematics, and (2) how such images affect people (in particular, how they affect those who are stereotyped negatively). In this study, we considered the first of these issues, and we have shown that there are significant associations between extreme stereotypes (related to occupation, political views, and personality) and MI. No significant association was found between gender and MI.

It is worth noting that the format of the questions in the instrument in this study has some consequences. That is, the respondents in the study were asked: «compare X with Y; which one do you believe have the strongest MI?» They were not asked: «imagine a person with a strong MI; is the person an X or a Y?» Accordingly, it is crucial to make clear which inferences we can draw from the results presented in this paper. We conclude on the form «X are stereotyped as having higher MI than Y»; not on the form «persons with strong mathematical identities are stereotyped as X».

Having made this distinction, we maintain that the significance of the results presented in this paper depend on answers to the second issue, namely, how stereotype images of MI affect people. This is an issue on which we have no data. Nonetheless, we propose a working hypothesis for future research: stereotype images of MI cause similar effects as they do in nearby domains, namely, a stereotype-threat (ST) (Spencer et al., 1999) for people who are stereotyped negatively and a stereotype-lift (SL) (Walton & Cohen, 2003) for those stereotyped positively. In effect, ST and SL function as a self-fulfilling prophecy: If people belong to a group that is stereotyped as having a low MI, they will develop it as such, and vice versa.

We emphasise that SL and ST do not depend on stereotype images to be true (e.g., Spencer et al., 1999). Nevertheless, it is relevant to ask whether the stereotype images presented in this study reflect reality accurately. Suppose that persons with certain occupations, political views, and personality traits have stronger MIs than others. In that case, we might ask ourselves: how do we establish an educational system so that everyone can participate equally (and agree upon the standards) in complex questions that require scientific evidence: questions regarding equality, immigration, vaccination, and global warming? Alternatively, if the students in this study were inaccurate in their descriptions, we might wonder why some groups in society are falsely portrayed as having a lower MI than others and what the practical implications of these images are.

Although we have no empirical answers to these issues, we maintain that future research on stereotypes and mathematical identity should confront the most urgent issues in what seems to be an increasingly polarised society.

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Emotions and decision-making in handling pivotal teaching moments

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The present study focuses on teachers' emotions in relation to their decision-making in handling Pivotal Teaching Moments (PTMs), adopting an Activity Theory perspective. We study a case of an experienced mathematics teacher, who participated in a professional development (PD) program. Data comes from videotaped lessons, PD sessions and interviews with the teacher. From the first interview we identified teacher's motives and goals. From the other three interviews and the videotaped lessons we identified relations between PTMs, emotions and actions. According to the results ten PTMs appeared, accompanied by negative, positive, mixed or neutral emotions. Teacher's emotions seemed to reflect his sense of possibility of achieving his goals while his actions move towards his goals' achievement.

Keywords: Teachers' emotions, decision-making, activity theory, pivotal teaching moments.

Introduction

Emotions set the context for teaching, affect social relations, and reveal effects of power in the classroom (Zembylas, 2004). Emotions has occupied much of the international literature, but very little research focuses on teachers' emotions (Hagenauer et al., 2015), and much more on emotions that arise in mathematics teaching. This is probably due to the methodological difficulties that lie in the qualitative study of emotion, due to its ambiguous nature, which is why most of the studies that have been carried out, mainly concern quantitative approaches. Teachers' emotions are related to the quality of teaching and are a key factor in teachers' decision making (Di Martino et al., 2013). Bishop (2008, p. 30) considers decision making at "the heart of the teaching process" and he argues that if we know about teachers' decisions, we can link teaching to a number of different aspects (e.g., objectives, intentions, children's attitudes, children's mathematical development) and so search for ways of improving its quality (Potari & Stouraitis, 2019). Zembylas (2004) highlights that teachers' decisions are influenced by emotions and reflect teachers' values and beliefs about teaching. Yet, the affective dimension of mathematics teacher decision-making is rarely the focus of research. Teacher decision making is triggered usually on what Stockero and Van Zoest (2013) call pivotal teaching moments (PTM). The tensions that arise during PTMs are considered stressful for teachers and usually negative emotions accompany their management (Pillen et al., 2013).

Therefore, we attempt deepening our understanding on teachers' on-the-moment decision-making while handling PTMs, and their emotions involved in the process. The study of mathematics teaching through these lenses can offer new perspectives on the management and the quality of teaching and consequently on student learning. Our research question is: How do emotions relate to teacher's decision-making in handling PTMs?

Theoretical Framework

In our study we see emotions being formed within the context of the activity in which the individual participates. We adopt the Activity Theory (AT) framework where the activity (mathematics teaching) with the teacher as the subject, is directed towards an object (students' learning) and is expressed through actions with conscious goals. Engeström (1999) takes joint activity as the unit of analysis focusing on the process of social transformation and emphasizes upon the conflictual nature of social practice. The activity is dialectically related to subject's actions: the motive of the activity is concretized as the goal of the action and the action is the one that affects the object of the activity. Action is the way in which the subject engages with the object, that is, the way in which the individual performs the collective activity. Within this context, emotion is a holistic expression of the subject's current state in relation to the object, and the subject's sense of the possibility of success in realizing the object/motive he/she has accepted (Leont'ev, 1978). According to Burkitt (2021, p. 13), emotions are integral components of social interactions, and "they function in complex ways, not only as internal signals to one's self, but also as signals to others, which are frequently spontaneously expressed in the moment without full consciousness of our intention". Thus, the emotion moves together with the activity as a whole, and is one of its manifestations in the actions of the subject in the effort to achieve his conscious goals. Instability, tensions and contradictions, that may occur, may affect subject's emotions in relation to its goals, and thus his actions, reforming the whole activity's context. According to Ekman and Cordaro (2011) the basic (seven) emotions are discrete physiological responses to fundamental life situations that have been useful in our ancestral environment. Plutchik (2001), in his three-dimensional model, presents the characterizations of the primary emotions' intensity and also refers to the "primary dyads" emotions, that are mixtures of two of the primary emotions.

Stockero and Van Zoest (2013, p. 127) define a pivotal teaching moment (PTM) to be "an instance in a classroom lesson in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction, to extend/change the nature of students' mathematical understanding". When a PTM occurs, the teacher should at first recognize it and then decide how to handle his/her interruption. Five types of PTMs were identified: extending, incorrect mathematics, sense making, mathematical contradiction and mathematical confusion.

The teachers trying to manage the PTMs that occur in their classrooms, they are called to take on the moment decisions. According to Engeström (2001, p. 281) "decisions are not made alone; they are indirectly or directly influenced by other participants of the activity (e.g., students in the classroom, other teachers). Decisions are typically steps in a temporally distributed chain of interconnected events". In terms of responsibility and power, decisions have moral and ideological underpinnings. Decisions shape the future of the broader activity system within which they are made, not just the ostensible problem or task at hand (Engeström, 2001).

Methodology

The context of the study and the case of the teacher

The present study is a case study of one high school mathematics teacher in Greece with 30 years teaching experience, a master's degree in special education and rich professional development (PD) experiences. This study is carried out in the context of a professional development program EDUCATE (http://www.ucy.ac.cy/educate/en/general-information/the-project), aiming to support teachers to balance differentiated learning and mathematical challenge. In the context of EDUCATE, he worked in a video club setting with five other high school teachers to conceptualize the meaning of differentiation and mathematically challenging tasks and enact them in the classroom teaching. In particular, seven two- hour PD sessions took place along a period of a school year (2017-2018) and in between the teachers had to plan lessons, enact in their classrooms to address the focus of EDUCATE and share video-excerpts from their lessons with the other teachers in the PD sessions. The teacher of our study designed three lessons two for grade 10 class and one for grade 11 class (mixed ability classes). The aim of balancing differentiation and mathematical challenge, to which the teacher is called to respond, is a fertile ground for creating PTMs and the emergence of emotions as it is new and demanding enough for him. The above process was followed by all the teachers in the group but in this article, we focus on the teacher with the characteristics we described above.

Data and Data Analysis

The data has been generated from the three videotaped lessons, the seven PD sessions and four semi-structured interviews with the teacher. Initially, we analyzed the videotaped lessons to identify the PTMs, teacher's emotion(s) and his actions dealing with each incident, drawing additional data from teacher's reflections on his lessons during the PD sessions. The interviews were conducted by the first author. The first interview's aim was to outline the teacher's teaching profile and his overall teaching goals. For each one of the other three interviews, the teacher was asked to watch the videotaped lessons and identify incidents that he considered important and/or that indicate emotions from his side. In the first part of each interview, we discussed the moments chosen by the teacher, while in the second part the moments chosen by the researcher, in relation to his emotions and actions. Some of the questions addressed to the teacher, when discussing about each incident, are: Why did you choose to discuss this incident? How do you feel at that moment? Why do you feel that way? How you usually feel in similar moments? How did you act/react? Why did you act that way?

Table 1: Coding PTMs, emotions and actions

<table>
<tr><th rowspan="2">Emotions</th><th>Basic (and intensity)</th><th colspan="2">Dyads</th></tr>
<tr><td>Joy (Pleasure, Enthusiasm),
Sadness (Disappointment), Fear, Surprise, Anticipation</td><td colspan="2">Pride (Pleasure + Satisfaction), Delight (Joy + Surprise),
Anxiety (Fear + Anticipation), Disapproval (Sadness + Surprise)</td></tr>
<tr><th>PTMs</th><td>Extending (E), Incorrect Mathematics (I.M.), Sense-making (S.M.), Mathematical Confusion (M.C.), General Confusion (G.C.)</td><th>Actions</th><td>Extends and/or Makes connections, Pursues student(s) thinking, Continues as planned, Ignores/Dismisses, Giving extra time to students</td></tr>
</table>

To identify the PTMs and teacher's actions during teaching we used an expansion of Stockero and Van Zoest's (2013) framework. To identify teacher's emotions during the PTMs, we used semiotic tools such as gestures, facial expressions, gazes (see Ekman & Friesen, 2003) and voice pitch. Then we transcribed the interview data and grounded analyzed them in order to verify teacher's emotions and find relationships between the PTMs, decision-making and teacher's emotion(s). Our emotion characterization was based on a synthesis of Plutchik's (2001) and Ekman and Cordaro's (2011) models. Through this process, the coding schemes used for the analysis of PTMs, teacher's actions and teacher's emotions, are presented in Table 1. Categorizing the data about "why" the teacher feels or acts in this way, we captured the categories that are presented in Table 2 along with their meanings.

Table 2: Coding the "why"

Why feels that way?	*Expecting goal:* the teacher expects the goal to be achieved *Moving away from goal:* the teacher feels that he is moving away from achieving his goal *Slow on goal:* the teacher feels that the process of achieving the goal is delayed *Conflicting goals:* when his thought is attracted in different goals at the same time *Achieving goal:* the teacher thinks he is approaching his goal *Contributing to goal:* when he contributes more than intended to achieve the goal
Why acts that way?	*Redirecting to goal:* when acting so that the lesson progresses back to achieving his goal *Holding on to goal:* when acting in such a way that the flow of the lesson doesn't deviate from the achievement of the goal *Processing goal:* when a goal has been achieved and he extends it through his actions *Globalizing goal:* when he had a goal about one student and momentarily makes it a goal for all the students of his class

Results

Motives and goals

Through the initial interview some information was gathered about teacher's motives and goals. Below is an excerpt from the interview with the teacher:

Interviewer: What characterizes your teachings?
Teacher: Students' engagement.
Interviewer: What do you mean by that?
Teacher: It's obvious. It makes no sense to just copy a solution from the board into their notebook or "parrot" procedures and operations to be able to solve an exercise. It is necessary for them to get involved in the resolution process….to think. Only then is it possible for them to make sense of what they are doing, to see some utility, and I am referring to the mathematics they use.
Interviewer: I see that this is of great importance. But it is also difficult to succeed. How do you do that?
Teacher: I improvise a lot. Depending on where the flow of the class will throw me or I will think what I want to do.
Interviewer: So?
Teacher: (pause) So, I try to play, challenge them with questions and arouse students' interest. It is important because the student will be bored, and secondly, I am bored too talking all by myself (laughs).

The above extract illustrates teacher's image of the object (motive) of the activity which is students' engagement and making sense of mathematics. To achieve students' engagement, he poses two main goals, to attract students' interest and to involve as many students as possible. He rejoices when the majority of students interact with each other and with him during the lesson as he considers learning a collective process. He wants even for the students that won't follow a mathematical orientation, to develop a mathematical way of thinking. It is also important for him to build a safe classroom environment for the students to feel free to express their ideas.

Emotions and Decision-making: Approaching "why"

In our data ten PTMs appeared: four are accompanied by negative emotions (I.M., M.C., G.C and E.), four by positive emotions (S.M., M.C., G.C. and E.), one by positive and negative emotions (S.M.) and one by neutral emotion (M.C.). In Table 3 we illustrate our findings about five out of the ten PTMs, the most representative of each category (based on the intensity of emotions' expressions).

Table 3: PTMs, emotions, actions and "why"

<table>
<tr><td colspan="4">1st PTM (Incorrect Mathematics): The teacher asks why we use the root symbol. Students give insufficient or incorrect answers. The teacher seems a bit irritated. He listens to all students' answers without correcting them. After summing them up, they come to a correct conclusion.</td></tr>
<tr><td rowspan="2">Negative Emotions</td><td rowspan="2">Anxiety
Disapproval</td><td>Why?</td><td>Interview's Excerpt</td></tr>
<tr><td>Expecting goal
Moving away from goal</td><td>"it is a crucial point what the students will answer for the development of the lesson" … "this knowledge should have been acquired in previous classes".</td></tr>
<tr><td>Actions</td><td>Extends and/or Makes connections
Pursues student(s) thinking
Ignores/Dismisses</td><td>Redirecting to goal</td><td>"It is important to hear all the views of the students so that there is a "game" between them." … "when something is heard from their classmate, it is more easily accepted by the students because it is not something "alien sounds" that is heard by the authority".</td></tr>
<tr><td colspan="4">2nd PTM (Extending): A student presents an alternative (correct) way of solving that goes beyond the mathematics that the teacher had planned to discuss. The teacher doesn't seem to fully understand it at that moment and skips it.</td></tr>
<tr><td rowspan="2">Negative Emotions</td><td rowspan="2">Anxiety</td><td>Why?</td><td>Interview's Excerpt</td></tr>
<tr><td>Slow on goal</td><td>"Because then I didn't fully understand what the student were saying". Also, "there was no time to devote to it" and had to remain in the original lesson plan.</td></tr>
<tr><td>Actions</td><td>Ignores/Dismisses</td><td>Holding on to goal</td><td>"You want to move forward, you also have a flow, a path of thought and you skip it".</td></tr>
</table>

<table>
<tr><td colspan="4">3rd PTM (Mathematical Confusion): There is interaction between the students. They ask questions, they are confused about the concepts of decimals with infinite number of digits and irrational numbers. A student says “it is not possible to draw a length of 1.4… and infinite decimals, right?”, he is troubled. The teacher tells him that if he gives him a ruler, he can draw it. After a few minutes of talking with the students, he states that this problem has preoccupied the Pythagoreans, and that in another lesson they will prove/construct the line segment with length $\sqrt{2}$.</td></tr>
<tr><td rowspan="2">Positive Emotions</td><td rowspan="2">Delight</td><td>Why?</td><td>Interview’s Excerpt</td></tr>
<tr><td>Achieving goal</td><td>“This shows that the lesson works, there is interaction, the students have entered the “game” and it starts working in their mind, the student has something to say and the teacher has something to teach”.</td></tr>
<tr><td>Actions</td><td>Extends and/or Makes connections
Ignores/Dismisses
Continues as planned</td><td>Holding on to goal</td><td>“I didn’t want to escape tangible planning. The goal for this course was to reach the definition of the ninth root, so time would not be enough.”.</td></tr>
<tr><td colspan="4">4th PTM (Sense Making): A student unfolds his thinking and explains everything very correctly and in detail. The teacher does not confirm anything and asks for the others students’ ideas.</td></tr>
<tr><td rowspan="2">Mixed Emotions</td><td rowspan="2">Joy
Disappointment</td><td>Why?</td><td>Interview’s Excerpt</td></tr>
<tr><td>Conflicting goals</td><td>“Because the student has understood from a previous lesson what he says and reproduces it now… I don’t want it to end like this for the other students”.</td></tr>
<tr><td>Actions</td><td>Ignores/Dismisses
Pursues students’ thinking</td><td>Globalizing goal</td><td>“If I had verified that this was the correct answer, the other students would have never tried to do something like that”.</td></tr>
<tr><td colspan="4">5th PTM (Mathematical Confusion): In classroom they complete the root-definition (on board) and a student does not understand why the constraint θ≥0 is needed since it is in the square. Then the teacher asks them: if $\sqrt{a} = -5$ then $a = 25$ and then $\sqrt{25} = -5$? He uses a non-example so that the student can see the error in his reasoning. He had “expected the students to have difficulty with the constraint”.</td></tr>
<tr><td colspan="2" rowspan="2">Neutral Emotion</td><td>Why?</td><td>Interview’s Excerpt</td></tr>
<tr><td>Achieving goal</td><td></td></tr>
</table>

Actions	Pursues students' thinking	Processing goal	"I don't feel anything. The use of a constrain is a common confusion-point for the students, you know that too. Every year with each class we have the same talk about constrains and their use."

The first and the second PTMs are accompanied by negative emotions which are seemed to be created due to the unexpectedness of these incidents and their potential to make the teacher deviate from the initial lesson planning and his goals. Nevertheless, teacher's actions seem to target on redirecting or holding on to his initial goals. Teacher's positive emotions in PTMs seem to be created due to the positive outcomes of his interaction with the students in relation to his goals. Although in the third PTM it could be expected that his feelings were negative, he feels surprised and pleased because his students' mathematical confusion is a sign for him that the lesson he planned, triggered their mathematical thinking, something that is of extreme importance for him. In the fourth PTM, on one hand he is joyed due to his student's understanding and rigorous explanation but on the other hand he feels disappointed because he thinks that if he confirms the correctness of this answer, the other students' engagement would be minimized. His decisions once again align with his main goals and are oriented in confronting that negative emotion. The neutral emotion that accompanies the fifth PTM may be due to the fact that the teacher had been expecting this to happen. This also can be confirmed by his readiness to deal with this incident in such procedural way.

Discussion

In our study we try not just to describe the teacher's emotions and decision-making in handling PTMs, but also to find a way to make sense of his on-the-moment decisions through his emotions. Our teacher's case indicates that his emotions aren't directly dependent upon the type of each PTM (e.g., 3rd PTM in Table 3), but they reflect his sense of possibility of success in realizing the motive he has accepted (Leont'ev, 1978). Our findings agree with what Schoenfeld (2011, as cited in Potari & Stouraitis, 2019) claims that decisions are consistent with the teachers' goals consciously or unconsciously. Through his decisions, the teacher seems to be led by his need to refocus his thinking and actions on the aspects of his environment (classroom interaction and management) they could change for achieving his goals, but also to be able to push back on the unpleasant emotions and try to maintain a positive disposition. A number of appraisals such as valence, goal congruency, expectedness, or controllability are assumed to influence human emotions; they can be classified as situational appraisals and appraisals about the self (Scherer et al., 2001, as cited in Schukajlow et al., 2017). The teacher's long teaching experience may justify his flexibility in how he moves towards his goals (Westerman, 1991) and in handling his emotions, but further research into this aspect is necessary. The study of teachers' emotions in relation to their decision-making in more cases of teachers could give us insights for understanding better how these interrelate in the complex context of mathematics teaching and learning.

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Preservice teachers' interest and self-efficacy beliefs while posing problems

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Problem posing is considered to have great potential to foster students' motivation because it provides the opportunity to create problems on the basis of individual interests and abilities. However, whether learners indeed use this opportunity is an open question. The aim of the present study was to investigate the role of interest and self-efficacy expectations in problem posing. In interviews, we asked preservice teachers (N = 7) why they decided to pose a certain problem, and we focused on their interest and self-efficacy expectations when we analyzed the data. The most important factor was preservice teachers' interest in the answer to the posed problem. Self-efficacy expectations have been also revealed as important for their decision to pose a certain problem. Preservice teachers wanted to be able to solve the problem or to master problem posing by posing multiple problems or a problem with adequate difficulty.

Keywords: Problem posing, interest, self-efficacy, real-world problems, preservice teachers.

Introduction

Problem posing has recently been receiving greater attention from the scientific community. Researchers have emphasized that problem posing has the potential to help students learn mathematics and may be particularly beneficial for students' motivation (Cai & Leikin, 2020). Problem posing conveys the experience of autonomy and control by allowing students to generate problems that fit their own interests, needs, and abilities. Consequently, learners' interest and self-efficacy expectations can be expected to be important for their problem posing processes. There is little research on the relationships between interest, self-efficacy, and problem posing, and prior studies have rarely addressed how interest and self-efficacy can influence the posing of problems. The present study is aimed at gaining a better understanding of the roles that interest and self-efficacy play for problem posing by investigating learners' beliefs about factors that affect the development of self-generated problems.

Theoretical background

Problem Posing

Problem posing is typically defined as the generation of new problems and the reformulation of given problems (Silver, 1994). It is considered a powerful learning approach for improving students' motivation and problem solving abilities and is also an important learning goal in itself (Cai et al., 2015). Even if some progress has been made in recent years, problem posing is still a rather young and evolving field in which scholars are just beginning to develop an understanding of the processes that take place and the benefits problem posing can have for the learning of mathematics. Here, we look at problem posing from the perspective of modelling and applications in mathematics education. We focus on modelling-related problem posing, which we define as the generation of real-world questions that can be solved with the help of mathematics on the basis of stimuli that are connected to the real world (e.g., textual descriptions of real-world situations, pictures, or artefacts from the real world) (Hartmann et al., 2021). An example of problem-posing stimuli used for modelling-related problem posing is depicted in Figure 1.

Cable Car

For more than 90 years, the cable car *Nebelhornbahn* took numerous guests up into the heights. Now it can go into well-earned retirement. Starting in the summer of 2021, a new cable car will transport enthusiastic outdoor fans to the mountains on the Nebelhorn. The aim of the project is to avoid long waiting times, provide seated transportation with an optimal view at every seat and increase the carrying capacity.

Technical data of the old cable car:

Model:	Large-cabin aerial tramway
Weight empty cabine:	1600 kg
Weight full cabine:	3900 kg
Height valley station:	1933 m
Height top station:	2214.2 m
Horizontal difference:	905.77 m
Speed:	8 m/s
Carrying capacity:	500 persons/h
Power unit:	120 PS

Figure 1: Real-world situation used as a stimulus for modelling-related problem posing

Problem posing and interest in solving self-generated problems

Motivation is an important variable that affects students' behavior, decisions, and performance (Middleton & Spanias, 1999). Closely related to students' intrinsic motivation is students' interest, which is defined as a person-object relationship that refers to both the state of attention and affect toward a particular topic (situational interest) and an enduring predisposition to reengage with a topic over time (individual interest) (Hidi & Renninger, 2006). Learning environments can foster the development of individual interest by triggering situational interest and building upon prior individual interest (Hidi & Renninger, 2006). Problem posing provides opportunities to generate problems according to a person's own interests, and thus, incorporating problem posing in class has the potential to improve students' motivation (Cai & Leikin, 2020). Prior research has shown that teaching interventions that include problem-posing activities positively affect students' individual interest in mathematics (Xia et al., 2008) and their situational interest in the topic that is being addressed in class (e.g., algebra) (Walkington, 2017). However, the extent to which students' interest influences their decision to generate a certain problem is an open question. Learners' affect and more specifically learners' interest could refer to different objects (Schukajlow et al., 2017). For modelling-related problem posing, they could either center their interests on the real-world context or on an inherent intramathematical topic. In addition, the people whose interests are being evaluated might differ. If teachers pose problems for their students, they might refer more to what they think their students find interesting than to their own interest. Preservice teachers might already identify with their role as teachers, but it is also possible that they still identify more with the student role, particularly because they do not yet have students of their own to relate to.

Problem posing and self-efficacy in solving self-generated problems

Self-efficacy is defined as the perceived ability to successfully perform an action in the future (Bandura, 1997). Prior research has demonstrated that self-efficacy is positively related to performance and is an important factor that influences students' decisions while learning (Pajares, 1996). For problem posing, the criteria for estimating students' success varies according to the goals of constructing a problem. Voica et al. (2020) showed that self-efficacy in problem posing occurs in relation to several goals. The preservice teachers investigated in this study reported on their self-efficacy to construct a large number of problems, difficult problems, original problems, or problems that matched the abilities of a problem-solver. Further, a person's problem solving self-efficacy could also influence his or her decision to pose a certain problem because learners might generate only the kinds of problems that they are confident they can solve. This could explain why learners tend to pose simple and closed-ended problems (English, 1998; Hartmann et al., 2021). The present study is aimed at shedding some light on this potential reason. As the participants of our study were preservice teachers, it is also possible that their expectations refer more to the success of their students than to their own expectations of success with respect to being able to solve the problem. However, it is also possible that preservice teachers consider it as particularly important that they are themselves able to solve their self-generated problems, because they might imagine their future selves as teachers. The study of Marschall (2021) shows that preservice teachers' self-efficacy appraisal is closely related to narrative self-schema and the development of professional identity. Thus, it is important to take into account the different perspectives they adopt when evaluating their problem posing process or the potential success of a student in solving the problem.

Research questions

The aim of the present study was to investigate the role of interest (RQ1) and self-efficacy expectations (RQ2) in problem posing. In particular, our research questions were:

RQ 1a: What are the objects of interest that preservice teachers refer to when posing problems?

RQ 1b: What are the perspectives of preservice teachers when they evaluate interest?

RQ 2: What are the goals that preservice teachers' self-efficacy refers to?

Method

Sample and data collection

Seven preservice teachers (three women, four men; between the ages of 20 and 26 years) from a German university participated in the study. The preservice teachers participated voluntarily in the study and were selected under maximum variation sampling criteria in order to gather information rich data and to analyze the existence of patterns across the variation. As selection criteria, we focused on their mathematical performance levels, their experience with problem posing and solving, and on their study programs. Five of the preservice teachers studied for a teacher profession at high track secondary schools and two of them for a teacher profession at middle track secondary schools. Four of them were in a master program and three in a bachelor program. Pseudonyms were used for the names of participants. In a laboratory setting, each of the participants received three real-world situations (Cable Car, Fire Brigade, Chopsticks) with the request to first pose problems on the basis of the given situation by instructing them "Read the description of the situation aloud. Then you think about what mathematical question you can ask yourself about the situation." and afterwards to solve their problems. One of the situations is depicted in Figure 1. A shortened version of the other problems is presented in Table 1. After posing and solving the problems, the participants were asked in stimulated recall interviews about why they generated the particular problems at hand. The interviews were video-recorded, the video material was transcribed and sequenced into classification units.

Table 1. Shortened version of the additional problem posing stimuli

Fire Brigade. The Muenster fire department has a total of 16 locations in downtown Muenster, so that there is a maximum distance of 6 km from a burning house. On average, a truck manages to drive about 40 km/h in Muenster city traffic. A central component of the Muenster fire-brigade is a fire engine with a turn-ladder. The dimensions of such a fire engine with a 30 m turn ladder are specified in the fire department's guidelines. [Table with data about the fire engine]. Using a fire-engine the fire-brigade can rescue people from great heights. The rescue is carried out via a cage attached to the end of the ladder. […] So called HAUS-rules, in which minimum distances of the vehicle are specified are applied [List with distance rules]. [Picture of a fire engine]
Chopsticks. Lisa is looking online for a gift for her mother's birthday. […] Lisa decided to buy her mother chopsticks. She finds the following offer: [Picture with product details including the price, and the length of the chopsticks, and the number of product reviews]. Lisa finds a nice storage box for the chopsticks [Picture with product details including measures, weight, price and product rating]. To save some money, Lisa researches online for a discount promotion. She discovers a discount promotion where she gets a 10% discount on her entire purchase if her purchase is worth 20€ or more, and a 20% discount if her purchase is worth 30€ or more.

Data analysis

A qualitative approach was used for this study to gain in-depth insights about preservice teachers' reasons for posing a certain problem. We focused on preservice teachers' interest and self-efficacy, because these constructs revealed as important during the interviews. The transcripts were analyzed using a qualitative content analysis. A category system was developed using a data-driven approach. The sequences were coded with regard to the objects of interest, the perspective of interest, the goals of self-efficacy expectations and the perspective of solving expectations. The material was analyzed by a trained coder; the coding was discussed among the members of the research team using consensual coding principles and checking for reliability (Cohens' $\kappa \geq .662$).

Findings

An overview of central categories and sub-categories that were derived from the material is presented in Table 2.

Table 2. Categories for reasons that explained the self-generated problems

Main category	Subcategory	Description	Example

Objects of interest	Interest in the answer	Interest in the answer to the posed problem	…because I was interested in knowing whether the chopsticks were free.
	Interest in the real-world situation	Interest in specific aspects of the real-word situation	…because if I think about what I find most interesting...
	Interest in the mathematical topic	Interest in the mathematical topic that is related to the problem	…because I like to have tasks with functions.
Perspective of interest	One's own interest	Statement of interest refers to the preservice teacher's own interest	…what I am personally interested in.
	Interesting to the protagonist	Statement of interest refers to the protagonist's interest	…I thought about what Lisa wanted to know.
	Interesting for students	Statement of interest refers to the interest of the imaginary students who would work on the problem	…because I thought that would be interesting for the students to know.
Goals of self-efficacy expectations	Solving	Reasons that address the problem-solver's self-efficacy with respect to solving the problem	…that I am able to solve the problem.
	Posing	Reasons that address the problem-solver's self-efficacy with respect to mastering problem posing	…that I am able to construct a problem with an adequate level of difficulty

Preservice teachers' interest in posing problems

Interest was often mentioned by the preservice teachers when explaining why they decided to pose a certain problem. Thereby the participants refereed to different objects of interest. Most frequently they were interested in the answer of their self-generated problem (18 sequences, 5 out of 7 participants). For example, Anna posed the following problem to the Chopstick task: "How much money can Lisa save?" She explained that she posed this problem because she wanted to know whether or not the chopsticks would be free.

Anna: (Chopstick task) Hm, [.] Definitely interesting for me too, because [.] I was also interested in whether the chopsticks would ultimately be free, that is, were given for free.

Their interest in aspects of the real-world situation was also often mentioned by explaining their decision to pose a certain problem (9 sequences, 5 participants). For example, Theo said that he included his personal interests in his problem posing process.

Theo: (Cable car task) […] and then I have linked the [.] knowledge [.] [.] with what would be of further interest to me personally. [.] So [.] How does it look if I [.] in particular limit the people to the window seats [.].

In addition, interest in the mathematical topic behind the problem was mentioned by three participants (3 sequences).

Further, differences were found regarding the perspective of who might find the problem interesting (Perspective of interest). Most preservice teachers referred to their own interest or what in general could be seen as interesting (15 sequences, 6 participants). However, some participants took the perspective of the protagonist of the given real-world context and generated a problem that they thought would be interesting for this person to solve (7 sequences, 4 participants).

Theo: (Chopstick task) So practically I put myself in her [the protagonist’s] position [.] and thought I would like to save money on the one hand, but also have as many chopsticks as possible [.].

One participant (Lea) also took the perspective of an imaginary student who could work on the problem (2 sequences).

Lea: (Fire brigade task) Because I also thought that the students might be interested in this. Or because they understand why they / because they don't think: This is a useless task [..] well, because somehow you [.] understand why you want to know.

Preservice teachers’ self-efficacy when posing problems

Regarding self-efficacy, we found that the preservice teachers in our sample often referred to their self-efficacy expectations to explain why they posed a certain problem. In particular, their expectations that they would be able to solve the self-generated problems played a major role in posing the problem. (12 sequences, 5 participants).

Fabian: (Chopstick task) It was not a known task, but of course a task that I could solve myself, that / exactly. Um, now with this task, I already had my solution method in my head. So, I knew straight away how / how to / so I didn't have to think about the right solution for a long time but already knew the procedure, um how to proceed.

In addition, their self-efficacy expectations of mastering problem posing was also mentioned by two of the participants (4 sequences). Thereby, the preservice teachers set different goals that they wanted to achieve. One of the participants (Leon) wanted to pose multiple problems with increasing difficulties. For each situation, he constructed at least three problems, beginning with an easy problem and ending with a difficult one.

Leon: (*Cable car task*) Again from easy to difficult. In the beginning, it was [.] the simple task with the Pythagorean theorem [..] um, and then it should get more and more difficult with a variable [.] that has to be calculated.

For the cable car situation, Leon generated the following four problems (Figure 1): “How long is the distance between the valley and the mountain station? How much time does it take to go one way? What is the maximum number of times per hour that the cable car can run? How many people would be in the cabin on average?”

Lea had a different goal, for her it was important to construct problems with an appropriate level of difficulty.

Lea: (*Cable car task*) So it was a compromise between: It's not too difficult, it's not too easy, it's somehow feasible.

The preservice teachers' expectations for solving the problem mainly referred to their own self-efficacy (11 sequences, 5 participants). However, the one participant (Lea) who considered students' interest also took into account students' potential solving success and constructed a problem that she was confident the students would be able to solve.

Summary and discussion

In the present study, we investigated how interest and self-efficacy contribute to preservice teachers' decisions to pose a certain problem. Our findings support the idea that problem posing could be a powerful tool to foster motivation (Cai & Leikin, 2020). The preservice teachers in our study often posed problems to which they were interested in knowing the answer. This indicates that learners use the opportunity that problem posing provides to integrate their interests in their problem-posing processes. Their interest in the answer of the self-generated problem might have benefits for their perseverance and the effort they put into problem solving or might positively affect their interest in the math that is needed to solve the problem (Walkington, 2017) or might contribute to their interest in mathematics in general (Xia et al., 2008). Future studies should investigate how this potential of problem posing can be used in learning environments that are aimed at fostering students' motivation. Our findings highlight the importance to distinguish between objects of interest (Schukajlow et al., 2017). Besides interest in the answer and in certain aspects of the real world situation, interest in the mathematical aspects behind the problems was also a reason for preservice teachers' self-generated problems. Further, preservice teachers mostly referred to their own interest, but they also took other perspectives and generated problems that they thought would be interesting to the protagonist of the real-world situation or to the students who would be working on the problems.

In addition, we focused on the role that preservice teachers' self-efficacy expectations play in problem posing. We found that some of the learners stated that it was an important criterion for the self-generated problems that they were confident they could solve the problems themselves. This contributes to explaining learners' tendency to pose simple problems as found in prior studies (English, 1998; Hartmann et al., 2021). However, in line with the results from the study conducted by Voica et al. (2020), their self-efficacy referred not only to their expectation that they would be able to solve the problem but also to their expectation that they could successfully pose a problem. Thereby, constructing multiple problems and constructing problems with an appropriate level of difficulty were identified as problem-posing goals. This finding exemplifies the idea that preservice teachers' self-efficacy appraisal is linked to their different roles such as their role as future teachers (Marschall, 2021). In addition, it shows the importance to take into account different goals associated with problem posing. For further research it seems necessary to better understand these goals to develop evaluation criteria to assess problem posing performance.

As a practical implication, our study shows that it was important for the preservice teachers to construct problems that they were confident they would be able to solve. When teaching problem posing in class, it therefore seems necessary to encourage students to also pose difficult problems that allow for progress in their learning. In addition, teachers should be aware of the different goals that come up with problem posing (e.g., posing multiple problems) and make them transparent to the students.

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Actions Speak Louder than Words: Social Persuasion through Teaching Practice

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What a student believes about their abilities to do mathematics has a significant impact on the ways in which they then do mathematics. If they believe they can solve a problem they behave very differently than if they believe they cannot. As such, the development of self-efficacy needs to be paramount in the conversation of what it means to teach mathematics. In this paper, I look at the subtle ways in which students' self-efficacy is developed within the specific teaching paradigm of Building Thinking Classrooms. Results indicate that what a teacher does is just as important as what a teacher says when it comes to developing student self-efficacy in mathematics.

Keywords: Self-efficacy, affect, social persuasion, thinking classrooms.

Introduction

Henry Ford is famously credited with stating that "whether you think you can, or think you can't … you're right". Ford believed that beliefs in your abilities to achieve something are central determinants on whether or not you do, indeed, succeed. Believe you can—and you likely will. Believe you can't—and you probably won't. This maxim likely emerged from Ford's belief that success is the product of hard work. And in order to persevere in the face of the work that is needed to succeed you have to first believe that your hard work will be met with success.

Bandura (1997) refers to this belief in your abilities to succeed as self-efficacy and argues that these beliefs affect how a person thinks, feels, and behaves within a given situation. "It affects the choices they make, the effort they put forth, the perseverance they display in challenges, and the degree of anxiety or confidence they bring to the task at hand" (Rouleau, Ruiz, Reyes, & Liljedahl, 2019). And because these self-efficacy beliefs affect the way you act, they "can powerfully influence the level of accomplishment that people ultimately realize" (Pajares, 2006, p. 341).

Self-efficacy has long been seen as having a significant influence on student performance in mathematics (Hackett & Betz, 1989; Hannula, 2012; Skaalvik et al., 2015). As such, helping students to develop positive self-efficacy beliefs about their mathematical abilities should be an important part of what it means to teach mathematics. In this paper I look at how this is achieved through a teaching paradigm called Building Thinking Classrooms (Liljedahl, 2020).

Development of Self-Efficacy

Whether a person believes they can, or believes they can't, Bandura (1986, 1994, 1997) argues that these beliefs emerge from an individual's encounters with one of four sources—mastery experiences, vicarious experiences, social persuasions, and emotional/physiological reactions. The first of these, mastery experience, is the most influential of the four and is the result of one's previous experiences with success and failure. If you have had a lot of experiences of success you are more likely to believe that you can be successful in future endeavors. Likewise, if you have had repeated experiences of failure, you are less likely to believe that future efforts will be met with success. In essence, self-efficacy is strengthened by success and weakened by failure and that once a trend is established, single contradictory experiences are less likely to affect a person’s overall self-efficacy belief. That is, if a person is accustomed to succeeding and has, as a result, developed positive self-efficacy beliefs, a single encounter with failure is unlikely to affect their overall positive belief in their own abilities (Skaalvik, Federici, & Klassen, 2015). Conversely, a person with negative self-efficacy is unlikely to change their view of their abilities after only one encounter with success unless that encounter was an AHA! experience (Liljedahl, 2008) or occurred on a task others found especially challenging (Bandura, 1997).

The second source of self-efficacy is through vicarious experiences—seeing a peer experience success or failure through their actions. This source of self-efficacy is greatly heightened when they perceive a strong similarity between themselves and the peer they are observing. For example, a student watching a friend be successful at solving a mathematics problem will create a stronger belief in their own abilities than watching a teacher solve the same problem. The teacher is perceived as being too capable and too knowledgeable for their successes to be seen a reasonable approximation of what will occur when the student tries it on their own. A friend’s success, on the other hand, is a better proxy—similarity trumps expertise.

The third source of self-efficacy, social persuasion, draws on persuasive communication and evaluative feedback to build up beliefs about one’s abilities. Unlike vicarious experiences, social persuasion is enhanced when coming from a source perceived to be knowledgeable. That is, the words of a teacher expressing confidence in a student or reminding a student what they are capable of is more effective at building up their self-efficacy than the same words from a peer. For social persuasion, expertise trumps similarity.

The final source of self-efficacy is emotional/psychological reaction and comes from how a person perceives their own emotional reaction to a situation. For example, if a student works very hard to try to solve a problem, persisting in the face of much frustration and several failed attempts, they may perceive this experience in a number of different ways. They may focus on the time it took and how much of that time was spent being frustrated and how many failed attempts there were and conclude that they are not good at solving problems. Alternatively, they may focus on the fact that they endured, persevered through the hardship of frustration and failed attempts, and conclude that they are strong and capable. The same experience can result in different reactions which, in turn, can result in different self-efficacy forming. “It is not the sheer intensity of the emotional and physical reactions that is important but rather how they are perceived and interpreted” (Bandura, 1994, p. 3).

Building Thinking Classroom

Building Thinking Classrooms (Liljedahl, 2020) is a teaching framework that was developed in response to the realization that much of what happens during a mathematics lesson is not thinking. In particular, the baseline data that emerged from this research showed that in a typical lesson about 20% of students spend approximately 20% of the time thinking—8-12 minutes per hour—while the other 80% of students spend no time thinking (Liljedahl, 2020). Research has shown that the normative practices present in many classrooms are promoting, in both explicit and implicit ways, non-thinking behaviors such as mimicking among students (Liljedahl & Allan, 2013). These normative structures permeate classrooms around the world and are so entrenched that they transcend the idea of classroom norms (Yackel & Cobb, 1996) and can only be described as institutional norms (Liu & Liljedahl, 2012)—norms that have extended beyond the classroom and have become ensconced in the very institution of school and fabric of what it means to teach.

Much of how classrooms look and much of what happens in them today is guided by these institutional norms—norms which have not changed since the inception of an industrial-age model of public education. Yes, desks look different now, and we have gone from blackboards to greenboards to whiteboards to smartboards, but students are still sitting, and teachers are still standing. Although there have been many innovations in assessment, technology, and pedagogy, much of the foundational structure of school remain the same. If we want to promote and sustain thinking in the classroom, these norms are going to have to change (Liljedahl, 2020).

Over the course of 15 years, and through the conducting of thousands of micro-experiments with over 400 practicing teachers, a series of 14 practices emerged that break away from the aforementioned institutional normative ways of teaching and have been proven to get more students thinking and thinking for longer (Liljedahl, 2020). Each of these 14 practices is a response to one of the following 14 questions:

1. What are the types of tasks used?
2. How are collaborative groups formed?
3. Where do students work?
4. How is the furniture arranged?
5. How are questions answered?
6. When, where, and how are tasks given?
7. What does homework look like?
8. How is student autonomy fostered?
9. How are hints and extensions used?
10. How is a lesson consolidated?
11. How do students take notes?
12. What is chosen to evaluate?
13. How is formative assessment used?
14. How is grading done?

Although each of these 14 practices, on their own and in concert, have been empirically shown to increase student thinking in the classroom (Liljedahl, 2020) the visually defining qualities of a thinking classroom is that students work together to solve thinking tasks in random groups of three while standing at vertical whiteboards (see figure 1).

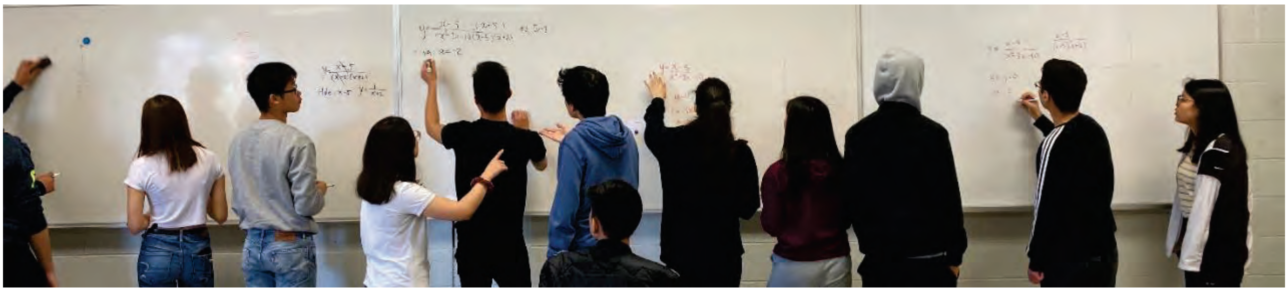


Figure 1: A thinking classroom

When put together, these 14 practices build a classroom ethos, routine, and culture of students thinking individually and collectively to do and learn mathematics. And it radically improves on the baseline data stated above. Rather than 20% of students thinking, we are now seeing upwards of 90%. And rather than thinking for 8-12 minutes students are thinking for 50 – 85 minutes.

Aside from getting students to think, building a thinking classroom has also been seen to fundamentally change other aspects of the student experience, one of which is their self-efficacy beliefs. The repeated and ubiquitous opportunities to persevere and be successful through the use of thinking tasks creates mastery experiences. In addition, the constant work in groups provides ample opportunities for vicarious experiences to occur. And the close proximity within which they work allows for the spread of positive emotional reactions.

What I am more interested in, however, and what is the phenomenon of interest for this paper, is the ways in which social persuasion manifests within a thinking classroom. Whereas social persuasion is most often seen as explicit verbal encouragement, in this paper I am going to look at the more subtle forms of social persuasion that are communicated through the thinking classroom practices.

Methodology

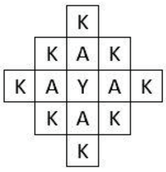
Data from this study are harvested from the aforementioned larger research project into building thinking classrooms that involved hundreds of teachers and thousands of students and took place across a wide range of grades (K-12) and settings (low socio-economic—high socio-economic, private—public, French—English). Regardless of grade and setting, however, the research followed a general research methodology that I refer to as rapid prototyping wherein the unit of analysis was a two-week micro-intervention. That is, for two weeks, a given teacher would enact a unique practice within their classroom and we would study the effect that this change in practice had on student thinking behavior—were more students thinking and were they thinking for longer than they had prior to the intervention? Based on the results of this micro-experiment, adjustments would be made to the intervention to try to increase the amount of thinking that was happening in the room and the teacher would enact this adjusted practice for the next two weeks. And so on. Once the data showed convergence towards an effective practice that practice was distributed to many different practitioners and longer studies were conducted (6 weeks – 10 months), and more adjustments to the practice were made. Despite the fact that we were focused on student thinking behaviors, a number of other aspects of the student experience were also captured—in particular, their reactions to the interventions. It is from these reactions, across a wide variety of micro-experiments that the data for this study were harvested. That is, the data does not come from any one study. As such, participants vary in ages from 5 to 18 and were in classrooms in varying socio-economic settings and geographic locations.

Results

In what follows, I use excerpts from these data to showcase the subtle and surprising ways in which three of the thinking classroom practices communicate social persuasion.

Thinking Tasks

If we want our students to think, we need to give them something to think about—something that will not only require thinking but will also encourage thinking. In mathematics, this comes in the form of a problem-solving task, and having the right task is important. The research (Liljedahl, 2020) revealed that when first starting to build a thinking classroom it is important that these tasks are highly engaging non-curricular problem-solving tasks. The tasks, the research showed, need to have a low-floor (accessible to everyone in the room), a high ceiling (evolving complexity), and be novel (because thinking is what we do when we don't know what to do). For example, see figure 2.



For this image, can you see a path from one letter to an adjacent letter that spells the word KAYAK? Can you see another one? Another one? How many unique paths are there that spell KAYAK?

Figure 2: A thinking task

When these types of tasks were used, we saw a significant increase in student thinking in the classroom. We also saw an increase in student self-efficacy as their efforts were met with success (master experiences). More subtle, however, was what began to emerge in the data after students' third or fourth experience with such thinking tasks. Consider this excerpt from week five of a three-month longitudinal study into the effects a teacher using the first three thinking classroom practices in a grade 8 (ages 13-14) classroom (thinking tasks, random groups, vertical non-permanent surfaces).

Researcher	Before you start, any thoughts about this task?
Kyla	Hmm! This one looks hard.
Researcher	Hard?
Kyla	Yeah! I mean, I don't even have a clue how we would start.
Researcher	So … what does that mean?
Kyla	It means we better start.
Researcher	Oh?!?
Kyla	Yeah. I mean, we almost never know where to start, but we always get there in the end. I'm sure it will be fine.
Researcher	How do you know?
Kyla	Our teacher wouldn't give it to us if she didn't think we could do it.

The repeated positive experiences with these types of tasks were not only building mastery experiences. These experiences were also building a confidence in their teachers' confidence in them. That is, the teacher was communicating to the students, through the use of carefully selected thinking tasks, that they were capable of solving the tasks at hand.

Visibly Random Groups

Once we have the thinking task students need someone to think with. We know from research that student collaboration is an important aspect of classroom practice because when it functions as intended, it has a powerful impact on learning (Hattie, 2009). How groups have traditionally been formed, however, makes it very difficult to achieve the powerful learning we know is possible. Whether students are grouped strategically (Dweck & Leggett, 1988; Jansen, 2006) or students are allowed to form their own groups (Urdan & Maehr, 1995), 80% of students enter these groups with the mindset that, within this group, their job is not to think (Liljedahl, 2020). However, when frequent and visibly random groupings were formed, within six weeks 100% of students entered their group with the mindset that they were not only going to think, but that they were going to contribute. In addition, frequent and visible random groupings was shown to break down social barriers within the room, increase knowledge mobility, and increase enthusiasm for mathematics (Liljedahl, 2014).

Visibly random groups gave lots of opportunity for both mastery and vicarious experiences to occur. But it also proved to provide social persuasion. Consider this excerpt from week two of a two-week micro-experiment into using random groups in a grade 7 (ages 12-13) classroom.

Researcher	So, the teacher has had you working in groups for a few weeks now. Any thoughts?
Sara	I love it.
Researcher	Why?
Sara	I love being in random groups. It’s like a new adventure every day.
Researcher	Do you think this is why she does it?
Sara	Sure. But I also think she does it randomly because it doesn’t matter what group we’re in. We’re all the same. We can all do it.

By forming the groups randomly, the students were interpreting this to mean that the teacher believed that they were all the same—that they were all capable. Incidentally, the students did not say the same things in classrooms where the teacher grouped students strategically. In those settings, the teacher's deliberate and careful selection of groups communicated the exact opposite—that some student were capable and some were not.

How we answer questions

In an institutionally normative classroom teachers answer between 200 and 400 questions in a day (Liljedahl, 2020), all of which fall into one of three categories:

1. Proximity questions: These are questions asked only because the teacher is close. Interestingly, in 90% of the cases, students make little or no use of the information they gain from the teacher’s response. They are only asking the question to show that they are on task—that they are being a good student.
2. Stop-thinking questions: These are questions student ask to get the teacher to help them avoid or to stop thinking. These are most often of the form “is this right”, “are we doing this right”, “are we going in the right direction”, or “is this what you wanted”. Thinking is difficult and if they can convince the teacher to help them, things would be easier.
3. Keep-thinking questions: These are questions that students ask so they can get back to work and are most often clarifying or extending questions. Often the students are in a hurry because they want to get back to the thinking.

To build a thinking classroom, only keep-thinking questions should be answered (Liljedahl, 2020). But this still leaves the question of how to answer proximity and stop-thinking questions. In the research, we found that the simplest way of dealing with these types of questions is to smile and nod as they are asking the question and then, when they are done, simply turn and walk away. At first, students hate this. But an interesting thing begins to happen after a few weeks. Consider this excerpt from week three of a three-month longitudinal study into not answering students' proximity and stop-thinking questions in a grade 9 classroom (ages 14-15).

Researcher	What was that? You asked her a question and she just smiled and walked away.
Morgan	Yeah. She does that a lot.
Researcher	She does!?! What does that mean?
Morgan	It means she thinks we can figure out for ourselves.
Researcher	What do you think?
Morgan	Yeah. We probably can. She's usually right about these kinds of things.

By smiling and walking away, it turns out that the teacher is communicating to the students that she believes that they are capable of figuring this out on their own.

Conclusions

The 14 building thinking classroom practices emerged out of empirical work wherein the goal was to increase the number of students who are thinking in class and for how long they are thinking during a lesson. Aside from achieving this, the research also produced a number of other results pertaining to how students experience mathematics, one of which pertains to their self-efficacy beliefs. Although this was not the intention of the research, data from across a wide variety of interventions and settings showed that aspects of the thinking classrooms improved students' self-efficacy beliefs. And although these changes are happening in part through the ubiquitous opportunities to have mastery and vicarious experiences, and to interpret those experiences with their peers, changes in self-efficacy is also happening through the non-verbal ways in which these practices communicate social persuasion—that the teacher has faith in the students' abilities. What these practices seem to communicate is not that "you can do it" but, rather, that "I believe you can do it". This is a subtle difference, but it intimates that before a student believes in their own abilities, someone else needs to believe in them. And the students need to see these beliefs. Before students can have confidence in their own abilities, they need to have confidence in their teacher's confidence in them.

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Values which facilitate mathematical wellbeing of Chinese primary school students: A preliminary study

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258 Grade 3 students from six classes taught by three mathematics teachers in a primary school located in suburban Chengdu, China were surveyed in 2021. The data reflect the values they embrace in mathematics learning as these are associated with mathematical wellbeing, namely, relationship, engagement, bliss, accomplishment, perseverance, meaningfulness, and learning. Although these values are similar to those identified in a similar and recent Australian study, the mathematics learning activities which embody the values are not all the same. Preliminary analysis also indicates that some of the seven values are embraced more than the others, and that school/teacher representation and teaching of values in China might be consistent across different classes.

Keywords: Chengdu, conation, mathematics education, values, wellbeing.

Student wellbeing in the context of mathematics learning

The wellbeing of students in mathematics education has been an ongoing concern for teachers, researchers and politicians (Clarkson, Seah, & Bishop, 2010; Hill, Kern, Seah, & van Driel, 2020). Students' general dislike of mathematics is often carried through into adulthood. In many societies, it is normal or even desirable to claim to be weak at mathematics. Also, mathematics anxiety is affecting increasingly younger children (Tomasetto, Morsanyi, Guardabassi, & O'Connor, 2021).

TIMSS 2019 (Mullis, Martin, Foy, Kelly, & Fishbein, 2020) results reaffirm the general lack of wellbeing amongst mathematics students worldwide. On average, across the participating economies, 20% of Grade 4 students and 41% of Grade 8 students "do not like learning mathematics" (p. 426). Also, 23% of Grade 4 students and 44% of their peers in Grade 8 were not confident in mathematics.

The COVID-19 pandemic – and the subsequent and sudden disruption to schooling all over the world – have affected students' wellbeing generally (O'Toole & Simovska, 2021; Schwartz et al., 2021). Thus, being able to effectively foster positive wellbeing in mathematics learning has assumed greater urgency, given that domain-specific wellbeing contributes towards general wellbeing.

Wellbeing as values fulfillment

There are traditionally two main perspectives of wellbeing, namely, hedonic and eudaimonic (Ryan & Deci, 2001). More recently, hybrid perspectives reflect how hedonism and eudemonism can both inform wellbeing at the same time. Australian researcher Julia Hill emphasises the links between values and wellbeing, citing Tiberius' (2018) values-fulfillment theory of wellbeing. According to this theory, "well-being is served by the successful pursuit of a relatively stable set of values that are emotionally, motivationally, and cognitively suited to the person" (p. 13).

In other words, student wellbeing can be seen as a function of fulfilment of what the student values. Fostering positive wellbeing in students' mathematics learning, then, would involve the provision of opportunities for students to realise their subjective and personal values, especially those which are related to mathematics and mathematics pedagogy. This domain-specific form of wellbeing is known as mathematical wellbeing [MWB], first proposed by Clarkson, Bishop, and Seah (2010). In this original conception too, there is association with mathematical and mathematics educational values.

Mathematical wellbeing and values

Hill, Kern, Seah and van Driel (2020) conducted a combined deductive/inductive thematic analysis of 488 Grade 8 students' responses in Melbourne, which confirmed that students' sense of wellbeing with mathematics learning was associated with the fulfillment of particular values. These values were grouped into eight categories, namely, *relationships, engagement, cognitive, accomplishment, positive emotions, perseverance, music,* and *meaning*. Examples of student-nominated learning activities which reflected these values are shown in Figure 1 below.

Themes & nodes	Student examples
Relationships	
Teacher support	*A supportive or good teacher*
Peer support	*Having friends to help me*
General support	*When I get help with my learning*
Engagement	
Interesting/hands on	*Learning interesting stuff*
Focused working	*Being absorbed in my work*
Independent/quietness	*When it is quiet and I'm by myself*
Music (engagement)	*Music helps me concentrate well*
Cognitive	*When I understand the material*
Accomplishment	
Good marks	*When I do good in a test*
Accuracy	*When I get the answers right*
General mastery	*When successful at learning something*
Completing tasks	*When I complete my work*
Confidence	*When I'm really confident*
Positive emotions	
Enjoyment/fun/happy	*If the maths class is enjoyable*
Relaxed/no pressure	*When there is no pressure*
Music (emotions)	*Music to listen to, to enjoy it more*
Perseverance	
Challenge	*Having work I find challenging*
Working hard/practice	*When I work hard*
Music (no reasoning)	*Listening to music in class*
Meaning	
Future skills	*Knowing these skills will help me in life*
Real world relevance	*I like when problems relate to real life*

Figure 1: Values and learning activities associated with mathematical wellbeing

More recently, further analysis has allowed for the valuing of *music* to be subsumed under the other seven values (Hill, personal communication). Nevertheless, the authors had called for further research to be conducted to assess the stability of this MWB framework. This includes the "extending [of sources of student input] to other year levels and populations" (Hill, Kern, Seah, & van Driel, 2020, p. 279). It is in this context that the current study has been set up, and it examines the following research questions in particular:

(1) What values related to mathematics learning do primary school students in Chengdu, China associate with positive mathematical wellbeing?

(2) To what extent are these values similar to the 7 values in the mathematical wellbeing framework constructed by Hill et al (2020)?

China as research context

China was chosen to be the research context for the current study, given that the East Asian cultural setting would be a contrast to the Australian mathematics classroom. Chinese students (and their parents) have been experiencing tremendous pressure and stress in schooling, including in mathematics, in part due to the high-stakes 'gaokao' national examinations which determine student access (or not) to prestigious universities around the most populous nation on Earth. In fact, the Chinese central government has introduced several measures in mid-2021 to reduce the negative wellbeing experienced by students in mathematics and other school subjects (The Economist, 2021).

As a preliminary study, only one school was chosen as data source. Since the Australian data were collected from secondary school students, the current study sought to gather data from a primary school. The volunteer school is a mid-size government school located in suburban Chengdu, reasonably popular amongst parents in the community. 258 Grade 3 students from six different classes taught by three mathematics teachers from the school provided the data for this study. Chengdu is the capital city of Sichuan province, located in southwestern China. With a city population size of more than 16 million occupying some 14,000 square kilometres, it is one of the 6 largest cities in the world by population or land area. Putting this in perspective, Chengdu has double the population and nine times the area of London.

Data collection

Data were collected via an anonymous student survey, completed during class time in 2021. Instead of using the learning moments such as those in Table 1, which would have been too leading, we listed 22 learning moments which were representative of Grade 3 Chinese mathematics classrooms, such as 'when you cooperate with friends to complete a mathematics task' and 'when your mathematics teacher praises you'. An 'others' item also allows for respondent-generated learning moments.

Each of these phrases was accompanied by a geometrical shape, which student respondents were asked to shade in yellow if it is associated with extreme happiness and wellness, and green if the happiness and wellness are not as intense. Given that 'mathematical wellbeing' might not be easily understood by the young children, the survey asked them to consider 'happiness and wellness' instead. The survey was presented in the Chinese language, and Figure 2 shows part of it with the content translated into the English language.

Of course, the survey items could have been presented in the form of a Likert scale. However, given that the student respondents were only 8 years of age, we had decided to design the survey so that it is aesthetically attractive, and such that it would be fun responding to the items through colouring in.

The Chinese teacher researcher in our team was personally present in all the 6 classes when the survey was administered, so that student queries were attended to on-the-spot.

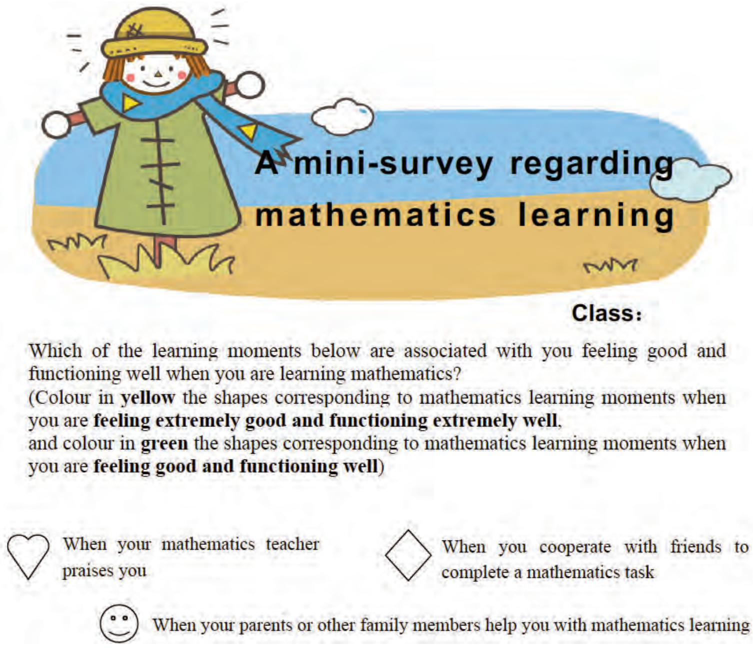

Figure 2: Part of the student survey (translated into the English language)

Data analysis

The 258 student responses on the hardcopy survey were entered into a Microsoft Excel spreadsheet, with the numerals 2, 1 and 0 used to represent yellow, green and no shading respectively. In other words, a score of 2 represents a strong association between the learning moment concerned with MWB, while a score of 0 represents the absence of any association.

Discussions within the research team were held to associate each of the 22 learning moments with an underlying value. For example, ‘feeling relaxed during mathematics lessons’ is regarded as reflecting the valuing of *bliss* or positive emotions. This discussion took place over several virtual, online meetings, given the prevailing ban on international travels resulting from the COVID-19 pandemic. Whenever general agreement could not be achieved for any learning moment, the team privileged the views of the Chinese teacher researcher, acknowledging that she possessed the relevant insider, cultural funds of experience to interpret the learning moments more accurately.

106 of the 258 students had shaded the ‘others’ item, with 84 of them nominating learning moments which were not amongst the given ones. During these discussion sessions, the research team also examined these nominations, allocating them to agreed-upon values. For example, ‘interacting with my mathematics teacher’ is associated with the valuing of *relationships*, and ‘helping my mathematics teacher out’ is associated with *accomplishment*. By the end of the discussion, seven value categories emerged. Based on these allocations, mean scores for the seven dimensions were

computed for each student, as well as mean scores for the seven dimensions for the entire student sample. A paired sample t-test was conducted with SPSS version 27 to determine if the differences in means between pairs of values were statistically significant.

Results

The seven value categories that emerged from the in-depth discussions are, namely: *relationship, engagement, bliss, accomplishment, perseverance, meaningfulness,* and *learning*. Comparing this list with the values in Hill, Kern, Seah, and van Driel (2020), correspondence could be observed for the two sets of seven values. In particular, the valuing of *relationship, engagement, accomplishment,* and *perseverance* are named similarly in the two lists. Whereas the Australian list has the value *meaning*, this is considered to be the same as *meaningfulness* listed in the current study. Similarly, *positive emotions* in the Australian list can be considered to be the same as *bliss* in the current study with Chinese data. We decided to adopt the term *bliss*, since it may be confusing for some people when considering *(positive) emotions* as a value. Last but not least, *cognitive* in the Australian list is regarded as being similar to *learning* as it is named in the current study. Similarly, we were concerned that considering *cognitive* as a value may interfere with people's understanding of the three aspects of mental processes, that is, cognition, emotions, and motivation (see Hannula, 2012).

It is important to note, however, that even though the same values might relate to MWB amongst Australian and Chinese students, the learning moments that operated in one culture may or may not be seen in the other culture. For instance, 'when my mathematics teacher gives us less homework' might be a learning activity that is hard to find in a typical Grade 3 Australian classroom. The same may be said of 'helping my mathematics teacher assess my peers' oral computation'. At the same time, we also looked out for learning moments which appeared to be unique to China, but which actually is due to the fact that the study has been designed to collect data from primary (instead of secondary) students. Thus, 'helping my group earn achievement stars' and 'interacting with my mathematics teacher' which are learning moments in the Chinese data may well be observed amongst Australian primary school students too, had the Australian study been conducted in primary schools.

The seven values are not equally associated with MWB. As shown in the radar chart (Figure 3), which maps how each of the 258 students valued the 7 dimensions, the arcs are concentrated away from the centre of the circle for *engagement, relationship, bliss* and *accomplishment.* Given that distance from the centre represents greater valuing, we can see that the Chinese students' MWB was more often associated with these four dimensions.

Indeed, results of the paired-samples t-test show that:

- *engagement* and *bliss* are valued differently at the .05 level of significance ($t=4.964$, $df = 257$, $n = 258$, $p < .05$, 95% CI for mean difference 0.076 to 0.176). On average *engagement* was about 0.13 point more valued than *bliss*,

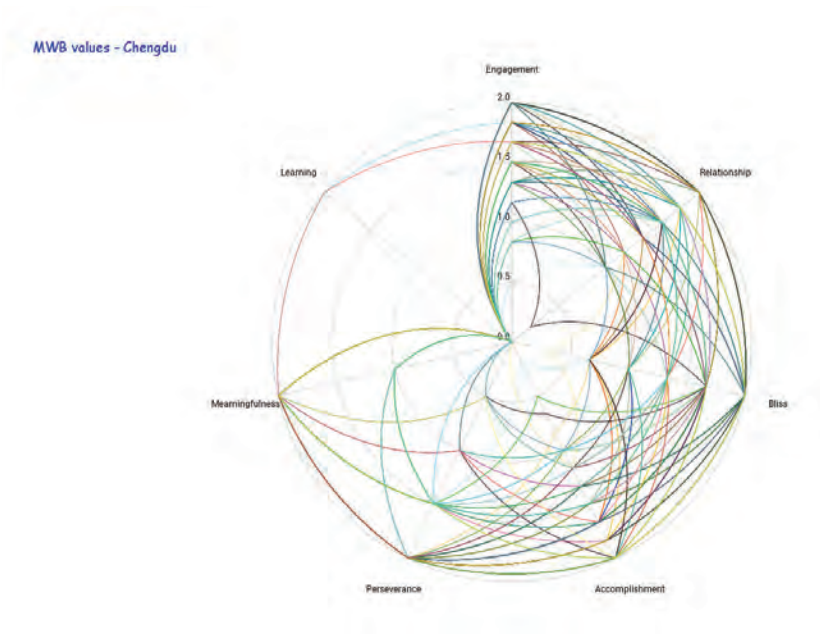

Figure 3: Differential valuing amongst Chinese primary school students

- *engagement* and *accomplishment* are valued differently at the .05 level of significance (t= 2.075, df = 257, n = 258, p < .05, 95% CI for mean difference 0.002 to 0.094). On average *engagement* was about 0.05 point more valued than *accomplishment*,

- *engagement* and *meaningfulness* are valued differently at the .05 level of significance (t= 2.442, df = 257, n = 258, p < .05, 95% CI for mean difference 0.018 to 0.165). On average *engagement* was about 0.09 point more valued than *meaningfulness*,

- *relationship* and *bliss* are valued differently at the .05 level of significance (t= 3.244, df = 257, n = 258, p < .05, 95% CI for mean difference 0.036 to 0.148). On average *relationship* was about 0.09 point more valued than *bliss*,

- *bliss* and *accomplishment* are valued differently at the .05 level of significance (t= -2.675, df = 257, n = 258, p < .05, 95% CI for mean difference -0.135 to -0.020). On average *bliss* was about 0.78 point less valued than *accomplishment*,

- *bliss* and *perseverance* are valued differently at the .05 level of significance (t= - 4.097, df = 257, n = 258, p < .05, 95% CI for mean difference -0.209 to -0.073). On average *bliss* was about 1.41 point less valued than *perseverance*,

- *accomplishment* and *perseverance* are valued differently at the .05 level of significance (t= -2.136, df = 257, n = 258, p < .05, 95% CI for mean difference -0.123 to -0.005). On average *accomplishment* was about 0.06 point less valued than *perseverance*,

- *perseverance* and *meaningfulness* are valued differently at the .05 level of significance (t= 2.754, df = 257, n = 258, p < .05, 95% CI for mean difference 0.030 to 0.183). On average *perseverance* was about 0.11 point more valued than *meaningfulness*.

The extent to which a student valued the seven values can be represented by the sum of the seven mean value scores, ranging from 0 (i.e. embracing none of the seven values) to 14 (i.e. embracing all seven values extremely). Given that students indicated their respective classes in the survey, and that the surveys were administered class by class, the 258 student data sets could be arranged according to class groups. When plotted against total mean value scores, we obtain the representation as shown in Figure 4. The graph suggests that the students' valuing has not been particularly strong (or weak) in particular classes (and taught by possibly different teachers). Thus, school/teacher representation and teaching of values in China might be consistent across different classes.

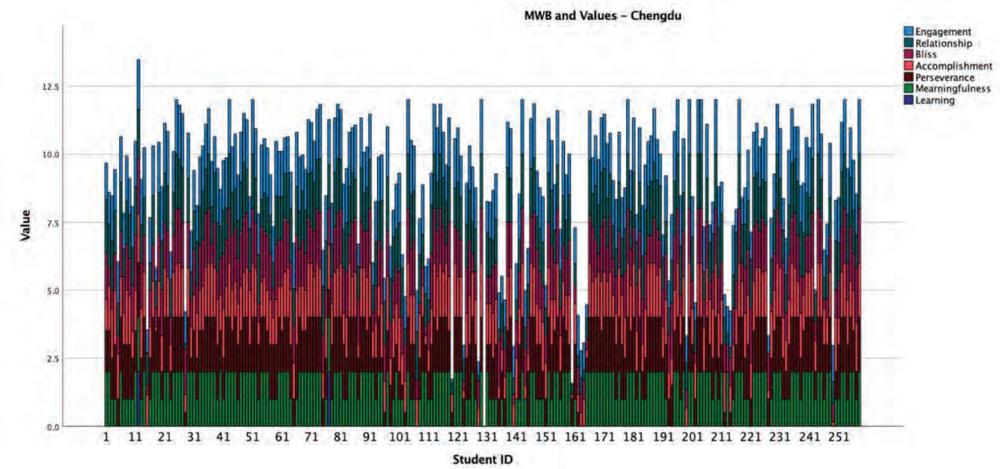

Figure 4: Total mean value scores for the Chinese student sample

Discussion

The current study responds to the call by Hill, Kern, Seah and van Driel (2020) to extend the investigation of values related to MWB to other grade levels and populations. The data collected from Grade 3 students in Chengdu suggest that MWB is associated with the students' fulfillment of the values of *relationship, engagement, bliss, accomplishment, perseverance, meaningfulness,* and *learning*. This thus answers Research Question 1.

Considering Research Question 2, the seven values identified in this study are similar to the seven values listed in Hill, Kern, Seah and van Driel (2020). The current study thus reaffirms the stability of the MWB framework proposed by the Australian study that was led by Julia Hill, in terms of the seven values which underpin it. However, these values could be expressed through different culturally specific mathematics learning moments in the two different contexts. It is important that teachers are aware of this attribute of values as they are embraced across different sociocultural settings.

It appears from our data that some of the seven values were embraced more than the others. Whether this was due to the nature of the local cultural traditions, of the mathematics education system and/or of MWB is unknown from the quantitative data. Further, qualitative research planned with selected students should be able to shed light on this observation. In particular, the paired samples t-test results suggest that *engagement* and *perseverance* were valued by the Chinese students more than they did with *relationship*, all of which were valued more than *accomplishment* and *meaningfulness*. *Bliss* is next, which the Chinese students appeared to value more than *learning* only.

Considering the extent to which the seven values as a whole were embraced by the 258 students across six classes taught by three different teachers, there was no evidence of class-level variations.

This suggests that values representation / teaching across different Chinese mathematics classrooms (in the same school) was consistent. This might facilitate values associated with positive MWB to be promoted amongst – and fulfilled by – students as they engage with mathematics learning.

Next steps

The intention of this paper is to report preliminary results and findings, providing teachers with some approaches to enriching mathematics teaching further with affective and conative considerations. The next stage of the study will be to both deepen our understanding of the available data and broaden the data sources, thus contributing to existing social theories of mathematics pedagogy.

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Developing and validating survey instruments for assessing beliefs and motivation in mathematics

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In this article, we describe the development and validation of two short questionnaires for assessing beliefs and motivation in mathematics that are (1) linked to Inquiry-Based Learning (IBL) and (2) short, concise and easy to administer quickly within the parameters of regular classroom instruction.

Keywords: Mathematical beliefs, motivation, IBL, scale validation.

Introduction

Research has shown that Inquiry-Based Learning (IBL) has the potential to improve students learning of and motivation for mathematics and science, as well as change students' beliefs about the subjects (Bruder & Prescott, 2013). However, many teachers report challenges with implementing IBL in their own classrooms (Engeln et al., 2013). The context of this study is a project called SUM, which aims to explore how IBL could be integrated in teachers' day-to-day mathematics teaching. One of the key research tasks of the SUM-project is to document and explain students' long-term development in terms of motivation for and beliefs about mathematics. In order to achieve this, the project needs survey instruments that can be quickly and easily administered to a large number of diverse-aged students in a classroom context. Several questionnaires have previously been developed to explore students' beliefs and motivation, but a review of the literature disclosed multiple reasons why there is a need for new instruments, particularly with respect to beliefs. First, we identified no instruments that have been conceptually linked to IBL. Second, most existing survey instruments were long with many items. Brevity is important when instruments are administered in classroom contexts, as the longer the inventory, the less students may take care completing it (Entwistle & McCune, 2004). Finally, earlier scales used to measure beliefs have mostly not been systematically validated, and according to Diego-Mantecon et al. (2019), there is still a need for a more thorough methodological analysis of these variables, particularly in the case of beliefs, using advanced psychometric tools.

In this paper, we describe how the aforementioned issues have been addressed through developing and validating a questionnaire assessing students' beliefs about mathematics, and adapting and validating existing scales measuring motivation. This resulted in the *Mathematical Beliefs Questionnaire* (MBQ) and the *Mathematical Motivation Questionnaire* (MMQ). Our main research questions are: a) What is the factor structure of the MBQ and the MMQ, and how can we achieve robust scales measuring the underlying constructs with as few items as possible? and b) Do the six proposed scales in the MBQ and MMQ have acceptable internal reliability and discriminant validity?

Theoretical foundation and initial development of instruments

Mathematical beliefs – the MBQ

The affective domain is characterized by many overlapping and related term such as beliefs, attitudes, values and emotions – that are not used in a uniform way (Philipp, 2007). As such, research on beliefs is complicated by numerous factors, including both methodological and conceptual issues. However, empirical work on an unclear concept can invite dialogue and serve to clarify it (Skott, 2015). Furthermore, although beliefs are not easily defined, reviews of the research literature have identified certain consistent features. In general, beliefs can be described as subjective knowledge (Philipp, 2007), and according to Skott (2015), beliefs are usually value-laden mental constructs that are relatively stable results of prior experiences. Beliefs are also organized in complex and quasi-logically connected clusters (Philipp, 2007). Although it is possible to group these clusters in a variety of manners, beliefs about mathematics have commonly been grouped according to beliefs about the nature of mathematics, beliefs about mathematics learning, and beliefs about mathematics teaching (Ernest, 1989; Beswick, 2012). Each of the three belief clusters can be mapped on a on a continuum that ranges from formal beliefs at one end to informal beliefs at the other (Collier, 1972). Formal beliefs about the nature of mathematics views mathematics as a hierarchically structured subject of procedures, rules, algorithms and formulas. Knowing mathematics is equivalent to recall and application of these procedures, and teaching mathematics is envisioned as a teacher-centered activity in which the teacher present and demonstrate the use of said procedures. Informal beliefs about the nature of mathematics identifies it as a creative subject of investigative processes tied to problem solving, proof, reasoning, communication, connections and making sense of the world around us. Knowing mathematics is displayed through active and successful engagement in these processes, and teaching mathematics is envisioned as a student-centered activity in which the teacher facilitates students' active knowledge construction through exploratory and open-ended processes.

In the development of the MBQ, we sought to relate each of the aforementioned three belief clusters to key characteristics of IBL. First, IBL is premised on an epistemological belief of mathematics as a dynamic and creative endeavor, where the focus is primarily on the process of mathematics and not necessarily on the product (Ernest, 1989; Artigue & Blomhøj, 2013). A belief cluster we may call Mathematics as a creative subject (Creative) points to the view of mathematics as a creative and human endeavor, where the purpose is to solve interesting problems and understand the world around us using creative and original thought. Second, a belief cluster we may call Mathematics instruction should be inquiry based (Inquiry) points to the view that teaching should be based on inquiry related activities, where students are provided with opportunities to explore and try out their own ideas (Artigue & Blomhøj, 2013). Third, a cluster we may call Mathematics is not an innate ability (Adventitious) rests on the premise that education should be for all, and, more importantly, that all students have the capacity for learning and becoming proficient in mathematics through effort and dedication (Kloosterman, 2002).

A literature review confirmed that there were no existing previously validated instruments for measuring students' epistemological beliefs about mathematics along these three domains. Therefore, a panel of researchers in the SUM-project generated a pool of items that captured aspects of each of the three aforementioned belief clusters. This item pool was generated based on an approach consisting of: a) identifying key characteristics of the theoretical definitions, and b) selecting and translating the most relevant items from existing literature (particularly Tatto, 2013; Collier, 1972) as well as supplementing this by formulating some new items. This process resulted in 17 items – significantly longer than the target length.

Motivation – the MMQ

Like beliefs, motivation has long been the focus of educational research, and there exists a number of theoretical approaches and associated concepts – many of them appearing related or partially overlapping. In the context of mathematics education, Schukajlow, Rakoczy and Pekrun (2017) identified expectancy-value theories and self-determination theory as especially important theoretical approaches, and the conceptualization of motivation in the MMQ encompass features of both.

The choice of motivation constructs in the MMQ was mainly influenced by the expectancy-value theory of Eccles et al. (Eccles & Wigfield, 2002), whereby students' motivation for engaging in an activity is thought to depend on both their expectations for success and how they value the activity. Specifically, our review of the motivation literature led to the conceptualization of students' motivation for mathematics through three constructs: Perceived competence, which relates to their perception of their mathematical ability and expectation for success, Intrinsic value, which can be linked to intrinsic motivation in self-determination theory (Ryan & Deci, 2000) and relates to the enjoyment they may experience when engaging in mathematics or the subjective interest the students' have in mathematics, and Utility value, which relates to how engaging in mathematics may be useful to the student, for instance with respect to pursuing future career goals (Eccles & Wigfield, 2002). These constructs are not only widely supported in the motivation literature, they can also be related to key characteristics of IBL. For instance, Fielding-wells et al. (2017) demonstrates the use of expectancy-value theory for developing insights into how engaging with IBL affects student's competence-related beliefs, sense of autonomy, and intrinsic valuing of mathematics, and calls for further research on how IBL may foster a perception of the utility of mathematics.

A number of scales for measuring aspects of student motivation exists, and the MMQ combines translated and adapted versions of items from the IMI - Intrinsic Motivation Inventory (Center for Self-Determination Theory) in order to address Perceived competence and Intrinsic value, as well as adapted items from the Usefulness scale in the Conceptions of mathematics inventory (CMI, retrieved from (Star & Hoffmann, 2005)) to address Utility value. This resulted in 20 items – again significantly longer than the target length.

Method

Many of the items in the item pool originated from well-known instruments, but in the MBQ and MMQ they are combined and adapted to our specific situation (IBL, mathematics, students aged 10-19). Furthermore, our aim is to achieve robust scales measuring the underlying constructs with as few items as possible. Therefore, we have conducted a validation study in order to explore the factor structure of these items and establish internal reliability and discriminant validity.

Participants and procedure

Data for the validation study was collected in the fall of 2019, and the participants were 377 students in the age range 10-19 (48% male) from 10 different schools in the SUM-project. Note that this constitutes a convenience sample where the students are nested in classes, and as one might expect a greater similarity between students in the same class, this is a limitation of the present study.

The participants responded to an electronic questionnaire consisting of 49 items, all answered along a 5-point Likert scale, as well as five background questions. The 49 items were separated into three sections: 1) 17 items from the MBQ, 2) 20 items from the MMQ, and 3) 12 items assessing students' past experiences in mathematics classroom. Before analyzing the data, missing values were handled by interpreting them as a "neither agree or disagree"-response and imputing the Likert-scale midpoint value 3. To assess the theorized scales in both the MBQ and the MMQ, the sample was randomly split in half and an EFA (exploratory factor analysis) was conducted on the first half (196 students) and a CFA (confirmatory factor analysis) was conducted on the second half (181 students). Small sample sizes can result in unstable factors, but ca. 200 is considered fair – especially if communalities and factor loadings are high.

Results

Exploratory factor analysis (EFA)

In line with Carpenter's (2018) summary of recommendations for non-normally distributed variables and possibly correlated factors, we conducted separate exploratory factor analyses using Principal Axis Factoring and oblique rotation to identify the latent factor structure of the MBQ and MMQ. In both cases the factorability of the items was deemed satisfactory, as the Kaiser-Meyer-Olkin measures were .74 for the MBQ and .93 for the MMQ, and the Bartlett's Tests of Sphericity were significant.

For the MBQ, five items were excluded to improve the interpretability due to low communalities and high cross-loading. The number of factors retained was based on several criteria. First, the Kaiser "eigenvalue greater than one" rule led to an initial extraction of three factors and the scree plot indicated the extraction of three or four factors. Second, a parallel analysis (100 datasets; CI 95%) indicated the extraction of three factors. Finally, the following criteria were also taken into consideration: the factors should be homogenous and theoretically meaningful, Cronbach alpha has to be sufficiently high (>.65), each factor should contain enough items (>2) with high loadings (>.50) and the total amount of variance explained must be sufficient (approx. 50%) (Carpenter, 2018).

A three factor-solution was preferred in the end because of its simple theoretical interpretation, sufficient number of items per factor, and general parsimony, along with sufficient total variance explained. Factor one (eigenvalue = 3.63, 24.70% of variance) was Mathematics as a creative subject (Creative). The three items retained all reflect seeing mathematics as a subject that is creative, humanistic and related to the world we live in (e.g.: C3 - "Mathematics is first and foremost about understanding the world around us"). Factor two (eigenvalue = 2.10, 14.13% of variance) was Mathematics instruction should be inquiry based (Inquiry). The three items reflect a belief that mathematics teaching and instruction should allow students to explore and understand mathematics in settings similar to how mathematicians work (e.g., I2 – "In mathematics class we should first and foremost experiment and try out our own ideas”). Factor three (eigenvalue = 1.95, 12.18% of variance) was Mathematics is not an innate ability (Adventitious). The three items retained in this scale reflect the belief that mathematical ability is the result of hard work and not innate (e.g.: A3 – “Everyone can become proficient in mathematics”). The three factors explained 51.01% of the total variance, and the factor loadings are summarized in Table 1.

Table 1: Factors and loadings for the MBQ and MMQ. Note: Factor loadings < .5 are suppressed

	Item / **Factor**	C1	C2	C3	I1	I2	I3	A1	A2	A3				
MBQ	1	.65	.62	.59										
	2				.73	.64	.53							
	3							.74	.68	.63				
MMQ	**Item** / **Factor**	IV1	IV2	IV3	IV4	IV5	UT2	UT6	UT7	UT8	PC1	PC4	PC5	PC6
	1	.84	.74	.84	.72	.95								
	2						.78	.76	.64	.63				
	3										.80	.84	.81	.87

For the MMQ, inspection of the communalities and the rotated pattern matrix led to the identification of 7 items with either low communalities (<.4) or relatively high cross-loadings. After a closer inspection of the wording of these items, we decided that it would be theoretically meaningful to remove them from the analysis. Most of the 7 removed items belonged to the proposed Utility scale, and referenced mathematics being useful to the student in the present (e.g. UT1 – “Mathematics is a useful subject in school”).

Repeating the EFA with the 13 remaining items indicated the extraction of three factors (see Table 1), together explaining 73.63 % of the total variance. Factor one (eigenvalue = 6.51, 50.07 % of variance) was Intrinsic value. The five items all reflect the notion that mathematics is a fun, enjoyable and interesting activity (for example, IV1 “I enjoy doing mathematics”). Factor two (eigenvalue = 1.87, 14.36 % of variance) was called Utility for future life. This is a slight narrowing of the original theorized scale, as the four items retained reflect the view that learning mathematics in school will be advantageous for the student in the future (e.g. UT6 “Mastering mathematics will help me get a job later in life”). Finally, the third factor (eigenvalue = 1.20, 9.20 % of variance) was called Perceived competence. The four items belonging to this factor all reflect a perception of oneself as being competent in the field of mathematics (for example PC4 “I have good mathematical skills”).

Confirmatory factor analysis

To assess the stability of the three-factor solutions, we conducted confirmatory factor analyses using the AMOS software. In both cases, the chi-square value in relation to the degrees of freedom gave p-values larger than .05 (MBQ: $\chi^2(24) = 28.61$ and $p = .24$, MMQ: $\chi^2(62) = 28.61$ and $p = .11$), indicating that the hypothesized models do not significantly deviate from the observed data. Model fit was further assessed using the goodness-of-fit index (GFI); the comparative fit index (CFI); the root mean square error of approximation (RMSEA); and the standardized root mean square residual (SRMR), with acceptable values near .95, for the two former and $\leq$.05 for the two latter (Byrne, 2010). The analyses indicated a good model fit in both cases, with fit parameter values GFI = .97, CFI = .98, RMSEA = .03, and SRMR = .05 for the MBQ, and GFI = .94, CFI = .99, RMSEA = .035, and SRMR = .035 for the MMQ.

Internal reliability and discriminant validity

We assessed internal reliability using Cronbach's alpha and the validation dataset. For the MMQ, the alpha values for Intrinsic value, Perceived competence and Utility for future life were .93, .92 and .76 respectively, indicating that the motivation scales were remarkably robust to our translation, adaptation and shortening. For the MBQ, the alpha values for Adventitious, Inquiry and Creative were .76, .68 and .70 respectively. Although alpha values greater than .70 is often referred to as a rule of thumb, this appears to be a rather arbitrary cutoff and lower values may be considered satisfactory for scales with few items. As such, we conclude that all six subconstructs have acceptable internal reliability.

Table 2: Scale intercorrelations. Coefficients marked with ** are significant at the 0.01 level.

	1	2	3	4	5	6
1 Creative						
2 Inquiry	.34**					
3 Adventitious	.20**	.08				
4 Intrinsic value	.56**	.04	.47**			
5 Perceived competence	.36**	.08	.37**	.66**		
6 Utility for future life	.37**	.13	.45**	.53**	.32**	

Table 2 shows the bivariate Pearson correlation coefficients between all the subscales. Significant correlations are observed between all scales except for Inquiry, and this is to a large extent expected as the subscales Creative, Adventitious, Intrinsic value, Perceived Competence, and Utility for future life all reflect optimistic beliefs and motivations about mathematics and mathematics learning, as well as mathematics being important for one's own future life.

Given the rather high correlation between some of the subscales, we assessed the discriminant validity using the Fornell and Larcker criterium (1981). The Average Variance Extracted (AVE) was calculated for each of the six subscales, and compared to the highest squared correlation with any other subscale within MBQ and MMQ respectively. For all the subscales, the AVE exceeded the highest squared intercorrelation, indicating that the subscales may be reliably distinguished.

Concluding remarks

Our analyses show that we can achieve robust measures of subscales pertaining to mathematical beliefs (Mathematics as a creative subject, Mathematics instruction should be inquiry based, Mathematics is not an innate ability) and motivation for mathematics (Intrinsic value, Perceived competence, Utility for future life) with relatively few items. Furthermore, these six proposed scales all have acceptable internal reliability and discriminant validity.

Acknowledgment

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Digital interactive storytelling in mathematics: an engagement structure enriched by the Guru stimuli

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This work means to highlight the reactions of the students when their interactions with a digital storytelling in mathematics are influenced by the feedback of an expert. That causes affective reactions among the students that, taking a specific role, are immersed in the digital story and act to build an original version of it. In particular, this research is interested in analyzing the students' affective pathway in connection with some Goldin's engagement structures that emerged from group discussions. The qualitative data emerged from a pilot study lead us to explore this aspect in order to define specific categories that include affective and cognitive aspects.

Keywords: Affective pathways, digital storytelling, engagement structure, teacher's role.

Introduction

In the recent period of pandemic emergency there has been much talk about Distance Learning, with teachers of all levels finding themselves transforming into distance learning teachers, with difficulties and constraints that this sudden change would mean from a pedagogical, didactic and social standpoint, and also with all problems connected with the greater or lesser technological competence of teachers and the extent of technological equipment available for students. In such a situation, we are called to reflect on the potential of technologies to integrate, improve and rethink some aspects of face-to-face didactics. Studies developed within the PRIN 2015 project "Digital Interactive Storytelling in Mathematics: a competence-based social approach "[1], go in this direction.

The basic idea is, on the one hand, to use technologies to involve the student in the story and make them interact with other teammates; and, on the other hand, to create problematic situations (mathematical) which the student-actor has to deal with during the story. Similarly, the student is faced with solving problems that may concern their character. On the basis of a plot prepared by the teacher, the students therefore become actors of a story that they themselves contribute to creating, also through choices made in problematic situations they progressively encounter and by interacting with their teammates.

The main objective is to allow, through a DIST-M (*Digital Interactive Storytelling in Mathematics),* for the active involvement of each student, who will have to think and act as a character in the story, having their own role, and taking on the emotional load that this condition entails. However, the whole group is the actor being in the narrative and in the mathematical challenges proposed. Each participant has a role and can make their own contribution; in fact, the student can be supported and is supposed to deal with peers, in a perspective that is not only inclusive, but also able to promote communication skills in mathematics. The study, whose first experiments were already presented

[1] This work is part of the project PRIN 2015 "Digital Interactive Storytelling in Mathematics: A Competence-based Social Approach", funded by MIUR, effective from 5 February 2017.

during TWG08 CERME11 conference (Albano et al., 2019), is based on a group of learners being engaged in a mathematical problem solving supported by an expert (Guru) who intervenes in the group discussion at various times, primarily when she feels the process is slowing or stops, acting as a motivator. So, through peer and expert interactions we attempt to answer the following two research questions: RQ1: How does an expert, that acts as a guide, a facilitator, know when to intervene in group learning? RQ2: What does feelings Guru's presence provoke in the students when they are playing in the digital story?

Starting from scholars such as Hannula (2002) and Cobb et al. (1989), who have investigated students' affect during problem solving-focused teaching experiments over a period of time, we mean to analyze the affective field in collaborative problems solving, such as digital storytelling, and how the affective field is addressed by the Guru's contribution.

Theoretical background

Most of the research on students' emotions in the field of mathematics education focuses on their role in mathematical problem solving activities (McLeod &Adams, 2012; DeBellis & Goldin, 2006). Indeed, mathematical problem solving activities also become fruitful ground for investigating the influence of affective factors in mathematical thinking processes (Gómez-Chacón, 2000; Pesonen & Hannula, 2014). In particular, DeBellis and Goldin (2006) talk about "*affective pathway"* as "established sequences of (local) states of feeling that interact with cognitive configurations" (p.134). They also underline the importance of meta-affect, defined as "affect about affect, affect about and within cognition about affect, and the individual's monitoring of affect through cognition" (p. 136). From a methodological point of view, the interplay between affect and cognition is quite a complex topic to be investigated. Such an interaction is far more present in problem solving situations where, given the nature of the same activity, students live experiences of impasse and construct new representations. In the context of problem solving activities, terms such as *engage* and *engagement* are used. In particular Goldin et al. (2011) refer to "engagement structure" as a tool for framing the analysis of the complex nature of affect, and particularly the interaction between individual and social aspects of students' problem-solving experiences in mathematics. In this kind of "framing" situation, affect, cognition and motivation interact to influence students' mathematical engagement in a social environment; the affective pathway is, therefore, one of the components of the structure. Goldin (2017) describes various examples of engagement structures (ES) constructed from observation. In this work we recall and adapt those that emerged from chat room discussions such as: *I'm Really Into This* (the desire to enter and maintain the experience of learning mathematics), *Help Me* (the desire to obtain help or support for solving a mathematical problem or understating mathematics). These ES relates the role mathematical affect has on ones' "doing" mathematics as a learner. Hence, our theoretical framework makes reference to Goldin's (2017) understanding of engagement while our approach is based on the use of collaborative scripts (King, 2007) that consider a sequence of Vygotskian activities (Vygotsky, 1980) embedded in a framework of digital storytelling. Eventually, the methodology involves the interaction among peers (Albano et al., 2019) on an e-learning platform (Moodle).

Methodology

The design of the activity

The macro-script of the story consists of five activities corresponding to stages of a mathematician's work during a problem-solving process (Boero, 1999). First we have an **Exploration** phase, where the problem is explored and first hypotheses of solution are proposed, followed by the **Conjecture** phase, devoted to refine the hypotheses (often in spoken language) to obtain a clear and complete conjecture. Third and subsequent phase is called **Formalization**, in which the "best" (or more promising) conjecture is translated into mathematical language; this represents an essential step to fully trigger the following phase, namely, the **Proof**, where all current mathematical knowledge is applied to prove (or confute) the mathematical conjecture. Finally, a **Reflection** phase, which allows mathematicians to possibly retrace the story from the mathematical point of view and produce a statement to be shared with the community. All these phases are translated and modeled with structured learning activities within a storytelling framework; each of them corresponding to an episode in the story. Along the above steps, various situations that a mathematician may face in solving a problem have also been identified: a) problem solvers observe the path for finding a solution; b) they take action to find a solving strategy; c) if they think they are failing or feel stuck, they try to reassure themselves in order to restart successfully; d) they try to find an insight or ask for help from an expert or from external sources (books, journals, etc.); e) they ask themselves whether the solution path is working or not; f) they look for evidence to justify their conjecture; g) in case they find something to be corrected, they give the new solution in the exact form. The sequence is not necessarily linear, and could be cyclical or could not include all of the situations mentioned above. However, in all these situations, we can identify an interaction among different forms of thought and discourse, which we have modeled as cooperation among characters.

The story and its characters

The key attitudes illustrated beforehand are "embodied" by 5 distinct characters (or roles) within our storytelling. Specifically, the characters will work as a team within the storytelling, each one contributing to a step: *Pest* keeps asking questions, trying to make everything clearer; *Blogger* records and shares all team's achievements in a written form; *Boss* keeps the group working and well-focused on the objective; *Promoter* suggests new ideas and searches for directions. In case of difficulty, the Promoter can ask the fifth character for helping them, *Guru* (an external expert, typically the teacher or a researcher) who, otherwise, would not be fully involved in the story. Guru intervenes during the interactions among the actors with the aim to let them clarify their comments and remarks or improve the communication. This type of personalized approach promotes first-person engagement within the story, while team working supports the social construction of (possibly new) shared knowledge. To sum up, we have a story, organized in 5 episodes, where students play as characters. They are given a proper role (which changes from one episode to the other) and specific actions to perform according to their role in the episode. Some tasks are individual, some other are collaborative and therefore require a good level of communication and coordination (students are not allowed to directly talk to each other). It is worth saying that students, based on our storytelling approach, do not "create" the story that they are playing (since by doing so it would be difficult to predict its directions and success).

The designed model allows to the students to participate in the story in two different ways. For each episode we have: 1) One group acts as Actors: each student in the group takes on the role of one of the characters described above; 2) The other groups act as active and aware Onlooker: each student takes charge of observing a specific character in the story and reflects on how the observed character behaves with respect to both the mathematical problem and the role she is playing.

The immersive aspect of DIST-M is strongly related to *I'm Really Into This* engagement structure: each student "comes into the story" as a character, as an Actor or as an Onlooker. Associated to *Help Me* engagement structure could be considered the role played by the teacher in the story (Guru), who will provide (explicitly or implicitly) for a variety of ways in which students' needs can be satisfied, overcoming the impasse responsible for disrupting the learning process. So, she embodies the role of "wisdom" and represents a source of knowledge for students to draw upon when needed. In order to allow each student to take part and play within the story, we arranged all students in teams of 4 members (one for each of the above characters). In unstructured collaborative problem solving, students often remain in the role in which they feel most comfortable. In the proposed model, however, both the groups of Actors and Observers rotate, as do the roles within each group. For example, if in the first episode a group played as Actors, in the second episode they will play as Observers and a group that played as Observers will play as Actors. Also, if a student played as Boss in the first episode, they will play as an Observer of a different role than Boss in the second episode. In this way, each student experiences (as Actor or Observer) all of the cognitive functions, and this can lead to ownership of those functions by each. The constant change of perspective, moving from Actor to Onlooker and from one role to another are intended to help internalize all the roles and phases (Vygotsky, 1980).

The chosen problem

The mathematical problem proposed concerned an algebraic regularity of four consecutive numbers (Mellone & Tortora, 2015) as follows: "*Choose four consecutive natural numbers, multiply the two intermediates, multiply the two extremes, and subtract the results. What do you get?"*

The problem was presented within the story in a more implicit way by means of some quadruplets and operations. Students were expected to explore the situation, to formulate conjectures, concluding that the result was always 2, and, thus, to prove it. At the end of the story we submitted a questionnaire to the students in order to investigate their feedback on the whole experience. In this work, we are interested in analyzing the students' feelings about Guru's involvement and the way they experienced her interventions in the discussion group. The interactions among peers try to give us an answer to RQ1; while through the answers that the students have given to a final questionnaire, we try to answer to RQ2. The sentences, considered in our analysis, are the students' answers to a specific question: "*Did you feel comfortable enough or would you have felt more comfortable if she had been closer in age to you, but still more experienced? To what extent? Justify your answer*".

The experimentation

The DIST-M was experimented with 26 first-year high school students. All students, enrolled in a Moodle course, were divided into groups of 4, or 5 if the total number of students was not a multiple of 4. Thus, we had four groups of four students and two groups of five students. In the case of groups of 5 students, the role that is duplicated is Pest (and Pest Onlooker) since this role is necessary to encourage the production of conjectures, arguments within the group, necessary to reach mathematical formalization and, therefore, the proof. The role of the expert, Guru in the story, was played jointly by a researcher together with the teacher. The flexibility of Moodle provided useful tools for both educational activities and social interactions. For instance, we used Chat rooms for all informal communication among students in the same group and as a privileged channel between the Promoter and Guru. Forums and Questionnaires, instead, were used in all formal communications between Actors and Guru. This shift, which mimics the transition from spoken to written language, was also intended to promote the transition to literate registers problem.

Data analysis and Results

In order to find an answer to the research questions and to investigate the affective factors (emotions, attitudes, interest) at stake in the interactions within the digital storytelling, we used both information extrapolated by the e-learning platform and data of the final questionnaire submitted at the end of the experimentation. In this regard, we looked at the transcriptions of the conversations extracted from the group chat room among all the characters and from the private chat room between Promoter and Guru. That provides us with information about possible interventions made by the Guru at various times of the group discussion, and primarily when she feels that the process is slowing or stops, acting, consequentially, as a motivator. Let us read excerpts of a Chat room discussion within Episode 2. In the first excerpt (lines 1-6) we can easily observe that Guru (named Gianmaria in the story) does not intervene during the discussion. He simply reads and observes, postponing her intervention to a later moment (line 7).

1	Promoter:	I told Gianmaria that we have found a formula
2	Pest:	I don’t think
3	Promoter:	He’s telling me that now
4	Promoter:	Eh…but I’m stuck
5	Promoter:	How should we proceed?
6	Blogger:	Let’s summarize: if we subtract the product of medium and of extremes of 4 consecutive numbers, the result is always 2
7	Gianmaria:	Fede, are you there?
8	Gianmaria:	Look, I have an app that can help you with words
9	Promoter:	A lot of useful words appear to explain the formula
10	Boss:	Yes
11	Promoter:	We have to reconstruct a theorem from those words

The students, starting from the discussion between Promoter and Guru, try to formulate the conjecture, showing some difficulties. Blogger (line 6) tries to summarize, but her attempt is not enough to overcome the impasse. Guru, then, observing the discussion, decides to activate the digital tile app so as to direct the students towards the construction of the conjecture. The intervention of the Guru (as confirmed also by teacher/expert of the class) has conducted to a reflection about her role: her facilitator function does’ not mean providing some specifics contents but guiding the students in the discovering of the solution. The role of facilitator was also recognized to the students, as some of their statements show:

41	Pest:	I think she is the most important character in the story.
42	Promoter:	I think Gianmaria was helpful in solving the puzzle because she was able to prop the guys up

43	Boss:	I think Gianmaria' s character was helpful and is a good character.
51	BLO1:	I think he was very helpful, since she has made us deepen our ideas.
52	BLO2:	She is very good. She helped us when the solution was not correct.
61	BO1:	She has been very good at removing doubts and making changes to the solution when necessary.
63	BO3:	She is very nagging, and it is lot of fun to answer her questions.
71	PO1:	Nice idea, but when she asks questions it feels like taking a real test; she should involve the characters more and make the questions simpler.
72	PO2:	I think that she is the main component of the story, although she could be a bit faster:)".
81	PRO1:	Thanks to her involvement we could engage with a person who knew more than us on the topic and was able to back us when we asked for help.
83	PRO3:	I think that sometimes Gianmaria asks too many questions, but that is acceptable since her job consists in finding the right solution.
84	PRO4:	Gianmaria is a character who helps solving the enigma by asking questions to the interlocutors to help them understand better.

However, there are also some expressions of disappointment regarding the role played by Guru: boredom expressions "*she should involve the characters more*" (line 71); or expressions that evoke mixed feelings: "*she could be a bit faster"* (line 72) in opposition to positive aspects: "*Nice idea*" (line 71); "*Nagging…and a lot of fun*" (line 63).

At the conclusion of our analysis, performed by crossing the conversation in the chat of the characters with Guru (lines 1-11) and the answers to the question of all students 'statements (lines 41-84), a first consideration concerns the fruitful participation of the students into the story: indeed, the lively conversation in all episodes, either among the peers or between peers and Guru, evokes the desire of the participants to feel immerged into the story. Next extracts from students' answers give us a perception of the students' feelings regarding the influence of the Guru in the story.

100	Pest:	Being in contact with your peers gives greater freedom and understanding of dialogue.
101	Promoter:	Speaking to a person closer to my age would make me feel more at ease; I think I would be able to speak more confidently and in a more relaxed way.
102	Boss:	In some cases, I found it difficult to express my views to Gianmaria.
103	BLO1:	It wouldn't have changed anything because I basically don't feel comfortable talking to people I don't know.
104	BLO2:	I get along great with people older than me, I enjoy talking to them and comparing notes.
108	PO2:	Because Gianmaria was nice and could make us feel at ease.
109	PO3:	Because an age gap doesn't have much effect on me.
110	PO4:	Her age doesn't change anything since she is behind a screen, anyway.
111	PRO1:	I believe that it would not have changed anything, even if she had been my age.
114	PRO4:	Because, anyway, a person who is closer in age to us knows better our way of thinking/reasoning.

We want underline as the full involvement of the students in the story has really motivated the student. Referring to specific words used by themselves with the teacher, they affirm "*I enjoyed mixing math with stories*"; so this put in result as being immersed in an interactive story has motivated the student, that is in line with the aim of the ES "*I'm Really Into*".

If we look at the affective pathways of the engagement structure "*Help me*" we can observe alternate feelings of confidence and discomfort in connection with the presence of Guru in the story. The mixed feelings regarding Guru's suggestions could be linked to critical thinking development.

Discussion

The findings lead us to define, a posteriori, some possible categories that will explored in some future works:

Category 1*- confidence vs development of critical thinking.* The expressions used by the students in their interviews, such as "*I get along great with people older than me*" (line 104); *"Gianmaria was nice"* (line 108); "*an age gap doesn't have much effect on me"* (line 109); "*her age doesn't change anything*" (line 110) induce us to suppose that these students are not embarrassed by the presence of an expert. So, they probably recognize also the added value that the Guru's presence, in their interactions, could give to the cognitive aspects. The interactions among them and with Guru could generate new knowledge in a collaborative and individual manner and, for consequence, to develop argumentative competences. Moreover, the use of digital in the learning can go in the same direction; indeed, the used expression by a student "*behind a screen*" (line110) leads to thinking about how the use of digital in learning contexts can foster the development of knowledge.

Category 2- *anxiety vs use of communicative registers.* The expressions used by the students in their interviews, such as "*greater freedom"* (line 100);*" more confidently and in a more relaxed way"* (line 101); "*I found it difficult to express my views to Gianmaria"* (line 102) highlight linguistic difficulties in interacting with Guru, and, therefore, a need to use literate registers. For this kinds of the students the presence of an adult, in the chat among peers or in a Forum session, provokes feelings of anxiety, embarrassment, as also remarked in the line 114 "*a person who is closer in age to us knows better our way of thinking/reasoning*". Probably, these students perceive the difficulty of using difference communicative register. It is clear that communication is influenced by context: the context of the situation, i.e., space, time, the participants as individuals, the context of the culture, which concerns the beliefs and knowledge about the participants and the topics of communication (Ferrari, 2021).

Category 3- *indifference vs passive learning.* Few students express feelings of indifference respect to the new situation. The expressions used by the students in their interviews, such as "*I basically don't feel comfortable talking to people I don't know"* (line 103);*" I believe that it would not have changed anything, even if she had been my age."* (line 111) underline a poor interaction with the peers and Guru. However, this type of feedback is also useful because it allows the teacher to detect errors and misconceptions in a timely manner, to be taken up and discussed later, outside the activities of the story.

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Dual processes and cognitive reflection in mathematics reasoning and archievement

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Keywords: Dual processes, cognitive reflection, problem solving, belief system, mathematics.

Introduction

Mathematics problem solving is a complex cognitive task that requires solvers to activate their mathematical knowledge, cognitive and metacognitive abilities, as well as their positive math related beliefs. The present study addresses the relationship that reasoning abilities and math beliefs system have with problem-solving efficacy and academic achievement in Mathematics, focusing not only on problem solving accuracy but also on cognitive reflection and math beliefs measures. Secondly, we tested the predictive capacity of these variables on Mathematics achievement in secondary school students.

Different theoretical approaches have been proposed to explain mathematical reasoning. Among them, this study is based on the dual-processes theories of thinking and reasoning (e.g., Evans & Stanovich, 2013). These theories assume two types of thinking processes: (a) Type 1 thinking is fast, guided by data, intuitive and believe based; and (b) Type 2 thinking is relatively slow and reflective, guided by will and conscience.

The balance between intuitions and conscious reflection on the mathematical reasoning is key. The Type 2 thinking is particularly relevant in problem solving when problems combine complexity with novelty and require explicit conscious, effortful reasoning to solve, whereas people have to detach or disregard what they already believe. In this line, Cognitive Reflection Test (CRT; Frederick, 2005) constitutes a mathematical instrument that evidences the propensity to be reflective, to think analytically despite having what initially appears to be a suitable response.

Dual process theories have associated Type I processes with various kinds of cognitive bias. For the purposes of this research, the effects of beliefs based biases on reasoning are particularly relevant in the field of mathematical learning and problem solving (see, e.g., Gómez-Chacón et. al, 2014; Stavy & Tirosh, 2000). Particularly, we consider the role of students' math related beliefs (e.g., beliefs about learning and solving math problems, maths self-efficacy) in problem solving performance that requires analytic reasoning processes.

Research study

Hypoteses

We argue that cognitive reflection and maths related beliefs are involved in mathematical problem solving students commonly face at school and, at the same time, underlie academic achievement in Mathematics. Accordingly, we hypothesized that 1) measures of cognitive reflection and beliefs should correlate with measures of problem solving accuracy, and that all these measures should correlate positively with Mathematics academic achievement; and 2) we expected that those three variables should show evidence as predictive measures of mathematics achievement.

Method

The study involved 121 students attending the 2nd grade of secondary school (age 13-14), each of whom performed three testing instruments, corresponding to the three aspects of this study: (1) Cognitive reflection Test (CRT); (2) Mathematics-Related Beliefs Questionnaire (CreeMat; adapted and validated by Gómez-Chacón et al., 2014); (3) Math problem-solving task (MPST).

The CRT assess cognitive reflection processes in the context of problem-solving tasks. We measured percentages of correct and superficial/intuitive answers in CRT. The MPST evaluates participant´s ability to solve math problems that require analytic thinking. It was also measured a students´ estimation of self-perceived task difficulty. Concerning the CreeMat, several measures were considered: students' beliefs about mathematics, beliefs about learning and solving math problems, and students' beliefs about oneself. Academic achievement was assessed by student´s grades in Mathematics.

Results and Conclusions

Results in both cognitive reflection and mathematical problem solving tests showed a greater number of superficial responses than correct solutions (CRT: $t(120)=6.88$, $p<.001$; and MPST $t(120)=6.95$, $p<.001$). Considering the metacognitive component of CRT, participants underestimated the difficulty of the problems ($r= -.18$, $p<.05$). As expected, we found a pattern of correlations between the cognitive reflection, math beliefs and math problem solving measures are significant, corroborating previous findings (Gómez-Chacón et al., 2014). Regression analyses showed the specific roles played by CRT and self-efficacy on the mathematical achievement ($r=.21$, $p<0.01$).

Findings indicate Type I thinking processes are more commonly used than Type 2 thinking by students (Evans & Stanovich, 2013). The results confirmed the association of students´ reasoning processes, as measured by CRT, with problem-solving processes efficiency and, ultimately, with mathematical achievement at secondary school. They also indicate that the metacognitive processes are not efficient enough at this stage of development.

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Strategy-based motivation to use the drawing strategy. The relationships between self-efficacy, value, cost, and gender

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Motivation and strategies are important for students' learning. In this contribution, we investigated the motivation to use the learner-generated drawing strategy. On the basis of expectancy-value theory, we analyzed the relationships between three components of strategy-based motivation (self-efficacy, value, and cost) and gender differences in these three components of motivation. To address research questions and test hypotheses, we re-analyzed the data from a prior study (N = 402) that assessed low-secondary school students' motivation at two measurement points. Results indicate (1) a positive relationship between self-efficacy and value and a negative relationship between self-efficacy and cost and (2) lower self-efficacy and a higher value given to the drawing strategy in female students. One important implication is that gender differences should be taken into account in strategy-based motivation in research and practice.

Keywords: Motivation, gender differences, learning strategies, self-efficacy, diagram.

Introduction

Motivation is a central construct in theories of affect (Hannula, 2011). In a broader sense, motivation comprises reasons for human behavior (Middleton & Spanias, 1999). One of the most influential motivational theories in the area of educational psychology and education is the expectancy-value theory of achievement choice (Eccles & Wigfield, 2020). According to expectancy-value theory, expectations of success (including self-efficacy expectations), task value, and personal costs are crucial for achievement-related outcomes. Following the idea that affect can address different objects (Schukajlow et al., 2017), we address the concept of strategy-based motivation (SBM) in this contribution. SBM is motivation that derives from characteristics of strategies and their use. The strategy we chose for this study is the learner-generated drawing strategy, which has repeatedly been demonstrated to be very important for problem solving in mathematics (Hembree, 1992). The drawing strategy has specific importance for solving problems that require the construction of mathematical models that rely on the spatial structure of the problem, such as geometrical modelling problems (Rellensmann et al., 2017). Further, we were interested in gender differences in SBM. To the best of our knowledge, prior research has not addressed the relationships between the components of SBM, and we do not know much about how female and male students differ in SBM. The aims of this study were to analyze the relationships between self-efficacy, task value, and cost regarding the drawing strategy and to analyze gender differences in students' SBM to use the drawing strategy. To collect indications of the stability of our findings, we collected and analyzed data at two time points that served as pretest (T1) and posttest (T2) in the original study to uncover the effects of the strategy training on students' performance.

Strategy-based motivation and gender

Expectancy-value theory

Initial intentions in the development of control-value theory were to integrate findings from motivational theories and to clarify why women make different educational choices than men and rarely study STEM subjects (Eccles & Wigfield, 2020). Expectancy-value theory proposes that students' subjective beliefs are important for career choices, interest, performance, and strategy use. Expectancies are defined as beliefs about one's ability to perform an activity (here, making a drawing), and they are related to the self-efficacy expectations proposed by Bandura (2003) in social-cognitive theory. In the case of the drawing strategy, students with high expectations give a positive answer to the question "Can I make a drawing?" Task value is another component in expectancy-value theory. If students value a strategy, they will use this strategy more often. Students who value the drawing strategy give a positive answer to the question "Do I value making a drawing?" Three types of values are attainment value, intrinsic/interest value, and extrinsic/utility value. Attainment value implies that an activity (e.g., making a drawing to solve a problem) is part of an individual's personal identity. Interest value implies the feeling of enjoyment that is associated with performing an activity (e.g., enjoyment in making a drawing). Utility value refers to the students' view that an activity can help them achieve their goals (e.g., applying the drawing strategy will help them solve a problem). Personal cost is another component of motivation. Cost includes the amount of time and effort that a person will invest in an activity. Students who ascribe low cost to the drawing strategy give a positive answer to the question: "Am I free of blockages that prevent me from investing time and effort into making a drawing?" Originally, cost was suggested as part of value. There has recently been a discussion about whether cost should be a component that is distinct from task value and expectations on a theoretical and an empirical level (Eccles & Wigfield, 2020).

Expectancies, value, and cost were found to predict learning outcomes, such as educational choices and academic achievement in mathematics (Eccles & Wigfield, 2020). In our recent study, we found that self-efficacy expectations and cost with respect to the drawing strategy affected the quality of drawings and performance in solving geometry modelling problems (Schukajlow et al., 2021).

Relationships between components of motivation

Since the later eighties, researchers have intensively discussed whether the components of motivation in expectancy-value theory can be empirically separated from each other and can be assessed as distinct constructs. They found that even first-graders differentiate between expectancies and task values, and this distinction persists for school and university students (see an overview in Eccles & Wigfield, 2020). Although the magnitudes of the relationships between self-efficacy, value, and cost are related to the question of whether distinct assessments of these components are possible, these relationships have not yet been the focus of research and need further elaboration with respect to the objects of interest targeted in motivational constructs. In the case of the drawing strategy, one open question is whether students who believe they are able to make a drawing also value this strategy and assign lower costs to this strategy than students who are not sure about whether they can apply this strategy while solving a problem.

It is plausible to assume that students who value an activity are likely to engage in this activity, to gain higher proficiency in doing this activity, and to consequently report higher self-efficacy regarding the respective activity. Further, self-efficacy might be related to cost, as students who ascribe low cost to an activity may engage in the activity more often, may be more successful in this activity, and may improve their expectations. Expectancy-value theory posits what factors can affect students' task value and cost. For example, students' interpretations of their prior experiences (which are related to self-efficacy) influence their affective reactions (e.g., enjoyment and pleasure), which in turn affect task value and cost. A positive relationship between the value and cost components results from their close relationship in expectancy-value theory. Eccles and Wigfield (2020) argued that the overall value of a task depends on its costs. If students think that an activity is too time consuming and too difficult, the value of this activity decreases. For example, if a student assumes that making a drawing is not worth the time, he or she might value making drawing less than a student who ascribes low cost to the drawing strategy. Most studies have reported significant relationships between the three components of motivation. The relationship between expectations and the intrinsic value of mathematics ranged widely between .36 and .68 for students in Grades 1 to 9 in the United States (Gaspard et al., 2020). In a study of adolescent students in mathematics in Korea, cost was negatively related to expectancies (-.24) and to task value (-.23) (Jiang et al., 2018).

Gender differences in self-efficacy, value, and cost

Gender differences were one of the main starting points for the development of expectancy-value theory. According to this theory, gender differences in choices in STEM subjects can be explained by differences in expectancies and values and go back to gender stereotypes in society. Indeed, an analysis of public views in many countries confirmed inequities in expectations and achievement in mathematics between female and male students (Forgasz & Leder, 2017). Such inequities were in turn found to increase self-efficacy expectations in male students and to decrease them in female students. Consequently, there are well-documented differences between female and male students in self-efficacy in mathematics across many countries all over the world (Else-Quest et al., 2010). According to expectancy-value theory, students value activities and tasks that correspond to their social and personal identities (Eccles & Wigfield, 2020). By engaging in these tasks, students can express themselves. Given that mathematics is seen as a male domain in many societies, many female students do not see mathematics as part of their identity, and they value mathematics less than male students do. As a result, many students demonstrate a gender gap in expectancies and value. In PISA 2012, female students reported lower self-efficacy and lower interest in mathematics than male students in Germany (Cohen's d of 0.53 and 0.39 in Schiepe-Tiska & Schmidtner, 2012). However, we do not know the extent to which self-efficacy, value, and cost with respect to strategy use might differ between female and male students. As making a drawing to solve modelling problems is part of the solution process, one can have lower expectations for female students. A gender difference in anxiety with respect to strategy use supports this expectation (Schukajlow et al., 2019). For differences in value and cost, we do not have specific expectations. It might be possible that female and male students value the use of the drawing strategy to a similar extent and assign a similar level of cost to this activity while solving mathematical problems.

Research questions and hypotheses

On the basis of expectance-value theory and prior research, we investigated the following research questions in this study:

RQ1: Are the components of strategy-based motivation (self-efficacy, task value, and cost) related to each other with respect to the drawing strategy?

H1: Self-efficacy will be positively related to task value and self-efficacy and task value will be negatively related to cost.

RQ2: Do female and male students differ in self-efficacy, task value, and cost with respect to the drawing strategy?

H2: Female students will have lower self-efficacy for the drawing strategy than male students, but no clear expectations in gender differences regarding value and cost could be derived from prior research.

Method

Present study, sample, and procedure

To address the research questions and test the hypotheses, we reanalyzed a sample collected in the framework of the Visualizations While Solving Modelling Problems project. This project is aimed at clarifying the role of learner-generated visualizations in solving modelling problems, including the effects of motivation and emotions on strategy use, quality of drawings, and performance. In one of the studies, students ($N = 435$; 6 middle-track schools) in each class were assigned to one of four groups. Three groups were trained in knowledge about the drawing strategy, whereas one group practiced algebraic procedures (e.g., solving equations). A prior analysis indicated indirect positive effects of the strategy training on performance with strategic knowledge about drawing and quality of drawings as intervening variables (Rellensmann et al., 2021). Before (T1) and after (T2) the training sessions, students filled out a questionnaire on self-efficacy, task value, and cost, among other tests. At T1, they reported their gender. At T2, some students were missing, or students did not answer all the scales on the questionnaire. Thus, the overall sample included 402 students (46% female; 14.6 years of age) from all four treatment groups.

Scales on self-efficacy, value, and cost

We used 5-point scales on self-efficacy, value, and cost with respect to the drawing strategy ranging from 1 (*not at all true*) to 5 (*completely true*). We adapted the items for this study from a prior study on motivation and learning strategies in mathematics (Berger & Karabenick, 2011) by focusing the items on learner-generated drawings while solving difficult word problems (Schukajlow et al., 2021). Self-efficacy regarding the drawing strategy was assessed with three items (e.g., "I'm confident I can make a very good drawing to solve any word problem"; Cronbach's $\alpha = .79$). Task value was assessed via interest value (2 items; e.g., "I like making drawings to solve difficult word problems"), attainment value (4 items, e.g., "making a drawing to solve difficult word problems is an important part of who I am"), and utility value (3 items, e.g., "I believe that it is important to make a drawing because drawings help me solve difficult word problems"). Cronbach's α for the task value scale was good ($\alpha = .88$). Personal cost was assessed with three items (e.g., “It takes a lot of effort for me to create a drawing to solve a difficult word problem”; α = .70). A confirmatory factor analysis confirmed the structural validity of the motivation to use the drawing strategy scale (Schukajlow et al., 2021).

Results

Relationships between self-efficacy, task value, and cost

Our first research question addressed the relationships between the three components of motivation: self-efficacy expectations, value, and cost. To address the first research question, we calculated Pearson correlations. In line with our expectations, the analysis revealed a positive relationship between self-efficacy and value with respect to the drawing strategy at T1 (.394) and T2 (.482; *p*s < .001). We also found a negative relationship between self-efficacy and cost with respect to the drawing strategy at T1 (-.386) and T2 (-.154; *p*s < .001) as expected in our study. Value and cost with respect to the drawing strategy were negatively related at T1 (-.141; $p < .01$) and the two motivational variables were not related at T2 (.027; p = 573). These results partly confirmed our expectations.

Gender differences between self-efficacy, task value, and cost

We applied an ANOVA with the factor gender and the dependent variables self-efficacy, value, and cost to address RQ2 on gender differences in SBM (see the descriptive statistics in Table 1).

Table 1: Means and standard deviations for self-efficacy, value, and cost at T1 and T2

	Self-efficacy		Value		Cost	
	T1: *M(SD)*	T2: *M(SD)*	T1: *M(SD)*	T2: *M(SD)*	T1: *M(SD)*	T2: *M(SD)*
female	2.94(.77)	3.15(.90)	3.31(.75)	3.34(.75)	2.64(.87)	2.83(.94)
male	3.17(.85)	3.30(.82)	2.96(.79)	3.12(.76)	2.57(.90)	2.82(.91)

At T1, the ANOVA revealed overall differences between female and male students regarding SBM, $F(3, 432) = 18.319, p < .001, \eta^2 = .113$. Similarly, at T2, the ANOVA showed differences in SBM regarding drawings between female and male students, $F(3, 401) = 7.772, p < .001, \eta^2 = .055$. A post hoc analysis confirmed our expectations about low self-efficacy in the use of the drawing strategy in female students compared with male students at T1 and T2; T1: $F(df = 1) = 8.836, p < .01, \eta^2 = .020$; T2: $F(df = 1) = 2.287, p = .078, \eta^2 = .008$. Moreover, female students assigned a higher value to the drawing strategy than male students at T1, $F(df = 1) = 8.836, p < .01, \eta^2 = .020$. At T2, the value of the drawing strategy tended to be higher in female students; T2: $\mathrm{F}(df = 1) = 2.287, p = .078, \eta^2 = .008$. The cost of the drawing strategy did not differ between female and male students; T1: $F(df = 1) = .733, p = .392, \eta^2 = .002$; T2: $F(df = 1) = 2.287, p = .851, \eta^2 < .001$.

Discussion

The aim of the present study was to analyze the relationships between three motivational components (i.e., self-efficacy, value, and cost) and to examine the gender gap in motivation.

The theoretical basis of our study came from the expectancy-value theory of achievement choices (Eccles & Wigfield, 2020). We followed the theory about the importance of the object for the investigation of effects (Schukajlow et al., 2017) and addressed SBM in our research. As a strategy, we selected learner-generated drawings, which were found to be beneficial for solving geometry modelling problems in prior studies (e.g. Rellensmann et al., 2017).

We found that self-efficacy was positively related to value and negatively related to cost at two time points. This finding indicates that students who demonstrate higher self-efficacy with respect to the use of the drawing strategy value this strategy and assign it a low cost. Our result is in line with the assumptions of expectancy-value theory about the relationship between expectancy and value beliefs and indicates the possibility of a feedback loop between expectancies on the one side and value and cost on the other side. The mechanism behind the relationship between self-efficacy and value should be investigated in future studies. One way this mechanism might work is that value increases students' engagement, and engagement positively affects performance, which in turn affects self-efficacy. Similarly, students who assign a low cost to a task are more engaged in doing this task, improve their performance, and increase their expectations of doing well when performing this task in the future. This relationship was stable across the two time points. This finding indicates stability in self-efficacy's relationship to value and cost. Cost was negatively related to value at T1, but it was not related to value at T2. This result is partly in line with our expectations. Moreover, cost's relationships with self-efficacy and value were weak at T1. Indeed, the relationship between cost and value was also weak in a prior study (Jiang et al., 2018). Along with other findings, our finding indicates that cost should be addressed as a distinct component in future studies. This is important because self-efficacy and cost have been shown to affect the quality of the drawing strategy and performance in solving modelling problems (Schukajlow et al., 2021). Another important result is that the relationships between cost and the two other components of motivation were slightly weaker at T2, and the relationship between cost and value was not even significant, which might be an indication that motivation to use the drawing strategy is not very stable. Solving problems that require the application of the drawing strategy might change students' perceptions of the cost of this strategy, which in turn might influence the relationships between cost and the two other components of the SBM. Our findings add to prior research on the relationships between the components of motivation (Gaspard et al., 2020; Jiang et al., 2018). Moreover, a new theoretical contribution of this study is that it expands the predictions of expectancy-value theory to SBM. Addressing the motivation to use the drawing strategy and investigating relationships between the three components of motivation are novel contributions of this study and, to the best of our knowledge, have not been made before.

Our analysis of gender differences in SBM revealed overall differences between female and male students at two time points. Female students reported lower self-efficacy expectations compared with male students at both T1 and T2. These differences are in line with prior findings on female students' low self-efficacy expectations regarding mathematics and confirm the existence of a gender gap (Schiepe-Tiska & Schmidtner, 2012). Interestingly, female students tend to value the use of the drawing strategy for solving modelling problems more than male students. This result was significant at T1, and the difference just missed the significance level at T2. A higher valuing of the drawing strategy in female students was unexpected because prior research has revealed that female students often value mathematics less than male students do. Consequently, gender stereotypes are not transmitted to all problem-solving activities in mathematics, and this is an important theoretical contribution of our study. One explanation might be that doing mathematics in school and in society might be more strongly associated with calculations or numbers and less with drawing activities. An open question with possible practical implications is whether focusing on strategies in mathematics classes and offering a more complex image of mathematics in society—one that goes beyond calculations or working with numbers—can contribute to changing gender stereotypes and can decrease the gender gap. The cost of applying the drawing strategy did not reveal any gender differences in SBM. Summarizing our results regarding SBM, we see a mixed picture: Female students have lower self-efficacy expectations and place a higher value on the drawing strategy than male students, whereas female and male students ascribe similar costs to the drawing strategy. An important practical implication of our study is that it uncovered different ways in which female and male students' SBM can be supported in mathematics classrooms. To increase female students' SBM, teachers should pay more attention to self-efficacy regarding strategy use (e.g., by giving positive feedback on the quality of students' drawings), whereas male students might benefit more if teachers emphasize the value of a strategy (e.g., by emphasizing the positive effects of using drawings to solve geometrical problems.)

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Upper secondary school students' gendered self-evaluation in mathematics

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Self-evaluation is considered one of the key concepts when trying to understand motivation, and it is gaining more interest especially when looking at the age span 15-18 years. Previous studies in self-evaluation and mathematics tend to use data from international large scales assessments, arriving with rather ambiguous conclusions, and smaller studies tend to use only one measure without control factors. The aim of this paper was to test the hypothesis that boys are more confident than girls in mathematics, while using Swedish as a control subject. A questionnaire was handed out to 399 upper secondary school students from different regions in Sweden, both vocational programmes and programmes preparing for further studies. Using both non-parametric analysis and linear regression, the results support the hypothesis. The relationship to the idea of confidence gap is discussed.

Keywords: Gender, mathematics, self evaluation, upper secondary school.

Introduction

One of the key concepts in affect in general and motivation in particular is self-evaluation (Nagy et al., 2010; Pajares, 2005). It contributes to perceived self-efficacy (Bandura, 1997), and can be seen as students' self-perceptions of their competence or ability. Previous studies have concluded that students base their mathematics self-concept largely on their experiences and history of achievement (Usher, 2009), and sex differences in grades cannot explain the gendering of career choices (Dekhtyar et al., 2018). In addition, meta-studies have concluded that girls most often have higher grades than boys, with the largest difference in language and smallest difference in mathematics (e.g. Voyer & Voyer, 2014), meaning that differences in self-concept most likely are not only due to grades. On a micro-level, self-evaluation as a concept is relevant since studies indicate that when students are asked to self-evaluate their capabilities or progress in learning a particular task, it encourages them to develop a higher level of competence and their self-efficacy beliefs are strengthened (Ramdass & Zimmerman, 2008). On a macro-level, some gender patterns have been identified where the overall conclusion is that boys tend to report higher levels in measures of self-evaluation (OECD, 2013), but studies also report that gender differences in mathematics self-concept are smaller in more egalitarian countries (Goldman & Penner, 2016), and that students' mathematical self-concept was strongly linked to their mathematical achievement, and that students that have low scores were the ones who overestimated their mathematical competence (Chiu & Klassen, 2010). It appears to be no unified picture how gender, mathematics, and self-evaluation is connected. The specific age span (15-18 years) is also of interest since this is when children/ adolescents are developing their academic self-perception, something that is gaining more attention especially with respect to gender differences

(Nagy et al., 2010). One interesting case is Sweden, a country with a reputation of its gender equality: Sweden is a country that is both wealthy and egalitarian, and a possible conclusion is that students should either have little gender differences and indicate low self-concepts (e.g. Goldman & Penner, 2016), or close connections with self-concept and achievement (e.g. Chiu & Klassen, 2010). Previous research signal mixed finding (e.g. Frid et al., 2021; Sumpter, 2012), including indications of gender confidence gap (e.g. Zander et al., 2020). This paper aims to look closer at upper secondary school students' self-evaluation in mathematics with respect to grades. We test the hypothesis that boys are more confident than girls in mathematics, a hypothesis that functions as the research question.

Background

The starting point for the overview of the concept self-evaluation here is Festinger's (1954) theory of social comparison processes, where the focus is on social standards with the conclusion that there are no objective standards. The focus then was mainly on interpersonal comparisons, which was extended to include intergroup comparisons (Tajfel, 1974), that group membership provides a basis for self-evaluation. Other concepts describing similar aspects are self-concept (Shavelson, et al., 1976), and self-beliefs specific to one's perceived capability which includes, for instance, task-specific self-concept, self-concept of ability, and academic self-concept (Pajares, 2005). Here, the focus will be on self-evaluation, emphasising the process of evaluation: "the evaluation or judgment of 'the worth' of one's performance" (Klenowski, 1995, p. 146). Looking at gender, the chosen theoretical stance is that gender is a social construction, more than just a consequence of a biological sex (Connell, 2006). It means that gender is a pattern of social relations, which means that definitions of women and men depend on the context and under constant negotiation. In order to understand these patterns, one can divide gender into four different aspects: structural, symbolic, personal, and interactional gender (Bjerrum Nielsen, 2003). Structural gender covers social structures, and symbolic gender focus on the gender as attributed symbols and discourse. The symbols could be attributed in both ways. It can be that an object or an abstract concept that is considered male or female, such as the idea of mathematics as a male domain (e.g. Brandell & Staberg, 2008). It could also be about how men and women are perceived such as the 'the hard working female' and 'the male genius' (Leslie et al. 2015). Such symbols inform us what is considered normal and what is deviant (Bjerrum Nielsen, 2003). The third aspect, personal gender, focuses on how the individuals perceive the structure and the different symbols, which includes self-evaluations. The fourth aspect, interactional gender, covers interactions of individuals that take place within this context that comprises the structure and symbols.

Regarding the process of determine one's value, studies have shown that students are using multiple frames of reference when evaluating their mathematics ability and these self-evaluations are pretty robust (Skaalvik & Skaalvik, 2004a). Self-evaluations that were made using other students in class as an external frame of reference, and on comparison of mathematics achievement with achievement in other school subjects which function as an internal frame of reference, were both strong predictors to mathematics self-concept and self-efficacy. This implies that when studying self-evaluation in one subject, such as mathematics, using other school subjects as well calibrates the evaluations. Recent studies have shown that social economic status can play a role (e.g. McConney & Perry, 2010), but due to space the focus here is on gender. When following students from grade 7 and onwards, in the beginning boys expressed more positive self-concept and these differences persisted over time (Nagy et al., 2010). Studies on older students reported similar results (e.g. Skaalvik & Skaalvik, 2004a). Others confirm the gender confidence gap: despite having higher or similar grades, girls reported lower self-evaluation/ self-efficacy and self-esteem in mathematics (Brandell & Staberg, 2008; Sumpter, 2012; Zander et al., 2020), but there is also indication that there is no significant difference looking at students (age 15) at lower secondary school (Frid et al., 2021). Further, when comparing mathematics with language, studies has indicated that male students, with respect to mathematics, signalled not only higher self- concept, intrinsic motivation, and self-enhancing ego orientation but also higher performance expectations compared to the female students (Skaalvik & Skaalvik, 2004b). However, when the focus was on language, women expressed higher intrinsic motivation.

Methods

The data was generated through an online questionnaire that was part of a study of upper secondary school students and their work with a mathematical model exploring segregation (see Tsvetkova et al. (2016) for more information of the full study). In the study, a questionnaire was included with questions such as "How would you evaluate yourself in mathematics?" and a control question using an equally important subject, Swedish. The scale was Very good/Good/ Average/ Below average/ Weak, the same scale as in previous research allowing us to make comparisons (e.g. Brandell & Staberg, 2008; Frid et al., 2021). There was also a question about which grade they got in mathematics and Swedish in their latest course. There were also questions about factors related to their social economic status such as what their parents worked with and if they were planning to go to university. These questions are not analysed in the present study. In total, 399 participants (233 boys, 166 girls) from 20 upper secondary school classes were part of the study. The classes came from three different regions in Sweden (east, middle and west), covering all three grades meaning the age span was 16-19. Each class had between 13 to 25 students and they came from different educational programmes, both vocational ones and programmes preparing for university studies. The data were analysed in two stages. In the first stage we adopted a non-parametric approach, looking at the difference in proportion of combined answers to the two questions. For each self- grade, we calculated the proportion of girls and the proportion of boys who gave each of the five possible answers to the "How would you evaluate yourself in mathematics?" question. We then took the difference and expected a strong correlation between self-reported grades and self-evaluation, and by looking at differences within each grade division we could visualise differences for each grade independently. In the second stage, the data were analysed using linear regression to predict self-evaluation from self-reported grades, stated gender, and whether one parent (or more) was born in Sweden. In order to perform the regression, we converted both the self-reported grades and the self-evaluations to a numerical scale. Such a transformation is never entirely justifiable (hence the first stage of the analysis) but they can be seen as reasonable given the nature of the grade scale and self-reporting. Again, we expected that self-evaluation to be correlated with self-reported grades, so we included the variable in order to see how much additional predictive power gender has over and above this relationship. The background of parent was included as a control variable, allowing us to see whether one factor linked to social economic status could play a role (e.g. McConney & Perry).

Results

The first results are about the responses about students' stated grades in mathematics and Swedish, see Table 1:

Table 1: Distribution of stated grades in mathematics and Swedish, n(%)

Subject/ Grade	A	B	C	D	E/F	No reply	Total
Mathematics	79 (19.8)	73 (18.3)	100 (25.1)	72 (18.0)	57(14.3)/ 8(2.0)	10 (2.5)	399
Boys	43 (18.5)	39 (16.7)	53 (22.7)	47 (20.2)	43 (18.4)		233
Girls	36 (21.7)	34 (20.5)	47 (28.3)	25 (15.1)	22 (13.2)		166
Swedish	92 (23.1)	123 (30.8)	115 (28.8)	34 (8.5)	20(5.0)/ 1(0.3)	14 (3.5)	399
Boys	37 (15.9)	64 (27.5)	80 (34.3)	24 (10.3)	18 (7.7)		233
Girls	55 (33.1)	59 (35.5)	35 (21.1)	10 (6.0)	3 (1.8)		166

In Table 1, Grade E and F is joined since so few students reported F. Girls report higher grades both in mathematics and in Swedish. The distribution of the self- evaluation was the following (Table 2):

Table 2: Distribution of self-evaluation with respect to gender, n(%)

Subject/ Self-evaluation	Excellent	Good	Average	Below average	Weak	No reply	Total
Mathematics	60 (15.0)	134 (33.6)	147 (36.8)	36 (9.8)	12 (3.0)	10 (2.5)	399
Boys	39 (16.7)	75 (32.2)	84 (36.1)	20 (8.6)	7 (3.0)		233
Girls	21 (12.7)	59 (35.5)	63 (37.9)	16 (9.6)	5 (3.0)		166
Swedish	85 (21.3)	165 (41.3)	116 (39.1)	15 (3.8)	4 (1.0)	14 (3.5)	399
Boys	38 (16.3)	95 (40.8)	75 (32.2)	13 (5.6)	2 (0.8)		233
Girls	47 (28.3)	70 (42.1)	41 (24.7)	2 (1.2)	2 (1.2)		166

Table 2 illustrates that boys tend to rank themselves higher compared to girls in Mathematics, but vice versa in Swedish. To analyse the different distribution presented in Table 2, which is only descriptive, we looked at students' self-evaluation in mathematics and stated grades in relation to expressed gender. As a calibration of the results, Swedish is used as a comparison. The results are from the calculation of the proportion of girls and boys evaluating in each category for each grade, then taking the difference. The darker blue colour indicates a more common answer by boys, whereas a darker red colour means a more common answer by girls, see Figure 1:

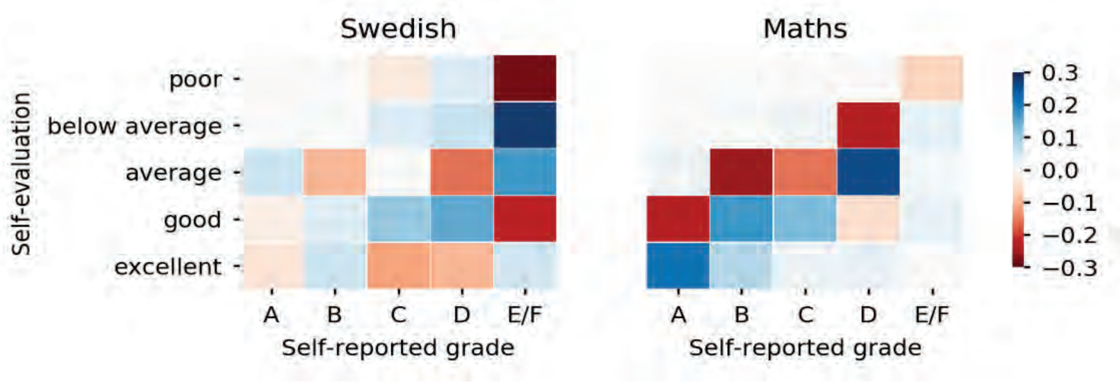

Figure 1: Difference in proportion of boys minus girls' self-evaluations for each self-reported grade

Figure 1 shows how the difference in self-evaluation between boys and girls depended on grade for both Swedish and Mathematics. For mathematics, we see that girls with a self-reported grade 'A' were more likely to rank themselves as 'good' than boys, while boys were more likely to rank themselves as 'excellent' than girls. Similarly, boys with a self-reported D were more likely to report themselves as 'average' than girls. The same pattern was repeated over all grades except E/F. In Swedish, the control factor, no such pattern was observed, with the possible exception of the self-reported grade 'E/F', where boys took 'below average' and girls took 'weak'. Based on this analysis, the conclusion is that girls under-valued themselves or boys over-valued themselves. The limitations are the small numbers, and therefore the next step is to validate these results using linear regression:

Table 3: Titles of tables, figures, diagrams, are in the style FigTitle, no dot at the end

Mathematics: Self-evaluation	Coefficient	Std. Error	t	*p*
Reported grade	0.512	0.031	16.20	< 0.001
Male	0.206	0.059	3.50	0.002
Swedish parent	-0.030	0.105	-0.28	0.782
Constant	1.83	0.127	14.32	< 0.001
Swedish: Self-evaluation				
Reported grade	0.454	0.045	10.04	< 0.001
Male	-0.015	0.076	-0.19	0.848
Swedish parent	-0.016	0.076	-0.20	0.841
Constant	2.20	0.210	10.47	< 0.001

The linear regression, Table 3, showed a strong relationship between self-reported grades and self-reported confidence in both mathematics ($t = 16.20$; $p < 0.0005$) and Swedish ($t = 10.04$; $p < 0.0005$). This reflects the correlation we see in Figure 1, model R-squared of 0.520. Only in mathematics did the 'boys' factor statistically significant in predicting self-reported grades ($t = 3.50$; $p = 0.002$). Table

3 provides non-standardized coefficients so we could reasonably interpret being male is associated with an average increase in self-evaluation of 0.206 points (where 1 point indicates a shift from average to good, or from average to excellent, etc.). In Swedish, being male was not statistically significant ($t = -0.19$; $p = 0.848$). The model R-squared was lower at 0.343. A control variable, whether the respondents had a Swedish-born parent, was not statistically significant in either regression. Taken together, these results strongly support the hypothesis that boys give themselves a higher level of self-evaluation compared to girls in relation to stated grades in mathematics.

Discussion

The aim of this paper was to study upper secondary school students' self-evaluation with respect to stated grade and gender. The overview of the grades show that they are higher for girls, both in mathematics and Swedish, which is in line what have been reported earlier (e.g. Voyer & Voyer, 2014). The hypothesis that we tested was that boys are more confident than girls in mathematics. Just as in previous studies, both national (e.g. Brandell & Staberg, 2008; Sumpter, 2012) and international (e.g. Nagy et al., 2010; OECD, 2013; Skaalvik & Skaalvik, 2004a), the results, from the non-parametric analysis and linear regression, confirmed that boys choose higher self-evaluation in relation to stated grades compared to girls. The results are to some degree in contrast to Frid et al. (2021), and compared to the PISA study (i.e. Chiu & Klassen, 2010), boys overestimated their ability at all levels or girls undervalued themselves. It indicates that several studies are needed in order to understand the concept self-evaluation from a gender perspective: since individuals are using multiple frames of reference when they judge their evaluations (e.g. Skaalvik & Skaalvik, 2004a), such discrepancies can be related to which frame that is in focus (e.g. Sumpter, 2012). Given that the present study used a limited scope of the definition of self-evaluation compared to Klenowski (1995), there are room for further development of the instrument that was used.

Nevertheless, combining the results from the present study and previous research, the conclusion is that boys appear to over-value themselves or girls tend to under-value themselves. This supports the so-called confidence gap (Zander et al., 2020). In the present study and the results from the non-parametric analysis, the confidence gap was visible mainly in mathematics and only partly in Swedish where in the group of self-reported grade 'E', boys more often picked 'below average' and girls more often took 'weak'. Here, the results differ slightly from what has been previously reported regarding girls and language (e.g. Skaalvik & Skaalvik, 2004b). This could be interpreted as an indication of patterns within a more general pattern in self-evaluation. Also, if students construct their mathematics self-concept on their experiences and history of achievement (e.g. Usher, 2009) and girls have better grades (Voyer & Voyer, 2014), the history of achievement appears to be cancelled out. If we, for instance, want to understand why we have gendered careers (e.g. Dekhtyar et al., 2018), we need to look beyond grades. The confidence gap is nonetheless an interesting phenomenon especially if used to blame girls for being not confident enough, given that they are already confident (e.g. Sumpter, 2012) or not having good enough grades (e.g. Dekhtyar et al., 2018). This is important when ideas such as 'the hard-working female' and 'the male genius' (e.g. Leslie et al. 2015) still exists.

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Attending to voices of women and racially/ethnically minoritized students: Conflicting perceptions of mathematical competence

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We present our findings from a pilot study that was conducted between Spring 2019 and Spring 2020. Our overall project goal was to gain a deeper insight into women and racially/ethnically minoritized students' experiences during the secondary school-tertiary transition in mathematics. In the report, we used aspects of the three-dimensional model of attitude (Di Martino & Zan, 2010). The model consists of three dimensions, which are vision of mathematics, perceived competence, and emotional disposition toward mathematics. We particularly focus on emerging mathematicians' conflicting perception of competence in mathematics and address the concomitant emotions. We documented voices of women and racially/ethnically minoritized students to address gender stereotypes and raise awareness about their struggles for a sense of belonging in mathematics from affective and sociocultural standpoints.

Keywords: Affect, gender, inclusion, women and minoritized students, perceived competence.

Introduction

Many scholars have researched the phenomenon of the secondary school-tertiary transition (STT), in mathematics considering cognitive, didactics, sociocultural (De Guzman et al., 1998), and affective dimensions. The nature of mathematics at university is different than secondary school mathematics, resulting in certain affect-related outcomes that prevent students from becoming competent mathematicians (Di Martino & Gregorio, 2019). Holding a positive attitude about ourselves is one of the key aspects of meaningful participation along with the ability to effectively communicate mathematically in becoming involved in a new community (Lave & Wenger, 1992). Women rate a lower mathematical self-conception than men even though mathematics has a slightly better gender representation in mathematics than other STEM domains (Sax et al., 2015). However, increased representation does not always provide women and racially/ethnically minoritized students with equal representation in public mathematical spaces, such as classrooms. Because of the strong beliefs of men on self-concept, they are more likely than women to be identified as full participants in mathematics environments (Mendick, 2005). It is documented that women in mathematics often experience challenges in a way that positions them at the periphery of mathematical participation (Solomon, 2007). Women have difficulties in acknowledging their potential in mathematics due to gendered stereotypes in many mathematical communities (Solomon, 2007). When women are exposed to environments with fixed ability views and gender stereotyping, their sense of belonging and perception of competence tend to decline (Good et al., 2012). STT research is limited in its ability to address the intersectionality of identities such as race, gender, or ethnicity in the examination of affective dimensions of mathematics learning. Therefore, we examined the obstacles in the STT by using affective aspects such as perception of competence in mathematics, particularly among racially/ethnically minoritized populations. We also highlighted perceived competence in mathematics (Di Martino & Gregorio, 2019) with an emphasis on emotional expressions, as well as the notion of belonging considering legitimate peripheral participation of women (Solomon, 2007). Further, we point out gender stereotypes and some concomitant emotional states play a role in how women construct their identities as they become members of mathematical communities (Solomon et al., 2011). Accordingly, our guiding research question is: *How do women and racially/ethnically minoritized students in the mathematics major position themselves as mathematics learners considering their perceptions of mathematical competence?*

Theoretical Perspectives

The three-dimensional model for attitude (TMA) characterizes attitudes in relation to beliefs and emotions (Di Martino & Zan, 2010). We adopted the TMA for our conceptualization of the relationship between beliefs and emotions as components of students' experiences with mathematics in university. The three dimensions are vision of mathematics, perceived competence in mathematics, and emotional dispositions towards mathematics. Students' vision of mathematics as being more relational or instrumental (Skemp, 1978) are linked to their beliefs of mathematics. Students' beliefs about themselves as mathematics learners, which are influenced by their vision of mathematics, can be thought of as perceived competence in mathematics (Di Martino & Gregorio, 2019). Finally, Di Martino and Gregorio (2019) identified associations between the vision of mathematics and perceived competence, which often manifested through emotional responses. In the study, we particularly addressed the component of perceived competence on students' mathematical experiences. We also discovered that the notion of belonging deserved more attention when it comes to the identity development of women and racially/ethnically minoritized mathematics students (Good et al., 2012). The TMA offered useful perspectives in analyzing the perception of competence as students reflected on their mathematical experiences from affective perspectives.

Method

Participants and Settings

We conducted a pilot study, the Women and Underrepresented Minorities in Mathematics (WURMM) study, from Spring 2019 to Spring 2020 in which 13 students participated. The pilot study focused on mathematics majors who were from racially/ethnically minoritized populations (e.g., students of color, women). The pilot study consisted of multiple data sources such as interviews, survey responses, students' reflection diaries, and seminar[1] artifacts. Considering our aim to address students' perceived competence, our paper focuses on the activity conducted in the second seminar which was held in Spring 2020 with four participants. We report data from three of the four participants (Table 1) for whom we have complete data profiles. Two students were pure mathematics (PM) majors and one student was a secondary mathematics teaching (SMT) major. The goal of this seminar was to gain insights into students' STT experiences from affective perspectives. We

[1] We use the term "seminar" to describe the intervention we included during the pilot study. These seminars (one in Summer 2019 and one in Spring 2020) consisted of four sessions each (each session was 2- to 3-hours in duration), and included mathematical tasks, discussions, and short presentations.

prompted students to talk about the attributes of mathematicians by addressing perceived competence in mathematics and mathematical identity during the seminar activities. The data sources for this paper consist of video recordings from four seminar sessions (two hours each) conducted in Spring 2020. We also collected data through students' written artifacts from the fourth seminar session's activity and post-survey responses. Finally, we draw on interviews conducted in Spring 2019, which included information about demographic, secondary school experiences, perception of mathematics, and perceived competence in mathematics. We were able to analyze the affective components of STT experiences by comparing secondary school to university experiences related to perception of competence and vision of mathematics.

Table 1: Demographic information of selected participants

Participants	Major/Year	Gender	Ethnicity
Dana	PM/Sophomore	Female	Hispanic/Latino
Manuel	SMT/Sophomore	Male	Hispanic/Latino
Sunny	PM/Sophomore	Female	Non-Hispanic/Latino

Session 4 activity: Exploring attributes of mathematicians

We designed an activity to capture the salient attributes of a mathematician's identity. The activity was useful in prompting students to articulate the attributes of a competent mathematician and accordingly to understand students' beliefs on their perceived competence in mathematics. Also, the activity enabled us to observe students' emotions while they talked about mathematical competence. We shared a Google document with students which was divided into four quadrants with a guiding question for each: (1) Who is a mathematician?, (2) What does a mathematician do?, (3) What does a mathematician say?, and 4) What do you consider a mathematician is not?

We asked participants to individually think about their responses to the questions and then each participant populated the Google document with their comments. Since participants were asked to use only short phrases or terms within the document, we also facilitated a discussion in which participants could more thoroughly share their perspectives. The activity helped elicit students' opinions on mathematicians' identity attributes and how they relate to emergent characteristics. We asked the same questions in the post-survey to provide students with a chance to express their personal opinions, without being influenced by responses from other participants.

Preliminary Findings

Using aspects of the TMA, we highlight our findings related to perceived competence and accompanying emotions. We present excerpts that reflect students' views on the attributes of a competent mathematician, as well as the emotions evoked when they reflect their own mathematical competence.

Attributes of a mathematician

Participants reflected on the attributes of a competent mathematician (Figure 1) including mathematicians' identity, discourse, and actions. Sunny used the expressions of "creative and logical" by stating that a mathematician communicates by "clearly indicating logical processes and reasoning as well as the goal of the problem" (Post-survey, 2020). Sunny valued logic and clarity as part of a competent mathematician's identity. Manuel similarly articulated that a mathematician is a "problem solver and logical" and asks "why?" Dana's reflection included: "The way in which they communicate automatically provokes thinking critically and stimulates conversation" (Post-survey, 2020). Dana's response underlined the importance of communication aspects. That is, she valued a mathematician who contributes to the common discourse within a community of practice.

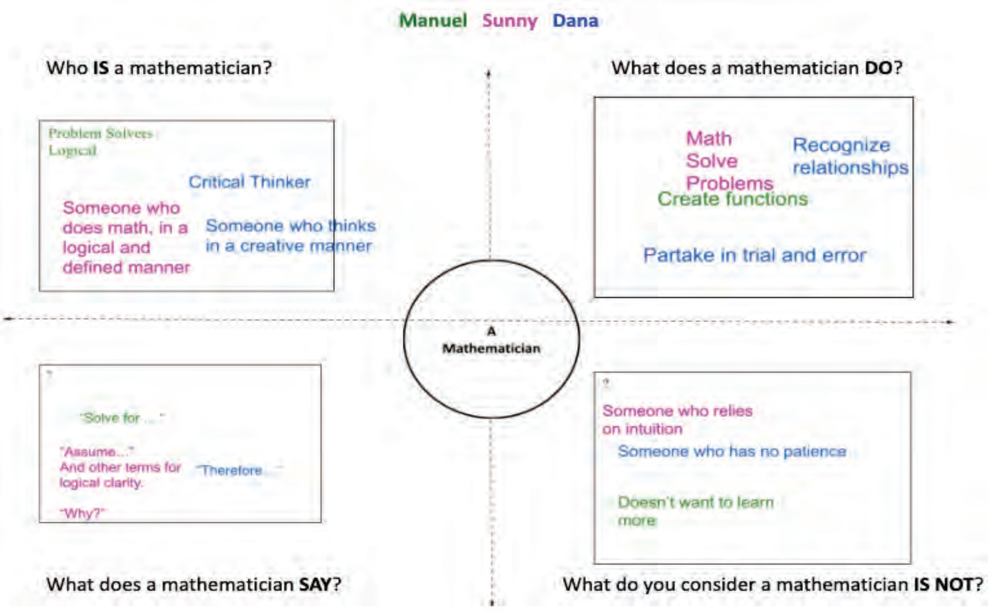

Figure 1: Students' artifact from the activity for exploring the attributes of mathematicians

Next, students elaborated on the prompt which was related to how mathematicians can change their confidence. Sunny reflected on the prompt from a personal standpoint and responded with "expanding her course work and exploring more areas of mathematics" as a way to help her to improve her confidence. Interestingly, Manuel referred to a mathematician's identity as a "title" rather than an "action" (Post-survey, 2020), which he thinks could help to improve how mathematicians do mathematics. We speculated that the basis of this idea of a mathematician might have its roots in the conventional view of a mathematician, which positions them as naturally capable in society regardless of their actions. Dana responded with: "by 'doing' more mathematics, and by surrounding themselves in a supportive environment that promotes intellectual conversation about mathematics without rejecting the ideas of others" (Post-survey, 2020), which aligned with her earlier comment during the seminar. Dana's responses reflected her perceptions of a competent mathematician that were closely related to social participation in mathematical communities. Dana valued a mathematician who is also precise in communicating mathematical ideas and creates a dialogue. From her perspective, the latter also stressed the significance of discourse, which was aligned with social aspects of a mathematician's identity.

Perceived competence in mathematics

We asked further questions about students' perception of their competence in mathematical communities. Dana's mathematical confidence was supported by her vision of mathematics. That is, she emphasized a key aspect of her mathematical confidence related to being able to develop relationships between abstract concepts as she progressed through the course work.

> I feel as if in a way... it has made me feel *more confident, but also less confident*[2] for different reasons. Like, *I feel more confident because of these higher-level math courses*. I'm learning about math as this abstract concept and the more I'm just learning about these different concepts and I'm able to connect different relationships that are allowing me to just form different connections. And so, in that way, *I begin to feel more confident*. (Seminar, 2020)

We also noticed certain negotiations concerning perceived competence in mathematics in Dana's articulations. It appears that Dana's perceptions of a competent and confident mathematician that she described earlier was conflicted with how she felt about participating in mathematical spaces. In the preceding excerpt, Dana outlined some perturbations in her perceived competence, including doubts when surrounded by "brilliant people" and questioning of her abilities in mathematics.

> …But also [I'm] just *less confident*. The more I'm surrounded by people that are just really brilliant and I just, I tend to… I don't know how to explain it. I feel as if, even though I am in the same classes as them and I do have a right to be there, I always just, *never built up enough confidence* to, like, for example, just ask a question or just be involved and I feel, not to turn this into a whole gender thing, but I feel as if… Because like STEM and mathematics still are more of a male-based subject, *it's hard for me to really voice my opinions and voice what I feel.* When I am in a group of all guys during a study group or like when I'm just doing partners, you know, I just *tend to get less confident*. (Seminar, 2020)

Dana's negotiations of her perceived competence in mathematics accompanied by emotional states and doubts about her belonging to her mathematics community. Affect was not only associated with vision of mathematics but also certain sociocultural factors in her experiences. When we raised the notion of societal norms that shape women's perceived competence in mathematics, her responses reflected a feeling of underrepresentation in mathematics classes starting in high school and dealing with norms and stereotypes regarding women's abilities attached to it. The next excerpt captures Dana's sensemaking of her participation in mathematical spaces, in which she reflected on her experience in high school and her ongoing negotiation of a sense of belonging in a mathematics community.

> ...I think we all know our capabilities and we all know that we deserve to be where we are in the mathematics department, but I also think, you know, we hear all these things and we hear these societal standards that *it's hard to just ignore those facts*, you know? And *it's hard to just push that aside*… I don't think as much in the university as it was in high school... in math classes where, at least for me, I would be one of the only few girls and sometimes the only girl in some of

[2] Italics are used to indicate expressions which we considered aligned with the emotional dimension of TMA.

my math classes. And I know that just *inhibited my ability to ask questions* and to, you know, *participate as much as all the other people* in my class because I just felt as if *I ...wasn't worthy enough* to ask these questions, but just *scared* and just that I don't know, I just *felt like always out of place* because of that. (Seminar, 2020)

Dana pointed out that gender-related factors hindered her participation in the mathematical discourse during high school mathematics classes. It is worth noting that Dana went to a small, private, and predominantly White high school in the southeastern United States. Her expression of discouragement to participate in such mathematical discourses focuses attention on the stereotypical positioning of women as dissociated from the mathematics discipline. Tensions surfaced during Dana's negotiation of her mathematical identity, specifically in the case of taking part in mathematical spaces. Her experiences in high school mathematics classes, such as being one of few females, made her question her belonging to mathematics. Moreover, Dana articulated her challenges with rejecting societal norms against women, which may have restricted her participation in mathematics classes despite her achievement.

Sunny also expressed her perception of competence in mathematics in emotionally laden ways, which raised questions regarding the definition of competence for students in the major. Considering Sunny had grades of over 90% (out of 100%) in her major, it was remarkable to detect conflicting patterns in her beliefs about confidence in mathematics during the seminar. The following excerpt highlights how Sunny attributed aspects of her self-concept to both the nature of mathematics being broad and unknown as discipline as well as to the social comparison.

Sunny:	*I don't think I'll ever be confident in mathematics*, to be honest, I feel like there's, it's just such a wide thing. And there's so much to learn. And there's so much that goes into it. I feel like I could like, have a Ph.D., and have like 10,000 awards and *still not feel confident* beyond it…
Interviewer:	And there is nothing to change that?
Sunny:	Oh, I don't think there is. I mean, in my case. I'm always going to feel like there's more to learn. And there are people who know more than I do.

Her negative statement about her confidence seemed to contrast with her previous view regarding improving her competence by expanding her course work. Though Sunny did not mention gendered discourses, her identity as a young woman in mathematics might have been shaped by the masculine nature of mathematics. Her reflections aligned with Dana's statement that being around smart people led to doubts about one's competence in mathematics. Finally, Manuel's reflection on the perceived competence in mathematics differed from those of Dana and Sunny regarding being surrounded by other smart people in the classroom. Manuel expressed that "I find more … drive and competitiveness about the idea that they know more than I do. I understand more" (Seminar, 2020). Evidently, Manuel distanced himself from his peers in his perception of competence in mathematics.

Discussion and Conclusion

We captured students' ideas of a competent mathematician and how those perceptions related to how they view themselves as competent in mathematics. We noticed that the nature of mathematics becoming an abstract and broad discipline had some favorable influences on how Dana and Sunny reflected on their competence. Talking about perceived competence in mathematics elicited some emotions for students such as being scared away from participation (Di Martino & Zan, 2011).

Moreover, we explored perceived competence of women and minoritized students in mathematics while illustrating the negative influences of gendered discourse on their identity development. We detected issues in the participatory nature of mathematical spaces, which seemed to exclude women from becoming full participants in it (Solomon, 2007). The emotional tensions we observed on competence could be an outcome of the patriarchal society's limited options for available identities as a woman and a mathematician (Solomon et al., 2011). While Sunny articulated her perceived competence in mathematics similar to that of Dana's, interestingly, Manuel, a male student, referred to becoming a confident mathematician as being subtly connected to one's personal qualities. Manuel possibly felt tasked to negotiate his masculine mathematical identity as a way to "prove" his abilities due to societal norms that value innate mathematical abilities (Mendick, 2006).

We also explored how women are challenged emotionally, which led to identity negotiations regarding their capabilities. While Dana believed that she was a capable mathematician (as evidenced by other survey data), gendered stereotyping (Leyva, 2017) may have created barriers for her towards participation in mathematics. In fact, Dana and Sunny were both emerging women mathematicians who possess superb mathematical ability and were extremely enthusiastic. Our analysis also uncovered alternative forms of becoming successful in the discipline of mathematics (Solomon, 2007) including legitimate peripheral participation (Lave & Wenger, 1992). In our investigation of perceived competence and emotions, we also gained insights on the need for belonging, particularly for women and racially/ethnically underrepresented groups participating in mathematical communities (Good et al., 2012). Belonging and perceived competence seem to be interrelated (Lahdenperä & Nieminen, 2020) and are specific to women and minoritized groups' participation and persistence in mathematics. Considering that small numbers of women and racially/ethnically minoritized students pursue mathematics related fields (NSF & NCSES, 2019), mathematics environments should foster inclusion for these populations (Pinheiro, 2021).

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Review of the methodology used in the research on the mathematics-related affective domain

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Keywords: Teacher attitudes, methods research, beliefs, literature review.

Introduction

Mathematics-related affective domain we understand using the framework of Hannula (2012) as having three different types of affect: cognitive (e.g. beliefs), motivational (e.g. values), and emotional (e.g. feelings). All these three aspects of mathematics teachers' affective domain are important to determine if they lead their students to an active and creative approach to mathematics learning (Boaler, 2015). Therefore, we decided to research key points during the university study, which influence affective domain of future mathematics teachers. In this poster we prepare a theoretical background for our research. We do this by reviewing methodology of related journal articles to identify methodological practices in the research on this topic. We hope that our paper will inspire discussion about the best methods to study mathematics-related affective domain and also possible collaboration with our team in this subject.

Literature Review

To obtain the journal articles suitable for our review we performed the search in the Web of Science Database on 18th July 2021. The search terms were 'mathematic*' combined with a Boolean operator AND with the terms 'teacher*' and with three words characterizing affective domain combined with a Boolean operator OR: 'attitude*', 'belief*', 'affect*'. The refined specifications of the search were that articles need to be open access and the publication years should be 2020 or 2021, to include just the newest and accessible papers. The papers were carefully studied, with the focus on the methodological parts and the finding were compared and discussed by both authors.

Results

Altogether, based on the search in the Web of Science database, 15 journal articles were included in this methodology review. One article was excluded because of focus on primary students instead of teachers. The list of the articles with complete references is because of the page limitation of the poster here: www.comae.sk/reference1.pdf. The most used research approach was quantitative (10 articles - for example Jeong & Gonzalez-Gomez, 2021, published in the journal *Mathematics*), followed by mixed approach (3 articles - for example Liebendorfer & Schukajlow, 2020, published in the journal *Educational Studies in Mathematics*). The least used approach was qualitative (2 articles - for example Lo, 2021, published in the journal *International Journal of Instruction*).

Following topics were researched in articles using **quantitative approach**: (1) self-efficacy - relationship with other constructs, impact on intention to leave profession, (2) (epistemological) beliefs about the nature of mathematics, (3) beliefs about teaching and learning mathematics, (4) development of attitude towards mathematics and (5) attitude to students' struggle when learning mathematics. In all 10 quantitative studies, a questionnaire was used as an instrument for data collection, containing various statements scaled by a Likert scale. Statistical methods were used to analyze the quantitative data, mostly analysis of variance and various correlation analysis methods.

Following topics were researched in articles using **mixed approach**: (1) perception of attitudes towards mathematics, (2) interest in mathematics and (3) attitude to student's struggle when learning mathematics. All three articles in this category used a questionnaire as an instrument to gather quantitative data. In these articles, additional qualitative data was gathered as a support either to ensure validity of the interpretations of the questionnaire data, or to uncover new themes (variables), which could be then analysed quantitatively. Open-ended items in questionnaires, written reflections and semi-structured interviews were used to gather such qualitative data.

Following topics were researched in articles using **qualitative approach**: (1) beliefs about mathematics and language and (2) beliefs about teaching mathematics. The only method used to gather qualitative data was a semi-structured interview. In both articles, the interviews were audio-recorded, transcribed and analysed either thematically or inductively.

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Inclusion and peer-collaboration in mathematics classrooms: The case of students' perspectives

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This study aims to highlight students' perceptions of team collaboration in mathematics classrooms. We focus on students with easy access to mathematics (GR1) and who admitted that they struggle with access to mathematics (GR2). Positioning theory is used to study participants' narratives about mathematics group work. Nine online one-hour interviews were conducted. The analysis of research data was done with grounded theory. The results indicate the following categories of participants' perspectives: Students' emotions while involved in peer-collaboration activities; Students' role in peer-collaboration activities; Participants' setting rules on how and with whom to collaborate; Participants' rationale for collaborating with their classmates. Similarities and differences between the two groups were identified as well as implications for promoting inclusion in peer-collaboration activities.

Keywords: Inclusion, peer-collaboration, mathematics, students' perspectives.

Introduction

Inclusion is built on the principle that all students should be valued for their exceptional abilities and included as important members of the school community (Civil, Hunter & Crespo, 2019). Recent studies argue that students' collaborative learning, where a small group of students work together to complete a mathematical task, could promote inclusion and equity in classrooms settings and affects students' identity as mathematics learners (César & Santos, 2006; Florian & Black-Hawkins, 2011; Kotsopoulos, 2014). Particularly, César & Santos (2006) argue that the role of collaboration in mathematics classrooms seems to facilitate students' positive attitudes towards learning and school achievement. Florian and Black-Hawkins (2011) suggest that teachers in inclusive practices should provide students with options to choose 'how, where, when, and with whom they learn,' in conditions that are designed to respond to their individual needs. On the other hand, students' experiences in collaborative learning activities may create roadblocks in the way they position themselves as mathematics learners (Kotsopoulos, 2014); while a 'distributed balance' between students' contributions in teamwork was identified by Dekker, Elshout-Mohr and Wood (2006). The latest findings could create inequitable learning opportunities and a sense of not feeling included in school activities for some students. Hence, the prerequisites for group work which are enacted in an inclusive manner still remain an open issue (Forslund Frykedal & Hammar Chiriac, 2018).

The current study focuses on how students are positioning themselves in peer-collaboration activities in mathematical classrooms. The participants are nine students, ranging from 11 to 16 years old, who are placed in two groups (GR1 and GR2). Participants in GR1 are five students who feel competent in mathematics and participants in GR2 are four students who feel that they struggle accessing mathematics. The two groups were distinguished in their own words. Additionally, some of them are certified by certain Greek institutions as gifted (GR1) or with mild disabilities in mathematics (GR2). We consider that all participants in this study are students with Special Needs in Mathematics (SEM)

(Scherer et al., 2019). The research questions are: RQ1: What characteristics do we identify in all participants' perceptions when talking about their experiences from peer-collaboration in mathematics classroom activities? RQ2: Which similarities and differences can we identify in the two group members' perceptions? By responding to the above research questions, we expect to identify if and how participants' experiences on peer-collaboration activities might affect their inclusion in mathematics activities. We view inclusion as equal opportunities for someone to participate and learn in mathematics classrooms (Esmonde, 2009; Civil, et al., 2019).

Theoretical framework

We view students' perceptions about their cooperative learning under the *positioning theory* perspective (Harré, 2005). Positioning involves the process of ongoing construction of the self through personal storylines that describe relations and interactions of people jointly engaged in an activity. This research also draws on related research in mathematics that considers the potentiality of positioning in studying students' conceptions of what doing and knowing mathematics entails (e.g., Evans, 2003). In particular, researchers argued that students' perceptions and emotions about their participation in any classroom activity may form their positional identities as mathematics learners. Esmonde (2009) argued that students' positioning and identity-related processes are just as central to their mathematical development as their content learning.

Students' perceptions about peer-collaboration in mathematics classrooms

Since students' sense of belonging in mathematical activities seems to be critical for forming their positional identities as mathematics learners, researchers tried to understand and compare the experiences of different students in peer-collaboration activities. In Mulryan's (1994) study perceptions of high mathematical achievers with low achievers were compared. She concluded that both groups focus on the social dimension of the collaboration experience while high achievers emerged as having a more complex understanding of cooperative small-group work than did low achievers. Webel (2010) when comparing the perspectives of two high school math students, showed that both students see collaboration as knowledge transfer from capable students to struggling peers. Kanevsky, Owen and Marghelis (2021) studied mixed ability students' considerations that influence their preferences for working alone or in collaborative settings. The results indicate that high ability students were more concerned about content characteristics (e.g., efficiency and quality of their work) while others were more concerned about social characteristics (e.g., potentiality for fun) in these type of school activities. Kotsopoulos' (2014) study illiustrated how one student (Mitchell) experienced his peers' rejection of his contributions when solving a mathematical task. Mitchell reacted to how his peers were positioning him by taking a 'silenct' stance. In this way, he created a wall between his mathematical thinking and his involvement in the peer-collaboration activity.

The above outcomes highlight the critical role of different students' perceptions about peer collaboration activities. In this study we attempt to highlight how two groups of students, who view in different ways their involvement in mathematical activities (struggle to gain access and have easy access) position themselves in peer-collaboration activities and how these positions might affect their inclusion in mathematics classroom activities.

Methodology

This study is a part of a PhD program which aims to highlight issues of inclusion in Greek mathematics classrooms. Even though in Greece, according to law 2817/2000, inclusive education is a legitimized school obligation to facilitate all students' educational needs for many Greek teachers, this is still a challenging issue when trying to accommodate different students' needs in mainstream classrooms (Avramidis & Kalyva, 2007).

Participants

The participants were nine students who voluntarily participated in this study. The nine students were placed into two groups. The first group, which we call GR1_stX (Group 1, student X) includes five students who felt that they are competent in mathematics, and they feel that they understand math quicker than their classmates. Three of them are certified as gifted by the Hellenic MENSA association. The second group, for which we use the abbreviation GR2_stX (Group 2, student Y), consists of four students who feel that they struggle to gain access to mathematics, and they think that their classmates are better than them. Three of the students are certified with learning disabilities from training and counseling centers of the Hellenic Ministry of Education. The rest of the students were placed in groups we formed based on the way the manner in which they talk about their access to mathematics (in their own words). The students varied from fifth to ninth grade (See Table 1).

Table 1: Participants' profile

Name	Sex	Age	Class/Grade	Certified or Words
GR1_st1	Female	15	9th	In their own Words
GR1_st2	Male	15	9th	Certified
GR1_st3	Female	16	10th	Certified
GR1_st4	Female	11	5th	Certified
GR1_st5	Male	12	6th	In their own Words
GR2_st1	Female	13	7th	Certified
GR2_st2	Male	13	7th	Certified
GR2_st3	Female	14	8th	In their own Words
GR2_st4	Male	14	8th	Certified

The researcher (first author) recruited the participants from three different institutions where she was working as a math tutor. Those institutions are the Hellenic Mensa Institution (certified students in GR1), a private study center (certified students in GR2), and a music school, and encompassed students that admitted, in their own words, that they have easy access or a struggle to access mathematics.

Research Data

Nine online interviews were conducted. The interviews lasted approximately one hour and were recorded after the researcher obtained permission from students' parents. In the beginning it was made clear to the participants that we are interested in identifying instances that they feel included or excluded in mathematics classroom activities. We used open-ended questions to build upon and explore their responses, so students themselves could reconstruct their experience within the topic under study. The interview consisted of three main areas of students' experiences: a) about specific mathematical lessons and tasks e.g., "*on which mathematical task did this happen*?" or "*give me more details about this lesson*"; b) about instances of mathematics teaching (e.g., "*Describe a case in which you felt you were actively involved in the lesson and how often does this happen?" "Are there cases where you think that you could be involved but you didn't have the chance?'; "What is your opinion of an ideal math lesson?"*; c) about peer-collaboration activities (e.g., *"Do you prefer to work alone or with your classmates?"; "What kind of working groups you want to be part of and why?" "If you were a teacher, in what ways could you form a group?"*). In this article, we focus on data in regard to the last area. Through students' answers, more close questions emerged about their perceptions or/and preferences in learning conditions.

Data analysis

Giving a voice to students is the main concern because discourse is socially constructed, and words have meaning when used by participants. The analysis of research data was done with the help of grounded theory. Specifically, our study lines up with Charmaz's (2006) approach in which feelings and views of the participants are emphasized and make sense depending on readers' and researchers' perspectives, practices, and purposes. The analysis of the data was carried out in four steps. Firstly, students' interviews were transcribed. In the second step of the analysis, we distinguished parts of the participants' interviews that refer to peer collaboration. Thirdly, we tried to focus on the main issues that characterize participants' positions about collaborating in learning activities. In this step, categories of participants' positions emerged. These categories were under negotiation among researchers until the final categories were established. In this way, we tried to respond to the first RQ. In the last step, we tried to focus on differences in the two groups' perceptions about peer collaboration and respond to the second RQ of our study.

Results

Participants' perspectives about peer-collaboration

Through analyzing students' answers, we came up with the following categories of participants' perspectives: *Students' emotions while involved in peer-collaboration activities*; *Students' role in peer-collaboration activities; Participants' setting rules on how and with whom to collaborate; Participants' rationale for collaborating with their classmates.*

We present each category and characteristic students' responses:

a) ***Students' emotions when involved in peer-collaboration activities.***

Most students expressed positive emotions like 'happiness' or 'satisfaction' while collaborating with their peers. These feelings are related mostly with a positive outcome from this collaboration like 'helping the others understand.'

Some characteristic examples of both groups' responses are: "When they understand, I feel happy" (GR1_st1); "It's a good feeling, it is a satisfaction that I know that and I can help others" (GR1_st2); "I like math and I like to show what I know and help others understand" (GR1_st3);"I like to explain math" (GR1_st5); "I feel enjoyment and satisfaction when they have understood them" (GR1_st4); "I feel glad that I helped my classmate"(GR2_st1).

Only two students expressed negative feelings like 'dislike', 'shame', or 'anxiousness': "I feel a little uncomfortable" (GR2_st1); "I don't want to…I feel ashamed. I'm a little anxious that the children will say something like 'look, you did it wrong' and stuff like that… this is what happened when I was in elementary school" (GR2_st2)." Student GR2_st4 expressed an insider in the community feeling while participating in peer-collaboration activity "I felt that my classmates like me, otherwise they wouldn't tell me anything…they wouldn't help me… this means that I have an active role at school."

b) Students' perceptions of their role in peer-collaboration

We identified two participants' perceptions of their role during collaboration activities: as *helpers* (e.g., 'I explain' 'I share ideas') or *help receivers* (e.g., 'I am asking for their help').

Students that position themselves as helpers said: "Usually, my classmates ask for my help (laughs) […] I try to explain it to them […]" (GR1_st1); "My classmates ask me about their homework and how we might solve this exercise" (GR1_st3); "Many of them who need help in math come to me" (GR1_st5); "If they ask for my help, I'll try to explain to them what they need, the best way I can, it doesn't happen a lot, It only happened once […]" (GR2_st4); "I just give them a little push when they have fallen behind" (GR1_st4). Other students position themselves as help receivers. Some characteristic examples are the following: "I am the one who asks for help a lot" (GR2_st1); "Yesterday I asked for their help since I couldn't figure out how exactly I would define x and y'' (GR2_st3).

c) Participants setting rules on how and with whom to collaborate

Some of the participants would prefer to work with others under certain conditions or rules. Students GR1_st1 and GR1_st5 expressed the *No competition* rule especially when collaborating in mixed-ability groups. Characteristic responses: "I wouldn't prefer it to be a competition, it just makes me feel nervous. […] because I don't like to fail." (GR1_st1); "Not to be competitive […] if you fail you get really upset (GR1_st5). At the same time, the above participants' *no competition rule* hides their perception that a mixed-ability group is likely not to have a successful outcome.

Students GR1_st1 and GR1_st2 expressed the *collaborating with those who really want to work* rule regardless of his/her mathematical competence.

GR1_st1: I wouldn't like to be in the same group with others who don't pay attention. I wouldn't mind if students who have difficulty were in my team, but not those who don't pay attention and want to copy my answers… in that case I just get angry because I would probably end up doing the whole project myself as it happens many times in other courses' teamwork.

GR1_st2: I don't see it only with students who have the ability to learn easily, but with the ones who have an interest to work. Even if they have never studied math, but they want to contribute to the team, that's ok for me.

Student GR2_st2 expressed the *collaborating only with friends* rule: “I like sharing ideas with each other […] but only with my friends.” No participant expressed rules for collaborating with students sharing the same math ability with high achievers in mathematics.

d) Participants' rationale for collaborating with their classmates

When students were asked if they prefer to work individually or in groups, they all tend to choose to work in groups and especially in mixed-ability groups. Here are some examples of their rationale for collaborating with their peers, which are linked with either *collective gains* (‘team progress’ or ‘team building’ or ‘socializing’) or *personal gains*. Analyzing students’ words, two dimensions came up: *collective with specific direction’s gains,* in which there are specific types of students that benefit from the others, and *general collective gains* in which each student can benefit from one another. On the other hand, personal gains seem to have three reasons which are: *making repetitions*, *listening to a new idea*, and *not losing teaching time* with the teacher’s explanation.

Students GR1_st3 and GR2_st3 seem to refer to collective gains when the students might benefit from each other, but this has a concrete direction i.e., the students who have access in mathematics to help the others who struggle to have access, or they are afraid to be unsheltered. Characteristic responses are: “Those of them who are good at math and understand them, should help the ones they aren’t’’ (GR1_st3); “When it is difficult for someone to understand, another one from the same group could guide the rest of them […] those who had questions and wouldn’t have the courage to ask” (GR2_st3). Furthermore, GR2_st4 “sees” that the whole team might help one member: “To work in groups, when a member has difficulty incomprehension, the team could help”.

GR1_st2 and GR2_st2 refer to collective gains when the students might benefit from each other but without concrete direction: “For me, it’s not the ability that matters, it has to do with someone’s interest to work. Even for someone who has never studied math, the willingness to help will somehow contribute to the team” (GR1_st2); “Not only the high achiever has some nice ideas but also the underachiever” (GR2_st2). The same students highlight the reasons ‘team progress’ and ‘helping each other’: “We try to make progress altogether” (GR1_st2); “To help one another” (GR2_st1).

GR1_st3 except the collective gains refers to socializing with peers. “In that way, we are able to talk to each other. Because there are times we don’t even talk in the classroom. I might haven’t talked to my classmates at all. That makes us feel gratified too.”

Three students prefer working in mixed ability groups but for *personal reasons*. GR1_st4 likes to help others because this is the only way (as she feels) she can move on and learn new things. “I was pleased that everybody understood because when they don’t, the whole group falls behind [in classroom teaching].*”* The other two participants take the opportunity to improve their mathematical knowledge *“*I do repetitions [on the classroom material] and I definitely understand it better when I am explaining something to someone else” (GR1_st2) or “actually I prefer working in groups whereas I can listen to others’ point of view since there might be another solution I haven’t thought about” (GR2_st1).

Comparing two group members’ perceptions

GR1’s participants feel positive emotions when they work with their classmates for both collective and personal reasons. They all perceive themselves as helpers and some of them expressed specific rules and conditions for participating in such activities. GR2's students feel positive too when they collaborate with their classmates, but they mostly perceive themselves as help receivers. They also prefer to work with mixed ability groups, like GR1, for collective gaining or personal reasons. They didn't set criteria and rules for participating in such activities except for one student who wants to work only with his friends.

Conclusion

This study aims to highlight students' perceptions of team collaboration. We focus on students who feel that they have easy access to mathematics (GR1) and students who feel that they struggle to gain access to mathematics (GR2). Comparing the two groups' perceptions about peer collaboration, we came upon the following observations: All participants prefer to work in mixed-ability groups in mathematic class and they highlight the inclusive dimension since no one considers math skills problematic in collaborating with their peers unless there is a competition among mixed ability groups. The last was identified in the case with students who have easy access to mathematics. Some participants expressed their options to choose their classmates in peer-collaboration activities (e.g., working with friends or students willing to contribute to the discussion). Students who feel that they have easy access to mathematics see themselves as helpers or knowledge-givers (Webel, 2010).

Viewing students' perceptions from an inclusion perspective we noticed that the participants who admitted that they have easy access to mathematics seem to implicitly position themselves as *directors of the peer-collaboration activity* since they consider themselves as the helpers, they set certain criteria with whom to collaborate while one of them is taking up the role of the teacher to move on with the lesson. Conversely, participants who admitted that they struggle to gain access to mathematics seem implicitly to position themselves as the *followers of the peer-collaboration activity* since they see themselves as the help receivers. The above two groups' perceptions may cause a 'distributed balance' between students' contributions in teamwork (Dekker, Elshout-Mohr & Wood, 2006) thus it challenges the fair distribution of all group members' opportunities to participate and learn in teamwork, which is a prerequisite for equity and inclusion (Esmonde, 2009). The above outcomes indicate the challenges mathematics teachers face when trying to include students who view differently their involvement in mathematical peer-collaboration activities. Managing equity issues among students' contributions in mathematics classrooms is a complex issue (Esmonde, 2009) but it is critical to establish meetings amongst and between students' differences (Skovsmose, 2019).

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Finding my way: a search for teacher identity

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Ashley, a lower secondary mathematics teacher has a personal goal for professional learning: to find a teaching method that feels her own. She participates in a 1-year problem-solving intervention that aims to increase pupil thinking and engagement in mathematics classrooms. The purpose of this paper is to explore the opportunities that Ashley has, to develop her mathematics-related teacher identity within the problem-solving project. The results show that while the problem-solving intervention creates rapid change into Ashley's classroom practices, it also succeeds to influence positively the development of her mathematics-related teacher identity.

Keywords: Professional development, teacher identity, tensions, mathematics.

Introduction

The descending trend of Finnish pupils' mathematics performance and interest to study mathematical sciences on a tertiary level has brought about national discussion about mathematics teaching in schools. While policymakers have slowly recognized the need to take actions to stop the alarming development, on a local level, mathematics teachers have witnessed the descending trend of pupils' performance and problem-solving skills for over a decade (cf. Vettenranta et al., 2016). The descending trend in mathematics performance, coupled with Finnish pupils' low emotions, attitudes and self-efficacy beliefs in mathematics (Metsämuuronen & Tuohilampi, 2014) challenge teachers in their everyday practices.

The needed changes in classrooms do not happen without motivated teachers. Generally, teachers who agree to participate in a professional development project can be considered as motivated to develop their teaching. However, previous studies indicate that teachers tend to receive new information as well as accept and adapt it into practice at different rate (see e.g. Laine, Näveri, Ahtee, Pehkonen & Hannula, 2018). This might be due to the different needs and goals teachers have for professional learning that also affect their openness to learn new knowledge and to adapt it into their classroom practices (Liljedahl, 2014).

This paper is about one teacher, Ashley, and her search for a mathematics-related teacher identity within a professional development project. Ashley takes part in a radical problem-solving intervention in lower secondary mathematics classrooms with four of her colleagues. All five teachers report to be highly motivated to try new methods in their mathematics classrooms (Viitala, accepted). However, they all have somewhat different goals for their professional learning within the project. While these goals in most parts are connected to enhancing pupil engagement in mathematics, one of the teachers, Ashley, emphasises also a more personal need: finding a teaching method that would become a natural part of herself (Viitala, accepted).

Ashley's personal need for the professional development project motivated me to look closer into Ashley's teacher identity and mathematics teaching within the project. I was curious of the reasons behind her feeling of incompleteness as a mathematics teacher, and how the problem-solving intervention succeeds to answer her needs for personal development. The feeling of incompleteness as a mathematics teacher connects affect to Ashley's mathematics-related teacher identity (see Lutovac & Kaasila, 2018) that in this paper is studied through view of mathematics (Op't Eynde et al., 2002). Furthermore, the feeling of incompleteness is a strong motivational aspect for Ashley to participate in the professional development project. This emotional strain can positively impact the transition from professional learning into the mathematics classrooms (cf. Andrá et al., 2019).

The purpose of this paper is to explore Ashley's mathematics-related teacher identity through her view of mathematics and changes in her teacher identity, as well as through the link between identity and teaching practices (see research themes in Lutovac & Kaasila, 2018). These issues are studied through affective lenses, that have seldom been done in the case of mathematics teachers, with an individual emphasis, a view that has often been neglected (Lutovac & Kaasila, 2018).

The paper aims to answer the following research questions: (1) *How does Ashley characterise the growth of her teacher identity over time?* (2) *What kind of tensions can be found between Ashley's experienced teacher identity and her view of good mathematics teaching?* (3) *How does the problem-solving intervention answer to Ashley's needs for professional development?*

Mathematics-related teacher identity

Many review papers on identity emphasise the importance of studying teacher identity in connection to professional learning but, at the same time, highlight the vast variety of views on identity and the lack of clarity in its definitions (see e.g. Lutovac & Kaasila, 2018). This realisation stresses the importance of situating the study within the field.

Following Lutovac and Kaasila (2018), in this study, the term mathematics-related teacher identity is used to emphasise the context (mathematics) in which teacher identity is discussed. Teacher identity is understood as a dynamic construct that changes over time. It is influenced by several factors, such as prior experiences, beliefs, attitudes, emotions, work environment and colleagues, and it develops through social interactions in different contexts.

Although identity is seen as a social construct that is developed in, and influenced by different communities of practice (Wenger, 1998), identity in this paper is studied from an individual perspective (cf. Lutovac & Kaasila, 2018). This means that the discussion about Ashley's mathematics-related teacher identity is mostly guided by Ashley's thoughts and feelings about who she is as a mathematics teacher, how she describes her professional development over time, and how her affective trait (Hannula, 2012) influences her mathematics-related teacher identity.

Teachers' mathematics-related teacher identity is also influenced by tensions between experienced and ideal mathematics teaching, and between view of oneself as a mathematics teacher and view of a good mathematics teacher (cf. Beijaard, Meijer & Verloop, 2004). In this study, tensions are considered as "pairs of contrasting forces that pull a teacher in two different directions" (Andrá et al., 2019, s. 3). These tensions are used to recognise the (conscious or unconscious) needs for the professional learning within the project, and together with Ashley's experiences from the project, evaluate the opportunities that the problem-solving intervention offers for professional development.

The problem-solving intervention

Together with four of her colleagues, Ashley takes part in a professional development project that, by integrating collaborative problem solving into everyday mathematics activities, aims at increasing pupil thinking and engagement in mathematics classrooms. To reach sustainable change in these mathematics classrooms, the traditional teaching and learning practices as well as the physical learning environments are challenged from several fronts, and the mathematics classrooms are treated as systems (Stigler & Hiebert, 1999).

The project utilised a teaching method called building thinking classrooms (Liljedahl, 2020). The teachers implement the new teaching method in their everyday practices in mathematics classrooms. The core of the method constitutes of three activities: Every lesson starts with a problem, the pupils work in visibly random groups, and they work standing on vertical non-permanent surfaces (Liljedahl, 2020). These core activities also resonate with the Finnish curriculum, which emphasises for instance collaborative learning and physical activities as opposed to sedentary lifestyle (see Finnish National Board of Education, FNBE, 2016).

The teachers in the Finnish problem-solving project implement these three core activities in their lower secondary mathematics classrooms from the beginning of the school year. The rest of the 14 teaching practices they implement into their teaching gradually throughout the school year. For this, they receive intensive support through workshops and informal meetings with the researcher.

Methods

Participant

The teacher, Ashley, is a mathematics, chemistry, and physics teacher at a lower secondary school in Finland. She has 3 years of teaching experience after graduating from a university, where she majored in mathematics. Ashley takes part in the project with two mathematics groups, one group from 8th grade and one group from 9th grade (13-16 years old pupils).

Data collection

The data were collected during the first semester of the intervention through two interviews, classroom observations and informal discussions with Ashley between the lessons. The interviews were audio-recorded, while other data supports on field notes.

To answer the first two research questions about Ashley’s teacher identity and view of mathematics, the data were collected from the first interview that took place in the beginning of the intervention. The interview was about Ashley’s educational background, previous professional development projects, her prior and current teaching, insights into her possible future teaching, and about her hopes and needs for the intervention (cf. view of mathematics in Op’t Eynde et al., 2002). She was also asked to explain how certain she is about her knowledge and skills in mathematics.

The data for the third research question about Ashley’s professional development were collected through the second interview about two months after the start of the project, classroom observations and informal discussions over the time of the first intervention semester. These data focused on the experiences from using the new teaching method in Ashley’s mathematics classrooms.

Alongside with implementing the new teaching method into her teaching, Ashley took part in workshops that were built around the building thinking classroom–method. She was introduced to the main ideas and prior results of utilising the method in mathematics classrooms and given hints on how to deal with challenging classroom situations that might occur during the intervention. It is important to recognise the possible influence of the workshops on Ashley's professional learning.

Data analysis

The data were analysed through data-driven content analysis. The first part of the analysis focused on the first research question about the growth of Ashley's teacher identity. The analysis followed the professional developmental path that Ashley describes in the interviews from university studies to the beginning of the problem-solving intervention. Ashley's narrative made it possible study the affective factors that are central for her (Di Martino & Zan, 2015). Moreover, the issues that she highlighted from her past were interpreted as instances relevant to the development of her mathematics-related teacher identity (Lutovac & Kaasila, 2018).

To answer the second research question about tensions, Ashley's mathematics-related teacher identity was contrasted with the view of herself as a mathematics teacher and how she experiences her own mathematics teaching (cf. Op't Eynde et al., 2002). Tensions between the experienced and ideal mathematics teaching, as well as experienced teacher self and ideal mathematics teacher, were interpreted as (perhaps unconscious) needs for professional learning (cf. Andrá et al., 2019).

To answer the third research question, Ashley's expressed goals for professional learning together with the needs arising from tensions are discussed in the light of the opportunities that the problem-solving intervention offers for professional learning, and what changes in her mathematics-related teacher identity Ashley experiences during the first semester of the intervention.

Results

Developing mathematics-related teacher identity over time

Ashley is a young mathematics teacher with a strong need to find a teaching style that feels her own. She has been pursuing towards this goal since being a mathematics teacher student at university. While the university failed to give her enough opportunities to rehearse her mathematics teaching in practice (she reports only having two practice lessons in mathematics), and thus help her to develop a strong mathematics-related teacher identity, she has been testing different teaching methods systematically in her own secondary mathematics classrooms.

By the time of the problem-solving intervention, Ashley has been teaching in this specific lower secondary school for two years. During this time, she has pursued towards more pupil-led teaching practices. She has tried different methods, such as task cards, to support pupil autonomy in learning. She has been guiding pupils to take more responsibility of their own learning and to study mathematics at their own pace. The pursued changes resonate well with the current national curriculum (see FNBE, 2016). She has also tried to differentiate learning especially upwards to create challenges also to high achievers. In Finnish lower secondary school, extensive learning support is given for low-achievers by law, but whether and how the high-achievers receive suitable challenges is dependent on the individual mathematics teachers.

After the first year of working individually towards developing her teaching, Ashely started to collaborate more with two other teachers in her school. This community of practice (cf. Wenger, 1998) has become an important resource for her. The teachers share ideas as well as plan and develop their mathematics teaching systematically together. On the classroom level, their goal has been to engage pupils more in learning mathematics. The three teachers also participate in the reported problem-solving intervention together.

Despite the many professional development projects, implemented individually or with colleagues, and even though she has found methods that engage pupils more in mathematics lessons, Ashley feels that these attempts have not been successful in the sense of finding a teaching method that ‘feels her own’ (her own words).

Tensions in mathematics-related teacher identity

There are many tensions in Ashley’s experienced and ideal mathematics teaching. She is aware of these tensions when she talks about them but, at the same time, it seems that despite the many attempts she has not been able to find tools to answer these needs in her mathematics teaching.

The first tension is between the teacher-led and pupil-led teaching methods. Ashley pursues towards pupil-led learning where pupils take a bigger responsibility in their own learning. In some groups, using the task cards is a step in this direction. However, she finds herself talking a lot in the lessons and explaining mathematical content to the pupils instead of guiding them towards more self-regulated learning.

The second tension is between the processes and the outcomes of solving tasks in mathematics lessons. Ashley explains how she tries to emphasise the solution processes over the correct answers. However, what she experiences in classrooms is that pupils concentrate heavily on the correct answer. Even a small miscalculation can lead to the pupil thinking that the whole task was incorrectly solved.

The third tension is about the number and quality of tasks solved in mathematics lessons. At the university, she realised that mathematics is more about the thinking processes and having a small number of good tasks, rather than solving a big number of routine tasks. As a mathematics teacher, however, she feels that she has moved back to teaching through calculations, as she was taught self in school. Even if all pupils have the freedom to select tasks, the emphasis is on calculating and not on thinking.

Ashley’s view of herself as a mathematics teacher is very close to her view of a good mathematics teacher. She explains that, similarly as a good mathematics teacher, she is easily approachable, she is genuinely interested in pupils’ learning, she is fair, and treats all pupils equally. Additionally, she likes mathematics, but is highly self-critical towards her teaching. The researcher’s notes from mathematics lessons and after-lesson discussions support these self-reflections from the interviews.

In the interview, Ashley emphasises that a good teacher also masters the content, adapts to different situations, is up to date, and can revise her own teaching. These are all elements that Ashley can be related to through her self-efficacy beliefs of mastering the mathematical content to the purposeful development of her own teaching. Hence, even though not explicitly said, my interpretation is that Ashley sees herself as a good mathematics teacher.

Learning from the problem-solving project

When asked explicitly, Ashley's goal for the problem-solving project is (free translation):

> To find a way to teach mathematics that motivates pupils, so that it would be nice to come to the classes, it would be nice to be there, I would enjoy being there, and that we get things done. And that everyone is included, everyone gets some kind of support even if not originating from the teacher, and everyone gets experiences of success.

While the explicit goal for the project is more on the affective level, similarly as the need to find a teaching method that feels a natural part of herself, the tensions stemming from the interview data highlight more concrete needs; Ashley wants her teaching to be more pupil-centred, tasks to concentrate more on thinking rather than calculations as well as to the processes rather than the answers, and learning to be more qualitative instead of rushing through a long list of tasks.

The goals reflecting the tensions in Ashley's mathematics teaching are well in line with the building thinking classrooms–method (see Liljedahl, 2020). Already after two months of implementing the building thinking classrooms–method into her mathematics classrooms, Ashley starts every lesson with a problem (pupil-centred learning that fosters thinking), divides pupils visibly into random groups using playing cards (social gains, among other things), and have pupils work standing on vertical non-permanent surfaces (blackboard, whiteboard tape and windows). She has also paid attention to answering pupils' questions, creating tasks that support group discussion and the processes of learning, how to level learning in classrooms, and how to organise the physical classroom environment to support the desired classroom activities.

The changes in Ashley's classroom practices have been rapid. Only after two months, Ashley explains that she has found the basic idea behind the building thinking classrooms–method, and she has been able to adapt the method into a natural part of her teaching. The building thinking classrooms–method has answered to her needs, especially those viewed through tensions. She also feels that the intervention has had a positive impact on pupils' learning and on classroom climate. Whether the change is sustained over time, we will see after the problem-solving intervention is over (delayed interview and classroom observations).

Discussion

The purpose of the paper was to explore Ashley's mathematics-related teacher identity through view of mathematics and changes in her teacher identity, as well as through the link between identity and teaching practices (see Lutovac & Kaasila, 2018). These issues were studied through affective lenses with an individual emphasis.

The first research question drew attention to the development of Ashley's teacher identity. This question revealed a severe deficiency in Ashley's teacher education that led her to systematically search for teaching methods that would fit to her view of good teaching. The question also revealed the importance of the social aspect in creating a mathematics-related teacher identity. When Ashley talked about her previous professional development projects, she highlighted the importance of other mathematics teachers at her school. This community of practice (Wenger, 1998) has been a great resource for her in developing teaching and mathematics-related teacher identity. As a result, Ashley has high self-efficacy beliefs and confidence towards classroom inquiries.

The second research question focused on the tensions between Ashley's experienced teacher identity and her view of good mathematics teaching. There were no tensions recognised between Ashley's view of herself as a mathematics teacher and a description of a good mathematics teacher. This indicates a quite well-established teacher identity. Ashley was very aware of her strengths and weaknesses in mathematics, and she recognised many tension in her mathematics teaching (cf. Pillen, Beijaard & Brok, 2013). However, what was not recognised was the tension between the verbalised goals for the intervention and the tensions in Ashley's mathematics teaching. This realisation takes us to the third research question about Ashley's needs for professional development.

While novice teachers traditionally are willing to rethink their teaching practices (Liljedahl, 2014), Ashley was ready to reject bigger parts of her teaching. Ashley's needs for professional learning were connected to the teaching and learning activities rather than to mathematical content (Lutovac & Kaasila, 2018). As a result, the problem-solving intervention seemed to have a very positive impact on Ashley's identity building and professional development. After a short period of implementing the new teaching method into her mathematics teaching, Ashley started to see changes in the classroom practices (addressing tensions) and in the emotional atmosphere in the classroom (addressing goals), resulting in a method that 'feels her own'. While these results are encouraging for the success of the intervention, they also confirm some previous research results and raise questions.

First, it seems that the tensions Ashley recognised were at least partly solved. This confirms prior research about the nature of tensions, that is, previously incompatible tensions can be solved (Andrá et al., 2019). Second, teacher identity is a social construct (Wenger, 1998) that is influenced not only by colleagues but also by pupils in the classroom. The positive impact that the intervention had on Ashley's teacher identity was highly influenced by the pupils' actions and reactions in the classroom.

The questions that raised from the study are connected to the interrelation between tensions (needs) and goals (wants) in novice teachers' identity building and professional development. Ashley reported being successful in terms of pupils' learning and motivation also in her previous interventions. What seems to be different this time is, that the intervention also addressed the tensions in her teaching positively. Indeed, what seemed to be a challenge for Ashley was to identify the tensions as concrete goals for her professional development. So, are solving tensions more powerful tool for professional development than fulfilling teacher's goals? There is a need for further studies on the interrelation between tensions and goals in professional development projects that aim to improve classroom practices and teacher identity.

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Cooperative learning in game theory activities

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The purpose of our paper is to analyse the effectiveness of group work and the role of the emotions in game theory activities. Such a setting, unfamiliar to most of the students, allows to build activities with more than one solution and more than one solving strategy. In particular, the activity concerning a cooperative game is reported and, following Kagel and Cooper's studies, it is highlighted how students involving in groups activities have a more strategic and successful approach than those who tackle them individually. One of the aims of this paper is to study the change of the vision of the mathematics after these activities, using the lenses of the Di Martino and Zan framework. Furthermore, this research looks at the metacognitive activities that are stimulated during the resolution and how they lead to the construction of shared knowledge.

Keywords: Cooperative learning, metacognition, game theory.

Introduction and theoretical framework

Problem solving and argumentation are essential skills, and they are one of the objectives of many educational systems (i.e., OECD PISA, 2021). These processes can be supported by collective work, that if it is done correctly, it triggers the process of collective metacognition that leads to the construction of shared knowledge. This study was carried out on game theory activities because this context allows us to analyse the students' ability to apply the knowledge acquired during their studies in unfamiliar situations (Antonini, 2019). This choice is justified by the desire to create a conducive environment to the development of the problem solving and the argumentation processes (Cramer, 2014).

Schoenfeld’s research shows that in order to solve a mathematical task correctly, there is a need to apply control processes. These processes involve understanding the text of the task, planning a strategy, controlling the situation, and managing one’s resources. Schoenfeld highlights the importance to be aware of one’s resources to be able to manage them. Furthermore, he analyses causes of failure or successful in activities of problem solving. The central elements are the beliefs and the individual’s perception of themselves. The beliefs depend on previous experiences with mathematics, and they influence the individual’s ability to use his or her knowledges (Schoenfeld, 1983). Beliefs, emotions, and attitude are very important elements in approaching mathematics; in fact, they are also essential in decision-making process. Emotional experiences are given by the combination of a cognitive and a psychological insight: the interruption of actions or a difference between facts and expectations can lead to an arousal (a general state of activation of the nervous system in response to internal or external stimuli). The emotional experience of the individual is given by the arousal and the formative evaluation of the experience. It is therefore clear that it is not the experience itself that arouses emotions, but the subject's interpretation of the situation, which depends on the individual's beliefs.

Di Martino and Zan highlight the importance of emotional component of the beliefs, in fact, the same belief can arouse different emotions in different individuals. They underline the need to use tools able to investigate the structure of the beliefs' system and to bring out the link between cognitive and emotional component. To analyse the data that emerged from their research, a model consisting of three dimensions was used (Di Martino & Zan, 2011): emotional disposition towards mathematics, perception of one's abilities and vision of mathematics.

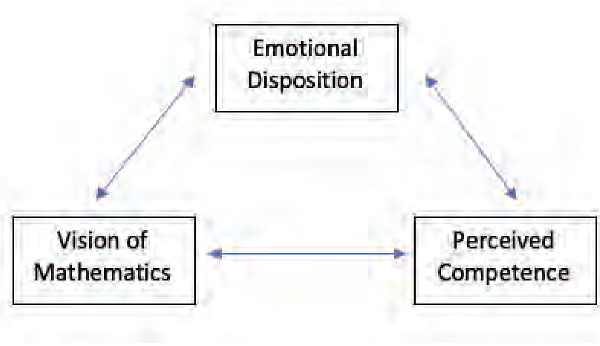

Figure 1: Di Martino-Zan three dimensional model for attitude (Di Martino & Zan, 2011)

It is possible to focus on the link between beliefs and emotions, in particular, negative emotional disposition, due to a negative approach with mathematics. A negative emotional disposition may be due to many characteristics that students assign to mathematics. This implies that there is a link between emotional disposition and vision of mathematics. The perception of one's abilities in mathematics influences the emotional disposition: in fact, the idea of successful in mathematics affects emotional disposition and thus the perception of one's competences.

There are many conceptions about idea of successful:

- successful identified with perception of the knowledge of rules and their correct application;
- successful identified with perceived knowledge of the meaning of the rules and their connection;
- successful identified with scholar successful.

The failure is often due to the link between negative emotional disposition and low perceived competence. Weiner classified the causes of the failure into (Weiner, 1986):

- local inside/local external: inside local causes depend on the subject, while external local causes depend on exterior features;
- stable/instable: whether or not it is possible to change them over time;
- controllable/uncontrollable: whether depend on the beliefs of subject.

Negative emotional disposition is a cause of failure in mathematics for some students. Low perceived competence and negative emotional disposition may be two independent dimensions: some students have a high perceived competence, but a negative emotional disposition and vice versa.

The development of activities in the game theory makes it possible to highlight how a collective performance promotes the activation of metacognitive processes to a greater extent. According to Kagel and Cooper (2005), those who tackle tasks in groups act more strategically than those who carry out them individually. Kagel and Cooper (2005) state that communication between individuals is a possible explanation for the more strategic approach taken by groups. Therefore, through confrontation with others, more ways to resolution were explored.

Cooperative learning is an approach in which small groups are created with students of different ability levels. The members of each group work together to achieve a common goal. The positive aspects of cooperative learning are group goals or positive interdependence, in which students must work together to achieve group success, and individual accountability. In other words, we mean the responsibility of everyone for the success of the group, which involves the motivation of each student to help to achieve the success of the whole group (Slavin, 1990). Efficient group work maximises the learning of each student.

Furthermore, organisational, cognitive, and metacognitive factors must be present for a group to successfully tackle problem solving activities. (Chalmers, 2009). Tuckman and Jensen's model of organisational factors consists of five phases that should be developed in the group activity: forming, storming, norming, performing and adjourning (Tuckman & Jensen, 1977). The first phase involves the setting up of the group work; in the second phase there is a comparison between the group members and the third phase is reached when a meeting point is found between the various ideas proposed. In the fourth phase the group members work together and in the last phase the work is reviewed. On a cognitive level, students need to develop shared knowledge in order to successfully complete the group work. Metacognition is defined as the student's awareness of her or his learning process; in the case of collective metacognition, each participant must be aware of his or her own and other group members mental processes through discussion and comparison. Collective metacognition requires that strategies of orientation, planning, monitoring, evaluation, reflection and elaboration are developed. Orientation is carried out before the problem occurs. Planning, monitoring and evaluation take place during the resolution of the task, while reflection and elaboration take place at the end of the task. These actions allow students to structure a solution strategy, monitor the processes involved, evaluate and interpret the results obtained and express the solution (Van der Stel et al., 2010). Comparison, along with the presentation of different points of view and questions from the group members help the construction of shared knowledge, allowing any individual difficulties to be identified and overcome. Each group member has their own solution strategy before determining the joint one, but previous studies have shown that groups perform better than they would have done individually (Frith, 2012).

By comparing with other group members, it is possible to produce different solution strategies to reach the final goal. Cooperative work, therefore, makes it possible to develop interaction between students, integration and to improve self-esteem (Slavin, 1990). Collective metacognition, moreover, helps to reduce the individual anxiety of failure by distributing it among all group members. The existence of a shared solution increases motivation to carry out such activities. Difficulties that may be encountered during a group activity are communication between individuals and cultural differences, aspects that may compromise collective metacognition (Chiu & Kuo, 2009).

Furthermore, one purpose of this research is to analyse, through two questionnaires, students' emotions and beliefs regarding mathematics and their vision about this discipline. It is important to understand the presence of a change about the student's vision on mathematics after carrying out activities.

Methodology

This experiment consists of two activities carried out within the game theory. One of the objectives of this research is to stimulate problem solving and argumentation processes, to make people take on different points of view, and to promote group activity and comparison between peers. At the end of each activity, an interview was conducted in order to capture the difficulties encountered during the task and some impressions regarding the group work. In addition, two questionnaires were proposed to the students: one at the beginning of the first task and one at the end of the second task. One of the purposes of the two questionnaires was to investigate the students' emotions and their vision on mathematics, and the second questionnaire also asked questions focused on the tasks carried out. The second questionnaire, moreover, analyses the change about student's vision on mathematics. Two questionnaires were created following Di Martino and Zan's researches (Di Martino & Zan, 2011): there were open questions to leave students free to express their emotions and closed ones to analyse specific aspects. Two questionnaires were compilated individually, while the interview was conducted during the final discussion about the activities' results.

Participants who took part in the experiment were 81 secondary school students, grade 9, 11 and 13, of the Italian education system (14-, 16- and 18-years old students). Three classrooms participated to experiment:

- classroom of grade 13 is composed by 29 students. In this classroom half of the students solved tasks individually;
- classroom of grade 11 is composed by 25 students;
- classroom of grade 9 is composed by 27 students.

In addition to these, the two activities were submitted to a "control group" consisting of mathematicians and non-mathematicians, to investigate the similarities and differences between the approaches adopted by high school students, mathematicians, and those with no specific mathematical skills. The control group was composed by people with different ages. In the control group there were 16 individuals, 10 students of master's degree in mathematics and 6 students of no mathematical course.

The experiment was carried out entirely remotely using platforms such as Zoom and Google Meet due to Covid-19 pandemic. Thanks to these, it was possible to create virtual rooms in which students could work in groups. The class of grade 13, due to classroom attendance regulations, was half in the classroom and half at home, so the task was carried out individually by those in the classroom and in groups by those remotely located. These roles were reversed in the second task. The control group took the task individually.

Each task took about two hours to complete. One hour was left for the students to solve the task in groups or individually. The remaining time was devoted to a collective discussion in which the students were able to compare and share the solutions they had arrived at and the reasoning for applying a particular strategy. During the discussion we noted students' statements to analyse their process of problem solving. In the following section we report some students' statements taken by interview's notes and answers to the questionnaires.

Analysis of the collected data

In this paper we focus on the data collected on the first task. The text is the following:

"In a shopping centre there are three shops, AltaModa, BlueJeans and CookLover, which need a new lighting contract. They have been offered several alternatives: if they take out the contract individually, they will pay €250, €200 and €350 per month respectively; if they decide to take out an overall contract, they will pay €600; alternatively, if they agree in pairs, the prices will be €350 per month for AltaModa and BlueJeans, €450 per month for AltaModa and CookLover and €420 per month for BlueJeans and CookLover.

Try to explain the offer and how the three shops could agree on the best offer. Give reasons for your answers."

The table below shows the solutions proposed by the test takers.

Table 1: Proposed solutions. A, B and C indicate expenses for A, expenses for B and expenses for C. % groups and % individuals represent the percentual of students' solutions.

Proposed solutions	**A**	**B**	**C**	**% groups**	**% individuals**
Equal division of the total contract	200	200	200	47%	65,5%
Proportional division of the total contract	187,50	150	262,50	31,5%	24%
Other solutions				21,5%	10,5%

The data comparison of this experiment shows, as argued by Kagel and Cooper (2005), that groups act more strategically than individuals. Students who took the test individually were unlikely to go beyond the first interpretation, namely the equal division of the overall contract. Analysing the response frequencies of both the students and the "control group", it emerges that less than 50% of the groups gave as their solution the equal division of the overall contract. On the contrary, among those who took the test individually, more than 60% supported this solution.

Equal division of the total contract is the most intuitive answer, to which everyone approaches at first stage; some individual students and most groups manage to move away from this first idea, planning other solution strategies.

Some students believed that the equal division was fairer because everyone pays less than or equal to the single contract, while others wondered if there was a division that provided savings for everyone, and this led to the formulation of new hypotheses for solving this task. In this way, those who reflected on these aspects showed that they implemented metacognitive actions such as monitoring and evaluation, as they were able to analyse and interpret the proposed strategies in the context of the problem. These considerations were more frequent in those who worked in groups. Groups that worked efficiently went through all the stages of Tuckman and Jensen model. The groups also took longer to reach a conclusion due to a more thorough analysis of the proposed strategies.

Kagel and Cooper research suggest that the groups should be able to arrive at an appropriate solution if it is proposed by at least one member of the group. The analysis of the group discussions shows that most of the times when a member proposed a strategy different from the first interpretation, the group was able to come up with a solution that deviated from the instinctive response, i.e., dividing the total contract equally. From the discussions in the groups, it is evident that the confrontation with others allowed the emergence of different points of view that led to various solution strategies. Sometimes some members of the group disagreed with the solution proposed by the team, on which occasions the students defended their ideas. At the end of the time allotted for solving the activity, most groups proposed an agreed solution.

Thanks to the interviews and questionnaires, it was possible to ask the students about their impressions of the interaction between groups. In the class in which the two activities were carried out by both individuals and groups, it was also possible to compare the two ways of carrying them out. For example, one request of questionnaires regarded the students' vision on mathematics. This question is proposed before and after the tasks. The why of this choice is justified by the needful to investigate the change of mathematics vision after carrying out the tasks. By students' responses emerged that their mathematics vision is changed thanks to this experience.

Some students stated:

Student 1: “I think my approach is changed: greater depth and focus on different strategies, without stopping at the first insights”;

Student 2: “We often think that mathematics is a strict science with certain rules, but in this activity, we were able to observe the existence of different points of view and different solutions”.

Moreover, in questionnaires, we asked them their impressions about interaction between groups.

Some students stated:

Student 3: “Group work helps, by exchanging ideas and thoughts you eventually reach the choice that is closest to the correct one”;

Student 4: “Every idea was made explicit to the whole group”;

Student 5: “initially I did not understand the problem very well, then the other members helped me to understand it”;

Student 6: “some ideas we had thought individually were changed by the analysis of the whole group, and others emerged thanks to the collective activity”;

Student 7: “it was a good confrontation, it was a ‘thinking together’ rather than a union of individual ideas: we perfected each other intuitions by helping each other.”

From the students own words, the comparison with others was useful for a better understanding of the problem: “...they helped me to understand it”. It also emerges that working together helped them to consider more strategies “...more emerged from the collective activity”. As Frith reports in his research (Frith, 2012), each student hypothesises his or her own solving strategy, but thanks to the collective activity, a better performance is achieved, by placing each proposed strategy under collective analysis. In some groups, the pupils advised their peers to identify with the situations or to consider the activities in a real context. This approach made it possible to concretely analyse the planned strategies and consider their possible implementation in real life.

Conclusion

In this paper we analysed capacity of students to act exploration of the problem, planning of strategies to be implemented, identifying objectives to be achieved. In particular, we focused the attention on the problem solving and the argumentation process and their implementation in group activity. In most cases, the collective performance of such tests ensured, the planning of a greater number of solution strategies and a more in-depth analysis of the procedures implemented and the results obtained. Each participant observed the work of the others, which made it possible to assess the effectiveness of the strategy. Through the exposition of different points of view and the analysis carried out by each component, almost all the groups managed to build a shared knowledge. Difficult students put their doubts to their peers and thanks to the explanations, sometimes repeated, of the other group members, they overcame their critical points. From the previous statements, moreover, we can see the implementation of metacognitive activities that led to the construction of shared knowledge. In fact, thanks to the support and the analysis of each group member, they were able to refine their initial ideas and make their peers understand things that were not clear.

The students themselves acknowledged that working efficiently in a group is better than working individually:

> Student 8: "I think that by working well in groups you can do more than you would do alone".

Those who had the opportunity to work on the activities once individually and once in a group stated that they enjoyed better working collectively. The development of individual metacognition, therefore, was supported by collective metacognition. In this way, thanks the collective work to achieve a shared solution, the motivation to carry out tasks was increased. In a few groups, the work was not conducted fruitfully, with little participation by some members. In such situations, however, collective metacognition was compromised, as students did not compare, preventing the construction of shared knowledge. The data collected through questionnaires allow to analyse subjects' emotional approach in these activities. From responses to the first questionnaire, some students stated that they had a good emotion regarding mathematics, but more than 60% of the students stated: "I like mathematics because I obtain good results". This highlights a strong link between emotional disposition and the idea of successful on mathematics (Di Martino & Zan, 2011). An interesting fact, obtained by responses to the second questionnaire, is the change of the students regarding the vision on mathematics. After these activities the students have more perception about the usefulness of the mathematics in the real life and they have had the opportunity to do mathematics in a different way.

As the tasks were solved, many groups stated that they were satisfied with the work done. Collective activity, in fact, helps to reduce the individual anxiety and fear of failure, distributing it among all the components. In fact, the answers given in the first questionnaire show that many students have contrasting emotions regard mathematical activities, but these students stated, in the second questionnaire, that they valued the group work positively. These results underline how cooperative learning is a positive element for a better approach with mathematics, but also for improving the emotional disposition towards mathematics. Observing the work and discussions of the groups, it emerged that in the classes where students are used to working in groups, there was more interaction and better organisation.

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Development of math anxiety in prospective elementary teachers

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Some people develop math anxiety whenever they are confronted with tasks that require mathematical skills. Math anxiety influences learners and their performance negatively. Furthermore, the math anxiety of teachers has an impact on their students. Therefore, one challenge of elementary teacher education is preventing and reducing math anxiety, especially at the beginning of the teacher training program. In this study, we investigate the development of math anxiety in prospective elementary teachers in Germany during the first semester at university. We provide the first results regarding the influence of math anxiety on dropouts at the beginning of the teacher training program. Finally, we describe differences regarding the development of math anxiety of prospective elementary teachers at three universities.

Keywords: Math anxiety, first-semester changes, transition, teacher training.

Introduction

Research on math anxiety covers a large field of work with different directions and focuses (Hembree, 1990; Zhang et al., 2019). Math anxiety is defined as "*a feeling of tension and anxiety that interferes with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations*" (Suinn & Winston, 2003; Visscher & White, 2020). The math anxiety of a prospective teacher not only affects her/his performance at university (Hembree, 1990; Pekrun et al., 2006; Zhang et al., 2019) but also as an elementary school teacher. Therefore, teachers' math anxiety affects learning situations (Visscher & White, 2020) and the performance of their students (Beilock et al., 2010). Whereas math anxiety and math teaching anxiety are two separate constructs, both are related (Brown et al., 2011). Math anxiety can lead to anxiety about teaching mathematics (Brown et al., 2011; Hadley & Dorward, 2011; Zhang et al., 2019). Because of the influence of math anxiety on teaching, elementary teacher training in mathematics is a big challenge for universities. Particularly at the beginning of the teacher training program since first-year students often exhibit a transition shock (Gueudet, 2008) which may increase the math anxiety.

Elementary teacher training in mathematics is supposed to substantially educate prospective elementary teachers and simultaneously prevent or even reduce their math anxiety. According to the definition, there are various dimensions of math anxiety, and the question arises which dimensions are crucial for elementary teachers. Based on the national curriculum in Germany, math education in elementary school is supposed to promote problem-solving, argumentation, and communication (KMK, 2004). Accordingly, elementary teacher education targets to qualify prospective teachers for problem-oriented teaching (Eichler et al., 2022). From this perspective, two dimensions of math anxiety are relevant for teacher training: problem-solving and explanation anxiety (Visscher & White, 2020). Hence, we focus on the development of both dimensions in our study.

In Germany, there are specific training programs for elementary teachers. Mathematics is a compulsory subject in most of those programs, and prospective elementary teachers are obliged to attend courses in math education. In our study, we focus on the teacher education of prospective elementary teachers and analyze the development of their math anxiety in the first semester.

Aims and questions

The presented study targets to answer three questions on math anxiety of prospective elementary teachers. Based on the negative influence of math anxiety we investigate the transition from school to university by examining those in their first semester at three German universities.

RQ1: Does the math anxiety of prospective elementary teachers change in the first semester?

RQ2: Does the development of math anxiety of prospective elementary teachers differ at different universities in the first semester?

RQ3: Do prospective elementary teachers who dropped out of the math course show a different level of math anxiety at the beginning of the study than those who passed the course?

The third question refers to those prospective elementary teachers who filled out the questionnaire at the beginning of the semester but not at the end. We do not know the exact reasons for these dropouts, but we assume most of these prospective elementary teachers did not complete the math course in their first semester. The question arises if there is a different level of math anxiety at the beginning of the semester in prospective elementary teachers who completed the course and those who dropped out.

Methodology

Instruments

For measuring math anxiety, it is relevant to use scales of math anxiety that align well with the kind of math we are asking teachers to teach mathematics in the classroom. Therefore, we collected data with the math anxiety scales from Visscher and White (2020) that include Likert scales for answers: ”Not at all”; ”A little”; ”A fair amount”; ”Much”; ”Very much”. Table 1 shows the original scales. For collecting data in German universities, we translated these scales into German. Table 1: Math anxiety items (Visscher & White, 2020).

Table 1: Items of math anxiety constructs

Factor 1 – Problem-solving anxiety		
1	Working on a math problem for which you are not sure where to start.	*Items pertaining to difficulty of problem*
2	Being given a math problem that does not look like any problem you have seen before.	
3	Being asked to solve a math problem when you are not sure which formulas to use.	
4	Working on a math homework problem and not making any progress for 5 minutes.	
5	Being assigned an extra-long math homework set.	*Items pertaining to length of problem set*
6	Beginning to work on a multi-page math worksheet.	

Factor 2 – Explanation anxiety		
7	When you are partway through figuring out a math problem, being asked to share your thinking with a classmate.	*Items involving internal doubt*
8	Sharing your solution with a small group of classmates when you are not sure it is	
9	Explaining your attempt at a math problem to a classmate, even though you are not very convinced that it is right.	
10	Describing to a small group of classmates how you went about a homework problem on which you received a perfect score.	*Items involving external validation*
11	After reaching an “aha!” moment on a problem on your math worksheet, being asked to explain your solution to a small group of classmates.	
12	Being asked by a classmate to go through your correct solution more slowly.	
13	Being asked to further justify why your mathematical solution is correct to a classmate who is not yet convinced.	*Items involving external doubt*
14	Having to convince a classmate that your different way of solving a math problem is equally valid.	
15	Continuing to explain your mathematical solution, even though a classmate doubts it is correct.	

Procedure

The study followed a pre-post design. Data collection took place in the winter semester 2019/2020 in math courses for prospective elementary teachers in their first semester (in the first and the twelfth lecture). We created a pseudonym for each participant to merge the paper-based questionnaires from the pre-and the post-test. Figure 1 shows the design and the differences between the three universities.

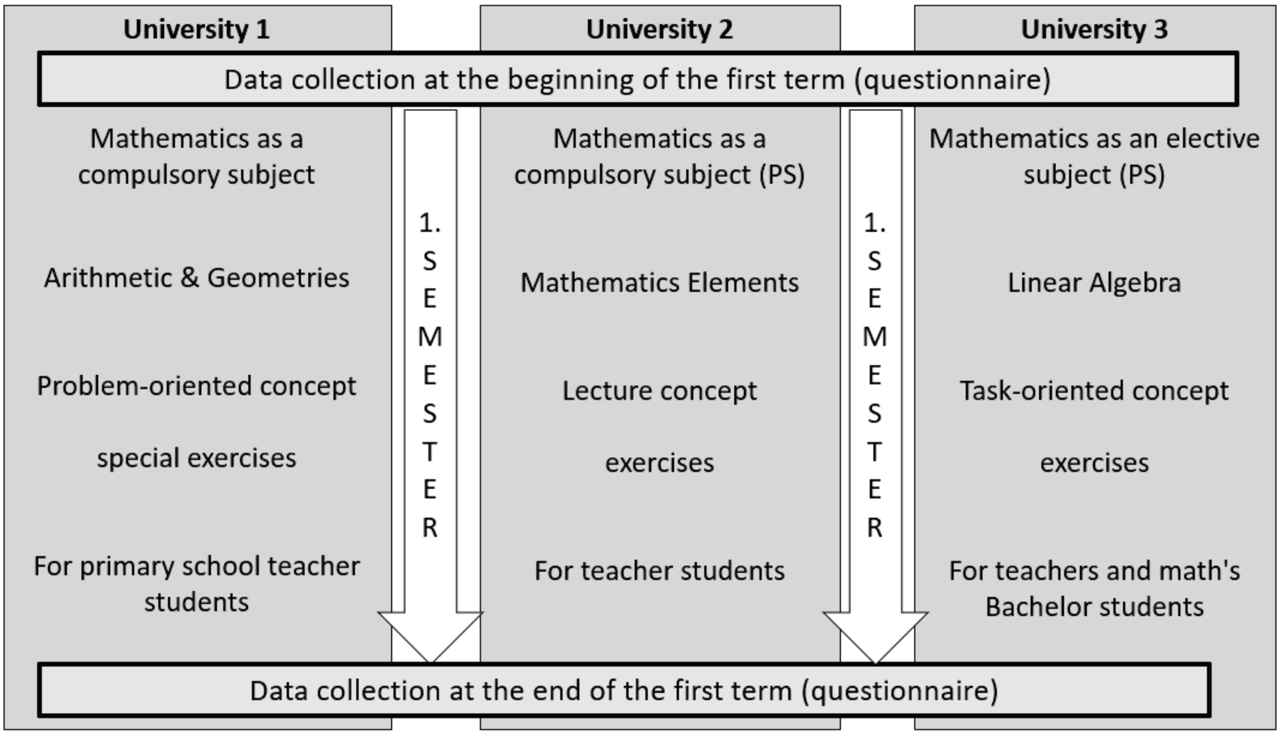

Figure 1: Design of the study and university differences

Sample

We decided to include prospective elementary teachers from three universities with three different math courses for beginners. The sample comprises 583 prospective elementary teachers; 400 participated in both surveys (University 1: 183; University 2: 150; University 3: 67); 183 only participated in the first survey, and we assume these are dropouts.

Analysis

There are several missing values in this data set which we excluded. If values are not randomly missed, deletion of cases with missing values biased the data. Despite a non-significant Little-Test (Little, 1988) (χ^2=*454,418, p = 0,251*) for item-nonresponse data, we assume the data is biased because of several unit-nonresponse data. We compared the scores of math anxiety of dropouts (missing data at the second survey time) with the complete data sets. The goal was to disclose differences between both groups and detect whether math anxiety at the beginning of the term can increasingly lead to the termination of the math course.

To check the quality of the data on an empirical basis (because we translated the items into the German language), we perform factor analysis. The Kaiser-Meyer-Olkin measure was *.823* and showed a relatively good factor analysis. Bartlett´s test of Sphericity was significant *(p < .001)*, indicating that correlations between items were sufficiently large for performing factor analysis. Based on the five dimensions of the scales of Visscher and White (2020), we used the maximum likelihood method and a varimax-rotated five-factor solution. Kaiser´s criteria and the scree-plot justified the extraction of five factors. One of the five factors has an eigenvalue just below *1*; four factors have eigenvalues *>1*. In total, these five factors explain *55,91%* of the total variance. Most of the items loaded highly on one factor. Only item 4 loaded on two of the five factors: comparably low on both. Therefore, it was excluded from the subscale about the difficulty of problems.

For reliability analysis, we calculated Cronbach’s alpha to assess the internal consistency (Table 2):

Table 2: Internal consistency of the math anxiety scales

	Problem solving anxiety			***Explanation anxiety***			
	Whole scale	*Difficulty of problem*	*Length of problem set*	*Whole scale*	*internal doubt*	*external validation*	*external doubt*
First survey Time	.80	.79	.75	.83	.84	.68	.78
Second survey time	.85	.82	.82	.87	.82	.70	.82

The first question was answered with a repeated measure t-test for contiguous samplings. To answer the second question, we use the data from prospective elementary teachers who participated in both surveys and perform a variance analysis with repeated measures. Regarding the third question, we assume all prospective elementary teachers who participated in both survey times completed the course regularly; all those who did not participate in the second survey terminated before the end of the semester. Therefore, we use the data of the first survey time and compare dropouts and prospective elementary teachers who complete the first semester in full. We consider this a limitation of our study since there are several reasons for not taking part in the post-test. Since the Levene-Test was not significant, we can assume variance homogeneity and perform the t-test for unpaired samples.

Results

Table 3: Development of math anxiety

	df	*t(df)*	*p*	*Cohen´s d*
Problem solving anxiety	399	- 2.82	.005**	0.14
Difficulty of problem	399	- 2.10	.036*	0.11
Length of problem set	399	- 1.53	.127	
Explanation anxiety	399	- 3.94	< .001**	0.20
internal doubt	399	- 3.02	.003**	0.15
external validation	399	- 1.93	.054	
external doubt	399	- 3.84	< .001**	0.19

Table 3 illustrates that the problem-solving anxiety decreased significantly over the first semester with weak Cohen's d effect sizes. Looking at the results of the problem-solving anxiety subscales, we see two opposing developments: a significant decrease within the difficulty of the problem but no significant development within the length of the problem set. The explanation anxiety decreased significantly over the semester, too. For the subscales, we can see that the internal and external doubts decreased highly significantly over the semester, while the external validation subscale barely reached statistical significance. The effect size of all significant developments of the subscales is weak, too. Therefore, we can speak of a weakly decreasing math anxiety in the first semester.

Table 4: Differences in the development of math anxiety

	df	*F*	*p*	*partial* η^2	*Cohen´s f*
Problem solving anxiety	2; 397	2.40	.092		
Difficulty of problem	2; 397	7.47	.010*	.036	.19
Length of problem set	2; 397	.21	.809		
Explanation anxiety	2; 397	.01	.986		
internal doubt	2; 397	.31	.388		
external validation	2; 397	.05	.952		
external doubt	2; 397	.81	.443		

Table 4 illustrates the differences in the development of math anxiety in prospective elementary teachers at several universities. While we found no statistical evidence of differences regarding the development of problem-solving and explanation anxiety among prospective elementary teachers from different universities and no differences in 4 of 5 subscales, we found significant differences in the development of the subscale difficulty of a problem. The Cohen's f effect size is *.19*, and therefore a weak effect. Bonferroni-adjusted post-hoc analysis revealed no significant difference between the universities in a pairwise comparison (University 1 vs University 2: *-0.001, 95%-CI[-0.17, 0.17]*)

(University 1 vs University 3: *0.127, 95%-CI[-0.0917, 0.35]*) (University 2 vs University 3: *0.128, 95%-CI[-0.10, 0.35]*). This determined the interaction effect, but the post-hoc test does not clarify which universities are different. Figure 2 illustrates that subscales about the difficulty of a problem increased at university two, whereas they decreased in universities one and three. For these differences in development, we did not find statistically significant results. However, the development of this subscale may not always be the same.

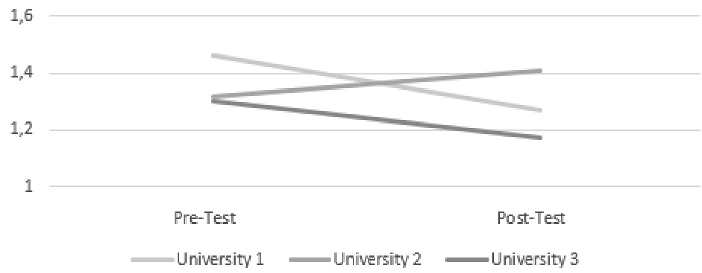

Figure 2: Problem-solving changes regarding the difficulty of a problem

Table 5: Dropouts

	Mean difference	*df*	*t(df)*	*p*
Problem solving anxiety	.013 95%-CI [-0.10, 0.13]	581	.22	.829
Difficulty of problem	- .025 95%-CI [-0.16, 0.10]	581	.39	.699
Length of problem set	.028 95%-CI [-0.13, 0.18]	581	- .35	.723
Explanation anxiety	.003 95%-CI [-0.09, 0.09]	581	.07	.947
internal doubt	– .047 95%-CI [-0.19, 0.10]	581	- .65	.515
external validation	.018 95%-CI [-0.06, 0.10]	581	.44	.661
external doubt	.021 95%-CI [-0.09, 0.13]	581	.36	.721

In summary, results reveal no differences in problem-solving and explanation anxiety at the beginning of the semester of prospective elementary teachers who terminated and those who completed the math course (see Table 5).

Discussion

The results show that math anxiety decreased over the first semester, despite the transition shock (Gueudet, 2008). We conclude that math courses in the first semester can help to reduce the problem-solving and explanation anxiety of prospective elementary teachers. In addition, results suggest that the directions of the developments are similar. Regarding the development of explanation anxiety, first-semester training helps to reduce both internal and external doubts, but explanatory anxiety does not decrease significantly after external validation. To summarize, the first semester can have various impacts on problem-solving and explanatory anxiety. Teachers generally avoid situations in the classroom that could cause anxiety for themselves or their students (Visscher & White, 2020). In this respect, our results are promising because lower problem-solving anxiety of teachers might lead to more problem-solving activities in school, and lower explanation anxiety might lead to more and better explanations in classrooms. Moreover, the transition between school and university, also called transition shock (Gueudet, 2008), does not impair the math anxiety of the prospective elementary teachers at the three universities. In this respect, it is possible the transition is either not perceived as a shock or has been mitigated and does not affect the math anxiety of prospective elementary teachers. However, reducing math anxiety in the first semester is desirable since it may indirectly support problem-oriented teaching.

We predominately see similar patterns in the development of problem-solving and explanation anxiety at the three universities. All three math courses similarly influence the problem-solving and explanation anxiety, despite the differences in content, learning concept, structure, lecturer, and participants. There is only a significant difference with a weak effect in the subscale of the problem-solving anxiety regarding the aspect difficulty of a problem.

In general, our results suggest the math course during the first semester influences the development of math anxiety in prospective elementary teachers, and it is possible to reduce math anxiety in this period. Due to the design of our study, the results only allow general statements about the development of math anxiety of prospective teachers, but no conclusion regarding the impact of a specific approach. However, the significant differences in the subscale "difficulty of a problem" might indicate that the different concepts of the math courses (see Figure 1) influence prospective elementary teachers' development of math anxiety differently. Since math anxiety can lead to anxiety to teach mathematics (Brown et al., 2011; Hadley & Dorward, 2011; Zhang et al., 2019), there is further research required to get detailed information about the effect of specific teaching approaches for university math courses.

Finally, the problem-solving and the explanation anxiety at the beginning of the math course has no influence on the progression during the first semester. Our results suggest the degree of math anxiety prospective elementary teachers exhibit at the beginning of the teacher training program is no crucial factor for passing the math course. Instead, the development of math anxiety within the first semester might have a higher impact on dropouts. However, based on our results, we only can draft this assumption but not empirically prove it. We do not know whether the participants who missed the second survey were dropouts or had other reasons, such as illness, individual problems, other appointments, delays, or refusal to answer the questions. This missing knowledge is a limitation of our study. To investigate the influence of math anxiety on dropouts, we need a study design for future research that measures math anxiety several times: at the beginning, in the middle, at the end of the semester, and one measure parallel to the final assessment. Possibly, this allows us to conclude the development of math anxiety and its influence on dropouts.

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Analysis of relations between attitude towards mathematics, prior knowledge, self-efficacy, expected and actual grades in mathematics

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The purpose of this study is to analyse relations between students' attitude towards mathematics, prior knowledge, self-efficacy, expected grades, and performance in mathematics among 115 first-year engineering students. We combine two statistical techniques to analyse the data we generated by questionnaires and two tests. First, item-level modeling, in terms of confirmatory factor analysis, which we use to compute the factor scores of construct-validated measures, and to control for measurement errors. Second, composite modeling, in terms of path analysis, which we use to test the research hypotheses. The findings show that both self-efficacy and expected grades have substantial effects on students' performance. Prior knowledge has a non-trivial effect on self-efficacy which, in turn, plays a significant role in students' grade expectations. All other hypothesised relations are not significant. We argue that these findings confirm some basic tenets of social cognitive theory.

Keywords: Affect, higher education, item-level modeling, path analysis, self-efficacy.

Introduction

Affect in mathematics education

Students' affect is critical not only to their well-being but also to their performance in mathematics. Following the lines of thought proposed by Hannula (2012), we define affect in mathematics education research as a general concept that encapsulates factors, other than purely cognitive ones, such as attitude, beliefs, emotions, feelings, goals, moods, motivation, norms, values, and self-efficacy. Thus, each factor that constitutes a unit of mathematics-related affect is regarded as an overlap between cognition, emotion, and motivation of varying stability with psychological, physiological, and social dimensions (Hannula, 2012). Prominent among the mathematics-related affect factors are attitude towards mathematics and self-efficacy. It is arguable that the former is prominent for its incoherent conceptualisations within mathematics research community (Di Martino & Zan, 2010) while the latter is prominent for its high predictive power of performance and its causal relation with students' mathematics performance (Roick & Ringeisen, 2018; Zakariya, 2021a).

Attitude towards mathematics

In line with the theoretical framework proposed by Hannula (2012), attitude of students towards mathematics (henceforth, attitude) shares a boundary between cognition (e.g., knowledge), emotion (e.g., likes and dislikes), and motivation (e.g., internal and external drives to approach or refrain from mathematics activities). It can be operationalised and measured using self-report psychometric research measures. Empirical evidence shows attitude is predicted by prior mathematics knowledge (henceforth, prior knowledge) and it predicts students' subsequent performance on mathematics tasks (Chen et al., 2018; Lipnevich et al., 2016). Students that belief in their mathematics ability, those that like mathematics, and those that approach mathematics with its pre-conceived utility for future aspirations are usually successful in mathematics tasks. On the flip side, students that do not belief in their mathematics knowledge, those that dislike mathematics, and approach mathematics with ill-conceived utility of mathematics for future aspirations are usually unsuccessful in mathematics tasks. Some researchers (e.g., Kiwanuka et al., 2020) have also shown that there is a reciprocal effect between attitude and performance in mathematics. That is, high achievers tend to develop positive attitude. In return, students with positive attitude tend perform well on mathematics tasks. Thus, attitude plays a crucial role in students' success on mathematics tasks.

Mathematics self-efficacy

Self-efficacy has its roots in social cognitive theory as propagated in decades of work by Albert Bandura. It entails "beliefs in one's capabilities to organize and execute the courses of action required to produce given attainments" (Bandura, 1997, p. 3). As it relates to mathematics learning, self-efficacy is students' self-evaluation of competence to proffer correct solutions to mathematics tasks (Zakariya, Nilsen, et al., 2020b). It is a combination of confidence in ones' capacity and an estimation of outcome that follows ones' effort. There are four sources of self-efficacy – mastery experience, vicarious experience, social persuasion, and affective states – among which mastery experience i.e., self-interpretation of previous attainment has the highest influence on self-efficacy (Zientek et al., 2019). An accumulation of evidence suggests that self-efficacy is one of the best predictors of mathematics performance. Students with high sense of self-efficacy have low mathematics anxiety, adopt deep approaches to learning, and perform well on mathematics tasks (Rozgonjuk et al., 2020; Zakariya, Nilsen, et al., 2020b). Evidence supports consistency of a model of reciprocal effect between self-efficacy and mathematics performance with generated data across 24 countries (Williams & Williams, 2010). Moreover, Zakariya (2021a) provides a tentative evidence for causal effect of self-efficacy and performance in mathematics. Thus, self-efficacy is a critical factor for students' success in mathematics.

Relations between attitude and self-efficacy

The relationship between attitude and self-efficacy coupled with their combined effect on performance has been sparingly studied. Yet, the results of the available studies are promising. Randhawa et al. (1993) using structural equation modeling show that attitude significantly predicts self-efficacy which in turn predicts students' performance in mathematics. More so, the effect of attitude on performance is partially mediated by self-efficacy in a non-trivial way. However, the study by Randhawa et al. (1993) is relatively old and focuses on high school students whose findings may not be of direct relevance to undergraduate students. In a more recent study involving seventh graders, Recber et al. (2018) show that there is a non-trivial positive correlation between attitude and self-efficacy. Further, both constructs are significant predictors of performance in mathematics (Recber et al., 2018). Regrettably, correlation between two variables has limited value in terms of inferential deductions and tangible conclusions. A similar limitation can also be ascribed to the study by Öztürk et al. (2019) who report a correlational analysis between attitude and self-efficacy, and their predictive effect on performance of middle school students in mathematics.

The present study

The intention of the present study is to provide an evidence-based model of relationship between attitude, self-efficacy, prior knowledge, and undergraduate students' performance (expected and actual) in mathematics. This study differs from the previous attempts in many ways. First, we approach the analysis from structural equation modeling (SEM) perspective, instead of correctional analysis, which avails us the opportunity to test theory-based hypotheses. Second, we focus on undergraduate engineering students, who have mathematics as a core subject but whose affect (i.e., relations between attitude and self-efficacy) appears not to be given much attention. The inclusion of other factors such as prior knowledge and expected grades in our model constitutes another difference from the previous attempts. In specific terms, the present study addresses the following research question: To what extent do attitude, self-efficacy, prior knowledge, and expected grades predict each other and undergraduate students' performance in mathematics?

To address the research question, we draw on theoretical foundations and some insights from literature to hypothesise that attitude, expected grades, and self-efficacy predict performance and are predicted by prior knowledge. We admit that there is a possibility of reciprocal effect between attitude and self-efficacy. However, we ensure temporary precedence by collecting attitude data eight weeks before collecting data on self-efficacy. As such, we hypothesise that attitude has a non-trivial positive effect on self-efficacy. Given that outcome expectancy is an integral part of self-efficacy, we hypothesise positive effect of self-efficacy on students' expected grades.

Methods

Participants and measures

One hundred and fifteen undergraduate students (90 males) voluntarily gave consent and took part in the study that lasted for a semester. These students, average age between 21 – 25 years, followed a first-year calculus course in a Norwegian university. They completed attitude towards mathematics questionnaire (AtMQ) and sat for a test of prior mathematics knowledge (TPMK) in the third week of the semester. On the one hand, the AtMQ is a five-item measure (sample item: I'm interested in what I learn in math) designed to expose a single construct on a four-point Likert scale from strongly disagree to strongly agree. On the other hand, TPMK is a 16 – item (22 subitems) test of basic algebra, functions, and geometry. Previous studies show that both AtMQ and TPMK demonstrate appropriate validity and have acceptable indices of reliability of .92 and .78, respectively (Zakariya, Nilsen, Bjørkestøl, et al., 2020; Zakariya, Nilsen, et al., 2020a). Further, we administered calculus self-efficacy questionnaire (CSEI) at the end of the semester. The CSEI is 13-item measure on which students are to rate their competence to solving presented exam-like first-year calculus tasks on a scale of 100 points, whose validity and reliability indices have been previously studied with promising results (Zakariya et al., 2019). As a measure of expected grades, an item was appended to CSEI that asks students to report their expected grades in forthcoming calculus exam, at the time. The students' final exam scores in the calculus course serve as a measure of performance in mathematics. The full measures are available as appendices in the referenced validation studies.

Data analysis

We analysed the generated data using some techniques of SEM in two stages. Stage one involves evaluating measurement models for AtMQ, CSEI and TPMK, using confirmatory factor analysis. The rationale for this analysis is to detect and correct misspecification errors in the measurement models prior to hypothesis testing. Simultaneously, we confirm construct validity of each of the measures and compute the factor scores. The second stage of the analysis involves testing the hypothesised relations between the research constructs. We evaluate the structural models using robust maximum likelihood (MLR) estimator. The models are assessed for their global fits of the generated data using a combination of criteria. The model exhibits an exact global fit of the data if the chi-square value is not significant. There is an excellent fit of the data if the comparative fit index (CFI), Tucker-Lewis index (TLI) are greater than or equal to .95, and root mean square error of approximation (RMSEA) is either ≤ .06 or its 90% confidence interval (C.I.) contains 0.06 (Chen, 2007). The model exhibits an appropriate global fit of the data if the ratio of chi-square value to the degree of freedom is less than 3, CFI and TLI are close to or greater than .90, and RMSEA is less than .08 (Bentler, 1990; MacCallum et al., 1996). Significant parameter estimates show that the model exhibits a local fit of the data. We run all the analyses in Mplus 8.5 software.

Results and discussions

Measurement models

The first set of results concern evaluations of measurement models for each of the measures. For both AtMQ and CSEI, we evaluated a one-factor model each using MLR estimator. Following the recommendation by Zakariya (2021b), we correlated disturbances of item 2 and item 4 of AtMQ to improve the model fit. In a similar manner, we correlated disturbances of item 9 with item 11 and of item 12 with item 13 to achieve a model fit as recommended by Zakariya (2021a). Further, we evaluated a one-factor model of TPMK using robust unweighted least squares (ULSMV) estimator. This estimator takes care of the categorical scoring of the TPMK. The best 17 out of the 22 subitems of the TPMK are used for this analysis as recommended by Zakariya, Nilsen, et al. (2020a). The results are presented in Table 1.

Table 1: Goodness of fit statistics of measurement models of the research measures

Global fit indices	AtMQ model	CSEI model	TPMK model
Chi-square estimate (χ^2)	7.846	99.151	129.769
Degrees of freedom (df)	4	63	119
p – value	.097	.003	.236
χ^2 / df	1.962	1.574	1.090
RMSEA [90% C. I.]	.091 [<.001 - .186]	.071 [.042 - .096]	.028 [.042 - .056]
CFI	.982	.905	.968
TLI	.954	.882	.964

The presented results in Table 1 show that there are exact fits of both AtMQ and TPMK models with the generated data. The non-significant chi-square values coupled with RMSEA, CFI, and TLI values that are within recommended ranges support the claim of exact fits of both AtMQ and TPMK models. That is, both AtMQ and TPMK measure the constructs (attitude and prior knowledge, respectively) they are purported to measure. More so, Table 1 reveals that CSEI model exhibits an appropriate model fit of the generated data. The chi-square value is significant but its ratio to the degree of freedom is less than 3. More so, the RMSEA value is less than 0.08 and both CFI and TLI are close to .90. These values support the appropriate fit of the CSEI model. That is, the CSEI measures the calculus self-efficacy of students it is supposed to measure.

Hypothesis testing (Addressing the research question)

After the evaluation of the measurement models of all the measures, we compute the factor scores of both AtMQ and CSEI using the default maximum of the posteriori distribution in Mplus because of the continuous nature of their datasets. On the other hand, Mplus uses maximum a posteriori method to compute the factor scores of TPMK because of the categorical nature of the dataset. Then, we saved the scores and use them to evaluate the hypothesised structural model of relationships between the research constructs. This evaluation avails the opportunity to test the research hypotheses and address the research questions. Figure 1 presents the goodness of fits statistics and the final evaluated model.

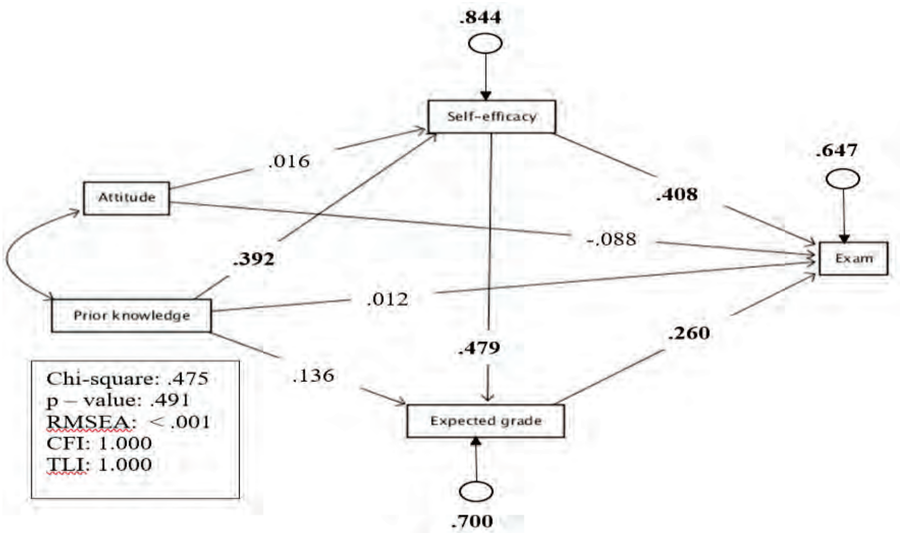

Figure 1: Evaluated hypothesised model of relationships between the research constructs with significant parameter estimates in bold faces

The presented results in Figure 1 shows some interesting findings. From the model fit perspective, Figure 1 shows that there is an exact model fit. That is, there is consistency between the hypothesised relationships and the generated data. In line with the postulations of the present study, Figure 1 confirms that self-efficacy and expected grades are significant predictors of students' performance in mathematics. That is, both high sense of self-efficacy and high students' expectations in the exams lead to high performance in mathematics. These findings corroborate previous studies (Rozgonjuk et al., 2020; Zakariya, 2021a) that have shown non-trivial relationships between self-efficacy and performance in mathematics. In support of the hypothesis of the present study, prior mathematics knowledge significantly predicts self-efficacy. This finding confirms a tenet of social cognitive theory that says that mastery experience (students' prior attainments) is an integral source of self-efficacy (Bandura, 1997). Figure 1 also provide empirical support for the non-trivial effect of self-efficacy on students' expected grades. This finding confirms the postulation of social cognitive theory that theorised outcome expectation as an integral part of self-efficacy (Bandura, 2012). Admittedly, it is logical that expected grade has a reverse effect on self-efficacy. However, we acknowledge this fact and take care of it by ensuring temporary presence with self-efficacy measure coming before the item on expected grade during the questionnaire administration. We recommend that future studies should be designed with this intention.

Contrary to the postulations of the present study, attitude fails to predict both self-efficacy and students' performance in mathematics. This assertion is deduced from Figure 1 that shows that the path coefficients (.016 and -.088) between attitude and the two variables (self-efficacy and exam) are not statistically significant. This finding that attitude does not predict mathematics achievement is aligned with some previous studies (e.g., Fernández-Cézar et al., 2021) although it does not support other studies that have reported substantial relationships between attitude and both self-efficacy and performance (Chen et al., 2018; Öztürk et al., 2019). It is possible that the findings of previous studies are not generalisable to our context. Another explanation for these unexpected findings could be a defect from the measure of attitude. Perhaps, the students had a different interpretation of AtMQ items from what the researchers intended. A future study may be designed to explore students' interpretations of these items. More so, Figure 1 shows that there is no evidence in the present study to substantiate non-trivial effects of prior knowledge on both the students' expected grades and performance in mathematics because the path coefficients (.012 and .136) are not significant. These findings are unexpected as well. A possible explanation could be a lack alignment between the knowledge assessed by PKMT and that of the current course. This observation requires further investigation. In sum, the results of the hypothesis testing address the research question by showing the extent to which attitude, self-efficacy, prior knowledge, and expected grades predict each other and undergraduate students' performance in mathematics.

Conclusion

We made some attempts in the present study to disentangle the complex relations between attitude, prior knowledge, self-efficacy, expected grades, and performance in mathematics among engineering first-year students. We combined item-level structural equation modeling techniques with composite modeling by using confirmatory factor analysis to compute factor scores which we further used in path analysis. This combined analytical procedure offers two advantages. First, we minimize biases from measurement errors by incorporating them in the item-level analysis. Second, we evaluate a complex model using a relatively small sample which would not have been possible, otherwise. The findings provide empirical support for substantial effect of self-efficacy and expected grades on students' performance in mathematics. They also confirm some theoretical postulations such as the crucial role of self-efficacy in students' outcome expectations on mathematics tasks. By implication, the findings support interventions on self-efficacy as a proxy to improve students' performance in mathematics.

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TWG09: Mathematics and Language

Introduction to the work of TWG09: Mathematics and language

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This paper is a summary of the work and discussions of Thematic Working Group (TWG) on Mathematics and Language. In this paper we provide an overview of both the presentations within the working group and of the main themes arising through the discussions. We highlight the diversity and richness of theoretical approaches, methodologies and research foci that lead to the rich discussions we have at CERME conferences. This richness also arose through a joint working session with TWG01 on argumentation and proof that brought different perspectives to the analysis of argumentation in interaction. We also consider the impact of the global pandemic on our data collection but also on the focus of our research within TWG09.

Keywords: Language, interaction, gestures, semiotics, written texts.

Introduction

The thematic working group 9 (TWG09) in CERME focuses on all aspects of language and mathematics, drawing on a diverse range of theories, methodologies and contexts. The importance of researching the role of language in mathematics is firmly established (Barwell et al., 2019; Erath et al., 2021; Morgan et al., 2021) and the working group contributions focus both on the mathematical content and the social, cultural or interactional context when researching the nature of this role. Over time the discussions of the working group have evolved and continue to evolve in line with developments in the field, and advance the work from previous ERME conferences. There is a great deal of richness and diversity in the range of methodological approaches, research foci, and contributions from the members of TWG09. This is partly as a consequence of the significant overlap the research included within the theme has with other working groups within the conference. Common connections include a focus on specific mathematical topics and practices such as multiplication, argumentation, or functions, as well as contexts of teacher professional development or mathematics teaching and learning in diverse and multilingual contexts.

Language is more than the medium through which the concepts and techniques of mathematics are taught and learnt, and this is widely illustrated in the work of TWG09. Research in this group is often characterised by the role that language can have in the learning of mathematics, in developing students' understanding of what it means to do mathematics, but also in terms of the barriers and obstacles it can raise for groups of learners. The research within TWG09 pays attention to how language, interaction and communication influences these processes of learning and teaching mathematics.

This diversity of perspectives, methodologies and connections both enriches the work of TWG09 but also poses a challenge for the group. This was already pointed out by Planas et al. (2019) with respect to the challenges related to presenting language-sensitive mathematics education research both during a conference and in papers and posters. CERME12 saw changes to the template that have made it easier to share transcripts in a way that respects the language of the participants, respects the theoretical approach used by the researchers, and made members of this working group able to more effectively share their data. However, research within this group relies heavily on the analysis of interactions; interactions between teachers and students, interactions within groups of students, and interactions between students and tasks. The global pandemic has seen huge changes in the nature of these interactions with large scale school closures, a move to online synchronous and asynchronous forms of interaction, alongside greater concerns over privacy and data protection. These changes pose challenges to both the collection of interactional data and the presentation of data to illustrate findings from the work within this group.

The relevance of language in the various domains of mathematics education research has also been previously highlighted (Planas et al., 2019). During the 2022 conference, the methodological domains of descriptive and interpretive research for understanding teaching-learning processes in more detail, design research studies aiming at both theoretical insights and teaching-learning arrangements, and intervention studies were addressed. In TWG09 on mathematics and language, there is often a clear focus on qualitative studies with only Quabeck and Erath reporting from a larger-scale quantitative study and Kortüm et al. giving an outlook on planned quantitative research in their poster. Looking at the presented research at the conference from the perspective of the age of the participants, we gain insights moving across primary and secondary schools, across studies in Higher Education, towards studies with teachers with varying degrees of classroom experience. This shows how language and interaction matter for the learning of mathematics right from early years to the point of learning to teach mathematics that continues throughout a career. This diversity of perspectives and foci is illustrated through the keywords of the papers as illustrated in Table 1.

Table 1: Keywords from the papers and posters in TWG09

Language theories, constructs and approaches	Communication, Dialogism, Discourse analysis, Discourse, Gestures, Interaction (analysis), (Interactional) Quality, Language as resource, Language diversity, Language, Lexicalization, Linguistic complexity, Literacy, Participation, Register, Revoicing, Semiotic mediation, Semiotic representations, Social semiotics, Syntax, Teacher talk, Verbal tools,
General Theories, constructs and approaches	Abduction theory, Conceptual learning, Diagnostic, Epistemology/Epistemics, Images of mathematics, Interpretive research (paradigm), Learning opportunities, Literature reviews, Operationalization, Professional Identity, Professional knowing, Progression, Qualitative research, Responsiveness, Video study
Mathematical content	Argument/Argumentation, Combinatorics, Diagrammatic reasoning, Dynamic Geometry, Explanations, Generalisation, Geometry, Graphs, Inquiry-based/inquiry-oriented learning, Multiplication/multiplicative structures, Problem-solving, Reasoning, Time telling, Word problems, Writing

Contexts	Artifacts, Bilingual education, Classroom/teaching practice, CLIC, Educational television, Learning disabilities, Multilingualism/ Multilingual classrooms/ Multilingual teachers, Pandemic, Pluriliteracy, Primary school/elementary school, Prospective/preservice teachers, Reading, Teachers, Textbooks, Web-based/Digital tools

The richness and diversity of research within TWG09 is not only visible in relation to contexts as well as the age and role of participants (see the Contexts in Table 1). It is also represented in the theoretical backgrounds of the presented studies as researchers drew from semiotic, dialogic, epistemological, pedagogic, sociological and interactionist perspectives (as shown in the range of theories, constructs and approaches in Table 1).and in some cases also connected different perspectives These theories and approaches have also been used to focus on different aspects of mathematics, with many focusing on specific mathematical topics and others focusing on mathematical processes and skills.

Organisation of TWG09's work

The working group included presentations of 18 papers and 6 posters, with a total of 47 participants from 11 countries. The contributions are characterised by a strong and fruitful diversity in the contexts explored, research questions considered, and the theoretical and methodological approaches taken.

In the 7 sessions the papers and posters were discussed and organised around some common themes, although some of the papers could have been included in more than one of the identified themes. In organising the themes, we strived to enable connections and thus to foster possible future collaborations. Furthermore, as the groups' diversity can also be a challenge for discussing each paper in depth (Planas et al., 2019), the organisation around common themes was intended to help the audience focus and think deeply on one (maybe not so familiar) aspect of language and interaction in mathematics education.

In each of these 7 sessions, emphasis was placed on the time for discussion for each paper individually with depth. This was realised with a short 5 minutes to highlight the key features of a paper by authors followed by a 10-minute reaction focused on raising questions and generating discussions. The in-depth discussion of the individual papers within each session was made possible through the breakout discussions enabled by the technology. However, this also resulted in the group splitting into four smaller groups and each member of the group choosing which one paper to discuss with the authors in more depth.

Furthermore, one session was dedicated to discussing larger themes across the specific papers or posters for identifying new trends, common challenges, ideas for future collaborations, etc.

Contributions and themes

We first present a brief summary of each of the contributions presented in the TWG during the conference. The themes running through the contributions include the multimodality of language, topic-specific language demands including argumentation, interactional perspectives, written and textual features of mathematics, and teacher education informed by language and mathematics research. We then conclude with some remarks on future challenges and directions highlighted during the discussions.

We opened with a paper from Huth that focused on the different functions of gestures in diagrammatic reasoning in primary school children's interactions. This was complemented with a poster presentation from Kimber and Smith on the relationship between speech and gestures in online teaching videos. Both these contributions focused on the different roles gestures can take in interaction in ways that both support mathematics learning, but also in ways that can constrain it. These different roles were also visible in the paper from Moutsios-Rentzos on multimodal argumentation in the joint session with TWG01 (Argumentation and Proof).

Several contributions from TWG09 focused on topic-specific language demands and semiotic representations that learners draw upon. Pacelli, Pellegrino, Carotenuto and Coppola reported on a project focusing on primary school learner's explanations when working with multiplication algorithms and the accompanying artefacts, presenting the specific case of Napier's bones. Also focusing on multiplicative reasoning in primary schools, Rønning distinguished between the multiplicative reasoning involved when working on different combinatorial problems that depended upon the semiotic representations used. The poster presentation by Şahin-Gür shifted the focus to calculus and learners aged 14 to 16 years by describing a design-research study to foster conceptual understanding in mathematical and language integrated learning-arrangements for qualitative calculus (i.e., understanding based on informal meanings of amount and change). Later contributions in the TWG sessions also included a focus on geometry in combination with digital technologies such as dynamic geometry software as discussed below.

The papers by Palm, Kapland and Bergvall and Planas focused on linguistic features and their role in the learning of mathematics. Palm et al. analysed how the changes in referents in textbooks for the final year in Swedish primary schools constructed opportunities for mathematical generalising. Planas drew on the notion of language as resource to argue for lexicalization as a potential resource for communicating mathematical meanings by reporting on her work with secondary school teachers. The papers in this session and the one that followed all focused on mathematics and language in interaction but in different ways. Bräuning and Feskorn focused on supporting interactional and communicative development of young learners in elementary school. Using an epistemological approach to the analysis of teacher-student interactions they illustrated how one teacher nurtured the students' development of mathematical interaction skills. Beck and Vogler continue this theme by examining responsiveness in interactions and the restrictions on primary school students' opportunities to participate resulting from particular teacher moves. Gíslason's focus on internally persuasive discourses also highlights the often restrictive nature of mathematical interaction, in that it often centres around visual or superficial features of a task rather than mathematical forms of argumentation and justification, this time arising from the students' interaction with dynamic geometry tasks.

Some contributions particularly focus on working with teachers with different levels of experience. Coppola, Ferrari and Miranda analysed student teachers' assessments of the written arguments made by undergraduate students which revealed a lack of attention to argumentative structures and a focus instead on precise use of language and the mathematical content. The poster presentation of Perlander offered a different lens focusing on newly graduated teachers use of mathematical reasoning demonstrated in classroom interactions. In a later session, this focus on teachers and teacher education was continued in a poster presentation from Dafnopoulou who looked at how teachers' professional identities are developed in multilingual mathematics contexts.

Quabeck and Erath shifted the focus to look at how quantitative approaches have been used to capture the quality of mathematics classroom interaction. Their analysis reveals that only a focus on discourse practices, that many of the qualitative studies in TWG09 had, is sufficient to capture the relevant differences in classroom interaction rather than a focus on teacher moves or analysis at the task level that are commonly focused on in larger-scale studies. Ingram also examines differences in approaches to the analysis of classroom interactions, illustrating differences in the use of the term revoicing depending upon the theoretical perspective taken that have consequences on how we understand the relationship between teaching and learning of mathematics.

ETC7 on Language in the Mathematics Classroom already highlighted that the inclusion of digital learning environments and tools is an upcoming theme in the international research on language, interaction and learning mathematics (Ingram et al., 2020). In addition to the paper from Gíslason above, two other papers considered the role of digital technologies in the relationship between language and mathematics. Meaney and Rangnes focused on digital tools including programming and the importance of how teachers use them with multilingual learners. They made a distinction between tools that acted as translators and tools that were used more broadly and consequently were more supportive in developing multilingual learners' mathematics and their language in mathematics. Baschek explored the use of PrimarWebQuests in Content- and Language-Integrated Learning (CLIL) settings in primary school illustrating the motivation of authentic contexts for students to use different working languages and offering an illustration of one such context in the webquests. This focus on CLIL was continued in the paper by Schüler-Meyer who suggested the notion of pluriliteracy as an alternative in CLIL with a shift in emphasis on learners becoming proficient in mathematics using multiple languages, rather than the common target language in focus in the CLIL settings.

In the final session the focus shifted to written mathematics continuing the development of this work across several ERME conferences. Malik and Rezat's paper focused on identifying specific linguistic features that cause learners difficulties in word problems by presenting a literature review. Teledahl, Ahl, Helenius and Kilhamn also analysed several research frameworks for assessing students' writing and illustrated by their literature review that further research is needed to develop research that attends to the different dimensions of students' writing. Kortüm, Meininghaus, Mentrop, Hußmann A., Hußmann St. and Nührenbörger also shared their work on developing a supportive diagnostic tool to assess reading and mathematical competencies and their interconnections. The TWG09 sessions ended with a final presentation from Tatsis and Maj-Tatsis that focused on the public discourses around educational television programs in Poland during the pandemic.

Argumentation and a joint session with TWG01

There was a joint session between TWG09 and TWG01 to bring together the overlapping focus of argumentation that often features in the papers of TWG09. During the virtual pre-CERME12 event, members of TWG09 discussed the idea of a joint session with another TWG at CERME12 as well as which other TWGs to approach. In this way, the members of TWG09 wanted to further embrace the CERME spirit and promote communication, cooperation and collaboration. Fortunately, the leaders and members of TWG01 on Argumentation and Proof were as enthusiastic about this idea as TWG09. Even though, there was always exchange between different TWGs, we are proud to be part of the first official joint session in CERME history and hope that there will be many more at upcoming conferences.

The joint session included two papers from TWG01 focusing on theoretical debates around the conceptualisation of argumentations and two papers from TWG09 focusing on examples of argumentation in practice. The paper of Moutsios-Rentzos began by challenging us to consider the multimodal aspects of argumentation and offering a tool to support the identification of specific ways in which verbal and non-verbal semiotic systems are both explicitly and implicitly used in mathematical argumentation. This paper drew on Toulmin's scheme of argumentation which is widely used by researchers in both working groups. In the next paper Cramer and Kempen challenged the extent to which this scheme can reveal all aspects of structures of argumentation, highlighting the limitations of the scheme as revealed through recent work within TWG01. The two papers from TWG09 focused on discourses of argumentation with Körner and Meyer illustrating the generalising process over time focusing on addition with zero and tracing one learner's development of argumentation during an interview. Toro and Castro then shifted our focus to mathematics teachers' argumentation during class discussions and how features and purposes of argumentation can be recognised through an interactional and communicative theoretical perspective.

The joint discussion in this session highlighted the different theoretical perspectives and approaches to the analysis of mathematical argumentation, and the nature of mathematical argumentation itself, yet with a common focus on identifying and recognising features of argumentation used by both teachers and learners.

Themes and challenges going forward

One session during the conference was organised without the presentation and discussion of particular contributions. Instead, it was organised as a collaborative working session with a focus on identifying and discussing themes and challenges going forward. The group members split in three smaller groups to focus on 1) how different theoretical perspectives can complement each other, 2) research issues challenging the field at the moment, and 3) implications for the professional development of mathematics teachers.

Over several CERMEs and ERME topic conferences there has been continuing focus on the use of different theoretical perspectives to research mathematics and language (e.g. Planas et al., 2019; Ingram et al., 2021). This is partly as a consequence of the range of fields research in mathematics and language draws from but also theoretical diversity is needed to address the complexity of the relationship between mathematics and language, particularly in the messiness of interactions.

Susanne Prediger illustrated this in her plenary presentation where she illustrated the range of theoretical perspectives and methodologies she has used in her work over a research agenda across several years (Prediger, 2022). Furthermore, as discussed in the joint session, since mathematics communication includes writing, talking, pictures, formulas, gestures, we need more refined theoretical tools than existing models (such as Toulmin's model) to capture its nature.

As already indicated at the beginning, one research issue challenging the field at the moment is that many researchers were not able to collect new data of language use and interaction in classrooms or even of small groups in schools due to school closures and restrictions. For researchers whose analysis often depend on video-taped interaction or who are interested in data from the same classroom in a more longitudinal perspective, this poses a huge challenge. Whereas more senior researchers may go back to “old but still rich” data, early career researchers and researchers starting new projects particularly struggle. In this context, it was also reported from different contexts, that (as schools are more and more challenged by the pandemic and other factors) access to schools becomes increasingly difficult, particularly in contexts in which, for example, teachers cannot be rewarded for their participation in research efforts. A third discussed point is that after a period of some political awareness for language and learning mathematics, other aspects are highlighted (as for example distance learning or digital tools) and more and more funding is directed towards subfields of mathematics education research without explicit connection to language and interaction.

The topic of teacher education has become particularly prominent in several recent CERMEs and topic conferences (e.g. Planas et al., 2019; Ingram et al., 2021), and is growing in importance as a focus of much of the research within this working group. Many of the participants are teacher educators and recognise the complexity of supporting teachers to recognise, identify and adopt linguistic features, teacher moves or interactional practices that support mathematics learning. Furthermore, it can be noticed that some projects on the classroom level reported in earlier conferences are now developed into a basis for the design of professional development programs focusing on language, interaction and learning mathematics.

Concluding remarks

The themes addressed by TWG09 show the variety of research questions, theoretical perspectives and methodologies that the papers and posters dealt with. Whilst this conference maintained this richness in the research and data shared, we have shifts in the focus of the research included. This may be a consequence of the nature of data it has been possible to collect and work on during the global pandemic, with fewer studies focusing on interactions within classrooms. In addition, there might be a general increase of attention to digital communication settings. It may also illustrate the interactions and connections with other working groups where many members welcome discussions that result from connections with these other groups. The joint session with TWG01 highlights how connections between working groups can add to this richness by offering different perspectives at the intersections of our research.

Acknowledgment

We want to thank all the participants of TWG09 who contributed through presentations and thoughtful discussions. We would also like to thank the participants of TWG01 who contributed to

the joint session leading to fruitful discussions around conceptualisations and illustrations of argumentation that cut across the two working groups. A special thanks goes to Alexander Schüler-Meyer who supported us during the conference and Aurélie Chesnais who supported as in the preparation of the conference.

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WebQuests in Content and Language Integrated Learning Classes on Primary Level

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In a research project on Information and Communication Technology usage in bilingual settings, pupils were observed by video recording while working with a bilingual PrimarWebQuest. The main research project focuses on the mathematical knowledge as well as linguistic aspects which pupils might acquire in CLIL-settings while using PrimarWebQuests. In this paper the pilot study is presented which focused on the interactions in such learning settings. It will be explained what Content and Language Integrated Learning (CLIL) means in a European context and to which components attention should be paid while planning the use of PrimarWebQuests in CLIL lessons. The fitting of CLIL and PrimarWebQuests will be described to outline the potentials of this approach. Its basic idea and its implementation in bilingual settings will be defined. Afterwards, the framework of the pilot study and first results will be described to conclude next steps of the main study.

Keywords: Bilingual education, primary schools, inquiry-oriented, web-based, time telling.

Research Aim

The advancing language competence of bilingually taught pupils has been proven, for example by Bechler (2014) or Kersten et al. (2010) and is caused by the higher frequency of language contact. In addition to *Basic Interpersonal Communication Skills* being in the front of traditional foreign language classes, bilingual classes deal with a *Cognitive/ Academic Language Proficiency* (Cummins, 1979). In a Canadian research project Bournot-Trites and Reeder (2001) could show that there are positive effects on mathematical learning when students are taught in two languages. Schubnel (2009), who investigated German-French speaking bilingual mathematics classes, suspects the additional linguistic access, which the pupils receive, as a reason for this phenomenon. Moschkovich (2002) describes a multiple-meanings perspective as one of three perspectives for describing bilingual students learning mathematics which seems to underline Schubnel's theory. The multiple-meanings perspective expands the complex role of language for learning mathematics as it highlights the idea of pupils developing and negotiating meanings of mathematical terms in bilingual settings.

As the pupils' everyday use of ICT, specifically the Internet, offers various possibilities to experience foreign areas, the research project investigates the combination of ICT and CLIL. Using ICT allows access to multilingual sources, which enables the pupils to work with authentic and multi-media material, including linguistic reality (Schmidt, 2013). The aim of the research project is to find out about the contribution of the PrimarWebQuest approach regarding the pupils' learning of a mathematical topic as well as the academic expressions which go with each topic. But for this pilot study the main goal was to describe and analyze the interactions between pupils, teachers and the material in a bilingual and digital learning setting for exploring the chances for CLIL lessons offered by PrimarWebQuests.

Bilingual Learning

Bilingual teaching and learning in Germany refer to educational situations in which a subject or even just a selected topic is taught in a different language for a fixed or an enduring time. The term *Content and Language Integrated Learning (CLIL)* is used as a generic term for bilingual learning models in Europe which can be implemented in different ways. It is the core philosophy of the European Union that each citizen of the EU should be able to functionally use at least two more European languages in addition to their mother tongue. This approach aims to connect content learning and language learning in an integrated way. It is seen as an effective method for learning a foreign language with the aid of authentic topics. The language abilities of bilingually taught pupils go beyond everyday speech, the so called *Basic Interpersonal Communication Skills (BICS)*. In bilingual classes pupils should acquire a more technical terminology, the so called *Cognitive/ Academic Language Proficiency (CALP)* which is necessary for demanding topics (Cummins, 1979).

Most studies about CLIL have been conducted at the secondary level. First results about CLIL at primary schools have been published (Elsner & Keßler, 2013). In Germany, the idea of CLIL-classes in mathematics has not been greatly accepted (Viebrock, 2013). There are only a few conceptual ideas in secondary schools as well as at primary level. There are even fewer teaching materials why it is necessary to fill this gap. One possibility for having access to more material seems to be the Internet.

In this pilot study, Coyles (2006) *4Cs Framework* was used for creating PrimarWebQuests for CLIL classes. In her framework she describes the four components *Content*, *Communication*, *Cognition* and *Culture* for planning CLIL lessons (see Figure 1). The *Content* of bilingual classes should be geared toward curricular guidelines of the subject, in this case Mathematics. The aim is a double and profound knowledge acquisition as well as the pupils' understanding with regard to the content. In Germany, the pupils' subject-based knowledge is defined by a curriculum. It describes mathematical skills and specific content areas pupils are supposed to achieve. In this study the pupils should deepen and expand their knowledge about times. *Communication* refers to extending pupils' language knowledge and abilities to qualify them for an academic interaction in class, like using the right expressions for talking about different parts of a day during mathematics classes. The component *Cognition* contains all cognitive abilities which can be established, such as strategy learning or metalinguistic knowledge which could be developed in discussions about different time telling expressions or during calculating activities of different time spans. Particularly in bilingual settings, pupils have to develop a reflexive attitude vis-à-vis their own and other *Cultures*. For behaving in an empathic way, it is important to think about various perspectives, opinions or feelings and to switch between them. The cultural component of the framework is touched by the task to help two people from another country by comparing German and English times for describing the differences to them.

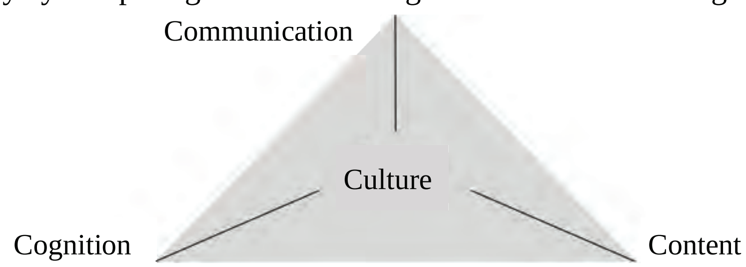

Figure 1: Curricular Framework for CLIL Classes according to Coyle (2006, p. 10)

PrimarWebQuests in Bilingual Classes

The method WebQuest, invented by Dodge and March in 1995, is an inquiry-oriented and web-based learning approach for using internet sources (Dodge, 1997). Even though it is offered on the Internet, the pupils can use online as well as offline sources which have to be chosen by the teacher in advance. WebQuests should give a structure for using the internet sources in an efficient and target-oriented way (Baschek & Schreiber, 2020). Schreiber adapted the method for primary school children in Germany. He called it *PrimarWebQuest*. A PrimarWebQuest contains various sources in order to deal with mathematical concepts and issues. It can set a focus on important aspects and reduce the complexity of available information which can be found on the Internet. The pupils learn in small groups how to conduct their research with online resources themselves (Baschek & Schreiber, 2020). Media competences like searching for, selecting and using online information as well as cooperating, presenting and reflecting can be trained with the small version of the original World Wide Web which is designed by a PrimarWebQuest. The requirements of a PrimarWebQuest should be made transparent for the pupils before beginning. They must be able to self-evaluate their learning process and the learning product (Schreiber & Kromm, 2020).

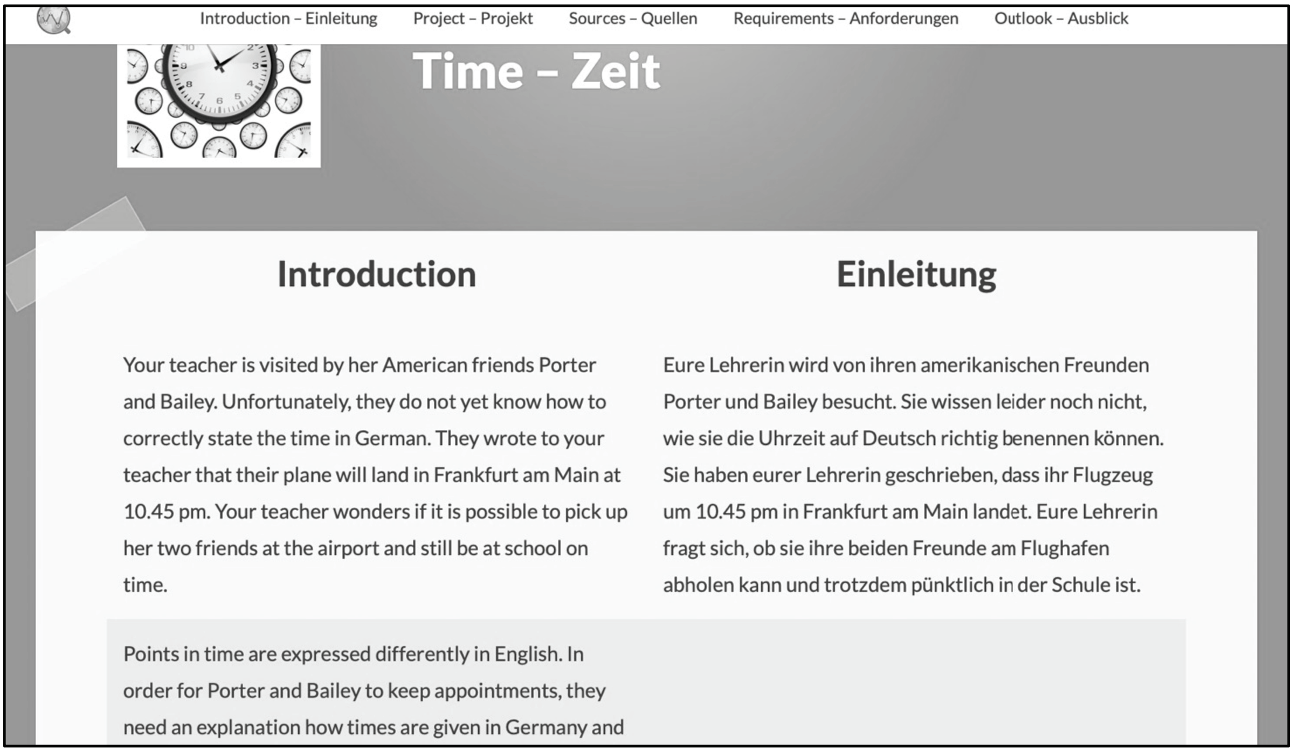

Figure 2: Part of the introduction of the bilingual PrimarWebQuest with the topic of times

Before starting their work, the pupils are split into small groups and get an introduction for using PrimarWebQuests. In the following sequence, the pupils look for more detailed information of their topic independently in their small groups. The teacher only supports them when she or he is asked to do so by the pupils. After the first working phases, there is a sequence for reflecting the working process up to this point. This sequence also offers the possibility to talk about crucial mathematical aspects. The pupils start to prepare their solution for the task and present their results at the end of the unit. When all presentations are done, the teacher reflects with everyone about their group work.

Generally, bilingual PrimarWebQuests (German/English) and monolingual German ones follow the same structure. On the web pages of bilingual PrimarWebQuests, pupils can read all information and instruction in both languages, despite the chosen sources. Because of two different language columns, they can choose the working language independently in every group. When opening one of the sites, the pupils only see the English text in the beginning. By clicking on an English paragraph, the German translation appears (see Figure 2). There are sources in both languages as well, so every group, regardless of which working language they choose, has the possibility to use both languages for research. The sources are mostly realistic web pages made by native speakers (Baschek & Schreiber, 2020). Bilingually designed PrimarWebQuests can motivate the pupils to switch between the languages. When the pupils are motivated to do so by the PrimarWebQuest or the teacher, they practice to work language-sensitively while switching. Additionally, the pupils can get into a collaborative dialogue because of the open-ended nature of the task. Discussing new terms with their classmates offers them the possibility to check their understanding of mathematical terms. Nevertheless, an adaption to the pupils' linguistic knowledge is necessary for a beneficial use of bilingual PrimarWebQuests and to ensure a successful learning experience for all (Baschek, 2020).

Pilot Study

The pilot study was undertaken for describing interactions in a bilingual and digital learning setting to discover chances for CLIL lessons offered by PrimarWebQuests. The group activities were video and audio recorded. Transcripts are and will be generated out of crucial sequences of the collected data, such as discussions about mathematical facts or language-sensitive activities, and will provide the foundation for the processes of analysis and interpretation. By using the interaction analysis (Krummheuer, 2002), the way, in which individuals create and negotiate taken-as-shared meaning, is reconstructed. The aim is to reconstruct any interactions in situations that are meaningful for the participants and to construct as many interpretations of these actions as possible. Every utterance is closely examined and analyzed in small groups for providing multiperspectivity. The utterances need to be kept in chronological order. Interpretations can only be justified and linked backwards as they can only be based on utterances made previously. These initial interpretations are then reinforced or rejected in order to ensure the most convincing interpretation of the episode.

A bilingual PrimarWebQuests[1] about *time telling* was tested with six third graders of a German private primary school offering bilingual classes (German/English) in multiple subjects. Mathematics is taught part-time in English and in German. The school follows the German curriculum and uses German and Irish textbooks. Most of the participants' mother tongue is German, one participant was raised bilingual. As not all pupils of the class were involved, the six participating pupils worked in two groups of three while their classmates did the regular tasks in their classroom. The two groups worked in one room at the same time and were supported by an English-speaking as well as a German-speaking teacher which were the researcher and an assistant. The pupils did not know this method before but were asked to use the bilingual PrimarWebQuest while the working language was chosen freely by them. The explanation for Porter and Bailey, who need the pupils' help for arrival and

[1] The bilingual PrimarWebQuest can be found here: https://math-primwq-bilingual.sd.uni-frankfurt.de

departure times in the PrimarWebQuest, had to be in English (see Figure 2). As the topic of time telling is dealt with mainly in second grade of German schools, on the one hand, this PrimarWebQuest should activate the previous knowledge of the pupils for deepening it. For instance, the difference of points in time and time spans turns calculating into a complex activity which needs to be practiced multiple times. Additionally, fractions, which are no topic of German primary schools, are used for time telling expressions. Instead of only learning the right expressions by heart, the pupils need enough time to think and talk about their conceptual meaning of those expressions for developing an understanding of fractions. On the other hand, using the shown task indented the pupils to learn the differences of time telling in the languages as an intercultural aspect. For example, using am and pm does not only mean to learn the right vocabulary. It also needs the pupils to think about the different time spans of every day and how times can be located in those parts of a day as this phenomenon does not exist in German time telling.

The different sequences of PrimarWebQuest usage offer several possibilities for supporting the mathematical language skills of bilingually taught pupils. To highlight possible learning opportunities of a PrimarWebQuest, pupil interactions are described and analyzed. During a group working sequence, the pupils discuss English and German vocabulary they will need for talking about different points of time. In the following example, the pupils know a German term and try to find the English translation together. Originally English utterances are typed in SMALL CAPITALS, whereas originally German utterances are typed in normal style. Gestures, activities and additional information are typed in *italics*.

Person	**Translated utterance**
Teacher:	OKAY. THE NEXT WORD IS THE BACKGROUND OF THE CLOCK WHICH IS WHITE. DO YOU KNOW HOW IT IS CALLED? IN GERMAN OR IN ENGLISH? GIOVANNA.
Giovanna:	Ziffernblatt?
Teacher:	Ziffernblatt IS THE GERMAN WORD. RIGHT *(attaches the word card to the black board)*. AND DO YOU KNOW THE TRANSLATION MAYBE? LUKAS.
Lukas:	NUMBER PAPER.
Teacher:	NO IT'S NOT THE DIRECT TRANSLATION. LAURA?
Laura:	FACE.
Teacher:	FACE IS RIGHT AND ONE WORD IS MISSING. LUKAS?
Lukas:	NUMBER FACE?
Teacher:	NOO.
Laura:	*(puts her hand in front of her face and whispers)* HAND FACE.
Teacher:	NO IT'S NOT A HAND FACE. WHOSE FACE IS IT *(points at the clock which is attached to the black board)*? LAURA?
Laura:	EH NUMBER NO EH CLOCK FACE?
Teacher:	*(nods)* CLOCK FACE. RIGHT. WELL DONE *(attaches the word card to the black board)*.

In this sequence it is possible to see how the pupils use both languages. Knowing the German term and supported by the teacher, they try to figure out the English translation. Lukas who was raised bilingual starts with an almost direct translation of the German term. Laura remembers *face* as a part of the translation. She could have gotten this information from the informational texts of the PrimarWebQuest. Listening to the teacher's hint, Lukas tries to build a word with *number* again and puts the two words together, calling it *number face*. The discourse seems to motivate him to use his metalinguistic knowledge. Laura instead thinks of further vocabulary by using the term *hand face* which probably refers to the hour and the minute hand and can be an expression she learned from the informational texts. Supported by the teacher, Laura finds the right translation *clock face*. As the specific terms of this topic are similar to everyday language, the main goal for the pupils to learn is using the right expressions. All pupils show a proper use of the terminology in both languages. In fact, they speak more German than English, but they think of the English expressions when writing down notes for their explanations. During the working sequences, the pupils read and interpret the given authentic sources and they need to gather and check suitable information details. It was possible to observe a subject-based discourse in both groups motivated by the PrimarWebQuest. Specifically, the pupils discussed a lot about the difference between am and pm because they developed an idea which parts of the day are addressed.

The following sequence shows how the pupils work together and discuss in a group spurred on by the PrimarWebQuest. They are divided over pm being the morning or the evening of a day and try to find an answer together. For solving their problem, they think of the information which the given texts contain and use a graphic which they copied from a text. The group work helps them to communicate successfully about new information and to get to a mutual solution.

Person	**Translated utterance**
Teacher:	YOU ALSO READ THE INFORMATION ABOUT AM AND PM, SO YOU CAN THINK ABOUT IT. WHETHER IT IS IN THE MORNING OR THE EVENING.
Julia:	IT'S IN THE EVENING. EH MORNING.
Laura:	So that means…Where is this piece of paper *(unintelligible)* it is here.
Julia:	Why was it there?
Laura:	We are, so PM IS IN THE MORNING *(points at the afternoon of a plan of a day with its times)*.
Julia:	No, PM IS EVENING.
Laura:	I see, right.
Julia:	IT'S IN THE EVENING.
Laura:	IT'S IN THE EVENING.
Teacher:	RIGHT.
Julia:	What is two o'clock in the evening?
Laura:	They could, she can come to school.
Eva:	Two o 'clock is in the morning.
Julia:	Two o 'clock can also be in the evening. Namely after one. At one o 'clock you eat and at two…
Laura:	But it is PM.
Julia:	Two o 'clock is fourteen o 'clock.
Eva:	Yes.
Laura:	Yes, but it also can be PM. It is PM anyway.
Julia:	It says PM, so it is fourteen o 'clock.
Laura:	Noo, PM is in the evening *(uses the wrong German article)*.
Julia:	Yes, therefore fourteen o 'clock.
Laura:	*(hits her head with her hands)*
Julia:	But it says, noo, fourteen.
Laura:	Yes okay, do what you want.
Julia:	Yes have a look Laura. Two o 'clock is fourteen o 'clock.
Laura:	Yes I understood that. Yes I understood that.

In the beginning, the three girls think that a pm time belongs to the mornings. The teacher needs to suggest to check their assumption. During the conversation with the teacher a change from one language to another can be observed. After the switch Julia does when the teacher comes to help the group, a multiple switching between the two languages can be observed. This shows that the pupils use their languages in a flexible way and adapted to their communicative intention as well as the task of the PrimarWebQuest. Set off by the group work, they need to describe their own thinking processes while comparing their assumptions and they can learn to verbalize their understanding of the informational texts on their own. In this sequence two different opinions can be observed, but the girls don't seem to be able to explain their assumptions. By naming examples, Julia tries to convince Laura and Eva. Working together, they seem to find out that 2 o'clock pm means 14 o'clock. The open-ended task of the PrimarWebQuest supports their subject-based discourse while preparing the explanation for Porter and Bailey because the pupils need to come to a mutual result.

Conclusion

To begin with, the pupils seem to feel safe because of the open choice of working language. They use the English language when talking to the English-speaking teacher or when preparing their explanation for Porter and Bailey. Therefore, the setting motivates the pupils to work with both languages in an authentic way. In retrospect, it can be considered the pupils learned some new vocabulary or rather time telling expressions and could deepen their existing knowledge. Both groups had the opportunity to improve their strategies in mediation, as most of their discussions are German and their explanation for Porter and Bailey has to be in English. Especially in the working sessions with both groups, most of the pupils showed advanced metalinguistic skills (*Cognition*).

The open-ended task allows an individual main focus during the working sequences. If the content of the sources is too complex for the pupils, the PrimarWebQuest can purposefully guide their work on the task by helping them structure their working steps. Specifically, the intercultural topic of using am and pm encourages them to think about the informational texts and to use their copied graphics (*Culture*). This unknown content motivates the pupils to negotiate new terms or knowledge with their working groups for developing new concepts (*Content*). This cooperation can support language learning as well as a proper language use. Additionally, the pupils seem to be able to communicate adequately to mathematics classes as they can explain their calculating of times and use expressions which contain fractions easily (*Communication*). Besides the language and mathematical learning, the pupils also worked on their media competences while multiple internet sources.

To conclude, using the method PrimarWebQuest can be advantageous for CLIL in mathematics classes. For many topics there is not enough material for CLIL lessons in mathematics. The Internet and its diversity can be a support. The method enables dialogues, offering the possibility to investigate the pupils' understanding of mathematical terms either for teachers or for researchers.

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Geometry lessons as a catalyst for the participation of all learners in mathematics lessons in primary school? Empirical studies on the responsiveness of interactions in geometric and arithmetic teaching situations

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Based on the thesis that it is easier to support the participation of all learners in classroom interactions in the field of geometry in primary school than in the field of arithmetic, this article focuses on the interactional support of children's participation by teachers in geometric and arithmetic classroom situations in the context of heterogeneous groups. Using the construct of responsiveness, it is worked out to what extent the opportunities of all children to participate in geometrically-oriented interactions can depend decisively on the content-related and linguistic design of the process of interactive negotiation and whether in the domain of geometry - as compared to arithmetic - interactive "moves" of the teachers which have a beneficial effect on the participation opportunities of the children can be reconstructed to a particular extent.

Keywords: Primary schools, geometry, responsiveness, participation, interaction process analysis.

Introduction

Geometry, after arithmetic, is the second domain in the curriculum of primary school mathematics both in Germany and in many countries (KMK, 2005; NCTM, 2000). In this area, orientation in space, the recognition, naming, and representation of plane geometric figures and illustrations, as well as the measurement and comparison of surface areas and spatial content are formulated. Geometry lessons not only help children to better understand arithmetic, they also address many of the things that interest them, and build on their previous experiences. In this way, geometry can become children's "strongest" mathematical discipline (van den Heuvel-Panhuizen et al., 2008). In geometry lessons, pupils further develop their spatial imaginations and can thus better orient themselves in our three-dimensional world. Probably due to the "accessibility" of geometric content for learners and its strong reference to everyday life, geometry teaching is mostly seen as a particularly suitable area for inclusive mathematics teaching. In a study by Korff (2018), teachers in Germany stated that creating joint learning arrangements in heterogeneous classes is more challenging in arithmetic than in geometric lessons. However, little research has been done on whether teachers actually "succeed" in supporting learners in the area of geometric learning opportunities in everyday mathematics lessons, against the backdrop of a heterogeneous student body, than they do in arithmetic lessons. From a co-constructivist perspective on learning through participation in discourse, this article approaches this research desideratum to make an empirically-based contribution to theory synthesis in this area based on the construct of responsiveness. The central research question is to determine to what extent geometric teaching situations in heterogeneous classes, in comparison to corresponding arithmetic situations, are characterised by a high degree of adaptive or responsive support for all children, to enable them to participate in co-constructive learning processes.

Theoretical background

Teacher support for learner participation in the context of a heterogeneous student groups

In recent years, in response to the UN Convention on the Rights of Persons with Disabilities, Germany and other countries have increasingly included children with special educational needs in mainstream classes at general schools (Sekretariat der Ständigen Konferenz der Kultusminister der Länder in der Bundesrepublik Deutschland, 2021; Westling, 2019). In these inclusive classes, the support of learners' participation is a central concern for enabling joint co-constructive learning processes for all students. In supporting these learning processes, the teacher as the “more competent other person” (Chaiklin, 2003, p. 41) assumes a key role. A common principle in mathematics didactics for supporting learning processes is scaffolding (Wood et al., 1976), in which this more competent other person supports a learner in the form of an interactive structure or interactive scaffolding, so to speak, by providing instructions and assistance in mastering (arithmetic or geometric) tasks. In this way, it is oriented towards the abilities of the learners. Over time, this interactive support, adapted to the progress of the learners, is increasingly “reduced” to enable the learner to participate more and more autonomously: the other person, who is more competent in the subject matter, consequently adapts his/her actions to the actions of learner and thereby “creates” a situation that is conducive to participation or learning. At the level of interaction, a successful scaffolding process and its interactive elements can be described with the construct of responsiveness (Koole & Elbers, 2014; Robertson et al., 2016).

Responsiveness as a construct for analysing (successful) scaffolding processes

Responsiveness is as a coordination process between a child and its person of reference, in which the person of reference orientates itself toward the child’s needs in order to respond to these in a prompt and coherent way. Transferred to teacher-learner interactions, responsiveness is an example of a successful scaffolding process (Beck, Vogler, & Vogel, 2020). For example, a teacher takes up a mathematical idea of a student’s contribution that was not/had not been anticipated. The teacher supports the learner in an adaptive way, e.g. by encouraging him or her to clarify and justify their utterance. So, responsiveness is distinguished by the fact that in the interactive interplay of “initiation, reply and evaluation” (Mehan, 1979, p. 37): the teacher adapts different interactive elements to the learners’ attainment level in terms of (1) mathematical content and (2) interactive discursive skills. In this conversation analytic approach, the aim is to empirically describe the extent to which a teacher’s actions in these asymmetrical learning situations are aligned with the assumed attainment level of a learner, in order to offer them the most productive participation opportunities possible through the appropriate supportive structuring of the interaction process (Robertson et al., 2016; Beck & Vogler, 2021). A high level of responsiveness generates interactional potentials (Vogler, 2019), which opens up possibilities for interpretation and action for the students, in which they can actively recognise or create networks between the different levels of representation (Beck et al., 2020). Both the teachers’ and the students’ subsequent actions are analysed in our observations, as the combination is necessary to reconstruct responsiveness.

Research question

As stated in the theoretical explanations, teachers will be confronted with ever greater heterogeneous learning prerequisites among the pupils in the early lessons, which they must take into account to enable appropriate participation opportunities in the lessons for all children. Teachers must therefore be highly responsive to pupils' expressions, both in terms of content and mathematics, as well as in terms of language and discourse. Geometry lessons seem to be more suitable for this than arithmetic lessons (Korff, 2018). This assumption will be empirically examined in the following under the following question: *To what extent are geometric teaching situations, compared to corresponding arithmetic situations, characterised by a high degree of responsiveness in supporting all children to enable them to participate in co-constructive interaction or learning processes?*

Data and methodology

To address the research questions and clarify the extent to which interactions in primary school arithmetic lessons are responsive, the comparative analyses in this section examine videographic data from the geometry lessons of a 1th and an arithmetic lesson of a 2nd grade class in Germany. The video of the first-grade class was recorded in the project "EVE", which focuses on the professional visions of prospective primary school teachers (Platz et al., 2018). The video sequence from the second-grade class was recorded as part of the project "The Next Level: Lehrkräftebildung vernetzt entwickeln" ("The Next Level: Developing Networked Teacher Training"), which is conducting diverse studies on the professional teaching perceptions of (prospective) teachers in heterogeneous learning groups. Interaction analysis is used to qualitatively analyse the sequences (Krummheuer, 2012) in which the statements are first interpreted individually, in the order in which they occur, to then be able to understand their relationships to each other, following conversation analysis (Sidnell & Stivers, 2010). This makes it possible to work out both the negotiated mathematical meaning and the characteristics of the turn-taking system. The construct of responsivity is used as a sensitising concept (Blumer, 1954) to describe the extent to which arithmetic lessons create opportunities for all children to participate in teaching-learning situations.

Empirical Case 1: Mr. Miller and the change of perspective

The sequence analysed below is from a mathematics lesson in an inclusive 1th grade classroom in Germany. In the following scene, Mr. Miller is interacting with seven children (Denise, Celina, Merve, Joshua, Can, Kostas, and Lars) at a table in a corner of the classroom. Two cuboids are forming an L-shape. Next to the cuboids is a yellow body that also has such an L- shape (Figure 2). Kostas and Can look at the front view of the L-building, and Joshua at the side view. Can and Joshua have described in advance what they can see from their perspectives:

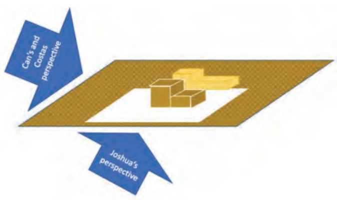

Figure 1: Group situation with Mr. Miller **Figure 2: Wooden cuboids and yellow L-shaped -model**

008	Teacher:	That means you both are right, ne. You have described it from **this** side
009		once.
010		*points at the place in front of Joshua*
011		This has Joshua done. Can has looked from this side.
012		*points at the table in front of Can*
013		that is, building components - #
014	Can	are the same
015	Teacher	are the same. Can but -
016		points at Can
017	Kostas	be the same.
018	Can	be different.
019	Teacher	look **different**. Depending on-(.) What does that depend on/ *shows with his*
020		*finger an arc from Joshua's to Can's place at the table*
021	Can	*raises his finger* row/
022	Teacher	hmm nee. What does that depend on/
(...)		
027		Kostas tell me
028	Kostas	We have just-
029	Teacher	What have you done now/ You are from there- *points at Joshua* to there
030		*points at Can*
031	Kostas	Well we have looked from the edge of the table, what Joshua and Can have
032		done.
033	Teacher	Exactly. That's another side.
034	Kostas	Yes
035	Teacher	Yes, this is another side. That is, the building components depending on
036		which side I look at them from, can look different.

In the selected sequence, the teacher and the students Can and Kostas talk about different views of the L-shaped building (see Figure 2). In line 08, Mr. Miller agrees with both of them and confirms the students' previous statements. In line 13, Mr. Miller begins to sum up: "That is, building components". He leaves his voice suspended for a moment and Can completes the sentence with the attribution: "are the same". Can's statement may refer to the fact that the L-building consists of two identical blocks. The teacher echoes this utterance in line 15 and completes it with a qualification: "can but-". In this turn, Mr. Miller also leaves his sentence hanging. In this way, he creates a universal turn that can theoretically be taken up by all children. This means exerts interactional pressure on the interactants involved to take up the turn. Kostas responds to this pressure and adds "be the same" <17>, again emphasising the equality of the two wooden blocks. Can completes the teacher's sentence with the contrary statement, "be different" <18>. Whether he agrees with the content of this statement or whether, in the interactional flow, he is merely processing the restriction imposed by Mr. Miller's utterance "but" <15> cannot be reconstructed beyond doubt. The teacher again follows Can's utterance and repeats "different" in line <19>. He adds the verb "to look" and thus places the attribution in the context of the appearance of the components. Taking into account all the partial aspects expressed by the interactants, the following content-related statement emerges: "That is, components are the same but can look different". From a mathematical perspective, this sentence has linguistic inaccuracies. In terms of content, however, it could be interpreted as follows: the statement that the same components look different can be interpreted from a spatial-geometric perspective on the situation as a thematisation of different views of the same spatial objects from different spatial perspectives. However, the different perspectives remain unmentioned until line 18. They are only addressed by the teacher following the statement "Depending on-", and the subsequent question.

"What does that depend on?" <19>. The associated gesture (arc) can be interpreted as an illustration of the children's change of perspective: the gesture can suggest that for Joshua, the components look different when he takes Can's perspective. However, Can alternatively interprets the gesture as a "row" <21> and seems unsure of this himself. The teacher evaluates Can's statement negatively <22> and repeats his question from <19>. In <29>, Mr. Miller again addresses the different perspectives by pointing first to Joshua and then to Can, accompanying this with the words, "You are from there- to there". Kostas reacts to these actions of Mr. Miller less regarding perspective than concerning the two boys (Can and Joshua), and expresses that they were looking from the edge of the table. The teacher evaluates this statement positively with "Exactly", and adds, "that's another side" <33>. After Kostas agrees, Mr. Miller finally sums up in line <35–36>: "Yes, that's another side. That is, the building components, depending on which side I look at them from, can look different." In this last statement, Mr. Miller adds the distinction of perspective to the statement from lines up to <18>.

Empirical Case 2: Mrs. Smith and the Number Greater than 90

The second sequence originates from the mathematics lessons of a 2nd grade class in Germany. The children sit in a semicircle in front of the blackboard. The teacher stands to the side of the blackboard. A number line up to 100 as well as seven different coloured balloons are attached to the board. Each balloon hangs above a certain number (see Figure 3). After the teacher has briefly referred to the last lesson, in which the class dealt with the structure of the number line, she begins to pose riddles to the children, e.g. "My number is greater than 90.", and the pupils have to find out what number the teacher is thinking of.

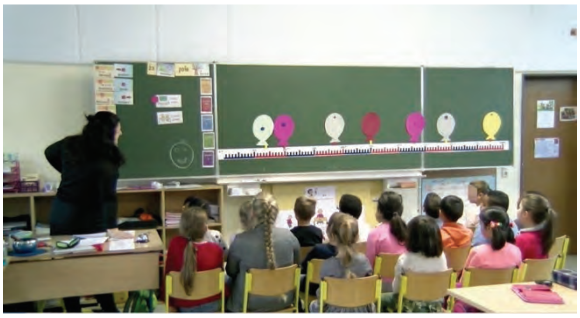

Figure 3: 2nd grade class and number line – Mrs. Smith

001	Teacher:	My number *raises right index finger* is larger than 90 (...)
002	Dennis	*raises the right hand*
003	Laura	*raises the right hand*
004	Teacher	What number is this?
005		*Seven more pupils raise their hand*
006	Teacher	Mhm Dennis please.
007	Dennis	*lowers his hand* Hundred
008	Teacher	Yes, but there the balloon is not pointing at aeh hundred, is it? Alica.
009	Alica	Ninety-five.
010	Teacher	Great Alica, would you please write this number in the balloon? *hands Alica*
011		*a pen*

Mrs. Smith tells the pupils that her number is greater than 90 and supports the relevance of her statement with a gesture <01>. She does not explicitly refer to the balloons or the number line and she has not explained the rules of the game beforehand, so there are potentially several possibilities for solving the task of finding a number greater than 90. On the one hand, any natural number from the number 91 onwards could be a solution. If (1) the number line or (2) the number line in combination with the balloons on the board are taken into account, the set of potential solutions is narrowed down: (1) the numbers 91–100 or (2) the number 95, above which a balloon is attached. After two children, who can either ask a question or name a solution, raise their hands, the teacher asks what the number is. Her question does not provide the pupils with any new insights or possible help. She may be indicating to the students through her question that they should now answer. A total of seven more pupils raise their hands and signal their willingness to participate in the mathematical discourse. The teacher then takes Dennis' answer, whereupon he puts his hand down and says the number "100" <07>. The teacher then voices her objection, "Yes, but there the balloon is not showing at aeh hundred, is it?" <08>, with which she refers to the balloons above the number line for the first time in the lesson. Although the number 100 is greater than 90, this is not the solution she is looking for, as no balloon "points" to this number. The verb "point" is presumably meant to describe the orientation of a balloon above a particular number, since balloons do not have hands or directional elements with which they can point to something. By calling on another pupil, the teacher reinforces her objection, as she can now name another answer or at least complement Dennis' statement <08>. Alica then says the number 95. The teacher positively evaluates this answer in line <11> with the adjective "great", and by requesting that the pupil write the solution in the corresponding balloon on the board.

Empirical Results

Both examples illustrate classroom conversations. In case of Mr. Miller the situation deals with spatial geometric and in case of Mrs. Smith with arithmetic content. In the situation with Mrs. Smith, a task is worked on in which different modes of representation are taken into account, such as the symbolic and iconic modes, which may be supportive for the learners. These modes of representation are supplemented by corresponding verbal impulses from the teacher. The representations on the blackboard and these impulses create different interpretative variations for the pupils that are not taken up. For example, it remains unclear why the mathematically correct answer "100" is rejected by Mrs. Smith. In case of Mr. Miller, similar interactional structures can be reconstructed. This is supported by actions on the material or gestures, in combination with verbal impulses from the teacher. Mr. Miller not only addresses the change of perspective through his statements but also illustrates it twice through different deictic and metaphorical gestures, which he accompanies linguistically. In addition, through his linguistic design, which resembles a cloze text, he repeatedly offers the learners opportunities to take over the turn and thus to participate in the discourse, but only in a fixed way, which was anticipated by Mr. Miller. He does not orient himself to their content-related meanings, such as equality of the two cuboids or Kostas' "row". The mathematical ideas introduced by the pupils and not anticipated by the teachers are devalued by them as undesirable. Based on these reconstructions, **a low level of content-related mathematical responsiveness** can be traced in the classroom conversations of both teachers. In this way, the interactional potential (Vogler, 2019) of the unanticipated contributions is not exploited. The interactive integration of the learners' contributions through the paraphrasing realised by Mr. Miller may initially seem supportive and conducive to participation from a linguistic perspective, but the reconstructed "cloze text" can be identified as a classic funnel pattern according to Bauersfeld (1995): the children's scope for participation is narrowed by Mr. Miller to such an extent that participation is only limited to the children "reciting" the answer anticipated by the teacher. Thus, the analysis has shown that refusing answers, as well as offering participation structures in a very restrictive way, as is the case with a funnel pattern (Bauersfeld, 1995), lead to a low degree of linguistic-discursive responsiveness.

Concluding remarks

This paper aimed to explore how primary school teachers facilitate rich participation opportunities for all pupils in the interactive interplay of arithmetic and geometry lessons. For this purpose, it was to be determined whether geometric teaching situations, in comparison to arithmetic situations, are more strongly characterised by a high degree of responsive support for all children, and therefore whether geometry teaching is possibly better suited for inclusive mathematics teaching. With the help of the reconstruction of responsive structures in two sequences from geometry and arithmetic lessons, it could be shown that the corresponding interactions in the area of geometry as well as in arithmetic show **a low degree of content-related responsiveness**, and that children in these situations are consequently not supported in participating in the content-related mathematical discourse. Likewise, both situations were characterised by a **low level of linguistic discursive** responsiveness. In addition, with the theoretical statements and the findings of Korff (2018), it may be assumed that geometry lessons initially seem to have more potential for the design of teaching-learning situations, but that such potential is possibly only rarely used in instructional discourse, as the analysis in this paper has shown. There seems to be a discrepancy between the perception of the potentials on the one hand and implementation in the classroom on the other. This divergence needs to be further investigated, for example, through collaborative discussion with teachers, or by addressing it in training programs under scientific supervision. Therefore, in the future, teachers will be observed in further arithmetic and geometric lessons to deepen the findings of our first paradigmatic examples.

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Learning How to Talk Mathematically: An Epistemological Approach to Teacher-Student-Interactions in Grade 1 and 4

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Since mathematics is universally abstract in a sense that other scientific disciplines are not, the teaching and learning of mathematics face major challenges. This is especially true at elementary school level when children start experiencing mathematics in formal instructional settings for the first time. One of these challenges – both for teachers and learners – is the development of mathematical communication skills to share ideas, create meaning, clarify understanding. Looking at two examples from grade 1 and 4 we aim at exemplifying how a teacher responds to this challenge. The epistemological analysis of the teacher-student-interaction reveals that the teacher constantly nurtures the students' development of mathematical interaction skills and that teacher and students undergo a mutual process of mathematical socialization from grade 1 to grade 4.

Keywords: Elementary school, language, epistemology, interaction, problem-solving.

Introduction: Mathematics and classroom discourse

Within the last decades, researchers have come to the conclusion that teaching practices highly influence students' learning opportunities and a paradigm shift in education from teaching to learning was found. This new rationale led to seeing the learning of mathematics beyond the mere acquisition of rote facts and procedures. Instead, a perspective that acknowledges students' mathematical learning processes and outcomes as tied to individual and reciprocal aspects came into being. Evidence shows that language is an integral part of this (Pimm, 1987; Erath et al., 2018) and communicative practices in the teaching of mathematics are required. Looking back at Cobb et al. (1997, p. 258), this idea does not seem to be new: "The current reform movement in mathematics education places considerable emphasis on the role that classroom discourse can play in supporting students' conceptual development." Still, the question remains how to establish ways that will cultivate content-related rich discourses. This appears to be especially challenging for three reasons: First, mathematical objects are abstract concepts that can only be accessed through concrete representations that do not speak for themselves; actual words are needed to interpret meaning. Second, elementary school students are only at the beginning of encountering those abstract concepts and depend on experienced others. Third, (mathematical) communicative skills are only learned by actually communicating (mathematically). These three aspects point out that (mathematical) language is both a goal in itself and a means for attaining the goal. The uniqueness of this manifold situation calls for closer attention and many researchers do so (e.g. Krummheuer, 2011; Erath et al., 2018).

The aim of our research project "PrimaL" (A Longitudinal Study of Problem-Based Mathematics Teaching in Inclusive Learning Environments) in this article is connected to take a more detailed look at teacher-student-interactions in the everyday mathematics inclusive classroom of a public elementary school. A peculiarity of the research project is the fact that the teacher does not undergo any training in planning communicational-sensitive math lessons. Furthermore, the lessons do not take place in a laboratory setting but reflect actual teaching and learning situations. On the theoretical basis of interactional (Krummheuer & Voigt, 1991) and epistemological (Steinbring, 2005) approaches, single case studies are analyzed to pin down the teacher's communicative factors that presumably lead to the construction of mathematical knowledge. For the purpose of this article, two task introduction episodes are chosen to investigate the following research question: What characteristics in teacher-student-interactions can be observed in grade 1 and grade 4 from an interactional and epistemological point of view? Differences between the teacher's communicative actions towards beginning and advanced learners of mathematics become obvious. Furthermore, a student-centered participatory classroom culture seems to emerge as the teacher carefully withdraws from the teacher-student-interaction.

Theoretical and methodical background: Teacher-student-interactions in the mathematics classroom

Generally, mathematical signs and symbols have a dual function: On the one hand, they are the mathematical content that students are required to learn. On the other hand, they constitute the central elements of communication. "In this regard, a learning student cannot be compared with a professional mathematician. The latter has many years of experience in mathematical communication [...]" (Steinbring, 2005, p. 15), while the former is only beginning "to develop and perfect such forms of mathematical communication with his classmates" (Steinbring, 2005, p. 15) and – as will be shown later – her/his teacher. From this point of view, the uniformity that is usually associated with mathematics cannot be taken for granted and learning processes are less uniform, but rather individual, divergent, and heterogeneous. Within those learning processes, (unfamiliar) mathematical signs and symbols need to be both interpreted and constructed. Due to Steinbring (2005), this interpretation and construction takes place in a triad of mathematical content, student, and teacher with communication as their linking element.

The most prevalent interaction pattern that is widely discussed in the field of (mathematics) education is the IRE pattern (teacher's initiation, student's response, teacher's evaluation) (Mehan, 1979). While many researchers describe this pattern as doing only little to enhance students' learning opportunities, some authors point out that a difference needs to be made between the pattern itself and how it is used. Wood (1998), for example, stresses the organizational benefit of the pattern (coining the "term recitation pattern", p. 172) while warning teachers to misuse it as an engrained way to wait for their intended answers to be given by the students. Instead, a series of IRE sequences can be used to orchestrate students' verbal contributions by e.g. highlighting, asking for clarification and reasoning or summarizing. On the surface, this may seem similar to the form of the initially proposed IRE pattern since the sequence teacher – student – teacher remains. However, the content of discourse changes as the students' thinking becomes the focal point of discourse. It is our intention to concretely shed light on this special discursive interplay between a teacher and her students.

Usually, when it comes to interactional research, mathematics teaching is observed and analyzed to describe empirical phenomena with the goal to develop empirically grounded local theories by understanding the actions of individuals (Krummheuer & Voigt, 1991; Krummheuer & Brandt, 2001). However, in these analyses the interactional perspective is the main focus (e.g. Jung, 2019) whereas the mathematical learning content fades into the background. Within this article, we try to include the *mathematical* perspective by considering the interplay of both interactional *and* epistemological aspects by using the analytical instrument "Formal-In" (Forms of teachers' mathematical interaction) (Bräuning & Steinbring, 2011). Within Formal-In, Bräuning & Steinbring (2011) seek to identify particularities in the teacher's verbal behavior that support children's interactive constructions of mathematical knowledge. They propose a methodical grid (see Figure 1) to analyze reciprocal teacher-student-interactions to "better understand how teachers can develop practices that foster mathematical communication" (Bräuning & Steinbring, 2011, p. 929). Although Formal-In was originally provided as a framework to analyze one-to-one teacher-student-interactions it also appears suitable for use on teacher-student-interactions within groups (Bräuning et al., 2020).

ID / ED	Instructive	Intervening	Explorative	Moderating
thing-like concrete use				
symbolic relational use				

Figure 1: The analysis grid Formal-In

The interactive dimension (ID) goes along with a teacher's orchestration of students' verbal contributions and consists of four patterns: instructive, intervening, explorative, and moderating. Interactions of the explorative and instructive type refer explicitly to the mathematical content in question whereas moderating and intervening interactions depict "the form and play" (Bräuning & Steinbring, 2011, p. 929) of communication. Furthermore, explorative and moderating interactions are understood as ways to find out about students' understanding of a mathematical phenomenon. In contrast to that, instructive and intervening interactions happen when interaction is "'transformed' into communication about the information intended by the teacher" (Steinbring, 2005, p. 77).

Because the learning of mathematics at elementary school level is initially bound to concrete material as embodied carriers of the non-perceivable and ideal structures, an epistemological dimension (ED) is included. In dealing with particular objects, both students and teachers can either stick to a thing-like concrete use with a focus on observable properties, which may hinder the interpretation of these objects as representations of something else, or move towards a more appropriate interpretation of symbolic-relational structures that become generalizable (Steinbring, 2015, p. 289).

Research context

The two sample episodes that will be presented in this article are taken from a first-grade and a fourth-grade classroom. Both classrooms are of particular interest as the learners have been taught by the same math teacher (female, in her late 30s, four years of teaching experience) from the beginning of their first day at school so that a look at the teacher-student-interactions in grade 1 compared to grade 4 seems to be possible in the form of a quasi- longitudinal study. The teacher-student-interactions were video- and audio-recorded throughout the school year. Building on the work of many other scholars concerned with the impact of students' language (proficiency) in the learning of mathematics (Prediger et al., 2018), it is important to mention that the school is located in a low-income and socially disadvantaged area. Most children are native German speakers whereas few are German learners. However, both native and non-native speakers deal with lacking vocabulary, limited language skills, and language-induced learning obstacles.

The following interpretative discussion of the chosen examples deals with the interaction that arose between the teacher and her students while approaching the following tasks: *Grade 1:* On a meadow, there are dogs and birds. There are 7 animals. Together they have 18 legs. How many dogs and birds are there? *Grade 4:* In a stable, there are horses and flies. There are 15 animals. Together they have 72 legs. How many horses and how many flies are there? Both tasks are identical in their structure, that is two animal species, the total number of animals, and the total number of legs are given while the exact number of each species is to be found out. Mathematically, the tasks represent Diophantine equations. Didactically, they can be classified as problem-solving tasks (Heinrich et al., 2015).

Analysis of transcript 1[1]: Grade 1 (end of the school year)

The students sit in a semi-circular arrangement in front of the blackboard. The task is visually presented on the blackboard. In addition to that, pictures of seven dogs and seven birds are put up on the blackboard. The teacher presents the task to the students but does not give any instructions[2].

The transcript (l. 1-89) can be divided into the following units of meaning (UM): *UM1* silent impulse and students' initial ideas (l. 1-7), *UM2* teacher reads out task and students come up with first ideas (l. 9-16), *UM3* clarification of total number of animals (l. 17-28), *UM4* total number of animals is seven (l. 28-33), *UM5* clarification of total number of legs (l. 34-45), *UM6* question about possible assignment for group work and summary of preceding task clarifications (l. 46-71), *UM7* teacher proposes a solution (l. 71-85), *UM8* teacher asks students to question her proposed solution (l. 85-89), *UM9* summary of task clarification and transition to group work with clear instructions (l. 89).

Using Formal-In as an analysis tool, the teacher takes a moderating role in UM1 by asking two students to read out the task. On an epistemological level, UM1 cannot be characterized since no mathematical idea is negotiated.

UM2 is characterized by an intervening interaction of the teacher:

9 Teacher: I will read out the task again. So everyone listens.

It can be assumed that she read out the task again to increase the students' attention. One student immediately says she knows the answer (7 birds and 7 dogs). However, it becomes clear that her answer is based on the number of animal illustrations attached to the blackboard. So in UM3, the teacher points out the illustrative function of the pictures but also offers the opportunity to use them as hands-on material to solve the problem. Therefore, UM3 is classified as an instructive interaction.

In UM4, the teacher intensively talks with a student, Zoe, in a one-to-one interaction. She asks Zoe to put seven animals on the blackboard but does not specify the number of dogs and the number of

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[1] The numbers used here are number lines taken from the original transcript. Full transcripts are available on request from the authors.

[2] We call this a silent impulse. The students (both in grade 1 and grade 4) are used to this way of starting a lesson.

birds. In this respect, the interaction is both instructive and explorative as the Zoe on the one hand is told to do something but on the other hand is not bound to detailed specifications. Zoe takes away all the birds so that seven dogs remain on the blackboard. UM 5 follows in which the teacher points out the discrepancy between the task (18 legs) and the number of legs that seven dogs have. She does not say how many legs seven dogs have. So Zoe points to two dogs and states that together they have eight legs. She moves on to the next picture and adds another four legs by counting in steps of one. She stops when reaching 20:

42	Teacher:	Do you have eighteen legs now?
43	Zoe:	No.
44	Teacher:	So there's something wrong here. There aren't eighteen legs.

In this phase, the teacher takes on a moderating and explorative role. She moderates the discourse but lets Zoe figure out on her own that seven dogs mean more than 18 legs. In the beginning, Zoe uses a symbolic-relational understanding in appointing eight legs to two dogs at once. Since first graders are usually only asked to master numbers up to 10, Zoe then moves on to counting the legs by ones which means that she uses the pictures to concretely represent numbers. However, when stopping at 20 the symbolic-relational mode can be assumed as Zoe seems to refer back to the given task.

In UM6, the teacher asks seven different students what is important about the task and what the task requires the students to find out:

53	Teacher:	What do you have to keep in mind? Fio?
54	Fio:	We are supposed to think about how many legs there are.
55	Teacher:	It's already written on the blackboard. Eighteen legs. Valentina?
56	Valentina:	We are supposed to consider so if we have dogs and birds, consider how many legs there are.
57	Teacher:	The task says eighteen legs. Seven animals with eighteen legs. What are you supposed to find out? Zoe?
58	Zoe:	We are supposed to think to find out how many animals have eighteen legs.

With her frequent questions, the teacher wants the students to set a mutual starting point. She highlights they mathematically relevant aspects (total number of animals, total number of legs) so that her interaction can be classified as instructive, explorative, and moderating. She constantly connects the students' statements to the given task (instructive) while also incorporating their understanding of the task (explorative, moderating).

In UM7, the teacher proposes an idea and puts five birds and four dogs up on the blackboard. One of her students says that there are nine animals now, and the teacher takes away two birds. By giving a concrete solution and asking for the mathematically relevant aspects of the task, an instructive interaction develops in which teacher and students seem to work in the sense of heuristic trial and error. It is followed by UM8:

85	Teacher:	The question for you to answer is how many dogs and how many birds are there. Student1 says that there are three birds and four dogs. Is that right? Is that the answer to the question?

Two students confirm that this is true but Zoe intervenes by saying that the number of legs still is not consistent with the task. The interaction is characterized as explorative because the teacher constantly requests from her students to think about the given solutions.

Finally, in UM9, the teacher transitions a group work. On the epistemological dimension, the teacher implicitly encourages her students to take a symbolic-relational perspective by highlighting again that the total number of animals needs to correspond to the total number of legs.

Analysis of transcript 2: Grade 4 (in the middle of the school year)

The students sit in a circle in the front of the classroom. The task is written twice on two sheets of paper lying in the middle of the circle. Moreover, there are two flies and two horses as illustrations. The teacher presents the task visually but remains silent.

The transcript (l. 1-17) can be divided into the following units of meaning (UM): *UM1* Silent impulse and Nelly's first idea (l. 1-2), *UM2* Jason's solution (l. 3-7), *UM3* Jonas' solution (l. 8-14), *UM4* summary of the task clarification as well as transition to group work with task assignment (l. 15-17).

UM1 begins with the silent impulse. Starting in grade 1, the students are socialized in such a way that one student, here: Nelly, reads out the task and already comes up with a first solution. Since the teacher directly asks Nelly to read out the task and Nelly shares her mathematical ideas with her classmates, this can be seen as an explorative-moderating interaction.

2	Nelly:	I think there are more flies than horses because flies have more legs than horses.

It becomes obvious that Nelly refers to a thing-like-concrete idea of counting the legs at first but then moves on to a symbolic-relational understanding.

In UM2, Jason offers a solution and Jeremy responds. He picks up Jason's idea and elaborates it:

4	Jason:	Fifteen times seventy-two?
7	Jeremy:	I would rather say multiply four until you reach seventy-two.

During the whole process, the teacher does not contribute to the discourse or interfere with the students’ ideas. She only calls the students by name. Therefore, the interaction is characterized as moderating and characterized as explorative since two students' ideas become present.

In UM3, Jonas comes up with a further idea:

8	Teacher:	Jonas
9	Jonas:	We can count the flies or horses first because we always have to calculate because I just did that in my head. Ten times four is forty. That’s just ten horses and there are fifteen animals and there and then I have to calculate the flies. So we still have to calculate the flies.
10	Nelly:	Huh?
11	Teacher:	What’s the matter, Nelly?
12	Nelly:	It just occurred to me if you now have two flies that make twelve and then you just have to add four more flies.
13	Student:	Five flies.
14	Nelly:	I mean five flies.

After 2 minutes, Jonas and Nelly are about to solve the task. Again, the teacher only moderates the interactions and lets the students express their ideas which obviously result in first tries to come up with a solution.

In UM4, the teacher asks the student to start working in groups to finally solve the task. She frequently points out that she expects them to think about the following:

17 Teacher: Can that be true? Why? Why not? Which possible solutions did you find?

This interaction is explorative and shows that thinking about and justifying different solutions are important aspects to the teacher.

Comparison of the two transcripts

When comparing the analysis results, it becomes clear that in grade 1 slightly more instructive and intervening than explorative and moderating interactions occur. This is also evident in the number and frequency of turn-takings: Throughout the transcript, the teacher and one student alternate consistently for the most part, so that the teacher assumes a controlling and focusing function. On the epistemological dimension, only a few phases can be determined. This is probably due to the fact that the first graders still have to get acquainted with the way mathematical tasks are talked about and mathematical meaning is negotiated. In grade 4, explorative or moderating interactions occur and are more dominantly linked to the epistemological dimension. The transcript shows that the teacher only says the students' names in order to orchestrate their verbal contributions whereas the students are responsible for the mathematical content. They work to a large extent independently and thereby almost come up with a final solution of the problem-solving task.

In general, the analysis tool Formal-In allows for comparing the transcripts with regard to the interplay of interactional and epistemological level. With the help of these two transcripts, a change can be shown: From grade 1 to grade 4, the teacher constantly withdraws from the interactions. With her interactive discourse movements, the teacher supports the students to move from the thing-like concrete towards the symbolic-relational understanding of mathematics. It is interesting to note that in the two transcripts the epistemological dimension occurs exclusively in explorative or moderating interactions. This is presumably due to the fact that the examples are the introduction to the lesson.

Conclusion

Our analysis of the two presented episodes and the extensive watching of the video recordings have led us to conclude that the teacher manages to establish a discursive classroom culture in which the students are encouraged to explore mathematical problems independently. Despite methodical limitations (small sample size, restricted contexts), we can outline that an explorative-moderating teacher-student-interaction pattern seems to be fruitful as it is also shown in Bräuning et al. (2020). Our investigations into this area are still ongoing. However, the findings presented in this article are promising to add to the existing body of research in the field of mathematics education. Our future work within the research project PrimaL will focus on finding more evidence for the characteristics of the explorative-moderation interaction pattern.

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On prospective teachers' evaluations of freshman students' written arguments

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In this paper we present a qualitative study in which we analyse how a group of prospective mathematics teachers evaluates written arguments produced by freshman students solving a problem involving graphs. The prospective teachers were attending a course of Mathematics Education for the second year of a master degree in Mathematics. In particular we focus on how prospective teachers in their evaluations look at mathematical content, language, and argumentative structure. The outcomes show, on the one hand, a certain attention to some linguistic aspects, on the other hand, a greater attention to the content than to the argumentative structure of the texts. We suggest that prospective teachers' models for linguistic education are often the standard ones, based on conformity to grammar or style rather than on adequacy with respect to goals.

Keywords: Evaluations, language, argument, prospective teachers, mathematical content.

Introduction.

Argumentation processes are growing more and more important in both research and curricula. In Italian curricula for the primary and secondary school level, argumentation is regarded as an interdisciplinary practice, related to mathematics but also to language, history, science and philosophy. Furthermore, it is given the function not only of representing what has already been learned but also of promoting learning. Its interdisciplinary character can be an obstacle to the development of appropriate teaching practices, as some teachers may stick to their subject and disregard the other aspects. Albano et al. (2019) found in an experiment that high school teachers evaluating students' arguments focused on the correctness of mathematical content only, and preferred concise though incomplete rather than more detailed and complete arguments. The goal of our study is to investigate more systematically how mathematics teachers (in this case, prospective teachers) evaluate students' written arguments. In particular, we want to highlight the aspects to which teachers give greater importance, what weight they give to material errors related to mathematical contents, to the lack of explicitation of the premises and logical connections, to the improper use of symbolic notations or verbal language. We expected attention to the argumentative structure, in accordance with school curricula, since prospective teachers have had some opportunities to reflect on argumentation in mathematics education as a teaching goal. The issue addressed in the paper can be seen as part of the more general problem of how teachers can promote students' argumentative competence.

Theoretical framework.

Argumentation in mathematics education has been widely studied in the last years, and a number of theories and models have been proposed, related both to purposes and form of argument. In this regard we assume that the main goal of argumentation in school is making the links between mathematical properties explicit, rather than convincing or warranting. Contrary to some current perspectives, which relate arguments to the application of external rules or constraints (e.g., logical rules), we assume that arguments in mathematics are based not on form but on meaning, and validity is a semantic property, according to Tarski's definition of *logical consequence* (Tarski, 1944). A statement P is a logical consequence of a set X of statements if P is true in all the interpretations that make all the statements of X true. A deep and insightful discussion on these topics has been carried out by Catarina Dutilh Novaes (2020). The idea of logical consequence is the core of classical mathematical logic and thus of mathematics. A good example of the role of meaning in mathematics is given by the way D. Hilbert formulates and comments the first axiom of the book in his Foundations of Geometry (Hilbert, 1909).

> I, 1. Two distinct points A and B always completely determine a straight line a. We write AB = a or BA = a.
>
> Instead of "determine," we may also employ other forms of expression; for example, we may say A "lies upon" a, A "is a point of" a, a "goes through" A "and through" B, a "joins" A "and" or "with" B, etc. If A lies upon a and at the same time upon another straight line b, we make use also of the expression: "The straight lines" a "and" b "have the point A in common," etc. (Hilbert, 1909, p.2)

Here Hilbert is not proposing the application of logical rules or other constraints, but a set of synonymous, interchangeable expressions. Expressions like "A and B lie upon A" and "a joins A and B" have different syntactical structure but the same meaning. So, what relates one expression to another is not form but meaning.

We adopt a semantic approach to argumentation for two main reasons. First, it is much nearer to mathematical practice, as it is impossible to describe any non-trivial argument according to its conformity to logical rules or other. Second, it is more useful in mathematics teaching, as the progressive achievement and refinement of meaning is a common feature of development of arguments at any school level, that any teacher can promote, recognise and evaluate.

Furthermore, an argument is, first of all, a piece of text, as argued by Bermejo-Luque (2011). So, language must be regarded as a relevant factor in the analysis of arguments. In this regard, we adopt a functional-linguistic perspective according to Halliday (1985). In this frame the focus is on the functions of a piece of text rather than on its form. In particular, we regard linguistic competence as the ability to critically and effectively use resources in different contexts rather than as conformity to grammatical or stylistic models. So, in our perspective an argument is the verbal description of the relationship of logical consequence between a set of assumptions and a conclusion. This means showing that in any interpretation where the assumptions are true, the conclusion is also true. Of course, a verbal description cannot be complete, as it is based on the meaning of words, that cannot be all defined. So, the degree of explicitation is never absolute but has to be related to the teaching goals and, possibly, negotiated in the class.

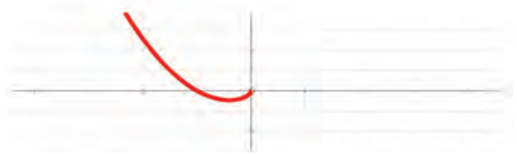
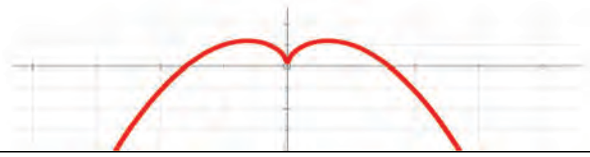
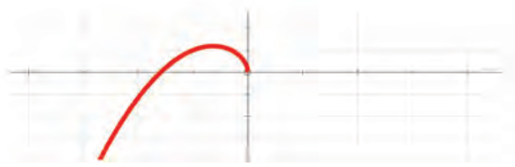
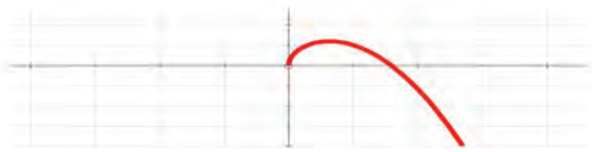
In this frame, we wonder how prospective mathematics teachers evaluate students' written arguments, in particular how much weight they give to the correctness of the contents or to the level of detail in the description of the logical connections.

Methodology.

The experiment.

The first step of our study consisted in collecting (and analysing) a number of written explanations produced by about 150 freshman students enrolled in a 3-year BSc degree in Biology (hereinafter, "students"). They were asked to solve the task in Table 1 and explain their answer. Then we selected some of the answers and we asked a group of prospective mathematics teachers (hereinafter, PTs) to evaluate these arguments. The PTs were attending a course of Mathematics Education for the second year of a master degree in Mathematics. The focus of this paper is on how prospective teachers evaluate the written arguments produced by students, while solving the task in Table 1.

Table 1: The task for the students

Let *f* be defined by: $f(x) = -x^2 + 2\sqrt{-x}$ a) Find the domain of *f*. b) Find *f'(x)* c) One of the following graphs corresponds to *f*. Which one? Explain.	
A) 	B) 
C) 	D) 

A priori analysis.

Problems of this kind have been presented by some of the authors in other occasions (see for example (Albano et al., 2019). They have proved suitable for triggering arguments because they involve different modes of representation (verbal, figural, symbolic), they require some basic knowledge only but cannot be solved just through the perfunctory application of methods learned at school. The crucial information to solve the problem is in the statement "One of the following graphs corresponds to f". Without it there is no way to give an answer and in any case reasoning by exclusion is necessary, since without further information it is never possible to conclude that a visualization of any graph corresponds to some equation. In some cases (such as here) it is possible to conclude that a graph does not correspond to an equation.

Freshman students' explanations.

Here is the English translation of a selection of five explanations provided by freshman Biology students. We are aware that some features of the text may be lost or distorted in the translation.

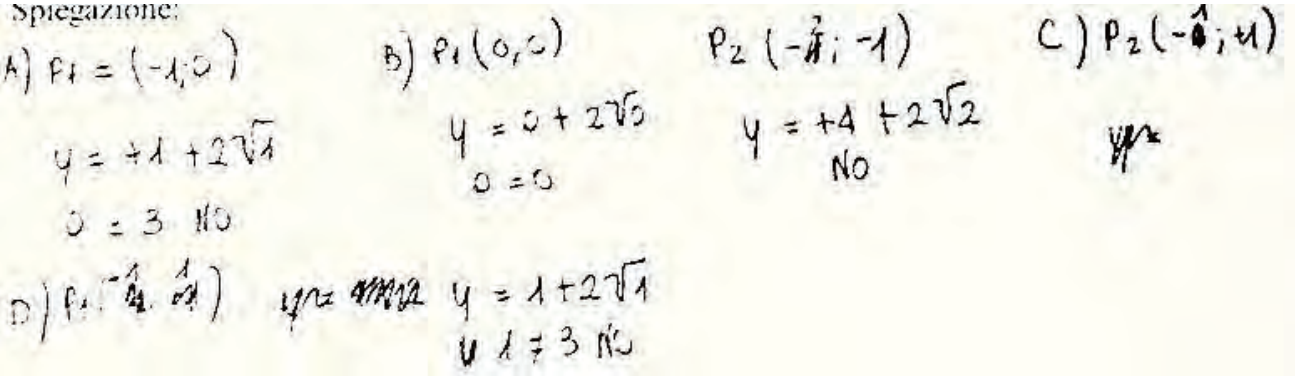
The selection of the arguments was made by taking those that were representative of the most frequent patterns. In particular we chose a detailed one (E1), a couple of concise ones (E2, E3), one without words (E4) and an incoherent one (E5).

E1: From the graphs given, observing the domain of the function f (x) (-∞, 0] we know that the function exists only for values from −∞ to 0, so we do not expect to have the function for positive values from 0, to + ∞; from this observation we can exclude graphs B and D, as they prove that the function exists for positive values (0, + ∞). Substituting in the function f (x) −1, then f (−1) ⇒ = −1, therefore values of x = −1, we obtain that y is equal to 1. The graph A can be excluded because for x = -1 the function in the graph has values y = 0. So by exclusion the graph corresponding to f (x) is graph C, as it satisfies the conditions.

E2: [Checks graph C on the diagram] I exclude the other graphs since B has domain in all R. Graph D has domain in x ≥ 0 and graph A has domain x ≤ 0 but for x = −1 y = 1 so I choose graph C.

E3: Graphs B and D are excluded because the domain is x ≤ 0. Graph A is excluded because f (-1) = 1. By exclusion, the graph corresponding to f(x) is C.

E4: [There is no verbal text, beyond three occurrences of "NO", so we present it as an image]



E5: Graphs B and D can be excluded for the domain. Studying the 1st derivative which is always negative because the domain, the function is always increasing so it is C. [1]

The sample.

The sample considered for our study consists of 50 prospective secondary school mathematics teachers, working in groups of 3 to 5. They were given copies of students' explanations and asked to evaluate them, in accordance with some guidelines, which were intended to make explicit the aspects we wanted to focus on: the weight given to mathematical content, to language, including symbolic notations, and to the argumentative structure. To this end, the guidelines drew the attention of PTs on the clarity of students' explanation, on their length (if they could be shortened or expanded in their opinion), on the mathematical correctness, on the linguistic appropriateness (possibly related to

[1] This is a good example of the drawbacks of the translations of students' excerpts from one language to another. The english translation could be mended by simply adding 'of' after 'because', whereas the Italian original text ("perché il dominio") requires the addition of some verb.

school level) and on the overall acceptability. Moreover, PTs were asked to indicate the critical points and add any comments they believed appropriate. It is worth mentioning that in the course of mathematics education PTs have had the opportunity to reflect on language, and on argumentation in mathematics education and on the importance of developing linguistic and argumentative competencies as a teaching goal.

To analyse the written evaluations provided by the PTs, we observed which elements they took into consideration. More in detail, we searched for finding if there were more references to mathematical content, e.g. material errors, or to the explicitation of the links between the parts of the arguments or to linguistic inaccuracies. We looked at which linguistic aspects the PTs gave weight and how they assessed, for example, improper use of symbols, or of language. For example, comments discussing the mathematical content of poorly formulated explanations without any remarks on language would show a strong focus on the content and a weaker interest in the other aspects. Furthermore, we also found it very interesting to look at the extent to which their attention is paid to other aspects such as completeness and coherence. Finally, we observed also the overall evaluation of each argument and also the models emerging from teachers' comments, in particular if there were expressions showing that they were more concerned about the conformity of arguments to some pattern or about their appropriateness related to some goals.

Data analysis.

We are focusing on the three main aspects outlined in the description of the task:

(1) Mathematical content
(2) Language (including symbolic notations)
(3) Argument (explanation of the steps, coherence, …)

What seems to emerge from the analysis of the protocols is that in most cases PTs, when dealing with the arguments produced by students, evaluate the content rather than the text with which it is communicated. In other words, the texts seem to be considered as carriers of mathematical content rather than as a component of students' productions to be evaluated. Some of them, for example, ascribe the incoherent organization of the text of the 2nd statement of E5 to the wrong calculation of the derivative. An extreme example is given by explanation E4 above: all the groups of the sample gave some interpretation of the procedure adopted by the student and, although there is almost no written text beyond formulas, about half of them describe the (presumed) argument as inadequate or lacking in rigor. According to a couple of groups this procedure shows some understanding of the mathematical ideas involved. Another group claims that this explanation could be appropriate if the student were at level 11. The reference to some school level as a term of comparison is another aspect emerging very often in teachers' evaluations, in most cases related to language. Arguments are considered more or less acceptable according to the (real or hypothetical) level of the student. The following comment, related to argument E1 is an example of this viewpoint.

> Although we reasoned exactly as [the student] did to proceed more quickly, we find this a sufficient but simplistic explanation of the problem for a student starting university. The answer would have been good if the student had been at a lower school level.

As far as language is concerned, most of the teachers who take it into consideration seemingly appreciate the conciseness rather than the completeness of the explanations. Moreover, they are much more focused on the use of mathematical terms or notations rather than on the overall construction of the argument. For example, although some groups appreciated E1 (which is probably the most detailed of the lot), some others claimed it is redundant. Most of the evaluations point out that expressions like ‘even graph’ are wrong, as (they claim) evenness is a property of functions and not of graphs, but on the other side a large part of the teachers was very cooperative (in the sense of Grice, 1975) when interpreting more ambiguous statements.

In general, most groups mainly focused on mathematical content and interpreted students’ productions in a very cooperative way, filling the gaps and neglecting some linguistic inaccuracy. In spite of their sensitivity to content errors, most groups interpreted them as occasional slips, ascribed to distraction or forgetfulness.

We report an evaluation of E2 where special attention is paid to the linguistic aspect.

> The explanation is extremely concise, but by no means incomplete. […] The calculation of the derivative is correct except for one multiplicative factor. We can reasonably conclude that […] the student did not arrive at the correct solution due to a distraction. The explanation […] reveals a fair knowledge of the topics covered […] Expressions such as ‘having domain in’ are not officially recognized nor in common use. The use of mathematical symbology is also poor, as can be seen in ‘for $x = -1$ $y = 1$’. The student omits the use of commas due either to sloppiness or to some difficulty in linking different propositions.

This comment on E2 is a bit different.

> Synthetic and not rigorous explanation, the student has the basic notions and can roughly solve the problem, but cannot adequately motivate it. The description of the domain is informally, but not formally, correct. The calculation of the derivative is incorrect. […] The language is not mathematical, and it is not appropriate for a student who should have mathematics maturity.

Also another group criticizes the use of language in E2 (but their symbolic, ‘meticulous’ representation of the domain is wrong).

> […] For example, a statement like “the graph X has domain Y” is a common error. We are sure that the student has understood the underlying concept but it is not the graph that has that particular domain but the function represented by the graph. There is also an error in the derivative, as a factor ‘2’ is omitted. We assume however, that this is just a simple, not trivial, slip, since the theorem of derivation of compound functions has been correctly applied, which in other answers has not been done. To be meticulous, the domain of the function could have been written more precisely, for example: $\{x \in R \vee x \leq 0\}$, however we believe it is quite a passable thing. As regards the use of Italian, there is a doubt about the phrase “I exclude the other graphs” placed at the beginning of the answer, since it is not clear whether it begins directly in this way (and in this case it would be unjustifiable, especially for one fifth high school or first year university student) or, having marked answer C as correct, she begins by directly motivating why the other graphs cannot be exact (in this case it is justifiable), precisely according to the answer already marked in the diagram. Not only that, we also add that in explaining why graph A could not be the exact one, one could be a little more precise by saying who the $x = -1$ $y = 1$ was referring to and not throw it there without adding any details. We believe that the answer is correct and suggests that the student has quite clear the basic mathematical concepts.

This group points out the improper use of mathematical notation as well.

> Apart from an error in the calculation of the derivative, the answer is mathematically correct and the thinking process is easily interpretable […] we consider it acceptable. The language could be

> improved, for example the student could have formalized the link between 'x = -1' and 'y = 1', however on the whole it is concise and clear and therefore adequate for his school level.

Some groups give a better evaluation to E2. However, all the PTs approved the reasoning in E2.

A fairly large number of PTs criticize reasoning by exclusion as in the next two evaluations.

> In this case an argument stronger than the simple exclusion would have been preferable, even a simple condition of belonging of particular points to the function would have been more adequate and appreciated.
>
> However, since the student has solved the question by exclusion, no certain confirmation can be drawn from it regarding the understanding of the "alternative" concepts which could be referred to for the direct resolution of the problem.

The following evaluation, even bolder than others, contains mathematically wrong suggestions.

> The derivative is wrong and as regards the choice of the graph, even if correct, it goes to the exclusion when it was enough to make the limit to $-\infty$ to identify the correct graph.

Discussion

Mathematical content

Prospective teachers were able to somehow interpret the mathematical content of the explanations, even when they were vague, inconsistent or even missing. In most cases they ascribed mathematical errors to slips or distractions. This is an interpretive pattern rather common among secondary school mathematics teachers and seems to depend on the interpretation of knowledge as separate from performance. The cooperative attitude of teachers is a known phenomenon that affects the attitude of students as well, who are aware of it and sometimes attribute failure in communication to the lack of cooperation of the teacher rather than the inadequacy of their product.

Language

Teachers in some cases did not consider language at all, in others they seemingly evaluated students' productions according to their conformity to some (generally unspecified) pattern rather than to their effectiveness in order to communicate their answers. A clear example of separation between effectiveness and conformity is the comment reported as the first indented citation of the section 'Data analysis': teachers rate the explanation as inappropriate, but admit they had followed the same pattern to proceed faster. Also the comparison between the text produced and the school level suggests a view of language as an (optional) tool to appropriately express already developed concepts rather than a fundamental one to promote learning. The other linguistic remarks are concerned with mathematical vocabulary: most teachers criticize the use of expressions like 'domain of the graph' or 'even graph' in place of 'domain of the function', 'even function'. Also in this case the use of 'graph' may not conform to mathematical vocabulary but is not a great danger for the clarity of the text.

Argument

In general, there have been very few remarks focusing on the structure and the effectiveness of arguments. Teachers have discussed the length of the explanations or the use of particular expressions but have not shown much interest in the explicitation of the relationships between the statements. E1, which is by far the most explicit, has been criticized for being redundant, whereas many groups have praised explanation E2 or E3. Actually, explanations like E3 (but also some of the others) are incomplete and do not give much information about the level of understanding of the student. Some teachers claimed that reasoning by exclusion is not the most appropriate way to solve the problem. We have remarked in the a priori analysis that there is no other way to answer the question. Reasoning by exclusion is based on a well known deductive rule (from 'P or Q' and 'not P' one can infer 'Q') and is perfectly correct from the viewpoint of logic. Most likely some teacher interpreted the explanations as occasions for students to display their mathematical knowledge. Again, there is some separation between knowledge and goals.

Summing up, the PTs involved in this experiment showed some more attention to language compared to the experienced, in-service teachers involved in the experiment discussed by Albano et al. (2019). Trouble is that their models for linguistic education are the standard ones, based on conformity to grammar or style rather than on adequacy related to goals. This suggests that language's evaluation by mathematics teachers is based on different criteria from those adopted by language teachers who follow the national curricula. Also mathematical notations have been considered as warrants for some unspecified rigour, to be applied in accordance with norms rather than semiotic tools capable of expressing meaning in relation to different purposes. This suggests that the semiotic systems of mathematics (verbal language, symbolic notations, figural representations) are to be regarded as such rather than as highways to get some already existing mathematical content not depending on them. So the use of such systems and in particular the conversions between them should be a topic for reflection for prospective teachers. Similar considerations hold for the argumentative aspects: the explicitation of the links among the statements has not been regarded as a relevant factor for evaluation. Apparently, the PTs expected students to write explanations to display their mathematical knowledge rather than to make their thinking process explicit and motivated.

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Multilingual mathematics teacher's professional identity in multilingual mathematics contexts

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Keywords: Language diversity, multilingual mathematics teachers, professional identity.

Introduction

Language diversity constitutes an overarching notion enclosing terms referring to the multiplicity or variety of language(s) used by teachers or/and learners at the school and societal level, such as multilingualism, bilingualism, second language, or varieties of the same language (Barwell et al., 2017). Multilingualism describes both the teachers and the context, at the current stage of the study, as multiple languages may be present in the school context even though they are not all used.

The need of broadening our understanding of teachers learning processes for teaching in multilingual contexts has recently been posed (Barwell et al., 2019). Even though teachers' teaching dilemmas in multilingual classrooms have been studied (Barwell et al., 2017), scares is the research on identity issues in cases of multilingual teachers in language diverse contexts, especially in Sweden (Delacour, 2020). Multilingualism is recognised in the Swedish educational system, while the role of teachers of all subjects, thus of mathematics, is to use and reinforce students' linguistic repertoires in the learning process (Skolverket, 2021). Therefore, the aim of this project is to provide empirical evidence and develop an understanding of the formation and development of multilingual mathematics teachers' professional identity and of their relation to school and multilingual mathematics classroom contexts. The following leading research questions being posed are: *How does teaching in multilingual mathematics classroom, as well as other past and present experiences influence multilingual mathematics teachers' professional identity? How do multilingual mathematics teachers' professional identities transform over time?*

Theoretical background

Conceptualising identity is viewed as more than self-image or reflection, and its dynamic nature is highlighted as being seen as 'a process of layering of events of participation and reification, in which our experiences and its social interpretation inform each other' (Wenger, 1998, p. 151). The theoretical framework of Patterns of Participation (PoP) (Skott, 2018) will be followed. Incorporating the work of Wenger and Holland and symbolic interactionism, it provides an understanding of teacher's professional identity in cases they are not part of predefined practices, like professional development program. PoP by focusing on teacher's interactions within their profession and in the classroom investigates how prior practices and figured worlds contribute to teacher's experiences of being, becoming and belonging as they engage with others in their profession and what changes may exist in the interplay of those over time (Skott, 2018).

Methodology

Being at the designing process, a multiple case study (Baxter & Jack, 2008) for a longitudinal period of 18-24 months, is proposed, understanding of teachers' professional identities in the long term. The study of three cases of multilingual mathematics teachers in lower secondary mathematics classrooms in Sweden are to be followed. The cases are proposed to be diverse school settings so as possible contrasts may provide multiple insides on those teachers' identities. Multilingual teachers will be selected according to the use of languages other than Swedish or English in different context, rather than their languages proficiency, while mutual linguistic repertoires with some of their students will be looked for. An ethnographic approach will be followed. Regarding the data collection process, multiple sources will provide a holistic view of teachers' identity (Skott, 2018). For that reason, semi-structured interviews and observations will be conducted regularly along the longitudinal period. Interviews will be used for informing about past and present experiences of the teachers about their education, teaching and language, also discussing issues emerging from the observations along the study. Observations will focus on the teacher, specifically teachers' interactions with multilingual students in instances that are related with language and mathematics. Following PoP, analysis will be conducted along with the data collection period, without predefined codes or categories, informing the data collection process.

The poster aims to discuss the contribution of such a project in the field and the methodological decisions for the specific aim, for the data collection process to start following the poster presentation.

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Persuasive moments, and interaction between authoritative discourse and internally persuasive discourse when using GeoGebra

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The purpose of this paper is to explore students' discourse when working in small groups on mathematical problems using GeoGebra. Specifically, the interest lies in to what extent the discourses are internally persuasive and to what extent they are alien to the students themselves. The data analysed are drawn from an upper-secondary class of students with histories of low attainment, focusing on functions and the Cartesian connection between algebra and geometry. Discourses of visual appearance were more present than academic mathematical discourse, and discourses of school as compliance counteracted students' appropriation of mathematical discourses into their own internally persuasive discourses.

Keywords: Classroom communication, discourse analysis, inquiry-based learning, dialogism, dynamic geometry software.

Introduction

Mathematics teachers and educators want students to integrate mathematical ways of thinking and communicating into their own thinking and communicating. According to sociocultural theories, the main goals of learning consist in expanding the students' action- and meaning potentials (e. g. Wells, 1999, p. 48), and other researchers have also argued for the motivating satisfaction that people derive from being able to "do something one could not do before" (Papert, 1980, p. 74). Unfortunately, for many students, mathematical ways of thinking and communicating stay alien to a large extent. Mathematics is only ever thought of as the words of authority, to be imitated in order to satisfy the requirements of the teacher, and ultimately, the school system. Many students thus rarely use mathematics studied in school in order to think or communicate about anything except school tasks.

The affordances offered by dynamic geometry software to visually perceive representations of mathematical objects, including the covariation of variables, have the potential to facilitate student's experiences of being able to do something one could not do before. Yet, there is a lot to learn about how students interact with such software in the classroom and the different types of learning made possible by different didactical designs. I therefore explore dynamic geometry problem solving discourses in an upper-secondary mathematics classroom of students with histories of low attainment. Here *discourses* refer to sequences of utterances.

Learning and authoritative and internally persuasive discourse

When we communicate, we are always responding to, and making use of context, which includes social and physical settings. What we assume to be our common ground, our shared assumptions about the world and the situation we are in and what it is we are trying to achieve, is crucial. When a group of people frequently interact in some sphere of shared activity, they develop patterns and types of utterances that are relatively stable. Bakhtin refers to these types as *speech genres* (1986, p. 60).

The goal of learning mathematics, from a dialogical perspective, is that students expand their discursive repertoires to include *mathematical discourse*—the historically established ways of communicating that competent users of mathematics employ (e.g., Sfard 2008). This constitutes the speech genre of mathematical discourse. But not all talk in mathematics class is directly related to mathematics. An important explicit theme as well as background assumption of communication in classrooms are the demands that the school and the teacher make on students. This I refer to as *schoolwork discourse* – utterances that refer to, or seem interpretable only in the context of the school as an institution that makes demands on students to finish certain work to some standard.

I relate learning to Bakhtin's concepts of *authoritative discourse* and *internally persuasive discourse*. The former is a type of discourse that demands acceptance, and derives its power from social authority, "independent of any power it might have to persuade us internally" (Bakhtin, 1981, pp. 110–111). For example, when mathematics functions as prescriptive rules to be followed, without justifications that are convincing to students, it is an authoritative discourse. On the other hand, a discourse is internally persuasive when it becomes tightly interwoven with "one's own word" (Bakhtin, 1981, p. 345). It is discourse that enters into an interaction and struggle with other internally persuasive discourses, which are all the other "available verbal and ideological points of view, approaches, directions and values" (Bakhtin, 1981, p. 346). It connects with and has an effect on our own discourses, being partly assimilated, partly modified, and always subject to our own creative intentions. In other words, it expands our discursive repertoires.

While a discourse can be internally persuasive without any observable indicators, some types of behaviour would imply that discourse is internally persuasive: when students explicitly make, test and modify mathematical conjectures themselves, they show that the mathematical discourse is interwoven with their own discourse and that mathematics interacts with their other internally persuasive discourses. In contrast, when students apply rules for calculation, without making sense of the rules themselves, after which they ask the teacher "is this right?", the teacher's (or the textbook) discourse has not become internally persuasive, it is only authoritative and remains alien to the students. The research question guiding this study is: How do everyday, mathematical and schoolwork discourses intertwine and interact in the problem solving discourses and to what extent do mathematical discourses become internally persuasive for students?

Method

This paper builds on my longitudinal case study of the classroom. In prior papers I described whole class discussions on contextual (paper-based) tasks (Gíslason, 2019), and a dynamic geometry task (Gíslason, 2021), while here the focus is on students in interaction with peers, while working on tasks.

The setting of the study is an upper-secondary school classroom in which most of the students have a history of low achievement in mathematics. The teacher of the class did not rely on textbooks available on the market, but rather found, translated, and adapted tasks from various sources, and made working with dynamic geometric software (GeoGebra) a centrepiece of classroom work.

The data analysed in this paper is drawn from two lessons, all involving the solving of tasks that are intended to make students aware of the Cartesian connection between algebra and geometry, more precisely on concepts of functional dependency and the interpretations of graphs. I recorded pairs of students selected at random, with a hand-held video camera that followed my focus of attention. My role was generally passive, but I was not invisible and the students sometimes addressed me, both to chat and to ask about mathematical tasks. All verbal utterances were transcribed verbatim in Icelandic and finally excerpts chosen to be presented in this paper were translated to English.

In analysing the data, I followed a dialogical approach. This means that I interpreted communicative actions (such as an utterance, gesture, or an input to a computer program) in context, as responses to what was said and what happened before and as initiations to further responses. The interpretations are also informed by my experience as a mathematics teacher and researcher and I began the exploration with an a priori theoretical distinction in mind, that of discourse being either closer to *everyday discourse* or closer to *mathematical discourse*. In the iterative process of analysis, it became clear that these often were intertwined, and the third major category became apparent, that of *schoolwork discourse*. Finally, I noticed that sometimes students seemed to be convinced by themselves that they had found an answer to a question, while at other times they sought external confirmations from an authority. I took this difference to correspond well to Bakthin's notions of internally persuasive discourse for the former and authoritative discourse for the latter.

Analyses of episodes

In the following I present two sequences of dialogue, selected to illustrate the three main types of discourse, how they intertwine, and function as internally persuasive and authoritative discourse. The students have all been given unique pseudonyms.

Whether to believe one's eyes: everyday speech and alienated mathematics (episode 1)

Ragna and Drífa have created a straight line using the line-tool in addition to the line that is modelled with the parameters *m* and *b* (which may be changed by moving sliders). The intention of the teacher was that students would use the sliders to change the line in accordance with task-directions. The first question asked the students to make it horizontal and go through a fixed point (that they had chosen freely at (4,2)). In this sequence the students address me, as I am recording them, treating me as an authority on the mathematics. Underlining indicates vocal emphasis.

1	Ragna:	So you want this to go [Addressing me, the researcher.]
2	Drífa:	through? [Ragna moves the slider for *m* back and forth, the line changes slope.]
3	Researcher:	We want this line to go through the point, and be … and be horizontal.
4	Drífa:	Okay. [The line now approximately going trough P but it is not horizontal yet.]
5	Ragna:	It isn't horizontal.
6	Researcher:	Then you must change it so that it will become horizontal.
7	Drífa:	You have to move this one. [Drífa points on the slider for *b* (y-intercept).]
8	Ragna:	Oh, okaaay. [Ragna moves the slider for *m* so that the line goes trough P, and then she moves the slider for b which moves the line in parallel off the point. This way does not work.]
9	Ragna:	"No can do". [In English.] [Ragna now adjusts the slope, until m=0, and the line is horizontal, but does not go through P.]
10	Drífa:	Yes! Yes like this. And then you move, no. [Ragna increases and decreases the slope (m), back and forth. Drífa points her finger to the point P.]

11	Drífa:	I think you should do …
12	Ragna:	Aargh. Difficult to own a mac. [Tinkers with the slope until the line is horizontal.]
13	Drífa:	Woah. [Swiftly moves the line, so that it goes through the point P.]
14	Ragna:	What is up!? [A phrase used to express surprise or joy.] [They open the note where they have copied the task-questions.]
15	Drífa:	Tada! ... And then m gives zero and b gives two. Doesn't it?
16	Ragna:	Yeah [They start typing in the note: m=0] Okay
17	Drífa:	Okay. Isn't that right?
18	Ragna:	Is it correct? Please tell.
19	Researcher:	Is this a horizontal line through the point P?
20	Drífa:	Yes.
21	Ragna:	It lies!
22	Researcher:	Yes.
23	Ragna:	Totally horizontal.
24	Researcher:	So why are you asking me?

At the beginning of episode 1, the students show that they want to comply with what the authority wants. In turn 1, they seem to assume that I, the researcher, is in the same authoritative position as the teacher and that they are responding to the perceived demand of the schooling situation. I respond with the "we" pronoun, as is common for teachers, possibly to try to frame the task as shared, something to be achieved together. I take this to be an example of schoolwork discourse, as the goal of "finishing the job" is an ubiquitous assumption of schoolwork.

In episode 1, students expressed joy when they managed to adjust the sliders to get a horizontal line (turns 13–15). Their first attempt was to move the slider for the variable b (constant term), until the line coincided with the fixed point, and then change the value of m (slope) to make the line horizontal (turns 4–9). They claim that this is impossible, perhaps perceiving that the "center of rotation" of the line is not in the fixed point. They then try the other way around; change m to make the line horizontal and then change the constant term to translate it (turns 10–14). In turn 15 Drífa reads the parameter values, $m = 0$ and $b = 2$, from the algebra window. By doing so, and noting "the answer", they show awareness that the mathematical object has both an algebraic representation and a graphical representation. They both express joy that they have found the solution (turns 16–17). However, as far as can be discerned in their talk and interaction with GeoGebra, they relied solely on slider manipulation and visual appearance, without any use of meaningful links between the algebraic and the graphical representations. For example, they did not mention that the slope should be zero to get a line parallel to the x-axis, nor did they express indications of recognising this after the fact. They achieved their goal using mainly everyday speech, more or less bypassing mathematical vocabulary and reasoning based on properties. Having found an answer, they were not fully persuaded by the visual appearance, and rather than linking their work to mathematical concepts themselves, they asked me, in turn 18, as an authority, to confirm the answer. In turn 19 and 24 I strongly imply that they *should trust* their own eyes, which can be interpreted as validating their visual, trial-and-error approach, rather than challenging them to explain their reasoning. It is possible that their experience with mathematics tells them *not to trust* their senses, and it is indeed important to reason on the basis of properties and not only from appearance. In summary, Drífa and Ragna did not make much use of mathematical discourse and it was not present in their internally persuasive discourses. Their everyday discourses were up to the task, and they produced a solution. At the same time they did not *fully* trust that they had an answer that would satisfy the teacher. Perhaps they had some awareness that appearance can be misleading, and therefore they sought additional confirmation for themselves through an authority.

Internally persuasive mathematical discourse (episode 2)

One main goal of the class was to get students to appropriate the language of variables. An experience of covariation can be made possible by creating a variable (represented by a slider) and linking that variable to a screen object, functionally dependent on the variable. The task text was as follows:

> Draw the following in GeoGebra:
>
> a) Make a square that can be enlarged and shrunk with a slider.
>
> b) Add a new slider that moves the square horizontally.
>
> c) Add another slider that moves the square vertically.

The link between a variable and a screen object is not given in the above task, unlike the task in episode 1. It is expected to be created by the student. The variable will create an interactive visual effect, closely linked to mathematical properties.

The teacher assisted students in constructing a dynamic square with vertices (*a*, *a*), (*a*, -*a*), (-*a*, *a*) and (-*a*, -*a*). In the following episode, two girls, Lilja and Anna, talk to each other and with the teacher, working on the second question, trying to create a slider that can move the square horizontally on the screen. In the first utterance Lilja suggests a modification, making it possible to move the vertex of the square via a slider determining a variable called *b*.

1	Lilja:	Plus x times b. [Might mean “add one unit, b times” to the x-coordinate, although a more streamlined way is to say “add b to the x coordinate”.]
2	Teacher:	Plus x times b? [Neither affirming nor rejecting, opening for further elaboration. He either does not follow or does not want to make the interpretation for the student.]
3	Lilja:	No [Shakes head, looks at the teacher].
4	Anna:	No... oh … I can’t remember which is x and which is y.
5	Teacher:	Okay the first number is always x and the second number is y.
6	Anna:	Okay should I then do .... aaah [Frustration, both hands waving].
7	Lilja:	We just want the x you know. [Referring to her knowledge that horizontal movement is described by a change in the x-coordinate.]
8	Anna:	Yeah the x is here, I am at the x you know. [Referring to the first coordinate as “x”.]
9	Teacher:	Yes.
10	Anna:	Okay, what should I do just ... plus?
11	Teacher:	Yes, yes what.
12	Lilja:	After the brackets. [Points toward the screen of Anna.]
13	Anna:	After the brackets?

14	Teacher:	Na then you add both to the x coordinate and the y coordinate if you do that. [The teacher knows that (a, a) + b in GeoGebra results in (a + b, a + b). It's unclear that this has an impact on the following turn.]
15	Lilja:	Then not, you should do before the second number ... hooo [Breaths in, throws head back, opens arms, visibly excited.] <u>Before the second number do plus b</u>! [Smiles, increased voice volume and much higher pitch.]
16	Teacher:	Okay that's y, then you move it to the y. [The teacher seems to interpret the suggestion as to write something equivalent to (a, b + a).]
17	Lilja:	No, that, before the first number. [She seems to sense what the teacher meant and the need to make it clear that she means (b + a, a) or equivalently, as she tried to express in turn 15, (a + b, a)].
18	Teacher:	Okay by the first number.
19	Anna:	But why plus b?
20	Lilja:	Because, because when.
21	Anna:	But there is an a there you know.
22	Teacher:	Yes, but yes but it…
23	Lilja:	<u>I got it, I got it</u>! [Visibly excited and joyful.]
24	Teacher:	Okay, show me.
25	Lilja:	Wait, wait.
26	Teacher:	You did a plus b.
27	Anna:	I did something wrong ... I first want to see that she can do it right, then I'll trust you ... first learn to do this plus.
28	Lilja:	Gurrrl gurrrl, gurrrl, gurrrl, gurrrl. Look … oh … gurrrl, gurrrl! [Outburst of joy, smiling and using a higher pitch and volume. Lilja now modifies her coordinates, her suggestion finally implemented as she has meant it, in the software.]
29	Anna:	Okay, uhm, how did you do it?
30	Lilja:	Look, gets bigger and smaller. Just do plus b after the second number.
31	Anna:	Plus b?
32	Lilja:	You know.
33	Anna:	Yes a plus b.
34	Lilja:	Just a plus, ... there ... minus you do also plus a, no, plus b. That's always the first number ... plus b. Now you won't flunk this class! ! [This is accompanied by pointing by her finger on the screen of Anna. She is directing Anna to type, first a + b in the first coordinate place and then -a + b in the first coordinate place of the next vertex. Then she gives a generalisation: always the first number (implied: of the first coordinate-place) plus b.]

Lilja now has a square that can be moved via the slider for variable b. Her square consist of the vertices $(a + b, a)$, $(-a + b, a)$, $(-a + b, -a)$ and $(a + b, -a)$. In episode 2 the students do not immediately solve the problem, which indicates that the mathematical symbolic language of variables and coordinates (as represented in the software) was not initially a part of their internally persuasive discourses. In turn 15 Lilja expresses her excitement in having grasped the nature of the connection between the symbolic slider-controlled variable and the visual behaviour of the screen object. She has not yet implemented her idea, which only happens in turn 28 when she modifies her coordinates and sees the results, verifying her solution. It is as if (this aspect of) symbolic algebraic mathematical discourse suddenly makes sense to her, using the Cartesian link to her own intent, incorporating it into her internally persuasive discourse. At first the teacher does not follow her "before the second number, do plus b" (turn 15), an everyday type of utterance, describing spatial arrangement. Lilja wants to replace (a, a) with (a + b, a) using the variable *b* to control a horizontal movement of the point. She uses everyday discourse to orient to the positions of symbols on the screen, "after the brackets", "before the first number", and the screen object "gets bigger and smaller", yet mathematical discourse is internally persuasive for her and is evident in her input to GeoGebra.

Lilja's partner, Anna, seems not to understand Lilja's description, and she seems to be more or less stuck at trying to imitate (Lilja's) authoritative discourse, grasping for step by step instructions. In turns 6, 8, 10, and 13 she seems to be trying to follow instructions (from Lilja and the teacher) as to what she should type into the software, without consideration of meaning. The mathematical symbolic system seems alien to her, it is only authoritative discourse that does not touch her own internally persuasive discourse. In turn 19, she asks "why plus b", which I see as her attempt to bring the authoritative discourse into contact with her own internally persuasive discourse. She wants the words that she has used to give commands to the computer to have meaning for her. Because what she has typed doesn't work, she expresses doubts as to whether Lilja has really "got it" (turn 27). Lilja describes her solution to Anna only on a syntactical level in everyday language (what symbols to type in and where) but never addresses her why-questions. Instead she tries to encourage Anna that she will not "flunk this class" (turn 34), reminding us that we are in school, talking in a voice from the schoolwork genre. Both Lilja and Anna are concerned to pass the course, but in this episode the mathematical content became internally persuasive only to Lilja.

In contrast with Drífa and Ragna in episode 1, Lilja does not need a confirmation from an authority. She is convinced that she has grasped the symbolic language and the link between that language and the visual representation. She communicates to the computer through the input text: "P = (a + b, a)" and experiences directly an expansion of her action potential. Her outburst of joy *preceded* her actual typing in of the command – it was not a response to seeing it work out (perhaps luckily, through trial and error, as was the case in episode 1). Afterwards, she was quick to also add a functional variable for vertical translation, generalising the method for translation of points in the coordinate system.

Conclusion

In the first episode the students were occupied with the task as a piece of schoolwork to be finished. While they expressed pleasure of having a result (achieved by trial and error), they made no explicit connection between the visual result and mathematical concepts, and also requested confirmation of their answer from an authority. Their satisfaction was due to having finished a job, not with having made mathematical discourse their own. Thus, the schoolwork discourse can be interpreted as being in this case not conducive to learning, or worse, actively working against learning.

In the second episode one student, Lilja, suddenly grasped the relationship between the mathematical symbolism and the visual representation. While she confirmed her answer visually, she was convinced that she knew what needed to be done *before* she gave any input to the software. I interpret her satisfaction as stemming from having made mathematical discourse internally persuasive. Her partner, Anna, did not show any indication of having made the discourse internally persuasive. The schoolwork discourse's assumption that students should finish the tasks set by the teacher frustrates Anna as she was concerned that she might fail the course. Lilja tried to help Anna finishing the task, but described only a step-by-step recipe, without reference to meaning. Lilja, therefore, was not hurt by the schoolwork discourse in this case, while Anna's learning suffered.

The two tasks worked on provide different opportunities to use mathematical discourse to achieve goals. In episode 1, the students manipulated ready made sliders to observe covariation of parameters and a visual representation. This did not make the *mathematical* relationship the center of attention. In episode 2 the students were expected to create sliders for variables and then define the mathematical objects themselves, using the variables. This required students to use mathematical discourse as a semantic tool, which means incorporating it into internally persuasive discourse. One of the students did so, while the other did not.

In their problem solving, students drew on everyday discourse to describe visual elements, both the visual representations of geometric objects and strings of symbols. They also assumed the everyday practice that to be persuaded of something, it is both necessary and sufficient to empirically check its appearence. Mathematical discourse was present in their discourses to a much lesser extent, and in a way that focused more on the surface (the syntax), rather than the conceptual meaning. Schoolwork discourse was always in the background if not explicitly apparent in talk about failing the course. Schoolwork discourse seems push students to imitate authoritative discourse, that is, using mathematical discourse without having made it their own. In other words, schoolwork discourse does not bring authoritative mathematical discourse into contact with internally persuasive discourse. Rather, it functions to keep mathematics only authoritative, and alien to the students themselves.

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Handmade diagrams – learners doing math by using gestures

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It is a well-investigated fact that human interaction implies a huge range of expressive modes. Unsurprisingly young learners although do so when doing mathematics together. In the last decade, more and more research has been concerned with this so-called multimodality in mathematical situations. Diverse modes and their interrelations were under investigation, like gestures, speech, or written expressions, sometimes actions. A systematic description of potentially mathematically used gestures by young learners as a part of this mix of modes is still missing. The paper is theoretically framed by an interactionist-semiotic approach wherein mathematics is seen as a social activity of using diagrams. The subject-specific role of gestures is to be clarified: Do gestures play a constitutive role in and for mathematical interactions, literally as a handicraft to work diagrammatically? The exemplary analysis reveals different functions of gestures in the diagrammatic work of the learners.

Keywords: Functions and forms of gestures, subject-specific gestures, diagrammatic reasoning.

Introduction

Observing young learners while doing mathematics together reveals the use of different expressions: Speaking, writing, using a tool (digital or analogue), gesturing or using facial expressions are perceivable as a kind of interwoven mix of modes. Theoretically, this mix of modes is often framed by the concept of *multimodality* (Arzarello, 2006; Radford, 2009). Radford (2009) states, "[…] that mathematical cognition is not only mediated by written symbols, but that it is also mediated, in a *genuine sense*, by actions, gestures, and other types of signs." (p. 112). This *genuine sense* of multimodality, and also of gestures for mathematics learning is often seen as given in advance (Huth, 2018, p. 219). A theoretically supported rationale for this assumption is missing, or at least not described in existing approaches to gestures in mathematics education. The present paper opens up a theoretically grounded view on gestures as signs in mathematical interactions by dint of the theoretical concept of *diagrams* and *diagrammatic reasoning* after C. S. Peirce. *Diagrammatic reasoning* in this sense includes the *construction of a diagram*, the *observation of relations* in that diagram and the *formulation of general conclusions*. It potentially leads to mathematical insights and further developed diagrams (Peirce, 1931, CP 1.54). The use of diagrams, which can be of various forms (Schreiber, 2010), is still perceptible at a very early state in the mathematics learning process. The question arises, if the use of modes in mathematical interactions of young learners and their use of diagrams are interwoven as well. For this purpose, the paper will first consider multimodality in mathematics learning, then offer a definition of gestures in mathematical interaction, and describes gesture as a sign after Peirce. Finally, the diagrammatic theory is used to focus the subject specific-role of gestures in the analysis. It should be investigated, whether gestures *represent* or rather (co-)*generate* leaners' diagrammatic reasoning by the following question: *Do gestures play a constitutive role in and for mathematical learner interactions, literally as a handicraft to work diagrammatically?*

Multimodality: A systemic feature of language – and of mathematics learning?

Radford (2013) claims, that thinking is deeply interwoven with bodily acting in the material world. In this sense, thinking in general, especially mathematical thinking, is *shaped and performed* by using different expressive modes. In line with these ideas, Arzarello (2006) advocates his theory of the *semiotic bundle* to describe a bunch of signs which are used by participants in mathematical interaction. This concept unites e.g., speech as a conventionalized, rule-based semiotic system, and also gestures as a more non-conventionalized mode. It intends to break down the rigid boundaries of considering exclusively semiotic systems in mathematical situations. Krause (2016), in line with Azzarello (2006), emphasizes the representativeness of gestures in mathematical interactions. She investigates mathematical occupations of upper school pupils and describes a "multimodal sign" (Krause, 2016, p. 49) of speech, gestures and inscriptions. The representative role of gestures is grounded in the view, that mathematical objects are not directly accessible because of their alleged abstractedness (p. 16) – a different perspective than it is supposed in the present paper. With regard to the concept of multimodality, Radford (2013) and Krause (2016) refer on the psychological theory of *sensuous* respectively *embodied cognition*. It emphasizes a mind-body-embeddedness in thinking, generating and experiencing the world. *Sensuous cognition* breaks down with a dualistic view of mind (internal and external). Body and mind are no longer separated, but related parts of human cognition.

In linguistic approaches, multimodality is grounded in the *system* of spoken language itself. Fricke (2012) proves a *systemic relevance of gesture* in spoken German language. In sentence constructions, gestures can be integrated in speech to fulfil syntactical positions (so-called *code-integration*, p. 75). In addition, gesture reveals features of language in their structural use, like e.g., *recursion*[1] (so-called *code-manifestation*). Some utterances show aspects of content exclusively coded in gesture. Sometimes, gesture is even obligatory e.g., for using some deictic expressions (p. 74). In addition, gestures can lead to conventionalized forms in interaction, using the same gestures for the same meant recurrently. *Multimodality* in this sense is seen as a kind of characteristic of the language system. This idea of a fundamental *grammar of speech and gesture* underlines the constitutive role of gestures for using language in human interaction. In psychological approaches McNeill (2005) admittedly assumes an "'unbreakable bond' of speech and gesture" (p. 24) and coined the idea of an integrative language system. However, he also states a language-gesture dialectic, where gestures can never be part of grammar in his view. In line with Fricke (2012), the present paper locates multimodality in the language-systemic sense, whereas gestures and spoken language construct this system.

Gestures: Definition, and its significance in interaction

To meet the plenty of manifold definitions of gestures, Andrén (2010) summarizes two main perspectives: The first includes nearly every body movement to be a gesture: movements of hands, arms, head, mimic or facial expressions, gaze, action, etc. In the second perspective, the gesture definition is more narrowed (p. 11). It is leaned on Kendon (2004) who investigates how body movements are *interpreted* by interlocuters in interaction and due to the situational context (pp. 14–

[1]With a finite number of elements and well-defined rules of using these elements one can generate an infinite set of language constructions (Chomsky, 1983, quoted after Fricke, 2012, p. 120).

15). He basically describes gestures as “visible actions” (p. 7) and with “features of manifest deliberate expressiveness” (p. 15). The gesturer is ascribed a “voluntary control” of his gestures which are “being done for the purpose of expression” (p. 15). Goldin-Meadow (2003) differentiates gestures from *functional acts at objects or persons* (p. 8), e.g., adjusting the glasses. In her view, such actions are not seen as gestures, even though they can be done parallelly to speech. She defines gestures based on two criteria: (1) Gestures are hand motions uttered while the *communicative act of speaking*, and (2) are *no functional act* on persons or objects (p. 8). The criteria appear to be mutually exclusive and obviously strictly separates actions from gestures. It can be questioned what is meant by *a functional act on an object or person* and if there could be rather such acts which *includes* a communicative act. A simple example is the (maybe wordless) placement of a chair to broaden an ongoing discussion group while a delayed person enters the room. Can’t we assume here a *communicative act*, although a *functional act* is performed on the chair? The placement is *readable* as a communicative act by the participants, e.g., as an invitation to participate in the discussion, the *where* and *how*. Theses interpretations are based on a culturally grounded commonly shared background and the actual situation in which the participants *act with* and *to* each other mutually and interactively. Thus, the *existence* of a communicative act attributed to an action or a gesture is rather the *result* of an interpretation and not independently given, or ascribed from an outer view. In mathematics education Sabena (2008) also focusses a separative definition of gestures while doing mathematic: She claims, they “*are not a significative part of any other action (i.e. writing, using a tool, …)*.” (p. 21). As a distinguishing feature, she constates, that actions are intentional and goal-oriented (p. 35). Gestures in Sabena’s sense obviously cannot be *manifest deliberate* like Kendon (2004) claims. They are to be seen strictly differently than writing or using a tool. However, Arzarello (2006) speaks of gestures that can be *fixed in written signs* in an ongoing mathematical occupation (p. 291). Harrison (2018) emphasizes the communalities of actions and gestures. He claims, that gestures are performed according to the material world and that some actions show comparable structures with gestures in their co-expressiveness with speech. A person sitting next to another person and performs a description of a direction may draw gesturally a kind of map on his or her thigh. The thigh is used as a paper to draw gesturally the map on. It is interpreted as a *part of that gesture* which in turn depends on this thigh-paper-map. With this in mind, it seems less useful to strictly distinguish gestures from actions or body touches. Their interwovenness is also reconstructed by Vogel and Huth (2020). For mathematical learning they found an intersecting action-gesture-use in *chronology*, the *semantic attribution of meaning*, and in *function* (pp. 241–243). Billion (2021) also takes a look at gestures and their intertwining with actions in mathematical situations to reconstruct diagrammatic interpretations of learners. This paper adopts Kendon's (2004) definition of gesture and draws on Harrison's (2018) view of gestures and actions: Gestures are those movements, which are ascribed the “features of manifest deliberate expressiveness” (p. 15) by the interlocutors. Material-use or exclusive body parts are no exclusive criterion. This definition can be linked to the considered interactionist-semiotic perspective on mathematics learning, where the interpretation of the participants is of core interest.

Mathematics learning: Diagrammatic reasoning in interaction

To focus the role of gestures in mathematical interactions of learners, the nature of mathematics learning has to be described in line with an interactionist-semiotic perspective. Krummheuer and

Brandt (2001) claim, that interactions in peers and with an experienced opposite constitute mathematical learning. From a semiotic view, the use of diagrams is the core of doing mathematics (Peirce, 1931). Both approaches combined, mathematical interactions provide a space for experiencing “a social practice with, on, about, and through diagrams” (Dörfler, 2006, p. 105). Accordingly, mathematics learning depends tremendously on being and interacting with others about, on and with diagrams to increase one’s own proficiency in acting mathematically. Learners gain a kind of autonomy in using mathematical signs and argumentations and develop from the role of participant to an autonomous actor (Krummheuer & Brandt, 2001, p. 20–21). In the ongoing process of negotiation, the interlocuters form together the content of interaction and commonly coordinated interpretations. These interpretations can be used as a kind of template to contrast and adopt one’s own insights. The common coordination tends to offer stabilization, and that is, what Schreiber (2010, p. 59) calls the *framing of the situation* in line with Peirce’ *ground of signs* (1932, CP 2.228). Consequently, Krummheuer and Brandt (2001) claim a social responsibility for utterances in interactions, even if one interlocutor produces its perceivable form. It emerges from the social. Utterances in multimodal designed mathematical interactions of learners can be described as *signs* of different kind with a triadic structure (Peirce, 1932): They consist of a *representamen*, an *interpretant* and an *object*. If a gesture is perceived as a *sign*, it can be described as a sign. It has a perceivable outer form – its *representamen*. The gesture sign evokes an interpretation in the mind of the sign reader – the *interpretant* – related to the probably referenced *object* of the sign (Schreiber, 2010, p. 32). The sign interpretation of the sign reader is shaped by his/her *ground* of the sign which can be named as the activated *framing* in the above-explained interactionist view. It includes theories, habits and experiences as an interpretation background. This concept of *framing* assigns a contextual, cultural and social dimension to the interpretation of signs. The interpretant of the sign is describable as the *effect of the sign* in the mind of the sign reader. It can be uttered again, and used as a new representamen. An endless process of semiosis emerges. In the Peircean semiotic, the concept of *diagrammatic reasoning* and the use of *diagrams* is central according to mathematical occupations. This approach shifts the view from supposedly abstract mathematical objects to mathematics as using signs and diagrams that are materially perceptible and manipulable by hands (Dörfler, 2006). Diagrams are rule-based fixations which show particular relations to each other and can be of different kind: written, created on a screen or even developed by material arrangements (Schreiber, 2010, p. 27). The relations and rules of the diagram are not given in advance, but are based on social negotiation (p. 41). They can be seen as a part of the ground of the sign or diagram interpretation.

Empirical example: Gesture use of learners while doing math

In the follwoing extract of analysis the main research question is: *Do gestures play a constitutive role in and for mathematical learner interactions, literally as a handicraft to work diagrammatically?* Two second graders, Maya and Dennis, ought to find all possible permutations of three elements, presented by plastic animal figures (white tiger, brown tiger, elephant). A hugh amount of paper cards with the animal’s faces are available for documentation. An accompanying person B offers the mathematical problem and the material. Maya and Dennis put in total 7 rows of paper cards (one of them twice) in front of them. They are finally sure *‘we found all of them’*. For the analysis, the situation was videotaped and transcribed: First, an interaction analysis (Krummheuer & Brandt, 2001,

p. 90) is applied. Based on this, a semiotic analysis (Schreiber, 2010) is conducted with the multimodal adopted *semiotic process-cards* (SPC) to map graphically the chronological sign process (Huth, 2018). Every utterance is pictured by two sign triads, one for speech and one for gesture respectively with R (representamen), I (interpretant) and O (object). The gesture-speech triads are connected via a common interpretant. Simultaneous utterances are mapped with a and b in one row.

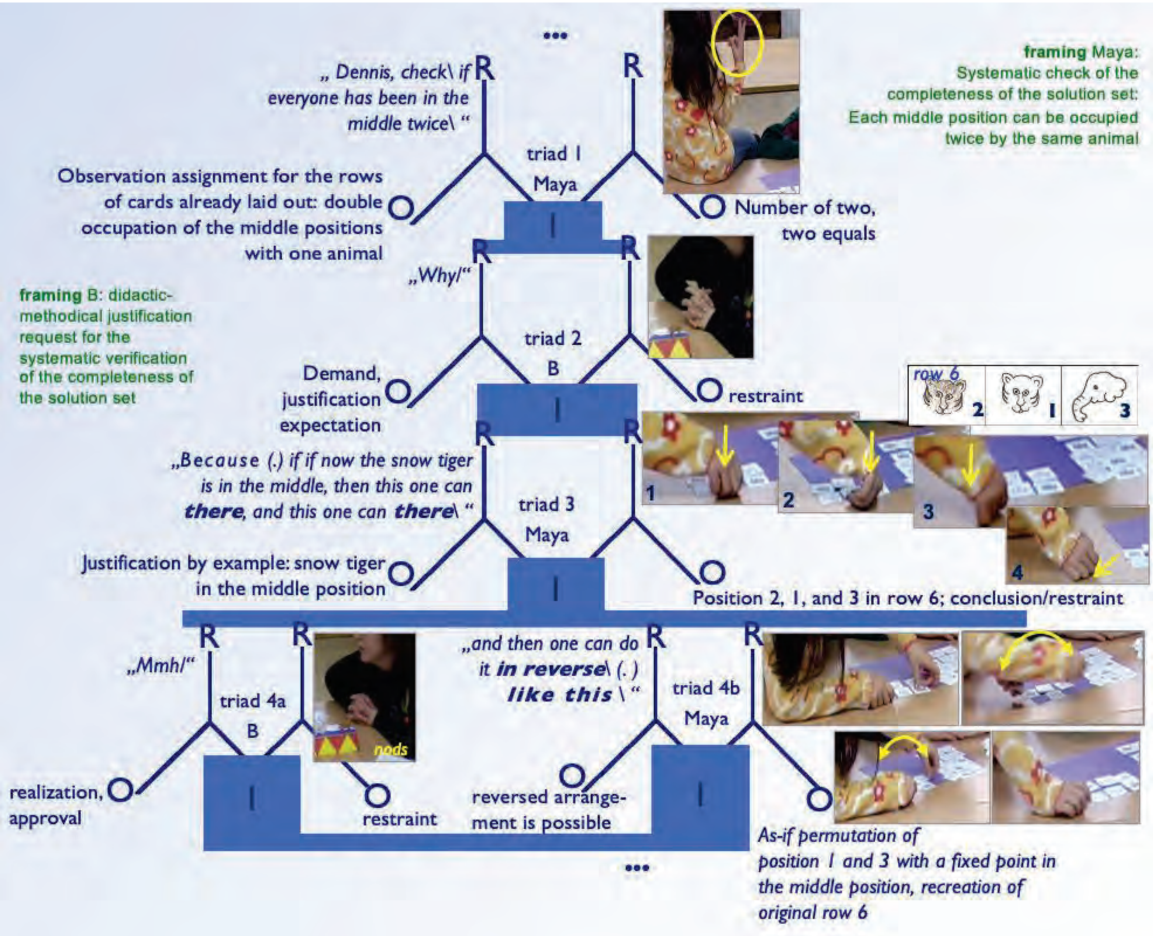

Figure 1: SPC, excerpt 1, Maya's observation instruction to Dennis

Maya starts with an observation order that refers to a first idea of a recognized generalization across all card rows: Dennis should check whether each animal has already been twice in the middle position (triad 1). Maya emphasizes gesturally the number with two similar fingers. She performs this gesture in relation to her speech in front of herself, not to the diagram of card rows, and refers to the mathematical idea of fixing points. In the SPC (Figure 1), it comes to the fore, how strongly gestures and actions are interconnected in the learners' diagrammatic reasoning: After a question of B (triad 2) the explanation of Maya is performed on the diagram of paper cards in both modes (triad 3 & 4b). The fixing point is emphasized gesturally by pressing on the middle position, and in action by leaving this position unaffected. The interchangeable positions are quickly typed or briefly swapped and returned. As activated framings we can assume a systematic review of the card rows based on a mathematical idea of fixed points. B's framing is additionally didactically overdyed. Figure 2 shows a later excerpt of the same SPC. Maya generates a new or more developed diagram mainly in her gestures (triad 23 & 25) after B asked why there are no more rows to find. Maya initially justifies her answer in the quantity of available animals. Then she uses the generated diagram of all rows to expand n to n+1, enriched with her idea of fixed points.

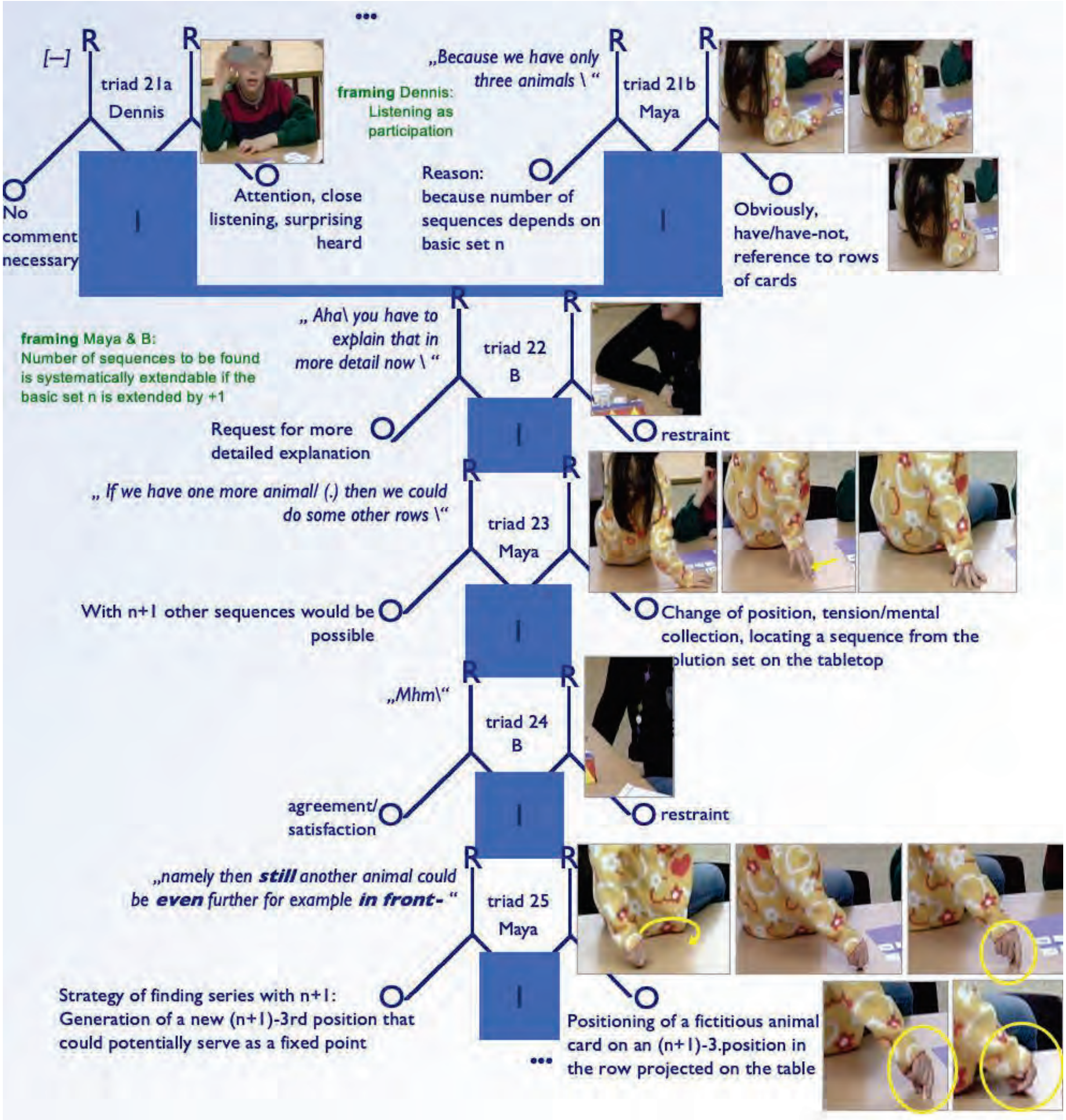

Figure 2: SPC, excerpt 2, Maya's gestural diagram construction and manipulation, rows out of n+1

Maya extracts gesturally one row out of the card-diagram to generate an example of a row (n+1). A new diagram next to the initial diagram is generated. In a Peircean sense, in her gesture she *creates a new diagram* and *generalizes the fixing point idea* to see, if it holds for all comparable diagrams. Maya gesturally outlines the contour of the new row (n+1), starting with an arched sliding gesture from the last to the first position over the table. Then she establishes a new first position in the outlined row, by using a kind of pantomiming action-like gesture with which she lays down exactly an animal card on the new front position (triad 25). With her speech, she indicates how to use the initial rows with n elements to generate rows with n+1 by referring to the concrete animal. Maya relates both diagrams to explain how to generate the second out of the first. Her gestures are used as mathematical signs to build a further developed diagram and to manipulate this gestural diagram again by gesture.

Dennis seems to honor Maya's ideas in a special way and marks it as important due to his ear-pinning listening gesture (triad 21a).

Conclusions

The example shows, that gestures play a constitutive role in and for the mathematical interactions of learners. They can be literally used as a handicraft to work diagrammatically on initially generated diagrams (Dörfler, 2006; Peirce, 1931). With Fricke's (2012) theory of a systemic relevance of gestures for language, one can conclude, that gestures in the Peircean sense have a comparable systemic relevance for acting mathematically: In the learner's diagrammatic reasoning, gestures are an *integral part of mathematical constructions*, like here the idea of fixed pints (mathematical *code-integration*). In addition, perceivable in the diagrammatic reasoning, the commonly negotiated use of *gesturally generated new mathematical diagrams* is observable (mathematical *code-manifestation*). In the shown example, the performed gestures forms range from an explication of speech (triad 1), to gestures, which create links between parts of the diagram (triad 3), until gestures, that are closely related to actions and the material world (triad 3 & 4b). The central mathematical idea is that of *fixing points* in permutations, which is first explained on *one row*, and then *extended* for the gesturally constructed new diagram. In line with Vogel and Huth (2020) gestures and actions are closely interrelated: Both modes are used in a *subject-specific sequence* to expand the generalization of a mathematical idea to n+1. Thus, with regard to the used gestures' functions, one can reconstruct a subject-specificity. They (1) *structure the mathematical interaction* (triad 21a & 21b), (2) which are *used to clarify the framing* (triad 1, 3 & 4b), (3) which *are used to present a mathematical idea* (triad 1), (4) which *manipulate the diagram* (triad 3, 4b, 23 & 25), and, (5) which are *used diagrammatically* to a general idea (triad 25). Considering that gesture mode is flexible due to its functions and forms, these descriptions should not be taken exclusive. In mathematics education research the focus should be shifted on such gesture functions in interaction to apply them to a larger database e.g., in different ages of learners. From this, suitable concepts can be developed to consider and foster the gesture use in mathematics classroom as a constitutive mode for *handmade diagrams* of learners and teachers.

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The status and negotiation of knowledge when teachers repeat students' words

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Teachers frequently repeat students' words and in doing so they are doing a wide variety of actions. In the mathematics education literature different names are given and distinctions made to the different types and forms of repetitions. This has led to ambiguity in the meaning and use of terms like revoicing, yet the distinctions made have important consequences for the negotiation of knowledge in mathematics classroom interaction. In this paper two examples are offered to illustrate one such distinction focusing specifically on the use of revoicing in the mathematics education literature, considering specifically the influence these distinctions have on epistemic rights and responsibilities within the interaction.

Keywords: Revoicing, repeating, epistemics, interaction.

Introduction

When teachers repeat students' words they could be doing a wide variety of actions. This repetition is also referred to by many different names in the literature, sometimes synonymously, sometimes drawing distinctions based on the form or the function of the repetition. These terms include for example: repeating, rephrasing, reuttering, reporting, recasting, reformulating, revoicing, rebroadcasting, repairing, and restating. These different terms often reflect the theoretical framing of the research study and are now often used without definition or explanation. Yet the differences between these terms are important when considering the function and role of repetitions in interaction in the management of epistemic access, rights, and responsibilities. In this paper I will offer some illustrations of a few of the distinctions made with a particular focus on revoicing as used by a range of authors within mathematics education, and the impact these distinctions have on the negotiation of knowledge in the mathematics classroom.

Revoicing

O'Connor and Michaels have defined revoicing as "at a minimum, this revoicing involves the teacher repeating or rephrasing some part of the student's utterance, and then opening up the next turn for the student to (tacitly or explicitly) agree or disagree with the teacher's revoicing" (2019, p. 167) building on their initial identification of this teacher move in their 1993 paper (O'Connor & Michaels, 1993). They describe revoicing as involving the teacher drawing an inference from the student's contribution, offering the student and other students the opportunity to validate or challenge this inference. They describe a range of interactional structures of revoicing including the use of discourse markers such as 'so' and turns returning to the students in contrast to the common IRE structure, but the emphasis is on the students' rights to accept or reject the inferences, and thus evaluate them.

Many researchers now use the term revoicing in their own analyses of classroom interactions, without definition or explanation (e.g., Takker & Subramaniam, 2019; Walshaw, 2017) suggesting that it has now become an accepted term within the field, but often to describe moves that only include some of the features originally described by O’Connor and Michaels (1993). The term is also being used synonymously with repeating, reuttering, rephrasing, reporting or reformulating (Forman, et al., 1998; Franke et al., 2007), recasting (Gardner & Forrester, 2010; Moschkovich, 2015), marking and rebroadcasting (Wilson et al., 2019), repairing (da Ponte & Quaresma, 2016; Hintz & Tyson, 2015), restating (Herbel-Eisenmann, et al., 2009) and building on or expanding (Forman et al., 1998; O’Connor & Michaels, 2019), as well as encompassing reformulations that involve moving from one named language to another (Enyedy et al., 2008). Some authors (and teachers (Herbel-Eisenmann et al., 2009)) make a distinction between repeating or reformulating and revoicing (O’Connor & Michaels, 2019; Walkington, et al., 2019) whilst others do not (Barwell, 2016; da Ponte & Quaresma, 2016).

This multiplicity and ambiguity in use can be explained by the different theoretical perspectives and constructs these researchers are drawing upon, including Goffman’s notion of reported speech (Forman et al., 1998; Goffman, 1981; Krummheuer, 2007), Stubb’s notion of meta-commenting developed by Pimm (1987), the notion of positioning as described within positioning theory (Harré & Van Langenhove, 1999; Herbel-Eisenmann, et al., 2015) and the notion of voices from Bakhtin (Bakhtin, 1981; Barwell, 2016). It can also be explained by the fact that when a teacher repeats what a student has said, they can do so in many different ways and achieve many different actions with this move. However, what is important is to consider the implication of these differences on students’ learning and experiences of mathematics.

One way to look at revoicing is to focus on the form of the teachers’ repetition of a student’s contribution. What is said is often not an exact repeat of the student’s contribution. Teachers can select, delete, paraphrase, and otherwise transform what the student said. The repetition can also include other discursive features such as the use of discourse markers like ‘so’, and direct or indirect references to the student whose contribution is being considered e.g., by using their name or using personal pronouns like ‘you’ combined with a laminating verb such as ‘think’ or ‘said’ (Herbel-Eisenmann et al., 2009; O’Connor & Michaels, 1993). Each of these features can alter the epistemic access, rights, and responsibilities within the interaction.

Many definitions or descriptions also consider the function or purpose of revoicing. Many of these include some way of foregrounding or drawing attention to some aspect of the student’s original utterance. These include enabling the idea to be heard (Hintz & Tyson, 2015) or appreciated by the rest of the class (O’Connor & Michaels, 2019), summarising or amplifying the idea (Hintz & Tyson, 2015), supporting mathematical practices such as abstracting, generalising or attending to precision (Enyedy et al., 2008; Moschkovich, 2015) and supporting the use of mathematical language (Battey, et al., 2016; Moschkovich, 2015). Other functions relate to the positioning and ownership of the mathematics under discussion and include giving students credit for the idea (O’Connor & Michaels, 2019), acknowledging a student’s idea (Hintz & Tyson, 2015), attributing authorship (Enyedy et al., 2008), competence or ownership (Battey et al., 2016), asking other students to agree or disagree with an idea (O’Connor & Michaels, 2019), positioning in relation to other students’ ideas or in relation to the academic task being undertaken (Enyedy et al., 2008; O’Connor & Michaels, 1993), facilitating student debate (Enyedy et al., 2008), as well as positioning in relation to accepted mathematics or sociomathematical norms (Alibali et al., 2019; Enyedy et al., 2008). Other descriptions include focusing on misunderstandings, creating a sense of engagement, giving teachers more time to think or make decisions about what to do next (O’Connor & Michaels, 2019), or establishing the role of contributing to the construction of knowledge within the classroom (Forman & Ansell, 2002) and maintaining and demonstrating common ground (Alibali et al., 2019).

Yet the key structural feature emphasised by O’Connor and Michaels (1993, 2019) is that the teacher’s reformulation (or repetition, recasting, etc.) is followed by the opportunity for the student to accept, affirm, reject, refute, deny, or clarify the teacher’s interpretation as evident in their reformulation. Only a few of the examples offered in subsequent literature illustrate this aspect (e.g., Heyd-Metzuyanim, et al., 2019; Ingram & Riser, 2019; Moschkovich, 2015). This diversity in use of the term leads to the question of whether some of the distinctions being made matter with respect to the negotiation of knowledge in mathematics classrooms.

Interactional management of knowledge

In this paper an Ethnomethodological Conversation Analysis (EMCA) approach is taken to examine the negotiation of epistemic status, rights, and responsibilities where teachers repeat students’ words in mathematics lessons. From this perspective, the negotiation of epistemic issues is made visible in interactions between teachers and students. For example, Hellerman (2003) and Macbeth (2004) used EMCA to show how the prosody and modifications of teachers’ repetitions of students’ answers to questions influenced how this repetition was treated in the interaction as accepting the student’s answer or as rejecting the answer.

EMCA examines how actions such as eliciting or access to knowledge are managed in interaction (Heritage, 2012, 2013; Ingram, 2020, 2021). Epistemic asymmetry can be considered to be a defining property of classrooms (Solem, 2016), with teachers usually asking questions to which they already know the answer and also evaluating students’ responses to these questions. The analysis below draws on Stivers et al.’s (2011) three dimensions of epistemic management: epistemic access, epistemic primacy (or rights) and epistemic responsibility. Epistemic access refers to who has access to the information. Epistemic primacy refers to who has the rights to tell, inform, assert, or assess something, and these can vary between classrooms depending upon the norms of interaction within them. Epistemic responsibilities can also vary between classrooms and refer to what information participants have the rights or obligations to know.

Data

The extracts used in this paper are taken from a corpus of videos of secondary mathematics lessons that were naturally occurring. The lessons were recorded by the teachers themselves and shared with the author who transcribed the whole-class interactions using Jefferson transcription (1984) (simplified to adhere to the publication guidelines).

Analysis

In this paper two examples are offered of interactions where the teacher has repeated students’ words. Each of these examples shares some of the structural features of revoicing but not all, and these similarities and differences have interactional implications for the negotiation for epistemic access, rights, and responsibilities.

In the first example the lesson is focused on linear sequences. The students have been asked to find the twelfth term of a sequence that begins 5, 9, 13, 17, 12, ….

Example 1

177	Tanya:	why?
178	Stevie:	you add twenty-eight to it and then that
179	Sid:	what?
180		(1.0)
181	Stevie:	no no, 'cause if you times four by seven, we have seven until we reach like the number at the top, seven ‘til we reach twelve, and then 'cause you're adding four every time, it's four times seven, like that, so that's twenty eight.
182	Tanya:	so four times seven is,
183	Stevie:	twenty-eight
184	Tanya:	so you think the answer is there's twenty-eight sticks, do you
185		(1.9)
186	Stevie:	u:m:=
187	Tanya:	=no, we agreed there was actually forty-nine sticks.

In this first extract we can see two partial repeats of Stevie’s answer by the teacher Tanya. The first of these in turn 182 repeats only a small part of what Stevie said in turn 181 and uses a *designedly incomplete utterance* (Koshik, 2002) which Stevie completes in turn 183. Whilst Tanya’s turn is a partial repeat and includes the initial inference marker ‘so’, it does not invite Stevie to agree or disagree with the repetition, rather it invites Stevie to add the missing part of this repetition. This repetition shows that Tanya has heard what Stevie said, but not necessarily that Tanya has understood in that there is no interpretation or inference drawn as part of this turn (Sacks, 1992; Ingram, 2020). However, the ownership of the contribution remains with Stevie.

The next repetition in turn 184 is doing something different. The twenty-eight is repeated but additional details are added by Tanya including that she has understood Stevie as saying that the answer is twenty-eight and connecting this ‘answer’ to the original question by adding the qualifier ‘sticks’. The inclusion of ‘you think’ means that the ownership of this answer again remains with Stevie, however the addition of ‘do you’ at the end, as well as the prosody of the turn, indicates that there is a problem with this answer. The lengthy pause and Stevie’s hesitation following this turn show that Stevie also treats there as being a problem. Whilst the ‘you think’ attributes the answer to Stevie, this problematisation brings into play the teacher’s expertise and knowledge to make an evaluation of this answer, even though there is no explicit evaluation. The negative evaluation is then made explicit in turn 70, where the use of ‘we agreed’ attributes the knowledge that the answer is 49 sticks to the whole class.

In the first example turn 184 has many of the structural features of revoicing described by O’Connor and Michaels (1993) and can also be described as summarising, enabling the rest of the class to hear what Stevie has said, acknowledging his contribution and giving him authorship of what is being said, and positions Stevie in relation to the academic task. Yet in turn 187 the teacher closes down the potential to build on, facilitate a student debate, or offer the opportunity for other students to agree or disagree with Stevie’s turn. So, whilst turn 184 shares many of these structural features, its actions within the interaction itself treat the students’ contribution as problematic, rather than as an idea to be used and argued with in a way that gives students some agency over the mathematics. It does not give Stevie ‘a voice’ in the ongoing interaction. Later in the interaction it turns out that Steve’s contribution answers one of the questions the class has been working on, but is the answer to how many *more* sticks are needed (28), rather than how many sticks in total are needed (49).

Example 2

In this example the students are analysing a mechanics problem in a revision lesson focusing on moments, where a simple diagram is drawn on the board. The students have already suggested that the question is about taking moments and the teacher is now asking where the students should take moments about.

455	Tess:	…so where do you think would be a good place if this is what I'm trying to find, to decide to consider the moments about that particular place. (.) just pick something and if it's wrong we can talk about
456	Sam:	if you're measuring from A,
457	Tess:	from A, so take moments about A. okay what you think about that Skylar.
458	Skylar:	er would B be easier, as we know the distance is one.
459	Tess:	so you want to take moments about B
460	Skylar:	yeah
461	Tess:	Sian:
462	Sian:	if you take moments about A you can ignore the reaction on A. so it's probably better to take about A
463	Tess:	because what would happen if we do what Skylar says.
464	Sasha:	then you would have another unknown,
465	Tess:	so then you'd have to do a bit more work because then you'd need another equation, and then you'd have to solve them simultaneously. so what are you suggesting, are you agreeing with Sian Sasha.
466	Sasha:	yep

In turn 457 Tess reformulates Sam’s suggestion in a more complete, and mathematically appropriate way within the context of this task and this classroom. The turn includes the discourse marker ‘so’, but the next turn is not offered back to Sam to affirm or deny that they are suggesting that moments should be taken about A. Instead, Tess invites another student to comment on that suggestion. In turn 459, Tess reformulates again into a more mathematically appropriate way and in a way that is consistent with her reformulation in turn 457, again using the discourse marker ‘so’, but this time she returns to the turn to Skylar to affirm her interpretation (turn 460). Sian then offers an argument for why they should take moments about point A. Tess then prompts for a justification for why Sian would not choose Skylar’s suggestion of about point B, which Sasha gives, before adding additional reasons to Sian’s argument, using the discourse marker ‘so’ and the pronoun ‘you’. Tess ends this turn by returning the turn to Sasha to affirm or deny whether they agree with Sian.

In this second example none of the teacher’s turns have all of the structural features of revoicing described by O’Connor and Michaels (1993) but do have many of the functions such as supporting mathematical argumentation, supporting the use of mathematical language, giving students credit for ideas and attributing authorship and ownership, and asking other students to agree or disagree with the idea being revoiced. The teacher also positions students in relation to other students’ ideas and in relation to the academic task being undertaken, as well as positioning in relation to accepted mathematics.

Conclusion

The precise features of teacher repeats of students' words influence the negotiation of epistemic rights and responsibilities within the interaction. Revoicing is different from rephrasing, repeating or reformulating in the epistemic roles and responsibilities teachers and students have. With revoicing the responsibility for the knowledge in interaction remains with the student(s), but also the students have the right and the responsibility to evaluate the knowledge that is being repeated. This is in contrast to rephrasing, repeating or reformulations where the teacher takes on this role and responsibility. There are consequently important differences between revoicing and other terms which have consequences for the status and negotiation of knowledge in mathematics interaction.

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Teachers’ talk and gestures in online teaching videos about graphs

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Keywords: Language, gestures, graphs.

Study background

Developing understanding of graphs and change involves alteration in students’ mathematical discourse and teachers play a critical role in developing this through their own language and gestures and their responses to students (Schleppegrell, 2007). Prediger and Şahin-Gür (2020) have explored changes in students’ language as they worked on qualitative calculus tasks involving complex situations such as decreasing positive rates. In pre-calculus contexts, research has found persistent difficulties with students confusing high rates with high values and in grasping that a graph represents a relationship between variables (Watson et al., 2013). In this pilot study we combine techniques from Systemic Functional Linguistics (SFL) and gesture analysis to explore teachers’ discourse in online teaching videos about graphs. Our research question is ‘How are graphical concepts constructed through teachers’ choice of language and gestures?’

Theoretical Framework

SFL characterizes written or spoken text as the product of choices in a system of language and practice (Halliday & Martin, 1993). SFL techniques permit analysis of the ideational, interpersonal and textual functions of a text. In our context, these functions tell us, respectively, about the nature of graphs, and how teachers and students interact around that knowledge, via discrete texts. SFL is a way to identify features and functions of the mathematical register, such as characteristic grammatical structures and technical terms (Halliday & Martin, 1993), and to analyze school texts (Morgan, 1998).

Mathematics teaching videos combine language, diagrams, symbolism and gesture and can therefore be viewed as multimodal, dynamic texts. We are interested in how moving between modes supports use of different representations in building mathematical arguments. We are also exploring types of gesture to help analyze mathematical reasoning, particularly pointing, depictive gestures and tracing gestures (Alibali et al., 2013) and how language and gestures together contribute to the ideational, interpersonal and textual functions of these multimodal texts.

Method

Even before the rapid growth in online teaching due to COVID-19, there was a growing phenomenon of online videos published by mathematics teachers and watched by thousands of teachers and students. These videos offer an opportunity for researchers to explore teachers’ language and gestures when introducing particular topics or presenting worked examples in a planned environment. The videos provide information about what the teachers deem important in explanations and may indicate what difficulties they anticipate students might have.

Two teachers gave consent for us to study five of their published videos in pre-calculus and calculus courses. The videos provide simultaneous access to speech, gestures and annotations. We used SFL to analyze transcripts alongside the videos to retain their multisemiotic nature. We then analyzed gestures accompanying each clause and how they related to diagrams or annotations. Here we present our approach to analysis, with examples from a video about estimating the gradient of a curved graph.

Preliminary findings

From the video transcript we identified participants and processes, which contribute to the portrayal of mathematics. The teacher's language choices interweave mathematics as human activity, such as *We want*, *Do you remember…?*, *Let's pick* (in which humans participate in mental or material processes), with mathematical objects participating in material and relational processes: *a line which touches the curve*, *the gradient's becoming quite steep*. We conjecture that changes from human participants to mathematical objects mark shifts from episodes of demonstration to mathematical argument and vice versa. We also found human activity and explanation interjected within relational processes concerning mathematical objects: '***the gradient** // which we sometimes use m to denote // **is going to be the change in y** // so these are the y values //… // **over the change in x** //…*' This may be a feature of a *teaching register*, concerned with connecting what the teacher is doing to the mathematical register, since this episode concludes with a densely-packed nominal group '*So [the gradient at [this particular [point on this curved graph]]] is 3*', which is more typical of the mathematical register.

Gestures can expand the meaning of text. Here we noted their use alongside repeated speech. For example, '*x is 2*' accompanies pointing to an equation and is repeated while pointing at 2 on the *x*-axis and tracing vertically up to the curve. This may emphasize the shift in representation as well as creating a point on the curve from a correspondence perspective, so has textual and ideational functions.

Our poster will elaborate our analysis of teachers' multimodal discourse on graphs.

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An interpretative analysis of generalization processes using abduction theory: The case of addition with zero

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This paper focuses on the reconstruction of generalization processes in interactions. By coordinating different theories, a way to reconstruct these processes by means of interpretative research is presented. This allows the reconstruction of implicit elements beneath explicit language. An interaction revolving around the number zero and addition with zero is analyzed. First, the different types of generalizations are presented and related to the theory of abduction. Second, the interpretative paradigm is detailed to show the procedure of the analysis of an interaction and the spoken language. Last, the results of the analysis are presented, they include reconstructed generalization processes and the tracing of the development of the generality within an interaction.

Keywords: Generalization, abduction theory, qualitative research, interpretative research, interaction analysis.

Introduction

Generalizing is a process that is deeply embedded in mathematics and mathematical thinking. For Dreyfus (1991) generalizing is to infer "from particulars, to identify commonalities, to expand domains of validity" (p. 35). It can also be seen as the application of an argument in a broader context (Harel & Tall, 1991). This paper focuses on the role of generalization in mathematics learning and offers a possibility to reconstruct generalization processes in interactions. Tall's (1991) and Haral & Tall's (1991) considerations of generalization provide the theoretical basis. Coordinated with Meyer's (2010) *theory of abduction*, their theory of generalization can be made accessible for the interpretative research. Therefore, the contribution can also be seen as an example of the connection of different theoretical approaches. Following the networking strategy of coordinating theories (cf. Prediger & Bikner-Ahsbahs, 2014), a conceptual framework is built. Coordinating these theories and following the interpretative research paradigm (Voigt, 1984) allows the explication of implicit elements of spoken or written language concerning processes of generalization. This will be demonstrated in this paper by analyzing an interview of a German first grader that focuses on the number zero and the addition with zero.

Theoretical Framework

Different Types of Generalizations

Harel & Tall (1991) made out three types of generalizations and refer to "schemas" when describing cognitive knowledge of an individual. They propose to distinguish between the following types of generalizations:

1. *Expansive generalization* occurs when the subject expands the applicability range of an existing schema without reconstructing it.

2. *Reconstructive generalization* occurs when the subject reconstructs an existing schema in order to widen its applicability range.

3. *Disjunctive generalization* occurs when, on moving from a familiar context to a new one, the subject constructs a new, disjoint, schema to deal with the new context and adds it to the array of schemas available. (Harel & Tall, 1991, p. 39)

Mitchelmore (1994) considers generalization processes in a comparable way. He does not speak of “applicability range” but of a set of applicable *situations*, which is more fitting for the scope of this paper and pairs better with the *theory of abduction* described in the next section. Therefore, the term *mathematical situation* will be used in the following. In this paper, mathematical situation refers to engagements with enactively, iconically or symbolically (cf. Bruner, 1960) presented mathematical objects of interaction. For example, an action, a material, or even an utterance in a mathematical context can be interpreted as a mathematical situation.

The *expansive generalization* is characterized by the fact that there is no change or reconstruction of the schema. If there is no explicit change, it can be difficult to distinguish expansive generalizations from a renewed confrontation with an already known mathematical situation, i.e., it can be difficult to determine whether the subject is dealing with a known applicable mathematical situation or with a new mathematical situation. However, when interpreting an interaction, this could be clarified, for example, by considering the mathematical situation against the personal background of the subject as well as the context in which they encounter the mathematical situation. Thus, the question whether the learner could have realistically dealt with the mathematical situation before, has to be considered while interpretating.

The *reconstructive generalization*, is characterized by a reconstruction of an existing schema. In this case, a modification of the schema enables the subject to consider situations that were previously not applicable, to be subsequently recognized as applicable. The range of the schema becomes larger, since more mathematical situations can be recognized as applicable. From a purely theoretical point of view, this process can be assumed to be often more cognitively demanding than the process of expansive generalization. Thus, in an interaction, reactions of the subject such as hesitation, pausing, checking, or even inquiring could be interpreted as indicators of reconstructive generalizations. Nevertheless, a deeper interpretative procedure is necessary.

In *disjunctive generalization*, no connection to an already formed schema is recognized. Instead, a new schema is formed. A connection between the different mathematical situations remains unrecognized. A disjunctive generalization may well lead to successful solving of a math problem, but is of little value for the subject (Tall, 1991).

Generalization is often considered an advanced mathematical thinking process (Dreyfus, 1991; Tall, 1991). Nevertheless, generalization plays also a role in learning elementary mathematics (Mitchelmore, 1994), which will be shown further in this paper.

Abduction Theory

Meyer (2010) focused on the analysis of discovery processes and, in doing so, elaborated Charles Sanders Peirce's logical-philosophical modes of inference for use in mathematics didactics. Since then, their use has become established, especially in interpretive research of mathematics education (e.g., Rey, 2021). Since discovering common properties of different mathematical situations is central to generalization processes, this theory lends itself as a further basis for interpreting generalization processes manifested in interactions. Meyer (2010) describes the three logical-philosophical inferences *abduction*, *deduction,* and *induction* as well as their interplay. Central in this context are the terms *case*, *rule*, *result* (resp. *phenomenon*), which are related to each other in different ways depending on the mode of inference.

A deduction is the inference from a concrete case and a general rule to a concrete result. The case and the general rule form the premise, the result is the inferred consequence. The rule is considered general, since it is possible to infer from different cases with the same rule. Deduction plays a crucial role in mathematics, since it is the only inference that guarantees certainty. (Meyer, 2010)

According to Meyer (2010), constructing new knowledge, which is not already implicated in former ones, is only possible by means of *abduction*. The generation of knowledge therefore always presupposes an abduction. It is a discovery process in which an individual generates a rule while exploring and explaining a phenomenon (Meyer, 2010). When confronted with a phenomenon $R(x_0)$ worth explaining, the individual infers a general rule and the specific case $C(x_0)$ that could explain the phenomenon (see Figure 1). The case and the rule go together and are acquired simultaneously; they represent a possible cause of the phenomenon. After the generation of a rule, the phenomenon becomes a *result* of that rule. Thus, the premise of this inference is exclusively the phenomenon; rule and case represent the consequence (Meyer, 2010). This is illustrated by the position of the horizontal line in Figure 1.

Phenomenon (Result):	$R(x_0)$
Rule:	$\forall i: C(x_i) \Rightarrow R(x_i)$
Case:	$C(x_0)$

Figure 1: The general pattern of cognitive abduction (Meyer, 2010)

Abduction is by no means an inference that provides certainty; an abducted rule is only one of many possible rules that could explain a phenomenon with the corresponding cases. Meyer (2010) states that in this context the term *rule* has to be understood in a very broad sense. Rules do not need not be true propositions in the mathematical sense. They might be only plausible or even false. The only conditions that rules must satisfy are that the concrete phenomenon/result can be logically deduced from them (as a consequence of the rule) and that the case forms a concretized antecedent of the rule (Meyer, 2010).

Coordinating the theories: On the role of abductions and rules in generalization

The reconstruction of inferences, especially the consideration of potential rules, offers the possibility to look at generalizing in process. For this purpose, the following section shows how we fitted together elements from the considered theories (cf. Prediger & Bikner-Ahsbahs, 2014). In particular, the concepts *abduction*, *rule*, *mathematical situation*, *schema* as well as (*expansive/reconstructive*) *generalization* are set into relation to each other. Also, these concepts are used for the reconstruction of mathematical communication in order to show the empirical fitting as well as the fruitful use of the concepts that allows to reconstruct also implicit aspects of spoken language.

The theories pointedly compared, following statements suggest themself: The recognition of a connection or the identification of common properties presupposes abductions and the formation or modification of schemas. For Meyer, abducted rules are central at this point; for Harel & Tall, common properties of the mathematical situations under consideration are central. These two considerations are comparable if one interprets the central common properties of the mathematical situations as being made out by abducted rules starting with concrete consequences of the rule (the former phenomena) and therefore sees the schema as being anchored in these rules.

Depending on the reconstructed type of inference, mathematical situations can be compared to the (contents of) phenomena or, after a successful abduction, as (contents of) results of the rule, after the phenomena have been explained. Depending on the generality of the rules in which a schema is anchored, an individual may recognize more or fewer mathematical situations as applicable. These situations can be identified as the x_i in Figure 1. The more general the rule, the larger the set of mathematical situations that can be recognized as applicable by expansive generalization.

If a new mathematical situation cannot yet be recognized as applicable, a reconstructive generalization can take place. The subsequent modification of the schema also presupposes a change of the associated rule. I.e., these rules become more general or the learner grasps the generality of an existing rule. At this point abductions take place, since knowledge is generated (Meyer, 2010).

Methodology

The data analyzed in this paper originates from an interview with a German first grader (age 7) that was conducted as part of the bachelor thesis of Mirjam Jostes. The topic of the interview is the number zero and addition with zero. During the interview a custom board game and other materials were used. The interview was video recorded and transcribed. The transcript is analyzed with regard to generalization processes. Guided by the theoretical framework outlined above, an attempt is made to make statements about generalization processes and their products by analyzing and interpreting the given interaction. The analysis follows the interpretative research paradigm of Voigt (1984), which is widely used in the German mathematics education community. The basis of the interpretative research paradigm is *symbolic interactionism* (see Blumer, 1986) and *ethnomethodology* (see Garfinkel, 1967). Central steps of the Voigt's interpretation method are:

Step 1: At the beginning, the interpreter paraphrases the situation with "his common sense" (Voigt 1984, p. 111, our translation). The goal of the first notation is not to eliminate these previous experiences, but to become aware of them. This pursues the goal of being able to explicitly distinguish oneself from the interpretations in the following; the interpreter thus forces themselves to see more than what was previously clear.

Step 2: Subsequently, the text is divided into episodes and the first utterance is "extensively interpreted" (Voigt, 1984, p. 112, our translation) in order to include as many interpretive hypotheses of the utterance as possible.

Step 3: The interpretation of single turns enables predictions about the further course of the interaction. These are subjected to a test on the subsequent utterance(s): They are either corroborated, when the predictions occur, or the hypotheses have to be falsified.

Step 4: By repeatedly following an extensive interpretation of the subsequent utterance (step 2) as well as a test on the subsequent utterance (step 3) until the end of the episode, the initial variety of interpretations is both expanded and increasingly restricted.

The resulting interpretations are then linked to the corresponding theoretical concepts according to the theoretical framework and checked again on the transcript. The interpretations, which are repeatedly corroborated in the transcript and also enable a maximum gain in knowledge from the point of view of the theory, represent the result of the analysis as interpretive hypotheses. In the following, only corroborated interpretative hypotheses are exemplified. By following the interpretative research paradigm of Voigt (1984) and reconstructing inferences in the interaction, one does not make statements about the cognitive structure of an individuum. One reconstructs inferences that a learner like the individuum, acting in an interaction like this, could make. The interpretation is based on the original German transcript. For this paper, the transcript was translated and adjusted to increase readability.

Analysis and Results

The interview begins with introductory questions. Subsequent to introductory questions, a board game is played. The game board is a path with 10 stepping stones. The first stone is labeled "Start" the subsequent stones are numbered 1 to 10. A custom dice that also has empty faces (represented by in the following, esp. Figure 3) is rolled and a game piece is moved accordingly. Each round Luisa writes down the current position of the game piece, the dice roll, and the new position of the game piece into a table (see Figure 2). The final column of the table is labeled "calculation". It is filled in after the completion of the game. For example, Luisa writes down in row 8 of the table: „current position 6, dice roll 0 ( ), new position 6", when she rolled a zero and did not move the game piece.

In preparation for the game, the interviewer asks the question "What are the special features of doing sums with zero" (T39). Therefore, "addition with zero" is considered the thematic schema, since it is explicitly mentioned by both the interviewer and Luisa and it plays a central role in the interaction.

T40	Luisa	Doing sums with zero. (*4 sec*) if I do one plus zero, it is one.
T41	Int.	mhm-…
T42	Luisa	because the zero, … is actually nothing.
[…]		
T46	Luisa	If I have one stone and add zero stones, it still remains one stone. Because, you can't see the zero.
[…]		
T49	Int.	ok-, and does it work with other numbers as well or only with the one?
T50	Luisa	It also works with other numbers
T51	Int.	why?
T52	Luisa	Because, if I do ten plus zero, it is still 10 (*nodding*) … because the zero actually fits every number.

Luisa responds to the interviewer's question by naming an example (T40), which in turn can be interpreted as the disclosure of a mathematical situation that she recognizes as applicable to the thematic schema. Besides the pure numerical example (T40), she paraphrases an action of adding

(T46). This could indicate that she also recognizes action-bound mathematical situations applicable. In T50, Luisa initially only says that it also works for other numbers. When asked, she gives another example in T52, (10+0=10), and concludes her statement with "because the zero actually fits every number". This linguistic expression suggests that her rule is quite general already.

Luisa's statements can be interpreted in such a way that she sees her schema “addition with zero” applicable to mathematical situations, when a number known to her (n) is the first summand. Thus, her rule allows mathematical situations of the form n+0=n, where n are numbers she knows. Since Luisa seems to already recognize action-bound mathematical situations as applicable, the rule could also admit all mathematical situations of the form nM+0=nM, where nM stands for the quantity n of a material/object M, e.g., one stone. Note that a reference to a material/object M seems to be optional for Luisa. At this point, it can be assumed that she deduces with a rule like “If I calculate nM+0, then the unaltered first summand represents the sum”. The special role of the follow-up question of the interviewer becomes apparent. Without it, it would be impossible to reconstruct the minimal generality of Luisa’s rule as being so developed.

Following the dice game, that took about 10 minutes, the focus is once again directed to the addition with zero. The interviewer asks Luisa to fill in her calculations into the table (see Figure 2). Luisa does not proceed chronologically. She starts by filling in row 4 and row 5 independently without hesitation. In row 6 (position 4, dice roll 0 ( ), new position 4) Luisa stops. This can be seen as a confrontation with a new mathematical situation for Luisa.

Nr.	position	dice roll	new position	calculation
1	Start 0		1	0+1=1
2	1		01	1+0=1
3	1		2X	1+1=2
4	2		3	2+1=3
5	3		4	3+1=4
6	4		4	4+0=4
7	4		6	4+2=6
8	6		6	6+0=6

Figure 2: Luisa’s table and calculations

T80	I	Tell me what you are thinking right now
T81	Luisa	so. (*slowly*) four- (*points at the current position 4*)… hm. plus four (*points at the new position 4*) (*5 sec*) (*faster*) four plus zero. (*begins to write “4”*)
T82	I	mh, you just changed your mind. Why is that?
T83	Luisa	Because the result has to be four. (*continues writing “+0=”*)

Luisa's hesitation and the formulation "four plus four" (T81) can be interpreted in such a way that she does not yet see the row 6 of the table as a mathematical situation that is directly related to “addition with zero”. The five-second pause and subsequent correction “four plus zero” (T81) can be interpreted as an indication of a generalization taking place. Here, a reconstructive generalization could take place, leading to a change of the rule associated with “addition with zero”. The new rule seems to be applicable for calculations of the form nM+0 as well as of the form nM+ . Theoretically, this would lead to a widening of the representational options of the schema “Addition with zero”. The schema Addition with zero can now be linked to more mathematical situations. An abduction that Luisa might have performed is reconstructed in Figure 3. Since a generation of the abduction is hypothetically considered here and not exemplified by Luisa, the inference is made from phenomenon to result and case simultaneously. The inference is therefore indicated by the line directly after the phenomenon (cf. Meyer, 2010). Luisa's expression "Because the result has to be four" (T83) can be interpreted as a *phenomenon in need of explanation*. Following this reconstruction, she would infer from the fact that the first number of the calculation and the result are equal that it has to be an "addition with zero."

Phenomenon:	The result is 4 and 4 appears in the first place of the calculation.
Rule:	If I calculate nM+ or nM+0, then the unaltered first summand represents the sum.
Case:	I calculate 4+ .

Figure 3: Reconstructed Abduction

This interpretative hypothesis is corroborated in the further course of the transcript, especially in Luisa's subsequent confrontations with very similar mathematical situations. For example, Luisa looks at table row 8 (current position 6, dice roll 0 ( ), new position 6) and says "funny" (T88), but then directly states "because I like such calculations with zeros" (T92). Building on the above interpretation, an expansive generalization could be reconstructed here. The rule in which the schema "addition with zero" is anchored would now be general enough so that the new mathematical situation 6+ can be recognized as applicable (see Figure 3). Moreover, in the subsequent discussion of table row 2 (current position 1, dice roll 0 ( ), new position 1), which she had initially skipped, Luisa directly refers to the previous task 6+ , "like down here with the six" (T104), which suggests that it belongs to a common schema. As Luisa seems to determine the order herself and does not want to or could not process row 2, but would be able to do this from a mathematical point of view, further strengthens the interpretation of a completed reconstructive generalization in the course of the interaction that allows subsequent expansive generalizations to take place. Part of the corroborated interpretative hypothesis is, that Luisa's rule depends on the position of zero in the calculation, i.e., zero has to be the second summand. Hypotheses where this was not the case, for example one hypothesis was that Luisa uses the commutativity of addition, were falsified during the course of the analysis, when she was confronted with the first table row.

Discussion and Conclusion

The scene shows how generalization processes can take place in an interaction and how they are built interactively through language. The analysis has shown how the theory of Harel & Tall (1991) can be made accessible to the interpretative research by utilizing the theory of abduction described by Meyer (2010). Without the possibility to reconstruct rules that manifest in a real interaction, statements about the set of mathematical situations an individual's schema applies to, would be empty. On the basis of the theoretical framework, it was possible to trace a conceivable development of the generality of rules and schemas in Luisa's language and actions. This provides insights into generalization processes that students like Luisa could carry out in an interaction like this.

Furthermore, the analysis has shown how challenging it is to make statements about the generality of a rule or the applicability range of a schema. However, the used method allows statements about what mathematical situations could be considered applicable and about possible dimensions of generality of a rule. Mathematical action is based on rules which are generated, applied or tested by abduction, deduction or induction. But they are seldomly made explicit. In classroom reality, a real time reconstruction of rules or of the set of applicable mathematical situations is inconceivable, even in smaller learning groups. This underlines the importance of explicating rules in the classroom and providing students with language means to do so. Asking for explicit formulations of an individual's rules and to explore the applicability range of schema together with the learners can stimulate further generalization processes.

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Support-Related Diagnostics in Mathematics Including Reading - FORMULA

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Keywords: Diagnostics, learning disabilities, mathematics and reading.

Introduction

The project *Formula* is funded by the German Federal Ministry of Education and Research. It aims to explore the relationship between the mathematical participation of students in inclusive education (in transition from primary to secondary school) as well as their reading skills and how these can be combined. The overall goal is to develop a highly valid, supportive diagnostic tool that can be used to assess reading and mathematical competencies as well as their interconnections. In combination with the associated remediation this is called the *Formula*-Concept. In this way, the project *Formula* responds to a problem of many already existing diagnostic and remediation instruments: They exclusively focus on subject-specific competencies and are neither linguistically appropriate nor linguistically sensitive.

Theoretical Background

Research findings confirm the connection between linguistic and mathematical skills - even for students with learning difficulties and developmental disabilities (Mann Koepke & Miller, 2013; Martin & Mullis, 2013). Students' language competencies are essential for the development of mathematical understanding and are considered as a predictor of learning growth (Prediger et al., 2019). Reading comprehension is regarded as a key skill in all subjects, including mathematics (Borasi & Siegel, 2000; Grimm, 2008). Reading promotion is a necessary component to be taught not only in language lessons but in all subjects. Moreover, this can be important for learners with special needs in particular.

As most recently shown by the empirical studies PIRLS 2016 and TIMSS 2019, a significant proportion of children in Germany at the end of primary school show both below-average mathematical competencies (25% below proficiency level 3; Selter et al., 2020) as well as an insufficient level in hierarchically higher reading processes (19% below proficiency level 3; Hußmann et al., 2017). However, it is unclear how reading skills affect the development of mathematical competencies and which relevance lower- and higher-hierarchical reading processes play in this context.

Research design

In our project we are addressing five research questions using a combination of a research design approach and an intervention study (see Table 1).

Table 1: Connections between research questions, methods and expected results

research questions	methods	expected results
1) How do students' learning processes proceed? What difficulties do they face and what kind of potentials becomes visible? (learning processes) 2) What does a supportive diagnostic concept and design look like? (design) 3) What are the local theories about the relationship between mathematics and reading competencies? (theory)	design-oriented developmental research approach, e.g. cyclic sequential design experiments, eye-tracking	detailed results concerning the correlation between mathematical competencies and reading comprehension
4) How do reading and mathematical competencies influence each other during the mathematical learning process? (relation math + reading)	pre-post interventional study	individual strategies of students, obstacles, potentials
5) How do teachers in elementary and secondary education use the Formula-Concept in their mathematics lessons to integrate reading promotion? (teaching profession)	design-oriented developmental research approach, survey	conditions for a sustainable transfer

The knowledge aims concern the differential evaluation of the competencies of reading and mathematics competencies. However, this is related to the presentation and meaningful context of mathematical tasks. Therefore, appropriate combined support measures for mathematics and reading should be developed, distinguishing between representational and contextual tasks. Tasks from the BASIS-MATH 4-+5 (Moser Opitz et al., 2016) are further developed and specified with regard to the competencies considered in this project. The resulting diagnostic tool will be pre-tested in individual sub-areas on smaller samples first (on average N=60/item), then piloted as an overall diagnostic tool (N=150). Additionally, students' reading comprehension will be assessed at the word, sentence, and text levels with help of the diagnostic test ELFE II (Lenhard et al., 2017).

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Linguistic features of word problems that cause difficulties for learners across the curriculum: A literature review

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Word problems can be challenging for learners of all ages on different levels. In this paper, specific linguistic requirements and features of word problems contributing to the difficulties for learners will be discussed. For this purpose, first results of a literature review are presented. The literature was analysed according to the model of genre features (Rezat & Rezat, 2017) as a theoretical framework. In the course of the literature review, a system of categories was developed to differentiate the linguistic features addressed in the respective studies. The studies that were assigned to the feature of linguistic complexity are discussed in detail with the aim of identifying specific linguistic features of word problems that cause difficulties for learners. The curricular context and the mathematical content of the studies will also be discussed.

Keywords: Word problems, literature reviews, syntax, linguistic complexity.

Introduction

In the sense of the spiral curriculum, applications are regarded as a topic that should be treated in an appropriate way at different grade levels (Müller & Wittmann, 1984, p. 4193). Word problems are a central type of task of teaching applications of mathematics at all ages. Therefore, they can be seen as a specific genre of mathematics education. A genre is defined as a "pattern of cultural-social interaction in a particular context" (Rezat & Rezat, 2017, p. 4193). In general, language and language use made for didactic purposes is an issue in every subject (Schleppegrell, 2004). Schleppegrell (2004, p. 82) points out that "[s]chool is a culture with its own expectations for particular ways of using language. [Therefore,] students need to learn about genres of schooling and the purpose for which they are useful."

Research has shown that learners at all ages have difficulties with word problems. Daroczy et al. (2015) point out that those features of word problems, which generate difficulties for learners can be at mathematical, factual-semantic, and linguistic-syntactic levels. However, it is unclear which specific challenges these are, in which grades and for which mathematical contents these are relevant, how they develop within the curriculum, and whether difficulties in one mathematical context can also be transferred to other mathematical content areas. Furthermore, Daroczy et al. (2015) highlight that a comparison of previous research results related to word problems is usually not possible. Verschaffel et al. (2020) reinforce this statement. They point out that various (international) research projects on word problems focus on different perspectives about the research topic. Therefore, there is a need for a systematic analysis of pre-existing research results that addresses the requirements and the characteristics of word problems that generate difficulties for learners from a curricular perspective. The systematic analysis of the research findings presented in this paper will focus on the results related to linguistic features of word problems and how they impose difficulties on learners. The following research questions will be addressed:

Q.1: What are linguistic features of word problems that cause difficulties for learners across primary- and secondary[1] level curriculum?

> Q.1.1: What specific linguistic features of word problems have been investigated in previous research?
>
> Q.1.2: Which of these specific linguistic features of word problems affect the learners' solution processes and quality of solutions?

Theoretical framework of the literature review

In order to be able to systematically represent the research results with regard to linguistic task features, the *model of genre features* (Rezat & Rezat, 2017) was used as an instrument. It is grounded in the pragmatic linguistic approach. In the model, a genre is described as a connection between the *type of act* and the *text type*. The authors describe the relationship of both main categories in the model as follows: "The type of act directs the expectation of the text type while features of the text type, especially the formulation patterns, indicate the type of act." (p. 4196). The main category *type of act* was defined based on Halliday's (1985) modelling of the situation of the text, integrating the three social functions of language: field, tenor and mode. Structural features (*structure of the text)* and linguistic features (*language features*) are assigned to the main category *text type.* The main category *text type* is used as a framework in the literature review to systematically identify studies in which linguistic and structural features of word problems are discussed. On basis of the systematic literature review, the genre-specific language features of word problems are determined. Language features that cause difficulties for learners are defined as features that impair learners' solution processes and/or quality of solutions.

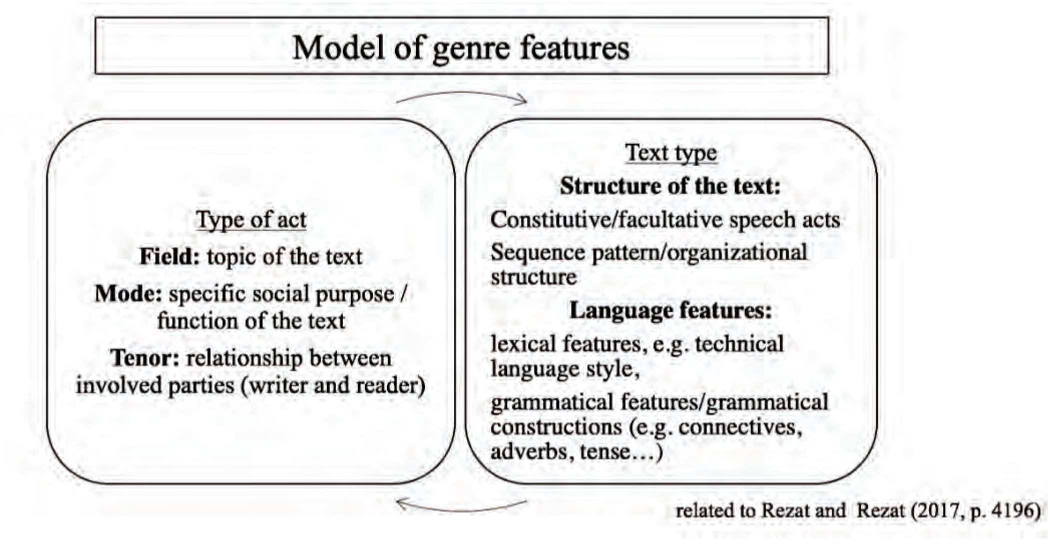

Figure 1: model of genre features

[1] The differentiation between primary and secondary school is based on the international standard. Primary level is understood up to and including grade 6.

Methodology of the literature review

In this section, the procedure of the literature review, which was conducted systematically is presented. Based on a specific literature search via the online literature database ERIC (https://eric.ed.gov)[2] a corpus consisting of 354 scientific articles (peer reviewed) was identified. The identified corpus was analysed based on qualitative content analysis according to Mayring (2015). The main category *text type* of the *model of genre features* with the subcategories *structure of the text* and *language features* was used to systematise the results. The category *language features,* which is the focus of this paper, comprises studies that analyse specific *lexical or grammatical features* of a text. Considering these criteria, a review of scholarly publications was conducted to determine the extent to which structural and language features, especially the *lexical* or *grammatical features* of word problems were addressed. For the review of the whole corpus, exclusion rules were also set and used for the analysis. This led to the removal of contributions that are not relevant in connection with the research questions. Consequently, papers that discuss the performance of learners with word problems (presentation in first and second language) and do not discuss the influence of linguistic features of the task, that analyse teachers' behaviour with word problems, and those that present interventions of teaching methods were excluded. Furthermore, studies that focus on interaction processes between learners, studies that address word problems from preschool or adult education and studies that address language/mathematical difficulties of learners with disabilities are excluded. The remaining corpus was analysed using the deductively developed categories and, if necessary, was extended by inductively formed categories.

Results

By means of the procedure described above, a total of 20 scientific publications were identified in which specific *language* and *structural genre features* of word problems in different grade levels are addressed. Of these 20 papers, 14 papers were assigned to the subcategory *language features*. The following section presents the results related to research question 1.1, followed by the results related to research question 1.2.

Results of the literature review in terms of investigated language features

The categories of language features provided in the model of genre features (Rezat & Rezat, 2017) were too general to allow for a differentiated view of the specific linguistic features of word problems that have been addressed in previous research and how they impose difficulties on learners. Therefore, the 14 scientific publications assigned to the category language features were inductively classified into different subcategories in order to be able to differentiate the particular language features that were investigated in these studies. In total, 5 subcategories could be developed.

Table 1 shows the subcategories describing the *language features* addressed in the respective studies and their definitions as well as the grade levels, in which the language features were investigated.

[2] The following search term was uses: ("Word problems" OR "Story problems") AND (language OR syntax OR "text structure" OR "word order" OR lexicology OR "visual aids" OR "illustrations")

Table 1: Inductively formed category system of the literature review

Inductive category	Definition of the category: Studies, …	Number of studies	Grade
(In-) consistency of language	… in which the influence of (in-)consistent language in the task is addressed in relation to the solution or comprehension processes.	5	1, 2, 3, 4 and 5
Personal pronoun	… in which the influence of personal pronouns in the task is discussed in relation to the solution or comprehension processes.	2	1 and 4
Vocabulary	… in which the influence of the vocabulary of the word problem and the vocabulary of the learners are addressed in relation to the solution or comprehension processes.	3	1, 7 and 8
Linguistic complexity	… in which the influence of lexical and/or grammatical features and/or features of the information structure related to the solution or comprehension processes are addressed.	5	1, 2, 5, 7, 8, 9 and 10
Deictics	… in which the influence of deictic expressions in the task is discussed in relation to the solution or comprehension processes.	1	10

The category syntactic/linguistic complexity comprises studies in which a combination of several individual linguistic features was varied at once in order to determine the influence of this variation on students' solution processes and/or quality of solutions. Since this feature is linguistically conceptualised differently, an overview of which linguistic features were (systematically) varied in the related studies is given to discuss this category in more detail. The linguistic modifications of word problems that were made in the study of Abedi and Lord (2001, p. 221) included: Unknown or rare words were changed, passive verb forms were transformed to active ones, long nominals were shortened, conditions were replaced by separate sentences, or the order of conditional and main clauses was changed, abstract or impersonal utterances were made more specific, complex questions were simplified by using simple question words. Leiss et al. (2019) increased the linguistic challenges of word problems by modifying four features: vocabulary, use of connectives, complexity of grammatical structures and information structure. In another study, a higher linguistic complexity was created by combining three original sentences of the tasks into one complex sentence (Muth, 1984). The following linguistic characteristics were varied individually in the study of Walkington et al. (2019): number of sentences, word hypernymy, pronouns, word concreteness, consistency of sentences, the topic of the task. Ambrose and Molina (2014) see linguistic complexity at the level of syntax and vocabulary.

Summarising this overview, it becomes apparent that on the one hand there are studies that focus on individual linguistic features of word problems, while on the other hand some studies focus on a combination of linguistic features. These studies generally relate to syntactic/linguistic complexity.

However, the notion of syntactic/linguistic complexity is defined differently in the studies and comprises different combinations of individual linguistic features.

Results of the literature review in terms of how language features impose difficulties on learners

In the next step, the results of the studies in the different categories were interpreted in terms of how the language features affect students' solution processes or quality of solutions. The studies that have been assigned to the category *(in-)consistence of language* show controversial indications of the extent to which this feature influences the solution processes and quality of solutions. The two studies assigned to the category *pronouns* show no evidence that this feature influences the solution processes and the quality of solutions by learners. The overall view of the studies that were assigned to the category *vocabulary*, however, suggests that there is a fundamental translation problem on the procedural and conceptual level on the part of the learners and that possible difficulties cannot be explained by the feature *vocabulary* alone. The study that was assigned to the category *deictics* showed evidence that a certain type of error occurs particularly frequently in the field of algebra when the tasks contain indexical expressions. Since only one study was assigned to this category, a more comprehensive view of this language feature is not possible.

This broad overview already suggests that the hypothesis that difficulties of learners working on word problems can not necessarily be attributed to individual linguistic features. This is also supported by the study by Walkington et al. (2019). They analyse the influence of linguistic complexity by varying individual linguistic features of word problems separately and systematically. In this study, in total 451 American students from different school districts from grades 7 to 12 participated. The students were asked to complete different task variations that were presented on a free online homework platform. For this purpose, the word problems from the content area of linear functions were used. The tasks were systematically varied with regard the six different linguistic features mentioned earlier. Each feature was also varied in terms of three levels of difficulty. Students were not given any special instructions that they were taking part in a research study. The accuracy of the solutions and the reaction time were determined with the help of the digital platform. Overall, the authors were able to show that there is little evidence that individual linguistic features have a significant influence on students' performance in solving the word problems in the different grades. Therefore, the results of the studies classified as investigating linguistic complexity will be discussed in a differentiated way with regard to research question 1.2. Table 2 provides a more detailed overview of the studies in which the feature of *linguistic complexity* is discussed.

Table 2: List of studies on linguistic complexity of word problems classified by grade levels

Studies on linguistic complexity of word problems				
Grades 1 and 2	Grades 3 and 4	Grades 5 and 6	Grades 7 und 8	Grades 9 und 10
Ambrose and Molina (2014)		Muth (1984)	Leiss et al. (2019) Abedi and Lord (2001) Walkington et al. (2019)	Walkington et al. (2019)

Abedi and Lord (2001) investigated the influence of language in word problems on learners' performance. For this purpose, the authors conducted two separate field studies. The first study was a perception study. For this reason, in total 36 eighth graders at four school sites in the Los Angeles area were presented original NAEP (National Assessment of Educational Progress) maths items and parallel revised tasks in simplified language. The mathematical requirements were kept constant. In a standardized interview, learners were asked which tasks they would choose in an exam situation. Each learner had to answer this question for four pairs of tasks (original and modified). This study shows that the majority of the learners preferred the modified tasks to the original tasks. In the second study, the Accuracy Test Study, 1174 American students in grade 8 from 39 classes were given a paper-and-pencil math test. The test included 10 original NAEP math items, 10 items with linguistic modifications as described above, and five noncomplex control items. Two different test booklets were developed where the original items are modified were developed comprising original and modified test items. Here, the students' scores on the original and linguistically modified items were compared. In principle, the linguistic modification also led to a significant difference in the mathematical performance of the learners. The modified tasks could be also handled better by the students, certain groups of learners particularly benefited from the linguistic modifications.

Leiss et al. (2019) reconstruct and investigate learners' comprehension processes when solving word problems and, among other things, address the question of what role task characteristics (complexity of the situation, linguistic complexity) play for the comprehension process. In their study, 55 seventh-grade German-speaking students attending three comprehensive schools were asked to complete three reality-based tasks on the topic of linear functions, which were systematically modified in terms of the complexity of the situation model and the linguistic requirements. The sessions took place under laboratory conditions and the students were required to solve the tasks by using the Think Aloud Method (p. 137). In this context, it was shown that the linguistic complexity influences the learners' comprehension more than, for example, the context into which the tasks are embedded.

The studies by Abedi and Lord (2001) and Leiss et al. (2019) showed that English-speaking learners are more likely to work on tasks with lower linguistic complexity. This means that the variations in the linguistic complexity (as defined by the authors) of the tasks resulted in a better understanding of the mathematical requirements by learners. In this context, it was also observed that tasks with lower linguistic complexity yielded higher rates of correct solutions. The results of Leiss et al. (2019) showed that the linguistic complexity of the tasks influences the comprehension of German-speaking learners more than the context of the task. At this point, it can be assumed that higher linguistic complexity (of different linguistic characteristics) can be understood as a feature that creates difficulties for learners.

In the two other studies, the feature of linguistic complexity was investigated using task examples from primary level. The main purpose of the study by Muth (1984) was to determine the relative importance of computational ability and reading ability to the solution of arithmetic word problems. Furthermore, Muth (1984) investigated how learners deal with different requirements imposed on them by different word problems. In this context, 200 sixth graders from two middle schools were asked to solve word problems that were modelled on NAEP tasks. Adding irrelevant information to the word problems forms a computational/problem solving demand in this study (p. 206). The variation of syntax represents a reading-related demand (p. 206). These tasks were used to test learners' skills in addition, subtraction, multiplication, and division. Four versions of the test were created by developing two versions with variations in the additional information and two versions with variations in the complexity of the syntax. The results showed that the variation of syntax had no influence on the accuracy of the solution. Ambrose and Molina (2014) were able to show in a study that it were not so much the superficial linguistic features of the text, including syntax and vocabulary, that hindered the children's successful interpretation, but rather the situations and the limited information provided about them that affected comprehension. The study involved 18 first grade Spanish/English bilingual learners who were asked to retell and solve word problems in their first and second languages. Addition/subtraction and division tasks served as the mathematical content here. This contradicts the above-mentioned results by Leiss et al. (2019). There, the context of the task did not have as strong influence on the understanding of the tasks as the linguistic complexity for learners in seventh grade. A possible interpretation of these contradicting results of the two studies might be that learners in higher grades have more in-depth contextual knowledge than primary grade learners. The latter may need more detailed explanations on the context of the tasks in order to develop an appropriate understanding of the factual situation to be able to solve the task.

In summary, studies that varied single linguistic features show no clear influence on the learners' performance (Ambrose & Molina, 2014; Muth, 1984; Walkington et al., 2019). Studies in which several linguistic parameters were varied show evidence that lowering linguistic complexity leads to higher rates of correct solutions (Abedi & Lord, 2001).

Discussion

Summarising the results of the literature review to answer research question 1.1, it was possible to provide an overview of linguistic features that were investigated in previous research on word problems. Within the framework of this review, the following language features could be found: (*In-) consistency*, *personal pronoun*, *vocabulary*, *syntactic/linguistic complexity* and *deictics*. From these categories, *linguistic complexity* is conceptualized in different ways in the studies.

To answer the research question 1.2, first insights from the review of all studies were summarised and a differentiated review of the studies which were assigned to the category linguistic complexity was conducted. The language features *(in-)consistency*, *personal pronoun*, *vocabulary* and *deictics* appear to be features that are defined and used in the studies in a widely consistent way. Even if the research subjects in the studies differ to some extent, it is possible to draw careful conclusions about features that cause difficulties for learners. Considering the studies on linguistic complexity, which are linked to secondary school content, multiple variations were made (Abedi & Lord, 2001; Leiss et al., 2019). The results of these studies suggest that the features that generate difficulties cannot be attached to individual linguistic features, but rather the language as a whole system controls how mathematical demands of the task need to be perceived and mastered by learners. It has become apparent that learners perform better with reduced linguistic complexity (Abedi & Lord, 2001). The studies in which selected linguistic features were modified seem to confirm this hypothesis (Ambrose & Molina, 2014; Muth, 1984; Walkington et al., 2019). Linguistic complexity as a genre feature seems to be a phenomenon that takes on different forms.

In terms of limitations, it should be noted that only a small number of studies were considered in this paper and the conclusions need to be substantiated by further studies. In the further course of this project, the analysis presented in this paper will be used as a starting point to carry out a task analysis with the aim to describe the connection between the language and the mathematical content in a more differentiated way with a particular focus on how the mathematical and linguistic demands of word problems develop across primary and secondary mathematics curriculum.

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Using digital tools in language diverse mathematics classrooms

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In this paper, interviews with teachers and preservice teachers are analysed to understand their views about using digital tools in mathematics classrooms, connected to language-diverse students': communication; potential for learning; and available identities. When digital tools were seen only as providing ways to utilise the home language so that existing mathematical knowledge could be used to complete tasks in the language of instruction, then language-diverse students' available identities were reduced to becoming like their non-immigrant peers. In contrast, when digital tools were considered as providing opportunities for utilising a wider range of their language resources, then preservice teachers broadened their views about language-diverse students' potentials for learning and available identities.

Keywords: Digital tools, multilingual classrooms, teachers, preservice teachers.

Introduction

Researchers have suggested that digital tools, both software and hardware, can facilitate language-diverse students learning mathematics (Le Pichon et al., 2021; Freeman, 2012). However, most research, such as that of Le Pichon et al. (2021) and Freeman (2012), used individualised computer programs, which provided instruction in the home language. The rationale for these studies was often based on teachers needing support in interpreting students' mathematical work, because with the "variety of languages present in the classroom, teachers do not always have the necessary tools and expertise available to appeal to every pupil's home language" (Van Laere et al., 2017, p. 98). These approaches raise two concerns: teachers became almost redundant in the teaching process; and only the student's home language was considered a resource for learning mathematics.

Yet, as Libbrecht and Goosen (2016) stated, "the introduction of ICTs into mathematics teaching brings different ways to express and perceive mathematical activities, concepts, and phenomena" (p. 217). For example, dynamic geometry programs, such as GeoGebra, are considered as providing easy manipulation of multiple representations, "the use of dynamic representations promotes geometric thinking and provides visual, algebraic and conceptual support for the majority of students" (Dockendorff & Solar, 2018, p. 67). Thus, digital tools could utilise language-diverse students' range of resources for learning mathematics, because the different modes, such as graphs and symbolic algebra, contribute to meanings about a mathematical idea being developed. Nonetheless, little is known about how teachers use digital tools to support language-diverse students to activate their repertoire of resources. In this study, we analyse interviews with teachers and preservice teachers about using digital tools to support language-diverse students to learn mathematics.

Knowing what teachers do and the possibilities that they see for using digital tools to teach mathematics in multilingual classrooms is important for designing appropriate teacher education courses and professional development experiences. In Norway, concern has been expressed that teacher education is not providing adequate support for preservice and inservice teachers to incorporate digital tools into their teaching (Søby, 2013). A survey by Guðmundsdottir and Hatlevik (2018) of newly-qualified teachers found that nearly half felt that their professional digital competency was poor because of poor initial teacher education. Yet, in the most recent Norwegian mathematics curriculum "digital skills" are highlighted as one of five "basic skills" for students, with programming being a required part of mathematics education in schools (Kunnskapsdepartementet, 2019). As well, concern has been expressed about mathematics teacher education, in Norway, providing adequate support for preservice teachers to facilitate language-diverse students to utilise their language resources (Rangnes & Meaney, 2021). Therefore, there may be a mismatch between what language-diverse students need from experiences with digital tools, both for learning mathematics and for later life, and with what teachers know how to provide. Thus, designing appropriate courses for preservice and inservice teachers is important based on what they already know as well as what they do not know.

In this paper, we explore how teachers and preservice teachers (PTs) consider digital tools could be used in multilingual mathematics classrooms. As so little research has been done in this area, there is a need to scope the field and identify what kinds of connections can be made between language-diverse students, digital tools and mathematics teaching and learning. The data is from five individual teacher interviews and one focus group interview with five preservice teachers. For the analysis, we use a theoretical model from earlier research (Rangnes & Meaney, 2021) to describe some of the complexity of teachers' views.

Theoretical framework

The "multimodalities, signs and signs-maker in a social context" (MSSM) model was developed as part of a larger project exploring how preservice teachers learn about teaching argumentation for critical mathematics education (LATACME) in multilingual classrooms. This model is designed to understand how respondents, such as teachers, describe the possibilities they see as being available to another group, such as students, when particular resources, such as dynamic geometry and Minecraft, are considered to be vehicles for carrying meaning – semiotic resources.

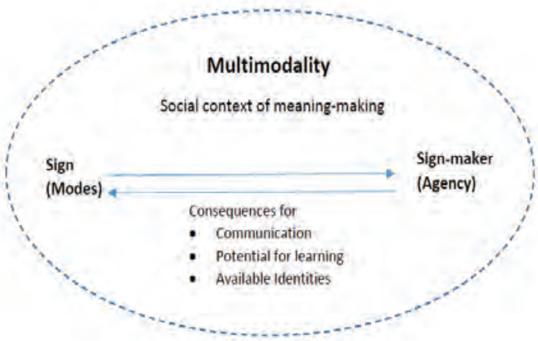

Figure 1: Multimodalities, signs and signs-maker in a social context (Rangnes & Meaney, 2021)

MSSM was built on Bezemer et al.'s (2012) work on multimodal social semiotics. In Figure 1, the oval symbolizes the social context, in which interactions, involving language-diverse students and digital tools, are embedded, as described by the (preservice) teachers. In any interaction, semiotic resources carry meanings. The potential for semiotic resources to be considered as appropriate meanings becomes recognisable, as a result of previous interactions. Therefore, the interaction is not separated from the wider context. Kress and van Leeuwen (2001) described how semiotic resources carry extra meanings, through provenance and connotation. Provenance describes how meanings are imported with the use of signs/semiotic resources from one context (culture, social group, etc.) to another, such as when Minecraft is seen as a play activity in a mathematics classroom, because of students' home experiences. As a result, the kind of meanings that students consider that Minecraft can carry is limited to what is usual in the home situations. The connotations connected to a sign, such as a block house in Minecraft, through the use of a particular mode, the graphic representation, are the socio-cultural values and backgrounds that provide extra information, such as a house being associated with somewhere to live. Participants in an interaction use the meanings from the provenance and connotation of a sign to make interpretations in an interaction, even if the sign maker had not intended for it to be understood as having these extra meanings.

In Rangnes and Meaney (2021), preservice teachers described the sign makers' (the students') choice of modalities, such as spoken or written language, to produce and interpret signs that carried meaning for the students in mathematics lessons. Interpreting PTs' descriptions of the use of different modes in the interactions provided insights into how PTs considered the students' *communication*, *potential for learning* and the *available identities* affected and was affected by the social context of the mathematics lesson. *Communication* occurs in interactions between people and between people and artefacts. Communication is linked to *learning potential* as it is through interactions that there is a potential to meet and engage with new ideas. In our study, we analysed (preservice) teachers' views on how language-diverse students could produce mathematical meaning, using different digital tools as a way of identifying learning potential. Similarly, we consider *available identities* to be the kinds of identities that the teachers implied were available to language-diverse students, when making use of digital tools as semiotic resources.

Methodology

The data was collected as part of larger, Norwegian-wide, research projects that both authors participate in. The focus group interview with preservice teachers was done by project colleagues and the five individual teacher interviews were collected by master students. Both the preservice teachers and the teachers taught grades 5-10 in Norwegian schools. The semi-structured interviews used a similar set of questions about a wider range of topics to do with using digital tools in mathematics classrooms. In this article, we only investigate responses about multilingual classrooms, which came towards the end of both sets of interviews. The format of the interviews and the different interviewers means that although the questions about digital tools and multilingual classrooms were similar in intent, they were not the same. However, as our intention is not to compare the results but to identify issues of interest, we consider that the interviews are sufficiently compatible to do a similar analysis.

In the previous research with this model, the preservice teachers had discussed specific episodes where students engaged with mathematics outdoors (Rangnes & Meaney, 2021). In the interview data, the (preservice) teachers sometimes described specific examples, but often described the potential with digital tools at a general level. Therefore in the analysis, we considered how the (preservice) teachers described aspects of modes, communication, available identities and potential for learning at a general level. The modes were the different kinds of representations that the (preservice) teachers identified as being important in multilingual mathematics classrooms. We identified comments to do with communication as those which talked about the purpose of communication in multilingual mathematics classrooms. Available identities were comments that (preservice) teachers made about language-diverse students as mathematics learners, while learning potential were comments about the kinds of learning that was made available in the mathematics classroom to language-diverse students. By considering how each of the aspects interacted together, we were able to consider how the (preservice) teachers considered the complex relationship between mathematics, digital tools and language-diverse students.

In the next section, we first describe our analysis of the teacher interviews and then the focus group interviews with the preservice teachers, by considering the modalities, the communication, the available identities and in the discussion and conclusion section, we discuss the results across the two data sets and what it means for teacher education programmes.

Teacher interviews

Modalities

The modes that the teachers highlighted in the interviews were mostly spoken and written language, including symbolic mathematics. Generally, mathematics was considered to be universal in these modes, "there are a few other words for things. But otherwise, it is exactly the same mathematics. It is generally valid" (Teacher2). As a result, the teachers focused on digital tools which translated instructions or translations of the students' answers. This would be through audio input or videos in the students' home language, which would enable the students to make sense of instructions illustrations and calculations, presented in Norwegian. T4 summarised that "the strength of the digital is that it is visual and a bit with being able to translate things". The benefit of visualisation was mentioned also by T1 as good for developing students conceptual understanding, but here visualisation seemed to just be referring to watching a replacement teacher on a video, not visualisation as described by Dockendorff and Solar (2018). The focus on these modes may have been because the teachers expected students to learn from listening to a teacher and reading a textbook. These expectations, or provenance (Kress & van Leeuwen, 2001), produced from earlier experiences, of mathematics teaching could lead to the assumption that if the student was unable to understand the teacher or the textbook, then they would be unable to learn mathematics.

Communication

In the interviews, the teachers highlighted the purpose of communication as being to gain access to the mathematical meanings that the students conveyed in their written work, with an assumption that the teachers needed to evaluate it, "it is a pity that one (the student) does not get to show their mathematical competence just because it is the wrong language" (T4). T2, T3 and T5 described how they could talk with students who spoke English as a second language, but, as T2 stated, students who used Polish would be problematic because the teachers would not understand. Consequently, T2, T3, T4 and T5 valued translation programs, such as Google Translate, or the opportunities in Excel, GeoGebra to change the language, because the students were then able to demonstrate their knowledge. For example, in written argumentation, the students could use translation programs to write in Norwegian well enough for the teacher to understand. To achieve this purpose, translation opportunities were highlighted by all the teachers, particularly when the students had limited Norwegian or English, which the teacher could understand.

The teachers also identified the purpose of communication as being for students to understand what was expected of them in class. Translation programs were considered as supporting this to happen, "it's a bit like doing nothing to actually work" (T4). T2 provided a story about a successful mathematics student who used it to understand what he had to do. However, T2 also noted that translation programs were only useful, "if they can then write in their own language, then you have to be good at math, but weak in the language", with "the language" referring to Norwegian.

Potential for learning

The teachers identified potential for learning as occurring when the students could follow the in-class teaching and completing textbook exercises. Digital tools were not discussed as supporting students' mathematics learning except for the access they provided to material provided by the teacher or the textbook. As a result, digital mathematics textbooks were valued because they had less text (in Norwegian) than ordinary textbooks (T3), making them easier to follow for language-diverse students. T2 and T5 described how students could watch YouTube videos on a mathematical topic in their home language, "I do not know if they (on the video) do it right, but they (the student) can see the video and I can see the calculation" (T5). The written calculation allowed the teacher to evaluate the students' learning. However, when language-diverse students had not previously learnt the same topics to the same degree as the other students in the class, then the potential for learning was limited. They considered that language-diverse students who had gaps, in relationship to what they were supposed to do in Norwegian classrooms, were not able to use translations programs or video lessons to bridge into the current mathematics classroom learning.

Available identities

The teachers were focused on bridging language-diverse students into existing classroom practices. Consequently, the students were described as having available identities either of being high achiever in mathematics from their previous countries or being low achievers with gaps in their understandings. As T5 stated, "to learn mathematics you have to have a foundation consisting of concepts and a language, and then be able to build on it". The high-achieving group could benefit from using Google Translate. However, language-diverse students with gaps in their mathematics education and limited Norwegian, were unable to take advantage of translation programs and so digital tools were not considered to provide any possibilities for supporting their meaning making,

> Often they are not necessarily bad in maths, they just have a lot of gaps and have to work with completely different things, things that they should have learned in 5th grade. Some have not got instant recall of basic facts to ten, then it becomes very difficult to add twelve and 18. It takes a long time (T2).

When language-diverse students came with such gaps, then digital tools were not considered as providing possibilities for supporting mathematical learning.

Preservice teachers focus group interview

Modalities

In the focus group interview, although they talked about different digital tools when they came to the questions on multilingual classrooms, the preservice teachers were asked specifically about programming. They described mathematics and programming as types of "language", where language had an implicit taken-for-granted meaning of natural languages. However, there was no details about whether they considered written set of instructions as programming or the modes that were available when the programs were run.

Communication

The preservice teachers' focus on mathematics and programming seemed to be about expressing ideas easily, "But as if there are people with both English and Norwegian, different word choices and in general. So I think it's easier, maybe? With programming because it's so distinctive" (PT2). Yet, what the meaning were that programming could provide were not discussed. Instead, programming was described as more precise and distinctive, making it easier to carry meanings. As well, the receivers of the meanings, expressed through programming, were not identified. It could be the teacher, or it could be other students who were the expected recipients of the communication.

Potential for learning

The preservice teachers discussed very generally what language-diverse students would learn from using programming. The preservice teachers had recently experienced learning programming and recognised that programming languages were unlike English or Norwegian, "so the language they use is also different. So, we became a little more aware of what that might mean" (PT3). As a result, they considered that programming, as another kind of language, was potentially easier for language-diverse students to learn. When they began programming, they used "a lot of trial and error". As well, they had been in a classroom and saw that programming was well received by the students "So that they wanted more challenges, and then programming had to be good" (PT1). As learning programming could be done through trial and error, the preservice teachers considered that the students had agency to control what they were doing, in contrast to "complete all the tasks in class" (PT1). Therefore, the potential for learning could be seen as being on a metalevel, learning about learning strategies rather than about learning specific mathematics or programming skills.

Available identities

As with the teachers, the preservice teachers also considered language-diverse students to be students with "language difficulties" (PT2). Yet, the precision of the programming language meant that students, who were not fluent in Norwegian, or had "language difficulties" could be challenged to learn programming. As well, given that few students had experiences with it, then all students would be learning it on an equal level:

> Yes, and so maybe most students are equal then, in programming. That not everyone has that much experience in it. So, one is in a way on an equal level, or, yes. It is equally unknown, equally known to all. And then you are not so outside. (PT2)

These reflections provided the PTs with the possibility to consider students as wanting to be challenged and not just complete tasks in the textbook and as having equal possibilities to learn programming.

Conclusion

Digital tools have been promoted as supporting students to learn mathematics by being able to convey meanings in a variety of ways (see for example, Dockendorff & Solar 2018). However, when the purpose of communication was to access students' mathematical knowledge, then the possibilities for using digital tools were reduced to having the students use translation programs. These programs provided the students with ways to show what they could do mathematically while they were learning the language of instruction. The use of translation programs provided opportunities to determine whether or not the students could become like the Norwegian students in the class, by showing they were capable mathematics students. On the other hand, using some digital tools, such as programming, allowed language-diverse students to be considered as capable learners, regardless of their fluency in the language of instruction. The features of programming language, although not discussed in any detail, seemed to provide opportunities to see language-diverse students as being similar to their classmates, in that they wanted to be challenged and they had the same capabilities to learn, perhaps more so because they had experiences of learning other languages.

The use of the MSSM to analyse the interviews provided insights into how the purpose of communication was connected to the potential for learning and the available identities for language-diverse students. However, it was more difficult to analyse the focus group interviews than it was to analyse the individual interviews because of the level of details provided by participants.

For teacher education, there are some not-so-simple findings from this research. The teacher interviews suggest that if programming is introduced into all classrooms, as the Norwegian curriculum (Kunnskapsdepartementet, 2019) now demand, and teachers only see the main role of communication in the classroom as interpreting and evaluating students' work, then it is unlikely that language-diverse students would be viewed in alternative ways. The provenance (Kress & van Leeuwen, 2001) connected to mathematics teaching, as being predominantly done through textbooks, would not allow for alternatives to be noticed. Similarly, even if learning programming does provide teachers with ways to see alternative identities for language-diverse students, but connections are not made to students' potential for learning mathematical content then the outcomes for these students will be limited. Unless teachers gain insights into how to make connections to mathematical ideas and value these connections, then programming can take on the connotation (Kress & van Leeuwen, 2001) of being game-like and so not a legitimate way for teaching mathematics to any student, including language-diverse students.

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Quality of students' explanations in describing multiplication algorithms: the case of Napier bones

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We describe a didactic cycle involving the Napier bones algorithm, included in a wider research whose aim is to investigate whether and how the use of different kinds of artifacts can affect students' explanations, supporting the understanding of some aspects of the multiplication. We present and discuss some data regarding group and individual tasks that allowed us to investigate how students' explanations were and what semiotic representations they chose when using Napier bones.

Keywords: Semiotic mediation, explanations, multiplication, artifacts.

Introduction

Algorithms for mathematical operations, which students experience at school, are not the only ones. In particular, for multiplication there are many alternative procedures associated with graphic, symbolic and/or material artifacts. Often, children, after knowing the traditional algorithm (multiplication in column), apply it in a mechanical way, without reflecting on the meaning of the foreseen steps, as a result of the fact that teaching multiplication is frequently based on memorising the multiplication table rather than on a meaningful understanding of the concept and related procedures (Vohra, 2007). Some studies (e.g., Haylock & Cockburn, 2008) suggest that children develop their mathematical understanding through language, symbols, concrete experience, images and manipulatives, which make mathematical concepts and operations more comprehensible to students (Swan & Marshall, 2010). In this sense, we believe that knowledge and use of different algorithms, instead a unique one, could help students to better understand some aspects of the multiplication. Knowledge about algorithms could change from being only instrumental to being also conceptual.

Students also find themselves having to use representations and symbols –typical of each algorithm– that, in addition to everyday language, are interpreted as carrying some meaning. At the same time, however, it should be kept in mind that semiotic mediation may not work. It depends on both the individuals and the characteristics of the tool involved in the mediation (Hasan, 2005). Explanations, produced by students during mathematical activities, could be a useful means for teachers both to see whether the mediation has worked and to observe whether understanding has been achieved.

The aim of our overall research is to investigate whether and how the use of different kinds of artifacts can affect students' explanations, and, therefore, the understanding of some aspects of the multiplication. We conducted a study in a primary school, involving 25 students who participated in mathematical activities focused on 5 different multiplication algorithms (multiplication in column, Chinese multiplication, Arabic multiplication, Napier bones multiplication, Genaille-Lucas rulers multiplication), each characterized by specific symbolic, graphic and/or material artifacts. In this paper we only focus on the Napier bones algorithm, answering to the following research questions: *what are the qualities in the students' explanations and what semiotic representations do the students' explanations draw upon when using Napier bones?*

Theoretical framework.

Our research is framed within the semiotic mediation paradigm, key aspect of the sociocultural theory of cognition, and developed in the work by Hasan (2005) and in that by Bartolini Bussi and Mariotti (2008). For Vygotsky (1978) cognitive development occurs through a *social activity* consisting of a progressive *interiorization* of strategies for using tools and mediation forms, allowing the child to attribute meaning to signs. The interiorization is carried out through *semiotic processes*: the construction of individual knowledge requires shared social experiences involving the production and the interpretation of different systems of signs (such as everyday language, images and symbols). Hasan (2005) refers to *mediation by semiosis*, i.e., mediation through the use of sign systems acting as an abstract tool in changing human activity, and emphasizes the semiotic function performed by language for the development of thought. Bartolini Bussi and Mariotti (2008) elaborated the Theory of Semiotic Mediation (TSM) in order to study links between artifacts, tasks, signs, and the mathematical knowledge to be mediated in the classroom. The teacher assumes the role of *cultural mediator* (Hasan, 2005; Bartolini Bussi & Mariotti, 2008): she mediates mathematical meanings to students through an artifact, exploiting its semiotic potential in relation to the tasks she has designed.

Admitting a link between language and thought, we can hypothesise that the quality of the explanations a student produce is related to the quality of her thinking (Albano et al., 2015; Ferrari, 2017). Within the field of mathematics, the explanation is closely related to 'mathematical understanding' and has to perform different functions: to clarify aspects of one's mathematical thinking; to describe the steps of a procedure used; to tell how one can arrive at a solution; to convince oneself or another person of some claim; to expand students' mathematics learning, as they communicate existing thoughts and can also generate new thought objects. In this way, explanations can be considered as constituent elements of the arguments (Levenson & Barkai, 2013) and, in our view, it is essential to analyse them in order to understand the development of knowledge that occurred during a semiotic activity involving artifacts.

Methodology.

The research is qualitative with an exploratory and descriptive aim: it is limited to collecting, analysing and interpreting data emerging from the specific context under investigation. In this study a single-group *pre-experimental research design* (Campbell & Stanley, 1963) was used, i.e., only the experimental group was considered, without a control group.

Context and overall research design.

The investigation involved 25 students from a fourth primary class of the Istituto Comprensivo 'Calcedonia' in Salerno (Italy), aged between 8 and 10 years.

The overall research design reflected the approach of the TSM (Bartolini Bussi & Mariotti, 2008). Researchers designed group and individual tasks, implemented in 5 didactic cycles, each one lasted about 1 hour, focused on a particular multiplication algorithm. In the first didactic cycle the work groups were formed and they remained the same for each cycle: the class teacher gave the role of ‘captain’ to five students with a good basic level and, then, four children were randomly assigned to each of them. Teacher was always present in the classroom, but her role was that of an external observer. The activities were conducted by one of the authors, a pre-service teacher who was doing her internship at that school. Each cycle was articulated in 3 phases: a *group activity* by means of tasks designed with the aim of exploring the artifact; a *collective discussion* on the structure and the way of using the artifact, based on the explanations produced by the students in the first phase; an *individual activity* by means of tasks designed to verify the understanding of the algorithm.

The didactic cycle n°4: Napier bones multiplication algorithm.

Napier bones multiplication algorithm is based on a material graphic artifact (Figure 1) consisting of 10 strips numbered from 0 to 9 and a further strip, called ruler, marked with the symbol ×. Each strip is divided into 9 squares, each—except the ruler—cut by a diagonal line running from top right to bottom left, in which are shown the multiples of the number marked at the top. In each of the 9 squares of the ruler there is a number from 0 to 9. To calculate the product between two multi-digit numbers, the bones of the digits of the first factor are placed side by side with the ruler on the right, creating a table with nine lines. Then, you have to consider the lines corresponding to the digits of the second factor, one after the other, starting from the units one. Each line will correspond to a partial product, obtained by summing along the diagonals. The desired product will be obtained as the sum of the partial products (Figure 1).

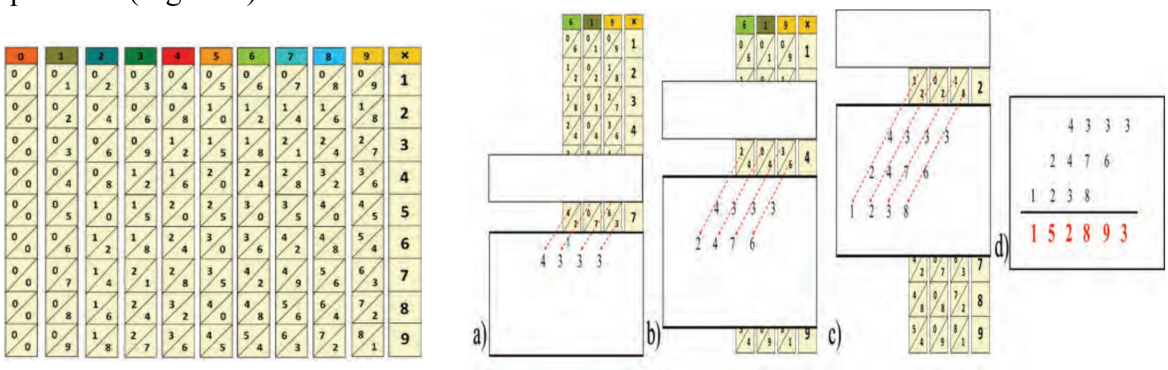

Figure 1: Napier bones (on the left) and four steps calculation of 619×247 (on the right)

The students were not familiar with this artifact, but it was introduced after they had already experienced multiplication in column, Chinese and Arabic algorithms, as part of the educational path. We expected not too many difficulties with the Napier bones, since some algorithm features are similar to those of Arabic multiplication. In the following we focus only on group and individual activities.

Group activity consisted of two tasks: Task 1 and Task 2. Each group was given a paper on which Napier bones were illustrated, without saying what they were. **Task 1** consisted of 3 questions (Bartolini Bussi & Mariotti, 2008): 1) What are they?; 2) What do they look like? You can use not only text, but also drawings to describe the objects; 3) What can they do? Justify your answer. The first two questions were strictly aimed at exploring the artifact and bringing out the students’ prior knowledge, on which the new mathematical content would be anchored; the third question involved highlighting the utilization schemes of the artifact, which would then become a tool[1].

Then the students were told that the strips were named Napier bones and were asked to cut out the paper strips so that they could manipulate and use them for **Task 2**, which consisted of 3 questions:

1. Imagine you only have Napier bones available to carry out multiplication. Describe how you could execute the operation 123×6.
2. Describe how you could execute 436×72 multiplication using Napier bones.
3. What similarities and what differences do you notice between Napier bones multiplication and Arabic multiplication?

Starting from the assumption that Napier bones derive from Arabic multiplication, Question 1 aimed to investigate whether students, in exploring and using the artifact, were able to explain how to calculate a multiplication with a one-digit factor; it also aimed to observe what kind of semiotic resource the students used in their explanations. Question 2 had the same aim, but in the case of multiplication with a non-consecutive two-digit factor, which involves adding the two partial products, as in the traditional algorithm. The aim of Question 3 was to investigate whether the students were able to make a comparison between Napier bones algorithm and Arabic multiplication.

For *individual activity* was designed and implemented only **Task 3** consisted of 4 questions:

1. In a third-year class, the math teacher gives each child Napier bones without explaining what they are or what they are used for. Luca can't quite understand how to use them. Explain what they are and show him a few examples, describing step by step how to use them.
2. Luca still shows some difficulty. Execute the same multiplications as in your examples using multiplication in column and explain to him the differences between the two methods.
3. What are the advantages of using Napier bones? And what disadvantages?
4. In your opinion, does the use of Napier bones make multiplication easier than traditional calculus? Justify your answer.

In Question 1, by asking for examples, we wanted to find out if the students were able to explain how to do multiplication using the bones—highlighting algorithm features—and what representation they used to support their answers. The aim of Question 2 was to further investigate whether the students were able to explain the mathematical properties underlying both the traditional algorithm and the Napier bones. The Questions 3 and 4 were aimed at understanding how they perceived a possible use of bones in daily school practice.

Data collection and analysis.

The data were collected by means of *observation* during each phase of the didactic cycle, *structured group/individual tasks*; *field notes* regarding the collective discussions based on students' production.

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[1] According to the instrumental approach (Béguin & Rabardel, 2000) a subject, engaged in a goal-directed activity, can build schemes of instrumented action for an artifact. Thanks to the visible signs elaborated by the solver (e.g., text, words, representation), we can make inferences about the schemes she is developing for the tasks.

In this paper we consider only the explanations provided by the students in *structured group/individual tasks*. As we have just described, for each structured task, students were asked to explain their answers and to use different types of representation. In each group/ individual task, requests such as “Justify your answer”, “Explain how you arrived at the solution” or “What differences do you notice?”, “What are the advantages/disadvantages?” were included in order to stimulate explanations and continuous comparison. Once the data collection was completed, the name chosen by each group (Fulmine, Vento, Tempesta, Pioggia and Neve) was recorded for all group activity protocols. The protocols of the individual activities were numbered from 1 to 25, assigning the same number to each student in the different activities.

For the analysis of the explanations the attention was focused on the different types of representation used by the students: *graphic, verbal*. In addition, for analysing their qualities we were inspired by the criteria as used in the work by Cusi et al. (2017), namely *correctness*, which refers to the absence in the explanation of mathematical errors; *clarity*, which refers to the comprehensibility of the explanation by an interlocutor ; *completeness* refers to the description of all the steps of the algorithm, including the final step leading to the solution. As an example “12 and 17 are 29” can be considered *correct*, *no clear* and *no complete* because the description of the procedure is missing and a classmate might not understand that ‘and’ refers to an addition. Thus, if an explanation is correct, clear, and complete, it is able to perform the functions that Levenson and Barkai (2013) attributed to it.

Results.

In this paper we present only some results from data concerning Question 1 and Question 2 of Task 2 carried out in the group activity and Question 1 of Task 3 carried out in the individual activity.

Task 2 - Group activity.

Concerning **Task 2,** all groups provided an answer to **Question 1**. The analysis has revealed that all of them used *verbal representation* to explain their answer. Almost all of the explanations, however, was not qualitatively adequate, in relation to the chosen criteria.

In fact only one group (Fulmine) produced a *correct, clear and complete* explanation in that, even though it made use of a colloquial register, it explained with care the procedure to be used to do multiplication, showing that the group identified some similarity between the calculation of multiplication with Napier bones and Arabic algorithm.

Fulmine: We took the 1-2-3-X sticks (the one that contains the numbers up to 9) then we did 6+1=7, 2+1=3 and then 8+0=8, and came up with 738 using Arabic multiplication.

The other 4 groups, even though they did not describe the correct procedure and did not specify the various steps to be taken, still provided an explanation that, for 2 groups (Tempesta and Pioggia) was quite *clear*, while for the other 2 (Vento and Neve) was difficult to understand (*no correct, no clear, no complete*). We propose, as examples, the answers given by Pioggia and Neve.

Pioggia: To solve with Napier bones you have to put them in a row, counting only the numbers that you have to multiply. We put a 1-stick, a 2-stick, and another, 3-stick, then put sheets on the other numbers and add up the results of each number. The result is 18.

Neve: To do the 123×6 multiplication we have to put the 6 in the 3, then the 6 in the 2 and finally the 6 in the 1 then we do the result and at the end the result is 7938.

The data regarding **Question 2** show that all groups provided the answer but the quality of the explanations worsened. None of the groups gave a *complete* answer and 3 out of 5 groups described the use of bones in a *no correct, no clear, no complete* way. As an example, the quality of the explanation provided by Fulmine seems to worsen with respect to the Question 1: the group described the procedure in a fairly *clear* way, but made an error in the sum of the partial products and, consequently, in the final result and they skipped some steps of the whole process.

Fulmine: We took the 4-3-6-X sticks (the one that holds the numbers 1-9) then we did 436×7 first and then we did 436×2 and finally we added up the two results and it came out 30520.

More *clear and correct* (even if the final result is missing) is the explanation produced by Tempesta, as the group did not explain how to arrange the sticks, nor how to calculate the two partial products.

Tempesta: First we looked at the numbers coming up in the 2's line and then the 7's line and finally we added up.

Task 3 - Individual activity.

Question 1 was completed by 21 out of 25 students and only 12 of these answered to the first part of it (e*xplain what they are*). As for the second part of the question (*show him a few examples, describing step by step how to use them*) only 16 out of 21 students provided explanations (Figure 2).

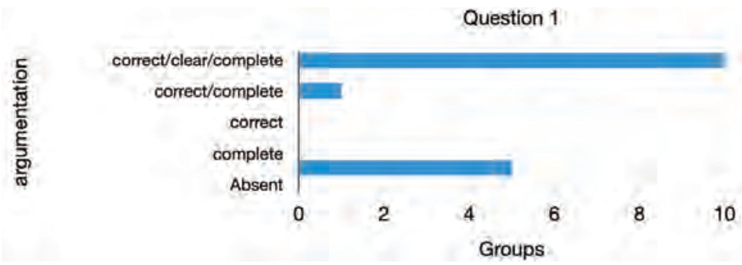
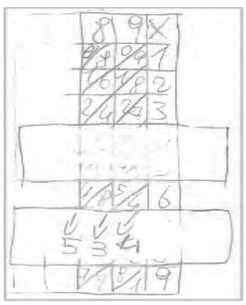


Figure 2: Task 3 - quality of the explanations in Question 1

Each one reported a single example and, 6 students used *verbal representation* to explain*,* describing in a more or less *correct and clear* way the various steps to be performed; 10 students, instead, used only the *graphic representation* by drawing the sticks they had to use to do the multiplication chosen as an example (Figure 2).

	Prot. n°12: I'll give you an example, 54×5; take the ruler and the 5-stick and 4-stick. Take a sheet of paper and divide it in half, you will come out with two equal sized sheets. Put the 5, 4 and ruler next to each other, take one half of the sheet and put it (ruler part) on 1-2-3-4 and the other on 6-7-8-9 and count the numbers in the diagonals that are created, like in Arabic multiplication.
Figure 3: A *correct/clear/complete*	

explanation (Prot. n°11) with graphic representation	

Regarding the quality, more than half of the sample (10 out of 16 students) provided *correct, clear, and complete* explanations, with some students using *verbal representation* (e.g. Prot. n° 12) and the majority using *graphic representation* (Prot. n°11 in Figure 3).

5 students out of 16 provided *no correct, no clear, no complete* explanations, using, often, a convoluted syntax. Only Protocol n°25 explained in a *correct and complete* way: using *verbal representation*, she explained all the steps to arrive at the result of the chosen multiplication, but, as for most of the class, she used a not very clear colloquial register. In addition, she was the only one to specify that it is important to move the second partial product one place to the left, because it is the product of the tens of the multiplier by the digits of the multiplicand.

Prot. n°25: Luca first you need to add up the numbers in the diagonals to get the result. If the multiplication is 172×47, to get the result you have to isolate first the second number i.e. 7 and then the first number i.e. 4. Finally you add it all up and the result will come out. But we must remember that the product of the tens must be under the other tens.

Discussion and Conclusions.

In this study we explored the qualities of students' explanations and what semiotic representations students draw upon in their explanations when using Napier's bones, in two different situations. Although this algorithm derives from Arabic multiplication, previously experienced, the analysis of the responses to Question 1 and Question 2 of Task 2 highlighted that the quality of students' explanations did not meet the chosen criteria. In addition they are all based on verbal representations. For the former only one group explained in a correct, clear, and complete way; for the latter only one group described the procedure of Napier bones algorithm in a correct and clear way. From the data analysis regarding Question 1 of Task 3 the quality of the explanations, with prevailing graphic representations, even if in colloquial register, seems to be improved according to the chosen criteria: 10 students provided correct, clear, and complete explanations and one student explained in a correct and complete way. This may also be a consequence of the collective discussion—not analysed in this paper—orchestrated by the preservice teacher (Vygotsky, 1978; Hasan, 2005; Bartolini & Mariotti, 2008) and based on the students' explanations collected in Task 2. In this way, students began to make sense of what they were doing and improved the quality of their explanations. Furthermore, the data seemed to suggest us that the use of graphic representations of Napier bones, in students' explanations in the individual activity, had helped them to perform multiplication correctly, to make connections with previous knowledge (Prot. n°. 12) and to understand some operation properties (Prot. n°. 25) (Swan & Marshall, 2010).

The study is at an early stage: data from other tasks, from other didactic cycles on different algorithms, and from collective discussions, still need to be analyzed. Nevertheless, we believe that it can be a starting point to reflect on artifacts mediation of a meaningful understanding of some aspects of the multiplication (Haylock & Cockburn, 2008).

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Generalizing mathematically through changes in referents

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In this pilot, opportunities to engage in mathematical generalization were identified in a section of a textbook from year 6. From a social-semiotic perspective, we explored how these opportunities were constructed linguistically. While passive verb forms and nominalizations constructed an independent character of mathematics, logic expansions constructed limitations for the generalizations. Changes in referents constructed opportunities for generalizing actions.

Keywords: Mathematical generalization, social semiotics, transduction, textbooks, primary school.

Background

Mathematical generalization (henceforth MG) has been raised as a big idea in mathematics, for example through scaffolding algebraic reasoning, functional thinking and many other mathematical activities (e.g., Dörfler, 1991). It is also known that students often have difficulties with, among other things, expressing generality and using generalized language (Mason, 1996). Generalized language may here be interpreted as a part of disciplinary literacy (Shanahan & Shanahan, 2008), as specialized texts and literacy skills are expected of students at more advanced levels of studies. However, big ideas are also a part of primary school mathematics (Madej, 2021). Therefore, the expected literacy skills and specialization are important to scrutinize also in texts for primary school.

Despite its importance MG is not in focus in the 2011 Swedish steering documents. The course plans hardly mention aspects of generalization (Hemmi, Lepik & Viholainen, 2013), and it has been shown that Swedish primary school textbooks after 2011 contain very low proportions of generalized arithmetic (Bråting, Madej & Hemmi, 2019). Still, textbooks largely seem to organize the teaching in Swedish primary school classrooms (Koljonen, 2020). This makes Swedish textbooks an interesting starting point for understanding opportunities to engage in MG.

Different aspects of language have been compared for various school years in Swedish textbooks in various subjects (e.g., Österholm & Bergqvist, 2013). Aspects of language have also been studied in textbooks in other countries (e.g., Alshwaikh, 2016). While the degree of abstraction and generalization has been studied in Swedish educational texts for social science, natural science and literature (Edling, 2006), to our knowledge so far, the ways in which MG is presented have not been investigated from a linguistic point of view. Therefore, this pilot study aims to explore and describe linguistic aspects of MG in a section of a textbook. The research questions are:

- What opportunities for mathematical generalization can be identified in the textbook section?
- In what ways are linguistic resources used to construct the mathematical generalization?

Mathematical generalization and a social semiotic perspective on mathematics

MG entails both generalization as an object or conclusion, and generalizing as an act or process (Ellis, 2007; Harel & Tall, 1991; Mason, 1996). It has been regarded as e.g., mental processes or social interaction across agents and within specific contexts, and may be expressed through gestures, images and other semiotic resources, as well as formal symbols or words (Dörfler, 1991; Harel & Tall, 1991; Radford, 2018). Ellis (2007) takes an actor-oriented perspective and empirically identifies different ways that a learner may engage in MG. In her taxonomy, these ways are described as the *generalizing actions* of relating, searching and extending, and the *reflection generalizations* of identification or statement, definition and influence. Generalizing actions are inferred through activity and talk, while reflection generalizations are inferred through statements or the use of a result of a generalization. Since textbooks are always oriented towards the students using them, Ellis' actor-oriented perspective seems useful for the present study. However, it is not what students actually do, say or write when using textbooks, or students' mental processes, which are in focus here. Adapting Ellis (2007), we look at explicit opportunities in the textbooks to engage in generalizing actions, to read or state reflection generalizations or to read or use a result of such a generalization.

To understand how MG is constructed through written language and other semiotic resources in textbooks, we build on a social semiotic perspective and Systemic Functional Linguistics, SFL (e.g. O´Halloran, 2005). Central in this perspective is that in any act of communication we make choices of language in order to construct a certain meaning (e.g. Halliday & Matthiessen, 2004). This is also the case for semiotic resources such as mathematical notation and images (O´Halloran, 2005). Further, a change in the semiotic mode, the process of *transduction*, includes ontological shifts (Kress, 2010). Through the "re-articulation of meaning from the entities from one mode into the entities of the new mode" (Kress, 2010, p. 125), we believe such shifts may be important for the ontological construction of MG. In this paper, various semiotic resources are analysed with SFL to understand opportunities to read and write MG in a primary school textbook section.

Methods

A textbook for year 6, the final year of Swedish primary school, was considered suitable since MG is often considered rather difficult for students to master (Mason, 1996), and therefore, it can be expected that a book for older students will contain the largest proportion of opportunities for MG. The text analysed in this study is taken from a year 6 book translated from Finnish into Swedish. This particular book seemed to include a large variety of opportunities for MG, compared to others.

In this pilot, we test the methods of analysis. We do not identify all textbook sections where MG is offered. Therefore, we selected an initial section on patterns, which was spontaneously deemed to offer MG. The selection is small, but we do not seek to generalize the findings in this pilot to a larger body of texts. To answer the research questions, two analyses were conducted. Both analyses were conducted by one of the researchers, checked by the other, and then discussed until agreed.

The analysis of mathematical generalization

To identify the opportunities to engage in MG, we used a taxonomy of mathematical generalization (Ellis, 2007). It is presented in our version adapted to textbooks in Table 1.

Table 1: Questions for identifying opportunities to mathematical generalization

MG (Ellis, 2007)	Questions posed to the text
Generalizing action	*Do the students have the opportunity to …*
Relating	• relate situations through the formation of an association between two or more problems or situations? • relate objects through the formation of a similarity between two or more present objects?
Searching	• detect a stable relationship between two or more objects. • test if a procedure remains valid for all cases? • check whether a detected pattern remains stable across all cases? • determine if the outcome of the action is identical every time?
Extending	• expand to a larger range of cases than that from which the phenomenon originated? • remove particulars to develop a global case? • operate on an object to generate new cases? • repeat an existing pattern to generate new cases?
Reflection generalization	*Do the students have the opportunity to write (in tasks) or read (in introductory text) …*
Identification or statement	• the identification of a property? • a statement of commonality or similarity, or of a general phenomenon?
Definition	• a definition of a class of objects all satisfying a given relationship, pattern or other phenomena?
Influence	• an implementation of a previously developed generalization or an adaption of an existing generalization to apply to a new problem or situation?

The SFL-analysis

Linguistic generalizations may be constructed through choices of referents and nominalizations (Halliday & Matthiessen, 2004). These are the main features explored in this analysis.

Referents are experiential elements in the text. Different types of noun phrases, and how they change, indicate different ways of using language (Halliday & Matthiessen, 2004). Referents can have an everyday character or be technical, and they may be presented in a variety of semiotic modes (Kress & Van Leeuwen, 2006). They may also be placed on scales between physical and abstract, specific and general (Edling, 2006). A move in the text from a specific to a general referent may realize a linguistic generalization in the text, whereas a move from a concrete to an abstract referent instead may realize a linguistic abstraction in the text (Figure 1). When a referent is changed into a different semiotic resource it is called a transduction which at the same time re-articulates the ontological meaning of the referent (Kress, 2010). For instance, a picture of 5 apples which then is followed by the number 5 entails an ontological shift from concrete to abstract meaning. Linguistic changes of referents including transduction may thus indicate opportunities in the textbook for MG. This will be explored in the analysis.

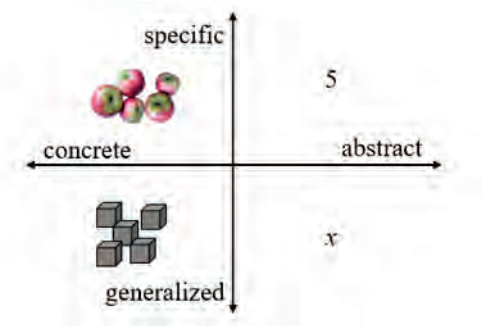


Figure 1: Abstraction and generalization in different referents, adapted from Edling (2006)

A *nominalization* is an incongruent expression, or grammatical metaphor (Halliday & Matthiessen, 2004), where a noun (e.g. *subtraction*) is used instead of the corresponding verb or adjective form (e.g. *subtract*). Nominalization is a means of constructing generalization because an operational process, expressed by a verb, is reified into a general concept by the nominal form. Operational processes are mostly tied to specific situations where calculations are needed, whilst general concepts describe mathematical relations without necessarily linking to specific situations. Nominalization also hides human participation, since there is no explicit human agent carrying out the action. This downplaying of human agents further accentuates generalization. Therefore, generalization may also be indicated through a *passive verb form*, e.g. in the phrase “can be used” instead of “you can use”. Our analysis explores in what respect nominalizations and passive verb forms contribute to MG. To avoid overlooking other important linguistic features, the instances of MG in the textbook section were read, reread and discussed with respect to the semiotic resources used in them. For instance, expansions of clause complexes show in what ways a text may be developed (Halliday & Matthiessen, 2004); this might contribute to MG. However, the analysed text mostly consists of main clauses.

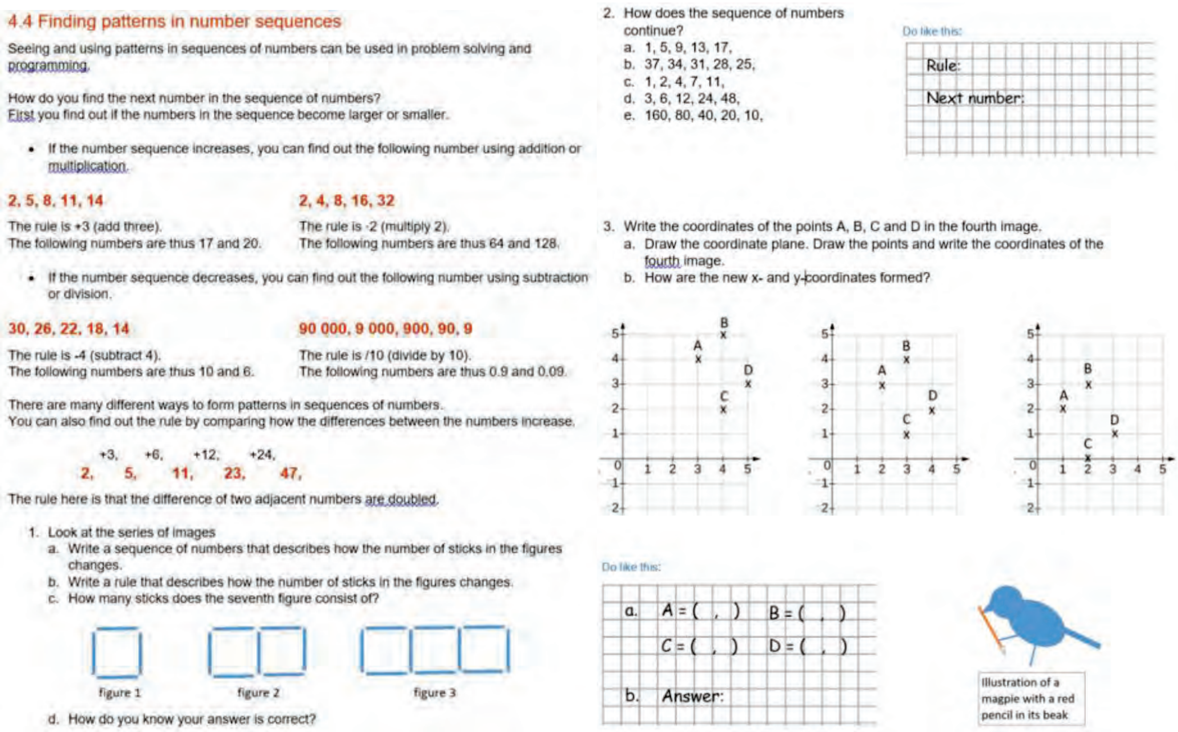
Finally: In a certain sense, textbook tasks comprise one half of a dialogue where the textbook author asks questions and the students answer. We acknowledge that in such a dialogue, any feasible reflection generalization will be constructed in the answer and not the question. Since this study does not look at actual answers, opportunities for reflection generalization are therefore not linguistically analysed in the tasks, only in the introductory text.

Results

Opportunities to engage in mathematical generalization, MG, are given in the form of generalizing actions and to some extent in the form of reflection generalization. To answer the research questions, the results are structured by what is identified in introductory text and tasks respectively. All translations from the textbook section are ours.

Nominalizations and passive verb forms constructing an independent character

The introductory text provides opportunities to engage in *generalizing actions* and read *reflection generalizations*. The top of the page question and the two following sentences (Fig.2) constitute a prompt to search for relationships, and thus models *searching* in the form of detecting a stable relationship between objects. A similar statement is made for decreasing number sequences using subtraction and division. Finally, "You can also find out the rule …" models *extending* to other ways of investigating number sequences. All three statements "If the number sequence [increases/ decreases] …" and "You can also ..." are descriptions of general strategies for investigating patterns and therefore examples of *reflection generalization* as *identification or statement*. Hence, in the introductory text, the generalizing actions and reflection generalizations seem to coincide



4.4 Finding patterns in number sequences

Seeing and using patterns in sequences of numbers can be used in problem solving and programming.

How do you find the next number in the sequence of numbers?
First you find out if the numbers in the sequence become larger or smaller.

- If the number sequence increases, you can find out the following number using addition or multiplication.

2, 5, 8, 11, 14
The rule is +3 (add three).
The following numbers are thus 17 and 20.

2, 4, 8, 16, 32
The rule is ·2 (multiply 2).
The following numbers are thus 64 and 128.

- If the number sequence decreases, you can find out the following number using subtraction or division.

30, 26, 22, 18, 14
The rule is -4 (subtract 4).
The following numbers are thus 10 and 6.

90 000, 9 000, 900, 90, 9
The rule is /10 (divide by 10).
The following numbers are thus 0.9 and 0.09.

There are many different ways to form patterns in sequences of numbers.
You can also find out the rule by comparing how the differences between the numbers increase.

+3, +6, +12, +24,
2, 5, 11, 23, 47,

The rule here is that the difference of two adjacent numbers are doubled.

1. Look at the series of images
   a. Write a sequence of numbers that describes how the number of sticks in the figures changes.
   b. Write a rule that describes how the number of sticks in the figures changes.
   c. How many sticks does the seventh figure consist of?

   figure 1 figure 2 figure 3

   d. How do you know your answer is correct?

2. How does the sequence of numbers continue?
   a. 1, 5, 9, 13, 17,
   b. 37, 34, 31, 28, 25,
   c. 1, 2, 4, 7, 11,
   d. 3, 6, 12, 24, 48,
   e. 160, 80, 40, 20, 10,

Do like this:
Rule:
Next number:

3. Write the coordinates of the points A, B, C and D in the fourth image.
   a. Draw the coordinate plane. Draw the points and write the coordinates of the fourth image.
   b. How are the new x- and y-coordinates formed?

Do like this:
a. A = (,) B = (,)
C = (,) D = (,)
b. Answer:

Illustration of a magpie with a red pencil in its beak

Figure 2: The textbook section Find patterns in number sequences

The written referents in the three statements in the introductory text mentioned above, e.g. "patterns in number sequences" and "the following number" change to the specific numbers e.g. "2, 5, 8, 11, 14". The ontological meaning in this transduction shifts from generalized to specific. The specific number sequences in the introduction thus function to explain and unpack the *reflection generalizations*. The nominalizations in the four arithmetic operations in these statements construct the number sequences as structures: it does not matter which specific numbers are added, multiplied, subtracted or divided, because it is the operations in general which are relevant, not their results. The passive verb form "are doubled" constructs *searching* as something independent from human beings: the activity does not render different results (e.g. as tripling instead of doubling), depending on who performed the investigation. This independent character is a quite fundamental aspect of MG. Finally, the introductory text is expanded logically in the reflection generalizations "If the number sequence [increases/decreases] …” constructing a condition for when the procedure described in the main clause is valid. These expansions thus limit the range of the generalizations.

Generalizing actions in tasks constructed through changes in referents

Task 1 prompts a re-articulation of the number of sticks in a given pattern into a number sequence. Through forming a similarity between the two objects, an opportunity to *relating* is constructed. The ontological meaning is re-articulated through a transduction from the generalized-concrete sticks to specific-abstract numbers. An opportunity to engage in *searching* is given through writing the rule for how the pattern changes. Here, the ontological meaning shifts from the specific-abstract numbers to the generalized-abstract rule, formulated as “+3 (add three)”. 1c prompts to continue the number sequence and generate a new case, the “seventh figure”. Here, the transduction re-articulates the generalized-abstract “a rule”, to the specific-abstract number of sticks which is supposed to be calculated. In this way, the ontological shift supports a control of the identified rule and the test for new cases, thus *extending* by continuing the pattern. The last part of task 1 gives the opportunity to write an *identification or statement*. However, this opportunity could not be analysed linguistically.

Task 2 is fairly similar to task 1. It prompts to re-articulate specific-abstract number sequences as generalized-abstract rules for the sequences and thus gives an opportunity to *searching*. The prompt in the last part is to re-articulate new cases of specific-abstract numbers, which gives an opportunity to extending. Since no concrete figures are given, this task does not comprise *relating*. The picture of the checkered piece of paper in the task and what is written on it, explicitly models how to engage in searching and extending the number sequences. In task 1 and 2, *searching* is then constructed through transduction from specific-abstract to generalized-abstract, whereas *extending* is constructed through the ontological shift back from generalized-abstract to specific-abstract meaning.

Task 3 gives the opportunity to a *reflection generalization* as an *influence* since the previously developed generalization of how to work with patterns is adapted to a new situation, consisting of coordinates which constitute successive sets of ordered pairs. The first step prompts to “[w]rite the coordinates … in the fourth image”, so focus is on extending through continuing the pattern, to generate a new case. *Relating* and *searching* are thus not supported ahead of extending, as in the previous tasks. 3a prompts “draw” and “write”. It could be argued that these requests include relating the points A, B, C and D in the sequence of graphs to their more abstract re-articulations as sets of ordered pairs of numbers. However, it is only the points and “coordinates of the fourth image” which are explicitly asked for. In 3b, the question “How are the new x- and y-coordinate formed?” could include searching for a stable relationship between the points A, B, C and D in the sequence of graphs. Just as in 3a, only the “new” coordinates are asked for. The picture of the checkered piece of paper in task 3 is different from in task 2. It supports writing down the coordinates for one set of points A, B, C and D, and to write the “Answer”.

To solve task 3, learners thus need to recognize the steps of relating and searching the pattern in the first three graphs without a prompt to do so. *Extending* to the fourth picture is the only explicit request. The steps include transductions from the graphical mode to numbers (3a), and from the numbers to formulating the rule of the pattern (3b), respectively. As in task 1 and 2, *searching* seems to be constructed through an ontological shift from the specific-abstract to the generalized-abstract. The graphs are interpreted as concrete referents in this study since they have a spatial extension. *Relating* in task 3 is thus constructed through an ontological shift from concrete to generalized meaning.

Concluding remarks

This pilot analysis of a section of a mathematics textbook has revealed opportunities to engage in *generalizing actions*, as well as reading and writing *reflection generalizations*. However, no opportunities to read or write *definitions* are given, nor to read a reflection generalization as *influence*. Swedish textbooks have low proportions of generalized arithmetic (Bråting et al, 2019) and mathematical generalization, MG, include many aspects (Ellis, 2007). We therefore believe that a forthcoming comparative study of textbooks would reveal differences in opportunities for MG. Further, different proportions of *searching* might be a distinguishing feature since searching has been prominent in the textbook example analysed in this study. A study of students' solutions or work with the textbook section may contribute to the understanding of what MG actually takes place.

In the introductory text, opportunities for reading *identification or statement* are constructed through nominalizations of the four arithmetic operations. The range of two *statements* about patterns are constructed through logical expansions in the text. Moreover, passive verb forms and nominalizations construct patterns as structures, independent of human agents. In this sense, the *identification or statement* does not only model reflection generalization, but also expresses the general character of mathematics. These linguistic features may be interpreted as a part of the disciplinary literacy (Shanahan & Shanahan, 2008) which is expected of students, in order to understand MG. Finally, the reflection generalizations in the introductory text are unpacked and explained through the ontological shift which occurs in the transduction from written text to number sequences. In this way we can see that various semiotic resources are used to express MG.

Opportunities for generalizing actions in the analysed tasks are mainly constructed through different transductions, i.e. changes in the referents' semiotic mode. When *relating*, the ontological meaning shifts from concrete to more abstract; when *searching*, it shifts from specific-abstract to general-abstract. When *extending*, the ontological meaning shifts back from generalized-abstract to specific-abstract referents. Therefore, for a textbook to support opportunities for learning *generalizing action*, we believe that transduction may be a key feature. Further, to enhance opportunities for MG attention should be paid not only to the change between concrete and abstract referents, but also between referents which are specific and generalized, and referents presented in different semiotic modes.

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Communicative projects about mathematical reasoning

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Keywords: Mathematical reasoning, dialogism, classroom practice.

Background

Mathematical reasoning as concept in education research comes in various shapes and is used with different purposes (Jeannotte & Kieran, 2017). Regardless of definition, mathematical reasoning is included in many curricula as part of being mathematically competent. Now, in Norway, after the curriculum renewal in 2020, (Norwegian Directorate for Education and Training, 2020), mathematical reasoning is linked to students' capability of using mathematical language, of engaging in posing and answering questions and of using mathematical concepts in different situations. The responsibility for creating opportunities for students to develop an understanding of and ability to carry out mathematical reasoning lies with the teacher. However, creating and maintaining a discourse that promotes the development of these competencies, seems to be challenging for teachers (Højgaard et al., 2010).

As part of my PhD work, the study presented here focuses on the interactions in the classrooms of newly graduated teachers when communicating mathematical reasoning. The study contributes to the understanding of what participation in mathematical reasoning activities looks like as well as aspects of how learning experiences influence each other over different timescales. The research questions are: What kind of communicative projects are established in the classroom when the new teacher and students are communicating mathematical reasoning? How do these projects evolve over time?

Theoretical framework

For this research I have chosen a dialogical approach. Dialogical aspects of students becoming active members of the mathematical discourse in the classroom have been addressed in several research studies. For example, studies have shown how participating in conversations about mathematics promotes students' understanding of mathematical concepts or a development of formal mathematical language (Barwell, 2016; Schleppegrell, 2007). Following Linell, dialogism is a theory of sense-making where "action, communication and cognition are thoroughly relational (or inter-relational) and interactional in nature" (Linell, 2009, p. 14). This relational aspect of communication puts focus on both sense-making as a situated joint construction and on communicative acts as sequentially ordered. In the negotiation of meaning there is a flow of initiatives and responses forming a dialogical interdependent relation between single acts and more overarching activities. Similarly, activities form larger units of communication, "communicative projects", interrelated to other goals or doings in, and across situations. This interdependence connects acts in the present to dialogical contributions prior in time and to future contributions (Lemke, 2000). Analysis of what happens in communicative projects, how they are organized or on what timescale they are carried out can be a way to understand a locally produced discourse (Linell, 2009). The dialogical principles of sequentiality, joint construction and act-activity interdependence also imply that contexts and discourses are constantly shifting, interdependent with what others do, have done and potentially could be doing in the future.

Methodology

The study focuses on communicative projects about mathematical reasoning and how they evolve over time. To identify communicative projects, I will do observations of a sequence of 10-12 consecutive lessons. This enables analysis of interactions both within a lesson and between lessons (Lemke, 2000). The observations will be done in three to four classrooms at upper secondary level using video recording. Also, the teachers of those classes will be interviewed. The teachers will have up to three years of experience teaching mathematics since graduating from teacher education. Video recordings and interviews enable analysis of how communicative projects evolve. Capturing actions such as gestures, the use of physical objects or use of words can connect different timescales. Interviews will give information about how the teacher planned and adjusted activities. Both methods can be used for finding shifts in organizational aspects (Linell, 2009) such as the use of time and space in the classroom or the function of reasoning activities. I will use the mathematical reasoning framework presented by Jeannotte and Kieran (2017) for identifying topic activities on mathematical reasoning. Although there are challenges in finding communicative projects and connecting interactions to a relevant timescale, the dialogical approach makes it possible to discuss how teachers handle shifting learning opportunities connected to mathematical reasoning. For the poster I hope to have some initial data and I want to discuss using dialogism as theoretical framework for the analysis.

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Elaboration and use of sentences for specialized mathematical meanings in classroom teaching talk

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Language as resource is a challenging research approach in mathematics education because it examines whether, how and why language can function to support mathematics learning and teaching. The approach originally started to develop in response to discourses of non-mainstream languages and cultures as problems or obstacles to mathematics teaching and learning. While in the past language as resource has been mostly examined with respect to the languages and cultures of school learners of mathematics and more recently to the orchestration of mathematical discourse practices, a mathematical-linguistic focus is proposed for extending the notion/approach in the study of and work with mathematics teaching talk. For this, a verbal tool in language –lexicalization or giving sentences with specific meaning potential into a content register– is presented and discussed in relation to school teaching and developmental work with mathematics teachers.

Keywords: Language as resource, mathematics teaching talk, developmental work, lexicalization.

Introduction

Planas (2018) built on two related arguments to explain the complexity and importance of the language as resource approach, namely: 1) all languages of learners own the potential of producing mathematical meanings; and 2) obstacles and supports to the realization of such a potential are critical to the generation of mathematics learning opportunities in classroom interaction. The view of the language complexity of mathematics learning was not specifically linked to the view of the language complexity of mathematics teaching. The current widening to include the languages of mathematics teachers in teaching talk represents substantial progress. The communication of mathematical meaning in classroom teaching talk is not always sufficiently explicit or precise. Communication in the mathematics classroom, as in all other language contexts, is made up of communicative intent and intended meaning on the one hand, and communicative function and interpreted meaning on the other. Mathematics teaching and teachers need to support learners in their communication of the intended mathematical meaning, but also need themselves to successfully resolve their communicative intents of mathematical meaning. Given the challenge with mathematics teaching talk that the intended content meaning be always communicated as clearly as expected, a question is: *How a focus on language at the sentence level can support research and work on mathematics teaching?*

Language in/for the study of mathematics teaching

In this first part, I address the critical accomplishment of content communication of mathematical meanings in classroom teaching talk. In Planas (2019, 2021), I initiated this discussion for algebraic contents through the examination of instances of two teachers' talk. That discussion was built on the identification of specialized meanings within the algebra of equations and of their communication in classroom teacher talk. In the teaching of the equation concept, explicitness of specialized meanings for algebraic equivalence and equal sign was particularly revealed as crucial and critical, and some tools in language were preliminarily explored. With inspiration in the Mathematics Discourse in Instruction frame (MDI; Adler, 2017, 2021) and in the Systemic Functional Grammar theory (SFG; Halliday, 1978, 1985), the next section offers some progress on the theorization of the verbal tool in language called lexicalization. The mathematical-linguistic focus in this report is not to say, however, that the study of content mathematics teaching can be limited to the study of vocabulary and grammar. Words and sentences are tools interacting between them, within the large system of tools and modes in language, and with the many discourse practices ongoing in any social setting. The selection of a mathematical-linguistic focus to illustrate an extended interpretation of language as resource is part of a more general attempt to rectify the imbalance I see in recent classroom research on mathematics teaching and language, where varied interests in discourse practices or strict interests in vocabulary and word names, have led to a certain disregard of what is linguistically said and how in mathematics teacher talk. Even though learners and teachers listen to and talk to each other in the classroom, and hence teaching is discourse in interactional practice (Erath & Prediger, 2021; Ingram, 2021), words and sentences in teacher talk mediate mathematical discourse practices with and between learners. Figure 1 puts together, not in opposition, content-specific verbal language of the teacher and mathematical discourse practices in the talk with learners. I see these two dimensions as made of related nodes and itemised into diverse discourse practices and verbal tools.

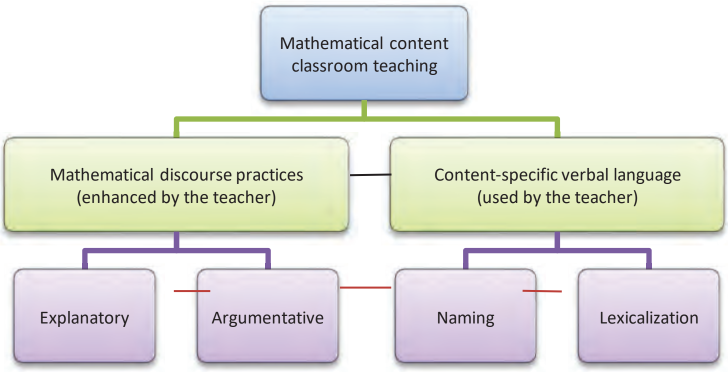

Figure 1: Elements for/in content-specific school mathematics teaching

Content teaching talk at the sentence level

With forms that have become lexical units or words and sometimes names in a language, we can produce combinations which, in turn, can be grammatically correct and at the same time semantically open. ‘Raining cats and dogs’ is grammatical and meaningful (i.e. ‘raining heavily’) in English; nonetheless, when beginning learners of English first listen to this sentence, they can find the form familiar but semantically open or even absent of meaning because its use in the English language system has not been learned yet. The form remains only a group of words as long as nothing more is known about it. Lexicalization is then the reductive process, from a semantic point of view, in which a grammatical form or sentence becomes lexical or with meaning into a precise language or register. In brief, grammaticalization is about structured forms (i.e. combination of words according to established rules or operations in a language system), and lexicalization is about specialized meanings (i.e. combination of words in a grammatical way and with precise meanings attached beyond those embedded into individual units or words) (Halliday, 1978, 1985; Lehmann, 2002).

Rather than a ‘defect’ in mathematics teaching talk, semantic openness is a consequence of the ubiquitous plurality in language, as well as an opportunity for the exploration and realization of mathematical meaning potentials. Here, lexicalization is the reductive process for the communication of precise mathematical meaning within languages of mathematics. Like the learning of word names and of specialized meanings behind them in naming, lexicalization involves more than learning the forms of grammar in a language. In Planas (2019, 2021), the frequent naming of the word name in Catalan for the equation concept without talking mathematical qualities and connections behind the name revealed a number of hidden conceptual meanings in lessons from the participant teachers. Paying now attention to the angle concept and given the option in language of producing grammatical forms with the word name for angle that are semantically open, lexicalization into geometry for conceptual teaching of angles is an issue. There are always sentences in teacher talk not intended to support conceptual understanding, with unfocused or implicit specialized meanings behind word names, functioning to promote pedagogic issues (e.g. ‘Look at the angles in the table’) or routine procedures (‘Always add the angle type’). There must be, however, teacher talk aimed at providing specialized (newer) meanings for mathematical objects encoded in (un)known words and sentences.

Just as how the MDI frame refers to naming in instructional research and to word use in developmental work with teachers, lexicalization –or lexicalizing to emphasize the process over the result– can be seen as the verbal tool available in language interpreted, in the work with teachers, as the elaboration and use of sentences with the potential to make mathematical meaning explicit and precise in classroom content talk. That is, work oriented to discuss how particular sentences function or not for content teaching, and how they might more ideally function to do so. ‘Angles made of a vertex and two rays’ is grammatically correct and it encodes specialized meanings into static plane geometry for the angle concept, but it could more ideally function to do so. While it is not a linguistic or grammatical obligation, the additional choice of the complement ‘… and the space between them’ would function to communicate the relevance of the region contained. More complements would still support meanings for ‘between them’ in relation to the two regions in principle delimited by the two rays. In ‘Angles from which the view is good’, for example, verbal complements for elaborating on angle view would function to encode language accounts of meaning into spatial geometry.

Table 1 summarizes the thinking of lexicalization from the use of meaning-focused sentences in languages for communication to its more ideal use in languages for communication of specialized mathematical meanings in classroom talk. There are bigger lexical unities above words and sentences in talk as well as multimodal accounts of meaning in communication; content communication cannot thus be theorized by the linguistics of words and sentences only without considering the wider communicational multimodal system and the representations of the contents at play. At the same time, the macro analyses of lesson teaching talk in search of communicational patterns do not tend to provide the grain detail of instances of lexicalization of content meaning. Words and sentences totally conforming talk, and accounts of meaning conformed at the expense of verbal representations made of words and sentences are both illusions. The intricacies and limitations of underestimating the effects of these illusions will require more study. For now, the current stage of theorization of the function of lexicalization orient the second part of the report on the function of language as resource in developmental work on mathematics content teaching with teachers.

Table 1: Lexicalization in mathematics content teaching

MATHEMATICAL LEXICALIZATION – SENTENCE USE	
General function *What does lexicalization do …* *in language for communication?*	Specific function *What does lexicalization do …* *in language for mathematics content communication?*
It encodes **- more or less complex clusters of meanings** for words and forms of grammar	It encodes - **specialized mathematical meanings** for words and forms of grammar within content-based mathematical languages

Language in/for work on mathematics teaching

Over the last decade, in order to improve the impact of mathematics professional development on classroom practice, increasing attention has been given to work with teachers guided by their teaching needs (e.g. Kazima, Jakobsen & Kasoka, 2016). It is an assumption that work with teachers on teaching is especially productive in terms of professional learning when the teachers take responsibility for the identification and interpretation of challenges. The four teachers (Jana, Maia, Anna and Roc) in the second round of the research and developmental project partially reported in Planas (2019, 2021) expressed various concerns with the teaching of the angle concept. They all had several years of mathematics teaching experience, and worked in the same secondary school at the time of the collaboration. Their professional knowledge and my mathematical-linguistic view of the MDI and the SFG frames were the points from which we explored angle-specific teaching. Although they were very much focused on their individual classrooms, had prepared lesson plans together and experimental tasks with dynamic geometry software. The results of the learners in the tests, however, had continued to show views of angles as static only, in disconnection with space, openness and inclination and as means for classifying figures. Anna especially linked the poor learning of specialized meanings for angle to common difficulties in the later understanding of classes of triangles with equal values for trigonometric functions, and of the slope of functions in calculus.

My response to the demand of the teachers was to interrogate their talk in the school lessons on angles. In most of my experiences of work with teachers, they do not normally feel that the mathematical richness of the classroom practices can be hampered by under specificity in talk. My response, though, was not a surprise, at least for Jana and Maia since they had participated in the first round of workshops about noticing choices in talk for teaching the equation concept (Planas, 2019, 2021). We agreed on exploring the improvement of angle teaching through improving angle teaching talk. Four of the 90-minute five sessions (S1 to S5) were task-driven workshops. Even though the four teachers graduated in mathematics, S1 was for revising mathematical knowledge on polysemous meanings for angle as: 1) static shape and dynamic turn; 2) object and means; and 3) region or sector and inclination or openness. S2 and S3 were for work on the language-based tasks, and S4 and S5 for work on explanatory tasks (Adler, 2017, 2021) of comparing angles, and of expanding the Pythagoras theorem to the obtuse/acute triangle. Developmental work thus supported mathematical knowledge, more ideal ways of choosing and using words and sentences for specialized communication, and mathematical discourse practices. Given the focus in this report, I present one of the language-based tasks. Task 1 was designed on the thinking of word names and sentences as related resources for mathematically situating the angle concept in teaching talk. The goal was to compare and eventually give sentences with angle-related word names, forms of grammar and meanings. Each workshop was organized to include the presentation of the task, the discussion of the teachers in group, and the final reflection with me. The pandemic circumstances recommended the synchronic online mode for all sessions. In the next section, the illustration of Task 1 is exemplary towards the argument that the realization of content teaching talk can be revised and practised throughout participation in workshops on mathematically and linguistically informed choices and uses of words and sentences.

A developmental task for discussion of angle-focused sentences

In S2, the participant teachers were given Task 1. An English version is offered in Table 2, with word names in the geometry of angles and pairs of related sentences written to represent those in Catalan, the source language, without pretending to be word-to-word translations or exactly equal in meaning. In the preparation of Task 1, I particularly drew on my knowledge of mathematics and of students' challenges in learning angles (for this, I built on my past experience as school teacher of mathematics, but also on Devichi & Munier, 2013), as well as on conversations with the teachers about their teaching needs and the local curriculum for low secondary mathematics specific to angles. On this basis, I considered emphases on dynamic turn vs. static object, object vs. means, and inclination or openness vs. region or sector, as well as angle-related word names that low secondary school learners usually know (e.g. point, vertex, orientation, measure) and which have extremely rich meaning potential in Catalan everyday languages. The original sentences came from talk of the teachers (T) in online recorded lessons at the pandemic time in 2021, and were chosen to illustrate instances with the potential of encoding specialized meanings for the concept. In all of them, the word for angle was named but important angle-related specialized meanings remained implicit or unclear. I invented the paired sentences to offer alternative sentences in the sense of more ideal, amongst many other possible, for the communication of specialized (newer) meanings for angle and related word names.

Alongside the centrality of the mathematical content in the design of Task 1, the linguistic particularities of Catalan, the language of instruction in school at my place and of the participant teachers, were also central. This is in line with what Halliday tells us (1978, 1985), as well as mathematics education research (e.g. Kazima & Adler, 2006) on the centrality of the form-meaning relationship in meaning making. When looking for angle-related word names that had been said by the teachers in the online lessons, I selected ten name words (first column, Table 2) in Catalan whose meaning inside and outside languages of mathematics was shaped by the words as put in interaction with other words. Triangle was one of these names. In a lesson in which Roc had asked his learners to draw an example of angle, some learners drew a triangle and then indicated with an arrow an internal angle of the figure. The view for angle as measure and means for describing triangles, and as part of a whole, was present in lessons from the other teachers, too. Since the name of triangle was common in the lessons aimed at the conceptual teaching of angles, I looked for opportunities in the languages of the teachers to relate triangles and angles in mathematically meaningful ways through a number of possible verbal complements. For this, I selected "A triangle contains one hundred and eighty degrees." I can imagine other word names and candidate sentences in an English version of the task and the study. As far as I know, in this language and regardless of being or not common in teacher talk, the choice of angle as verb is available, and specialized meanings for angle as action can be encoded through sentences about placing an object at an angle or about changing its direction.

Table 2: English version of the worksheet with Task 1

Task 1. Word use into angle-focused sentences Which are the meanings behind the word names and the sentences?		
Words	*What does T say?*	*What could T say?*
Angle	The angle rotated around the point.	The segment line rotated around the point makes the angle.
Turn	One turn of an angle is the circle.	One complete turn of a segment line around a point draws a circle.
Point	An angle is an angle at a point.	An angle is a rotation of a segment line around a fixed point in a plane or in space.
Vertex	Angles are made of a vertex and two rays.	Angles are made of a vertex, two rays, and the space between them.
Sides	Sides are not important in angles.	The length of the sides is not a characteristic of angles.
Orientation	The angle is different because of the orientation.	The angle is different because its rotation has opposite orientation.
Measure	The angle is different because of the measure.	The angle is different because the measure of the rotation is different.
Right	Perpendiculars intersect at a right angle.	Perpendicular lines intersect at four right angles.
Straight	Straight angles look like zero.	Straight angles and zero angles look similar but the turn implied is different.
Triangle	A triangle contains one hundred and eighty degrees.	The three internal angles in a triangle add to one hundred and eighty degrees.

Since the four teachers accepted recording the working sessions, I could take notes at different moments in time. They were given the document with the task and were asked to comment on each word name and the sentences two by two, and to reflect on encoded specialized mathematical meanings possibly different to those already known by the school learners. On the one hand, the teachers noted that many words could be assumed to be known by learners although not necessarily in relation to the angle concept. The Catalan word name for ‘right’, *recte*, was particularly discussed. *Recte* is used in the everyday language to qualify objects with no curvature; the angle-related meanings for *recte* cannot thus be understood through the meanings behind the word that may have been learned and, in addition, the word is mathematically polysemous in the geometry of angles as it can function to encode the ninety-degree measure and the notions of inclination and perpendicularity. On the other hand, the teachers noted that any of them could have said the original sentences (second column, Table 2) in their teaching. They became engaged in distinguishing what was said from what was possibly intended, and so in assessing these aspects separately. Regarding “Sides are not important in angles”, for example, Anna said to miss some explanation of the reasons for why they are not important, and hence found the alternative sentence more ideal. We all agreed on the need to produce teacher talk aligned with mathematical discourse practices of content explanation.

Two of the teachers had referred to, “The angle rotated around the point” (first column, Table 2) in their lessons. This sentence was selected for Task 1 to pay attention to the specialized meaning as dynamic turn encoded in the name for angle. Attention to meanings behind some preposition words, however, soon led to enhanced talk about the choice of spatial prepositions such as at/around (a point) and on/around (a segment line) as words with implications on the communication of angle-related meanings. The four teachers knew that the Catalan compound word *al voltant de* (of which the English one-word ‘around’ might be a feasible counterpart) can express spatial or plane relationships, or relative location and directional motion of the subject object. The Catalan version of “The angle rotated around the point” is then grammatically correct and mathematically meaningful but the basic meanings implied are not about the angle as dynamic turn as in the Catalan version of ‘The segment line rotated around the point.’ In the former sentence, we work with angle as means to produce in the plane or in space other mathematical objects included the solid angle (which is not a curricular content of low secondary mathematics); in the latter, we work with segment lines to produce Euclidean plane angles. At the end of the developmental session, the mathematical meanings encoded behind prepositions in interaction with other word meanings had been discussed intensively, and related to the choice of words and sentences in content teaching. Anna, for example, introduced the discussion of how differently it was ‘around the point’ from ‘rotated around the point’ in that the two expressions encoded two different sets of paths or sequences of positions enclosing the point.

Language accounts of meaning matter but so do other accounts

The interpretation of ‘more ideal’ in Task 1 is not independent of the circumstances in which the sentence might be said in the school classroom, of the discourse practices ongoing, and of the broader talk and modes of communication accompanying the teaching. The inseparability of all these aspects was addressed in S4 and S5, in the discussion of promoting the communication of angle-related meanings more largely across mathematical discourse practices resourced by languages of content teaching. The fact that these are classrooms with multilingual learners is not insignificant although it was hardly brought up by the teachers in their comments over the workshops. Like the efforts here to reproduce in English word names and meaning-focused sentences originated in Catalan without missing the mathematical-linguistic points intended, Jana, Maia, Roc and Anna teach angles, equations… by moving across forms of academic and everyday Spanish and Catalan in ways that may favor the communication of some specialized mathematical meanings in one language only at a time. These teachers flexibly use their languages as resources in their content teaching of mathematics with the possibility of encoding different mathematical meanings in each language.

Behind the arguments of improving the clarity and coherence of teaching at the sentence level of language, there is the Vygotskian assumption that opportunities of mathematics learning are generated through exposition and attention to specialized mathematical talk and texts, or by active listening to others who are more knowledgeable. Making available specialized mathematical ways of talking is critical for school learning, and hence there are consequences of opening and of closing opportunities for mathematics learning through teaching talk, be it an ever present and highly visible or almost a disappearing feature of classroom communication. These arguments also apply to multilingual mathematics teaching with multilingual learners and those belonging to cultures that have developed non-verbal ways of communication. The nodes and tools in Figure 1 fall short if teachers are confronted with the communication of specialized mathematical meanings for names and grammars to learners who are learning some of the words and sentences as used in the language of instruction, and who can well recognize some of the specialized meanings when encoded in other linguistic forms or semiotic modes. Teachers cannot assume that multilingual learners who are learning the (sign) language of instruction do not know some of the mathematical meanings involved in classroom interaction, and teaching thus needs to facilitate their participation and sharing through mathematical discourse practices entailing the use of the various semiotic modes in language.

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Measuring quality interaction: how much detail is necessary? Results from a quantitative video study on the conceptual dimension

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Quantitative video-based investigations of quality interaction raise challenges due to their complexity. In this paper, three different operationalizations for measuring mathematically rich interaction are derived from larger teaching quality studies that vary in detail. Those heterogeneous operationalizations are task-, move-, and practice-based. We apply all three operationalizations to video data from 49 small group interventions (210 students, age 10-14 years) that aim at fostering students' conceptual understanding of fractions. Qualitative and quantitative results indicate that when capturing interaction quality, measuring the quality by task- and move-based operationalizations is not sufficient to capture all relevant differences in classroom interaction.

Keywords: Interaction quality, video study, operationalization, conceptual learning.

Introduction

Qualitative studies repeatedly provide empirical evidence that quality interaction is crucial for students' access to mathematical learning opportunities (e.g., Walshaw & Anthony, 2008). The transition towards quantitatively researching quality interaction, as every quantitative research, entails considering *how* to transfer theoretical conceptualizations into measurable operationalizations (Praetorius & Charalambous, 2018). However, operationalization is a major challenge in time and effort, especially when assessing the quality of classroom discourse as part of quality interaction (Pauli & Reusser, 2015). Therefore, in this paper, operationalizations for quality interaction that differ in their operationalization bases (i.e., the way to simplify the measurement based on tasks, moves, or practices) are derived from larger teaching quality studies (e.g., Charalambous & Litke, 2018; Pauli & Reusser, 2015). They are applied to video data from a language-responsive intervention study (Prediger et al., in press) to investigate empirically in how far the ways to simplify measurements when operationalizing quality interaction matters.

Background: Quantitatively measuring the quality of interaction

There is strong empirical evidence that underlines the importance of quality interaction for students' access to rich mathematical learning opportunities (Cai et al., 2020; Walshaw & Anthony, 2008). At the same time, qualitative studies show that quality interaction that aims at students' conceptual understanding (rather than procedures and facts) is strongly intertwined with teachers' initiating and continuously supporting students' rich discourse practices such as explanations, argumentations, or justifications (e.g., Erath et al., 2018). However, quantitative evidence with a focus on this relation is still too rare (see overview in Erath et al., 2021).

For conceptualizing quality interaction, Erath and Prediger (2021) suggest disentangling the talk-related, discursive, conceptual, and lexical quality dimensions for teachers' activation and students' individual participation. Because the teachers are particularly responsible for the quality of interaction

(Walshaw & Anthony, 2008) and mathematically rich learning opportunities consist of teachers' addressing and supporting students' conceptual understanding (Hiebert & Grouws, 2007), this paper concentrates on the teachers' enacted *conceptual activation*. Conceptual activation means the degree to which teachers engage the students in rich mathematical (e.g., conceptual instead of procedural) activities in classroom interaction. Other quality dimensions are excluded here due to space limitations.

For operationalizing conceptual activation, researchers use heterogeneous operationalizations in video studies that all vary in the level of detail. As our literature review will show, capturing *conceptual activation* is addressed by *task-, move-, or practice-based operationalizations* and differs with regard to measurement decisions (counting time, turns, or rating highly inferential).

High or low demanding tasks that intend challenging or less challenging mathematical activities might serve as a precondition for conceptual activation but are not necessarily identical to their enactment (Cai et al., 2020). Larger teaching quality studies therefore entail both, task demand and their enactment. In PYTHAGORAS, a *task-based operationalization* (rating the quality of demands and supports posed by tasks) is capturing the length of teachers' enacting more or less demanding tasks (Hugener et al., 2006, p. 165). Repeatedly, *task-based operationalizations* capture the quality of classroom interaction together *with move-based operationalizations*. *Move-based operationalizations* are all teacher actions in classroom interactions, for instance, if teachers' moves address mathematically more or less demanding types of knowledge or students' activities (TIMSS video, Stigler et al., 1999). We often identified *move-* and *practice-based operationalizations* that jointly assess the quality co-constructed in the teacher-student interaction. For instance, the classes' conceptual activation can be assessed by rating to what extent teachers' moves and enacted practices explicitly address and support the connection of representations and mathematical concepts (MQI, Charalambous & Litke, 2018, p. 447). Further, we identified approaches to capture conceptual activation that are based on task-, move-, and practice-based operationalizations, for example, the level of task enactment (e.g., TALIS, OECD, 2020, p.10). However, especially for the highly inferential ratings, it often remains unclear to what extent which operationalization is decisive for rating high or low quality (in conceptual activation). A major challenge for all studies is measuring aspects of teachers' and students' discourse in the enacted classroom practices, as it necessitates time and effort due to its complexity, interactivity, and multidimensionality (Pauli & Reusser, 2015). Therefore, researchers utilize *practice-based operationalizations* that focus solely on the teachers' contributions (e.g., TIMSS, Stigler et al., 1999), turn-related assessments of whether contributions contain reasoning but omit the length or the contribution's aim (e.g., PYTHAGORAS, Hugener et al., 2006), or by rating the quality of discourse in less details on scales like in MQI or TALIS video (Charalambous & Litke, 2018; OECD, 2020). To sum up, researchers use different approaches *how* to operationalize conceptual activation, based on *tasks, moves, or practices.* Given the immense workload, the assessment of practice-based operationalizations capturing teachers' and students' discourse must always be simplified in a certain way.

Following the results from this short literature review, we derive the need to compare more systematically how the simplification to task-, move-, or practice-based operationalizations for capturing conceptual activation matters. Therefore, we use operationalizations according to the three different bases systematized from existing quantitative studies (see columns of Table 1). Also, we decided to assess conceptual activation in a high level of detail by capturing the amount of time descriptively first and not mixing quality and amount (which is criticized by Praetorius & Charalambous, 2018). For increasing the comparability of the different operationalizations, we relate all to the total time on task of the group (including times of writing or silence). Thus, in our framework, the quality dimension *conceptual activation* is operationalized in three ways (Table 1).

Table 1: Operationalizing conceptual activation by three different operationalizations

	Task-based operationalization	**Move-based operationalization**	**Practice-based operationalization**
Conceptual activation	Relative length of group time spent on conceptual tasks	Relative length of group time spent on conceptual moves	Relative length of group talk spent on conceptual practices (incl. teacher)

The entries of Table 1 show the operationalizations that are applied: *The task-based operationalization* is measured by the relative length of group time spent on conceptual tasks; the move-based operationalization is captured by the relative length of group time spent on conceptual moves; *the practice-based operationalization* is measured by the relative length of group talk spent on conceptual practices (incl. teacher). Contrary to other studies' operationalizations, the practice-based operationalization entails the teachers' and students' discursive realization with rich discourse practices (i.e. the length of an individual student's explanation) and not only what teachers intend for students' contributions.

By operationalizing conceptual activation systematically with different operationalizations, it becomes possible to quantitatively investigate their relation. We contribute to the question of whether *task-based operationalizations* (with much less coding effort and complexity) can already capture relevant aspects of conceptual activation or must be extended to more complex and detailed measurements of *move-,* or *practice-based operationalizations*. By applying three *operationalizations* to video data from 49 small groups working on developing a conceptual understanding of fractions, the relation can be empirically investigated. Hence, the pursued research question is *how are the different task-, move-, and practice-based operationalizations of conceptual activation related?*

Methods: Research context and data analysis of types of operationalizations

Research context. The research question is pursued in the larger mixed-methods intervention study MuM-MESUT 2 (pre-post-follow-up design, see Prediger et al., in press). Instruction aiming at developing a conceptual understanding of fractions was organized in small groups of 3 to 6 students (10-14 years old) and one teacher per group (pre-service teachers or PhD-students). It spans over 5 sessions of 90 minutes each. All small groups' teaching is based on identical curriculum resources, although some small groups received additional integrated lexical support.

Data corpus. The analysis presented in this paper is based on all sequences of small groups' interaction in phases of knowledge inquiry, consolidation, and organization for reinventing the concept of equivalence of fractions (49 small groups, 210 students, ~30 hours of video).

Data analysis. The analysis of the three *operationalizations* was conducted in five steps. First, five descriptive low and high inferent basic codings were conducted on the curriculum resources, transcripts, and video recordings. Basic codings entail (1) time-on-task (time spent on selected tasks, excluding breaks and off-topic time); (2) individual length of students' as well as teachers' utterances; (3) *task-based operationalization* (time spent on conceptual tasks, task demand categorized by their intention in a previous study); (4) *move-based operationalization* (length of the sense-making units when teachers' moves aim at the construction of conceptual knowledge, see Wessel & Erath, 2018), and (5) *practice-based operationalization* (person-related, second-specific length of engaging in rich discourse practices like explaining of mathematical concepts). (3)-(5) were coded in sense-making units that start with teachers or students initiating a problem or question and end when the problem or question is marked as being finished. Second, interrater reliability for the high inferent basic codings was controlled in R and shows good results with an overall $\kappa = 0.87$. Third, according to the entries in Table 1, operationalized criteria were calculated as the *percentage of time of enacted task demand in time-on-task, percentage of time in enacted move demand in time-on-task,* and *percentage of time in enacted discursive realization in time-on-task*. Forth, descriptive statistics were calculated for determining the means, standard derivations, and Pearson's correlations.

Empirical results: Conceptual activation assessed by tasks, moves, and practices

This section provides an example for large differences of conceptual activation assessed by different *operationalizations* in two small groups with comparable conditions (students from higher-tracked schools and the same curriculum material). This example motivates the analysis of correlations between task-, move, and practice-based operationalizations.

Comparing the assessment of conceptual activation in two small groups

Table 2 shows the scores for conceptual activation in all small groups as well as in the two small groups Q and U chosen for the illustrations, here.

Table 2: Distribution of teachers' conceptual activation relative to the time-on-task

Conceptual activation	Relative length of group time spent on conceptual tasks	Relative length of group time spent on conceptual moves	Relative length of group talk spent on conceptual practices (incl. teacher)
Mean (SD) of all small groups	77.4% (10.9%)	35.3% (12.9%)	23.5% (12.3%)
Mean of small group Q	80.2%	52.7%	18.0%
Mean of small group U	82.0%	61.1%	35.4%

Concerning *the task-based operationalization,* Teacher Q and Teacher U both invest a similar relative length of group time on conceptual tasks (both >80%). This high proportion might seem surprising but can be explained by the intervention design that aims at developing students' conceptual understanding (of fractions) and therefore contains more conceptual and less procedural tasks. With respect to *the move-based operationalization,* Teachers Q und U also engage students in conceptual moves for (jointly) constructing conceptual knowledge in more than half of the time-on-task (Teacher U slightly more), which is distinctively above all groups' mean. However, assessment of quality varies *considerably* in the two small groups' *practice-based operationalization*, despite the similar task-based and move-based assessment. While in group Q teacher's and students' contributions are in 18% of the time-on-task talk on conceptual practices, group U has almost double as much enacted, rich discursive realization. Thereby group Q falls below all groups' mean and U exceeds all groups' mean.

Qualitatively analyzing exemplary situations from these two groups' teacher-student interactions illustrates what these different proportions in *the practice-based operationalization,* thus discursive realization, mean for students' engagement in conceptual practices. The comparison takes place when Teacher Q and U work on the same tasks with their small groups: Switching and connecting symbolic, contextual, and graphical representation of equivalent fractions in the bar board (for details on tasks see Wessel & Erath, 2018). When orally discussing results, Teacher Q and Teacher U at first use similar moves to intend an explanation of why the drawing in the bar board fits the respective contextual and symbolic representations, thus intending group talk on conceptual practices. However, Teacher Q's moderation quickly switches to less demanding discourse practices or even non-discursive practices (like one-word answers or half-sentences) when asking her students questions like "How much do you have to color for the teachers' share?", "How often did the teachers hit the goal?" or "How many pieces do you have to colour?". It becomes apparent that she does not succeed in engaging students in conceptual practices like explaining-why that are necessary for developing conceptual understanding and meaning. Contrary, Teacher U holds up the enactment of demanding *conceptual practices* by asking the students to explain "Why did you chose this bar" or jointly creating explanations like "This bar fits this group because …" in orally discussing the results. The students in group U take up the discursively demanding talk on conceptual practices and jointly construct explanations which meet the discursive needs for developing conceptual understanding.

Summarizing, the varying quality in conceptual activation in small group Q and U was only visible in the *practice-based operationalization*, i.e., when also considering the quality of teachers' and students' contributions, not only *task- and move-based operationalizations.* Hence, it is essential to operationalizing the enacted discursive realization by *practice-based operationalization* also with respect to teachers' and students' rich discourse practices.

Quantitative dependencies between assessments by three operationalizations

The last section qualitatively underlines that it is necessary to distinguish operationalizations for assessing conceptual activation. In the following, we generalize this idea by quantitatively investigating the association between assessing conceptual activation in *task-, move-, and practice-based operationalizations* in the whole data corpus. Table 3 shows Pearson's correlation coefficients for all small groups' coded data.

The correlations between three *operationalizations* of conceptual activation show that there is a relation that is low to moderate (0.204-0.448).

- Contrary to expectations, the association between *task- and move-based operationalization* is low (0.204). This implies that the teachers' enacted relative length of group time spent on conceptual moves is only weakly associated with the relative length of group time spent on conceptual tasks (in our data). Two interpretations are possible for the data corpus in which all teachers use the same tasks: Either (some) teachers facilitate conceptual tasks with (several) procedural moves and thus do not meet the aim (like Teacher Q) or (some) teachers' enacted moves also address conceptual knowledge in non-conceptual task demands. This finding must be interpreted carefully and might not apply when teachers use their own tasks with more varying demands. The *move-based quality assessment* might then be closer connected to the *task-based quality assessment*.

- The association between *task-* and *practice-based operationalization* (0.151) is similarly low. This means, how much time teachers spent on *conceptual tasks in time on task* is almost hardly related to a rich practice-based enactment concerning the talk within conceptual practices in time on task (in our data).
- The association between *move-based and practice-based operationalization* (0.448) was anticipated from existing research since initiating conceptually demanding discussion (by *moves*) requires teachers' and students' explanations, argumentations, or justifications. This shows that what teachers ask is more closely connected to the groups' actual *practice-based realization.* However, Teacher Q's *discursive realization* that partly fails to meet the *enacted move demand* serves as an example for explaining that the association is much lower than 1: Not every conceptually demanding situation is also *enacted* discursively rich.

Table 3: Pearson's correlation coefficients between operationalizations in all 49 small groups, * for significant correlations ($p < 0.05$)

Conceptual activation (relative to the time-on-task)	Task-based operationalization	Move-based operationalization	Practice-based operationalization
Task-based operationalization	1	0.204*	0.151*
Move-based operationalization		1	0.448*
Practice-based operationalization			1

Discussion

By including tasks, moves, and practices as operationalization bases for assessing conceptual activation, we were able to investigate their relationship empirically. We show quantitatively that quality assessments of conceptual activation based on *task-, move-, and practice-based operationalizations* are low to moderately associated with *move- and practice-based operationalization* in closer connection. This is in line with researchers emphasizing that the teachers enact cognitively demanding tasks in varying quality (Cai et al., 2020; Walshaw & Anthony, 2008). Also, the qualitative example of Teacher Q and U underlines the great importance of operationalizing practice-based: Teacher U's moderation leads to a substantially more challenging and sophisticated group talk on conceptual practices, even though both teachers enact similar quality of interaction when assessing with task- and move-based operationalization. Thus, it seems insufficient to capture the conceptual activation by task- and move-based operationalizations only. Beyond this, assessing the *enacted discursive realization by practice-based operationalizations* can better capture students' access to and engagement in rich mathematics.

The presented empirical results point to the importance of methodological considerations for future quantitative studies aiming at investigating quality interaction in the dimension of conceptual activation. *How* to turn theoretical conceptualizations into measurable *operationalizations* must be discussed carefully. This is a major challenge and at the same time a huge gap in research on teaching quality, as many researchers mix measurements of amount, quality, and frequency (as criticized in the survey by Praetorius & Charalambous, 2018).

The empirical results presented in this paper are limited due to the specific data collected and analyzed: Because of comparable conditions in the intervention and identical curriculum resources, the findings cannot be immediately transferred to whole-class settings and everyday mathematics classrooms. Also, we cannot conclude which type of operationalization is more valid to capture classroom interaction. However, our next step in the larger project is to investigate the predictive power of the three *operationalizations* for the growth in students' conceptual understanding of fractions in the dimension of conceptual activation, but also in the other quality dimensions (Erath & Prediger, 2021) as this is identified as research gap in mathematics education research (Cai et al., 2020; Praetorius & Charalambous, 2018). We especially aim at generating quantitative empirical evidence of the relation between discourse quality and student learning (Erath et al., 2021, Pauli & Reusser, 2015).

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Choice of representations in combinatorial problems

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This paper is based on classroom sessions where Norwegian 9-year-old (Grade 4) children work on combinatorial problems. The classroom sessions are part of a four-year long research project where the topic of multiplicative structures was central. I will investigate to what extent the pupils recognise the combinatorial problems as multiplicative and identify possible connections between the semiotic representations chosen in the solutions and in the formulation of the problems.

Keywords: Combinatorics, multiplicative structures, register, semiotic representations.

Introduction

This paper is based on data from the project Language Use and Development in the Mathematics Classroom (LaUDiM)—a four-year collaboration project between researchers at the Norwegian University of Science and Technology and two primary schools. In this project, a central theme was multiplicative structures, and a recurring question was to investigate connections between the formulation of a problem, the pupils' choice of semiotic representations to solve the problem, and to what extent they recognised a given situation as multiplicative.

In this paper, I study pupils in Grade 4 (nine-year-olds) working with two combinatorial problems (see Figures 1 and 2). The problems were presented with no previous instruction that could give an indication about what mathematical knowledge and techniques that would be helpful for solving the problems. Pupils worked in pairs, and work in selected pairs as well as in whole-class sessions were video recorded. I pose the following research question: *In what ways can the context of a situation be seen to influence the choice of registers in the solution, and how can the chosen registers provide evidence about the extent to which the situations are perceived as multiplicative?*

Theoretical framework

The concept of *register* is used in somewhat different ways by different authors. Duval uses the term register to mean a semiotic representation system (e.g., natural language, symbolic systems, graphics) and emphasises that "[c]hanging representation register is the threshold of mathematical comprehension for learners at each stage of the curriculum" (Duval, 2006, p. 128). I follow Duval's usage of the term, which is also in accordance with the usage by Prediger and Wessel (2013) in their model concerning changing and relating registers. This model entails a transition between different registers - a concrete representational register, a graphical representational register, different verbal registers and a symbolic-algebraic or symbolic-numeric register (Prediger & Wessel, 2013, Fig. 1, p. 437). This resembles the process described by O'Halloran when she writes that language is used to introduce and describe a mathematical problem, later to visualise the problem, and then the problem is solved using mathematical symbolism (O'Halloran, 2005, p. 94).

Before discussing the classroom situations, I will define what is meant by *a multiplicative structure*, or *a multiplicative situation*. Steffe defines a multiplicative situation as a counting situation where "it is necessary to at least coordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit” (Steffe, 1994, p. 19). This is the basis for my discussion of multiplicative structures. There are several different classifications of multiplicative structures to be found in the literature (see e.g., Greer, 1992). I will rely on the classification given by Vergnaud who splits multiplicative structures in three classes: *Isomorphy of Measures*, *Product of Measures*, and *Multiple Proportions* (Vergnaud, 1983, p. 128). The latter will not be discussed here. *Isomorphy of Measures* is defined as a structure involving a direct proportion between two measure spaces, M_1 and M_2 (Vergnaud, 1983, p. 129). Situations like equivalent groups and multiplicative comparison (Greer, 1992) fall into this category. Further, *Product of Measures* is defined as a structure involving a mapping from a product of two measure spaces into a third measure space, $M_1 \times M_2 \rightarrow M_3$ (Vergnaud, 1983, p. 134). Combinatorial problems (Cartesian products) and rectangular area problems fall into this category. Rectangular *array* problems (Fosnot & Dolk, 2001), to find the total number of items laid out in a row-column pattern with a certain number of items in each row and each column, may look similar to area problems but they actually belong to the class *Isomorphy of Measures*, with M_1 = [number of rows (columns)] and M_2 = [number of items in each row (column)]. Unlike many other Isomorphy of Measures-problems, these are *symmetric* (Rønning, 2012). In combinatorial problems, the measure space M_3 may not be initially present. M_3 is where the counting unit is situated, and that this space may initially be unknown, represents a challenge when solving such problems. This is also connected to the phenomenon that the counting unit is of indefinite quantity and that there is not always a clear strategy to determine when the problem is actually solved (English, 1991; Shin & Steffe, 2009).

The tasks given to the pupils

The first task given to the pupils is presented, in its simplest version, in Figure 1, the second task in Figure 2. The tasks were given on two different days of the same week. No instructions were given on how to solve the tasks, and what strategies that might be helpful could also not be inferred from what the class had worked with immediately before the sessions where these tasks were presented.

> How many different gingerbread biscuits can we make if we have cutters in these four shapes ☆ ◇ ○ ♡ and we have white, green and red icing?

Figure 1: Task 1

> Ms. Hall has 3 pairs of trousers and 5 sweaters. The trousers are in the colours blue, black, and grey. The sweaters are in the colours blue, red, black, green and purple. She will use one pair of trousers and one sweater each day, and she will combine different pairs of trousers with different sweaters. How many days in a row can Ms. Hall wear different outfits?

Figure 2: Task 2 (Ms. Hall is the teacher in the class)

After agreeing on a solution, each pupil in the pair was asked to produce a written account of the method used. Both tasks induce a mapping $M_1 \times M_2 \rightarrow M_3$, with M_1 containing shapes in Task 1 and trousers in Task 2. M_2 contains colours in Task 1 and sweaters in Task 2. In Task 1, M_3 contains coloured shapes (biscuits). It could be argued that since M_3 in Task 1 contains coloured shapes, the measure space M_3 is not really new, it is a variation of M_1. A more precise representation of the mapping in Task 1 could therefore be $M_1 \times M_2 \to M_1^*$, where M_1^* denotes *coloured shapes*. In Task 2 one may think of a mapping $M_1 \times M_2 \to M_3 \to M_4$, where M_3 contains outfits and M_4 contains days of the week. M_3 and M_4 are isomorphic, so this transition would be expected not to be challenging.

In the language of Steffe (1994), one can say that it makes sense to distribute the composite unit from M_1 over the elements of M_2, or the other way around, which means that the situation is symmetric (Rønning, 2012). Both problems can be seen as a matrix product **c** of two vectors, **a** and **b**, where **a** and **b** represent the composite units from M_1 and M_2, respectively, and $\mathbf{c} \in M_3$ (see Figure 3).

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \mathbf{b} = [b_1 \quad \dots \quad b_m] \text{ and } \mathbf{c} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} [b_1 \quad \dots \quad b_m] = \begin{bmatrix} a_1 b_1 & \dots & a_1 b_m \\ \vdots & & \vdots \\ a_n b_1 & \dots & a_n b_m \end{bmatrix}.$$

Figure 3: Matrix structure of a combinatorial problem

Each element, $a_i b_j$, of the product matrix **c** represents one possible combination (composition). This representation shows that the dimension of M_3 equals the product of the dimensions of M_1 and M_2.

The discussion above shows that although the problems in the two tasks are computationally equivalent, the mappings between measure spaces are somewhat different.

Design and method

The project LaUDiM was based on interventions consisting of two-three classroom sessions dealing with the same mathematical topic, preceded by planning meetings where teachers and researchers met. Between and after the sessions, reflection sessions were held. The design of the classroom sessions was based on the Theory of Didactical Situations (Brousseau, 1997).

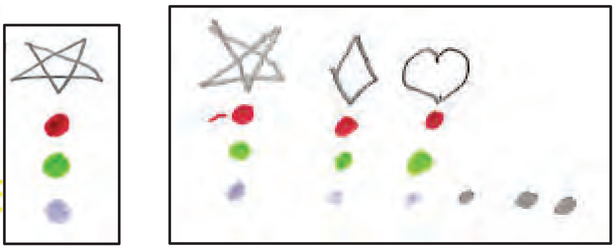
The classroom sessions contained whole-class activities and activities where pupils worked in pairs with given tasks. For each session, the work of two pairs was video-recorded, as were all whole-class activities. Attempts were made to choose pairs to be recorded differently for each session, so that the pupils should not feel that only a few participated in the project. The pairs were determined by the teacher, based on her expectations of who would collaborate and communicate well. The school from which data for this paper come, lies in a well-established, middle-class neighbourhood. The pupils all have Norwegian as their first language. Data consist of video-recordings from the sessions, as well as pupils' written work, collected from all pupils, not only those who were video-recorded.

The analysis is based on the thematic development of the dialogue in the pairs, as well as the written work, including work from the pupils not video-recorded, in order to identify statements that show the choice of the representational registers and also serve as evidence for the pupils' possible perception of the situation as multiplicative. Parts of the video-recorded discussions have been transcribed and translated into English. In the analysis I will follow Naomi and Roger in Task 1 and Naomi and Filipa in Task 2. This means that I present data from one pupil (Naomi) in her work with both tasks. Therefore, I will pay most attention to her work in the pairs.

Analysis of the work in pairs

Task 1 (Naomi and Roger)

Naomi and Roger start looking at the task and Roger's first suggestion is that there will be seven different possibilities since there are four shapes and three colours. Then Naomi starts drawing the four shapes in one row and she colours the heart red. She indicates that she can continue to draw new rows with the same shapes and change the colour for each row. She does not complete the drawing in detail but on the video recording it can be seen that she indicates three rows with four shapes in each row (a matrix structure). Then she counts, one-two-three, four-five-six, seven-eight-nine, ten-eleven-twelve, tapping on the drawing column by column as she is counting. I interpret from her gestures and utterances that she has identified a countable unit. She indicates groups of three, but still she counts the shapes one by one. She now considers herself finished with the task and Roger does not object. Since Naomi and Roger solved the first task so quickly, I challenged them to find out what would happen if they had eight shapes and seven colours. They cannot really think of eight different shapes, so Naomi just draws eight circles in a row and fills in with more circles below these. They start to colour each row in one colour (purple, blue, red, …), until they have used all seven colours and hence got seven rows. The result is shown in Figure 4[1]. To the right is shown the calculation the pupils wrote on their worksheet. The drawing shows that they have marked four groups of 14 circles (dots). In the calculation I interpret the first line (14 14) to represent the first two groups, added to get 28 (second line). Then another 14 is added to get 42 and finally another 14 (not written) to get 56.



a) b)

Figure 4: Naomi and Roger's solution **Figure 5: From Naomi's solution in "Our method"**

As part of the task, each pupil should fill in a sheet with the heading "Our method for the biscuits task". Naomi wrote (referring to a version of the task with three colours and six shapes):

> We thought that we took a star like this with the three colours beneath [indicated by an arrow pointing to Figure 5 a)]. Then we did the same with all shapes, like this [indicated by an arrow pointing to Figure 5 b)]. Finally, we counted all the dots. We could also take them in small groups like this 3+3+3+3+3+3=6+6+6=18.

In Figure 5 b), Naomi has not drawn all the shapes. Based on her text quoted above, I assume that she has imagined the last three shapes, without a need for drawing them.

[1] They did not colour the first row of circles, so they put in eight green dots at the bottom. The three encircled shapes at the bottom of the drawing do not belong to this solution.

The representational register used in the formulation of Task 1 consisted of a text and the iconic representation ☆ ◇ ○ ♡. This drawing, endowed with colours, formed the starting point for the work in all the pairs. The pairs worked more or less systematically but the solutions shown in Figures 4 and 5 are representative for many of the pairs, as evidenced by the worksheets. The preferred representations show a matrix structure where each entry consists of one particular shape and one particular colour, mimicking the matrix product in Figure 3. Each entry has the form $a_i b_j$ where a_i represents a shape and b_j represents a colour. An emerging multiplicative structure can be seen, represented graphically as well as numerically. In the solution shown in Figure 4, Naomi and Roger have made groups of two and two columns, and in her description, Naomi writes "[w]e could also make them in small groups like this 3+3+3+3+3+3=6+6+6=18", indicating six groups of three or three groups of six. It is not clear what reasoning lies behind the representation "3+3+3+3+3+3=6+6+6=18", since the only evidence is what Naomi has written. It could be that she groups each 3+3 into 6 and then gets three groups of six. Another possibility is that she sees 3+3+3+3+3+3 by counting on the columns and 6+6+6 by counting on the rows (Figure 5).

Task 2 (Naomi and Filipa)

The two girls start by drawing five sweaters and three pairs of trousers and then they colour each piece, using different colours for each piece in the same category. On the video can be seen that Naomi draws a line from the red sweater to the grey pair of trousers and writes "man" (Monday) above this line. Filipa connects the red sweater with the brown pair of trousers and writes "tir" (Tuesday). The girls continue in the same way, ascertaining that for each new combination they find, it is not already taken. After having written Tuesday for the second time, a break can be observed on the video, and it seems that they struggle to find new combinations. Gradually, they find new combinations and when they have found 14, they think they are done.

1	Naomi:	All the trousers on this [points to the brown sweater] because this has three lines. [Looks at the sheet] I think we have made it.
	Filipa gets a new sheet. Naomi looks further at the drawing.	
2	Naomi:	Oh, we can have one more. [draws a line between the green sweater and the blue pair of trousers]
	Naomi writes 2 weeks and 1 day.	
3	Naomi:	I will ask if it is correct.
	One of the researchers comes to the table and asks if the pupils have found a solution.	
4	Naomi:	We think we have figured it out. We *think* it is two weeks and one day

Naomi puts emphasis on 'think', which I take to mean that they are not sure, and they ask the researcher for confirmation. When the researcher is reluctant to give an answer, the girls call upon one of the other researchers, and the following conversation takes place.

5	Researcher:	Is that what you found? How did you find that out?
6	Naomi:	We took everything together. So there are three lines for each outfit.
7	Researcher:	Three lines for each outfit? On each sweater and each pair of trousers?
8	Naomi:	No, for each sweater and each pair of trousers … For the trousers, it will be … five lines.
9	Researcher:	Are you sure you have drawn lines between all? Have you found all the outfits?
10	Naomi:	I think so. Is it correct?
11	Researcher:	You have to try to convince me. How are you thinking to make sure you have found absolutely all? It can be easy to forget to draw a line, right?

12	Filipa:	Yes.
13	Researcher:	Have you found a strategy to be sure that you really have taken all the sweaters with all the trousers?
14	Naomi:	It is not so easy to see if we have taken all.

The result of Naomi and Filipa's work can be seen in Figure 6. The lines are marked with abbreviations of the weekdays and to the right is written "2 weeks and 1 day" (2 uker og 1 dag).

Figure 6: Naomi and Filipa's solution

Figure 7. One of Frances' five groups

What appears from the pupils' discussion of Task 2 is a less systematic approach, uncertainty about whether they have identified all possible outfits, and a solution based on counting one by one. There is some evidence of grouping when Naomi says "three lines for each outfit" (#6) and later adjusts to three lines for each sweater and five lines for the trousers, after being questioned by the researcher. Still, there is no evidence of seeing the problem as a situation of five groups of three or three groups of five. The representation chosen by Naomi and Filipa in Task 2 (Figure 6) is much further away from a matrix structure than Naomi's representation in the solution of Task 1 (Figure 5). As an example of a grouping emerging also in Task 2, I show a solution produced by Frances (Figure 7). She made five groups, each group containing one coloured circle representing a sweater and three coloured circles representing three pairs of trousers. One of these groups is showed in Figure 7. Inside each group she had written "tre dager" (three days). She also wrote "take three times 3 times 3 times 3 times 3, which is 15". Hence, she got the correct answer but wrote "times" instead of "plus". Other indications of an emerging multiplicative structure are shown by Roger and Nora, when they say that they have "five lines for each pair of trousers" and "three lines from each sweater", and then they say that they take "all the sweaters with all the trousers and all the trousers with all the sweaters".

Discussion

Prediger and Wessel's (2013, p. 437) model shows a relation between a concrete representational register, a graphical representational register, different verbal registers and a symbolic-algebraic or symbolic-numeric register. Both tasks start with a verbal representation, in Task 1 also a graphical representation. In both tasks, all the pupils made use of a graphical register in the solution process, (evidenced by the worksheets), but also other registers could be identified.

The formulation of Task 1 used a verbal register and an iconic graphical register (Duval, 2006, p. 110), the picture of the shapes. This picture turned out to be instrumental in the pupils' solutions. All pupils started by copying the picture of the shapes and then they started to colour the shapes. Almost all pupils ended up with a matrix structure similar to what is shown in Figures 4 and 5. Most pupils got several examples to work on, with different number of shapes and colours, and the representations developed into being more systematic and refined for each new example. I described Task 1 as involving a mapping $M_1 \times M_2 \to M_1^*$, with M_1 containing shapes, M_2 containing colours, and M_1^* containing coloured shapes. The elements of M_1^* are, in a concrete sense, a product (combination) of the elements of M_1 and M_2: "shapes times colours gives coloured shapes". For combinatorial problems, an issue is that the target measure space may not be present from the beginning, and that it is not clear when to stop counting (English, 1991; Shin & Steffe, 2009). The strong relation between the measure spaces in Task 1 may have reduced this challenge.

Task 2 involves a mapping $M_1 \times M_2 \to M_3 \to M_4$, where M_3 contains outfits and M_4 contains days of the week. Here, the connection between the measure spaces is weaker than in Task 1. Task 2 was formulated purely in a verbal register, but the dominating register used in solving the task was an iconic graphical register. The solutions were heavily based on drawings of the clothing items. The lack of a stopping strategy was evident in Task 2, as exemplified by Naomi and Filipa: "We *think* it is two weeks and one day" (#4) and "It is not so easy to see if we have taken all" (#14). They stopped because they were not able to find more possibilities, not because they were convinced that the solution was correct. There are some occurrences of statements "three times five" in the data material, but with no clear reasoning about *why* three times five is a representation of the given situation.

Although a graphical register was used in both tasks, the representation chosen for Task 1 was much closer to a symbolic-algebraic representation (matrix) than was the case with Task 2. In Task 1, Naomi also introduced a symbolic-numerical representation by writing 3+3+3+3+3+3=6+6+6=18. A similar representation as in Figure 6 could be found in many of the pupils' worksheets, and some indications of grouping could be found in the graphical representations (Figure 7) as well as in the discussions. However, there is a significant difference in the chosen representations and in the extent to which the situations are perceived as multiplicative. Despite utterances like "three lines for each sweater" and "five lines for each pair of trousers" in Task 2, there are very few indications of grouping and counting of composite units. The counting is done one by one, sometimes using tally marks when a new connection between a sweater and a pair of trousers was discovered. I interpret this difference to be due to two aspects: The difference between the representations in the formulation of the tasks, and the nature of the target measure space. It turned out that in Task 1, many of the pupils could generalise to other numbers, whereas no such generalisation was observed in the work with Task 2. Mathematical symbolism, seen by O'Halloran (2005) as the final stage in a solution process, is generally used to a very limited extent.

In Task 1 the target measure space M_1^* is a variation of the measure space M_1. This makes the situation close to a rectangular array situation (Fosnot & Dolk, 2001), with shapes (biscuits) laid out in a row-column pattern. Therefore, Task 1 is not a genuine *Product of Measures* situation, but more like an *Isomorphy of Measures* situation, and hence less challenging (Vergnaud, 1983).

In this paper, I have shown how the choice of representations and the difference between the measure spaces can influence pupils' solution strategies. In Task 1, the pupils found a systematic solution strategy, which they also could generalise to larger numbers. The nature of the measure spaces in Task 1 made this situation closer to a rectangular array problem than was the case with Task 2.

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Content-specific teaching practices to enhance learners’ concise language use while dealing with amount and change

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Keywords: Teaching practice, content-specific, explicit language use.

Content-specific teaching practices to increase conciseness

In their research survey on language-responsive approaches, Erath et al., (2021) observe that so far, planning and interaction dimensions of scaffolding have been investigated separately and call for investigating the mutual relationship: “Future research should strengthen the efforts to combine the analysis of dimensions that have formerly been researched separately” (p. 258). In particular, little is known about how macro-scaffolding is supported by micro-scaffolding strategies as Pöhler (2018) outlined while addressing this question to some extent by investigating the impact of micro-scaffolding prompts on intended teaching-learning processes. Even if a few studies already show that students’ language development can be enriched by teaching practices (e.g., Smit et al., 2013), further studies are requested (Erath et al., 2021).

One part of my PhD thesis will contribute to reducing this research gap by showing the impact of spontaneous support through adaptive micro-scaffolding practices on students’ language use while dealing with amount and change and their relationship.

In previous empirical studies we have already shown that conceptual challenges in dealing with amount and change, e.g., confusing the levels f, f', f'' (Nemirovsky & Rubin, 1992), can be overcome successfully when a concise meaning-related language is developed (Prediger and Şahin-Gür (2020) differentiated conciseness topic-specifically into explicitness of level references and preciseness of change process comparisons). In addition, we have observed that teachers' prompts for making references explicit are crucial for developing conciseness of language. These have given us important hints for the further development of the teaching-learning arrangement. At the same time these insights emphasize the need to provide opportunities for enhancing students’ explicit and concise language while dealing with deep mathematical concepts such as amount and change.

So far much is known about students’ processes of extending their individual lexicon, but too little is known about explication and preciseness and even less is known about the effective support of more and more explicit and precise language expansion. This allows me to present a poster sketching the research on the following research questions:

What teaching practices repeatedly emerge in relation to learners' increasing explicitness of language use while dealing with amount and change? And how are these teaching practices related to varying degrees of explicitness and particular concepts of amount and change?

Methods

Data collection. The data collection for the complete PhD study comprised design experiments with 16 pairs of 10 and 11th graders (14-16 years old), and the author as the design experiment leader. The research reported in the poster focuses on data from design experiments in laboratory settings with nine pairs of 11th graders. Two sessions of 45-60 minutes each were completely video-recorded for each pair (around 1000 minutes of video material).

Data analysis. In order to analyze the connection between teachers' practices and students' topic-specific explicitness and preciseness of their language use, the qualitative data analysis proceeds in three steps needs to coordinate two theory-driven deductive codings:

Step 1. Locating students' and teachers' utterances in the topic-specific degree model for conceptual conciseness (explicitness and preciseness) (Prediger & Şahin-Gür, 2020)

Step 2. Depicting the chain of utterances (Prediger et al., 2020), marking and connecting the pathways

Step 3. Based on these coordinated, theory-driven codings, the teachers' questions and prompts are coded and typical recurring practices are identified in sequences or questions and prompts, following the analytic procedure suggested by Prediger et al. (2020).

Conclusion

The necessity to elaborate vague student language has already been identified in other transcripts from qualitative calculus (Prediger & Şahin-Gür, 2020). The poster will reveal first content-specific insights on what is crucial to pick up and connect in students' thinking (e.g., identifying and explicating the levels) by identifying relevant teaching practices.

Outlook for the physical poster. The poster will list in bullet points the theoretical approaches and show the research question. I would like to spend a large part of the poster on empirical findings, illustrated in an exemplary transcript and the related analytic scheme of the identified practices.

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Pluriliteracy as a framework to conceptualize CLIL mathematics classrooms

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In an increasingly globalized world, European politics pushes for schools to accommodate multilingualism and to facilitate students in becoming proficient in the subjects in multiple languages. As a result, many schools in the Netherlands begin to offer mathematics in a Content- and Language-Integrated Learning (CLIL) English framework. However, CLIL is often realized in ways that limit students' use of their home languages in favor of maximizing students' exposure to English, which does not support students in becoming proficient in mathematics in multiple languages. To establish alternative didactical approaches to CLIL, this theoretical paper introduces the notion of pluriliteracy to mathematics. Pluriliteracy emphasizes students agentic use of representations during practices of reading and writing mathematically. The notion of pluriliteracy is illustrated with examples.

Keywords: Multilingualism, pluriliteracy, literacy, CLIL.

Introduction

Traditional Content- and Language-Integrated Learning (CLIL) classrooms follow an ideology of maximizing exposure to the target language and minimizing students' use of their home language(s), with the idea that such maximized exposure would benefit students' learning of the subject-specific target language. This traditional approach to CLIL education is prevalent in the Netherlands. However, translanguaging questions this ideology, as it highlights that students' multiple languages are tightly intertwined and form one shared resource for cognition and communication (García et al., 2017). In fact, there is convincing evidence that translanguaging is most beneficial for multilingual students' mathematics learning (Schüler-Meyer et al., 2019b; Uribe & Prediger, 2021). Hence, the old ideology of maximum exposure to the target language needs to be replaced with new principles for conceptualizing CLIL in mathematics classrooms in terms of translanguaging.

Beyond addressing recent research in multilingual mathematics education, innovative CLIL classrooms also need to address contemporary issues of digitalization and globalization. With respect to digitization and its associated increase in textual communication, there is a need to prepare students for reading and writing a wide range of multimodal mathematical texts (website or newspapers, enriched with diagrams or tables). With respect to globalization, there is a need to prepare students for accessing their mathematical knowledge in multiple languages. Inside school, globalization is already reflected in the superdiversity of classrooms (Vertovec, 2007). To address these two developments as well as recent findings of the relevance of translanguaging, CLIL classrooms should focus on facilitating students' pluriliteracy (Coyle & Meyer, 2021). In this theoretical paper, I introduce pluriliteracy in mathematics and discuss its benefits for conceptualizing CLIL mathematics classrooms, with the research question of how to conceptualize discourse practices of translingual textual communication in terms of pluriliteracy.

A model of pluriliteracy in mathematics classrooms

Pluriliteracy in mathematics can be defined as the agentive engagement in multilingual discourse practices of textual communication of mathematics across languages, genres and representational registers. Figure 1 represents this definition in the form of a three-dimensional model of mathematical pluriliteracy, with the central assumptions that languaging, literacy (conceptualized as proficiency with mathematical genres) and representational registers are intertwined elements of pluriliteracy. Hence, pluriliterate practices can be considered as points in this three-dimensional space, and particularly, the cubes in the model denote such practices on different levels of proficiency.

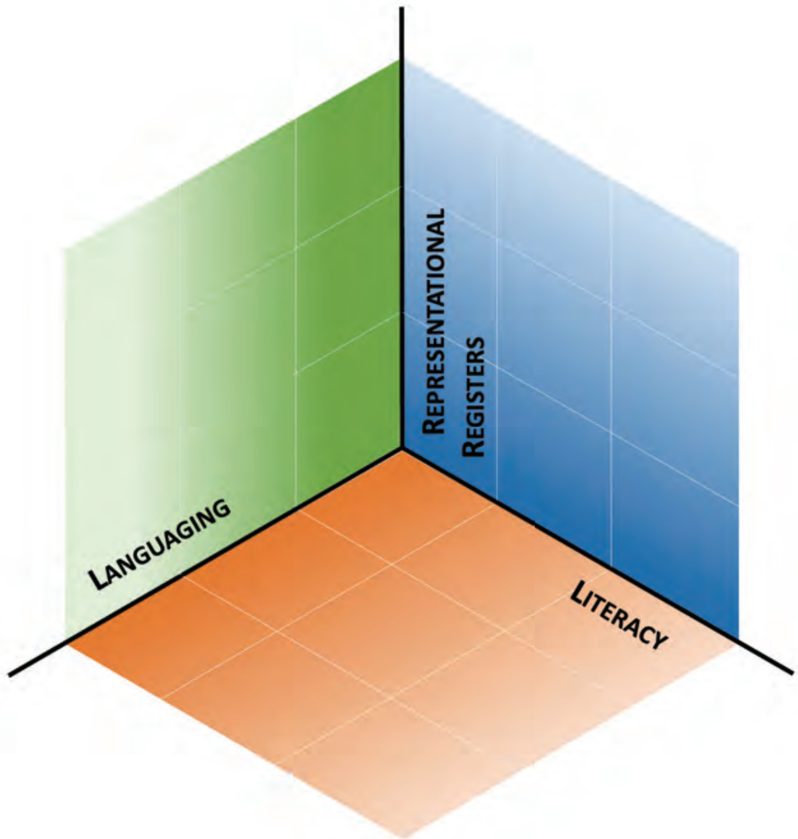

Figure 1: three-dimensional model of pluriliteracy in mathematics

The model of mathematical pluriliteracy builds on a monolingual definition of mathematical literacy which conceptualizes mathematical literacy as the proficiency to agentively "create, use and interpret representations of mathematics concepts, utilizing a variety of verbal, visual, gestural and multimodal 'texts'" (Way & Bobis, 2017, p. 1). Texts in mathematics education can be recursively conceptualized as linguistic artifacts that initiate a subject's meaning-making through inviting specific relations to previous text (Rojano et al., 2014, p. 390). Agency is a crucial element in this definition, as students should be able to be literate outside of and after school, so they need to be able to initiate, for instance, the use of representations to make sense of a multimodal text.

While the definition of literacy does not consider multilinguality, it can be extended into a definition of mathematical pluriliteracy based on the notion of multilingual discourses. Moschkovich (2015) argues that the notion of discourse is central for literacy in a multilingual context, as it provides a broader view of proficiency that “not only [involves] oral and written text, but also multiple modes, representations (gestures, objects, drawings, tables, graphs, symbols, etc.), and registers (school mathematical language, home languages and the everyday register).” (p. 44). While pluriliteracy adopts this broader view of proficiency, it focuses on a specific element of discourse, namely textual mathematical communication (Fig. 1, text as communication). Accordingly, pluriliteracy emphasizes mathematical practices that are particularly concerned with producing and interpreting mathematical texts (Fig. 1, reading, writing and using representations).

Pluriliteracy in mathematics classrooms

In the following, I will give examples of pluriliteracy in mathematics classrooms, highlighting the specific pluriliterate discourse practices students might need to engage in when developing their pluriliteracy. The examples from the students Joris and John are from a bilingual Dutch-English 8th-grade CLIL classroom in an urban middle school in The Netherlands, where pluriliteracy practices were implemented for the topic of percentages and diagrams (van der Aalst et al., 2021). In this classroom, master students designed learning activities to promote pluriliterate practices to support students who aim for a lower degree, to democratize access to CLIL mathematics education.

Example 1: Reading as pluriliterate discourse practice

Reading a text with mathematical information across languages and across academic and everyday registers is an example of a pluriliterate mathematical discourse practice. Many multilingual students will be confronted with a multimodal mathematical text in their home language outside of school, requiring them to interpret mathematical information by utilizing both, their home language and the language of instruction. Outside of school, mathematics often appears in multimodal texts, possibly in another language, such as websites or advertisements. For instance, deciding where to buy an electronic product – when visiting family in Mexico or Turkey or at home in the US or Germany/Netherlands – involves comparing prices and discounts as well as customs rates on websites in different languages. Hence, reading as pluriliterate discourse practice requires all, work with texts (literacy), representations (representational register) and languaging in multiple languages.

Reading a multimodal text with mathematical information across languages, genres and registers is demanding and involves actively asking questions to the texts, on both the academic and colloquial level. These reading demands become evident when considering how students who learn mathematics in English might read an advertisement of a Dutch supermarket (Figure 2). The advertisement presents a discount that is dependent on the number of pizzas bought. The new prices are presented in orange squares which grow in size, perhaps presenting a growing stack (“stapel”). In an academic reading, the squares could remind of a step function. Furthermore, only the biggest square is labeled with the percentual discount that results from the “stapelkorting”. In a colloquial reading, these 62% are an attractive offer, because in the context of other frequent advertisements in Dutch supermarkets which advertise 50% discounts (buy one product, get one for free) it can be considered as better. In an academic reading, one could identify that the advertisement refers to pizzas of different price categories, and indeed, the 62% only apply if one buys four of the most expensive pizzas for 3.99€. The advertisement is an example of a multimodal text that integrates language and pictures through a (culture-specific) “visual grammar” (Kress & van Leeuwen, 2006).

Stack discount

Summer weeks

Up to 62% discount.

2 pieces 4.00

3 pieces 5.00

4 pieces 6.00

(Pizza brand)

Deep frozen

All types, combinations possible.

[price] per piece from ~~2.25~~ – ~~3.99~~

Figure 2: Advertisement in a Dutch supermarket, found at https://static.wekelijkse-folders.nl/image/item/albert-heijn/8030/img001.jpg (Accessed at Aug.-31-2021)

Engaging with the multimodal text in the advertisement could elicit and facilitate several further pluriliterate discourse practices. Firstly, *Reading the text from different perspectives:* With respect to a colloquial reading, students have to be able to identify relevant factual information from the advertisement, such as the price and the respective discount ‘model’ for different numbers of pizzas. With respect to an academic reading, particularly, the use of representations, students should find a mathematical model that allows them to calculate the discount in both, percentage, and absolute numbers. If the “stapelkorting” is assumed to continue for arbitrary numbers of pizzas, a step function with an increase of 1 for each step is an interesting model to explore. Secondly, *formulating questions about the advertisement:* To critically read the advertisement in an academic way, students could formulate their own questions, such as: “How much do I save each time when I only buy the cheaper pizzas, which cost 2.25€ per piece, in absolute numbers, but also in percentages?” “Is the advertisement correct or perhaps misleading?” “What should I do if I want to buy 5 pizzas?”. Thirdly, *communicating results across colloquial and academic registers:* To answer their questions, students have to communicate their results in the form of a text (or presentation). For that, students have to use academic mathematical English, for example concepts such as “amount” of “decrease”, but also their colloquial language, for instance as “to buy 5 pizzas with a price of 2.25€, you should buy them as two “stapels” of 3 and 2 pizzas, as well as representational registers and a suitable genre.

Example 2: Using representations as pluriliterate discourse practice

Mathematical representational registers are crucial for multilingual students to make sense of mathematics, as they enable students to activate everyday meanings in their home language (Moschkovich, 2002) or to mediate between their languages (Uribe & Prediger, 2021). Accordingly, representational registers enable pluriliterate discourse practices. For instance, specific representations are better suited to convey a specific message for a specific audience in a suitable format – hence, they support the (trans-)languaging dimension, but at the same time require specific genre choices. Thus, from the perspective of preparing students to communicate mathematical information across languages and across academic and everyday registers, it is key for student to learn to agentively and purposefully use representations.

The use of representations can be conceptualized as practices of translating between a situational ('colloquial') representation and a suitable formal ('academic') representation (Prediger et al., 2016). Such practices of translating between representations are crucial for multilingual students (Moschkovich, 2002; Schüler-Meyer et al., 2019a). However, little is known about how multilingual students become agentive in connecting informal and formal representations to support their mathematical reasoning across multiple languages. To the authors knowledge, there is only circumstantial research on this issue, and only in monolingual contexts. For instance, in Early Algebra, students' informal drawings can be a vehicle for introducing variables (Schliemann et al., 2010).

In a summative self-assessment for the topic of percentages (8th grade), students in a multilingual Dutch-English CLIL-classroom attempted to connect formal and informal representations, supported by links to everyday contexts (Figure 3). Here, students are supposed to link informal contexts with formal representations. The percentage bar acts as model and can support linking informal and formal representation, e.g. linking the situation of how much remains to download (bottom right) with the formula for calculating the file size with the help of the download bar (top right). The following episode highlights how two students, John (Polish and Dutch as home languages) and Joris (Dutch as home language), struggle to make such connections.

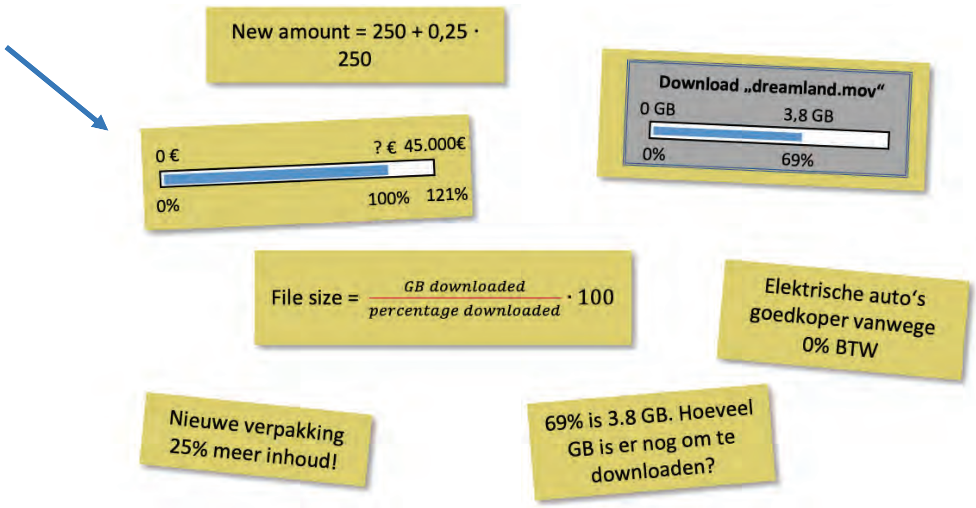

Figure 3: Cards that students have to link to generate tasks, in the context of a self-assessment

Joris: Maybe, maybe this one's belong together then [points at card with arrow]. But I'm not sure.

John: Hm. Ik twijfel. Ik weet niet, is dit increase of decrease zeg maar? *[I doubt that. I don't know, is this increase or decrease, so to say?]*

Joris: I think this is a, I don't know because it is already given the number. It could be increase because 100 percent is normally the max of something so if 21 percent joins it, it would be an increase. I think this is a, I don't know because it is already given the number. It could be increase because 100 percent is normally the max of something so if 21 percent joins it, it would be an increase.

Joris suggests connecting the download bar with the card below that presents a formula for calculating file size. John attempts to interpret the download bar in terms of the concepts decrease and increase. These two concepts have been repeatedly discussed in the previous learning-teaching unit, and they stem from the target repertoire of English terminology. Joris follows this line of reasoning and interprets the card as an increase, but he is conflicted by a "rule of thumb" that percentages cannot be higher than 100%. Hence, students see through the representations towards underlying concepts. For that, they use their multilingual repertoires as communicative resource to express doubt.

Despite regular use of the percentage bar in the preceding teaching unit, the students struggle to make sense of its colloquial meaning, that is, with respect to what situation the percentage bar could re-present (that electric cars do not pay value-added tax /'BTW'). In earlier attempts, students used pattern matching to connect informal and formal representations, that is, searching for cards with the same numbers. These struggles with the percentage bar are in line with the general observation that the students in the unit rarely initiate the use of representations themselves and struggle to connect representations across colloquial and academic registers.

As representations are a crucial tool for multilinguals to connect their multiple languages and representations have been frequently used in the preceding learning-teaching unit, this finding is unexpected. Accordingly, more research is needed on the representation-dimension of pluriliteracy, that is, the agentive creation, use and connection of representations.

Example 3: Pluriliterate writing as example of the literacy dimension

Mathematical writing is a crucial discourse practice for communicating mathematically. In school mathematics, mathematical writing has been shown to be beneficial tool for consolidating and reviewing knowledge but has rarely been investigated as a discourse practice that needs to be explicitly taught. Particularly, the demands of writing in a second or third language, which are quite extensive (Manchón & Matsuda, 2018), have seldomly been explored, so the demands of a pluriliterate practice of mathematical writing are unclear.

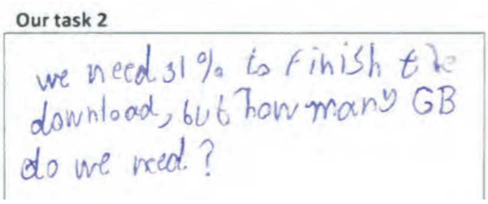

Figure 4: Joris and John's writings in the context of summative assessment

Joris and John’s attempts at mathematical writing in Figure 4 are an example of how students use previous texts to write mathematically, which results in a bricolage of texts from the assessment tasks. When required to formulate a task based on the given cards (Figure 3), students write a task in English (Figure 4). Joris and John’s task resembles the card at the bottom right in Figure 3, which translates into “69% is 3.8 GB. How many GB are there still to download.” Accordingly, the words “download”, “how many” and “GB” are revoiced from the cards. However, to formulate their question students write from their own perspective in terms of “we need”. Hence, the students’ writing practice reminds of the notion of flow (Barwell, 2020) where previous instances of a specific text uses inform students’ later use of these text pieces in their writings.

Discussion and Summary

This article has introduced a three-dimensional model of pluriliteracy as a means to identify discourse practices of textual communication across languages and modes. The model allows to innovate CLIL classrooms with respect to two major developments of the 21st century. Firstly, globalization increasingly requires students to access their knowledge in multiple languages in their future life. Secondly, digitalization makes communication and information increasingly textual and multimodal. Thus, facilitating pluriliterate discourse practices in the CLIL classrooms could enable students to engage with mathematics in their future life in a globalized and digitized world.

The model of pluriliteracy builds on established research in multilingual mathematics education, for instance to substantiate literacy through discourse practices Moschkovich, 2015) and the importance of connecting registers and representations for students to use their multilinguality for epistemic purposes (Uribe & Prediger, 2021). While that allows to identify relevant discourse practices that constitute pluriliteracy, it also points to further research needs to understand translingual discourse practices of textual communication, particularly with respect to interpreting real-live texts in different languages or to produce texts in writings or presentations while activating multiple languages as epistemic and communicative resources. While such discourse practices should be informed by the use of representations, the data presented here suggests that students struggle with an agentive use of representations during reading and writing. Hence, more research is needed to investigate pluriliterate discourse practices.

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Images of mathematics: The public's reactions to educational television programs in Poland during the 2020 lockdown

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Keywords: Images of mathematics, educational television, pandemic.

During the Covid-19 pandemic, we have experienced the broadcasting of educational television programs in Poland, which were offered as an alternative to classroom teaching in all subjects, especially for the students who did not have access to online courses. "Szkoła z TVP" ("School with TVP") was such a program; a total of 1,600 lessons were broadcasted within this program, with a total duration of 1,440 hours. Concerning mathematics, this program has sparked a lively debate among different agents, including academics, mathematics teachers, journalists, parents, as well as students, who were the target group of these series. The aim of the present study is to analyse the images of mathematics, as they emerged in discussions among the aforementioned agents. The data came from 1202 texts published in educational fora (closed Facebook groups), social media, emails, websites (articles, interviews, and comments) and communication platforms. We view the image of mathematics as consisting of: stated attitudes, feelings, description/metaphor for mathematics, beliefs about the nature of mathematics, views about mathematicians and their activities, beliefs about mathematicians' ways of knowing and warranty of mathematical knowledge, description/metaphor for learning mathematics, aims for school mathematics, memories of best/worst mathematics lessons, beliefs about mathematical ability and beliefs about sex differences in mathematical ability (Lim & Ernest, 2000, p. 195). Based on these, we performed a thematic analysis (Boyatzis, 1998) by assigning codes to the utterances contained in the texts. The codes were then combined, to identify emerging themes, which constituted the different images of mathematics.

Results and selected examples

Our analysis led us to four overarching images of mathematics: a) mathematics is unambiguous, b) mathematics teachers are expected to act as motivators, c) mathematics lessons are expected to be interesting and lively and d) mathematics teaching should not be based on rote memorisation. Due to space limitations, we present examples only of the first two images.

Mathematics is unambiguous

This image emerged from two codes: the first one referred to the teacher-presenter's mathematical errors (ERR), while the second referred to the nature of mathematics (MATH):

> Without a doubt, the content that has reached us contains gross errors. (ERR) Basic mathematical skills and concepts are so clearly defined that there is no doubt that there are no two points of view concerning the correctness of the content contained in the program. (MATH) ("Nauczycielka o lekcjach matematyki w TVP: To jest przestępstwo popełniane na umysłach

naszych dzieci” / “Teacher on mathematics lessons at TVP: This is a crime committed on the minds of our children”, 2020)

Mathematics teachers are expected to act as motivators

This image emerged from codes related to the teachers-presenters’ verbal (QUEST) and non-verbal behaviour:

> The lessons are such that there is a person or two in front of the camera and a whiteboard. Some people act as if they stand/sit in front of a normal class, ask questions to non-existing children (QUEST), listen, and then answer by themselves... (Papuzińska, 2020)

We may relate this to the ‘Teacher as motivator’ role described by Munter (2014): “the teacher must be energetic and captivating so that students will be sufficiently motivated to learn with no mention of what the teacher should do with respect to content” (p. 600).

The images identified in our study are related to images that appear in the literature. It seems that the television programs have enhanced specific images of mathematics teaching, especially those related to rote memorisation. Bakker and Wagner (2020) mention that “Several colleagues worried that quick adoption of new technology will lead to falling back to less favorable pedagogy” (p. 2). Our study’s contribution is twofold: towards the improvement of educational programs (and eventually online courses) and towards the improvement of the public’s images of mathematics.

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Quality in students’ writing in mathematics

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We present a literature review with the aim to identify and characterise research frameworks that propose standards for assessing the quality of students’ mathematical writing. Eleven studies that investigated students’ solutions to problems, journal writing, essays, or instructions, and proposed frameworks for analysing such writing, were identified. The analysis sought to characterise what the frameworks attended to in students’ mathematical writing and whether they described a progression with a reference to quality of the writing. The main result is that the frameworks mainly attend to the content and that there is little attention to form. Further, quality is most often regarded as a question of content, if the text contains for example an explanation, then the explanation is seen as a sign of quality with no reference the form. Our review concludes that research rarely describes levels of quality in students’ mathematical writing, something we believe further research should do.

Keywords: Writing, quality, progression.

Introduction

The universal practice of having students present their solutions to mathematical problems in writing and using this writing to draw conclusions about students’ proficiency, has been common for centuries (Morgan, 1998). Reporting mathematical work in writing requires knowledge on how to represent mathematical investigations, conclusions, justifications, and arguments. In many curricula communication is presented as a competency and as such it can develop so that students over time could exhibit a progression, for example in the quality of their mathematical writing. In this paper we have investigated the question of quality in mathematical writing through a literature review where we set out to answer the questions: Does research provide standards for assessing the quality of students’ mathematical writing (MW) and if so, what are these standards?

A theoretical point of departure is the idea that competence in mathematical communication, in our case in writing, is a skill that can be analytically separated from other skills. This idea is captured by the concept of communicative competence (Hymes, 1972), which suggests that in a particular social situation all things said are not communicated with equal competence (Craig & Muller, 2007; Rickheit et al., 2008). The simplest criteria for what can be considered competent communication are efficiency and appropriateness (Rickheit et al., 2008). Efficiency relates to the idea that communication takes place to fulfil a goal and if you achieve your goal then the communication is effective. Appropriateness connects with the social aspect of communication: to communicate well in a social situation is to be familiar with all the ways of making meaning within that situation. Wittgenstein referred to this social localisation using the metaphor of a language game. He suggested that competent communicators are those who know and abide by the rules of the language game that the social situation constitutes (Wittgenstein, 1953/1986). It can thus be argued that what is considered good communication changes with the situation. This idea is fundamental to the movement labelled Writing Across Disciplines which has advocated the differentiation of writing skills in relation to different academic disciplines of which mathematics is one (Bazerman et al., 2005).

In line with the ideas presented above it can be noted that modern communication in mathematics is a special kind of communication which has developed over two centuries (Solomon & O'Neill, 1998) and is defined by its multi-semiotic nature (O'Halloran, 2005). Because mathematical concepts are often difficult to articulate in ordinary language, mathematical symbolism has developed to express meaning beyond what can be expressed by ordinary language (Schleppegrell, 2007). It has been proposed that MW can be decomposed into a formal logical structure in which we find definitions, theorems and proofs, and a complementary informal structure in which motivation, analogies, examples, and explanations have their place (Steenrod et al., 1973). The logical structure relies on mathematical notation and symbolism which owe their strengths to their universality and independence of context (Sfard et al., 1998). The complementary structure relies on natural language, which owes its strength to its flexibility, offering a nearly endless number of ways of conveying meaning through multiple modes.

Mathematical writing takes place in relation to different objectives and prompts. Students use writing to personally make sense of, record or explore mathematical ideas, but also to document processes, explain, describe, and make arguments in social situations with peers or teachers (Casa et al., 2016; Morgan, 2001; Stylianou, 2011). Within these different categories there is a difference between writing in mathematics and writing about mathematics (Bosse & Faulconer, 2008). Writing to personally make sense of mathematics and writing for others such as teachers and peers represent two very different processes, but commonly, they are concurrent in school mathematics (Morgan, 1998). The definition for MW used in this paper is borrowed from Casa et al (2016) and encompasses both these categories. MW is understood to be writing to reason (personal) and to communicate (with others) in an educational and mathematical context. We assume that standards for assessing the quality of students' mathematical writing apply to either one specific form of writing or several forms.

Review methodology

We conducted a literature search in ERIC (EBSCO) where we searched abstracts using the following search words: [writing AND (mathematical or mathematics) AND (pupil* or student* or children*)]. The search words were chosen with the intent to capture research in which students' mathematical writing is treated as an object separate from communication in general (Schleppegrell, 2007; Steenrod et al., 1973). The search was performed on the 24th of May 2021. We found 806 peer-reviewed articles fitting the inclusion criteria (see table 1).

Using the software *Covidence* to manage our literature review, each paper was screened by two reviewers. The screening was made in two steps. First, we screened the title and abstract to forward relevant studies to full-text review. In cases where we were hesitant, we chose to forward the paper to full-text screening. In the full-text screening, papers were included if they contained frameworks or guidelines on how to write, or explicit descriptions of progression or levels of writing.

Of a total of 141 papers, 50 remained after the full-text screening (see table 1). These were categorized according to what was investigated in the paper in terms of: Studies that compare different frameworks for analysing mathematical writing (1); Studies that create a framework for analysing mathematical writing, but where the research question concerns something else (the framework created is a means to study for example the way digital technologies influence students' writing or how writing affords learning) (26); Studies that use a pre-existing framework for analysing mathematical writing (the framework is a means) (12); and finally, Studies that result in a framework for analysing mathematical writing (11). In this study we have chosen to focus on the latter category to investigate only research that focuses exclusively on students' mathematical writing as an object.

Table 1: Screening process

	Inclusion Criteria	**Exclusion Criteria**	**Excluded**
Title and abstract screening 806 studies imported	Mathematical writing Students Frameworks for assessing writing or identifying aspects of writing	Journal writing without guidance for assessing writing No instruction or identification of aspects of writing or frameworks for assessment	665 studies irrelevant
Full-text screening 141 studies imported 50 remained for data extraction	Frameworks for how to write Guidelines for how to write Explicit ideas for progression Descriptions for progression or levels of writing Meta-reflections on the benefits of writing	Not a research article No criteria for analysing or assessing mathematical writing Full text not available. Not about mathematical writing	57 21 8 5

The literature review is a configurative review (Gough et al., 2012) aimed at identifying, interpreting, and characterising research frameworks that propose standards for assessing the quality of students' mathematical writing. After an initial review of overarching data such as age groups and types of writing (Bosse & Faulconer, 2008; Casa et al., 2016), the analysis focused on the proposed frameworks. The first part of this analysis sought to characterise the frameworks in relation to what they attended to, and the second part was concerned with investigating hierarchies or descriptions of progression in relation to quality, within the identified frameworks.

Results

The types of mathematical writing that was investigated varied across the studies in the sense that the writing samples were collected through different prompts. They were, however, all examples of accounts of writing that *communicated* mathematics, i.e., the texts were created with the understanding that they were supposed to be read by someone other than the student herself. The most common were prompts related to documenting and reporting solutions to mathematical problems. This type of writing was found in six of the eleven studies that were part of the review (Hughes et al., 2019; King et al., 2016; Kosko & Zimmerman, 2019; Morgan, 2006; Pugalee, 2004; Teledahl, 2016). The remaining prompts concerned proofs (Lew & Mejía-Ramos, 2019), journals (Clarke et al., 1993), essays (Stonewater, 2002), and instructions (Kline & Ishii, 2008; Shield & Galbraith, 1998). The writing samples were collected from different age groups, three from students in higher education, two from students aged 17-19 and the remaining from different constellations of students aged 6 - 16.

In our characterisation of what the frameworks attended to, we were originally interested in questions of form and content, inspired by previous research in which content and form in students' MW emerged as contrasting qualities. In our analysis, content and form proved to be useful categories although they failed to capture everything the frameworks attended to. Beyond these categories, we identified attention to the meta-functions of language as the focus of one of the frameworks. Our final characterisation therefore consists of the categories meta-functions of language, content, and form, which will be presented below.

Meta-functions of language

Only one of the proposed frameworks attended to the meta-functions of language and communication (Morgan, 2006). Morgan suggested using tools from systemic functional linguistics (SFL) to analyse students' MW. Such an approach would enable us to understand how students represent their understanding of who does mathematics, what kind of objects are involved in mathematics and what relationships are constructed (the ideational aspect); who the participants are in the written mathematical communication including their relationship to each other and to mathematics (the interpersonal aspect); and what the written text does (the textual aspect).

Content

The most common question that the frameworks attended to was *what?*, i.e., there was a concern with content in all but one of the studies. A typical framework lists several textual elements that should appear in the MW. In the studies investigating problem-solving accounts or solutions, such elements could include recounts of the steps of the problem-solving process, explanations, accounts of reasoning, and referrals to context (Hughes et al., 2019; King et al., 2016; Kosko & Zimmerman, 2019). Mathematical language or the use of mathematical terms can also be viewed as elements of MW that are either present or not in the text, hence it becomes a question of content (King et al., 2016; Kline & Ishii, 2008). One of the studies characterises the content of the MW as descriptions of sub-processes in the problem-solving process (Pugalee, 2004). There is an orientation element, an organizational element, an execution element, and a verification element, all of which correspond to processes in students' problem-solving behaviour. The studies that describe types of writing other than solutions to problems also list content. These include descriptions of context, descriptions of steps, exemplars, kernels, goal statements, justifications, links to prior knowledge, dialogue, summaries, and recounts (Clarke et al., 1993; Kline & Ishii, 2008; Lew & Mejía-Ramos, 2019; Shield & Galbraith, 1998; Stonewater, 2002).

Some of the content elements described in these frameworks are described with a reference to quality. The most common quality articulated is *mathematical.* The term mathematical is attached to textual elements such as steps, descriptions, explanations, language, and reasoning. There are, however, no descriptions of how a textual element qualifies as being mathematical.

Form

Three of the proposed frameworks address the question of mode or form. One of the studies (Teledahl, 2016) lists empirically identified modes of communication in MW together with their apparent function in the MW. Two of the studies investigating proofs (Lew & Mejía-Ramos, 2019) and essay writing (Stonewater, 2002) attend to students' use of mathematical notation and in the case of poofs, and in particular their adherence to rules of academic language and the formality of the proof.

Hierarchies or progression

Only four of the eleven studies address the question of hierarchies or progression in students' MW. The remaining seven either do not mention any differences in quality or treat quality as a question of whether the MW contains a desirable element or not or how often it appears. The two studies that address quality describe three (Clarke et al., 1993) or six (Kosko & Zimmerman, 2019) levels of writing and details the differences between these. In the case of Kosko & Zimmerman the analysis, and subsequently the quality criteria, details the difference in quality between six different ways of expressing an argument in students' MW. Two other studies describe elements of MW that were highly valued by respondents in relation to writing an instruction (Kline & Ishii, 2008) or an essay (Stonewater, 2002). The latter is the only study that mentions form as something that affects the quality of the MW. Stonewater suggest that "*Highest scoring writers use both a greater number of* algebraic, numeric, and graphic representations […] *and a greater variety of these representations to augment their written work than do the lowest scoring writers*" (2002, pp. 330-331).

Discussion

The review aimed to identify research-based frameworks that propose standards for assessing the quality of students' mathematical writing. The results indicate that while there were several frameworks that in different ways describe students' mathematical writing there are very few that do so with a focus on different levels of quality. Of the four frameworks that describe quality beyond the inclusion of different elements in the MW, only one investigated accounts of problem solving. This framework however focused on a specific part of such accounts: the expressing of an argument. The other three studies investigated other types of mathematical writing.

If quality is thought of as something that can be decided based on which elements are included in MW, then there are more examples of frameworks. The choice whether to include specific features in MW can be related to what prompted a student to write the text or to her idea of the discourse in which the text is written. The simplest of the criteria for communicative competence are appropriateness and effectiveness (Rickheit et al., 2008) both of which depend on knowledge about the discourse or, to use Wittgenstein's words, the rules of the language game (1953/1986). Adherence to rules and conventions regarding the form and content of MW is connected to the appropriateness as well as the effectiveness of the communication. If you communicate in ways that let readers recognise the writing as mathematical discourse, then this is appropriate and if readers understand what is communicated then the writing is effective. In the written discourse of mathematics there are conventions and expectations (Schleppegrell, 2007; Sfard et al., 1998) particularly in mathematical domains such as proofs that rely heavily on mathematical notation (Steenrod et al., 1973). The result of this review, however, indicates that in school mathematics such conventions and expectations are, if not missing, then at least not described in research, neither regarding content nor form. It is possible that this is connected to the fact that communication in school mathematics, because the students are young and/or still learning, rely more on natural language than other modes of communication (Sfard et al., 1998).

In preparing for the review, we assumed that we would find research that had taken an interest in how the use of different representations or forms of communication could contribute to an increased quality of MW. Not only did we encounter very few frameworks that addressed the issue of different levels of quality at all, but we only found one that mentioned form. The study that mentioned form, however, proposed that a greater number of different representations together with a greater variety of forms of representations was a sign of high-quality writing without addressing the appropriateness of the representations in relation to the context. How students communicate in writing is, from what we can conclude, not an issue connected to the quality of the writing. If we instead turn to content, we still encounter a lack of agreement on the features that signal quality in accounts of problem solving. The idea that the quality of MW increases if it contains arguments and/or justifications is only visible in frameworks that investigate other types of writing. Considering the small number of papers in our study it is likely that our scope was too narrow and that our study will yield different results when we, in the future as part of our longitudinal study, include also studies that propose frameworks for analysing MW while investigating something other than the quality of the writing itself. But this issue pinpoints our concern regarding the separation of MW as a means from MW as the primary object of interest. Our aim was to identify studies in which the focus was on the MW itself, as an object worth investigating in detail.

Considering that the studies investigated different types of writing from different age groups, one might ask if we should expect agreement or universal ideas on quality in student MW? Is writing not intimately connected to different curricula, different contexts, and different levels of mathematics? We believe there could be universal ideas on some elements of MW, for example the inclusion of arguments and/or justifications in students' accounts of problem solving. It is our hope that future research will attend to such ideas.

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Understanding mathematics teacher's argumentation: considerations during classroom tasks discussion

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This paper presents the research results whose aim is to understand the mathematics teacher's argumentation during classroom tasks discussion. Aspects of a theoretical stance are used, on the one hand, to show how the teacher's argumentation can be described in the mathematics classroom and, on the other hand, to specify aspects of the discourse in the mathematics classroom. The analysis is illustrated with one episode in a tenth-grade classroom, where the teacher and her students discuss the solution of a task about trigonometric ratios. The results allow recognizing features of the teacher's argumentation features, purposes, and conditions that activate it.

Keywords: Argumentation in mathematics classroom, discourse in mathematics classroom, learning opportunities, professional knowing.

Introduction

The argumentation of the mathematics teacher during teaching requires specifying aspects about the participation and discussion in mathematics class. On the one hand, it can be observed how research on and with mathematics teachers has grown in the last two decades, which in addition to being multifaceted, has a broad scope for teaching and learning mathematics (Chapman, 2016). On the other hand, interest in discussion and participation in mathematics class, particularly in the discussion of tasks, is associated with the way in which learning is conceived, this is from a participatory perspective (Krummheuer, 2011), where learning is conceptualized as participation in classroom discourse. The interest here is focused on the teacher, in what some authors have called the Teacher's mathematical discourse (e.g., Planas et al., 2016), that is, consider the teacher's communication of mathematical content in his interaction with his students. In addition to investigating the learning opportunities that these pragmatic considerations can promote; it is interesting to recognize links between the teacher's argumentation and student participation.

Given this concern and considering that: (1) argumentation can be used to deepen the decisions and practices of teachers (Metaxas et al., 2016); (2) there are elements or qualities in the communication between teacher and students that are important for learning mathematics (Drageset, 2014); (3) to describe the discourse in the classroom, detailed frameworks with categories and concepts are needed to describe individual turns (Drageset, 2015), and (4) argumentation in the educational field can be conceived as a social space and discursive (Ayalon & Hershkowitz, 2018). This paper proposes to answer the question: how is the argumentation of the mathematics teacher during class task discussion? The data analysis is illustrated through one episode of a mathematics class lesson in a tenth-grade teacher (15-16-year-old students) in the city of Medellín (Colombia).

Theoretical background

This section comprises two sections; the first one presents considerations on argumentation and the second one presents concerns on class discourse.

Considerations on argumentation

Like different authors, the definition of argumentation presented by van Eemeren and his colleagues (2014, p. 7) is taken up:

> Argumentation is a communicative and interactional act complex aimed at resolving a difference of opinion with the addressee by putting forward a constellation of propositions the arguer can be held accountable for to make the standpoint at issue acceptable to a rational judge who judges reasonably

This consideration regarding argumentation seeks to attend to complex interactions in mathematics classroom, where the teacher and his students argue during the development of a lesson regarding a certain task. It also implies that the object of this research, the mathematics teacher's argumentation, tries to isolate itself from the classical position, in which argumentation is assumed as a set of premises and conclusions formulated with the help of formal symbols, to assume a position closer to language and communication. Considering the argumentation under this theoretical assumption, consists of seeing the argumentation as a type of activity with purpose or intention, so that the activity is recognized as a process whose representation is the use of language and, therefore, the structure of the constellation. of propositions must be analyzed as speech acts that are part of the resolution of differences of opinion.

The difference of opinion does not necessarily take the form of a disagreement, dispute, or conflict, but there is one party that has a position and another that doubts whether to accept that position (van Eemeren et al., 2014). In mathematics class, it is possible that there are doubts regarding a statement, indication or explanation of the teacher, doubts about an answer or procedure different from the one presented by the teacher, or different answers to a task in the students' work, where the teacher's argumentation is required.

This position regarding argumentation also requires considering the process and the product of argumentation. In this document, the process is analyzed based on three dimensions, the first is the communicative dimension, which refers to what the teacher says and why he says it, that is, statements, questions and purposes; the second is the interactional dimension, which refers to the place where he says it and to whom he says it; and since the argumentation takes place in an educational context, the epistemic dimension is considered, which refers to how he says it and why he says it. The product includes each of the episodes selected for the analysis, which begins with an argumentative intervention, in which the difference of opinion on the part of the teacher or a student is made explicit and ends with the closing. In the episode, the professor seeks to convince his students from his point of view, for which he draws on his knowledge and professional experience.

Given the purpose and object of research of this study, an adjustment is made to the conditions that should occur in the mathematics class for the development of the argumentation proposed by Solar and Deulofeu (2016). In this way, the following conditions are recognized: (i) Communicative and interactive strategies, (ii) Focus of the lesson, (iii) Task focus, and (iv) Professional knowledge. The indicators for each dimension emerge from the respective analysis.

Considerations on discourse

Given that the argumentation is expressed mainly orally and by a group of participants (Knipping & Reid, 2015), it is pertinent to consider some class discourse elements. Discourse and its terms are frequently used in studies in Mathematics Education (Lim et al., 2020). However, the term discourse is usually related to different approaches and traditions, which implies that no single interpretation is used (Ryve, 2011). However, as other researchers (e.g., Moschkovich, 2003) and given the perspective in which this work is inscribed, we adopted the notion of discourse presented by Gee (2008, p.161):

> A Discourse is a socially accepted association among ways of using language and other symbolic expressions, of thinking, feeling, believing, valuing, and acting, as well as using various tools, technologies, or props that can be used to identify oneself as a member of a socially meaningful group or “social network,” to signal (that one is playing) a socially meaningful “role,” or to signal that one is filling a social niche in a distinctively recognizable fashion

Discourse is considered something more than speaking or writing (Moschkovich, 2003), considered the language in use since it can be interpreted differently depending on the context. Furthermore, discourse refers to multiple processes through which people communicate (Planas et al., 2018), which implies considering it as a means and objective (Gee, 2008). For example, a discursive means of the teacher in class around the discussion and resolution of tasks and a goal is teaching-learning objects (Planas et al., 2018). This position is consistent with the purpose of the research since it is interesting to know what the teacher says and how he says it, the identity he takes when he says it, and the acts accompanying it.

In this paper, mathematical discourse is understood as the interventions by the whole class or in small groups, where the teacher and his students discuss mathematical tasks that take into consideration the understanding of concepts, operations, procedures, and their interrelations (Walshaw & Anthony, 2008). The mathematical discourse also includes “not only ways of speaking, acting, interacting, thinking, believing, reading, writing, but also mathematical values, beliefs and points of view” (Moschkovich, 2003, p. 326).

Likewise, this study explores the teacher’s mathematical discourse, which is considered an essential component of educational mathematics practice. It is understood as the selection, sequencing, explanation, adaptation, and *argumentation* of multiple situations. The teacher communicates with his students during the solution of a task in class to raise generality mathematics [italics added] (Planas et al., 2018).

It is not possible to characterize discourse as a series of individual actions, but rather as a social practice, where each intervention is related to previous interventions (Drageset, 2014). Therefore, the typification of teacher reactions to student intervention proposed by Ruthven and Hofmann (2016) is used, which includes: Approve, Disapprove, Repeat, Restate, Translate, Redirect, Probe, Expand, Revert and Devolve.

Method

This research corresponds to a study with a qualitative interpretive approach, where observation is used as a tool for data collection. It is intended to explore and describe environments and situations in mathematics class and produce in-depth interpretations to analyze the individual and collective actions of the mathematics teacher.

This article reports on Emma's class (pseudonym), a tenth-grade class with 32 students (pseudonyms used) whose ages range from 15 to 17 years (female group). In preparing her lessons, Emma follows the curricular plan designed by the educational institution, which is consistent with the statements of the Colombian National Curriculum Standards. The data correspond to six lessons guided by Emma. The episode presented in this article corresponds to one of these lessons, where Emma and her students discuss the following task: *Finding the value of trigonometric ratios of notable angles*. The task was developed during two lessons. In the first lesson, Emma, together with her students, finds the value of the sine, cosine, and tangent. In the second lesson, there is an autonomous work by the students accompanied by Emma interventions. The episode that is presented take place during the second lesson. Each episode begins with an argumentative intervention and ends with the closure. The argumentative intervention is preceded by turns that contextualize it. Not always, there is an argumentative intervention and a closure. More than one argumentative intervention or more than one closing may occur.

For data analysis, discourse analysis techniques are used in two actions: fragment and connect (Boukafri & Planas, 2018). Fragment to obtain more manageable units and connect to discuss data and results that have been treated separately. According to Boukafri and Planas (2018), the reiterations of fragmenting and connecting lead to three units of analysis: turn, episode, and lesson. At first, the analysis episodes in each of the lessons are identified, for them a tracking is made in each of the turns, both teacher and students, of the argumentative interventions, and of the respective closings, which indicates the beginning and the end of the argumentation. Then, in the unit turn, the reactions to the students' interventions are identified in response to the teachers turn, using the framework of Ruthven and Hofmann (2016). Responses are taken either when the teacher's turn starts the episode or when a student starts it. The turns are analyzed in the three dimensions: communicative, interactional, and epistemic. The actions of the communicative dimension relate to the framework of Ruthven and Hofmann (2016). Regarding the episode unit, argumentative interventions and closings are retaken, to identify purposes of the teacher's argumentation. And in a third moment, in the lesson unit, the adaptation to the proposal of Solar and Deulofeu (2016) is used to identify the conditions that activated the argumentation in the different episodes within a specific lesson.

Data analysis and findings

The analyzes are exemplified from an episode, where Emma is explaining the procedure that allows us to calculate the value of tan30 °. Together with her students they have reached the expression 1 / √3, the students seem to realize that they must rationalize, to which Emma asks them why they do this, marking the beginning of the episode. Given the interventions of the students, it seems that there is a certain level of understanding of the procedure to follow. Then, however, there is an intervention by Sofia and Mia, which reveals difficulties and requires the teacher's attention.

253	Emma:	[...] At what point do we rationalize? Why do we have to rationalize?
254	Students:	Because there is a root below.
255	Emma:	Because the root can't be left below and what is that called below?
256	Students:	Denominator.
257	Emma:	Denominator. Well, then it would be a tangent of 30 ° ... here it would be equal to 1 per root of 3, how much does that give me?
258	Students:	Root of 3.
259	Emma:	Root of 3 over the root of 3 by root of 3.
260	Sofia:	3 root of 3.
261	Mia:	Root of 6, right?
262	Emma:	Well, let's look at what did we say, last class? Turn back in the notebook.
263	Alice:	Root 3 square.
264	Emma:	Root of 3 square, root of 3 by root of 3 gives me root of 3 squared and what happens here?
265	Students:	They are canceled.
266	Emma:	What is canceled?
267	Bianca:	The exponent in the root.
268	Emma:	The exponent in the root? ... The exponent with the root. I have a tangent of 30 ° is equal to the root of 3 over 3

In this episode there are two situations that deserve attention within the teacher's argumentation. The first one, the anticipation of a difference of opinion through Emma's questions in [253], also considered as the first argumentative intervention. The responses of the students in the interventions [254, 256] allow Emma to identify a certain level of appropriation in said procedure, her intervention in [225, 257] consists of supporting the justification of the students and therefore presenting a partial closing. And the second situation, based on the interventions [260, 261] of Sofia and Mia respectively, show a difference of opinion and therefore correspond to the second argumentative intervention, since in addition to presenting different answers, they warn errors, before which Emma does not explicitly declare the error, but directs the justification for the students to realize it and convince themselves of the expected response, presenting the closure in [268].

Regarding the communicative dimension, it is identified as a question from Emma, with which she seeks to probe the appropriation of what it means and implies rationalizing an expression, marks the beginning of the episode. This question is preceded by interventions from the students, before which Emma raises statements with which she approves, translates, restates, or reverts. Even though in a previous episode the notation of irrational expressions seemed to have become clear, a certain procedure takes place in turn [257] before which the students express in [258] $\sqrt{3}$ as an answer. Emma intervenes in [259] to restate it, to which Sofia raises $3\sqrt{3}$ as an answer, and Mia $\sqrt{6}$ as an answer. In the following interventions, Emma uses questions and assertions, with which she takes up procedures that have already been discussed. Also, in the intervention [255] distinguishes another type of reaction: request, which, given the purpose of this research, makes it necessary to continue expanding the table proposed by Ruthven and Hofmann (2016).

In the interactional dimension of this episode, participation, media, and class norms, convincing and discussing, stand out as characteristics. The intervention [262] seems to be interesting, in which Emma refers to previous lessons by inviting the students to review her notebook. In addition to involving them in the answer to a question, she attends to a question of the students in handling with roots, using an indication to be followed by all, because she knows that it is a question that had been discussed.

On the other hand, they are characteristics of the epistemic dimension: treatment of the mathematical object when requesting clarity [255], taking up other lessons to verify the use of a certain procedure [262], error handling [262] and, procedures and answers to verify [253, 255, 257, 264, 266, 268] and validate a given answer [268]. Emma's actions described in this dimension allow us to suggest how her experience in this degree of schooling allows her to justify to the students the use of a certain procedure, insist on when and why it should be done and anticipate possible difficulties with the treatment of the same.

Before describing the purposes of the professor's argumentation in this episode, the particularity of it is highlighted, in it an argumentative intervention is identified [257] during an explanation process, which alludes to the fact that within the professor's mathematical discourse also of explanations there is also argumentation, and it reaffirms the consideration of explanation as a different process from argumentation. Emma seeks to justify the procedure for solving the task, to achieve this, she poses a question [253], considered here as an auxiliary argumentative intervention, which is preceded by interventions by the students, in which answers to the question [255, 257] and therefore a first closure. However, the episode does not end there, it is only until the intervention [268] that the closing of the episode can be recognized, when Emma states the answer to the task. In this way, the purposes in this episode are highlighted: to clarify the properties of the mathematical object, the root, involved in the solution procedure of the task [253, 255, 257, 264, 266], to clarify the solution procedure of the task [255, 257, 262, 264, 266, 268], and dealing with different points of view that do not match the expected response of task [262].

In relation to the conditions that triggered the argumentation, the following are identified: (1) The communicative and interactive strategies, the questions associated with the task solution procedure draw attention, in which Emma seeks to retake procedures, so much so that they were treated in the same lesson or in previous lessons [257, 262, 264], which seems to be related to the statement of the task "Finding the value of the trigonometric ratios". It seems that the students still have difficulties in handling procedures: rationalization, root management, and fractional operations, necessary to respond to the task. (2) The approach to the lesson, the argumentative intervention refers to understanding, that is, Emma observes the work of her students and begins the argument by asking the reason for a procedure [253]. And (3) professional knowledge, Emma seeks to link the work done in previous lessons [262], since she seems to be aware that, to respond to this type of task, the students should be able to handle different concepts and procedures. In addition, it is repetitive the action of inviting students to name mathematical objects in an appropriate way [255, 266].

Conclusions

We can affirm that the argumentation of the mathematics teacher constitutes a complex formed by three articulated dimensions: communicative, interactional, and epistemic, whose objective is to educate students in mathematics. The primary intention of the teacher is for students to understand mathematical objects, and for this she puts into play resources that are in these dimensions. The use of the teacher's reaction typology for the identification of own actions in each characteristic stands out as a success, and the contribution of this research by expanding said typology is worthy of note. It is important to point out how communicative actions allow observing the participatory perspective of learning, to which the research alludes, where not only the teacher's intervention is recognizable, but also that of the students.

We recognized how Emma links her students in answering questions or situations in a class lesson, how she raises justifications to convince students of a certain answer to a task or question, and how they use students' concerns to open the space for discussion and participation, and how she seems to be interested in the students not only correcting an answer but also being participants and aware of the errors when carrying out a procedure for solving a task. The link between the actions of the epistemic dimension with professional knowledge is also evident, since Emma's actions indicate her experience, which is, it can be corroborated in how she raises justifications for certain procedures, insisting on when, how and why they should be made.

The purposes of the argumentation warn how Emma in addition to presenting the solution of a certain task, she is interested in having her students participate in the class lesson. It was useful to recognize the argumentative interventions and the closings in each episode, which in addition to delimiting said episodes, allowed us to recognize situations in math class lessons where the teacher argues. The conditions that activate the argumentation are recognized both in the interventions of the students and of Emma, they account for specific moments of a class lesson where the teacher should be attentive and prepared to face them.

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TWG10: Diversity and Mathematics Education: Social, Cultural and Political Challenges

Introduction to the work of TWG10: Diversity and Mathematics Education: Social, Cultural and Political Challenges

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Scope and focus

Thematic Working Group 10 has been active since CERME3 in 2004 and is focused on discussing mathematics education within the realms of *the cultural, the social and the political*. TWG10 builds on the premise that mathematics education is always more than an encounter between an individual and a mathematical object in a classroom setting. Instead it views such encounters as shaped, produced and reproduced in the context of wider cultural and societal contexts that are inherently social and political (Black et al, 2021). At the same time such encounters are viewed as contributing and constituting the contexts in which they are embedded in ways that reproduce, challenge or disrupt power relations. Research in this group is characterized by multiple efforts to reflect its own double-role in analyzing, shaping and reconfiguring mathematics education practices.

The group is specifically interested in research that investigates how diversity and difference is produced through mathematics education and how this process affects the possibilities, opportunities, obstacles, privileges and disadvantages associated with mathematics education. This includes issues of gender, race and ethnicity, language, socio-economic status, social class, disability, life opportunities, aspirations, worldviews and ideologies, school systems, governance structures, space, and settings. Additionally, diversity and difference may occur in relation to who is doing the research and who is being researched, posing methodological issues of an ethical, ontological and aesthetic nature. As all these multiple diversities and differences intersect, a reflective approach is expected in reporting what might be the effects of specific mathematics education reforms but also in discussing the effects of particular theoretical frameworks that attempt to frame and discuss mathematics teaching and learning in praxis. The group strives to unravel and contradict power relations between fields and how research depends on both theoretical and empirical assumptions in practice. Hence, to decenter oneself as a researcher is a strategy and joint endeavor in the team's collaborative work during the conference.

Organisation of TWG 10's work

Understanding research as a practice that is situated within diverse cultural, social, and political contexts has implications for practicing research *in situ*. During the conference, we organized the group to work in a way that 1) cultivated a change of perspective and fostered reflexivity and 2) created awareness about the effects of power relations that are always embedded in efforts to understand, theorise and research diverse practices in mathematics education. Hence, we began our work in the group by posing core questions that are ongoing and have been a theme throughout the years in TWG10:

- What forms of exclusion or inequality are being created through mathematics education and how their visibility or invisibility becomes framed or narrated?
- What possibilities or opportunities are there for disrupting inequalities or exclusion in mathematics education?

Due to the rapidly changing landscape in education following from the pandemic, we added the questions:

- What new forms of exclusion or inequality has the COVID 19 pandemic created or made visible for mathematics education? Or are existing inequalities merely amplified?
- What possibilities or opportunities has the COVID 19 pandemic created for disrupting inequalities or exclusion in mathematics education?

In an attempt to make poster contributions visible to the whole group, they were also presented in the first session. This potentially stabilized the hierarchical distinction between papers and posters by ensuring they were reported and discussed by the group.

The development of reflexivity was sought by following the principle of *peer presentation*, namely that authors do not present their own paper, but give a short (5 minutes) presentation of a colleague's paper. This peer presentation included a description of the main ideas from the perspectives adopted in the paper and the formulation of questions from the presenter's own perspective. This was followed by a discussion in smaller break-out rooms between the author and reader - but also other TWG participants joined and added their reflections to the discussion. We finally held a joint discussion on interesting, important and challenging topics to put forward. In this way we sought to recognize research as a collective assignment that takes place in a network of social practices of dissemination, reflection, writing and problematizing as we shared and developed ideas, methodology and theory.

In order to encourage and also facilitate drawing connections between papers, they were grouped in sessions that were broadly thematic in some way. A number of papers focused on mathematics in a range of out-of-school settings provoking us to think about how localized mathematical practices relate to the mathematics curriculum. Ferrarello et al. presented findings from their project on Mathem-ethics in a prison setting in Italy, Solares-Rojas & Goizueta looked at the embedded mathematics utilized by Hñañu women embroiderers in Valle del Mezquital, Mexico and Francois & Vandendriessche reported on their ethnographic study of local activities described as string figure making in Northern Ambrymese society, Vanuatu. These papers raised debate regarding the paradox of validating mathematical knowledge from marginalized communities using academic mathematics and whether this really legitimates embedded mathematics or simply marginalizes in a different way.

Another common focus across the papers was teachers' understandings and experiences of marginalization and diversity. Gildehaus & Liebendorfer highlighted how a group of pre-service teachers often experience being positioned as less valued in comparison to mathematics majors on university mathematics courses. Xenofontos et al. explored teachers' perceptions of the causes of marginalization in school mathematics in Scotland highlighting the dominance of social class in teachers' perceptions. Hummel & Bohlmann reported on pre-service teachers' understandings of diversity and their desire to acknowledge diversity in their future mathematics teaching but with limited knowledge of how to do so.

A third commonality between some of the papers was the recognition of diversity between students and how this might be both a challenge and a resource within classroom practices. In relation to gender, Foyn & Solomon focused on the challenges faced by a high performing girl, Sarah, whose experiences in the mathematics classroom are dominated by male performances of 'smartness'. The paper by Tiedke et al. focused on factors that influence the construct of low attainment prescribed by teachers - also highlighting the role of gender, in addition to self-concept and the quality of classroom management. A third paper by Ay highlighted differences between privileged and non-privileged students in their approach to modelling tasks outlining how more privileged students are able to unpack real world assumptions more readily when engaging with such tasks. Two papers also focused on recognizing differences between students as a resource for generating social transformation and change. Carrijo identified racial differences as a resource for investigation in the mathematics classrooms so that students may see their own lived realities in their mathematics activities. The paper by Ryan et al. focused on multilingual students' relocating of academic school mathematics across the home-school boundary - which, they argue, is a useful focus for pedagogic approaches that try to recognize home and community practices as a resource for learning mathematics.

Assessment was another theme that was addressed in two papers. Makrakis looked at how time and speed in national high stakes mathematics tests in Greece produces exclusion from mathematics. By contrast, Nieminen focused on an alternative framework that emphasizes students as co-designers of assessment (Universal Design for Learning), and explored how assessment frameworks may be designed to increase rather than hinder participation in mathematics and open up access for students with disabilities.

A larger group of papers investigated how research in mathematics education can produce social transformation both within the classroom and in society. Steflisch discussed teachers' perceptions of innovation in the mathematics classroom and categorized their responses into three types. The paper argues that those who struggle to stick to their pedagogic ideals rather than reverting to traditional pedagogic strategies may offer the most potential in terms of bringing about change. Lo Sopia et al. also focus on teachers' perceptions - but in relation to creativity in problem solving activities in the context of schools where there are high levels of student drop out. In addressing resistance to pedagogic change at a local level, Reinholz et al. discuss EQUiP - an observation tool which offers teachers/mathematics faculty with data on the link between social demographics and student participation in their own classroom as a tool for professional development. Plunger highlighted the necessity of learners' reflective processes for using mathematics to critique society - particularly, in relation to context orientated reflection. Buttitta & Di Paola discussed the concept of cultural transposition as a means to decentralise a didactic practice from a specific social and cultural context. Finally, Wright introduced the concept of socio-mathematical agency to critical mathematics education, which he defines as "the ability to use mathematics effectively to argue collectively for social change".

Another theme focused on developing critical thinking through mathematics education in ways that question socio-political bias and inequalities. Steffensen et al. presented findings on students' views of the pandemic that demonstrate their ability to identify and use mathematics-based argumentation as a means to question a range of social inequalities. Andersson et al. highlight the challenge in doing this, outlining how discourses regarding the necessity of mathematics to democracy and citizenship make the development of critical thinking with mathematics difficult. Kollosche focused on questions regarding the epistemic status of mathematical knowledge itself. He proposed the ‘styles of reasoning’ framework as useful to critical mathematics education since it can help highlight the socio-political bias of mathematical knowledge without dismissing its objectivity altogether.

Finally, several papers discussed the COVID 19 pandemic and the way it has made visible hidden inequalities produced and reproduced through mathematics education. Vosbergen highlighted how the pandemic created a mathematics teacher shortage in the Netherlands which manifests a break down or blurring of the distinction of public and private education leading to questions regarding the quality of teaching and de-professionalization. Abtahi et al discussed the ethical issues made visible by the pandemic in doing mathematics education research. Lastly, Applebaum et al. pinpointed the pandemic as an example of a dystopic crisis that should be embraced by critical mathematics education suggesting the need to appropriate the tools of dystopia for local and indigenous struggles.

Common conclusions and open questions

TWG10 historically is orientated towards perspectives and methods that are more visibly located in other related disciplines of reference but not yet established within the field of mathematics (Abreu et al.). There is a strong emphasis on critical social theories and the questioning and deconstruction of concepts that are often taken for granted in mathematics education more broadly. CERME 12 was no different in this respect - group discussions on the above papers led to questions around the epistemic status of what we might term as ‘academic mathematical knowledge’ and how mathematics circulates across institutional boundaries with everyday practice. What are the power hierarchies at work here? This led us to consider whether the pandemic has created further in/out relations in mathematics education which linked back to the first session and indeed the conversations held within TWG10 in the CERME11 ¼ pre-conference meeting.

Additionally, a key tension in the group was around modelling and its function in critical mathematics education. Clearly, global crises such as the COVID 19 pandemic and climate change are generating more interest in modelling as a way to develop awareness and action for social justice. But the group also questioned how far modelling a role plays in the hegemonic reproduction of injustice and inequality and how we might prepare teachers to discuss this with students. This leads to a broader question: are we, as mathematics educators creating the problems of injustice that we are trying to solve?

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Ethical awarenesses arising from data collection in mathematics education at the time of the COVID-19 pandemic

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Ethics is an important aspect of any mathematics education research. The two projects referred to in this report were designed to explore the experiences of migrant and/or Indigenous students and teachers in mathematics classrooms. Both were disrupted by the COVID-19 pandemic's shift from in-person to online data collection. The disruption led us to become aware of ethical aspects of the research that were not previously evident to us. We present and discuss several such moments of awareness, including the "shielding" role of school making some injustices invisible, and challenges of participatory online interaction.

Keywords: Research ethics, migration, ethical awareness, mathematics learning, mathematics teaching

Introduction

The presence of migrant or Indigenous pupils in mathematics classrooms can create educational and cultural vulnerabilities, as well as opportunities for transformation and adaptation. Generations of Indigenous students may have experiences of tensions in classrooms through acts of assimilation and colonization. Students and teachers can experience tensions of transformation and adaptation as they engage in classroom practices on the basis of what they know about their own and the community's history and culture (Gutiérrez & Rogoff, 2003). Within mathematics education, there is a long history of research focusing on challenges and possibilities experienced by migrating pupils (e.g., Planas & Civil, 2013) and Indigenous pupils (e.g., Meaney et al., 2012). In our two projects, in Canada and in Norway, we study the impact of migration and the history of colonialization on mathematics learning from the perspectives of students and teachers. Each project aims to examine potential richness afforded by students' multiple cultural, linguistic, and educational experiences of mathematics (education): e.g., different ways of doing, learning, and thinking about mathematics. The aim of the Canadian project is to promote intercultural dialogue. The Norwegian project aims to develop strength-based pedagogies. We will describe the projects and then elaborate on the data collection methods and how they changed as a response to the pandemic. Finally, we discuss some of the ethical issues which emerged in relation to this transition.

The Canadian project: Migration in Mathematics Classrooms (MMC)

More than 500,000 children in Canada come from migrant backgrounds (Statistics Canada, 2011). The Canadian study defines migrant students as those who, as a result of a change of residence, experience differences of culture and language in school. Thus, migrant students include those whose families have moved for economic or employment reasons, refugees, and Indigenous students whose families have moved between Indigenous settings and an urban school. We ask: how does migration affect students' and teachers' experiences of school mathematics? We designed the study to create multivocal, intercultural dialogue among families, children, teachers, and school systems, in relation to the learning and teaching of mathematics, by highlighting "unfamiliar" mathematics for teachers and for students following Tobin (e.g., 1989). Initially there were to be two phases of data collection. In the first phase, our research group would visit ten schools and work with five students in each school. These students would take the role of co-researchers, gathering accounts of their prior and current experiences of mathematics, from their parents, and in their mathematics classes. Near the end of our meetings, students would synthesize their collective experiences in the form of collages to communicate students' experiences with teachers. At the end of the group meetings, a round of interviews with each individual student would take place. The second phase would focus on teachers' experiences. We would share each collage with teachers from the school, to understand their responses to students' experiences of migration. Teachers, as co-researchers, would collect observations of their own practice, noting shifts arising from their new understanding of their migrant students' experiences. At the end of this phase, teachers from the ten participating classes would compare the collages created in their schools, their responses to them, and resulting shifts in practice.

The project was interrupted by school closures from the COVID-19 pandemic. We redesigned our data collection, shifting it online, but continued with a multivocal, dialogic, layered approach. In the redesigned project, the two data collection phases are maintained but their character has changed. In the first phase, we planned ten virtual math clubs, each with six children in, meeting. virtually for five sessions and engaging in both synchronous tasks and "take home" activities to work on with parents/siblings to share at the next session. At the end of each math club, we hold interviews with each student and, if they wish, a family member. The second phase will include six virtual groups of mathematics teachers who will meet twice. We will share and discuss the student participant profiles with teachers, as well as examples illustrating children's cultural-historical repertoires with respect to mathematics. Teachers will be asked to reflect on information shared, their reactions, and implications for practice. So far, we have collected data from two groups of children: 11 sessions of online mathematics activities with 7 children ages 8 to 12.

The Norwegian project: Mathematics Education in Indigenous and Migrational contexts (MIM)

The Norwegian project investigates challenges in mathematics education in times of societal changes and movements. The aim is to promote education responsiveness to diversity, and through participatory research methodologies develop and evaluate strengthening pedagogies. These are research-based pedagogies building upon participants' strengths and assets. They may be identified by examining past positive experiences; encouragement of hope and optimism and development of emotional satisfaction with the present (Seligman &

Csikszentmihalyi, 2000) moves away from cultural-deficit orientations and instead promotes achievement for all students. We address how different languages and cultures may either challenge or support the mathematics learning of two groups in Norway, Indigenous and migrated students.

Migrated students perform lower than their Norwegian peers on national tests in mathematics (SSB, 2019), as do Sami (Indigenous) students (Udir, 2018). Sámi mathematics teachers are decolonizing their school mathematics through developing culturally based examinations (Fyhn et al., 2016), interdisciplinary teaching (Nutti, 2013) and activities building identity and pride based on cultural heritage combined and language revitalization (Huru, Räisänen & Simensen, 2018).

We had planned to collect data in four rounds including: (a) conversational interviews, (b) video- and audio- recordings of planning and enacting of various practices and pedagogies, (c) collection of artefacts from discussions and classrooms (e.g., student work), and (d) collective discussions to reflect on the work together. Starting with the students, then teachers, and if required, school leaders, community members, we would start with the “core” and work our way out to the periphery. Moreover, because history and context are central to this work and to shed light on the other findings, the final phase of the project would include collection of relevant policy documents and media analysis.

This plan was interrupted by school closures in the pandemic. Our researchers were not allowed to visit schools for at least the first year of the project. Thus we flipped the order of data collection to start with a larger media analysis and online interviews with school leaders and community members. Participatory work features a democratic model that challenges traditional ideas of who can produce, own and use knowledge. It honours and grapples with tensions from bringing together ways of knowing and generates new forms of knowing (and being) (Morales, 2016). The pandemic interfered with this important principle for data and analysis.

Ethics in mathematics education

In each project, the research team grappled with the disruption of the pandemic and became aware of moments of interactions that led to ethical perplexity. A conversation among the teams emerged, focusing on ethics. Varela (1999) explains that acquiring ethical know-how is a:

> skillful approach to living […] based on the pragmatics of transformation that demand nothing less than a moment-to-moment awareness of the virtual nature of our selves. In its full unfolding, it opens up openness as authentic caring. These are radical ideas and strong measures for the troubled times we have at hand, and the even more troubled ones we are likely to have. (p. 75)

In the troubled time of the pandemic, we made changes to our practices. To remain ethical demanded ‘nothing less than our moment-to-moment awareness’ of ourselves as researchers and of our practices. In doing research, our approach to ethics is pragmatic and relational and promotes equity. It is pragmatic because we understand that rules and standardized procedures can lead to inequity and can even be unethical. As researchers, being ethical as we conduct research involves working to develop an ethical awareness and to make decisions with this awareness (Varela, 1999). The ethics are relational because our research involves relationships with participating students and teachers. Moreover, we understand that our practices as researchers are produced in relation to our understanding of our participants, and their responses are similarly in relation to us. In this sense, research ethics are dialogic.

Some ethical awarenesses

In this section, we report on some ethical awarenesses about relationships that arose as a result of the pandemic's disruption—in some cases, relating to new approaches to data collection, but in other cases, relating to broader aspects of the research. By nature, crises are unique. Thus, the ethical dilemmas arising for mathematics education researchers in this pandemic are unprecedented in many ways. Nevertheless, there has been some discussion (probably not enough) considering research in contexts of social conflict. Vithal & Valero (2003) pointed to the impact of conflicts on research questions and the theories that relate to the questions. So far, the pandemic has not changed the fundamental research questions in either of our studies, though it is changing the order in which questions are addressed in Norway. Further, we know that when we write papers we often adjust the research questions from the ones we foresaw. Thus, we expect some adjustment to our questions. Nevertheless, Vithal and Valero suggested a move toward critical theories and participatory research in conflict situations, and both of our projects already take these approaches.

Relationships with children and their families no longer mediated by school

In any research project, relationships are built between researchers and participants. We hope that these relationships extend beyond what is necessary to obtain data. The transition to online data collection made visible schools' roles in facilitating such relationship-building. If we had been visiting schools, the schools would mediate the relationship between researchers and students and their families. In Canada, we are now attempting an online approach that reaches out to families directly using the connections we have in the group—at least one participant school had an established relationship. The Norwegian project started later and has not been able to establish such relationships.

Given that the children are minors, it is appropriate that researcher interaction with them is mediated. Schools are well-positioned to do this normally because the system has experts who are at least somewhat familiar with educational research. The families and children may trust the researchers on the basis of their trust in the teachers who invited the researchers into their classrooms. When the mediation of the school is removed, then who plays this important role? Community organizations may feel ill-equipped to mediate in this way due to their unfamiliarity with this kind of research. In Canada, participating students have joined the project through their parents. What if students were not as interested in the math club as their parents?

Questions arise when schools do not mediate research relationship: What is the nature of a school's mediating role in relationships between researchers and children/families? If the school is not there (e.g., in this pandemic) how can/should researchers manage relationships with families and children?

Different mathematical knowledges are valued differently

Whose mathematical knowledge is highlighted? The Canadian data collection methods entail doing mathematical activities with students with a migration background. The Norwegian research addresses minoritized students which includes Norwegian Indigenous children whose voices rarely, if ever, are heard. While designing, conducting, and reflecting on mathematical exchanges, a tension emerges about whose mathematics is highlighted.

In the original Canadian project, the goal was to get insight into migrant students' experiences of the 'standard' mathematics taught in school. By participating as researchers in their own classes, their account of these experiences would be important data (rather than our own observations of the class). In the revised project, the mathematics activities are facilitated by members of the research team. The distance in the original design is lost. We acknowledge that these tensions cannot be resolved. We all bring our ideas and experiences of mathematics (our repertoires) to the project, and we 'see' participants' repertoires through our own. In the revised project, we are now more entangled in the interaction between different mathematical repertoires.

In the Norwegian project, the plan was for participatory research with schools in northern and southern Norway. We planned for close collaborations with students, practicing and becoming teachers and school leaders. We had planned to start the participatory work as the heart of the research, in close partnerships with students and teachers in their classrooms. However, the pandemic forced us to reconsider. Instead of being present in the classrooms: noticing, interviewing and observing for positioning and storyline analysis (e.g., Andersson & Wagner, 2019), we changed our plans and started from the periphery. We decided to analyze the societal discourses first through a media analysis, and second, zoom-interviews with all school leaders and some some of the teachers. Through the media analysis we identified positionings that were made available to migrated and Indigenous mathematics students in this public discourse through seven prominent storylines about youth from minoritized cultures and/or languages in Norwegian news media (Andersson et al., 2021). The ethical question we ask ourselves is, how will our knowledge about these storylines in the public discourse influence our data collection and analysis when we as researchers now move from the periphery towards the center, into these schools in the post-pandemic Norway? Careful ethical considerations are needed in every stage of the research to make it possible for us to avoid reproducing the public storylines in our analysis and research texts.

Relationships online

In both projects, we became aware of how the online environment has the potential to change relationships. In the Canadian project, as the location of data collection changed from school to the virtual space (as students and researchers are physically at their home), aspects of the role of school are made visible to us. School conceals differences among students; the four walls of the school appear to be shielding students from these differences and to offer a sense of security. Inspired by Fasheh's (1998) question, "Which is more fundamental? Outward peace or being true to our humanity?", we ask: What sort of an "outward peace" is created by schools and what becomes invisible by this portrayal of peace? What deeply rooted inequalities are given permission to be ignored in an "equal school"?

In Norway, zoom-interviews also show this approach. Some of our researchers are insiders, or at least known, in some of the northern and Indigenous communities where we do our research, getting interviews has been relatively easy. In (most) Indigenous communities where we do not have those types of connections this has been harder. This reminds us of the importance of establishing trust in research in tight communities. Experiences of colonization enhances the need for members of the communities to know where we as researchers are coming from and understand at least some of the underlying history and storylines (see Battiste, 2007). Our research is based on participatory practices, and we wonder how we can be truly participatory online? Are we able to establish partnerships and trust in the schools and communities during a pandemic and in post-pandemic periods? We also wonder to what extent the research incorporates the insider's perspective online, especially in the participatory context and with participants as co-researchers. How do we build trust in a video-conference medium?

We note that answers to this question will help us think about how we build trust, incorporate insiders' perspectives and take care of those participating in face-to-face interaction too. But now, we are interested in trust-building in the new mediums. How do we as researchers understand the context of participants if we cannot physically observe the context? As both projects' original plans were interrupted and we do not know the extent of online and virtual data collection in the new 'normal', we need to be concerned with how our understanding is impacted. We are becoming more aware of the impact of our observations outside the formal data collection on our interpretation Practices.

Concluding thoughts

Through this presentation, we have described some of the ethical awarenesses that have arisen for us as we reworked two research projects in the context of the pandemic. We have focused on awarenesses about relationships in mathematics education research, focusing on the role of the (absence of) schools, the impact on how different knowledges are valued, and the effect of the online environment on relationships. We hope that the conversations with the CERME12 community will not only help us reflect on the ethical dilemmas that we are facing, but will also open up a discussion about ethical issues related to data collection in cyberspace and beyond.

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Mathematics curriculum discourses on democracy: critical thinking in the age of digital traces

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The use of people's online digital traces has given rise to concerns for democracy. The digital traces may affect the individual's life in unexpected and negative ways. Such traces may also be of importance for understanding the spread of disinformation and the like. This paper reports on a Foucault inspired discourse analysis of the Swedish upper secondary mathematics curriculum. Two discourses are construed in the intersection of critical thinking, democracy, and this new technology. Skovsmose's concept of mathemacy is used to identify what is critical knowledge and what is not. The first construed discourse is, "With knowledge in formal mathematics, critical thinking on democracy will follow." The second is, "Rather a personal career than a critical citizenship." Neither of the discourses promotes a need for mathematics education to change due to new technology with regards to critical thinking.

Keywords: Critical mathematics education, digital traces, democracy, discourse, curriculum.

Introduction

We leave digital traces of our actions online. They are used in algorithms that update the feed in social media and provide us with search engine results. This enhances experiences and makes them more personalized. However, most users do not know the exact mechanisms behind what is seen on the screen. In conjunction with research across various fields, this gives rise to concerns about citizens' ability to critically assess mechanics of democratic elections, economic exploitation of vulnerable social groups, and the flow of (mis)information, etc. Our interest in this paper is in how this critical gaze is highlighted in school and in governing school documents, especially in mathematics education.

This paper reports on a discourse analysis of the current Swedish mathematics curriculum for upper secondary school, adopted in July 2021. The analysis is inspired by the viewpoint that teaching mathematics is a political activity due to its contingency and sociological impact (Gutiérrez, 2013; Valero & Orlander, 2018). Since society is in a state of constant change, there is always a need to assess whether the curriculum is in tune with contemporary societal challenges. The analysis takes a particular interest in potential issues for democracy related to the new technology of collecting and mining people's digital traces. By democracy, we mean a way of living together in a society with a shared belief in values such as equality and equity, and where different opinions can meet in debate without any hampering interference from misinformation. One incentive for writing this paper is to contribute to an exploration of how critical thinking for democracy can be nurtured within mathematics education, as in e.g., critical mathematics education (Skovsmose, 2005). The specific aim of the paper is to investigate curriculum aspects of the current situation of critical thinking in relation to democracy and this new technology. This could inform future explorations of what can be improved (C. H. Andersson, 2021). The research question is: what discourses can be construed in the intersection of critical thinking in relation to democracy and the changing role of mathematics in society due to the new digital technology?

Digital traces and democracy

The following examples are intended to illuminate some topics that we view as possible to draw on in a version of mathematics education which includes a focus on critical thinking in relation to digital traces and democracy.

Digital trace data is processed in algorithms to extract information that was never explicitly there. This fuels the new technologies of machine learning and artificial intelligence that are increasingly becoming a driving force in the economy (Villani et al., 2018). Data can be, and is, used to prey on vulnerable social groups with targeted advertising (O'Neal, 2016). If the workings of the new economy are opaque to most citizens, democracy has few regulatory controls to alleviate such ramifications. With knowledge of how targeted advertising works, citizens could also start to think critically about it in new ways. When machines learn from human behaviour, there is a risk that undesirable attributes flaws such as racism and sexism may be embedded in the algorithms and lead to discrimination (Villani et. al., 2018). If this happens to the algorithms that steer and select information online, these algorithms could become a structure that reproduces social injustice. The risk is exacerbated if the algorithms are believed to be neutral.

Social media can be used to change voting behavior (Bond et. al., 2010), which has been attempted on a large scale to support a particular candidate through the use of digital traces (Sabbagh, 2020). This means that the digital trace you leave can act as someone else's tool without your knowledge, and furthermore be used against your interests. It is debatable to what extent this is successful, nevertheless the strategic use of digital traces has become an integral part of election campaigning. Attempts to swing public opinion can also operate through personal digital traces since they cluster people into groups with similar interests. Such groups are used as levers in disinformation campaigns where group members' views are cultivated and directed (Starbird et al., 2019), demonstrating another way for the digital traces to act as tools for others.

Digital traces are relevant to take into consideration when understanding search engines. One example is how these attributes more relevance to clicked links (Shah, 2021). When sensational (mis)information is uploaded, the resulting clicks out of users' curiosity increase its visibility in new searches, which then result in even more clicks, and a relevance feedback loop that changes further search results. The interaction between human behavior and the algorithms can propel the visibility of the sensational, rather than the relevant. This knowledge may be the basis of a new perspective and critical thinking in relation to news found with search engines, for example.

Previous research in mathematics education

Critical Mathematics Education (CME) has a long history through the work of Freire, Frankenstein, Skovsmose, and others, of engagement in how mathematics education could support democracy (e.g., Skovsmose, 2005). There is research addressing topics around new technology and democracy, for instance how people meet new technology albeit being unaware of how mathematics operates in it (e.g., Straehler-Pohl, 2017), the need for mathematics education to address computer science (e.g., Borovik, 2017), and students' ethical reasoning when data science is discussed in mathematics education (Register et al., 2021). However, we have been unable to locate any research with a specific interest in mathematics in relation to critical thinking on the ramifications for democracy of digital traces, the specific interest of this paper.

Democratic values aimed at in governmental documents in Sweden

Democracy occupies a salient position in the Swedish Education act (Sveriges Riksdag, 2021), stipulating that education shall convey and anchor the "fundamental democratic values "on which Swedish society rests" (4 §, chapter 1, our translation). It is not specified, however, if this means merely conveying democratic values as such, for example how education is organized, or whether students furthermore should attain *specific knowledge* that supports the sustainability of democracy, such as through developing argumentative speech skills. Other steering documents and government agencies have the option to add such levels of detail. Concerning search engines, Sundin (2015) showed that the Swedish primary and lower secondary school curriculum treated them as neutral, and thereby hindered critical media literacy. More recently though, the Agency of Education has in different ways started to express the need for compulsory school pupils to be informed on how the algorithms of search engines work (C. Andersson, 2021). The mathematics curriculum subject to analysis in this paper is thus enacted in a time of increasing awareness of the importance of online algorithms, and is a part of a collection of documents that have the overarching of aim of conveying and anchoring democratic values.

Theory

This paper takes a dynamic view of theory, meaning that theory adapts to the questions asked. The paper coordinates (Prediger et al., 2008) Foucault's (e.g., 1995) framework with CME (e.g., Skovsmose, 2005). Coordination is achieved through using Foucault's framework for how knowledge, practice, and the dynamics of structure and agency are intertwined within institutions; while CME is used to pinpoint the characteristics of mathematical knowledge from mathematics education that, if the discourse allows, could benefit the development of critical thinking. Coordination of the two frameworks takes place only within the frames of this paper's research question.

Foucault's (2002) framework defines discourse as language use, norms, habits, artifacts, institutional praxis, etc. A discourse is a collection of such matters together with the discourse's rules of formation that act as gatekeeper for the creation of objects (including abstract) that can be a part of the discourse. There is also an enunciative function of statements which is the action performed by them (Foucault, 2002). Through the limits of what is said and acted, including what cannot be said within the discourse, discourses portray structure-agency dynamics and how knowledge and power are produced and linked. This can be related to agency in educational institutions or within an educational system (Boistrup, 2017).

The second theoretical framework has a CME perspective, since we draw on Skovsmose (2005) to introduce *mathemacy* as a competence that is a "reference to mathematics, in the broad sense of the term, but also reference to a notion like democracy interpreted as a way of living … [and] a capacity to modulate, and to see a situation as open to change" (pp. 187). Critical thinking is similarly defined in this paper as the *competence to think beyond the most common practice of the local milieu, realize the contingency of situations and evaluate alternatives*. The similarity enables us to use characteristics of mathemacy for critical thinking. Skovsmose (2005) addresses three distinct types of knowledge related to mathematics, rephrased by us as (1) mathematics knowing itself; i.e., dealing with mathematics notions (2) pragmatic knowledge; i.e., applying mathematics notions in different situations, and (3) critically reflecting on such applications including the consequences of different mathematical decisions in people's lives. The third knowledge type, reflective knowledge, is an important characteristic for the development of mathemacy, whereas the other two are not always required. This has affected the analysis, which is described below.

Methodology and Process

Data

The 27-page national mathematics curriculum for upper secondary school in Sweden begins with a preamble followed by the general aim of the subject. Then follow 11 course titles, each of which is described by two sections: *central content* to be taught and the *knowledge requirements* for each grade (The National Agency for Education, 2021). The 11^{th}, and last, course's central content does not stipulate any specific content but just give examples of possible content, and is excluded from our analysis. The ten remaining courses are divided into three parallel tracks, vocational education, social sciences and art, and science and technology. Many words and phrases in the central content are repeated across courses, or in the descriptions of grades within or between courses. The document is in Swedish. Quotes in this paper are translated to English by the authors.

Method of analysis

The analytic procedure was a back-and-forth process between three steps, though, for clarity, they are described as three steps following each other. The first step was to formulate codes. Codes were allowed to be intersectional to encompass interplay between concepts. The code *individual* was used throughout the analysis corresponding to when students were mentioned separately and in subgroups. Two codes were related to the first two knowledge types identified by Skovsmose (2005), i.e., (1) mathematical knowing itself and (2) applying mathematics. The code *to-do* encompasses limited and specific tasks within the realms of either of the knowledge types (1) or (2). The code *to-know* concerns mathematical knowing, either in (1) mathematics itself, or (2) in relation to applications of mathematics. Two codes were needed to pinpoint Skovsmose's third knowledge type, critical reflections of mathematical applications (called reflexive knowing). One of these codes is *to-judge*, which concerns all instances where there is a judgment of mathematical methods or other mathematical acts. The other code needed for fulfilling the reflexive knowing is *society*, which concerns instances where mathematics is connected to the wider context beyond school.

The second step was to use the analysis program Nvivo. Using the program's tool *cases*, the document was divided into overlapping sections: preamble, aim of the subject, central content, knowledge requirements, and the different educational tracks. Every piece of coding was thereby labeled with at least one case referring to its locality in the document. The outcome was analyzed by both quantification and interpretation. An example of the former is statistics on the simultaneous appearance of codes. An example of the latter is investigating whether codes tend to appear close to each other, e.g. in the same sections of the text. Another example of the latter is reinterpretation of statistics due to the repetitive nature of the text.

The third step was discourse analysis inspired by Foucault. This analysis was based on the first two steps of analysis, which gave information about patterns of what is written and what is not. This informed us about what the rules of formation of objects for a construed discourse could be. At this stage there were many alternatives for such rules and, consequently, for possible construed discourses. To reduce this complexity, we followed Foucault's (2002) suggestion to start with what appear to be tensions in relation to our research question. Discourses are broad and general in the sense that both sides in a conflict within an institution or a discipline can use them, or they can remain invariant across other kinds of tensions. Consistency was achieved by identifying an enunciative function for the statements, so that what first appeared to be a tension or contradiction was dissolved in light of how the rules of the particular discourse work. Each such consistent view on the research question, based on rules of formation of objects and specified enunciate functions for statements, became a construed discourse named with a phrase capturing its central theme.

Main findings and construed discourses

The construed discourses in this paper are (1) *with knowledge in formal mathematics, critical thinking on democracy will follow*, (2) *rather a personal career than a critical citizenship*, and (3) *society as an aim but not to be assessed.* Since the third discourse differs from the others with its focus on assessment, it is outside the scope of this paper and will not be described in the findings.

The first discourse, *with knowledge in formal mathematics, critical thinking on democracy will follow*, has formal mathematics as a vehicle to have mathematics education updated to the new digital technology and any related issues with democracy. The curriculum's preamble posits, "as society is digitized, mathematics is used in increasingly complex situations" (The National Agency for Education, 2021, pp. 1). However, the document never explicitly follows up on what this change means for mathematics education, nor does the curriculum make any statements that can be construed as being about critical thinking. One of the codes for critical thinking is *to-judge*, and it has a textual separation from the other codes. For instance, it has few simultaneous appearances with other codes, and it is separated by punctuation from other codes in a way that is not typical of the rest of the text. A similar textual separation of codes exists for the code *society*. This does not only mean that *to-judge* and *society* are separated from other codes, but it also means that they are separated from each other. Hence, the text does not connect the markers whose simultaneous appearance could have indicated critical thinking on a societal level. There is only one instance of a simultaneous appearance of *to-judge* and *society*: "[f]urthermore, the teaching must contribute to the students developing knowledge about the significance and use of mathematics in other subjects as well as in a professional, social and historical context" (The National Agency for Education, 2021, pp. 1). The coding of *to-judge* is too weak here for critical thinking, and derives only from the word "significance" (Sw. betydelse) as a value-laden word. Instead of explicitly describing critical thinking, or how the new digital technology is affecting mathematics education, the curriculum describes mathematics education in other ways. It is described as something that the students shall be able *to-do* (307 occurrences) and *to-know* (145), and to a lesser degree *to-judge* (57). When "judging" occurs, it is typically found in students assessing the plausibility of their answers in problem solving, which further emphasizes *to-do* as the most prominent since it concerns the outcome of students doing. The closest the text ever comes to students questioning, not only answers but also methods, is in the central content: "[e]valuation of properties and limitations of mathematical models" (pp. 3). However, there is no discussion of alternative models or any arguments for choosing between them. Overall, the text mainly describes mathematics education as something that concerns separate individuals and their ability to perform tasks. This can be seen as situated within Skovsmose's description of mathematical and pragmatic knowledge, with little reflective knowledge. This is underpinned by the analysis of the tandem appearance of codes, which clearly shows a strong link between *to-do* and *individual*. The occurrence of this pair is most common in the knowledge requirements. The absence of any explicit statements on critical thinking in relation to democracy, and the absence of anything on new technology after its mention in the preamble, have the same root in this discourse. They are interpreted as redundant by the in lieu salient position of performing mathematical tasks. In this discourse, it is sufficient for students to know formal mathematics. This knowledge will provide them with a sufficient toolbox, and since formal mathematics does not change, no further pointers on how to use it in the context of democracy or new uses of digital traces in society are needed. The rules of formation do not allow the creation of the object *student's mathematically guided critical thinking on democracy*, because that object is subsumed into the object *student's knowing mathematics*. The discourse does not in any way dispute the need for mathematics education to engage in critical thinking or democracy; on the contrary, the enunciative function for statements on mathematical tasks is to show how to do this. Formal mathematics resolves the issue!

The second discourse, *rather a personal career than a critical citizenship*, is orientated around tensions within the text. One tension from the viewpoint that steering documents may concretize overall aims, is that the code *society* first has a clear presence in the preamble, is reduced in the aim of the subject, only mentioned sparsely in the central content, and is non-existent in the knowledge requirements. Navigating through the steering documents from the overarching aim in the educational act down to individual students' learning goals seems to leave a trace of diminishing pronounced societal aims. The only other topic in the text that goes beyond the classroom setting is in mathematics for professional life. In contrast to the societal aim, however, the notion of mathematics to support a career is still clearly present in the central content. There is even a separate subsection only for this in the courses for vocational education; for other tracks there is specific content included that resonates with their education track, such as geometric sums (useable for calculations on loans) for economically oriented study-programs, and vectors (for physics) for science and technology. In relation to our research interest, this discourse *does not* promote the need for mathematics education to engage in critical thinking related to democracy, including in relation to any new technology such as use of digital traces. When the text mentions society these functions as lip service to the overarching aim in the education act. The enunciative function in this discourse for society thereby creating a hindrance to any actual engagement with it.

The two discourses described above can both be connected to previous research. The discourse, *with knowledge in formal mathematics, critical thinking on democracy will follow*, has similarities with Valero and Orlander's (2018) description of an implicit assumption in mathematics education of the intrinsic power of mathematics that automatically gives students access to powerful and universal reasoning. They describe it as an often dominant view in curricula formulation, and note that a tension has been identified in research between this view and views that center more on uses and applications of mathematics. The discourse, *rather a personal career than a critical citizenship*, resonates with Jablonka's (2003) description of mathematical literacy for developing human capital, which emphasizes economic growth for both the student and society.

Discourse analysis is an interpretive process that can take different routes. The analysis in this paper provides merely one way of viewing the mathematics curriculum. In this case, the construed discourses challenge the notion of a curriculum that is regularly updated due to new uses of mathematics in society. Taking the (self)regulation induced by discourses into account, a change towards a more vivid ambition to nurture critical thinking for the benefit of democracy in a rapidly changing society would therefore be held back by the inertia of these discourses. Therefore, it would be interesting to investigate if teachers and students also engage in these or similar discourses when reflecting on digital traces, and critical mathematics education, vis-à-vis democracy. In that case, it might be possible to identify if, and how, they engage in other discourses which may alleviate some of the official resistance to change.

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Actionable Mathematics Education Via Dystopia

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We suggest that Mathematics Education theory and practice can find new directions through recognizing their dystopic characteristics, and embracing them as both the source of challenges and method of response. This contrasts with the generally utopic approach of most scholarship. We offer critical ethnomathematics education as a model for such an approach, since it has its own origins in lingering dystopic legacies of colonialism. A perpetual hopelessness and disempowerment is one implicit curriculum of contemporary mathematics education, where the mathematics once learns might help to describe things, yet hardly assists in transforming the reification of power and agency in society. Embracing dystopia rather than trying to circumvent it helps us see that it is more crucial to re-think curriculum than pedagogy.

Keywords: Critical ethnomathematics, curriculum studies, ethnomathematics.

An Introduction

We began our study with the recognition that school mathematics does not necessarily prepare society to respond to the most pressing crises of our epoch: severe weather and climate change, refugees from war and climate change, human trafficking and global economic injustice, etc. In early 2019, the world seemed unprepared for the COVID-19 pandemic, whether individually or via social policy. In particular, the situation created by the pandemic transformed many aspects of social reality globally (Możgin, 2020), as well as school reality (Bond, 2020; Niemi & Kousa, 2020), spreading fear and insecurity, and making people feel unprotected. In general, we live in a time of acceleration (Virilio, 2012; Rosa, 2013), with precarious scenarios of evolution; yet the daily experience of the COVID-19 pandemic, in and out of school, leapfrogged doomsday pronouncements away from mythology or Revelation, and into a visceral idea of the end of the world. The focus on the pandemic can also be understood as a primarily "first world problem," in the sense that there were and continue to be far more urgent crises outside of the highly industrialized nations, in what is sometimes referred to as the "global south." This aspect of crises helps us to think simultaneously about whether school mathematics prepares humanity for anticipating and responding to global crises, and at the same time whether it enables humanity to recognize the legacies of colonialism that influence what is even appreciated as a crisis in the first place.

We apply strategies from "critical ethnomathematics education" for understanding parallel concerns of equity (opportunity and outcome) and curricular content choices. We do this because critical ethnomathematics has already demonstrated techniques for educators to theorize paradoxes of global crisis. "Critical ethnomathematics education" makes sense of mathematics education, both for preparing people to anticipate and respond to global crises, and for practicing pedagogies that address seemingly insurmountable, “dystopic,” crises in our environment, geopolitics, and the future of our planet as ever-present yet necessarily livable.

Social and Political Mathematics Education as Context

Ole Skovsmose (2021) proposes three types of relationship between mathematics and crises: Mathematics can (a) picture a crisis, (b) constitute a crisis, or (c) format a crisis. Are there parallels for mathematics *education*? Even as numerous mathematics educators would agree that significant cultural, historical, and political knowledge is needed to make sense of complex global issues (e.g., population movements of refugees and migrants (Xenofontos, 2015; Planas & Gorgorió, 2004), climate change (Barwell, 2013; Appelbaum & Stathopoulou, 2020), species extinction (Radakovic, et. al, 2018), local community problems (Gutstein, 2006; Hunter & Sawatzki, 2019)), typical school curricula worldwide is primarily formal, decontextualized (academic) mathematics, a manifestation of Western thought, Cartesian logic and ‘rationality’ rooted in a logic of domination and human-centric thinking, sometimes termed a logic of domination (Warren 1990). In this conception, purpose lies in generalizability, wide applicability to decontextualized concepts/methods, and in structures that manipulate variables across specific cases. This bird’s-eye view of human experience and deeply-rooted cultural patterns undergirded by Western assumptions of domination over nature are increasingly recognized as creating interrelated challenges of climate change, extreme weather, food production, and species extinction, demanding new directions of relation centred in social justice and alter-global social movements (Warren 1990, Appelbaum, 2018). Persistent dichotomies (e.g., people of Western culture(s) and Others) project one of the pair as the epitome of progress (despite its ignorance of other knowledge traditions and practices): mathematics education is seen in this was as a tool of power, oblivious of its failings (Gerofsky, 2010). Educational experiences curate forms of knowledge and exclusion, function as processes of normalization and epistemicide (Santos, 2007; Paraskeva, 2020), and structure the identification of differences across teacher, student, family and community cultures.

The catastrophic COVID-19 emergency demonstrated these processes of normalization and epistemicide more clearly than pre-COVID mathematics simply because it placed ongoing learning and teaching practices in a new context. Initial discussions -- politicians, media, medical experts, educators -- often amplified feelings of confusion and disempowerment. Some well-intending mathematics educators seized on the “teachable moment.” Remote education simplifies some aspects of instruction, complicates others. Some mathematicians and mathematics teachers used the virus to make mathematical concepts and skills accessible (Sullivan et al., 2020). An unprecedented number of blogs, websites, news articles, Tik-Tok videos, Instagram feeds, etc., used visual representations and analogies to explain exponential growth, the nature of meaningful evidence, model mathematical inquiry, or demonstrate the importance of mathematics in the study of a global pandemic. That is, we lived through parallel experiences: for school mathematics, the questions mostly became a matter of how teaching will be continued through distance learning, not a moment of curricular reconsideration; for public pedagogy, this was an explosive moment of graphs, metaphors, and a contestation of knowledge, demonstrating the superiority of social media over school in making mathematics relevant. Although public pedagogues successfully provided resources, and although there are many ways in which they created examples of how mathematics can help the public to understand their situation, feel informed and to witness themselves as in control of knowledge, we ask whether this was nothing more than a shifting of focus from school to popular culture of a more insipid and disempowering form of education, constructing a dystopian version of knowledge and knowing. We claim most of our many resources on using mathematics to solve real-world problems are caught in the trap of social and cultural reproduction, despite their claims to a certain overarching 'natural goodness' (Swanson, 2017). COVID-19 is an example: the majority of school and public pedagogy mathematical lessons focused on mathematical models of the behaviour of the epidemic, and not on a broader framework for interpreting the models of the world, or our experience of it (regarding modelling and problem solving, see English & Watson 2018; Kaiser & Brand 2015; Houston et al. 2010; Lingefjärd & Meiter 2010). Such public pedagogy (Appelbaum, 1995) mirrored standard textbook approaches –simplified, artificial models, glossing over details, confounding variables -- even as it dressed up key concepts and relationships in engaging video and animation. The more general observation is that the focus on the pandemic distracted from the enormous crises around the world that existed pre-pandemic, and continue to this day, inadequately addressed. This latter point indicates one more way in which mathematics education and its impact on problem generation and solving can have far-reaching consequences for what becomes the focus of attention, reproducing and amplifying global legacies of power, as well as related assumptions about what is a universally agreed-upon "urgent need."

Embracing Dystopia

Critical mathematics education embraces coloniality as both the problem and the method of social change; mathematics education can embrace coloniality and dystopia as its problem and method. Rather than searching for solutions to the legacy of colonialism, critical mathematics education recognizes the dystopia of coloniality as here to stay, and appropriates both western mathematics and the coloniality of school mathematics as its own tool, not for "dismantling the master's house," but for accomplishing local and indigenous goals of dignity and reconciliation (Appelbaum & Stathopoulou, 2020). This contrasts with the pursuit of utopian dreams. Those working to implement curricular reform imagine a post-dystopic vision; they try to overcome dystopia, and are doomed to failure no matter what gestural leaps they attempt. As Stein, et al. (2020) argue, "decolonization is increasingly treated as a site and subject of consumption and accumulation, not only of material benefits, but also of knowledges, relationships, experiences, and even critique itself" (p.44). This is why we urge avoidance of "decolonial critiques,' as is fashionable academic currency, arguing instead against solutions and alternatives to colonization within existing paradigms, regimes of property, and comfort zones. We understand "colonial patterns of relationship and colonial habits of being are reproduced at the very moment they supposedly become unsettled … when efforts made under the umbrella of decolonization are re-routed back into the same desires and entitlements that produce colonization in the first place," so that "the transformative possibilities and ethical responsibilities of decolonization are eclipsed, and decolonization itself becomes weaponized as an alibi to continue colonial business as usual." Fantasizing a possibility of decolonization is a fallacy, as is curricular reform. It is better to appropriate methods and resources of dystopia for alternative, local goals. In has sometimes been coined 'creolization' (Appelbaum, 1995, 2008).

This approach critiques reform efforts in general as typically enacting a utopian-fueled fantasy of leaping out of the current dystopia (see Popkewitz 2018, Lim & Apple 2016). A caricature of reform efforts would describe policy-makers as saying, “Oh, this didn’t work. Let’s try something else, which would be so great!” Instead of imagining utopias, designing them and promoting new curricula, teachers, and students to act as utopian characters, we urge the following: start with acknowledging that the current curricula, teachers, students, and policy-makers are currently actors in a dystopia. Instead of trying to escape that dystopia, we could appropriate tools of the dystopia for local and indigenous struggles. The dystopia is both the problem and the method. Even as we pursue local and indigenous appropriation of tools and practices of coloniality for locally-identified purposes, we recognize the problematic term “indigenous” as preserving and maintaining distinctions of colonialism. One might say that we are perpetuating the dystopia. A ‘composite’ definition of both terms, utopia and dystopia, understands how most utopias are linked by their commitment to a form of enhanced sociability, or a more communal form of living, sometimes associated with ideals of friendship, while their dystopian counterparts are substantively connected by the predominance of fear, and the destruction of ‘society’, as a polar opposite of friendship. Perhaps we can reframe indigenous in this sense of an ‘enhanced sociability;’ where enhanced sociability has been maintained for some period, “utopia” has been lived to some extent (Claeys 2013); and where the opposite occurs, “dystopia” is the relevant descriptor. Fundamentally, utopia and dystopia are mutually determining.

Mathematics Education in the time of war in Ukraine, in the time of COVID, in the time of mass migration from severe weather and famine … are here to stay, as is mathematics education in the time economic inequality, the breakdown of democracies ... Mathematics and mathematics education are at the heart of each crisis, serving at once as forms of knowledge with which we describe and come to know each aspect of the global crises together, and in erasing alternative forms of knowing and coming to know about our world and its future. Inherited from critical mathematics education is the key concept, “abyssal gap of coloniality” (Santos 2007) -- that separation literally and epistemologically between metropolitan and colonial societies. Even today, people who live in or whose origins are in former colonial countries -- and women, refugees, etc. -- are framed as inferior by structures of coloniality. This distinction also concerns people’s knowledge, a distinction that ignores the intrinsic value of various bodies of knowledge in favor of dominant social, political and economic structures. This knowledge is excluded and essentially erased, because the people who produce this knowledge are excluded as creators and finders of knowledge. Epistemological clashes between different kinds of knowledge, in particular, between scientific and non-scientific forms of truth, are only recognized once one takes a critical stance. One kind of knowledge—counting as true—is on the one side, while the ‘other side’ is relegated to mere “beliefs, opinions, intuitive or subjective understandings” that at best are issues for scientific inquiry. In this way “abyssal thinking” consists of distinctions and dichotomies that construct a divided world (Santos 2007, p. 45). It is supposed that people stay in a static situation – the dystopia does not change placement relative to the abyssal line. Yet people are also different, even if they share elements that define them on one side or the other (López, 2019; Benson, 2019). “Kinds of people are cemented through research and administrative apparati, but also through uprising and revolt" (p. 164). People are constructed in these kinds of ways for governmentality's purposes, but at the same time, since people are not just the object of static nominalism (Hacking, 2006), and not merely "passive receivers of imperial administration and control," they react in resistant ways.

Mathematics education expresses coloniality and other characteristics of power relationships through languages of accountability and global economic competition. Mathematics education is both experience and cause of this dystopia. We can explore ways to open imagination, giving learners spaces for creativity and knowledge of self, and in using problems posed by the students themselves; such a utopian imaginary of mathematics education can challenge the perspective of globalization, exploring for example alter-globalization (Appelbaum, 2018)–where there is space for solidarity, participation, self-determination, dignity, and reconciliation. However, there is no magic in pursuing utopia, as in contemporary rhetoric and its dreams of "Mathematics for All", "Life skills", "Citizenship", "Problem Solving," and "Mindsets." Language such as deregulation, climate, and inequality, relegated to those ways in which people "solve problems," establishes the abyssal gap between politics and life (Latour 2018). a language of culture, survival, justice, existence, land, and land reform might describe what is at stake with necessary clarity. Such language is central to critical ethnomathematics. What would mathematics education look like if culture, survival, justice, existence, land and land reform replaced numeracy, life skills, problem solving and mindsets in our rhetoric, framing of research and practice, policy documents, to dwell in dystopia rather than fantasies of various utopias?

Latour (2020) proposed a within-dystopia response to the pandemic: "Let us take advantage of the forced suspension of most activities to take stock of those we would like to see discontinued and those, on the contrary, that we would like to see developed." We advocate an analogous approach for mathematics education through the following questions, paraphrased from Latour.

What are the mathematics education activities, in and out of school and remote school learning, now suspended, that you would like to see not resumed?

Describe why you think those activities are harmful/ superfluous/ dangerous/inconsistent, and how their disappearance/suspension/substitution would make the activities you favour easier/ more consistent. (Make a separate paragraph for each of the activities listed in the first question).

What measures do you recommend to ensure that the workers/employees/agents/entrepreneurs who will no longer be able to continue in the activities you are removing find support for their transition toward other activities?

Which of the now-suspended activities would you like to develop/resume or even create from scratch?

Describe why your newly developed or resumed activities seem positive to you, and how they make it easier/ more harmonious/ consistent with other activities that you favour, helping to combat those that you consider unfavourable. (Make a separate paragraph for each of the activities you list).

What measures do you recommend to help workers/ employees/ agents/ entrepreneurs acquire the capacities/ means/ income/ instruments to take over/ develop/ create these favoured activities?

In those places in the world where COVID-19 was experienced as the most urgent crisis, there were three main mathematics-related trends: (a) The greater need than we might have previously realized for the wider public to comprehend and interpret the mathematics behind models, graphs, etc., related to the pandemic, and in general as preparation for any crisis. (b) Perpetuating the same curriculum that did not prepare people for understanding the crisis in the first place even as schools focused during the pandemic on how to make (mathematics) teaching more accessible. (c) Inconsistencies between what the public needs and what schools are doing. To understand why these trends unfolded, we analysed school mathematics and mathematics education as dystopia. Critical ethnomathematics is a model of how to embrace dystopia rather than to try to overcome it or avoid it. Critical ethnomathematics education is an approach that addresses dystopic elements of contemporary mathematics education practice while centering attention on coloniality and the need to exploit traditional school mathematics in ways that serve local cultural, political, and environmental needs, in a broader, ethnomathematical commitment to local and indigenous mathematical practices.

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Social (in)equality through mathematical modelling? – Results of a case study

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Solving realistic mathematical tasks with multiple possible solutions can require many competencies. At the same time, they can allow students to engage with a situation mathematically according to their own preferences. Previous studies seem to indicate that – through socialisation – socio-economically privileged students are more likely to acquire skills dealing with such tasks. This paper approaches the described issue qualitatively by comparing modelling processes of privileged and unprivileged student pairs. It turns out that privileged pairs, on average, spend more time on making real-world assumptions and they show a broader spectrum of assumptions compared to unprivileged pairs. Thus, it is discussed to what extent differences and similarities can be traced back to students' habitus and how modelling tasks may thus increase both social inequality and social equality.

Keywords: Social (in)equality, mathematical modelling, qualitative content analysis, Bourdieu.

Introduction

In a variety of studies social background is associated with school success (e. g., OECD, 2019). This can be explained, among other things, by parents who are more or less likely to support their children's learning financially, culturally, socially and psychologically (OECD, 2019). Moreover, certain characteristics tend to be taught in socio-economically privileged households, such as argumentation and communication, discussing learning strategies or perseverance in problem solving (e. g., Weininger & Lareau, 2009). In the German educational standards for mathematics (KMK, 2004) such characteristics are described as competencies students are meant to develop. It seems that privileged students internalise a habitus through socialisation (Bourdieu, 1994/1998), which rather finds acceptance in school. Furthermore, in mathematics class students should apply their knowledge in real-world situations in order to be able to view natural, social and cultural phenomena from a mathematical perspective (KMK, 2004). The role of the social background when dealing with real-world tasks is, thereby, controversially discussed. While, for example, Piel and Schuchart (2014) show social class differences to be more likely to occur for reality-based tasks than for purely mathematical tasks, Ay et al. (2021) find socio-economic status to be less strongly related to the solving of modelling tasks compared to other tasks. This paper focuses on this field of research from a modelling perspective, by analysing and comparing modelling processes of privileged and unprivileged student pairs qualitatively (Schreier, 2012). Using Bourdieu's habitus, it thereby shall be discussed to what extent differences can be explained by students' socialisation and in what way results from other fields of research can be confirmed.

Theoretical framework

This raises the question of why socio-economic status can be related to certain school behaviours. For instance, the number of books on the shelf cannot yet explain how social inequality occurs.

According to the sociologist Pierre Bourdieu, there is a connection between one's socio-cultural position and the individual lifestyles. A mediator, Bourdieu calls *habitus*, stands between social position and individual patterns of thought and action. Hereby, one's habitus represents the disposition towards the world, such as habits, ways of life, attitudes, values, aesthetic standards, etc. According to the habitus-theory, most actions of individuals do not pursue an intention, but are an expression of their acquired dispositions (Bourdieu, 1994/1998). The habitus creates a scheme that individuals use to classify and judge other situations, objects, persons, actions etc. – and ultimately themselves. However, individuals usually do not consciously activate these schemes. Clearly, people from the same social class do not completely match in their habitus. Nevertheless, they are most likely to have had more similar experiences and, thus, match more in their behaviour than people from other classes (Bourdieu, 1972/1977). Thereby, a person's habitus can often be detected in little things.[1]

Social (in)equality and the processing of tasks

When addressing social (in)equality, a variety of factors such as migration background, language, gender and school systems, can be relevant. In this study, socio-economic status is considered as a central distinguishing characteristic since it measures family resources and their social position and thus aligns with Bourdieu's concepts. For some time now, empirical studies based on sociological theories regard how children process tasks. In one study, children are given 24 pictures showing food. They are asked to group the pictures so that they fit together well. It turns out that the unprivileged children sort the pictures more often according to their own experience, such as 'tastes good', while the privileged children are more likely to sort the pictures according to more abstract criteria such as 'vegetables' (Holland, 1981). Similar results can be found in other studies. Students are shown sketches of two tables showing pizzas and seats. On one of the tables, there is one more pizza, but also two more people can be seated here. Students are asked which table they would join and why (Lubienski, 2000). In her study, the unprivileged students are more likely to focus on real-world concerns (e. g., arriving late) instead of using the task to learn generalizable methods (using relations to make comparisons). The author states that this experience-based orientation could hinder unprivileged students in understanding the mathematical ideas behind the situation. On the other hand, the author finds several instances of unprivileged students being concerned with getting the algorithm that solves the task, getting frustrated and giving up more quickly when facing barriers.

However, looking at social diversity in a fruitful way also entails considering the needs and strengths of unprivileged students. Previous research suggests that it might just be topics that are problematic, communicative and relevant to students' real world that enables them, regardless of social background, to participate in the classroom according to their own abilities and experiences (Nasir & Cobb, 2007).

[1] Bourdieu's habitus-theory is often accused of determinism. Bourdieu argues that individuals can be free and creative within a certain frame. According to him, certain life courses are not predetermined, but more or less likely.

Mathematical modelling

Mathematical modelling can be defined "as the solving of a realistic problem" (Maaß, 2010, p. 288). Modelling problems are often accompanied by authentic situations, missing relevant information and multiple possible approaches. During a modelling process students need to identify and collect relevant information, translate a respective situation into mathematical terms, structures and relations, work within the mathematical model, interpret and check results with respect to the corresponding situation (KMK, 2004). This distinguishes modelling from embedded tasks, as embedded tasks use contexts from the real-world, but "have no real relation to reality. The factual context is of no importance regarding the solution" (Greefrath et al., 2017, p. 933). Following the rules of the game of embedded tasks, students can usually be successful in the mathematics classroom if they ignore the context of a situation, use recently learned formulas and don't question the motivation as well as the action of involved persons (Verschaffel et al., 2000). When approaching modelling tasks, these rules hardly apply. Instead, real-world~~s~~ assumptions and everyday knowledge are crucial for modelling.

Synthesis and research questions

Now, why should one draw on Bourdieu's concepts of habitus when discussing social (in)equality in mathematical modelling? The metaphor of a game might clarify this. When playing a game, players need a practical sense or a *feel for the game*. That is the "mastery of the logic or of the immanent necessity of a game – a mastery acquired by experience of the game, and one which works outside conscious control and discourse" (Bourdieu, 1987/1990, p. 61). This construct coincides to some extent with the game in mathematics classroom (Verschaffel et al., 2000). Both have rules to be followed in order to be successful, have an inherent logic that is not revealed and are acquired through experience. Looking at previous empirical findings, it seems that many unprivileged students lack a feel for the game when they draw tasks with (seemingly) multiple possible solutions on everyday experiences rather than more abstract constructs such as the mathematics beyond (Cooper, 2007). According to the "habitus as the feel for the game" (Bourdieu, 1987/1990, p. 63), different immanent rules apply in different fields so that habitual behaviours are manifested in social classes. Bourdieu (1984) describes the habitus of lower social classes as a *taste of necessity*, focusing in everyday life on practical, functional and technically necessary things, on conformity and on immediate satisfaction of needs. Such taste may thus become apparent when observing students organising food according to their desires (Holland, 1981) or discuss planning to meet their friends (Lubienski, 2000).

It remains unclear, to what extent these concepts can be applied to mathematical modelling. While the inclusion of reality-based experiences is highlighted in other studies as being disadvantageous for task processing, it might just be seen as an advantage in modelling. For example, assumptions are essential and, thus, a real-world model is still being developed. Therefore, characteristics attributed to unprivileged students might be advantageous here. Still, a wide set of competencies which can be seen as useful for modelling has been attributed to privileged households in various studies (e. g., Weininger & Lareau, 2009). This leads to the following research questions:

To what extent do socio-economically privileged and unprivileged pairs differ in making assumptions in the modelling process? To what extent can conclusions be drawn about students' habitual behaviour?

Methodology

A qualitative approach is chosen with the aim of understanding, reconstructing and interpreting contexts and processes. 24 tenth-grade students (around age 15) from two urban secondary schools in western Germany belonging to four different classes partake in this study. The student body of both schools is considered culturally diverse and heterogeneous in performance due to schools' locations and concepts. Pairs of students are shown a picture of a giant pizza (www.bit.ly/2WRXOSU) and get the information that they are planning a party for 80 people and the task to figure out how many pizzas to order. The context comprises a realistic situation and the giant pizza can be ordered in the students' home region and, thus, may have a connection to their everyday life. To enable drawing conclusions, the pairs are put together according to their socio-economic status. Therefore, student and parent questionnaires are carried out for measuring students' HISEI (*Highest International Socio-Economic Index of Occupational Status of both parents*). The HISEI is determined by the professions of the parents and takes income and educational level into account (Ganzeboom et al., 1992). Students who fall into the upper quartile of the nationwide comparison of the HISEI are considered socio-economically privileged. Reversely, students from the lower quartile are considered unprivileged. The survey is divided into the three phases of observation, stimulated recall and interview. A qualitative content analysis (Schreier, 2012) serves as foundation for the data evaluation. Therefore, all phases are being recorded, transcribed, and coded. The evaluation follows a quantitative-qualitative approach by comparing categories quantitatively, identifying qualitative differences and supporting concise differences by means of transcript excerpts. To ensure coding quality, codings were carried out and compared by two independent persons. In addition, inductive subcategories were developed based on the research interest. For example, processes that serve to develop a real-world model are divided into subcategories like *simplifying*, *organising* and *assuming*. Simplifying contains e. g. repeatedly reading text segments or identifying missing information. Organising is often reflected by students drawing or measuring, and assuming entails that assumptions and premises are set or estimated using everyday knowledge. Whereas simplifying represents a surface-level processing strategy, organising and assuming represent deep-level processing strategies (Schukajlow et al., 2021). This paper will present some results on these subcategories. For readability and recognition, privileged students are assigned a three-syllable name and unprivileged students a two-syllable name.

Results

The modelling task Giant Pizza allows multiple possible approaches and solutions. The students can decide individually, which meaning they attach to the photo, to what extent they estimate using everyday knowledge and objects of comparison and to what extent they use mathematics to solve the task. Accordingly, a wide variety of approaches are found. Firstly, there are five pairs (Dominik & Krystian, Vivien & Oliver, Tobias & Benedikt, Samuel & Nathalie, Dawid & Leon) who choose a mathematical approach by estimating the diameter of both the giant and an ordinary pizza, determining their areas, comparing them and generating a real result. Secondly, some pairs (Julia & Florian, Samuel & Nathalie, Michael & Paulina, Kaia & Mila, Ronja & Hürrem) choose an extra-mathematical procedure by dividing the giant pizza visually and concluding the number of giant pizzas to be ordered. Thirdly, there are pairs (Amba & Bahar, Lena & Pia, partly Sofi & Aram) where the result is rather guessed. Regardless of their approach, all couples come to a final solution.

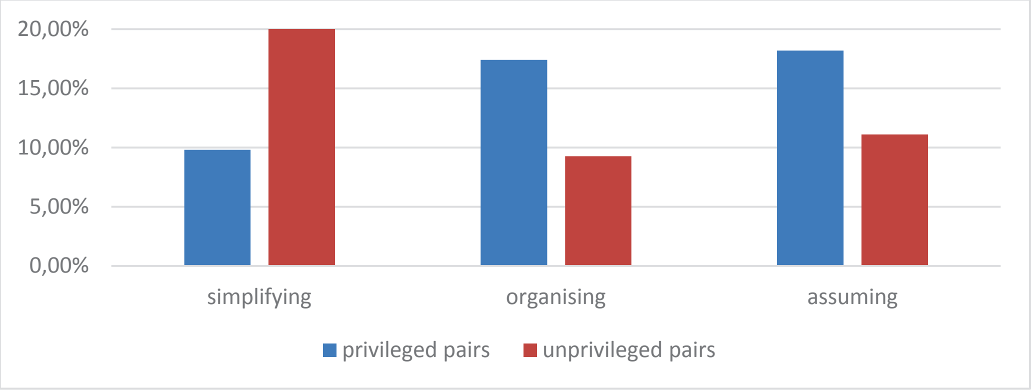

Figure 1: Contribution of the categories

Figure 1 shows that privileged and unprivileged pairs differ regarding their processes. On average, the unprivileged pairs deal more extensive with simplifying processes, whereas privileged pairs put a stronger focus on organising and assuming. In many processes, estimations play an important role, although not all students use them as part of a solution approach. Only the five initially mentioned pairs use everyday knowledge to estimate the diameter of the giant pizza. As objects of comparison, they use parameters such as the height of people or the dimensions of trailers. Additionally, there is a variety of other estimates and premises that pairs consider relevant (see Table 1).

Table 1: pairs' everyday knowledge and objects of comparison

… observed only in unprivileged pairs	… observed in both groups	… observed only in privileged pairs
size of the angle of a pizza slice	diameter of an ordinary pizza; pizza per guest; width of a person; gender / age of the guests; comparison with the school class	height of a person; size of a family pizza; size of a salami slice; length and span of a hand; length of a forearm; size of a pizza plate; dimensions of a trailer; size of a bun; thickness of the giant pizza; duration of a party; other foods at the party

On average, the privileged pairs do not only spend more time on assuming, they also show a broader spectrum of assumptions. Some are used to compare sizes, others for visualisation and others for validating the results. Five pairs use visualisations to generate a real result. Two of them (Julia & Florian, Michael & Paulina) thereby express objects of comparison, such as the hand of the women and the size of a pizza plate, to develop their model. Taking a look at the process of Kaia & Mila shows that they also use visualisation as basis for their approach. "How many pizzas do you think you can fit in here, normal ones? […] 1, 2, 3, ((tracing circles)) 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, shall we just say 20?" (Kaia, 04:42) As other students, Kaia carries out the visualisation without sketching. The pair does not estimate the diameter of the giant pizza and does not express objects of comparison to find out the number of ordinary pizzas that match the giant pizza. In the interview Kaia explains: "And there's the picture, then I have a rough idea of it and can put a guess in there accordingly.” (01:18) Even though she does not include any objects of comparison explicitly, she seems to have a sense of the size. Instead, she expresses other assumptions:

02:40	Kaia:	[...] if you're planning a party […] you have to assume that people are our age.
02:54	Mila:	Okay.
02:55	Kaia:	[…] if I we’re planning a party (.) I would do about half boys, half girls, right? […]
03:07		Let's say then that women usually eat a little less than men […]

Kaia thus relates the task to a situation that is close to her life: “It says YOU and a party with 80 people. […] people at a young age, they certainly invite people from their grade or from some hobby area, football or something else.” (03:07, stimulated recall) Other pairs are rather guessing a result. Amba & Bahar, for example, keep discussing what the intention of the task might be. Bahar explains what to do instead of calculating. “I don’t think we’re supposed to calculate anything there [...] we’re just supposed to say how many pizzas, whether that’s enough.” (03:22) Asking Amba about this during the stimulated recall she explains: “it’s a task where you just have to estimate and not really calculate. You can tell by feeling how much you need or don’t need. […] You don’t have to think much.” (01:16) Although they make a few assumptions and visualise to some extent, they mostly go back to repeatedly reading the task to find some hidden information. In the end, they guess a result.

Discussion

The aim of this paper is to uncover differences and similarities in the modelling processes of socio-economically privileged and unprivileged students as well as to draw conclusions about habitual behaviour. For this, 12 pairs of students work on the modelling task Giant Pizza. All pairs are organising or assuming in some way, for example sketching or using everyday knowledge. Nevertheless, the pairs differ noticeably in how deeply they engage with these processes and whether those lead to a further development of their models. The work assignment of the modelling task does not indicate that estimation is to be done using everyday knowledge or that mathematisation should be carried out. It stays hidden, to what extent reality is to be considered. Five pairs choose to calculate and compare the areas of the giant pizza to an ordinary one using estimation and everyday knowledge. Four of those pairs are assigned to the socio-economically privileged group. Five pairs divide the giant pizza visually and conclude the number of pizzas to be ordered. Here, all of those who expressed objects of comparison for their visualisation are socio-economically privileged. This is also reflected in the quantitative comparison. The unprivileged pairs engage more in reading repeatedly and talking about the relevance of the text elements, whereas the privileged pairs spend more time organising and assuming. Further, the processes of privileged pairs are characterised by a wider range of estimates and objects of comparison, which they use to develop their models and verify their results. It seems like surface-level processing strategies (Schukajlow et al., 2021) are rather used by unprivileged pairs while the privileged pairs tend to focus more on deep-level strategies. Relating these results to Bourdieu's game (1990), there are inherent necessities or logics in the modelling task. It seems that the privileged pairs are more likely to recognise them (as stated by Cooper, 2007). These necessities are contrary to the rules that usually apply for embedded mathematical problems (Verschaffel et al., 2000), where the situation has no relevance for processing. Those school socialised routines do not apply here, and thus habitual differences may be meaningful in explaining the observed differences.

With regard to the taste of necessity of lower social classes (Bourdieu, 1979/1984), the process of Kaia & Mila in particular stands out as one of the few unprivileged pairs who intensively deal with assumptions and everyday knowledge. While most privileged pairs estimate measurements and use objects of comparison, Kaia & Mila make assumptions regarding the organisation of their own party, for example the gender balance, the age of the guests and the leisure activities people are invited from. In accordance with the findings of Bourdieu (1984), Holland (1981) and Lubienski (2000) one could say, that the approach of Kaia & Mila conforms to everyday useful purposes. A notable difference to Lubienski's study however is that she considers it less appropriate when students refer to everyday experiences. Here, using everyday knowledge is not inadequate, but conversely, of central importance. Yet, differences are apparent in the use of everyday experiences.

Nevertheless, socio-economically unprivileged pairs appear across all described approaches. Besides, even though they are less likely to bring in estimates, a few seem to have a sense of dimensions of objects of comparison. Also, all pairs still achieve a final result, although the approaches vary in depth and adequacy. Unprivileged students who are more likely to give up (Lubienski, 2000), cannot be observed – despite frustration. Further, three unprivileged pairs use adequate mathematical or visual approaches to find a result. In addition, also pairs who follow a guessing approach show organising and assuming to some extent. Due to mathematical errors or misconceptions, such processes remain partly infertile or get stuck.

At this point, some aspects should be discussed critically. Bourdieu does not operationalise his constructs essentially, and his empirical findings refer to France in the 1960s. Thus, his constructs run the risk of being overinterpreted and his work can hardly consider more recent phenomena such as educational expansion. Nonetheless, current empirical studies can confirm Bourdieu's findings to some extent and, thereby, make his constructs more tangible. Bourdieu's theories and constructs are primarily sociological in nature and not directly designed for application in didactic research. Other factors that may be relevant here but are not (or can't be) controlled include other characteristics of social background and the individual school context in which the students find themselves. Moreover, it could also be fruitful to analyse this data through the lens of Bernstein's work on realisation rules and language codes. For methodological reasons, a dichotomization of socio-economic status is carried out into privileged and unprivileged pairs. This represents a simplification of reality, as socio-economic background is a complex construct that entails individual pathways in specific cases (Weininger & Lareau, 2009). In addition, more cases need to be studied to confirm the results.

In this paper, systematic differences and similarities between socio-economically privileged and unprivileged pairs of students in mathematical modelling can be identified. Thereby, behavioural patterns become apparent which may be attributed to students' socio-economic background, be it in relation to Bourdieu's habitus or to more recent empirical studies. At the same time, many fertile approaches are evident regardless of socio-economic status. Overall, mathematical modelling tasks seem to contain aspects that are rather difficult for unprivileged students. At the same time, the case analysis provides evidence that all students in some way can engage in the context and achieve a result (for a quantitative comparison see Ay et al., 2021). This paper, thus, provides indications that mathematical modelling contains aspects that may increase social inequality as well as social equality. In addition to investigating these findings further, it needs to be discussed what must be done so that all students can benefit from mathematical modelling regardless of their background.

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Learning together with immigrant students: differences as resources in mathematics classes to fight racism

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Several situations of racism and xenophobia indicate the need for the school to consider means to promote the recognition of diversity and foster inclusive learning practices considering immigrant students. Racism fosters prejudice and intolerance based on stereotypes that can limit the foreground of the students. Fighting racism requires recognizing differences. Meeting amongst Differences in mathematics classes call immigrant students to collaborate, and differences can be resources for investigation in mathematics classes. They can choose themes referring to their own realities. This is related to students' motives for learning mathematics for social transformation.

Keywords: Foregrounds, Racism, Mathematics education, Immigrant student.

Introduction

According to the United Nations International Migration Inventory, in 2019, around 3.5% of the population on the planet are migrant people[1]. Education systems across the world are becoming increasingly more diversified and are being called upon to rethink their policies and structures in the understanding that migration is a human right that includes the right to education.

The debate on immigration and its impact on education are not new. However, it is a relevant and complex theme that demands constant reflection. Several situations of racism and xenophobia indicate the need for the school to consider means to promote the recognition of diversity and foster inclusive learning practices considering immigrant students.

Education is a political act because it is at the service of certain proposals (Freire, 1972; 1998). It is therefore not disjointed, disinterested and neutral and can take a democratic or authoritarian position. Education can consider the demands of subordinate groups such as people with disabilities, women, LGBTQIA+ people, black people, immigrants, and other members of marginalized groups. It can be exercised in the sense of anti-racist practices.

Skovsmose (2020) argues that mathematics can shape society, and this requires considering an ethical dimension in the face of a series of social implications. Mathematics education has a political and social dimension and must propose reflections that express concerns about oppression and exploitation, poverty and economic inequalities, racism, democracy, and ecological crises.

Consequently, mathematics education is also not neutral and apolitical. It can give power to a group of people by making them more accepted in society's decision-making abilities. This fact is directly related to racism, not only in relation to access to mathematical knowledge, but also to the different types of knowledge that are considered or not in school. Also because of the misuses of

[1] Retrieved April 4, 2021, from https://nacoesunidas.org/estudo-da-onu-aponta-aumento-da-populacao-de-migrantes-internacionais/

mathematical knowledge and skills in the hands of exploiters to keep privileges of certain groups according to racist assumptions.

This essay is a result of doctoral research reflections. It is in the process of qualitative research in the phase of data production through interviews with immigrants and teachers who teach mathematics. The research is structured around understanding the possibilities of mathematics education for the inclusion of immigrant students (Carrijo, in progress—a).

This essay aims to foster the debate on the possibilities and challenges that contemporary migratory processes present to the field of mathematics education. Therefore, it seeks to problematize, reflect, and signal the learning possibilities that migrations and cultural diversity provide.

For presenting a theoretical discussion on themes addressed in my doctoral research, I will, in the first section, discuss racism and how it can shape the *foregrounds* of immigrant students. Next, I discuss *meeting amongst differences* as a learning opportunity for mathematics classes. Then, I address the necessity to include *themes with political and social implications* in the process to consider immigrant students' differences as a resource for mathematics education with antiracism proposes.

Immigrant students and racism

Racism is part of a larger, structurally based racial system and it is constantly being brought up to date. It is a systematic form of discrimination through practices that produce disadvantages or privileges for individuals, depending on the racial group to which they belong. Not only black people are victims of oppression of a racialized system. Latinos, Arabs, Persians, Gipsies, Jews, Asians are included in this system (Bonilla-Silva, 1997; Marinucci, 2018; Romero, 2008).

In terms of a globalised world, the inequalities are intense and based on ideas of belonging and not belonging, and of favouring and disfavouring. This includes precarious public services, social and political instability, and social and economic vulnerability. In this context, populations in migratory movements experience several forms of exclusion and violence centred on racialisation. In other words, racism fosters prejudice and intolerance based on stereotypes and produces xenophobic practices that perpetuate the oppression and subordination of migrant populations. In the social context of exclusion, differences can be used to degrade, a variant that combines fear and contempt.

Racialized societies reflect the ways in which the school reflects privileges and exclusions. Thus, racism permeates the way diversity can be understood as a problem and is often ignored or fought. In culturally diverse environments with the presence of students of different nationalities, this may be even more evident.

The denial of diversity in the educational context can mean barriers to learning. Vandenbroeck (2013) points to the following misconceptions: *Believing that all students should all be treated the same way*: This can put the teacher in a position to consider the criteria of a “standard” student, an ideal student. In this case, this “standard” student may be in the teacher's imagination a student who corresponds to a white, middle-class child who lives in a traditional family. *Reduce the student to a nationality or a cultural group*: it denies the enormous diversity present in cultures. There is a risk of having the stereotyped idea that the child of African origin will be a good dancer or that they will eat with their hands, for instance.

Because of stereotypes, students can be excluded from mathematics opportunities. Barros (in progress) has noticed that social stereotypes might shape and limit the *foregrounds* of groups of people. A *foreground* is constituted by the person's hopes, priorities, perspectives, aspirations, and possibilities in life (Skovsmose, 2014). In this sense, racism can limit the *foregrounds* of immigrant students by belittling and oppressing them.

About schools in the USA, Danny Martin (2019) states that black children routinely experience different forms of systemic violence in mathematics education. For Black students, teacher evaluation was nearly as powerful a predictor as math performance. The white frames and white imaginations of Black children's intellectual inferiority also emerge to produce injurious intellectual violence against Black children.

Also, for Carneiro (2005), the epistemicide acts beyond the annulment and disqualification of the knowledge of the subjugated peoples, but also by the restriction of access to quality education, by the processes of intellectual inferiority, underestimation of the cognitive capacity and discrimination present in the educational processes.

Fighting racism requires recognizing differences. And this demands educational processes that challenge and deconstruct homogenizing practices that do not recognize differences. For this, it is necessary to create spaces, actions and activities that encourage the overcoming of exclusion and social injustice.

In this context, the challenge for mathematics education is to mobilize inclusive ways of connecting to diversities. This includes seeking forms of understanding them as a learning possibility. That is, transforming differences into a pedagogical advantage and creating spaces, actions and activities that encourage overcoming exclusion and social injustice.

Meeting amongst differences for learning together

Racism is a narrative based on the oppressor-oppressed power relationship. The oppressed are transformed into almost a "thing" and subjected to domination, alienation, or marginalization. The dehumanization of non-white people guaranteed white supremacy and justified exploitation and social inequalities.

In this sense, the humanization process through education must raise awareness of people who are capable to reflect, analyse, critically position themselves and make ethical decisions in society. Mathematics can bring possibilities to understand the world critically, pointing out that situations of social inequalities need to be transformed.

According to Skovsmose (2019), the *Meeting amongst Differences* in mathematics classes highlights the meeting between people with different life experiences, cultural backgrounds, different fulfilled and frustrated dreams. Also, with different hopes, priorities, opportunities, perspectives, and aspirations. It is about the meeting between students with different *backgrounds* and *foregrounds.*

The meeting between immigrants and nationals in school and non-school environments demands respect for diversity. It requires valuing all the knowledge and experiences involved. This meeting is a possibility to learn about cultural, linguistic, religious, physical multiplicities, and so on, through mathematics.

Gutstein (2006) points to the possibility of *reading and writing the world with mathematics*. It means involving students in issues related to social justice. This can help to create space for human rights reflections and students can discuss reality and engage in social change. In this sense, immigrant students, through mathematics, can interpret their life contexts. Furthermore, they can recognize themselves as agents of change and use mathematics to change their realities into a sense of belonging. When writing about the world, the students can use mathematics to find solutions. When students want to understand, for example, the situation of labour market, they can identify themselves or members of their communities, schools, or churches as potential victims of work analogous to slavery. They can engage in denunciation and understanding the needs to combat precarious work circumstances, for example.

In a social racialized system mathematics can be used as a tool for oppression and dehumanization. Also, the access to mathematics, and hence to economic opportunities, is selective in protecting white and male privilege rather than being truly democratic in nature (Martin, 2013).

But on the other hand, the *Meeting amongst Differences* in the mathematics classroom together with *reading and writing the world with mathematics* can create an environment in mathematics classrooms to learn collaboratively. It is a possibility for students who are culturally and linguistically different, poor students, students with disabilities, and immigrant students or children of immigrants, for example, to have mathematics learning opportunities. This space is invited for *Dialogue Across Differences*.

Dialogue Across Differences supports place across differences by establishing dialogues between different worldviews. The participation of students in the educational process takes place in a dialogic educational process in which each student can express their worldview, can be heard, and be considered. Any kind of differences with respect to cultural backgrounds, religions, nationalities, economic conditions, or abilities is considered to provide learning environments where all students can learn together (Skovsmose, in progress).

Moura (2020) reports in his research the possibility of meeting between deaf and hearing students in mathematics classes. This context provides to students the movement to see the other, to wanting to be together with each other and to learn from each other, promoting cooperation and building equity.

Considering diversity at school goes through valuing and using student diversity as a resource for education and taking responsibility in the search for social justice and equity (Akkari, 2018). In this process of being together and being able to see the other as possibilities to learn together, the differences that students bring to the mathematics classroom are considered an important contribution to the construction of knowledge.

Immigrant students need to be considered in mathematics classes. Respect for diversity goes through recognition. This is the way to combat racism that mutually feeds practices in society and at school. The themes addressed in mathematics classes are also important bridges between differences.

Political and social implications issues in mathematics classes

Thinking about the issues that can create barriers in the mathematics learning of immigrant students goes through many issues. A widely debated point involves mathematics classes in linguistically diverse contexts. Barwell et al. (2016) point out that diversity in language can create hierarchies and exclusion since not speaking according to a standard form of the local language can put students in the position of "less educated". One must consider that each participant in a mathematics classroom, including the teacher, brings their own combination of languages, varieties and modes of speech that must be considered to strengthen dialogue.

On the other hand, speaking the local language is essential in the learning process context but does not guarantee that immigrant students are considered as equals in mathematics classes. Although the immigrant students speak the local language without any indication of foreign accent, it does not prevent them from exclusion and to be labelled as "foreigners" (Baber, 2007).

I consider it is also important to include *themes with political and social implications* in mathematics classes to support an inclusive environment. These themes can involve issues of interest of the students and mathematics can help in understanding and reflecting on different realities. Themes such as distribution of wages and wealth; racism and xenophobia; migratory contexts; housing and living conditions; or about representation in social media, are some examples of real contexts that can be explored. Immigrant students are called to collaborate, and differences can be resources for mathematics classes.

Skovsmose (in progress) proposes the landscape of investigation *Erosions of Democracy* focused on some specific issues. Students can imagine the functioning democracy of many countries with extreme differences between rich and poor, investigate the distribution of welfare and share particular interests. Also, Britto (2013) planned mathematics lessons that relied on the investigation into the visibility of black people in Brazil. Through a landscape of investigation, he proposed an investigation to some students about the presence of black people in magazines. Students had to look for pictures of children and consider whether these children were being registered in positive or negative contexts and reflect the visibility of black people.

These two examples illustrate the possibility of proposing investigation in mathematics classes focused on themes with political and social implications. Students can look and reflect on these questions and understand the reality in a broader way. Mathematics is a tool to understand society's injustices and adopt a critical stance towards inequalities. In classes with immigrant students, these themes can refer to their own realities.

However, considering social and political issues goes beyond simply bringing certain issues into the planning of mathematics classes. It is crucial when considering topics referring to the context of immigrant students, not put them in the place of "folklorization", of exotics. Also, the mistaken perception of “foreigners” as inferior intellectually and culturally, would contribute to the exclusion process.

According to Akkari and Maleq (2020), citizenship refers to belonging to a global community. Education must include demands for local and global challenges. In this sense, it is important to consider the global context in the themes addressed in the classes. In the landscape of investigation *Global Visibility Matters* (Carrijo, in progress—b), I consider Global Citizenship and the necessity to enable students to understand global issues in mathematics classes. It includes an awareness of other perspectives, a perception of oneself as part of the global community. It can help to develop a sense of social responsibility and solidarity towards less privileged groups of people. This perception is also part of the process to fight against racism.

Such an education demands a critical and transformative approach. It means considering and respecting diversity in mathematics classrooms, ensuring inclusive and equitable quality education, and promoting learning opportunities for all. Such an education must also value diversities in mathematical knowledge, which means considering and learning together with others, and acquiring different kinds of mathematics knowledge.

Concern about how racism can create exclusions within and outside the school context must be part of the teaching and learning process with immigrant students reflecting on their expectations for the future. I see the concerns of mathematics education moving towards combating racism by using social-political themes as tools to *read the world with mathematics*. But beyond that, you must *write the world with mathematics*. And this is related to students' motives for learning mathematics as it relates to their own future possibilities for life and social transformation.

Concluding Remarks

In a society strongly marked by great inequalities, the coexistence of differences in the same educational space has been a challenge in building education systems that truly consider everyone.

Only the access of immigrant students to school, supported by legislation, does not guarantee support to immigrant students in the process of belonging that is crucial to immigrant people.

Racism can create a learning barrier for immigrant students because of stereotypes that can limit their *foreground*. They can feel excluded in the mathematical learning process, or they don't see that mathematics plays any significant role in their futures.

Mathematics education has a political and social dimension and must propose reflections that express concerns about racism. It is necessary to embrace differences and consider the contexts and knowledge that students bring to school, developing actions that can impact the inclusion of immigrant students.

Meeting amongst differences for *reading and writing the world with mathematics* together makes it necessary to address political and social issues. These themes can refer to the immigrants’ reality and by investigation, students can learn together.

The next steps of the doctoral research will bring an analysis of reports of experiences lived by immigrants and also discussions with teachers who teach mathematics in Brazil about possibilities in mathematics classes with immigrant students.

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Problems with variation in teaching/learning Geometry: an example of Chinese Cultural Transposition

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This paper discusses some theoretical/methodological observation and some qualitative results coming from a Cultural Transposition experience, implemented in the Italian school context (grade 8), according to the methodology of variation, as one of the most significant problem solving approach in Chinese schools. The framework of the Cultural Transposition and the methodology of variation are presented as an important condition for "decentralizing" the didactic practice from a specific social and cultural context. We argue that looking at different teaching/learning mathematics strategies coming from East-Asia cultures can favor some cultural contaminations at school and allow students to a significant and unusual thought about "inclusion" and "diversity" in mathematics. Our variation problems are designed on 3D Geometry and are aimed to guide students in discovering the relationship between pyramid and cone areas/volumes.

Keywords: Cultural Transposition, Chinese problems with variation, Teaching Geometry.

Introduction

In a society dominated by cognitive zapping (Veen & van Staalduinen, 2009), which is constantly changing due to socio-cultural transformations, school is also changing. Learning and teaching trajectories are in fact, in many cases, varying in all countries. Looking at the new scenarios that the school is experiencing in recent years, the classroom realities that teachers and students observe are in fact nowadays changing, enriched by new approaches, new stimuli, new routines, new didactical processes that come from inside and outside social and cultural classroom contexts. These scenarios are in many cases complex and difficult to study for the mathematics education Communities. One pioneering work (Bishop, 1988) highlighted the importance of recognizing mathematics practices as social phenomena that are embedded in those cultures and those societies that generated them. D'Ambrosio (2010) pointed out that taking care about cultural and social issues into mathematical practices contributes to the understanding of cultures and the mathematics itself. Nowadays the awareness of taking into account cultural and historical contextualization in Math classroom teaching practices and the crucial assumption that culture permeates mathematics education practices is well known by all the mathematics education Communities. As Radford underlined, it has to be clear for teachers that the "configuration and the content of mathematical knowledge is properly and intimately defined by the culture in which it develops and in which it is subsumed". (Radford, 1997). Barton's research (Barton, 2008) reinforced the importance of a cultural perspective to study mathematics education phenomena, looking for example at different languages, and cultural manners. Agreeing to this perspective, we argue that culture gives students an opportunity to engage in mathematics focusing on concepts of inclusion, integration … to discover diversity as a resource! In some cases, the participation of many countries in international student competence tests (such as PISA) gives also the opportunity not only to compare different counties obtained results (Schleicher, 2019), but also to reflect on the educational practices, social, political and cultural environments that determined these performances. The last good results (as the previous ones) obtained by students from East Asia, especially those from Hong Kong and Shanghai, have prompted comparisons to reflect on the "reasons" for this educational success. It is well known (Spagnolo & Di Paola, 2010) that in these countries some learning approaches that compared to those of their Western peers produce different mathematical skills, in many cases more useful. According to Di Paola (2016) and Mellone et al. (2019; 2021) in these learning/teaching approaches there are many cultural factors linked to different cultural assumptions, educational practices and related cognitive styles (Bartolini Bussi & Martignone, 2013; Mellone et al., 2019). The same feeling is nowadays claimed by many teachers and researchers around the world: they in fact confirm how, in many cases, East Asian students, already educated in their own country, show more developed knowledge and skills in many mathematics subjects.

Theoretical framework

Bartolini Bussi & Martignone (2013), studying didactical phenomena discussed in the previous paragraph, emphasized the possible correlation between mathematical knowledge and the cultural context, cultural beliefs, in which they are and have been inserted and in which, therefore, mathematical knowledge is elaborated, assimilated and transmitted. In recent years, some other research works, through qualitative and/or quantitative approaches (Bartolini Bussi et al. 2017; Di Paola 2016; Mellone et al., 2019) pointed out how the *cultural diversity* could become, in this sense, an opportunity in mathematics Education (Kaiser, 2018). Researchers and teachers, coming into contact with educational practices adopted in other cultural contexts, are able to *deconstruct* (Derrida, 1967) them, reconsidering the themes of educational intentionality defined as background of their educational practices. Mellone at al. (2019), inspired by the Skovsmose's (1994) approach, defined the framework *Cultural Transposition* as "a condition for decentralizing the didactic practice of a specific cultural context through contact with the didactic practices of different cultural contexts". *Cultural Transposition* (CT) is a perspective that can allow the meeting between different mathematics education school practices/approaches, coming out from different cultural contexts, and define a potential space for reflection and awareness, and also development for researchers, educators, teachers, students (Mellone et al., 2021). CT involves those who implement/observe math teaching practices, coming from other cultures, to a "*deconstruction*" process useful to a re-interpretation of their own thought and consequently a possibility to change/improve personal (cultural) beliefs, values, and didactical principles. As Derrida stated defining the deconstruction, "*an analysis of the different levels in which a culture is stratified*" (Derrida, 1967). According to this perspective it is possible to recognize, valorize and include possible differences coming from "other" cultural Communities, linked to different values, principles, beliefs systems, happens, for example, in schools Communities (Mellone et al. 2021). To "get in touch" with different educational practices, coming out from different social and cultural contexts, can help researcher and teachers not only to become more aware of their social and cultural paradigm in regards to the classroom teaching practices but also to deconstruct their thought decentralizing their cultural expectation and assumptions and rethinking mathematics educational practices (Bartolini Bussi et al, 2013) in terms of "inclusion" and significant us of possible social, cultural "diversities". Of course this "changing process" is complex and needs more and more opportunities for reflection and *contamination* (Bartolini Bussi et al., 2017; Jullien, 2006). From the school student's perspective, *diversity and difference* in learning mathematics strategies appears, in this sense, a great opportunity, a good chance for something that maybe should not be favored without this "revolution" of prospective and without this change in the cultural system of reference. This could permit us to look at diversity and difference in mathematics and in mathematics teaching within the realms of the cultural, the social and the political. Spagnolo & Di Paola (2010), a pioneer of this kind of subject, presented this approach as a continuous open dialogue between cultures, societies, histories … useful to *cross* the didactics of mathematics. In the last years several researches (e.g. Bartolini Bussi et al., 2013; Di Paola, 2016; Mellone et al., 2021; Spagnolo & Di Paola 2010) discussed different CT experiences in Western school contexts. Several of them look to Chinese culture and in particular the use of Chinese practices of "*problem with variation*" for an early approach to Algebra in Primary school. What is a rather new (Leung, 2003), research subject discussed in this paper, is the use of this methodology in teaching/learning mathematics in Middle school and, in particular, in teaching/learning Geometry.

"*Problems with variation*" is considered one of the most significant mathematics education tools used in Chinese Primary school (e.g. Bartolini Bussi et al. 2017; Fan et al., 2004; Mellone et al., 2019; Sun, 2011). In the last twenty years, many researchers underlined the importance of the variation approach as a necessary condition for deep learning (e.g. Marton & Booth, 1997; Sun, 2011) and in particular for the learning of mathematics. variation is typically expressed by Rowland (2008) as a practice in structured exercises which varies considerably from country to country and from text to text. Sun (2011) underlines the use of variation problems as a tool "to discern and compare the invariant features of the relationship among concepts, solutions and contexts, and provide opportunities for making connections, since comparison is considered the pre-condition to perceive the structures, dependencies, and relationships that may lead to mathematical abstraction." (p.107). In Chinese language it is *Bianshi* (變式,) where *bian* stands for "*changing*" and *shi* means "form.". Yakes and Star (2009) looked to variation as a critical means for comparing and developing flexibility for learning mathematics already from the first school years. In China, the variation approach to problem solving is linked to many disciplines and used in all school levels from the first grades. The issue of variations in problems perfectly reproduces one of a Chinese proverb: "*no clarification, without comparison*", and it is "*in contrast*" to the assumption used in many textbook in different countries: "to consolidate one topic, or skill, before moving on to another" (Rowland, 2008). According to Rowland, comparing this problem solving approach with the one used, in many cases, in the Italian school (textbooks and/or practice), it is in fact possible to point out a strong difference, in some cases, common in several Western countries and related to the use of isolated problems/exercises, organized in progressive steps, strongly partitioned between them and so not very useful to define a possible abstract thinking looking relationships between concepts, strategies, algorithms (Cai & Nie 2008; Spagnolo & Di Paola, 2010). These considerations can be useful to underline the significant opportunity to proceed with *contamination experiments* (Jullien, 2006) aimed to get in touch with different "good practices", coming from different cultural contexts and linked to different language; different historical tradition; ideologies; school systems; governance structures… In this sense, the *contamination* appears as an important condition for teachers to "decentralize" their own didactic practice of a specific cultural context to something more wild, different, and in many cases more useful for their own mathematics students and could be included. Bartolini Bussi et al. (2013), Di Paola et al. (2016), Mellone et al. (2021) discussed this approach in Italian schools for an early approach to Algebra in Primary school. As we already discussed, very few research papers refer instead to the inclusion and the use of "*problem with variation*" in teaching/learning mathematics in Western Middle school (grade 8) and, in particular, in Geometry. The CT discussed in this paper is aimed to look at a different teaching/learning mathematics practice, coming from East-Asia cultures, as an important opportunity for a significant and unusual (for the Western culture) 3D Geometry problem solving. In particular the proposed variation problems are aimed to guide Italian students in discovering, almost autonomously, the relationship between pyramids and cones area/volume. According to the declared aim, our lens was focused on students' reaction to this approach and, in particular, the research question to which we tried to answer was: *What kind of process do grade 8 students of Italian (Western) culture show in variation geometric problem solving?*

Methodology

In this section we discuss our CT in teaching/learning Geometry, implemented in the Italian school context (four classrooms of grade 8), according to the methodology of variation. It is important to underline that our intention was not an attempt to *translate*, or even worse *import* a Chinese practice/teaching strategy into the Italian culture and more in general in the Western one. On the contrary, the educational path we analyze, developed by the author of this paper and the group of teachers and researchers with whom we have collaborated for some years, is aimed to create a real CT, with an interesting inclusion of the different Chinese model into the Italian didactical practice in grade 8. With the aim to reply to the research question expressed before, our CT was particularly focused to help students to transfer their mathematics knowledge from a well-known context to another apparently different context, using the variation approach. The two contexts refer to 3D Geometry and in particular to the concepts of pyramid, cone and their area and volume. Authors of this paper were involved in all phases of the education path, the design, the implementation of it with school teachers and the analysis of the collected data. The grade 8 students (around 100) were engaged for almost 20 hours. All of them were conscious about experimenting with a pedagogical method from another culture; during the CT path they explored it, initially by themselves, after being guided by the Authors of this paper.

According to the declared aim and what the literature discusses about educational practices based on the same research subject (e.g. Bartolini Bussi e al., 2017, Mellone et. al., 2019), the sections of the CT path referred to the proposed mathematical subject were designed following this frame:

1. Pre/Post-questionnaire about 3D solids knowledge (pyramid and cone and their area/volumes),
2. Single students resolution of different variation problems about pyramid and cone solids.
3. Students interview about their own variation problem solutions (difficulties and mistakes personally implemented CT).

Before to start the first phase, with the aim to propose useful tasks in all research phases of the implemented educational path, we provided a structured and very fine *a-priori* analysis of Chinese and Italian textbooks (schemes, images, writing, task …) and of methodological implicit assumptions that variation approach could have had on the involved students about the inclusion of this didactic methodologies in the one commonly adopted by Italian teachers. All research phases were video and audio recorded. These collected data were qualitatively and quantitatively analyzed (with cluster and implicative analysis). Researchers and teachers studied these data step by step during all CT path phases in order to eventually redefine and redesign the path frames. According to the contamination theory, protocols and interviews were examined with a specific focus on possible interesting used of different sings (words, pictures, arrows, colors, …) useful to describe students approach to variation and, in particular, to underline possible difficulties or readiness to define relationship and among concepts, solutions and contexts. Cluster and implicative analysis gave us the possibility to put in evidence possible stable behaviors in the analyzed students sample. In this paper we don't present these data; we are referring only to some qualitative findings.

An example of an implemented variation problem task (the first triplet) is shown in Figure 1. Starting from a problem concerning only the pyramid solid and its area (first problem on the left), we asked the students to solve two more problems on the cone solid (regarding its area and volume). According the variation problems theory and the contamination one, discussed before, the defined triplet was structured in order to favor the possibility to find, independently, the relationships (similarities and differences) between the two proposed texts and the images of geometric solids, without an explicit "presentation" by the teacher of the second one (about the cone).

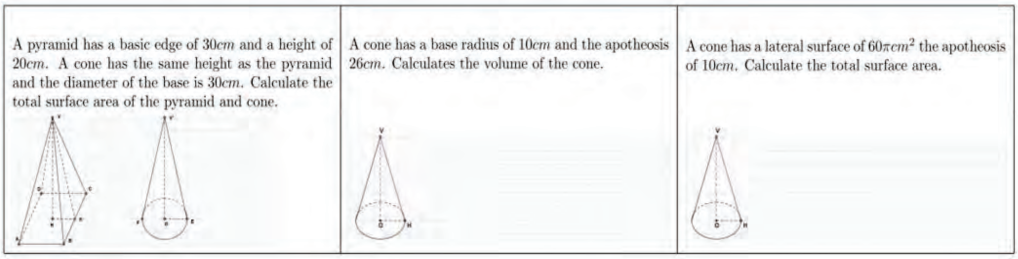

Figure 1. Our CT: an example of a first triplet of variation problem

As we also mentioned before, all students knew the concept of pyramid and its area and volume calculation and properties; none of them knew (we formally tested it) the cone solid. Students, to solve this task needed to autonomously "transfer" (using variation approach) knowledge and skills acquired on the pyramid solid, to the cone. In the three problems the graphical representation has been also inserted in order to help students to better focus on the relationships and possible links in area and volume calculation. This choice retraces what Chinese textbooks and Chinese teachers commonly propose in classrooms.

Empirical findings and first conclusion

In order to give an overview of the obtained results and related discussion, in this section, we briefly present some finding, come from the analysis of students protocols referred to the phase 2 of the implemented CT path and some data coming from a qualitative analysis of the same two students interview (phase 3 of the CT) about their own variation problem solutions.

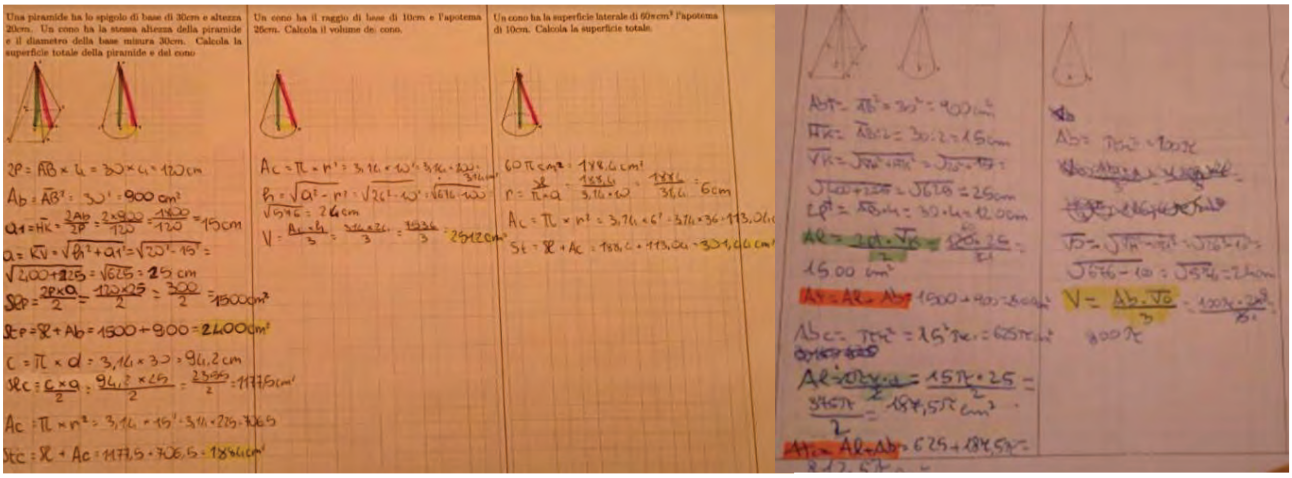

Figure 2. Maria's and Valentina's variation problem protocols

Maria's and Valentina's protocols appear interesting for replying to our research question. In both students sheets it is, in fact, possible to observe an noteworthy and autonomous use of the colors in the proposed variation problems. Maria used colors to highlight analogies and relationships between part of solid. A solid that she knows and a solid that she doesn't know. Valentina did the same but taking into account the solving strategies and the formulas, instead of the graphic solid representation. Thanks to the Chinese variation approach, she autonomously discovers a relationship between two used formulas. In both cases (we found similar approaches in many student protocols) the opportunity to be *contaminated* by the Chinese variation methodology (Mellone et al., 2019, 2021) guided them in discovering in the problem solving activities something new as the relationship between pyramids and cones area/volume. Their approach to variation problems and their "surprise" to the possibility to autonomously learn "new mathematics' ' (Maria used these terms during her interview), clearly emerged also during their subsequent interview. The following part of the dialogue between Maria and her teacher, appears interesting:

Teacher: Could you better explain to me what you mean?
Maria: I highlighted with the same color what behaves in the same way.
…
Maria: Here I have something that I know and something unknown into the same problem. I tried to look, to search for analogies and relationships. It's a interesting problem. I like it.
Teacher: What did you find?
Maria: All are right triangles, the first two are just the same ... same numbers! So I highlighted them the same way.
…
Maria: Prof., could we do the same argumentation between Prisms and Cylinders? We studied them… We could find a general "rule", we could use less memory and more variation. Problem solving can become more "simple".
Teacher: You can try, …
Maria: It is great. I didn't know anything before today about it. We have to look also to other countries (smiling) …

As Maria, almost all students benefited from the Chinese contamination of variation problems. they declared that working with this methodology, they became more aware of their knowledge and they were able to construct step by step a general solving approach to this kind of problems (the same finding in Cai & Nie, 2008). Of course a few of them (Giovanni, Marco and Francesca speeches are some examples) didn't immediately reach up to this expertise.

Giovanni: We haven't studied the cone solid, the teacher was wrong … that's why I only solved the pyramid problem.

Marco: I don't know the cone and I don't remember the pyramid, sorry.

Francesca: We have not studied the cone, but the teacher gave us a task with the pyramid and the cone; we don't know anything about the cone ... I tried to use what I knew about the pyramid but I couldn't find the way.. why did the teacher put the pyramid next to the cone? I will try to think about it. We didn't do this same kind of problems before

As we said before, this "changing process" is complex and long (Bartolini Bussi et al., 2014); Giovanni, Marco and Francesca need more and more opportunities for reflection and *contamination*.

According to Spagnolo & Di Paola (2010), Sun (2011) and Mellone et al. (2021) findings, our students approach underlines how the implemented CT path gave them a good chance to stimulate and improve a possible abstract thinking, finding relationships between concepts, strategies, algorithms in variation problems. The defined variation problems gave them an important opportunity to deepen abstract geometrical thinking - structure, strategies, relationship, concepts … - (Leung, 2003) than does typical isolated problems on the two investigated solids as, in many cases, happens in Italian (but also in other Western countries) school context. They discovered a "new", "diverse" culture and different related problems solving approaches that enrich their teaching practices and, according to our finding, their mathematics learning (Barton, 2008). Thanks to the CT path they had the opportunity to engage in mathematics through a cultural lens, helpful to discover and use diversities as resources for their future teaching. This gave us, as researchers, the chance to rethink our own cultural expectations and assumptions about possible important cultural, social and political issues in Mathematics Education research.

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Mathem-Ethics in prison: how mathematics can enhance social skills

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The aim of this paper is to investigate the potential power of mathematics in a socially difficult environment, as a high-security prison. More, we ask whether mathematics can enhance social skills (ethic skills, in particular), not only in prisoners: we present a mathematical experimentation involving prisoners, as those who, by means of mathematics, first learn ethic skills and then teach ethic skills to people from outside the prison, with a methodology we call "Mathem-Ethics". The theoretical framework we work with is Horizontal Teaching, which provides a way to learn for both learners and teachers. We briefly discuss some results of the experimentation, in terms of its efficacy on people from inside and outside the prison.

Keywords: Mathematics education, ethics, social change, horizontal teaching.

Introduction.

"Mathematics is bad for society", said Arvid, an 11 years old student, in a challenging tone (Ryan et al., 2021). What Arvid was brave to say, is perhaps, shared by many students of different ages. As Wright (2019) highlighted, quite often mathematics is reduced to a set of rules and procedures to memorize and apply (OFSTED, 2012; Foster, 2013), resulting in many children (and adults) continuing to exhibit alienation from mathematics (Nardi & Steward, 2003). Nowadays education aims more and more towards functional mathematical literacy, that refers to the capacity of creating and applying mathematical knowledge when required in real life (Jablonka, 2003), and many efforts are taken from researchers and teachers to enhance mathematical skills in students that are useful for developing citizens able to solve real life problems that require mathematical literacy. But still, it is a common idea among students that these mathematical skills are only technical, and are not useful in social life. If mathematics is not bad for society, at least it is seen as useless for society, not connected to ethics and not useful to contribute to a life change. "Discussions about the connections between mathematics and democracy amongst the general populace have not been explicitly well rehearsed, other than to either assume that mathematics has nothing to do with anything political, being neutral in form and practice, so it has nothing to do with democracy, which is something political; or that it is implicitly democratic." (Swanson, & Appelbaum, 2012, p.1). However, mathematical literacy is relevant to both social and economic needs and to an individual's participation in today's democratic society (Jablonka, 2003; Skovsmose, 2007). If there are those who think mathematics is a tool to obtain social training in obedience, an industrial trainer (Ernest, 2019), Swanson and Appelbaum ask whether disobedience to the evocative power of mathematics could be itself a democratic action. Democratic teaching practices may have a positive influence on students' learning outcomes in the mathematics classroom. This may be because successful democratic teaching and learning is conceived as situations where "individuals are able to think for themselves, judge independently, and discriminate between good and bad information" (Dewey as in Orrill, 2001, p. xiv). Therefore, there is a need to rethink teaching, taking into account the social dimension. Both Vithal (1999) and Aguilar and Zavaleta (2012) pointed to the need for empirical studies to experiment with existing theoretical ideas on the connection between mathematics education and democracy. Daher (2019), for instance, presents a study to assess four democratic practices (freedom, equality, engagement and justice) in the mathematics classroom. If mathematics education can be used to develop citizens' skills to sustain a democratic society (Aguilar and Zavaleta, 2012), it should be even more important to develop it in a context where the students are prisoners. This has not been investigated so much and it is the focus of this paper. Several mathematical activities with students that are inmates have been successfully carried out in Swedish prisons (Helenius and Ahl, 2017; Ahl and Helenius, 2020). The authors highlight that successful mathematics education in prison can play a role in producing identity change increasing opportunities for re-entry in the society. In this paper we present a project, called *Vietato non toccare (Don't not touch)*, carried out in a high-school in a high-security prison in Sicily (Italy). This project aimed to enhance mathematical skills in inmate students through a laboratorial approach, in order to improve motivation of students and their sense of self-efficacy, i.e. confidence of success in handling a problem (Bandura, 1997). More, we aimed to use mathematical education for ethical purposes, with a methodology that we call *Mathem-Ethics*, experimenting with a way to use mathematics to achieve a change of life. We call Mathem-Ethics a methodology to teach/learn mathematics, giving precise ethical meanings to mathematical concepts. It is possible to design mathem-ethical activities, with the precise target of enhancing both mathematical and social skills. Our research questions are: "Is it possible for inmate adults to acquire ethical skills by means of a mathem-ethical path that will help them with a real life change?" (RQ_1), "Is it possible to enhance social skills in people by means of a mathem-ethic path in prison?" (RQ_2). In this paper, we first present the theoretical framework as the basis of our activity, Horizontal Teaching. Afterwards, we briefly describe the whole project and the methodology. Then we present the mathem-ethic path that was carried out. We end with some discussion and conclusions regarding the efficacy of the experimentation for those involved.

Theoretical framework.

Ethics or moral philosophy is defined as that branch of philosophy that involves systematizing, defending, and recommending concepts of right and wrong behaviour. Ethics is born, already with Socrates and many Greek philosophers, to answer questions of human morality by defining concepts such as good and evil, right and wrong, virtue and vice, justice and crime. It is a common opinion that social sciences, like literature, art, history, are didactically useful for ethics investigations in students. It is our opinion that mathematics too can be used to learn concepts useful for students' lives, in general, and to ethics, in particular. The theoretical framework we use to achieve ethical goals is based on Horizontal Teaching (HT) (Ferrarello et al., 2013). In teaching/learning environments, the actors are the one who teaches, the teacher, and the one who learns, the student. These two figures generally have distinct roles and, in traditional environments, the teacher transfers his/her knowledge to the student, we might say in a "vertical way", top-down. In HT, on the other hand, there is a teacher's awareness: the teacher is willing to challenge him/herself and expand his/her knowledge by entering the sphere of the student's knowledge. In fact, in HT, the two sets of knowledge, the one of the student and the one of the teacher, are placed at the same level and initially have an intersection (Figure 1a); the teacher must be able to enter into this intersection and expand it, so that the knowledge shared is greater than at the beginning of the process, expanding not only the student's knowledge, but also his/her own (Figure 1b). The expansion of the teacher's knowledge takes place not only in terms of content, but also with respect to the students' experiences, their interests, their learning styles. The expansion of students' knowledge is not only about content, but also about the way they learn, contributing to enhance their mathematical literacy (Jablonka, 2003). So it is not only the student who absorbs from the teacher's knowledge but also the teacher expands his/her knowledge: this is the fundamental characteristic of Horizontal Teaching.

Figure 1: Horizontal Teaching

In the HT model, learners are seen in relationship with themselves, but also with the other learners and the world, just as in Maheux & Roth (2011), where the authors describe a way of thinking about knowing, namely relationality, starting from the biological theory of cognition, (Maturana & Verden Zöller, 2008). Human beings are theorized as complex biological "learning systems" that coordinate with the co-emerging environment that they "bring forth" (Maturana & Varela, 1998) and in knowing mathematically, the learner and the knowledge are not independent entities. The novelty of HT, with respect to this model, is the role of the teacher, who becomes a learner too, sharing with the other learners (the students) their participation in creating a new world of knowledge.

Description of the project and Methodology.

Vietato non toccare (*Don't Not touch*), https://sites.google.com/view/dontnottouch/home, is a project conceived in 2017, when one of the authors won the *Italian Teacher Prize*, promoted from the Italian Ministry of Education. It was carried out with adult inmates in the high-security prison of Bicocca, within the high-school "K. Wojtyla" of Catania. It aimed to create an exhibition of mathematical objects in prison, and it consisted of three steps: 3D-printing of some objects, (Step 1); mathematical training of inmate students on the created objects (Step 2); guided tours of the exhibition (Step 3). The exhibition consisted of four sections (see Table 1).

Table 1: Sections of the exhibition

Section 1	**Not orientable surfaces**: Möbius strip; Klein Bottle.
Section 2	**Pantographs for geometric transformations**: Pantograph for homothethy; Pantograph for axial symmetry; Pantograph for central symmetry.
Section 3	**Conicographs**: Cone with conics' sections; Ellipsograph with antiparallelogram; Ellipsograph with rhombus; Hyperbolograph with rhombus; Parabolograph.
Section 4	**Archimedes' machines**: Parabolic mirror; Coclea; Lever.

Most of the objects of the exhibition are mathematical machines (Bartolini Bussi & Maschietto, 2006). The authors led the training course for the inmate students (Step 2), to help them understand the functioning of all the objects: 15 students were enrolled, just 3 became guides of the exhibition, because the others were moved to other prisons or got out of the prison. The training consisted of 20 meetings, held according to the methodology of the *Mathematics' Laboratory* (ML) (Anichini et al., 2004). In the ML, students are guided to discover and construct the concepts, supported by the teacher and their peers, in a collaborative and/or cooperative way. In fact, the ML is a set of activities carried out by students and teachers aimed at the construction of meanings of mathematical objects. In the ML, students do not study mathematics, but rather do mathematics, dealing with a problem, exploring, conjecturing, proving, applying. During the training (Step 2), every object was introduced by means of a problem, typically: "What does it do? And why?". The object was manipulated by students working together and under the guide of the teachers they developed an understanding of what the object does and why and they constructed the mathematical meaning lying within the mathematical object. At the same time, they were introduced to the ethical meaning of each section (see next paragraph). Learners are not only invited to touch, but rather they have to do it, in order to make visible and comprehensible the mysterious operating forces of mathematics, often remaining *unrevealed* (Roth & Maheux, 2015). Once the students finished the training course, they became guides of the exhibition and the gates of the prison were opened (Step 3). Visitors were students from school and university, teachers of all school grades, both mathematics teachers and not, university professors, both mathematics professors and not. Also the exhibition was organized as a mathematical laboratory: visitors were invited to touch the objects and manipulate them in order to understand their functioning. Together with the guides, visitors were assisted by the authors of the paper, who explained the ethical part of the various exhibition sections. Being in a high-security prison, we could not videotape the students or the visitors. We conducted a qualitative analysis, based on the spontaneous comments of students. They did not gain any kind of advantage in taking part in the project and they were not forced to write those comments. As for visitors, we administered a questionnaire with two questions. The first one differs according to whether the visitor was a student (question Q_S) or a teacher (question Q_T). Q_S: "Has your visit to the exhibition had or do you think it will have an *impact on the way you conceive mathematics*? (e.g. in relation to the social role that mathematics can play)"; Q_T: "Did your visit to the exhibition have or do you think it will have an *impact on the way you teach*? (e.g. with regard to methodologies, attitude towards students, topics etc.)". The second question was the same for both (Q_{ST}): "Did or do you think the visit will have an *impact on the way you behave in society*?"

The Mathem-Ethic path.

At the guided tours of the exhibition, all sections were presented according to their mathematical meanings and ethical meanings (Ferrarello et al., 2021) that we briefly describe here. Section 1- "Not-orientable surfaces: when *inside* and *outside* merge together": The objects we see every day have generally orientable surfaces, i.e. they have an inside and an outside, separated by a boundary, which we are forced to cross whenever we want to pass from inside to outside or vice versa. In life, we often classify things and people, by placing them inside or outside a certain set, with a very clear boundary. What we discovered, working with 'bad-by-definition' people in prison, is that, indeed, these boundaries are not as clear as we paint them, but rather blurred. Objects in this section were symbols of the meeting among people from inside (the prisoners) and from outside (the visitors) in the neutral field of mathematics. Section 2 - "Pantographs for *transformations*: transform yourself by remaining true to yourself": In geometry, it is possible to transform a figure into another one, maintaining certain properties. A task of education should be to make students grow and evolve, by changing some things about them (transform yourself) and maintaining some of their features (remaining true to yourself). The school does not replace the students with new people, but rather transforms them. Objects of this section were symbols of the transformation of students and visitors. Section 3 - "Conicographs: Conditions for being in a *locus*". Conic sections are curves that can be described as geometric loci, i.e. we can describe a mathematical condition for a point, satisfied if and only if the point belongs to the curve. Being in a locus (a geometric one or, in a metaphor, in a real place) therefore depends on the conditions that one sets. Our behaviors (the conditions, in the mathematical metaphor) determine the effects on our life (the geometric locus, in the mathematical metaphor). If you really want to change the locus, then you have to start by changing the conditions. The ethical meaning of this section is the importance of self-determination to encourage students and visitors to own their own lives, taking them in hand and setting new conditions, to achieve new loci. Section 4 - "Archimedes' machines: Sicily, land of mathematics!". The place where the project has been carried out, Sicily, is often covered by prejudices associated with facts, people and ideas, which are not universal in Sicily: it is also the land of a millenary culture. Our exhibition is pleased to host a section of machines attributed to the Sicilian genius of Archimedes. In this section we want to remember Sicily for other facts, people, ideas and those that have contributed to our scientific culture: it is not only the land of mafia, but also the land of mathematics: like a person is not only a prisoner, but also, in this case, a guide of a mathematics exhibition (and much more).

Brief discussion and conclusions.

In this paragraph we briefly discuss the efficacy of the HT in the third step of the project: the guided tours of the exhibition, hosting people from outside the prison. In the following, we report in Italics some parts of the spontaneous comments from prisoners and answers from visitors. We underline where, as per the HT framework, students (blue set in Figure 1) can be also teachers (let us recall that visitors were teachers and students from outside), while teachers (pink set in Figure 1) are inmate students after becoming guides of the exhibition. In order to answer our first research question, RQ_1 ("Is it possible for inmates adults to acquire ethical skills by means of a mathem-ethic path to help them in a real life change?") we take into consideration the teachers' expansion, i.e. what the inmates, as guides of the exhibition, learned. We report some of their spontaneous comments. *This project brought into my life something unique and unimaginable. I didn't believe I could reach so much, and this makes me understand that also I have the skills and possibility to yearn for a future rich of occasions, and to give a turning point to my life*; and more. *This project involved a cultural change for us, but above all a personal one. At the beginning we were very enthusiastic but not very motivated, just because we gave limits to our potentialities, to our knowledge and intelligence. But by attending classes with commitment and passion we managed to get great results, making calculations, conjectures and evaluations. Until a few years ago, we would never have imagined debating on mathematical concepts. …* [We thank the teachers who succeeded] *to illuminate the way of our journey to a better future, believing from the beginning in* *us, teaching us that in life we must look beyond our expectations. But above all, thanks to mathematics, we know who we are and who we will be. Remembering that mathematics is not only made of numbers and sums, but also of much more! Thanks again for believing in us, and that we believed that the only thing you can't do in life is to divide by zero!*. We underline that the inmates we worked with are all incriminated for mafia association (n. 416 bis of the Italian criminal code). Some of them are, by "family tradition" involved in crime and also did not have school education until they entered prison. Some, even if they are Italians, had difficulty also in speaking proper Italian, because they used to speak dialect. Moreover, inmates did not gain anything in taking part in the project. Little by little, they made *our* project *their* project and felt proud of being part of it. It is too early to see long term effects of this adventure, but for sure, from their word, we can say that a change already started. The 3 guides just finished high-school in prison and they are planning their future, even enrolling to university. In order to answer our second research question, RQ_2 ("Is it possible to enhance social skills in people by means of a mathem-ethic pathin prison?") we take into account the answers to the questionnaire by the visitors. As for Q_S, students visiting the exhibition expanded their knowledge seeing mathematics in a different way: they recognize that *it is possible to make mathematics also with a social role, a role of unity and education* because it can be a *tool for educating, growing, learning,* not *only for well-educated students,* but for all. *The myth of the social role entrusted mostly to social sciences has been dispelled*. As for Q_T, teachers visiting the exhibition had the opportunity to test the efficacy of the ML, whether they were math teachers or not. *Only those who play an active role in the learning process really learn: it is fundamental to let students having the pleasure of discovery. Asking the right questions, leaving space for silence, listening, observing together, providing tools, stimulating curiosity, are much more important activities than simply providing data and ready-made answers. Student as actors and not audience are happier and more effective* (from A. and P., not math-teachers that already work in a laboratorial way: the visit confirms their belief). We want to underline the position of N., a math teacher, who is revising his teaching practice, and he claims that the visit *has accelerated this revision process*. The teacher learned that *you cannot set a standard model to which the students must adapt, but the reverse process is needed: the teacher must adapt to the students.* Then all teachers benefit from visiting the exhibition expanding their knowledge in terms of teaching methodology. Let us move to Q_{ST}. Teacher P. wrote that *the way of behaving towards society has become more conscious*. The project *breaks down many commonplaces about prisons and prisoners* because the project shows that *the polarity of reality is a scam: true/false, good/bad, ignorant/knowledgeable, right/wrong, the polar view of reality is limiting, reductive, excluding* (target of Section 1 of the exhibition). Visitors (students and teachers) *discover wonderful people with a great desire for redemption*, and they found that *everyone needs to be recognized: trust, sincere appreciation, a smile, kindness, can help everyone grow much more than through disapproval and punishment* (target of Section 2 of the exhibition). Thanks to education, and also to math education *not always those who have committed a crime are bad people and can never change, [*they*] improve or try to take a new and better path* (target of Section 3 of the exhibition). Visitors *saw in their eyes* (the prisoners) *the desire to escape from a past that, now thanks to the experience made with the project, was "narrow"* (target of Section 4 of the exhibition). This led them *not to be prejudiced and do not judge too quickly, because the social, economic, cultural* *situation of each person varies according to the society in which they are born and grow up and they cannot choose that. While it is all of us who together can do something different and we can benefit from it all together!* Visitors are inspired to do their best in a view of *I care*. A last consideration is the reciprocal relation amongst the people involved in the project. At every tour, the discourse always started with mathematics, and became more and more mathem-ethic. During the flow of the tour visitors and guides entered more and more into a confidential conversation. Usually, at the end, there was no more distinction between "inside people" and "outside people". One of the guides said that he *succeded to enter in to a relation with people, this was always my flaw*. The same happened to visitors. In the intersection of blue and pink sets (Figure 1), we find no difference between teachers and learners: a full Professor, visiting the exhibition, said "*I have had the pleasure of feeling ignorant in a singular place, where the last prove to be the first*". We hope that future mathem-ethics paths taken in prisons could open the doors to the development of social skills both in prisoners and in math educators. More to come!

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Surprising everyone but herself with her good results: the twin dynamic of invisibility and failure to see

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In this paper we explore the co-construction of identity in a Norwegian lower secondary school mathematics classroom. Focusing on one high-performing girl, Sarah, we analyse the role of male-dominated performance of "smartness" in her positionality in the figured world of Class A. While Sarah can be simply understood as making herself "invisible" in this dynamic, her teacher's account draws our attention to the impact of gender performance on what she sees and values in her students. We argue that Sarah's positionality is the result of a twin dynamic between girls' cultural invisibility and her teacher's failure to see, indicating a need for greater awareness of girls' situation in mathematics classrooms, particularly where - as in Norway - gender is seen as "no longer an issue".

Keywords: Classroom dynamics, gender performance, invisibility, equity.

Background and literature

Although Scandinavian countries are often seen as a "beacon" of gender equity, women's participation in science, technology, engineering and mathematics (STEM) lags behind other countries (Talks et al., 2018), particularly in Norway: only 1 in 3 STEM graduates are women (Confederation of Norwegian Enterprise NHO, 2018). Talks et al. (2018) suggest that part of the problem is the common perception that Scandinavian countries have "fixed" the problem of gender inequity. In this paper, we explore girls' experiences of mathematics in the crucial years before they choose their final educational pathway, in a setting where gender equity in the sense of a level playing field is assumed to follow from an emphasis on equal opportunities. We find that, on the contrary, being a successful student is marked by a highly gendered performance within a classroom dynamic that goes unquestioned by all participants, including the teacher. Focusing on the case of Sarah, a consistently successful girl, we argue that the gendered performance of "smartness" in the classroom renders her invisible, and contributes to her teacher's failure to see her achievement: everyone is surprised by her marks except Sarah herself.

The link between classroom culture and students' mathematical identities is well established. Black (2004a, 2004b) notes the role of interaction between teacher and students in the construction of mathematics knowledge in a British primary school, with particular implications for girls. Rather than engaging with the girls about mathematics, the teacher "somehow negotiated with these girls a coping mechanism where they stayed silent on the periphery of the classroom in whole-class discussions, but were praised for neatness and presentation elsewhere" (Black, 2004a, p.49). Girls who laid claim to a higher profile were "positioned out" as in the case of Sian, a girl whose ability was publicly acknowledged, but was exploited by the teacher to work as a "pace-maker", contributing minimal responses which enabled the boys to continue in more productive dialogue with the teacher: "it is because of Sian's compliance with the teacher's agenda ... that the [high performing] boys … were able to engage in more dialogic talk .... it involves using the right kind of input (or response from the teacher) to signal and be recognised as 'high ability'" (Black & Radovic, 2018, pp. 280-281).

Similarly, Foyn (2021) investigates how being good at mathematics is performed in a Norwegian lower secondary school. She argues that gender is refracted through a cultural model of "smartness" signified by "effortless" work, interrupting the teacher or challenging her mathematical competence, leading to a collectively held claim that there are no gender differences, but that the best students are boys (since they act in this way). Hence the gendered nature of high achieving girls' self-censorship away from activities that are connected to the performance of smartness goes unchallenged. In an earlier study, Foyn et al. (2018) focused on "clever' girls" positionality in an upper secondary school classroom in Norway, finding that they subtly positioned themselves as clever without performing in the same way as the boys in the classroom. However, their performance was restricted by a gendered discourse in which they both "policed" each other and "self-policed" in order to (re)enforce particular rules for combining being female and being good at mathematics: performing in terms of visible "natural ability", flair and competitiveness was unacceptable as "feminine" behavior, and indicators of ability (e.g. ability group membership, marks) were expected to be noticed but not commented on. The case of girls' visibility in mathematics is also elaborated on by Walls (2009), who identifies gender differences in the way students express their response to mathematics. She argues that, in order to survive, girls and women in mathematics "are required to don a cloak of invisibility that affords them temporary status as honorary males in a male domain" (Walls, 2009, p.47).

The teacher's role in how students develop their positionality is illuminated by Jaremus et al. (2020). They found that teachers assumed three main categories of students: the gifted, characterized by their perceived natural ability, speed and achievement; the "dedicated", characterized as hard working; and the utilitarian, having specific career goals (mostly "masculine") which required mathematics. These subject positions "were not equally available to girls and boys" (p. 226): the utilitarian and gifted groups were predominantly male, while the dedicated group was mostly female. Jaremus et al. argue that the "naturalization" of mathematics as masculine excludes girls from the "gifted" subject position, whereas the normalization of effort makes the dedicated position available to them; the utilitarian position is available only if they can subscribe to the normalized aspirations to male-dominated careers. Both Walls' and Jaremus' research took place in Australia, while Black worked in Britain. As noted above, Scandinavian countries, Norway included, lay a strong claim to gender equity, yet Foyn's earlier work questions this. In this paper we explore the twin dynamics of girls' (self-) imposed invisibility and teachers' assumptions about their capability in the Norwegian context further. Hence our research question is: What are the dynamics of gender (in)visibility in a Norwegian classroom?

Theoretical framework – positionality in a figured world

In this paper we draw on Holland et al.'s (1998) theoretical framework to see the mathematics classrooms as a figured world, a "socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (Holland et al., 1998, p. 52). A figured world is "a social reality that lives within dispositions mediated by relations of power", where actors see themselves as having "more or less influence, more or less privilege, and more or less power" (p. 60). Importantly, though, figured worlds are not independent of other worlds, and the structuring effects of major discursive forces such as class, gender and ethnicity which underpin power and privilege in surrounding worlds impact on local worlds too. Thus,

> social categories also can have meaning across many figured worlds. [...They] separate those who are routinely privileged from those who are not. Cross-cutting markers tend to become stereotypically associated with these social categories, if not actually demanded (p. 130).

In the Western world at least, gender as a positional force may lead some female students to see themselves as not having access to significant acts in the classroom such as participating in discussion about mathematics: "gendered dispositions to participate, or not, in given activities, develop in places where gender participation in activities is treated as a claim of gender specificity" (Holland et al., 1998, p. 143). Actors in a figured world get to know "their" position in relation to others as they participate in its everyday practices; in a mathematics class, students (and the teacher) learn to live out the figured world in terms of what they are "allowed" to say or do, what is expected of them, and what is valued. To understand girls' positionality in a mathematics classroom, we need to notice the mundane activities of the classroom, its norms, rules and habitual acts:

> They come to have relational identities in their most rudimentary form: a set of dispositions toward themselves in relation to where they can enter, what they can say, what emotions they can have, and what they can do in a given situation (Holland et al., 1998, p.142–143).

Thus students' acts in the classroom are based on a blend of figurative identity - "signs that evoke storylines or plots among generic characters" - and positional identity - "acts that constitute relations of hierarchy, distance, or perhaps affiliation" (Holland et al., 1998, p. 128). Hence being a female "clever student" or a male "clever student" is "normally" played out differently in the same figured world, and position becomes disposition, ways of being that are frequently unconscious and "out of awareness" (p. 139). Habitual acts may thus lead towards situations of exclusion and inclusion of which actors in the figured world are unaware. As Holland et al. point out,

> even in situations where all students are admitted to the arena of learning, learning is likely to become unevenly distributed .... Teachers will take some students' groping claims to knowledge seriously. … Others, whom they regard as unlikely or even improper students of a particular subject … are less likely to receive their serious response (p. 135).

The mundanity and ordinariness of acts of exclusion or inclusion mean that noticing, resisting and countering these norms is unlikely or difficult, because "the everyday aspects of lived identities . . . may be relatively unremarked, unfigured, out of awareness" (p. 140). In this paper, we focus on the ordinariness of acts of inclusion/exclusion in the mathematics classroom on the basis of gender, and how this mundanity appears to prevent such acts from becoming visible.

Methodology

The data for this paper derive from a larger ethnographic study tracking a Norwegian lower secondary school mathematics class (“Class A”) from grade 8 to grade 10 (Foyn, 2021). Just outside Oslo, the school is in a high socio-economic status area with a fairly homogenous population mostly comprising native Norwegians and native Norwegian speakers. The school prides itself on its high grades. Students are assessed in the national system with grades from 1-6, where grade 6 is the best. Grades 5 and 6 are considered to be at a “high level”. Grades awarded correspond with the goals set for each grade, so achieving grade 5 in two consecutive years requires making the improvement required to meet the higher goals of the later year. All students follow the same curriculum. We draw on a variety of data collected by the first author in this ethnographic study: fieldnotes from participant observation; focus group interviews with the students in 8th and 9th grade; individual narrative interviews at the end of 10th grade; interviews with the teacher, Miss A, at the end of 8th and 9th grade; and copies of the teacher’s assessment record and students’ diary notes.

In this paper, we focus on one case study student, Sarah, whose 10th grade interview about her grades and performance, work effort and experience of mathematics is of particular interest because of her comments on her marks. However, the ontological implication of taking a figured world approach means that it is not possible to investigate any act, event, or statement in isolation, because it occurs between people in a context over time. Thus analysis focuses on story structure, collective storying and their connection with collectively spoken and enacted norms and values.

Sarah’s and the other students’ stories were analyzed in terms of narrative structure. Operationalisation of important concepts are exemplified in Table 1.

Table 1: Operationalization of narrative concepts

Concept	Definition	About	Operationalization
Positional utterances	About relations to groups or actors	Positionality	Describing/explaining oneself in relation to subgroups/persons
Flow	Narrative structure, choice of incidents, combination of ideas	Style of authorship	Sequencing of events, connections
Contradictions	Conflicting/contradictory issues in the talk	Ruptures	Contradictory claims/voices

Interviews with Miss A and the students take place within the figured world of Class A. Table 2 illustrates the operationalization of the central concepts of norms/rules and values in the figured world.

Table 2: Operationalization of central concepts of figured worlds

Concept	Definition	About	Operationalization
Norms/Rules	Expected actions or moves	Habitual events	**Observations** - what is repeatedly observed? Expected actions based on previous experiences in observations **Narratives** - descriptions of characteristic actions in the class **Co-constructions** in focus group interviews

Values	Perceived importance of actions or objects	How artefacts are employed, positional acts	**Observations** –acts that seem to give credit to the actor **Narratives** – description of what is important to do in order to claim status **Co-constructions** in focus group interviews

In addition, the analysis draws on fieldnotes and observations of student movement within the classroom to gain a broad access to the figured world of Class A. Interviews were transcribed in Norwegian and translated into English. In our translations we have aimed to keep as close as possible to our understanding of intended meaning in the original Norwegian. This study followed the research ethics practices of the Norwegian Centre for Research Data, and all names are pseudonyms.

Analysis: Sarah in Class A

> *SARAH: Is seen as mediocre by the teacher. She keeps on working. Enjoyed maths in primary school**. I think she is overlooked by the teacher**, because when I talk to Sarah in the classroom, I get the feeling that she is getting the concepts and does understand the connections. Easygoing, natural way of acting. Is improving her grades, got a rock solid 5 in the final test this year.*

(Fieldnotes, end 9th grade)

This summary impression of Sarah remained unchanged through 10th grade, with reference to both the way she acted in the classroom and her assessment record: she performed steadily at grade 5 throughout the year. Not being noticed seemed to be her "destiny" in this mathematics classroom, and her interview at the end of 10th grade revealed that Sarah was aware that this was the case.

Sarah's story – everyone is surprised by her good results, except herself

A typical feature of Sarah's story is her straightforward attitude when she describes her work in mathematics. Even though she is ambivalent about the way they work with mathematics, she just gets on with it: *"In 10th grade we kind of had to learn it quickly and then it wasn't as much fun because I didn't quite get it, but I learned it"*. It seems that Sarah tends not to like the fast pace, but she accepts the situation and goes along with it. However, this doesn't affect her performance, because her marks indicate improvement in Miss A's assessment protocol and Sarah says that this will continue: *"I think it might go upwards if I'm working to make it go up"*. Despite this confidence, she hesitates to position herself among the students who are doing well in mathematics, instead positioning herself as ordinary: *"I guess I've always been somewhere in the middle in maths, really. I find something difficult while something is very easy, surely like most of the other students, so like many or most of them, actually"*. Furthermore, she declines to query her marks: "*I don't often dare to say that I deserve a higher or lower grade, I'm more that what she gives me is what I get*".

Given this apparent acceptance of the situation, the most striking moment in Sarah's story is her account of her teacher's excited response to her final test score. Mid-sentence, she suddenly mimics Miss A: [Excited voice, imitating the teacher's bright tone] "*Wow! This is really good, aren't you surprised?* [Continues in her own tone, with indignant emphasis] *I was just like, 'no thanks!'* ***I** wasn't surprised"*. Her rejection of Miss A's storying of her results as a surprise returns when she is asked how she thinks the others in the class, including the teacher, see her: *"I think she [the teacher]is a bit like the others in the class, who think I'm a bit dumber or not as good as I am"*. She continues; *"people might think I'm going to get slightly worse grades than I get, or they go like* [parodying puzzlement]*'are you smart?'"* Clearly aware of her positioning by others, Sarah's resistance goes no further than parody. To understand further, we turn to an analysis of the figured world of Class A.

The figured world of Class A - Sarah's position within the performance of smartness

Although the students describe Class A as a unit storied with a 'we' in which everybody does their best and works together despite differences, this image cracks the moment achievement is mentioned. Eva admits that *"There are some who get 6s in every single test, also there are quite a few who are average, and also some who can't do it, the special group"*, while Elias says "*There's a group that is quite a lot better than the others, at a higher level than the others ... They do more difficult tasks, help others a bit more, give explanations and discuss a bit more with Miss A"*. Sarah is aware of this group as well. Asked who is very good at mathematics in Class A, she replies, "*I feel boys or people think that. At least in our class, the guys are the smart ones good at maths, but I think that it differs from class to class"* She adds: *"We have a lot of very extraordinarily smart boys, at least, who are doing maths for upper secondary school and things like that, so I think a lot of people think they're smart"*. She describes the boys as smart, but it is notable that she doesn't accept this argument unreservedly - twice she says this is "what people think". She goes on, perhaps reflecting her own experience: "*But I think the girls are keen to do well, maybe, more than the boys too"*.

Miss A's account adds to this complex picture of how things are seen in Class A. In her interview at the end of 8th grade she is asked if there is any subject the students connect to status. She replies: *"In this class we have a whole bunch of special boys …, who are very interested in mathematics and science. And getting good grades in mathematics is high status"*. As for the girls, she says: *"I have the impression that they like to do well, but I haven't picked up any indication that mathematics is particularly significant.* She goes on: *I think maybe they are thinking a bit more in the direction of language, for those who like to write"*. She repeats this account of the boys in her 9th grade story of Class A: "*I have to mention this group of boys, "the smart boys"; they are a driving-force, academically. They easily affect others in a positive way"*. Miss A's comments on the girls' assumed favoring of language over mathematics are by her own admission speculative, and appear to be based on the fact that the girls do not *act like* the boys. Both teacher and students described how this group of boys performed smartness through acts which have particular significance in Class A: acting as "assistant teachers", engaging in discussion with Miss A and so on. Miss A stories the girls very differently. Only two high performing girls are presented as high achievers in mathematics alongside the "smart boys" in Miss A's narrative of the class, but they have a less prominent position than the boys, being mentioned in either the 8th or the 9th grade, but not both. Neither are described as particularly interested in or focused on mathematics, and they appear in Miss A's narrative as stereotypical girls in mathematics While the "smart boys" are presented as enjoying discussion of a subject they are interested in -*"Erik and Ross, they are the same, they think that the subject is interesting and like to enjoy it and discuss"*, Emilia is presented as hard working - *"I have to say Emilia works extremely hard and tackles challenges head on and wants to stretch herself"*. Equally successful, Kine is portrayed as lacking in confidence: *"Kine can feel a bit of performance pressure … when she really trusts herself and comes up with something it's really great"*.

Although Sarah is also a consistent high performer, she is not mentioned alongside the "smart boys". In 8th grade Miss A describes her as being in the group of students who are in the middle, both in terms of achievement and how they work in mathematics. She comments that some of the girls, including Sarah, are doing better in other subjects: *"I think I can say that Sophie, Maya, Kine, Josephine and Sarah are all typically better in social science and religion"*. At the end of 9th grade, Sarah is still not mentioned among the best students, even though we know that her achievement has improved. Instead, repeating her emphasis on hard working girls, Miss A places her among a group of girls she labels "the sporty hard-working girls", characterized as *"just chatting girls, laughing […] they spend a lot of time together in their spare time. Also, they are sporty girls"*. Sarah, whose results in mathematics are improving all the time, is barely mentioned in Miss A's storying of Class A, even though achievement is clearly important in this classroom culture. It seems that good marks are not enough for Sarah to be recognized as a good student in mathematics, since she is positioned outside of the highly gendered performance of "smartness".

Discussion: the twin dynamics of invisibility and failure to see

In this paper we have focused on just one girl, who interested us because of her critical parody of her teacher's surprise at how well she had performed on a test. Our analysis is not intended as a basis for generalization about girls' experiences – other girls in Class A have different experiences. However, Holland et al.'s (1998) framework emphasizes that Sarah's positionality cannot be understood in isolation from her context; it takes place within the dynamics of the classroom as a figured world, hence our research question: "What are the dynamics of gender (in)visibility in a Norwegian classroom?". We have seen how Miss A fails to see her achievement as worthy of the label "good at mathematics", but Sarah herself doesn't resist her positioning as a mediocre student beyond a private parodying of the teacher and the other students, even though she is aware and resents the fact that her competence in mathematics is not recognized. For us, Sarah's positionality is a double bind: she is caught between others' failure to see - students and teacher are blinded by the "smart boys" performance of smartness - and her invisibility in that she is unable or unwilling to perform smartness.

Holland et al's (1998) theory provides tools which enable us to understand the dynamics behind this double bind. We argue that Sarah's positionality and the fact that her mathematical competence goes unrecognized in Class A are two sides of the same coin. Miss A and Sarah both act within the norms and values of the figured world of Class A, caught by the same dynamics of power and privilege in connection to the smart boys' performance of smartness. It is as though this performance is a significant marker of being good at mathematics which goes beyond results. As Holland et al. (1998) point out, actors in a figured world get to know "their" position in relation to others as they participate in its everyday practices; they learn to know what they are "allowed" to say or do, what is expected of them, and what is valued. These relations take place within the mundanity of the classroom, its rules, norms and habitual acts. In Class A, the performance of smartness is inextricably linked to gender – we see that not just from Sarah's account but from the other students and Miss A herself. Its habitual nature means that exclusion is hidden in plain sight, since Sarah can only make herself visible by her good results, and these go unnoticed, remarked on only with surprise as though this was an unusual event. Without access to those acts which signify smartness - interest, discussion, helping the teacher, Sarah is invisible, and described at best as "hard working", echoing Jaremus et al.'s (2020) finding that female students are excluded from the position of "good at mathematics". We argue that in fact she has no means for countering these norms in Class A - Foyn et al. (2018) drew attention to how difficult and even risky it can be to break out of gender dynamics, or challenge gendered norms in the classrooms, since "discourse border guards" ensure that gender lines are not permeable.

As Holland et al. (1998) emphasize, the mundanity of everyday lived identities makes them difficult to challenge. It might be argued that Sarah could change her behavior in order to publicly resist her positionality, but Sarah's double bind means that this isn't easy. This is not to say that change is impossible, but making Sarah's situation visible goes beyond Sarah's and Miss A's acts alone - it requires a collective recognition and action. The implications of Sarah's story in the figured world of Class A are that gender dynamics in mathematics classrooms need to be discussed in classrooms, school departments, and teacher education; arguably this is particularly so in a country such as Norway where gender inequity is assumed to be in the past, and mundane classroom practices go questioned.

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A philosophical investigation of local mathematical practices

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In this article we explore the possibility of implementing local mathematical practices in the current mathematics curriculum of the studied area. This research relies on two theoretical frameworks that focus on the value of culture in relation to the study of science and scientific practices in relation to the learning process. The analysis is based on an empirical long-term ethnographical investigation in the Northern Ambrymese society (Ambrym Island, Vanuatu, South Pacific). The local activities investigated in this empirical research can be described as string figure-making. By improving our understanding of the implementation of local mathematical practices in the current mathematics curriculum, we will show the added value and provide a guideline for best practices.

Keywords: Ethnography, local mathematics curriculum, practical turn, social turn, learning theory.

Introduction

Research on local mathematical practices and the implementation of these practices in the main school mathematics curriculum relies on two theoretical traditions. The first of which is the practical turn within the philosophy of sciences; the second, the social turn in learning theory. Both traditions share a similar concern for the fusion of both scientific practices and learning processes in a given sociocultural environment. In the theoretical section, we shall conduct a brief overview of the way in which both traditions evolved over the past decades and how the research we are reporting on is a continuation of these traditions. In the empirical part, we report on the ethnographical investigations of the local mathematical practice of “string figure-making” and its historical, cultural, social and political connections to argue for the value of implementing these practices in the current school curriculum of mathematics.

A practical turn in philosophy of science

The relation between mathematics and culture was one of the first investigations (Powell & Frankenstein, 1997) undertaken, mainly in non-western culture. The topic also became of interest in the study of western mathematical practices (Larvor, 2016a, 2016b) with a large research community (e.g. Association of the Philosophy of Mathematical Practice (APMP) founded in 2010). The practical turn was further developed in situating knowledge related to the body. Haraway (1991) speaks in this context about situated knowledge as knowledge that is constructed from a particular and specific embodiment. It is based on these partial perspectives that an “objective” vision can be constructed. Specifically related to mathematics, Lakoff and Núñez (2000) set the paradigm of the embodied mind holding that mathematics results from the human cognitive apparatus. It was further investigated by de Freitas & Sinclair (2014) by emphasizing a non-reductionist role of the body in constructing knowledge, as we elaborated on in François (2017). The body is part of a mathematical practice that produces mathematical knowledge as specific kinds of “transmaterial assemblages”. This way of building knowledge, as a co-

construction based on an embodied practice can have an important implication when it comes to the teaching of mathematics.

A social turn in learning theory

The second aspect of our article focuses on the implementation of local mathematical practices in the school mathematics curriculum. The theoretical background of this research topic relies on the social turn within learning theory and the emphasis on the fusion of the learning process into a sociocultural environment. Social sciences have developed during the last decades three main learning theories, each of them with a specific focus on the learning process (François & Pinxten, 2017). Whereas (i) behaviorism focuses on the input-output mechanisms ignoring the black box in between (ii) genetic psychology mainly focuses on the black box and what is going on in the student's mind. The (iii) sociocultural theory considers the environment in which the learning process is taking place. The social turn within the learning theory became of interest in the field of mathematics education (Bishop, 1985) mostly as a reaction to the huge dropping out of pupils from a specific socio-cultural background. Even in countries with high scores in international comparative research (which doesn't imply our support of these tests) one can easily conclude that migration and language backgrounds are an important determinant of failure in mathematics (OECD, 2014). Sociocultural theory elaborates on the learning theory of the Russian Lew Vygotsky (1978) and his concept of a *zone of proximal learning* (ZPL). The concept can be understood as the cognitive field of the pupil, which can be spotted at the fringe of the background knowledge and the out-of-school worldview. It is the zone of learning where the pupil will be able to connect insightfully to new knowledge because of the intrinsic relation between background knowledge and new inputs. Background and out-of-school knowledge are integrated in formal learning as a stepping-stone for acquiring new knowledge, new meanings and new mental frames. The concept of ZPL was an inspiration for later developments of socio-cultural learning theory and such central concepts as the pupils' background and foreground (Vithal & Skovsmose, 1997; Skovsmose, 2005). Lave & Wenger (1993) elaborated on the ZPL concept and developed a more specific notion of *legitimated peripheral participation* (LPP). They emphasize that the learning process is always a situated learning that considers the student holistically. The student is an agent who is active within a specific world context and all these aspects are mutually constitutive for the learning process. Learning is not perceived as the reception of factual knowledge or robust information. It is a social activity that takes place within a community, at first legitimately peripheral. Later, it increases gradually in engagement and complexity. Participation in social practices is the fundamental form of learning. It implies more than connecting the immediate context to the instruction. It is even more important "to consider how shared cultural systems of meaning and political-economic structuring are interrelated, in general and that they help to constitute meaning within communities of practice." (Lave & Wenger, 1993, p. 54). This concept of LPP will be an important tool to analyze our data on informal learning practices in the local communities and to understand the importance of implementing them into the formal schooling and learning environment. François & Pinxten (2017) state that we have to consider out-of-school knowledge and skills the child possesses and uses when first coming into contact with mathematics education. We have to investigate the background knowledge that children actually bring to school and how we can introduce their learning context in the teaching of mathematics at school. Blindness and ignorance concerning the local culture, local practices and knowledge may well explain the gap between success and failure, in a formal mathematics classroom. In the following section we will examine the possibilities of implementing local practices within the school curriculum.

String figure making as a local mathematical practice

For over a century, string figure-making practices have been observed by anthropologists in many regions of the world, especially within "oral tradition" societies (Paterson, 1949; Maude, 1978; Braunstein, 1992). Some mathematicians have also regarded string figure making as a worthy topic within their discipline. At the beginning of the 20th century, Cambridge mathematician W. W. Rouse Ball (1850-1925) devoted a chapter to string figures in his popular book on mathematical recreations (1911) which is—to our knowledge—the first attempt made by a mathematician to demonstrate the connection between mathematics and string figure-making (Vandendriessche, 2014a). Thereafter, a few mathematicians have developed mathematical and modeling tools in order to formalize this practice (Amir-Moez, 1965; Storer, 1988).

In the Vanuatu Archipelago (southwestern Pacific), string figure making was first documented in the 1920s by anthropologist L. A. Dickey (1928). Called the New Hebrides by the British navigator James Cook in the 1770s, this archipelago—consisting in a chain of 80 inhabited islands, located 1,750 kilometers east of northern Australia—was managed through a Colonial French & English Condominium from 1906, until its independence in 1980 and the foundation of the Republic of Vanuatu. Consequently, this country has been engaged in a (long and difficult) process of decolonization, and particularly in the field of Education. For about a decade, the Republic of Vanuatu has undertaken a reform of the National Curriculum (inherited from the colonial period), with the goal of taking more into account the various local cultures *i.e.* the various vernacular languages as well as traditional knowledge and practices.

Conducted in Northern Ambrym (Central Vanuatu), since 2006, Vandendriessche's ethnomathematical project is devoted to the study of different practices with a mathematical character such as the making of string figures. It is based on a long-term ethnography (around the village of Fona) and carried out in collaboration with both local people (practitioners and local educators in particular), and actors of the "Vanuatu Cultural Centre" (national institution working for the preservation and the promotion of different aspects of Vanuatu's culture). In François, Fantinato, Vandendriessche & Mafra (in press) we investigated the specific topic of "The researcher as the 'other'" as well as the ethical issues raised while conducting ethnomathematical field research, and collaborating with local people and institutions. We will therefore not elaborate further on these fundamental issues here.

In Vanuatu, making a string figure requires creating a loop by knotting the ends of an approximately two-meter-long string—which is made with a thin slice of a pandanus tree leaf. The activity then consists in applying a succession of operations to the string, using mostly the fingers, and sometimes the wrists, mouth, or feet. This succession of operations, which is generally performed by an individual and sometimes by two individuals working together, is intended to generate a final figure, whose name refers to a particular being or thing. Our ongoing ethnomathematical project aims at collecting various types of data: 1) the procedures leading to the various figures (using an original symbolic writing system for noting/recording them) 2) the vernacular (technical) terminology linked with the studied practices 3) the oral texts and/or discourses which are sometimes associated with the latter. At a second stage, the collected data are analyzed—and put in perspective with other ethnographical sources—to comparatively analyze the mathematical dimension of the latter procedural activity in their relationships with other forms of knowledge (Vandendriessche, 2014b). Indeed, a string figure can be seen as the result of a "procedure" (or "algorithm") consisting of a succession of "elementary operations". Most of these operations can be defined as "geometric" operations whose purpose is to modify one configuration/state of the string to transform it into another (see database in progress http://emergences.huma-num.fr/items/show/13, Vandendriessche, 2015).

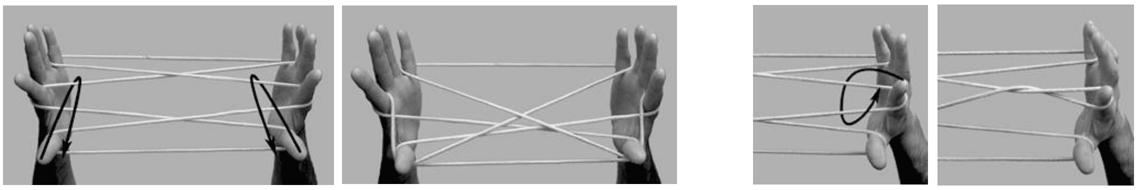

Figure 1: Elementary operations. Left: "picking up" a string. Right: "twisting" a loop

The activity of creating new string figure procedures can be regarded as mathematical at different levels. Their production very likely required an intellectual task of selecting the elementary operations and organizing them in procedures and "sub-procedures" (*i.e.* ordered sets of elementary operations either iterated within a given procedure or repeated identically within several different string figure algorithms of the same corpus). Based on an algorithmic practice, the production of string figure algorithms is also of a "geometrical" and "topological" order, insofar as it is based on investigations into complex spatial configurations, aiming at displaying either a 2-dimensional or a 3-dimensional figure. Several recurrent phenomena confirm this point: the concept of "iteration" (iteration of a pattern or a sub-procedure) and the concept of transformation (of the final figure "geometry" *i.e.* combination of motifs) are ubiquitous in this practice. Finally, some Ambrymese string figures suggest that practitioners have elaborated some procedures by altering one or several operations involved in the making of another string figure.

Local mathematical practice as knowledge transmission

In Vanuatu, there are no less than 120 different vernacular languages, corresponding to different cultural areas in the archipelago. A large number of string figures can be found in various linguistic areas, whereas a few string figures seem to be more locally practiced. However, there are significant linguistic variations (from one area to another) related to string figure making, in the names given to the activity as well as to the final figures, and in the use of technical expressions. In Northern Ambrymese society, string figure making is locally termed using the vernacular expression "*tu en awa*" (literally "to write with a string"), suggesting that this activity is perceived in this society as an encoding of information. Some other vernacular expressions are used by practitioners to refer to the (basic) movements involved in string figure making. In particular, the (elementary) operations implemented to the string are designated through action verbs; the subjects being the finger names (for instance, *pokolam hu pokokiki*, the thumb picks up the little finger *i.e.* implicitly a string running from the little finger). A few (short) "sub-procedures" (ordered sequences of a small number of elementary operations) are also named. These technical expressions are used in instances of transmission from one person to another, although not consistently. The "*tu en awa*" procedures are indeed taught or shown most often without any technical comments. However, the existence of these expressions is an indicator of the perception by the actors of orderly sequences of operations, suggesting a local perception of the notion of "elementary operations" and "sub-procedures" revealed by ethnomathematical analysis. Furthermore, vernacular terms explicitly express the property of "symmetry" (shared by a number of these figures) and the "iteration" of a pattern or a sub-procedure. It seems that string figure making is mostly transmitted to children by their mother and/or grandmother. The long "sub-procedures" (made with more than three elementary operations) are nameless for the Ambrymese practitioners. However, they spontaneously relate one string figure procedure with another when both "*tu en awa*" procedures share an "ordered set of basic movements". The practitioners do so by pointing out that from a given stage of a string figure procedure, "You do it in the same manner as you do it in another procedure". While working with Ambrymese children, we often noticed their ability in making such links between string figure procedures. This suggests that "sub-procedures" play—for these practitioners—a major role in the process of memorizing the making of string figures. In Ambrym (and more generally in Vanuatu, and even in Melanesia), the practice of string figure making is—or was—meant to record, memorize and/or express a particular knowledge of mythology, cosmology, geography, social rules, and ritual prescriptions (Vandendriessche, 2014b, 2015). For instance, the Ambrymese string figure named *bulbul algon* (literally "canoe lizard") is related to the story of Yaulon, one this society's mythical heroes. *Bulbul* designates this hero's canoe, while *algon* (lizard) recalls the symbol of one of the seven grades of the chieftainship system, whose conception is attributed to this local mythological hero. In this society, string figures are preferably performed during the yam harvest (from February to July), while their usage is prohibited outside this period, the making of such figures being perceived as having a negative impact on the growth of the plant's stem winding around the stake: it would favor the entanglement of the stem, slowing down the plant's growth. The practice of string figure-making can thus be analyzed as a method for the organization and the transmission of knowledge (mythological, cosmological, etc.), involving the use of (ethno-) mathematical concepts. When learning how to make string figures, Ambrymese children become acquainted with a technical activity (with a geometric and algorithmic character), requiring dexterity and concentration, and, at the same time, they develop their knowledge of their cultural environment.

A pedagogical application

Based on social learning theory and more specifically on the concepts of ZPL as developed by Vygotsky (1978) and later on elaborated on by Lave & Wenger (1993) as outlined in the theoretical sections, we can now analyze the added value of implementing local mathematical practices in the school curriculum. From social learning theory, we have evidence that pertains to the pupil's background information (Vithal & Skovsmose, 1997; Skovsmose, 2005), which is essential for the learning process. Pupils attending classes need to be apprehended in their entirety, as agents who develop extracurricular skills, and who live and interact in a complex environment (Lave & Wenger, 1993), where they share common cultural systems and social rules. Most of which are handed down through informal learning systems by the community. The Vanuatu National Curriculum (VNCS, 2010) is a good example of what we mean with the integration of local practices into the formal curriculum. It recognizes the value of traditional knowledge and practices—such as string figure-making, mat making, and sand drawing in particular—for calling upon the latter practices in formal education. Teachers should begin with the children's background before introducing unfamiliar knowledge, skills and attitudes. The VNCS (2010, p. 52) claims that "Teachers need to breathe life into the curriculum and demonstrate its relevance to children and students by using local examples whenever possible". It is implicitly recognized that some of these traditional practices are sometimes in decline and prompt the members of each community to "assist the children in learning about these art forms and make simple mats and other objects". Therefore, the Vanuatu Cultural Centre has given its (mandatory) assent for our project in ethnomathematics providing it leads to pedagogical applications. Beyond this institutional incentive, another motivation for undertaking such educational research is that Northern Ambrymese people generally welcome with interest the idea of using their traditional practices in the curriculum. Whereas the purpose of ethnomathematical (theoretical) research is clearly not perceived as vital as we think it is, indeed, its educational valorization makes sense for these people because they consider it as a way of preserving their local culture. Aware of these practices' decline in their community and asserting that young people are no longer interested in traditional knowledge, they consider this valorization as a way of preserving their local culture. In this context, pedagogical materials will be elaborated in an attempt to help local teachers in experimenting with the use of culturally related mathematical string figure-making practices "as such", in and of themselves. A pedagogical mathematical sequence (say in Northern Ambrymese 6th year classrooms) related to string figure-making could start by the collection of the string figures (and their vernacular names) that the pupils do remember. They should be prompted to use their local vernacular language for expressing the various operations involved in the making of these figures, as well as the symmetries and the iteration of patterns. The set of string figures thus collected could be then completed with other "*tu en awa*" procedures known to the elders of the community. Previous studies on string figures' value in mathematics education suggest that practicing string figure making may develop vital skills necessary for practicing mathematics—such as concentration, self-evaluation, spatial relation consciousness, or conducting step by step ordered sequences of instructions (Moore 1988; Murphy 1998). Beyond this analogy between string figure-making and mathematical practice, the pedagogical sequence might continue through an in-depth analysis of a set of procedures, bringing to light the (elementary) operations involved, their impact on the string configurations and their organization in sub-procedures. The teacher might induce pupils to reflect on how a string figure has sometimes been transformed into another, and how the iteration of an ordered set of operations may allow the iteration of a given motif. Ambrymese string figures sometimes differ on one—and only one—elementary operation. This remarkable property implies a methodology to create new string figures (Ball, 1911; Murphy, 1998) by altering a few operations within some "*tu en awa*" procedures, the pupils would become creators of new string figures themselves. This pedagogical (ethno-) mathematical sequence on string figure-making will not be isolated from the other forms of local knowledge imbedded within this activity. One way to do so, would be to include this sequence in an interdisciplinary pedagogical project, bringing into the classroom the cultural and cognitive complexity of this practice.

Concluding remark

In this concluding remark, we will emphasis the added value and we will offer a guideline to implement local practices in the school curriculum. As we could see from the empirical example, local mathematical practices are embedded in the social and cultural environment. They take part in the communities' social networks, their histories and their languages. They enmeshed with their natural environment and have a deep connection with nature, with a circular timeline, and with seasons, e.g. string figures are performed during the yam harvest (February–July). The idea of sustainability is implicit. Local practices are also part of the social interactions, social relations and of the informal learning processes in these communities, outside of school. String figure making is practiced by children of the Ambrymese community to play without any deeper meaning given to this practice. Connections to the broader cultural tradition are analyzed in the case of string figure making, e.g. knowledge transfer, storytelling–related to mythical heroes of that society, and encoding information. Out-of-school practices can be of interest in the formal mathematics curriculum. String figure making, with an emphasis on the mathematical analyses of the procedures (e.g. symmetry, iteration) shows how a local practice can be analyzed as a method for the organization and the transmission of knowledge. Implementing these practices in mathematics and school curriculum acquaint pupils with a mathematical (e.g. geometric, algorithmic character) activity. At the same time, pupils develop knowledge of their own cultural environment and they learn to value local traditional practices and knowledge as a means of preserving their culture. This way we create a continuum between informal and formal learning which relates to the socio-cultural learning theory, the concept of a *zone of proximal learning* (ZPL) and the concept of *legitimated peripheral participation* (LPP). The local practice we studied is performed in a situated context, it is connected to the daily life experiences and related to broader cultural traditions and transmission of knowledge. The guideline we take from this analysis is that local practices can serve as a tool to connect to pupils. When implemented in a formal school curriculum, they remain connected to the pupils' background and their out-of-school worldview. Pupils will be able to connect insightfully to new (mathematical) knowledge because of the intrinsic relation between local practices and the mathematical procedures that underlie the local practices. Even new pedagogical materials shall never be isolated from the other forms of local knowledge they are imbedded in. This clears a path for interdisciplinary activities and the integration of mathematical knowledge in a lively and meaningful context. A situated learning process considers the student as a whole person as is the case in the out-of-school transmission of the local practices we studied. The challenge for further research will be the application of this study to a variety of cultural environments and how the implementation of local practices in formal (mathematics) curricula should work.

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How to educate preservice mathematics teachers? - Identity perspectives in a mixed concurrent setting

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Different systems (e.g. concurrent and consecutive), knowledge (e.g. CK, PCK, SRCK, PK) and interventions (e.g. bridging courses) in teacher education are widely discussed. Facing a loss of interest in mathematics and often claimed irrelevance of subject specific content, we investigate preservice teachers' participation in university mathematics from an identity perspective, taking their social context into account. Three group interviews of 14 preservice teachers in a mixed concurrent setting (one course specific to preservice teachers, one general mathematics course for major students as well) were analyzed, showing that they experienced a position of being less valued in the mixed course. We identify different ways of students' self-authoring, which partly explain their disaffection and consequent learning behavior. Consequences for institutional settings are discussed.

Keywords: Preservice teachers' education, university mathematics, identity, participation.

Educating preservice mathematics teachers

Although there are many different school and training systems internationally, the discourse about content or pedagogical knowledge, pedagogic content knowledge or school related content knowledge (Dreher et al., 2018), scientifically or practically oriented teacher training is evident in almost every context – especially for preservice higher secondary teachers. While there are additional or dual training programs in some countries, the main form of teacher education in most countries remains the preservice education at universities (Durand-Guerrier et al., 2010). These trainings can be organized differently, mainly in an either concurrent or consecutive way. The concurrent model of initial teacher training involves a study program that combines general, theoretical, and practical training from the outset. Students gain a teaching specific degree, mainly with two different subjects. This is the case for example in Austria, Poland, Sweden, or Turkey. In the consecutive model, preservice teachers first obtain a general qualification (e.g. a university degree in mathematics) followed by further studies to gain an additional qualification for teaching. They often (but not always) study one single subject. Examples for consecutive systems are France, Georgia, Italy, Malaysia, or Singapore (see Tatto et al., 2012 for details).

Largely independent from the concrete organization, the often-challenging transition from school to university mathematics and the overall faced double discontinuity (Klein, 1932) reveal further challenges for preservice mathematics teachers. It seems, that many students worldwide question the relevance of their studies, being dissatisfied and feeling treated as second-class students which are no "real mathematicians" (Tatto et al., 2012; Bauer & Hefendehl-Hebeker, 2019).

In this paper, we focus on the mainly concurrent system for preservice higher secondary teachers in Germany with its rather mixed setting: In contrast to the consecutive system, preservice teachers enroll in specific teacher training programs at the beginning of their studies and have to choose two subjects. However, they usually attend some subject specific courses together with major students of the respective subject – sometimes also subject-specific courses designed specifically for preservice teachers. Additionally they have, courses on subject-specific pedagogical content knowledge as well as general teacher training courses on pedagogical content knowledge together with preservice teachers of other subjects. Hence, preservice mathematics teachers share part of their courses with mathematics major students.

Differences in cognitive prerequisites between mathematics majors and preservice teachers are rather small at the beginning of their studies (Bauer & Hefendehl-Hebeker, 2019). However, an overall disaffection and different participation, based on a perceived exclusion and lack of relevance, can be seen for some preservice teachers, after the first semester: they lose interest in university mathematics more than the major students do, describe themselves as being excluded from the mathematics community, especially in courses that are shared with mathematics major students (Ufer et al., 2017; Liebendörfer, 2014), they think about dropping out more often, report to both copy homework and use surface learning strategies more often. They report to be very disaffected with their study content, criticize university mathematics as being irrelevant for their future profession, and demand more practice-related content (Gildehaus & Liebendörfer, 2021 with further references).

To better cater to preservice teachers' demand for more practically relevant content, many universities adapt or supplement their teaching, e.g. with specific tasks connecting university mathematics with school mathematics. However, the main critique often sustains, right after specific tasks are finished (Bauer & Hefendehl-Hebeker, 2019). Making the relations to school content more visible may raise students' utility value but not their interest or personal relevance (Rach, 2020).

To better understand preservice teachers' disaffection with and perceived exclusion from mathematics, we suppose to broaden the perspective by including the social context: Preservice teachers need to negotiate and integrate their own competing interests and self-images between their subjects and pedagogy facing competing subject cultures and corresponding attributions. This may explain specific values and motivation that could explain different participation, as well as an overall rejection of university mathematics content (Gildehaus & Liebendörfer, 2021). We therefore take identity as a theoretical perspective to examine preservice teachers more closely within the social context of university mathematics (Graven & Heyd-Metzuyanim, 2021). Investigating their sometimes-perceived exclusion from university mathematics in line with different participation, we focus students' positioning and authoring.

Identity in figured worlds of university mathematics

"Identity is a concept that figuratively combines the intimate or personal world with the collective space of cultural forms and social relations" (Holland et al. 1998, p. 5). Hence, identity covers specific career aspirations and interests of preservice teachers as well as the social context of university mathematics and the exclusion mentioned above.

According to Holland et al. (1998), identity is mainly produced and developed in figured worlds which are “socially and culturally constructed realms of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others” (p. 52). University mathematics as a figured world possibly includes major students, preservice teachers, professors, and tutors as relevant actors, where specific outcomes (e.g. knowledge, solutions of tasks) would be valued.

The transition from school to university mathematics can be seen as a phase of identity development, a situation where a new figured world is entered and orientation of specific actors, roles, positions, and practices are sought. Students may search for “new guidelines” (Hatt, 2007) within this new world of university mathematics. Certain roles within this figured world help to define who one is. The so called “positional identity is a person’s apprehension of her social position in a lived world: that is, depending on the others present, of her greater or lesser access to spaces, activities, genres, and, through genres, authoritative voices, or any voice at all” (Holland et al., 1998, p. 151). Along with day- to-day social interactions, one claims position and positions others by social interaction and on the ground relations of power (Hatt, 2007). In terms of the experienced exclusion that some preservice teachers described and the feeling of being perceived as “less valued”, it may thus be a question of their position and positioning within the figured university mathematics world.

Based on Bakhtin’s thoughts on dialogism, the concept of “authoring” refers to how individuals construct their own identities (Urrieta, 2007). Since one’s positioning within a world may also determine one’s space of answering and negotiating one’s own identity, Holland et al. (1998) introduce the “space of authoring”. Hereby, authorship is not a choice, because within social situations every action or inaction would be a reaction and response to the situation (Urrieta, 2007). Thus, the interactions between characters and actors (e.g. preservice teachers, major students, professors, and tutors) of the figured world, might indicate insights of the individual’s space of authoring and negotiating of their perceived position. E.g., the described position of preservice teachers feeling “less valued” may be integrated or negotiated within students self-authoring process.

Research questions

Facing the disaffection of preservice teachers in mathematics from an identity perspective, we aim to examine their position and authoring within the social context of a mixed concurrent model:

RQ1: How do preservice mathematics teachers describe and experience their perceived position and positional identity in a mixed concurrent teacher training system?

RQ2: How do preservice mathematics teachers author their own spaces and voices in this situation?

Method

Our study is located at a German university where in their first semester, preservice higher mathematics students attend one course (linear algebra, LA) together with mathematics majors and one course specific for preservice teachers (introduction to mathematical thinking and working, IC). This latter course is designed as a bridging course, which aims to facilitate the transition from high school mathematics to abstract, formal mathematics at university. Thus, its overall content requirements are lower than the ones in the LA.

Three semi-structured group interviews with four to five preservice mathematics teachers each (n=14; 8 female, 18 to 23 years old) were conducted in winter term 2019/20 at the University of Paderborn two months after the first semester had started. The interviews took place in person, after students answered an open call voluntarily. The call addressed groups of students already e.g. they had been working together on homework before. This was chosen to comfort them to freely discuss their perceived position and limit the influence and power of the interviewer. It also gave the chance to directly contrast different authoring between the students. They were guaranteed anonymity and in no further contact or relation with the interviewer. The participants represented diverse second subjects (e.g. sciences, foreign languages, sports or social science).

The interviews focused on the students' experiences and their identity. The guide included questions like: "To what extent do you identify yourself with university life here? How much do you feel as mathematicians? How would you describe to your parents what mathematicians are and do?" Students were not explicitly asked whether they perceived differences between the two courses. Only during data analysis, this occurred to be a dominant theme and was therefore included.

For data analysis the interviews were coded deductively and inductively (Saldana, 2016), firstly identifying passages along students perceived positions (e.g. their scope of action, their ways of participation), as well as their authoring (e.g. their reasoning narrating themselves). The different themes that emerged within these categories (e.g. writing to the board, asking questions) were then combined into broader categories (e.g. being valued, being competent, and access to spaces) and later analyzed with the specific focus on the different courses. Keeping and gaining different perspectives in the interpretation, themes were frequently discussed with researchers from different backgrounds (e.g. mathematicians and preservice teachers, males and females). In the following, all presented quotes were translated from German by the first author.

Results

RQ1: In line with the separate study programs, the perceived position of preservice teachers was described as "different" and divided from the mathematics major students. This difference was present in all spaces, e.g. during the mixed courses, but also during general learning at university:

> Student 3: So, computer scientists and mathematicians always sit together and preservice teachers always sit together somehow. And I think that it just doesn't come together at all.

Consistent with their further career aspirations, large differences were reported concerning the value of explaining and teaching mathematics to other people:

> Student 4: So, math is fun to me, as a teacher. It's not that I'm happy in my math world like a mathematician, but I would like to teach it to someone.

Within the two courses, they described differences how these attributes were valued in the different courses:

Student 3: In IC, I think, there are only people who are really teachers and you can say that they (…) really try to explain things to you (…). In LA there is just less, yes, somehow motivation to explain.

In line with this, they felt relatively respected and accepted in their position in the IC, even though they were mainly passive receivers of knowledge and realized that the content was "far from what is done in school already":

Student 3: Well, I think it works to a certain extent in IC. So, you can still keep up well.
Student 2: IC gives a good feeling, especially in contrast to LA.

Within their LA course, however, and in general in comparison with the major students, they described to feel less valued and as second-class students. Mainly they referred to other students:

Student 1: I think it's always a pity when people come and say, oh yeah, you're only doing a teaching degree.

Similarly, they referred to the lecturer of the LA course:

Student 6: But you always have the feeling that he doesn't really think much of teachers either. // He doesn't say that precisely, but always
Student 4: So, you always notice that you're talked down to.
Student 6: Like: "Yes, we do that so that the future teachers sitting here can teach their students that, so they can then solve the math problems of tomorrow." And then you think to yourself: "are we too stupid for that or what?"

Preservice teachers' access to activities and spaces was also limited in comparison to mathematics majors, who were remarkably more present in terms of what "counted" in this figured world, hence in relation to the specific power of knowledge (even though they were fewer people): They were for example using the whiteboards in the university, visibly discussing mathematical problems. They were described as "asking questions in the lecture" and "making jokes with the professor", while the interviewed students did not see such voice for them:

Student 5: They [mathematics majors] are more on the professor's level than on ours.
Student 6: Well, they always get along really well with him, and he also makes jokes with them.
Student 4: But we just don't understand it.

Their institutionally different and divided positioning became also clear, as they were not attending the second course for major students "Analysis I":

Student 3: And then he says yeah, you might know that from Analysis (…).
Student 4: Then you sit there and think to yourself, well, I'm studying to be a teacher, I don't have Analysis in parallel (…). How am I supposed to know that?

Hence, some described feeling misplaced, since they felt not addressed or valued by the professor in the LA course:

Student 5: I think it's true that you sometimes have the feeling that you really notice that the lecture is tailored to mathematicians. So that he always addresses them, the mathematicians rather, and you as a preservice teacher are just like (…) okay, somehow I'm a bit out of place here.

RQ2: The most dominant topic in students' ways of authoring was to find a voice at all, since they all felt more or less overwhelmed in their current situations. However, we found remarkable differences between some students:

One way of authoring was that of quickly accepting the perceived position of being less valued and use this image of “just a teacher” to legitimize less participation and lower performance in the LA course, including the acceptance of being perceived as less competent in mathematics:

Student 1: I would never call myself a mathematician, for that I also see more and more in the homework that it’s just not like that (…) I’m just a freshly started teacher student.

Student 1: So, my entire first half of LA [referring to necessary point in exercises] is not based on my own knowledge, but that is just sometimes a thing.

This was underlined with the idea, that the LA course would not be very relevant, and one would somehow do the same in the IC course anyways. Hence, the IC course was seen as very relevant and useful and perceived competence was taken out of this:

Student 1: I now have at least one lecture where I can still follow (…) and once you have that in a somewhat weakened form, it really doesn’t hurt, I think.

Other students, in contrast, struggled with their perceived position and tried to renegotiate it, mainly referring to their self-image of “always been good in mathematics”. These students tried to find a voice and participate, but realized that their actions were not valued:

Student 6: When they [tutors in the LA course] roll their eyes when you ask something.

Student 5: There you already don’t dare to ask a question.

What followed this failed participation was a rejection of the mathematical content of the LA course, as being irrelevant and major students and tutors of this course being unsocial (see Gildehaus & Liebendörfer, 2021 for details). While those students participated successfully in the IC course, they used it to legitimize the irrelevance of the LA course. They authored themselves as mathematics teachers, who are just not competent in the LA course, because it did not suit their specific needs as preservice teachers.

In addition to these two ways of authoring, there was one more way presented by a rather exceptional student: She agreed on being positioned as a (less valued) preservice teacher but tried to renegotiate her position to be recognized as full mathematics student as well. She valued both courses for their individual strengths and reported to participate successfully in both courses (e.g. she was asking questions in the tutorials for both courses and sometimes chatting with the tutors). Thus, this student was also exceptional in that she was mathematically able to participate in both courses. At the same time, she clearly distanced herself from the teaching profession:

Student 2: A lot of people here are like, yes, teacher is my dream job, but I’m not like that. (…) I don’t know yet if I’ll eventually be a computer scientist instead of a teacher, I don’t know. Why can you only be one thing, // why can’t you be everything?

Discussion

Preservice mathematics teachers are often disaffected with their studies, question the relevance of the content, and aim for more practical relevance. Using an identity framework, we expanded the theoretical perspective from the individual to the social context of university mathematics. We analyzed group interviews of first-year preservice teachers in a mixed concurrent setting in Germany, where they attend one specific preservice teachers’ mathematics course and one mathematics course together with major students.

RQ1 investigated the perceived position of preservice teacher within the social space. Even though preservice teachers were the vast majority in the mixed course, they felt positioned as a marginalized minority: they described exclusion (insider jokes, mismatching addressing) and being less valued and perceived less competent by professors, tutors, and major students. RQ2 focused students' different ways of self-authoring along this perceived position. One way of negotiation was to accept the position including lower performance and less participation in the LA course. Another way was to reject being less valued despite finding no voice in the LA course, followed by a rejection of the content and staff of this course. A third way was finding an exceptional position claiming to be both a preservice teacher and a mathematician.

The first two presented ways of authoring eventually lead to accepting and integrating the perceived position of "just being a teacher" legitimating students' minor engagement (e.g. copying homework, as Student 1 describes) with university mathematics contents, as they consider the contents as not relevant for them. Yet, students following the second way were clearly seeking participation in university mathematics. Their rejection and claimed irrelevance followed after not finding a voice.

Despite initially small differences (Bauer & Hefendehl-Hebeker, 2019) preservice teachers are soon positioned as less competent than mathematics majors. Their performance might in fact be lower as they study a second subject and miss synergies with the Analysis course. However, within the mixed setting preservice teachers are so strongly positioned as less valued group that even exceptionally able students can hardly escape attribution to the teachers' group only. The positioning as preservice teacher builds on very visible categories like attending the Analysis lecture and students' physical positions in the LA lecture and thus becomes very dominant. Our research also confirms that almost only high-performing students may legitimately participate in the university mathematics discourse (Solomon, 2007). In concurrent teacher education with mixed settings, preservice teachers may then have severe trouble participating equally. This calls for a new pedagogy that allows legitimate participation like asking "stupid" questions without tutors rolling their eyes.

Specific bridging courses seem double-edged: They help students participating in a discourse. However, the courses may also be used to legitimize the irrelevance of lectures like LA, since they are not "real teaching lectures" and thus not important for "just teacher students". Thus, an intervention to accommodate students in their transition to university mathematics, can also lead to additional distancing.

We conclude that if we want to address the dissatisfaction and alleged content-irrelevance reported by preservice teachers, we should not focus (only) on the usefulness of the content for the profession but find ways for students to participate mathematically on an equal level with any other group attending the same courses. Taking these results into the wider context of concurrent and consecutive teacher training systems, the consecutive model, not having an institutional distinction, could reduce the perceived differentiation between the major and teacher students. At the same time, individually (as well as systemically) different values and images of "becoming a teacher", "valuing explaining" remain and might not prevent from similar positioning and related self-authoring.

Limitations and further research implications

The presented results mirror one specific institutional system, with the data highly pending on their specific social context. Future research should include different institutional systems in an international perspective to draw conclusions on preservice teachers' identity formation if they have only shared courses with mathematics majors or only their own courses. Including the perspective of mathematics major students and teaching staff would also help depicting preservice teachers' identity formation in the shared social space. It remains unclear how major students, that might struggle with the contents could find a voice to participate, as the "just teaching" image is obviously not available for them.

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Diversity-aware teaching of mathematics: an explorative study on pre-service teachers' views

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Preparing future teachers to be aware of students' diversity and enabling them to teach responsively in order to create equal opportunities is considered an important part of teacher education. Realizing this for mathematics education is commonly associated to various approaches to promote equity. In this contribution, we focus on the innate views of pre-service teachers on diversity-aware teaching of mathematics to investigate their initial understanding. We present an explorative study that surveyed the views of 105 primary school pre-service teachers by means of a questionnaire. Those views were categorized by a qualitative content analysis. The results show that pre-service teachers most likely want to be and act aware of diversity when teaching, but rarely show sensitivity or knowledge of the particularities of school mathematics.

Keywords: Pre-service teachers, diversity-aware teaching, mathematics teacher education.

Introduction

Notions like 'diversity' or 'equity' have gained in significance in the last twenty years in the realm of education as well as in other discourses and society in general. Despite conceptual and terminological vagueness around these terms, in education they are commonly linked to the idea of creating an *education for all,* i.e. overcoming socio-economic inequalities and providing all students with equal opportunities to participate and succeed in the education system. This idea results in certain demands for teacher education as well as research attempts to conquer the lack of theoretical and practical knowledge how to reach this goal. Responding to students' diversity is associated with knowledge, awareness and acceptance of other cultures (including demographic factors such as race, gender and age) and results in responsive teaching to and through students' personal and cultural strengths and experiences (Gay, 2010). Considering that within the German education system the term of 'cultural awareness' is highly associated with intercultural learning – which does not necessarily provide an intersectional perspective on all markers of difference in students – we used the terms 'diversity awareness' and 'diversity-aware teaching' to represent the associated meanings as stated above.

Preparing future teachers for diverse classrooms is a complex issue which includes, among others, practical, theoretical, philosophical, ethical and affective aspects. In this paper we want to focus on the aspect of *diversity-aware teaching*, which (for us) combines being aware of differences among students, appreciating those differences as well as taking actions to promote equity. These aspects play into the perception what teaching a diverse classroom could look like and are tangent to reflecting on the relation between teaching and (structural) learning disadvantages. This process can be labelled as 'doing pupil' and is highly normative (Schönknecht & de Boer, 2008, p. 255).

Apart from certain general characteristics of diversity awareness, every school subject has specific peculiarities that result in subject-related attributes of diversity-aware teaching. Since mathematics education is widely known as a gatekeeper to successful participation in school and society, it is attributed with a significant role in creating equal opportunities (e.g., Stinson, 2004). We regard diversity-aware teaching as central to a pedagogy that seeks to overcome social inequalities and see special importance in investigating diversity-aware teaching in the realm of mathematics education. However, there is no consistent concept for diversity-aware teaching of mathematics and there is still little known about how to develop according dispositions. Considering these two aspects of unclarity we seek to investigate pre-service teachers' innate views on diversity-aware teaching for mathematics. This, in turn, should give us information about how to improve mathematics teacher education in order to enable pre-service teachers to reflect on their own view on diversity-aware teaching before engaging with a certain approach and eventually develop responsive dispositions.

In our study we surveyed the views of 105 primary school pre-service teachers by means of a questionnaire. We will first discuss the state of research on pre-service teachers' views on diversity-aware teaching of mathematics, before we present our empirical study on the pre-service teachers' views.

Literature review

Terminological and conceptual aspects of diversity awareness and diversity-aware teaching of mathematics

Although the concept *diversity awareness* is considered as central for teaching (Mason, 2008; Turner et al., 2012), it is rarely defined or described explicitly. Notions that are used as synonyms or underlie a similar understanding are *diversity sensitivity* or *cultural awareness.* Furthermore, there are parallels to the concept of *culturally responsive teaching*. Besides the fact that there is no common understanding or widely accepted definition, we find a terminological inconsistency whether diversity awareness is an attitude, a set of attitudes, a competence, or a belief. Despite the terminological inconsistency and conceptual ambiguity, there is consensus that diversity awareness is not a passive mindset of being tolerant. It moves beyond merely tolerating differences and is rather about appreciating, understanding and valuing personal differences. This includes perceiving differences as normal and reflecting on standards and norms (such as heterosexuality, Christian socialization or whiteness) as well as developing agency to promote equity. All of these aspects of diversity awareness feed into potential conceptualizations of diversity-aware teaching. In accordance with Eickhoff and Schmitt (2016) diversity-aware teaching requires

- to recognize differences (and commonalities) between individuals,
- to accord everybody equal rights,
- to recognize unintentional attributions, stereotyping, stigmatization or discrimination and work against them,
- to perceive existing inequalities, to name them and finally to reduce them.

Due to its historical development and its special attributes, school mathematics inheres structures and practices that result in inequitable access to achievement and participation for certain groups of students. It is in particular non-native speakers, female students, students with disabilities, immigrant students, students of colour and socially or economically disadvantaged students who face difficulties in school mathematics. Diversity-aware teaching in the realm of mathematics education requires knowledge and reflectivity about discriminating structures that are specifically inherent in school mathematics. This includes, among others, the fact that mathematics is primarily an abstract science, which is hierarchically organized and requires prior knowledge. Mathematical knowledge is often framed as intellectual with little integration of the body and of physical learning. Furthermore, high school mathematics is often characterized by problem solving activities which is difficult to reconcile with the desire of female students who tend to prefer clearly structured activities (Budde, 2009).

Conceptual frameworks for mathematics education referring to, and thus being aware of students' diversity in international literature are often presented as 'equity in mathematics education' (e.g., Civil, 2007; Gutiérrez, 2012; Turner et al., 2012), 'culturally responsive mathematics education' (e.g., Parker et al., 2017) or 'teaching mathematics for social justice' (e.g., Gutstein, 2006). Implementing those concepts in teacher education as well as addressing the contradictions within researching social injustice and mathematics education (e.g., Aguirre et al., 2017; Bartell et al., 2017; Hauk et al., 2021) are vivid contributions. However, although significant contributions around equity in mathematics have been done (see, e.g., Atweh et al., 2011 for a compilation of international work on that topic), there is a need of more research on how to achieve equity and social justice in mathematics education.

Studies on pre-service teachers' views on diversity-aware teaching of mathematics

In our study we investigate pre-service teachers' innate views on diversity-aware teaching of mathematics in order to provide a reference for the development of mathematics teacher education. Following Felton-Koestler (2017), we focus on and use the term 'view' for a number of reasons. First, encouraging pre-service teachers in unpacking their views about teaching is a key component to developing reflective practitioners. Second, our data consists of written responses to a questionnaire, with a small number of selected questions. The answers might enable us to capture students' views, but not something like beliefs, which are often framed as relatively stable and difficult to access (Felton-Koestler, 2017). For the same reason we did not focus on the interplay of pre-service teachers' identity and their views, even though views on diversity as well as on diversity-aware teaching are always connected to the perception of oneself and personal schooling experience (e.g., White et al., 2020). We perceive 'view' as a manifold influenced mode of looking at or regarding something and/or as an opinion or judgment.

To our knowledge there are no studies that focus on pre-service teachers' innate understanding of, or views on diversity-aware teaching of mathematics. Bitterlich and Jung (2019) found that pre-service teachers see major challenges in responding to students' diversity, especially in mathematics, and that they are missing concrete positive experiences they could associate to this diversity (p. 620). These findings deliver further evidence that diversity-aware teaching of mathematics is seen as very difficult, which highlights pre-service teachers' valid concerns and feelings of unpreparedness, but do not present how pre-service teachers view diversity-aware teaching of mathematics initially.

In line with these findings, the need to prepare pre-service teachers for diverse classrooms is a focal point in related research, whereas most studies discuss pre-service teachers' development of views or skills before and after attending a certain course. These pre-post-study designs are based on specific conceptualizations or approaches for equitable mathematics education, instead of determining pre-service teachers' innate views.

Connecting children's mathematical thinking and their cultural funds of knowledge can be seen as one approach of diversity-aware teaching which works towards equitable mathematics teaching. Turner et al. (2012) focused on the development of pre-service teachers' learning trajectories for engaging with children's *multiple mathematical knowledge bases*, considering and including the diversity in community or culture-based experiences and knowledge. Pre-service teachers framed drawing on community-based knowledge for mathematics learning as a possibility to position "children, their families, and communities as valuable knowledge resources" (p. 74). This proposes a view on diversity-aware teaching which purposefully incorporates multiple mathematical knowledge bases in instruction and was thereby consistent with the learning trajectories goal to promote equity. But other comments of pre-service teachers showed deficit-based views of children's families or communities, e.g., by framing certain characteristics of some families as detrimental to children's learning. Those results picture diversity-aware teaching more as an approach that has to compensate for collectivized deficits. These findings highlight an important notion to diversity awareness, since *being aware* does not necessarily equal *appreciating* diversity.

White et al. (2016) implemented a cultural awareness unit for pre-service teachers aiming to develop awareness of the role of culture in the teaching and learning of mathematics. One goal of the unit was for pre-service teachers to explore strategies to teach mathematics to all students (and therefore to reflect on their own view of diversity-aware teaching). They were assigned to search for journal articles that discussed the teaching and learning of mathematics with students that were culturally different from themselves. Stating that most pre-service teachers were drawn to a "particular culture or strategy for teaching diverse students" (p. 167) when choosing an article, the findings suggest that pre-service teachers' tend to break down diversity-aware teaching into specific or even isolated perspectives on diversity. These findings relate to the importance of acknowledging how difficult the conceptualization of diversity-aware teaching of mathematics is and was taken into consideration while attending to the pre-service teachers' answers in our study. The approach to elicit pre-service teachers' perspective of teaching mathematics for equity through literature-based reflective tasks can also be seen in Jackson and Jong (2017). They stated that even though students were aware of "the necessity to incorporate students' backgrounds in the classroom […] they did not realize the importance of doing this in a mathematics class" (p. 76). 35% of participating pre-service teachers had not thought about how to connect culture with mathematics education.

Summarizing, these findings led us to our research interest to what extent pre-service teachers view diversity-aware teaching of mathematics as linked to social constructs and socio-political context.

Methodology: data collection and analysis

The aim of our qualitative study was to examine pre-service teachers' views on diversity and diversity-aware teaching of mathematics. For this purpose, we designed an online questionnaire with a small number of open questions on diversity and inclusion in the context of mathematics education. The study was conducted in April 2021 with a group of pre-service teachers from Saxony (a federal state in the eastern part of Germany and former GDR) who were in the third year of their academic teacher training. The student population in Saxony is predominantly white, middle class and without confession or of Christian denomination. 105 students finished the questionnaire (gender identity: 1 diverse, 11 male, 93 female; average age: 23,5 years). Our investigation is to be seen as an explorative study, which serves as a starting point for further research, and additionally as impetus to modify the ways and contents of our own teaching. In this paper we focus on two questions of the questionnaire: 1) What is your understanding of diversity? and 2) What is your understanding of diversity-aware teaching of mathematics? The data consists of written responses to each question which differ in length between single words up to several paragraphs (approx. 160 words).

As a tool to analyse the data, we used a qualitative content analysis. This approach aims at the establishment of categories that are supposed to be developed while doing the text analysis. In our study we deployed an inductive category formation as there is no specific theoretical background that could be used and as we wanted to analyse the data as open as possible. The inductive ongoing "aims at a true description without bias owing to the preconceptions of the researcher, an understanding of the material in terms of the material" (Mayring, 2014, p. 79). We applied an open coding procedure, formulated and revised the categories while working through the text.

Results

First, we will focus on the pre-service teachers' views on diversity. The analysis showed differences in the conceptual understanding of diversity. We could identify a large number of vague and descriptive answers, simply pointing to differences (among students) or mentioning supposed synonyms (such as heterogeneity or variety) (88 out of 105 responses). Some of the pre-service teachers specified those differences in terms of individual characteristics (special skills, appearance, needs, interests etc., stressing the individuality of every single person) or in terms of group-related characteristics (social or cultural background, first language, race, gender, religion, etc.). Some of the answers mentioned that the notion *diversity* implies valuing differences, mutual respect or seeing it as a chance and enrichment for a classroom and/or society (26 responses). Only few students additionally pointed to a reflective mindset, e.g., considering diversity as normal or desirable or questioning structural inequalities as result of a constructed norm (13 responses).

Second, we will look on the pre-service teachers' view on diversity-aware teaching of mathematics. Here, a large number of the answers referred to a teacher's mindset towards diversity. Most of the participants stated the need to be aware of differences and/or students' individuality as well as to react to it (63 responses). Segments we coded accordingly can be seen as unspecific (e.g., not suggesting any implications for adapting the teaching), and do not allow a deeper analysis of the (conceptual) understanding of the awareness for differences or how it could affect teaching. About 10 % of the answers showed a positive reference to diversity for teaching/learning interactions, e.g., claiming appreciation or promotion of diversity as well as valuing individuality. Here we could clearly recognize links to the pre-service teachers' conceptual understanding of diversity.

Another 20 % suggested a reflection on what is perceived as diverse or as normal (referring to students' characteristics, identity or learning pathways) as part of diversity-aware teaching. Some of the pre-service teachers also mentioned the need for teachers to reflect on their own perceptions and pointed to the influence of attitudes or of normative assumptions for classroom interaction. Almost a third of the answers were coded as *socio-political perspective*. Here we found reflections on internalized stereotypes within teachers, reproduced stereotypes in narratives or teaching material or on structural relations between educational disparities and social/societal disadvantages.

44 responses referred to certain methods, specific tasks or organizational adjustments for teaching a (however defined) diverse group of learners in order to respond to students' diversity. We summarized these answers as *teaching strategies*. Most of the strategies aimed at reducing inequalities or at valuing diversity within teaching/learning interactions. If specific methods, tasks or organizational adjustments could be identified as typical or specific for mathematics teaching, we coded them as *specific structural aspects of mathematics education*. Here we found references to the negatively attributed reproduction of stereotypes or discrimination within certain materials, narratives or organizational aspects that are characteristic for mathematics education. For instance, some of the pre-service teachers mentioned the particularities of word problems whose narratives run danger to reproduce stereotypes or do not represent the lifeworld experiences of certain groups of students. Others were concerned with gender-related differences and called for methods, activities and contexts that enable especially girls to successfully participate in the mathematics classroom. We could also find references to mathematical tasks that allow for different approaches and/or different results (such as natural differentiation), or to linguistic challenges and necessary support in teaching and learning mathematics. Some mentioned specifically 'fast calculators' or students 'with poor counting skills'. However, only 19 % of the answers showed such references to specific structural aspects of mathematics education.

Discussion

The majority of the pre-service teachers in our study presented a view on diversity-aware teaching which simply put *being aware of differences in and individualities of students* as an essential part of teaching. Most of them presented themselves as well aware of the diversity in current and future classrooms. Significantly less answers framed diversity-aware teaching within a view that explicitly valued or mentioned the promotion of diversity. The transfer of being aware of diversity into teaching strategies seems to be a major obstacle, since not even half of the pre-service teachers mentioned any strategies or approaches to do so. This tendency becomes even more apparent when focusing on the elicited views on diversity-aware teaching which could specifically be related to mathematics teaching and learning: Only 20 out of 105 pre-service teachers linked diversity-aware teaching to specific structures or characteristics of mathematics education. Within those subject-related answers most stated some sort of appreciation of diversity and referred to (responsive) teaching strategies specifically for mathematics. This highlights that pre-service teachers most likely want to be and act aware of diversity when teaching mathematics (whether due to an appreciative attitude or due to the sheer necessity to do so), but rarely show sensitivity or knowledge of the particularities of school mathematics. This also points out another tendency within the study: If a connection between diversity and mathematics education was made, it predominantly referred to tangible interactions or strategies of teaching rather than to structural contingencies. None of the given answers reflected on mathematics as a gatekeeper or on the influence of mathematics education for paving educational pathways. These findings are well in line with current research reflecting on mathematics teacher education (e.g., Jackson & Jong, 2017) and deliver implications for our own further teaching.

Within the stated tangible strategies for diversity-aware teaching of mathematics, the expressed connections between mathematics teaching and diversity awareness mainly referred to the very apparent level of representation. More than half of the elicited views on diversity-aware teaching that specifically referred to teaching mathematics emphasized the need for the representation of diversity in teaching materials (e.g., illustrations in textbooks or chosen contexts for word problems) or they stated the need to avoid stereotypes or discriminating language when phrasing tasks. Making a deeper connection between diversity-aware teaching and structural disadvantages in mathematics education could only be identified in four answers. These answers presented a view on diversity in students (cultural, community or family) backgrounds and experiences as relevant for mathematics education. This view can be interpreted as a first step to recognizing the complex interdependencies of mathematics education and socio-politics. For us, these conceptions represent a higher reflective level when debating diversity-aware teaching of mathematics.

Addressing mathematics pre-service teachers views on diversity can be seen as a key component in preparing them for diversity-aware teaching. Eliciting those initial views was one of the goals in our study. To us, the results highlight the importance of referring diversity-aware teaching explicitly to mathematics education and emphasize the need to discuss the role of (school) mathematics as a gatekeeper in teacher education – despite or even because of the vagueness when pinning it down. Even though there are numerous significant contributions in mathematics education research on equity (e.g., Atweh et al. 2011; Gutiérrez, 2012; Aguirre et al., 2017), there is no shared or well-defined understanding of ‘equity’ and how it can be achieved in mathematics education. Following the call of Gutstein and colleagues (2005), that each of us “has a responsibility to both think about and act on issues of equity” (p. 98), we consider becoming aware of one’s own understanding of diversity and diversity-aware teaching as one of many steps to deepen the understanding of equity in mathematics education. In addition, according to the identified categories in this study, a theoretical framework combining socio-political and subject-specific aspects for diversity-aware teaching of mathematics seems to be a necessity in future research.

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Study guidance in Arabic in mathematics – tutor resources

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Keywords: language as resource, mathematics, epistemology, study guidance in mother tongue.

Background

Due to migration many students attend classrooms where they cannot learn mathematics in their home languages because of prevailing monolingual curricula and classrooms norms. This is the case in mainstream mathematics classrooms in Sweden where Swedish is the language of instruction and where this study was conducted. It is regulated in Skolförordningen [the Swedish School Ordinance] (SFS 2011:185 chapter 5 §4), that a student can receive study guidance in the mother tongue (SGMT), if the student needs it. Thus, SGMT in mathematics is grounded in the idea of the home language as a resource for multilingual students' mathematics learning. SGMT is provided by SGMT tutors. There are no formal educational requirements for tutors. According to Skolverket [Swedish National Agency for Education] (2015) it is desirable that tutors have subject knowledge, knowledge about the students' prior and current educational systems and pedagogical practices, and a developed linguistic awareness about differences and similarities between Swedish and the mother tongue. This places special demands on tutors' resources. Further, it means that they need to handle epistemological dimensions of language and mathematics in their everyday work as SGMT tutors in mathematics. Thus, it is of interest to explore, with an epistemological focus: How are a mother tongue study guidance tutor's languages and knowledges employed (or not) as resources for multilingual students' mathematics learning in a mainstream school?

Theoretical approach: The Language as resource model

To explore how SGMT tutors' resources can be employed as resources in mathematics activities from an epistemological perspective we use "The Language as resource model" (The model) (Ryan, et al., 2021). It builds on previous research concerning the language-as-resource idea, more specifically the epistemological potentials of multilingual language use and mathematics identified in this research (see Ryan & Parra, 2019). The model has two axes that make up an interface, which holds epistemological language-as-resource potentials that move from separating to synthesizing languages and mathematics (see Figure 1). For example, potentials of translation activities are activities in which languages are used in a separated manner, as separating first- and second languages, whereas *translanguaging* (García & Wei, 2014) activities are examples of activities in which languages are used in a synthesized manner. That is languages are used without clear boundaries and students use their full range of language resources, that is all words, grammatical structures, idioms *etc*. that are available to a speaker. Actualizations of language-as-resource potentials in multilingual mathematics activities move dynamically over this surface, that is among the language-as-resource ideas' epistemological potentials for language use and mathematics. The *x*-axis displays a continuum of epistemological potentials of mathematics that move from separating ways of knowing mathematics to synthesizing plural ways of knowing mathematics, for example from separating informal and formal mathematics to synthesizing a plurality of mathematics acknowledging mathematics as cultural activity. The *y*-axis displays the continuum of epistemological potentials of multilingual language use that move from separating named languages to synthesizing new language practices. The surface holds two named potentials; *the 'lever' potential* that can move students from informal mathematics talk in their mother tongue to formal mathematics talk in Swedish, and the *'one-new-whole' potential* that constitute prerequisites to produce new ways of languaging and knowing mathematics which requires interknowledge among the languages and plural mathematics. These two potentials are implicit in Rosén et al.'s (2020) conclusion that SGMT can be viewed as support through the student's mother tongue but rather should be regarded as a translanguaging practice that challenges linguistic and cultural boundaries.

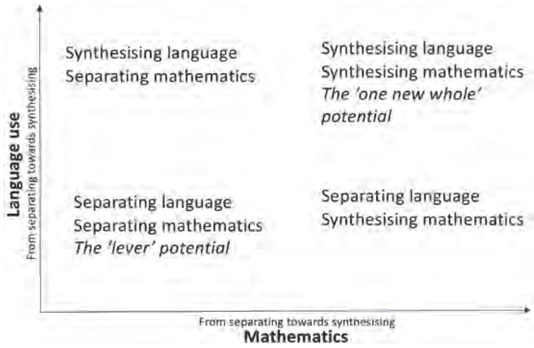

Figure 1: Framework for epistemological potentials in multilingual mathematics activities based on the language-as-resource idea.

The poster presentation

In the poster we present an example of an analysis of an excerpt from an interview with a SGMT tutor (in Arabic) to illustrate how the model can be used for analysing how SGMT tutors' resources can be employed as resources in mathematics activities from an epistemological perspective.

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2+2=4? Mathematics lost between the pitfalls of essentialism and alternative truths

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This essay problematises the epistemic status of mathematical knowledge. It is based on the observation that essentialist epistemologies provide no solid basis while relativist epistemologies have not yet convincingly succeeded to explain the objectivity of mathematical knowledge. I will start with three examples from popular media which illustrate that awareness for the problem discussed here has already reached the interested public. I will shortly address popular answers to the problem, only to refute them. I will end the essay by discussions which stay close to the example of 2+2=4, ending with the presentation of possible directions for further understanding and research.

Keywords: Mathematical reasoning, epistemology, relativism, alternative truth, styles of reasoning.

Debates on 2+2=4

Surprising as it may seem, the question whether 2+2=4, which so prominently demarcated the border between 'truth' and 'doublethink' in George Orwell's novel *Nineteen Eighty-Four*, has recently become the object of public attention. One example is the award-winning short film *Alternative Math* (Maddox, 2017), which is an alternative-facts parody, where the mathematics teacher's rejection of a student's claim that 2+2=22 kicks off a chain of events in which the teacher has to face outraged parents, a public mob and undiscerning officials. It begins with the teacher showing the primary school student his test with 2+2=22 marked wrong and with the teacher explaining 2+2=4 with markers to the disavowing child. In the next scene, we see the teacher and the boy's parents discussing the issue. When the parents ask the teacher who says that 2+2=22 is 'not the right answer', the teacher replies 'says maths', whereupon the father asks 'Who are you to say that your answer is right and his is wrong?' Later, when a board of officials asks the teacher to proclaim that she is 'open to the possibility there might be multiple correct answers', the teacher replies 'there is only one correct answer'. In the end, the teacher is fired but manages to demonstrate the consequences of allowing 2+2=22, at least for the school's finances. While the comedy of the short film builds on applying the logic of alternative facts to something as consensual as mathematical facts, the irony is that the teacher offers little more than authority and dogmatism to legitimise the epistemological claim made, which eventually does not dispel but invite further scepticism. Why does 2 plus 2 equal 4 and nothing else?

A different episode was brought to us by James Lindsay, a graduated mathematician, blogger and critic of post-structuralism. As a parody of the logic of post-structural narratives, Lindsay had posted the following on Twitter: '2+2=4: A perspective in white, Western mathematics that marginalizes other possible values.' What followed were intense debates on social networking services whether 2+2=4 had to be accepted as a true statement and if 2+2 could equal 5. Lindsay (2020) provided an (obviously factionary) overview of the discussions and rejected different proposals for 2+2 not equalling 4. The proposals included non-trivial mathematical interpretations of the expression 2+2 (e.g., in a ternary numeral system, where we might note 2+2=11, or in the residue system modulo 3, where we might note 2+2=1), as well as peculiar contextualisations, for example, interpretations of addition as the bringing together of animal populations which might result in procreation or in the consumption of some specimen. Such proposals may be useful philosophical exercises that help us to explain what we mean by 2+2=4, but in all cases the expression 2+2 has been changed to mean something different than intended, thus losing its relation to the original problem.

A last example comes from the United States and even made it into the newspaper and television news in Germany and Austria. In early 2021, the Oregon Department of Education offered a course for in-service teachers based on the materials provided by the project ‘A pathway to equitable math instruction’ (Cintron et al., 2000). Most uproar was caused by the question whether mathematics was racist, but debates also kindled over the question whether or not mathematics provides unambiguous knowledge. Cintron et al. stated that ‘the concept of mathematics being purely objective is unequivocally false, and teaching it is even much less so’, that ‘upholding the idea that there are always right and wrong answers perpetuate objectivity as well as fear of open conflict’, that focussing ‘on getting the “right” answer’ is an instance of ‘white supremacy’, and that teachers should instead encourage students to ‘come up with at least two answers that might solve this problem’ (p. 65). Consequently, the material was interpreted by many by claiming that mathematics does not produce unambiguous knowledge, and that claims that it would were establishing mathematics as a racist endeavour. Meanwhile, the project leaders clarified on their homepage that this is not what they meant and changed their material, avoiding the problematic passages. The echo in the media with many rather emotional than epistemologically educated responses however indicates that there is a growing fear that the objectivity of mathematical knowledge may be called into question sooner or later.

Two popular explanations and their problems

Lindsay (2020) presented himself as a mathematical realist and argued that mathematics deduces propositions such as 2+2=4 from fundamental premises (axioms) which are ‘relatively simple and connect to the real world in a very obvious way’. Considering the axiom of induction from Peano’s axiomatisation of natural numbers, such fundamental premises do not at all appear to be ‘relatively simply’, nor do they ‘connect to the real world in a very obvious way’. But there is a more fundamental problem: Axioms for arithmetic were provided by Peano only at the end of the nineteenth century. Before that, it was impossible to prove the statement 2+2=4 in the way described by Lindsay. So, does the validity of the statement 2+2=4 really depend on mathematics as a deductive discipline? In fact, we could argue that we constructed mathematical theories in a way that provides for the truth of 2+2=4 (and Lindsay implicitly does so as well when arguing ‘if we chose to start with different fundamental assump¬tions, we’d have a different mathematics that doesn’t seem remotely interested in reality at all’).

A different explanation is that 2+2=4 is abstracted from experience. (Remember the teacher explaining 2+2=4 with markers in the above-mentioned short film.) We see that two and two apples make four apples, that two and two cows make four cows, that two and two fingers make four fingers, and by induction we presume that 2+2=4 holds true in general. But even if induction was the reason that we hold 2+2=4 to be true, how about 74+26=100? Surely, we would not claim that we saw 74 of something and 26 of that same thing make 100 so often that we hold the abstract 74+26=100 to be true in general. Even worse, Greer (2005) made a strong point that 'whether a situation is appropriately modeled, or not, by the equation 2+2=4 [is] a matter of complex interpretation' (p. 297). Already Frege (1884) had established that the use of arithmetic presupposes a specific interpretation of one's perception, and Bishop (1988) informed us that the corresponding epistemology of what he calls 'objectism' has to be understood as a Western particularity. Consequently, it is not convincing either to establish the truth of basic arithmetic through induction.

The epistemological problem in more general terms

The discussion whether or why 2+2=4 holds true is intimately (although often not explicitly) connected to an unresolved struggle in epistemological enquiry between essentialism and relativism, which can only be presented here in a brutally abridged version: Robertson & Atkins (2019) proposed that 'essentialism is the doctrine that (at least some) objects *have independently of how they are referred to* (at least some) essential properties'. They add that such views have been fundamentally objected lately. In this sense, Lindsay (2020) argued that 2+2=4 is a truth that exists independently of humans and can only be discovered. In contrast, relativism supposes that 'things have the properties they have […] only relative to a given framework of assessment' (Baghramian & Carter, 2019).

In the philosophy of mathematics, the most prominent essentialist position is Platonism, where it is assumed that mathematical objects have a mind-independent and unalterable existence, that they provide shapes for the composition of our perceivable world, and that their properties can be discovered by the human mind. Linnebo (2018) reported that 'platonism has been among the most hotly debated topics in the philosophy of mathematics over the past few decades' and that 'a variety of objections to mathematical platonism' includes that 'abstract mathematical objects are claimed to be epistemologically inaccessible and metaphysically problematic' (p. 1).

In mathematics education research, the best-known objections against essentialism come from radical constructivism and post-structuralism. While the former stressed that humans lack the sensual apparatus to create an unbiased understanding of the world, and that knowledge is therefore necessarily relative to the limits of perception and interpretation (Glasersfeld, 1995), the latter was more interested in the refutability of truth claims and the historical deconstruction of their relations to power. For example, I discussed elsewhere how the laws of logic are far from natural or necessary but cultural products, which allied with specific social interests (Kollosche, 2014). So, is 2+2=4 a 'perspective in white, Western mathematics that marginalizes other possible values' after all?

Interestingly, scepticism based on post-structuralism is not only guiding progressive research on equity in mathematics education and beyond, it also laid the epistemological ground for the phenomenon of alternative truth. As McIntyre (2018) documented in much detail, the intellectual pioneers of alternative truth explicitly adapted the post-structural claim that every discourse is necessarily constructed and socio-politically biased to a technique that then allows to refute any truth claim. The very real effects of this development can be observed in the course of the denial of anthropogenic climate change or the COVID-19 pandemic by leading politicians. MacMullen (2020) argued that this epistemological scepticism is no longer only the academic background of the alternative truth phenomenon but an explicitly held position of some citizens. Elsewhere, I claim that this problem is not appropriately addressed in mathematics education research (Kollosche, 2021a).

Koertge (2017) regarded the phenomenon as a result of fundamental problems of relativism. She agreed that ‘all observations are laden with theory’ but questioned whether this means ‘that there is no objective/impartial perspective from which we can appraise and compare the truth-value of claims’ (p. 809). She warned that, ‘if it really were true that scientific assessments of truth and falsity could never be objective and could never be more than warring opinions, then we would be left with nothing but a clash of civilizations’ (p. 810). Indeed, it is one of the central problems of relativist epistemologies that they cannot sufficiently explain how we can avoid a situation where anybody can claim anything, how some sort of epistemic commitment can be guaranteed, and how specific ‘frameworks of assessment’ become accepted as scientific while others do not.

Skovsmose (2012) was early in warning the community of mathematics education researchers that post-structural deconstruction might be insightful but not productive in legitimising knowledge. In the light of the spread of alternative truths in public debate, Marcone et al. (2019) recently wondered if the fight of some scholars in our discipline against ‘the uncritical faith in mathematics’ and ‘the ideology of certainty’ has played a part in the post-factual ignorance of scientific facts (p. 186). They felt that ‘our arguments against universality and neutrality [of mathematics] have been trivialized and turned back against its original intention’ (p. 187). But have they? The fact that relativist arguments have been adopted by post-fact politics neither solves nor disvalues the original philosophical problem. It invites us to rearticulate it with greater urgency. Thus, we may ask: How can we explain the objectivity of mathematical knowledge without falling back to essentialist epistemologies?

Getting back to 2+2=4

In a way, the above question is typical for Wittgenstein’s *Remarks on the Foundations of Mathematics* (1956), which I cannot address any further in the limits of this contribution. Recently, I have only seen the question further discussed by Azzouni (2006). I recommend to read his chapter, but this is not the place for me to discuss his sometimes fruitful and sometimes problematic ideas. Instead, I want to return to my initial example of 2+2=4. Staying with such a basic example allows us to ground our complex philosophical thoughts in a matter that stays somewhat simple.

Let me shortly present my own understanding of why 2+2=4 or 74+26=100 holds true in a non-essentialist epistemology: We experience that there are several situations in which two of something and two of that same thing make four of that thing. There may also be situations in which this addition does not make sense. In the course of abstracting our experiences to the statement 2+2=4, we also learn in which situation it makes sense to see 2+2, and in which it does not. However, we do not arrive at the knowledge of 74+26=100 this way. Instead, we arrive at the knowledge of 74+26=100 through a line of algorithmic argumentation. From our empirically-based handling of small numbers, we abstract a certain logic of calculation, which is suitable to inhabit our most simple and inductively true calculations and can be extended to quantities well beyond our experience. The study of the teaching and learning of arithmetic is a prominent field in mathematics education research and investigates such processes, in which the handling of large numbers is again and again made sense of through our handling of small numbers. We could say that calculation is a kind of reasoning that allows us to establish 74+26=100 as a true claim, even though it is no longer directly connected to experience, nor are its truths established deductively.

A more sophisticated description of such an understanding has been provided by Piaget (1971) through the distinction of empirical and reflecting abstraction. Lensing (2018) discussed the importance of this distinction for mathematics education research and provides a good introduction. In short, empirical abstractions abstract from observables, while reflecting abstractions abstract from human action. In this sense, 2+2=4 might be an abstraction from empirical cases of counting and adding, but the logic that lets us arrive at 74+26=100 has been gained through an abstraction from our actions with numbers in easier cases. Piaget’s distinction and its application to mathematics education shed light on the complex interplay of different epistemic processes that lead to the justification of knowledge. I will end this paper by a presentation of a more complex description of such processes and their interplay not from a psychological but from an epistemological perspective.

Styles of reasoning as a way out of the epistemological dead end

Here, we might pause and wonder whether there are fundamentally different ways to establish the result of an addition of natural numbers as a true claim. Such results might be abstractions from empirical observations, they may we won through the manipulation of material models such as markers, they might be products of an algorithmic technique, and they might be theorems proven in axiomatic theories. These ways to establish a truth claim such as 2+2=4 reach the same judgement, but it is interesting that they arrive there by very different activities, which, each for itself, has the power to establish a claim as true. If we do not want to allow ourselves to fall back to essentialist explanations, this power and the consensus reached here are curiosa that demand an explanation.

Elsewhere, I have proposed to elaborate on Hacking’s (1992) framework of *styles of reasoning* in order to arrive at a possible explanation (Kollosche, 2021b). A style of reasoning includes methods of argumentation for reaching truth claims, therewith also a set of statements whose truth-or-falsity can be decided scientifically, therewith also a set of objects that enter the scientific discourse in the first place. Hacking (1992) drew on the work of science historian Crombie (1994) who described the historical development of six distinct styles of reasoning: the postulation style, the experimental style, the modelling style, the taxonomic styles, the statistical style, and the genetic style.

It is already interesting that nearly all styles (maybe not so much the genetic style) are used in mathematics or use mathematics themselves: postulation in the axiomatic-deductive theories of mathematics, experimentation in experimental approaches to mathematics and in experiments on the suitability of mathematical models for specific applications, modelling for the understanding of our world through mathematics but also for gaining knowledge of mathematics objects through models such as diagrams, taxonomy as a principle for concept development in mathematics, and statistics as a mathematical technique itself. Further, Hacking (1992) proposed to add ‘algorismic’ reasoning to Crombie’s list (p. 8), and was open to further additions or changes. We can recognise the postulation, the experimental, the modelling, and the algorismic styles of reasoning in our above-mentioned explanations of why 2+2=4, which already shows the potential of this framework.

Here, I want to return to more general philosophical considerations one last time. Hacking’s (1992) framework of styles of reasoning clearly proposes a relativist epistemological position. However, it does not appear completely defenceless against the objections issued against epistemological relativism. Let us address Koertge’s (2017) concern that, in relativist epistemologies, ‘truth and falsity could never be objective and could never be more than warring opinions’ (p. 810). Note that opinions are individual views that do not require any justification (although opinions can be elaborated upon). I may hold the opinion that twelve is the most beautiful number, and may even present my thoughts associated with this opinion, but this does not establish ‘twelve is the most beautiful number’ as any kind of justified knowledge. When Koertge warns that relativism would degrade truth claims to opinions, she fears that we would be lost in opinions and that shared and accepted knowledge would no longer be possible. In the styles-of-reasoning framework, this is not the case. Here, we have a limited number of historically established frameworks of assessment, which are shared amongst people and used to justify knowledge on an interpersonal level. Especially, styles of reasoning do not allow to present just any opinion as knowledge, just as a mathematical theory does not allow to proof any statement whatsoever.

Admittedly, one might not be completely satisfied with the merely historical explanation of why Hacking’s (1992) styles of reasoning are accepted as truth-makers in science. Should there not be reasons specific to the respective styles which explain why they are acceptable for scientific inquiry and why other frameworks of assessment such as astrology are not? It is not easy to give an answer here. Hacking underlines that relativism implies that there is no objective ground from which the suitability of a framework of assessment for scientific enquiry can be decided. This means that answers can only be found in the styles and their interplay. He furthermore stresses that the acceptability of frameworks does underly historical changes, that all styles have appeared sooner or later in history, the statistical style being the most recent, that once-accepted styles such as the logic of resemblance as described by Foucault (1966) appear utterly ridiculous to us today, and that we cannot foreclose that new styles may emerge and gain acceptance. It may be productive to focus on the interplay of different styles of reasoning. Ruphy (2011) introduced the concept of ‘ontological enrichment’ to capture that styles of reasoning do not only create new objects of scientific inquiry (in the way that Peano’s axiom system created numbers as axiomatic-deductive entities), but that these new objects mirror already-known objects (such as numbers as algorismic entities). Despite these ontic differences, we act ‘as if’ we were speaking about the same object in both styles, which allows us to gain understanding by studying this object in the various styles. Is it a fruitful hypothesis that new styles of reasoning gain acceptance through reproducing the knowledge gained by already-accepted styles while adding some new possibilities for understanding? For example, see how the postulation style mathematics of Ancient Greece as documented in Euclid’s *Elements* reproduced the already existing algorismic and diagrammatic mathematical knowledge of the time while adding new possibilities of expressing, justifying, ordering and interrelating knowledge – at least until incommensurability problems provoked serious doubts concerning the suitablity of this acting ‘as if’. I assume that the history of mathematics would be a good field to study the fruitfulness of this hypothesis, not only for the philosophy of mathematics itself but for epistemology in general.

Back to mathematics education

While the above considerations have been very philosophical and theoretical in nature, the mathematics classroom demands specific action. I cannot claim that I have thought-through recommen¬da-tions for the mathematics classroom, but I have pressing questions, which mathematics education research might want to consider:

- While the styles-of-reasoning framework proposes that mathematics is an integral part of many styles of reasoning, the public image of mathematics and research in mathematics education focuses on one specific style of reasoning, namely on mathematical proof (Kollosche, 2021b). In how far does this bias hinder us to study mathematical reasoning more broadly in research and to address different forms of mathematical reasoning in the classroom? In how far is this bias socio-politically functional, for example for upholding an ‘ideology of certainty’ around mathematics?
- As the styles-of-reasoning framework allows to preserve both the ideas of relativism and of justified knowledge, may it be helpful, for example in the context of studies on what US-Americans call ‘race’ and mathematics, to articulate a critique of the socio-political bias of mathematics without questioning its objectivity altogether?
- In how far would a differentiated understanding of mathematical reasoning in the framework of styles of reasoning allow to deconstruct exactly which epistemic practices may interact with the preferred styles of knowing of possibly disadvantaged social groups without questioning the epistemic value of mathematical reasoning in general?
- Assuming that the growing public scepticism towards scientific knowledge leads to demands that schools should provide some sort of an epistemological education, that is education about the justification of knowledge, then would mathematics education not be a privileged place, given that mathematics is intimately connected to nearly all styles of reasoning? What would such an epistemological education in the mathematics classroom look like?

Eventually, answers to be above problems might also help to articulate in how far mathematics education helps adolescents to develop their reasoning skills – a claim that reoccurs in mathematics education research and educational policy in connection to mathematical argumentation, mathematical modelling, and problem solving in mathematics. Rethinking these fields from the perspective of a relativist epistemology can allow to articulate their theories more clearly, while the relativist framing would allow to more easily include a socio-political perspective in these fields of research.

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Mathematics teachers' creativity for fostering inclusion and preventing early school leaving

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In this paper we investigate mathematics teachers' creativity in offering problematising activities and promoting inclusion and positive attitude toward school in territories with worrying dropout rates. This is an initial study in which we interviewed four teachers working in this kind of territories and very active in designing and implementing inclusive and visionary mathematics activities. From the analysis of the answers to two particular questions, we outline the perceptions of these teachers on the sources of their educational creative work, identifying peculiarities and communalities. In particular, a common intention to achieve citizenship education objectives emerges.

Keywords: Mathematics teacher, creative teaching, inclusion, problematising education.

Introduction

Equity, inclusion, and fair opportunities were felt as crucial issues by researchers in mathematics education already before the pandemic and have become even more urgent issues during it (Bakker et al., 2021). In particular, the pandemic, as "a magnifying glass on issues that were already known" (p. 1), revealed even more clearly the inability of the school education to properly include pupils in disadvantaged contexts and its responsibility in creating several unacceptable phenomena of social marginalization and exclusion. Indeed, nowadays, the educational outcomes of younger generations are still determined to a large extent by the socio-economic background of their parents rather than by their own potential. Promoting inclusion[1] and equity in education and training is thus fundamental in breaking these patterns. In the suburbs of many cities, school lacks resources to properly include children and teenagers experiencing socio-cultural disadvantage and ends up losing its institutional value in the eyes of pupils and their families. Hence, there is risk of early school leaving and, in some particular contexts, also the consequent serious danger for children and teenagers to be approached by gangs and organized crime. For this reason, it is of fundamental importance that educational research reflects on the endogenous factors of early school leaving (Morgagni, 1998), i.e. factors that are internal to the educational system. An effective struggle against this phenomenon should aim to create the conditions for every student to create her own identity within the school. In particular, this possibly implies the need that the mathematical activities proposed at school should (re-) build a positive attitude towards mathematics (Di Martino

[1] Here, the term "inclusion" is used in a general sense, as in the Italian educational system. By inclusive educational environment we refer to the participation in mathematical activities both by students with claimed disabilities and by students with special educational needs. The latter group also includes students with very critical cultural disadvantages, such as limited Italian language proficiency or lack of cooperation, if not hostility towards the school, by their families.

& Zan, 2011) for the students which are most in difficulty. In socially disadvantaged contexts, this means proposing to pupils tasks which are meaningful in their eyes and improving their perceived competence in mathematics.

For several years our research group has been engaged in educational projects in collaboration with schools to implement highly inclusive mathematics education environments, aimed at preventing the phenomena of early school leaving. In these contexts, inspired by the vision of active pedagogy, we have tried to design mathematics education activities as engaging and captivating as possible, for example by activating students body movement inside schools, but and also in outdoor environments; by creating narrative frames that allow students to link the exploration of the mathematics worlds to the exploration of cultural and naturalistic city sites (such as museums, art galleries, parks, site of urbanistic interest and so on) and critically connect pupils to the territory in which they live. The theoretical background of reference for these mathematics education activities design has been the informal mathematics education (Nemirovsky et al., 2017), which concerns non-compulsory activities, with very fluid boundaries between disciplines and with no formal grading.

Working on these educational projects, we have also tried to involve the school teachers in the design and implementation of these visionary informal mathematical activities. Indeed, our main research interest in informal mathematics education projects is to investigate how to create an intertwining between the highly engaging approach of mathematical activities implemented within them and curricular teaching. We recognize the centrality of school teachers in these transformation projects: they are called to be guides capable of creative inclusive mathematics education activities, in a continuous dialogue with the territories in which they work. With this gaze, in this paper we present a first research attempt to capture and describe the kind of mathematics education creativity needed for teaching in socio-cultural disadvantaged contexts, and elsewhere. Our research aims to investigate teachers' creativity, both in design and implementation of mathematics activities responsive to students' interests, in order to foster inclusion and thus prevent early school leaving.

Theoretical framework

Our research is framed in active pedagogy, born at the end of the 19th century, that considers the student as the active protagonist of the educational process and the relationship between the subject and the environment as central. Therefore, according to this point of view, it is essential to organise educational and school contexts in such a way that the environment can foster creativity, plurality of opinions and the freedom of pupils to experiment on their own. Moreover, the educational experience cannot disregard the everyday life in which the subject lives.

In this scenario, the educational proposal of Paulo Freire was to create an emancipatory pedagogical process aimed above all at that part of society that lives in a situation of oppression (Freire, 1970). According to Freire, the action of the educator is characterised by two essential moments: the preparation of the lessons and the meeting with the students; it is by analysing these two practices that it is possible to distinguish a *depositary education* from a *problematising education*. In the first case, the educator, during the preparation of her lessons performs an act of knowledge in relation to the knowable object. Then, during the meeting with the students, she narrates and discusses the object on which she has performed her act of knowledge. The task of the students, in this perspective, is to know or learn by heart what is narrated and delivered by the educator (student seen as a patient or a docile recipient of content). Depository education does not allow the students to perform a true act of knowledge, since what is delivered or deposited to them is a possession of the educator and not a mediator of the critical reflection of both (Freire, 1970). In the case of problematising education, on the other hand, there is not a clear distinction between the moment of preparing the lessons and the moment of meeting the students. The latter is characterised by dialogue, which is fundamental for the educator in order to review – through the re-visioning of the students – what was objectified in the preparation of the lesson. Problematising education becomes an act of knowledge: both when preparing the lesson and when meeting the students, the educator is a *knowing subject*. With the problematising practice, the educator makes the students become *critical researchers*, in a permanent dialogue with the educator, who is a critical researcher herself. According to Freire (1970), it is only through a problematising education that it is possible to carry out an emancipatory pedagogical process that pushes the students for learning to face their world by putting knowledge into play, developing creativity and constant reflective critical capacities. The educator, therefore, has to reflect on what to propose to her students and, from the constant dialogue with them, imagine original themes to be returned as concrete problematic situations, challenging and motivating for the students, which allow them to reflect on significant dimensions of reality, promoting a critical understanding.

In this frame, our research aims to explore teachers' creativity in the strictly intertwined processes of designing and implementing mathematical problematising education activities (Freire, 1970) in socio-cultural disadvantaged contexts. In literature, there are several studies on the mathematical creativity of professional mathematicians. For example, Hadamard (1945) in his seminal work collected the perceptions of contemporary mathematicians on the mechanisms by which they produce new mathematics. He outlined the existence of unconscious mental processes and tried to capture the phenomenon of illumination in doing mathematics. Later, Liljedahl (2004) updated Hadamard's work, focusing on the specific context of mathematical problem solving. In particular, he confirmed the experience of mathematical illumination, perceived as caused by a sudden coming to mind of an idea. He analysed preservice teachers' logbooks, together with reflective anecdotal accounting from undergraduate mathematics students and prominent mathematicians of our times. Here our focus is on mathematics teachers' creativity and very few studies can be found about it (see for example Levenson, 2021).

Methodology

Participants

In order to investigate the particular kind of creativity of the mathematics teachers we are interested in, we identified two Italian informal education projects that offered very innovative mathematical problematising education activities (Freire, 1970) to primary and secondary students from socio-culturally disadvantaged areas of two different cities, Naples and Turin. Within these projects, we involved four teacher-researchers, as they are responsible for educational design: Perla, Nadia, Riccardo and Claudia (pseudonyms). The first two are high secondary school math teachers in

Naples (Italy) and partners of the cultural association “Matematici per la Città” (MPC). The association was born in 2013 with the aim of realising urban math walks through the suggestive and fascinating streets of Naples, bringing mathematics out of the classroom, in a less formal, more comfortable and enveloping scenario. Currently, MPC collaborates with several primary and secondary schools in Naples, often located in disadvantaged districts of the city. It promotes extra-curricular educational paths with the aim of stimulating and encouraging children and teenagers to observe reality with curiosity and critical sense. In all its activities mathematics is strongly intertwined with other disciplines, such as art, music, history, chemistry, and above all with citizenship education. The other two teachers, Riccardo and Claudia, are lower secondary school math teachers in Turin (Italy) and both took part in the design of the educational activities of the “Next-Land” project. The project, promoted by the cultural association Next-Level, started in 2020 and is still underway. So far, it has involved lower secondary school students from the most disadvantaged Turin’s neighborhoods. Next-Land offers educational paths on STEM disciplines, which are intertwined with the arts. It is characterised by a widespread, laboratorial educational action, which is hosted in the city’s museums, open spaces, research places and in digital environments. Among the transversal aims of the project there is the fight against the gender gap in study and work contexts.

Data collection and analysis

Data collection consisted of oral and semi-structured interviews, inspired by the Explicitation Interview method[2] (Vermersch, 1994), audio-recorded and transcribed. Due to space constraints, in this paper we present only the analysis of the responses to two particular questions, which we consider to be the most interesting in the context of an exploratory study of teacher creativity. Here we refer to the selected two questions as Question 1 (Q1) and Question 2 (Q2):

> Q1. In designing a teaching activity, do you feel more like ideas emerge all of a sudden or that they originate from a process of investigation?
>
> Q2. Referring to your design choices, do you think that, in general, they also consider your personal background and the social and cultural dimension? If so, to what extent do you think that the personal, social and cultural dimension can influence the creation of educational activities? Could you refer to a specific anecdote or a particular moment to support your thinking?

Q1 is inspired by a particular question used by Hadamard (1945) and later by Liljedahl’s (2004). This question refers to the dichotomy, already emerged in the community of mathematicians at the beginning of the 20th century, between a vision of mathematical activity that emphasizes intuition and another one that tends to emphasize the rigour and formalism of deductive reasoning. In our context, in which we are not reflecting on mathematical activity but on the design of teaching activities in mathematics, we ask teachers whether they attribute their creative acts to a spontaneous arising of ideas or to a work of reflection that we might consider more methodical and goal-

[2] The Explicitation Interview is a method based on techniques for the formulations of the re-launchings aimed at facilitating and attending the a posteriori verbalization (in the sense of putting into words) of a particular experience (Vermersch, 1994).

oriented. By this question we want to collect the teachers' perceptions on how the creative acts at the basis of their educational work originate.

Q2 aims to investigate teachers' perceptions about the causes behind such creative acts, and in particular whether and how they would connect them to the social and cultural dimension of their lived lives. The answers to this question could provide some information about the teacher's personal and professional experience, which she relates to her work in instructional design. Therefore, with this question we expect to outline some aspects that are at the origin of the creativity of the interviewed teacher, which determine her uniqueness.

At this stage of the research, we have analysed the answers, highlighting commonalities and peculiarities. Next, we will move on to enlarge the sample of teachers and, alongside this type of analysis, we will analyse the recurring themes.

Findings

Answers to Question 1

Q1 asked to choose between two alternatives: the teacher could attribute her creative acts to a spontaneous arising of ideas or to a more methodical process of investigation. All four teachers chose the first alternative and all shared, as an argument to their answer, what they believe to be the sources behind what is currently a spontaneous arising of ideas in their design activity. As we will show shortly, the reasons that the teachers identified are all different from each other. We draw attention to the fact that the other question, Q2, was formulated precisely for the purpose of investigating the origin of the creativity of the interviewed teacher. However, in the analysis we preferred to keep the answers to Q2 distinct from the argumentations accompanying the answers given to Q1, because of the characteristic of spontaneity of the latter

> Perla: So look, I'll tell you that I usually get ideas all of a sudden [...] with an association, which is often instinctive, of elements that are apparently very far away. [...] I am quite convinced that this fact, which is quite easy for me, more than anything else I find it instinctive to put together apparently distant worlds, is a consequence of my mathematical education, in the sense that it is a bit like when you have the data of a problem or the hypothesis of a theorem and you have to put them together and arrive at the final result or proof of the theorem, I have the feeling that when I come up with a project idea, in the end, I actually think in this way [...]

Perla attributes what she considers a "sudden" and "instinctive" emergence of ideas in her design work to her mathematical education, which taught her to put seemingly distant elements into communication with each other, as when from the data of a problem one seeks a solution or from the hypotheses of a theorem one arrives at its thesis.

> Nadia: [...] Some conditions have been created, even with the association I belong to [MPC], to design together, to reflect together, so when I find myself alone in designing an activity, I don't know how much it is accidental or how much it is actually the result of this continuous process of sharing and building together and so, sometimes, I say: "Is it mine? Did it come to me spontaneously?" and maybe, even this approach of building activities in this way, in my opinion, leads me to think spontaneously about things, but it is a spontaneous in some way conditioned.

Nadia talks about a "conditioned" spontaneity. This conditioning leads her back to the experiences of collective reflection and shared planning, within the association MPC. Moreover, in the interview, Nadia links her design choices also to post-university education. In particular, she refers to courses for teachers or conferences on mathematics education that over the years she has carefully selected because they gave space to collective design activities and because they promised to provide first-hand experience of laboratories.

> Riccardo: Most of the ideas that come to me come out of the blue while I'm doing something else. [...] while I'm shopping, while I'm jogging, while I'm reading a book that has nothing to do with anything, I get ideas and then I try to develop them later. So, I can't tell you whether actually, I mean whether to classify them as ideas that come out of the blue or that are the result of a process. They are certainly, in some way, the result of a process of which, however, I am not always aware here. It seems to me that they happen suddenly, but I also wonder how much this suddenness is due to the stimulations I had, to the thoughts I had [...].

Riccardo relates the spontaneous arising of ideas, during the preparation of the lesson or the meeting with the students, with a process of which he is not always aware, but which he recognizes as being conditioned by external stimuli and previous reflections.

> Claudia: It depends! Maybe we need to distinguish between the planning of the mathematical activity itself (a longer process of investigation, which leads me to search among books, websites, ideas from notes taken during conferences...) and the [laboratory] methodology to be used (more improvised creative ideas).

Finally, Claudia prefers to distinguish between the spontaneity of some ideas, which initiate the planning – which at the beginning of the interview she recognizes to be connected with the places or the artefacts she has available – and the work of defining the activities, which she considers a work of investigation for which she uses different sources.

Answers to Question 2

In reference to Q2, the answers of the teachers interviewed return some salient aspects of their personal and professional experience.

> Perla: [...] my dimension of life has a profound influence on the creation of the activities, that is, in those activities there is really me! This is a certain fact and often, I try to put in what I think is important for children to discover, to know, that is, that can form them in some way as citizens. [...] the example of the project on the enlargement of the map of municipalities, where did that come from? It came from the fact that over the last ten years, for personal reasons, I have interfaced a great deal with the administrative bodies of my city [...] and I have realized how important it is for a citizen to know how the administrative machine works [...]

In answering Q2, Perla asserts that what she proposes to students is strongly determined by personal interests and curiosities – the books and articles she reads or the movies she watches – but also by a range of social and cultural experiences that she has collected over time. The excerpt contains one of the examples she gave us, in which she recounts how her political commitment led her to instructional designs in which mathematical goals were intertwined with goals of citizenship education she recognized to be of crucial importance for her students.

> Nadia: All the activities that we realize with the association [MPC] always try to work on breaking distances in some way: breaking distances within the city, within a territory, within a class, this is a climate that I feel I always carry inside. I experience it externally in the places I frequent and it also influences the type of activity.

Nadia, as in her answer to Q1, refers first of all to her activity within the association MPC. She claims that the sharing of “educational responsibility” and the possibility of experiencing it as a “collective process” offered by the association influences the way she relates to her students and the dialogue she strives to create in the classroom. She also finds that her teaching choices have been influenced by her movement from a small town to a big city like Naples, where she currently lives and works. Naples gave her the opportunity to relate to a broader cultural environment and meet people from different cultures, drawing her attention to issues such as migrant rights, gender equality, or what it means to be a citizen. As shown by the excerpt, this influenced her goals as an educator and in particular her desire to educate to “break the distances” among individuals.

> Riccardo: Certainly, personal background comes up a lot in the design of activities. [...] To make students feel good in school, to make them feel good together with others and to make them feel free to express their opinion [...] is something that I, in some way, trace back to my personal background, to my life experience, to my school experience as a student, [...] but maybe also to the very motivation for which I decided that I would be a teacher in life. More than teaching math, really more about having the opportunity to interact with students to help them bring out their potential to interact with others.

Riccardo believes that his personal experience as a student and his own motivation in wanting to become a teacher- which is characterized by the desire to educate students in comparison with others and in collective living – greatly influences his design choices.

> Claudia: I have been a researcher in the history of mathematics, and most of the activities created by me take cues from my knowledge in this discipline and my passions. I love history! And I really enjoy doing activities that can leave my students surprised, because I myself love finding myself fascinated by mathematical content when it has been presented to me in a “creative” way!

As shown in the excerpt, Claudia believes that her previous experience as a researcher in the history of mathematics and her personal tastes as a student, fascinated by creative and surprising didactic actions, have influenced her design work the most. Moreover, answering to Q2, Claudia recognized that her creative attitude in didactical design was influenced by the successful professional experience in a particular school, situated in a poor neighbourhood of the city of Turin, which is characterized by the presence of many families of recent immigration. She said: “Once I figured out how to ‘catch’ these guys (each in their own way, of course), my creative streak increased, because I could see how much my effort paid off.” Finally, she noted that the presence of students from different countries motivated her to design mathematical activities involving different cultures, such as one on number-words to reflect on the decimal number system.

Although in different ways, all four teachers believe that their personal backgrounds and their social and cultural dimensions influence the work of creating teaching activities.

Discussion

In this first study we tried to inquire the way in which these four mathematics teachers, which are responsible for the educational design inside two very innovative informal mathematics projects, tell about their experience of creating mathematical activities and how they link it with their personal and cultural background. Looking at the interviews, in particular at the answers to Q1, it is possible to recognize in their description that the way in which the design of the mathematics activities emerges is characterized by illuminating moments, but also by a “conditioned” spontaneity, as Nadia says. Moreover, the answers to Q2 helped to catch how the different teachers’ personal, professional and cultural experiences play a key-role in the realization of the very innovative mathematical activities they propose, both during the preparation of the lessons and during the meeting with the students (Freire, 1970). Perla referred to her personal interests and to her experience of collaboration in the administrative bodies of her city. Nadia told about the influence of her social commitment, through the militant activity in the association MPC, and beyond. Again, Riccardo referred to his school experience as a student and his educational objective of educating for collective living, which he sees as a priority. Finally, Claudia told about her research experience in the history of mathematics field and about her professional experience in a challenging reality, which required great effort on her part to engage the students. The answers to Q1 and Q2 show a very varied picture on these teachers’ perceptions regarding the sources of their creative educational work. Nevertheless, the same answers reveal a point of contact in the teachers’ pedagogical goals. All of them aim at educating at democratic citizenship through mathematics. Perla spoke about a project exploring with students the administrative bodies of the cities; Nadia referred to her (and her association) action of breaking distances, at all social levels; Riccardo declared his desire to educate at a collective living; finally, Claudia told how she used mathematics to create meeting among different cultures. In this sense it is worth to emphasize how Freire’s ideas were of great inspiration for the field of critical mathematics education (see, e.g., Skovsmose, 1994). One of the most recognized pedagogical aims in this field is the intertwined development of critical citizenship consciousness and mathematical competencies:

> […] mathematics should be a vehicle for students to deepen their grasp of the sociopolitical contexts of their lives, and through the process of studying their realities -using mathematics- they should strengthen their conceptual understanding and procedural proficiencies in mathematics. (Gutstein, 2007, p. 109).

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Students’ thoughts on time and speed in high-stakes mathematics tests

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Time and speed are important aspects of mathematics education and assessment. Students are expected to produce results in prescribed time frames, which means that they have to perform at a specific pace. How does this affect students’ engagement with a mathematics test? How can a high-stakes strictly-timed mathematics test increase the focus on speed? How can these conditions produce exclusion in mathematics assessment? We discuss these questions through an interview with a student a few weeks after she took the mathematics test of the Greek national exams of June 2021, which is a nation-wide, high-stakes and strictly-timed mathematics test.

Keywords: Time in education, speed in mathematics, time in assessment, mathematics assessment.

Time and speed in mathematics assessment

Time is an important dimension of education and affects every aspect of it. There are very specific time frames for every school activity and there is a predefined pace at which students and teachers are expected to work. In assessment, students are expected to work within very specific time limits and also at a specific pace (Adam, 1990).

In mathematics assessment, research has studied the effect of speed on students’ performances. Some results show that students who are more mathematically anxious (Boaler, 2014) may be affected by the time limits of mathematics tests more than students who are less anxious (e.g. Hembree, 1987; Tsui & Mazzocco, 2006). There are conflicting reports on this issue and it has not been studied enough.

Boistrup (2010) studied mathematics assessment through the analysis of assessment acts. One assessment discourse which she analyses is “Do it quick and do it right” (p. 166), which describes emphasis on a quick and correct final result of the student’s work in an assessment task rather than on the thinking process that produces this result.

For high-stakes mathematics tests with strict time limits, the issue of the effect of time on students can be more complicated. High-stakes tests influence teaching, students’ preparation and learning very much. Teaching shifts from issues that focus on knowledge and meaningful skills towards a narrower focus on the tasks which are regularly included in a specific test, a phenomenon that is described in research as “teaching to the test” (Cankoy & Tut, 2005, p. 235). This also affects learning as

> more time spent focusing on procedural skills such as drills, test taking, or practice with tests from prior years, with little connection with conceptual understanding and qualitative reasoning, can distract students and encourage them to memorize procedures and to search for a single path to a single answer (Cankoy & Tut, 2005, p. 242).

Research on mathematics assessments suggests there are important non-content factors that affect students’ performances, for example, test familiarity (Hembree, 1987). Furthermore, high-stakes testing increases the phenomenon of shadow education or private tutoring, which increases inequalities, because low-income families cannot afford it (NESSE, 2011).

These high-stakes tests are, often, nation-wide mathematics tests that states need for assessing and ranking masses of students with the minimum cost and in a manageable time frame. But, mathematics tests which focus on the mathematical content and not on other factors, usually, have to be longer, and require more time to design and to score, and more time from the students to complete (Suurtamm et. al, 2016). This potentially creates a tension. Firstly, time limits are imposed that, sometimes, can be very strict and, secondly, speed is imposed as an important skill in mathematics literacy. Students may be interpellated by this double imposition and form themselves as governable (Kollosche, 2018) subjects who are supposed to find speed with mathematics important and act accordingly. Thus, speed becomes a characteristic of informal social norms or rituals (Lundin & Christensen, 2017) of mathematics education. It, also, may be related to the value of efficiency that is highlighted in contemporary society and economy (Chronaki, 2017). These impositions provide forms of exclusion which everyone agrees on without consciously knowing and which, ultimately, are necessary for the economic and ideological function of mathematics education (Pais, 2014).

There are questions raised by these issues about the position of the students in this framing (Makrakis, 2021). How are students affected by time limits in high-stakes mathematics tests? How do time limits of mathematics tests affect students' preparation for them? How do students view the fact that speed is expected of them in a strictly-timed mathematics test? How are these issues connected to the sociopolitical character of mathematics education?

Methodology: Data collection and framework of analysis

I will discuss these questions presenting preliminary results of interview data from a broader study that I am carrying out. I focus on one interview of a student that had just participated in the mathematics test of the Greek national exams of June 2021. Greek national exams are taken just after high school graduation and largely determine a student's admission to higher education after students' performances are compared to each other. Their grade in mathematics may contribute even more to their final score. Greek national exams in mathematics present some general regularities and trends as far as the preferable tasks are concerned (Thomaidis, 2021). The Greek national mathematics exam has a three-hour time limit, which is often insufficient (Mavrogiannis, 2017). The test always contains four tasks, task A asks about theory and the other tasks B, Γ and Δ are mathematical problems.

This semi-structured interview was taken, recorded and transcribed a few weeks after the exam through Skype (for COVID-19 safety) and a whiteboard was screen-shared. I interviewed a 17-year-old student, Litsa (name changed), who performed well at the test. I have taken interviews with 13 students for my general project and Litsa was one of the typical cases. The interview questions were about time, speed, anxiety, and her decisions during the test in relation to the time limit. The interview was conducted and transcribed in Greek and translated by me for the needs of this paper.

I will briefly discuss excerpts of the interview. In my comments I will include some elements of discourse analysis particularly, from approaches that are derived from Lacanian Discourse Analysis, which view discourse not as a closed structure but study of the emergence of elements from the Real (Lacan, 2006) behind the signifying structure, which may, also, act retrospectively. They view discourse in reference to the issue of the agency of the split subject and they study not only what is being said, but, also, latent elements, which are present through their absence (Frosh, 2014) and, also, nonsensical elements of discourse (Parker, 2014).

Results

Nikos: You said "concerning your preparation". Did you mean (…) Did you have tests of prior years in order to prepare for the time limit?

Litsa: Yes, yes, I did have tests of prior years. I, also, took mock tests held by the tutoring school which I attended and I timed my completing the test, uhm, in order for me to use less than 3 hours for sure.

Litsa thinks that completing the test in time needs special preparation. She did past exam papers in order for her to be more familiar with the circumstances of the actual test. Tutoring schools organize mock tests (which they call "simulation tests") which simulate the test of the Greek national exams. Litsa evaluates the fact that she took such a test as important for feeling prepared for the time limit. Apart from the timing of the test which the tutoring school held, she says that she also timed herself, imposing a limit of her own for less than 3 hours, in order for her to be sure that, when she takes the actual test, then the three-hour limit will be enough. Taking the initiative to time yourself is a different experience than having a time limit imposed on you, because it means that you have internalized the need for synchronization and performance in the expected time frame and you posit yourself as the one who regulates it. She tries to fit herself to the standards expected of her, synchronize herself and become the one who delivers the expected performance on time.

Nikos: How anxious did you feel before and when you were taking the test?

Litsa: All in all, before taking the test I was feeling only a bit anxious. I am not anxious in general. I knew that I had done the appropriate preparation. OK, I knew that it is mathematics – you can get stuck somewhere. But that didn't make me feel anxious. When taking the test, I wasn't anxious, I just kept writing, I didn't even have the time to feel anxious. So OK.

Litsa evaluates the fact that she had done the "appropriate" preparation as important for feeling not too anxious. Preparation for the test is somewhat specific. Every student is expected to have done past exam papers and to have solved most of the tasks that are included in the official school textbook and one of the 3-4 most popular books which are written to prepare for the Greek national mathematics exams. So, Litsa, having done this expected preparation, feels not too anxious. She, also, recognizes that, in mathematics, having done what is expected may not be enough, as you may get "stuck somewhere". That is, in a task you may not think of something to do in a given time, which you cannot predict beforehand. She, also, says that this did not make her feel anxious as she "kept writing" and "didn't, even, have the time to feel anxious". She, maybe, recognizes that she has reasons to feel anxiety, she recognizes that she cannot do anything about those reasons, and then she just has to perform. She knows that time may not be enough if you think too much, or if you feel too unsecure. She can cover those reasons by "just writing". She, also, identifies herself as someone who is not anxious "in general". So, she acts and poses as a person who is not one of those that get anxious, but one that just keeps writing, because she is supposed to and prepared to do so.

Litsa: Surely, I tried to prepare to complete the test at least half an hour earlier. So, I started with huge rush, let's say, to complete as soon as possible task A and B, in order to have time for Γ and Δ.

Nikos: OK. What do you mean by saying "huge rush"?

Litsa: As quick as possible. I wasn't thinking much during tasks A and B. I did them (…), I tried to do them mechanically. That is, for them to be something I had already seen, so as not to waste time on this.

Litsa says that she set a personal limit of 2-and-a-half hours to complete the test and half-an-hour to check again her answers. She evaluates different tasks as needing different amount of time to be spent on. Task A on theory needing the least along with task B which is considered the easiest problem and tasks Γ and Δ, which are considered the most difficult, needing the most time. So, Litsa thinks that she had to work at different paces during the test.

Litsa says that she has to be quick, especially at some tasks. When asked to explain her "rush", she says that she has to do them "mechanically". She is describing herself as having to work very quickly on something so familiar that she does not even have to think about it, because she must do so in order to be on time and because she is prepared to do exactly that. She makes the sentence "to be something I had already seen, so as not to waste time on this" which is a syntactically incorrect collation of two phrases in order to just show that they are connected in her mind. She evaluates having seen similar tasks as very important in order for her not to spend time thinking how to solve them, because this would have been a "waste" for her, as it would result in her spending less time at tasks Γ and Δ, as they, actually, require thought. She evaluates writing as the only thing that she is expected to do at tasks A and B, instead of both writing *and* thinking.

Nikos: Task B3 asked you to compute the asymptote line and compute a limit. Do you maybe remember which of the ways a., b. or c. did you try in order to compute it?

Litsa: Huh, I don't remember. I think, huh, just a minute, because I remember nothing. I think b., I'm not sure. I moved (…) I wanted to get rid of this the fact that this expression is indefinite. I moved one of those to the denominator, so (…) I don't remember how I thought of it (…)

Nikos: Did you do it on the first or the second try?

Litsa: On the second. I think. I'm not sure.

Nikos: (…) If you tried it with way b, then it couldn't be computed. It could be computed with ways a. and c. So, you began with way b, and then moved on to a.

Litsa: I did it for sure. For this kind of tasks I begin using one way or the other and then I see if it can be computed. If not, then I do the other way.

Nikos: OK. Do you think that there is a way to think which of those two ways can produce the outcome? Or do you choose by luck?

Litsa: By luck mostly, but… because e^{1-x} (…), I don't remember now, when you compute the derivative, I think that it cannot produce the outcome, right? Do you move that to the denominator? I did this mostly by luck, I didn't think about this with enough (…) mechanically. I didn't remember.

Nikos: So, then, you choose a way by luck, but you knew that it may not be able to produce the outcome, and then you would try another way, right?

Litsa: Yes, yes, yes. That's what I do usually.

Here Litsa was asked which way she chose in order to try and compute the limit $\lim\limits_{x\to+\infty} xe^{1-x}$. The three possible ways were: a. $\lim\limits_{x\to+\infty} \frac{x}{\frac{1}{e^{1-x}}}$ b. $\lim\limits_{x\to+\infty} \frac{e^{1-x}}{\frac{1}{x}}$ and c. $\lim\limits_{x\to+\infty} \frac{xe}{e^{x}}$. Only ways a. and c. can produce the required outcome. She remembered that she tried way b. and then moved on to try way a. and finally did manage to produce the required outcome. When asked if she chose this specific way by luck, she agreed. She is aware that for this task she could not know which way would produce the required outcome before making the necessary computations. So, she says that she had to choose by luck and if the way she had chosen failed, then she had to try another one and not think about this any further. Here, she makes a syntactical error just to squeeze the word “mechanically” into her sentence. She thinks she is not supposed to think about this point, but just to choose a way, and if she hits a dead-end, she tries again, in an automated process, in order for her to be on time. She says that she usually does so. She expects this to happen and she isn’t surprised by it. She shares no thought that this approach may make her lose time.

Nikos: Do you think that in order for someone to succeed at the mathematics national exam it is required of them to be quick at doing mathematics, and why?

Litsa: Uhm (…), I would say yes. Because national exams tests have a narrow time frame and if you get stuck somewhere (…) uhm (…) and you do not move on quickly to the next tasks, you won’t be able to spend the time that you want on it. So, generally, yes. To be fast to complete the first tasks quickly, because the next ones really require to put thought on. So, if you don’t have this time, then you can’t solve it.

Here Litsa evaluates speed as important and relates it to the “narrow time frame”. Since you are expected to operate in this narrow time frame, you really have to be quick and not “get stuck somewhere”. For her, being quick means not getting stuck, because if you do get stuck, then you spend more time than what you are supposed to. And this is not the same for each task, as she repeats that she feels that the first tasks require more time than later ones. So, if you do not do the first ones quickly, then you will not have the time “you want” on the later ones. She says “you want”, but who is the one that wants? Is it herself? Is it what is wanted of her by others or, maybe, what she thinks is wanted of her?

She says that the last tasks are the ones where you are required to “put thought on”. For the first ones you are not, you just have to complete them as fast as possible and this means without thinking. Therefore, no thought somewhere is described as the precondition for thought elsewhere, with time being the key here.

The tasks where you do not have to think, tasks A and B, assess theory and basic problems. If you complete them without mistake, then you get half marks and you score about average. But Litsa is a good student who expects and is expected to perform very well. So, she may not have to “put thought on” the tasks that get you just average performance. She may think that it is almost self-evident for her to do them without thinking. And just to do them quickly in order for her to “put thought on” the tasks that really matter for a student of her expected performance, tasks Γ and Δ, which are always more difficult.

Nikos: Do you think that in order for someone to be good at mathematics, they have to be, also, quick at doing mathematics and why?

Litsa: OK, maybe. In the sense that (…) I don’t know, actually. Yes, I usually correlate it with quickness, so OK, someone can be good and solve a task for many hours and so on, but I find better someone who solves them fast. Uhm (…), it comes to their mind automatically. It doesn’t require much time and so on. Which means that they have done more practice. They have worked on them more, so, consequently, they can solve more complex problems faster.

Nikos: Uhm, so you are saying that being good relates to speed at doing math in your opinion.

Litsa: Yes, it relates to a point, in my opinion.

Nikos: And you say that displaying speed, they show that they have practiced enough on what they are solving and this is an element which you say that it shows that they are good. Is that what you mean?

Litsa: Yes. Of course, this doesn’t go for every case. There are people who find it more difficult and need more time. Or people who have not practiced at all and they solve the tasks quickly. But, on average, I think that the one that solves the tasks quickly is the one that has practiced a little, just a bit more than the rest.

Nikos: You said that [the one that] has practiced more, but, also, the one that shows that they have practiced more. Right? So you believe that it matters to be shown that they have practiced more. Is that what you mean?

Litsa: What do you mean? In quickness? To show that they have practiced more?

Nikos: For them to show their familiarity with what they are dealing at that moment.

Litsa: Yes, certainly. When someone is solving a problem fast and so on, yes, it is shown that they are pretty familiar with it. They go qui (…), their mind is inside this process. It works mathematically. So, reasonably, they are more familiar.

Nikos: OK. What do you mean by saying "their mind works mathematically"?

Litsa: Generally, every mathematics problem has a certain process to be solved. So, a deductive (…), a deductive way of thinking. You begin. You draw conclusions. You analyze them. Then, you connect them in order to find the solution to the problem. Surely, it is something that you can acquire very difficultly, let's say. And it is the most difficult part in mathematics, for you to get in the process of thinking in this way and analyzing every piece of data that you are given and drawing conclusions. So, this particular part is that which need the most work and which shows who has acquired a mathematical way of thinking.

Nikos: And for you, do you believe that this is shown when they do it faster, also? That this is something that helps you do that?

Litsa: Yes, this is an indication, yes.

Here, Litsa says that she considers someone is better at doing mathematics when they solve something faster than someone who actually solves it but takes many hours. She says that the reason for this is that the first solves it automatically and it comes to their mind "automatically". Automatically, like mechanically, means that they do not have to do something, they do not have to think or process something, they just solve it. Here, in her descriptions of the difference between someone who is and is not good at mathematics, we can find a gap. She repeats that being faster at solving a task means being better at doing mathematics, because this shows that you have spent more time practicing. Having practiced shows as familiarity and quickness, and this allows you to be considered better at mathematics. Is there a possibility that someone has practiced a lot, but this doesn't show as quickness? She says yes, but "on average" no. "On average" is a statistical measure, but in common speech it can mean other things, too, like "normally". It may be used as a manifestation of a norm, which may have exceptions, but still counts.

She, then says that the person who does mathematics quickly and shows being familiar with the test indicates that "their mind is inside this process" and "works mathematically". She explains this as having deductive thought. She uses the Greek word "Παραγωγική", which, apart from "deductive", also means "productive". When she describes it, of course she does not describe deductive reasoning, but another process. She describes that a student facing a task may start by making some initial results that can be drawn firstly by the data. After that, they have to find ways to connect these results in order to achieve the requested outcome of the test. In order to do that the student has to have experience and also to try out different things which may fail. The process of course takes time, so this is in contrast with what she says earlier. Students who take less time, are those who actually take the time to do what is asked. Here she appears to describe a situation in which the students that are supposed to solve it, are the ones to actually solve it. She says that "this particular part is that which need the most work and which shows who has acquired a mathematical way of thinking". So, she thinks that there are students that have acquired this "mathematical way of thinking" and others that have not. And speed at doing mathematics is an "indication" of that.

Discussion

The timed conditions of high-stakes, strictly-timed mathematics tests shape the way students view and prepare for them. Students may feel that the performance expected of them requires very specific preparation according to time. Students may feel that they have to fit the pace by which they solve tasks to a required time frame. They may view this as fitting a norm which is considered familiar and relevant to mathematics a type of social synchronization in doing mathematics. Students may view their speed at doing mathematics as a characteristic that defines their identity as students of mathematics, i.e. as those who perform in the required time frames or as those who do not. Students are interpellated by this framing.

This, maybe, produces a fantasy of the ideal student of mathematics assessment, who does not only know mathematics, but also preforms mathematics in a quick and automated way, in a minimal time frame, and this is reflected directly and objectively in their assessment score. Students who do not fulfill this fantasy may be excluded, and this fantasy itself retrospectively justifies their exclusion. That is, mathematics assessment tests are supposed to assess if you are good at mathematics. But if there is a latent belief that being good at mathematics also means fast, then some mathematics tests assess speed, too. So fast students will, mostly, succeed and then the fantasy of the ideal quick mathematics student will be fulfilled retrospectively. This fantasy acts as one of the ideological presumptions, which normalize the exclusion produced by mathematics education and assessment and, also, make possible its reproduction in the future.

If doing mathematics is described as a ritual influenced by social and discursive norms, then this ritual is, also, a ritual that happens in time. In mathematics assessment tests with strict time limits, the line that shows what is mathematical and what is social is even more difficult to recognize. This affects the identifications that students make in relation to mathematics. And this produces exclusions which have less to do with mathematics than we commonly think.

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Lessons learnt from history: Socio-cultural perspectives on mathematics education reform in Egypt

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Keywords: socio-cultural capital, power dynamics, reform, memorisation, history.

Introduction

The study of educational reform initiatives is not a neutral enterprise. It needs to take into consideration – among other factors- the social and cultural heritage in which the given reform is situated (Ball & Lewis, 2008). In other words, for a reform in mathematics education to be socially accepted and practically effective, it needs to be bought into by the actual stakeholders in closest relation to learners and by the learners themselves. Hence, we can consider these social stakeholders as pillars of capital that make or break the reform initiative enactment. Building on this contextual view of educational reform, this poster is underpinned by a historical investigation of reform patterns in Egypt. It utilises a widened perspective on forms of capital, in an attempt to analyse patterns of societal buy in that have evolved over the years and that can be traced as part of the recent reform initiative in mathematics education in Egypt.

Theoretical Underpinning

This poster is based on the forms of capital theory, in which Bourdieu (1986) distinguishes between three types of capital; namely: economic, social and cultural capital. The latter two refer to the power of societal buy in and cultural heritage to contextually determine how a given initiative gets received and practiced. These three forms of capital are underpinned by what Bourdieu (1986, p.256) refers to as "the historical baggage" of a given context. "The historical baggage" is marked by eras that have shaped the cultural and societal topology of power dynamics. For the scope of this poster, based on the historical inquiry process (Hicks & Doolittle, 2008), I present a work-in-progress literature review covering three main eras of educational reform along the modern Egypt educational historiography, namely the post-colonial era (1914-1922), the republican era (starting 1967) and the era of global and open economy (1971 onwards). Patterns noted in these times serve as a lens for better investigate the current social tension around the current educational reformation, which aims to move away from a longstanding tradition of memorisation based instruction (Megahed, 2018).

Three eras of mathematics education in Egypt

Centered around the depiction of a historical baggage, the poster uses the historical inquiry process to illustrate the aforementioned three selected eras of societal transition in educational policy in Egypt, which show a repeating trend of societal power dynamics, presented more elaborately below.

Mathematics education in post-colonial Egypt: Two-tier structure

During the British colonisation, the modern Egypt was exposed, over an extended period of time, to a westernised education system in mathematics, which focused on creation and problem solving. This philosophy was socially foreign to the local eastern educational culture at the time. The westernised curriculum was only available to the elite, which resulted in the mainstay of a societal and cultural divide. This divide in educational ethos triggered the first, so called two-tier structure of education in Egypt which was largely sustained even after the colonisation (Heyworth-Dunne, 1968).

Mathematics education in the Republic of Egypt: Centralised Governance Model

In an attempt to offer equal opportunities for all, the government of Egypt, in 1967 took the radical step of unifying the curriculum across contextual affordances. The curriculum was centrally controlled by the government. The idea was to ensure the same curricular enactment is replicated irrespective of educational context, thereby closing the longstanding socio-cultural gap, which was created by the two-tier structure and safeguarding the system against extremist stimuli (Sayed,2006).

Mathematics education in the open economy of Egypt: Back to two-tier structure

President Sadat practiced a relaxed policy of open economy, which in turn gave rise to western investment in national education. Despite being heavily controlled by the national agenda for education, the elite had the opportunity to be educated differently; a societal trend that again gave rise to the return of the two-tier structure. Memorisation based instruction was again the ethos of the mainstream channels for education, while problem solving was taught selectively (Sayed, 2006).

Discussion

With this toggling trend of access, the mainstream of educational stakeholders have over the years solely been exposed to memorisation based instruction. This explains the reality which fostered the mainstay of the socio-cultural divide. Shifting suddenly to a different mathematics instructional model is understandably perceived as foreign to the mainstream and unwelcomed by the elite. The growing need to include Egyptian students, often refugees, at European schools gives rise to the importance of understanding the contextual reality that these learners come from and how they relate to western educational practices. This makes this poster relevant beyond the case of Egypt.

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Universal Design for Assessment in mathematics

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Universal Design for Learning (UDL) is a commonly used framework for designing accessible learning environments. While UDL has been reportedly applied to testing situations, much less is known about how classroom assessment (e.g., formative assessment) could be designed accessible to support the learning of all students. In this conceptual study, the previously introduced idea of Universal Design for Assessment (UDA) is reformulated in the context of mathematics. It is argued that in the test-driven assessment culture of mathematics, UDA holds specific promise; recent studies have noted that mathematics assessment does not enable students with disabilities to participate fully due to inaccessible practices. The proposed framework discussed how UDA could promote the following guidelines in mathematics assessment: i) partnership, ii) diversity, and iii) dialogue.

Keywords: Mathematics assessment, Universal Design, accessibility, diversity, ableism.

Introduction

It is a publicly known secret that classroom assessment has been unable to meet the needs of students with disabilities. In assessment, disabilities are not seen as something to be celebrated but as something to be overcome. Assessment with both its summative and formative purposes largely draws on individualised assessment accommodations rather than on inclusive practices; students with disabilities are seen as the problem to be fixed, not assessment itself (Nieminen, 2021).

While recent contributions have critically examined how mathematics education constructs disabilities through inaccessible teaching practices (e.g., Lambert, 2015; Nardi et al., 2016; Tan et al., 2019), much less attention has been given to assessment. This is surprising given how test-driven the assessment culture of mathematics is. As shown by Nieminen (2020), mathematics assessment plays a crucial role in *disabling* students. The test-driven culture of mathematics does not only create barriers for learning but for inclusion and participation by excluding students with disabilities from other mathematics learners both physically and socially (Bagger, 2022).

In this conceptual study, the commonly used framework of Universal Design for Learning is used to rethink mathematics assessment as an inclusive endeavour. This study draws on earlier critical work to understand ableism in mathematics education (Padilla & Tan, 2019): how assessment produces an ideal of certain normality and then excludes students who do not fit this ideal of a *normal, able* student (Nieminen, 2022). Rather than focusing on the pitfalls of current assessment practices, this study reaches further by formulating a framework for Universal Design for Assessment to guide future research and practice in mathematics education. First, Universal Design is introduced.

Universal Design for Learning

Overall, Universal Design refers to accessible design for everyone, originating from the field of architecture. In education, Universal Design has been largely promoted through the pedagogical framework of Universal Design for Learning (UDL). UDL refers to accessible pedagogical design that "proactively builds in features to accommodate the range of human diversity" (McGuire et al., 2006, p. 173). Due to such underpinnings, UDL is often connected with the social model of disability that does not understand disabilities as a deficit to be cured but instead sheds light on educational practice that actively *disable* students. While UDL has been most commonly used to design accessible practices with disabilities in mind, recent contributions have expanded the notion to address, for example, racism (Waitoller & Thorius, 2016). The UDL model encompasses three main principles, as formulated by CAST (2011):

- **Engagement** (the 'why' of learning): Multiple ways for stimulating interest.
- **Representation** (the 'what' of learning): Multiple ways for representing knowledge.
- **Action & Expression** (the 'how' of learning): Multiple ways to express knowledge.

UDL builds specifically on the very idea of design: rather than drawing on retrospective, individual adjustments it instead shifts our gaze to careful design of learning before the learning process itself. While UDL has been largely promoted in educational policies and practices, thus far such designs have been rarely reported in mathematics education.

Universal Design for Assessment

While UDL has been widely promoted in education, its implementation in assessment has received less attention. Accessible design in *test item design* has been noted, and indeed Universal Design for Assessment (UDA) has so far focused on how to design accessible large-scale exams (see Ketterlin-Geller et al., 2015). It is fitting that the seminal study by McGuire and colleagues (2006) only referred to UDL in assessment in terms of large-scale national testing.

Some more holistic conceptualizations have been offered. Ketterlin-Geller (2005) defined UDA as "an integrated system with a broad spectrum of possible supports so as to provide the best environment in which to capture student knowledge and skills" (p. 5). Ketterlin-Geller and colleagues (2015) discussed UDA in terms of target and access skills in assessment. According to the authors, assessment is intended to 'measure' certain skills and abilities (*target skills*), while other skills might also influence students' performance while demonstrating their mastery (*access skills*). Through careful pedagogical design, the interference of access skills can be minimized (e.g., a large font size ensures accessibility in a test so that the test item measures the intended mathematical skill).

However, to date, earlier studies have not built a critical framework to guide the design of classroom assessment in all its diversity beyond test design (e.g., self- and peer assessment) (Nieminen, 2022). Moreover, there is a need for a mathematics-specific UDA framework to address the ableism and inequity related specifically to mathematics assessment.

A socio-political, mathematics-specific framework for UDA

In this conceptual study, UDA is reformulated in the context of mathematics education. Recently, there has been a call for critical approaches to challenge ableism in mathematics education regarding students with disabilities (Tan et al., 2019). This is exactly the approach taken in this study. In fact, the UDL framework has been criticised for its focus on pedagogical design over challenging ableism and injustice, trading disability activism into an 'activation of neural networks'

(Nieminen & Pesonen, 2020). In the words of Hamraie (2016), UDL has become “emblematic of a depoliticized orientation toward disability” while largely ignoring “systems of oppression such as racism, sexism, or ableism” (p. 302). Following Hamraie, this study uses UDA as an inspiration but ties it with a critical approach. The UDL framework is reformulated as a novel UDA framework for mathematics assessment (Table 1). While introducing the UDA guidelines in the following sections, the main issues in mathematics assessment are introduced from the viewpoint of equity and disability rights. The study mainly focuses on the viewpoint of students with disabilities, but the framework holds promise for intersectional work too (Nieminen, 2022; Waitoller & Thorius, 2016).

Table 1: The reformulated framework for Universal Design for Assessment in mathematics

Original UDL principle	Engagement	Representation	Action & Expression
Revised UDA principle in mathematics	Partnership	Representation & Diversity	Dialogue

UDA principle 1: Partnership

The issue: In test-driven assessment cultures, students are merely the targets of assessment rather than active agents. In other words, students are *objects* in assessment, not *subjects*. Even when ideas such as Assessment for Learning are promoted, the process tends to be dominated by teachers’ actions and choices. As students are not enabled opportunities to co-design assessment practices, they might learn to be dependent on teachers’ actions rather than to truly ‘own their own learning’. The effects of such unilateral idea of assessment might be more prevalent for students with disabilities who have historically been dependent on teachers’ actions (Nieminen, 2021) and in mathematics education (Lambert, 2015; Tan et al., 2019). As the medical model of disability dominates in assessment, students are dominantly seen as the objects of support services determined by others.

The first UDA principle draws on the ideal of democratic education that understands learners’ rights to take part in actions and decisions that concern themselves. This is achieved through the principle of *partnership* that provides students with opportunities to act as co-designers of educational practices (Cook-Sather et al., 2018). Matthews and colleagued (2021) noted that while co-design practices has been reported widely in educational literature, such approaches have been rare in assessment. This highlights the urgency of the first UDA principle, especially in the test-driven context of mathematics assessment. The first UDA principle taps into the design-based roots of UDL (McGuire et al., 2006). Traditionally, UDL has emphasised that accessible educational design benefits everyone. The design process should hear the voice of the end users: designing *for* students with disabilities is not enough as assessment needs to be designed *with* them. This was noted by Nieminen and Pesonen (2020) who reported a university mathematics course whose design drew on UDA. As the design process only heard students’ perspectives after the course design, the process was certainly not inclusive; a worthwhile lesson for both the authors and the readers! Importantly, the first UDA principle emphasises that all students need to be heard in assessment design processes, but marginalized students in particular to enable accessibility (Cooker-Sather et al., 2018; Matthews et al., 2021).

In practice, the first UDA principle means that students are enabled possibilities to design the mechanisms of assessment. The principle disrupts the individualizing nature of assessment by rendering the mathematics assessment design process as a collaborative, communal project (see Matthews et al., 2021). Traditionally, it is the teacher who determines the learning goals and then designs assessment accordingly. While seeing students as partners, they need to have their voice heard in terms of which assessment practices are used and how. Experiences of co-design might be especially powerful for students with disabilities as this way they could feel sense of agency in how their mathematical skills are assessed: they indeed become subjects in assessment.

Students might co-construct a rubric with the teacher and, in the process, engage in a discussion about the relational standards regarding mathematical skills and knowledge. Students could design novel assessment practices, perhaps drawing on accessible digital technologies. Students with disabilities have been shown not to be able to fully participate in mathematics tests (Bagger, 2022; Nieminen, 2020); students could co-design other, more accessible forms of both summative and formative assessment. Even tests can be communally co-designed as reported by Rapke (2016) in the context of university mathematics. In Rapke's study, students have an opportunity to co-design a mathematics exam, as sitting an exam was required by the regulations of the university. This way this summative practice became a communal process instead of an individualised practice in one given time.

Assessment co-design processes (e.g., students taking part in constructing digital assessment forms) could promote students' understanding of mathematical knowledge and how this could be validly assessed: this is *assessment literacy* in action. As students learn the 'hidden mechanisms' of mathematics assessment, they also learn to examine their own assessment actions reflexively. For example, Nieminen and Lahdenperä (2021) discussed how mathematics students' preference for traditional assessment practices resulted from an assessment culture that undermines students' assessment literacies. Instead, students could be trained to critically examine how mathematics *is* and *should be* assessed, and what their active role in the process could be. Fostering assessment literacy is especially important for students with disabilities. This way students can learn not to only determine themselves as mathematical learners through the assessment information provided by others.

UDA principle 2: Representation & Diversity

The issue: Mathematical knowledge is most often presented in the form of text. While graphs and graphical illustrations are an important part of presenting mathematical knowledge, in the end, what is considered as the most powerful form of representation is abstract mathematical text and notations. In mathematics assessment, *time* has been another crucial determinator of mathematical knowledge. This is most imminent in controlled testing situations. These very boundaries of text and time are not accessible for all learners (Bagger, 2022; Thomas et al., 2015). While inaccessible representation of knowledge can exclude learners from mathematical communities in overt forms (e.g., by dividing students with hearing and vision impairments to segregated classrooms), covert forms are also present. For example, students with dyslexia might feel they do not belong in mathematical communities due to the dominance of visual text format (Nieminen, 2020).

Much like the original UDL principle, the second UDA principle promotes the idea that mathematical knowledge can be represented through a variety of ways. This UDA principle promotes multiple forms of media in presenting mathematics. For example, mathematical knowledge could be presented in the forms of images, videos, and embodied ways such as dance, gestures, signs and movements. In assessment situations, and especially in summative assessment tasks, enough time should be provided for students for whom time management itself might be an access skill (see Ketterlin-Geller et al., 2015). Mathematical concepts are often presented in unnecessarily complex ways, especially in high schools and universities. The UDA principle reminds about the ableist underpinnings of abstract mathematical text and the preferred fast pace of mathematical learning and assessment. Sometimes it is simply colors and images that are enough to provide accessibility; even such simple ideas might disrupt the ableist norms of mathematical representation. This UDA principle ensures that the need for individual assessment accommodations is lowered, supporting not only students with disabilities but also, for example, those needing extra support with language (Thomas et al., 2015).

In order to be successful, UDA needs to disrupt the ableist norms of mathematics assessment (cf. Nardi et al., 2016; Padilla & Tan, 2019). Nieminen and Lahdenperä (2021) showed how mathematics assessment sets boundaries for what counts as mathematical knowledge, thus setting epistemic boundaries for who can *know* mathematics and how. Even if teaching practices would promote conceptual understanding of mathematics through multimodal ways, assessment might override such representations by reminding what is truly important: individual performance textual summative assessment. Indeed, students in Nieminen and Lahdenperä's (2021) study expressed that the knowledge produced through self- and peer-assessment is *invalid*. According to the students, such formative assessment practices could be used, but exams show what one's *real* mathematical skills are. Such a hegemonic role of tests is ableist, as students with disabilities are excluded not only socially but epistemically. They are taught to understand themselves as the 'others' who are not able to fully participate in one of the most sacred rituals in mathematics education: the exam.

The second UDA principle holds promise for many marginalised forms of knowledge. Mathematics assessment always privileges certain forms of knowledge over others. What students with disabilities learn in mathematics assessment is that their personal epistemologies – the knowledge about themselves, their very personhood through which they operate as mathematical thinkers and doers – are something to be overcome, not celebrated. This of course aligns with the harmful idea of disabilities as deficits (Lambert, 2015). Thus, this UDA principle calls for assessment practices that allow students to inclusively use their "cultural repertoires, identities and out-of-school activities" in assessment (Waitoller & Thorius, 2016, p. 384). In this way, marginalized forms of knowledge can be valued in assessment by offering students various ways to understand themselves as mathematicians. For example, both formative and summative assessment tasks could draw on the language (in its both verbal and nonverbal forms) and cultural knowledge of students themselves.

UDA principle 3: Dialogue

The issue: As noted, mathematics assessment is globally built around testing. Even when other forms of assessment are introduced, tests still remain in the very centre of assessment and grading mechanisms of mathematics. Yet it is tests in particular that causes barriers for students with disabilities (Bagger, 2022). To foster inclusivity, mathematics assessment needs to diversify its practices to enable all learners to show their skills and capabilities.

The UDL principle of Action & Expression promotes various actions through which students can demonstrate their skills and knowledge. The second UDA principle is built on this premise: it reminds that all students have the right to be assessed through a diverse menu of practices (e.g., self- and peer-assessment, portfolios, group assessment…). Such practices should also be provided through multiple forms of media (cf. UDA principle 2). UDA principle 2 focused on the presentation of knowledge, but the third principle emphasises that diverse assessment should be used to widen the very idea of what it is to do mathematics. The principle draws on earlier work that has promoted very similar ideas in terms of mathematical tasks (e.g., Nardi et al., 2016).

This UDA principle promotes *dialogue* as the main purpose of assessment. When assessment draws strongly on summative practices, assessment becomes a monologue. The concepts of 'dialogic assessment' and 'dialogic feedback' have been used to emphasise how the learning potential of assessment is best achieved when students have an opportunity to *use* feedback (Steen-Utheim & Wittek, 2017). This means that assessment is not primarily used as the last word but that students could utilise feedback to enhance their mathematical work and understanding further. Importantly, such feedback could be produced in a dialogue not only with the teacher but with other students as well – or even with non-human actors such as computers (e.g., through automatic digital feedback).

The learning benefits of dialogic feedback and assessment are discussed elsewhere (Steen-Utheim & Wittek, 2017), but here the focus is on how such practices promote inclusivity and accessibility. As summative and formative assessment are both understood as forms of dialogue, focus can be shifted from only discussing the 'validity' and 'reliability' of assessment (as is often the case in mathematics assessment). Just as in any dialogue, the importance of content should surely not be neglected: for example, inaccurate feedback does not promote learning nor inclusion. Viewing assessment as a dialogue it becomes possible to notice all aspects of interaction and dialogue, such as expression of mathematical language (Thomas et al., 2015) and bodily expressions such as gestures and signs. In a dialogue, students with disabilities need to have their voice heard: what novel assessment innovations are yet to be discovered (cf. UDA principle 1)? Could mathematical knowledge be demonstrated through a dance? Tests might have their place in mathematics assessment, but only as a part of dialogue. For example, perhaps students might wish to back up their test results with a digital portfolio where they could save evidence of their learning in various forms (video, images, social media posts…).

Dialogue in mathematics classrooms rarely happens only between the teacher and a student. The third UDA principle also includes the idea of communal interaction within the whole learning community in the classroom and beyond, extending to families and school communities. In mathematics assessment, students often produce artefacts only for the purposes of assessment (e.g., tests or essays). However, through communal assessment it is possible to challenge the epistemic idea that mathematical abilities are purely *individual* (Nieminen & Lahdenperä, 2021) by producing something concrete and *useful* as a part of the assessment task (Nieminen, 2022). For example, it is possible to conduct assessment in the form of a communal real-life project. Perhaps students might want to demonstrate their statistics skills by conducting a survey about inequities in their school context. In such communal projects, all students can participate through their personal ways of communication and interaction, everyone working inclusively toward a communal goal.

Just as any dialogue could take multiple and sometimes even surprising turns, mathematics assessment now becomes a risky business. As students with disabilities learn to use assessment and feedback for their own purposes (call it 'assessment literacy' or 'critical thinking'), the results might not be what educators wanted it to be in the first place (students might even decide they do *not* want to engage with mathematics at all!). This is the beautiful risk of democratic education. A sustainable dialogue cannot be dominated by any actor, which also holds true for mathematics assessment.

Conclusion

UDA offers a valuable framework for mathematics education to strive for accessible ways to assess students' mathematical skills. While offering practical tools, UDA is, above all, a way to examine mathematics assessment through a critical lens. It offers mathematics educators a way to render assessment – traditionally a major source for inequity (Bagger, 2022; Nieminen, 2020, 2021) – as a tool for inclusion. Thus far disabilities have been understood as deficits in assessment: if we wish to celebrate diversity in mathematics classrooms, assessment simply must be rethought.

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Mathematics, ethics and climate change- ethical awareness in mathematical education.

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Keywords: Mathematics, Ethics, Climate change, Critical math.

Background

The overall goal of the research project presented here is to investigate and produce knowledge about how students and teachers are learning mathematics when presented with mathematics in the context of climate change. Climate change offers a timely and to many people relevant problem, and this problem consists of amongst other things ethical considerations. There are no easy technical solutions to the problem of climate change and the norms and values to address the problem differs around the world (Block, 2019). These norms and values are based on ethical standpoints. It is these that will be discussed and exemplified in mathematical examples in a series of lessons in school-based research.

School-based research

For my coming school-based research, a series of lessons will be created to exemplify how ethics and mathematics are intertwined and, in the classroom, there will be discussions about ethical considerations concerning climate change. Different ethical standpoints and perspectives will be presented to the students, illustrating the complexity of the problem of climate change. Articles dealing with the ethical implications concerning climate change are multiple in the research community. Gardiner (2011) argues that climate change plays a fundamental role due to decision making, which affect animal and future generations. Gardiner (2011) also point out some reasons why it is hard to be ethically sound. Local emissions have a global effect. A decision in one place may have implications in a totally different place in the world. Raymond (2004) argues for an ethics of commons. The atmosphere is a global common good and he proposes that the emissions should be allocated between nations. It could be done by using the principle of equal burden. Meaning that nations should reduce their emissions based on the burden of this reduction. Another approach based on equal human rights would be to allow an emission level per capita. Shue (1999) concludes that "whatever needs to be done by wealthy industrialized states or by poor non-industrialized states about global environmental problems, the costs should initially be borne by the wealthy industrialized states" (p.111). He then describes the reasoning behind this conclusion and how proportional and progressive burden can be explained. All these suggestions mentioned above are based on different ethical assumptions, and are examples of what can be presented and discussed in a mathematics classroom regarding ethics and climate change. They also deal with mathematical concepts, for instance proportionality and progression, that can be used as examples. The mathematical examples the teacher make to illustrate climate change, through graphs, diagrams and tables influences how the discourse is formed and has implications on the finding of possible solutions to climate change. These choices and implications are something both the students and the teachers must be made aware of.

Methodology and analysis

In the upcoming project, data will be collected through interviews with the students and the teachers. The analysis of the data will be performed using *actor-network-theory* (ANT). It is a methodological approach to study social phenomena where everything exists in a continuously changing network of relationships (Latour, 2005). As Latour (2005) describes, everything that happens in a social situation takes place on the same level. So, for instance, humans as well as objects have agency, and both play a role in creating a social situation. In ANT there are two central concepts, *mediators* and *intermedieries*. *Mediators* "transform, translate, distort, and modify the meaning or the elements they are supposed to carry". (Latour, 2005, p.39). *Intermedieries* on the other hand, is what transports meaning without transformation: "defining its inputs is enough to define its outputs" (p.39). But how do we distinguish between mediators and intermediaries? Latour mentions that to learn ANT is nothing more than to "become sensitive to the differences in the literary, scientific, moral, political, and empirical dimensions of the two types of accounts" (p.109). I will look at the student as an actor and mathematics as an actor and how the two-trough translation is changed into a mathematician - an actor-network - a new entity with new agency. What are the necessary interactions for this translation to be successful?

Discussion questions

- What are suitable mathematical examples based on ethical standpoints that can be useful when teaching and learning mathematics in a climate change context?
- What should be the main focus when using ANT as methodology in this research, is it the students, teachers as actors or maybe Climate Change as a token, and how these are changed by the interaction in the classroom?

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Mathematics' learners engaging in context-oriented reflection. An exploratory investigation to promote more differentiated views on mathematical concepts.

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This paper presents an investigation in which eight-grade students (14 to 15 year olds) are encouraged to engage in context-oriented reflection related to the mathematical measurement of geometric shapes. The contribution of such reflections to societal and socio-political challenges is not primarily aimed at critiquing society or mathematics. Rather, the objective of this study ends one step earlier by providing students with a better grounding for such critique (be it positive or negative). By means of interviews, data are produced to provide a deeper look into the students' reflective processes. Evaluation results show that, in their work, students address the context of buildings and their furnishings. This and other aspects are used to analyze reflection processes and results of varying profundity.

Keywords: Reflection, mathematical literacy, geometric shapes, calculative measurement of sizes.

Introduction

Context-oriented reflection is advocated by various authors (e.g. Skovsmose, 1998; Schneider, 2019). For the present investigation, the definition of context-oriented reflection is based on that of Schneider (2019), in whose research project I am involved. Here, context-oriented reflection means reflecting on mathematical concepts and the effects that arise from using them in different contexts of our world and society. Thereby, reflecting means thinking about the characteristics, connections and relationships that cannot be read off by the given fact (Schneider, 2019; Plunger, 2021). What can be read off by given facts or not depends on the prior knowledge of the reflecting individual. Context-oriented reflection thus has to be an individual process or experience which focuses on new insights related to effects, characteristics and connections of mathematical concepts in their application to different contexts. The product of this process should be a fund of knowledge, which might simply be communicated. However, the mere repetition of such knowledge cannot be a sign of reflection. For different perceptions of mathematical literacy, several types of reflection seem to be an essential contribution to a mathematical education for societal participation and the development of society (Plunger, 2021). Context-oriented reflection enables diversity in access to mathematics learning, especially as a contribution to societal and socio-political challenges. First, it is about examining mathematical content for its effects and functions in society and characterizing this activity itself as relevant content for mathematics education. Second, the activity of reflecting forms a counterpart to the widespread calculus-oriented, repetitive activities in learning mathematics: insights into characteristics, connections and effects which initially seem to be hidden in the given facts, are generated through independent reflection. Third, learners are given the opportunity to make reflections related to society on the basis of personal, subject-related experiences of learning mathematics and supra-subject-related experiences from their lifeworld (including school) with the thematized mathematical concept. The reflection process should provide insights into the extent to which the mathematical concept contributes to our life together, or what would change without it.

In this paper, I will rely on data collected within the framework of my PhD project. The intention was to use special tasks to encourage students aged 14-15 to engage in specific types of reflection on different mathematical contents. Here, the results of the task that aimed at context-oriented reflection related to the mathematical measurement of geometric shapes are discussed. In this regard, three questions will be addressed in the following sections. (1) What contribution can context-oriented reflection make to mathematics learning from a theoretical perspective? (2) How can students be encouraged to engage in context-oriented reflection? (3) What context-oriented reflections do students engage in when they are encouraged to do so through specific tasks and settings?

Theoretical Perspective

In this section question (1) is addressed. In 2012, Fischer presented his ideas about (mathematics) teaching in our society based on the division of labour, where it is not possible to be an expert in every discipline. Thus, according to him, the goal of general education should be to enable learners to make independent evaluations and to be able to form appropriate judgments for themselves, relating to mathematical content. Calculus-oriented mathematics education, as described by Kollosche (2019, p.105), seems to work against such a purpose, because it seems to prepare students “to repeat the same procedure again and again, also following rules which are set by others and whose purpose might not be fully understood” (Kollosche, 2019, p.107). The challenges facing our societies today require an education that empowers critique (Steflitsch, 2021). Critical awareness involves the ability to perceive multiple facets, both positive and negative. Skovsmose (2011) contrasted the modern, very positive conception of mathematics with a critical one (p.60). In order to give students access to such a critique, he made them take a closer look at applications. In doing so, he encouraged a series of reflections that focus on the effect of mathematics in this concrete application (e.g. ibid., pp. 72-75). Fischer’s approach to mathematics is a more neutral one. In 2005, he stated that mathematics is indispensable in modern society “because it materialises abstract things and thus makes them accessible to individual thought as well as to communication” (p. 293). Furthermore, he argued that such materialisations offer a dialectical potential in the sense that they can be used to hold on to what is described as well as to overcome it (Fischer, 2005, p. 24). In both cases mathematics is seen as a tool. According to Skovsmose such a tool brings with it very specific effects in a certain situation, which should be analysed. According to Fischer mathematical concepts provide the chance to materialise certain aspects of a situation to make them accessible for discussion.

In my conception of context-oriented reflection, mathematical concepts are also interpreted as tools, but they are dissociated from one specific context insofar as different contexts of application for one of these tools are taken into consideration. The comparison between these different contexts should help to compare the distinct characteristics of this tool and might provide an insight into what is gained through mathematization, but also into its limitations. This ultimately should enable one to make a more informed judgement about the use of this tool on the one hand, in relation to other concrete situations, but also in general to better estimate how and when this tool can be used. With this type of reflection, the reflective activity ends before there is the concrete occasion to engage in criticism related to the subject or society. Rather, the mathematical concept is seen as a kind of tool and an awareness is created that its use can bring along a range of effects. This raises the following questions: Which of these effects are grounded in the concrete materialisation (mathematization through the tool of the mathematical concept)? Moreover, which ones are grounded in interpretations of this mathematization? Thus, through context-oriented reflection deeper insights may emerge related to the mathematical concept and to the contexts under consideration. However, the essential interest of context-oriented reflection lies in the effects that are made possible by the mathematical concepts. Thinking about such effects should afford a differentiated stance between the two poles of an exclusively positive and an unreflectively negative conception of mathematics. This creates a basis that contributes to more well-founded subject and social criticism.

Data production by means of interviews

This section discusses the methodology used to produce the data in response to question (2) How can students be encouraged to engage in context-oriented reflection? In my PhD-project I focused on model- and context-oriented reflection and decided to investigate this question by means of interviews with pairs of students aged 14 to 15 years; I conducted interviews with eleven pairs of students who participated voluntarily and from three different schools (four different classes) in Carinthia. Each pair participated in three interview sessions focusing on varying mathematical contents (arithmetic, statistics, geometry). It was ensured that the required content was already addressed through school lessons, but the concrete lessons of the different classes were not further analyzed. On each interview date students were asked to deal with two tasks in partner-work, the first task aiming at model-oriented reflection, the second task aiming at context-oriented reflection. At the end of each interview a short discussion followed about affective issues related to the processing of the task. The interviews are documented by audio recording and its transcripts. The task for the context-oriented reflection on the content of geometry is reported in Figure 1; It was presented to the students on the first or third interview date, after a task that was intended to stimulate model-oriented reflection which dealt with the determination of the air volume in a classroom by means of a cuboid.

Geometric Shapes

In mathematics lessons you have learned to calculate lengths, areas or volumes of different geometric shapes.

What are the advantages of being able to calculate lengths, areas or volumes of different geometric shapes?

Figure 1: Task, headed "Geometrische Figuren und Körper", translated by C.P.

The intention of the task in Figure 1 is that students think about the consequences that (can) result from the mathematical determination of various geometric shapes. Possible lines of thought are (a) that geometric shapes have the potential to be seen in many objects of our environment, and the possibility of calculating certain sizes from such objects can be helpful. (b) Geometric shapes can be used as models for buildings, e.g. a swimming pool, which can facilitate planning since necessary material for the construction can be calculated in advance, as well as the water capacity. (c) If geometric shapes are used as models for buildings, there is the possibility of testing different solutions, e.g. with regard to material consumption: a cube-shaped-house offers more space with less material consumption than one with an elongated base. (d) Lengths can easily be measured physically, but this is not true for areas and volumes. (e) Computational methods can provide more exact numbers for lengths than physical measurement methods in selected cases (e.g. for the diagonal of a unit square).

The task itself may stimulate reflection, but reflection also needs an environment that invites it. In conducting the interviews, it was important to me to create such an atmosphere. The students should understand that it is their thoughts, their reflections that matter. By noticing and feeling that there is real interest in their personal thoughts, they should be encouraged to reflect. Additionally, I tried to apply some reflection-inviting features, but they seem to be dialectically related to each other. *Working in pairs vs. individual work:* communication seems to be essential for reflection because through communication new perspectives and ideas can be exchanged and further developed. On the other hand, individual reflection is also a prerequisite for successful reflection in communication, and reflection is also possible individually, for example, when one enters into an inner dialog. In the interviews individual work is scheduled to grasp the task and first deal with it for the first 2-5 minutes. This also should increase the likelihood that both partners have individual ideas that can be exchanged and discussed. Working in pairs additionally provides the possibility to get closer to the thoughts of the students, because a conversation with a classmate should favour a natural and spontaneous exchange about the results and thoughts that led to them. A one-to-one dialogue between student and scholar seems to inhibit such natural exchange. *Speaking vs. writing:* aiming for a predefined written product may inhibit reflection, on the other side writing would demand more clarity and thus deeper reflections. During the work with the tasks, the students were given paper and pencil for notes but were not required to create a written product. *Free conversation vs. interventions:* the interest in the thoughts and pair-working processes was emphasised in the introductory phase of the interview and a conversation between the two students was encouraged above all in the beginning of the working process. Yet, interventions provide the opportunity to ask for more detailed explanations of statements or to stimulate alternative directions of reflection. As an Interviewer I used guided and free interventions focusing on general or content-related aspects. The first interventions usually consisted of general questions such as *Look at the question again - did you answer it? Is it possible to approach the question in a different way? Try to explain how you came up with the different ideas/aspects.* Only in a second step, interventions should become more content-specific in order to obtain a scale for the evaluations. Content-specific interventions for the task discussed here were, e.g. *Try to consider this question separately for lengths, areas and volumes. What possible alternatives are there to mathematical calculation? Try to address this task in the sense of what would be the consequences if there were no way to calculate lengths, areas or volumes.* Such interventions were chosen freely during the interview, depending on what I as interviewer considered the students' processing required. This allowed me to ask for more detailed explanations or review the processing as well as stimulating alternative ways of reflection where this seemed appropriate. Nevertheless, it remains a challenge to maintain the balance between interventions for the purpose of focusing the core of the given task and a natural exchange of thoughts in a conversation between students.

Reflection processes of learners

This section addresses question (3), What context-oriented reflections do students engage in when they are encouraged to do so through specific tasks and settings? In general, in the interview-parts that refer to different tasks on content-oriented reflection it became evident that students found it very unfamiliar to engage in such reflections. They were able to draw on appropriate contexts, nevertheless they had difficulties to concentrate on the core of the question. Depending on the mathematical concepts on which the task focuses, positive but also negative effects of the mathematical tool are observed by the students. Across the different tasks it can be noticed, that the students do not take a critical look at their results, hardly putting them into perspective or considering the possibility of differentiation. Such attitudes might hinder the understanding of mathematical concepts but especially deeper-founded critique towards mathematics or society.

Regarding the task presented in this paper, three characteristics that are consistently observable in the corresponding interviews are described in more detail. For their reflections, learners draw on the context of buildings, especially houses, and their furnishings. In doing so they have difficulty differentiating where length measures and where area measures are required. An effort to find positive examples is recognizable, while negative aspects are not illuminated. In the following, the reflection processes of one interview pair, Klara and Joelle (all names changed) are examined in more detail and accompanied by spotlights from other couples. This allows an insight into how these three characteristics develop and how differentiated the underlying statements are.

Klara and Joelle initially had different approaches to the task. Joelle saw the geometric shapes and their calculability as personally relevant to her because she wants to become an interior designer. She justified this importance for interior designers with the fact that one must know the surface area in order to know what fits into this space, Joelle's first comment on the task (all transcripts translated from German by CP) was:

Joelle: […] So, what is important for me, for example, because I want to become an interior designer, is exactly this kind of thing, because if you have to put something in a room, you also have to know the surface area and everything, so that it really FITS in the room, because well, you can't, I don't know. [Into] a three meter long room you can't put a four meter long bed, ok that's maybe a bit exaggerated, but that would just be an example, you can't put it in, because. For that it is once practical.

This indicates one aspect of thought line (a), that it can be helpful in a situation to determine the area of a particular object. Implicitly, it must be assumed that (a) geometric figure(s) can be seen in this area. However, the further description of such a situation (does the bed fit into the room?) reveals that even more differentiated considerations would be possible here, Joelle herself indicated that this is not a well-suited example. In the ongoing conversation, the importance of the computational determination of geometric shapes for furnishing was put into perspective, presumably from personal experience with furnishing (Klara mentioned that sizes like length, width, height are usually indicated), but not explicitly related to what these measurements can and cannot contribute (e.g. that a number that expresses the area of a room is not helpful to determine whether an object like a bed fits well in it). Later in the conversation, Klara justified the area calculation of a house or apartment by knowing “how much living space a person has in the house”. Klara’s first contribution to the task addressed precision as an important element of calculating, but together they failed to come to a coherent justification for themselves.

Joelle: And if you just calculate it could be even more exact, because you have, for example, decimal numbers or something, because that’s what you have when you measure normally, yes, ok, it could be, but. Yes, hmh. That was not so now, smart {Both laugh}.

In this conversation it is repeatedly recognizable that the two had difficulties in differentiating where mathematical measurement methods are possible and where physical measurement methods are necessary:

Klara: Yes, such a house is something that should be beautiful. Especially because people are going to live inside it and

Joelle: Yes. And.

Klara: It is perhaps simply important that one can calculate all the areas and so, hmm, because I would not go to a house now, the tape measure from one corner to the other and then perhaps even in the height. It is perhaps intelligent if you know the dimensions and can then calculate how many rooms will fit into the house and.

Even in other interviews, for example with Chris and Vedran, problems with differentiating the need for physical measurements and possible calculations of sizes arose. Following a statement that the benefits of calculating are being more precise, taking less time and effort I requested them to explain in more detail what this means through a dialogical conversation. Vedran told Chris to start by gauging a length, a width and a height. Taking these sizes would take about 30 minutes, while these few numbers could be calculated much faster using mathematical calculations. Then the volume of the room could also be calculated to know how much space there would be for a cupboard and find something that would fit in the room. Chris did not contradict this, on the contrary, he affirmed several times throughout these remarks and anticipated Vedran’s statement that afterwards furnishings that fit the room have to be found. In this case again, we see that the two do not take into account that a number expressing volume is not suitable for finding fitting furnishings. Further, like other pairs, they do not recognize that a physical determination of lengths is a prerequisite for being able to determine concrete numbers for areas and volumes in which geometric bodies are seen. In any case, Chris and Vedran do not relate these facts to the example they outline. Mark and Nicolo, on the other hand, recognized that when they thought of the need to determine lengths, only examples where lengths can be determined physically came to their mind. This is probably based on the fact that lengths are easier to measure physically than areas and volumes – an approach to line of thought (d). Nevertheless, an intervention from my part with the question of how areas or volumes can be determined without calculating, and their adequate response to it, did not help to address aspect (d) explicitly. They continued to look for situations in different contexts where a computational determination of lengths seems necessary. They mentioned very complicated constructed situations where different lengths had to be added and geometric figures played only a secondary role. Other examples referred to the fact that desired areas or volumes were given and suitable lengths had to be found.

For other students the fact (d) that areas and volumes cannot easily be determined by other means was a starting point:

Gina:	But I think that you might not be able to estimate a volume very well, if you could not
Hannah:	No.
Gina:	determine it mathematically like that, because or at least I can't.
Hannah:	(Uncomprehensible.)
Gina:	And that this is indeed helpful.
Hannah:	It is indeed very helpful, yes.

Almost all pairs took up the context of buildings and their furnishings for their reflections without noticing that the specification of a number for the area is not very helpful. As can be seen from excerpts of the interview with Klara and Joelle, both tenable examples and those that require a closer examination are mentioned together during one interview. An explicit evaluation of such examples is rather not done by the students and occasionally these examples are kept until the end of the interview. All in all, critical aspects are rather faded out, whereas benefits are often mentioned with emphasis, without backing up them by reasonable statements.

Discussion

The interview results show that although the students take up similar contexts on the one hand, they come to different depths with different approaches and at different points. What all pairs of interviews have in common is that they have difficulties in keeping their focus on the core of the question, which is the *calculative determination* of sizes of *geometric shapes*. Nevertheless, they make a great effort in the interview to engage with the question and to work on it with their reflections. It is remarkable that the students only look for positive aspects and tend to ignore critical aspects or at least do not examine them in detail (e.g. suitability of an area measurement for the equipment of a room). On the one hand, this may be due to the task, which explicitly only asks for advantages, but in the interview I did ask for more precise explanations at such points. On the other hand, it may also be due to the selection of the participants for the interview, who were willing to engage in such interviews to reflect on the use of mathematics. Such student might have a more positive attitude towards mathematics in general.

According to the theoretical perspective context-oriented reflections should contribute to independent evaluations, appropriate judgements and the analysis of effects emerging from the use of mathematical concepts. At the same time, these aspects provide a kind of solid groundwork for a well-founded subject and social criticism, because they seem to facilitate the ability to adopt a critical stance. This would also include differentiating considerations or careful examination of arguments, which regrettably could not be largely observed in this study. However, it must be recognised that this is only a first step in the demand for more reflection in learning mathematics. It investigates the reflection processes of students who had little experience with this in their mathematics classrooms. The students are not involved in a systematic classification and differentiated consideration of their reflections outside the interviews. This is not possible within the framework of my PhD project. The students themselves see a need for further processing insofar as some of them ask for feedback in the follow-up conversation or express an interest in exchanging results with other pairs. From a teacher's perspective, the students' reflections would offer potential to look at the results on this question even more deeply and to develop a more differentiated picture of the collected "advantages". This seems to be significant in relation to this one mathematical concept of the calculability of geometric figures and solids, but it would also promote a further development of the ability to reflect and a transfer of this ability to other contents.

In the follow-up conversations to the interviews, almost all participating pairs described their processing of the tasks as being meaningful. This indicates that the students are willing to engage in the analysis of the use of mathematics in contexts they are familiar with - what seems to be a very important prerequisite in the development of a critical attitude. The positive effect of reflection for the students should be used as a potential in learning mathematics in order to draw a more differentiated picture for the use of mathematical concepts, especially as a basis for a well-founded and if necessary differentiating critique towards society or mathematics.

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Moving beyond gatekeeping: Using data analytics to overcome resistance to pedagogical change

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Postsecondary mathematics education often plays a gatekeeping role in STEM higher education. Accordingly, research highlights the need for pedagogical change to promote more equitable mathematics classrooms. However, mathematics faculty often resist change, grounded in rationalizations such as "math is just hard." In this manuscript, we offer a concrete approach to overcoming such resistance. The approach uses classroom data generated by the observation tool EQUIP. We provide reflections from one participating mathematics faculty member on his experiences engaging in this process. We discuss implications for using this approach at scale.

Keywords: Change, classroom observation, data, instructional change, pedagogy, reflection.

Introduction

Mathematics is widely recognized as a gatekeeper to higher education, especially within STEM fields (Martin et al., 2010). Cultural narratives position mathematics as largely white, masculine, and able-bodied (Reinholz, 2021). As a result, mathematics plays an integral role in the persistent marginalization of students based on race, gender, disability, and other identities (National Center for Science and Engineering Statistics, 2021). Mathematics instruction is typically guided by a number of purportedly neutral logics, which in fact disproportionately impact students based on their identities (Leyva et al., 2021). Logics include, "mathematics as universal and objective," "math is challenging," "some people are simply not math people," and that the role of mathematics instruction is to "weed out incapable students so the best ones remain" (Reinholz & Dounas-Frazer, 2017; Seymour & Hewitt, 1997). Given that such logics are regularly deployed in mathematics instruction, how can the field make progress towards more equitable and anti-racist teaching practices?

In this manuscript, we offer an approach for making progress despite this seemingly intractable problem (cf. NCSES, 2021). Our approach is organized around classroom observation and data analytics. First, we document concrete events in a context that is relevant to an instructor – their own classroom. Next, we use these as grist for interrogating taken-for-granted cultural logics. To do so, we use the classroom observation tool EQUIP (Reinholz & Shah, 2018), which provides data analytics describing patterns of inequity in classroom participation, disaggregated by social marker identities. By providing such data to an instructor, we first demonstrate concretely how inequities arise in everyday classroom interactions. Next, we work with instructors to change their teaching practices, and accordingly, demonstrate that changing such participation patterns is within an instructor's locus of control. This serves for the basis for discussions of why such inequities existed in the first place, and why they align with what one would expect based on historical patterns of marginalization in mathematics. To illustrate this approach and learning process, we provide reflections from a single mathematics faculty member who participated in an EQUIP learning community in Spring 2020. Our goal is to spur conversations on how this approach can be scaled up and used across contexts.

Background

Participation in classroom discussion is highly valued across mathematics education (Hufferd-Ackles et al., 2004). The reason for this focus is that participation in discussion contributes both to learning and identity development for students (Reinholz & Shah, 2018). It follows then, that students who participate are more likely to succeed and persist in mathematics. Yet, simultaneously, research shows that these patterns of participation are inequitable across identities such as race and gender (Ernest et al., 2019; McAfee, 2014). Thus, participation is one concrete site within which inequities arise in the mathematics classroom. Moreover, patterns of participation are relatively easy to capture through classroom observation, and once captured, changing such patterns is within an instructor's control.

We used EQUIP to capture participation (Reinholz & Shah, 2018). To efficiently facilitate the use of the EQUIP protocol, we used the free, open source and customizable EQUIP web app (https://www.equip.ninja). EQUIP describes classroom participation at multiple levels – individuals, groups, and the whole class – through a variety of data visualizations. The unit of analysis in EQUIP is a contribution, which consists of a sequence of talk from a single student. Each time a new student talks, a new contribution begins. For each contribution, a coder (in this study, a graduate student observer who was trained to use EQUIP) codes several aspects of the contribution, such as the length (1-4 words, 5-20 words, or 21+ words) and type of student talk (why, how, what, or other), and an instructor's questions (why, how, what, other, or N/A). Because all these facets of classroom practice are attached to specific students, analytics can be disaggregated at a student level. By including demographic information, a user can then generate analytics about particular student groups (e.g., Black women, Latinx men). The specific demographic categories and codes used can be customized according to the local context. In this way, EQUIP represents a particular methodology for tracking classroom participation, which is instantiated according to local needs.

Method

Participants and Context. We report on work that took place at a large, research intensive Hispanic Serving Institution (HSI) in the US.[1] Participants were recruited through an open call to the university. The second author of this manuscript (Professor C) participated in a cross-disciplinary learning community during the Spring 2020 semester. During Spring 2020, instruction began in a face-to-face modality, and transitioned online mid-semester in response to the global coronavirus pandemic.

[1] HSI is a federal designation that denotes at least 25% of the student population is classified as Hispanic.

The second author is a member of the Mathematics Division, whereas the first author is a member of the Mathematics Education division. This difference in affiliation manifests in a strong difference in professional obligations between the two authors, with the second author expected to publish regularly in highly specialized technical journals, pursue external funding, supervise theses in mathematics research, and promote an overall culture of “competitive” mathematical ability within the department with an eye towards maintaining standing in the larger international research community. This role directly reflects the second author’s educational background and acculturation within mathematics. In large part, the role of the second author then is to find and select the “best” students to follow the same professional trajectory at the second author. This institutional pressure differs from an approach that would instead focus on helping all students succeed as much as possible.

Focal Course. Professor C taught *Introductory Real Analysis* (also called *Advanced Calculus*, in the US), which met three times weekly during 50-minute sessions. The student population in this course is diverse in terms of race, gender, and mathematical focus. Some students are focused primarily on becoming mathematics teachers, whereas others are working towards STEM-intensive careers. In either case, Real Analysis is a critical course for all mathematics majors and is seen a difficult course. Students enroll in this course during their 3rd or 4th year, after completing three pre-requisite courses. For students transferring from local community colleges, this may be their first mathematics course in a new learning environment. Given all these factors, the student population is very heterogeneous.

There were 33 students in the course (22 women, 11 men). The racial demographics were: 12 White (36%), 5 Asian (15%), 9 Latinx (27%), 1 Pacific Islander (3%), and 6 Unknown (18%). These data were collected from a survey of students, and missing information was filled in based on the instructor’s perceptions (which is the reason for unknown results in the race category). The racial demographics are somewhat less diverse than the campus as a whole. We recognize that instructor perceptions may not always align with student self-identification, but we note that an instructor’s biases are most likely to align with their own perceptions, so they were still useful for an intervention to reduce bias and promote equitable teaching. The gender demographics were notable because 3rd/4th year math courses tend to skew heavily towards men but did not in this case.

On the target campus, the deadline to withdraw from classes is two weeks after the semester begins. Given that this date is so early in the semester, students may realize that the course does not fit into their schedule or work/life balance only after this deadline. As a result, there are sizable populations of students that essentially “drop” the class by ceasing to participate for the majority of the semester. These students can be become especially frustrated at their lack of success and finding effective strategies to keep them engaged or get them reengaged in the course is nontrivial.

Design. The six faculty participants in the learning community met on a regular basis (typically every few weeks) to work collaboratively to improve equity in their classrooms. The learning community was organized around iterative reflection cycles. In each cycle: 1) instructors were observed teaching (through a video recording), 2) the teaching was coded and feedback was provided, and 3) there was a feedback meeting to debrief and plan next steps. By including multiple reflection cycles over the course of a semester, instructors had multiple opportunities to reflect, change practices, and observe changes in the data. There were five debrief meetings throughout the semester. In this manuscript, we focus on a Professor C, who was the only mathematics faculty member in the learning community.

A total of four lessons were video recorded and coded by a graduate student who was trained in the EQUIP protocol. In the beginning of the semester, instruction was recorded in the classroom using a video camera, which was later coded. In the second half of the semester, Zoom virtual meetings were recorded and coded. In the classroom recordings we could only capture whole-class discussions (due to data collection limitations), whereas in the Zoom recordings we captured both whole-class discussions and breakout rooms. Instructional practices were coded along several dimensions, including the length and quality of student talk, and the type of instructor question. The coded data from each observation were provided to Professor C in a written report, which outlined key highlights in the data, as well as possible suggestions for revision to practice. Detailed notes from each meeting and well as records of feedback were retained to support data analysis and interpretation.

Analyses. In this brief manuscript, our analyses focus primarily on the Professor C's reflections (the second author), with some reference to the analytics generated in the process. Prior work has documented changes to instructional practices through EQUIP communities (Reinholz et al., 2019, 2020). Here, our goal is different. We are focused on longer-term changes to an instructor's logics, and how EQUIP could support such self-reflection.

Results and Reflections

Initial Approaches. Even though Professor C was a member of the mathematics division, he had a personal commitment to do his own studies of feminist and anti-racist literature. This is somewhat unusual for core mathematics faculty. The impetus for this reading was grounded in his general worldview and attention to equity. This background work provided him with a foundation to start thinking about students' intersecting identities with varying degrees of existing privilege. However, most of this reading was in more general settings and not specific to mathematics. This also meant that he did not have specific instructional moves that he could apply to mathematics teaching. This was a primary motivation for joining the learning community. While he was aware of issues, he was not sure how to approach solving them. Moreover, he was less aware of how his own demeanor and particularly privileged identity (white, hetero-passing, cis-male) influenced the nature of discussions in his classrooms.

There is a general perception within the department that students should be "mathematically mature" by the time they enrolled in *Advanced Calculus*. Given these broader narratives within the department (and mathematics writ large), Professor C initially felt that if a student was not "ready" by the time they enrolled, then there was little chance of them doing well. Professor C reflected on this bias of his and its origins in his own educational experience. Even as he questioned his own assumptions, there was still the larger issue of developing effective strategies and alternate pedagogical approaches to help enhance participation and success for all students in his class. One strategy that Professor C did use from the beginning of the semester was breaking students into groups of 3-4 students who would work collaboratively on problems. A rationale was that "stronger" students would serve as the de facto liaisons with Professor C, providing a kind of cover for students less willing to engage in what had been described as a "coldness", "lack of patience", or "condescending attitude" from Professor C (quotes taken from prior student reviews). As we note below, Professor C later came to question this idea of stronger students being the ones to support their peers, and he looked for ways to engage more students directly in the class.

Data Analytics. We begin with data from two of the four EQUIP observations that focused on in-person teaching. Across the two observations, what is perhaps most notable is that 23/33 (70%) of students logged no interaction at all. This issue of such a marked absence of classroom participation then became a topic of focus during the learning community meetings lead by Professor R, which occurred after observations. In addition, of the 35 interactions logged, they were disaggregated by the following racial demographics: 18 White, 14 Unknown, 2 Latinx, and 1 Asian. Thus, while only 36% of the class was White (12/33), 51% of contributions were from White students (18/35). By gender, they were: 11 by men, and 24 by women, which largely matched the class demographics of 11 men and 22 women. This shows that there were clear patterns based on race, but not so much based on gender. Largely underlying these patterns was the fact that 18 of the contributions belonged to just three students (a white man, a white woman, and a woman of unknown race, with 6 contributions each). The fact that a few individual students could dominate the discussion became a focal point of discussion within the learning community. Notably, the presence of a dominant subset of students was consistent with the notion of "de facto liaisons" to the professor. What is important from the perspective of anti-racist teaching, is recognizing that this small subset of dominant students often belongs to privileged racial groups in the discipline, and thus better distributing their participation would be a move towards racial equity.

The final two observations were conducted during virtual teaching. Whereas the initial observations focused on whole-class discussions, the primary teaching method in the online setting was through breakouts, which were easy to code using the virtual medium. Across these observations, there were 61 contributions logged, from 18/33 students (55%). The breakdown of contributions by race was: 18 White, 27 Unknown, 11 Latinx, and 5 Asian. By gender, it was 13 by men, and 48 by women. It's notable that the patterns of participation were more equitably distributed by race, with far more contributions from Latinx and Asian students than in the original two observations. This may partially be explained by the closer interactions between instructor and student being documented in breakout rooms. In addition, we suspect that strategies developed (e.g., using student names directly) provided Professor C with tools to engage more students and increase racial equity. This allowed him to shift the patterns that were present in the first two observations.

Transformed Practices. By the end of the semester, through looking at the EQUIP data and interacting and receiving feedback from the other faculty and classroom monitors in the discussion group, Professor C began to develop a better sense of the need to use more personally identifying features of students, such as first names, when engaging in classroom/Zoom discussions. These was particularly profound in a subject matter like mathematics, which is typically seen as objective and depersonalized. Likewise, he developed an emerging sense of trying to ask questions related to tasks at hand that were not only directed towards getting the answer, such as "What were some difficulties that came up while trying to solve problems?” or “What was the most interesting part of the discussion for you?” to give more students an opportunity to participate and thereby integrate themselves into the classroom dynamic. This was important because mathematics is usually seen as right or wrong, and the goal is to get the right answer as quickly as possible. Changes in questions can be seen in Table 1.

	Why	How	What	Other	N/A	Total
Face-to-Face	3 (8.6%)	1 (3%)	3 (8.6%)	7 (20%)	21 (60%)	35
Zoom	2 (3%)	0	10 (16%)	38 (62%)	11 (18%)	61

Table 1: Transformation of teacher question types

There are two notable shifts. In the first two sessions, 60% of questions were coded as N/A, which meant students shouted out answers without a specific question from the instructor. This decreased dramatically in the Zoom sessions, reflecting the instructor's increased role in managing the discussion. This is important, because prior EQUIP work has shown that when students simply shout out answers, dominant students tend to dominate the discussion. Also, there was a larger increase in "other" questions (from 20% to 62%), reflecting the alternative types of process questions the instructor was asking.

Overall, Professor C developed a much greater sense of and strategies for addressing the need to keep as many students as possible actively engaged in classroom dynamics as a means towards enhancing equity in the classroom. He recognized that without such strategies, students who were from dominant racial and gender groups were most likely to dominate. Thus, these general strategies of involving students who may not be participating are an important step towards being able to monitor racial and gender equity in an ongoing way. In future classes, if similar patterns become visible, Professor C was now equipped to utilize strategies to engage individual students within particular identity groups to better bring them into the classroom discussions.

Reflecting One Year Later. Since the time Professor C participated, he taught *Calculus 2* via Zoom to 130 students. Admittedly, the online environment did not readily facilitate implementation of the techniques explored and cultivated in the Spring 2020 EQUIP discussion group. However, he will soon take over the role of coordinator for the course and will be responsible for the educational experience of approximately 450 students. This involves Professor C coordinating across two to three Instructors as well as seven to eight Teaching Assistants. Having seen the inadequacies of online instruction for what is for many students an especially demanding and stressful course (e.g., in limiting meaningful interactions between instructors and their students), from his experience with EQUIP, he is keenly aware of the need for implementing strategies to keep students engaged in as equitable of a fashion as possible, and that doing so is critical to student success. This will include

1. Engaging in discussion with Teaching Assistants and Instructors about the need to encourage different groups of students to participate in class and to find novel strategies to facilitate that participation. Such strategies include soliciting replies to questions which do not directly deal with the mathematical problems at hand, or explicitly finding ways to elevate the voices of students that otherwise might be quiet.
2. Developing the capacity for Teaching Assistants and Instructors to personalize interactions using names or other means of identification to reinforce students' personal involvement and investment in the classroom environment. For students who might not feel as though they belong "by default," instructors need mechanisms to explicitly create a sense of belonging.

3. Facilitating conversations among Teaching Assistants and Instructors about the role implicit bias may be playing in their teaching and ways in which they could work to address said bias. This is especially salient with regards to social markers such as race and gender, which are visible in the classroom and are also known sites of inequity in mathematics education.

These implications for practice and coordinating the learning experiences of students were informed by Professor C's opportunity to reflect on his data and his own biases in teaching with EQUIP data. Outside of the classroom, Professor C also reflects on the role of larger movements. For example, in response to the Black Lives Matter movement as well several other social justice movements involving Latinx, Asian, and Native American peoples, Professor C views part of his role as a tenured member of his department and as a course coordinator as involving anti-racist work, which he views as essential to ensuring equitable access to success in STEM. His participation with EQUIP provided invaluable tools and insights for performing this kind of action in concrete ways in the classroom. It helped bridge the gap from theory to practice. In this way, although the EQUIP community was ostensibly focused on classroom teaching, it provided impetus, support, and strategies that could be transferred to other venues such as course coordinating, departmental policy, and civic engagement.

Discussion

Mathematics long has a history of gatekeeping and weeding out students. There is also a history of divide between Mathematics and Mathematics Education faculty within mathematics departments. In this paper, we share preliminary work that aims to bridge both divides, by applying mathematics education techniques to teaching mathematics, and by building a meaningful partnership from faculty in both branches of the department. The approach is grounded in data. Especially for faculty working in Applied Mathematics, there is a facility and disposition towards using data that makes the approach particularly appealing.

Here we illustrate how the data were helpful in overcoming inertia associated with the weed-out culture of mathematics. Initially, Professor C was coming from the perspective that some students simply would not succeed, so it was important to center efforts on those who would most likely succeed. Working with concrete data and a supportive community, this shifted over time to building greater awareness of the sociological aspects of teaching and how to develop concrete strategies for bringing more students meaningfully into the mathematical discussions.

This preliminary work has the potential to scale. Given that EQUIP is a fully customizable tool, it can be used to consider equity issues relevant to any context. While some of the issues related to race are specific to the US context, racial dynamics can play out differently in other places, and there are other forms of hierarchy (e.g., language, immigration status). Another issue to consider in scaling is the process of observing and facilitating professional learning. As the work continues, we will explore further models for self-study and self-reflection, which provide a greater variety of opportunities.

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Multilingual students' talk about their work to relocate school academic mathematics in home-school transitions

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We interviewed groups of students in a language diverse school, where the prevailing language norm was Swedish only, to answer the question; What do students say about relocating school academic mathematics in transitions between home and school? The students mentioned relocating school academic mathematical concepts, problem solving and arithmetical methods from home to school and vice versa. The relocating work provided resources for mathematics learning and feelings of being smart and mathematically knowledgeable and the opposite. We conclude that pedagogical designs that enhance students' first languages and home cultures as resources may benefit from considering students' work with relocating school academic mathematics to enhance opportunities for mathematics learning.

Keywords: multilingualism, mathematics, transitions, home, school.

Introduction

Multilingual students are far from a homogeneous group either in, or across contexts. However, multilingual students are often from marginalized groups whose home languages and cultures may not be acknowledged in mainstream society in general or in mathematics learning in particular (see for example Källberg, 2018). First-language-as-resource and culturally responsive and relevant pedagogics are pedagogies that have been advocated for some time. They are designed to enhance mathematics learning opportunities that recognize non-dominant languages and cultures as resources for mathematics learning. The latter approach often emphasizes colloquial, non-academic mathematics as part of non-dominant languages and cultures. Experiences with out of school academic mathematics may also be significant resources for multilingual students' mathematics learning (Abreu et al., 2002). However, for multilingual students to use school academic mathematics experiences as learning resources, transitions between in and out of school may require work to relocate school academic mathematics in conversations outside of school and vice versa (Abreu, 2008). We are interested in what students say about relocating school academic mathematics in conversations about mathematics in and out of school since relocating school academic mathematics influences multilingual students' relations with and expectations of mathematics education (Meaney & Lange, 2013). This influences their opportunities for mathematics learning and ultimately their future prospects. Therefore, in this paper we investigate the question; What do students say about the work they do to relocate school academic mathematics between school and home and vice versa? We use the metaphor work because to re-locate something in the physical world requires work. In the same vein we think that relocating school academic mathematics requires work.

Transitions between different cultures of school academic mathematics

In Sweden and elsewhere, language and cultural diversity is rapidly becoming the demographic norm in society and hence also in mathematics classrooms (Meyer et al., 2016). In most places it is possible to identify a dominant language and culture to which non-dominant cultures and languages often are positioned as subordinated. This connects to asymmetric power relations which motivates us to investigate multilingual students' work with relocating school academic mathematics in transitions between home and school. Mathematics learning is often perceived of as trajectories from informal, colloquial cultures of mathematics articulated in home languages to formal school academic cultures of mathematics articulated in the dominant language (see for example Webb & Webb, 2016). In this paper we consider cultures of formal mathematics as cultures of school academic mathematics. Cultures of informal mathematics we think of as colloquial every-day mathematics. In processes of transitions between informal and formal mathematics, students may find some of their everyday out of school experiences and languages valued and useful while they may find other experiences and languages not valued or even rejected (Crafter & Abreu, 2011). Despite that experiences with non-hegemonic cultures and languages (e.g. indigenous, immigrant or working class) can provide rich resources for mathematics learning (Huru, et al., 2018; Planas & Setati-Phakeng, 2014), they are often viewed as deficits (Gutiérrez, 2008; Källberg, 2018) which may stigmatize transitions and ultimately diminish some students' opportunities to learn mathematics. In addition to experiences with informal mathematics, multilingual students may have experiences with school academic mathematics out of school. Consequently, when mobilizing resources for thinking and doing mathematics, multilingual immigrant students may move among language and cultural resources where the dichotomy informal-formal may apply, but they may also move among various cultures of formal school academic mathematics. Albeit still debated, the notion of mathematics (as a plural noun) as cultural products is widely accepted. Different cultures have developed mathematical practices that share resemblance, but hold distinct features (Ryan & Parra, 2019). School academic mathematics could be considered one type of cultures of mathematics among other mathematical cultures. Transitions between cultures of school academic mathematics is not a straightforward matter. To relocate school academic mathematics as resources among sites require intellectual work (Prediger et al., 2019). Multilingual students often need social and pedagogical support to successfully conduct such intellectual work (Abreu, 2008; Meaney & Lange, 2013). Moves among different school academic mathematics are not neutral. Rather, they signify values of normative appropriateness (Abreu et al., 2002). Consequently, who is appreciated as mathematically knowledgeably may shift among such moves. This means that multilingual students may find themselves positioned as mathematically knowledgeable in one location while being positioned as less knowledgeable in a different location when for instance using the same algorithm. Such experiences may evoke stigmatization and is ultimately an issue of social justice (Meaney & Lange, 2013).

Theoretical framing: migration and re-location work

To capture the work that students do to relocate school academic mathematics we recognize that in migrational contexts, motion and mobility are metaphors to think about identities, cultures and societies. Mobility suggests discontinuous states of being and knowing that dialogue with where one came from (Keating, 2009). This means that issues of dislocation and re-location that pertain to here-and-now and there-and-then are present in conversations for instance when multilingual students talk about mathematics in and out of school. People speak in and from spaces and times that project particular values, social orders, authorities, and affective attributes (Dong & Blommeart, 2009). As people move in spaces (physically or metaphorically) of for instance school academic mathematics, they may find themselves or others as highly capably to think and do mathematics in some spaces, while they may find themselves as incapable in other spaces. This is not because of lack of competences, but because the values and norms that organize the spaces in particular ways have changed (Dong & Blommeart, 2009). Therefore, time and space are constitutive because they shape how people connect to each other and to mathematics. When mathematics is discussed in families with migrational experiences, family members may speak about mathematics from times and spaces that are located prior to migration. Their talk may project values and norms that connect to times and spaces prior to migration that relate to for example curricula content, social classroom norms, methods, and to what is valued as mathematical knowledge. In conversations for example about homework these values are relocated and experienced against the values and social orders that organize the time and space of the student's experiences with mathematics at school. Simultaneously, the student's experiences with mathematics at school that project the values of the dominant societies' school mathematics are relocated and experienced against times and spaces before the family's migration. We use the expression "experienced against" to echo Garcia and Wei (2014) who discuss multilingual translanguaging as a re/production of language, in which the "enaction of language practices that use different histories … are experienced against each other in [multilingual] speakers' interactions" (p. 21) because we see the enaction of different mathematics practises in the same vein. Paying attention to what language and mathematics norms are experienced against each other; who is positioned as knowledgeable; and what affective attributes are present in acts of relocation allow us to illuminate some work that multilingual students do to relocate school academic mathematics in and out of school.

Methodology

Three mathematics teachers at a school located in a city center in south Sweden invited the authors to participate in a project about enhancing multilingualism and language responsiveness as resources for mathematics learning. As part of the project the authors conducted three group interviews to learn what students themselves say about thinking, doing and talking about mathematics in multilingual contexts in and out of school. At the school most students were multilingual but quite a few students spoke only Swedish, the language of instruction. The three teachers selected students who were willing to share their experiences for the interviews, while at the same time aiming for as diverse groups as possible with respect to gender, languages, experience of migration and mathematics achievements. When asked, most students in the interviewed groups claimed that their school was a good school and that they learnt a lot. We asked about languages that were part of the students' everyday life. The students in the three groups mentioned Arabic, Bosnian, Kurdish, Polish, Spanish, Swedish, and Turkish. The Grade 4 group (aged 10-11) consisted of one boy and three girls. The Grade 7 (aged 13-14) group consisted of one girl and four boys. The Grade 8 (aged 14-15) consisted of three girls and two boys. Most of the students in the groups were multilingual. In the groups some students had experiences of attending school in other countries, some were born in Sweden by immigrant parents. The interviews that lasted about one hour were semi-structured (Patton, 2002). We (the researchers) had a battery of issues that we wanted to talk about, but we were sensitive to peruse issues that appeared to be crucial to the students during the interviews. School academic mathematics transitions between home and school were one of the salient issues that the students wished to discuss. We started by discerning excerpts that comprised talk of transitions of school academic mathematics between home and school. We found 26 excerpts, 9 of them were selected for in depth analysis based on that they focused students' intellectual and/or affective work to relocate school academic mathematics in transitions between home and school in conversations with others. By intellectual work we mean for example work to move among languages or mathematical methods. By affective work we mean moving among different kinds of emotions. Each of the 9 excerpts were analyzed with respect to *norms and values*, who is *positioned as knowledgeable* and to *affective attributes*.

Multilingual students' work to relocate school academic mathematics

We present findings on transitions from school to home and then from school to home.

Transitions from school to home

One student says[1]:

Student 1: If you learn math in Turkish, but Swedish math, it may be difficult for Turks, because they are not used to it. They do not have the same mathematics as us.

This student of Turkish descent talks about moving between two spaces of school academic mathematics (the Swedish and Turkish) in connection to internet conversations with his Turkish cousins. The quote shows that it is the different kinds of mathematics (problem solving methods in this case) that are at heart. When this student experiences Swedish and Turkish school academic norms against each other he finds that:

Student 1: I do not know Turkish mathematics. I have not studied their mathematics. I have only seen it [Turkish mathematics] when my cousin does it. When he read [a textbook problem] I did not get a thing. They have a whole different way of solving the problem.

This student seems to be aware of how moving between spaces means encountering different school academic values and norms. He appears to experience that because he doesn't know the norms and values of Turkish school academic mathematics, he cannot engage in the problem solving. By saying "I did not get a thing" it seems as if he finds himself unknowledgeable in the Turkish school academic space. Although that he does not explicitly mention affect, experiences of not understanding mathematics are usually not positive ones.

Another student shares that:

[1] To recognize the equal value of all languages we wanted to provide readers with quotes in the original language together with English translations. This was not possible for space reasons.

Student 2: Here in Sweden you learn in one way, but in our home country we learnt in a different way. Sometimes there are faster ways than those that you find in the textbook, so then I have a plus that he [the student's father] wants to help me. He says "Yes but you could just do this".

To move among different spaces of school academic mathematics norms seem to be a resource for this student. The relocation of at-school mathematics to home allows her to add the methods that her father shows her to her mathematical repertoire. Here both the father and the daughter appear to be positioned as mathematically knowledgeable. The daughter positions her father as a knower of mathematical methods that she grasps. The affective attributes in the excerpt shows that she is positive about the situation and finds her father's mathematical knowing beneficial (a plus) to her learning.

One of the students talks about using mathematical concepts in Arabic at home. Her parents encourage her to speak about mathematics in Arabic at home to maintain her Arabic language skills. In a sense, norms and values that relate to spaces and times prior to the family's migration are relocated to the here and now space and time. On the one hand this student seems to be positive about that and mentions for example that knowing several languages is a plus when applying for jobs in the future. On the other hand, she says that she speaks Arabic with her parents "because I have to":

Student 3: I speak Arabic with my parents because I have to. They want me to improve and extend my [Arabic] vocabulary. When I do it [talk about school academic mathematics] in Arabic then I feel a bit less smart. I always use some Swedish words to be able to do it [talk about school academic mathematics] in a good way.

This student works intellectually to relocate school academic mathematical concepts to the Arabic language that her parents want her to use. She explains that since she learns mathematics in Swedish, she does not know all the mathematical concepts that she knows in Swedish in Arabic. This influences how she sees herself as mathematically knowledgeable. When she talks about school academic mathematics in Arabic, she says that she feels less smart. To appear more knowledgeable in conversations about mathematics with her parents she "always use some Swedish". It is not because of lack of knowledge but because of moving between different (language/conceptual) norms and values that this student finds herself positioned as mathematically less knowledgeable (than at school) in conversations with her parents.

Transitions from home to school

Some students share experiences with school academic mathematics transitions from home to school. In the quote below the student talks about methods for problem solving that her parents taught her which they learnt in their home country. When she brings those methods to school the teacher seems to reject the student's work to relocate school academic mathematics methods that she brought from home:

Student 3: And when I show it [the methods that his parents taught him] [the teacher's name] says that it is wrong. He says "It is this way that you should count. It [the student's method] is not wrong but it is this way not that.

Here to appear mathematically knowledgeable in the classroom space, the student's work to relocate school academic mathematics from spaces and times prior to migration, to school requires that she ignores the mathematical methods that she learnt at home. It seems like the teacher indirectly positions the parents' mathematical knowing and the school academic mathematical norms of their home culture as subordinated to those of the majority society. To show that both methods could be appreciated the interviewer comments that it is good that the student knows both methods. The student replies in a downhearted tone:

Student 3: Yes, but that will not help me.

In contrast, another student reports positive feelings with relocating school academic mathematics methods from the home country to school. He says:

Student 4: It is always a plus because then you show that you have different methods for calculating. The teacher might ask a question, and then you can say several different ways to calculate it.

This teacher seems to welcome different methods in the classroom and appears to consider knowing a variety of methods as an indicator of mathematical knowledgeableness. Hence, the teacher's approach seems to position the student's home school academic mathematics as equally important and appreciated as those of the majority society. This appears to support the student's cognitive work with relocating school academic mathematics. Potentially the relocation work is a resource for his learning and a source for positive feelings about himself as mathematically knowledgeable.

Implications and closing remarks

In this paper we investigated what students say about the work they do to relocate school academic mathematics in transitions between home and school and vice versa. One reason for that is that the students that we interviewed were interested in sharing their experiences with relocating school academic mathematics. This may be because some of the students had experiences of schooling in countries where they were born. We noted, in line with the findings of for example Prediger et al. (2019), that students with migrational experiences in their families need to conduct intellectual work to relocate school academic mathematics among sites. We found that the students needed to do affective work because school academic norms and values change when they talk about and do school academic mathematics in mathematics spaces that connects to times and spaces before and after migration. For some students this meant that they felt mathematically knowledgeable in one school academic space while at the same time finding themselves unknowledgeable in another school academic space. Although that the notion of transitions and the dichotomy informal-formal mathematics consider students' out of school experiences with colloquial home cultures and languages (Abreu, 2008; Webb & Webb, 2016), it does not fully grasp the intellectual and affective work that multilingual students engage with when relocating school academic mathematics in transitions between home and school. Language as resource and culture responsive teaching approaches may benefit from including attention to students' work with relocating school academic mathematics. Lessons could include discussions in small language homogenous groups about the mathematical concepts that are at stake. In such discussions the home language is both a resource for mathematics learning in the language of instruction and a learning object in its own right. The discussions could be summarized on the board in a table that shows the concepts in the languages present in the classroom. In the same vein different arithmetic or problem-solving methods could be focused. In this way the intellectual work that students do with relocating school academic mathematics may be a resource for learning rather than evoking stigmatization as for example discussed by Meaney and Lange (2013). In addition, it would give a broader range of students the opportunity to perform and be viewed as mathematically knowing subjects in the classroom (see for example Ryan et al., 2021). It could mitigate experiences of being positioned as unknowledgeable at school and/or at home. This is important because as Crafter and Abreu (2011) mentioned, multilingual students' mathematics learning transcend the physical school building. Ultimately attention to students' work with relocating school academic mathematics is a sociopolitical issue because it revolves around encounters between dominant and non-dominant languages and cultures (Planas & Setati-Phakeng, 2014). Pedagogical designs that appreciate and make explicit both dominant and non-dominant school academic mathematics norms recognize that multilingual students need access to the dominant language and mathematics to be recognized as mathematically knowledgeable in the dominant society where they may conduct future studies and compete on the labor market. However, to merely ask multilingual students to (re)produce dominant mathematics may cause harm (Le Roux & Rughubar-Reddy, 2021) because their relocating work is then ignored or rejected. This may cause stigmatizing feelings which diminish multilingual students' opportunities to learn mathematics.

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Mathematically significant knowledge of Hñañu women embroiderers from the Valle del Mezquital: a possibility for an encounter and dialogue

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We present some results of an ethnographic perspective-based study of mathematically significant knowledge of Hñañu women embroiderers from Valle del Mezquital (Mexico). In this contribution, we focus on an interview throughout where one embroiderer and the researchers encounter and dialogue about the realization of embroideries ordered by the clients (the researchers). This encounter (Radford, 2021) gave rise to a joint discourse built through the discourses from the culture of the embroiderers and that of the researchers. Based on these results, we reflect on the possibility of constructing encounter opportunities in classrooms between school mathematical knowledge and knowledge about embroidery. We consider that the construction of such encounters requires the active participation of members of the – historically marginalized - indigenous communities so that the school will not be a space for the domination and homogenization of people and their knowledge.

Keywords: sociocultural studies, indigenous knowledge, ethnomathematics, cultural education.

Introduction

The discussion we propose takes place in the diverse context of studies regarding the knowledge of cultural groups and how they relate to mathematics and school. These studies are plagued with encounters and ruptures between the Western mathematical tradition and knowledges from other cultures and peoples (D'Ambrosio, 2014, 1985). The Western mathematical tradition is often presented as a parameter, setting out development levels and promoting a deficiency-based approach to other cultures' knowledge (Albanese et al., 2017; Bishop, 1991; D'Ambrosio, 1985; Radford, 2020).

The need to overcome this deficiency-based approach has led several authors to reconceptualize the idea of mathematical knowledge (Ascher, 1991; Bishop, 1991; D'Ambrosio,1985). In turn, mathematics education research has developed several proposals seeking to integrate these forms of knowledge into schools (D'Ambrosio, 1985; Kisker et al., 2012; Lipka et al., 2019; Stathopoulou, 2020; Pradhan, 2020; Sharma & Orey, 2020; Albanese et al., 2017; Gracas & Marinho, 2015; Verner, 2019). However, integration processes into school practices are not simple; they are entangled in a complex mesh of continuity and discontinuity between the activities and knowledge of cultural groups and those specific to schools (Solares et al., 2016; Trinick et al., 2017).

We based our study on critical approaches to cultural-centric positions to widen our understanding of mathematical knowledge and to include, among others, forms of knowledge involved in practical activities recognizable from the Western mathematical tradition (Bishop, 1991; D'Ambrosio, 1985, 2014), which we have named *mathematically significant knowledge.*

This contribution gives an account of local expressions of mathematically significant knowledge of Hñañu women embroiderers from Valle del Mezquital (Mexico). Through the analysis of interviews, we give an account of the dialogue between the embroiderers and the researchers in which they negotiate and discuss meanings related to embroidery and mathematics knowledge. We have named this polyphonic and incipient dialogue as a *discourse-for-us* (Venegas-Thayer, 2019), a jointly built discourse that emerges from the discourses of the cultures of the embroiderers and the researchers.

Theoretical framework

We chose to articulate complementary theoretical notions from the Ethnomathematics Program (D'Ambrosio, 2014) and the Theory of Objectification (Radford, 2021). According to D'Ambrosio (2014), we understand *mathematical knowledge* in a broad sense as the "bodies of knowledge elaborated based on qualitative and quantitative practices, such as making comparisons, ordering, classifying, inferring, and the code systems of measure, weight, and quantity [numbers], which have been accumulated through generations, in certain natural and cultural environments" (p. 102). Thus, when we speak of Western mathematical knowledge, we mean that which has been developed – and is still being developed – in what today we call the Western culture.

This broad conception of mathematical knowledge allows us to refer to the explanation and practice systems that shape embroidery and are recognizable from the Western mathematical culture. Thus, for example, we say that an embroiderer counts, adds, or subtracts stitches, reflects a motif according to a symmetry axis, etc. In other words, we associate the knowledge mobilized by the embroiderers to perform specific actions with knowledge that is part of Western mathematical knowledge. It is this knowledge, developed in this specific region and culture and mobilized by the embroiderers, that we call *mathematically significant knowledge*.

Concerning the specific manifestations of these explanatory and procedural systems in a culture, we take up the distinction between *knowledge* and *knowing* proposed by Radford (2021) in his Theory of Objectification. Knowledge is not simply present in a group and transmitted to its new members or something each subject constructs when facing reality. Knowledge is a social, historical, and cultural product. It is accessible and shared by groups of people who use it in recognizable and significant ways inside the group. Knowledge is potentiality, instantiated in specific moments and circumstances through actions performed by particular persons. Following Radford, we call these instantiations knowing. Thus, knowing is not an atemporal construction of the subject but an instance of knowledge that occurs inside the cultural group and activities that give it meaning (Radford, 2021).

Methodology: A Study in Two Phases

The methodological design of the study included two phases. In the first phase, we describe and classify many traditional embroideries of the Hñañu culture of the Valle del Mezquital through plane isometric transformations that allow creating their geometric motifs (Barquera y Solares, 2016). During this first phase, the contact with the embroiderers has allowed us to become conscious of the distance between how we understood the embroidery activity and its products and how the embroiderers do it. This understanding led us to design the second phase, in which we adopted an ethnographic perspective (Lave, 2011) to understand the knowledge the embroiderers mobilize in their activity.

We used the results from the first phase as a starting point for the second; mainly, to design the interviews with a family of embroiderers and field observation guidelines. In this contribution, we focus on the results of our interview with Alejandra.

Interview Design

At the time of the study, Alejandra was 24 years old and had completed high school studies. Her first language is Spanish and, although she understands Hñañu, she does not speak it. The interview with Alejandra took place in two parts: in December 2016 and in March 2017. Here we focus on the second interview, organized around a few embroideries we asked her to do as clients.[1] Figure 1 shows the embroideries we ordered (*el encargo*, in Spanish): a tortilla napkin and a shirt.

Figure 1a

Figure 1b

Figures 1a and 1b: Embroideries ordered in the interview with Alejandra

We chose motifs related to translation, rotation, and proportionality tasks, typical of school mathematics. For each of the ordered embroideries, we performed an *a priori* analysis from the perspective of academic geometry. That is, the mathematical reference knowledge we wished to relate to the mathematically significant knowledge observable in Alejandra´s activity. Table 1 presents the embroidery ordered for the napkin and the task we asked her to perform.

Table 1. Embroidery Task of a Tortilla Napkin with Geometric Motifs

Task: to embroider a tortilla napkin with geometric motifs	
Academic mathematical knowledge involved: axial symmetry, translations, reflections	
Task formulation during the interview:	Vertical direction in which this motif is not usually continued

Horizontal direction in which the motif is usually continued

[1] An important part of the activity of the embroiderers from Valle del Mezquital is selling their embroideries. The order, fabrication and sale of the embroideries guide a significant amount of the activities they perform.

"I want a napkin with this embroidery at the center but repeating it four times, in this direction (pointing horizontally with the hand) and in this direction (pointing vertically)."	

Also, informed by our previous field observations, we anticipated the possible procedures and difficulties Alejandra might find when doing the embroidery as a basis for discussion during the interview.

Interview Analysis.

What follows is an analysis of some of the dialogues we had with Alejandra concerning the napkin embroidery.

Reflection and Little Rows: Embroidery Knowledge Encounters Academic Geometry.

Based on the *a priori* analysis, we believed the task could be solved by reflecting or translating the central motif (Figure 1a). However, when presented with the task, Alejandra immediately noted that "this cannot be continued because it ends in a little flower, it ends in a tip," pointing at the tip of the motif's flower. Alejandra explained that the problem making the embroidery was that the motif's flower tips would end up stuck or joined together. This is not relevant from the – academic and formal – point of view of axial symmetry, and, thus, we did not think it would be so for the task. However, for Alejandra, this aspect was fundamental because the flower tips should not be joined, possibly responding to aesthetic criteria. Thus, we had contradictory ways of understanding the solution for the task – doing the embroidery. Alejandra believed it could not be done. In contrast, we believed it was not only doable but that the "obstacle" Alejandra anticipated was not understandable in terms of our academic knowledge since all geometric objects in the plane can be reflected on an axis.

After several minutes of explaining and despite the anticipated problem, Alejandra agreed to fulfill the order and try to satisfy the clients' petition. First, we must note that this negotiation between the clients' request and the embroiderer's work is common practice in embroidery. As Alejandra put it: "It is like they say, if it is what the client wants, we have to do it."

About one month later, we returned to get the embroidery. Alejandra told us she had managed to do it and then explained how she did it.

Dialogue. The Little Row as a Solution

Alejandra: Yes, because, in fact, I tried once or twice, and I could not do it. Then I undid it again and restarted it, and I said: "You know what? I will leave one thread, a little row". So, I left a strip of thread here, so I could start again, to do it again, repeat the same [motif]. [She pointed out in the napkin one small space she introduced between the motif and its copy].

Interviewer: There is a space here? [Pointing out at the space between the motif and its copy].

Alejandra: Uhuh, that is why it seems like a little hole here in the middle. Because I was trying to join it to the other, but it was going to look weird, it would look ugly. So, I thought: “I had better leave one [space] and then start the other one so that it will look well [...]”.

Alejandra: Now, if you placed a mirror here, it would reflect the same one over here, like this. Also, it is the same thing [using her hand as a make-believe mirror, placing it on the space she left between the motif and its copy; using her right hand to point out the copied motif].

In contrast with the possible reflections and translations we anticipated, Alejandra introduced a novel solution to perform the requested embroidery. It was an unexpected and non-academic response to the task. Furthermore, Alejandra displayed a specific knowing we were not expecting: the introduction of space between the motifs – the *little row* (la *filita*, in Spanish). This allowed her to do this unique embroidery keeping the motifs from being conjoined and, thus, "looking weird, looking ugly". Simultaneously, this solution satisfied the buyer’s request and her aesthetic criteria.

When describing how she performed the task, Alejandra used the *mirror metaphor*[2]. Although we do not know if this metaphor is part of the embroiderers’ knowledge or if they use it in their everyday activities, after its introduction, both Alejandra and the interviewer used it on different occasions to continue talking about the embroidery and how it was done.

These dialogues, together with the words and metaphors – such as *little row* and *mirror* – introduced and negotiated throughout the interview, account for the “voices” of the other. Alejandra and the interviewer gradually established a negotiated discourse that allowed them to talk about embroidery and how it is done, despite the differences between their respective knowledge. This is the discourse we call *discourse-for-us* (Venegas-Thayer, 2019).

In the interview, they continued discussing the motifs that “can be continued and those that cannot.” For Alejandra, it was essential to make it clear to us that some embroideries “can be continued” and some cannot, both materially (i.e., is it feasible to do them with embroidery techniques) and aesthetically (i.e., they do not look ugly or weird). According to her explanation, some motifs “can be [continued] in both directions,” that is, in the direction of the fabric’s both *weave* and *warp*[3] (which, from the point of view of academic geometry, would be equal to a tessellation of the plane).

To explain which motifs can be continued and which can’t, Alejandra resorted to a classification: there are “cut-off” motifs, and there are “finished” motifs.

Dialogue. Cut-off and Finished Motifs

Interviewer: And how do you know it is possible [to continue a motif]?

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[2] It’s a metaphor commonly used when studying axial geometry in elementary schools in Mexico. You imagine the use of a mirror perpendicular to the plane of the motif you wish to reflect, placed over the reflection axis.

[3] The embroidery is done on a base fabric, created with two perpendicular groups of thread interwoven in a regular pattern. Regarding the embroiderer’s body, the threads that run from the right to the left form the fabric’s weave, while the series of threads that run perpendicular to them form the warp. Thus, *weave* and *warp* shape an orthogonal grid.

Alejandra: Because here you can see one half [Figure 2a]. Pretend we only have one half of this here [she points out half of one of the motif's diamonds; Figure 2a]. It is as if it had been cut-off like this [she points out the halves of the diamonds, in the directions of both the weave and the warp]. Then it can still be continued upwards, downwards, and sideways.

When you see it is cut-off, then it can [be done].

However, for example, these ones are finished. Therefore, they are finished [motifs from Figure 2b].

Figure 2a. Example of a cut-off motif

Figure2b. Examples of (horizontally) finished motifs

According to this explanation, cut-off motifs (*motivos mochados*, in Spanish) are those we can imagine as continuing but have been cut off, like the diamonds in the motif in Figure 2a. These motifs can be continued precisely in the direction in which they were cut off. In contrast, some motifs are already finished and cannot be continued. For example, the motifs in which the figures – flowers, stars, etc. – appear complete or finished, there is no need – or aesthetic possibility – to continue them. In this case, Alejandra uses embroidery knowledge to explain why some can be continued and some cannot, proposing a classification of embroideries.

Discussion and Final Reflections

The interview with Alejandra was designed as a space for an encounter (Radford, 2021) between the knowledge of embroidery and academic geometry. The order had the double purpose of creating a conversation around the act of embroidery and facilitating the emergence of mathematically significant knowledge. This interview's analysis allowed us to showcase some tensions between the knowledge of embroidery and academic geometry concerning the carrying out and the classification of the embroidery motifs. During the conversation, these tensions led the participants to introduce and articulate different semiotic resources (words, gestures, metaphors, etc.), which reflect/refract their knowledge rendering it intelligible.

In this encounter, with its communicational possibilities and limitations, both the embroiderer and the interviewer were able to talk and make progress in their conversation about embroidery categories in a significant way, from the point of view of both embroidery knowledge and Western mathematical knowledge. We observed the interlocuters progressing in their conversation, progressively negotiating and providing common meanings for the terms, which allowed them to bring out their ideas, accepting each other in a joint reflection of the subjects that interest them.

Based on these results, we wonder about the possibility of building bridges between academic mathematical knowledge and embroidery knowledge. For example, is it possible to create in schools encounters between academic mathematical knowledge and embroidery knowledge in a way that benefits educational activities?

Even acknowledging the importance and validity of the knowledge of the embroiderers and other cultural groups, taking it to school is not trivial. Taking the tasks of the embroiderers from Valle del Mezquital to the classrooms would imply many reformulations and adaptations of the conditions and demands characteristic of the educational institutions and communities (Traoré y Bednarz, 2009). From the point of view of school mathematics we wonder, how to acknowledge and dialogue with the mathematically significant knowledge of embroidery? What is it desirable and possible to take to school to create new academic meanings?

We believe these encounters between the school knowledge and the knowledge of other cultural groups require changes in our ways of understanding, teaching, and learning mathematics, changes in the current curricular organizations, teachers' education, and the creation of didactic proposals. To create these proposals, we need active and significant participation from indigenous communities, who have been historically marginalized, so that schools are no more a space for the domination and homogenization of people and knowledge. In this way, these encounters will allow the "the creation of reflexive and ethical subjects who critically position themselves in historically and culturally constituted mathematical discourses and practices, and who ponder new possibilities of action and thought" (Radford, 2021, pp. 15-16).

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Students' discussions of cultural, social, economic, and political aspects of the COVID-19 pandemic

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Worldwide, students have experienced the impacts of the COVID-19 pandemic. This paper investigates students' reflective discussion of the consequences of measures taken during the pandemic in Norway. Socio-critical mathematics perspectives are used as a theoretical framework when analysing 13–14-year-old students' discussions of the pandemic. The students read, discuss, and present findings from media articles provided by their teacher. We found that the students identify and use mathematics-based argumentation when reflecting on cultural, social, economic, and political aspects from their own real world. They ask questions, investigate, listen to peers' argumentation, show awareness of, and discuss challenging issues such as mental health, increased violence during home-schooling, unemployment, loss of income, and rising inequality in society. We suggest that mathematics education can promote students' reflective discussion of wicked problems.

Keywords: Socio-critical mathematics, mathematical representation, discussion, wicked problems.

Introduction

The ongoing COVID-19 pandemic is an example of complex real-world problems often referred to as wicked problems. Characteristics of wicked problems are that time is running out, no optimal solution, conflicting interest, no central authority, and policies discount the future irrationally (Levin et al., 2012). During the pandemic, citizens worldwide dealt with a highly mathematics-based language and mathematical representations. For instance, the Reproduction (R) number showing how many people one infected person infects on average is central in describing, understanding, and predicting scenarios in Norway. Kollosche and Meyerhöfer (2021) describe the R-number as having "formatted our perception of the status of the pandemic" and "justified specific political measures" (p. 10). They highlight that although substantial measures are taken partly based on this number, the significance of the R-number remains questionable.

In the Norwegian mathematics curriculum (MER, 2019), it is stated as vital that students critically reflect on models, statistical representations, and mathematical argumentations to make justified stands in essential issues in their lives and society. In this paper, we are inspired by Nakling's (the third author) master thesis, involving 13–14-year-old students' mathematical argumentation about risks of infection control measures of COVID-19. The students used media articles to discuss risks and impacts for groups in society. In this paper, we use the empirical data to consider how students can engage in wicked problems. Specifically, we investigate *How do students apply mathematics to critically reflect on the consequences of measures taken during the COVID-19 pandemic?*

Socio-critical perspectives in mathematics education

The use of models in the pandemic is often connected to statistics, for instance, in reporting numbers of infected. Weiland (2017) argued that students should learn “to tackle complex sociopolitical issues in conjunction with learning powerful statistical concepts and practices in an effort to be able to read and write both the word and the world with statistics as critical citizens” (p. 45). Reflecting critically on societal issues in authentic applications of mathematics in real life can involve empowering students in discussions about the consequences of using mathematics, statistics, or mathematical models in society (Barbosa, 2009; Blomhøj, 2009; Gibbs, 2019). Barbosa (2009) emphasised that students’ reflection of the nature, criteria and consequences of mathematical models in society is crucial and should be the primary goal for socio-critical discussions. Relatedly, Blomhøj (2009) described that critical reflection may concern “the societal issues where mathematical modelling and models are used as means for analyses and critique of political decisions or societal phenomena” (p. 13). However, Gibbs (2019) found limited empirical research on students’ socio-critical perspectives and reflexive discourse during modelling. She argues that such discourse is crucial to move from modelling as a school activity to understanding the role of mathematics in society and that reflecting through mathematics can make it clear that a reduced representation can be an incomplete representation of the social phenomenon.

When students engage in socio-critical modelling activities, they “make assumptions about variables and quantities needed for their models” based on their own identity and history (Brunner et al., 2021, p. 139). However, the foundation, variables, and quantities may be invisible when using existing models or statistical representations. Therefore, when working with mathematical representations, it is essential to consider the properties of models or the statistics, what variables are included, and perhaps more importantly, what is not included. For instance, the R-number gives important estimates nationwide of health issues regarding changes in numbers of infected, hospitalised, etc. However, it does not include the impact of the pandemic on variables such as mental health issues.

A democratic society needs to develop competence in the general population “to critique mathematical models and the ways in which they are used in decision making” in society (Blomhøj, 2009, p. 11). The prevalence of mathematical applications in the pandemic, and the use of these in decision-making, call for students’ awareness of consequences of these applications so that they can critically reflect on such as the foundation, who has chosen the variables, what kind of variables are considered as important, and what does the models, statistics, and numbers tell us. Hattori et al. (2021) describe how socio-critically open-ended problems could foster critical mathematical literacy. They emphasise that students should develop abilities to critically perceive mathematics according to the situation and make social decisions based on values using mathematical thinking. They described how students discussed problems related to the pandemic in groups and argued for solutions based on personal values and mathematical argumentation.

Rosa and Orey (2015) highlight mathematical modelling as a teaching methodology to develop students’ social-critical efficiency. They encourage investigating familiar problems to the students so that the content becomes the stimuli for students’ critical reflections on the role of mathematics in society, showing that mathematics applied is not neutral. Cultural, economic, environmental, political, social, and natural aspects of reality are contents they suggest as essential for socio-critical modelling. The COVID-19 pandemic is an issue with multiple aspects, as regulations have a range of impacts. One could say that pandemic regulations are based on a model of reality, where different facts and considerations are taken into account. However, it is only partly mathematical, as sub-models, numbers and statistics, together with some weighting of concerns, as elements of a rather unarticulated model. While Rosa and Orey (2015) emphasize the importance of engaging students in the whole modelling process, it is also of value to critically reflect on their suggested aspects through numbers, statistics and modelling presented in the media together with the role of these numbers. In this paper, we ask how students apply mathematics to critically reflect on the consequences of measures taken during the COVID-19 pandemic. We investigate students' utterances in light of Rosa and Orey's (2015) aspects and pay attention to aspects that may be in conflict with each other.

A Norwegian context during COVID-19 pandemic

The Norwegian Government has various policies to limit and delay COVID-19 outbreaks, such as restrictions on travels and group meetings, prohibiting public events, and closing schools and national borders. In the initial stages of the pandemic, the government designed a strategy based on keeping the R-number below 0.9, the so-called suppress strategy. This strategy was partly based on the capacity of hospitals and uncertainties regarding treatment, potential vaccines, and how the virus would affect people and the nation. Closing schools and borders severely impacted citizens and the national economy, and later, a shift in the policy focused on social consequences such as loneliness and unemployment. The Norwegian population shows significant trust in authorities, e.g. they trust the information provided by the government, and there is an estimate of 90% vaccine coverage.

Teaching in Norway varied between digital home-schooling and physical attendance at school, with varying restrictions on social distancing. The students in the empirical data experienced a long period of home-schooling during their last year of primary school. In their first semester at lower-secondary school, teaching varied much between home and school, and when at school, students were grouped into so-called cohorts. Their mathematics teacher, Nakling, did his master thesis in his class.

Method

To investigate students' discussions, we used empirical data from Nakling's master thesis. Nakling designed the lessons, and Hauge and Steffensen participated in the collection of data. The teaching took place during a two-week project and consisted of four lessons, each lasting 2 x 45 minutes. The students were 13–14 years old, the class consisted of 30 students, and they worked in groups of five, a total of six groups. Their teaching plan can be described in three parts. During the first part, the students worked with graphs to ensure students' understanding of what they represented. For example, the teacher showed a graph that simulated the progression of hospitalised infected individuals over time compared to the capacity of hospitals, with and without societal interventions. The students were prompted to consider potential consequences whether measures were taken too early or too late. In the second part, the student worked in groups. Each group was assigned one role from society: nurses, parents, unemployed, psychologists, university students, and teachers. The teacher had selected 3–4 media articles for each of the groups, who were to present the situation of their assigned role based on mathematics-based content. The third part was a plenary activity. Here, each group first presented their findings orally, followed by a plenary classroom discussion about the pandemic and risk assessment for groups in society.

The students' utterances were video recorded during group work and in the plenary part. Only students who had given consent were recorded, resulting in data from four of the six groups. When analysing students' utterances, we took starting point in the six aspects of reality described by Rosa and Orey (2015). The cultural, social, economic, political, environmental, and natural aspects were used to identify and categorise students' utterances. While some of the utterances overlapped different categories, others did not. The idea behind this categorisation was to get an overview of the width of the student's reflections. The categories are used to structure the discussion part. The theoretical described above are used to understand, interpret, and support the discussions.

Students' discussion on cultural, social, economic, and political aspects

In the following, we analyse excerpts from the students' group work, plenary presentations and plenary discussion to show students' reflexivity related to Rosa and Orey's (2015) aspects of reality. We connect these to mathematical models, here broadened to include applied mathematics and statistics in general. We further discuss this in relation to how their experience with the pandemic possibly has influenced their reflectivity.

Cultural and social aspects

The media articles provided by the teacher included a range of cultural and social aspects as a consequence of the handling of the pandemic, such as increased violence towards children during home-schooling and mental health issues. In the group discussing risks for parents, a researcher asks if they have considered any risks. Chris starts by saying it is relatively easy to handle the pandemic and suggests that laws could regulate people's behaviour and that people who violate these could get punished. He declares: "If you say you get the death penalty if you walk out the door", indicating that if people just stayed at home, the virus would not spread. In Norway, relatively few were fined for violating COVID-19 regulations. On the contrary, many restrictions were encouraged rather than forced, basing restrictions on people's trust in authorities and their wish to contribute to the common good. This reflects a cultural and social aspect of Norwegian society. However, this careful approach was criticised in the media when the number of infected people increased. Chris' only concern seems to be to stop the pandemic without introducing other concerns.

As a response to Chris, Kari says she has read about some of the risks of home-schooling. She argues it will affect children because parents get strained during lock-down:

Kari: Some take the stress out by being violent to their children [...] primary students are less followed up by teachers […] 1 of 3 parents was angrier at their children. It was more violence, and parents had anxiety, depression and stress symptoms.

Kari’s argumentation challenges Chris “simple” solution that everybody should stay at home. She brings forward new perspectives, namely that if you impose people to stay at home, this can have severe consequences. The numbers are taken from one of the media articles her teacher has provided to support her argument. She further reflects on youth managing home-schooling better than children in primary school. Kari connects this to her and her peers’ own reality and identity, as highlighted as relevant by Brunner et al., 2021, when making assumptions about real-world situations. Kari’s critique of Chris’ claim is partly supported by numbers and is related to humans’ well-being and what is considered appropriate behaviour. She thereby draws on cultural and social aspects.

Later in the group discussion, Chris says:

Chris: I now think the problem is that it is hard to justify a choice if we do not have all the facts or know all the risks. If you compare them, which is most important if there is a great danger of many dying if we go for it? But it can have such big consequences. Then we have to find out which one weighs the most. And in that way, we can decide. But if we do not know all the risks, it can be very difficult.

Chris problematises the group discussion, including his previous claim. He critically reflects on decisions based on limited facts and knowledge of risks. Perhaps he still believes in a shut-down and penalties for leaving home without permission, but he realises there are competing concerns that must be weighted and with unknown consequences. Critical reflections on facts and uncertainties are essential in modelling, and when Chris uses his hands to visualise the weight of each risk, he communicates a model for decision-making. The pandemic is surrounded by uncertainty about facts and perceived risks, so Chris points to a highly relevant political aspect: what, how and who to choose.

Ingeborg raises concerns about mental health issues when presenting her group’s findings in the plenary activity (from the psychologist group):

Ingeborg: Almost 40% between 16 and 25 year- olds state they are unhappy with life in the pandemic. About 1 of 3 Norwegians feel lonely during the pandemic. During spring 2020, there is an increase of 45% in pills against anxiety, depression, and insomnia.

Ingeborg uses mathematics-based language, emphasising numbers such as 40%, 45% and 1 of 3 when she presents the magnitude of concerns. Statistics is thereby used to critically reflect on a social issue. Her tone of voice is serious, and she keeps eye contact with her classroom peers. Later, in the plenary discussion, students continued to elaborate on mental health issues. For instance, Anna (from the student group) said: “It was rather lonely, really. They just said ‘don’t be with others, just stay home’, and it came rather suddenly and we didn’t get any information”. She relates to the numbers expressing loneliness and voicing her own experience. Other students expressed it as important to be allowed to visit some friends during lock-down. In the plenary discussions, the teacher asks the students what they would argue as appropriate measures in society if they were decision-makers? Iris (from the nurse group) replies: “It is a bit difficult, but cohorts in schools are good because it keeps people apart, and at the same time, people don’t need to develop anxiety and depression”. When referring to mental health issues, she nodded towards the psychologist group, perhaps as a potential solution to the issues they described. She probably referred to cohort regulation as a way to still have social relationships, basing this on personal experiences and a way of making social decisions based on values and using mathematical thinking as described by Hattori et al. (2021).

Economic and political aspects

In the group discussing risks related to unemployment, Miriam says to her group members: "Many struggle and many borrow money to survive. Many have become unemployed due to the pandemic, and struggle with such strict measures". Miriam uses the word "many" multiple times to emphasise the problem of unemployment. Because she has read articles presenting numbers on unemployment, "many" refers to these numbers and communicate that she is concerned about the economic aspect. The numbers have thereby contributed to pointing at risks as a result of restrictions. Miriam seems to have forgotten the specific numbers when Haavard asks curiously:

Haavard: Can you tell us a little about those who have lost their jobs?
Miriam: Then I will find some numbers.

Miriam starts looking for numbers. In the following discussion, they use phrases such as "8% have difficulties", "1 of 4 households are affected", and "15% have no savings left". There is a distinct move from the less precise "many" towards a mathematical-based langue. They also give a more nuanced picture of what kind of struggles there are. The move to precise numbers is probably due to the task given by their teacher and that they are going to present their findings to the rest of the class. Miriam's use of "many" can be regarded as her interpretation of the situation is severe.

They use both the Internet and the provided articles, which brings up conflicting numbers. In her initial search, Miriam found that 25% of household is economically exposed. A moment later, she exclaims, "Oh, no, sorry. Now I entered another website. This is from October tenth". After examining together with Haavard, she continues: "So, 18%, or about 430 000 households, are still exposed. This is in December". After some discussions, she concludes, "The number of economically exposed households has thus decreased". The conflicting numbers resulted in critical reflections on the relevance and validity of the numbers in play, which are crucial in developing and considering statistics and mathematical modelling. It is also part of reading and writing the world and the world with statistics, as described by Weiland (2017). This focus on relevance related to updated statistics is seen when they later investigate the issue of people's savings.

When asked if they had noticed any risks, Haavard replies: "It may be a bit odd saying this, but they risk starving to death. Honestly. Or, it's unlikely, but they actually can". So far, the discussions have focused on the number of people losing their jobs or struggling without reflecting on how this struggle could manifest. Now, Haavard is bringing up the risk of starving. Miriam adds, "But 5% need to borrow money for food". As previously, she provides numbers to strengthen her argumentation. Although Norway has a fair welfare system, with no cases of extreme poverty, there is an increase in low-income families. Typically, people with low-income jobs took the biggest blow financially in the pandemic. The students' focus on the risks of unemployment brings awareness to these aspects.

The economic consequences were later connected to political aspects. Miriam suggests that in their presentation, they should say that, "It will create inequality in society for a long time". The issue of long-term perspectives of inequality is new, and Haavard struggles to formulate Miriam's utterance on their PowerPoint slide. He asks her about it, and she reformulates: "This crisis can lead to bigger differences. And the economy can be destroyed. It will take a long time to recover". She focuses on inequalities and long-term perspectives. In their critical reflection on the consequences of measures, they do not explicitly connect and critique the nature of the numbers, statistics, or mathematical models themselves, as described by Barbosa (2009) and Blomhøj (2009). The topic of inequality and differences in society can, for some students, have strong proximity and for others be at a more distance. However, an awareness of this aspect is essential for all students. During their investigations and discussions about numbers and statistics of unemployment and people struggling economically in the pandemic, they have started to explore the important issue of inequality in society.

Concluding comments

In this research, we have investigated students' discussion of the consequences of COVID-19 measures. The students develop a critical understanding of real-world problems through mathematics-based information by showing awareness of and discussing the measures' cultural, social, economic, and political aspects. Rosa and Orey's (2015) environmental and natural aspects might have been included in their discussion if other articles had been chosen for the students. For instance, more biological aspects of the virus or reduced CO_2-emission due to less traffic and fewer flights. The cultural and social aspects in the students' discussions involved mental health issues and family violence during home-schooling. The numbers identified by the students represented mental health issues, which Anna, Iris and other students associated with feelings of loneliness. When students reflect on these problems with peers, mental health issues can be normalised and make them less challenging to discuss. The use of numbers showing the extent of people struggling with mental health issues during the pandemic may have contributed to forward discussions on this topic. Undoubtedly, the pandemic severely impacted the economic and political aspects of society. When governments closed restaurants, hotels, etc., and maintained strict travel restrictions, many people lost their jobs and income. Miriam and the other students started to explore some of the statistics and numbers related to loss of income and increasing inequality in society. Social inequality is a concern in critical mathematics education, and inequality seems to have increased during the pandemic (Borba, 2021).

Although students identified and used the numbers to discuss wicked problems and consequences of measures taken, most of the time, they did not critically question statistics or numbers in itself. Thus, implications of this research can be that teachers can have an important role by asking questions leading to such inquiries. They could encourage them to critically examine what is behind the numbers, statistics, and mathematical representation, what they consist of, sources, its use, and the consequences of mathematical application in society. Perhaps students could have made a model themselves to inform decision-making. Such a critical gaze can bring further awareness of the limitation of mathematics, but still how mathematics permeates wicked problems like the pandemic. Engaging students in these issues facilitates their understanding of different perspectives and as critical mathematical citizens. While this research is limited to a few students in a short period, it provides insights into how students relate to mathematical argumentation during a time of crisis.

Other characteristics related to the student's discussions involved questioning, listening, investigating, and changing perspectives due to others' argumentation. They moved from giving simple solutions to wicked problems towards sharing more nuanced reflections like Chris did when reflecting on risk and potential measures. When society deals with wicked problems and decision-making, it is impossible to predict all consequences. At the beginning of the pandemic, concerns were raised about the capacity of health institutions, and mathematical representations heavily informed governments in their decision-making. Later, concerns for young children and mental health issues were given more significance. Other values came into play. When measures are implemented in uncertain times, it is crucial that uncertainty is conveyed and risks are presented openly to give citizens reason to follow given advice, as doubting the science behind COVID-19 can be fatal. However, wicked real-world problems need multi-disciplinary approaches where it is essential to consider cultural, social, economic, and political aspects, to understand the complexity of the problem and avoid misuse of information, mistrust and polarisation in society. Although other school subjects might have added perspectives and depth to the students' discussions, the activity shows that mathematics education has a vital role in involving students in investigating mathematics-based information in times of crisis, discussing aspects of their own reality.

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Math teachers’ visions of an ideal math class: What do they tell about bringing innovations into the classroom?

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Various social and political factors within the school environment influence how teachers ultimately design their teaching and also their motivation to bring innovation into the classroom. The paper presents results of math teacher interviews about their vision of ideal mathematics teaching. From these answers, conclusions can be drawn about teachers’ potential for bringing innovation into the classroom and the factors that might hinder them. Three different types have been derived, with varying levels of motivation to participate in in-service teacher training and to further develop the teaching. Teachers that like to teach in the way “it has always been done” show the least potential, in contrast to the idealists among teachers, who align their teaching with their high standards to design lessons that they believe best for their students’ learning. To narrow the theory-praxis gap, the focus should be on the third type of teacher – the struggling ones – who need support to stay true to their ideals and to avoid adapting to more traditional patterns.

Keywords: Teacher socialization, mathematics teachers’ beliefs, theory-practice relationship.

Introduction

Working with in-service teachers in professional development courses is not always easy, and participants are often reluctant to acquire new knowledge and develop their teaching. “We do not have time for this” or “I am experienced enough after teaching for 20 years, I don’t need this anymore” are just two statements given at the kick-off event for an in-school professional development course for high school mathematics teachers that aimed to elaborate the concept of “critical mathematics education” and possible implementations in the classroom.[1] These comments were made before the teachers even knew in more detail what the course was supposed to be about. From the very beginning, one could notice a general aversion to any participation in the professional development course. The intended courses at this school were canceled after the kick-off due to the lack of teacher motivation.

Continuing professional development throughout the career is seen as an essential part of the teaching profession. Teachers need to constantly develop their teaching practices to be effective in today’s classrooms. In-service teacher training is a necessary prerequisite for innovations to be implemented in the educational system and the classrooms, as it is not enough to rely on next-generation teachers to bring about changes. It requires the participation of experienced teachers (Mayr & Neuweg, 2009). However, it seems that the awareness that teaching is a life-long learning process is not very pronounced for some teachers. Some of them might feel that they are “conclusively trained teachers” after completing their degrees and do not see a necessity to professionally develop further, which is why they articulate their resistance to development work (Körkkö et al., 2020). Such a mindset has long been fostered in Austria by the political and

[1] All quotes from my interviews are originally in German and have been translated into English by me.

structural requirements placed upon teachers since, until a few years ago, high school teachers had no obligation to undergo further training after completing their studies. Such regulations might therefore influence the image that teachers have of their profession, the demands placed upon them, and ultimately the extent to which innovations find their way into teaching practice. As teacher educators and researchers work to constantly improve teaching and learning and are eager to bring new scientific insights into practice, such attitudes are worrisome. This is especially the case as teachers are the ones who are supposed to engage students to evolve, learn new things, rethink old habits, and keep up with the times to meet future challenges.

The theory-practice gap in teacher education – complex reasons

The issue of the theory-practice gap is an ongoing one and has been discussed intensively in the research community. Reasons why it is especially challenging to bring about changes in the educational system are complex and multifaceted.

Teacher education programs might already cause some trouble in letting pre-service teachers link theory to practice and might not be structured in a way that counteracts eventually unfitting attitudes about the teaching profession. Most of the student teachers start teacher education programs a short time after they leave high school. This means they bring a vast amount of experience into the system they now want to work in and have strong preconceptions about what it means to be a teacher. Changing these preconceptions within teacher education seems to be very hard (Joram & Gabriele, 1998). Research shows that new teachers are often highly influenced by how they learned the subject matter themselves when they start teaching (Stofflett & Stoddart, 1994). This influence seems to go deep as even experienced teachers prefer to use teaching styles they were getting used to as students. Moreover, as they might think they already know what it means to be a teacher (as they have observed teachers more than half of their lives), many do not understand the need and the usefulness of the knowledge and competencies they acquire within teacher education. Their preconceptions influence the way they understand new knowledge (Korthagen, 2010).

Even if pre-service teachers' attitudes shift in the course of the teacher education program, it is not certain that innovations will find their way into the classroom. Different studies across different contexts describe that novice teachers experience a "practice or transition shock" after entering the teaching profession (e.g., Corcoran, 1981). As many do not feel well enough prepared and experience frustration within their beginning years, they are not using the theoretical knowledge and competencies acquired in teacher education. Many then fall back into what they experienced in their school carrier as students. Most new teachers tend to adjust their focus to rules and practices in school rather than on recent scientific insights. Especially young teachers experience the pressure to acquire the school's culture, following old patterns and standards that other teachers are using, which discourages modernization and innovation in teaching. Experienced teachers might pass on their attitude that theoretical knowledge and competencies acquired at university cannot be put into practice and is just something you needed to learn to get your degree. This "teacher socialization" often causes a shift away from initial ideals, as it might seem difficult for an individual to influence these existing patterns (Brouwer & Korthagen, 2005). Within this process, even highly motivated teachers with the will to change classroom practices towards more innovative and research-driven teaching practices might resign and comply with "how it has always been done". Resisting these dynamics and insisting on one’s ideals usually needs more individual effort. Moreover, it often goes hand in hand with resistance from different sides (colleagues, students, parents, principal) as schools often have hidden hierarchies keen on maintaining the self-created power structures. Consequently, efforts to change often encounter the “school system’s immune system” and are either rejected or absorbed and defused (Heintel & Krainz 1998). This already illustrates the complex interrelationships and the wide variety of socio-political influencing factors that affect a teacher and thus also his or her teaching.

Nevertheless, some teachers are still eager to design classroom practice in the best possible way for students, even if this sometimes requires creative ways to not adhere to all the guidelines or get into conflict with more traditionally set colleagues. These are the ones most willing to develop further to meet new demands and are therefore also the ones with the most potential to bring changes into the classroom.

Nowadays, mathematics lessons can be carried out in a variety of ways: they can be very traditional and exercise-oriented or very application-oriented, the focus can be more on individual performance or group work and discussion, the lessons can always be similar or, depending on the topic, always different – in short, teachers have many possibilities to design their mathematics lessons. If you ask mathematics teachers about the “ideal” mathematics lesson in which they think children can best learn mathematics, it becomes apparent that opinions differ widely. Based on the teachers’ answers, one can see whether this ideal teaching is implemented in reality or whether they only have an idea of it and do not carry it out for various reasons. The paper aims to present results of interviews carried out with middle and high school math teachers, who described their view of an ideal mathematics class. Looking more closely at their visions of mathematics teaching can also provide insights into their motivation to bring about changes to their teaching and the factors that might cause resistance.

Data collection and analysis

Twelve Austrian mathematics teachers who take part in the professional development course about implementing critical mathematics education approaches participated in the semi-structured interviews. Ten of them are teaching at a middle school, and two of them are teaching at a high school and a middle school (together, they form the entire mathematics teacher team of a middle school in Klagenfurt).[2] The teacher group ranged from participants with only one year of teaching experience to participants with over 30 years of experience. Five of the participants were male, leading to a nearly balanced gender distribution. The interviews were part of a larger study on how teachers deal with bringing critical mathematics education approaches into their classrooms. One of the questions focused on how the teacher would design mathematics classes to make them the best possible for their students to learn mathematics. Participants were asked to describe their vision of it (or what they would change compared to current practices to make it an ideal math class) without

[2] Pre-service teacher education differed between middle and high school teachers in Austria until some years ago. High school teachers needed to complete at least five years of studies at university level, while middle school teachers participated in a three-year program at the educational college. Therefore, theoretical background about teaching and learning mathematics might differ considerably between these two groups which might also influence their answers.

thinking about school guidelines or other regulations. Audio records of the whole interviews were transcribed and coded, letting themes emerge from the data. Within the process, codes across teachers' answers have been compared in order to find similarities and differences, which led to three different themes regarding the question of an ideal mathematics class.

Findings: The ideal mathematics class – three different approaches

Results indicated that there are three different approaches to answering this question. From the answers given, it can be concluded that depending on which approach the teachers tend to correspond to, they also show different potential to bring innovations into the classroom. In general, it was striking that many of the teachers needed some time to think about the question. Some articulated that they have never thought about it before. Therefore, they had some difficulties answering the question, and it was challenging for them not to think about school regulations. That already indicates that many teachers do not think beyond their usual image of teaching and are quick to comply with the school's guidelines or "hidden" agenda of their school without reflecting on it. As one teacher with 20 years of teaching experience replied after thinking about it for some time:

> Hm… Depends on the children. Maybe I would use more visual materials and maybe more time. I would really like to explain everything in more detail, but that is not possible in the lessons. Perhaps also more examples from real-life… I don't know. It would probably be good to have more math lessons per week… but that is not realistic anyway. We already have four math lessons a week, so from that point of view… I don't really know.

It becomes apparent that she has never really thought about it and also doesn't exactly know what to answer. In between, she keeps thinking and throws in new ideas, which, most of the time, she then discards because the regulations in the school system, in her opinion, don't allow it anyway. She does not elaborate on any of the ideas mentioned but simply lists different approaches. This suggests that she does not exactly know how she envisions an ideal mathematics class and that she might have never reflected about the socio-political agenda of her teaching as well. It appears that this teacher has adapted to the school's culture and teaches within this framework without feeling much need to change or evolve her mathematics lessons. Therefore, it might be assumed that she sees less need for regular in-service teacher training.

That might as well be the case if you as a teacher meet your own standards or ideals of teaching and school guidelines fit you well. A younger teacher (7 years teaching experience) responded very quickly and short: "I wouldn't change anything. It suits me just the way it is." He then explained that for his lessons, he often follows the two school books to structure his teaching, using the one with explanations for elaborating new content and the other one with examples for giving homework. That seems to work well for him and, in his opinion, also for his students. However, as school books often mainly focus on exercises and do not foster reflections about the use of mathematics, it might be assumed that students will mainly acquire operational and procedural skills in such a way of teaching mathematics.

These examples show that some teachers seem to always have had a clear image of what it means to teach mathematics and do not feel that changes or adaptations are needed. As a result, they likely are the ones with less motivation to participate in training courses, and when they do participate (which is obligatory for middle school teachers but not for high school teachers who started teaching before the school year 2019/20), they might make little use of the new knowledge provided in these courses. Therefore, the potential of this group of teachers to bring innovations and new scientific insights into the classroom is rather low. As these teachers do not reflect much about their teaching practice and are quite happy with their traditional form of teaching mathematics it might also be assumed that they will not easily see a necessity for using more critical approaches in their mathematics lessons and might not really be aware of a connection between mathematics and socio-political issues themselves. They might not change their attitudes towards professional development and the teaching and learning of mathematics unless they experience an event that causes them to do so (Pehkonen, 1994).

Other teachers struggle more with meeting their own ideals of teaching mathematics, have a clear idea of how it should look but are unsure how to realize it. They articulated their struggles between meeting the school systems' demands and their standards of teaching as an excerpt of the interview of a young high school teacher (3 years of experience) shows:

> In any case, [my ideal mathematics class would be] very application-oriented, where you can maybe also try things out... That you can touch certain things or you do projects or just always have such small fields of application […] Working in a more open framework, where you can also work across subjects and not be stuck to mathematical content only. But you are a little bit caught in this concept of school, and you need to prepare students for the [standardized] matriculation exam. Nothing has happened in this [more open] direction in the last years. Everything is already so stuck. But for me [the ideal mathematics class is structured so] that the teacher should act more as a coach than a preacher, who stands in front and presents everything. That would be the ideal case... (is thinking) … But there is again the question if and when you can implement that so easily... I am asking myself whether you can do so much as an individual.

It gets clear that she is not genuinely practicing what she would see as an ideal mathematics class, pointing to constricting structures in the school. Her struggles show how much influence the (hidden) structures within the school system and, above all, the school culture itself and the associated expectations placed upon her ultimately have on her teaching practice. She is obviously experiencing a notion of a discrepancy between her ideals developed during teacher education and the pressure of more traditional patterns in school. As Dann et al. (1978) indicated, these "discrepancy experiences" lead to a decline in using more innovative teaching practices with whom teachers got in contact in pre-service teacher education programs. As a result, they rely more on traditional teacher-centered instruction. Others, like Brouwer and Korthagen (2005), indicated that this might also cause these teachers to start to doubt (again) whether it is possible to put the theoretical and research-based knowledge from education programs into practice. Even if you as a teacher are ultimately the one responsible for what happens in your classroom, the guidelines of the principal, the attitudes and practices of your colleagues, the expectations of parents and students (which are often generated by a certain school culture) might ultimately strongly influence how you design your classes.

Even though these struggling teachers experience obstacles, they do show the motivation to change classroom practice. How much of these ideas will find their way to their students might depend on how deep their beliefs are settled and how much support they will get to realize them. This is especially important in the first years of teaching as the answer of a novice teacher, who only started to teach some months before the interview took place, shows:

> I would say that it [the ideal math class] should be a bit experimental, with group work, problem-solving, discussing how you find a solution, and where everyone should be able to benefit from each other. […] But I still have to get more into it, because you also need all the materials, you still have to get everything. But so far, my teaching hasn't been quite the way I imagine it, I was a bit more reserved in the first year, but it should go in that direction if everything is possible.

This young teacher has an idea of how she would like to design her math classes but hasn't really dared to carry that out in her first year, was still "reserved", and doesn't yet exactly know whether it's possible to realize it. This is probably because she first wants to get to know the habits of the school and her colleagues to see how far her ideals are from current practices, which is in line with research on teacher socialization (Brouwer & Korthagen, 2005). It gets clear that novice teachers orient their practices strongly on their colleagues and cannot easily realize their individual visions about teaching, which also underlines the power of the socio-political environment in maintaining the status quo. However, the motivation of this group of teachers to participate in professional development courses might be pretty high (especially at the beginning of their struggles), as these courses can be one way of supporting and strengthening their ideals. Moreover, they might get to know ways to realize them within their teaching and find others who have similar visions. Though, if there is no support from other teachers or from their institution to realize the ideals, these struggling teachers may adapt to prevailing rules and patterns and scale back their initial ideals after a few months or years. At the same time, their beliefs that theoretical knowledge and competencies acquired in teacher education can be useful in practice might fade, and their motivation to participate in professional development courses might decrease. This might be especially the case for teachers whose ideals were not yet firmly established but only began to change in the course of their teacher education.

Others with stronger convictions might find creative ways within the system to realize what they feel is the best possible way to teach mathematics or might otherwise even leave the profession when they are not able to do so.

> I arrange my mathematics lessons within the school system in a way that I think the students can take away the most – otherwise, I simply couldn't work there. […] For example, vocational orientation is, in my opinion, very important, and that also happens in my math classes, just like in other subjects. References to everyday life are always included […]. Because just going in [the classroom] and simply calculating examples, then it's also boring at some point, and then you [as a teacher] are just happy when vacations begin again. And I don't want to be part of that [group of teachers], and I only want to do the job as long as I really like doing it!

For this teacher with about ten years of teaching experience, it seemed natural to always arrange mathematics lessons in a way that he felt would benefit students the most. He includes content that is not demanded by the mathematics curriculum if students are interested in it (e.g., cryptocurrency), setting a focus on building relationships to real-life situations. He clearly distinguishes himself from the group of teachers who, in his opinion, equate mathematics lessons with calculating examples without really thinking about what students are interested in. Moreover, he articulated that he is keen on constantly developing his teaching further and that he would find it a good idea to observe math classes of his colleagues to learn from each other.

It is apparent that these teachers are more eager to participate in professional development courses and are the most likely to bring new ideas and innovations into the classroom. They seem to manage to find ways to deal with organizational struggles as well. As their teaching ideals are strong, they are willing to put more effort into designing lessons that meet those ideals. But even if they are willing to work more on an individual basis, most of them will need support in the school community after some time. If this type of teacher finds that, despite great effort, they cannot make their ideas a reality and nothing in their environment is improving, they may even leave the profession. Therefore, the ones with the greatest innovation potential are also the ones who resign first when they realize that nothing is changing and that they, as individuals, cannot do much about it.

Discussion

In education, there is still a large gap between theory and practice, and as the distribution of types of teachers in this small sample shows, a majority of teachers are within the first two types – so they are either not seeing any reason for changes or are struggling to implement them. Only two of the twelve teachers participating in the study could be categorized into the third, more idealistic type. Even though the three approaches were derived from a small sample of 12 teacher interviews, it can be assumed that most teachers can identify themselves with one of these categories. Besides, these cannot be considered static approaches; instead, teachers can find themselves in several of these within their professional life.

Moreover, it seems that there is a noticeable difference between middle and high school teachers when it comes to openness for development work. The initial excerpts in the introduction came from experiences with high school teachers with whom it was first planned to carry out the professional development course on critical mathematics education. However, as they showed no willingness to participate in any professional development course and even expressed a discouraging attitude towards anyone not directly anchored in the school system, a new school had to be found for the research project. From the beginning, middle school teachers were more open-minded towards the professional development course and were immediately interested in what it will be about in more detail. Even though the interviews show that also among middle school teachers, some do not offer much potential for change, these teachers still get involved in in-service training on new topics voluntarily. The fact that there are such noticeable differences in this respect between these types of teachers might be attributed to a wide variety of reasons: Teacher education has been structured differently and anchored at different institutions until two years ago in Austria. Regulations concerning professional duties differed as high school teachers had no obligation to undergo further training after graduation. This might have led to an image that this is not part of their job but more something like a hobby for the over-idealistic. In contrast, middle school teachers are accustomed to investing at least 15 hours per year for this purpose (which is still no guarantee for developmental work as you only need to be physically present to get your certificate). Moreover, the student clientele differs greatly in terms of social background, which makes collegial cooperation in middle schools all the more necessary to cope with possible challenges. The list can certainly be continued. Still, it does already show that socio-political guidelines and school conventions greatly influence the success of professional development initiatives and how much innovative, science-based knowledge actually ends up in the classroom. However, as these described differences between high school and middle school teachers derived from the data and experiences with a rather small group of teachers of only two different schools, generalization of findings can only be done cautiously, and there might be different results in other areas and contexts.

To narrow the gap between theory and practice in the future, it seems particularly important to focus on the struggling teachers and strengthen those who have ideas but do not put them into practice for various reasons. Much potential for innovation will remain unused if these teachers are not supported in ways that allow them to realize what they learned is best for their students. Since the reasons for the theory-practice gap are manifold, there will not be one single way to solve the issue. Rather, interlocking initiatives at different levels could help move in the right direction and, above all, support those open to development and innovation. After all, it became clear that the extent to which innovations find their way into the classroom thus depends on a wide variety of interlinking governmental, political, cultural and social structures. The individual teachers' vision alone will (most of the time) not be enough for getting new research-based insights into practice. Decisions on how teacher education programs are structured, how cooperation between university and school institutions is promoted, how traditional the school structures are, and the legal framework conditions for teachers can strongly influence how much of the vision is carried out in reality.

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The construction of low attainment in mathematics – why are primary school children selected for intervention programs? Results from a meta-analysis of case studies

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This paper presents a study based on a meta-analysis of 43 case studies on primary school children participating in an extracurricular support program for low attaining children named MaKosi ('Mathematische Kompetenzen sichern'). Whereas this program particularly aims at identifying low attaining children and supporting their arithmetic skills, this paper investigates social factors that can be identified as shared and comprehensive to this group of children attending the support program. As part of the case studies, interviews were conducted with the children, their parents, and (mathematics) teachers in order to examine factors that may lead to or influence the assumed low attainment in mathematics. The results indicate an interplay between individual, social, and school-related factors, such as gender, self-concept, and the quality of classroom management.

Keywords: Low attaining children, case studies, social factors.

Introduction

Teachers, parents, and students typically assume that there are children who easily learn mathematics – and that there are those who do not. In psychological approaches learning difficulties are described as a consequence of dyscalculia or individual learning disorders that need to be 'cured' or, at least, need intervention (Kuhn, 2015). Some mathematics educational researchers conceptualise dyscalculia or learning disorders as the result of inappropriate teaching and individual difficulties and, therefore, call for improvement of assessment and the quality of teaching. In addition, training or intervention programs, e.g., in universities have been implemented (in Germany at the universities of Münster, Wuppertal, Bielefeld, and others) linking the identification and support of children, the training of student teachers, and research on learning difficulties. However, in this paper we follow Uwe Gellert's assumption that the characterisation of a group says more about the characteriser than about the characterised (Gellert, 2013). The phenomenon of low-attaining children in this understanding is not a natural or biological description of reality, but constructions that emerge in social and political contexts (ibid.). Low attainment can be problematised as a consequence of inequality. Consequently, socio-political perspectives on mathematics education focus on the contexts of the construction problems, e.g., in classroom interactions (Gellert, 2008; Heyd-Metzuyanim, 2013), as a matter of identity (Andersson et al., 2015) or (auto-)exclusion (Kollosche, 2019). A research desideratum addresses the question of how other agents involved such as teachers and parents come to the conclusion that there are learning difficulties that cannot be solved in the standard lessons. Complementing these studies, we ask, *which social factors can be identified regarding primary school children's attendance at an* *intervention program for low-attaining children.* We would like to change the perspective by not examining what these learners need or what their (lacking) competences are, but why they are perceived as children who have difficulties and, therefore, are selected for intervention and in what way social factors affect this selection. To answer these questions, we draw on data of 43 case studies constructed by student teachers about children who are characterised as low attaining learners especially in arithmetic and attend an intervention program. First, we give a brief overview of the theoretical frameworks with respect to explanations for low attainment in mathematics and its construction in classroom interactions. After that, the study's methodology is outlined and justified against the theoretical frameworks. Finally, the results are presented and discussed.

Theoretical frameworks

As there are many definitions, classifications, and understandings of having problems in mathematics such as a (mathematical) learning disability, special needs in mathematics, mathematical learning difficulties, or dyscalculia (Scherer et al., 2016), in this paper we use the term low attainment with focus on arithmetic contexts. The term low attainment means that children do not show the teacher's expectations of mathematical performance. It involves individual aspects of learning as well as social and cultural factors and, thus, can be seen as a complementing perspective on other (psychological) classifications (for an overview see Fritz et al., 2019; for a social-political perspective see Straehler-Pohl et al., 2017).

Low attainment in mathematics (LAM) between school, family, and the child

From a mathematics education viewpoint (unlike a psychological one) low attainment is conceptualised as a psycho-social interplay. Gaidoschik (2017) states that there are no actual evidence-based *causes* of LAM but numerous factors that make LAM more likely. These factors can be seen in the realms of school (lack of assessment, teaching mistakes, discontinuity, etc.), family (lack of support, anxiety, drill, etc.), or the child itself (self-concept, motivation, etc.). Benölken (2016) developed a model of LAM that connects inter- and intrapersonal 'risk factors'. Whereas family, peers, and the context of school are *inter*personal factors, there are further possible *intra*personal risk factors such as difficulties in concentrating, unfavourable mathematics-related self-concepts, or self-efficacy expectations, often as a result of multiple experiences of failure. Psychosocial approaches emphasise that such risk factors can exist not only within the individual, i.e., due to intrapersonal determinants or processes but also outside the individual, i.e., in their social environment: Family conditions are unfavourable, for example, due to an environment that is poor in stimulation and experience with regard to points of contact with mathematical content. Until now, there is a lack of empirical research on these risk factors and how they are connected to learner's development of LAM, but one can find hints in studies that research learner's identities or give reasons for (auto-)exclusion from mathematics education. Among others, Andersson et al. (2015) find out, that the identity of being a 'math hater' and corresponding disengagement in class can change over time. They state, that the contexts of task, situation, school organisation, and the socio-political context matter and that identity narratives change in relation to available contexts. Kollosche (2019) asks why learners reject mathematics and describes that auto-exclusion is motivated by the organisation of mathematics education and closely linked to the subject of mathematics itself and, therefore, also represents a didactical problem.

In all these considerations of what may cause LAM, in these approaches it is clear that there is an unsatisfactory performance in mathematics and that this must lead to support (Scherer et al., 2016). Other approaches, instead, question whether this shown performance corresponds to ‘reality’ at all or if it is a result of interactional (co-)constructions.

Construction of LAM in classroom interactions

Uwe Gellert and colleagues use Basil Bernstein’s theoretical frameworks of pedagogic codes and their modalities of practice:

> When the mathematics teacher poses a problem, students need to respond in a manner that is seen as appropriate. They must be able to recognise that particular responses are expected, and they must be able to produce a desired response. (Gellert, 2008, p. 218)

These abilities are distributed unevenly with respect to the different socio-economic backgrounds of the children. Gellert shows, that achievement is not primarily based on childrens’ mathematical abilities, but on their differential rule recognition responses (Gellert & Straehler-Pohl, 2011). In a case study, Heyd-Metzuyanim (2013) shows that ‘learning disability’ is not an individual characteristic, but an interactional co-construction between teacher and learner, relying on the interplay of following rules and routines in the classroom. In this case, the learner responded to the rules of participation, but the mathematical content was mostly inappropriate so that she could not negotiate mathematical meaning. She had no choice but following routines without understanding. “The implications of these findings lie in highlighting the necessity of taking into account the social and affective, as well as the cognitive, aspects of learning difficulties in mathematics.” (Heyd-Metzuyanim, 2013, p. 362).

These studies reconstruct the emergence of disparity while participating in classroom interactions. In contrast, Straehler-Pohl and Pais (2013) reconstruct mathematics educational failure as a consequence of very low academic expectations and, therefore, provoke learner’s resistance. As non-participation is no legitimate option for children in everyday lessons, it leads to exclusion. Within these perspectives, LAM appear to be an interactional construction caused by practices and routines in school. In addition, however, this is criticised as the reasons for inequality can also lie outside the school context.

Researching the connection between social factors and the construction of LAM

Gutiérrez (2012) reminds us that “[…] learning is intricately connected to the contexts in which it occurs” (p. 18). She argues that there is a need to reclaim space for studies that focus on learning in context. Researchers must consider the complexity of the phenomenon. Pais criticises that “[a]ll the complexity of the social and political life of the student is wiped out of the research focus. The student is reduced to a biological entity, likely to be investigated in a clinical way.” (Pais, 2012, p. 53). Though we do not agree with his overall assumptions, we emphasise taking social factors and contexts of learning into account. Our theoretical assumption is that performance in class is not the only reason for being assigned as low attaining but it is also a result of social constructions. Yet, the process of these constructions is not alone in the hands of teachers. We assume that there is an interplay between the teacher's and the learner's views, between the quality of teaching, and the learner's situation. We seek to explore these complex interplays.

Methodology: Data collection and analysis

This exploratory study is based on a meta-analysis of 43 case studies on primary school children who participated in a support program for low attaining children named *MaKosi* ('Mathematische Kompetenzen sichern' – 'ensure mathematical competences', translated from German). The long-term project *MaKosi* was conducted from 2014 to 2018 under the supervision of Ralf Benölken at the University of Münster (Germany) in cooperation with a primary school (Benölken, 2016, 2017). It was organised as a 'learning-teaching-laboratory', i.e., a project seminar that links student teachers' theoretical and practical education by working with children. It mainly aims at developing student teacher's professional competences and supporting children characterised as low attaining in mathematics. In each semester the program took place in the afternoon at the primary school once a week over a period of four months. Each 90-minute-session was divided into three parts: at the beginning and at the end playful problem tasks respectively games were offered to support children's self-perception and joy of engaging in mathematics. The main part of the sessions was the 60-minute diagnostic and support unit in which one student teacher and one child worked together in one-to-one-interactions in established teams. During these sessions, the children worked on various tasks and the student teachers noted down the children's ways of thinking as well as aspects, that stood out to them and that they considered particularly important and relevant, in an observation log. Teachers and parents decided on the children's participation in the program: First, teachers were given information about the program and the theoretical framework. They elected children providing a written rationale. Then, parents were asked to fill in a consent form. The data we refer to in this study is drawn from the individual case studies that student teachers produced following the project *MaKosi* as part of bachelor's (in total 18) and master's (in total 25) theses on the children they worked with in the project over the full period. The case study approach provides a profound, multi-faceted appreciation of an issue, which is intended to paint a holistic and realistic picture of the social world (Lamnek, 2010). In case studies, triangulation of different methods, e.g., participant observation or (guided) interviews, is often used in order to capture all significant dimensions and facets of an issue and to be able to gain a more precise insight into how the diverse factors interact (ibid.). The case studies were primarily aimed at reconstructing the child's difficulties as well as risk factors that could promote these difficulties. In addition to the above-mentioned observation log on the children's way of thinking and their task completion, the students also used guided interviews with the children, their parents, and the mathematics teachers as data collection instruments. From these data, the students worked out how the child's development (including physical and academic development) progressed, how his LAM manifested itself, and what aspects they perceived as risk factors for LAM.

As stated in the introduction, we would like to change the perspective by not examining what these learners need or what their (lacking) competences are, but why they are perceived as learners who are low attaining. In other words, our aim is to reconstruct factors, especially social ones, that lead to these children being characterised as low attaining. Therefore, we conducted a meta-analysis that includes 43 case studies written as part of the project *MaKosi*. Using qualitative content analysis (Mayring, 2015), the risk factors described in the case studies were first coded using three categories deductively derived from theory: *child*, *school*, *social environment* (Benölken, 2016; Gaidoschik, 2017). Subsequently, five subcategories were formed from the data in conjunction with the theoretical models of Benölken (2016) and Gaidoschik (2017) for each of the three superordinate categories: (1) The factors relating to the category *child* included the subcategories *work habits* (e.g., lack of independence, low perseverance or difficulties in concentrating), *developmental factors* (e.g., difficulties in motor skills, perception or language), *affective-motivational characteristics* (e.g., unfavourable mathematics-related self-concept or negative attitude towards the subject of mathematics), *general school-related insecurity* (e.g., discomfort, feelings of inferiority, or emotional reactions), and *relevance* (especially not recognising the importance of the subject for life). (2) For the category *social environment* the five subcategories *lack of stimulation* (e.g., an environment that is poor in stimulation and experiences for dealing with mathematical topics), *learning environment at home* (e.g., lack of support, difficult and unsettled family circumstances), *negative role models* (family members who have also been characterised as mathematically low-attaining or have unfavourable mathematics-related self-concepts), *lack of participation* (and of interest of parents in school-related matters), and *emotional stress* (e.g., due to divorce of parents or pressure) were formed. (3) The subcategories *discontinuity* (e.g., class repetition or frequent change of teachers), *relationship* (especially a negative relationship between child and mathematics teacher), *situation of the class* (e.g., restless classes with many learners or a negative atmosphere in the class), *classroom management* (quality of teaching, individual promotion) and *cooperation* (e.g., between teachers and parents or of different professions) are subsumed under the category *school*.

For each child, it could now be noted whether each factor was perceived as a relevant factor in the context of the case study. It was only asked whether the respective factor played a role and not how strongly it was perceived, i.e., no weightings were applied.

Results

Across the 43 case studies, the comparison of the categories showed that the subcategories belonging to the category *child* were most often perceived as risk factors for the children's LAM. The factors *work habits* (in total 36 times), *developmental factors* (32 times), and *affective-motivational characteristics* (28 times) were each described in a majority of the case studies. *Emotional stress* (20 times) and *lack of stimulation* (17 times) were the most frequently perceived factors in the category *social environment*, and in the category *school*, these were the two subcategories *classroom management* (26 times) and *discontinuity* (21 times). *Relevance* (7 times) and *cooperation* (4 times), on the other hand, were rarely assessed as relevant factors. When comparing the case studies with each other, the wide range of combinations of the fifteen subcategories perceived as risk factors is striking. For example, while only three relevant factors were identified for one child (all can be assigned to the category *child*), there are children for whom up to ten different factors were observed. Furthermore, the distribution of the perceived factors in the three areas of *child*, *school*, and *social environment* varies depending on the case study. In this respect, Figure 1 shows a possible typification of the individual cases in which, depending on the focus of the factors described, they were either primarily assigned to one of the three categories or located at the interface of two or all three categories.

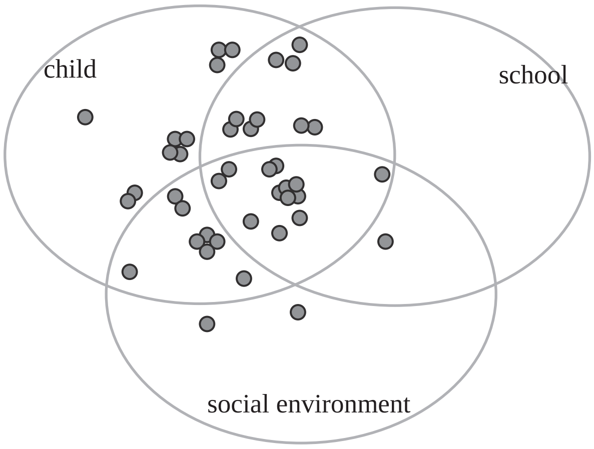

Figure 1: All cases

As can be seen in Figure 1, most of the individual cases are located in the centre or the upper-left area of the mapping. The focus of reconstructed factors for most of the children was, thus, on intrapersonal factors, i.e., category *child*, or at the interface of this category with one or both of the other categories. Furthermore, it can be seen that for none of the children in our case studies a clear focus on *school* factors was described. Nevertheless, as shown above, *classroom management* was one of the most frequently mentioned factors. The meta-analysis also showed that a large proportion of the individual cases considered (a total of 30 out of 43, i.e., almost ¾) were female. If we now look at the focal points of the factors described in the case studies for girls and boys separately, as shown in Figure 2, we can see that the individual case studies of boys are quite scattered.

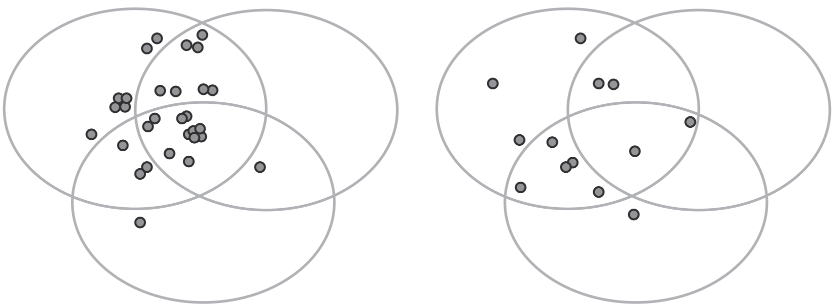

Figure 2: Cases split by gender (on the left: girls; on the right: boys)

Rather often, however, they include types that combine factors that can be assigned to the area of the *social environment*, but have fewer *school*-related factors. Many girls, on the other hand, show a combination of factors belonging to the categories *child* and *school*, whereby an accumulation of affective-motivational characteristics is particularly noticeable. Thus, 23 girls were perceived to have an unfavourable mathematics-related self-concept. In summary, it can be stated that a combination of different factors from the three categories seems to be prevalent in many children, whereby in the case studies the students mainly reconstructed factors regarding the category *child*. In the other two categories, individual factors were dominant, for example classroom management or emotional stress.

Discussion

The results confirm that there is a complex interplay between perceptions of LAM. Especially girls with a negative self-concept and unfavourable motivational factors were selected. Two points seem important to us: first, inequality with respect to gender is a well-known and unsolved problem in Germany. Cultural factors seem to play a role as there are other countries where these differences do not exist to the same extend. Second, the question of individual factors corresponds to the assumption of Heyd-Metzuyanim (2013, p. 363): "The problem lies in the permanence of the disability title and the apparent disregard for the social and affective processes that may be (at least partially) responsible for its development in the first place." The girls are labelled as having problems and see themselves similarly, which, again, has consequences for their performance. In addition, the children's education is characterised by discontinuities. Many of them repeated or were repeating a school year. However, Gellert's (2008, 2013) results show precisely that being perceived as high- or low-performing is related to following rules of discourse. If children do not have access to these rules, repeating a school year does not change that. Benölken (2016) and Gaidoschik (2017) focus on risk factors that lead to LAM. We also found a striking relation between classroom management, emotional stress, and school-related insecurity in general. In addition to Gaidoschik's and Benölken's views we argue that these social factors may indeed have an impact, but especially they make a categorisation of children more likely. In our case studies we found a low attaining girl who scored high achievements in other mathematical assessments conducted by student teachers. All in all, we identify several aspects that seem to influence teachers' and parents' decisions for selecting children – at least gender and self-concept play a role – that appear in mathematics lessons as performance or attainment. These social factors seem to be 'hidden' under the construct of low attainment. At the same time, we recognise a low quality of mathematics lessons and teaching, discontinuities in didactics, and an unfavourable learning atmosphere. None of this will change by learners attaining intervention programs because the lessons and school contexts themselves are problematic. When it comes to supporting learners, it seems adequate to focus not only on mathematical, but broader facets like motivation and self-concept (which is indeed intended in *MaKosi*). Surprisingly, we could not find any relation to socio-economic factors. This leads to the limitations of our study. As the program was conducted at *one* school in a relatively privileged region, there were few economically disadvantaged children. We see our study as work in progress; more children with various backgrounds and at different schools need to be included.

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How the structural crisis in the labour market impacts the quality and identity of mathematics teacher students

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This paper analyses the impact of two structural context factors on mathematics teacher students. First, the Netherlands is coping with a massive mathematics teacher shortage. Second, the Dutch knowledge-economy feeds the private tutoring sector. The impact on young teacher-students is tremendous; they start working as a teacher too early. Besides successful studying, broader professionalization and quality of mathematics education are in jeopardy. A quick-fix for mathematics education might do more damage than foreseen.

Keywords: Teacher students, teacher shortage, shadow education, identity, quality of education.

Introduction

This paper contains an overview of the current situation in the Netherlands when it comes to mathematics education and mathematics teacher training. Many teacher training students start working as an unqualified teacher before they finish their teacher training. The shortage of mathematics teachers and private forms of education lure teacher students into the work field earlier than preferred. Research on this situation seems to be primarily focused on pupils and teachers and not on teacher training students.

Quality of mathematics (teacher) education

There is no single definition of quality in higher education. Harvey and Green (1993, as cited in Onderwijsraad, 2015) distinguish five different definitions of quality. One of them is quality as transformation. It stands for a mutual process between provider (teacher training) and participant (teacher student). This process adds value to the participants and empowers the participant to influence their own transformation into a professional. The educational work field can be seen as a connection between teacher student and teacher training through traineeship, in alignment with the intentions of both teacher training and teacher student. All from the perspective of safe and solid development of professional mathematics teachers.

Biesta (2020a) states quality is not the right word to use in relation to teacher training, because who is against quality? Quality stands for measuring and scoring, which fits the tenor of the knowledge economy in which scoring as high as possible is the most important goal. Commissie Beleidsevaluatie Lerarenopleidingen (2013) emphasizes it is crucial to keep a broad perspective on the development of teacher students. Thinking in competences in teacher education may keep the focus on the technical aspects of professionalization and may lead to a 'training' model instead of a broader 'forming' model. Competencies are important, but should be seen in a broader normative professionality. This 'forming' model and the model of quality of transformation come near to

Biesta's (2020b) vision about education. Education should not be only about knowledge and competencies employed from the perspective of the current social order. The pedagogical mission of education contains socialization; developing identity and subjectification; being invited to be a subject in and with the world, being a 'self', all in interaction with and interruption by 'the other'.

If mathematics education should involve more than a focus on competencies and qualification, it is important to give mathematics teacher students the opportunity to experience mathematics, mathematics education and the role of mathematics (education) in society in as much depth as possible. The Dutch Knowledge Base is used by universities of applied science to guarantee quality in mathematics teacher training and can be found on the website 10voordeleraar.nl. It is composed by mathematics teachers and teacher trainers and describes the minimal knowledge an incipient teacher should have about mathematics. It states that besides mathematical knowledge (being able to do the math), a teacher should develop general mathematical skills. This means for example being able to use ICT, specify how mathematics is used in other school subjects and 'indicate the impact of society on the development of mathematics'. A broader perspective would be the addition 'how can mathematics be used in other subjects and how does or can mathematics have impact on developments in society'.

In 1980s and 1990s a reform of the Dutch secondary mathematics education curriculum was initiated and implemented. The new Realistic Mathematics Education curriculum (RME) had the aim of making mathematics meaningful for everyone, being connected to reality and being relevant to society (Hoogland, 2020). Students should be supported by constructing their own knowledge and developing mathematical insights (Gravemeijer et al., 2016). In the 1980s the market was led by mathematics textbooks inspired by RME. The tendency of teachers and textbooks to think of instruction in terms of individual tasks and their focus on procedures that generate answers quickly are reasons RME-goals weren't reached (Gravemeijer et al., 2016). Besides that, Dutch mathematics teachers have the tendency to depend on their textbooks (Daemen et. al, 2020). A new curriculum for secondary education is in development. It seems mathematics education might be stuck in a focus on qualification for now, despite the original intentions of RME.

Mathematics teacher education

The Amsterdam University of Applied Science (AUAS) houses the bachelor-study for becoming a mathematics teacher for junior secondary school (age 12-16). The teacher student population is diverse. There are students from urban and rural areas, students with a bi-cultural background, students with a refugee-background, first-generation-students and students who enter from a vocational school or enter from another university. If a teacher-student is successful, the student manages to finish within 4 or 5 years. Unfortunately, not all teacher students are successful. To illustrate this: from the 63 teacher students starting in 2018, only 43% passed the first year course after 2 years of studying. Only half of the 30 full-time teacher-students who started in 2015 with the main phase of their study, a 3 year program, graduated 4 years later. Student teachers who take more than 5 years to study are called 'slow students' (langstudeerders).

Different obstacles cause students in general to need more time to finish their study. Elffers a.o. (2018) distinguish formal obstacles and informal obstacles. A formal obstacle could be not passing exams and informal obstacles are for example being a first generation student or studying with a disability. Student-engagement and thorough guidance of students are important factors for study-success, especially in the first year of studying (Diepen & Elffers, 2019). Although improvement is always possible, pre-teacher-education at AUAS is investing in student engagement. The question arises if there might be other reasons why a substantial number of students are dealing with a substantial study delay. At this point it is time to include factors relating to shortage in the teacher labor market and private tutoring.

Teacher shortage: unqualified teachers, freelancers and growing inequality

The Netherlands has a shortage of qualified mathematics-teachers. A quick search at ‘www.meesterbaan.nl’, a national vacancy-website for education, shows 85 vacancies for mathematics teachers for secondary education (learners age 12-18) at the beginning of the school year 2021/2022. Assuming these are vacancies at different secondary schools this means 4,7% of all secondary schools in the Netherlands are searching for a mathematics-teacher. Schools are allowed by government to hire unqualified teachers, like guest teachers, teacher students or teachers in another subject to fill the gaps. It is one of the reasons why it is hard to define the exact shortage of qualified teachers (Adriaens et al., 2017). Still the education inspectorate states that in 2016 mathematics was the school subject with the highest number of unqualified teachers, namely 7,4%.

The question of supply and demand on the teacher labour market causes a growing number of teachers to start freelancing for financial reasons and to avoid workload. The Chamber of Commerce claims that the number of freelance teachers in general almost tripled in the past seven years. Freelance teachers are more expensive for schools to hire and freelancers have the opportunity to make demands. For example one can come to an agreement not to attend meetings or seminars at school, do surveillances during examinations or start extra educational projects. This undermines schools being a community or a system in which teachers are caring for the development of their students and each other not only during class, but also outside classrooms. Or put another way, it will reduce schools to institutions for qualification, with less opportunities for socialisation and subjectification.

The number of vacancies is concentrated in the western part of the Netherland. This is the densely populated area of the Netherlands and has more urban regions. Figure 1 shows the expected teacher-shortage in the Netherlands in 2025 compared to 2019. Mathematics (wiskunde) has by far the worst prognoses. De western areas (NH and ZH) will have a shortage of respectively 57 and 73 FTEs. A tight teacher labour market increases turnover in highly urban areas and schools with disadvantaged children. In urban areas there are less young teachers, which could indicate it is difficult to find replacements for leaving teachers (Dijkslag, 2019). When it comes to shortages in urban areas: teaching in a superdiverse context requires extra skills. Teaching in general has the characteristic of being a lonely job and requires full responsibility from day one (Snoek, 2016). The complexity of teaching in a superdiverse environment causes starting teachers to choose 'less complicated' schools (Dijkslag, 2019; Gaikhorst et al., 2019).

The shortage of qualified teachers in these areas has a negative effect on equality of opportunities for students (Elffers, 2019a). The education inspectorate states on their website that students in areas coping with a shortage of teachers receive their lessons more often from non-graduated teachers. Besides that, they have more teachers in one school year than schools in other areas. A teacher shortage enhances inequality of opportunity, stimulates a group of teachers to start freelancing and the number of vacancies for mathematics teachers rises.

	GR	FR	DR	OV	FL	GD	UT	NH	ZH	ZL	NB	LB
Nederlands	2	2	1	2	1	5	4	9	13	1	4	2
Duits	10	7	4	12	6	27	19	37	53	3	21	10
Engels	0	0	0	1	1	2	1	2	3	0	2	1
Frans	6	6	3	10	4	20	14	35	46	3	22	9
Biologie	0	0	0	1	0	1	1	2	2	0	1	1
Informatica	2	2	1	5	2	8	7	20	19	2	11	5
Natuurkunde	6	5	3	8	5	15	11	25	31	3	16	6
Scheikunde	6	4	3	6	4	12	10	23	33	3	12	5
Wiskunde	13	8	5	14	11	27	25	57	73	6	25	11
Aardrijkskunde	0	0	0	1	0	1	1	2	2	0	1	1
Economie	0	0	0	0	1	2	1	1	3	0	1	1
Geschiedenis	0	0	0	0	0	1	1	1	2	0	1	1
Levensbeschouwing	0	0	0	1	0	2	1	1	2	0	1	0
Maatschappijleer	0	0	0	0	0	1	1	1	2	0	1	1
Techniek	0	0	0	0	0	1	0	1	1	0	1	0
CKV, Kunstvakken	0	0	0	1	0	1	1	2	3	0	1	1
Gezondheidszorg en Welzijn	0	0	0	0	0	1	1	1	2	0	1	0
Klassieke Talen	3	1	1	2	2	6	5	13	19	1	8	5
Lichamelijke Opvoeding	0	0	0	0	0	1	1	1	2	0	1	1
Overige Vakken	0	0	0	1	0	2	1	1	2	0	1	1
Praktijkonderwijs	0	0	0	0	0	1	1	1	1	0	0	0
Totaal	51	38	22	66	40	136	107	238	314	26	133	61

Figure 1: Extra vacancies teacher per subject and province in 2025 vs. 2019 (Adriaens et al., 2017)

Shadow education: inequality of opportunity

The Netherlands has a knowledge economy, which brings focus on qualification. Parents put their effort into ensuring their children are as highly educated as possible. This has contributed to the rise of shadow education in the Netherlands. The term shadow education stands for privately funded extra educational activities for learners after school with the intention to improve performance at school (Elffers & Jansen, 2019). Examples are homework guidance, extra private lessons in school subjects, training for exams and summer schools. Mathematics, being an obliged subject in most routes in Dutch secondary schools preparing for university and university of applied science, has been doubtless a profitable school-subject for shadow-education in the Netherlands. Figure 2 shows the spending on shadow education increased eightfold in almost two decades (Elffers, 2019a). Shadow education can cause inequality of opportunity. Not all parents are financially able to pay for private education after school. Teachers might spend less time to give extra guidance to their students and refer to extra lessons (Elffers & Jansen, 2019).

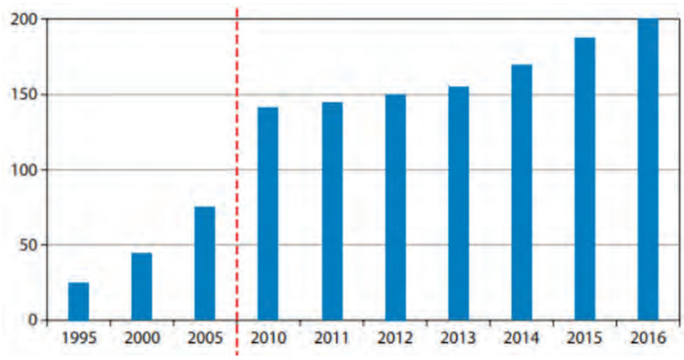

Figure 2: Spending on shadow education in the Netherlands (Elffers, 2019a)

After corona: public and private forms of education can start dating again

Because of the corona-pandemic and its negative influence on education, the Dutch government released 8 billion euros to support national education. This ‘Nationaal Programma Onderwijs’ gives secondary schools two years of funding to help learners get rid of disadvantages due to the corona-crisis. Money can be spent on interventions that are proven to be effective. This can be for example: extra lessons, extra teachers, teacher support, summer programs and weekend schooling. This support plan is the government’s open invitation for public and private forms of education to start to cooperate even more. Shadow education entering public schools is not a totally new phenomenon (Bisschop & de Geus, 2017). The market for shadow-education has been growing due to the Dutch knowledge-economy. (Elffers, 2019a). It seems, when it comes to private education in public schools, in most cases even before the corona pandemic, schools bore the costs or parents were asked for compensation. (De Geus, W. & Bisschop, P. 2017) Both Elffers (2019) and De Geus and Bisschop (2017) conclude more research has to be done on the effects of shadow education on teachers and pupils. An important group of participants is missing in this conclusion, because who are filling the gaps?

Mathematics teachers education students fill the gaps

Since government allows non-qualified teachers to teach at schools in combination with the current teacher shortage, teacher training is coping with schools offering teacher students a job or asking them to do extra educational activities, even before they have finished their teacher-training. This phenomenon is called green picking (groenpluk) (Diepen & Reumerman, 2018).

The AOb, a Dutch teachers union, conducted research with 606 teachers younger than 35 years old by bureau Investico with the topic ‘starting teachers’. The article indicates that almost 60% of the respondents had a job as a teacher before graduation.

Reasons for students to accept a teaching job include: financial incentives, being ‘honored’ and having a ‘good connection with the school’. Not all schools provide proper guidance for starting teachers. About 30 % of the teachers quit after 5 years, because of workload and burn-outs. (Pol & Tunali, 2021).

A short inventory by the author about shadow education and green picking among 24 teacher-students in their first year of study (2020-2021) is shown in Table 1. 12 out of 24 students state they have been active in some form of shadow education. 20 students have been active in some form of private education, which could be, for example, helping a neighbor’s pupil for a fee.

Table 1: Results inventory on activities in education by first year students

Question	Number of students	Percentage
Have you been working as a mathematics teacher besides your traineeship?	7	29 %
Have you been offered a job as a mathematics teacher?	6	25%

Did you receive more tasks besides your internship?	6	25%
Did you give exam training through a tutoring company?	8	33%

So while public education and private forms of education compete and cooperate for employees in the teacher labor market, mathematics teacher education students are filling gaps and put effort in to getting students to access the highest levels in the knowledge economy.

Possible consequences

More research on this topic needs to be done, but still possible consequences can be mentioned. Of course a positive effect might be that teacher education students gain more experience and have context to connect information and activities from their teacher training with their daily practice. However, the way in which the Dutch mathematics curriculum for secondary education has been handled over the past 30 years does not show an inspiring example for a rich mathematics education. Early adoption by the educational field and early development of identity as a mathematics teacher might make it more difficult to stay open to other ideas and concepts. Because teacher students are busy working they will have less time to be creative and innovative and they will have little time to reflect on their experiences being a teacher. The importance of reflection and personal professional development might shift to the background, already being 'on board'. Even more, hiring teacher students as an unqualified teacher might make them think they are ready to be a teacher. Like a student stated in van Diepen en Reumerman's research (2018, pp. 19): *'Thirty hours a week for four years: what is that about? My trainee school says I can be a teacher by the end of this year.'* Just like their employer they will be focused on fast qualification. This might endorse the mindset of 'finally finishing the training' instead of the concept of a lifelong personal and professional development. A generation of teachers may stay underdeveloped while being absorbed by the field.

Conclusion: narrow teacher training, narrow mathematics education

Mathematics teacher shortage and shadow education impact the broader development and professionalization of mathematics teacher students. It doesn't matter if students are 'slow' or 'fast'. Either way they might have the tendency to be focused on their own quick qualification. Meanwhile, they already serve the knowledge economy without room for creative input or any healthy criticism. Since the professional development of teacher students seems to narrow because of this structural crisis on the mathematics teacher labor market, mathematics education itself will stay narrow. This is an alarming situation. Mathematics teachers should not be able to just 'indicate the impact of society on the development of mathematics'. In current times mathematics education should take part in addressing challenges like climate change, data-abuse or inequality of opportunity and contribute actively to personification and subjectification, a quick-fix for mathematics education is creating the opposite.

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Conceptualising and operationalising socio-mathematical agency

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In this paper, I introduce a new theoretical construct of 'socio-mathematical agency' (SMA), which I define as the ability to use mathematics effectively to argue collectively for social change. I present a conceptualisation of SMA which embraces the need to generate powerful mathematical knowledge, and which draws on critical mathematics education in foregrounding the need to attend to learners' individual and collective agency. I propose that developing SMA in learning mathematics can make a significant contribution towards cultivating the collective knowledge and critical understanding needed to address environmental, economic and social challenges facing global society. I present some suggestions for how SMA might be operationalized in the classroom, which I hope will generate further debate about the efficacy and possible future development of SMA.

Keywords: Socio-mathematical agency, powerful knowledge, student agency, social justice.

Introduction: What is 'socio-mathematical agency' and why is it needed?

In this paper, I introduce a new theoretical construct of 'socio-mathematical agency' (SMA), which I define as the ability to use mathematics effectively to argue collectively for social change. The need to develop SMA amongst students is highlighted by recent calls from intergovernmental educational policy-making organisations for a more humanistic school curriculum that cultivates the collective knowledge and critical understanding needed to address the environmental, economic and social challenges facing global society (Organisation for Economic Co-operation and Development [OECD], 2018; United Nations Educational, Scientific and Cultural Organization [UNESCO], 2015). I propose that focusing on the development of SMA in learning mathematics can make a significant contribution towards these aims. The benefits of SMA are exemplified by the impact of a report by the Imperial College COVID-19 Response Team (2020), released soon after the initial outbreak of the pandemic, which used mathematical modelling of coronavirus infections to predict over half a million deaths in the UK if existing precautions were not strengthened. The report led to a significant change in public attitudes towards the virus, prompting the UK Government to introduce additional measures, including social distancing, which prevented an even greater death toll (Skovsmose, 2021).

Mathematical skills are widely recognised as essential for solving real life problems. There is growing consensus among mathematics education researchers that, to develop powerful mathematical knowledge (needed to solve real life problems), students need to be given opportunities to experience processes mathematicians go through in generating new knowledge (Mason et al., 1985; Schoenfeld, 2012). These include working collaboratively (most new knowledge is generated by mathematicians working in teams), posing questions, conjecturing, reasoning, explaining, justifying. Gutstein (2006) argues that engaging in complex mathematical tasks, through a curriculum that emphasises reasoning, communication and problem-solving, is essential for the empowerment of mathematics learners.

The recent Covid-19 pandemic has highlighted how understanding mathematical concepts, such as exponential growth and moving averages, is necessary for individuals to make sense of data presented in the media and hence arrive at rational decisions about behaviours that take account of the risks to health. Mathematical knowledge is important in helping generate a wider understanding of the many crises currently facing humanity. Coles et al. (2013) highlight how gaining an appreciation of the shape of a normal distribution curve, and how this relates to the mean and standard deviation, can help to explain why global warming is also associated with more extreme cold spells. However, mathematics does not always serve the public good. Skovsmose (2021) highlights how mathematics sometimes (as in the examples presented above) helps to 'picture' a crisis, but at other times can actually 'constitute' a crisis, e.g. in situations where complex mathematical algorithms have led to a collapse in the stock market or to automated reactions of an aeroplane that prevent the pilot from averting a disaster. Mathematics can also 'format' a crisis, e.g. the choice of mathematical models we use to predict climate change can influence how we interact with the climate in future. Careful thought therefore needs to be given to the type of mathematical knowledge, skills, behaviours and dispositions that learners need to develop if they are to make effective use of mathematics to advocate and bring about future social change for the public good.

Theoretical framework: Conceptualising socio-mathematical agency

The content of school curricula has invariably proved contentious, which is hardly surprising given the competing ideological perspectives of those different interest groups involved in their development (Wright, 2012). Even amongst those who champion the empowerment of learners, there is disagreement over how this should be achieved. Social realists, such as Muller and Young (2019), claim that some types of knowledge, particularly those which are formal and specialised, are inherently powerful. They draw on Bernstein's (2000) contention that abstract knowledge is powerful in that it can extend learners' horizons by allowing them to think 'the unthinkable' and the 'not-yet-thought'. It should be noted that Muller and Young's view of 'powerful knowledge' includes an appreciation of 'disciplinary meaning' (how new knowledge is generated within the discipline) as well as understanding abstract concepts, thus endorsing the generally accepted notion of powerful mathematical knowledge described in the Introduction. Muller and Young (2019) blame the involvement of politicians in curriculum-making for an increasing tendency to prioritise the nurturing of skills and competences that are seen as contributing towards economic growth at the expense of powerful knowledge. They claim that knowledge is consequently viewed primarily as an individual asset, rather than for the public good.

In contrast, critical realists argue that abstract knowledge alone should not be considered powerful, since its power largely depends on the agency of the learner (Manyukhina & Wyse, 2019). Given the possibility that mathematics can constitute or format a crisis (Skovsmose, 2021), particular attention must be given to developing learners' agency, to ensure that mathematical knowledge is used for the public good. Before going further, it is important to clarify what is meant by 'public good' and 'social justice' in this paper. I draw here on Tawney's (1964, cited in Reay, 2012) notion of 'the good society' which strives to eliminate all forms of special privilege within education and within society more widely. A socially-just society is one based around cohesion and solidarity in which individuals share a common interest and treat others in the same way they would like to be treated themselves. Similarly, a socially-just education system is one which aims to secure for all children what a wise parent would seek for their own child.

Recent events, such as 2020 US Presidential election and the Covid-19 pandemic, have raised awareness of how misleading statistics and media reports can influence the voting habits and behaviour of millions of people (Alderson, 2020). This, in turn, has refocused attention on the school curriculum as a means of fostering the type of critical understanding and collective knowledge needed to promote human rights, equality and social justice (UNESCO, 2015), and to address the social, economic and environmental challenges facing global society (OECD, 2018). Those who misrepresent powerful mathematical knowledge as purely abstract and apolitical and ignore the role that agency plays in empowering learners may be diverting attention away from tackling these challenges. Locating the power required to advance social justice primarily within abstract knowledge is an 'epistemic fallacy', which ignores the reality that such power is dependent on the agency of the 'knower' and rests on the false assumption that school is a level playing field (Alderson, 2020).

Manyukhina and Wyse (2019) describe learners' agency as having two dimensions: 'sense of agency' (a feeling of control over their own learning) and 'agentic behaviour' (exercising control through making decisions and taking actions). Both dimensions need to be present if students are to be empowered as autonomous learners. Instilling learners with a sense of agency is of little use if they are not provided with real opportunities within the curriculum to exercise that agency. Manyukhina and Wyse argue that the structure in which learning takes place and the agency of the learner have mutual causality. Providing a context-sensitive learning environment in which students have space to explore and to be creative helps develop their sense of agency. Conversely, allowing students to exercise their agency and become actively involved in their learning promotes academic achievement and impacts positively on learners' views of themselves and their place in the world. There is a danger that knowledge-based curricula, such as the current National Curriculum in England, which place too much emphasis on acquiring disciplinary knowledge at the expense of shaping learners' identities, neglect the development of the values and attributes students need to contribute towards the public good:

> … it is critical to support young generations in developing the capacity to think critically and independently, engage in autonomous decision-making based on informed choice, and act effectively in a manner that ensures the essential balance between individual and societal interests and priorities. (Manyukhina & Wyse, 2019, p.239)

Skovsmose (2011) highlights how mathematics teaching around the world tends to be dominated by an 'exercise paradigm', in which the teacher presents the solution to a closed mathematical problem on the board before inviting students to complete a series of almost identical problems. Given the status of school mathematics as a gatekeeper qualification, such an approach may be empowering in a pragmatic sense, as it is assumed to help learners acquire the qualifications that they need to access higher-paid employment. However, it is disempowering in a socio-political sense as it stifles opportunities for learners to develop their mathematical agency, i.e. the ability to apply powerful mathematical knowledge in solving real-life problems. Skovsmose proposes an alternative ‘critical mathematics education’ in which students reflect ‘through’, ‘with’ and ‘on’ mathematics by: participating in meaningful investigations in which they make their own decisions, pose their own questions, interact and communicate with other learners; carrying out mathematical inquiries which deepen their understanding of their social, cultural, political and economic situations; questioning the nature of mathematics and how it can be used to make decisions affecting them and others.

For mathematical knowledge to be used to advance the public good, consideration needs to be given to collective, as well as individual, mathematical agency. Freire (1974) contends, in his theory of ‘education for critical consciousness’, that genuine understanding can only be achieved through learners developing a critical awareness of their own situations and how these relate to their studies. From Freire’s perspective, the purpose of education should be to meet the collective needs of the community (or society), rather than for individuals to achieve success within the system, through raising awareness of, and challenging, structural inequalities. Emphasis should be placed on mobilising solidarity with those who are marginalised or oppressed and engaging in collective action to challenge exploitation. Freire’s (1972) notion of ‘praxis’, i.e. “reflection and action directed at the structures to be transformed” (p. 96), is used by Gutstein (2006) in proposing a framework for ‘reading and writing the world with mathematics’, in which students use mathematics to “investigate and critique injustice, and to challenge, in words and actions, oppressive structures and acts” (p. 4). Through generating an understanding of power relations, and how these relate to their own lives and experiences as mathematics learners (both in terms of how they may be exploited themselves as well as being complicit in the exploitation of others), students develop their sense of social agency and self-efficacy, i.e. a belief that they can influence or change society. However, such an approach requires a fundamental shift in students’ orientations towards mathematics and in the relationships between mathematics teachers and learners.

In conceptualising SMA, i.e. the ability to use mathematics effectively to argue collectively for social change, I have argued in this section that it is necessary to consider students’ development of powerful mathematical knowledge, including an appreciation of disciplinary meaning, as well as their ability to apply this knowledge in solving real-life problems (see elements 1 and 2 below). However, SMA must also involve a readiness of students to use mathematical inquiries to deepen their understanding of exploitative power relationships within society and a disposition towards using mathematical arguments to expose and challenge injustices they encounter (see elements 3 and 4 below). Finally, SMA needs to enable learners to foster a sense of collective agency (see elements 5 and 6 below). Therefore, I propose the following conceptualization of ‘socio-mathematical agency’ (SMA) which incorporates six elements drawn from the theoretical frameworks presented above (Gutstein, 2006; Manyukhina & Wyse, 2019; Muller and Young, 2019; Skovsmose, 2011):

1) An appreciation of disciplinary meaning (how new knowledge is generated) in mathematics.

2) An ability to apply abstract mathematical concepts in solving meaningful real-life problems.

3) A readiness to use mathematics to explore and develop understanding of social justice issues.

4)A disposition towards using mathematics to expose/challenge exploitation and social injustice.

5) A willingness to work with others in using mathematics to construct an argument for change.

6) Confidence that it is possible to influence society through mathematical argument and action.

Classroom practice: Operationalising socio-mathematical agency

In this section I consider what SMA might look like in the classroom and some strategies/conditions that are likely to promote its development. Firstly, it is worth stressing that I believe that there will always be a place in the mathematics classroom for explaining abstract mathematical concepts and practising routine mathematical procedures. However, there are far more engaging ways of doing this than resorting to the 'exercise paradigm' (described earlier), such as making use of richer tasks that provide opportunities for extensive practice through 'mathematical etudes', which have proved just as effective for attaining procedural fluency (Foster, 2018). Having said that, developing SMA necessitates students engaging regularly with open-ended tasks in which they are given greater control over their own learning through making their own decisions about the direction this will take. They should be encouraged to review their own learning, e.g. by reflecting on any errors they make and non-productive paths they explore along the way. Students should also be provided with regular opportunities to work collaboratively, explain and justify their mathematical reasoning to others, listen to and respect each other's points of view, and appreciate the fallible nature of mathematics in which new knowledge is generated through conjecturing, argumentation and arriving at consensus (Hudson, 2018). Frequent opportunities should also be created for students to generate mathematical models to solve meaningful real-life problems, which involve making simplifying assumptions, choosing which mathematical procedures to apply, and considering the limitations of the solution in relation to the initial assumptions (Schoenfeld, 2012). SMA might then be demonstrated through students discussing these solutions and presenting their findings to others.

Findings from the Teaching Maths for Social Justice (TMSJ) research project (Wright, 2017; 2021) demonstrate students' enthusiasm for exploring social justice issues (such as voting systems, Fairtrade and measures of inequality) in the mathematics classroom. Identifying and building on the strong links that exist between mathematical concepts and social justice issues helps students to develop their understanding of both areas simultaneously and to appreciate the legitimacy of such explorations in the mathematics classroom. These links also provide starting points for teachers to bring social justice issues into the mathematics classroom, whilst navigating the pressures they face in getting through an often-demanding scheme of work. An example of an activity that provides such a starting point is investigating how various methods for counting votes (including Borda Points) can be applied in determining the outcome of an election and then considering which method is 'fairest' (Wright, 2016). Note that Borda Points are based on assigning terms from arithmetic or geometric sequences to different preferences for candidates in an election, which means this activity could easily be attempted as part of a unit of work on sequences (students might go on to explore other types of sequence and consider whether applying these might be 'fairer'). Findings from the TMSJ research project suggest that, as well as developing understanding of social justice issues and related mathematical concepts, such activities also have a positive impact on students' overall engagement with mathematics, as they become more aware of the relevance of the subject to their own lives and society in general (Wright, 2017; 2021).

Developing SMA necessitates going beyond merely exploring social justice issues and requires students to use the increased awareness they gain by doing so to expose and challenge exploitative power relationships. Another activity generated during the TMSJ research project highlighted the challenges teachers might face in doing this and how it might require a re-evaluation of the relationships between teachers and students. The activity involved investigating the proportion of the price paid for a chocolate bar that goes to the cocoa producers and other parties (retailers, importers, etc.) and comparing differences in these proportions between Fairtrade and non-Fairtrade chocolate. One teacher researcher was initially frustrated following a heated class discussion in which students began to question the validity of Fairtrade, as they considered the 4% of the price for Fairtrade chocolate that went to the producers to be grossly unfair (despite this being eight times as much as for non-Fairtrade chocolate). On reflection, however, the research team saw this as a positive development, with students beginning to challenge and question commonly held assumptions about Fairtrade (which might perhaps be more accurately described as 'less unfair trade'), and subsequently decided to change the title of the activity to 'How fair is Fairtrade?'

In Freire's (1974) terms, the events described above might be interpreted as students moving away from a position of 'naïve' awareness towards one of 'critical' awareness, as they begin to question unequal power relationships within Fairtrade production, which might be indicative of the development of SMA. The role of the teacher is crucial in such situations. Freire would argue that the teacher should adopt a 'radical' perspective by promoting debate and reflection, working with learners to develop critical awareness, and helping them to arrive at their own solutions (rather than imposing her/his own views). This approach complements the adoption of open-ended and collaborative tasks referred to above that aim to develop mathematical agency. Such a 'radical' stance also requires teachers to reflect critically on their own positions of power, their own views of social justice issues and the extent to which they, themselves, may be privileged. Sticking with the same scenario, a 'radical' teacher might further cultivate students' SMA by prompting debate around whether Fairtrade (despite being only slightly less unfair than conventional trade) might still have strategic benefits, e.g. in putting pressure on chocolate companies to make modest improvements to the conditions of producers, and the potential for the collective power of students (as consumers purchasing Fairtrade products) to impact on the lives of producers in less wealthy countries.

Collective agency, which involves students developing confidence that collaborative action can bring about change and working with others in using mathematics to generate powerful arguments, is an essential element of SMA. This was apparent in the 'Making a Change' activity developed during the TMSJ research project (Wright, 2016; 2017; 2021). In this activity, students worked in small groups, to choose an issue of interest to them, research it and develop a mathematical argument to support a change they would like to see made. Finally, they were asked to present their argument to the rest of the class, which prompted further debate amongst students around the full range of social justice issues explored. The activity prompted exceptionally high levels of engagement amongst students, including those who had previously appeared disinterested in the subject. Students were excited by the opportunity to use mathematics to explore an issue that was of particular interest to them, which was something of a novelty for most students.

The role of the teacher is again of crucial importance in such situations, with the need to maintain a balance between allowing students space to follow their own lines of inquiry and providing them with the support needed to develop a powerful mathematical argument. During the ‘Making a Change’ activity, the support provided by teachers included discussing issues to be considered in collecting and analysing statistical data on other students’ opinions on the chosen issue, and discussing the difference between mathematical and non-mathematical statements, such as “School absence rates have fallen recently” and “The percentage of students with persistent absence (defined as missing at least 15% of school days) fell from 12.5% in 2006/07 to 8.4% in 2011/12” (Wright, 2016, p.45). Encouraging students to plan in advance, and to evaluate their approach after completing such activities, is important for developing SMA. For the ‘Making a Change’ activity, students might be asked to reflect on questions such as: ‘Is your suggestion for change achievable? How effectively did you use mathematics to strengthen your argument? How well did you work together as a group?’ Establishing genuine collaboration within groups of students, based on solidarity, trust and assigning value to communal effort (Radford, 2012), is an essential aspect of SMA. Boaler (2008) offers various strategies for encouraging students to respect each other’s views and to take responsibility for everyone’s learning (related to the notion of ‘relational equity’), including allocating group roles and recognising the achievements of all students.

Concluding remarks

In this paper, I have outlined a new theoretical construct of ‘socio-mathematical agency’ (SMA) and reflected on some classroom practice that provides a useful starting point for its operationalization. I hope this will stimulate debate amongst researchers and curriculum makers around how mathematics teaching can contribute towards the development of the collective knowledge and critical understanding needed for today’s learners to address the environmental, economic and social challenges facing global society. I plan to work collaboratively with teachers in conducting research in schools to further develop and refine the conceptualisation and operationalisation of SMA presented in this paper, and to explore the potential of SMA for cultivating the collective knowledge and critical understanding needed to address the environmental, economic and social challenges facing global society. I aim to report on the findings of this research in the near future.

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Scottish teachers' perceptions of marginalisation in school mathematics

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Several studies highlight significant differences between the mathematical performances of white middle-class boys and several other groups of children with other demographic characteristics. Across different countries, discussions about who is marginalised vary. In Scotland, marginalisation is typically associated with social class and children's socioeconomic backgrounds. In this paper we explore Scottish teachers' perceptions of the causes of marginalisation in school mathematics. 29 teachers from different school levels participated in individual semi-structured interviews. All teachers' responses reflected the social-class discourse of policymakers. Few teachers recognised other marginialising variables (i.e. gender, English language competence) as well. We conclude that the intersectional character of marginalisation needs to be promoted more explicitly in both initial teacher education and continuous professional development programmes.

Keywords: Teachers' perceptions, marginalisation, school mathematics, Scotland.

Introduction

In recent years, mathematics education has taken a more explicit socio-political turn (Gutiérrez, 2013), by raising and examining questions regarding, inter alia, who decides what is included in school curricula (Appelbaum & Davila, 2007), who gets excluded from school mathematics (Xenofontos, 2015), and how concepts like equity and social justice have their place in the field's discourse (Xenofontos et al., 2021). This paper draws on data from a wider project in Scotland examining teachers' perceptions of marginalisation and equitable teaching practices, as well as teachers' understandings of concepts like equity, diversity, inclusion and social justice in relation to school mathematics. Here, we explore the following research question: *What are teachers' perceptions of marginalisation in relation to school mathematics?* Before we present findings from our work, we turn our attention to the international literature on marginalisation in mathematics education and the importance of transitioning from one school level to another.

Who is marginalised in mathematics education?

In an initial conceptualisation of equity, Gutiérrez (2002, p. 153) envisioned a stage at which mathematics education stakeholders would be

> unable to predict student patterns (e.g., achievement, participation, the ability to critically analyze data or society) based solely on characteristics such as race, class, ethnicity, sex, beliefs and creeds, and proficiency in the dominant language. Being unable to predict mathematics patterns based solely on certain student characteristics addresses issues of power. Rather than expecting that mathematics reform will lead to middle-class White men falling out of power only

to be replaced by another group (e.g., students in poverty, Black women), an equitable situation is when no group oppresses another.

Sadly, the current reality is still far from such a stage. Several studies, typically from North America and many European countries, indicate large discrepancies between the mathematical performances of white male middle-class pupils (the so-called dominant group) and pupils from marginalised groups. These groups include, but are not limited to, girls (Stoet & Geary, 2018), LGBTQI+ children (Leyva, 2017), those whose home language is other than that of school and instruction (Chronaki & Planas, 2018), children with intellectual, emotional, and kinaesthetic disabilities (Watson & Gable, 2012), and those from low socioeconomic backgrounds (Gates, 2019). In fact, things become more complicated when marginalising variables are considered in intersectional manners, like, for instance, the intersection of gender and race (for example, Joseph et al., 2019, write about black girls and their struggles in the white, patriarchal structures of mathematics education).

It appears that different countries and educational systems, not least at the level of policy-making and research, adopt specific lenses through which marginalisation in school mathematics is examined (Graven, 2014; Xenofontos, 2019). In the US, the focus is almost exclusively on race and ethnicity, specifically in relation to the Black and Latinx communities; in several European countries (i.e. Spain, Cyprus) these issues are typically approached in relation to immigration and other-language learners, while in China and many Latin American countries discussions mainly revolve around rurality. In the UK, marginalisation is typically seen through a social-class lens (Gates, 2019), with Scottish policies being no exemption. Specifically, the Scottish educational system explicitly uses the term *poverty-related attainment gap* (Scottish Government, 2018), to describe discrepancies regarding performances and participation rates between pupils who live in economic deprivation and those from affluent backgrounds. To identify the extent to which an area is deprived, the Scottish Government uses measures across seven domains (income, employment, education, health, access to services, crime, housing), and calculates a relative measure of deprivation, called the Scottish Index of Multiple Deprivation (SIMD). All areas in Scotland are given an integer SIMD value, from 1 (most deprived) to 10 (least deprived). Schools receive direct additional financial support, based on how many children from low SIMD's attend each school.

Transitioning from one school level to another

Studies around the world indicate that, as children transition from one school level to another, there is a general decline in their engagement with mathematics (Martin et al., 2015), a decline of their self-efficacy beliefs, motivation, and performance (Deieso & Fraser, 2019), and a reinforcement of stereotypes regarding gender-based mathematics performance (Denner et al., 2018). These observed differences are typically attributed to factors such as teachers' self-efficacy beliefs (Midgley et al., 1989), teachers' and parents' emphases on goal (Friedel et al., 2010), as well as teachers' different approaches in using instructional materials (Fan et al., 2013).

The Scottish Government (2016) makes clear that central to its political agenda is to "improve educational outcomes in communities with a high concentration of children living in poverty" (p. 25); therefore, "[e]nsuring effective transitions between primary and secondary education is particularly important” (p. 14), especially for children from less affluent backgrounds. A deeper understanding of the extent to which teachers’ perceptions converge or diverge will help us pinpoint continuities and discontinuities in children’s lived school experiences, and consider how to better celebrate or address them.

Methodology

Participants

The General Teaching Council of Scotland (GTCS), the body responsible for teacher registration, registers teachers either as primary or secondary (www.gtcs.org.uk). Primary teachers are generalists and work with students of ages 3-12, in nurseries and primary schools. Anecdotally, some primary teachers self-identify as early-years teachers, due to their preference of working with children of ages 3-7. Those using the primary teacher label prefer working with children of ages 7-12. Secondary teachers teach their specialist subject area and work with students of ages 12-18.

Participants in this study were teachers working in the Central Belt of Scotland, a large region with areas of different affluence levels, but also with those areas with the lowest SIMDs in the whole country. Volunteer teachers were sought via the networks of local authorities, our own professional networks, and on social media. Teachers who expressed interest passed the details onto other potential participants, in the form of snowball sampling. In total, 29 teachers were recruited, 8 of whom identified as early-year teachers (EY), 11 identified as primary teachers (PT), while 10 were secondary mathematics teachers (ST). Other than one teacher in early-years, two primary and one secondary, all other participants had more than five years of professional experience.

Data collection and analysis

All participants were invited to an individual semi-structured interview. Each interview was audio-recorded, lasted approximately 45-55 minutes, and was held at each participant’s school. As part of a wider project, the interview protocol included questions about teachers’ perceptions of and experiences related to (a) marginalisation and the attainment gap in mathematics, (b) equitable mathematics teaching practices, and (c) concepts like equity, inclusion, diversity, and social justice. Below, we present some sample questions, to give a sense of the interviews’ content:

1. As you may be aware, here in Scotland there is extensive discussion on the *attainment gap*, especially in mathematics/numeracy. How do you understand this attainment gap?
2. Could you give any examples from your own professional experiences where you observed gap(s)? How does it impact your day-to-day life as a mathematics teacher?
3. Why do you think some children do not perform as well as others in school mathematics? Why are some children sent to the margins?

A thematic data analysis was employed. Following Braun’s and Clarke’s (2021) suggestion, we moved away from a need to achieve data saturation; we rather aimed at dwelling “with uncertainty and recognise that meaning is *generated* through interpretation of, not excavated from, data” (p. 201). Due to the exploratory nature of this study, no predetermined coding scheme was utilised. The two authors worked separately and together, to generate codes and later collate them to generate themes, in similar ways, grounded theorists discuss moving from open to axial coding (Scott & Medaugh, 2017). As a result of focusing on the richness of our data, and not data saturation, we were not particularly interested in quantitative measures (i.e. frequencies, percentages, number of respondents) in presenting our themes. Instead, we followed a phenomenographic approach. Phenomenography explores variation in the ways a phenomenon is perceived by a group of people (Cope, 2004), by taking a second-order perspective, mapping people's experiences and attempting to see the world through the eyes of those experiencing it (Marton, 1981).

Findings

Data analysis brought to surface two themes related to teachers' perception about the causes of marginalisation. Specifically, all participants talked about social class as the main marginalising variable in school mathematics, while very few made scatter references to other variables. Below, these themes are presented in more detail.

Social class as the main marginalising variable

A great homogeneity was observed regarding what teachers distinguished as the main marginalising variable for children's participation and attainment in mathematics. In ST9's words, "students from more affluent households are more willing to give things a go, perhaps get it wrong, mess it up a bit" compared to their classmates from deprived areas. Using vocabulary that reflected national policies, *all* 29 teachers across the three school levels referred to poverty, low socioeconomic status, and SIMDs. Some representative examples are presented below:

> "I suppose the attainment gap reflects the, I don't know what you would call it, the affluence gap. I don't know. Monetary gap. Economic gap. In my experience, they have reflected each other almost identically." (ST7)

> "The school that I teach in has a lot of pupils from low SIMDs. Most of the children are level 3 in the indication mark-up. There's a high level of students who get free school meals and as a result, our school gets quite a lot of funding from the National Improvement Framework. […] And a lot of parents are proud, they don't want to tell you that they're struggling or whatever. But you're aware that these children are not getting proper meals." (PT2)

Nevertheless, some participants emphasised the important role of teachers and schools in addressing all children's learning needs as they navigate through the challenges caused by poverty. The quotes below are representative of an early-years teacher and a secondary mathematics teacher:

> "What's really nice about the nursery here is that I've seen children really flourish, who come from a really poor background, a really poor housing area and yet they are doing absolutely fabulous work in the nursery just because the educators have got the right way of doing things with them and give the right direction, if you like." (EY1)

> "We're certainly not a school that feels sorry for itself and where we're situated. […] We know where the school is situated, we know the catchment area, the SIMD values, but that's it. Nothing is mentioned beyond that. […] We're a school that strives to be the best, it doesn't matter where we are. […] A lot of it comes from leadership, from the staff but often the

> pupils, too. If there are discrepancies in behaviour or homework, then they are minorities. They understand that that's not how you're expected to behave here. That's not the standard that we expect from you." (ST10)

In summary, teachers' responses put poverty and socioeconomic status at the center of their perceptions of the causes of marginalisation. This is, in a way, unsurprising, as it reflects the adoption of a social-class perspective on marginalisation in education expressed by UK policies in general (Gates, 2019) and Scottish policies in particular (Scottish Government, 2018).

Scatter references to other variables

Few references were made by teachers across school levels about other marginalising variables. These can be grouped in two broad areas. The first concerns *family-related issues/dynamics*, like parents' lifestyles, family structures, and family-school relationships. According to PT6, typically "we consider poverty as an indicator, but for some children why they are not attaining has nothing to do with poverty", therefore, "what's most important is the school's knowledge of children's families". From this perspective, teachers pointed out that low attainment can be a result of "the chaotic lifestyles that some families have" (EY1). As EY3 commented, "I've seen parents' anxiety and mental health be so prevalent that they won't allow their child to go to school because they can't be without them".

The second area is related to students' *individual differences*, such as gender, disability, English language competence, and other developmental issues. PT8, for instance, pointed out mathematical difficulties faced by children with English as Additional Language (EAL). In her own words, "I've got children who are in that gap because they're EAL. They can literally do their simple 1 + 1, but they can't do word problems". Likewise, ST3, a female secondary teacher, discussed many girls' low confidence as opposed to that of their male classmates:

> "I feel like girls have a very low confidence of maths and they don't want to put an answer down because they don't want it to be wrong. […] Boys are overly confident and therefore don't study and do worse than they should do, and girls have very low self-esteem, get very anxious and don't want to attempt." (ST3)

Interestingly, most references to family-related issues were made mainly by early-years and primary teachers, while discussions of students' individual differences were made by primary and secondary teachers. This move from socio-cultural to cognitive/affective concerns appears to be in accordance with general trends of mathematics education research on transitioning from one school level to another. For example, many recent studies examining transitioning from pre-school or kindergarten to elementary school focus on the impact of family-related factors (i.e. Niehues et al., 2021). Conversely, recent studies focusing on the elementary-secondary transition are typically more interested in cognitive/affective issues and students' individual differences (i.e. Cantley et al., 2021).

Conclusions

In a sense, it was not surprising that *all* 29 teachers in our study pointed out that social class, poverty, and pupils' socioeconomic status constitute variables associated with pupils' mathematical performance and participation. One the one hand, we consider the fact that few teachers talked about other marginalising variables (e.g. gender, English language competence, disability) encouraging, as those teachers demonstrate some awareness that reality is more complicated than focusing exclusively on social class. Nevertheless, we are aware that simple references to other variables do not necessarily mean that teachers approach marginalisation from intersectional perspectives. As several studies inform us, many teachers have social and political intuition; they sense the interplay between school mathematics and political issues, but do not always know how to put intuition into practice in ways that help their students (de Freitas & Zolkower, 2009; Xenofontos, 2016). Our findings stress an urgency for more intersectional approaches in examining marginalisation in mathematics education, something that could originate from research in teacher education (both initial teacher education and continuous professional development), aiming at having an impact on policy-making, teachers' practices, and students' lived school experiences. Besides, as Freire's (1970) writings have taught us over 50 years now, instead of wait for systemic changes to happen "miraculously", those of us involved in education and share values of equity and social justice could start by initiating small projects in their immediate professional environments.

To conclude, our findings concur with ongoing calls for employing more intersectional approaches, and critical and nuanced discussions on how inequities and marginalisation are constructed and/or even normalised in mathematics education. Intersectionality, as an analytic framework, allows scholars in different fields to explore, inter alia, the structural interplay of variables such as race, class, gender, sexuality, and disability. Yet, we need to acknowledge that we cannot always capture social experience with a finite number of marginalising variables, to describe the intersectional identity of a person (Appelbaum, 2002). At best, we can have an approximation based on important characteristics of how identity is read by others in a social situation. Teacher education initiatives should be designed to address more sophisticated understandings of concepts like marginalisation, equity, and social justice. Hence, mathematics teacher education, we believe, must challenge the current simplistic understanding of marginalisation and provide targeted support to teachers so to rethink their narratives around practices that aim at helping children regardless of background.

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TWG11: Algorithmics

Introduction to the papers and posters of TWG11: Algorithmics

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Keywords: Algorithms, algorithmic thinking, teaching and learning of algorithms.

In CERME12, our working group "Algorithmics" started its work as a newly established TWG. Since algorithms have always been at the heart of mathematics and their importance has been steadily increasing since the beginnings of theoretical computer science, the design and analysis of algorithms – called *algorithmics* (Traub 1964, Knuth 1985) – lies at the intersection of mathematics and computer science. For this reason, on the one hand, various algorithms and algorithmic activities have their traditional place in mathematics curricula at all levels. At the school level, mathematics and computer science have interacted since the 1980s, when many schools set up labs with computers equipped with programming software. On the other hand, many questions arise in the context of teaching and learning algorithms: a first, more applied group of questions aims at algorithms in mathematics education and curricula, a second, more theoretical group of questions seeks to clarify the concepts of algorithm and algorithmic thinking.

Conference presentations

Due to the Corona pandemic, the conference was held as a virtual event. Nevertheless, a total of 11 papers and 7 posters were presented remotely by their authors at the conference, with a total of 24 group participants from 11 countries. The contributions were considered in four themes, as follows.

Theme 1: Beliefs and domains in which algorithmic thinking occurs

A first group of papers focuses on the place and importance of algorithms in mathematics in general and arithmetic in particular. They assess the beliefs of experts about the role of algorithms in mathematics and mathematics education or their role in mathematics courses.

- Lockwood, DeJarnette, Thomas and Mørken offer three perspectives on algorithms, particularly in computational settings: an algorithmic approach in a mathematical example, the view of a mathematician, and the view of an undergraduate students taking a course in mathematics.
- Geraniou and Hodgen interviewed two mathematics educators who had experience using technology to solve mathematical problems, and they, too, shared very different views on algorithms in mathematics education, one even not seeing the use of algorithms as a mathematical activity.
- Kortenkamp analyzes an arithmetic course for pre-service primary teachers. He identifies several algorithmic activities in the topics covered in the course, such as designing algorithms, specifying algorithms, performing algorithms, proving their correctness, and comparing algorithms.

- Leifeld and Rezat's poster provides a thorough analysis of the possibilities of certain arithmetic algorithms for addition and subtraction to deepen students' understanding of inverse operations.

Theme 2: Teaching and learning of algorithmic thinking at primary level

Another group of papers focuses on teaching and learning algorithmic thinking in primary school. They use different tasks with different goals: Some use algorithmic thinking as a means to an end (in the sense of learning a new mathematical concept), some use algorithmic thinking as a goal (in the sense of understanding a given algorithm, or developing an algorithm to solve a problem):

- Crisci, dello Iacono and Ferrara Dentice explore how primary school children can be stimulated to learn new mathematical concepts by Scratch. In their report, they present a specially designed task that required visual programming to complete a given figure so that it becomes axially symmetric. The children developed two different strategies to solve this task. For example, they found out that points that are axially symmetrical to each other must be equidistant from the axis.
- Funghi and Ramploud are interested in how to teach the standard long-division algorithm so that children understand why it "works." To this end, they had fourth graders compare the optimized, digit-by-digit long-division procedure with the procedure in which the divisor is repeatedly subtracted from the dividend. Their analysis of class discussions suggests that this approach could actually result in less rote learning, but in more conceptual learning.
- Zindel's study wants children to acquire algorithmic thinking without using computers. In her papers, children are instructed to decrypt and encrypt certain words. They had to articulate the necessary steps themselves and record them in writing. Although these texts show great differences, the author succeeds in reconstructing some constituents of algorithmic thinking.
- In Gaio's study, too, children are asked to develop algorithms in the sense of systematic procedures, without the help of computers. Here, the tasks given to children of different school levels (3rd to 8th grade) are in the context of sorting problems. As the author reports, he can see traces of classical algorithms in the procedures that the children have worked out cooperatively.

Theme 3: Teaching and learning of algorithmic thinking at university level

Concerning the development of algorithmic thinking at university, the discussions showed two big issues: the development of algorithmic thinking and algorithmics in mathematics, for students, independently of their projects, and more specifically, the development of algorithmic thinking in mathematics for future teachers, and in particular future primary teachers.

Four papers dealt with algorithmic thinking in advanced mathematics, three at university, and one concerning an education program for gifted students. Above them, three were interested in links with discrete mathematics, combinatorics, graph theory, which illustrates the specificity of those mathematical fields, at the interface with computer science:

- De Chenne and Lockwood explore the use of programming and computer science in solving basic counting/combinatorics tasks in college, and how the knowledge of student in computer science can influence their solving strategies and support their learning.
- Medová, Milicic and Ludwig study the competencies involved in algorithmic thinking for university, and in particular abstraction, modelling, and visualization skills which are difficult to master for students, and questions the development of computational thinking in mathematics.

- Bóra and Gosztonyi analyze the place given to algorithms and what could be seen as algorithmic problem solving, in Hungary's advanced mathematics programs, questioning what can be considered traditionally as algorithmic in mathematics and its place according to mathematical culture of the country.
- The paper of Calor, Palha and Kubbe concerns at the same time advanced mathematics and pre-service secondary teachers' education. It deals with analysis, and in particular developing instructional material concerning differential equations for algorithmic thinking and programming. First results show that students did indeed develop algorithmic thinking in their work.

The two other contributions dealt with algorithmic thinking in and for primary teacher training:

- Weber's paper examines primary teachers' use of loops to solve a geometrical problem and their conceptions of the loop construct. It elaborates some challenges in their conceptions and some misconceptions that require deepening their understanding from a teacher training perspective.
- Dobgenski and da Fontoura's poster presents and reflects on an experience of making pre-service primary teachers deal with computational thinking using Scratch.

Theme 4: Concepts related to algorithmic thinking: computational thinking, algebraic thinking, problem solving, and mathematical literacy

The last group of contributions deals with different, no less relevant aspects of algorithmic thinking:

- Rafalska's paper illustrates how tasks could be constructed in order to lead children in mathematics lessons (without the use of computers) to algorithmic thinking in the sense of developing a solution strategy and which individual learning processes can be triggered by these tasks.
- Pohlkamp and Lengnink's paper takes a different look at algorithms: It discusses algorithms that make decisions and are thus socially relevant. Addressing and studying them in the classroom would mean taking more seriously the educational mandate to teach social skills as well.

Finally, two poster proposals deal with two concepts related to algorithmic thinking:

- Rekstad and Rasmussen investigate the question to what extent aspects of computational thinking mentioned in the literature are also reflected in teachers' beliefs when asked about the role of computational thinking in mathematics education.
- The relationship between algorithmic and algebraic thinking is the subject of Müller-Späth, who plans to investigate how algorithmic thinking (realized by an app) affects the development of the ability to generalize and thus of algebraic thinking.

Conference discussions

As mentioned earlier, our working group has just begun its work, and a common understanding of the concepts has yet to be developed: What does algorithmics mean in the context of teaching and learning mathematics? What is algorithmic thinking? To this end, after the presentations in which quite different views were expressed, we worked on the following three questions:

Question 1: Which mathematical algorithms could stimulate algorithmic thinking?

The discussion of this question revealed relatively unanimously five mathematical types of algorithms: i. Algorithms based on the place value system (standard algorithms for addition etc., algorithm for calculating logarithms), ii. graph-theoretic algorithms (shortest path problem, Königsberg problem), iii. approximation algorithms (Heron's algorithm, Newton's method, etc.), iv. sorting algorithms (heap sort, bubble sort, etc.), and v. miscellaneous (Gauss's Easter algorithm, etc.). We were not in agreement of whether each procedure is also an algorithm. For example, everyday procedures (tying shoes, making jam sandwiches, etc.) were not considered by all participants to be suitable for addressing and promote algorithmic thinking in its "proper sense" because they show only one characteristic feature of algorithms: the order of steps.

Question 2: Which mathematical topics could promote algorithmic thinking?

The discussion of this somewhat broader question also yielded five topics from which tasks could come to stimulate algorithmic thinking: i. number theory (arithmetic, prime number tests, factorization), ii. discrete mathematics (graph theory, combinatorics, counting problems, etc.), iii. geometry (transformations, algebraic geometry), iv. computer science (cryptography, etc.), and v. games and puzzles (Rubik's cube, tower of Hanoi, etc.). One participant's question about what properties these fields would have in common was discussed intensively and controversially.

Question 3: What (human) activities with algorithms can we think of?

The activities discussed suggest a wide range of possible activities to deal with algorithms: i. creating (developing algorithms, improving algorithms, debugging algorithms, optimizing algorithms, transferring algorithms to an analogous situation, etc.), ii. analyzing (effectiveness and proof, efficiency, complexity, stability, similarity etc.), and iii. comparing (comparing different algorithms for the same problem, comparing analogous algorithms for different problems, classifying algorithms, etc.). Although executing an algorithm without any reflection would be a possible activity with algorithms, most participants do not want this to be understood as algorithmic thinking.

Surely the reader can think of further examples or answers to these questions. In other words, the three questions need to be discussed further and their answers are still quite open.

Outlook

As the overview of the contributions as well as the first answers to central questions show, there is a great variety of approaches (theories, methods) and views (topics, perspectives) in our working group. Given that we are entering a young (or at least long-neglected) area of research in mathematics education and that we have just begun work in our TWG, it was to be expected that the results would be disparate and sometimes controversial. However, in terms of a first step towards a robust and sustainable understanding of concepts, this diversity makes us confident that there are many more questions around the challenging topic of algorithms and algorithmic thinking that are worth working on. With this in mind, we look forward to CERME13 and hope for a fruitful continuation of the work we have begun – and that it can then be carried out again as a physical conference.

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The role of algorithmics in a Hungarian mathematics education program for gifted students

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Keywords: Hungarian mathematics education; gifted students; series of problems; algorithmics; discrete mathematics.

Introduction

Algorithmics is not an autonomous domain required by Hungarian mathematics curricula. The introductions of the latest curricula consider the development of algorithmic thinking as an aim of mathematics education (together with logical thinking and problem solving), but don't develop this intention further. The teaching of algorithms and algorithmic thinking is included in another discipline, informatics, an autonomous discipline since the 1990s (Kőrösné 2009), where algorithmics is associated with programming.

However, algorithms and algorithmic thinking seems to appear frequently in Hungarian mathematics education, even if these appearances are not explicit and not associated with programming or computers. The identification of (implicit) appearances of algorithmics in Hungarian curricula and teaching practices needs further systematic research. Here, we show a more obvious example, issued from a (well-known and highly influential) mathematical program for gifted students, developed by the mathematician Lajos Pósa. In this program, different problems related to discrete mathematics play an important role and several of them can be clearly associated with algorithmics.

Pósa math-camps

Lajos Pósa has been developing his talent-nurturing method since 1988. He created an environment where talented and motivated students can develop their problem-solving skills. The main goal of his mathematical "camps" is that students could experience the joy of problem solving. His method can be described as a mix of discovery and guided learning approach (Győri & Juhász, 2018). Students enter the program typically at grade 7 and leave around grade 12.

In the Pósa math-camps problems are not stand-alone problems. They are carefully designed and organized into complex series of problems (Gosztonyi, 2019) constructed by various problem-threads. Problem-threads are sequences of problems organized into an intertwining structure of problems. Each problem can be a part of multiple sequences. The nature of connection can be various such as: topical connection (the topics of problems are related) or the core-ideas of a solution are similar.

Problems analyzed

The following 2 problems from the “scale” problem-thread (targeting gifted 12-14-year-old students) will be analyzed in the poster. Ideas related to algorithmics appear quite clearly in this thread. It starts with a quite well-known problem. In the poster more examples will be represented exploring the connection between problems within this specific thread and within the program as well.

1.a We have some weights; one is slightly heavier than the rest. Can you find which one is heavier with the help of a balance? Use the least possible number of steps.

3. We have 8 weights, six of them are 1000g, one is 1010g and another one is 1020g. Can you find, with the help of a scale, which weight is 1010g and which is 1020g? (Scales are different from a balance. They have only one plate and they indicate the weight of the items on it.) Use the least number of steps possible.

Conclusion

As our examples will illustrate questions related to algorithms appear repeatedly and to a certain degree explicitly in Pósa’s program. Algorithms must be constructed in order to solve certain problems. Questions related to the efficiency of algorithms are raised. Related proofs appear, often in form of impossibility proof. Coding of information is necessary in certain cases. However, computers are used in no case in Pósa’s program. We have good reason to think that ideas related to algorithmics appear recurrently in ordinary mathematics education in Hungary as well.

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Students think algorithmically with differential equations

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Keywords: Algorithmic thinking, mathematics, in-service teachers, secondary school, higher education.

Aim and rationale of the study

In the past decade, computational thinking (CT) has been a hot topic in educational research and practice (Grover & Pea, 2013; Selby & Woollard, 2013). CT might have a positive influence on learning mathematics (Barcelos et al., 2018). The aspects of CT include algorithms and algorithmic thinking (AT) (Shelby & Woollard, 2013).

AT is a skill that is not easy to develop for students in secondary and higher education. In-service teachers should be better prepared to teach in ways that involve AT. There is a lack of instructional materials to implement AT in in-service teaching trainer programs. The aim of our research is to develop and investigate tasks that enhance in-service teachers' knowledge about AT.

The rationale of our study relies on a system of assessments originally used in Computer Science Education by Grover et al. (2015, pp 209). It describes eight ideas for a high school curriculum involving algorithmic problem solving, two of which involve AT. These are 1) algorithms and pseudocode and 2) algorithmic flow of control, particularly sequence, serial execution, and loops. We understand AT as 1) understanding how an algorithm works, 2) being able to describe an algorithm as pseudocode, 3) knowing how simple loops work, and 4) understanding how commands are executed in sequence.

Method

Tasks for AT in Dynamical Systems course

In the Dynamical Systems course of our in-service teacher training program, students learn to find exact solutions for differential equations. However, sometimes exact solutions do not exist, and solutions need to be approximated with numerical algorithms. To introduce AT into higher teacher education, we replaced an optional part of the Dynamical Systems course (fractals) with the subject of numerical methods. It is important that students understand numerical algorithms. Students should also be able to implement the numerical algorithms in a computer program and test and debug the program. In other words, students must learn CT skills, in particular learn to think algorithmically, to move from a mathematical model (the differential equation) to a reliable approximate solution. The assignment involves Euler's method to solve a first-order initial value problem. The goal is to make students understand that the steps made with Euler's method form an algorithm, which can be described with pseudocode that includes loops.

We developed a number of tasks that aim to elicit AT. The first design principle regards the context in which the AT takes place. Because we wanted to investigate AT in its natural context (in-service teacher training programs), we chose a content topic that was part of the higher education program and a subject that involved the use of algorithms in a meaningful way. Therefore, we chose a course in differential equations (Dynamical Systems). The second design principle concerns the nature of the tasks. We developed tasks existing of activities that require AT, such as creating a pseudocode involving serial execution and/or loops (Grover et al., 2015). In this study, we investigated how one in-service teacher solved these tasks and engaged in AT in the first design cycle.

Analysis

We investigated how the task (the finishing of the pseudocode in Euler's method) can elicit AT through task-based interviews and the thinking out loud method (Schellings et al., 2006). In the thinking out loud method, the student is asked to constantly articulate his or her thoughts. The students were selected by the teacher of the course on the basis of their mathematical knowledge level (two with an average mathematical knowledge level and two with a higher-than-average mathematical knowledge level, and in each pair of students, one student was chosen at random). Each student was interviewed individually by one of the authors of this paper, and the interviews were video recorded and then transcribed. Analysis of the transcripts was based on the categories of Grover et al. (2015) Here, we report on the results of one student.

Preliminary results and further research

Analysis of this student's think-aloud protocol while solving the task revealed aspects of AT. The results showed that the student was able to explain the Euler algorithm and create the pseudocode (including loops), which are forms of AT according to our analytical framework. The assignments therefore seem to have potential for the development of AT. However, there are aspects that need to be improved. The next steps are to further develop our coding scheme for AT and to investigate how our research can contribute to teachers' work on algorithms in their practice.

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A digital artefact based on visual programming for the learning of axial symmetry in primary school

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In this work we used visual programming in order to mediate the learning of axial symmetry in primary school. More in detail, we designed and implemented an a-didactical activity in the Scratch programming environment in which students have to symmetrize figures with respect to oblique axes. We tested the learning activity with fifth-grade students with the aim of understanding to what extent algorithmics and visual programming influenced the rise of new solving strategies by students. Preliminary results from a qualitative analysis seem to show that students were able to improve their strategies in the programming dimension and to evoke some aspects of axial symmetry.

Keywords: Elementary school mathematics, programming, symmetry.

Introduction and theoretical background

In recent years, several national and international institutions promoted the introduction of algorithmics and computer programming in the school. For example, the Italian Ministry of Education (MIUR, 2018) and the European Commission (https://ec.europa.eu/education/education-in-the-eu/digital-education-action-plan_en) encouraged the introduction of computational thinking, coding and the creative and critical use of digital technologies since primary school. Already in the 1980s the research in education was oriented towards the development of new programming languages and techniques, increasingly accessible to primary school students. In those years, Papert (1980) created the LOGO programming language, used, for instance, by Bideault (1985), Salem (1988) and Clements et al. (2001) for learning geometry in primary school. More recently, Bartolini Bussi and Baccaglini-Frank (2015) used the programmable robot Bee-Bot in order to deal with geometric aspects. Tchounikine (2016) underlines how visual programming and algorithmics are suitable tools for conveying content from different disciplines. In this view, many authors (for example, Benton et al., 2018; Forster et al., 2018; Zhang & Nouri, 2019) used the Scratch visual programming language (https://scratch.mit.edu/) for learning mathematical concepts. Scratch was also used in the French project EXPIRE (https://expire.univ-grenoble-alpes.fr/) to deal with mathematical contents in primary school (Chaachoua et al., 2018).

This work arises from the context described above. It is a part of a broader research project whose aim is to investigate students' difficulties related to symmetries (Dello Iacono & Ferrara Dentice, 2020), and to design suitable learning activities to try to overcome them. In particular, in this work, we decided to use Scratch programming language to mediate the learning of axial symmetry in primary school. Indeed, axial symmetry, like any non-identical geometric transformation, is associated with the idea of movement, as underlined by Ng & Sinclair (2015). Also Jagoda & Swoboda (2011) emphasize the importance of associating symmetries with movement, and they invite to provide students with tasks and tools that allow them to manipulate objects and to experience the action of geometric transformations and their results. In this regard, Scratch programming language could be a suitable tool as it allows students to interact dynamically with the environment. So, we designed an a-didactic situation (Brousseau, 1986) in the Scratch programming environment that involves the use of algorithmics in the students' task. Students have to work on a digital artefact, with figures to be symmetrized with respect to oblique axes. The choice of the oblique axis of symmetry wants to create a cognitive conflict in students (Fischbein, 1989), who are often accustomed to working with horizontal or vertical axes of symmetry. In this way, the intuitive model of an object symmetrical with respect to a horizontal or vertical axis can be replaced by the rigorous mathematical model of an object symmetrical with respect to an axis, regardless of the position of the latter with respect to a reference system such as a sheet of paper or a squared grid.

We experimented our a-didactic activity in a fifth-grade class. In this work, we show the preliminary findings of a qualitative analysis whose aim is to understand to what extent the algorithmic and the visual programming underlying the activity we designed influenced the emergence of new solving strategies by students, linked to the construction of symmetrical images with respect to an axis.

Design of the digital artefact

We designed an a-didactic activity requiring students to work on a digital artefact in the Scratch programming environment. The scene involves three students from a dance school, Piero, Isabella and Giada, moving across a stage to create choreographies for a show. Piero and Giada's movements are already pre-established, while students have to guide Isabella's movements through the creation of a program by dragging and encapsulating some available instruction blocks (see Figure 1).

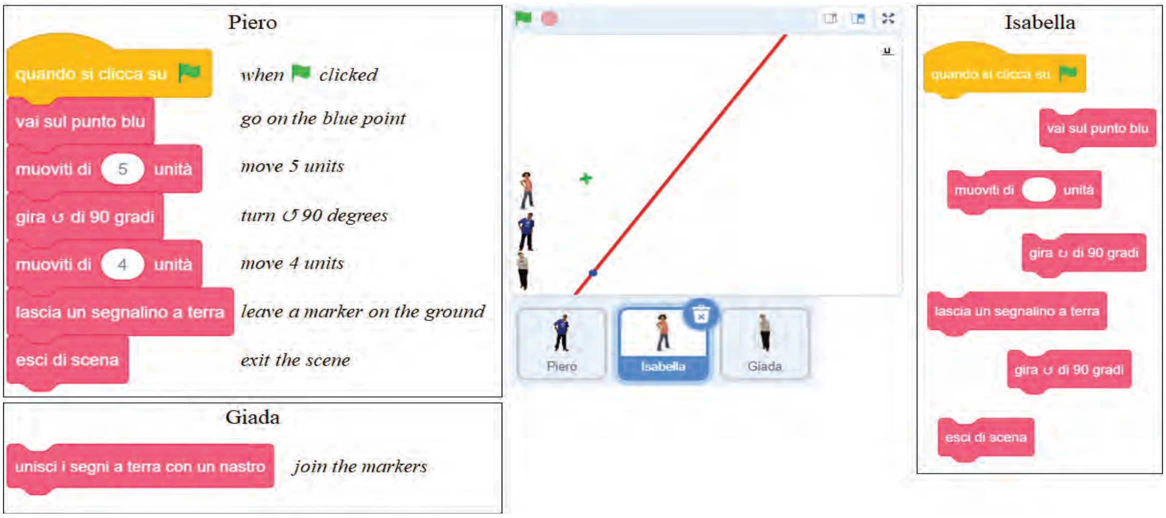


Figure 1: The Scratch visual programming artefact

The activity begins by clicking on the flag and executing Piero's program. Piero moves and performs a sequence of steps, leaving some markers on the stage (the markers represent the points to be symmetrized). Through the digital artefact, students have to provide instructions to Isabella, so that she can perform a choreography symmetrical of Piero's one with respect to a line crossing the stage transversely. By clicking on the flag again, students simultaneously display Piero and

Isabella’s movements on the stage. In this phase, they can carry out a first visual check on the correctness of the program she created for Isabella, by observing if the simultaneous Piero and Isabella’s movements create symmetrical choreographies. Finally, students execute Giada’s instruction “join the markers”. So, the artefact returns in feedback an image obtained by joining the markers left on the stage by Piero and Isabella with a broken line. In this way, students can verify if the geometric figure is symmetrical or not with respect to the line (see Figure 2). Therefore, they can check (in two different moments and in two different ways, that is by looking at the characters’ movements or by visualizing the figure obtained by joining the markers) if the instructions given to Isabella’s character are correct or not. If students are not satisfied with the program created for Isabella, they can make changes at any time.

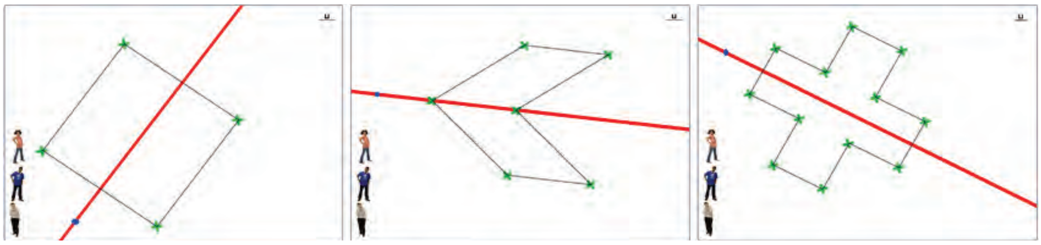


Figure 2: Examples of figures to be obtained on the stage at the end of the activity

The instruction blocks were created in the Scratch environment specifically for this activity by collecting different standard instruction blocks, in order to relieve students of the aspects related to programming and to not significantly affect their cognitive load. The text in the instruction blocks was designed to bring out mathematical meanings related to the concept of axial symmetry.

Methodology

The a-didactic activity, involving the use of the digital artefact described above, was experimented with 21 students of a fifth-grade class of the primary school of the “Istituto Comprensivo San Giovanni Bosco”, near Benevento (Italy). They had already carried out classroom activities with their teacher on axial symmetry, but only with vertical/horizontal lines, and they had never handled activities related to computer programming.

The experiment took place in the classroom, during 3 curricular lessons of 90 minutes each, in the presence of a researcher and the mathematics teacher of the class. It was conducted during the Covid-19 pandemic and the students respected the rules of distancing and the use of masks. In the first lesson, the students became familiar with the digital artefact. The researcher presented the characteristics of the Scratch environment by means of a video projection and she invited the students, divided in groups, to manipulate Scratch on the class computer. In the next two lessons, the students were divided into pairs based on their closeness in the classroom (only one group consisted of 3 members) and each student worked on her laptop.

Each lesson consisted of two main moments: an a-didactic moment and a moment of collective discussion, at the end of which the researcher carried out the institutionalization. During the a-didactic moments the students acted on the digital artefact and, for each choreography, they carried out the following task: “Create a Scratch program for Isabella to replicate Piero’s movements on the other side of the red line. Then ask Giada to join the markers”. Students communicated with each other to verbally validate their strategies.

Afterwards, the researcher delivered to each group a paper sheet with the screenshot of the stage on the PC as it appeared immediately after the execution of Piero’s program, that is, with the markers left on the ground by Piero. The students, communicating with each other to agree on shared answers, first carried out the following task: “Draw the marker(s) left by Isabella on the other side of the line. Then join the markers”. Then, they answered the following open questions: “What do you observe looking at the figure?”, “What do you observe looking at the programs of Piero and Isabella?”. In answering the questions, they could visualize the work done with the digital artefact.

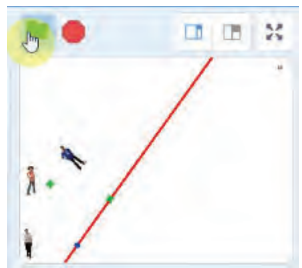
We collected the following data: the video recordings of the screens of the PC used by the students; the paper sheets relating to each choreography filled in by students after the activity with the digital artefact; for each group, audio recordings of the whole activity with the digital artefact, as well as of the collaborative work moments related to the filling in of paper sheets; the audio recordings of the collective discussions guided by the researcher; the notes collected in class by the researcher and the teacher. To analyze video/audio data, we identified critical events, and transcribed and coded them to construct the storyline (Powell et al., 2003).

Preliminary findings

In this section, we show the preliminary findings of a qualitative analysis aimed to point out the students’ programming strategies during the learning activity. In the analysis, we took into account explicit references to the digital artefact - or its characteristics - appeared when the students worked with it, or in students’ oral and written productions. Out of respect for ethical requirements, in the following analysis students’ names are fictional.

The first programming strategy we observed is as follows: the students first visualized the execution of Piero’s program, then they created a program for Isabella and, finally, they simultaneously performed both Isabella’s and Piero’s programs. Only once the students thought they had obtained the right programs, they clicked on the flag in order to visualize their execution. This strategy was employed by most students in the initial phase of the activity with the artefact. Table 2 shows an example of application of this strategy realized by Sabrina.

Table 1: The first programming strategy used by Sabrina

			
min. 2.02: Sabrina clicks on the flag and she	*min. 6.06*: She clicks on Isabella's icon.	*min. 10.33*: She creates Isabella’s program.	*min. 11.11:* She launches both Piero and Isabella’s

visualizes the execution of Piero’s program.			programs.

Sabrina visualized the execution of Piero’s program only once (see Table 1, min. 2.02) before constructing Isabella’s program (from min. 6.06 on). Later, after completing Isabella’s program, she ran both programs simultaneously (see minute 11.11). Nevertheless, after checking the execution of Isabella’s program, she understood that the program she created did not satisfy the required task (Isabella’s movements on the stage did not correspond to Piero’s ones on the other side of the line). At time 11.43, her groupmate Arianna, who had already completed the task, intervened in order to help Sabrina in programming, and, while observing Sabrina’s screen, she suggested a new strategy:

1	Arianna:	Let’s compare it with Piero, I copied from Piero, but… just I changed turn in that way, because she has to turn here, do you see? So, go on Piero and look that these do not have to be here *(she refers to some instruction blocks of Isabella’s program)*... So, go on Piero… [...]
2	Sabrina:	*(Sabrina clicks on Piero’s icon)*
3	Arianna:	Well, do not touch Piero’s program… *(Arianna clicks on the flag and she performs Piero’s program)* This is Piero’s program… Now you have to do the same thing with Isabella… Here it is *(Arianna scrolls the code area until she finds Piero’s program)* Here it is… I copied everything from here, just I changed...
4	Sabrina:	But so did I...
5	Arianna:	Here it is, now go on Isabella…

The strategy suggested and clearly explained by Arianna was the following: to “copy” the instructions included in Piero’s program, by dragging them, in the same order, in the code area of Isabella, reversing the direction of rotations (‘I changed turn in that way, because she has to turn here’, see line 1). After Arianna’s explanation, Sabrina seemed to focus on the aspect related to the reversing of the direction of rotations (line 4), by saying that she considered this aspect during the construction of the program, too. Afterwards, still interacting with Arianna, Sabrina edited Isabella’s program, by using the strategy suggested by her groupmate.
Table 2 shows an excerpt of application of this second strategy.

Table 2: The programming strategy used by Sabrina discussing with Arianna

		
min. 13.03: Sabrina clicks on Piero’s icon and she visualizes Piero’s program.	*min. 13.24*: Sabrina, interacting with Arianna, modifies the number of steps in Isabella’s program.	*min. 14.09*: Still interacting with Arianna and visualizing Piero’s program, Sabrina correctly builds Isabella’s program.

In this excerpt we see that Sabrina switches from viewing Piero’s program to correcting/constructing Isabella’s program. This passage from one character to another is repeated several times until Piero and Isabella’s programs appear “symmetrical”. Later in the activity, all the students in the class adopted this programming strategy, considered more effective.

In applying the first programming strategy (see Table 1) students focused on the visualization of Piero’s movements on the stage during the execution of the program. The absence of a grid made this strategy expensive, as the students had to establish the number of characters’ steps without a visual reference. On the other hand, the second strategy (see Table 2) was based on the direct visualization of Piero’s program, regardless of its execution.

As a result of using this second strategy, explicit references to the Scratch programs created by the students appeared in several students’ answers in the paper sheets delivered to them after working with the digital artefact. In particular, some students explicitly referred to some sentences displayed on the instruction blocks, such as “turn ... 90 degrees”, also reporting in written form the symbols of the arrows (see Figure 3). Furthermore, from some written productions relating to the question “Look at the program you created for Isabella and compare it with Piero’s program. What do you observe?”, it emerged that the students recognized that there was a sort of symmetry between the two programs.

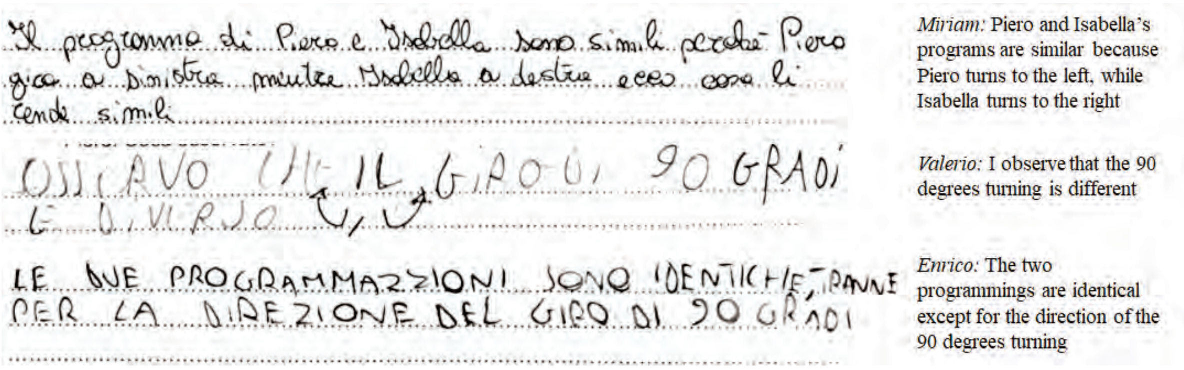


Figure 3: Some students’ written productions

As Figure 3 shows, the students spoke of “similarity” (e.g. Miriam) or even of “identicalness” (e.g. Enrico) between the programs of the two characters, except for the direction of rotations. Such observations became more and more frequent in the written and oral productions of students.

The use of the second strategy could indicate a purely reproductive (and not productive) students’ attitude. However, the reflection on aspects related to computer programming has allowed them to identify new strategies, better than the previous ones and more effective. In fact, the students, in using the first strategy, were not able to produce “symmetrical images”. It was precisely the failure of the strategy that led them to look for a new one, which allowed them not only to produce “symmetrical images” but also to identify aspects related to the definition of axial symmetry, such as the equidistance of corresponding points from the axis. This emerged, for example, during a moment of collective discussion, when a student, Francesco, referred to the movements of Piero and Isabella using the expression “the same distance run on one side and on the other side”.

Discussion and conclusions

This research aims to contribute to study how algorithmics and visual programming can be integrated into teaching practices as tools for learning mathematical concepts. Specifically, we designed an a-didactic activity based on the use of a digital artefact, realized in Scratch visual programming environment, for the learning of axial symmetry in the primary school. The task requires that three characters, Piero, Isabella and Giada, perform some choreographies in a show. The students visualize the predetermined Piero's movements, and they have to create a program for Isabella, by manipulating some suitable instruction blocks, so that Isabella realizes a choreography "symmetrical" to Piero's one with respect to an oblique line. The visualization of the execution of the predetermined Giada's program returns figures that students can verify as being symmetrical or not w.r.t. the line.

We experimented this activity with students of the fifth grade of primary school, who never handled classroom activities related to computer programming. The aim was to understand how the a-didactic activity, involving algorithmics and visual programming, influenced the rise of new solving strategies by students, linked to the construction of symmetrical images with respect to an axis. The qualitative analysis took into account the programming strategies adopted by students and the references to the algorithmic dimension in their oral and written productions. The preliminary findings showed the emergence of two main programming strategies. Firstly, most students created Isabella's program only by visualizing Piero's movements, and not the computer program which generated those movements. Such a strategy soon proved to be expensive and ineffective, in the sense that only few students were able to perform programs responding to the task. This encouraged the rise of a new strategy, also due to the collaboration among the students. The new strategy consisted of replying Piero's program for Isabella, by paying attention to reversing the directions of the rotations. Indeed, Piero and Isabella's programs had to allow their respective characters to move symmetrically w.r.t. the line. So, the directions of rotations had to be the one the reverse of the other. The emergence of this second strategy was linked to the identification of a sort of "symmetry" between the two programs. The new strategy was more effective than the previous one, both from an informatic point of view (i.e. students were able to realize "correct" programs) and, most importantly, from a didactic point of view (it allowed students to figure out some aspects linked to the axial symmetry, as, for instance, the equidistance of corresponding points from the axis). Therefore, visual programming could be a valuable learning tool, able to mediate mathematical meanings. Moreover, it could enable students to devise (new and more effective) solving strategies. In the future, we plan to test the a-didactic activity also with lower and higher secondary school students.

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Listing algorithms for combinatorics problems with variable parameter values: a case study

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In this paper, we explore how a student used computer programming to solve counting problems with variable parameter values. We present a case study where a student, Allen, used computer programming as an aid in finding a closed-form solution for C(n,k)*, the number of ways to select* k *objects from a set of* n *distinct objects. We document some of his difficulties, as well as aspects of his solution that were key to his success. We discuss implications for future research that examines listing algorithms to find closed-form solutions for general problem types in combinatorics.*

Keywords: Combinatorics, student thinking, algorithmic thinking.

Introduction

Combinatorics is an increasingly relevant area of mathematics due to its applications in fields such as computer science and information science. Yet, research in combinatorics education indicates that students at all levels struggle to learn combinatorics (e.g., Batanero et al., 1997; Eizenberg & Zaslavsky, 2004). A repeated conclusion is that attending to the objects being counted in a problem (i.e., the set of outcomes) is productive for students (Lockwood, 2014), and there has been attention to how listing the outcomes can impact student understanding of underlying counting principles (Lockwood & Gibson, 2016; Wasserman & Galarza, 2019). Some of this work specifically focuses on students writing computer programs to list the outcomes (Lockwood & De Chenne, 2020; Medova & Ceretkova, 2021), where the students solve a counting problem by exhaustively listing and counting the outcomes using fundamental tools of Python, such as nested for loops and conditional statements. Because this line of research allows for students to construct and reason about algorithms for the purpose of better understanding an important area of mathematics, there is motivation to examine how similar methods can be applied to other topics in combinatorics.

In our previous work we have explored students writing computer programs in Python (e.g., Lockwood & De Chenne, 2020), and in these studies we have only examined students' solutions to counting problems with fixed parameter values. In these solutions, some of the parameter values are encoded as an intrinsic part of the structure of the code, such as the number of loops. However, many combinatorial problems, such as binomial identities, have variable parameter values and are almost never stated with fixed values. A next step is to examine how students can write computer programs to solve counting problems with variable parameter values. This next step has many anticipated difficulties, and in particular computational solutions to variable parameter problems often require recursion while their fixed-parameter-value counterparts do not. This can be a barrier for researchers and students, and researchers might look for alternative ways of addressing problems with variable parameter values that do not include the use of additional tools or techniques.

In this report, we analyze case-study data of a student who wrote a sequence of computer programs to aid in finding a closed-form solution for $C(n,k)$, the number of ways to select k objects from a set of n distinct objects. In doing so, we distinguish between combinations, which are unordered collections of objects, and permutations, which are ordered collections. By presenting these data, we seek to better understand (1) how the utility and development of algorithms for problems without fixed parameter values differ from those in existing literature, and (2) some of the difficulties and successes of the student that are unique to problems without fixed parameter values.

Literature Review and Theoretical Perspective

Student difficulties in combinatorics are well documented, and they include difficulty with distinguishing between problem types (Batanero et al., 1997) and with verifying that solutions are correct (Eizenberg & Zaslavsky, 2004). Lockwood (2014) has proposed a set-oriented perspective to counting, which is “a way of thinking about counting that involves attending to sets of outcomes as an intrinsic component of solving counting problems” (p. 31). Sets of outcomes refer to the collections of elements being counted in a counting problem. This perspective includes examining and using properties of individual outcomes, as well as structure in the entire set of outcomes, and is reinforced by Wasserman and Galarza (2019) describing how the encoding and conceptualization of sets of outcomes can influence solutions for category I and category II combination problems.

Additional research has investigated combinatorics in a computational setting, such as Medova and Ceretkova (2021) quantitatively assessing the relation between computing ability and combinatorial reasoning. In line with such work, we have examined ways to use computer programming as a means for students to interact with individual outcomes, and to write algorithms that would list the set of outcomes (e.g., Lockwood & De Chenne, 2020). We adopt the term ‘listing algorithm’ to refer to an algorithm that exhaustively lists and counts the set of outcomes, with the intention of being used to design a computer program. We are cognizant that algorithms are far broader in general and enumeration algorithms in combinatorics need not be used in a computer program. Hence, our use of listing algorithm is not intended to encompass all types of enumerative algorithms in combinatorics. When creating a listing algorithm, students must first decide on an appropriate way to encode the outcomes so that every outcome is represented, and no outcome is represented more than once. Then, the students must decide on an algorithm to list the set of outcomes. Hence, students have the opportunity to reason about outcomes individually, as well as the set of outcomes collectively, which reflects important aspects of a set-oriented perspective. Although there are numerous ways to write listing algorithms, the primary tools in this report are for loops and conditional statements in Python.

Medova and Ceretkova (2021) have reported on computational solutions to problems with variable parameter values done by students in a Programming 2 course, but it is unclear how students with less programming experience may approach similar problems. We have previously only had students write computer programs to solve problems with fixed parameter values, while many common combinatorial problems have variable parameter values. For example, a fixed-parameter-value problem might be “How many ways can you flip a coin 5 times if exactly 3 of the coins landed on heads,” while a problem with variable parameter values is “How many ways can you flip a coin n times if exactly k of the coins landed on heads." Designing listing algorithms for problems with variable parameter values presents additional difficulties, which may include learning new computational skills such as recursion. We are motivated to examine ways that students can engage in generalizing solutions from problems with fixed parameter values.

Methods

We present a case study of one student, Allen (pseudonym), who wrote computer programs to aid in finding a closed-form solution for $C(n,k)$ during a set of task-based interviews. These data come from a study conducted in the United States that examines the role of computing in combinatorics education. Allen was recruited from an introductory computer science class; he was selected based on a survey, which indicated that Allen had not taken a discrete class, and although he had been exposed to some counting problems in high school, he did not appear to recall formulas from that period. He also indicated that he had taught himself how to write programs in Python in high school and was interested in pursuing a degree in computer science. We chose Allen as a participant because he exhibited a strong computer science background. We interviewed Allen in person over three 90-minute sessions, and we focus on the last session in this paper. We chose these data because they provide an example of a student using computational techniques to solve a problem with variable parameter values. We do not claim that other students would produce similar results. Indeed, Allen's mastery of fundamental computer science ideas indicates that most students would not produce similar results, but we nevertheless feel our findings have useful theoretical implications about using programming to solve problems.

Task

These data occurred after Allen compared an incorrect solution to a correct solution of the Book Problem, which states "Suppose you have eight books and you want to take three of them with you on vacation. How many ways are there to do this?" The solution to this problem is $C(8,3) = 56$. This was the first time the authors presented a problem involving combinations (rather than permutations) to Allen, and his original solution was $P(8,3) = 336$. After the authors asked Allen to list the first ten outcomes that a listing algorithm would produce, Allen noticed that his solution (which he had developed for permutation problems in previous interviews) was incorrect because it produced the same outcome more than once. While Allen was able to create a listing algorithm that gave him the correct answer, he was unable at first to justify an appropriate mathematical expression by hand. The data we present in this report is of Allen finding a closed-form solution for $C(n,k)$ by writing a sequence of computer programs to find specific values of $C(n,k)$ and comparing those values to $P(n,k)$, which he could produce by hand. These data were unanticipated by the authors, who did not ask Allen to find a closed-form solution for $C(n,k)$, but in the course of the interview decided to allow Allen to pursue a solution. Allen's work occurs in three stages: writing a listing algorithm for $C(8,3)$, writing a listing algorithm for $C(n, 3)$ for various values of n, and writing a listing algorithm for $C(n,k)$ for various values of n as k increased.

Allen had previously worked on problems involving Cartesian products, arrangement with repetition, and arrangement without repetition. We provided Allen access to a Jupyter notebook on a desktop computer while he was being interviewed, which allows users to write computer programs in Python as well as write prose in Markdown. Jupyter notebooks are arranged into cells, where computer programs can be written into individual cells in the same notebook. This allows for a single notebook to include multiple questions and listing algorithms simultaneously, so that the user can easily reference prior work. During the first session, we included a brief Python primer that described how to use fundamental tools in Python, such as lists, loops, and variables. Allen had already written listing algorithms for problems involving Cartesian products, arrangement with repetition, and arrangement without repetition. We provided printed problem statements and paper to write on.

Data Analysis

We chose this episode because it demonstrates how a student used computational techniques to aid in the development of a closed-form solution for a problem with variable parameter values. The first author identified key moments where Allen used his computer programs to reason about the relationship between *C(n,k)* and *P(n,k).* She then created an enhanced transcript of the episode by including pictures of Allen's work and screenshots of the computer programs he wrote. This analytic process allowed us to make a narrative of Allen's reasoning about *C(n,k)* as it relates to *P(n,k)*, facilitating our articulation of Allen's case and the role of listing algorithms in his work .

Results

We categorize the results into three sections: listing algorithm for *C(8,3)*, listing algorithm for *C(n,3)*, and listing algorithms for *C(n,k)*; this reflects the chronological order of Allen's progress, and informs us of the difficulties Allen encountered, and how he resolved them. Further, this progression demonstrates distinct steps Allen took as he generalized his listing algorithm.

Listing Algorithm for *C(8,3)*

Allen's first solution to the Books Problem was $8 \times 7 \times 6$, which is the number of ways to arrange three of the books. However, this expression accounts for different arrangements of the books as distinct outcomes, whereas the question does not. A correct solution is $8 \times 7 \times 6/3 \times 2 \times 1$. We asked Allen to verify if his solution was correct by describing a listing algorithm, and then writing down the first ten outcomes the algorithm would produce. Allen described an algorithm that would list all arrangements of three books (which he had implemented in Python for previous problems), which is incorrect but consistent with his solution. Allen then wrote the outcomes in Figure 1.

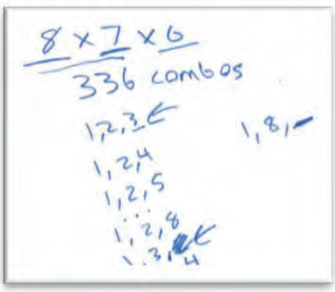

Figure 1: Allen's Partial List of Outcomes for the Books Problem

In Figure 1, the last outcome in the list was 1, 3, 2, which Allen erased and replaced with 1, 3, 4 after he realized that 1, 2, 3 and 1, 3, 2 represent the same outcome. This demonstrates that Allen's error was not due to misunderstanding the question; he was aware that the problem counted unordered combinations of books, and not arrangements of books. Allen realized that his solution was incorrect, but he was not able to find a numerical expression that accounted for different arrangements of the books being the same outcome. He then decided to implement a listing algorithm that differed from his previous algorithm by requiring that the books be placed in increasing order in any outcome. For example, the outcome 1, 3, 2 would not be counted in this new listing algorithm because the numbers are not in increasing order. He justified that enforcing the increasing order would still produce every outcome, and that every outcome would be produced only once because there is exactly one way to place the numbers in increasing order. His implementation of this listing algorithm is shown in Figure 2. Allen explains the logic of his code as:

Allen: Say I pick book number 5 as my second book. I don't need to pick book 1, 2, 3, 4, or 5 [as the third book] in that case, so I'd only have three books to choose from.

After running his code, Allen realized that the correct solution was exactly one sixth of his original but incorrect solution; that is, that *P(8,3) = 6*C(8,3)*. His additional work sought to justify this ratio.

```
In [7]: count = 0

        for i in range(1,9):
            for j in range(i+1,9):
                for k in range(j+1,9):
                    count = count + 1
                    print(str(i)+" "+str(j)+" "+str(k))
        print count
```

Figure 2: Allen's Listing Algorithm for *C(8,3)*

Listing Algorithm for C(n,3)

After Allen realized the correct solution was one sixth of his original solution, he investigated how the ratio would change as the number of books increased. He said the following:

Allen: That's interesting, 8*7 is 56, I'm fairly certain of that. That is interesting, though. Completely the third value [the value 6 in 8*7*6, his original solution] didn't even really matter it looks like … So this last one was equal to 8*7. I just want to see if bringing the total number of books down by one, does that make it equal 7*6? Is there a relationship there?

To create a conjecture for the value of *C(7,3)*, Allen reasoned about the mathematical expression *8*7*, indicating that he was attempting to find an empirical pattern in the expressions themselves. He seemed to reason that the correct solution would be to take his original solution, and remove the last term in the product. We take this to be an instance of empirical pattern matching, where he was hoping to recognize a pattern in the numerical solutions without reasoning about why that pattern is reflected in the set of outcomes. He was aware that his pattern matching might be incorrect, indicating that dividing by *6* "might just be luck." After changing his computer program to solve the new problem, he observed that the correct answer was *7*5*, which is one sixth of his original answer, Allen decided to investigate the ratio *P(n,3)/C(n,3)* for other values of *n*. This ratio is 6 for every *n*.

To investigate the ratio as *n* increased, Allen wrote a new computer program. Because the number of books in his listing algorithm for *C(8,3)* was only represented by the 9 in each of his for loops (in Python, range(1, n+1) produces the numbers 1 through *n*), he created a variable *book* that could be changed. By replacing the number of books with a variable, he could create a program to calculate *C(n,3)* without writing a new program for every value of *n*. Importantly, this change did not require restructuring the algorithm to accommodate a variable parameter. Allen wrote the function in Figure 3, which inputs the number of books and returns the ratio *P(books,3)/C(books,3*). By iterating through multiple values for *books*, Allen verified that the ratio was 6 for every value of *books*.

```
In [23]: def fun(book):
             count = 0
             for i in range(1,book+1):
                 for j in range(i+1,book+1):
                     for k in range(j+1,book+1):
                         count = count + 1
                         #print(str(i)+" "+str(j)+" "+str(k))
             return (book*(book-1)*(book-2))/count
```

Figure 3: Allen's Listing Algorithm for *C(n, 3)*

Listing Algorithms for *C(n,k)*

After finding that *P(n,3)/C(n,3)=6* for every value of *n*, Allen decided to see how this ratio changed as *k* increased. Here the number of books being selected was represented by the number of nested for loops. This is not easily changed, and creating a function that returns *C(n,k)* for any value of *n* and *k* would require a significant restructuring of the code. Allen's goal was not to create a listing algorithm for *C(n,k)*, but rather to determine a closed-form solution for *C(n,k).* While a listing algorithm for *C(n,k)* may have aided him, he was able to pursue other means of generalizing his algorithms.

Rather than write a single function for *C(n,k)*, Allen wrote two different functions for *k=4* and *k=5*. For each of these, he made conjectures about *P(n,k)/C(n,k)* before he observed the actual ratios. For both values of *k*, Allen's computer programs showed that the ratio was constant. Specifically, his programs returned that *P(n,4)/C(n,4)* = 24, and *P(n,5)/C(n,5)* = 120. Although his conjectures seemed to express numerical pattern recognition based on empirical data rather than conjecture based on mathematical analysis of the counting problem, his persistent comparisons between *P(n,5)/C(n,5), P(n,4)/C(n,4),* and *P(n,3)/C(n,3)* showed that the ratio between the ratios followed a pattern. In describing this pattern, he states:

Allen: What I noticed is each time you go up, you multiply by the next number. So 6 times 4 equals 24, which multiplied by 5 equals 120, which multiplied by 6 equals 720.

After formalizing this pattern, Allen used his knowledge of a closed-form solution for *P(n,k)* to conjecture a closed-form solution for *C(n,k)*, as shown in Figure 4.

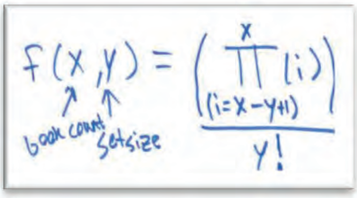

Figure 4: Allen’s Closed-Form Solution for *C(n,k),* or *C(X,Y)*

In this expression, *X* is the number of total books and *Y* is the number of books being selected. Allen asked the interviewers if there was a symbol for repeated multiplication, and we described the notation in the numerator. The numerator of the right side is Allen’s known solution for *P(n,k),* and the denominator is the reciprocal of the conjectured ratio between *P(n,k)* and *C(n,k).* While Allen found this expression through numerical pattern recognition based on empirical data, he was able to justify the closed-form solution by stating that Y! is the number of ways to arrange the selected books, so dividing by Y! essentially removes all repeated occurrences of the outcomes.

Discussion

In these data, Allen appeared to have more difficulty finding an expression for *C(n,k)* than writing a computer program that listed the outcomes to a counting problem. And while Allen used his computer program to find such an expression, there were essentially three big ideas that he reasoned about: (1) generalizing and implementing his computer programs, (2) conjecturing about the values of *C(n,k),* and (3) formalizing and justifying a mathematical expression for *C(n,k).* The first idea was the one that seemed to pose the least challenge to Allen, after he realized that his original solution was incorrect. We differentiate between ideas (2) and (3) because they occurred at different points in the data, and because his ways of reasoning about them were different. As Allen wrote his different computer programs for *C(8,3), C(n,3), C(n,4),* and *C(n,5),* he conjectured about their values based on empirical patterns that mostly concerned how the values were written. For example, when Allen saw that *C(8,3) = 7*8,* and his original solution was *8*7*6,* his conjecture for *C(7,3)* was *7*6* because his original solution would have yielded *7*6*5*. That is to say, his conjecture was that the correct solution would be to remove the smallest term from the incorrect solution, and his reasoning for this was based on the observations from his computer program. In this way, Allen was projecting forward so as to anticipate future values and see if his conjectures were true, but his conjectures were based on the cardinalities of the sets of outcomes rather than the sets of outcomes themselves. It is difficult to say how much of his conjecturing would have remained the same if he had not written the computer programs himself, and instead were given a table of values for *C(n,k),* but we feel that a contribution to his success was repeated conjecturing followed by writing a computer program to verify if the conjecture were true. In contrast, when reasoning about (3) Allen reflected on the data he had already found to formalize a mathematical expression, and then he reasoned about the sets of outcomes as a means to justify that expression.

We do not intend to criticize Allen for his empirical conjectures during stage (2), but as mathematicians it is easy and often justifiable to place more value his reflective behavior during stage (3). However, we are also aware that stage (3) may never have occurred if not for stage (2), and we recommend other researchers to allow times where students can engage in empirical conjecturing or reasoning.

Another difficulty with writing programs to solve combinatorial problems with variable parameter values is that the output of the program is not a symbolic mathematical expression, yet mathematical analysis typically involves symbolic, closed-form expressions. The outcomes of a computer program and mathematical analysis are not the same, so they can be difficult to compare. In our data, Allen was able to create an expression for $C(n,k)$ by comparing the value to a known similar value, $P(n,k)$, as one parameter varied at a time. Future research could examine ways of using and analyzing computer programs with variable parameter values other than as a way to compute fixed instances of parameter values. While Allen used his computer programs to compare the values to other known values, an alternative method would be to examine the behavior of the program as the parameter values changed.

Allen created and generalized a listing algorithm for a problem that he could not solve by hand. The utility of the computer program in Allen's case was not reinforcing known mathematics, but the ability to verify conjectures about unknown mathematics. Because Allen decided on how to proceed during his work, we hypothesize that he was partially motivated because he was given agency over his mathematics. Future research could examine a computational environment as a means of solving unknown mathematics or mathematics that is unfeasible to compute by hand. In combinatorics education, this might mean using the computer to reason about sets of outcomes that students can create listing algorithms to produce, but that are difficult to count with conventional formulas and expressions. In such cases, the end goals might not be to find a closed-form solution that counts the set of outcomes, but to reason about how the size and structure of the set of outcomes changes as parameter values change.

Acknowledgment

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A practical experience in pre-service teacher education focusing on computational thinking

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Keywords: Pre-service teacher training, mathematics education, block-based programming.

Introduction

Computational thinking (CT) is a recent topic in most Brazilian schools since it was included in the National Common Core Curriculum – NCCC (*Base Nacional Comum Curricular* – BNCC) in 2018. NCCC aims to guide school curricula and provide for the development of CT through Mathematics teaching (Ministério da Educação, 2018). The purpose of this research is to present how elementary pre-service teachers apply CT skills when they design a lesson to teach elementary mathematics.

Theoretical framework

In this research we consider Selby and Woollard's definition (2013, pp. 5) of CT: it "is a focused approach to problem solving, incorporating thought processes that utilize abstraction, decomposition, algorithmic design, evaluation, and generalizations." This approach combines the four CT pillars showed by BBC Learning (n.d) and the Reference Curriculum in Technology and Computing (Raabe et al., 2020, p. 19): abstraction, algorithms, decomposition, and pattern recognition.

Code programming using block-based programming languages is a popular form to develop CT skills (Hsu et al., 2018). Brennan and Resnick (2012) identified computational concepts by studying activities in the Scratch online community. These concepts are common in many programming languages and include sequences, loops, parallelism, events, conditionals, operators, and data.

Methodology

This research is qualitative in nature and seeks to address the following research question: *how do pre-service teachers include computational thinking concepts when they design elementary mathematics activities from NCCC's perspective?* To discuss this research question, we analyzed an activity developed by three of thirteen participants of a pre-service teacher training process, which took place over four months, with one 3-hour session per week, in a Brazilian private university in 2021. Due to the COVID-19 pandemic, all classes were remote, synchronous, and occurred via a virtual meeting program. The aim of this training process was to discuss CT connection with elementary mathematics teaching. Through a teaching experiment, we proposed tasks to the students in which they had to think how to design mathematics lessons using unplugged and plugged CT for their future primary education classes. Data were collected from pre-service teachers' protocols, training's video recordings, and Scratch's programming code from plugged activities designed. Data were analyzed taking into account CT definition and concepts mentioned before. The analyses of qualitative data sources were divided to identify the frequencies of CT concepts and of aspects of CT definition.

Statement and discussion of results

The proposed lesson had to consider one of the elementary mathematics ability previewed in NCCC, which introduces the concept of geometric orientation to children. Pre-service teachers explored Scratch's programming language and proposed a pre-made scenario presented in Figure 1. They made a small program to provide pupils only with a yellow butterfly flapping the wings.

Figure 1: Plugged activity proposed by students

In the teachers' code programming, we found sequences as expected: 16 loops (repeat and forever), 5 events that occur when the green flag is clicked or a space key is pressed, and 5 parallelism actions, three of which begin simultaneously when the green flag is clicked and two happen when the space key is pressed. Teachers did not consider conditionals, operators, or data in their code.

Pre-service teachers' description about how they constructed this activity reported they did not make the code in one step; they had to split it in small parts (like move to the right, move up, move down…) and synchronize Scratch's actors' movements. Analyzing all data source, we found evidence the participants used all the CT pillars to conclude the task (abstraction, algorithms, decomposition, and pattern recognition).

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Synergy of two division algorithms in 4th grade: opportunities and challenges

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This paper aims at showing the potential of the synergistic use of algorithms as artifacts for the development of mathematical meanings with pupils of primary school. Specifically, we consider two division algorithms introduced in 4th grade and we show how a specific task design, that involves a comparison between the two different algorithms performing the same division, can generate a mathematical discussion. In such a discussion we can identify several situated signs potentially useful for the development of mathematical meanings related to the algorithms' functioning.

Keywords: Sinergy between algorithms; Semiotic Mediation; Artefacts; Canadian division algorithm; TI×- division algorithm.

Introduction

This article presents a study on teaching multiple algorithms for calculating divisions, carried out within a larger project (see www.percontare.it) aimed at providing Italian teachers with a repository of educational activities in Mathematics for primary school that pay particular attention to inclusiveness (Baccaglini-Frank, 2015). A fundamental design feature of the activities is their aim to help students reach a solid construction of mathematical meanings through the use of artefacts, following the Theory of Semiotic Mediation (TSM) (Bartolini Bussi & Mariotti, 2008). This aim is prominent also within the set of activities concerning the introduction of different calculation algorithms for arithmetic operations. In this study we will focus on the teaching and learning of division between natural numbers through two algorithms in fourth grade (age 9-10).

The first algorithm we consider is the "Canadian algorithm" (Lisarelli, Baccaglini-Frank & Di Martino, 2021; Boero & Ferrero, 1988), consisting in a repeated subtraction of the divisor from the dividend. The other one is *"TI×- algorithm"* (Karagiannakis, 2018), that is similar, from a mathematical point of view, to the long division algorithm (see the *Methodology* section). The main difference between these two algorithms can be expressed through the *transparency* construct with respect to the meanings of the division between natural numbers. We want to extend to these algorithms the definition of transparent and opaque representations of numbers introduced by Zazkis & Gadowsky (2001): "A transparent representation has no more and no less meaning than the represented idea(s) or structure(s). An opaque representation emphasizes some aspects of the ideas and structures and de-emphasizes others" (p. 45). We can say that the Canadian algorithm is transparent with respect to the meaning of division, conceived as a progressive distribution, while TI×- algorithm is opaquer with respect to this meaning. Our research hypothesis is that children can make sense of the algorithms, understand *why* they work, and gain deep understanding of division of natural numbers, by becoming fluent with both of them and then comparing them and discovering what is behind the opacity of an algorithm like TI×-. Our main research interest is to test this hypothesis. In the study, we report on our attempt at promoting a mathematical discussion overcoming the opacity of the TI×- algorithm.

Theoretical framework

An *algorithm* can be considered as a cultural product, designed to solve a given class of problems. Schmittau (2003), discussing the role of algorithms within Davidov's curriculum, expressly talks of them as a "symbolic trace of the meaningful mathematical actions required to solve a problem" (p. 240). In this perspective, an algorithm develops historically and is configured as a cultural tool that mediates an individual's knowledge and understanding of Mathematics (Ebby, 2005). The reliability of an algorithm rests on a body of knowledge that is not always visible to those who use it. Traditional algorithms, in fact, are the result of a historical-cultural evolution that has often favored the efficiency of algorithms in a mechanical sense rather than their transparency with respect to mathematical meanings underlying each step (Bass, 2003).

In this perspective, we can interpret didactical activities on algorithms for the arithmetic operations through the lens of a whole tradition of studies which have shown how it is possible for students, through the use of artefacts to accomplish a task, to develop meanings linked to the knowledge incorporated in the artefacts themselves (e.g., Bartolini Bussi & Mariotti, 2008). Starting from a Vygotskian perspective, Bartolini Bussi and Mariotti emphasize the crucial role of social interaction as an engine for student learning, with a focus on the semiotic processes that can occur in the classroom starting with an activity with an artefact, triggered and supported by the teacher. Nevertheless, unlike most of the studies informed by TSM - which concern the use of only one artefact - in this study we chose to use two different algorithms as artefacts. This choice is supported by recent studies that have begun to investigate the possibility of introducing more than one artefact having a common potential with respect to the development of the same mathematical knowledge (e.g., Faggiano et al. 2018; Maffia & Maracci, 2019). These studies confirm that in specifically designed didactical activities, the introduction of more than one artefact can result in a synergy, which can increase the didactic potential of the activity with respect to activities involving a single artefact (Faggiano et al., 2018; Maffia & Maracci, 2019).

Therefore, conceiving algorithms as artefacts can be useful to make visible to the students mathematical meanings related to the body of knowledge that make the algorithms reliable. From the perspective of TSM, starting from the use of algorithms to carry out specific tasks, and participating in an explicit discussion on this use - intentionally orchestrated by the teacher - students can develop knowledge on the nature of the algorithm and the properties of the operations to be performed. More specifically, we present a didactic intervention whose aim was precisely to develop students' knowledge of the mathematical meanings underlying two different division algorithms.

Research questions

1. What signs in the discussion can evolve in the direction of the discovery and understanding of the mathematical meanings underlying the functioning of TI×- algorithm?
2. Can (and if so, how can) the synergy of artefacts foster the emergence of signs related to mathematical meanings in common among the two algorithms involved?

Methodology

According to the TSM framework, the didactic activity was designed to include a part of interaction with artefacts and a subsequent part of mathematical discussion, to be developed in Distance Education (DE) due to the persistence of the SARS-COV-2 pandemic. As showed in other studies (e.g., Ramploud, Funghi & Mellone, 2021), in order to face the constraints of DE, teachers have sometimes chosen to adapt the framework of TSM to their new conditions. In this case, the teacher chose to separate the part of interaction with the artefacts from the discussion: in a first moment students had to calculate the division 874:7 with both algorithms, as homework; then, the class was divided into 4 heterogeneous groups of 7-8 students each, and the next day for each group a lesson in DE of 40 minutes was dedicated to the mathematical discussion starting from the different solutions of the students. The lessons were all video recorded and fully transcribed.

The two algorithms considered were specially chosen in analogy with other studies (e. g., Lisarelli et al., 2021) to allow "to discover various mathematical meanings behind the long division algorithm [in our case, the TI×- algorithm] and their role in unveiling the whys: the role of place value, the hidden powers of ten, [...] the meaning of each digit of the quotient, how each remainder is obtained." (ibidem, p. 3). We describe below the two algorithms for the division presented in the PerContare project.

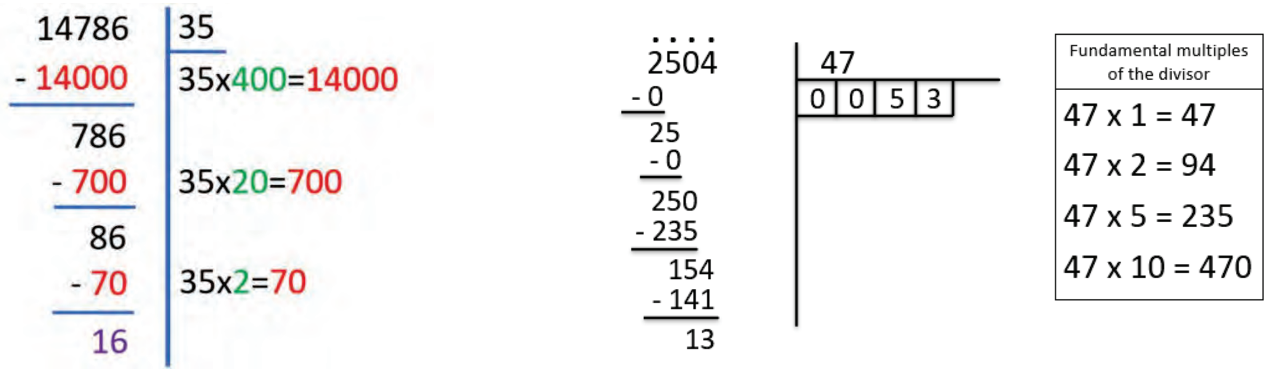

Figure 1: An Optimized Canadian algorithm applied to the division 14786:35 (on the left); TI×- algorithm applied to the division 2504:47 (on the right)

The first is the Canadian Algorithm in its non-optimized version (Boero & Ferrero, 1988). It works like this: the solver identifies a multiple of the divisor that is less than the dividend; then, s/he subtracts this multiple from the dividend. S/he repeats the same reasoning starting from the result of this subtraction (i.e., s/he identifies a multiple of the divisor, which is less than the result of the subtraction, and so on) until s/he obtains a number that is less than the divisor (it could be 0). The left diagram in Figure 1 shows an optimized version of the Canadian algorithm applied to the division 14786:35, where the multiple of the divisor to be subtracted at each step is chosen among those that are also multiples of the highest possible power of 10 (i.e., at the first step 35 ×400 is subtracted, then 3502×, then 352×). The second division algorithm is TI×- algorithm. To illustrate it, we can start from this example: 2504:47 (see Figure 1 on the right). The first step to be carried out consists in writing some useful multiples of the divisor, which we call *fundamental multiples* (on the right in Figure 1). The multiples chosen are ×1, ×2,× 5, ×10, since they are the simplest multiples to calculate (the multiple ×5 can be obtained calculating half of the multiple ×10) and they are sufficient to obtain all the others applying the distributive property (e.g., 47×3=47+94). At this point, we can set up the diagram in Figure 1. We can observe that the number of spaces reserved to the quotient digits corresponds to the number of dividend digits, regardless of the development of the division (in our example, 2504 consists of 4 digits, so the diagram provides 4 spaces for the quotient digits). We therefore carry out the steps according to the acronym «TI×-», thus explicating its meaning:

- **T**ag the first dividend digit from the left, making a dot above it (in our example, the digit 2);
- **I**nsert the number of times the divisor is contained in the tagged digit in the first space dedicated to the quotient digits in the diagram. In our example, 47 is contained in 2 0 times, so we write 0 in the first space reserved for the quotient from the left;
- At this point (we are at the "×" in the acronym) we have to multiply the divisor with the number found, in our example 47×0 = 0;
- Finally, we subtract (we are at the "-" in the acronym) what we obtained from the tagged digit, in our example 2–0 = 2.

We then repeat the same procedure "TI×-". In our example, we now have to tag the digit 5, and transcribe next to the result of the last subtraction carried out, so that we now consider it as the number 25. Now, 47 in 25 is contained 0 times, so we transcribe 0 into the second space dedicated to the quotient, and then we calculate 47×0=0. At this point we subtract 0 from 25, obtaining still 25, and so on. We are therefore able to complete the division when we have tagged all the digits of the dividend, obtaining 53 as quotient, and the remainder 13 as the result of the last subtraction.

At the moment of the discussion we analyze here, participating students had worked since grade 2 on Canadian algorithm, while TI×- algorithm had been introduced for about a month: the students had learned to execute it correctly, but no time of the previous lessons was dedicated to the deepening of the mathematical meanings underlying its functioning.

This work focuses on the analysis of the signs that emerged in the mathematical discussion. We will distinguish between *situated signs, mathematical signs,* and *pivot signs* (Bartolini Bussi & Mariotti, 2008). Situated signs are signs that arise during the activity with the artefact, so they are contextual and understandable only to the participants to the activity at that time; mathematical signs, on the other hand, are the formal signs referring to the mathematical knowledge at the basis of the task designed by the teacher. Finally, pivot signs are particular artefact signs that the teacher can use to support a possible evolution from artefact signs to mathematical signs. We coded the transcripts classifying the signs with the labels “situated signs” (SS), “mathematical signs” (MS), “pivot signs” (PS). We focus especially on signs that could be related to mathematical meanings underlying the two algorithms, in particular those related to positional value of dividend digits and to the meaning of the sign to tag dividend digits in TI×- algorithm.

Data Analysis

To answer to our research questions, we chose to analyze two excerpts of the mathematical discussion of the first lesson, that we believe to be particularly significant to show the potential of synergy for the discovery of relationships between the two algorithms. The mathematical discussion was started by the teacher, who showed on her shared screen the operation 874:7 carried out with the two different algorithms (see Figure 2).

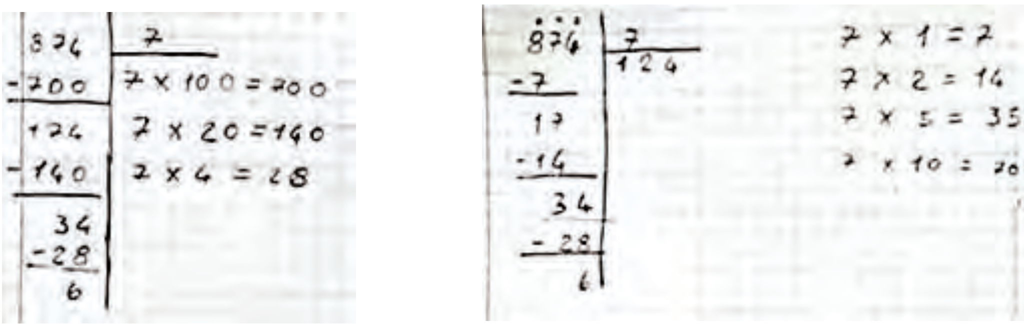

Figure 2: Optimized Canadian algorithm (on the left) and TI×- algorithm (on the right) to calculate the division 874:4

1	P1:	[...] I think that [...] it is not that we do faster in one algorithm, for example in the Canadian algorithm with respect to the TI×- algorithm, because in the Canadian algorithm we subtract bigger numbers, but in the TI×- algorithm we subtract the numbers... [...] 7, 14, 28, though... [the number 7] counts like the hundreds and then it is removed from the hundreds [of the dividend]...
2	Teacher:	Wait, I'll try to repeat what you said [...] in the TI×- algorithm, when I subtract 7, for example… what happens, to this 7 here? What do you mean, P1?
3	P1:	It's like... it's like a hundred [he means 7 hundreds], because ... it's below 8, but not because it is below 8, because it is used as... hundreds.
4	Teacher:	So, you are saying that this 7 is worth 7 hundreds... okay? And in the other algorithm what do we have in the first step, P1? […]
5	P1:	In the first step we have 700 [to subtract].[1]
6	Teacher:	Minus 700, ok.
7	P1:	It is because it involves all the numbers [the Canadian algorithm], but actually I think the TI×- algorithm is faster because you have to write fewer things.

In this first excerpt, the comparison of the two algorithms applied to the same operation allows P1 to make an interesting consideration, linked to positional value of dividend digits in TI×- algorithm (see the reference to MS "hundreds" in lines 1 and 3). In line 3, P1 tries to express something about the dependence of TI×- algorithm from some formal choices, such as tagging and respecting digits' vertical alignment. P1 seems to describe - still at an intuitive level - that the value of 7 subtracted from the dividend digit 8 should not be inferred from its vertical alignment, but from a certain "use" of 7 as hundreds, not better specified. It is only with teacher's intervention (line 4) that the identification of subtracted 7 in TI×- algorithm and subtracted 700 in Canadian algorithm becomes explicit (line 5), so that in line 7 P1 realizes a further step describing a more general difference between the two algorithms – namely, TI×- algorithm's articulation digit-by-digit, absent in Optimized Canadian algorithm.

[1] It is necessary to specify that, unlike in English where 7 hundreds and 700 are both pronounced as "Seven hundreds", in Italian 7 hundreds and 700 are pronounced as two different words, "sette centinaia" and "settecento" respectively. So, we note that in Italian the correspondence between 7 hundreds and 700 is not as transparent also for linguistic reasons.

8	P2:	I think they are both correct and I also noticed another thing: in the Canadian algorithm I see that you will always add a number... if you remove the two zeroes from 700 it becomes 7; if you remove the 4 from 174 it becomes 17. It is as if in Canadian algorithm you add a number [i.e. you write a digit at the end of each subtrahend].
[...]		
9	P2:	I was just thinking of the end [i.e., to the last subtraction 34 – 28 = 6 present as the last step in both algorithms], because when I thought that 700 becomes 7 when one removes the zeroes, it seemed strange to me that the end was identical [in both algorithms] and that you don't remove 4 from 34, so that it doesn't become 3.
10	Teacher:	Then you are saying: the last operation is the same.
11	P2:	Yes, because it is as if the division made different calculations but it has to arrive at the same point, where there is no other way to get to the result, so you have to do that ... that reasoning to get to the result.

In line 8 P2 describes a connection between the two algorithms, but differently from P1 she does not talk in terms of place value. Instead, she uses SSs "remove the zeros" and "add a number [digit]". In the expression "it is as if in Canadian algorithm you add a number", P2 formulates a simile that constitutes evidence of her attempt to describe something she has intuitively grasped but that she is not completely able to express. In this perspective, "add a number" and "remove the zeros" have the value of PS, with the potential to lead P2 to discover mathematical meanings at the basis of TI×- algorithm - in particular, positional reading of dividend digits and of numbers to be subtracted in the various steps. As we see in lines 9 and 11, this observation triggers P2 to deepen the relationships between the two algorithms. Indeed, P2 moves her attention to their last part. In the expression "it seemed strange to me that the end was identical and that you don't remove 4 from 34, so that it doesn't become 3", P2 identifies the last subtraction ("the end") as a common aspect of the two algorithms, but she also observes that this correspondence breaks her previous expectation on how TI×- algorithm works. The expression "you don't remove 4 from 34" is a PS, which recalls the previous sign "remove the zeroes" in line 8. Indeed, also this sign - "you don't remove 4 from 34" - has the potential to lead to the discover of positional reading within TI×- algorithm: the teacher can exploit this sign to support a reflection on the fact that the number 34 does not respect the correspondence identified by P2 because it must be interpreted in term of unities, differently from the other subtracted numbers in the previous steps of TI×- algorithm. SS "the end was identical", moreover, signals a further step towards a recognition of the common process at the basis of both algorithms, with respect to what P2 observed in line 8. In that line, she was still focusing on the perceived distance between the two algorithms; in line 9, instead, the identity of their last subtraction elicits a feeling of "strangeness", pusheing P2 to formulate in line 11 a more general conjecture on the similarities between the two algorithms, moving from a formal description towards meanings. Two significant SSs in this respect are "different calculations" and "same point": these are PSs with the potential to evolve towards a discovery of distributive process common to both algorithms, presented through different formal steps.

Discussion and conclusions

Data analysis shows how the designed task allowed the emergence, in the discussion among the involved children, of situated signs potentially significant for a progressive development of mathematical meanings that are crucial to explain the two algorithms' functioning, especially regarding TI×- algorithm. As we observed, in fact, this second algorithm is opaquer than Canadian one, with respect to the progressive distribution process underlying to both. Regarding the first research question, terms such as "hundreds", "remove the zeros" and "add a number" emerged as potentially crucial signs for this development, since they can lead to a reflection on the dependence of TI×- algorithm on place value of dividend digits. Situated signs as "different calculations" and "same point", instead, are significant because they were used to describe a similarity regarding the general mechanisms at the basis of both algorithms. These signs, in a TSM perspective, could be useful to the teacher to manage the discussion, in order to allow students discover the progressive distribution process underlying both algorithms. Regarding the second research question, the signs "different calculations" and "same point" emerged in relation with the issue of the identity of the last step of both algorithms. As shown by research with a similar approach to the discussion of division algorithms (see Lisarelli et al., 2021), this can be one of the key considerations to develop a "backward" reasoning to build a real argumentation of the two algorithms' functioning, using the more transparent algorithm to shed light on the steps of the opaquer one. This is particularly relevant for the argumentation of the opaquer algorithm's functioning (for Lisarelli et al. it was DMSB algorithm, for us is TI×- algorithm) and the discovery of mathematical meanings at its basis. Therefore, the designed task and the following discussion allowed the emergence of situated signs related to mathematical meanings common to both algorithms involved. It is necessary to underline that, both in our study and in that by Lisarelli et al., the synergy of artefacts could be seen as a substantial identity of distributive process underlying the involved algorithms: the opaquer algorithm can be seen as more refined and efficient on a formal level, through an appropriate recourse to the digits' vertical alignment and their reading according to their place value. Using a metaphor, we can say that this choice of algorithms as artefacts used in synergy transforms one algorithm into a sort of "can opener" of mathematical meanings for the other one. We believe that the analysis presented here contributes both to research concerning the introduction and the use of artefacts in mathematics classroom and to research concerning the teaching of algorithms in primary school. Our study contributes also to discussion about the potential of comparing algorithms and procedures as means of development of students' conceptual and procedural knowledge (e.g., Rittle-Johnson et al., 2017; Weber, 2019), since we highlighted the powerfulness of synergy of algorithms as artefacts, especially when among them intercours a relationship such as the one we described. Further studies are needed to confirm this, and to explore if there are other conditions determining which synergies are useful to develop mathematical meanings.

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Re-discovering sorting algorithms: the importance of cooperation

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Keywords: Algorithms, Discrete Mathematics, Cooperation.

Introduction and Context

In the growing field of research of elementary discrete mathematics education, the research wants to focus on some aspects of young students, grade 4 to 6, learning about sorting algorithms. Moreover, elementary computer science is gaining attention both from the CS education research community and from the elementary school teachers and heads (Franklin et al, 2015) and new trends in teaching CS in primary education are recently emerging. From a Mathematics Education perspective, we are thinking of our work as able to enhance the study of teaching and learning skills of mathematical practice through discrete mathematics problems, both general skills, such as reasoning, modeling and problem solving skills and others particular to discrete mathematics, such as algorithmic and recursive thinking skills. Here, in particular, we want to underline aspects of cooperative dynamics emerging from some sequence of tasks regarding the above.

The tasks we are going to present are part of a preliminary research that were proposed to various schools and age groups. In a design research paradigm (Plomp and Nieveen, 2007), the activities were tried out many times, always with an a priori analysis together with the teachers and with a retrospective look after each lesson. Students from two different schools were involved: a total of 22 classes in grades ranging from 3rd to 8th. Before this, we did a preliminary survey among teachers (Gaio & Di Paola, 2018). Teachers, especially at lower levels, admit not to have the necessary knowledge to teach discrete mathematics and algorithmic topics in school and are really open to explore new possibilities on the topic.

Background Theory and Methodology

Teaching methods follow the model of Realistic Mathematics Education (Gravemeijer, 1994) and Guided Reinvention of mathematics (Brousseau). Focus is on the activity, on the process of mathematization (Freudenthal, 1973). Realistic Mathematics Education (RME) is an instructional design theory which centers around the view of mathematics as a human activity (Freudenthal); "*The idea is to allow learners to come to regard the knowledge that they acquire as their own private knowledge, knowledge for which they themselves are responsible.*" (Gravemeijer).

The research methodology is that of design research. For the purpose of this thesis, the developmental approach is taken into consideration (Plomp and Nieveen, 2007), the goal being to design and develop a, research based, intervention and constructing design principles in the process of developing it. The goal is to explore new learning and teaching environments, to verify their effectiveness; to develop somehow new methods, instruments, and teaching actions to further improve in the field of problem solving and logical thinking, using somehow unusual topics as algorithms and cryptography are for primary school students. The design experiment is a classroom experiment in which the researcher (or researchers) cooperates with the teachers in assuming teaching responsibilities.

Tasks and data

As stated above, in a longer sequence of tasks, we want to focus on some tasks regarding sorting. The content goal is to teach something about sorting (numbers, quantities, objects), algorithms mode of operation, their speed, and the fact that there might be different algorithms leading to a solution. In some activities basic computational complexity is emerging, even if not formally introduced to children. We want them to re-discover the algorithm (or algorithms) which is good to solve a predetermined problem and get a sense of what it means to order something. Ordering themselves in groups, ordering object using a balance scale. At the end of each task a discussion follows.

Data collected is represented by events selected from the video recordings available. We used an inductive approach in video selecting, beginning with viewing the corpus in its entirety and focus on details later. Some events which were particularly relevant were isolated. We are, with this approach, analyzing selected episodes focusing on the same happening and constantly revising our finding and new hypothesis.

Results and perspectives

As some examples of the results obtained, we have a video-analysis of groups sorting themselves into a given order. Looking at the collaboration between students, one can observe that two different algorithms translate into two totally different group dynamics. There could be one or two students acting as leaders and just giving direction to others in the group, or more cooperative oriented groups, or also when a leader is absent. While using sorting networks, for example, the importance of discussion and the process of dialectic with peers allows students to discover important points in the mathematical situation they are facing, such as a sense of comparison between different numbers, according to Vygotsky's perspective on the zone of proximal development. On a similar example, students find out that if they are greedy - or we might say use a greedy algorithm - then the group might not success in the final goal and the collaboration with the others is important in mathematics, as in many other aspects, to reach (faster and better) results.

Finally, the belief is that this kind of activities can be relevant, beyond a CS perspective, also from a mathematics point of view as described briefly above and deserve a deeper investigation in research.

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An exploratory study on mathematics teacher educators' beliefs and understandings about computational thinking

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This paper reports an investigation of mathematics teacher educators' views and perceptions on computational thinking (CT) and its impact on mathematical learning. We conducted semi-structured interviews with experienced mathematics teacher educators, all of whom have extensive experience with the use of digital technologies for mathematical teaching and learning and report on data from two of them. Our aim is to offer insights into how CT is perceived and understood by them, to support them in self-assessing their possession of CT practices, and how to support mathematics teachers and students in gaining CT. We offer ideas regarding the promotion of CT and its integration in mathematics teaching and learning.

Keywords: Computational thinking, mathematics teacher educators, digital technologies.

Introduction

Over the past decade, there has been upsurge of interest in the teaching and learning of computational thinking (CT). The proponents of CT conceive of CT as a critical skill for all that will enable humankind to harness the power of computers for the common good (e.g., Wing, 2006). As a result, many countries are in the process of introducing CT into their school curricula, either as a new dedicated subject, a cross-curricular theme, or integrated within an existing subject, such as mathematics (see, e.g., Bocconi et al., 2018; see also, Royal Society, 2018). The relationship between CT and mathematics has been of particular interest. Indeed, some see CT as offering the potential to transform school mathematics (eg, Perez, 2018), but realising this potential will be a challenge, for students, mathematics teachers and mathematics teacher educators (MTEs).

Teacher education will be critical in enabling mathematics teachers to realise the potential of CT to transform mathematics. Yet, to date, educational literature on CT, or computational competency or the "new digital age competency" as sometimes is referred to, (e.g., Grover & Pea, 2013, Li et al., 2020) has mainly focused on students' CT. In this paper, we address this gap by investigating MTEs' CT and their computational practices. We present initial findings from an exploratory study, investigating the views of two experienced MTEs, who both have extensive experience with the use of digital technologies for mathematical teaching and learning, including specifically with teachers. We discuss their views on CT from a practitioner and a research perspective, debating about assessing the possession of CT and how to support mathematics teachers and students in gaining CT. We conclude by offering ideas about the promotion of CT and its integration in mathematics teaching and learning.

Computational Thinking and Mathematics Education

CT was first mentioned by Papert (1980) in his seminal *Mindstorms* book, it gained more momentum when Wing re-introduced it in her 2006 pivotal paper making the case for CT as a critical skill for all. Wing (2010) defined CT as "*the thought processes involved in formulating problems and their solutions so that the solutions are represented in a form that can be effectively carried out by an information-processing agent*". (Cuny, Snyder & Wing, 2010). Many researchers (e.g., Brennan & Resnick, 2012; Grover & Pea, 2013) have attempted to clarify this definition. This work emphasises that CT is less about the use of technology and computers and more about the concepts, practices and processes involved (e.g., Lodi, 2020). In Shute et al.'s (2017) terms, CT is "a way of thinking and acting, with or without the assistance of computers" (p.143).

The consensus of research (e.g., Shute et al., 2017) is that, whilst there are practices in common, CT is a distinct and separate discipline to mathematics. However, CT involves practices that are also required in mathematics, such as "decomposition, abstraction, algorithm design, debugging, iteration, and generalization" (Li et al., 2020, p.156). For the purposes of this paper, we will draw on Perez's (2018) summary of the literature on CT practices and dispositions as outlined in Table 1.

Table 1: Perez's (2018) categorization of CT practices and dispositions adapted (see Figure 2, p.428)

Computational Thinking Practices	
Problem Solving	*Abstraction & Generalisation*
Assessing and pursuing different approaches and solutions to a problem	Collecting, organizing, manipulating, and representing data
Generalizing and transferring problem-solving processes to other situations	Abstracting the essential elements of a situation or task
Using incremental and iterative approached, decomposing tasks into smaller pieces	Thinking in levels andunderstanding relationships within a system
Reusing, remixing, and innovating	Choosing effective tools and models for working with data
Efficient and effective combinations of resources, testing and debugging	Developing algorithms and automations
Formulating problems so that they can be analyzed using programs and other tools	Designing and using computational models and simulations
Computational Thinking Dispositions	
• Confidence in dealing with complexity • Persistence in working with difficult problems • Tolerance for ambiguity • The ability to deal with open-ended problems • The ability to communicate and work with others to achieve a common goal or solution	

Teachers' Computational Thinking

Some research has examined the teaching of CT both in general (see Grover & Pea's, 2013, review) and specifically in mathematics and other STEM subjects (see, e.g., Lee et al., 2020). A particularly fruitful avenue of research has investigated the use of Scratch programming in mathematics (e.g., Sung et al., 2018) with some potentially promising results (e.g., Boylan et al., 2018). However, the efficacy of such pedagogic initiatives is dependent on mathematics teachers' knowledge, beliefs and attitudes about CT. Some small-scale research has begun to examine the issues and challenges in this area. Sands et al.'s (2018) survey of 74 US teachers' views of CT suggests that one challenge may be teachers' limited understanding of CT. For example, they found that most respondents viewed CT as synonymous with doing mathematics and using computer or technology. Angeli and

Jaipal-Jamani (2018) examined the effects of an intervention on 21 preservice science teachers' CT in a small-scale study. They found that the use of scaffolded programmed scripts resulted in increased CT skills amongst the preservice teachers. However, they found that these increased skills were limited to lower levels of CT and, hence, their study highlights the difficulty of developing higher order CT skills such as generalisation and abstraction. Israel and Lash (2020) carried out a study in one US elementary school examining the integration of CT into mathematics teaching, mostly using the Scratch environment with some lessons using Code.org materials. They found that, despite a very strong commitment from the school and its teachers to CT and integrating CT and mathematics, relatively few of the lessons showed evidence of an integrated approach to teaching CT and mathematics. Chevalier et al.'s (2020) study suggests ways in which these challenges can be addressed. They found out that it is important "not only having a good understanding of CT (e.g., not focusing exclusively on acquiring programming skills), but also thinking and planning carefully in developing and implementing educational activities to develop students' CT" (as cited in Li et al., 2020, p.154).

Teacher Education and MTEs' Computational Thinking

As Lee et al. (2020) observe, teacher education and professional development are critical to the development of effective CT teaching (Weintrop et al., 2016). Yet, we are unaware of any research examining MTEs' knowledge, beliefs and attitudes about CT. To help us reflect on MTEs' CT practices, we decided to have initial discussions with MTE, who have extensive expertise in teacher education and research in the use of digital technologies for mathematical teaching and learning. Such MTEs' beliefs can support our investigation on what CT is, what CT practices are, what the relationship between CT and mathematical thinking is, how CT practices can be promoted among mathematics teachers, why CT practices are useful (or not) and what they offer to mathematics education.

The Exploratory Study

Our aim was to answer the following Research Questions: What are MTEs' CT practices? What are their views on mathematics teachers' CT processes and the link between mathematical thinking and CT? In order to gain insights to these questions, we carried out an exploratory study and interviewed three MTEs with expertise in digital technologies and mathematics. In this paper, due to constraints of space, we present vignettes of only 2 of those MTEs, whom we will refer to as Carole and Naomi. Both Carole and Naomi have school teaching experience, but also lengthy experience as MTEs (Carole over 10 years and Naomi over 7 years). They both have a doctorate in mathematics education and their research interests lie in the use of digital technologies for mathematics teaching and learning, but also in mathematics teacher knowledge.

The interviews consisted of 2 parts. In the 1st part, the interviewees had to solve a task, using the Think-Aloud protocol (Güss, 2018). We asked interviewees to reflect on (a) the programming aspects, (b) mathematical definitions, (c) the structure of the mathematical and the tool's language, and (d) the algorithms. Given that CT is a relatively new area of interest, we wanted a task that would enable the interviewees to articulate various aspects of CT practices. Hence, we chose a task that they were familiar with and involved a digital tool of their choice. This has an advantage of generating a range of ideas in a relatively short space of time, but has some limitations in terms of comparing the MTEs' beliefs. In the 2nd part, we asked them for their own definition of CT and how this relates to the definition by Cuny, Snyder & Wing (2010), which was mentioned earlier and states that CT is "*the thought processes involved in formulating problems and their solutions so that the solutions are represented in a form that can be effectively carried out by an information-processing agent*". Then, we asked them to reflect upon their own approach to solving their task in relation to the CT practices as presented by Perez (2018) in Table 1, identifying which CT practices they used. We asked them to (a) Reflect on mathematics teachers' CT: You have used this task with teachers, what aspects would you highlight to teachers in terms of what is different to pen-and-paper mathematics? How would you design a task to make teachers aware of key elements and features relevant to CT?; (b) Reflect on CT and mathematical thinking, knowledge, teaching, learning, pedagogy. How is CT linked to mathematics? What pedagogical strategies regarding CT would you use with teachers?

Carole's Vignette

A mathematical task

We asked Carole to tackle the 'Rich Aunt' task, a problem that she has used frequently in teacher education at M-level. In this problem, students are asked to decide between one of three gift schemes:

a) £100 now, £90 next year, £80 the year after, and so on;
b) £10 now, £20 next year, £30 the year after, and so on;
c) £1 now, £2 next year, £4 the year after that, and so on.

To solve the task, Carole used Excel to create a table using a formula and dragging across cells to create a table and a graph comparing the annual value of each scheme. Carole uses this task to enable teachers to "experience the power of Excel", because, without Excel, it would be more "time-consuming". However, one doesn't always need to think mathematically: "That's the problem with excel you are not even aware that you actually typed in a formula. … Which is slightly different to having an awareness of a formula behind it, the mathematical formula behind that."

Carole's definition of Computational Thinking

Initially, Carole conceived of CT as synonymous with computer programming such as Scratch and Logo and didn't consider the Rich Aunt task involved CT, saying that she had engaged Excel as a tool using "a simple formula to generate numbers by copying and dragging cells" rather than "thinking the problem through" as would be required for programming. When presented with the list of CT Practices [Table 1], she identified several practices that she had used in the task: pursuing different approaches, using incremental and iterative approaches, innovating, debugging, formulating problems for analysis using tools, organising and representing data and using computational models. She did not consider that she had used algorithms or automation, despite her use of formulae in Excel. She defined CT as "adapting your mathematical thinking to the tool at hand which has got computational power … you think of the mathematics first and then how do I go about using this tool …. You almost have a plan of how you're going to investigate the maths problem and really the tool is just a tool that helps you execute that plan."

Carole said that largely CT is part of mathematics, although she felt that algorithms and automation are "*not necessarily*" to be part of mathematics. She considered CT to be more about working with what is already known in contrast to mathematics which enables one to work with the "*unknown*".

Naomi's Vignette

A mathematical task

We asked Naomi to tackle the task presented in Figure 1 further below, which she has given to mathematics teachers for research purposes. In this task, teachers were asked to explore the 3 diagrams presented in GeoGebra and discuss how to use them when teaching one of the Circle Theorems ("the angle at the centre of the circle is twice the angle at the circumference") to a mathematics class. When discussing this task, Naomi commented on the teachers' keenness to avoid exploring complex diagrams, demonstrating lack of confidence in dealing with complexity. Naomi reflected upon the value of looking at a simpler case to support mathematical thinking and work towards proving a conjecture. Choosing such a special case to make a conjecture simpler to think about, could be considered as decomposing the given task into smaller pieces, which is a CT practice as presented in Table 1. Naomi also reflected upon the importance of task design considering the constraints of the tool in use. GeoGebra may have been chosen due to its dynamicity and the benefit of exploring many different cases, but Naomi argued that you may find some level of dynamicity when using pen and paper. In fact there is some rigidity and inflexibility in GeoGebra as it follows certain mathematical rules, referring to rounding errors as well as the fact that the theorem 'broke' for certain extreme cases. She referred to a tension between Geogebra as a mathematical tool and a tool for mathematics pedagogy and suggested that this tension may prompt productive mathematical thinking.

Naomi's definition of Computational Thinking

Naomi defined CT as "not just about programming, it is about a wider understanding of using computers, but even more broader than that digital technology to solve problems […] kind of using the software tools for an investigative process" to solve a mathematical problem in our case. She later on added that CT is "part of problem-solving or in other words, it's about incorporating another tool into your problem-solving kit". After being presented with the definition as shared in the literature, Naomi reflected: "You need to appreciate that you put things in order… so that could be perhaps tending to be an algorithmic process (putting things in order) so that GeoGebra then becomes a useful tool, as you construct things". It is also worth mentioning Naomi's view on what the "information processing agent" is and that it might be restrictive. This phrase may seem to imply the use of a digital tool, but Naomi offered her own personal definition mentioning any tool that can support the problem-solving processes. "That decision making about what's the right tool to use […] is that fuzzy boundary of computational thinking [and] other kind[s] of tool-based thinking".

When presented with the list of CT Practices [Table 1], Naomi identified several practices that she had used in the task: efficient and effective combinations of resources, testing and debugging, formulating problems so that they can be analyzed using programs and other tools, abstracting the essential elements of a situation of task, thinking in levels and understanding relationships within a system, choosing effective tools and models for working with data. Naomi considered some of the practices to be poorly defined. For example, she argued that "thinking in levels and understanding relationships within a system" is difficult to interpret as she is not sure what is meant by "levels", although she speculated that it might be related to "different levels of abstraction". She commented that "Reusing, remixing and innovating" didn't make sense to her. She argued that "testing and debugging" should have a more prominent role in CT. To justify this, she gave as an example the GeoGebra diagram and when teachers dragged the points to explore the diagram, 'testing' special cases, such as 360 degrees or 0 degrees, and addressing the pedagogical challenges created by rounding errors in GeoGebra.

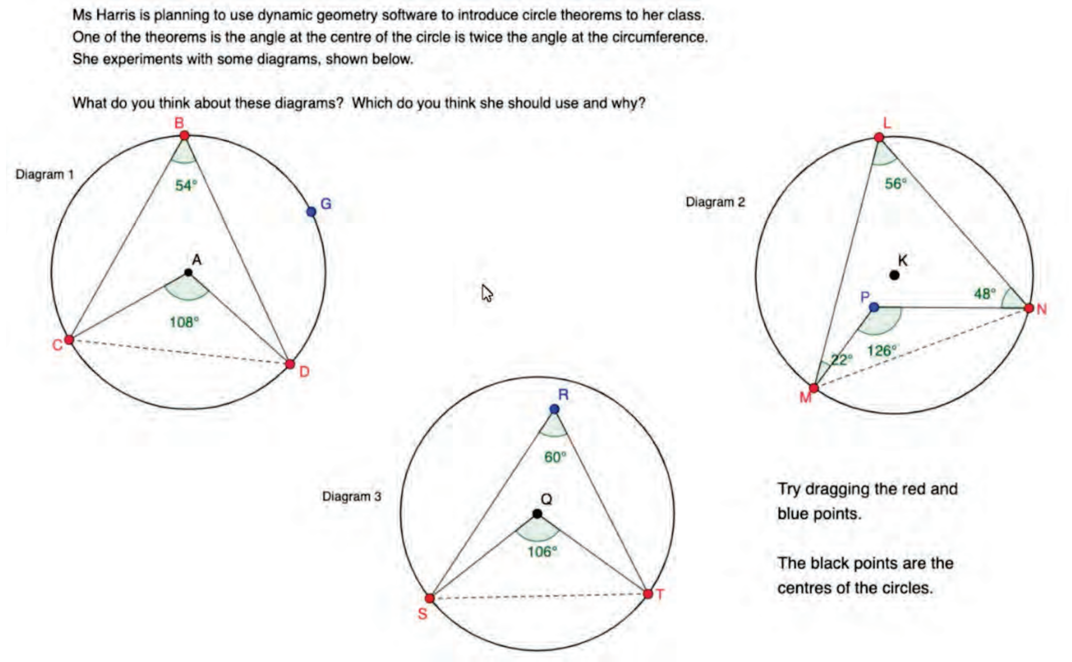

Figure 1: The GeoGebra task Naomi used for research purposes and discussed during her interview

Conclusion

Our study indicated that both Carole and Naomi were skilled in CT and mathematics to solve and discuss a familiar task. Unlike the teachers in Sands et al.'s (2018) neither viewed CT as synonymous with doing mathematics nor simply using digital tools to do mathematics, although both appeared to view CT as closely tied to computers and other digital tools. Nevertheless, their understandings and beliefs about CT were somewhat different. Naomi considered CT to be distinct from mathematics and involving an understanding of how to use digital tools to investigate problems in mathematics and beyond. As such her beliefs were broadly in accordance with the consensus of the academic literature, albeit she appeared to believe CT to be closely related to digital tools. In contrast, Carol appeared to view CT largely as part of mathematics and synonymous with programming. Indeed, she did not consider that using a spreadsheet such as Excel involved any CT. We note here that Carol's views may have been influenced by the particular task she discussed in that the automation and iteration involved very well-understood mathematics: addition, subtraction and multiplication. Despite Perez's (2018) claim that the practices identified in his review represent a concensus in mathematics education, the two MTEs found some of the practices identified to be unclear and poorly defined. This is despite both MTEs being highly skilled in the use of digital technologies in mathematics. Whilst Naomi appeared to believe that constructing algorithms provided a focus for thinking mathematically, Carole appeared to believe that using algorithms meant that mathematical thinking was no longer necessary. This highlights a tension between a common purpose in mathematics education of using digital tools to outsource some of the mundane and well-understood mathematical work, whereas in CT, it is crucial to consider, construct and adapt some of the less mundane mathematical processes such as analysing, generalising and abstracting (Pérez, 2018). Crucially, in CT, such processes are designed to be carried out by the information processing agent, which may be, but is not necessarily, a digital tool.

Our study suggests that MTEs would benefit from opportunities to explicitly engage with the nature of CT and its relationship to mathematics and to the application (and non-application) of technology. In doing so, a critical research task is to articulate the nature of Computational Thinking Pedagogical Content Knowledge (CTPCK) as a new term and its relationship to existing work, such as PCK in mathematics and the TPACK (Technological Pedagogical Content Knowledge, Koehler et al., 2013), which we hope to discuss in our future papers. Our future work entails the investigation of mathematics teacher educators' perspectives on what CT is and assess their CT skills, which elements in particular mathematics teachers possess and which ones they should acquire to enrich their mathematics teaching practice.

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Algorithmics in Arithmetic: Revealing algorithmic activities in a first-year arithmetic course for preservice teachers

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The science of algorithms, that is, the design of algorithms and the analysis of their correctness, complexity or efficiency, is at the intersection of mathematics and computer science, as outlined in the scope and focus of the new thematic working group 11 of CERME12. However, one might question whether algorithmics is indeed a relevant topic for mathematics education. This paper investigates whether algorithmic activities are included in a first-year mathematics course for preservice elementary school teachers. Indeed, for 25 of 26 lectures in the course algorithmic activities could be identified.

Keywords: Arithmetic, preservice teacher education, elementary school mathematics, mathematics education.

Introduction

The new founded TWG11 that meets for the first time at CERME12 is focused on *algorithmics*, which is, according to the description of the TWG, the science of the design and analysis of algorithms. In this article, I try to give reasons for including such a group for a conference on mathematics education, by identifying algorithmic activities (to be defined later) in a lecture that is clearly addressing the needs of prospective elementary school mathematics teachers.

Before we can describe *algorithmic activities*, we have to explain what we mean by *algorithm*. In mathematics education, some prominent algorithms usually serve as defining examples, like the long division. More generally, according to Cormen (2009)[1], "an ***algorithm*** is any well-defined computational procedure that takes some value, or set of values, as ***input*** and produces some value, or set of values, as ***output***. An algorithm is thus a sequence of computational steps that transform the input into the output" (emphasis as in the original text). Of course, the output is not just any output, but should be the answer to the problem that is encoded by the input. Thus, the long division algorithm is an algorithm that receives a dividend and a divisor as input, and gives the quotient and remainder as output. An algorithm that produces the correct answer to any feasible input is called *correct* and is said to solve the corresponding computational problem (Cormen, 2009).

An algorithm needs to be created or *designed* by someone in order to be used. It has to be formulated or *specified precisely* in some language, for example in a natural language like English, in (pseudo code) as a description in or almost in a programming language, or in hardware that represents the algorithm (Cormen, 2009). If there is such a specification, it can be used to *carry out* the algorithm either manually or by a machine (usually a computer).

[1] While there is a vast amount of literature on algorithms in theoretical computer science, I refrain from citing all possible sources. For further sources that support the general statements on algorithms, please use the overview in the notes for Chapter 1 of Cormen (2009).

There might be several algorithms that solve the same computational problem. This calls for *comparing algorithms* for the same problem with respect to their efficiency in terms of *time* (number of computational steps needed to solve a problem) or *space* (usually the amount of memory needed to store intermediate results), or both. Here, it is important that these measures are dependent on the input, and some algorithms might be better in certain instances than others, even if they solve the same problem.

Other measures that might be surprising at first are *elegance* or *simplicity*. The latter can be defined by the number of different computational steps (or *elementary operations*) used by an algorithm, for example. *Elegance* is not as easy to define in general, as it is often the case for solutions to problems in mathematics that are called "elegant". In Aigner & Ziegler (2010) this is discussed briefly for proofs of theorems that are worthy to be included in "The Book, in which God maintains the perfect proofs for mathematical theorems," – and they just refuse to define or characterize what constitutes such a proof. At the same time, they claim for many of the proofs given, that they are "elegant", again without a proper definition of elegant. For this paper, we can follow their lead and just conclude that it is possible to compare algorithms with respect to various measures.

In theoretical computer science there is much more work about algorithms that we cannot cover in this paper, but still we were able to identify several *algorithmic activities* in the last paragraphs:

- Design of algorithms: Creating an algorithm applicable to a class of problems
- Specification of algorithms: Describing the algorithm in a (formal) language
- Carrying out algorithms: Following the specification of an algorithm step by step
- Proving the correctness of algorithms: Finding arguments why the algorithm indeed solves the class of problems, either formally or pre-formal
- Comparing algorithms with respect to time (number of steps while carrying out), space (memory needed for bookkeeping during execution), elegance, or simplicity

In mathematics education, algorithms are often associated with mindlessly carrying out algorithms. Not only it is possible to *carry out algorithms* consciously, but this constitutes only a fraction of the possible algorithmic activities. For example, Krauthausen (1993) claims that it is a worthwhile activity to compare different approaches to written algorithms for multiplication, and Lisarelli et al. (2021) describe how 6^{th} graders compare division algorithms.

There definitely is a common ground for algorithmics both in mathematics and computer science. In Knuth (1985) it is discussed to what extend mathematical thinking and algorithmic thinking coincide, without a final conclusion, but highlighting several occurrences of algorithmic thinking in a sample of mathematics books. Knuth chose nine math books and analyzed the content on page 100 of each book, which is a very intriguing methodological approach.

In our research we want to find out, which algorithmic activities are relevant to mathematics education. As a first approach to this question, we analyze a lecture that is part of the study program for prospective elementary school teachers. So, to refine the research question, we ask: *Which algorithmic activities can be identified in a first-year arithmetic course for prospective primary school teachers at a German university?*

Context

The lecture that will be analyzed is part of the B.Ed. study program “Grundschulpädagogik” (elementary school pedagogy) at the University of Potsdam, Germany and has been created by the author and a colleague in 2010 at the University of Education Karlsruhe, Germany. It was held every year by author since then, and has been adapted to various settings in elementary teacher study programs at Karlsruhe (2010-2011), Potsdam (2014-2021) and the University of Halle-Wittenberg (2012-2013). The number of students in the course varied between 100 and more than 200, and it was always compulsory for the students. In Halle-Wittenberg, also prospective secondary school teachers were attending the lecture, and in Potsdam students of inclusive education both for elementary and secondary school have to attend the lecture. Starting 2021, the lecture has been made available as an Open Educational Resource (OER)[2].

The universities in Halle-Wittenberg and Potsdam are the only universities involved in elementary teacher education in their respective states, so all elementary school teachers in the states Sachsen-Anhalt (Halle-Wittenberg) and later Brandenburg (Potsdam) who studied mathematics during these years should have been affected by the algorithmic activities contained in the course.

The course curriculum matches the regulations of the module in the accredited study programs. For accreditation of the programs it is necessary that they cover the topics and competences as given by the Standing Conference of the Ministers of Education and Cultural Affairs (Kultusministerkonferenz, 2019), which are based on the more detailed information of the German mathematical societies (DMV et al., 2008). While we only consider a single course as opposed to surveying all arithmetic courses in Germany, we still have reason to believe that the results are in line with other courses at other German universities, as the course adheres to the national standards, has been created by two lecturers with different biographies, was used in at least three universities and has been based on existing literature.

The course itself has been designed without an explicit focus on algorithmics. As such, it stands for a generic first-year mathematics course in teacher education, or any other elementary mathematics course in mathematics teacher education. Our goal is to identify algorithmic activities in existing courses, and not to design a course that includes algorithmic activities, thus proving the fact that algorithmic activities are relevant in mathematics teacher education.

Data and Methodology

The data used for the study are the lecture slides that are provided as PDF files as accompanying material (1156 pages). For further details, also the original Keynote presentation (1446 slides) in the version of the academic year 2020/21 was available. Due to the Corona pandemic, the complete course has been recorded in short video clips of about 5-15 minutes that could be used asynchronously by the students. The complete video material (13+13 lectures for winter and summer semester, altogether 213 video clips with a total playing time of 28 hours, 19 minutes and 10 seconds,

[2] Currently available at https://openup.uni-potsdam.de/course/view.php?id=65 through a guest login. A full version of the course in German will be published later.

was available for in-depth analysis in cases where the presentation slides did not show the activities in detail.

As a second data source, the accompanying exercises and the interactive materials used in demonstrations and for self-explorations by the students is available. However, this study focuses on the content that was presented through the video lectures.

The material was reviewed by the author and each part of the lecture was classified depending on whether any algorithmic activity as described above could be identified. As our research question is not of quantitative nature, we refrained from asking a second person to code the data. Also, we are aware of the limitation that the author of the course and this paper is the same person.

The data was coded on a per-slide basis, using the slides and videos for clarification in some cases. For each video it was noted whether algorithmic activities could be identified on the slides. If so, they were categorized in the categories *Design (D), Formulation (F), Carrying out (CAR), Correctness proof (COR)*, and *Comparison or Analysis (COM).* Both the activity and the page number in the PDF slides were noted. The involved algorithms were recorded as well, and in some cases additional remarks. After coding, the data was aggregated for each episode in both series.

Results

The number of clips that contained algorithmic activities is shown in the table below, together with a short description of the algorithm involved in these activities.

Table 1. Algorithmic Activities identified in the lecture on arithmetic (S=Series, E=Episode)

S	E	German title of episode	Algorithms used	D	F	CAR	COR	COM
1	1	Mengen	Set Product		2		1	
1	2	Aussagen	Boolean operations / variables / loops	3	3	3	2	2
1	3	Beweisbedürfnisse und Beweistechniken	Find number not divisible	2	1	1	1	
1	4	Was machen wir gleich?	Change of representation / Completing a relation to an equivalence relation			2		
1	5	Everybody needs somebody to love - Abbildungen und Zahlen	Functions in General	1				
1	6	Zählen von Kardinalzahlen mit Ordinalzahlen	Counting and Peano	2	2	3	2	2
1	7	Vollständige Induktion	Induction / Zone theorem	1		1	2	1
1	8	Rechnen - Addition und Subtraktion	Addition by counting	2	3	1	2	1
1	9	Rechnen - Multiplikation und Division	Multiplication as repeated addition / division in various ways	4	4	2	4	2
1	10	GZSZ: Große Zahlen, Schöne Zahlen	Repeated bundling, base change	2	4	3	2	1
1	11	Rechnen in Stellenwertsystemen I: Addition und Subtraktion	Base change	1	4	4	2	2

S	E	German title of episode	Algorithms used	D	F	CAR	COR	COM
1	12	Mach mal Schräggitter & Wir bauen uns einen Computer	Written multiplication / hardware addition	4	4	3	2	3
1	13	Division in Stellenwertsystemen	Long division	3	3	5	2	3
2	1	Die ganzen Zahlen $\mathbb{Z}$	Multiplication of whole numbers (sign rules)				1	
2	2	Teilbarkeit, Teilen und Reste	Divisibility rules	3	6	2	4	4
2	3	Der Hauptsatz der Arithmetik	Euclidean algorithm etc.	4	3	3	4	2
2	4	Der Chinesische Restsatz	Chinese remainder theorem	3	2	1	4	1
2	5	Von $\mathbb{Z}$ nach $\mathbb{Q}$	Solving linear equations	1	1		1	1
2	6	Wie viele Rationale Zahlen gibt es?	Counting fractions	1	1	1		
2	7	Bruchrechnung in der Schule – Bruchzahlen	–					
2	8	Bruchrechnung in der Schule – Addition und Subtraktion	Addition	1	1	1		1
2	9	Bruchrechnung in der Schule – Multiplikation und Division & Dezimalzahlen	Division of fractions	1	1		1	
2	10	小数と分数 – Eine Reise durch die Klasse 3-6 in Japan	Multiplication algorithm	1	1		1	
2	11	Alles wird eins – Arithmetische und Geometrische Folgen	Representation of fractions	1	2	3	3	1
2	12	Reelle Zahlen	Approximation using Heron and nested intervals	2	4	2	3	2
2	13	$\mathbb{R}$ gebe mir ein $\mathbb{C}$	Algorithmic view on coordinates		1			1
		TOTAL		**43**	**53**	**41**	**44**	**30**

The data shows, that indeed all but one lecture feature algorithmic activities. In S2E7 fractions and their representation in school are introduced and discussed. In that lecture, no operations with fractions are used, so it is difficult to describe an algorithm that is based on several steps using operations.

Most algorithms are designed prior to their formulation. Instead of coming out of the blue (or the textbook), algorithms are depicted as something that is created by someone. We see that it is hardly the case that algorithms are just used without a proper introduction, that is, a focus on their design. Also, most algorithms are discussed for their correctness, at least in part.

The comparison and analysis of algorithms is the activity that happens the least. This can be related to the fact that in many situations standard algorithms exists.

In S2E7–S2E10 less algorithmic activities can be seen. These three lectures focus on teaching fractions in school, both in Germany and Japan. They rely on typical content found in textbooks, and do not introduce new mathematical content, but mostly subject-specific pedagogical content. Still, algorithmic activities are highlighted whenever possible.

An activity that was found in the additional material is a homework project called "Maths around the world". It takes place between the first and second semester of the lecture, and students are asked to find as many as possible interesting ways to do written calculation. Usually, they ask people from other generations (their grandparents) or friends from all over the world, or they have a migration background themselves and can report about their own experience in school. This activity is focused on analyzing and comparing algorithms, and shall compensate for the fact that most of the content in the lecture is being presented instead of experiencing it in a constructivist manner.

Discussion

As can be seen from the results, algorithmic activities are indeed integrated into all lectures but one. This gives rise to the question whether S2E7, the lecture on fractions and their models, could be enhanced with algorithmic activities, too. One possible activity could be an algorithm that creates models from fractions (or vice versa). Further inspection shows that this would be in line with a digital activity that is available in the moodle course and has been used to create the lecture slides as well. Figure 1 shows a Cinderella (Richter-Gebert & Kortenkamp, 2012) based interactive construction that students can use to explore fractions. The construction of such a representation can be described through an algorithm, which can be discussed in the lecture.

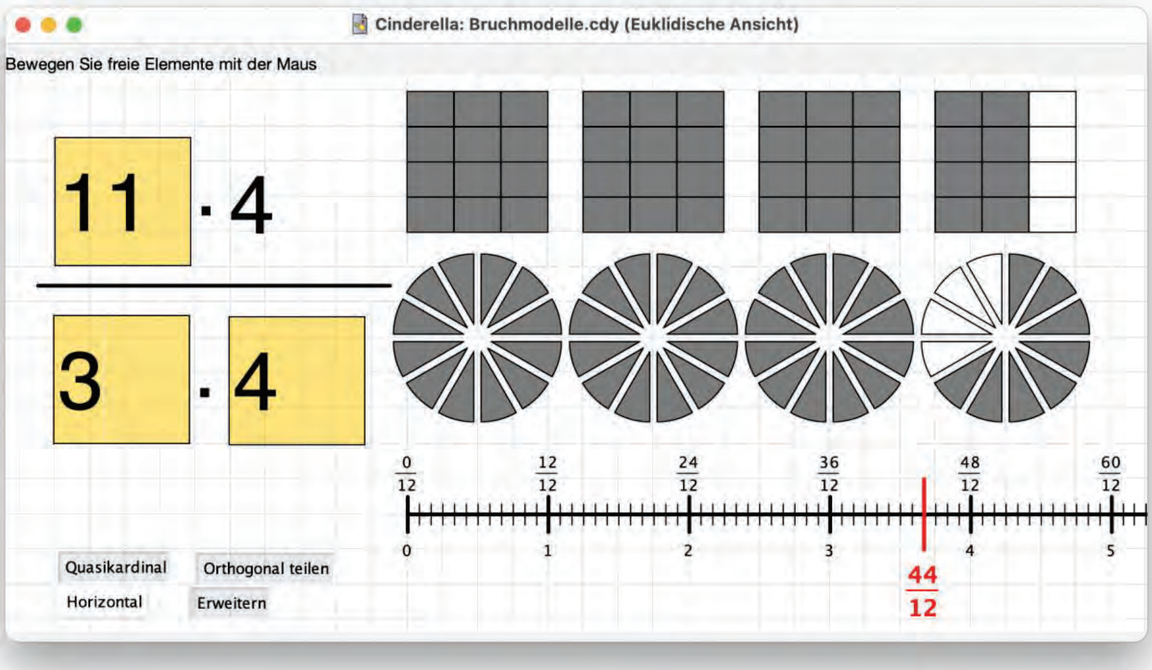

Figure 1. A dynamic representation of various fraction representations used in S2E7. The yellow fields can be changed using the mouse or keyboard and the representations change accordingly.

Although there are lots of algorithmic activities that are already incorporated in the lecture, we ask whether it is possible do more comparison and analysis. The comparison of algorithms is a very important element of the "Maths around the world" activity, where a lot of different algorithms for the same tasks are collected by the students. The question whether all algorithms lead to the same result concerns the correctness, but students identify differences like a faster execution, a better understandability, or a more general or more specialized approach. This also leads to the important insight that they should appreciate and embrace approaches that their future students in the classroom already know from their parents or by their own invention. Another very prominent algorithm comparison takes place in S2E12, where square roots of numbers are approximated through Heron's algorithm and a traditional nested intervals approach. Students experience again that both algorithms lead to the same result (and thus are equivalent), but Heron's method finds an approximation with a given accuracy much faster. This introduces students to questions about the efficiency of algorithms that constitute an important part of computer science and shows that not only the problem itself can have a certain complexity, but it also depends on the algorithm that is used to solve it. So, it would be great to find more opportunities for comparison. Unfortunately, this introduces another problem, as the teacher students would have to learn more algorithms and might complain about additional content for their examination.

Conclusion and Future Work

All in all, we can state that algorithmic activities are indeed an integral part of the arithmetic course we analyzed. While this could have been caused by background of the author of this paper, who also created the course, it still shows that algorithmic activities can be included in arithmetic organically.

The results from this study will be used to redesign the course, again. In particular, the homework assignments will be redesigned to include more algorithmic activities that are carried out by the students themselves, complementing the algorithmic activities that they experience during the lecture or in the video clips.

As a further step, we will apply the same methodology to other lectures in the B.Ed./M.Ed. program in mathematics education, to find out whether the results are specific to arithmetic – due to its computational roots – or can be generalized to other parts of mathematics as well. It is worthwhile to analyze other courses, in other subject areas like geometry, stochastics or algebra, from other authors and to see whether they already contain algorithmic activities, and to see how the inclusion could be achieved or strengthened based on the examples found in this course.

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From written addition to written subtraction: A proposal to develop and understand algorithms through algorithmic thinking

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Keywords: Written algorithms, elementary school mathematics, algorithmic thinking, subtraction.

Children have to be prepared to participate in an increasingly digitalized society and education is regarded to be the elementary key for children to participate in digital change. In Germany, the Standing Conference of the Ministers of Education and Cultural Affairs (KMK) has launched a strategy for education in the digital world. This strategy comprises a framework of competencies to be acquired by students during primary and secondary education. One of these competencies relates particularly to problem solving by using and/or developing algorithms.

Due to the traditional role of algorithms in mathematics, mathematics education is one area, besides informatics education, where thinking about and the development of algorithms may be promoted. Mathematics education research on written algorithms has shown that the written algorithms are on the one hand preferred by learners, but on the other hand are mostly carried out without understanding (Fischer et al., 2019; Jensen & Gasteiger, 2019). This has generally led to a critical consideration of the role of algorithms in primary education (Selter, 2000; Hurst & Huntley, 2018).

The notion of algorithmic thinking is associated with the two previous mentioned problem contexts: It is regarded as “a key element of the new digital literacy”, and “it can contribute to a deeper mathematical understanding” (Stephens & Kadjevich, 2019, p. 117). Algorithmic thinking comprises cognitive abilities such as decomposition and abstraction and is required whenever an algorithm is to be understood, tested, improved or designed (Stephens & Kadjevich, 2019).

In summary, in the context of algorithms in primary mathematics education three main strands can be identified and need to be followed:

1. Political demands starting from the role of algorithms in society (Use/development of new algorithms).
2. Teaching and learning algorithms with understanding in mathematics education
3. Algorithmic thinking as a possible link between problem context 1 and 2.

Research Question

The question is how these strands can be connected in mathematics education. There is a need for learning environments related to the learning of algorithms that contribute to a deeper understanding of algorithms and also foster algorithmic thinking. Therefore, our research question is, how to design a learning environment for the learning of written algorithms in primary school that fosters algorithmic thinking and contributes to students understanding of the algorithms.

Theoretical framework

We present a proposal of a learning environment, as part of a design-research project that connects the three strands related to algorithms in elementary school. The main idea of the learning environment is that algorithmic thinking is fostered by deriving written subtraction from the algorithm of written addition.

Students learn the difference between “putting together” (addition) and “taking away” (subtraction) as inverse operations in first and second grade. While written arithmetic is widely taught through demonstration (Jensen & Gasteiger, 2019), our instructional approach is to use the commonalities of the written addition and subtraction algorithms. Both algorithms are similar in the way that they calculate each place value separately and proceed from the smallest place value to the biggest, i.e. from right to left. However, the operation (add/subtract) to be carried out with each place value and the carry-over are inverse in addition and subtraction algorithms. Addition carries ten units of a place value to the next larger one, while subtraction works the other way around and borrows one unit of juxtaposed larger place value and regroups it into ten smaller ones.

The idea is that students decompose the iterative steps of written addition and invert them separately, and thus derive the written subtraction algorithm from written addition. This way, they carry out the two main aspects of algorithmic thinking. The students should discover that the steps of both algorithms are almost identical and only have to be inverted with regard to the operators "add/subtract" and "carry/borrow" (abstraction).

Consequently, the students use the discovered structure of the algorithm of written addition and refine it to written subtraction as step-by-step instructions to solve a subtraction problem with the derivative of an already known and formulated algorithm (developing new algorithms).

In a next step the learning environment will be tested with students who have just learned written addition in order to improve it with the goal to foster students’ understanding of written subtraction through algorithmic thinking.

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Mathematical algorithms in civic contexts: mathematics education and algorithmic literacy

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Mathematics is used to address civic problems, and mathematical skills also are also a basis for a critical attitude of responsible citizens. As algorithms are more and more presented as solutions to social problems, it is necessary to highlight how mathematics education can contribute to a reflected handling as well as an evaluation of the topic. For this purpose, exemplary algorithms from civic applications such as proportional representation, fair distribution, and algorithmic decision-making systems are the starting point for an analysis. The question is how discussing relevant algorithms promotes a deeper mathematical and contextual understanding. Linking both enfolds a mathematics-specific empowerment in an issue of general interest. As a conclusion, this results in a sketch of what algorithmic literacy could look like as an educational ideal of teaching algorithm in mathematics.

Keywords: Algorithmics, citizenship education, critical literacy, mathematical applications.

Motivation from the perspective of education theory

In a digital world where a lot of data emerges algorithms are presented as solutions to social problems, e.g. sentences based on algorithms as *fair* decisions. So, algorithms are too important to leave the concept obscure to most citizens who have a right to transparency and a say in the basic structures of society. According to Fischer (2012), citizen participation in a dialogue with experts requires basic knowledge and reflection skills on the part of lay citizen. How can mathematics education contribute to such basic knowledge and reflection skills regarding algorithms in civic contexts?

Considering the mathematical-contextual dichotomy of civic issues, a promotion of mathematical enlightenment must analyse both mathematical and contextual characteristics as well as their social relevance (Winter 1990). This paper will therefore explore how addressing civic extra-mathematical applications can foster a deeper understanding of algorithms and stimulate reflection on their use. Mathematics cannot only be used for a description of reality, but also for defining public quantifications, e.g. poverty line or tax rate. Since a major aspect of citizen empowerment through mathematics involves addressing the "formatting power of mathematics" (Skovsmose 1998, p. 197), the examples in this paper will focus on a prescriptive, hence non-descriptive, use of mathematics and algorithms. In a first step, classical algorithms are examined in their mathematical-contextual dichotomy regarding algorithmic literacy and citizen empowerment. An outlook will suggest why algorithmic decision-making systems pose an even greater societal challenge and raise new questions for this perspective. Beyond the reflection on algorithms currently envisaged in the context of algorithmics (Lagrange 2019, p. 33), there is a need for a fundamental awareness for the relevance of algorithms in everyday life, that is so necessary that "algorithmic literacy" (Oldridge 2017) refers terminologically to this ideal.

Comparing algorithms within mathematics and in applications

A very accessible definition for an algorithm would be that of a problem-solving tool that given a certain input, produces an output using a finite set of successive instructions, but it lacks the complex and constructive traits of algorithms (Modeste et al. 2010, pp. 53–54). From an educational perspective the broader definition provides important insights, but the specifications have enlightening potential, too. The constructedness of algorithms directly leads to the responsibilities of their application as part of a socially constructed reality. While intra-mathematical algorithms, e.g. the Euclidean algorithm as a paradigm, provide purely mathematical results (ibid., p. 58), extra-mathematical algorithms are based on a real issue that has been mathematised, and their output has a meaning outside of mathematics. An example of experiencing, learning, and testing such extra-mathematical algorithms is the fair distribution of an amount less than the sum of two individual claims. The Talmud suggests the following algorithm (Young 1994, p. 67): Distribute the resource equally until one receives half of one's claim. Distribute the resource to the other one until the missing amount to the full claim equals the correspondent opponent's missing amount. Now continue to distribute the resource equally. The algorithm terminates as soon as the resource is spent. If e.g. a resource is worth 300 and Mara is entitled to 300 and Nele to 200, we obtain the distribution (200, 100). For comparison, the proportional approach would deliver (180, 120).

Comparing the Euclidean algorithm with the Talmud algorithm reveals an important difference for the reflection of algorithms: In the first case, the algorithm is there to extract a well-defined result and to do so as elegantly as possible. In the second case, the result is not so clearly defined: What does *fair* mean? In addition, the calculation of the proportionality shows that the extra-mathematical meaning of an algorithm's output cannot only be vague in advance, but also ambiguous as different mathematical approaches (the alternative here is just a calculation) produce divergent solutions.

Proportional representation as a starting point for mathematical investigations

The importance of a mathematical investigation of different algorithms for a social problem can be illustrated by the example of proportional representation. If one allows only integer allocations, proportionality is not a solution. Under this constraint, there are several methods to get the output of a parliament composition, considering the share of votes as the input (e.g. Balinski & Young 2001). Three of them (Hamilton, Jefferson and Webster) are currently used in different German parliaments: Why is there no unitary solution and where are mathematical differences?

In order to distribute M seats, the Hamilton method starts with the rule of three and gives each party i the rounded down quota $q_i = M \cdot \frac{v_i}{V}$ (v_i: votes for party i, V: total votes); if there are still n seats that haven't been distributed yet, each of the n parties with the largest remainder gets one more seat. The divisor methods, on the other hand, are based on the following algorithm: Each v_i is divided by the same random starting divisor and the quotient is rounded with a certain rule (the Jefferson method rounds down, while the Webster method rounds to the nearest). If the sum of all these results is greater/less than M, the past step is repeated with a greater/smaller divisor. If the sum equals M, each party gets the rounded quotient as the number of seats (if M is missed due to the same rounding limit for several parties, the decision between these parties is made, for example, by drawing lots).

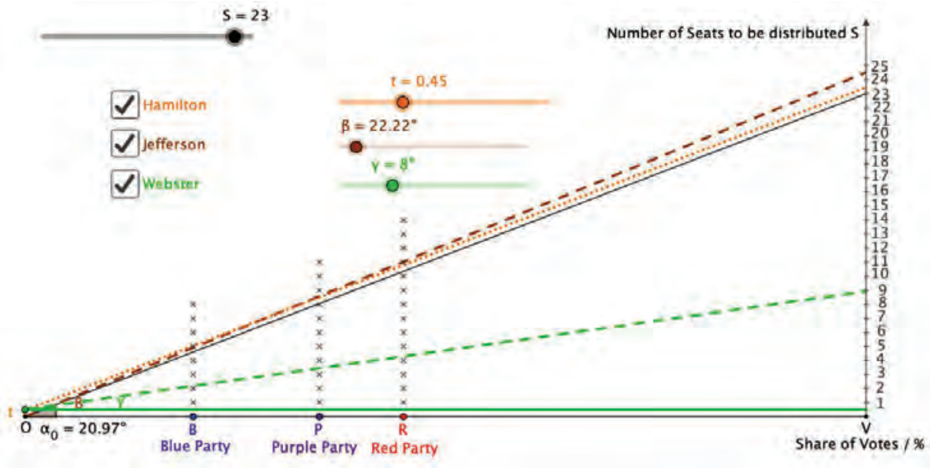

Figure 1: Visualisation of the algorithms of proportional apportionment

The obviously significant contextual consequence of a different parliament motivates the mathematical comparison of the algorithms. While their outputs can be easily contrasted, it is more difficult to compare the procedures of the algorithms. The intercept theorem offers a geometrical, dynamical visualisation (Figure 1). The length from O to the point of a party in relation to the horizontal leg corresponds to the party's vote share. Therefore, the length of the vertical leg through the point of a party indicates proportional seat share of this party. As there are normally less crosses for integer seats under the hypotenuse than seats to be distributed, the algorithm of the Hamilton method is consistent with the *translation* of the hypotenuse. The *rotation* of the hypothenuse can easily be derived from the visualisation as the common principle of the methods of Jefferson and Webster. This initiates a discussion on mathematical characteristics of the algorithms as well as contextual effects: The divisor methods preserve the original proportions, while in Hamilton method the triangle resolves in O. What does that mean mathematically? When does a method (not) favor larger parties? In which way can such a bias be desirable?

Since all algorithms try to approximate the mathematical concept of proportionality with integers, a comparison with mathematical means like the intercept theorem is possible and accessible. Nevertheless, this doesn't lead to an obvious best solution. It is a human decision which method should be prescribed in the electoral law. The importance lies in the fact that the methods can lead to slightly different results, which can even be decisive for the government in the case of narrow majorities. The existent pluralism in Germany document that the political choice between the methods is not so unequivocal as well. The fact that there are also other possible calculations of the divisor methods, each of which leads in each case to the same result, is not to be deepened here, even if it does enable a discussion on the result equivalence of algorithms.

Difficulty of the distinction and transparency: algorithms for fair distribution

With recourse to the distribution of a scarce resource, there is another algorithm with a variant of the „first come, first serve"-principle: Firstly one calculates all permutations of the different claims. For every permutation the resource is allocated along the order, until it is spent. The actual allocation is the arithmetic mean of all these allocations: If a resource is worth 300 and Mara is entitled to 300 and Nele to 200, we obtain the distribution (300, 0) if Mara is first; otherwise, it is (100, 200). The solution

Table 1: Talmud (Shapley) solution to the claims of three wives (Young 1994, pp. 71–72)

	$c_1 = 100$	$c_2 = 200$	$c_3 = 300$
$R = 100$	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
$R = 200$	50 $(33\frac{1}{3})$	75 $(83\frac{1}{3})$	75 $(83\frac{1}{3})$
$R = 300$	50	100	150

of this so-called Shapley value would be equivalent to the Talmud algorithm. Applying both algorithms to other examples with two claims leads to the correct hypothesis that this algorithm coincides with the Shapley solution for two claimants (Young 1994, pp. 67–70). However, the Talmud presents a second example of a fair distribution, but this time with three claims and varying resource values (Tab. 1). Obviously, this distribution doesn’t follow the proportional logic ($R \neq 300$), nor is the Shapley algorithm behind it ($R = 200$). The algorithm behind this centuries old Talmudic allocation rule has only been discovered in 1985 (Young 1994, p. 72). Its implementation for two claims has been described above, although there is a simpler and older solution in this special case, that doesn’t work for more than two claims. The long time of not knowing the algorithm exemplifies how difficult and lengthy it can be to reconstruct an unknown algorithm, even if the known input and output are rather accessible. At the same time, it becomes obvious how dissatisfying it is to obtain only the result without insight into the process or even an explanation.

Further conclusions can be drawn: Firstly, for certain inputs it is not possible to differentiate between some algorithms (Shapley vs. Talmud algorithm for two claims), so the constraints for this limited result equivalence should not be neglected. Secondly, even for a supposedly simple distribution there is no single solution, and one must choose between the alternatives. As there is no algorithm for objective fairness, the extra- and intra-mathematical reasons for a particular algorithm and against the other are significant. All ends up to the idea that both the mathematical and contextual aspects of socially relevant algorithms should be potentially transparent for interested citizens. But we have seen a degradation of transparency so far. In the case of proportional representation, the idea of propor-tionality determines the various algorithms and can easily be communicated. However, fairness has not such a clear mathematical landmark. Thus, the results appear even more subjective and debatable. Algorithms are used to create a fair process, in the sense that there are no arbitrary changing rules. The results are legitimated only by the *incorruptible* algorithm. But the choice of an algorithm itself is subjective and ambiguous. This contradicts the view that algorithms are neutral, which is ultimately a deduction from the myths about mathematics (Hersh 1991). For the aim of a critical citizenship, the daily experience of being confronted with established algorithms and its results can be facilitating by exemplarily implementing algorithms in a concrete context. An online game about fake news has shown that simulating the producer’s perspective promotes awareness for a further reflected consum-mation of fake news (Roozenbeek & van der Linden 2019). In analogy, students should formulate and compare different algorithms to solve social issues. In this way, they become sensitised for the important mathematical and contextual aspects in the face of existing or newly deployed algorithms. Until now, algorithms have been a tool that helped humans respond to civic problems. As seen above, the use of algorithms does not automatically lead to one suitable solution though. To find appropriate solutions to a problem, one must consider the mathematical properties and contextual consequences of the algorithm. Therefore, this aspect of algorithmic literacy refers to 'literacy through mathematics' since mathematics helps to understand the algorithm. Moreover, citizens should critically reflect the use of algorithms in the world and their stance on the issue. This requires 'literacy towards mathematics'. A simple example would be that if two friends must share a scarce resource, they can dismiss all algorithmic approaches and decide to not use algorithms or even mathematics to solve their problem. A more elaborate position would comprise the awareness, that the use of an algorithm doesn't form itself an indisputable legitimacy and that certainty in computation is not equivalent with unambiguity in an extra-mathematical context.

Automated decisions through algorithms: playful insights

It is precisely because of their constructive character and the clear step-by-step instructions that algorithms are potentially transparent and assessable. The use of artificial intelligence and algorithmic decision-making systems (ADMS) changes this. Nowadays, ADMS take over the decision and share only the final results without disclosing the finding process. Examples are decisions from banks, insurances, employers as well as from the justice system; common hope is to improve the fairness of the respective processes (MacCarthy 2019). These non-transparent techniques can no longer be completely retraced, but in a democracy a majority should be able to control and accept at least their fundamental principles. It includes the critical evaluation of cultural practices involving algorithms in ADMS. The didactic problem is that there can be no enlightenment if nothing can be made transparent. One approach is to analyze the functionality in easy, transparent examples and then to reflect further on the consequences if not all processes were longer traceable. One good teaching unit to do so is the Good-Monkey-Bad-Monkey-Game under creative common license (Lindner & Seegerer n.d., pp. 4–7). The basis is a data set of images of monkeys with different characteristics (smiling, eye shape etc.) and the information whether this monkey bites or not (for examples of such images see Figure 2).

By means of a training set, students must construct a decision tree as an algorithm to determine whether a monkey bites. The algorithmic decision trees are then checked on a test set. Even for the simplest version of the game, more than one decision tree is a fitting solution. Therefore, students are confronted with the ambiguity of a possible algorithm. In the extended version of the game, it is more difficult, first, to find a splitting for the training set, that, second, fits in the test set. In addition, there is one monkey in the test set that doesn't fit because of different, so far unknown properties. This undecidable case shows the limitations of this method, especially concerning the quality of the training and test set, from which reality may still differ. Instead of deterministic predictions as "every non-smiling monkey with x-shaped eyes always bites", such decision trees rather result in good probabilities. Figure 2 shows a constructed variation where x-shaped eyes still implicate biting, but there is only a stochastic correlation between smiling and biting (original data in the last column of Figure 2). A non-smiling monkey without x-shaped eyes bites with a probability of $\frac{2}{3}$: should it be

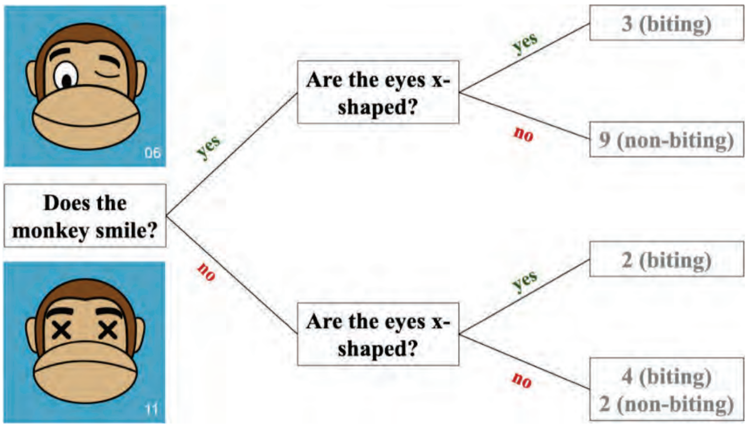

Figure 2: Decision tree for a non-deterministic variant of the monkey-game

isolated or incarcerated? Mathematically, the mistake is smaller, if such an animal is treated as a biting one. Socially, one can argue that punishing an innocent monkey is faultier than sparing the troubled one. If one omits the distinction of the eyes in this decision, a non-smiling monkey bites with a probability of $\frac{3}{4}$. The probability of punishing an innocent has decreased, although we have less information and fewer steps in the algorithms to the detriment of more biting monkeys that could be spared. Generally, less information or fewer steps go hand in hand with a greater uncertainty, but less work could justify a greater deviation regarding the cost-benefit calculation. E.g., not distinguishing the eyes of smiling monkeys leads to a $\frac{3}{4}$ chance of non-biting instead of a deterministic statement: Is this good enough in the related context? However, the characteristic of the x-shaped eyes is more compelling than smiling. If a monkey has x-shaped eyes, it is without any further distinction and without doubt clear that it bites. Hence, a decision tree starting with the x-shaped eyes would need less steps than Figure 2. The efficacy of an algorithm could be theoretically verified – in practice, the permutation possibilities grow fast with additional information –, but how can an algorithm take the other arguments into account?

Via such decision trees students deal with the construction and validation aspect of algorithms. By thinking in terms of case distinctions and conditional probabilities, they experience how a computer generates decisions themselves. While the algorithms so far have been independent from the context (e.g. the parliament size doesn't change the principle of the algorithms), the decision tree relies on the data of the context. In a second zoo with other monkeys the decision tree could be totally different. This raises the question of how objective and unambiguous such algorithms are for automated decision-making. With reference to statistical literacy, the critical evaluation can be extended to the origin of data (Gal 2002, p. 11): Were the monkeys with the x-shaped eyes hungry or have they been mistreated because of existing prejudices? The algorithm can't be neutral if the reference data is biased. This may not seem so central to the example of the biting monkeys, but what if job applications are pre-sorted using ADMS or recidivism is predicted for criminals?

Algorithmic literacy: towards a reflection of algorithms and ADMS

Algorithms and ADMS are used to solve real civic problems. This can be a starting point for a detailed mathematical analysis. It becomes clear that algorithms are neither unambiguous nor objective, e.g., a proportional representation by different algorithms can lead to different compositions of the parlia-

Table 2: Categorisation for a critical evaluation of socially relevant algorithms

<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="2">instruction independent from context</td></tr>
<tr><td>+</td><td>–</td></tr>
<tr><td rowspan="4">well-defined output</td><td rowspan="4">+
–</td><td>Euclidean algorithm</td><td rowspan="2">Good-Monkey-Bad-Monkey</td></tr>
<tr><td rowspan="2">Algorithms for
proportional representation</td></tr>
<tr><td rowspan="2">unknown-
unknown</td></tr>
<tr><td>Algorithms for
fair distributions</td></tr>
</table>

ment. Mathematical explorations (such as geometric representations) can help to answer the question of whether two algorithms produce the same output and how they differ. The extra-mathematical interpretation draws students' attention to the existence of such a pluralism.

Following Jablonka’s (2017, p. 44) categories of explicitness for the relation between mathematics and the context as well as for the intra-mathematical model, we would like to classify the discussed examples in terms to the explicitness of selected aspects of algorithms. The vertical axis is used to categorise the relationship between mathematics and context: while the greatest common divisor is explicitly defined as a mathematical output, ‘fairness’ of a distribution is a fuzzy concept. This category is a continuum rather than a dichotomy, since proportionality is well-defined within mathematics, but its application to seat distributions with integer seats is ambiguous. For the monkey game, ‘well-defined’ means whether the monkey bites (or with what probability), the consequences of how to deal with the monkey are yet not so clear. The second axis makes a difference whether an instruction is independent of the context: the Euclidean algorithm works the same for every two natural numbers. Decision trees, as in the monkey game, are based on data; with new information about other monkeys, the rule for decision making will change. The societal challenge is the fourth category of the “unknown-unknown” (ibid.). Indeed, there are more dimensions to what is unknown. Furthermore, one can differentiate whether humans don’t see the algorithm behind a decision or are not able to understand it. Maybe an understanding of some ADMS and its application is even not possible in a semantic way. To raise awareness for the inaccessible unknown-unknown, the only possibility is to discuss the other categories and to imagine losing their defining feature.

Teaching algorithms must thematize the consequences of implementations in social contexts. Zweig et al. (2018) propose general competencies for such a necessary algorithmic literacy. Respective mathematic-specific competencies with a citizen empowering potential are concretised, analogously to statistical literacy, in “worry questions” (Gal 2002, p. 17), e.g.: (1) What is the specific output and purpose of the algorithm and who has developed it? (2) Where did the data come from and what happens if the input changes? (3) How is the result derived? What is the underlying mathematical model of the algorithm? Are there alternative or equivalent models? (4) What is known and what is *not*? Is this adequate? In the sense of the mathematical-contextual dichotomy, mathematics education can contribute to an algorithmic literacy by focusing on the intra- and extra-mathematical characteristics of algorithms in civic contexts and on the interdependence between mathematics and the context.

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Exemplifying algorithmic thinking in mathematics education

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In this paper, we consider the construct of algorithmic thinking in mathematics education. We are convinced that algorithmic thinking is an invaluable way of thinking for students to develop, and so we are motivated to promote theoretical discussions in the field about the nature and utility of algorithmic thinking. We present three examples of algorithmic thinking – a mathematical example, an example from a mathematician interview, and an example from an undergraduate student interview. We then briefly review some relevant literature related to algorithmic thinking, and we conclude with some avenues for future research. Our goal is to further conversations about and refinements of characterizations of this important topic.

Keywords: Algorithmic thinking, Computing, Mathematicians, Undergraduate students.

Introduction and Motivation.

The phrase "algorithmic thinking" (AT) has appeared intermittently in mathematics education literature for the past several decades (e.g., Abramovich, 2015; Knuth, 1985; Schwank, 1993; Stephens, 2018), and it seems to be gaining a resurgence of interest with corresponding attention in computation and computational thinking in mathematics settings. In this article, we explore the construct of algorithmic thinking in mathematics education research. We are motivated by observations from our research with students and mathematicians, as well as in our own mathematical experiences. We have noticed a certain kind of algorithmic approach to problems, particularly within computational, machine-based settings, which reflect the presence of algorithmic thinking. These approaches, and the reasoning that underlies them, seem to be useful and valuable, and thus they represent a phenomenon that we want to better understand. In addition, we are motivated by the increasing presence of the term AT in mathematics and CS education literature, and we want to facilitate consistency and coherence for how it might be used in mathematics education.

We have two goals in this paper. First, we aim to exemplify AT within computational settings. We provide examples of such approaches in three ways: in mathematical examples, in interview data with mathematicians, and in data with undergraduate students. These examples allow us to illustrate what we mean by algorithmic approaches, hopefully facilitating better communication and discussion about the construct of AT. Second, given the existence of such algorithmic approaches and their reflection of AT, we briefly situate these ideas within math education literature and suggest ways to explore AT in future work. Ultimately, we aim to establish a shared understanding of algorithmic thinking, which could contribute to broader interests within the mathematics education community about the nature of thinking and problem solving in an increasingly computational world.

Broadly, an algorithm can be characterized as a set of steps to accomplish a given task. This broad definition allows for the term algorithm to include non-mathematical tasks like making a cup of coffee, as well as encompassing processes like the bisection method presented below. Rasmussen, Zandieh, King, and Teppo (2005) offered a definition of "algorithm" that serves as a useful starting point and highlights one way to characterize a distinction between an algorithm and a procedure in mathematics. Rasmussen et al. noted, "we use the term 'procedure' to indicate steps used to solve a particular task, and the term 'algorithm' as a reference for a generalized procedure that is effective across a wide range of tasks" (2005, p. 63). Implicit in this characterization is that there is often an underlying, generalizable approach of way of thinking on which an algorithm is based. Gravemeijer and van Galen (2003) made an analogy between mathematical algorithms and facts, using the example of the formula for area of a triangle (p. 115). To *know* a formula such as $A = ½\, b\, h$ implies that one can apply that formula to calculate area across a range of examples. This is consistent with the notion of a generalized procedure, and this is the general perspective of algorithms that we take in our work. Further, we acknowledge that any process could technically be considered algorithmic if it involves steps of any kind, but we are interested in approaches that foreground the *development* of an algorithm as opposed to only the performance or implementation of algorithms.

The phenomenon that we discuss in this paper is the ability to develop, explain, and iterate the steps of an algorithm in a mathematical context. Our examples highlight algorithmic approaches within a computational setting because such a setting tends to foreground the value of such approaches.

Examples of Algorithmic Approaches.

We offer examples of algebraic approaches in three contexts: a mathematical example, an example from an interview with a mathematician, and an example from an interview with an undergraduate student. In each of these cases we explain what makes the approach algorithmic, and we also discuss potential affordances of such an approach. Our aim is to demonstrate what we mean by algorithmic approaches, and also to make the case that such approaches arise in a variety of settings and situations.

An algorithmic approach in a mathematical example – the bisection method.

We begin with a mathematical example. We focus on an example of solving equations, which is a classical and central topic in mathematics. The standard approach to solving $f(x) = 0$ is to apply a sequence of algebraic operations to both sides of the equation, with the ultimate goal of isolating $\boldsymbol{x}$ on one side. This works well for certain classes of equations like polynomials of degree at most four and some equations involving trigonometric, exponential, and logarithmic functions. In more general situations it is usually not possible to find an explicit formula for the solution. However, the Intermediate Value Theorem tells us that a solution does exist even in very general situations: "Let f be a continuous function defined on the real interval $[a,b]$. If $f(a)$ and $f(b)$ have opposite signs, then f must have a zero at some real number c in (a,b)."

The Intermediate Value Theorem says nothing about where in $[a, b]$ the zero is, but it can help us to develop a strategy for computing approximations to the zero. Perhaps the most obvious approach is systematic guessing, and the simplest guess is the midpoint $m = (a + b)/2$. If we compute $f(m)$ and determine its sign, we see that we can limit our attention to a smaller interval $[a_1, b_1]$:

1. If $f(m) = 0$, we have stumbled upon the zero c and there is nothing more to do so we set $[a_1, b_1] = [m, m]$.
2. If $f(m) \neq 0$, then f has opposite signs at the two ends of either the subinterval $[a, m]$ or the other subinterval $[m, b]$. In the former case we set $[a_1, b_1] = [a, m]$, in the latter case we set $[a_1, b_1] = [m, b]$.

This process can be repeated with the new interval $[a_1, b_1]$, and we then obtain a new interval $[a_2, b_2]$. If we repeat again and again we obtain a sequence of ever smaller intervals $[a_3, b_3]$, … , $[a_n, b_n]$, … . We note that c is located inside each of the subintervals, but the length of the intervals is halved each time. We can then conclude that if we stop this process after n steps, we know that the current midpoint satisfies $|c - m_n| \leq (b - a)/2^n$. This is a very different approach toward solving an equation than balancing the two sides of the equals sign. Rather, this approach, which is known as the bisection method, is inherently algorithmic, and the big idea is that by iterating steps in this process, we can ensure that we can find as close an approximation to the actual zero as we would like. In fact, this idea can be adapted into a proof of the Intermediate Value Theorem. In addition, this idea can be made more specific in the form of pseudocode that describes the steps more precisely, as in Figure 1.

$a_0 = a; b_0 = b;$
for $i = 1, 2, \ldots, N$
    $m_{i-1} = (a_{i-1} + b_{i-1})/2;$
    if $f(m_{i-1}) == 0$
        $a_i = b_i = m_{i-1};$
    if $f(a_{i-1})f(m_{i-1}) < 0$
        $a_i = a_{i-1};$
        $b_i = m_{i-1};$
    else
        $a_i = m_{i-1};$
        $b_i = b_{i-1};$

$m_N = (a_N + b_N)/2;$

Figure 1: Pseudocode for an algorithm that represents the bisection method

This algorithm may be converted into code in a suitable language, and we then have our own substitute for the "solve-button" on advanced calculators. This algorithmic approach to solving equations naturally raises some questions, such as *Is this a valid way to solve an equation? What is gained by converting the algorithm into code and running the resulting program on examples?*
What do students gain from understanding such an approach to solving equations? We would consider the approach of iteratively finding the zeros of the function via the bisection method as we described it as inherently algorithmic because it involves first designing and then implementing an

iterative series of steps, which could generalize regardless of the function. The iterative, algorithmic process is foregrounded in this approach.

An algorithmic approach from a mathematician interview – summing primes.

As another example of an algorithmic approach, and the importance and potential usefulness of such an approach, we draw on an excerpt from a conversation with a mathematician, pseudonym Michael, about the role of algorithms in teaching mathematics at the post-secondary level. The excerpt comes from one of a set of interviews we conducted with research mathematicians about how they use computation in their work. In the first of these interviews, the interviewer (the first author of this paper) noticed the mathematician returning to the importance of algorithms and what he called "algorithmic approaches," so the interviewer asked him to expound on these ideas. Michael, a mathematical biologist, began with an example from a mathematics course that he regularly taught:

Michael: It's been interesting to observe that there are some people who just don't get how to do algorithms…So for instance, a really simple example that I always have at least one student ask me in the computational course—I ask them to sum the first hundred prime numbers using MatLab, and MatLab has this function called Primes. And the way that function works is you put in a number, and it returns all the primes less than that number. And I always have students ask me, 'but I only want the first hundred. I don't know which prime is the hundredth one.' So, and it's funny because other people just go, obviously, 'Oh pick a big number and just take the first hundred…And so that second one was a series of steps: pick a large number, plug it in the function, take the first hundred, and you're done. The other ones are like, I don't know if they don't, can't translate the question into a series of steps like that.

To us, Michael articulated an algorithmic approach that some students were able to leverage. Notably, he seemed to be describing a difference between students who do or do not have such an approach at hand (or, we would interpret, do or do not think in such a way on such a problem), and he was also implying some practical ramifications for not being able to think in such a way. That is, without the algorithmic approach he described students get stuck on the problem and do not know how to proceed. We consider this a description of an algorithm and not a procedure because it suggests an underlying approach that could be generalizable regardless of which prime is being considered. Thus, it is important to help students gain access to these approaches and to try to make some progress on these ideas. This example sets up a distinction in this professor's mind that there is something akin to thinking algorithmically (or to think in such a way as to leverage that approach), there are affordances to such an approach, and not everyone automatically uses that approach.

An algorithmic approach in a student interview – articulating "the way a computer thinks."

As a final example, we offer an example from an interview with a pair of undergraduate engineering majors who were enrolled in a vector calculus class, Corey and CJ (pseudonyms). These data came from a paired teaching experiment that occurred for a total of 12 hours over nine 60-90-minute sessions, during which the students wrote Python programs to list outcomes and solve combinatorial problems. We present an episode from the fifth session, in which the students were solving the Marbles problem: *Suppose you have six different marbles in a bag. Write out all of the* *possible ways you could pick two marbles out of the bag, without replacement.* They were asked to write code that would solve the problem. CJ listed the outcomes by hand as seen in Figure 2.

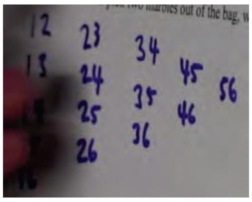

Figure 2: CJ's list of outcomes for the Marbles problem

Confident of their list, the students began to think about how to code the problem. CJ had an insight based on the list he had written by hand: "If we're just trying to list – the second one it looks at has to be bigger than that one." The interviewer asked him to follow up on this. CJ wrote the code in Figure 3, which lists pairs from the set of marbles via nested looping, where the conditional statement only prints outcomes in which the second term in the pair is bigger than the previous term. The *total* here acts as a counter, which is incremented each time and is printed at the end (yielding 15).

```
marbles = [1, 2, 3, 4, 5, 6]
total = 0

for i in marbles:
    for j in marbles:
        if i != j and j > i:
            print(i, j)
            total = total + 1
print(total)
```

```
1 2
1 3
1 4
1 5
1 6
2 3
2 4
2 5
2 6
3 4
3 5
3 6
4 5
4 6
5 6
15
```

Figure 3: CJ and Corey's code and output for the Marbles problem

Int. 1: CJ, you observed something in there about the second column.

CJ: If it just goes through **systematically the way a computer thinks**, looking at the next one and then using it; then our second column always has to be bigger than our first column. If *j* is bigger than *i*, then it won't ever print 2 and then 1. […] The *i* not equal to j that just makes it so it can't be 1, 1; or 2, 2. If *j* was bigger than *i*, then print it. These are all the orders, it won't ever do the same combination twice because it won't choose 1 and 2; then 2 and 1.

Corey: [Corey adds to the code in Figure 3] And *j* is greater than *i*.

Int. 1: Cool. If you do it, can you describe how many you think you'll get and what you think the outcomes will look like? Do you think – ?

CJ: I think it'll go through how I did it.

Corey: I think it'll go exactly the way he did it.

Int. 1: Great. What about your program makes you think it'll run through in that way?

CJ: Looking at *i* first, it's gonna go through everything that could have 1, and everything that's bigger than 1. Then it's gonna go to the next part; then it's gonna go to 2; then it's gonna look at everything that's bigger than two and print everything with that.

CJ reasoned about an algorithmic approach in this problem. This is seen when he discussed going systematically "the way a computer thinks," which suggests and underlying, generalizable approach rather than just one procedure. He was thinking about what algorithmic process the computer might adopt to list the outcomes. He connected his listing process to what the computer might do and correctly asserted that their code would list the outcomes as he had. We argue that the students adopted an algorithmic approach in writing their code, and they thought about what process the computer might complete to accomplish a task. In this sense, kind of algorithmic thinking we are describing is important students' engagement with computational practices like programming. In addition to helping students think about computational practices like programming, such an approach can also useful in highlighting combinatorial ideas. Lockwood & De Chenne (2020) explored how students' reasoning about conditional statements like "if j != i" and "if j > i" supported students' reasoning about permutations and combinations, respectively. Thus, the algorithmic thinking used to reason about programming can then also help to enrich students' mathematical understanding as well.

To summarize this section, we highlighted three instances of algorithmic approaches that emerged in our data and our mathematical experiences. These examples are meant to illustrate what we mean by algorithmic approaches and also to motivate our focus on better understanding such approaches. Each example shares common features – the construction of a systematic, step-by-step process which could (though does not have to) be programmed into a machine. We suggest that there is some way of thinking that underlies these algorithmic approaches, and this is what we want to try to characterize and understand. AT could be the construct that underlies algorithmic approaches, and, ultimately, these examples compel us to explore what AT might entail and why it might be important. Having exemplified what we mean by an algorithmic approach, we now briefly review mathematics education literature for existing characterizations of AT.

Algorithmic Thinking in Mathematics Education.

There are some perspectives in math education literature that frame AT as characterizing *a way of thinking* rather than thinking about algorithms themselves. This is in line with the approaches we exemplified in this paper. The phrase "algorithmic thinking" appears occasionally in mathematics education literature, and the construct is used in a variety of ways. Knuth (1985) proposed a distinction between "algorithmic thinking" and "mathematical thinking" as an effort to distinguish the thought processes of computer scientists from those of mathematicians. Ultimately, Knuth concluded that there is no such singular concept as mathematical thinking, nor is there for algorithmic thinking, but rather each is comprised of a set of modes of thought. Many of these modes overlap (e.g., formula manipulation, abstract reasoning, information structures) with a few exceptions. Notably, "mathematical thinking" includes the infinite while algorithmic thinking does not; and algorithmic thinking accounts for problem complexity (i.e., the "cost" of running an algorithm), which does not typically surface in mathematics. Knuth's characterization of algorithmic thinking seemed to contribute primarily to highlighting some of the overlap—and some of the distinctions—of the disciplines of mathematics and computer science.

The NCTM handbook reported on differences between algorithmic thinking and recursive thinking. Although this is an interesting characterization, we note that their definition is quite broad and could apply to almost any process or mathematical situation. They define algorithmic thinking as follows:

> *Algorithmic thinking* is a method of thinking and guiding thought processes that uses step-by-step procedures, requires inputs and produces outputs, requires decisions about the quality and appropriateness of information coming in and information going out, and monitors the thought processes as a means of controlling and directing the thinking process. In essence, algorithmic thinking is simultaneously a method of thinking and a means for thinking about one’s thinking.

Schwank (1993) investigated what she referred to as different mental models that students might apply to algorithmic thinking, although she does not define algorithmic thinking in this paper. The types of tasks that she used to characterize algorithmic thinking were more aligned with typical computer science or computer programming tasks. In this work, the term “algorithmic thinking” implicitly referred to the types of thinking necessary to construct algorithms, although Schwank’s primary argument was that there are multiple ways of doing so. More recently, Abramovich (2015) used the phrase “algorithmic thinking” somewhat analogously to how other researchers have used “procedural knowledge” in conversations about the relationship between procedural knowledge and conceptual understanding for learning mathematics. Abramovich described how problem posing—the cyclical act of extending a concrete (often procedural) problem to a more generalized case—could promote a link between procedural skills and conceptual understanding. There continue to be new contributions to the discussion of algorithmic thinking and how it can be integrated meaningfully in to the mathematics curriculum (e.g., Stephens, 2018; Stephens & Kadijevich (2020)). Stephens (2018) argues for AT as one type of reasoning and suggests that we consider ways to leverage powerful developments in programming to improve mathematics education.

Another line of research emphasizes activities and practices related to the creation of algorithms, such as *algorithmatizing,* which is an approach through which students develop algorithms through carefully chosen contextual problems that require students to model a particular situation, reflect on their solution procedures, and develop increasingly sophisticated models and procedures that can translate to other situations (Gavemeijer & van Galen, 2003, p. 114). The process of learning to use and understand algorithms implies their construction—or reinvention—on the part of students, rather than the acquisition of algorithms as existing objects. At the post-secondary level, students in a differential equations class reinvented Euler’s method for approximating solutions to differential equations, following the careful selection of tasks that facilitated the generalization of procedures (Rasmussen & King, 2000; Rasmussen et al., 2005). By engaging in algorithmitizing, which involves reasoning about and developing algorithms, students must necessarily think critically about what is entailed in that algorithm. Thus, algorithmitizing can be thought of as an external activity that reflects thinking about an algorithm.

To summarize, in the mathematics education research literature, the term AT has been used and construed in a variety of ways. There are similarities in terms of presenting a general approach toward solving problems, but we think there is room to specify a more consistent and coherent characterization of AT in math education. (Regrettably, due to space, we do not elaborate AT in the computer science education literature, although that is another rich resource and set of perspectives).

Future Work Toward Characterizing Algorithmic Thinking

Having exemplified algorithmic approaches that we feel represent AT, and having briefly reviewed some relevant literature, we conclude with some ideas for more work that needs to be done to explore and elaborate the construct of AT. There is not clear consensus among research communities about what AT is or how it should be characterized or defined, and more work should be done to study explore what is entailed in AT. Once characterizations of AT are established, empirical studies can be designed to explore how students, instructors, and mathematicians engage with and employ AT. One potential future area of research is to explore how students interpret, evaluate, and/or debug existing algorithms. In addition, we acknowledge that AT should be framed within other kinds of thinking – such as computational thinking (Wing, 2006, 2008), mathematical thinking, or recursive thinking – and we believe that more work is needed to relate such constructs and terms. In this brief paper we do not have space to explore AT's relationship to those other kinds of thinking, but there is potential for rich theoretical investigations. It would also be worthwhile to explore the computer science education literature as it relates to AT and to compare and contrast the development and use of algorithms in computer science versus mathematics and other fields.

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Graph problems as a means for accessing the abstraction skills

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Computational thinking is an important 21st century skill. The ability to design own algorithm, i.e. algorithmic thinking is its integral part. Graph algorithms seem to be a promising mathematical content contributing to development of algorithmic thinking. However, in order to apply the corresponding skills to the problem at hand, first a corresponding representations has to be found. This step of abstraction is crucial for the application of skills to unknown situations and can be seen as a prerequisite for the algorithmic thinking. Solutions of the representation problem of 58 undergraduate students were analysed. Most students chose the diagram as a representation of the situation, only three students used the adjacency matrix and no students chose the incidence matrix or adjacency list, the other known representations. This may indicate that more activities are needed for enhancing students' ability to represent the graph, either by matrices or by a diagram.

Keywords: Algorithmic thinking, computational thinking, abstraction, graphs, graph theory.

Introduction

With the continuous digitalization of our daily life the development of corresponding digital competencies is crucial for future generations. The importance of "[...] thought processes involved in formulating a problem and expressing its solution(s) in such a way that a computer - human or machine - can effectively carry out" (Wing, 2017, pp. 8) is more and more recognized. Thus the corresponding competencies are increasingly integrated into the current curricula referred under the term of Computational Thinking (CT) (Bocconi et al., 2016) including concepts such as logical reasoning, abstraction, decomposition, generalisation or algorithmic thinking, understood as "[…] the ability to think in terms of sequences and rules as a way of solving problems or understanding situations." (Csizmadia et al., 2015, p. 7).

The abstraction is the one crucial aspects of the CT describing the process of "… reducing the unnecessary detail" of a situation, problem, or artefact (Bocconi et al., 2016, pp. 18). Certain proficiency in abstraction is a crucial aspect during the process of problem solving, for mathematics as well as for computer science (Ferrari, 2003). Without the ability to simplify a given problem or situation in the process of abstraction resulting to a corresponding mathematical or computational representation, the remaining problem-solving process is bound to fail, nor can other maybe suitable solving strategies or algorithms be applied, as the corresponding requirements cannot be checked. This may lead to an unfeasible choice of problem-solving process for the given problem, resulting inevitably in an incorrect solution.

Graph problems offer a good starting point to address and foster abstraction (Milicic et al. 2021) as some applications of graph algorithms are easy to be explained and offer the suitable environment for mathematical investigations with low demands on previous knowledge, e.g. constructing Eulerian trail (Geschke et al., 2005), minimum spanning tree (Vidermanová & Melušová, 2011) or estimating chromatic number of graph (Prayitno et al., 2022). In order to obtain a graph from the application or a problem from the real world, firstly the redundant and unnecessary information have to be identified and removed. Redundant information are details of the given problem not contributing in any way to the problem-solving process and can therefore be omitted. Unnecessary information are details not related to the problem which should be solved. Graph problems offer therefore many possibilities to address and foster abstraction as the first step and following, algorithmic thinking subsequently. Despite existing algorithmic thinking skills, the process of abstraction can still be challenging for students (Wetzel et al., 2020), emphasizing the importance of addressing especially this aspect of CT in the school environment.

Like in other fields of mathematics, e.g., arithmetic or algebra, using multiple different representations and linking between them can enhance conceptual understanding of given topic (Hodnik Čadež, 2018; Griffin, 2004). Moreover, the graph representations have to be handled differently while implementing graph algorithms in programming languages and also lead to different space complexity of the produced instantiations.

The earlier findings (e.g. Hazzan & Hadar, 2005) shown that students often overestimate the value of diagrammatic representation of graph. This results to decrease of the level of abstraction while dealing with graph theory concepts and may cause the difficulties related to the recognition of the details of graph algorithms (Dagdilelis & Satratzemi, 1998). However, the recent study of Prayitno et al. (2022) reveals that students are able to come up with both, diagrammatic and matrix representation of the graph while solving novel problems stemming from graph theory and both types of representation can lead to correct solutions of the problem in algorithmic graph theory.

In this paper we present the initial results of an exploratory study with university students. It was conducted in order to identify any possible challenges students face when they are asked to abstract a given graph problem and condense the corresponding information in the suitable representation. As graphs have different representations, it is not clear which of them is preferred by students. We formulated the following three research questions:

1. To what extent are the undergraduate students able to use abstraction in order to represent the relation given by the computer model by the means of graph theory?
2. What representations do they prefer?
3. What kinds of mistake can occur there?

Methodology

To conduct this research, we analysed the submitted solutions of the 58 undergraduate students. All of them were in year 1 of the bachelor study programme Applied informatics and took the paper and pencil test focused on algorithmic graph theory as a part of assessment of the course Discrete mathematics 2 focused on combinatorics and graph theory. The test itself consisted of 6 problems. One of them is analysed in further details in this paper. The written informed consent was requested and collected after the students got their whole evaluation and passed/failed the course. The all of these 58 students attempted to solve the problem and provided their consent.

We expected that the participants are equipped with a decent knowledge of programming and therefore possess adequate algorithmic thinking skills, as they have passed the introductory course in programming in the previous semester, covering work with variables including arrays, sequences, conditionals, loops with numbered (for loops) and conditional repetition (while loops) of instructions. The test was administered at the end of the second semester of their study before the exam in regular programming course comprising the use of procedures, recursion, dynamic variables (FIFO/LIFO structures) and object-oriented programming. The different representations of graph, diagrams, incidence matrix, adjacency matrix and adjacency list were integral part of the discrete mathematics course.

The task of finding a graph that represents which area shares an edge with each other in the two dimensions is a frequently occurring problem when teaching and learning graph problems. Using the means of augmented reality (Buchner, 2018), we extended this problem by another dimension by not using planar areas, but some three-dimensional objects, see Figure 1a and 1b as one of the three different situations used in the exam.

We derived this task using the elements from the SOMA cube puzzle (Peter-Orth, 1985). The task was formulated as follows: “Represent the adjacency of the all parts of the kit (consisting of the 3 or 4 cubes) from which the ‘sofa’ is assembled. The tiny gap between the parts is only for better clarity.” The students were asked to scan a respective QR code and open a webpage inside a browser on their mobile device or with their web camera. No additional hard- or software aids were thus necessary in order to solve the task. By pointing the camera at the marker, the object as seen in Figure 1a and 1b was visible using Augmented Reality (AR). The students could turn the object around on their screen and shrink or enlarge it using common gestures with their fingers on the screen[1]. A solution using a diagram as representation is given in Figure 1c.

(a) (b) (c)

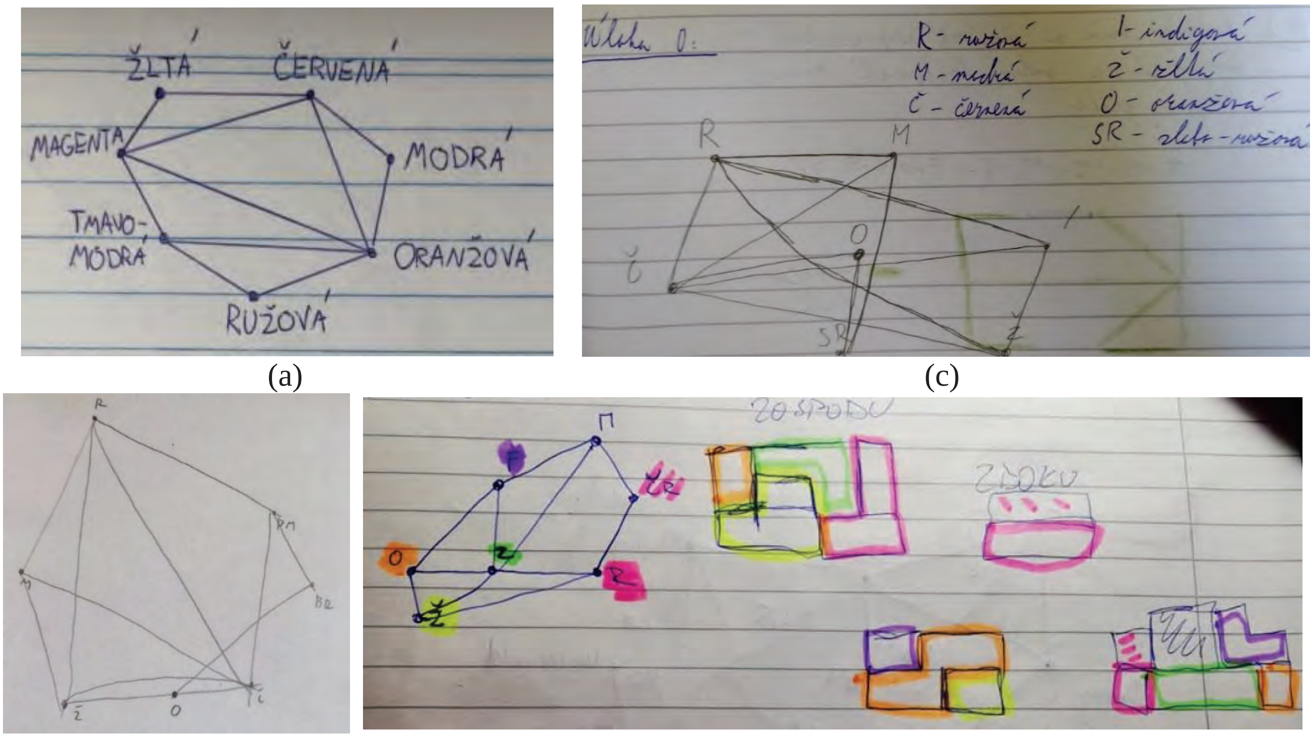
Figure 1: Two views of the sofa consisting of SOMA cube elements (a and b) and diagram of its graph representation (c)

[1] See https://colette-project.eu/AR/somas.html for the used AR-marker as well as the presented setting (Soma Sofa 1).

Findings

Out of the 58 solutions submitted by the students, 45, i.e., 77.6 % were correct. Although the students were not instructed which representation they should use, only the 3 students used the form of adjacency matrix, one of them used the adjacency matrix and the diagram, while other used the representation of the graph by a diagram. No students used incidence matrix or adjacency list. This may result from the dominant use of the diagram representation of graph during the lectures and problem-solving sessions throughout the whole semester.

Different names of vertices were used among the correct solutions. Only one student labelled the vertices as v_1, v_2, etc. and on top he provided a table which label means the solid of what colour. Some students used the full colour names (Figure 2a), some just the abbreviations (Figure 2b) and some also included a list of abbreviations (Figures 2c and 2d). Most solutions have the vertices placed in shape of regular heptagon (Figures 2a and 2b), only occasionally the vertices were placed in different configurations (Figure 2c). One student also used the colours and drew the bottom and side view (Figure 2d) on top of the graph representation.



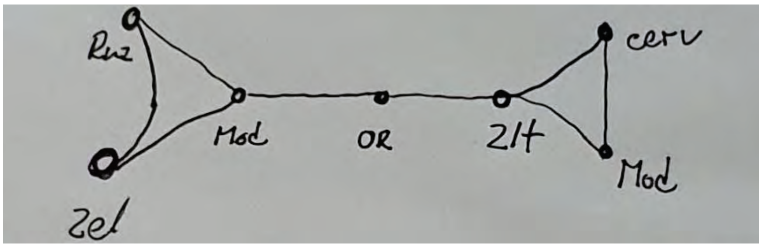
(a) (c)

(b) (d)

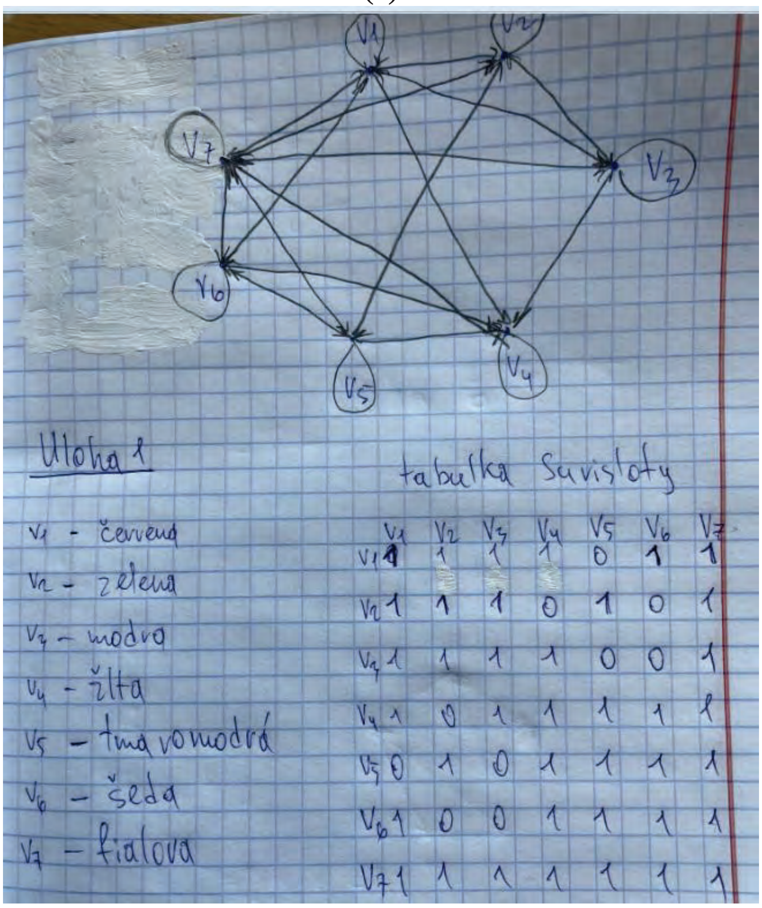
Figure 2: Students' solutions with different labelling of vertices

Ružová = pink; modrá = blue; červená = red; indigová = indigo; žltá = yellow; oranžová = orange; slabo-ružová = light pink; tmavo-modrá = dark blue. BR probably states for bledoružová = light pink and BM for bledomodrá = ligh blue. Zospodu = bottom view; zboku = side view.

The majority of incorrect solutions can be considered as flips when one or more edges were missing. The solution in Figure 3a is an example of solution where quite a lot of edges were missing. Furthermore, the student did not distinguish between the two shades of blue. On the other hand, the solution in Figure 3b represents a solution with excess edges. In addition, the student considered the relation of being adjacent as reflexive.



(a)

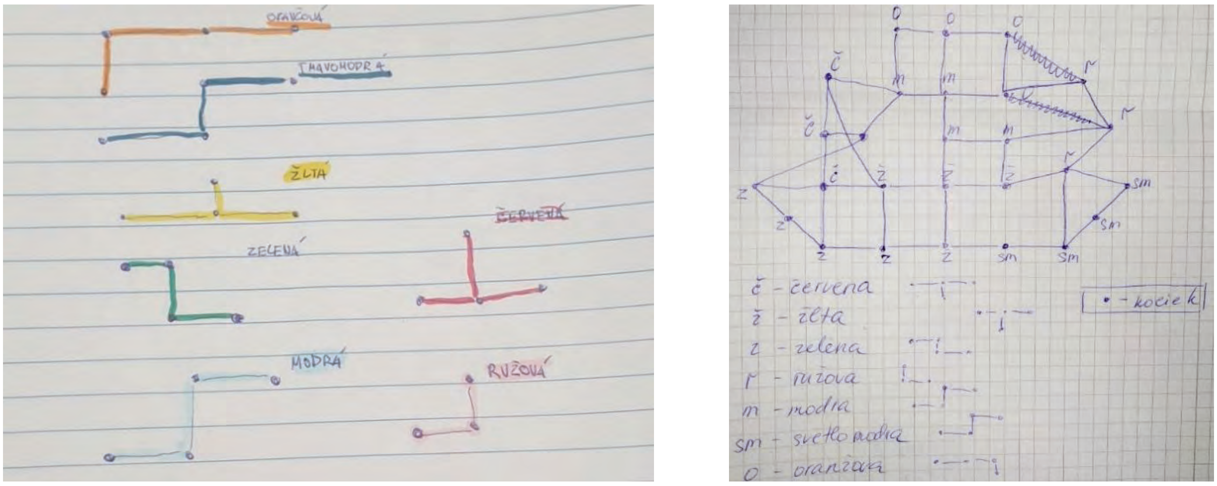


(b)

Figure 3: Examples of incorrect students' solutions

On top on typical solutions in Figure 2, there were several solutions demonstrating unique approach to the representation. For instance, two students considered the small unit cubes instead of the tricubes and tetracubes as vertices (Figure 4). One of them (solution in Figure 4a) further concluded that the graph is not connected as he considered the tiny gaps between the parts of the given problem as separating. The second unique student's solution (Figure 4b) is a correct one for this choice of representation. The corrections made by student demonstrate that it was quite challenging to keep tracking of so many (27) vertices.

The matrix representation of graph was preferred by only three of 58 participating students, two of them were correct. The incorrect representation by adjacency matrix (Figure 3b) was the only case when student used two different representations. Even though the diagram fits the matrix, the represented graph does not fit the adjacency of part of the SOMA 'sofa'.



(a) (b)

Figure 4: Solution of students' representing the cubes as the vertices

Discussion

The rate of correct solutions was much lower than expected in case of the students who passed the introductory course of programming and the course focused on the graph theory specifically. The observed success-rate was even lower than the success-rate of a similar group of students solving modelling problems in algorithmic graph theory (Medová et al., 2019). One of the possible explanations vests in that during the course and in ample cases of textbooks the problem situation is already given by diagram representation of graph and the abstraction job as such is already done by the author of the task (Fojtík, 2021). Also, pre-prepared 'suitable illustrative graphs … using colours' are often considered an effective mean 'to emphasize the characteristics of the concepts' (Milková, 2009). However, the informed choice of suitable representation is one of the crucial aspects when tackling unstructured open problems (Swan & Burkhardt, 2014) particularly in algorithmics, and influenced by the level of abstraction of the solver. Therefore, the process of abstraction itself, regardless the type of representation, is essential and crucial for solving problems in the situations requiring to omit the unnecessary and redundant information (Wetzel et al., 2020), most frequently in cases of real-life applications. The students should estimate the necessity of the information based on the definition of the particular problem.

Our findings suggest that the students used mostly the diagrammatic representation of graphs. It complies with the observation by Hazzan and Hadar (2005) than students tend to over emphasize the visual aspect of graph theory. The students may use the diagrammatic representation because the problem was not set to any complex problem situation, because, as stated by González et al. (2021) even the students with low level of reasoning in graph theory should be aware of limitations of the different known representations for different purposes. On the other hand, as we stated in our previous work (Milicic et al., 2021) any different representations of graphs can lead to different variables for representing the graph in the computer and it may be easier to analyse or use for subsequent tasks and solving processes. Adjacency and incidence matrices can directly lead to use of arrays and adjacency list is just a small step to the FIFO list. In contrast to diagram, the representations of graph by matrix and list permit to get the information about the adjacency of two vertices in constant time but the instantiations vary in space complexity. Even though the diagrammatic representation cannot be used for computer processing, it seems to be suitable for students while learning the principles of algorithms (Melušová & Vidermanová, 2011).

Conclusions

The tasks to represent the relations between the parts of geometric shapes are often used in graph theory education. The technology of augmented reality enables us to use some three-dimensional geometric shapes instead of usual planar problems without any additional hard- or software aids. The task to omit the redundant information and come up with an abstract representation caused some difficulties even in case of students equipped with their decent programming capability and experience. It seems that more attention should be paid to different representations of graph during the course, particularly to creating the representation by students instead of pre-prepared graphs.

However, the extent to which is the level of abstraction connected to the ability to solve (modelling) problems in algorithmic graph theory is still to be investigated and estimated.

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An app-based learning environment for building algebraic competencies through algorithmic thinking

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Keywords: Algebraic thinking, algorithmic thinking, programming app, primary school education.

Introduction

In recent years, the demand for the implementation of computer science (CS) content in German primary schools has strongly increased, so that there are several projects investigating its implementation. However, since a stand-alone subject CS in German primary schools is not a realistic scenario, relevant topics and competences should be integrated into existing school subjects (Schmid et al., 2018), e.g. mathematics.

Algorithms, Arithmetic, and Algebra

The connection of algorithmic, arithmetic and algebraic thinking can be realised by primary school children through the use of algorithms during programming activities (Agatolio et al., 2018). Kilhamn and Bråting (2019) also see an intersection between algebraic and computational thinking. However, this should be investigated especially with regard to whether programming activities support or limit the development of students' algebraic thinking. In their paper, they conclude that there are programming activities that have an impact on students' algebraic thinking (especially with regard to symbolisation). As an interesting open research area they define the question of whether structures embedded in algorithms can lead to "the development of algebraic thinking in terms of an increased focus on structure and generalization" (Kilhamn & Bråting, 2019, pp. 571–572).

In my PhD project I will examine intersections between primary CS and mathematics education with the focus on how the structures of an algorithm can influence generalisation in understanding an arithmetic topic. Therefore, a learning environment including an app (see Figure 1) with additional supportive tasks is developed. With this, primary school students can use and study the similarities of arithmetic and algorithmic structures with the goal to come to algebraic generalisations.

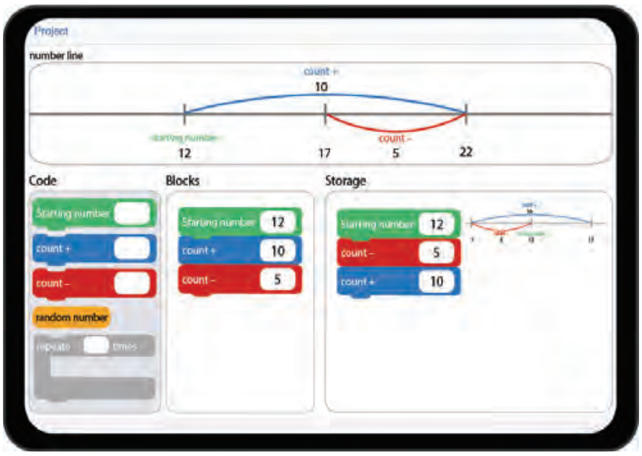

Figure 1: App 'Rechenketten'

Research Question

The learning environment and the research study are structured by the following research question: *To what extent can (1) understanding and analysing, (2) modifying, and (3) developing an algorithm help to obtain algebraic discoveries in terms of generalised arithmetic structures?*

Design of the Learning Environment

To explore and generalise arithmetic structures as a transition to algebraic thinking (e.g. Kaput, 2008), students use the arithmetic task format "Rechenketten" (arithmetic Chain, see Figure 1) to explore several arithmetic structures such as the property of commutativity, the distributivity or multiplication as repeated addition. The app is developed with the intention that these structures can be seen simultaneously in the arithmetic and algorithmic representation[1] so that a linking of representations is possible.

Therefore, three corresponding tasks are structured along the three aspects (1)–(3) of the research question: Task series 1 aims at understanding and analysing similarities and differences between the representations (1). Task series 2 aims at modifying e.g. inappropriate representations (2). Task series 3 encourages the development of algorithms as solutions to problem-containing arithmetic tasks (3).

Design of the Research study

In the sense of developmental research (Prediger & Link, 2012), a learning environment for the use of the app is to be (further) developed. In order to investigate learners' cognitive processes in terms of our research question, fourth graders will be interviewed after having used the app. The interviews will be videographed, transcribed and analysed using qualitative methods such as Grounded Theory.

For this purpose, a piloting of the app and the tasks was carried out in the autumn 2021. The insights gained from this will flow into further development so that the first design experiments can be carried out in spring 2022. These will then be evaluated to provide new insights for theory and practice.

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Task design for promoting pupils' algorithmic thinking in problem-solving context without using computers

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The paper presents the task design aimed at the emergence of the algorithmic thinking of pupils in the problem-solving context without using computers. Discussing the results of the experimentations carried out at the grade 5th, it aims to offer an account of the conditions and constraints for effective algorithmic problem-solving as well as to identify mechanisms that emerge.

Keywords: Algorithm, algorithmic thinking, problem-solving, task design.

Introduction

(Re)appearance of algorithmics and programming in school curricula of many countries has impulsed research addressing the development of pupils algorithmic thinking. In existing literature, including the one that relates to mathematical education, this concept is usually studied through the lens of programming (Hickmott et al., 2018; Bråting, & Kilhamn, 2020). To date, there has been limited empirical research about the conditions and constraints for the emergence of algorithmic thinking in a non-programming context. In this paper, we take a step toward addressing this need for research by studying algorithmic thinking via analysis of pupils' algorithmic activity in problem-solving context without use of computers. More precisely, we are interested in the pupils' activity related to the conception and implementation of an algorithm in order to solve a problem as well as reflecting on solutions through the analysis and proof of the algorithm. As a definition of an algorithm, we use the following one:

> Algorithm – a problem-solving procedure that in a finite number of constructive, non-ambiguous, effective and organized steps produces the answer to the given problem for all instances of this problem. (Modeste, 2012, p. 25).

This paper reports on research work in progress that has several goals. The first one is to design a set of tasks and the associated sequence of lessons aimed at the emergence of pupils' algorithmic activity. The second one is to examine whether the design is effective. In particular, to check whether and to what extent it favorize the construction, analysis and proof of the algorithms by pupils. The third one is to contribute to the development of a theory that would help to explain the mechanisms that emerged during the scenario implementation.

Theoretical framework

This study is a design-based research (Barab, & Squire, 2004) based on the hypothesis that unplugged problem-solving (i.e., problem-solving without using a computer (Bell et al., 2009)) constitutes a favorable context for the emergence of pupils' algorithmic thinking.

The task design draws upon principles from the Theory of Didactical Situations (TDS) (Brousseau, 1997). More specifically, the tasks were designed as *a-didactical* situations, where the pupils' interactions with the organized *milieu* are supposed to lead to the construction of the algorithms that are the optimal solution of the given problem. This implies that pupils' "basic knowledge" should be sufficient to start the work. The *milieu* must provide pupils with feedbacks to their trials in such a way that they can tell without teacher interferences whether they succeeded or failed. Pupils in a situation should be able to experience the limitations of their strategies developed at the earlier stages of problem-solving and make them evolve towards the algorithms that constitute a solution for the given problem.

In our choice of mathematical problem for task design, we based ourselves on the notion of fundamental problem for algorithm (FPA) proposed in (Modeste, 2012) with reference to the concept of fundamental situation from TDS (Brousseau, 1997). FPA could be seen as a problem that evokes the concept of algorithm and is adapted for the tasks design aimed at engaging the pupils in algorithmic activity. As the criteria of FPA, Modeste (ibid.) proposes the following: it should be algorithmically solvable; the notion of algorithm is indispensable for its solving; it evokes the problems that handle the algorithm as a tool (aspects of effectiveness and problem) as well as an object (aspects of complexity, proof, theoretical models). The key value of FPA is that it could be used for design of tasks/didactical situations for a class. These situations should be aligned with the criteria of FPA mentioned above. In particular, they should evoke an algorithm not only as a step-by-step procedure but also as a general problem-solving method that can be applied to all instances of the problem as well as to provide a favorable *milieu* for provoking questions about proof and complexity of the algorithm at stake.

Methodology

The research we report in this paper concerns the first cycle of the design that included the phases of development, *a priori* analysis, implementation, *a posteriori* analysis and refinement. The phase of development was carried out in parallel with the analysis of the strategies susceptible to appear during problem-solving. The experimentation phase used to collect data was followed by the analysis of pupils' strategies that appeared in the class. On the basis of the comparison between anticipated strategies with actual ones, we concluded about the potentialities of designed situations in promoting algorithmic activity of pupils and whether the task design should be improved.

In order to identify the mechanisms that emerge during the implementation of the developed sequence of lessons, we analyzed pupils' decisions and actions with material *milieu* as well as their discourses produced at the private and social levels. In particular, we paid special attention to explanations and arguments expressed inside a group and during the collective discussion regarding the question about the reasons why the strategies work.

In what follows, we first describe the problem of list sorting that, as we claim, is a fundamental problem for algorithm. Then, we present the set of designed situations, explain their link to the problem of list sorting as well as the analysis of anticipated solving strategies. We conclude the methodology section with a description of the designed sequence of lessons, experimental conditions and data collection. In the subsequent section, we present and discuss the results of the implementation of the developed lessons sequence in a class. The last section is devoted to conclusions and perspectives of the research.

The problem of list sorting

Formulation of the problem. In a model of computation with the only allowed operations of comparison and exchange, sort a non-empty list $L = (e_k), k = 1 \dots n$ (of objects comparable two by two) according to an order relation $\preccurlyeq$.

We use the term "model of computation" as a set of allowed operations in reference to the computation and complexity theory (Bilardi, & Pietracaprina, 2011). The relation of order $\preccurlyeq$ in a list is a binary relation that permits to compare the lists' elements between them in a coherent way. To sort a non-empty list $L = (e_k), k = 1 \dots n$ according to the relation of order $\preccurlyeq$ means to find a list $L' = (e'_k), k = 1 \dots n$ with the elements that are a permutation of the elements of L such as $\forall\, i,\, j = 1 \dots n,\; i \leq j \Rightarrow e'_i \preccurlyeq e'_j$. The "*operation of comparison*" is an operation that permits to receive the answer "yes" or "no" to the question "Is $e_i \preccurlyeq e_j$, i,j=1…n ?". The "*operation of exchange*" could be defined by the transposition of the elements e_i and e_j, $e_i \neq e_j$, $i = 1 \dots n, j = 1 \dots n$ in the list L that allows to obtain a list $L'' = (e''_k),\; k = 1 \dots n$ where $e''_k = e_k \;\; \forall k \neq i, j,\;\; e''_i = e_j$, $e''_j = e_i$.

The formulated problem responds to all criteria described above. Indeed, it is algorithmically solvable and many sort algorithms exist. Problem-solving includes the phases of elaboration and validation of an algorithm. The existence of many sort algorithms raises the questions of their comparison from the point of view of their complexity and finding the optimal one. This provides a possibility to study an algorithm not only as a tool but also as an object. On the basis of the problem of list sorting, we designed a set of problem-solving situations that we describe in the following section.

Set of problem-solving situations

The designed sequence of problem situations includes three games using the playing cards as manipulatives. In each game, a pupil has thirteen playing cards of the same suit and a cardboard grid with predefined ten places situated in one line. At the beginning of every game a pupil shuffles the cards, chooses randomly ten cards (without looking at their values) and puts them up-side down on the grid (one card per place). The rules of the games vary from one game to another, but the goal is the same, i.e., to sort the cards in ascending order, using only the three allowed operations: "take two cards", "put the two cards in ascending order", "put the cards on the places".

Game 0 (individual). A pupil should sort the cards individually, turning no more than two cards at a time and using the three allowed operations. There is no restriction for the number of comparisons to make (a pupil can turn the cards two by two as many times as he wants). Once the pupil thinks that all cards have been sorted, he can turn them to check this.

Game 1 (in pairs). For this game, the pupils play in pairs. Only one grid and thirteen cards of the same suit are needed. The first pupil (which we call in the following "operator") shuffles the cards, chooses randomly ten cards from thirteen and puts them on the grid without showing their values to another pupil. The second pupil ("player") must sort the cards which values he doesn't see, giving the instructions to the operator that correspond to the three allowed operations. The operator should execute the commands precisely and formally (without giving any information to the player). When the player thinks that the cards are sorted, he says "stop" and the operator turns the cards. The player wins if all cards are in the right order. The pupils exchange their roles once the player's attempt to sort the cards is finished.

Game 2 (in small groups). The pupils play in small groups of four persons. At every time there are one player and three operators that manipulate the cards on their grids. The rules of the game are the same as in the previous game, but, this time, the goal of the player is to sort the cards of all operators at the same time giving the instructions composed by the three allowed operations. The pupils exchange their roles once the player's attempt to sort the cards is finished.

Game 3 (in the whole class). This game follows the same rules as the previous one but this time, the goal is to sort the cards of all pupils of the class at the same time using the minimum possible comparisons. Each group send a delegate who has only one attempt.

Being a didactical variable, the number of cards to sort in the games could vary. But it should not be too small in order to provide a *milieu* sufficient to foster the development of sorting algorithms by pupils and not equal to the total number of cards (in order to avoid the "trivial" algorithm where one can deduce the final position of a card from its value).

As a possible prolongation of the proposed games, we may consider the problem-solving situations where the pupils investigate the possibility to sort more than 10 cards (11, 12, 20, 100 cards) and eventually any number of cards. The last refers to the proof of an algorithm (by recurrence, by invariant) and offers the possibility to study the complexity of an algorithm as a function of *n*. The notions of complexity in time and space could be also introduced considering the programming context.

Link to the problem of list sorting

The games 0 to 2 can be seen as the ten elements list sorting problem in the model of computation where a number of allowed operations changes from one game to another one. Thus, in game 0, the fact that the pupils can see the values of the cards can be considered as the case where the model of computation (CM0) contains the operations of comparison, exchange, identification of the index of an element with a given value and identification of the value of an element with a given index. In game 1, the fact that a player can see if the cards were changed by the operator or not, correspond to the model of computation (CM1) with the available operations of comparison, exchange, identification of the indexes of elements that have been exchanged and the operation that provide the response "yes" or "no" to the question: "Was there an exchange of the cards during the last comparison?". The goal of game 2 to sort the cards of three persons at the same time was retained in order to prevent a player to use the information about the cards' exchanges in order to put him in the model of computation (CM2) with only available operations of comparison and exchange that corresponds to the list sorting problem formulated in the previous section.

Expected solving strategies

The game 0 aims at familiarizing pupils with the manipulatives and the rules of the games. In particular, the pupils need to understand the meaning of the three allowed operations and the actions that are forbidden (for example, to glide the cards on the grid). It is expected that the pupils will memorize the values of the cards and their positions in order to sort them (strategy S_m in Figure 1).

Thus, the strategies that can appear will be rather the instance algorithms (that allow to sort only particular cases of the cards placement).

In game 1, due to the change of rules, the pupils cannot see the values of the cards, but they still can observe if there were exchanges or not. The strategy of memorisation of the positions of cards that were exchanged is tedious. Therefore, it is expected that the pupils will engage themselves in the research of a more economic strategy. Between the possible strategies we can imagine those that use the information about the exchange S^e during the last comparison of two cards and those that don't S^g. In the first category, it is possible to distinguish the family of strategies S^e_{st} that use the information about exchanges as a condition for termination and the family S^e_{ch} where the information about exchanges is used for choosing the next pair of cards to be compared but the termination of the strategy is determined by the end of the systematic process of comparisons. For example, to the first category belongs the strategy S^e_{lr} that compares cards two per two from left to right and backwards until there are no more exchanges and $S^e_{rdm}$where cards pairs are chosen randomly until there are no more exchanges. As examples of strategies that belong to the category S^e_{ch}, we can think of S^e_{in} that is based on the idea to create a subset of the cards with installed local order (constituted of one card at the beginning) that will extend at each step by insertion of one card from the non-sorted part. The insertion is made by comparing a card from the non-sorted part with all cards of the sub-set starting from the most right one until a card with which there is no exchange. Another example of the strategy of the family S^e_{ch} is quick sort S^e_{qs}.

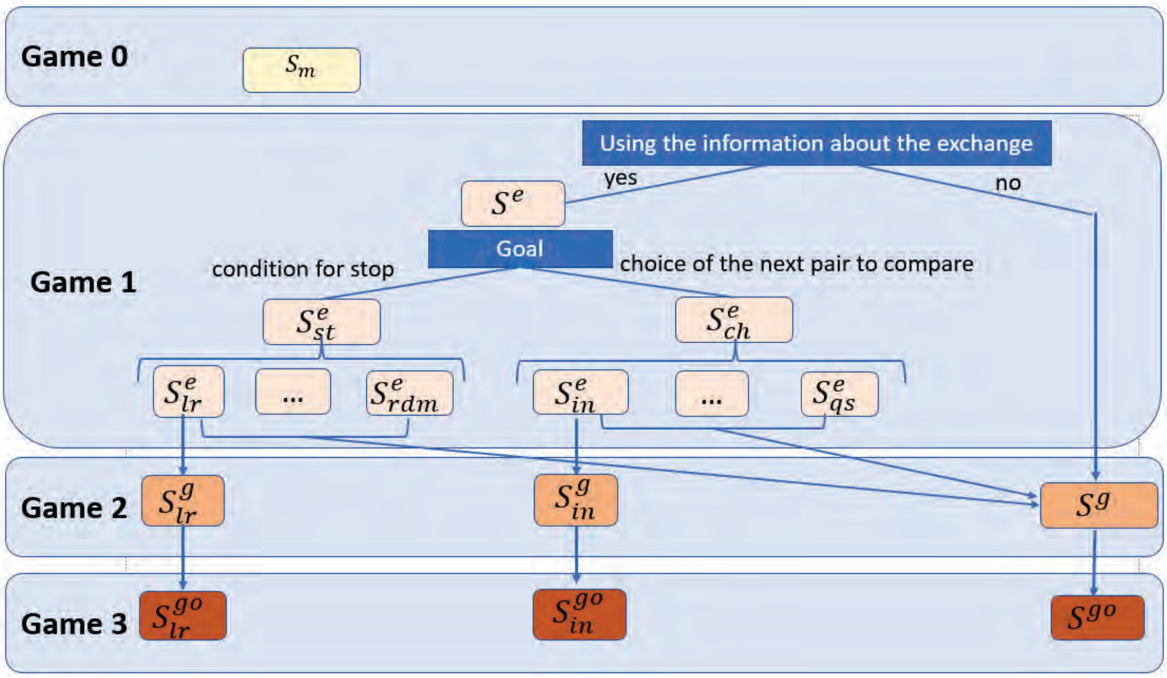

Figure 1: Evolution of expected pupils' strategies during problem-solving

If pupils don't use the information about the exchange, they are *de facto* placed in the next problem situation (game 2) with CM2. The strategies of the class S^g in this case could be based, in particular, on the idea to bring one card by one on their definitive positions considering the global order (as in the selection sort, for example).

For pupils who elaborated the strategies of the class S^e in game 1, game 2 introduces in the *milieu* the question about the worst case of cards placement. Thus, it is expected that certain of these strategies will evolve, the others will be abandoned and replaced by strategies of class S^g. For example, the strategy S^e_{lr} is susceptible to become the cocktail sort algorithm and the strategy S^e_{in} can be evolved into the insert sort algorithm. The strategy S^e_{rdm} (that becomes too costly in the game 2) will be abandoned as well as quick sort algorithm (which cannot be executed in CM2).

Game 3 aimed to introduce in the *milieu* the question about the possibility to use the strategies elaborated in game 2 for a bigger number of operators. It was also expected that the challenge to use the minimal number of comparisons possible would allow pupils to pass from the simple observations that the strategies work to analysing their properties. This may result in an optimization of the strategies (which we showed on Figure 1 using superscript "go") or their replacement by more efficient ones from the point of view of the number of comparisons needed to solve the problem.

We hypothesize that the set of developed problem-solving situations could be used at different school levels for promoting pupils' algorithmic activity. In this paper, we are interested in the implementation of the lessons sequence at the upper primary school level drawing the following research question: "Does and to what extent the tasks design favorize the emergence of algorithmic activity of pupils of this level?"

Lessons sequence and its implementation in the class

On the basis of the developed set of problem-solving situations, we elaborated the lesson sequence in collaboration with a teacher associated with the research. It included three lessons that were implemented in the class of 5-th grade (that corresponds to pupils' age of 10–11 years) in the city centre of Lyon, France. In the first lesson of one hour, the pupils played in the games 0 to 2 according to the rules described above. The developed strategies were formulated only at the private level (pupils described them on the paper after game 1 and 2) in order to give enough time for all pupils to do their research and to elaborate the strategies without influencing on this process by ideas produced by others. The teacher didn't not intervene in the pupils' problem-solving.

At the second lesson of two hours, the teacher introduced a new challenge: to sort the cards of all pupils of the class at the same time. The pupils were asked to think in groups about the strategies to apply in case of seven cards on a grid and to count the number of the comparisons needed to solve the problem. Thus, it suggests that the assignment given in the class differed from the initial formulation of game 3 presented above. The attempt of each group to sort the cards of all pupils of the class was followed by the general discussion led by the teacher who invited pupils to explain why the proposed strategy worked (or why they thought that it would work) and if it could be optimized from the point of view of the number of comparisons. The third lesson of two hours was devoted to the following analysis of elaborated strategies. In particular, pupils may explore the possibility to sort more than 10 cards (11, 12, 50, 100, any number of cards) using the developed strategies as well as to discuss the termination of the elaborated algorithms.

Data collection

We collected the videos of three lessons filmed by one camera at the back of the class, videos of work of three groups of pupils at each lesson, pupils' written descriptions of developed strategies. All videos were transcribed and analyzed as well as the pupils' productions.

Data analysis and findings

Due to the space restrictions, we report only about a part of the results that concern strategies development at first two lessons. The *a posteriori* analysis of the game 0 and 1 showed the consistency with the *a priori* analysis. In game 1 we noticed the appearance of the strategies of the two families S^e and S^g. The first family was represented, in particular, by the strategies S^e_{lr}, S^e_{in} and S^e_{rdm} described above. The family S^g was represented by a selection sort algorithm developed by the pupils who didn't use the information about the exchange of the cards. Comparing the anticipated strategies with those that appeared in the game 2, we point out new elements not envisaged in the *a priori* analysis. This concerns the case of one pupil named Arthur that we present in the following.

In the result of all games, three groups proposed the selection sort algorithm (one group used 42 and two groups used 21 comparisons to sort seven cards). In one of these groups, we also noticed the appearance of the bubble sort. Moreover, one group proposed the insert sort with 21 comparisons, one group stopped their research on the strategy S^e_{lr} with 43 comparisons and two groups didn't succeed to develop general algorithms.

Analysis of Arthur's case of strategy development

Arthur in the game 1 elaborated the strategy S^e_{in} described above. When the rules of the game 2 were announced by the teacher, Arthur resisted to abandon the usage of the information about the cards' changes and took the decision to apply simultaneously the strategy switching from one pupil to another (which means that every pupil executed the same strategy three times). Thus, Arthur generalized the strategy in a trivial way. However, from his actions we can infer that Arthur anticipated that implementing the strategy only one time is not sufficient and that execution of several times the same strategy will not change the order of already sorted cards.

The new challenge to sort the cards of all pupils of the class lead Arthur to understanding that his strategy would be too costly from the point of view of the number of needed comparisons. He took the decision to try to sort the cards of the pupils of his group with closed eyes. Arthur's tentative to come off the information about the cards' exchanges eventually helped him to generalize his strategy in CM2 in a non-trivial way and to obtain the insert sort strategy with 21 comparisons.

For the teacher's question if the strategy will work for the whole class, Arthur gave the positive answer referring to the experience with three pupils and added the following:

Arthur: In my head … it is logical. We make the cards number 4 and 5 … imagine that one person doesn't exchange, there are other people that could exchange … because we don't know what they have … therefore it is necessary to come back to the beginning of the grid. Because, if, for example, here [shows the place number 4 on the grid] there is an ace … it should be brought back to the beginning.

In this excerpt, Arthur explains the decisions ta.ken during strategy development. He refers to the cards number 4 and 5, but the chosen cards are used only as means to illustrate the "local" worst case (when an ace is placed on the right extremity of a subset of cards) identified by the pupil. The given example has a general character and used by the pupil to validate the elaborated strategy.

Conclusions and perspectives

The *a posteriori* analysis showed that the designed set of tasks supports to a large extent the development of sorting algorithms by pupils of upper primary school level. The identified individual cognitive path enriched *a priory* analysis. The generalisation in the game 2 could be made trivially, but the goal to sort the cards using the minimal possible number of comparisons in game 3 puts a pupil in a situation where he should search for a more efficient way to generalise the developed strategy in CM2. This case also permitted to point out the ways to improve the design for the next cycle of the design experiment. For example, the assignment of the game 2 could be modified by asking the operators to make manipulations with cards (exchanging or not exchanging) under the table or use the method of "closed eyes" for a player used by Arthur.

Besides, we revealed several mechanisms that emerged during implementation of the lessons sequence. The process of strategies development involves conjecturing, taking decisions and generalization. Artur's case showed the capacity of pupils of this level to consider the "local" worst case of the cards' placement that is crucial for evolving S^e strategies in general sorting algorithms.

In most cases, the pupils' experience to sort cards of three operators in game 2 seems to be decisive for retaining the strategy as the one that would work for the whole class in game 3. The arguments about strategies validity produced by pupils in the private level has mostly a pragmatic character and are highly anchored in the actions made during strategies development. More research needs to be carried out to identify the favorable conditions for fostering the algorithms validation process. Future work is also likely to compare the results obtained from the experimentations at different school levels in order to study the impact of "external" conditions (age, pupils' knowledge repertoire, etc.) on the pupils' algorithmic activity in the developed situations.

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Teachers' beliefs on computational thinking in school mathematics

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Keywords: Computational thinking, mathematics teachers, compulsory education.

Introduction – positioning and focus

In this poster proposal we present an aim to investigate which aspects of computational thinking that emerge in teachers' beliefs on computational thinking in school mathematics. Several researchers have commented that there is a lack of empirical research on programming and computational thinking in didactic practices (Forsström & Kaufmann, 2018; Grover & Pea, 2013; Weintrop et al., 2016). Forsström and Kaufmann call for a discussion regarding the influence of the role of the teacher in students' learning processes in activities such as programming (2018). By analyzing reports written by 12 lower-secondary and secondary school teachers from Norway who planned, implemented, discussed, and reflected upon a classroom session in mathematics involving computational thinking, we want to reveal insight in which aspects of computational thinking that are prominent in teachers' beliefs on computational thinking in school mathematics. This research will have a teacher perspective.

An OECD report links the intention with computational thinking in school mathematics to expectations of a more general understanding of the role of technology in society (2019). The OECD report also refers to a report on computational thinking that provides a comprehensive overview and analysis of recent research findings on computational thinking in compulsory education. This report summarizes concepts and skills of computational thinking and presents a framework for computational thinking in compulsory education (Bocconi et al., 2016, 2018).

Results from an analysis in the first author's master's thesis show that there is a relationship between aspects of computational thinking that emerge in activities, and what three teachers state about computational thinking. However, the research in the master's thesis reveals that there is no apparent correspondence between the intention of computational thinking in school mathematics, and what emerges through activities in mathematics and in the three teachers' beliefs of what is important in computational thinking in school mathematics. These results indicate that the three teachers focused more on skill development than problem solving. Based on this, two new dimensions of computational thinking emerge. One dimension emphasizes the development of skills, and the other dimension highlights problem solving. These dimensions do not exclude each other, but rather show what is emphasized (Rekstad, 2021).

Methodology

In order to illuminate which aspects of computational thinking that emerge in teachers' beliefs on computational thinking in school mathematics, we have collected data from 12 mathematics teachers who are enrolled in a teacher development program. The data consists of reports about their developmental work on computational thinking in their own classrooms. The teachers are to plan and implement classroom activities that develop computational thinking, and afterwards place the activity in the framework of Bocconi et al. (2016, 2018).

We plan to apply content analysis where we evaluate and code the teachers' reports and beliefs. The analysis will afterwards be discussed according to the framework of Bocconi et al. (2016, 2018). This framework addresses core skills, approaches, and dimensions of computational thinking with or without technology, in which computational thinking are used.

Implications

We have gathered data, but not yet carried out the analysis. There are two elements that might be interesting to see. Firstly, it will be interesting to see which aspects of computational thinking are prominent in the teachers' planned session, reflections, and discussion about their own experiences with teaching computational thinking. Secondly, it is interesting to look at whether the activities mentioned in the teachers' reports fulfill the criteria of the problem-solving or skills-development dimension of computational thinking. Based on the results from the masters' thesis, we expect to see more of the dimension that emphasizes skills development.

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Advantages and disadvantages of using loops in algorithms: conceptions of pre-service primary teachers learning Scratch

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The purpose of this paper is to explore pre-service primary teachers' conceptions of loops for drawing squares in their early stages of learning Scratch. Twenty-six pairs of student teachers, all with some experience in teaching mathematics at primary school level, explain the advantages and disadvantages of using loops in Scratch. Qualitative content analysis of their written explanations revealed three categories: (1) the higher efficiency of loops is seen as a clear advantage, (2) their narrower range of applicability as a clear disadvantage, and with regard to (3) the cognitive demand of loops, the responses are contradictory. In addition to some misconceptions, we could identify a key fact that student teachers must know in order to competently handle loops in Scratch, that is different outputs may occur from the same instruction if the instruction is inside a loop.

Keywords: Algorithmic thinking, loops, pre-service teachers, primary mathematics, Scratch.

Introduction

In computer science, imperative programming is a programming paradigm that uses sequences of instructions to control the flow of a program. The *loop* construct is a specific instruction type used to manage and control flow: it denotes a sequence of instructions—the loop body—that are specified once but may be carried out several times in succession. The purpose of this study is to find out how pre-service primary teachers who are novice programmers think about loops and how they use loops for solving a geometrical problem. Although loops are indispensable for algorithms and thus essential for algorithmic thinking, empirical research shows that the use of loops is challenging to novices. So, how do pre-service primary teachers think about the value of the concept of loops?

In contrast to the traditional approach of examining the loop programs of novice programmers, we examine their responses to a question which asks about the advantages and disadvantages of the loop concept in Scratch. For this purpose, our student teachers had first learned how to develop an algorithm that draws a square using the programming environment Scratch. We chose this approach because our interest is a better understanding of primary teachers' thinking in loops, with an aim of better understanding algorithmic thinking more generally. Second, drawing regular polygons such as a square is a common task in primary school textbooks. Third, due to its block-based and visual nature, Scratch seems to be appropriate not only for representing algorithms, but also for introducing primary school children and their teachers to programming. Furthermore, Scratch's instruction "repeat (number)" for count-controlled loops does not require a loop counter variable (which is difficult to master for novice programmers, see Cetin 2015 or Lagrange 2020). In a self-paced learning unit designed for primary teacher training, our student teachers had to solve several geometric problems and represent their algorithms using flowcharts (on paper) before realizing them using Scratch (on the computer). As a result, they should learn some central concepts of algorithmic thinking such as thinking in loops. The learning materials contained several tasks, and also metacognitive reflection questions on their learning. In the last task of the learning unit, they were asked to set up an algorithm to draw an arbitrary regular *n*-gon (*n* is to be entered as input). Consequently, to solve this general geometrical problem, they had to use loops not with a fixed number of repetitions (e.g. 4) but with a variable upper loop bound. Since generalization and algebraic thinking is beyond the scope of this paper, the corresponding results cannot be investigated and discussed here.

In this study, we address the following question: *What are primary student teachers' perspectives on the advantages and disadvantages of the loop concept, after they have been introduced to it?*

Thinking in loops as a constituent element of algorithmic thinking

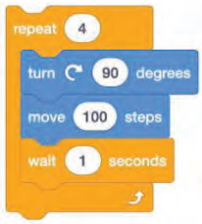
As theoretical computer science shows, a system of data-manipulation rules which follows the imperative programming paradigm must offer three constructs so that all theoretically computable problems can actually be solved by an algorithm: sequence, repetition (loops), and selection (conditionals) (Böhm-Jacopini-theorem, see Curzon et al. 2019, p. 525). In addition, two types of loops are distinguished, count-controlled loops (modeled by "repeat (number)" in Scratch or "for" in other programming languages) and event-controlled loops ("repeat until (condition)" or "while").

Since algorithms are mathematical objects that solve computable problems, *algorithmic thinking* is a specific way of dealing with algorithms, namely the design and analysis of algorithms. To do this at a sufficiently explicit level, algorithms must be accessible in a suitable symbolic representation, such as in a programming language. For example, as Lagrange points out, if "[...] formal notation for algorithms [...] is a vehicle for abstraction rather than for execution on a computer" (2020, p. 45), then algorithmic thinking is involved in the development of an algorithm that draws a regular polygon, but not in its implementation in a programming language. *Since loops are a constituent element of algorithms, thinking in loops can also be considered as a constituent element of algorithmic thinking*. Accordingly, to introduce students to algorithmic thinking, we can familiarize them with the loop concept and have them solve tasks such as drawing a regular polygon. In the corresponding algorithm, only the simple loop type of count-controlled loops is needed, because the number of iterations is already explicitly known at the time of entering the loop.

As some authors point out, algorithmic thinking is quite important for primary school mathematics (e.g. Benton et al. 2018; Gleasman et al. 2020). Therefore, primary teachers should be able to introduce their pupils to early algorithmic thinking by, for example, teaching topics such as multiplication as repeated addition or place value through programming. At the same time, many studies have shown that loops cause various difficulties for novice programmers, and not only when variable is involved in iteration or when loops are nested (for loop errors in text-oriented programing languages see Cetin 2015, for loop misconceptions of school students working with Scratch see Grover & Basu 2017, for a systematic overview see Swidan et al. 2018).

Context and method

A total of 52 primary student teachers participated in the study, 43 (82.7%) are female and 9 (17.3%) are male. All participants had some experience in teaching mathematics, but no one had prior knowledge of programming or computer science. All of them were enrolled in two primary teacher education classes in Switzerland and had chosen mathematics as their individual focus towards the end of their studies. As a part of this program, I have offered a course called “Standard Written Algorithms and other Algorithms”. One of its components was a self-paced learning unit called “Algorithmic Thinking in Mathematics: Introduction to Scratch” (approx. 6 to 8 lessons). By working through a series of activities in which several geometric figures (squares, further specific convex and non-convex regular polygons, etc.) had to be constructed using Scratch, the student teachers were gradually introduced—by the learning unit material, not by me in person—to different representations of algorithms (flowcharts and code) and to the three control concepts of algorithms (sequence, iteration, and selection). Thus, to learn the concept of loops, one of the first activities was to examine the effects of a piece of code (see Figure 1, left). In the following activity, the concept of loops was introduced by a (fictional) pupil who proposed to replace the four sequences of repetitive instructions with the (new) loop instruction. The task then asks the student teachers to explain the advantages and disadvantages of using loops (see Figure 1, right). This kind of task has been shown to be highly effective for learning especially for building comprehension-based, conceptual knowledge, whether used retrospectively or, as in this case, integrated into learning materials (Bisra et al., 2018). Since the the advantages and disadvantages of loops were not discussed in class while working on the unit, we can analyze these students’ explanations to get a fuller picture of their conceptions of the loop concept than if we were to examine their loop programs for errors.



Activity “Introduction to the loop concept”:

A pupil of your class suggests that instead of using the script on the left, use the script above: What advantages and disadvantages do you see in this so-called loop?

Figure 1: An activity to introduce the loop concept

In order to obtain high quality responses, all 52 participants worked through the activities of the learning unit in pairs (Gleasman et al., 2020; Robins et al., 2003). Since they had to write their answers for each activity, the data set consisted of 26 written responses, each listed several advantages and disadvantages of loops. In view of the research question, these responses will be analyzed here. Since our goal is to generate hypotheses about the algorithmic thinking of primary teachers, we did not to compare or weight them in our analysis such as by counting frequencies. Rather, we have analyzed the answers according to the qualitative content analysis (Mayring 2015). Three main categories, *efficiency* of loops, *cognitive demand* of using loops and *applicability* of loops were inductively reconstructed by repeatedly summarizing the answers with descriptive terms.

Results

We present two main findings. First, three main categories of answers emerged when the students reflected on the advantages and disadvantages of using loops (see Table 1). While some of the sub-categories relate to the process of programming, others refer to the final program. Second, a key fact was identified that must be known for the successful mastering of loops, as well as two learning misconceptions. We will now provide details on each of these findings.

Table 1: Categories of advantages and disadvantages primary student teachers list when learning loops

Category 1: Efficiency of loops …		
… in terms of programming	Using loops allows *fewer programming actions*. As a consequence, it requires *lower expenditure of time* for writing a program.	Seen as advantages
… in terms of the program	Using loops allows *fewer lines of code*, and is thus more *space-saving*. As a further consequence, the size of the square is controlled via *a single instruction*.	
Category 2: Cognitive demand of using loops …		
… in terms of programming	The use of loops *increases* or *decreases the cognitive demand* on the *writer* of the program.	Mostly seen as disad-vantages, sometimes as an advantage
… in terms of the program	The use of loops *increases* or *decreases the cognitive demand* on the *reader* of the program.	
Category 3: Applicability of loops …		
… to problems to be solved	*Not every problem* can be solved with the use of loops.	Seen as a disadvantage

Types of characterizations

There were three main student teachers' characterizations of the advantages and disadvantages of loops: (1) the *efficiency of loops*, which students related to the process of programming or to the final program, (2) the *cognitive demand of loops*, which students again related to programming or to the final program and (3) the *applicability of loops*, in terms of what kind of problems can be solved using loops. For an overview of these main categories with their subcategories, see Table 1.

Category 1: Efficiency of loops

All student teachers expressed unanimously that loops bring a gain in efficiency. We can identify two subcategories of answers, depending on whether efficiency refers to the programming process or to the final programming output.

Subcategory 1.a) A first sub-group of student teachers' answers relates efficiency to the act of pro-gramming, that is, to the process and the effort to move it forward. The student teachers argued that using loops results in fewer programming actions (such as "drag the block from the block palette into the code area") or leads to a lower expenditure of time for writing a program. The following two statements illustrate this first sub-category:

> The moves and rotations need to be programmed only once, instead of four times. (P17)

> The loop is a useful tool to speed up the programming of repetitive sequences. (P01)

A few students mention both effects, that fewer actions and less time are needed thanks to loops:

> One advantage is that it takes much less blocks to program the script. One is thus faster. (P12)

Subcategory 1.b) Quite differently, some student teachers related efficiency to the final program, that is, to the product and its length. They suggested that the use of loops results in fewer lines of code. Some answers of this sub-category conclude that the runtime of the program will be shorter:

> The efficiency of the algorithm is increased, since fewer instructions need to be executed, therefore the square can be drawn faster. (P20)

Other student teachers saw that the program therefore requires less space, and sometimes mentioned in the same breath that the shorter length also makes the program clearer:

> The proposed loop makes the script more compact. Due to its compactness, it takes up less space and is therefore clearer. (P26)

Unexpectedly, a group of student teachers formulated another advantage resulting from the reduced number of instructions, namely that the size of the square can now be manipulated more easily:

> The length of the sides of the square can be modified at a single instruction. (P17)

With Cetin, this answer might indicate that its authors see loops as a whole, as a new object, and thus have reached the level of the "object conception of loops" (2015, 159). This would mean that they no longer see the loop as a step-by-step arrangement of four individual sides, but as a single, integral object depending on one parameter (the size of the square) only.

Subcategory 1.c) A third group of answers relate efficiency to the criteria of both sub-categories, i.e. to the programming process as well as to the final program. For example:

> The advantage of a loop is that it must be set only once and will still be executed four times. Thus one saves time when programming. Since one does not have to enter the instructions several times, the program is shorter and therefore clearer. (P07)

Category 2: Cognitive demand of using loops

As the statement "less space […] therefore clearer" (P26, see above) shows, some students also considered the intellectual challenge that the new concept poses to them, or its cognitive demand. While most student teacher pairs found an adequate use of loops more demanding than the use of sequences of instructions, there are also few who stated that loops place less cognitive demand. Again, we can identify two subcategories which refer either to programming or to the program.

Subcategory 2.a) A first group of answers concerns the process of using loops and the associated cognitive demand, which student teachers judge partly as higher, partly as lower. To illustrate, here is an answer that explains how loops are a challenge:

> An additional block [the “repeat ()”-instruction, see Figure 1, C.W.] is introduced, making programming more complex. When a mistake happens, it happens everywhere (on all sides of the square) and it is more difficult to determine the origin of the mistake. (P18)

This statement reveals a challenge that one must master in order to competently handle loops: The *one-to-one relation between output* (in the ‘stage’ window) *and instruction* (in the code area) *must be broken*. This fact applies because while instructions that are not part of a loop body show (at most) one effect on the output side, any instruction within a loop can have multiple effects (at most as many effects as often the loop is run). Therefore, in the case of an unexpected output, it is not straightforward to determine from which instruction(s) of the loop body it originates.

A rather contrary, but positive view on the cognitive demand is expressed in the following answer:

> Fewer errors can happen because you program fewer instructions. (P03)

Subcategory 2.b) A second group of student teachers’ statements refers to the final program, as they comment on the cognitive demand of reading loops. Again, it was not considered unanimously within the data set. A corresponding response is as follows:

> One disadvantage is that with loops it is harder to see what exactly is being repeated. In contrast, with the full script you can follow every step. (P02)

The following response illustrates even better why loops can be demanding:

> It is more difficult to follow the individual operations or steps. You can't see the individual steps that make up the square. (P06)

As already mentioned, one student teacher pair surprisingly found it easier to understand instructions located in the loop body and gives the following argument as an advantage of loops:

> For schoolchildren: It is more difficult to follow when the various program steps are one after the other, instead of within a loop. (P17)

This is the only response that looked at the cognitive demand of using loops for their future pupils.

Category 3: Applicability of loops

In the responses for this category, student teachers looked at the problems that can be solved with the use of loops. They expressed that instructions inside a loop do not allow as much modification or variation as those outside. In this respect, the use of loops does not allow to solve all possible problems. The following response shows this perspective very clearly:

> Not all changes can be made to the shape, for example, you cannot change the thickness of each side. The loop repeats the same thing 4 times. This means that it is not possible to change anything in every single repetition that is made. The loop causes that the same instructions are repeated, so modifications in between are not possible. (P13)

Here the instructions of the loop body are considered fixed and cannot be modified. As a result, some shapes are judged to be non-drawable (which is sometimes true, but sometimes not):

> A disadvantage of loops is a lower flexibility: if a rectangle or an irregular triangle is to be drawn, a loop is of no use. (P21)

The following response is somewhat more nuanced, pointing out that an instruction of the loop body may well have different consequences, at least to a certain degree ("regular" differences would be possible, others not). It focuses not on a mathematical but on a design aspect of the shape:

> The thickness of the sides can be increased or decreased, but this is done regularly. If we want the sides to be of different thicknesses, we can't do this with loops. (P15)

The following response suggests how this issue could be overcome with the help of variables:

> The sequence in the loop is fixed. In this case the instructions are repeated four times stubbornly and cannot be varied so easily. With the help of variables, that would be possible. (P01)

Unfortunately, this statement is not further elaborated or illustrated it with an example.

Discussion

The study is related to the question of what constitutes algorithmic thinking in mathematics and the assumption that thinking in loops is a constituent element of algorithmic thinking. It explores learning to use count-controlled loops in 26 pairs of pre-service primary teachers solving a geometrical problem utilizing Scratch. Unlike previous studies (Cetin, 2015; Grover & Basu, 2017; Swidan et al., 2018), we are not interested in student teachers' errors in their loop programs, but rather in how they explain the advantages and disadvantages of the loop concept. The reconstructed categories relate to *efficiency*, *cognitive demand*, and *applicability*. Although the student teachers involved had just been introduced to the loop concept, they recognize its efficiency as a first advantage. However, they only refer to surface features (fewer instructions are needed, the program is shorter), which is characteristic for novices. Second, as future specialists in teaching and learning, students address the cognitive demand of the new concept for solving problems. However, not every student pair assesses it the same: While most students found it more difficult of keeping track of the interaction of an instruction within the loop and its effect on the output, some find it easier to follow sequences of instructions structured by loops. The statements referring to the third category, the applicability of loops, mention this aspect only as a disadvantage. At this early point of their learning process, students still feel constrained by the construct, again in terms of surface features (design possibilities, possible shapes of the figure). Since loops with variable upper bounds were only brought up at the end of the learning materials, it is not surprising that no student took the expert point of view yet (only thanks to loops, a geometric problem like drawing arbitrary regular *n*-gons can be solved).

The analysis of our data set also provides two misconceptions: The claim that "[…] fewer instructions need to be executed, therefore the square can be drawn faster" is not correct, since a smaller number of instructions does not necessarily lead to a shorter runtime (instructions in the loop body can be executed several times). Also the claim "if a rectangle or an irregular triangle is to be drawn, a loop is of no use" is a misconception, since iterative elements are also present in polygons such as rectangles. In addition, a key fact was identified that students must be aware of: they need to consider that instructions within loops can have not just one but multiple effects on the output, so that two different outputs do not necessarily refer to two different statements. *In this way, thinking algorithmically would mean not only seeing the advantages and disadvantages of loops that go beyond surface features, but also breaking the one-to-one relation between output and instruction*.

We are aware of the limitations of our study, such as the small number of students involved, the type of tasks they had to do, the programming language employed, or the analysis of written responses without subsequent in-depth interviews. However, under these limited conditions, the study provides some qualitative results that complement and extend previous studies (Cetin, 2015; Grover & Basu, 2017; Swidan et al., 2018). Since the category system makes preliminary hypotheses, it might contribute to developing a theory of learners' thinking in loops and therefore of their algorithmic thinking. For example, future research could address the following questions: To what extent would our category system predict the advantages and disadvantages that student teachers of higher grade levels would see? Could our category system even serve as an empirically supported description of the learning trajectories in learning to use loops to solve a mathematical problem?

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Developing algorithmic thinking without programming by designing instructions for encryption

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Algorithmic thinking is an important part of digital literacy, enabling learners to navigate the digital world with confidence. As part of a Design Research study with 22 learners in grades 5 and 6 so far, it was investigated to what extent the children learn to think algorithmically when they discover an encryption algorithm without use of a programming language. It was found that some learners can intuitively and independently develop certain algorithmic thinking ideas (such as sequencing and loops) when writing instructions (=developing an algorithm). Hurdles show up in the degree of precision and the degree of generality of the instructions.

Keywords: Algorithmic thinking, algorithms, encryption, design research.

Introduction

Digital literacy is important in order to be able to navigate the digital world in a responsible manner. Many IT systems in the world around us can only be used in a responsible manner if the basic ideas behind them are understood. Wing (2006) coined the term "computational thinking" in this context, "that represents a universally applicable attitude and skill set everyone, not just computer scientists, would be eager to learn and use" (Wing, 2006, p. 33). Computational thinking includes processes such as "solving problems", "using abstraction and decomposition when attacking a large complex task", or "thinking recursively" (ibid, p. 33). Hence, computational thinking is a different skill than programming, as "programming" relates to concepts regarding the syntax and semantics of programming languages. As one example, the "CS Unplugged" project (Bell, Rosamond, & Casey, 2012) presents several resources developed to promote computational thinking without a computer.

Up to now, there have been very few learning opportunities on this topic for children in Germany, as computer science is not a compulsory subject throughout the country or even offered at all (cf. Pasternak, Hellmig & Röhner, 2018). This is due, among other things, to the way education is promoted. Computer science is often only offered as an elective subject from grade 9 onwards. Digital literacy is politically regarded as a task for all subjects, and thus especially for mathematics education. Therefore, it is also due to the general conditions that, due to the general educational aspects, algorithmic thinking should be promoted in mathematics classes as well. Besides, the opportunities should be used that exist due to the topics that are covered anyway.

Theoretical background

Although Wing (2006) proposes the term *computational* thinking, this type of thinking should not to be considered as restricted to dealing with computers. Instead, Wing (2006) stresses that computational thinking is an activity of humans, not of computers. It is a foundational type of thinking that allows humans to attack problems in a way so that they can ultimately be solved using a computer.

Such considerations are not unknown to mathematics. Here, the idea of algorithm proves to be much older than that of computer (Möller & Collignon, 2019). Many algorithms are part of the mathematics curriculum, although their algorithmic nature often remains implicit. That is, certain algorithms are made a topic of discussion (e.g., written arithmetic procedures or solving equations), but it is rarely made explicit that this is an algorithm. This also leads to the fact what constitutes an algorithm is not defined and ideas such as "explicit sequence of steps to solve a problem" or similar have not been an explicit learning object in mathematics education so far.

In order to reduce the complexity of computational thinking as a learning subject, and at the same time to emphasize more the role of the algorithm, this study proposes to use the term *algorithmic thinking* (AT) for capturing the kind of thinking that is necessary when designing and reflecting algorithms (in contrast to technical programming). There are different conceptualizations for specifying what is required for students to possess algorithmic thinking. Following Harlow et al. (2016, p. 340), five algorithmic thinking ideas are distinguished in this paper:

- "**Sequencing**: Creating an ordered list of instructions to complete a task"
- "**Breaking down actions**: breaking an event into smaller parts"
- "**Looping**: Repeating a set of instructions multiple times"
- "**Event-driven programming**: identifying how one circumstance triggers another"
- "**Message passing**: coordinating actions across code or characters"

This paper examines the extent to which AT ideas can be developed. One promising way to promote the development of algorithmic thinking is to engage students in writing instructions for self-discovered algorithms. After all, writing is not only a learning goal in its own right, but also an important thinking tool (learning to write and writing to learn, cf. Morgan, 1998). Austin and Howson (1979, p. 167 f.) also already dealt with the connection between language and concept formation. Heller and Morek (2015) distinguish three different functions of academic language (AL) both oral and written: (1) "AL as a medium of knowledge transmission (communicative function)"; (2) "AL as a tool for thinking (epistemic function)"; (3) "AL as a ticket and visiting card (socio-symbolic function)" (ibid., p. 175). This suggests that writing promotes deeper engagement with the content and thus fosters deeper understanding. At the same time, instructions for algorithms can be conceived as a separate text genre that students need to learn (similar to geometric construction texts, cf. Rezat & Rezat 2017).

In this paper, the idea is to develop algorithmic thinking by inventing an encryption algorithm without directly focusing on programming (similar to the examples of Bell et al., 2012). Encryptions are suitable because, on the one hand, they are an important topic in everyday life. On the other hand, they are suitable because even without programming, the algorithm must be described very precisely and accurately so that someone else can understand or reverse it. The basic idea of a symmetric encryption procedure can be experienced and learned by reinventing the Caesar Cipher, which is described in more detail in the design section below. In reality, of course, this is currently even more complex. However, since the teaching-learning arrangement should be suitable for grades 5/6, the Caesar Cipher was chosen as an example.

This leads this study to the following research question:

> RQ: To what extent do children in grades 5/6 show AT ideas when writing down instructions (algorithms) for encryption and decryption? What hurdles are encountered?

Methods

The methodological framework of this study is design research (Gravemeijer & Cobb, 2006), whose aim is on the one hand the development of a teaching-learning arrangement in several cycles and on the other hand the analysis of the learning processes initiated by the teaching-learning arrangement, i.e., in particular typical learning pathways and hurdles. In this paper the first results of the first cycle are presented.

So far, 22 learners in grades 5 and 6 have participated (a total of about 1600 minutes of video material). The design experiments were videotaped and passages relevant for the analysis were transcribed.

To select relevant passages, those involving writing the algorithm were identified. To show the range of written products, four cases, some of which are very different, are contrasted below. For data analysis, a qualitative content analysis (Kuckartz, 2019) of these products was conducted and the reconstructed AT ideas were marked by ||...||. For this purpose, a deductive-inductive procedure drawing on the categories from above (||sequencing||, ||breaking down actions||, ||looping||, ||event-driven programming||, ||message-passing||) was chosen to reconstruct which AT ideas learners address when it comes to the elements of instruction (algorithms) and which learning trajectories and hurdles occur.

Design

The learners have been working with the Caesar Cipher (cf. Allen, 2017). In this process, letters are shifted forward in the alphabet by a fixed key (Table 1). For example, the plaintext "hello" and key 3 provides the ciphertext "khoor", as every character is shifted by 3 characters.

Table 1: Clear- and walking alphabet to illustrate the Caesar Cipher (here with key k=3)

plaintext	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
ciphertext	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c

Table 2: Letter number table

a	b	c	d	e	f	g	h	i	j	k	l	m
0	1	2	3	4	5	6	7	8	9	10	11	12
n	o	p	q	r	s	t	u	v	w	x	y	z
13	14	15	16	17	18	19	20	21	22	23	24	25

When numbers are assigned to the letters of the alphabet as in Table 2, the encodings can also be expressed mathematically by the basic arithmetic operations or modulo calculations. The encoding of a plaintext letter can thus be represented by the calculation $y = (x + t)\ mod\ 26$ where $x, y \in \{0, . . . ,25\}$ is the number of the plaintext and ciphertext letters, respectively, and $t \in \{1, . . . ,25\}$ is the key number. As an alternative to the modulo computation, a bifurcation could also be considered: if $x + t < 26$, then set $y = x + t$, otherwise set $y = x + t - 26$. Formulas for decryption using this method are obtained analogously. The plaintext letter is obtained from the ciphertext letter using the calculation $x = (y - t)\ mod\ 26$, which can alternatively be described by the branching "if $y - t > 0$, then set $x = y - s$, otherwise set $x = y - t + 26$".

The teaching-learning arrangement consists of three sessions of 45 minutes each. Here, the focus lies on the learners' instructions they wrote down at the end of the first and the beginning of the second session. In the first session, the learners are first confronted with an encrypted message that they need to decrypt. However, they are not given explicit instructions on the Caesar Cipher, but rather discover the procedure by themselves. Afterwards, the learners are tasked with writing down instructions for their discovered procedure, for encrypting as well as for decrypting messages. The writing assignment demands a lot of precision, or even the ability to commit. This is different from oral language, where it is sometimes rather fleeting and children may contradict each other in the next sentence. By writing it down, a commitment is demanded, and the AT ideas in the learners' instructions are made explicit.

In addition, two design principles were implemented to initiate AT ideas. The design principle "Using your own instructions again" was implemented in such a way that at the beginning of the second sessions the children were asked to encode and decode again according to their own instructions from the first design experiment session and then to reflect on whether they had really only used their instructions. The design principle "confrontation with algorithms (instructions) of other children" was implemented in such a way that they were then confronted with another instruction from another group that addressed other aspects in order to sensitize them to this and to encourage the learners to sharpen their instructions again.

Empirical insights

Four of the instructions written down at the end of the first session are presented below (Table 3). They were chosen because their differences prove instructive for illustrating the range of written products.

Florian and Aaron ||sequence|| their instructions by making several steps explicit ("pick a number," "move each letter in your text by its respective place," "as soon as the alphabet ends, start over"). Moreover, they perform ||breaking down actions|| in the process, as they sequence into several substeps. It should also be emphasized that they are one of the few who also intuitively implement ||looping|| directly ("as soon as the alphabet ends, you start again from the beginning", "then continue with all the letters like this until they make a meaningful sentence"). Finally, they also show the idea of ||event-driven programming|| by formulating conditions ("when you have figured out the number of shifted digits").

Karin and John also show the idea of ||breaking down actions|| by aptly describing the shifting pattern of encryption and decryption ("In encryption you go letters forward", In decryption you always go letters back"). Thus, they also address a step of ||sequencing|| at the same time, but fold the steps less widely than Florian and Aaron. No ||loops|| or ||conditions|| are explicitly formulated. The instructions of Anna and Rebekka are very similar. They too focus on the central shift pattern and thus address ||sequencing|| and ||breaking down actions||, respectively. Unlike Karin and John, however, they do not formulate the instructions in general terms, but for a specific key, namely key 2. Marcel and Jonas, on the other hand, do not address the algorithm per se, but rather focus on the input-output idea by stating that "you have to give the other person the key and then you can write the text". This most closely addresses the AT idea of ||message-passing||, although the statement remains very general.

Table 3: written products for instructions on encoding and decoding of three learner pairs (literally translated from German) and reconstructed AT ideas.

	Encryption	Decryption	AT ideas
Florian & Aaron	Encryption: Pick a number from 1-25 and move each letter in your text by that number. *As soon as the alphabet ends, start again from the beginning.	Decryption: Look first how the letter is called if you shift it by all possible places. When you have found out the number of shifted digits (then continue with all letters) until they form a meaningful sentence.	||sequencing||, ||breaking down actions||, ||looping||, ||event-driven programming||
Karin & John	In the case of encryption, you move letters forward.	When decrypting, you always go back letters.	||sequencing||, ||breaking down actions||
Anna & Rebekka	You get the encrypted letter by moving two letters to the right.	You get the decoded letter by moving two letters to the left in the alphabet.	||sequencing||, ||breaking down actions||
Marcel & Jonas	You give the key number to the person you want to write to and then you can write the text.	You should drag the first word through all 25 possibilities and then use the correct principle to decode the other words.	||message-passing||

In summary, algorithmizing works quite well for two written products (||sequencing||, ||breaking down actions||). However, only one instruction also contains the idea of looping and ||event-driven programming||. The third treatment is more on the surface or usage level and, apart from the ||message-passing|| aspect, does not really address the algorithm itself. So, there are some ideas available as resources among the learners. However, not all AT ideas are activated by all learners by themselves. In addition, with ||sequencing|| also - when it occurs - the degree of precision needed is partly unclear.

The lack of precision also becomes clear when looking at the learners' oral statements about it. When asked by the teacher how John and Karin proceeded with an encoding task, John answers "We have always gone one further" (#2) and, when asked again by the teacher (#3), justifies this with "Because when you encrypt you always go forward" (#4).

1	Teacher	Can you briefly describe how you proceeded?
2	John	We have always gone one further.
3	Teacher	Ok, can you still explain why you always went one further?
4	John	Because when you encrypt you always go forward.

With regard to ||sequencing||, John remains as general as they have formulated it in their instructions by only mentioning the step of going one character forward. So, he does not name any further steps verbally either. Afterwards, the teacher gives the order to work on the decoding task “EQTQPC” (plaintext is “CORONA”). John suggests in #10 to try everything from 1 to 25 and starts directly with the first letter of the ciphertext ("E"), which he moves to a "D".

10	John	Ok. Then we would actually have to try out everything from 1 to 25. Then one back from E the D.
11	Teacher	You can write in there again and take notes.
12	John	Ah, yes. D. Ah I didn't click on it. D. Then back from Q, that's P. Then back from T one, that's S. Then back from Q one, that's P again. But that doesn't give a real word (laughs) You can see that now.
13	Teacher	Mhm. What could you do now?
14	John	Then two back, three back and then four back and then 5 back.
15	Teacher	Karin, what would you do now? What is your idea?
16	Karin	I've already tried a few combinations and I'm just about to try A, so. (long drawn out) A. (long drawn out) M. N.
[…]		
22	Teacher	So describe what you're doing right now. What exactly are you really doing now?
23	John	The letter back and try that until it makes a correct word.

In #12, John further unfolds this process using the example. By performing the shift in steps for each individual element, he basically performs a ||loop||. This is also reflected in #14 when, after shifting one place in the alphabet, he now suggests as a further procedure "then two back, three back, and then four back, and then 5 back" (#14). This is a change from the written product, as ||loops|| were not made explicit in the instructions before. This shows that he uses the idea of ||loops|| intuitively as a structured procedure, but that he does not yet describe this process in general. Possibly he lacks appropriate linguistic means for this. A possible linguistic device would be "do … until …". He uses this in #23 to indicate the termination condition for the algorithm as a whole: "until it results in a correct word".

The following is another brief look at what situational potential the design principle of "confronting instructions from other children" can have. Anna and Rebekka formulated their own instructions in an example-based manner. After Anna is confronted with Florian and Aaron's instruction, she notes that the instruction is "much more general" (#101).

101	Anna	This instruction is also much more general. So, I think we understood it last time in such a way that we should really formulate instructions for this example. So, with the right instruction you really come from each task also really to the solution. Both encryption and decryption. And ours is more related to a specific example.

In summary, it can be stated that hurdles can be reconstructed in the *degree of precision* of the instructions or in the *degree of generality*. Under certain circumstances, as seen in the two scenes, these can be resolved by a renewed reflection on one's own instructions by executing them on another example or by a contrasting comparison with instructions of other children.

Discussion

Algorithmic thinking is important for mathematics education and first ideas can be initiated in grades 5/6, as shown here. The initial results of this Design Research study show how diverse the starting points for possible learning processes for algorithmic thinking are. Learners intuitively activate different AT ideas. It should be emphasized that the learners develop AT ideas in a setting that does not draw on any ideas of programming, but rather on the general discovery and description of algorithms. The empirical insights illustrate possible working mechanisms of the design principles investigated in this study, namely “Using your own instructions again” and “confrontation with algorithms (instructions) of other children”, which are suitable for eliciting further AT ideas

The results presented here also make a first contribution to the discussion whether algorithmic thinking or programming should be learned first. It is possible that learners develop AT ideas even without programming, which are thus starting points for further learning processes. It would therefore be wasted potential to simply prescribe the programming language and so without first giving the learners the opportunity to develop the AT ideas themselves. This could provide a promising starting point for the developing of algorithmic thinking, although it is likely that some aspects of algorithmic thinking require programming to become fully developed. The results show that general principles (such as looping) could possibly first be learned in a general way, and then be applied to different programming languages or programming environments such as Scratch. The idea behind this is that the children learn and understand not only the specific handling of a tool, but the principles behind it.

What is challenging is to find an appropriate level of precision and generality when writing instructions for algorithms. Algorithms could indeed be thematized without programming. However, until learners understand how a computer works, they lack a kind of alphabet of what is allowed as ingredients for creating an algorithm. So, it is not trivial to find or name the smallest building blocks that are considered valid for describing an algorithm. Here, learning opportunities are needed. The design principle "confronting with algorithms of other children" has revealed situational potential in this respect. Based on this, the children developed AT ideas on their own.

In the next step, it will be examined to what extent implementing the instructions with Scratch building blocks promotes the degree of precision and to what extent further AT ideas are sharpened or developed. In addition, it is conceivable to check for other (mathematical) topics to what extent AT ideas can be developed in the respective area.

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TWG12: History in Mathematics Education

CERME 12: Thematic Working Group 12
History in Mathematics Education

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Introduction

History of mathematics *in* mathematics education, and history *of* mathematics education continue to receive much attention. Although empirical research and coherent theoretical/conceptual frameworks within this area have emerged relatively recently, there exists an increasing interest in these lines of work. The purpose of this CERME TWG is to provide a forum to approach mathematics education in connection with history and epistemology.

TWG12 welcomes both empirical and theoretical research papers, and posters proposals related to one or more of the following issues – although any paper or poster of relevance to the overall focus of the group is taken into consideration. All educational levels can be considered, from early-age mathematics to tertiary education and teacher training:

1. *Design of teaching/learning materials using the history of mathematics, preferably with conclusions based on empirical data.*
2. *Research on the existing uses of history or epistemology in curricula, textbooks, or classroom practice.*
3. *Research on the history of mathematics education, on local, national or international level.*
4. *Connections between mathematics education, history of mathematics, and history of mathematics education: Theoretical and methodological issues.*

Even though the inception of this TWG is fairly recent – it started in CERME6 (2009) – it has deeper institutional roots within the mathematics education research community. The *History and Pedagogy of Mathematics* (HPM) study-group (http://www.clab.edc.uoc.gr/hpm/about%20HPM.htm) was created at the 1972 ICME conference; it has been organizing satellite conferences to the ICME meetings since 1984, and has several active regional branches (HPM-Americas, European Summer Universities).

In CERME12, TWG12 accepted 13 papers and 1 poster. One paper was withdrawn due to financial problems, one participant could not present because of illness. The Covid pandemic was the reason that CERME12 was delayed until 2022, however, even in 2022 all sessions were on-line. Thanks to the excellent virtual platform created by the Free University of Bolzano and the effort and enthusiasm of the organization and participants, TWG12 was again a success, though with participants from a more restricted range of countries than in previous years

Before going into any details, it should be stressed that TWG12 has four general but distinctive features which give these meetings their specific flavour. Firstly, its topic lies at the intersection of different fields of research – mathematics education research and history of mathematics – which requires versatility and methodological vigilance (Fried, 2001; Chorlay & Hosson, 2016). Secondly, the strength of the historical and the HPM community varies greatly among countries, and these meetings play a crucial role for researchers working in relative isolation, and with difficult access to resources in the field. Thirdly, the scope of TWG covers both history in mathematics education and history of mathematics education, which are two significantly different research topics (TSG 27 and 55 in ICME14); connecting the two lines of investigations is a constant challenge. Fourthly, since the topic of TWG12 is neither specific to one level of the educational system (from primary education to teacher-training) nor to any single mathematical topic (be it fraction concepts, algebra, proof, etc.), the work in TWG12 intersects that of most other TWGs. It should be noted that, as in former editions of CERME, there was little intersection with what was covered in TWG8 (Affects and the teaching and learning of mathematics), TWG10 (Diversity and mathematics education), in spite of the fact that it is not uncommon for outsiders of the HPM research community – among which most policy-makers and curriculum-designers – to ascribe such goals to the historical perspective in teaching.

These four features make these meetings not only useful but also challenging and exciting. As the final discussion made clear, the general feeling among the participants was that one of the main outcomes of this meeting is that we actually learned a lot from the one another, both from their papers and from the lively discussions.

Some significant features of the 2022 conference

From an organizational viewpoint, due to the relatively small number of contributions, the TWG12 team decided to offer three work formats for the sessions. In addition to the two traditional formats – oral presentations of a paper (talk) and discussions of topics of general interest – we also organized two 1h30-hour workshop sessions (with two parallel presentations each). A distinctive feature of TWG12 is that it potentially covers all mathematical topics, all historical periods, and many different research perspectives (history of education, historical roots of a given concept or practice, analysis of teaching resources and curricula, task design, analysis of teaching practices, training design etc.), usually making it difficult for the participants to make the most of short oral presentations and 8-page papers. The workshop format enabled some of the participants to actually share some material, and discuss both research questions and methodological issues. Out of the 13 contributions four were found particularly suitable to be presented in a workshop: two contributions on history *of* mathematics education (Hamann & Schmidt-Thieme, Goemans & De Bock) and two on history *in* mathematics education, either at the secondary level (Spies & Junker) or in teacher education (Arnal-Bailera, Beltrán-Pellicer & Oller-Marcén). The participants of TWG12 agreed that the new format was enjoyable and should be kept for upcoming CERME conferences, whenever possible.

As to content, there was a general balance between the two main lines of investigation: 6 papers on the history of mathematics education, and 7 papers on history in mathematics education. Let us highlight a few features of the Bolzano edition of CERME, without aiming for a thorough survey.

As far as history of mathematics education is concerned, a significant share of the papers bore on the New Maths movement in Europe, thereby shedding light on this complex historical phenomenon from a variety of perspectives: connections with advanced mathematics (in terms of content and of interactions among different social groups and stakeholders); controversies at the time of the reform and long-term effects (if any); impact on curricula, teaching resources, and actual teaching practices; specifics of different countries (Belgium, Nordic countries) and of different segments of the education system (e.g., vocational training).

As far as history in mathematics education is concerned, the "traditional" line of work on task design was represented by several papers (Surroca, Spies, Oswald). In spite of differences in terms of mathematical contents and segments of the educational system at stake (from primary to tertiary education), these papers illustrate how the HPM perspectives requires that researchers combine inputs from two different fields of research, mathematics education and history of mathematics. They also showcase a practice which lies at the heart of the HPM perspective: the use of original historical sources in the classroom (Chorlay, 2016).

Over the last decade, teachers have become a central topic of interest for researchers working in the HPM community, with inputs from many theoretical frameworks. Teacher education was the main focus of the papers presented in CERME 12, with analyses making use of a variety of theoretical constructs and frameworks from mathematics education: the "four S"-model (van den Bogaart), Zhang's the ADTRE model, and Godino's "epistemic configurations" (in the paper of Oller-Marcén et al.). By contrast, the analysis of "ordinary" teaching practices in primary or secondary education is still an emerging topic, with one paper in CERME 12 (Chorlay), with theoretical inputs from the Documentational Approach to Didactics (Gueudet, Pepin, Trouche) and the Didactic and Ergonomic Double Approach (Robert, Rogalski, Vandebrouck).

Challenges for the future

The number of papers and posters presented in TWG12 was modest, and half of the participants of this edition of TWG12 were first-timers. This reflects the fact that, in spite of the fact that the HPM community has a stable international basis and ramifications in many countries, it remains somewhat marginal within the mathematics education community. Some of the reasons accounting for this situation are structural and beyond the scope of our actions, such as the modest role of history of mathematics in curricula, and the necessity for researchers in this field to be knowledgeable in two different areas (history and mathematics education). However, the importance of the inclusion of history of mathematics in teacher training seems to be gaining more recognition in many countries, thus providing opportunities to further and disseminate research. Also, the fact that research carried out from an HPM perspective touches on virtually all mathematics education topics should pave the way for exchanges and collaboration. While this is true at a theoretical level, it remains a challenge to strengthen and stabilize these connections. In the spirit of CERME's three Cs (Communication – Cooperation – Collaboration), it is one of the aims of TWG12 to reach out and generate fruitful interactions.

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A suitcase full of logs – an experiment with the use of history of mathematics in teacher training

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A group of students in a Dutch teacher-training-program were given a collection of fragments from different textbooks, to devise a lesson-plan for the introduction of the concept logarithm. Some of the fragments in the collection contained information on the history of mathematics. The lesson-plans were analyzed to determine how the history of mathematics was incorporated. Some students only used biographical information on the inventor of logarithms, others used exercises and a logarithmic table to enhance the comprehension of the concept.

Keywords: History of mathematics, logarithms, lesson design, empirical study, teacher training.

Introduction.

This paper describes an empirical research activity that I carried out as part of a larger study. In this study the way history of mathematics can be used by teachers to improve mathematical cognitive demand is investigated. In this paper I will focus on an assignment on the history of logarithms. The research question is: in what ways do in-service teachers use the history of mathematics in a lesson design for the introduction of the concept of logarithms?

The structure of the paper is as follows. First, I will shortly present formats for presenting history of mathematics in curriculum materials. Then I will describe the design and results of a teaching experiment at a teacher training program at the Amsterdam University of Applied Sciences (AUAS), where participants were asked to design a lesson plan for the introduction of logarithms, using a collection of preselected fragments. Finally, I will discuss the results of the teaching experiment and make some remarks on how this is embedded in the larger study.

The 4S formats.

The history of mathematics (HoM) can be used in mathematics education in many ways. Both Tzanakis et al. (2000) and Jankvist (2009) have written important texts on how this can be done and reasons for doing so. In a recent article (Agterberg et al., 2021) we describe a categorization of ways how HoM can be presented in curriculum materials or classroom activities. This categorization is meant to be used as a tool in what we call a design-oriented approach. With the help of this tool, teachers should be enabled to select and adjust curriculum materials that use HoM or even design classroom activities themselves. We distinguish between four formats for presenting HoM in curriculum materials. We chose format names that all start with the letter S: *speck*, *stamp*, *snippet* and *story*. The term speck was chosen to indicate the smallest format. The word stamp refers to the notion of a small, separate box of information, alongside the main text. The term *snippet* is used for the average size piece of historical information with substantial relation to the text in which it is incorporated. The fourth and last format story relates to larger historical packages that transfer a story of mathematics. Table 1 summarizes the main characteristics of the formats. A complete description of the formats can be found in (Agterberg et al., 2021).

Table 1: Four formats for presenting HoM in terms of size, content, location and functions

Format	Speck	Stamp	Snippet	Story
Size	Very small	Small	Medium	Large
Content	Text only	Text with or without illustrations	Text with or without illustrations, related to instruction or task	Text with or without illustrations, related to instruction or task
Location	Arbitrary location in curriculum material	Arbitrary location in curriculum material	Particular location in curriculum material	Particular location in curriculum material
Goal	Transfers historical aspects of the discipline	Transfers historical aspects of the discipline	Can transfer historical aspects of the discipline	Can transfer historical aspects of the discipline
Tool	Primarily affective, motivational	Primarily affective, motivational	Primarily support of learning of mathematics itself	Primarily support of learning of mathematics itself

In the next section I will show some concrete examples of these formats from Dutch mathematics textbooks. These examples were used in the teaching experiment that is mentioned earlier and will also be described in the next sections.

A suitcase full of fragments.

In February 2021 I taught a course on history of mathematics at the mathematics teacher training program at the AUAS, together with two colleagues. The students that participated in this course are all in-service teachers. They had already finished a teacher training program for lower secondary education and are working as mathematics teachers. The course on history of mathematics is part of a part-time teacher training program for higher secondary education. The goals of the course are to increase the participants' knowledge on history of mathematics and to teach them how to use history of mathematics in the classroom. Due to Covid-19 the class was taught completely online.

In the first session the students were instructed to do an assignment we called 'suitcase assignment'. For this assignment we used a virtual suitcase, containing fragments from three chapters from different Dutch mathematics textbook series, in which the concept of logarithms was introduced. Figure 1 shows one of the pictures that was in the suitcase (coded P2). It is a photo of some handwritten long multiplications, used in the original chapter to illustrate how calculations were done back in the 16th century. No wonder that Napier wanted to create a shortcut, seems to be the message of this picture. In didactical terms one could say this picture can be used to demonstrate the usefulness of logarithms, so the history of mathematics serves as a tool for understanding why a certain concept is developed. In terms of the formats described in the previous section, this picture can be regarded as a stamp: a small piece of historical information that can be placed at relatively arbitrary position in a chapter on logarithms.

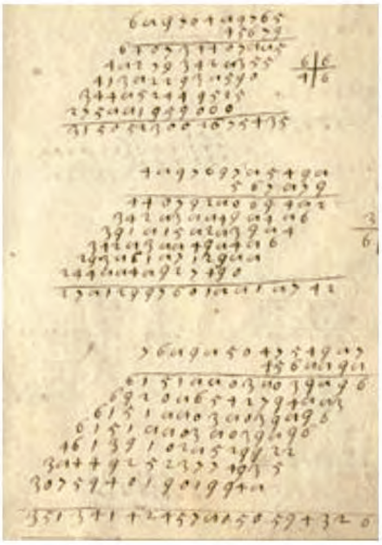

Figure 1: handwritten multiplications (cTWO, 2011, p. 21)

Figure 2 is a translated text from another chapter, stating some information on John Napier, the inventor of logarithms. In the original chapter the text is accompanied by a picture of a statue of Napier, which was left out in the suitcase assignment. This fragment contains only biographical information and does not give any mathematical information on why or how the logarithms were invented. This fragment can also be categorized as a stamp.

John Napier, also known as John Neper (Edinburgh, 1550-1617), was a Scottish mathematician who made his name mainly with his invention of logarithms. John studied for some time at St. Andrews University but also spent considerable time in other European countries. He was a convinced Protestant and particularly passionate about theology. In 1593, he published a religious work entitled Plaine Discovery of the Whole Revelation of St. John, which was translated into Dutch, French and German, so that he also became known on the continent. He practised mathematics mainly as a hobby.

Figure 2: translated text on John Napier (Wageningse Methode, 2015, §7.2)

Figure 3 is a translation from an exercise from the same chapter as picture P2 (figure 2). In this exercise the pupil learns by some examples how to use a logarithmic table to calculate specific multiplications, through some examples. The pieces of the logarithmic table that are necessary for performing this calculation are also shown on the right-hand side. In the original chapter, there is a full logarithmic table available as well. This table was also put in the suitcase, as picture P4.

How can you use these tables to approximate 456 ×12,3?

1. Search the table for the number closest to 4,56 → 4,574
2. Find out which power of 10 equals 4,56 → $10^{0,66}$
3. Find out which power of 10 equals 456 → $10^{2,66}$
4. Search the table for the number closest to 1,23 → 1,230
5. Find out which power of 10 equals 1,23 → $10^{0,09}$
6. Find out which power of 10 equals 12,3 → $10^{1.09}$
7. Find out which power of 10 is the answer: $10^{2,66} \times 10^{1,09} = 10^{3,75}$
8. Find out which number this is: 5,623 ×1000=5623

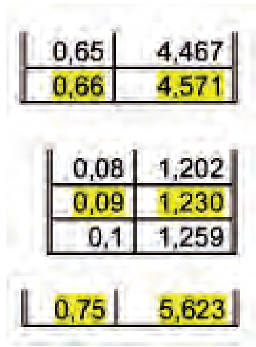

Figure 3: translated exercise on the use of the logarithmic table (cTWO, 2011, p. 22)

Table 2 lists the complete contents of the suitcase and specifies the contents of all historical fragments. All pictures, texts and exercise were separated from the original structure in the chapter and put together randomly in a table. The texts were retyped and renumbered, with identical fonts, colors and layout, so the fragments that belonged together in the same chapter could not be recognized on external features.

Table 2 Contents of virtual suitcase with fragments of chapters on logarithms

Type of fragment	Number of fragments	Historical fragments	Name of fragment	Contents of fragment
picture	4	3	P1	face of Napier
			P2	handwritten long multiplication
			P4	logarithmic table
text	8	3	T2	short biography of Napier
			T3	reason for developing logarithms
			T5	explanation of logarithmic table
			T8	long biography of Napier, including mathematical content
exercise	10	3	E5	use of logarithmic table to find specific powers of 10
			E8	use of logarithmic table to convert numbers into powers of 10
			E10	use of logarithmic table for multiplication and division

The students were given time to read all the fragments in the suitcase, then select the fragments they wanted to use in their lesson plan and adjust them if they wanted. Finally, they arranged the fragments in an order of their choice and wrote an explanation of their choices and purposes. They were instructed to select at least one fragment that contained, in their opinion, something related to the history of mathematics.

Results.

Twenty students handed in a complete lesson plan after the session. Their designs were scored on what historical fragments they used. The results are in table 3.

Table 3 Amounts and combination of selected historical fragments in suitcase assignment (P=pictures, T=texts, E=exercises, #=frequency)

#P	#	selection	#	#T	#	selection	#	#E	#	selection	#
0	1	-	-	0	1	-	-	0	8	-	-
1	10	P1	5	1	9	T2	4	1	8	E5	4
		P2	1			T3	1			E8	4
		P4	4			T8	4	2	4	E5 & E8	3
2	7	P1 & P4	5	2	8	T2 & T8	2			E8 & E10	1
		P2 & P4	2			T3 & T8	3				
3	2	P1, P2 & P4	2			T5 & T8	3				
				3	5	T2, T5 & T8	2				
						T3, T5 & T8	3				

Almost all students chose to incorporate at least one piece of biographical information. Three students did not include it at all (and four students chose both texts T2 and T8 that contained biographical information). The logarithmic table P4 was chosen thirteen times, but two of those lesson plans did not contain an explanation and/or exercises to accompany it. This way it was merely used as an illustration, a bit like illustration P2: an ancient long multiplication, showing how arithmetic was a tedious activity back in those days. The reason for developing logarithms is mentioned in fragments T3 and T8. Six students chose neither of them (among them two of the students that also did not incorporate any biographical information).

Twelve students selected one or more historical exercise. One of these students did not include the logarithmic table (P4) in his lesson plan, so it is questionable if this student really understood the exercises himself. From the eight students that did not select any historical exercise, four only selected fragments that contained biographical information (only T2 and/or P1) and the other four also incorporated some historical mathematical information (at least one of T3, T8, P2 or P4).

The way the students incorporated the fragments in their lesson plan was also categorized according to the 4S formats. The smallest format speck was not found amongst the lesson plans. The fragments in the suitcase were not that small and no student deliberately adjusted a chosen fragment in a way that reduced it to a speck. The eight lesson plans that did not incorporate exercises were all categorized as stamps. The remaining twelve lesson plans were mostly categorized as using history of mathematics in one or more snippets. Two students arranged their chosen fragments with history of mathematics to a complete story. The results of all twenty lesson plans are shown in table 4.

Table 4 Lesson plans categorized according to 4S formats

Format	**Frequency**
speck	0
stamp	8
snippet	10
story	2

Discussion.

In the previous section the research question "in what ways do in-service teachers use the history of mathematics in a lesson design for the introduction of the concept op logarithms" was answered with regards to type of fragment (picture, text, exercise), contents (biographical, mathematical) and finally format (speck, stamp, snippet, story).

The choice of fragments and design of lessons by the students will certainly be affected by the constraints of the suitcase assignments: limited time for reading, selecting, arranging and reporting, the assignment was completed during a (virtual) classroom session instead of a homework assignment, unfamiliarity of students with the fragments from different series of textbooks. The concept of logarithms can be presented in different ways (Kuper & Carlson, 2020). The students could possibly have had problems with the concept of logarithms themselves.

The historical development of logarithms is complex, but also provides promising opportunities to use it in class (Panagiotou, 2011). Even if the explicit historical information is left out, the geometric and arithmetic sequences from Stifel, Napier and others can be used for constructing logarithms through covariational reasoning (Ferrari-Escolá et al., 2016). The teacher can use this as an approach for constructing the concept during instruction, or have the pupils develop it themselves through a form of guided reinvention.

The students that did the suitcase assignment had received some instruction on how to use history of mathematics in their classrooms prior to this course, but this depended on the classes they took in previous years of teacher training. The fact that most of the students had little prior knowledge of the history of logarithms has probably also affected their choices considerably. Finally, knowing that the lessons were only planned and did not have to be enacted in class, could also have impacted the design. If the lessons had to be performed with a group of pupils, the students might have altered their plans.

The course on history of mathematics will be taught again in the beginning of 2022. The suitcase assignment with fragments from textbook chapters on logarithms will remain in the course, as a starting point for discussing the possibilities of using history of mathematics in the classroom. We intend to have the students enact the lessons they designed and reflect on this during the course.

This will surely provide us with a lot of interesting information from the classroom perspective, that we can learn from, to improve the course on history of mathematics itself and incite new research activities.

Acknowledgments.

I would like to thank my AUAS-colleagues Peter Lanser and Lidy Wesker-Elzinga. We taught the history of mathematics course, in which the suitcase assignment took place, together. They are very supportive to my research activities and it is inspiring to work with them.

The suitcase assignment is one of three assignments that were carried out by our students during the course on history of mathematics. In each assignment the students designed a lesson plan and they also filled out questionnaires on what they think about using history of mathematics for classroom activities. All lesson plans and questionnaires are collected as data and I am analyzing them as part of a larger study. I want to thank my PhD-supervisors Ron Oostdam, Fred Janssen and Bonne Zijlstra, who guide me in this research project.

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The effect of a reading of Clairaut on prospective secondary mathematics teachers' instructional design

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The use of original sources in teacher training is a rather well established research topic in the HPM domain. However, analyzing its explicit impact on classroom practice is still a relatively unexplored topic. This work addresses this topic by analyzing the tasks designed by 21 prospective secondary mathematics teachers just after participating in a series of activities structured around a fragment of Clairaut's Éléments de Géométrie. The possible effect of their reading of Clairaut is assessed in two ways. First, comparing the participants' epistemic configurations with that of Clairaut. Second, determining the steps of instructional design in which they considered to have been influenced. To do so, we use some elements of the onto-semiotic framework as well as the ADTRE instructional design model to analyze the tasks designed by the participants and their answers to an open ended question.

Keywords: Original sources, teacher training, epistemic configuration, instructional design.

Introduction and objectives

Over the last decades the use of history of mathematics in the context of Mathematics Education has become an intensive worldwide area of research (Clark et al., 2018). From early-age mathematics to tertiary education and teacher training, there are many reasons and many ways to introduce a historical dimension in mathematics education (Jankvist, 2009).

Regarding the use of history of mathematics in mathematics teacher training, Clark (2019, p. 49) proposes the following question as worth exploring: "Are the outcomes of teachers' study of history of mathematics seen in their classroom practice in explicit ways, and if so how? If not, why? Are the implicit ways equally meaningful?" In the particular case of original sources, the previous question is clearly related to the use of history as a resource by the teachers.

Consequently, the following research question naturally arises: Does the reading of original sources have an effect on teacher's professional practice? This question does not seem to have received much attention in the literature (Arnal-Bailera & Oller-Marcén, 2020), so this work aims to partly address this research question by analyzing the activities designed by a group of prospective secondary mathematics teachers after their work around a fragment of Clairaut's *Éléments de Géométrie*. In particular, our main objective is to determine the possible impact of their guided reading of Clairaut on their subsequent instructional design decisions.

Theoretical framework

Instructional design is an essential component of teachers' professional competence. There exist different instructional design models but all of them involve, at least, task analyzing and designing. In Figure 1 we show the elements of the so-called ADTRE model (Zhang et al., 2018).

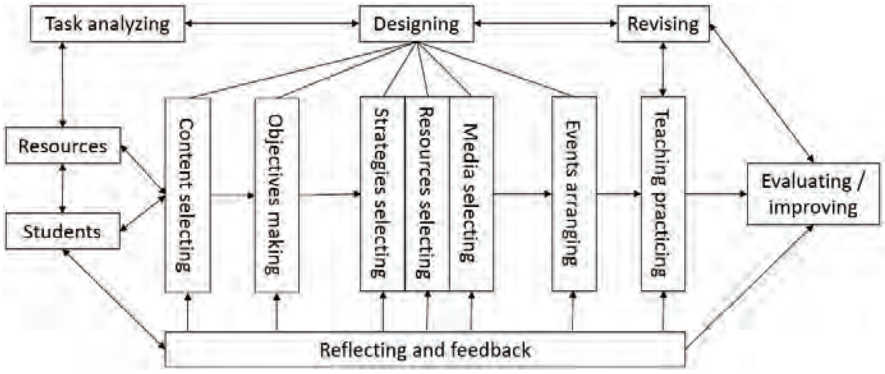

Figure 1: The ADTRE instructional design model. Adapted from (Zhang et al., 2018, p. 1078)

In the analyzing phase "instructional tasks are determined as a result of textbook and related curriculum material analysis and learner analysis" (Zhang et al., 2018, p. 1077). Even if these authors do not consider it explicitly, it seems clear that this phase may eventually involve reading and analyzing relevant historical texts or original sources.

Every theoretical framework dedicated to the knowledge and competencies of mathematics teachers considers how complex it is to define what makes a good teacher. These frameworks tend to agree on the fact that knowledge alone is not enough to do so, and for that reason the notion of competency is introduced. For example, a teacher should be aware of what is happening in the classroom from a specialized mathematics point of view, and act in consequence. Godino et al. (2017, p. 100) call this "competence for analyzing and managing didactical configurations." Didactical configurations take into account different aspects of instruction (epistemic, cognitive, affective, etc.).

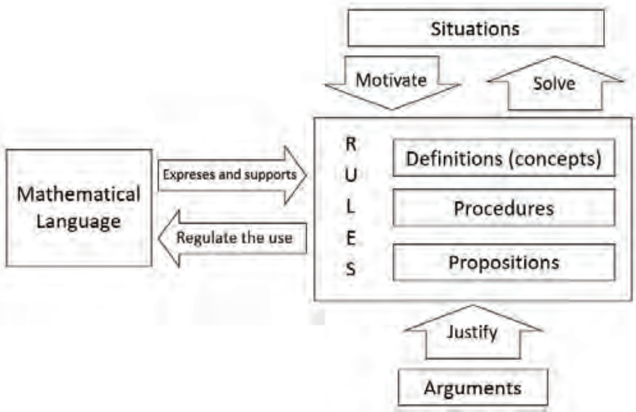

Figure 2: Components and relations of an epistemic configuration. Adapted from (Font & Godino, 2006, p. 69)

The Onto-Semiotic Approach to Research in Mathematics Education (OSA) assumes a pragmatic definition for the meanings that emerge within a system of mathematical practices (Godino et al., 2007; 2019). These meanings are established in terms of epistemic configurations (Figure 2), involving a set of primary objects: situations, languages, concepts-definitions, propositions, procedures, and arguments. Font and Godino (2006) show that the epistemic configuration is also a useful theoretical construct in order to analyze mathematical texts. In fact, these authors consider that mathematical texts can be seen, in some sense, as a system of mathematical practices.

The notion of epistemic configuration is central to the OSA and it has been extensively used in the research carried out from its perspective. This includes works focused on teacher education, in which this notion is used, for instance, to analyze teachers' practice or task design (Giacomone et al., 2018; Hummes et al., 2019).

Method

The experiment was carried out with 24 participants (13 men and 11 women) of the Masters' Degree in Secondary School Teaching. In particular, it took place within the course "Design, organization and development of activities for the learning of Mathematics" during the academic year 2018–2019. It consisted of three different activities:

1. A guided reading of a fragment by Clairaut (1741, pp. 125–128) in which the participants, working individually, had to complete a series of five tasks. In them, the participants had to identify relevant mathematical contents and processes, discuss them, and provide their personal point of view about different aspects of the text.
2. A classroom session in which the participants worked in pairs in two tasks, arising from certain claims by Clairaut and essentially related to the definition of tangent to a circle.
3. A homework individual task in which the participants were asked to design a 50 minutes session for grade 8 students, using GeoGebra, and related to the mathematical contents covered in Clairaut's text.

The selected fragment by Clairaut (which will be analyzed in detail in the next section) deals with the definition of the tangent to a circle at a point and the proof of what we call nowadays "alternate segment theorem". The interesting features from the point of view of Mathematics Education of Clairaut's work, and of this fragment in particular, were discussed by Barbin (1991) and Chorlay (2015), respectively. A complete description of the first activity, and the obtained results within the MKT framework can be found in (Arnal-Bailera & Oller-Marcén, 2020). In this paper, we focus on the third activity.

As we have already pointed out, the third activity was related to instructional design. In particular, the participants had to provide:

- The objectives of the session.
- The definitions of the mathematical concepts that they would introduce.
- Methodological indications including the intended use of GeoGebra.
- A sequence of tasks with an estimated timing.
- A GeoGebra file to be used in the session.

In addition, the participants were explicitly asked about the influence (or not) on the design resulting from their work with Clairaut's text.

Only 21 out of the 24 participants completed the activity. Each of them submitted a written document and at least one .ggb file. In this work we focus on the written productions. Taking into account the objective of the paper, our work was organized into three steps.

1. We carried out an *a priori* analysis of Clairaut's fragment.
2. We analyzed the participants' session design.
3. We analyzed the participants' answer to the question about the influence of Clairaut's text.

For the first two steps we have used the epistemic configuration as an analytical tool in order to compare the participants' configurations with Clairaut's, looking for possible similitudes or differences between them. For the third step we have adopted a deductive content analysis approach to the answers using elements of the ADTRE model as a tool.

Epistemic configuration of Clairaut's fragment

As a first step in our work, we used the notion of epistemic configuration (Font & Godino, 2006) in order to analyze the selected fragment of Clairaut's *Éléments de Géométrie*.

- Situation.
  The text aims to prove that, in any circle, an angle between a tangent and a chord passing through the point of contact is equal to half of the central angle defined by the chord.
- Concepts-definitions.
  In the fragment, Clairaut explicitly defined for the first time the following concepts:
  - C1: Tangent to a circle at a point, which is defined as a line touching the circle at a single point.
  - C2: Alternate-segment angle, which is defined as an angle between a tangent to a circle and a chord passing through the point of contact.
- Procedures.
  For a given inscribed angle, the chord opposite to its vertex is fixed, and then the vertex is "moved towards" one of the endpoints of the chord.
- Propositions:
  The so-called inscribed angle theorem (an angle inscribed in a circle is equal to half of the central angle that subtends the same arc on the circle) is explicitly stated in the text. In fact, it is introduced and proved just before the considered fragment.
- Arguments.
  Clairaut's proof relies on two arguments. The first one is explicitly stated, whereas the second one is just implicitly assumed:
  - A1: When the vertex is "moved towards" one of the endpoints of the chord, the inscribed angle becomes the alternate-segment angle because the secant line turns into the tangent.
  - A2: The alternate-segment angle is equal to the inscribed angle because during the process, the "moving" angle remains constant.
- Mathematical language.

- Graphic: The text is accompanied by a figure which is to be found in a sheet at the end of the book. This diagram suggests movement and supports a dynamic reading of the configuration, which is congruent with dynamic geometry.
- Symbolic: Points are represented by capital letters (A), angles, arcs and circles are represented by three points (ABC), segments and straight lines are represented by two points (AB).
- Verbal: There are two levels. The “main” discourse and marginal notes that are used to remind or to emphasize ideas.

The Participants’ configurations

Firstly, we analyze the participants’ epistemic configurations and compare them with the configuration underlying Clairaut’s fragment, which was described in the previous section. Due to space restrictions, we will particularly focus on the elements which, we think, are more innovative in Clairaut’s text: the procedure and the arguments (see the previous section).

First of all, 4 participants followed exactly the same procedure found in Clairaut’s text, considering a chord, and moving one of its endpoints. In the case of 5 additional participants, the dragging of points in GeoGebra played an important role in the procedures included in their proposals, even if it was not used to construct the tangent line as Clairaut does.

Regarding the arguments, the idea that the secant lines ultimately become the tangent line as a result of a dynamic process (A1) was clearly identified in the design of 8 participants. However, argument A2, which essentially involved considering the continuous limit of a constant magnitude, was found only in 4 of the designs.

The participants’ point of view

Now, we turn to the participants answers to the open question about how Clairaut’s work influenced their design. Only 4 out of the 21 participants who completed the activity answered that Clairaut’s text had not had an impact on their design. One of them did not provide any explanation. The remaining three participants provided arguments related to difficulties reading the text or to not finding the contents of the text useful for their purposes. In Table 1 we provide two examples of statements made by some of these students.

Table 1: Arguments of students for not being influenced by Clairaut

Participant A	“… since it was easier to me to see it in my way […] I think it is the same for others. The text is cumbersome…”
Participant B	“… [the text] is interesting to teach history of mathematics […] as a pedagogical object [sic] I didn’t like it.”

Now, the other 17 participants declared that the text had had some kind of influence over their decision making process. After categorizing the participants’ answer according to the elements of the ADTRE model, we provide in Table 2 the frequency of the identified ones.

Table 2: Elements of ADTRE model identified in the participants’ answers

Content selecting	Objectives making	Strategies, resources and media selecting
14	4	15

As we can see, many participants perceived an influence over their decisions about content selection. As an example, one of the participants stated that:

> Clairaut’s text was useful for me in order to introduce the concepts, in the way he did so

Also, a good number of participants informed about an influence over their strategies, resources and media selection. The following are two illustrative examples

> The criterion to choose this definition [Clairaut’s] was that it was easily implemented in GeoGebra

> We used the same idea as in Clairaut’s text […] we used an interactive software allowing to modify the angle […] this avoids the students to make the abstraction required when reading the text

However, very few participants made explicit a possible impact over their decisions regarding objectives making. One such example is the following:

> It served to me to […] involve the students in aspects related to proof in mathematics

Finally, it was particularly noteworthy that no mention to aspects related to the arrangement of activities was mentioned.

Conclusions

The direct influence of the activities carried out by de participants around Clairaut’s text was less important or, at least, less clear than we expected in advance. In fact, a few participants even explicitly claimed to not have been influenced at all. In addition, taking into account their own perceptions, the eventual influence was mostly related to content and strategies selection (see Table 2). Note that these elements require, in principle, a lower level of re-elaboration by the participant because contents and strategies can be more easily reproduced. This tendency to reproduction over more elaborated ways of making use of the text was already noticed by Arnal-Bailera and Oller-Marcén (2020).

Some authors (Biza et al., 2008) point out that the students’ initial conceptualizations of the tangent line in a Euclidean Geometry context have a deep impact on their understanding in more general contexts, such as Analysis. We have seen that Clairaut’s text, which in some sense brings together both contexts (Chorlay, 2015), has promoted among some of the participants the use of definitions, procedures, and arguments in their design that can be more easily transferred from a static geometrical setting to a more dynamic analytical setting.

It is noteworthy that about one half of the participants did not include Clairaut’s dynamic procedure and arguments in their proposals. In some cases, they seem to consider these arguments to be too complex for their students (see participant A in Table 1, for instance). Since this statement is merely based on the participant’s opinion, it illustrates the important role of beliefs in this type of activities, and the need to address them in some sense. In other cases, the participants' designs do not provide explicit information about the mathematical arguments that they would use in the classroom.

Even if scarce, the arguments provided by the participants that did not feel influenced by Clairaut's text were rather rich. Some of them contained elements related to the difficulties inherent to reading original sources (Jahnke et al., 2000), while others can be related to the fact that those participants did a shallow reading of the text in their previous work with it (Arnal-Bailera & Oller-Marcén, 2020). There were also some comments reminiscent of ideas exhibited by Siu (2007). It is noteworthy that no participant considered the possibility of including the explicit use of the original source in their session design.

Finally, we found some interesting comments that may foster further research. For example, one participant stated that:

> The classroom discussion […] was useful to focus the first part of…

This comment suggests that the role of the teacher educator is crucial when implementing this type of activities and has an impact over their outcome. Using original sources in a teacher training content is different from doing it in a school context (Jankvist et al., 2020) so the practice of the teacher educator might also be an interesting object of study.

On the other hand, statements such as

> The text made me reflect about which definitions and activities were more adequate to present the concepts at this level

point out the fact that the work with original sources might promote the prospective teachers' self-reflection about elements related to their didactic-mathematical competence (Breda et al., 2017). This opens an interesting line of research that we plan to explore in the near future.

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Investigating the potential of a historical document for task-design

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Abstract: This paper is a progress report which presents the context, the outline, and the starting point of an ongoing research project. In the context of a reform of the French high-school curriculum, teachers were unexpectedly required to use history of mathematics in the classroom, without being instructed how to, and with almost no standard resources – such as official guidelines or textbooks – to work with. This provides an opportunity to study the work of teachers as designers of teaching material. We set out a research protocol in which five teachers agreed to (independently) design teaching sessions starting from the same document, namely an extract from Euler's Elements of algebra on the numerical approximation of square roots. This paper aims to establish the relevance of this document in this research context.

Keywords: History, task-design, teacher as designer, cognitive demand.

Rationale

This paper is a preliminary report which aims to present the rationale and starting point of an ongoing research project which lies at the intersection of several important yet under-researched (or hitherto independently researched) topics in mathematics education.

First, research on the use of history of mathematics in the mathematics classroom have convincingly argued that, from both cognitive and didactical viewpoints, tasks based on historical sources are particularly suited for non-routine mathematical work (Fauvel & van Maanen, 2000; Jankvist, 2009; Chorlay, 2016). However, whether or not this *potential* for cognitively demanding classroom activities translates into practice calls for empirical studies on actual classroom practices. Such studies are still quite rare in the History and Pedagogy of Mathematics research community.

Second, the interaction between teachers and resources for teaching are studied from a variety of perspectives (Adler, 2000; Remillard, 2013; Choppin, McDuffie, Drake, & Davis, 2018), but mostly in contexts where teachers deal with ready-made teaching material such as textbooks. Arguably, contexts in which teachers and researchers jointly engage in *task design* have also been studied (Jones & Pepin, 2016), but contexts in which teachers work as *autonomous* designers are largely understudied, with some exceptions usually related to the use of ICT (see, for instance, (Trouche et al., 2018)).

Third, in the wake of Mary K. Stein's work on cognitively demanding tasks and the distinction between low-level (or routine) tasks and high-level tasks (seen as characteristic features of "doing mathematics"), many papers have studied how actual teaching practices maintain or lower the demand-level of tasks (Henningsen & Stein, 1997; Boston & Smith, 2009). To the best of our knowledge, these theoretical tools have not been used to study tasks

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designed on the basis of historical documents, along with their actual classroom implementation.

In what follows, we will explain how a recent reform of the French educational system provided an opportunity to set out a research protocol enabling us to empirically study how teachers interact with historical sources and engage in task design. Among the many features of the teachers' work as autonomous designers (and implementers) of tasks, we will focus on the observables pertaining to the level of tasks actually selected by the teachers (i.e. cognitive demand), and to the level of uncertainty which they are willing to leave room for in the classroom (Zaslavsky, 2005).

Outline of the research project

Institutional context: A new high-school curriculum (2019)

The French educational system has a national syllabus which the Ministry of Education changes approximately every eight years, for reasons which are usually not explicit, and as the result of a non-transparent process. In 2019, new high-school syllabi were implemented in all subjects, in the context of a large structural reform of high-school (grades 10 to 12). As far as history of mathematics is concerned, this new syllabus stood in sharp contrast with former syllabi. In 2010, for instance, the previous syllabus for science majors (grades 11-12) mentioned history of mathematics in passing, in an introductory paragraph on the varieties of forms of students' engagement with mathematics.

On the face of it, the 2019 syllabus deepened the same ideas:

> It can be judicious to enlighten the content of the mathematics course by a historical, epistemological or cultural contextualization. Indeed, history can be seen as a source of problems which clarify the meaning of some notions. The passages labelled "History of mathematics" point to some possibilities along this line. Teachers can implement them by relying on the study of historical documents (MEN, 2019. Our trans.)

In both syllabi, history of mathematics is seen as a tool rather than a goal (Jankvist, 2009); historical activities should be weaved in the general fabric of the course (should they be considered). Although this tool is versatile, the main goal is to foster conceptual understanding and meaning-making, rather than – for instance – motivating students, finding real-life or extra-mathematical applications of mathematics, or showing that school mathematics reflect the cultural heritage of humanity as a whole (with inputs from many cultural areas) etc. (Jankvist, 2009). Also, the use of historical sources in the classroom is not mandatory. The difference between the 2010- and the 2019-syllabi does not lie in the general intentions, but in the strength of the "recommendation" that teachers select and study historical problems – and, possibly, historical documents – with their students, as one of the many ways to support students' engagement with mathematics. In the 2010-syllabus, the above quoted short passage was the only mention of this topic in a 7-page document, making it a rather elusive suggestion. By contrast, the 2019-syllabus is lavishly peppered with paragraphs explicitly labelled "history of mathematics", which makes this topic one of the few guiding threads of the whole syllabus (along with "proof" and "algorithmic thinking and programming"). Some of these paragraphs are carefully worded and give specific suggestions, as in: "The history of probability theory provides a framework to explain the mathematization of chance. An instance is given by the correspondence between Pascal and Fermat on the problem of points – also known as Méré's problem –, along with the ensuing works of Pascal, Fermat, and Huygens. The problem of the Duke of Tuscany, or Leibniz's works on games of dice can also be mentioned." (Grade 10). Other paragraphs are rather pithy, leave much room for interpretation, and do not even hint at specific means of implementation; as in: "One can mention the slow elaboration of the notion of function, from Antiquity up to today's codification by Dirichlet, by foregrounding some important stages: Newton, Leibniz, Euler. The importance of the algebraic notation should be stressed." (Grade 10)

These features of the new syllabus came as a surprise to both teachers and teacher-educators, and the long list of rather cryptic yet heavy-handed historical "suggestions" understandably bewildered most. Moreover, the traditional resources on which teachers usually draw to meet the requirements of a new curriculum were wanting: firstly, up until the 2010 reform of teacher-training, very few teachers had any academic background in the history of mathematics, nor any experience of its inclusion in the classroom. Secondly, the syllabus to be implemented as from September 2019 was published shortly before, leaving very little time for commercial publishers to develop textbooks (there are no official textbooks in France). Since the historical suggestions were rather new, unexpected, and required that some textbook authors be well-versed in history (which is not usually the case), many textbooks failed to meet the challenge. Thirdly, when a new syllabus is published, it is customary for the Ministry of Education to publish guidelines for its implementations along with on-line teaching resources. In 2019, such resources were published to scaffold the implementation of the "proof" and "algorithmic thinking and programming" transversal features of the syllabus, but none regarding the historical suggestions.

Research protocol

The publication of the 2019-syllabus came as a somewhat pleasant surprise and unexpected endorsement for the IREM network (Institutes for Research in Mathematics Education). Indeed, these state-funded structures in which academics (mathematicians, mathematics educators, and, occasionally, historians of mathematics), teacher-trainers, and teachers collaborate to develop resources for teaching and teacher-training has a subcommission focusing on history and epistemology of mathematics. Since the 1980s, both in the French national context of the IREM network, and in the international context of the HPM Study Group, this subcommission has been working along the exact same general lines mentioned in both the 2010 and 2019 syllabi, arguing that many historical documents provide opportunities for a genuine engagement in mathematics, i.e. to "do mathematics". Moreover, it has accumulated a significant collective experience of resource development (Fauvel, 1990; Chorlay, 2016, Barbin, 2018). This subcommision reacted to the publication of the 2019 curriculum by launching the collective development of a book meant to provide high-school teachers with a range of thought-through and empirically tested (at least once!) classroom activities based on historical documents and compatible with the new curriculum.

This collective project is the context of our study. A study which involves two types of participants – five high-school teachers and a researcher in history of mathematics and mathematics education. A study which weaves together two distinct projects: one is the development of a classroom session, with a view to contributing a chapter to the new IREM book; one is a research project, carried out by the researcher and bearing on the work of the five teachers as they engaged in the resource-development project. It was agreed from the outset between each teacher and the researcher that both projects were distinct but compatible. The following protocol was agreed upon:

1. The researcher would select one historical document which he thought fit for the resource-development project. All teachers would be given the same document.
2. A first meeting would take place between the researcher and each teacher, individually. The historical document would be read together; its mathematical content would be discussed; possible connections to the curriculum would be discussed, in a "brainstorming" mode. The goal of this session was to generate a shared understanding of the didactical potential of the document. "Shared" meaning: shared between the researcher and each of the teachers; and similar for all teachers, even though they did not communicate with one another. In this meeting, no specific choices of implementation would be made or even discussed. The meeting would be recorded.
3. Each teacher would work independently from both the other teachers and the researcher in order to design some teaching session compatible with the resource-development project. For research purposes, teachers were asked to endeavour to keep a record of their work: personal notes, draft versions of the final documents, etc.
4. Each teacher would implement the session(s) she/he designed. The sessions would be recorded, either in audio or in video form.
5. Two short interviews would take place: (1) shortly before the actual session(s), the researcher would carry out a semi-guided interview bearing on (a) the teacher's self-recollection of the design process, (b) the choices which the teacher made along the way, (c) the goal(s) of the session(s), (d) the expected or possible difficulties, to be experienced either by the students, or by the teacher. (2) shortly after the session(s), an informal debrief would focus on topics (c) and (d).
6. This would be the end of the research project. The resource-development project would enter new phases: exchange of information among the teachers; possible engagement in a new task-design cycle (alterations, then implementation of the altered sessions); writing of a chapter for the IREM book. These new phases would involve both the teachers and the researcher in a collaborative way.

Let us underline a few specific features of this research protocol. First, in the research phase (stages 1 to 5), there was no communication among teachers, so we are not studying an instance of collective or collaborative task design. Second, the nature of the teacher-researcher interaction was of the "clinical partnership" type (Wagner, 1997). Consequently, this should not be considered an instance of teachers and researchers working as "partners in task design" (Jones & Pepin, 2016). Third, we are not studying how teachers interact with curriculum *material* such as textbooks: the two documents which set the stage for the teacher's task design activity are the national syllabus on the one hand, and a historical document on the other hand. Whether or not the teachers would look for and use other documents in the task design process is one aspect of their engagement in this process that would be studied.

A priori analysis of the didactical potential of the historical source

We selected a three-page extract from the 1774 French edition of the first volume of Euler's *Elements of algebra*. The teachers were told that this book is not a research treatise but rather a didactic work, covering algebraic topics ranging from the very elementary (operations with fractions and directed numbers) to the rather advanced (solutions of algebraic equations up to the fourth degree). The extracts were taken from the final chapter. The teachers were given a three-page document, but the researcher explained that they would focus on the first part consisting of paragraphs 784 and 786, the rest being provided mainly for context. The document below is taken from the 19th century British edition, which we altered slightly both to stick to the mathematical notations used in the 1774 French edition and to restore a numerical error which got corrected in later editions:

Document 1: Extract from (Euler, 1828, pp.677-680)

CHAP. XVI. Of the Resolution of Equations by Approximation.

784. When the roots of an equation are not rational, and can only be expressed by radical quantities, or when we have not even that resource, as is the case with equations which exceed the fourth degree, we must be satisfied with determining their values by approximation; that is to say, by methods which are continually bringing us nearer to the true value, till at last the error being very small, it may be neglected. Different methods of this kind have been proposed, the chief of which we shall explain.

(…)

786. We shall illustrate this method first by an easy example, requiring by approximation the root of the equation $xx = 20$.

[Footnote by J. Bernoulli: This is the method given by Sir Is. Newton at the beginning of his Method of Fluxions. When investigated, it is found subject to different imperfections; for which reason we may with advantage substitute the method given by M. de la Grange, in the Memoirs of Berlin for 1767 and 1768.]

Here we perceive, that x is greater than 4, and less than 5; making, therefore, $x = 4 + p$ shall have $xx = 16 + 8p + 16 = 20$; but as pp must be very small, we shall neglect it, in order that we may have only the equation $16 + 8p = 20$, or $8p = 4$. This gives $p = \frac{1}{2}$, and $x = 4\frac{1}{2}$, which already approaches nearer the true root. If, therefore, we now suppose $x = 4\frac{1}{2} + p$; we are sure that p expresses a fraction much smaller than before, and that we may neglect pp with greater propriety. We have, therefore, $xx = 20\frac{1}{4} + 9p = 20$, or $9p = -\frac{1}{4}$; and consequently, $p = -\frac{1}{36}$; therefore $x = 4\frac{1}{2} - \frac{1}{36} = 4\frac{17}{36}$.

And if we wished to approximate still nearer to the true value, we must make $x = 4\frac{17}{36} + p$, and should thus have $xx = 20\frac{1}{1296} + 8\frac{34}{36}p = 20$; so that $8\frac{34}{36}p = -\frac{1}{1296}$, $322p = -\frac{36}{1296} =$

$-\frac{1}{36}$ and $p = -\frac{1}{36\times 322} = -\frac{1}{11592}$; therefore $x = 4\frac{17}{36} - \frac{1}{11592} = 4\frac{4473}{11592}$ value which is so near the truth, that we may consider the error as of no importance.

The goal of stage 2 of the research protocol was to reach a shared understanding regarding some critical features of the text. In the list of features below, we italicized those contents which are explicitly mentioned in the high-school curriculum. Although the notion of *didactical potential* of a document calls for a more refined delineation, for now we will consider that connections between the content of the document and explicit curriculum goals testify to this potential. Consequently, this potential is not a property of the document *per se* but is highly context-dependent. Below, we also added (and italicized) the grade(s) in which these goals are mentioned (except for the "algorithmic thinking and programming", for which the same goals hold across all three high-school grades).

- This text provides opportunities to carry out routine calculation (*grades 8-10*): expansion of $(4+p)^2$, calculation with fractions, solving first degree equations, solving equation $x^2 = 20$. Even though the procedures are routine, the numerical values involved quickly become difficult to operate upon by pen and paper calculation only.
- This provides opportunities to carry out comparisons of specific numbers, either by comparing the successive approximations 4, $4\frac{1}{2}$, $4\frac{17}{36}$ … to a numerical approximation of $\sqrt{20}$; or by comparing the squares of 4, $4\frac{1}{2}$, $4\frac{17}{36}$ … with 20. These are all *grade 10* goals.
- The text weaves together two *genres* of mathematical texts: the exposition of an algorithm, and a heuristic argumentation providing some warrants for claims regarding key steps of the calculation.
- The meaning of the main warrant ("*pp* must be very small, we shall neglect it") is ambiguous, and the text does not provide any proof-type justifications for it. Several interpretations are possible. A static interpretation: $|p|$ being less than one (a claim which also calls for warrants!), p^2 is less than $|p|$ (*grade 10*), also less than $|8p|$ etc. But Euler wrote that, as the algorithm unfolds, one is *ever more* justified in neglecting p^2, thus possibly pointing to an asymptotic interpretation such as: when p is less than 1 and tends to 0, not only is $|p|$ less than p^2, but it becomes infinitely less since the ratio $|p|/p^2$ tends to $+\infty$.
- The text mentions or points to several topics in number theory. First, the introductory paragraph mentions a classification of numbers (some are "rational" while some other necessarily involve "radical quantities"), and, implicitly, $\sqrt{20}$ belongs to the second category (*grade 10*). Second, number-theoretic considerations provide an answer to a key question regarding the algorithm: it will not stop, should the condition to be met be the production of the exact value of $\sqrt{20}$. Indeed, starting from a rational input (such as 4), the algorithm will yield only rational numbers, thus leaving $\sqrt{20}$ beyond reach.
- The text displays the first steps of a method, but the claims as to the scope of this method are implicit. Is Euler claiming that the four values from (4 to $4\frac{4473}{11592}$) are

increasingly better approximations of $\sqrt{20}$ (*grade 10*)? That an iterative interpretation of the algorithm leads to a sequence of numbers with limit $\sqrt{20}$ (and with a strictly decreasing distance between the sequence and its limit) (*grade 12*)? That this "method" works for *all* square roots? Or even for all polynomial equations?

- Jean Bernoulli's footnote reminds the reader of the fact that Euler is merely expounding Newton's method (actually in Raphson's version (Bailey, 1989; Ypma, 1995)). To the expert reader, this should bring to mind the topics of tangents and derivatives; topics which, on the face of it, do not play any part in the text. However, the linearization of the equation amounts to considering a tangent instead of the curve, and the relevance of these ideas (algebraic or geometric linearizations) reflects the relevance of the concept of derivative as provider of local linear approximations. It so happens that, when one deals with polynomial functions only, derivatives can be defined and worked out purely algebraically. Hence Euler's text can be seen as presenting a very interesting special case (theory of derivatives in a polynomial context) of a very important general concept (the derivative as provider of local linear approximations) (*grades 11 and 12*).
- The text illustrates the first steps of what is clearly an iterative algorithm. Identifying this text as presenting an iterative algorithm, extracting the algorithm by editing out the heuristic parts of the text (parts which may also contain algorithmic steps, e.g. expand $(x_n+p)^2$, solve linear equations), and, possibly, implement it in a programming language, all these are tasks which fit exactly the "algorithmic thinking and programming" strand of the whole high-school syllabus.
- In *grade 10*, other algorithms for the approximation of the solutions of numerical equations – such as bisection – are to be studied and implemented. Since Euler presents a different method, it could be interesting to compare these methods in terms of efficiency. In *grade 12* it can be proven that the convergence of Newton's method in the case of square roots is quadratic, hence much faster than the linearly convergent dichotomy. Also, implementing Newton's method requires that a first rough approximation of a solution be taken as starting point (here $\sqrt{20} \approx 4$), which, in itself, calls for another (maybe more elementary) algorithm.
- Euler's text can be criticized, or at least questioned, as to rigour. In particular, he used the same letter x to denote different numbers; same for p. At least two reactions could be mathematically and didactically relevant: one could either realize the fact that this iterative method generates recursively defined sequences, and introduce notations such as $x_{n+1} = \frac{1}{2}\left(x_n + \frac{20}{x_n}\right)$ (*grade 11*); or consider that the letters represent programming variables and not mathematical variables (*grade 10*). In this second context, some of the "=" symbols should be read as value-change operators and not as mathematical equalities.

The research protocol (stages 1-5) was implemented in the 2020-2021 school-year, with five teachers; the data are being processed. The goal of this initial progress report was to establish that the historical document selected by the researcher has the potential to generate cognitively demanding classroom sessions, and leaves considerable leeway for the teachers to engage in task design and implementation.

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The reception of modern mathematics in Belgian technical education: Adhering or resisting?

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Belgium was one of the first European countries to be involved in the modern mathematics movement of the 1960s. The reform was led by a university professor (Georges Papy), supported by a number of enthusiastic teachers of general secondary education. The reformers showed almost no interest in technical education. In the mid-1960s, however, it became clear that also the mathematics programs of technical secondary education would be modernized. We examine the diverse reactions of stakeholders of technical education to the announced reform. Emile Ridiaux and like-minded teachers prepared and attempted to use modern mathematics for technical applications. Others, e.g. inspector Charles-François Becquet, resisted the upcoming reform, but his organization called MATEC, could not stand up to the well-oiled machine of Papy and his disciples. Consequently, from September 1, 1969, modern mathematics was generalized in all technical secondary schools in Belgium.

Keywords: Belgium, MATEC, modern mathematics, reform movement, technical education.

Introduction

Belgium was a pioneering country in the modern mathematics movement of the 1960s. A first experiment with modern mathematics began as early as 1958, one year before the landmark Royaumont Seminar. The experiment was carried out with prospective kindergarten teachers and was led by Frédérique Lenger and Madeleine Lepropre (De Bock & Vanpaemel, 2018). For technical assistance with follow-up experiments, the experimenters enlisted the help of Georges Papy, a professor of algebra at the University of Brussels. However, Papy did not limit himself to consulting: From 1959 he started his own experiments and would completely dominate the movement. In May 1961, he founded the Belgian Centre for Mathematics Pedagogy that grouped the country's most prominent modern mathematics enthusiasts. The Centre coordinated all aspects of the reform movement: It edited new curricula, set up experiments, developed course materials, retrained teachers, established local working groups, etc. The interpretation of modern mathematics was quite radical. Sets, relations, and algebraic structures formed the basis; logic, deductive reasoning, and proof were central. With appropriate political support, Papy's modernization efforts would lead to a mandatory implementation of modern mathematics in the general sections of secondary schools from September 1968.

Papy and his team members, who typically had an academic or general education background, showed little interest in technical education. No experimental programs or experiments were anticipated for this type of education. From official directives of the mid-1960s, however, it became clear that technical schools would also be involved in the modernization of the mathematics programs, a reform that prioritized intellectual formation over practical utility. A circular letter issued by the *Nationaal Verbond van het Katholiek Technisch Onderwijs* [National Association for Catholic Technical Education] on November 15, 1966, stated that every mathematics teacher in technical education should urgently inform himself about modern mathematics. To this end, these teachers were strongly advised to attend the classes given at the initiative of the Belgian Centre for Mathematics Pedagogy, and school boards were asked to give their teachers the opportunity to do so. (Holvoet, 1968). We find a similar recommendation in the 1966 mathematics program of the *Enseignement technique et professionnel féminin de l'État* [Technical and professional education for women in the state], as cited in Holvoet (1968): "All teachers are strongly advised to increase their knowledge by all means and in particular to attend regularly the courses organized by the Belgian Centre for Mathematics Pedagogy[1]" (p. 101).

In this paper, we search for answers to the following research question: "How have teachers and other stakeholders in technical education responded to an announced reform that seemed diametrically opposed to the goals of mathematics in technical education?" To answer this question, we have analyzed the available reform documents and all Belgian journals for mathematics teachers from that period: *Mathematica & Paedagogia*, *Nico*, and *Matmo* (see below). Moreover, we systematically searched *BelgicaPress*, a digital database of historical Belgian newspapers up to 1970[2]. This study is part of a line of research on the history of the modern mathematics movement in Belgium that recently led to the book *Rods, sets and arrows: The rise and fall of modern mathematics in Belgium* (De Bock & Vanpaemel, 2019). However, the reform in technical schools and the role of key actors in it remained underexposed. The present study aims to fill this gap in the existing literature on the Belgian modern mathematics movement. The scope of this type of research is, however, not just "purely" historical: Researching mathematics education of the past can provide historical insights that support the future, or as Van Bendegem (2021) stated it:

> This research is not merely historically interesting but is also relevant to understanding what present-day STEM (and I would add STEAM) initiatives are really about and what the potential pitfalls and dangers can be, and what the underlying educational aims and goals are (or should be) vis-à-vis society at large. (p. 604)

Ridiaux's actions to welcome modern mathematics

Emile Ridiaux (1924–2006), trained as an engineer, was a mathematics teacher in the secondary technical section of the *Université du Travail* [University of Labor] in Charleroi, an important industrial center in the French-speaking part of Belgium. With a view to preparing mathematics teachers in technical schools for the arrival of modern mathematics in their classrooms, Ridiaux founded a new teachers' journal, titled *Matmo*, a reference to "modern mathematics", and subtitled *Revue des ensemblistes du secondaire* [Journal for enthusiasts of sets in secondary schools]. *Matmo* was published between 1964 and 1968. A total of thirty issues appeared: Ten in the first volume (1964–1965), ten in 1966, five in 1967, and five in 1968. Layout and style of the journal were very

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[1] All translations were made by the author.

[2] Collections of the Royal Library of Belgium (https://www.belgicapress.be/). Keywords used: "Becquet", "Matec", "Matmo", "Papy", "Ridiaux".

basic and each issue had only 16 stapled pages. The journal did not question the upcoming reform as such, on the contrary. In the first *Editorial* (September 1964), we read:

> We do not claim to be scholars! We know the scholars, because they have come down to us, to our teaching! At the secondary level, even at the primary level, we want to apply their advice, because we have confidence, and, with them, we have hope in a much better mathematical future.

Browsing through the different issues of *Matmo*, one notices that most of the contributions were written by Ridiaux himself and some of his colleagues. Mainly lesson ideas for lower (technical) secondary education are presented, on "modern" topics such as sets, relations, and structures (inspired by Papy's proposals), but also "classical" topics from algebra, higher arithmetic, and geometry are considered from a modern point of view. Sometimes student reactions are included in the lesson plans, showing that the lessons had already been trialed. Attention is also paid to applications of mathematics, especially from physics (e.g., electrical circuits), but also from chemistry and biology. *Matmo* regularly informs its readers of new developments related to mathematics education, both in Belgium and abroad (e.g., with echoes from France and the USA). Specific Belgian reform initiatives, in particular the in-service courses of Papy's Centre, receive ample attention. The emergence of an opposition movement in 1966 (see the next section) is also reported and limitedly documented, but already in the first issue of 1967, Ridiaux writes that although he welcomes remarks and criticisms, "We would like to avoid a quarrel between the ancients and the moderns, especially among teachers..." (in capitals, no pagination). After that, *Matmo* will not include any news about the opposition movement…

It is worth mentioning that in 1969 Ridiaux published a textbook on a remarkable method in which arrow graphs, one of the favorite representational tools of Papy and his collaborators, were used to transform algebraic formulas (Ridiaux, 1969). This technical skill was not considered important by modern mathematics reformers, but it was fundamental to (technical) applications of mathematics. In Ridiaux's method, a path of arrows, representing operations, is drawn from a variable to be found to a given variable. By tracing the inverse arrows, which represent the inverse operations, the requested variable can be calculated (see Figure 1). Sometimes, Ridiaux also visualized the different steps in this process via film strips, a didactic tool known from Papy's *Mathématique Moderne* (De Bock & Vanpaemel, 2019). In Ridiaux (1970), the author explained how this method can also be used to solve (systems of) linear equations. Ridiaux's method is no longer used in Belgian schools today, but it is still recommended, albeit in a modified form, by some mathematics educators of the 21st century (see, e.g., Noël, 2003). The case illustrates how Ridiaux tried to didactically modernize mathematics for technical education, his way of conforming to reformers' expectations.

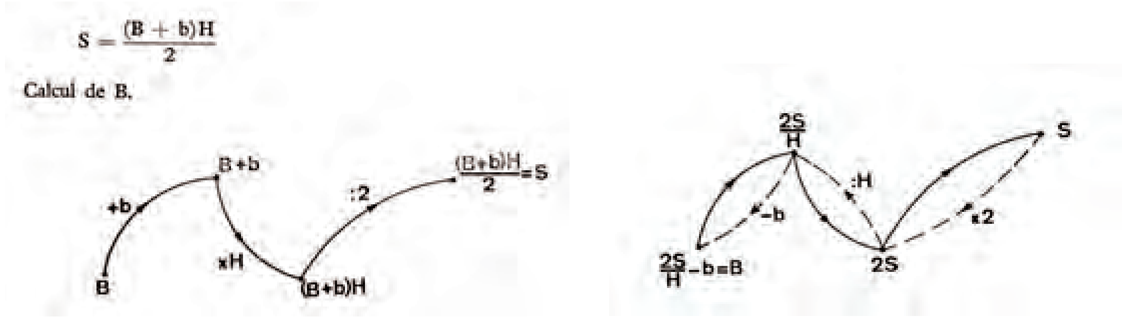

Figure 1: Example of Ridiaux's method for the transformation of formulas using arrow graphs; the variable *B* has to be calculated from the given formula (Ridiaux, 1969, p. 18)

Becquet and the action of MATEC

From Charles-François Becquet (1915–1987), a totally different sound was heard. Becquet was a master of mathematics (University of Liège, 1940), a teacher and later an inspector of technical secondary education organized by the state. In addition, though less relevant here, Becquet was a political author and militant in the Walloon Movement during and after World War II. In July 1963, Becquet had published an article entitled *Réformons les mathématiques* [Let's reform mathematics], in which he argued for a reform but one different from that proposed by Papy and his Centre (Schwilden, 1968). More specifically, Becquet advocated an approach that "makes better use of intuition, repetition, and new methods, such as those used in programmed courses". Becquet did acknowledge, however, that it would be desirable "to integrate the useful parts of new theories, taking into account the various levels and objectives of education" (Schwilden, 1968, p. 7). He also insisted that the reform be carried out slowly and gradually, and that the old not be swept away at once:

> Basically, the first phase would be focused on setting the traditional in order and the second phase on implementing the new theories. In this way, there would never be a deep conflict in the continuity of the curriculum, but a slow and secure evolution. The teachers themselves would not feel like pawns, like toys in the hands of curriculum developers. (Becquet in Schwilden, 1968, p. 7)

As early as September 1963, Becquet and some like-minded mathematics teachers at schools for technical education launched their own experiment to modernize the mathematics curricula for 12- to 18-year-old students in these types of school (Gadeyne, 1968). Eventually, 35 schools would follow Becquet's experimental method (Vermeulen, 1968). Becquet considered his experiment successful: A survey of participating teachers revealed that the method better established basic knowledge and facilitated understanding of mathematics. As a result, fewer students failed mathematics than before (*Le Soir*, April 24, 1968). In support of his actions, Becquet had founded a new teachers' association, *Mathématique et Technique* (MATEC) [Mathematics and Technique], in the first half of 1966. *Matmo 04/66* recorded a short press release about the initiative:

> About 30 school principals and teachers belonging to the various networks of technical education met, under the presidency of Mr. Inspector Becquet, at the *Institut d'Enseignement Technique de l'État*, in Namur, to constitute an association Mathematics and Technique, whose aim is the

> defense of technical education and its illustration in the framework of mathematics. (no pagination)

A Board of Directors was elected consisting of Becquet (president), J. Loomans (general secretary), and G. Benoit (administrative secretary), the latter two being principals of a technical school of, respectively, the official and Catholic network. Nine working groups were also established, each to address a specific problem related to the mathematics programs, and the relationships between technical education, on the one hand, and other forms of education and industry, on the other (*Matmo 04/66*, *05/66*). At a session on October 12, 1966, MATEC's Board of Directors adopted the following resolution:

> The Board of Directors of the teachers' association Mathematics and Technique expresses its strongest reservations regarding the consequences for the future of students in technical education resulting from the introduction of systematic teaching of set theory in the lower grades of technical and vocational secondary education. (*Matmo 10/66,* no pagination)

MATEC thus explicitly opposed the systematic introduction into the lower grades of technical secondary education of set theory, "a gateway to philosophy and logic but not to technical calculation", as Becquet had declared in an interview in *Le Soir*, a leading newspaper in the French-speaking part of Belgium (*Le Soir*, April 24, 1968, p. 6). But what did the technical schools fear would be lost, and what specifically did they propose as an alternative? The main concern, perhaps, was that in Papy's curriculum proposals mathematics would not be helpful to and would even isolate itself from other courses in technical education. In a letter dated November 1966 from Becquet addressed to the principals of the technical schools we read:

> The technicians insist on the practice of calculation. A solid knowledge of the fundamental operations involving integers, decimal numbers, and fractions is essential. The study of the metric system and its elementary applications, initiated in elementary school, must be continued. [...] Vocational training, the practice of technical drawing, and the technology course require real aptitudes in the above-mentioned subjects. […] In technical schools, the first-year drawing program includes geometric drawing and, already, an introduction to industrial drawing through the representation of simple parts. Teaching recipes alone would have no educational value. […] The drawing activity must therefore, in principle, be based on a prior knowledge of geometric notions acquired by experimental or intuitive means in the course of mathematics. It should be noted that the first-year geometry course should, more than in general education, introduce students to the knowledge of the common plane forms and solids and train them to see in space, because these notions and this skill are indispensable for the drawing of parts in the wood and iron workshop. It follows that the preservation of an elementary practical teaching of geometry, but always with a formative character, is essential. (*Matmo 10/66,* no pagination)

As an alternative, MATEC proposed that the lower grades of (technical) secondary education would study arithmetic, both calculation techniques and problem solving, the metric system, the basics of algebra (first degree algebra and the roots of a quadratic equation), plane geometry, trigonometry with right triangles (perhaps extended to non-right-angled triangles), and the first elements of solid geometry. Becquet did not oppose, at some later stage, the integration of some elements of vector and matrix algebra in schools for technical education. As stated by Noël (1993) and evident from the above listing, MATEC considered mathematics primarily as a technical tool and not as gymnastics of the mind. In the textbooks for Becquet's experiments, developed under his supervision by a group of teachers from technical schools involved in these experiments, the difficult chapters were split into two parts to separate the meaning of operations from the practice of calculation. Needless to say, the work of MATEC was strongly criticized and even ridiculed by Papy ("Mr. Becquet, the courses of Belgian Centre for Mathematics Pedagogy are open to you. Learn! You will be forgiven a lot", Papy, 1968, p. 34).

Vain hopes and math wars in the late 1960s...

The political decision to mandatorily introduce modern mathematics in the first years of secondary education from September 1, 1968, was made by Henri Janne, Minister of Education in an outgoing government, and announced in a circular of May 14, 1965 (Janne, 1965). Janne was a former rector of the University of Brussels, a socialist and political friend of Papy. On April 11, 1968, Janne's decision was confirmed by Frans Grootjans and Michel Toussaint, the Ministers of Education at the time, but their decision only concerned the general sections of secondary education (Grootjans & Toussaint, 1968). Thus, no formal decision was yet made about modern mathematics in technical secondary education. Moreover, on June 17, 1968, a new government was formed with two new Ministers of Education: Piet Vermeylen and Abel Dubois. The newly appointed Ministers soon allowed some schools of general secondary education "not yet ready to implement the new curricula" to postpone the introduction of modern mathematics by one year (Noël, 1993). Could the decision of Grootjans and Toussaint be reversed? A time of uncertainty followed for Papy's partisans, a time of hope for his opponents. This uncertainty led to a fierce math war in 1968–1969, especially in French-speaking Belgium (De Bock & Vanpaemel, 2019).

During that war, MATEC, which in the meantime had grown to a grouping of more than 400 school principals and teachers (*Le Soir*, April 24, 1968), further developed into a more or less structured opposition movement against Papy and the Belgian Centre for Mathematics Pedagogy. Already in 1966, it united forces with the *Association des Docteurs et Licenciés en Sciences Mathématiques sortis de l'Université de Liège* [Association of Doctors and PhDs in Mathematical Sciences from the University of Liège] (*Matmo 10/66*). Toward the end of the 1960s, an umbrella organization of mathematics teachers opposed to Papy's reform was established—the *Association des Professeurs de Mathématique de l'Enseignement Secondaire* [Association of Teachers of Mathematics of Secondary Education] (Derwidué, 1969). In reaction to this opposition, a group of teachers who supported Papy's reform project for the secondary level was created—the *Comité pour la Promotion de l'Enseignement Mathématique* [Committee for the Reform of Mathematical Education]) (Genaert, 1969). In 1968–1969, both organizations mobilized all those involved in the reform, including parents, for large-scale information meetings, hearings, and colloquia held in major Belgian cities (Figure 2).

Figure 2: Invitation flyer for a gathering of parents in Liège on June 23, 1969

The headlines of articles in the Francophone press of the period were telling: “La guerre des maths aura-t-elle lieu?” [Will the maths war take place?] (*Spécial*, March 6, 1968, p. 16), “Sur le front des maths” [On the maths front] (*Pourquoi Pas?*, August 29, 1968, p. 105), “Des cobayes pour les Papystes” [Laboratory animals for Papy’ists] (*Spécial*, April 9, 1969, p. 15), “À quand un cessez-le-feu et une commission d’armistice? Le pénible spectacle offert par la ‘guerre des math’ ” [When will there be a ceasefire and an armistice commission? The painful spectacle offered by the “math war.”] (*Le Soir*, April 27–28, 1969, p. 7), “Nouvelles maths: pour ou contre?” [New maths: for or against?] (*La Libre Belgique*, June 12, 1969, p. 5). The leftist press was mostly sympathetic to the reform initiated by Papy and his Centre, not only because Papy was a socialist, but also because there was a vague belief in the emancipatory power of the project.

The war was eventually won by Papy and his proponents. Minister Dubois, also a socialist, stuck to the decision of his predecessors. In an address delivered at the teaching college in Nivelles on April 27, 1969, he stated:

> The new curriculum is the Belgian version, very pragmatic and very adaptable, of a mathematical conception which is now being introduced in all industrialized countries; ... it constitutes a clear obligation for all schools run by the state; no one, my predecessors nor myself, has ever envisaged reconsidering it. (Dubois, 1969, p. 3)

When from September 1, 1969 all students of the first years of secondary education, both in the general and in the technical sections, without any exception, were subjected to modern mathematics, the late 1960s math war ended quickly. The official programs of the technical sections were similar to those of the general sections except for a few details. For Dubois, this was a deliberate decision, even a matter of principle, viz. an occasion to eliminate, maximally, divisions between different types of education (and thus to upgrade the status of the technical schools).

It would be unacceptable if not [the same programs were applicable] and that, with the introduction of the new program, the harmful division between general and technical education, each in their respective fortresses, would be continued. (Dubois, 1969, p. 3)

Conclusion

In the Belgian modern mathematics movement of the 1960s, technical secondary education entered the picture relatively late. The technical schools did not ask for the reform, but when it became clear that they would be involved, reactions were mixed. Ridiaux and some like-minded teachers adhered and tried to make the best of it, explaining in a newly founded journal *Matmo* how to incorporate modern elements into mathematics for technical education; others, such as Inspector Becquet and the members of MATEC, chose to resist. After a bitter math war in 1968–1969 between supporters and opponents of Papy, especially in French-speaking Belgium, modern mathematics was also implemented in technical secondary education, one year later than in general secondary education but with an almost identical program.

Future research could clarify the specificity of the Belgian modern mathematics movement in an international context. However, despite increasing scholarly interest and some single analyses with a specific scope (e.g., Kilpatrick, 2012; Vanpaemel & De Bock, 2019), comparative research on the international "New Math" phenomenon is still in its infancy.

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The “best” axiom system for teaching geometry to secondary school students: A source of controversy in the early 1960s

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Jean Dieudonné’s “Euclid must go!” provoked controversy among participants at the 1959 Royaumont Seminar. Although Dieudonné’s outcry led to misunderstandings, many contemporaries shared his view that geometry education needed to be modernized. On how to proceed, there was less agreement. At international meetings after Royaumont, no clear and supported solution could be proposed. For Dieudonné and other (Bourbaki-oriented) mathematicians, linear algebra was the royal road to geometry, but even then opinions differed widely. The controversy peaked in 1964 with the publication of Gustave Choquet’s L’Enseignement de la Géométrie and, shortly thereafter, Dieudonné’s Algèbre Linéaire et Géométrie Élémentaire. We carefully analyze these two seminal books and the role they played in the debates on modern mathematics in the early 1960s. Key differences are revealed and put into perspective with the similarities identified.

Keywords: Axiom system, geometry education, Gustave Choquet, Jean Dieudonné, modern mathematics.

Introduction

The 1959 Royaumont Seminar is widely recognized as a turning point in the history of the modern mathematics movement of the 1960s (Bjarnadóttir, 2008; Skovsmose, 2009). The Seminar brought together American and Western European reformers for the first time, and it also marked the start of this movement in several countries not previously involved. Although in recent years several researchers have argued that very diverse positions were taken at Royaumont, for example about the role of applications (De Bock & Zwaneveld, 2020) and of systematic psychological research (Schubring, 2014a, 2014b), in the collective memory of mathematics educators, Jean Dieudonné’s sharp attack on the traditional teaching of (Euclidean) geometry, for which his battle cry “Euclid must go!” was symbolic, continued to resonate.

A lot of ink has also been spilled about the exact meaning of Dieudonné’s slogan. Whence this vehement attack on “Euclid” while he deeply admired the achievements of the Greeks to mathematics (“I consider their creation of geometry perhaps the most extraordinary intellectual accomplishment ever realized by mankind”, Dieudonné, 1961, p. 35)? First, Dieudonné criticized what he considered a fossilized (French) educational tradition in geometry education, or as Hans Freudenthal (1967, p. 745) formulated it:

> The inventor of the slogan “A bas Euclide” a follower of Euclid? It looks odd, but it does so simply because few slogans have been misunderstood as badly as this one. Partly, it was Dieudonné’s own fault. When he cried “A bas Euclide,” he actually meant “A bas ‘Euclide’,” (viz. the Euclid of French lycée textbooks), but in oral discussions it is a hard thing to pronounce quotation marks.

Almost everything to which much attention was paid in this tradition, in particular “the triangle”, had, according to Dieudonné (1961), “as much relevance to what mathematicians (pure and applied) are doing today as magic square or chess problems!” (p. 36). However, Dieudonné was clearly in favor of an axiomatic approach to geometry, but not the old one of Euclid with its fuzzy distinction between “axioms” and “postulates”, but of a “modern” one based on linear algebra. Dieudonné’s views caused “sharp controversy” among the Royaumont participants, yet all agreed that a modernization of geometry education was needed. There was, however, less unanimity on how that modernization should be achieved. In a number of follow-up meetings in the early 1960s, the international mathematics education community attempted to come to some form of agreement, though without success. The debate quickly rigidified into the search for the “best” axiom system for secondary school geometry, and further hardened with the publication of Gustave Choquet’s *L’Enseignement de la Géométrie* and Jean Dieudonné’s *Algèbre Linéaire et Géométrie Élémentaire*, both in 1964.

In this paper, we first provide a brief overview of the development of the geometry debate in the first half of the 1960s. Second, we carefully analyze the aforementioned books by Choquet and Dieudonné in order to unravel the differences and commonalities in vision of these two scholars. To this end, inspired by the topic model for discourse analysis of Jacobs and Tschötschel (2019), we first manually compiled a list of topics (such as for instance “droite” [straight line], “symétrie” [reflection], …) that appear in the books, and this page per page. Then, we counted (using the freely available online tool of browserling on https://www.browserling.com/tools/word-frequency), how many times a topic is mentioned in each book. We compared how the topics that ranked highest in this count were dealt with in the two books. In comparing the manipulation of the different topics, we paid specific attention to the differences caused by the axiom system chosen. In the final section of this paper, we discuss how it came to a settlement, at least between Choquet and Dieudonné.

The geometry debate in the aftermath of Royaumont

The first half of the 1960s was particularly rich in international activities on the modernization of school mathematics. In addition to the annual meetings of the International Commission for the Study and Improvement of Mathematics Teaching (CIEAEM), important international meetings were held in, among other places, Aarhus, Denmark (1960), Zagreb-Dubrovnik, Yugoslavia (1960), Bologna, Italy (1961), Budapest, Hungary (1962), Athens, Greece (1963), and Frascati, Italy (1964). These meetings were initiated by one of the following international bodies, either in partnership or not: The International Commission on Mathematical Instruction (ICMI), the Organization for European Economic Cooperation (OEEC; in 1961 joined by nations outside Europe to form the Organization for Economic Cooperation and Development, OECD), and UNESCO. A detailed discussion of the objectives and conclusions of these meetings is beyond the scope of this paper and for that we can refer the reader to, for example, Furinghetti and Menghini (to appear). To indicate the international context and “climate” in which the polemic between Choquet and Dieudonné occurred, we discuss in a nutshell some key elements related to the debate on geometry education.

Although the 1960 Aarhus meeting was intended to have a broader scope, the debates concentrated on "Modern teaching of geometry in secondary schools with particular emphasis on ways of treatment opened up by developments lately, in particular by the algebraization of mathematics" (Behnke et al., 1960, p. ii). No less than five different proposals for "new" axiom systems for secondary school geometry were presented, including by the Royaumont lecturers Dieudonné and Choquet. Dieudonné reiterated his position that geometry education should be approached from the study of two- (or three-) dimensional vector spaces over the field of real numbers, equipped with an inner product. Choquet proposed his axiom system for affine and metric plane geometry that he would adopt in *L'Enseignement de la Géométrie* (1964). Although views differed, there seemed to be a consensus that Euclidean geometry was too difficult for 12–13-year-olds, even in the form of *Bewegungsgeometrie* [motion geometry], a German approach rooted in the work of Felix Klein and proposed at Royaumont by Otto Botsch. There were timid attempts, by Freudenthal, Willy Servais, and Tullio Viola, to include psychological aspects in the debate on school geometry, but this led to no less than a quarrel between Freudenthal and Dieudonné, with the latter stating, "*la psychologie, je m'en fiche*" [psychology, I don't care] (Behnke et al., 1960, p. 104).

In Zagreb-Dubrovnik (1960), a group of experts was commissioned to work out a detailed synopsis for modern secondary school mathematics "in the spirit of Royaumont". The experts agreed on the introduction to set theory, algebra, analysis, probability theory and statistics, but regarding geometry education, the outcome was an ambiguous compromise. For the final years (15–18-year-olds), an axiomatic and structural approach—heavily influenced by Choquet's vision—was recommended, while in the early years (11–15-year-olds), the emphasis would be put on a more intuitive approach (OEEC, 1961). The attention to intuitive geometry was largely due to the intervention of the Belgian Paul Libois, a lifelong advocate of intuition-based geometry teaching, but a dissident voice in the 1960s debates (Vanpaemel & De Bock, 2017).

The 1961 meeting in Bologna was intended to discuss the results of Aarhus and Zagreb-Dubrovnik, which had not led to any form of agreement. The discussion in Bologna led to some kind of compromise around a proposal by Emil Artin to define the Cartesian plane axiomatically as a vector space of dimension two with an inner product, but Freudenthal, Libois, and Viola partly disagreed. According to the first, the geometry programs proposed in Aarhus and Zagreb-Dubrovnik were based on an "anti-didactic inversion", expressing that an end product of mathematical activity, the most recently composed structure of mathematics, is taken as a starting point for mathematics teaching (Freudenthal, 1963).

At the meetings in Budapest (1962), Athens (1963), and Frascati (1964), results of early experiments with modern mathematics could already be presented. In particular, the audacious approach of the Belgian Georges Papy, combining mathematical rigor with an innovative pedagogy, received considerable attention and appreciation.

Two notable books shaping the debate of the mid-1960s

When reading Choquet's *L'Enseignement de la Géométrie* and Dieudonné's *Algèbre Linéaire et Géométrie Élémentaire*, one immediately notices the points of agreement in the objective Choquet and Dieudonné aim to achieve with their work; similarities in their views on the teaching of geometry are already apparent in the introduction to their books. Indeed, both authors state that they mainly target secondary school teachers. Moreover, they agree that the teaching of geometry for young children (age less than 13) should be based on observation and experiments, while in the last two or three years of the secondary school, linear algebra should be the basis of geometry. Between these two age groups, children should be gradually familiarized with deductive reasoning.

As (pure) mathematicians, Dieudonné and Choquet were "*Bourbakists*". *Bourbaki*'s approach to mathematics was axiomatic-deductive, formal, and uncompromisingly rigorous, deliberately excluding diagrams and external motivations. Both our protagonists were convinced that the *Bourbaki* model for structuring mathematics as a science was also the "best" model for structuring mathematics education, in casu geometry education, with the axioms of linear algebra as a fundamental model.

Before presenting their "best" axiom system for teaching geometry to secondary school students, Choquet and Dieudonné digress on how it should not be done, both referring to colleagues along the way, anonymously or not. Choquet seeks to minimize the number of axioms:

> mais je ne pense pas qu'il soit désirable, comme l'ont préconisé certains professeurs, de prendre au départ de très nombreux axiomes [but I do not think that it is desirable, as some professors have advocated, to take very many axioms at the start[1]] (Choquet, 1964, p. 10)

Dieudonné denies the claim that the axioms of linear algebra are too abstract, and refers to Choquet on the way:

> partir d'un *autre* système d'axiomes, […] le plus connu est sans doute le système d'axiomes proposé récemment par Choquet, d'une remarquable ingéniosité qui témoigne du grand talent de son auteur, mais que je tiens pour parfaitement inutile et même nuisible. [starting from another system of axioms, […] the best known is undoubtedly the system of axioms recently proposed by Choquet, of a remarkable ingenuity which testifies to the great talent of its author, but which I consider to be perfectly useless and even harmful.] (Dieudonné, 1964, p. 17)

However, both authors agree on the fact that the "best" axiom system should separate affine and metric notions, or as Dieudonné phrases it:

> Je pense surtout ici à la distinction (clairement sentie depuis Poncelet) entre propriétés géométriques de nature « affine » et propriétés de nature « métrique ». Il est particulièrement choquant, du point de vue logique, de voir mélanger en une incroyable salade ces deux types de propriétés dès le début de la traditionnelle « Géométrie euclidienne », mettant exactement sur le même plan des notions aussi différentes que celle de parallèle et celle de perpendiculaire [I am thinking here above all of the distinction (clearly felt since Poncelet) between geometric properties of an "affine" nature and properties of a "metric" nature. It is particularly shocking, from a logical point of view, to see these two types of properties mixed together in an incredible salad from the very beginning of the traditional "Euclidean geometry", putting exactly on the

[1] All translations were made by the authors.

> same level notions as different as that of being parallel and that of being perpendicular] (Dieudonné, 1964, p. 13)

Likewise, both Choquet and Dieudonné first treat the two-dimensional geometry of the plane before generalizing the concepts to the three-dimensional geometry of the space.

Faithful to his opinion that geometry should start off with a limited number of strong axioms, Choquet immediately sets off with axioms on a straight line and on a set of straight lines. He defines the *plane* as a set, equipped with a structure, induced by a set of subsets, called *straight lines.* Then he defines *parallel* straight lines and he states *axioms* on the existence of a unique straight line connecting two different points in the plane, and on a unique straight line parallel to a given straight line and passing through a given point of the plane (the so-called *parallel postulate*). Being parallel is an equivalence relation on the plane, leading to the concept of the *direction of a straight line* as the equivalence class to which the straight line belongs.

Dieudonné, however, first dedicates some pages to the necessary basic properties of real numbers and to the axioms of Euclidean geometry. He defines the *plane* as a set on which there exist three operations: addition of two elements, scalar multiplication of an element with a real number, and scalar product of two elements. Assuming the former two operations satisfy the necessary conditions, he calls this plane as soon as possible a *vector space*. Dieudonné then defines a *straight line* as an affine subspace with its *direction* fixed by the corresponding linear subspace (i.e., a straight line through the origin). Two straight lines are *parallel* if they have the same direction (that is, if they are both affine subspaces resulting from the same linear subspace). From these definitions, and using the properties of the underlying vector space, the proposition that there exists a unique straight line through two different points, and the *parallel postulate*, are proved.

In conclusion, because of the different choices for the axiom systems in the two books, properties that are put forth as axioms by Choquet can be proved as corollaries by Dieudonné, and vice versa.

In the further treatment of affine geometry in the books by Choquet and Dieudonné, their different starting points lead to only minor or mere formal differences. Noteworthy however, is the topic of the *addition* of two vectors. In Dieudonné's exposition, this addition is inherent to an operation on the elements of a vector space. Choquet, however, defines this operation by making use of parallelograms: The addition of two points x and y is the point z such that the origin, x, y, and z form a parallelogram. It illustrates Choquet's statement from his introduction, expressing how vectors are naturally at the foundation of geometry:

> L'axiomatique d'Euclide-Hilbert est basée sur les notions de longueur, d'angle, de triangle. Elle cache à merveille la structure vectorielle de l'espace […]. Le fait qu'un triangle soit la moitié d'un parallélogramme n'a pas empêché qu'on mette l'accent pendant plus de vingt siècles sur l'étude détaillée […] des triangles […]. On voyait le triangle, mais non le parallélogramme qui aurait pu conduire aux vecteurs. [The axiomatic system of Euclid-Hilbert is based on the notions of length, angle, triangle. It hides wonderfully the vector structure of the space […]. The fact that a triangle is half of a parallelogram has not prevented one to put emphasis for more than twenty centuries on a detailed study […] of triangles […]. One saw the triangle, but not the parallelogram which could have led to vectors.] (Choquet, 1964, p. 10)

Continuing with the definitions of an affine transformation, a translation, a homothety, and a dilatation, there are some differences in the order in which these concepts are treated and in the importance that is given to them. Loyal to his choice of the axiom system, Choquet first defines a translation as an operation on a straight line, before extending this definition to the plane. Dieudonné first defines these transformations as mappings on the vector space, even before defining a straight line. The definition of a dilatation is in Dieudonné's book left for the exercises, which should not be a surprise, since a dilatation is either a translation or a homothety.

Entering the world of Euclidean geometry, the consequences of the different choices for the axiom system, seem, at first sight, to manifest themselves again more firmly. At this point, Choquet first mentions briefly that only basic and well-known properties of real numbers will be used. He then posits in an axiom the concept of *distance* as a structure on a straight line. Later, in his chapter on the axioms of the *metric structure*, he first states an axiom on when two straight lines are *perpendicular*, followed by the definition of *perpendicular projection*. Only then, the notion of distance is used to define the *norm* of a vector. This concept of norm, together with the perpendicular projection, finally leads to the definition of the *scalar product* of two vectors.

Dieudonné reverses this order more or less completely. Since he starts with a vector space as underlying structure of the Euclidean space, he disposes immediately of the *scalar product* of two vectors. Similar to Choquet, however, he restricts this scalar product as an operation on a straight line. Applying the scalar product, the definition of a *norm* of a vector and the *distance* between two points, follow immediately. Only then, the notion of two *perpendicular* vectors is defined, again using the *scalar product* of these two vectors.

In his chapter on the metric structure of Euclidean space, Choquet mentions that the axioms he presented allow to show that the concept of distance that he introduced is equivalent to the concept of distance that is associated to the scalar product (which is the way Dieudonné defines distance), summarized in a Theorem:

> Théorème 96.3. Pour tout point $o \in E$, on peut définir sur E, de façon unique, une structure d'espace vectoriel d'origine o, muni d'un produit scalaire, dont les variétés affines de dimension 1 et 2 sont les droites et plans de E, et dont la distance associée au produit scalaire soit identique à la distance donnée sur E. [Theorem 96.3. For any point $o \in E$, one can define on E, in a unique way, a structure of a vector space with origin o, equipped with a scalar product, whose affine subspaces of dimension 1 and 2 are the lines and planes of E, and whose distance associated with the scalar product is identical to the distance given on E.] (Choquet, 1964, p. 148)

The treatment of the Euclidean transformations or isometries (i.e. translations, rotations, and reflections) is very similar in the books of Choquet and Dieudonné. Of special interest is that both authors first define a rotation, and then use the group of rotations to present a definition of an angle. As Choquet formulates it, a substantial part of geometry can be constructed without the use of angles:

> remarquons que nous avons pu construire commodément une grande partie de la géométrie sans parler jamais d'angles: La structure affine de Π, le théorème de Pythagore, la théorie des similitudes, ont été établis sans utiliser, ni angles, ni cas d'égalité des triangles. [note that we

> could conveniently construct a large part of geometry without ever talking about angles: The affine structure of Π, the Pythagorean theorem, the theory of similitudes, were established without using neither angles, nor similar triangles.] (Choquet, 1964, p. 97)

Only in a later stage they write about trigonometry and trigonometric formulas and about measuring angles. Both Choquet and Dieudonné end their exposition on the "best" axiom system by looking beyond secondary school geometry by examining in the appendices some extra topics, such as angles from a different viewpoint, non-Euclidean geometry, and quaternions and rotations.

A settlement between Choquet and Dieudonné

Papy's didactic approach, as elaborated in his *Mathématique Moderne* (Papy, 1964–1967), soon enjoyed considerable prestige in circles of mathematics educators, particularly in France (see, e.g., Walusinski, 1963). Also Dieudonné (1964) had praised "the remarkable and promising trials of our Belgian neighbors" (p. 17). Papy's approach to the teaching of geometry basically consisted of two stages. In the lower grades (12–15-year-olds), a system of (synthetic) axioms for affine plane geometry was gradually introduced, leading to the structure of the vector plane, an approach showing strong parallels with Choquet's. In the third year of secondary school (14–15-year-olds), this structure was equipped with an inner product, leading to the so-called Euclidean vector plane. At the beginning of the upper grades (15–18-year-olds) then followed what Papy called a "*retournement psychologique*" [psychological reversal]: The achieved goal of the first stage—the Euclidean vector space structure of the plane—was taken as a new and "unique" axiom for the second stage, i.e. the further development of geometry (of the plane and of higher dimensions) from a purely algebraic perspective.

At a meeting of the CIEAEM in Milano Marittima, Italy in 1965, and titled "The place of geometry in modern mathematical teaching", André Revuz tried to reconcile Choquet's and Dieudonné's points of view (Revuz, 1965). Revuz, who was esteemed by all parties, presented a statement about the role of geometry in the education of 12- to 18-year-olds, based on Papy's two-stages approach which was agreed upon by the CIEAEM members present. In the same year, on the occasion of an ICMI colloquium in Echternach, Luxembourg, this proposal was solemnly approved by Choquet and Dieudonné (De Bock & Vanpaemel, 2019).

In conclusion

Excesses of modern mathematics of the 1960s, such as the one described in this paper, have likely contributed to the current divide between the mathematics community and the mathematics education community, a divide that is unhealthy for both (Fried & Dreyfus, 2014). As concluded by Kilpatrick (2012), although modern mathematics is considered a failure, it did change mathematics education and the way it is treated by its stakeholders. Likewise, the books discussed in this paper were criticized (Freudenthal, 1967) and presumably not often used in secondary education, but they did stir the debate on geometry education in the 1960s and afterwards, geometry education did not return to the state it had before Dieudonné initiated the debate. Although the outcome of this debate on geometry education is in large likely due to the diplomatic qualities of Revuz, further research on the role of Papy's approach in this settlement could be of interest.

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Notabilia Arithmetica: a Jesuit mathematics school book

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In this paper we take a look at an arithmetic textbook from 1703 called Notabilia Arithmetica that belongs to the holdings of the old school library of a former Jesuit gymnasium in Germany and that was edited there, too. We analyse the book in regard to its content and possible integrated methodical aspects in order to find out more about mathematics education in Jesuit schools in the early modern period. The book shows us that – even though mathematics played a minor role in Jesuit education and was basically limited to Euclid in the general teaching rule – practical arithmetic must have been taught to a considerable degree, so that an arithmetic textbook was a necessary means for teachers.

Keywords: Arithmetic, Textbook research, Educational history, Jesuit education.

Introduction

Notabilia Arithmetica is the title of a small, pocket-sized book from 1703 that belongs to the stock of the Josephine library, that is the library of the former Jesuit gymnasium *Josephinum* in Hildesheim, Germany.

The research presented in this paper is part of an interdisciplinary research project aiming at a broad account of the library stock. So in the first place the focus is a local one. In the second place, though, we aim at embedding our findings in the history of classroom practice. Existing literature on Jesuit mathematics education has been thoroughly reviewed by Diaz (2009). However, research on mathematics education in Jesuit schools in Germany is extremely scarce. Furthermore, Diaz states a lack of research on classroom practice and that the "reality of teaching practices and curriculum […] need[s] […] further scholarship to understand the praxis of the Jesuit system." (2009, p. 65)

Following the presumption by other authors (e. g. Schubring, 1987) that teaching practices are determined by textbooks, we find the *Notabilia* a unique source. What makes it stand out from most other books in store of the library is that it has been edited by Hildesheim Jesuit mathematicians themselves. We may therefore start from the premise that the book was really used in classroom context and that it therefore serves as a source on actual Jesuit mathematics education.

In this paper we take a closer look at the *Notabilia Arithmetica* in regard to possible implications about mathematics instruction in a Jesuit gymnasium. We start by giving some general information on Jesuit education as well as on the history of the *Josephinum* and its library. Following this, we introduce the *Notabilia* itself. We describe its content and take a closer look at indications of purpose and methodical aspects, using the hermeneutic approach. Finally, we contextualize it by drawing comparisons to what other sources tell us about Jesuit education.

The Jesuits’ role in education

Following reformation and – as a consequence thereof – decline of Catholicism in the 16th century, the Jesuits drew on education in order to regain ground. Having officially been founded in 1540, the Jesuits soon started settling throughout Europe and the rest of the world and establishing colleges with associated *gymnasia*. Their education was characterized by a firm und uniform order, which was written down in the *Ratio atque Institutio Studiorum Societatis Jesu* in 1599. The Jesuit educational system became very influential and served as model for schools run by other religious orders, making it a relevant subject of research concerning mathematics education in the 16th and 17th century.

The *Ratio Studiorum* contains rules for the professors of the different subjects, and it is understood that it is due to the major influence of the famous Jesuit Christoph Clavius that mathematics has been taken up as a regular subject in there at all (Diaz, 2009). It was part of a two-year philosophy course, which comprised the subjects of the *quadrivium* and which followed on the grammar course and preceded the theology course. There was a clear focus on religious education whereas mathematics and the natural sciences did not play an important role within the curriculum (Hammerstein & Müller, 2005). This shows in the *Ratio*, as well: the paragraph containing rules for the professor of mathematics comprises less than one page, stating that three quarters of the physics course should be spent on Euclid and some geography or either astronomy in addition. Every month or every second month a mathematical problem shall be presented and discussed with a larger audience, and once a month the contents dealt with in recent lessons should be reviewed.

Generally, the primary aim of education was to have students who were capable of using their acquired knowledge on practical problems. Therefore, repetition, practice and rote learning were important and common parts of scholasticism. Typically, a new subject was first presented in a *lectio*, then further treated in different *quaestiones* and finally memorized and repeated in written exams and *disputationes*. The students’ achievements were regularly publicly displayed and rewarded with prizes. There was a catalogue of textbooks that were supposed to be used in the Jesuit gymnasiums and which were preferably written by Jesuits, too. Hammerstein & Müller (2005) name some of these standard textbooks but none for mathematics.

The *Gymnasium Josephinum* in Hildesheim

A short history of the *Gymnasium Josephinum*

The history of the *Gymnasium Josephinum* goes back to the year 815 when Louis the Pious – son of Charlemagne – founded the diocese of Hildesheim. It started out as a cathedral school located in the dome court, responsible for elementary education and being of high importance in the Middle Ages.

In 1587 the first Jesuit priest came to Hildesheim and by 1595 the Jesuits had established a residence and started a new gymnasium alongside the old cathedral school, first only teaching lower grammar classes. In 1601 the residence became officially a Jesuit college. Due to the Thirty Years’ War the Jesuits were expelled from Hildesheim in 1634 but returned already in 1643 and in the following years the school grew further. It is not before this time that we have knowledge of teachers for mathematics and the natural sciences. Around 1660 a vicar bestowed money explicitly meant for the constitution of four teaching positions for philosophy and mathematics and by 1664 there was at least one specialist teacher for mathematics (Gerlach & Seeland, 1950; Pilz, 1995). By the end of the 17th century the school had been given the name *Gymnasium Mariano-Josephinum*.

Since the Jesuit order was disbanded in 1773 the gymnasium was assigned to the diocese of Hildesheim again and it is still an episcopal secondary school located next to the dome today.

As to the students, already around 1700 there was what we would nowadays call a heterogeneous student body, regarding the region they came from as well as their estate. Besides students from noble or bourgeois origin there were also the poor ones, the *pauperes*, who obtained free housing and who especially the Hildesheim Jesuits were said to take a lot of care of (Pachtler, 1894).

The school library

As early as 1595, the year that schooling was taken up, the Jesuits of Hildesheim also took up establishing an associated library. When they were forced to leave Hildesheim they had to leave the books behind, as well, and when they returned in 1643 they found the library had been marauded in the meantime. They restored it and started collecting books again. Due to donations, gifts and inheritance the stock grew considerably up to about 20,000 volumes. The library got again endangered during the Second World War and indeed the old school building that had housed the library was destroyed in 1945. Luckily, most of the books had been evacuated before to a church outside of Hildesheim. They were only retrieved in the late 20th century, which is why their revision is still ongoing. Today the remains of the Josephine library comprise about 13,500 volumes, a considerable part of which dates back to Jesuit times. The library has been incorporated into the dome library and up to now only about 5000 volumes have been catalogued (Schmidt-Thieme, 2020).

Several mathematical textbooks are part of the stock, not least because Peter Heckenberg bequeathed his private collection to the library upon his death in 1695. Heckenberg who was a vicar and librarian of the Josephine library in the 17th century was himself the author of some of these mathematical textbooks (for further detail see Schmidt-Thieme, 2020). Nevertheless, we do not know the number of these books yet nor what role they played for practical mathematics education.

Standing out, though, are two booklets, both of them pocket-sized and especially noticeable as they were edited by mathematicians of the Hildesheim Society of Jesus themselves, leading to the assumption that these were produced for classroom work and therefore played a role in teaching practice. These are an arithmetic textbook by the name of *Notabilia Arithmetica* from 1703 and an edition of Euclid (*Elementa Euclidis*) from 1704. The latter still needs a closer look at. For what we found so far, it is most striking that we are dealing with an edition of Euclid’s Elements that lacks all the proofs (*sine demonstrationibus* is part of the full title) but all the propositions are basically reproduced. However, this paper deals with the *Notabilia Arithmetica* that seems to be an original work promising to give us further insight into the Jesuit mathematics classroom.

The *Notabilia Arithmetica*

General description

The book's full title is *Notabilia Arithmetica quae Omnem Arithmeticam, ejusque varium usum, cum in aliis multis, tum maxime in Geometricis operationibus succincte proponent* (Notable arithmetic that propounds briefly the complete arithmetic, its various uses, both in many other [things] and in most geometric operations), there is no author mentioned, but named as editors are mathematicians at the gymnasium of the Hildesheim Jesuit society. It has been printed in Hildesheim by Johannes Leonard Schlegel in the year 1703 and comprises 144 pages and 8 further uncounted sheets. Two of these added sheets differ slightly in the two copies that we worked with, so they might be drawn instead of printed. The *Notabilia Arithmetica* is written in Latin. The copy from the Josephine library shows a hand-written note, "biblioth paup 1709", on the title page, suggesting that it has been incorporated into the *bibliotheca pauperum*, the library for the poor students.

The book begins with a foreword dedicated to lecturers (*lectori*) and ends with some further suggestions for lecturers (*monitio ad lectorem*), so it is quite clear that it was written for teachers rather than for students even though it became part of the students' library. It is explicitly called a *compendium* in the foreword, containing only what is necessary, so it was supposedly not meant for the introduction of new topics. The reason given for the booklet's special format (approx. 8 cm x 13 cm) is that this facilitates compiling it with other mathematical textbooks, namely especially some of those written by Peter Heckenberg[1].

The copy found in the Josephine library is not the only copy that has persisted, there are at least seven more: three in the dome library in Hildesheim, one in Hannover as part of the private library of Gerhard Molanus[2], two in Münster and one in Trier, both of the latter being former sites of Jesuit colleges, as well.

Structure and content

Before we go into some focal points, we give an overview of how the book is structured and what mathematical subjects it contains. The content of the *Notabilia* is divided into four parts, each of which comprises several topics, rules and algorithms.

The first part deals with integers and their basic arithmetic operations. It starts with *numeratio* – that is introducing figures and their names – and goes on with introducing meaning and procedures of addition, subtraction, multiplication and division, followed by rabdology, which is calculating with the use of Napier's bones.

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[1] We have not found any books written by Heckenberg that share the same format yet, the remark might be an explanation, though, why the *Notabilia* and the *Elementa Euclidis* were both edited in this special format.

[2] Gerhard Wolter Molanus (1633–1722) was a professor of mathematics at the University of Rinteln, later abbot of the monastery in Loccum and acquainted with G. W. Leibniz, with whom he exchanged books from his private library.

The second part deals with fractions, starting again with *numeratio*, being followed by the rules for cancelling and the basic arithmetic operations for fractions.

The third part is about arithmetic rules and contains a vast variety of subjects. It starts with a relatively long chapter (*Cap. 11*) on the so-called golden rule, also known as *regula de tri* (rule of three), and goes on with a much longer chapter (*Cap. 12*) comprising 16 common and widespread rules (*Aliae regulae Arithmeticae in vulgari usitate*), such as the *regula falsi* (method of false position), but also rules on arithmetic (e. g. the so-called Gauss sum) and geometric progressions and some combinatorics. The latter indicates Athanasius Kircher[3] as source. The common rules are then followed by some shorter chapters on decimal fractions, some geometric rules (including trigonometry, measuring, planimetry and stereometry), astronomy, chronology (first of all *computus*), algebra (including extracting roots) and bookkeeping.

The fourth part contains various things and we could describe it as comprising miscellanea. It starts with the definition of a number as a manyness composed of unities and some more definitions of special numbers (e. g. odd, even and perfect numbers), which are followed by an overview of population figures (*Cap. 19*). Next are surveys containing currencies, measures of capacity and length, square measures, some geography and chronometry (calendar). For all kinds of measures the local Hildesheim measures are included.

These four main parts are followed by a synopsis of the whole book, condensed in *XII. Propositiones Arithmeticae* and twelve arithmetic paradoxes. The propositions are preceded by a short insertion that seems to be an announcement of a *disputatio* in the school. The booklet ends with an alphabetic index and the aforementioned hints for lecturers. Appended are three arithmetic tables and a geometric one, the latter alongside an explanation.

Quite some things are noticeable about the mathematical content of the *Notabilia Arithmetica*.

First of all it shows a vast variety of subjects, exceeding pure arithmetic by including topics like geometry, metrics, astronomy and geography. The inclusion of the latter can be explained with regard to the *Ratio Studiorum*, where astronomy as well as geography are explicitly named as possible subjects of lesson. As to geometry, there is no reference to Euclid but instead those areas are included that are less theoretical but have a relevance to practical applications. It appears that the same can be said about all the other topics and about the arithmetic chapters, as well. Especially chapters 11 and 12, which are the ones that deal with the rule of three and the other common rules and which therefore have a major relevance for everyday calculations, take up a big part of the whole work (altogether the two chapters comprise 20 pages). It appears that usefulness and applicability have been main criteria in choosing what subjects to include in the *Notabilia* and consequently – as it is called a compendium for teachers – in the mathematics classroom.

Having grown out of a cathedral school and with a religious order being in charge of the school, one could have expected the arithmetic course to stand in the tradition of medieval Christian education,

[3] Athanasius Kircher (1602–1680) was a German born Jesuit mathematician and seems a convenient source regarding that textbooks used by Jesuits should preferably be written by Jesuits, as well.

more precisely to refer to Boethian number theory. As we can see, this is not the case. The only exceptions are chapter 19, which includes some short definitions that resemble those we find in the works of Boethius, and chapter 11 on the rule of three where certain kinds of proportions are defined. Both parts added together comprise no more than five pages and especially chapter 19 stands out from the rest. Instead the content rather strongly resembles the content of the arithmetic books written by Renaissance reckoners.

This prioritization of content seems again plausible with regard to the aims of Jesuit education as mentioned above, namely the ability to apply knowledge and solve practical problems. In comparison to the *Ratio Studiorum*, though, it is striking that arithmetic is not scheduled there at all, let alone to such an extent. The existence of the *Notabilia Arithmetica* therefore serves as a proof that Jesuit mathematics education – at least as far as Hildesheim is concerned – has exceeded the standards by including arithmetic and further practical knowledge to a considerable extent.

Methodical references

Due to the *Notabilia* being a sole compendium, we cannot draw conclusions about the introduction of new mathematical concepts, procedures and propositions from it. But nevertheless, there are features that are interesting from a methodical point of view.

One thing that is noticeable is that several cross references can be found within the main parts of the book so we may assume it was one aim of the author(s) to write a book that was user-friendly. The existence of the alphabetic index and the comprehensive propositions at the end support the impression that the book was meant for practical use and possibly to serve as a reference book that was easy to use.

Another thing that attracts attention is that for some topics, e. g. the rule of three and calculating with fractions, we find guidelines regarding how to do exams and what theory needs to be proven. One example that we will now take a closer look at is chapter 11, the chapter on the rule of three. After different variations of the rule are presented a paragraph follows that is captioned *Regula aurea Examen & Theoria* (p. 40-41). As to exams it is said that the rule should be examined in all its variations, but with an emphasis on the simple rule in opposite to the *regula composita*. Regarding theory, we find standards that are to be followed. First, it is explained for the basic rule, how the figures that are involved in the calculation are interrelated by referring to their proportionality. The proportionality is then used to show how and reason why the fourth figure can be calculated from the other three. The passage ends with *Quod erat demonstrandum*, before the same argumentation is done for the inverted rule. As for the *regula composita* it is stated that it leads to the same problem already treated, a fact that will please the students. This bit creates the impression that a certain students' pleasure is granted. Finally, it is indicated that the majority of the common rules following in chapter 12 can be derived and proven from the rule of three and that it therefore suffices to learn the former.

In this paragraph we find confirmed that there are rules that are to be followed by all (at least all Hildesheim professors), even for an area that is not covered by the *Ratio*. We also find confirmed that exams are an important part of the curriculum, otherwise they would hardly be mentioned in a booklet that comprises only what is most necessary. The emphasis on the simple rules leaves several possible explanations. In the first place, one could take it as evidence for the minor significance attributed to mathematics within the Jesuit curriculum. It might just as well be an expression of time shortage, which is something we must assume considering the small amount of time that the *Ratio* allows on mathematics, even more as it is already exceeded by the variety of arithmetic topics. The explications also suggest another reason, namely the subject's internal material logic that allows us to deduce special rules to basic rules respectively to trace them back vice versa. Altogether the *Notabilia Arithmetica* shows us that despite the emphasis on practical knowledge Jesuit arithmetic education went beyond the pure memorization of algorithmic procedures. Even though a lot of the content reminds us of the Renaissance reckoning books, in this regard there is a considerable difference.

One thing that does not become clear from the remarks on demonstrations is whether they reside only with the teachers or are presented by students, as well. Exams were usually written but another part of the book gives us further insight, namely the aforementioned announcement of a *disputatio* (p. 136), which stand out in a curious way. There are no further comments about it and there is not even space between it and its adjacent sentences. We are only told that the disputation took place on the 5th of September 1702 in the school's public auditorium. Participants are named as *Domin[i] Antonio Fondeux* and *Leo[n]ardo Fischer* and as opponents *Joanne Robeck*[4], *Andrea Antonio Neuhaus* and *Johan. Angelo Schwartz*. We have no further information about the actual event but the announcement tells us that disputations on arithmetic were conducted at all. We may assume that these were events where students were meant to present propositions, proofs and explanations that they memorized from their teacher's demonstrations. The fact that the announcement is exposed in the *Notabilia* might nevertheless be a hint that we are dealing with an outstanding event and that arithmetic disputations did not take place regularly.

Conclusion

We find that the *Notabilia Arithmetica* has been written for teachers of mathematics, as a compendium of the topics from arithmetic lessons and as a reference book for necessary arithmetic knowledge. The *Notabilia* does not serve as a methodical manual for the introduction or development of mathematical procedures or concepts but it does include guidelines on what contents should be proved and how this reasoning was supposed to be done. In addition, it includes instructions on what was to be reviewed in exams. Even though the book was meant for teachers, it became placed at the poor students' disposal. Seemingly, some teacher or librarian regarded it as useful for them, too.

As for the content, the *Notabilia* shows a strong emphasis on practical applicability and computational abilities. This corresponds with the general aims of Jesuit science education but exceeds the standards for the professors of mathematics as we find them in the *Ratio Studiorum*, concerning both extent and topics. We may assume that there was a general need for usable arithmetic knowledge which was satisfied by the Jesuit educators or either a strong reckoning

[4] Johann R. Robeck (1672–1735) from Calmar, Sweden, was a philosopher who is best known for a treaty on suicide for theological reasons.

tradition going back to Renaissance reckoning masters. In any way, Jesuit mathematics education covered much more than Euclid's Elements and it can be assumed that the *Notabilia Arithmetica* arose from practical need for an arithmetic textbook that presented the subject matter in short and that served as a kind of teaching aid for lecturers, just as the *Elementa Euclidis* did for Euclidean geometry. This assumption is supported by the apparent absence of a standard textbook for mathematics and especially elementary arithmetic in the catalogues of such standard books. The fact that the book seems to have found its way into other Jesuit colleges, as well, might be another instance showing us that a textbook that summarized arithmetic was required for teaching practice.

This paper is just a starting point from which much more research on the topic needs to be done. Such future research must include analysis of and comparison to further books from the Josephine library and locating our findings in the broader context of Jesuit education. The latter comprises the more general context as well as special ideas concerning mathematics education, in particular those of Christoph Clavius (for more details on the role of Clavius see Diaz, 2009).

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Initiate reflection by original sources – Conception and empirical evaluation of a teaching project in calculus class

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One effect of working with original sources in mathematics classes that is repeatedly mentioned is the stimulation of reflection on mathematics and, as a result, a change in the students' mathematical beliefs. In the context of the teaching project presented here on working with Bernoulli's first textbook on calculus, this could be impressively confirmed empirically by means of a qualitative study. The article is dedicated to the conception of the teaching sequence, as well as first results of the empirical evaluation, and also poses the question of the conditions for dealing with original sources in a digital setting in a reflection-stimulating way.

Keywords: Beliefs, Calculus, Original sources, Reflection.

Working with original sources as a unique tool

The integration of the history of mathematics can not only contribute to the acquisition of process-related competencies and deeper understanding of the role of mathematics in the world as required by the curriculum, but also promises meaningful experiences with mathematics. Potential opportunities as well as difficulties depend strongly on the type of historical references used and also on the method chosen. Therefore, with regard to the teaching project discussed here, we first take a special focus on the motivations and problems specific for working with historical sources. Working with historical sources not only makes it possible to learn something about the history of the subject, but, if the source is chosen appropriately, it also offers the rare opportunity to gain insights into real (historical) research work in mathematics lessons. Thus, authentic insights into the thinking and working of researching mathematicians become possible. The ICME Study (2000) in this context lists three motivational fields: "replacement," "reorientation," and "cultural understanding" (Jahnke et al. 2000, p. 292). Thus, especially the work with historical sources is said to have the possibility to complement the learners' beliefs of mathematics by the aspect of mathematics as a process and human cultural achievement and thus to contribute to a more comprehensive picture of mathematics among the students. This initially theoretically expected outcome could also already be confirmed in various empirical studies and traced back to experiences within the history of mathematics (cf. e.g. Jankvist, 2015, or the approaches referred to in Bütüner, 2015). Furinghetti et al. (2006) implicitly take up this and add two further aspects to the three prominent points mentioned above: In addition to the opportunities for deeper understanding they also address potential problem areas. On the one hand, a historical text cannot be completely grasped on the first reading, and the interpretation is not clearly right or wrong, unlike what we are used to in mathematics classes. On the contrary, a hermeneutic, discursive approach like known from humanities is required here, which is clearly different from the ordinary experience in mathematics classroom and to which both the learners and the teachers must be prepared. This is also true for the different levels of language, which have to be switched between, if mathematics from the source is to be connected with today's school mathematics. The differences are to be made fruitful as an irritation that promotes understanding. The language competence necessary for this is certainly a high hurdle for many students, to which appropriate assistance must be provided. However, if these hurdles, created by the difference of mathematical representation in historical texts, are taken up and (at least partially) overcome, it is precisely these differences that offer the opportunity to reflect on the mathematics we have known so far and to enter into discussion:

> "Sources from mathematics history that defamiliarize the familiar arouse epistemic curiosity and trigger reflections in the classroom in a natural (intrinsic), effective, and performance-enhancing way." (translated from Glaubitz, 2010, p. 30)

This thesis particularly emphasizes the role of the source as an occasion for reflection on mathematics at various levels (see below).[1] To summarize the motives outlined here, mathematics history especially by working with original source-texts does not function first as a “tool” (among others) for learning mathematics (cf. Jankvist, 2009), but rather as a (possibly unique) tool on the way to a valid image of mathematics and to meaningful experiences in mathematics education.

With regard to teaching practice, at least two research questions are following: Is there empirical evidence for the claim that working with historical sources leads to increased reflective competence? And if so, which conditions for success can be named? In order to answer these questions, we will first present a project in calculus classes in a digital setting and after that discuss the empirical evaluation of student products with regard to different levels of reflection. Finally, first conditions for success for a reflection-oriented (digital) use of historical sources in mathematics lessons will be derived from this project.

A teaching project on Bernoulli's differential calculus

We are convinced that the opportunities described will not happen by themselves. Rather, good methodological framing is necessary.[2] This is especially true if the use of sources does not take place in face-to-face lessons, but must take place in a digital setting. How this can be done successfully is shown by the project presented below, in which the source work was rather closely guided and methodologically accompanied. As meant by the hermeneutic approach to original sources (cf. Jahnke, 1991) the project was organized as the ending of the “normal” calculus series in Q1 (Grade 11 in German Gymnasium) in an mathematics course of a manageable size with 18 students. The course is characterized in particular by the fact that almost all students have at least average mathematical competencies. Only three students deviate downwards in this respect. In addition, the course as a whole shows a high willingness to perform and a readiness to engage with unusual content and methods. In contrast to the teaching project on Bernoulli's differential calculus

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1 Reflection on mathematics and its relationship to environment, and the individual is repeatedly seen as a contribution to general mathematics education (cf. Bauer, 1990, Skovsmose, 1998 or Lengnink, 2006).

2 For a discussion on possible methods for introducing and reading historical texts in general see Junker & Spies (2020).

presented by Jahnke (1995), our project started by a specific introduction to the historical context of the source dealt with. In addition, hermeneutic reading of the source was guided to a greater extent than was intended by Jahnke. Due to the corona-related school closure, the project ultimately took the form of a digital guided source study in which the focus was on a sort of “interrupted reading” (cf. Metz et al., 2007) of the source excerpts flanked by work assignments.[3] These consisted of reproducing the newly learned historical procedures and tasks aimed at tracing Bernoulli's argumentation in detail and clarifying the unfamiliar historical terminology. This was accompanied by writing assignments. Due to the fact that there was no possibility for direct interaction while working, the formulations of the assignments were very small-guiding and pointing the students to the criucial points explicitly. Students were required to submit their assignments and each received individual feedback on their results. Questions regarding the content could also be directed to the teacher at any time via e-mail. At the end of each work block, explanations and answers were provided for everyone in various digital formats in response to questions asked and uncertainties observed. In this way, questions and uncertainties of individuals could be answered and at the same time productively turned around for the other students. By this at least indirectly the group was addressed despite the isolated way of working at home.

The project was introduced with a podcast that was supposed to introduce the students to the mathematical-historical situation around 1700 related to the beginnings of differential calculus and the scientific debate between Johann Bernoulli and his colleague Marquis de l'Hospital. The podcast did not yet contain mathematically relevant details, but served mainly to give the students an insight into the possibilities of scientific exchange and to point out the special situation between Johann Bernoulli and l'Hospital (cf. Jahnke, 1999). The format of a podcast was chosen in order to preserve, in the spirit of “StoryTelling”, the character of oral storytelling and its advantages over a read story (cf. Alchin, 2011 or Heering, 2016), even in digital format. Furthermore, the constellation described in the podcast could be taken up again at the end of the series by asking the students themselves to enter into a scientific exchange with Johann Bernoulli as l'Hospital. In order to prepare the students for the unusual fact that Johann Bernoulli did not use the representation of functions in a coordinate system to introduce the differential, an excursus was made on the possibility of constructing parabolas without a coordinate system. In several explanatory videos, the students were given the opportunity to reproduce such a parabola construction using only compass and ruler and to repeat relevant mathematical content (Thales' theorem and the geometric mean theorem). This was followed by an initial view in Johann Bernoulli’s Text - an excerpt in German translation from the work "Die Differenzialrechnung" (Schäftlin, 1924), which dealt with the introduction of the postulates fundamental to his work as well as the derivation of the differential of the quadratic function. In a small-step, task-guided analysis of the short source excerpt, the students should interpret the derivation of the "rule of derivation" for parabolas geometrically. On the one hand, this rather small-step approach to Bernoulli's geometrical way of reasoning is due to the experience that even advanced mathematics students showed great problems in leaving the context

3 Due to limitation of space we can’t present the tasks and materials in detail here. An english translation of some of the used texted in and classassignments is available online via https://www.uni-siegen.de/fb6/phima/mu/schulprojekte.html

of 'function, coordinate system, function graph' and following the geometrical arguments when working with this source (cf. Spies & Witzke, 2018). On the other hand, even in high school students usually have little experience in approaching a mathematical text independently, which is especially true for historical mathematical texts. The questions posed in the tasks thus exemplify to the learners how such a text can be approached, how deep one has to turn to each character with 'pen and paper' in order to really understand the mathematical argument. Through the questions asked, the repeated reading and the repeated comparison of what is read with one's own - in this case elementary geometric - (pre-)understanding, i.e. the reading in the hermeneutic circle is instructed.[4] This example shows, however, that the difference between "mathematical reading" and the reading of texts in the humanities or fiction, for example, can be experienced at least implicitly in the intensive examination of historical texts and thus a reading competence is trained that is indispensable for later (mathematics) studies. Because for Bernoulli's work, the notion of differential is of central importance, this initial source work was concluded with the task of writing a fictional entry for Diderot and d'Alembert's Encyclopedia of Knowledge on the meaning of the term differential. The genre of the encyclopedia article calls for the formulation of thoughts in an educational language, thus leaving behind both the level of (historical) technical mathematical language and individual linguisticization in the process of understanding. The change of language level not only initiates reflection on content, but is also considered an educational goal in mathematics education (c.f. Jahnke, 1991, p. 8). Besides, this task also points to the encyclopedia as a work that shaped the science of the Enlightenment and, with Diderot and d'Alembert, at least to two other greats of the time. In order to bring the students to a common level of learning for the following tasks, an explanatory video was created afterwards, in which the problems of understanding that had become apparent in the processing of the tasks, especially with regard to the concept of the differential, were taken up and attempted to be solved. Following the derivation of the differential, the first task from Bernoulli's textbook was to be reproduced, namely the construction of the tangent line to a parabola point. Here, too, we decided on a small-step, question-guided analysis of the source aimed the independent construction of a tangent to any parabola point. The tasks in this block are intended to sharpen the eye for the peculiarities of a geometrically reasoning calculus: While in school lessons, in which the slope of a tangent at a given point x0 is searched for or the derivative function is to be determined, Bernoulli solves the classical ancient, geometrically formulated tangent problem for a conic section. Thus, a second point is to be constructed for an arbitrary curve point in order to be able to construct the tangent line in this point on the basis of Euclidean geometry. The term "subtangent" mentioned Bernoulli‘s solution is an example of a mathematical term that was mathematics-common at the time and in the context of Bernoulli's argumentation, but which is unknown in (school) mathematics today. Here the students can also experience a development of mathematical terminology. By asking the students to construct the tangent line on their own at the end, it was at least possible to ensure that those students who were able to do so had a rough understanding of the theory. Just as after the first

4 For the students in the project it was the first time to deal with historical sources in mathematics classroom. Perhaps there is no such small-steped guidance necessary if there is more experience in hermeneutic reading and if the teacher has the possibility to interact directly with her students.

source work, an explanatory video was after this block, too, which was supposed to answer questions of the students that arose during the processing of the source work and which tried to solve problems of understanding that appeared in the students' results.

To conclude the project, the students should now take on the role of l'Hospital themselves and enter into a fictive scientific exchange with Johann Bernoulli. In a letter, they were to address their questions regarding Bernoulli's procedure for the determination of differentials and the construction of tangents to Bernoulli, by comparing their newly acquired knowledge with their previous ideas of differential calculus.[5] This task also served the purpose to ask last open questions for understanding. It is noticeable that the letter form was well received. Most of the students kept the letter form given at the beginning in their own texts and obviously enjoyed writing down their thoughts in this style. The questions and reflections touched on a very wide range of topics: They ranged from questions about concrete conclusions in the source excerpt read, to astonishment about the geometric method of constructing and discussing curves and the validity of Bernoulli's postulates, to the question of why this type of calculus is not part of the curriculum. At the end a response from the teachers again as a letter on behalf of Johann Bernoulli addressed all of these issues and collected questions from the students to address and clarify. Obvious misconceptions were addressed and could be cleared.

Levels of reflection within the students letters – Qualitative content analysis

The students' letters to Johann Bernoulli are written testimonies that in a certain sense document first effects of the source project. The "scientific exchange" in the letter format also leads to relatively open and personal formulations of the thoughts. The students' documents thus allow conclusions about the extent to which the source work actually led to reflections on mathematics. In order to examine the students' letters in a structured way the documents were read following the method of qualitative content analysis (Mayring, 2002). The units of analysis thereby resulted from the sections of meaning within the letters. Guided by the expectation to find testimonies of personal reflection on mathematics, the levels of reflection also formulated by Glaubitz (2010) following Bauer (1990) were first used as deductive categories: the object level, which contains both the reflection in the mathematical work and the reflection on the object, directed "to the essence of the discipline". Furthermore, there is the level of meaning, which includes the reflection on the meaning of working with mathematics as well as the reflection on the meaning and limits of mathematical thinking itself. And finally, the level of self-reflection, which contains the reflection about the effects of the occupation with mathematical contents on the possibilities of the own thinking and acting as well as the reflection about the role of mathematics for the own person and the own self-conception. As a result of a first review of the material, the object level was again divided by us into three levels and thus the category catalog was inductively expanded. On the one hand, there is the level in which mainly a comparison of different mathematical methods and approaches is made (level I). Furthermore, we distinguish here a level in which thinking about a transfer of mathematical approaches and methods to obvious questions and problems beyond the treated mathematical problem takes place (level II). And as a third level we have singled out the level that

5 While the previous are past-related tasks, this writing task is a present-related task in the sense of Schorcht (2018).

involves reflection on the question of the genesis of mathematical content and problems (level III). These sub-areas of the subject level correspond at the same time to different requirement areas of mathematical tasks. With this subdivision, we thus map the very different levels of object reflection that are visible in the letters. The analysis of the letters based on these categories show that each category can be found in several examples[6]. So one can be justified in hoping that the treatment of also historical perspectives and contents in mathematics lessons can reach the most diverse levels of reflection. Almost all of the students' letters contain expressions such as "exciting," "surprised," "impressed," or "interesting,". This shows that the confrontation with historical mathematics also had an emotional effect and that, similar to the reflection on aesthetic experiences with mathematics (cf. Müller-Hill & Spies, 2015), meaningful experiences are documented here as well. Furthermore, these are all clear indications that most of the students were able to use the series of lessons to restructure their own understanding of mathematics and its meaning, to add new aspects, and to expand their repertoire of ways of thinking about mathematical questions. Through the analysis of the letters to Bernoulli, we can thus assume that the students' engagement with the mathematical content was stimulated at various levels, some of which were high, and also at structurally different levels. This provides further empirical evidence for the theoretically assumed effectiveness of historical sources in mathematics education.

Suggesting conditions of success (not only) in a digital setting

The evaluation of the final letters shows that the students were stimulated to reflect deeply on a wide variety of levels through their intensive engagement with the historical source material. This finding is nevertheless astonishing, since often in particular the direct exchange about the readings of the learners among themselves and with the teacher as well as the openness of the task in the context of the hermeneutic method is described as a special moment, which stimulates the reflection in the first place. However, precisely these aspects of the source work had to be omitted in the digital implementation of the project. Therefore, in conclusion, some methodological conditions will now be discussed here, which allow historical source texts to become an occasion for reflection in mathematics lessons even in the digital setting.

Working with original sources requires a framework that can both convey a sense of the historical context and motivate work with the historical text (which is usually outside the curriculum). In a digital setting, this might be done with the help of a podcast, a historical documentary, or an introductory text. Depending on the format chosen, a more or less differentiated picture of the historical context is drawn. A deeper understanding of the source continues to be a necessary basis for successful reflection. This in turn requires that on the one hand the professional as well as the historical preconditions are given in order to be able to do something with the historical mathematics. In the project described, a first important step was to introduce the students to the geometric construction of the parabola in order to be able to recognize the geometric-analytical arguments in the source text as such. So the alienation effect of the source does not become a barrier to understanding. In a digital setting, such an introduction can take place, for example, via an

6 Cf. Junker & Spies (2020) for exemplary quotations for each category

explanatory video or a text flanked by tasks. On the other hand, a more in-depth individual engagement with the text is necessary. As described above, we believe that this cannot be initiated by a simple reading assignment, at least in a digital setting, but requires questions that guide the reading, encourage repeated reading, and draw attention to the central parts of the argument. Although such guidance in face-to-face classes can, of course, happen in individual exchanges, one advantage of the digital course emerged here: All students could (and had to) work intensively on the sources on there own. Thus, the work was done at the individual pace and each individual had the chance to experience that he or she is able to work out a completely unknown content and to comprehend the unfamiliar mathematical way of reasoning, since there was simply no possibility to have a contact person immediately in case of difficulties in understanding or to wait until the issue is clarified in the plenary. Even if the experience of competence and deeper understanding is certainly an important basis for a successful reflection, the direct exchange is naturally missing in a purely digital setting, which in turn would have led to fruitful discussions and reflection about the subject matter. In order to move from an in-depth reading of the source excerpt to reflection on what has been read, the mathematical argumentation must be left behind and the meta-level must be taken up. This also needs to be instructed. In the digital setting, this step can be stimulated, for example, by writing assignments, as it was done in the project presented here by the encyclopedia entry on the differential and the concluding letter. It is important to choose a form that stimulates the change of language levels in a natural way and also encourages the students to connect the historical argumentation with their previous knowledge. Again, the advantage of the purely digital setting is that own thoughts have to be formulated individually. On the other hand, there was no need for communication about the individual results, which would certainly have led to fruitful discussions, since ideas can be taken up directly, thought through further and then formulated together. The experiences in the presented project show that source texts can also become an occasion for reflection in a purely digital format if the discussed conditions for success are met. Above all, the experience of (guided) individual engagement with the text and writing about mathematics turned out to be important factors for successful reflection. The missing exchange about what was read could at least in part be taken up by direct feedback and interposed explanatory videos, even if here of course the potential can only be exhausted in real communication about historical mathematics. It can therefore be assumed that a combination of digitally guided intensive individual source work and the exchange about it in face-to-face lessons with a subsequent writing task to be worked on individually would be a format that could strengthen the observed effects and that the "foreign and bulky" of the source texts could be used productively in class.

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Concurrence of two mathematics worlds in the Netherlands, 1600 and beyond

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In the early 17th century at the University of Leiden, mathematics courses in the Liberal Arts and a mathematics course for surveyors and engineers in Dutch language existed peacefully next to each other, with some pathways between them. The reasons for this unique situation, the relation between the different mathematical courses and the characteristics of the Dutch language course are discussed. The design of the Dutch language course has some aspects in common with modern curricula. Two questions arise. Did this course influence the mathematics education in the next centuries? Was the Dutch language course a first step towards the development of mathematics as a school discipline? The second part of the paper attempts to find answers to these questions.

Keywords: History of mathematics education, mathematics for practitioners, engineering school, mathematics as school discipline, mathematics teachers.

Introduction

Towards the end of the 16th century two different worlds of mathematics and mathematics teaching may be distinguished in North-western Europe. One was the learned world of mathematicians at universities, at other institutes of learning, or working independently, whose object of study was the mathematics of the Greek mathematicians, their translators and commentators. It was a world of theory, in which an increasing number of ancient works became available for study, encouraging the development of mathematical theory. The language of communication was Latin, obligatory at all universities, the language of the men of learning. Mathematics teaching formed a part of the Liberal Arts, the general education as a foundation for the higher faculties. The students followed at least lectures on pure mathematics (number theory and Euclidean geometry), sometimes followed by mixed mathematics, subjects such as astronomy, mechanics, and cosmography.

The other world was the practical world of the common people, practitioners who didn't know Latin and who used some mathematics as a tool in their working life. Their grasp of mathematics might be extensive, but often it was very limited, without much theoretical background. In this practical world of surveyors, bookkeepers, building masters, military architects, traders, navigators, etc. there were as yet no teaching institutes. Private instruction by reckonmasters, self-instruction and learning at work were the main avenues to mathematical knowledge. The mathematics taught was primarily practical, without much attention to mathematical theory, confined to routines that seemed to work in a specific situation.

This changed in Leiden around 1600, at the university, which was established in 1575 as the first university of the northern Netherlands. The mathematics lectures, in Latin, in the Liberal Arts faculty were taught by Rudolph Snellius, while from 1600 a Dutch course for surveyors and military engineers flourished, under responsibility of the University and taught by university professors. This last course became known as Duytsche Mathematique (*Dutch Mathematics*). The reasons for this unusual situation, some characteristics of and the relations between the two mathematics form the subject of the first section of this paper. Two questions arise.

1) Are influences of the characteristics of Duytsche Mathematique recognisable in mathematics education in the Netherlands during the 17th and 18th century?
2) Was the engineering course an isolated incident or a first step towards mathematics as a school discipline?

Following (Chervel, 1988; Cardon-Quint & d’Enfert, 2017) a school discipline is taken as a subject taught in institutes, with specific teaching and learning goals, development of specific content partially independent of academic mathematical sciences, dedicated teaching methods, learning means, tests and examinations and taught by a body of professional subject teachers.

Mathematics teaching at Leiden University in the early 17th century, an innovation

The governors of the new Leiden University did their utmost to invite humanistic scholars with a good reputation to become professor in Leiden. As was the case in the older renowned universities, the theoretical framework for the faculties consisted of the works of Greek and Arabic authors, such as Aristotle, Euclid, Ptolemaeus, Hippocrates, Galenus, Avicenna and Averroës (Otterspeer, 2008). Justus Lipsius, Joseph Scaliger and Claudius Salmasius were some of the more famous scholars who came to Leiden, in return for a very good salary, free living and exemption of the obligation to give public lectures. Their names warranted quality of the university; they were expected to attract many students.

The first appointments were in the three main faculties: Theology, Law and Medicine, followed by Literature and Languages (classical and eastern) and Philosophy. Mathematics had a low priority; during the first years there was no professor for the undergraduate mathematics courses. In 1581 Rudolph Snellius (1546-1613) was appointed, a graduate in the Liberal Arts of the Calvinist University of Marburg. In Marburg he became an enthusiastic follower of Petrus Ramus (1515-1572), the French humanist and educational reformer. Ramus had proposed a shorter, well-structured curriculum, which combined theory with practical exercises. For mathematics he proposed to teach only those parts of Euclid and other authors which were useful to the students and to combine theory and practice (Wreede, 2007).

Snellius was not the only follower of Ramus at Leiden University, moreover he favoured teaching a variety of authors, both the ancient and modern. Some of the authors he taught were Ramus, Valerius (*Physica*), Euclid (*Elements*, *Optics*), Maestlin (a follower of Copernicus) on cosmography and astronomy, and classical authors such as Aratus, Proclus and Plinius (on geography). Being a Ramist meant he would be inclined to value mathematics used for practices on its own merits. Around 1605 Rudolph gave lectures on geometry, geography and optics, his son Willebrord assisted him with lectures on arithmetic and astronomy. Willebrord Snellius (1580-1626) was taught by his father, he also was a student of Ludolf van Ceulen (see below) and of Joseph Scaliger. He went on to become a highly praised and very productive mathematician, who succeeded his father in 1613 (Wreede, 2007).

Towards the end of the sixteenth century the commanders of the Dutch army, Maurits van Nassau (1567-1625) and Willem Lodewijk van Nassau (1560-1620), realized they would need many more mathematically trained engineers and surveyors to maintain their military advantage over the Spanish army. There were a few mathematically knowledgeable military engineers such as Adriaan Antonisz. and Samuel Crop and some practitioners such as Ludolf van Ceulen, who based their practices on sound theoretical mathematical knowledge, without knowledge of Latin, but they were few. Maurits, son of Willem van Oranje[1], had been a student at Leiden University, where he met Snellius and a fellow-student, Simon Stevin (1548-1620). Stevin, a good mathematician and engineer, became the tutor, quartermaster-general and principal advisor of Maurits. Together they convinced the curators of the University of Leiden to establish a separate course in Dutch language for surveyors and military engineers, the Duytsche Mathematique. It started officially in January 1600 with a curriculum prescribed by Stevin in the *Instruction*. Stevin designed an efficient curriculum, with the content, teaching methods and work forms prescribed. Theory preceding and alternating with practice; a method of study very much favoured by Stevin (Krüger, 2015). The course itself, the use of a written curriculum and the teaching language were an innovation.

The first professors who taught Duytsche Mathematique were Ludolf van Ceulen (1540-1610) and Simon Fransz. van Merwen (1548-1610). Simon Fransz. Van Merwen was a surveyor and burgomaster of Leiden. Ludolf van Ceulen was a respected mathematical practitioner, who taught mathematics and fencing and served on some committees advising the government. He didn’t know any Latin but was very accomplished in geometry. He is best known for his calculation of the first 35 decimals of π. Van Ceulen passed away in December 1610, whereupon his former student, assistant, surveyor and army engineer, Frans van Schooten, continued the lectures and the practice sessions in the field. At first as a temporary lecturer, from 1615 as professor Duytsche Mathematique (Krüger, 2018). His extensive and well-illustrated lecture notes[2] (BPL 1013) provide information on the content, structure, and didactic approach of the course. Van Schooten followed broadly the *Instruction,* with some adaptations and with deliberate didactics. In BPL 1013 calculations of roots, using decimal numbers is followed by some geometry and the principles and practice of surveying. This is followed by advanced surveying techniques, advanced geometry, calculations of volumes and wine gauging. Students who had come this far could take an exam to become an admitted surveyor, Rudolph Snellius had been an examiner on behalf of the university. The second part of the lecture notes is about fortification, defined by Van Schooten as the art to build and defend fortifications as well as the art to attack and conquer fortified places.

In 1646 Frans van Schooten Sr. was succeeded by his eldest son, Frans van Schooten Jr., a very talented mathematician, who edited a new edition of Viète’s work and translated Descartes’ *Geometry* into Latin. He used the lecture notes of his father, but also gave lectures on logarithms, algebra, and sundials (Dopper, 2014). Though Frans Jr. was more a scholar than a practitioner, he

[1] Willem van Oranje was the leader of the Dutch revolt against the Habsburg rulers, until his murder in 1584.

[2] BPL 1013, University Library Leiden, special collection

maintained contacts with surveyors and engineers, even if he was not himself a surveyor and engineer as his father had been.

Ludolf van Ceulen, Simon Stevin, Willebrord Snellius and Frans van Schooten Sr. all served as examiners for the admission of surveyors in the province of Holland (Krüger, 2014). In the academic lectures the professors taught the theory of mathematics as developed by Greek and Arabian scholars and their interpreters, with Latin as teaching language. The Duytsche Mathematique on the other hand taught which mathematics to use and how to use it in specific practical situations, with Dutch as teaching language, in interaction with practitioners. In Leiden, during the first half of the 17th century these two mathematics programmes, with different aims and content, existed peacefully together with links between them through the actors involved. Many factors, political, social, personal, were at the basis of this unusual situation.

Mathematics education during the 17th and 18th century

In 1681 the Duytsche Mathematique was abolished. It was reintroduced in 1701, as a theoretical university course, taught by a lecturer and it never regained the fame it had in the 17th century. Until well into the 19th century mathematics remained a rather unimportant part of the undergraduate program. However, the interest of university professors in technical applications increased. For example, civil architecture, hydraulics, steam engines, windmills, optics, navigation, and fortification were at one time or another subject of lectures at the universities of Leiden, Franeker and Utrecht (Krüger, 2014).

In society the notion that mathematics was relevant for many crafts and professions gained acceptance. In primary schools there was a gradual improvement in the position and quality of arithmetic teaching. Virtually every village and town had at least one primary school, for which the local government was responsible. In the more ambitious towns mathematics formed part of the comparative examinations for a teaching post, a mathematics journal for teachers was published from 1754 (Krüger, 2019). Between primary education and university education the contours of on the one hand general education, on the other hand vocational education became visible.

General education as a preparation of boys for university took place in Latin schools and Illustrious schools. In so-called French schools general education prepared the sons and daughters of well-to-do citizens for their position in society. Latin schools offered no mathematics or sciences, Illustrious schools offered a range of subjects in Latin, but some mathematical topics were taught in Dutch. Examples are fortification and navigation at the Illustrious school in Amsterdam from 1743-1762; geometry, arithmetic, algebra, and fortification were taught at the Illustrious school in Deventer from 1690-1727. The mathematics taught in French schools depended on the headmaster-owner. In all these institutes the content was mainly determined by the teacher and was theoretical, though teaching applications. There was no formal curriculum for any type of school.

The perceived relevance of mathematics for many professions and crafts was reflected in the relative abundance of textbooks being printed and in the existence of many small private schools and teachers offering private mathematics lessons (geometry, navigation, fortification, bookkeeping, etc.). Naval colleges were established to teach mathematical skills to navigators and from the late 18th century drawing schools and lessons in mathematics and drawing in orphanages became more common. Both in vocational training and in general education, mathematics teaching was haphazard, depending on initiatives and skills of individuals, with no written curriculum nor a system of examination. Neither was there a group of professional mathematics teachers (Krüger, 2019). However, from 1756 onwards the three institutes of the Foundation of Renswoude were comparable to the Duytsche Mathematique. In these institutes, in The Hague, Delft and Utrecht, talented orphans received a combination of professional training, based on mathematical subjects, and general education. From the start governors and mathematics teachers formulated a curriculum with subjects, aims, theory preceding practice and alternating with practice, examinations, textbooks, and other materials for the students. The first mathematics teachers had good contacts with many different practitioners, there were also contacts with professors at the Universities of Leiden and Utrecht (Krüger, 2013).

Unlike the situation in some other countries (Karp & Schubring, 2014) the Dutch government did not establish primary, secondary, or vocational institutes, until the end of the century (Krüger, 2019a).

Mathematics: a school discipline in general education

During the 18th century, under the influence of French authors, mathematics gradually was considered as formative for the development of the mind, to learn to reason clearly and with use of logic. As a result, when around 1800 education became the responsibility of the national government, mathematics was on the agenda. Primary education was regulated first, in 1806; arithmetic became a compulsory subject. Moreover, all teachers had to pass exams, including arithmetic, to be allowed to teach and the inspectorate became rather important in the acceptance of modern ideas with regards to content and methods. From about 1820 the development of mathematics as a school discipline becomes noticeable: textbooks specifically for use in (extended) primary schools, specialized journals, teachers specializing in mathematics and authors discussing teaching methods (Krüger, 2019). The authors might be teaching at the Royal Military Academy, at French schools and other private schools, at naval schools and other similar institutes or they belonged to the group of primary school teachers in the highest rank, who had specialized in mathematics. Those who became proficient in mathematics could improve their position (Smid, 2019). The regulation of secondary education had to wait until 1863. At least until then there existed a patchwork of private and council schools, extended primary schools and vocational institutes, all teaching some or quite a lot of mathematics, with textbooks written by some of the teachers.

From the start of the 19th century mathematics was seen by regulators, inspectorate and others as belonging to general education, with a formative value in sharpening reasoning faculties. On the other hand, for many people the main aim of teaching mathematics was to prepare for crafts and professions. In their opinion the emphasis of teaching mathematics ought to be on mathematics used in professions and how to use this. Tensions between these aims surfaced occasionally, as in the case of Jacob de Gelder, a highly praised mathematician and teacher, author of many textbooks (Beckers, 1999). In 1819 he left the School for Artillery and Military Engineers after serious disagreement with the director, a military man, about the amount and level of mathematical theory which was taught (Krüger, 2019a). Another example is the appearance of an arithmetic journal aimed at crafts men and farmers, as the calculations at primary schools were deemed to be too exact and rigorous for daily practice (Krüger, 2019). Indeed, most pupils would enter the workforce after leaving primary school and learn a craft. The development of mathematics as a school discipline, visible in the first half of the 19th century, received a strong impulse through the regulation of secondary education in 1863. For general education two new types of school were introduced, of which the Hogere Burger School[3] (HBS) would become the most influential (Krüger, 2014). At the HBS the students were taught a range of subjects, with sciences, mathematics and modern languages taking up many hours in the timetable. The school had as one of its aims preparation for the Polytechnic School, also established in 1863, the successor of the ailing Academy for civil engineers (Krüger, 2019a). In the early years of the HBS mathematics included subjects such as mechanics, use of technology, cosmography, and line drawing, next to arithmetic, algebra, geometries, and trigonometry. The development of a group of professional mathematics teachers got a strong impulse through the HBS. Gradually topics such as mechanics and cosmology disappeared from the mathematics curriculum. From the last quarter of the 19th century there were discussions about the content of the curriculum, the teaching methods, examinations and the merit of various textbooks. The teaching was theoretical, the use of mathematics in practice was delegated to the final years of Engineering Studies. At first, many mathematics teachers had a background in military or civil engineering, gradually most mathematics teachers had a degree in mathematics, but no experience with the use of mathematics in practices.

Discussion

Are influences of the characteristics of Duytsche Mathematique recognisable in mathematics education in the Netherlands during the 17th and 18th century?

As described in the second section, during the 17th and 18th century there was a growing awareness of the relevance of mathematics for numerous practices. Illustrious schools, small private schools and private teachers offered the teaching of mathematics to use in practices, or as part of general education. Mostly these programs or schools lasted at most a couple of years and details of the content are unknown. The only known example of a programme which showed some of the characteristics of the Duytsche Mathematique is the curriculum of the Foundation of Renswoude. The three institutes of this Foundation resemble Duytsche Mathematique in the combination of theory and practice, the structure of the programme, the interaction with practitioners and the relative success; moreover, professors at the universities of Utrecht and Delft occasionally served as examiner for the Foundation in Utrecht and Delft (Krüger, 2014).

Another sign of the awareness of relevance of mathematics for practices is the publication of numerous books on mathematical subjects, both the fundaments and the practices. Books on practices offered some mathematical theory; further research is necessary to determine the relevance and the depth of the treatment of mathematics in these books.

[3] *Higher Burgher School,* meant for the sons of middle- and upper-class citizens

Was the Duytsche Mathematique an isolated incident or a first step in the evolution of mathematics as a school discipline?

The engineering course, the Duytsche Mathematique, at Leiden University was at the time innovative, due to the teaching language, and the way it was designed. A written curriculum, prescribing content, teaching methods, work forms, theory preceding practice and alternating with practice, is reminiscent of mathematics as the present-day school discipline. However, while the teaching method as seen in the manuscript by Frans van Schooten Sr. shows strong didactics, directed towards the needs of the students, the content was mostly aimed at the practices of surveying. Not all of it, there are geometry examples which do not seem relevant for surveying or engineers but are mathematically interesting. Also, there are indications that the practice somehow disappeared from the course, Frans van Schooten Jr., an excellent mathematician, seems to have concentrated on teaching and expanding the theory of the subject, with theoretical exercises about the practice of surveying, application of theory, as opposed to exercises in the field, using mathematics. One could say that Duytsche Mathematique showed some characteristics of a future school discipline, however it may also be seen as a forerunner of Engineering Studies.

The first example of a possible development of mathematics as a school discipline in the 18th century is given by the Foundation of Renswoude. The programmes of the three institutes started coordinated, with some contact between the teachers in the three towns, with theory as the basis for practice. Interaction with practitioners was relevant for the programme, which could be described as a combination of general education with vocational training.

General education, with mathematics as a permanent factor, gained terrain from the start of the 19th century, in the form of primary education. This was followed in the second half of the century and in the 20th century by secondary education and the first years of vocational training. General education is theoretical, mathematics is about theory of the school subject, formulated in its own language and applied to theoretical examples. The use of mathematics, which mathematics to use and how to use it, is rarely or not at all experienced in general education and there is no interaction with professional users of mathematics.

Was the Duytsche Mathematique a first step in the evolution of the school discipline? Yes and no. It was more an example to follow than a first step; the Dutch society was not yet ready for more or less autonomous school subjects. A hundred years would pass before schoolteachers started to specialize in mathematics and another hundred years before a recognizable body of mathematics teachers became visible. They formulated opinions about the mathematics teaching programme, produced teaching materials and contributed to the formation of the school discipline, losing the interaction with professional users of mathematics in the process. Meanwhile the two worlds of mathematics from 1600 blurred and branched in many different ways, developing into a rich and varied universe of mathematics.

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‘Maths knows no races or frontiers’: the history of mathematics for an antiracist education

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Keywords: Antiracist education, Inclusive school, History of the Shoah.

Towards an antiracist education

While the awareness towards the Shoah had always been prerogative of humanities in educational context, only in recent years it emerged as unspoken potential of scientific disciplines. Hence, we would discuss the opportunity and the impact of introducing these issues in schools starting from a different perspective, the one of the history of mathematics and science. The national guidelines *Per una didattica della Shoah a scuola* (2017) identify the interdisciplinary and transcultural approach as the root cause that makes the didactics of racism one of the most complex educational challenges, since it involves different skills and specializations. Conversely, the cross-cultural dimension invoked is typical of the multidisciplinary approach of the history of STEM disciplines. Believing that the education to live together in diversity is everyone’s task, research in the history of STEM can become an important component of the contrasting action on modern racisms (Segre, 2018). A good teaching of the history of scientific disciplines can contribute to counter the cognitive-essentialist bias which underlies the racist views of new generations (Rutherford, 2020). Moreover, historical-scientific research becomes an educational tool aimed at ‘disarming’ false arguments, stereotypes and slogans which are often puerile but simple, and therefore effective. The plots of the internal history of STEM can make the future generations aware of the instrumental use that was made of such disciplines, which are usually considered – by their very nature – impervious to ideological conditioning.

A response to local needs

Unfortunately, the chronicle of many European countries, and specifically in Piedmont, has recently recorded many serious acts of racism and anti-Semitism, which bring back in vogue the most forbidden Nazi-fascist stereotypes. The perception that these intolerable facts are not isolated and the consequent concern about the surfacing of forms of hatred that make the past dramatically close, are reflected in the statistical data: from January 1, 2018, to the end of February 2019, 118 acts of racism were reported in Piedmont, 36% of which in Turin and its metropolitan area. In 8% of cases, they occurred in a school context. On the other hand, according to IRES Piemonte annual report for 2020, only 13.67% of the population considers racism and other forms of discrimination to be of concern and, due to Covid19 emergency, this percentage dropped to 9.5% in 2021. This is even more regrettable considering that the Piedmontese scientific community was among those most dramatically affected by the racial laws of 1938 (Luciano, 2020).

Educational activities in schools

Hence the idea to design educational activities and training courses in order to sensitize teachers and students of secondary schools to Memory, starting from the critical re-reading of some aspects and moments of research and teaching of mathematics and science during the fascist dictatorship (ideologization of mathematics, teaching of racism, etc.). For instance, the analysis of textbooks of the period is proposed in order to raise awareness of the instrumental use of theoretically ‘neutral’ subjects such as mathematics. At the same time, such interventions aim to promote integration and social and cultural inclusion of students from fragile backgrounds. During the school year 2020-21, three schools were involved in this project.

- At the IC “U. Foscolo” (Turin) two classes (grade 8) delved into the personal trajectories of scientists who were victims of racial laws, reading and commenting on the correspondence of that ‘dark’ period. To make them accessible to a wide public, they created a freely navigable multimedia content published on the school website (https://www.icfoscolo.org/wp-content/uploads/2021/01/MOOC_Giornata-della-Memoria.pdf) on the 2021 Holocaust Memorial Day.
- At the IIS “Santorre di Santarosa” (Turin) three classes (grade 11 and 12) with biochemistry study address opted to focus on the eighteenth-century debate on polygenism and racism and on the exploitable use of mathematics (statistics, demography) and science (biology, anthropology, medicine) to justify colonialism, racism and anti-Semitism, in modern and contemporary times.
- At the Liceo “G. Peano” (Cuneo) the Mission Memory path was pursued: teachers and students (grade 11 and 13) rediscovered the life stories of their colleagues who were persecuted for racial reasons, with particular attention to the local context: they focused on the reconstruction of the biography of Ugo Levi (1903-?), who taught mathematics and physics in Cuneo in the historical period considered, also interviewing some of his former students.

In conclusion, history of STEM becomes a resource for improving students’ consciousness: actually, “history is something that can make us aware of who we are, and how we have come to be the individuals that we are” (Radford, 2014, p. 89). We claim that history of mathematics and sciences provides an important means of combating ignorance, with the goal of defeating racial discrimination and intolerance, and contributes to identify appropriate antiracist resources to incorporate into school curricula. We hope that the ideas, insights and experiences illustrated can form a basis for new educational actions in this direction, which are especially needed in our society, crossed by currents of racial hatred and contempt for diversity.

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Concepts introduced in Norwegian textbooks before, during and after the New Math period

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For a nuanced discussion on the legacy of New Math, detailed knowledge of the content of mathematics before, during and after New Math is needed. In this article, we mirror a study of Icelandic textbooks, studying at what ages certain New Math-connected concepts were introduced in 14 Norwegian textbook series for age 7-15 published in the 1950s, 60s, 70s and 80s. As was the case with Iceland, set-theoretical concepts appeared with New Math and were mostly gone by the 80s, while axioms of the number field and the study of number were to a large extent present throughout the period studied. These were introduced at an earlier age in the 70s. We find that some New Math-related concepts were introduced in textbooks before New Math entered on a full scale, unlike Iceland. Like Iceland, many other concepts were introduced in the 70s and stayed in textbooks.

Keywords: New Math, textbooks in mathematics, mathematical concept

Introduction

New Math was an international movement that began in the mid-twentieth century. It has been strongly associated with the set concept. Since the set concept lost its place in textbooks quite soon, it is tempting to believe that the long-term impact of New Math was small. As Schubring (2021) points out, many believe that New Math affected all countries similarly. For informed discussions on this period within the history of mathematics education, more nuanced pictures of the legacy of New Math in each country are important. In this article, we present such a picture for Norway, mirroring a similar study for Iceland by Bjarnadóttir (2017).

In 1966, an international meeting for experts on mathematics teaching at the primary level was held in the UNESCO Institute for Education in Hamburg. In the report from this meeting, a list of basic topics was proposed that should be included in primary school. These included concepts from set theory (Williams, 1967, p. 47). However, set theory was just one of several topics in mathematics that were introduced to students either as a new topic or for younger students. According to Kilpatrick (2012), internationally many of the ideas of New Math are still part of school mathematics.

In this paper, our research questions are: Which concepts related to New Math were introduced in the textbook series for compulsory school from age 7 in Norway and to what degree did they survive into the 80s?

The starting point for New Math

The Royaumont seminar in 1959 was “the starting point for the coordinated international efforts to reform mathematics teaching based on the conception of ‘modern mathematics’” (Schubring, 2014,

p. 89). Teachers, school leaders and researchers from many countries, including Norway, Sweden and Denmark attended this seminar (OEEC, 1961). The starting point for New Math was a desire to teach mathematics that corresponded to modern society with its tremendous acceleration in technical, economic and social development (Christiansen, 1967). According to Christiansen, the curriculum in mathematics was previously narrow and the primary focus was to practice formal skills, while mathematical concepts and the interaction between these were often less emphasized.

The conclusions from the Royaumont seminar stressed that one must move away from memorizing facts and procedures to experimenting, discovering and making mathematics with physical objects. "This experimentation must lead to the abstraction of the quality of a set called number" (Schubring, 2014, p. 93). In the beginning one should focus on the ideas and later the mathematical concepts sets, subsets, mapping etc.

At the Royaumont seminar, it was also outlined how one could reform mathematics teaching in school. Regional cooperation was recommended. The authorities in Norway, Sweden, Denmark and Finland were positive about Nordic co-operation and in 1960 the Nordic Committee for the Modernization of Mathematics Education (NKMM) had its first meeting (NKMM, 1967). In 1967 a report was published on the work of this committee, providing a description of the current situation in the four countries, followed by goals for the schools' mathematics teaching. Several experimental texts for textbooks at different grade levels were developed and experimental teaching was carried out. The report also includes a proposal for a concrete curriculum for first to twelfth grade.

For the first three school years, two experimental texts were developed by NKMM, one Swedish and one Danish-Finnish, written by Bundgaard and Kyttä. "The basic ideas are the same in the two series, but they differ in many details" (Håstad, 1967, p. 99). For grades four to six, NKMM had trouble finding a suitable writing team and therefore they decided to translate the experimental texts published in the United States in the large project School Mathematics Study Group (NKMM, 1967, p. 108). For grades seven to nine, NKMM developed experimental texts in geometry and algebra.

Iceland did not participate in the Nordic collaboration, nevertheless the result of the collaboration influenced Iceland through, for example, the choice of a Danish-Finnish textbook series. The Bundgaard-Kyttä textbooks series was translated to Icelandic (Bjarnadóttir, 2017), together with textbooks for age 10-12 written by Agnete Bundgaard alone. We will call the combined series the Bundgaard series. Bjarnadóttir (2017) has studied how students in Iceland were introduced to certain mathematical concepts, often related to New Math, and at what age they were introduced to these concepts. In her study, the content of this textbook series is compared with the content of other series in use in Iceland before, during and after the introduction of New Math. In Iceland, there was just one textbook series in mathematics from 1939 to 1966 (Bjarnadóttir, 2017). The results of her study are that the topics that were new in the Bundgaard series were "the use of set theoretical concepts and the notation for building up the number concept and understanding of operations through repeated reference to the axioms of the number field" (p. 58). Negative numbers were not mentioned in it. Bjarnadóttir concludes that this textbook series went far in meeting the requirement that "mathematicians" had for mathematics in primary school. Here, she is most likely referring to those mathematicians who participated in the Royaumont seminar in 1959. The topics that survived in textbooks after the New Math period were primes and divisibility, mental arithmetic and approximation and estimation. The topics that did not survive were replaced by an introduction to statistics, probability, the use of variables and solving simple equations. These were new topics when the Bundgaard series was replaced by an Icelandic textbook series in the 70s.

Bjarnadóttir's study shows that there were more than set theory concepts that were introduced to the students with New Math. Some of these were also introduced for students in the textbooks in the period after New Math, not necessarily for the students at the same age.

Norway participated in the cooperation and was therefore naturally affected by this participation. But unlike Iceland, in Norway there were many publishers of textbooks for primary schools. After a temporary curriculum in mathematics with New Math was introduced in 1971, several textbook series were published with content adapted to this new curriculum. Soon, mathematicians as well as parents of school children protested. In the final curriculum in mathematics, which was approved in 1976, New Math content was greatly reduced (Gjone, 1985, p. VII:31; Solvang & Mellin-Olsen, 1980, p. 1:18).

"The residue of the new math era may be difficult to see in today's school mathematics, but it is there" (Kilpatrick, 1997, p. 956). Kilpatrick mentions inequality as a subject that came in with New Math in USA and has remained in school mathematics.

Method

As outlined above, Bjarnadóttir (2017) studied the content of textbooks that were published in Iceland before, during and after the introduction of New Math. Based on Bjarnadóttir's study, we have done a similar study of when students encounter the various concepts for the first time in Norwegian textbooks.

Bjarnadóttir first developed tables with mathematical concepts based on literature on New Math, then she analysed how and when students were introduced to these concepts in Icelandic textbooks. Bjarnadóttir chose the following categories *set theoretical concepts and notation; structure of the number field;* and *study of numbers.* We have chosen to use Bjarnadóttir's categories to be able to compare the Icelandic and the Norwegian situation.

The two authors of this paper have each studied each textbook series separately and noted where the students are first introduced to the various concepts. When in disagreement, we have studied the textbooks together and come to an agreement on where the students meet the concept for the first time. We have noted both the first implicit treatment of a concept and the first explicit one. For reasons of space, in the tables in this article, we only include the first explicit introduction. If the concept is only introduced implicitly, we have marked this by putting parentheses around the age. We exemplify what we mean by an implicit introduction of a concept through the commutative law for addition. When several examples and tasks in a row have the same numbers (as for example 2+3 and 3+2), without the commutative law being mentioned, we conclude that there is an implicit introduction to this concept.

After preparing tables that show the age at which students are introduced to a topic for the first time, we have analysed these tables to compare different periods, which also makes it possible to compare the result from Norway with the situation in Iceland.

In this study, we analyse the content of textbooks that were published before, during and after the period of New Math, that is the period 1950 to 1980s. Early in the 1970s, several textbooks with New Math were published. We have chosen to include all known series from this period. In the period before and after New Math, we have chosen series from Aschehoug and Cappelen, as these published textbook series for the whole period. They have partly the same authors as the books that were published in the 1970s. We also conjecture that they were used by many schools in Norway, although there exists no statistics to determine this. We have also included Tanum, during and after the introduction of New Math (see Table 1), to include one more textbook series for the 80s. None of the Norwegian textbook series published in the 1970s were directly linked to NKMM's experimental texts.

Table 1: Textbooks included in our analysis. Textbooks from 1970s are marked for clarity

	Publisher	Year	Age	Title	Author
A1	Aschehoug	1953-	7-13	Nå regner vi	Aam, Johannesen, Slaatto, Paulsen
A2	Aschehoug	1964-	7-15	A. regneverk for folkeskolen	Paulsen, Slaatto
A2+	Aschehoug	1971-	7-9	A. regneverk, mengder og tall	Paulsen
A3	Aschehoug	1980-	7-12	Vår matematikk	Harboe
A4	Aschehoug	1987-	7-15	Regnereisen	Venheim, Breiteig m. fl.
C1	Cappelen	1962-	7-15	C. regneverk: matematikk	Bue
C2	Cappelen	1971-	7-15	C. matematikkverk	Bue, Gjerdrum
C3	Cappelen	1980-	7-12	Jeg regner	Gjerdrum, Bue
D1	Dreyer	1972-	7-12	Grunnskolens matematikk	Eicholz, Arneberg
E1	Eli	1972-	7-15	Ny regning	Dentrup, Kjeldberg, Kjeldberg
G1	Gyldendal	1972-	7-12	Matematikk for grunnskolen	Myrmo, AAs, Grymer, Ridar
N1	NKI	1973-	7-12	Tal og teikn	Rudjord, Bjørklund, m.fl.
T1	Tanum	1970-	7-12	Min matematikk	Viken (red.)
T2	Tanum	1980-	7-12	Min matematikk	Viken

Dreyer (D1) was based on a US textbook series, adapted to Norwegian conditions. Eli (E1) was based on a collaboration between Norwegian and Danish teachers. Gyldendal (G1) refers in the teacher manual to NKMM's report but does not claim to build directly on it. Tanum (T1) was based on a Swedish textbook series and refers also to NKMM. The other textbook series (A2+, C2 and N1) were modernizations of previous Norwegian textbook series. A2+ was written as a supplement to an existing textbook series to introduce students to set theory. This supplement was only for students aged 7 to 9, older students used A2 textbooks. In our analysis, we have first analysed A2 (under the heading A2), then the combined series of A2 and A2+ (under the heading A2+).

When comparing our results with the findings of Bjarnadóttir (2017), we need to keep in mind that the two textbook series before New Math in her analysis were used in the periods 1922-37 and 1927-80. The textbooks we have studied were published much later. In addition, the Icelandic textbook series before New Math were for students from the age of 10 because children were expected to learn some mathematics at home before starting school at age 10.

Results

We will follow the structure of Bjarnadóttir (2017), looking first (briefly) at concepts from set theory, then axioms of the number field and finally what Bjarnadóttir calls "topics of number".

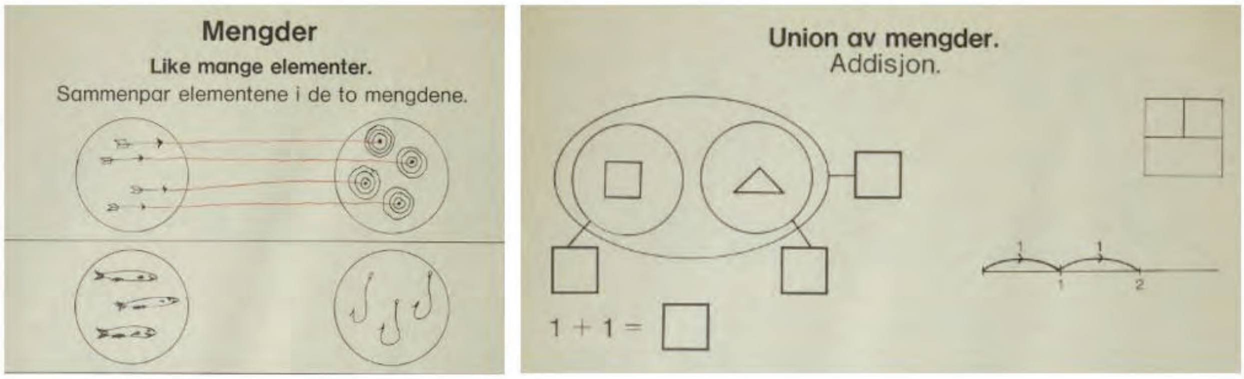

Figure 1: A section from A2+ from 7-year-olds. To the left with introduction to Sets (p. 10) and to the right with introduction to Union (p. 30)

Our analysis shows, not unexpectedly, that set theory entered the textbooks in the 1970s for the students aged 7 (see Figure 1 for the introduction in a textbook series) but were not included in the textbooks that were published after the New Math movement had passed. This is similar to what Bjarnadóttir found in Iceland.

For axioms of the number field (Table 2), the picture is different. Most of these were present in the 60s or even in the 50s, but in the 70s series, they were presented to younger students. In the 80s, it is more complicated – some of the axioms are left out, some are presented later, while some are even presented earlier.

The commutative law of addition is introduced implicitly for 7-year-olds in all time periods, from the 50s to the 80s. However, there is variation in when students are introduced to this law explicitly, from age 8 to age 13. It also varies when students are introduced to the commutative law of multiplication. If the law is first introduced implicitly, then it happens when students are 7 or 8 years old. It is often explicitly introduced when students are 9 years old. The same applies to the distributive law, but it tends to be introduced explicitly in the 70s (under New Math). The associative laws of addition and multiplication are introduced explicitly at the secondary level (from 13 years) in the 60s and at the primary level (under 13 years) in the 70s. The associative laws appear implicitly or not at all in the textbooks of the 1980s.

The additive identity is implicitly included for 7-year-olds throughout the period, and explicitly included later. Only in the 80s is it mentioned explicitly to 7-year-olds. The fact that addition/subtraction and multiplication/division are inverse operations is included in almost all textbook series, either explicitly or implicitly. But it varies when students are introduced to these connections. It seems that students were introduced to these connections at a younger age in the 70s.

In the 50s, the role of 0 in multiplication is introduced to 9-year-olds. In the 60s, this has been moved to when the students are 13-14 years old, and in the 70s and 80s it is again in most of the textbook series for 8-9-year-olds. The role of 0 in division is not mentioned in the 50s and 80s. In the 60s, it is mentioned in the same textbook as 0 in multiplication, and in the 70s it is mentioned at a higher grade level if at all.

Table 2: The age-groups of students for which axioms of the number field are presented

Topic\Textbook	**50s**	**60s**		**70s**							**80s**			
	A1	**A2**	**C1**	**A2+**	**E1**	**C2**	**D1**	**G1**	**N1**	**T1**	**A3**	**A4**	**C3**	**T2**
Commutative law of addition of multiplication	 (7) 13	 13 14	 9 9	 13 9	 9 9	 8 8	 8 9	 (7) (8)	 10 9	 (7) (8)	 (7) 11	 14 9	 (7) 9	 (7) 9
Distributive law	(9)	14	13	9	9	(9)	9	(9)	9	(10)		14	(8)	(8)
Associative law of addition of multiplication		 13 14	 13 13	 8 9	 9 9	 8 9	 7 9	 9 10	 10 10	 7 (9)			 (8) 	 (9)
Identity -additive -multiplicative	(7) (10)	(7) 14	14 14	(7) 9	13 13	(7) 	10 8	 8	12 12	(7) (8)	7 	7 9	(7) 	(8) (8)
Inverse -additive -multiplicative		13 	14 13	13 	12 15			(12) 12	(10) (12)	 12		13 15		
0 in multiplication 0 in division	9 	14 14	13 13	9 14	 (15)	8 	8 9	8 	9 12		9 	9 	8 	
Negative numbers		12	12	12	11	11	12	12	10	11	12	12	11	10
Inverse operations add./subt. mult./div.	 13 10	 14	 9 13	 13 9	 (9)	 (9) 10	 9 9	 9 (9)	 8 8	 10		 7 11	 (8) (9)	 (10) 10

Within study of number (Table 3), most topics were present already in the 60s. However, many of these were introduced far earlier in the 70s. Again, the picture is mixed as we enter the 80s.

While the number line is not included in the series from the 50s, it is included in most later series from the age 7. In some textbooks, especially in the 80s, it appears a little later. The connection between even numbers and odd numbers was introduced earlier with the New Math: for 7-8-year-olds students in the 70s as compared to 12-14-year-olds in the 60s. We see that the early introduction lasts into the 80s.

It varies how old students are when they are introduced explicitly to primes, factorization and divisibility (from 8 years to 14 years), but these concepts are often introduced simultaneously.

Students are introduced to bases other than ten in some of the textbooks in the 60s and 70s, but not in any of the textbooks in the 80s. Modular systems is never a topic in the textbooks studied.

In the 60s and 70s, students were introduced to equations at the age 7-9, while in the 80s they were introduced to equations at the age 10-12. This is different from the variable concept. It appeared in the textbooks for 11-12-year-olds in the 1960s, for 7-9-year-olds in the 1970s and again for 11-12-year-olds in the 1980s (except in one textbook series where students were introduced to this concept when they were 9 years of age).

Statistics is not included in the textbook series from the 50s, later, students are introduced to statistics between the ages of 8 and 11. In only two of the textbook series do we find probability, one of them in the 70s and one in the 80s.

Table 3: The age-groups of students for which topics on numbers are presented

Topic\Textbook	50s	60s		70s							80s			
	A1	A2	C1	A2+	E1	C2	D1	G1	N1	T1	A3	A4	C3	T2
Number line		7	9	7	7	7	7	7	7	8	11	9	7	9
Number relations:														
Even & odd		14	12	7	8	8	8	7	7	(8)	8	7	8	8
primes	11	14	12	14	12	12	9	11	10	12		10		11
Factorization	11	14	12	14	11	8	9	11	10	12		10		11
Divisibility	11	14	12	14	12	12		11	10	12		11		11
Bases other than ten		15		15			(12)		9	(7)				
Modular system														
Symbols as 7+2 for 9			14	7	(8)		11	7	7	7				9
Variables		11	12	7	8	9	9	9	9	8	12	12	9	11
Equations		12	7	7	7	8	7	8	8	9	12	12	10	11
Probability					14							12		
Statistics		11	11	9	11	8	10	8	8	11	10	10	10	9
Mental arithmetic	8	8	11	8	8		9	10	9	9	10	8		9
Approximation, estimation		13	11	13	11	9	9	10	10	9	12	9	9	9
Use of calculators												10		

Discussion and conclusion

Set-theoretical concepts were not included in the textbook series in primary school in the 50s and 60s in Norway, entered with New Math in the 70s and partly disappeared again in the 80s. As such, we can distinguish a clear "New Math" period in Norwegian textbooks. Several other concepts (for instance associative laws) were introduced to students at a younger age in the 70s than before. Of these, some of the concepts stayed in the textbooks for the same age group in the next decade, while others returned to textbooks for students of a higher age. While textbooks series vary, there is a tendency for more of the terms be introduced explicitly earlier in the 70s.

Our study gives a somewhat different picture of New Math's role than Bjarnadóttir's (2017) study. As there were no new textbook series in Iceland for decades before New Math, the change was abrupt. In Norway, on the other hand, several textbook series were published in the decades before New Math took hold. This contributed to a more gradual change. One major difference was that in Norway, many New Math ideas entered textbooks in the 60s, but New Math (including set theory) came in full force in 1971, with a new temporary curriculum. In Iceland, New Math (including set theory) was implemented in the 1960s, but some New Math ideas (such as statistics) only entered primary school in the 1970s as other New Math ideas waned.

Although New Math has been partly derided and ridiculed in later years, we do see that many concepts which are now taken for granted as a part of school mathematics, were introduced in the 1960s and 1970s. This applies, for example, to statistics, approximation and estimation, as Bjarnadóttir also concludes in her study.

When it comes to the textbook series that were published in the period of New Math, in Iceland and Norway, we see that there are some similarities in when students in the two countries are introduced to the different concepts. However, we also see large variations between the different textbook series published at about the same time. This variation is more apparent in our analysis than in Bjarnadóttir's, as there were fewer competing textbooks in Iceland.

Taken together, the detailed analyses provided by Bjarnadottir for Iceland and by us for Norway, provides a foundation for a more nuanced discussion on the legacy of New Math in the two countries. If such analyses are done for more countries, we may see even more what is special with how New Math has been implemented in different countries.

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Mathematic-historical example for university teaching of mathematics - how can methodical knowledge be taught as well as the culture of mathematics be made accessible?

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In our presentation, we introduce a teaching unit for university mathematics teaching, which is based on historical documents. Specifically, these are two articles by David Hilbert (1862-1943) and Adolf Hurwitz (1859-1919) on proofs of the transcendence of the number e and an accompanying letter exchange between the two authors. We illustrate that these archival finds are suitable to deepening students' understanding of both mathematical methods and mathematical culture.

Keywords: Historical documents, Proofs of transcendence of e, David Hilbert and Adolf Hurwitz, Mathematical culture

Introductory reflections on history of mathematics in university teaching

Incorporating the historical development of mathematics can be used as a didactic element in teaching mathematics. In textbooks, monographs and lectures, it may help mathematicians to classify the mathematics dealt with and thus at the same time enables the reader a systematic approach to the mathematical content.

In a case study on the use of so-called knowledge maps in a history of mathematics course in university teaching (Khellaf et. al, 2018), we found that students develop very individual and confident ideas and cross-references about the development of mathematical thinking. Figure 1 illustrates different examples of student views on structuring mathematical development steps.

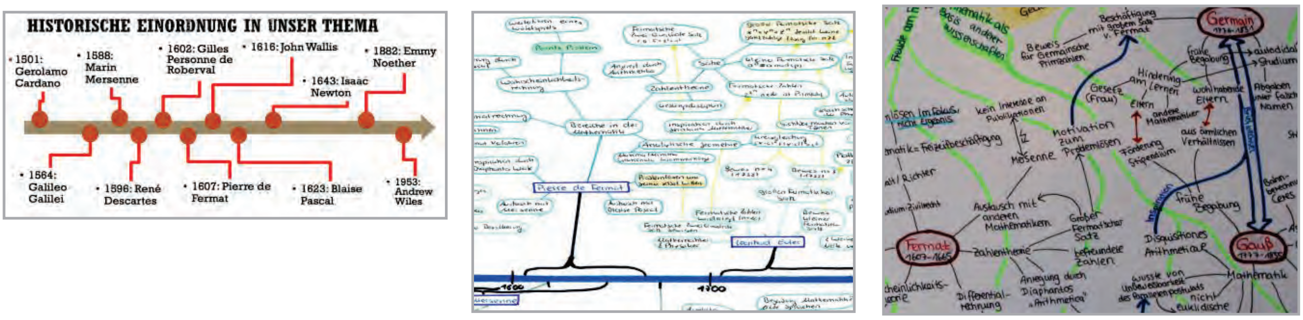

Figure 1: Excerpts from a lecture and two knowledge maps by students, created in the seminar "Development of mathematical ways of thinking“ (University of Hanover).

The importance of mathematics history in university teaching has been discussed at various points in the past. The mathematician Max Dehn (1878 - 1952), for example, took a particularly firm standpoint. In a statement on the teaching of mathematics at the University of Frankfurt in 1932, he named the history of mathematics as fundamental subject matter in addition to "the structure of analysis", "axiomatics in geometry, the foundation of arithmetic". He justified his view by writing that the treatment of mathematics history ensures "[...] to consider again and again the multiple connections between [mathematical topics], and this is more important than to bring no matter how [new] original results.“ (Dehn, 1932, p. 72) In addition, Dehn underscored the high value of a mathematics-history seminar, which was aimed particularly at faculty members, as a "kind of continuing educational class".

The mathematics didactician Gregor Nickel argues in his article “Zur Rolle von Philosophie und Geschichte der Mathematik für die universitäre Lehrerbildung“ (Nickel, 2015), translated "On the Role of Philosophy and History of Mathematics for University Teacher Education“, that mathematics history must be understood not as an accessory but as a "subject of teaching in its own right."

> "The historically reasonably adequate presentation of a topic from the history of mathematics certainly cannot be done incidentally, but requires time and attention, on the part of the teacher a certain mathematical-historical professionalism, and on the part of the students a solid prior mathematical experience." ((Nickel, 2015, p. 216) translated from German by the author)

In the following we show a practically oriented teaching unit on transcendence proofs of Euler's number e, which on the one hand is thematically suitable for the current teaching material in early university semesters (after two basic lectures on analysis), and which on the other hand can be built on a limited corpus of mathematics-historical documents.

Historical material for the construction of a teaching unit

The transcendence proofs of e of Adolf Hurwitz (1859-1919) and David Hilbert (1862-1943) from 1893, which are the subject of this work, can be juxtaposed with an exchange of letters between the mathematicians, which shall make both the genesis of the proofs themselves and the contemporary context tangible and raise questions about the "culture of mathematics". In the following, we will first discuss excerpts of this correspondence, then give a sketch of the proofs’ main ideas.

Correspondence between David Hilbert and Adolf Hurwitz

The correspondence in question essentially comprises five letters written by David Hilbert and Adolf Hurwitz between October 1892 and February 1893. The life and work of these two mathematicians was characterized by enormous and groundbreaking developments within the research and teaching of mathematics as well as the institutional conditions in the German-speaking world (see some aspects below in the introduction of the teaching unit).

Hurwitz and Hilbert (as junior scientists and professors in Königsberg, Zurich, and Göttingen) took an active part in developments of mathematics and exchanged views on them in a correspondence comprising a total of at least 198 postcards and letters, which is now kept in the Göttingen State and University Library (under the directories Cod_Ms_D_Hilbert_160 and Cod_Ms_Math_Arch_76). They had met when Hurwitz was appointed associate professor in Königsberg in 1884, where Hilbert attended his lectures as a committed and ambitious mathematics student. Since then they had an ongoing collegial friendship, which was maintained through regular written exchanges, especially after Hurwitz's call to the ETH Zurich (then still Polytechnikum). In their letters the topics ranged from general remarks on the situation in higher education and personal matters to very concrete research results and, where appropriate, mutual references and suggestions for improvement. In the latter sense, for example, the exchange of letters on new proofs of transcendence of the number e quoted below was written. This begins with a letter in which Hilbert presented his simplification of a known proof (by Thomas Stieltjes). In the following we will quote some relevant passages of the letters, which are mostly three to five pages long. The focus at this point will be on the nature of the communication regarding the proofs.

A new development of the proof was communicated by Hilbert in his letter of December 31, 1892:

> "Concerning my proof of the transcendence of e, I realized very soon after I wrote to you that one can still considerably shorten the same by omitting the whole Stieltje pointe. [...]"
>
> - The proof follows, in particular the choice of the polynomial f(z), see next section. -
>
> "[...] You see that hereby the proof also contains a different conclusion at all, in that not from the integral sum II but from the integer I the dissimilarity from 0 is shown. The use of this conclusion also gives the proof of the transcendence of π a simplification which seems to me not inconsiderable."

Hurwitz replied on January 10, 1893, again with a new proof and an idea:

> "Your scientific communication concerning the number e has, as you can imagine, interested me very much. [...] The matter did not let me rest and I discovered a further simplification, [so] that one can now bring the proof in the first hours of a lecture on differential calculus. [...] Have you already edited your proof? If so, please write me whether you have already sent it to Klein. I would then have the above further simplification printed in a short note behind your work. I would prefer it if we chose the Göttinger Nachrichten. [...] So please answer me quickly, even if only by card. That your punch line can also be applied to π is clear; but I haven't quite thought it through yet."

Within three days Hilbert replied (on January 13, 1893):

> "I have already worked out my proof for e and π during the Christmas vacations, and in the process - especially in the part dealing with π - a number of advantageous and simplifying things have emerged, so that the whole thing will now take up 4 - 5 printed pages, and my presentation is by no means brief. In your proof, of course, the integral is avoided; but whether the presentation of the proof becomes shorter and clearer is not yet quite clear to me. [...]. But it is my conviction that the proof with the help of the integral will always remain the clearest and most capable of development [...]"

Hurwitz wrote about a month later (on February 08 or 13, 1893):

> "For a long time I wanted to answer your dear letter of 13/I, but - as it goes - the answer was postponed from day to day. Today, as a guiding impulse, your transcendence note arrives, which I immediately typed into a café. You have written the note with Gaussian classicism. I hope that you will agree with the short note (2 printed pages presumably) in which I will make the modification of your proof, [...]. Felix Klein submitted it to the Göttingen Society on 4/II. As an advantage of my modification I see that it becomes clear in the proof that only the addition theorem and the differential equation [...] The idea of modifying your proof by replacing the integrals by limit values had also occurred to me. However, there still seem to be difficulties [...]"

The letters already clarify the core of the proof differences: The preference of integral or differential calculus, which still lay the foundation of every analysis course today. It is interesting that Hilbert and Hurwitz emphasize the respective greater "simplicity" of their proofs (and remain in disagreement).

Two proofs of transcendence of the number e

In the following, we will give a sketch of the two proof lines, published in (Hilbert, 1893) and (Hurwitz, 1893). We mainly limit ourselves here to the mentioned crucial difference in the approach of the proofs (the „differential analogue“), which is illustrated by excerpts from a lecture by students in a seminar (see more details in the next section).

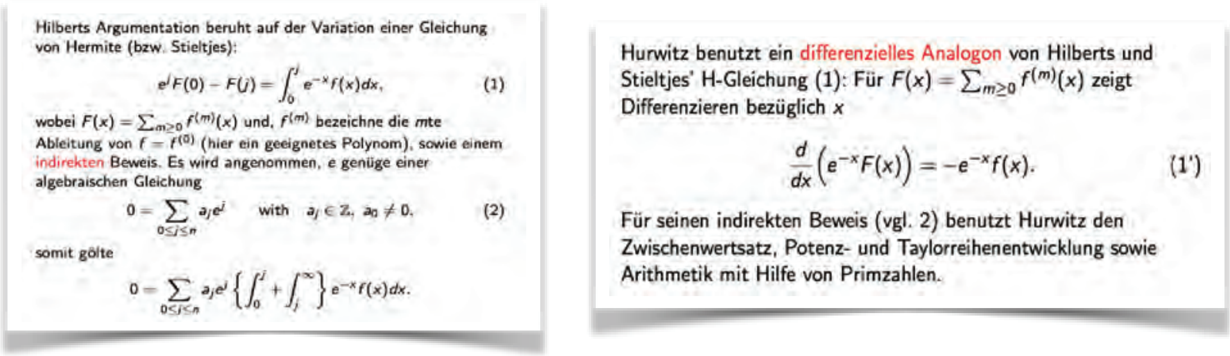

Figure 2: left: Excerpt from an introductory seminar lecture "Basic Idea Proof Hilbert, right: Excerpt from an introductory seminar lecture "Basic Idea Proof Hurwitz"

Figure 2 describes that Hilbert's argument is based on a variation of equation (1), where F(x) is defined as given and f^(m) describes the mth derivative of f^(0). Hilbert then goes on to give an indirect proof by estimating the individual integrals, respectively their arithmetic properties, and leading equation (2) to contradiction. In contrast, Hurwitz argues with a "differential analogue," see equation (1’). For his indirect proof Hurwitz further uses the intermediate value theorem, power and Taylor series expansion, and arithmetic of primes. We cannot go into detail through all the mathematical steps here, however, we want to state that the proofs are non-trivial, but quite short, and all methods are usually discussed in the first two analysis lectures of a university course.

In favor of the two proofs as well as on the corresponding historical documents for practical teaching purposes is the fact that the basic idea of the proofs provides an application of the fundamental theorem of integral and differential calculus. An integral proof is contrasted with a differential proof. Students thus learn, using a concrete example from modern history, that the methods and foundations experienced in their first university semesters are sufficient to understand and discuss seminal mathematics. In addition, the parallel use of original proofs and accompanying correspondence allows a variety of cultural aspects of mathematics or the mathematical community to be reflected, see next section.

Implementation in a teaching unit

Based on the historical documents, we constructed a teaching unit, which was so far conducted in two different seminars: A seminar on selected topics in analysis (10 participants; University of Würzburg) and a mathematics-history seminar on developments in mathematical ways of thinking (13 participants, University of Hanover). In both seminars (each 90 minutes per week), the content was divided into three parts: Mathematics-historical context and the letter exchange, the proof of David Hilbert, and the proof of Adolf Hurwitz before the background of the correspondence. The introduction was done by the lecturer, the proofs were presented by one student each. Time (about half an hour) was then allowed in each lesson so that the content presented could be discussed. In the following we will refer to the analysis seminar. The students were all in their third or fifth semester (Bachelor Mathematics or Computational Mathematics).

In the introduction, we first gave a brief historical contextualization of the seminal period of the late 19th century for mathematics in the German-speaking world: The first International Congress of Mathematicians was held in Zurich in 1897, journals and publishing houses were founded or changed their direction, lecture notes of mathematicians were published, polytechnics gained importance, interdisciplinary fields (for example, geometry of numbers, algebraic number theory, and algebraic geometry) were discussed and evolved, teacher training was professionalized - to name only a few contemporary aspects, some of which still influence our teaching and research practices in mathematics today. We then introduced the mathematicians Hurwitz and Hilbert, and their correspondence, which spanned a total of several decades. Finally, the letters referring to the proofs of transcendence of e were presented in detail. In the discussion that followed, the students were particularly interested in the influence of the mathematicians Charles Hermite (1822 - 1901) and Felix Klein (1849 - 1925). Furthermore, we focussed the way of discussing mathematics and developing proofs by letters - in contrast to today's mathematical culture and speed (e.g. by zoom meeting).

All students were given preparatory questions for the seminar sessions, and the presenting students in particular were asked to integrate these into their presentations. As an example, we show here the questions which especially refer to the difference integration-differentiation:

In your presentation, address the following questions:

(I) Historically, integral or differential calculus – what came first? Since when were these related to each other? Treat these questions discursively and be critical in your literature review.

(II) Fundamental theorem of differential and integral calculus: prove that Hurwitz's approach is indeed the „differential analogue" of Hilbert's approach.

(III) Partial integration: prove the Hermite equation. What is the differential analogue to partial integration?

The presenters were encouraged to prepare together to some extent. The questions served as a guideline. They were deliberately kept both general and concrete. Question (I) was directly stimulated by the correspondence and the question whether differential calculus or integral calculus should be studied first and why. Especially in their search for sources, the students became active: The discussion led us to different protagonists of the history of mathematics (from Archimedes to Newton / Leibniz) and their influence on our mathematics today. Questions (II) and (III) were more specific to the respective lecture. By differentiating Hilbert's integral, question (II) could be addressed; the differential analog (question (III)) to partial integration is the product rule used for Hurwitz's equation (1).

With all students we summarized: Both proofs so far rely essentially on the differential equation exp = exp' for the exponential function (or, equivalently, the integral equation for exp). This kind of reproduction property appears in the proof of Hermite's identity (1) as well as in its differential counterpart (1'). The similarity of the proof ideas is due to the main theorem of integral and differential calculus. The real differences of the two proofs lie mainly in the mathematical methods and tools used in the estimation of the arithmetic properties. Together with students, we have decomposed the proof into individual steps, see a summary of the results of the proof analysis in Table 1.

Table 1. Comparison of the steps of the proofs, highlighted with frames: integration-differentiation difference (blue), differences concerning mathematical tools (red)

Steps of the proof	Hilbert	Hurwitz	Background & Application
Basic starting point	Integral equation (1) Partial integration	Differential equation (1') Product rule	Duality in fundamental theorem of analysis Exponential function exp = exp'
Assumption	Algebraic equation = 0 (2)	Algebraic equation = 0 (2)	Concept of the proofs: Proof by contradiction
Defining ingredients	Definition of a polynomial f	Definition of the polynomial (7)	Comparing the polynomials: only slight differences due to prime numbers
Construction	Combining (2) and (1) (algebraic equation)	Combining (2) and (1') (algebraic equation)	Similar basic construction: Integration of equation (1) resp. (1') in algebraic equation (2)
Mathematical Tools	Investigation of Arithmetical Properties: Elementary formula (3) Arithmetical estimation (integers)	Investigation of Arithmetical Properties: Mean value theorem to receive equation (6) binomial theorem / Taylor series expansion Arithmetical estimation	Different proof lines: Tricky methods, Tools in principle elementary / understandable, Execution sophisticated
Conclusion	Assumption not possible	Assumption not possible	Conclusion: e is transcendental

In addition to the mathematical facts, further facets of mathematics were addressed. Especially with regard to students in their first semesters, a lack of understanding of the "culture of mathematics" is often thematized. This fuzzy term is used for very different aspects of the subject: For example, technical skills such as writing down a mathematical conclusion or the correct use of operators are addressed, or an understanding of the difference between examples and a rigorous proof is named. Especially the discussion of proofs is treated repeatedly and with different perspectives: The fact that mathematical facts can be proved in many ways (and by methods of different subdisciplines) and that it is ultimately a matter of perspective and context which proof lends itself or even "prevails" at which point is an integral (and challenging) part of understanding mathematics as an inquiry-based discipline. Looking at the related process of the creation of mathematics, its culture certainly includes the negotiation of the content itself: How is mathematics discussed and circumscribed, and by whom? Here again there is the possibility of taking different perspectives. In addition to the very concrete question of the use of terms and attributes themselves, it is also possible to look much more indirectly at the way in which results are published and thereby made known. Even science, which seek to be as objective as possible, is determined by subjective people.

One aim of the teaching unit was to initiate a process of understanding, that the culture of mathematics carries this richness of facets within itself and thus, despite the immanent formal rigor, can be thought much more creatively than perhaps suspected. Respective aspects were directly or indirectly addressed: Proofs in general, the need for variations of proofs, formal language and notations, mathematical methods and tools, meta-level: when is mathematics "simple"? or the evolution of mathematics in general.

Some conclusions concerning the culture of mathematics

In some concluding remarks we want to take up two of the mentioned aspects that have met with a particular response from students with reference to our teaching unit.

Proofs and variations of proofs

The fact that proofs represent a fundamental difference between school and university didactics and thus form a decisive hurdle for students, especially first-year students, has been noted in numerous works. Students in particular have difficulties "acquiring an understanding of the culture of proof and argumentation in scientific mathematics." (Jahnke & Ufer 2015, p. 350) Yet proofs represent the heart of mathematical culture (cf. (Grieser, 2015)), they should themselves be able to be seen as a means of developing knowledge and understanding. Georg Pólya already stated that "beginners [must] be convinced first of all that learning proofs is worthwhile, that they have a purpose, that they are interesting." (Pólya (1967, p. 195)) The question here is how appropriate motivation can happen. In some cases it is underlined that just an "upstream exploratory phase for understanding an assertion" [helps] in the "eventual formation of a need for proof and is indispensable for making out a proof idea." Accordingly, the need is expressed to give students "ample opportunity to develop problem awareness.“ (Grieser, 2015, p. 662)

Our thesis is that by including the letter exchange of Hilbert and Hurwitz in the formation of the two proofs of transcendence of e (as „exploratory phase“), this process of proof development may be made more comprehensible and tangible. In addition, we assume that a reflection process on proof variations in general can be triggered. In a qualitative survey in the seminar on selected topics of analysis, students gave preliminary feedback on this. Here, we give some excerpts from the answers to the questions about the meaning of variations of proofs (connected with the correspondence) and about their own perception of the discussed proofs of transcendence of e against the historical background: Proof variations were described by students as "representing developments within mathematics" and as "very important, especially that they become easier and more understandable", "through the correspondence the [...] background" was recognized and it became apparent that "through discussion (competition) [of Hilbert and Hurwitz] development" occurs. All students found the historical or personal reference to mathematicians interesting, and some even motivating. Students highlighted that the "writing became more understandable" and "clearer why [...] some methods were used". One student, who found the letter exchange interesting and motivating, evaluated that the proofs, however, did not become easier to understand. In two additional oral discussions with the students (each at the end of the respective seminar presentations), it became clear that it was precisely the comparison of the two proofs by the authors in the letters that made them reflect. One student, for example, brought up the general difference between "elementary" and „simple" mathematics, another remarked that Hilbert's integrals seem rather "scary", though mathematically more "universal". Together we then noted that in fact Hurwitz's more "elementary" proof was more often used in textbooks, while Hilbert's proof was more likely to be used or developed in more advanced research. Overall, students rated the difficulty of the mathematics as adequate for their prior knowledge (from lectures on analysis) and felt Hurwitz's proof was easier. Although we are of course referring to a small number of students and have no quantitative study to show, consequently, we see a tendency that our teaching unit has triggered an interesting thought process among students regarding the role of proofs and variations of proofs. Naturally, not all students have a strong interest in history of mathematics, e. g. only some indicated in the survey the wish to have a dedicated course on history of mathematics, however, all indicated they would like to see more mathematics-historical aspects in their studies (which is not provided for in their regular curriculum).

In summary, we would like to conclude that a teaching unit based on historical documents is able to broaden the students' horizons both from a mathematical as well as a cultural point of view. The documents presented above in excerpts are suitable for this purpose, since they allow not only a mathematical depth but also evolutionary access to the development of mathematics itself.

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Exploring a property of mixed product in the middle school inspired by Bézout's use of units

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In his arithmetic treatise, Bézout teaches arithmetic as grounded on quantities and units. It is from this perspective that we consider a property of a "mixed product", involving a multiplication on sets of quantities. In this note, we present the main lines of some didactical research on the use of this property with students of years 7 and 8 (12–14 years old). We focus mainly on the conception of units and related notions, as the multiplicative relation between units of the same kind.

Keywords: Bézout, quantities, unities, multiplication, mixed product.

Historical introduction.

Étienne Bézout (1730–1783) is well known in algebraic geometry for his theorem on the number of intersection points of two plane algebraic curves. However, we are interested here in his « Cours de mathématiques à l'usage des gardes du pavillon et de la marine», first printed in Paris, in 1764[1]. The French Royal Academy of Sciences praised his particular way of presenting the different notions in his « Éléments d'arithmétique »: "so M. Bezout has often treated these objects in a way which is absolutely his own & which makes them surprisingly simple." (Histoire de l'Académie Royale des Sciences, 1764, p.97). His success did not diminish until the end of the 19th century (Alfonsi, 2011).

In his treatise on arithmetic, Bézout presents an application that we find of great didactic potential. It is a multiplication in which the unit of measurement is made to appear. The property could be stated as follows: multiplying a quantity (the multiplicand) by a multiplier is equivalent to multiplying the number of units of the multiplicand by the multiplier, and is also equivalent to multiplying the unit of the multiplicand by the multiplier. This particular use of units was remarked upon by the Académie des Sciences as an innovation.

> These subjects, so often treated, become, by the manner in which he offers them, absolutely new & of the most luminous simplicity. We can put in the same rank what he says on the nature of the units in multiplicand, multiplier & product […] (p.98)

Bézout gives the following example:

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[1] The references in this text point to the edition printed by J.B.G. Musier fils in 1779, identical to the first one (Alfonsi, 2011) and the oldest that is available in a digitized text. Any quotation is our free translation from French.

> 112. Let us ask, for example, what is the value of 5/7 of a pound? Since the 5/7 of a pound is the same thing (96) as the seventh of 5 pounds, I reduce the 5 pounds to pennies (57) and […]. If one were to ask for the 5/7 of 24 pounds, it is obvious that one could first take, as we have just done, the 5/7 of a pound, and then multiply by 24 what this operation would have given but it is more convenient to multiply 5/7 by 24 pounds, which gives 120/7 pounds (107), and then to evaluate this last fraction which will be found to be worth 17 pounds 2 pennies 10 denarii 2/7. (Bézout, 1779, item 112. Our free trans.)

He also gives the following example involving fractions:

> 96. For example, in 4/5, 4 can be considered as representing any four things, 4 pounds, for example, which must be divided into five parts; for it is obvious that it is the same thing to divide 4 pounds into five parts in order to take one of these parts, or to divide one pound into five parts in order to take 4 of these parts.

Bézout doesn't justify this property in his manuscript. Nevertheless, he gives a meaning thanks to the use of the different kind of units and stressing the multiplicative relations between units of the same system. It is also through the use of units that Bézout gives meaning to the different numbers, including decimal numbers and fractions, and also gives theoretical justifications to the algorithms of arithmetic operations and meaning to the rules of calculation and to the properties satisfied by the numbers under consideration.

The heritage, in the sense of Grattan-Guinness (2004), of Bézout's work, and in particular his use of units as a didactic tool, and even as a theoretical tool, is part of the inspiration for the research of which we present some of the results in this paper.

Mathematical perspective.

From a contemporary mathematical point of view, the property quoted above is a *mixed product,* which involves two multiplications in different sets. First, the *external law,* which acts on the set of quantities, and secondly, the *internal law of composition* on the set of numbers considered (*e.g.* integers or rationals).[2] Nowadays, and contrary to Bézout's usage, we would write the first step of the calculation of the example (112) (before dealing with sub-units), using the multiplication and equality signs and parentheses, which provides 3 equalities, as follows:

$$5/7 \text{ x } (24£) = 24 \text{ x } (5/7\ £) = (24 \text{ x } 5/7)\ £.$$

In order to be demonstrated rigorously in some generality, it requires mathematical arguments that were out of use in Bézout's time (the modern axiomatics developed in the nineteenth century), and which are beyond the scope of the teaching curriculum of middle school today.

Theoretical framework and didactical motivation.

In this former example, the initial quantity is 24 £, in the term (24 x 5/7) £ the number of units is scaled, and in the term 24 x (5/7 £) is the unit itself that is scaled. We consider both the possibility

[2] This is different from considering only the multiplication in the set of rational/integral numbers with the commutativity and associativity properties of it, disregarding the set quantities (and the units).

of varying the number of units considered and the one of scaling the unit itself. We have three different expressions for the same quantity, so in each concrete case we can choose to use the first or the second way of proceeding, depending on what would, for example, simplify the calculations. This is one of the (technical) interests of this property. However, what motivates us was to better understand what competencies are the ones reflected when using the unit-scaling strategy. We also stress that the reasoning involved can be used to conceive multiplication involving fractions, as, *e.g.*, in item (96)[3]. Indeed, Behr et al. (1997) consider the rational number as an operator acting, either on the number of units, or in the unit itself of the operand. In their experiment, the students (preservice teachers) were reluctant to scale the unit as if this strategy were "cognitively more demanding" (p.65). The unit-scaling perspective relies on quantities and measures, and differs from the more studied one about "unitizing" and "composite units", as in Lamon 1996. Indeed, although she defines *unitizing* as the "cognitive assignment of a unit of measurement to a given quantity" (p. 170), it seems to us that her examples refer rather to a grouping notion. Nevertheless, she states that "The ability to form and operate with increasingly complex unit structures appears to be an important mechanism by which more sophisticated reasoning develops" and pointed out that this perspective has been shown to be successful in several mathematical teaching domains. Behr et al. (1997, p.50) also agree with the relevance of conceptual units in learning. See the references in therein.

We hypothesize that a perspective based on quantities with a broad approach to units, could support the development of the unit-scaling reasoning, as well as could perform the multiplicative relations between units of the same family, since they refer to the size of the quantity, instead of referring to the number of unities composing of the quantity. We focus on these concepts (quantities, units, multiplicative relations between units, quantity-scaling strategies) and the relations between them.

Chambris, Coulange, Rinaldi, & Train (2021) pointed out some other deeper potentialities of the mixed product property, related to the multiplication of fractions, the equivalence of fractions (in terms of "compensation theorem"), and the knowledge about metric units. They previously identified this property (in terms of "multiplicative version of the compensation theorem"). They show that the understanding and the teaching, of it could rely on the knowledge of the "related units". Indeed, Chambris (2021) have introduced into didactic research the notion of *related units*[4], based on quantities as a foundation for numbers. We can interpret these units as obtained by enlarging or reducing, let's say, a *standard* unit (*e.g.* pound, gram)[5]. For instance, to justify his calculation in his item 96 quoted above, Bézout uses a unit, and in item 112, we can see 24 pounds as a unit.

[3] Let us remark that a cognitive gap of abstraction is involved when considering 4/5 as 4/5 of the unit 1.

[4] Previously named relative units, she actually calls them *related units*, whose meaning is shown in this example: both the sizes (the weights) of "1 gram" and of "250 grams" can be used as units, units that are related one to the other.

[5] We prefer this point of view, to that of considering a *related unit* as the unit *composed* by many equal standard units, a notion that lead to a *grouping* approach, and makes it more difficult to consider a *related unit* smaller than the standard one, and then making obstacles for the multiplicative relations (in both directions) between units of the same family.

In the same article, Chambris, Coulange, Rinaldi, & Train (2021) stress that the *related units* are an implicit knowledge, mostly missing in the French curriculum. Furthermore, the lack of this knowledge seems to be in connection to some recurrent difficulties in the students, as for example, conceiving the multiplicative relations between different units of the same family, *e.g.* the ten is ten times smaller than the hundred, and the hundred is ten times bigger than the ten. Indeed, when working with metric units and "numeration units" (Chambris, 2021), Chambris, Coulange, & Train (2021) noticed difficulties in the students and the teachers in managing the possible links between

- *related units* (metric and *numeration*),
- multiplicative comparison relations (*e.g.* 1 cm is ten times smaller than 1 dm),
- the composition of these relationships (*e.g.* ten times smaller than a tenth).

Observe that these items appear in Bézout's work (with a different point of view), even though he uses the partitive approach to define fractional units and fractions, instead of the "fraction as comparer approach", as named by Freudenthal (1983), which leads to a better mastery of the multiplicative relations between units (Cortina et al., 2014).

On an ongoing project, we therefore sought to go further in identifying the lack in the competencies that is reflected in the missing of the mixed product. We asked ourselves what treatment involving units are needed, or at least useful, to support the quantity-scaling strategies in a meaningful way. In this note, we outline some explorations thought the interview of a middle school student.

A didactical reading of the use of units and multiplicative relations in Bézout's treatise.

Through some citations, we show here that the units are at the base of Bézout's arithmetic treatise, using them as a tool, as well as means of meaning. At the beginning of his treatise, Bézout defines units as "a quantity that is taken (usually arbitrarily) to serve as a term of comparison for all quantities of the same kind." (item 3). He then adds "The number expresses how many units or parts of units a quantity is composed of." (item 5). Bézout makes the distinction between numbers, which he calls *abstract numbers* (*e.g.*"three or three times"), and numbers expressed in units, which he calls *concrete numbers* (*e.g.*"four pounds")[6]. His multiplicand of item 112 is as well, from which we took as our first example. Bézout defines the sub-units of the simple unit as parts of it. About tenths he writes the following:

> 21. In order to evaluate in decimals the parts smaller than the unit, one conceives that this unit [...] is composed of ten parts […]

This is a partitioning approach. Nevertheless, he considers these sub-units as units in their own right, and hastens to mention one of the multiplicative relationships between them and the simple unit: " they [the tenths] are ten times smaller than this one [the simple unit] (item 21)", which will make possible, later on, an explanation of the proposed calculation techniques and the exposed properties, including that of the mixed product. Indeed, to mention the multiplicative relations bet-

[6] We observe that what he defines as a *concrete number* seems to be, nowadays, rather (the measure of) a quantity in a given unit.

ween units of the same system is recurrent is Bézout’s treatise, in particular when introducing the tens, hundreds,… and the decimal place-value system: “a number followed by two others, [...] marks a number a hundred times greater than if it were alone” (item 11); “ as one moves from right to left, the units of which each number is composed are ten times larger” (item 15); “these new units, ten times smaller than the tenths, will be one hundred times smaller than the main units” (item 22).

Furthermore, we find it interesting from a didactical point of view that before introducing decimal numbers, he first introduces another system of units, in which the ratio between one unit and the next is *not* division by ten. Indeed, the first example proposed is that of the pound (as a currency), “the pound is divided into 20 parts, which are called pennies, the penny into 12 parts, which are called denarii.” (item 17). He then quotes the ounces, the toise, the day and the marc, which were common units of volume, length, time and currency at the time, before introducing the divisions and subdivisions by decimals, the convenience of which he praises. A multiplicative relation is also quoted to explain the meaning of multiplication of fractional numbers, “multiplying the denominator 3 by 5 changes the thirds into fifteenths, *i.e.* into parts five times smaller” (item 106).

To describe the process of multiplication, Bézout uses the notion of unit twice. Firstly, in the multiplication algorithm itself, to name the different digits of the multiplicand and the multiplier “[...] and retain the tens, which are hundreds, to add to the next product which will also be hundreds.” (item 50). Secondly, he uses the units to give meaning to this technique (“[...] because the number by which I multiply is a number of hundreds.” (item 51).

The notion of unit, and in particular that of *unit fractions*, remains fundamental in Bézout's treatment of fractions. He then considers a second definition, based on division, with a “unit” point of view, “Another way of looking at a fraction is to consider the numerator as representing a certain quantity that must be divided into as many parts as there are units in the denominator.” (item 96), and uses the property of the mixed product to interpret it. We quoted the end of item 96 in our introduction.

Didactical exploration. The problem type. Interviews.

We want to explore whether, in specific problems where the didactic variables encourage it, the unit-scaling part of the mixed product property is used. We also look for the skills that are involved in it. The first author began her experimental approach with an interview between directive and semi-directive, with an interview guide with questions in a precise order, but left the possibility of asking other questions, depending on the interviewee's answers. On the one hand, we tested some knowledge on notions of units, conversion of units and measures, which are part of the curriculum. On the other hand, we checked the state of mastery of a certain point of view on units, less present in the teaching, such as the multiplicative relations between units of the same family. We finish with two problems, whose written productions we have collected. The two problems proposed aim to observe whether students are leveraging units to use the mixed product property by scaling the quantity and make calculations easier. Among the knowledge potentially involved in solving the problems this way, we sought to test which ones are mastered by the students, which ones are not, and then we wanted to know if the students use this property to solve the two problems.

The basic problem we propose to study is the following. Given a quantity of a certain magnitude whose measurement is expressed by the number **a** relative to a quantity **u** taken as a unit, multiply this magnitude by a number **k.** The situations to which the learner is exposed highlight the various multiplications **(k** x **a**, and **k** x **u**), as well as the fact that a unit, like any quantity, can be enlarged (the scalar **k** is greater than 1) or reduced (the scalar **k** is smaller than 1).

In our starting example,“Rice pudding at the Indian festival”, the initial quantity is enlarged.

Problem (Rice pudding). For an Indian festival, it has been decided to make rice pudding. The rice is sold in packets of 250 grams and 1 kilogram. We calculated the quantity of rice needed, according to the number of guests. We need 7 packages of 250 grams. Then we say that there will be 4 times as many guests. How much rice is needed in total?

This is a special case, because the initial quantity (7 times 250 grams) is measured in relation to another quantity, **u**, which is 250 grams, taken as a unit (here 7 stands for **a**). To take 4 times this initial quantity, we could proceed by multiplying 7 by 4, then computing 250 grams 28 times. More simply, we can also choose to first enlarge our unit by 4 times, which gives 1000 grams, and then take 7 times 1000 grams.

We chose to interview students from the Collège International de l'Esplanade, Strasbourg, during the school year 2020-2021, one in year 8 (13 years old) and four in year 7 (12 years old). Here we briefly present part of the analysis of the interview with the 8 year student, which lasted 18 minutes.

Analysis of Persephone's interview

After analyzing the interview, we present here some evidences on the skills of the student. We have sought to highlight whether the mathematical knowledge existed for the student, whether the student had to find it by herself, with some adaptation or not. We also point out some knowledge that seems to be lacking in her studies, or some competencies that are not fully developed.

Indeed, she knows several families of units with their different units (*e.g.* “meters, kilograms” (A019), “hectograms decagrams decigrams” (A021)), knows what quantities they measure (*e.g.* “meters distances” (A025)), and knows how to convert between units of the same kind: to the interviewer's question in A106, “one meter equals how many centimeters”, she answers “one hundred” (A107), then adds that “one millimeter equals zero point one centimeters” (A115). She also masters the ratio between units of the same kind: the researcher : “how do you go from ten to a hundred?” (A078), the student : “by doing times ten” (A079).

The student knows the multiplicative relations between hours and minutes (“[the hour is] sixty times [larger than a minute]”, in A087) and “[a minute is] sixty times [smaller than the hour]”, in A089), although she does not state them spontaneously. We can ask ourselves whether this is due to a lack of practice in these manipulations because she has not been taught, for example, that a minute corresponds to a unit sixty times smaller than an hour, that it is one sixtieth of an hour.

The student knows that one hour is sixty minutes (A081). In contrast, the opposite conversion appears more difficult to the student. To the question “one minute is how many hours?” (A082), she starts to answer with an order of magnitude that is right “one minute is / zero decimal point” (A083), then she hesitates, then makes a mistake. The question seems to be out of reach at this point for the student.

The interview also shows that the student is more comfortable with families of units whose ratio between them is ten (so are every example of units given by the student). The dozen seems to be more remote to her ("by tens it is much easier" (A057)), and she may not consider the ten as a unit ten times larger than the single unit, or the hundred as a unit ten times larger than the ten. Sub-units, such as the tenth, are not named, and naming the multiplicative relations between units of the same species, when this relation is less than one, is also more difficult for her as shown in what follows.

A118	Interviewer:	the centimeter is larger / and how many times larger than a millimeter?
A119	Student:	since there is a decimeter / zero decimal point // by no means / is bigger / ten times uh a hundred times

With questions about the change in the measure when the unit have been changed, the student succeeds in giving the right answers. However, when the researcher asks "had you ever thought about that?" (152) , she replies "yes, but it's still paradoxical" (A153), then adds "I wasn't taught that, we weren't told about that no / on the other hand it's still visible" (A155).

The student doesn't succeed to correctly solve the « Rice pudding problem ». Despite an initial impulse to enlarge the unit in the first problem, she abandons this strategy, and does not use the unit-scaling part of the mixed product property yet leading to simpler calculations. The student begins by answering the question about the quantity of rice needed:

A181	Student:	it will take um a kilogram of rice because we make two hundred and fifty times four so there are four times more and so twenty-one um not at all / twenty-eight packets of um rice of two hundred and fifty grams

Note that she starts by multiplying 250 grams by 4. Then she multiplies 7 times 4, which would be too much. Then she keeps this second product, giving as a result twenty-eight packets [...] of two-hundred-and-fifty rice, which is right, and does not take into account her first multiplication, that is 250 times 4, and which should have been multiplied by the number of initial packets, that is 7, to obtain a right final result. The researcher unsuccessfully tries to get her back to the first multiplication.

A194	Interviewer:	at one point you had multiplied the four here / by four times as many guest
A195	Student:	yes
A196	Interviewer:	by the two hundred and fifty grams of rice / and you got a kilo / isn't that another way of calculating?
A197	Student:	well, if you wait (silence) well, we don't know if it's the packets or if it's the grams // is that it?

Conclusion on Persephone's interview

The student interviewed has mastered all of the concepts related to the units taught in primary school and at the beginning of middle school. In contrast, the student does not manage to solve the problems correctly. In between, we identified some notions that are not completely out of reach, but that they don't appear as a solid background, and not taught at the school. We hypothesize that these are the notions that are lacking, or are at least useful, to give the mixed product a ground that can bring it meaning. This study shows that a broader view and knowledge about units and at least the following notions are lacking: families of units whose ratio between them is *not* ten, multiplicatives relations -in the two directions- between units of the same family, fractional units, as well as considering the metric units and *numeration units* as units in their own right. It seems to us that a point of view that Bézout gives in his treatise, where the conception of units is based on quantities and that the relations between their sizes are highlighted, could support the conceptualization of quantities into units and make closer the competency to apply it in a given task by using the quantity-scaling strategy.

Perspectives.

The aim of the interview was to test the knowledge of the students in the concepts involved in the application of the mixed product when *enlarging* or *reducing the unit*, as well as the ability to apply it. After a deeper analysis of the tasks, we sought to assess some of the involved concepts more precisely, before implementing a teaching experiment with students of year 7 (the results of which will appear in a forthcoming article). Indeed, the teaching experiment is based on the idea of better identifying the concepts at stake in the mixed product, by testing the students before and after exposing them to didactic situations with the aim of supporting them in the development of the missing competencies (including those highlighted in the interview), and in particular, the multiplicative relations between quantities of the same kind, as units of the same family and fractional units, through a "fraction as comparer approach" (Freudenthal, 1983), as well as handling tasks.

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Standing on the shoulders of Giants
– looking at the teaching subject “conics”

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Why do we teach parabolas? One possible answer is because they are in the curriculum. How does content get into the curriculum, whose responsibility was and is the design of curricula and textbooks in Germany? The history of conic sections as a subject reveals a rich and varied teaching tradition that not only calls into question ideas about “invariability” of teaching subjects, but also enables greater openness and joy in the design of mathematics curricula.

Keywords: Conic sections, curriculum development, Neuere Geometrie, Meraner Reform, history of mathematics education

Why should math teachers in Germany study the history of German math curricula?

How did the current curriculum of secondary school mathematics come about? Over the past millennia, a large number of (still-valid) mathematical truths have been discovered, discussed, and proven. Who chooses what to teach in class and on what grounds is such a decision taken? To what extent does the development of modern mathematics effect curriculum development? Students obtaining a teaching profession should ask themselves these questions also in order to recognize their own prospective responsibility in mathematics curriculum development.

However, the study of the historical development of curricula in Germany is complicated, as these differ in the different federal states. The pronounced federalism in the German cultural and educational system has a long tradition. Until 1871 Germany consisted of many independent feudal states and free commercial towns. These small states had their own cultural and educational policies. Even with the establishment of the German Reich in 1871, the competencies were not centralized, the states remained responsible for education and culture. The promotional principle continued during the Weimar Republic. Under the rule of the National Socialist regime, education and culture were centralized. The accompanying ideologization of science and culture through racial theories, as well as the indoctrination of young people in schools, can also be demonstrated in the maths curriculum and maths textbooks (Mehrtens, 1989). The ease with which the Nazi regime was able to influence schools and universities in the centralized system led to a special appreciation of the federal division of responsibilities after the end of the Second World War. This principle was also used by the allied victorious powers of the Second World War when developing administrative structures in the western zones of occupation. In the Soviet occupation zone, however, after the founding of the GDR in the 1950s, the federal structures were dissolved and centrally standardized curricula and textbooks were developed. As early as 1956, the guidelines for the mathematics curricula for the upper level in the GDR show a strong reduction in the treatment of conic sections to spheres and calculations using methods of analytical geometry and analysis[1], while the mathematics textbooks for the the upper secondary schools in the FRG still were based on the methodology to the Weimar Republic. The *EOS* (extended general education polytechnic high school) in the GDR began with the 9th grade, the upper level had only 2 years compared to the 3 years of upper secondary classes of the Gymnasium in the FRG. Accordingly, in the justification of the guidelines for the teaching of mathematics in the *EOS*, content reduction and pre-employment focus on calculation methods are in the foreground. After reunification, federal principles were also established in education policy in the new federal states. Today, according to the federal principle, educational policy in Germany is a matter of the federal state. Accordingly, curricula, textbooks, school types differ in the different states. Standardizations and coordination are planned and implemented by the *KMK* (Conference of Ministers of Education) and relate to specific structures, such as Exam formats. The educational standards set nationwide for Germany in 2011 provide guidelines on what students should be able to do and when; these requirements are based less on content than on skills. The shift from input-oriented content requirements for the main subjects and foreign languages in the form of educational plans and curricula to output-oriented standards is one of the measures of a general turn towards output orientation and the definition of central standards in Germany. Conic sections are no longer obligatory. The introduction of central exams had a strong influence on the content of mathematics education at the Gymnasium. After 2000, a new, nationwide trend towards the *Zentralabitur* (centralised A-level-examination) began, partly with reference to the PISA studies and the unexpectedly poor performance of German students. The ideologised and scandalized debate about central exams did not refer to existing experiences with central exams in the federal states of the former French occupation zone, nor to the central examinations of the GDR and the majority of the new federal states.

The discussion of educational policy, educational institutions, mathematics curricula and exam formats in Germany, including the time of the Nazi regime, the post-war period and the special features of the GDR's educational system, is of particular importance with regard to right-wing populist developments. The growing centralization efforts in the context of the digitization of schools, which are presented exclusively as an answer to new global challenges, also require cultural and historical contextualization. For the latter, the study of mathematics education is particularly suitable due to its algorithmic subjects (Weiss, 2019). The preoccupation of mathematics teachers with the history of mathematics teaching and the development of selected content as a subject can also be motivated pragmatically in Germany at present. In some federal states there are opportunities for schools to have more autonomy, e.g. through internal school curricula that are developed by the school's teaching staff. The content-related discourses are often restricted by current pedagogical trends, such as discovery learning or project teaching. Dealing with historical teaching materials gives one the opportunity to be inspired by alternative presentations and task formats without coming into conflict with current educational theories.

1 a collection of digitized GDR curricula can be found at https://bbf.dipf.de/de/sammeln-entdecken/besondere-bestaende-sammlungen/lehrplaene#0

Why should math teacher students in Germany study the history of conic sections as a teaching subject in German mathematics curricula?

During their university studies, the student teachers are confronted with the values of various communities of interest and common practices and thus various reasons for different selections of mathematical subjects and its concept developments. Most of the time, however, this happens implicitly through the enculturation in various communities of practice. The appearance and the disappearance of conic sections in mathematics teaching offer ample opportunities to prospective mathematics teachers to look into their own traditions. Dealing with different value systems from the historical perspective enables the prospective mathematics teachers to develop their own points of view and to structure and classify different value and norm systems without personal conflicts with actors of the different communities in their own environment.

In the following considerations we limit ourselves to the teaching practices of three communities that are or were important for the university education of German *Gymnasium* mathematics teachers: the community of researching mathematicians, the community of mathematics educators and educational scientists and that of historians and philosophers of mathematics. We also limit ourselves to the teaching of conic sections. Of course, there are lecturers and teachers who belong to several communities. Fundamental principles such as the historically determined close connection between research and teaching and the freedom of research and teaching in German university education are represented by the norms and values of all three communities. In the present article, however, the main focus will be on tensions and contradictions between these communities and their historical contextualization. For this reason, we also put contrasting values of the different communities in the foreground.

The teaching subject “conics” in the teaching practice of researching mathematicians

The education of Gymnasium math teachers in Germany is closely linked to the education of mathematicians. The mathematics diploma was only introduced in Nazi Germany.[2] To contrast the different communities, we refer in the following to university locations in which only teachers for upper secondary schools like the Gymnasium are educated (e.g. the universities of Bonn, Göttingen and Mainz). The separation of teacher education for Gymnasium teachers from that for the primary, elementary and secondary schools[3], corresponds also to the historical development. In the university education of mathematics teachers for Gymnasium, math courses are held by researching mathematicians and are usually shaped as common lectures with students studying only mathematics. The teaching activities of the community of researching mathematicians and mathematics lecturers are very much motivated by the care for the next generation of mathematicians. The value system of this community is characterized by the promotion of mathematical achievement. In many ways, values and norms result from the role of mathematics as

2 Information about the historical development of German mathematical institutes one can find in (Schubring, 1985)

3 Grundschule (from school year 1 to 4), Hauptschule (from school year 5 to 9) and Realschulen (from school year 5 to 10)

a competitive sport. The mathematics lectures focus on learning modern mathematical language and modern methods to solve open and important problems.

Conic sections appear in the mathematical lectures of math teacher students, if mentioned at all, in group-theoretical or projective contexts, but not associated with the elementary mathematical object "conic". Analytical geometry is taught in the form of linear algebra in which conic sections are used for classification implicitly. Mathematical research is highly specialized; the recognition of what counts as important knowledge is regulated by the appreciation of the international community. The predominant teaching method so far is the deductive, axiomatic. In the last few decades there have been efforts to introduce additional mathematical courses for student teachers that deal with elementary mathematical content. Such courses are usually designed by researching mathematicians.

The teaching subject "conics" in the teaching practice of historians of mathematics

The teaching and research area History of Mathematics has in Germany a long and worldwide recognized tradition (Purkert & Scholz, 2009, Weiss, 2020). In the GDR, courses on history, philosophical aspects, and the logical foundations of mathematics were compulsory in the curriculum for all students mathematics teachers (Schreiber, 1996). The extent to which mathematical-historical contexts are addressed in today's university education for student teachers depends on local conditions. In the community of mathematical historians, however, there is agreement that conic sections are a fundamental topic in the history of ideas in mathematics (see e.g. Struik, 2013). Both the Greek origins of the conceptual development of conic sections (e.g. Coolidge, 1968) and their outstanding role as a problem-solving method (e.g. Renn, Damerow & Rieger, 2002) were and are main topics of lectures on the cultural history of mathematics. However, there are also delimitations. The values and norms of teaching history of mathematics are linked to the use of historical scientific methods. The latter can differ from the use of the history of mathematics by mathematicians or math educators (Fried, 2001). Michael N. Fried and Sabetai Unguru demonstrate the explication of these contradictions using the theory of conics (Fried & Unguru, 2017). The possibility to limit oneself to conic sections when considering fundamental philosophical differences becomes apparent in Évelyne Barbins research on the philosophies or theories behind history and education (Barbin, 2015). The main examples for comparing the different methods and perspectives are conic sections (see also Barbin, 2012, Bartolini Bussi, 2015).

The teaching subject conic sections in the teaching practice of mathematics educators

In the context of subject matter didactics, the parabola, the hyperbola as examples of functions and the area and volume calculation of conics are topics of teaching. Here, too, it depends on local conditions whether connections to the content of higher mathematics or to topics of the history of mathematics are shown. As we shall see, the teaching history of conic sections is particularly suitable not only to show aspects of the historical development of the values and norms of the community of mathematics educators, but also their relationships to the other communities. In the 19th century, math teachers of the Gymnasium were simultaneously researching mathematicians and very often also historians. They developed the teaching of conics taking all three perspectives into account.

The history of teaching conics as a path to common traditions and values

Scientific development and the elementarization of mathematics

This contradiction between mathematics as a science and the exemplified elementary school mathematics was the basis of different fruitful discourses in the 19th century, which led to a transformation of the pre-university teaching that was to great extent shaped by Euclid's elements until then. The focus on the development of geometry teaching in the 19th century is especially illuminative since here the scientific development and the elementarization of mathematics happened parallel to each other at frequent intervals and often by the same people. The historian and maths teacher Max Simon for instance notes: "When you look at the elementary geometry of the 19th century, it is especially worth mentioning, how the great developments of science also come to light in elementary geometry." (Simon, 1906, translation by the author). These developments are among others descriptive geometry (Monge), Analysis situs (Carnot, von Staudt), geometrical constructions (Steiner), projective properties (Poncelet), barycentric coordinates (Möbius), linear algebra and algebra (Graßmann, Plücker), analytical geometry (Gergonne).

Flourishing teaching culture in *Neuere Geometrie*

Under the banner of *Neuere Geometry* (Newer Geometry), the research in geometry as a science and the educational reforms in mathematics teaching merge. "The New Geometry, seen from its genesis, is not as much in contrast to the geometry of the ancient than it is in contrast to analytical geometry… Analytical geometry as a subject is a continuation to the elements, but as a method, it is in contrast to the elements"(Pasch, 1882, S.1, translation by the author). The immediate junction of new developments in mathematics with teaching reforms is also fostered by the professionalization of the teachers, restructuring of the school system, development of new curricula as well as changes in the university system. In 1810, for instance, the examination of teachers for *higher schools* was introduced, which did not only require decent knowledge in philosophy and history but also in mathematics. In 1812, the deep knowledge of Euclid's books 1-6, 11 (spacial geometry) and 12 became a general requirement for the final examination (*Abiturprüfung*) at school.

With the so called *Süvernscher Normalplan* (1816) and a renewed lesson scheme for maths education, for instance the analytical approach to conic sections became a teaching subject in grade 10 and 11 (*Sekunda*, 16-17 years old) at the *Gymnasium*. While the conic sections were taken up in the curriculum, text books with different approaches to the subject appeared. For an impression about these different presentations, we recommend a look at the antiquarian or digitally available text books of this time. The theologist Johann Andreas Matthias (1813) for instance, chose the approach to conic sections along the Apollonian way. The mathematician Johann August Grunert (1824) however, used the analytical method to deal with conic sections in his teaching script with exercises and their demonstrated solutions. Also, the mathematician, philosopher, reform educator (*Reformpädagoge*), politician, school teacher (*Schulmann*) and founder of the Berlin Pedagogical Seminar, Karl Heinrich Schellbach, composed in 1843 a text book about conic sections (Schellbach, 1843). An impression of later teaching texts on the subject, which even took projective approaches into account, as well as a detailed analysis of the presentation of *Neuere Geometrie* is provided by

Sebastian Kitz in his dissertation on *Neuere Geometrie* as teaching subject for higher teaching institutions (*höhere Lehranstalten*) between 1870 and 1920 (Kitz, 2015). Examples of the appearance of modern mathematical developments in elementary geometry, as it was described by Max Simon, are also Hermann Hankel's (Hankel, 1875) and Jakob Steiner's (Steiner, 1876) synthetical treatises on conic sections.

The royal road to geometry

The expectations regarding the reforming power of *Neuere Geometrie* become apparent in Hankel's way to rephrase the well-known ancient anecdote: "There is no royal road to geometry. We, however, can add: The *Neuere Geometrie* is this royal road." (Hankel 1875, S.33, translation by the author). Despite these high expectations in the *Neuere Geometrie* and its rapid development as scientific discipline, the school reform initially experienced setbacks. The choice between synthetic and analytic geometry, between Euclidean and *Neuere Geometrie* was at first in the *Gymnasium* decided in favor of Euclidean geometry without the treatment of conic sections. Consequently in 1837, by a Prussian circular directive (*Preußisches Zirkularreskript, i.e. Runderlass*) of Johannes Schulze, the successor of Süvern, disposed a reduction of scheduled mathematics lessons and the removal of conic sections of the curriculum at the Gymnasium (Treutlein, 1911, pp. 37- 45).

Meraner Reform and *Anschauungslehre*

Only during the gathering of philologists in 1864 in Jena, it came to the foundation of a mathematical-pedagogical section and to the revival of the discussion on conic sections for the teaching at secondary school. In these discussions, the treatment of conic sections in analytical form was linked with the notions of variable and function and hence with the intentions of the *Meraner Reform* for the introduction of differential- and integral calculus (Schimmack, 1911). The proposals of the *Meraner Reform* did not only take those parts of the theory of conic sections with a direct relation to the notion of function into account, but also recommended to deal with conic sections in analytical and synthetical form – even with application to the elements of astronomy, albeit without exemplification of its implementation. Another source of the reformation of the Euclidean tradition of geometry teaching is the development of the *Anschauungslehre*, an education to an inner intuition and view. The geometry book in three volumes of Henrici and Treutlein (Henrici & Treutlein, 1981-1983) as well as Treutlein's *Anschauungslehre* (Treutlein, 1911) – called by Felix Klein "exceptionally noteworthy book" (Klein, 1925, p. 261) – give a good impression of a versatile pedagogically rich treatment of conic sections respecting the different approaches. Accordingly, Treutlein connects plane geometry with spatial geometry by geometrical transformations as reflections, creates references to applications and uses folding and models for the education of internal intuition and view (*Anschauung*) (Weiss, 2016). Also, descriptive geometry, that was only taught at *Realgymnasium* and *Oberrealschule* (secondary schools with a focus on science) can be found in the appendix of the third volume of the geometry text book of Henrici and Treutlein. Here we find (without exercises) an introduction in different projection methods and hence an integration of this approach. An other interesting textbook on the theory of conics, which implemented the perspective of geometric algebra, was the book by Hieronymus Georg Zeuthen (Zeuthen, 1882), a Danish historian and mathematician.

The dawn of conic sections

From the beginning of the twentieth century until to the New Math in the Seventies, one can find planimetric, stereometric, analytical, affine, perspective, projective up to group theoretical conceptions of conic sections, mostly close to the treatise of Walter Lietzmann's *Elementare Kegelschnittlehre* (Lietzmann, 1949). Until the end of the sixties, one can speak of a bloom of conic sections. The New Math brought conic sections in relation with differential and integral calculus as well as considerations with set-theory and geometrical transformations. Spherical geometry served as contextualization of contents and methods that were acquired in the theory of conic sections (Athen et al., 1967). Not well-known are the international endeavors in the New Math reform (De Bock & Zwaneveld, 2017) to strengthen the application side of New Math. Also, in general secondary schools of the GDR the basics of descriptive geometry where taken up in grade 7 and 8 when the school subject Technical Drawing was introduced. With the reform of the upper school in 1975 and unified examination requirements the conic sections were more and more reduced to linear structures in analytic geometry and in the analysis to the investigation of function graphs of parabolas (Schupp, 1988) and are nowadays reduced to the context of functions. The introduction of dynamic geometry software, in particular with the possibility of 3-dimensional dynamic representations, has not yet led to the high expectations placed on it with regard to the revival of traditional geometrical content. On the other hand, the new technical possibilities for visualizations and animations also arouse the interest of mathematicians who are teaching higher mathematics and perhaps open new doors for common practice.

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TWG13: Early Years Mathematics

Introduction to the papers of TWG 13: Early Years Mathematics:

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Introduction

Thematic Working Group 13 focuses on the mathematics learning of 2–8-year-olds. We acknowledge that this learning may take place in kindergarten, preschool, primary school, and other less formal settings. We pay attention to how mathematics is approached implicitly and explicitly by children and teachers, and other responsible adults. A wide range of topics is covered in TWG13, spanning different areas of mathematics and different theoretical and methodological approaches. This wide span is a characteristic feature of TWG13, a feature that has been documented at other conferences as well (e.g., the POEM conferences, see Carlsen et al., 2020) and by Levenson et al. (2018).

Overview

Thirty-six participants from fifteen countries presented 22 papers and 3 posters in TWG13 during the CERME12 conference. These were organized into eight themes, namely: theoretical frameworks, toddlers' education, teachers' actions and training, curriculum and inclusion, analyses of children's activities, measurement in early years mathematics, literature overview, and prospective teachers' education. Each session was devoted to one of these themes and contained paper and poster presentations, followed by a discussion on the presentations, as well as a general discussion on questions related to the theme of the session. Next, we present the main questions related to each theme and the main outcomes of the discussions that took place.

Theoretical frameworks

Two papers were presented in this session, by Wernberg and Johansson, and by Kleven. The questions discussed were related to the possibility and the plausibility of a 'unified' theory, used to provide a holistic view of a learning situation. The discussion led us to acknowledge that theories cannot be static; they need to be continuously developed and adapted, e.g., by empirical studies. Especially when analyzing kindergarten activities, it is difficult to restrict oneself to only the children, the adult, or the activity, since they are all interwoven. Therefore, several frameworks can be useful to grasp this complexity; these frameworks could be the part of a network of theories.

Additionally, qualitative and quantitative methodologies could be used complementarily, in order to reach well-verified theories.

Toddlers' education

Three papers were presented in this session, by Cooke, by Palmer and Björklund, and by Vanegas, Giménez and Prat. The questions discussed related to the outcomes of studies on toddlers' mathematical learning and on the limitation of such studies. We firstly observed that studies on toddlers' mathematical learning are becoming more frequent in TWG13. Additionally, we agreed that the analysis of the mathematical activities of such young children raises questions on whether and what kind of mathematics is actually involved in those activities. By not having access to the verbal utterances of these children, the risk of over-interpretation may be high. We also agreed on the importance of engaging or collaborating with preschool teachers, since their expertise and experience might be crucial in understanding the toddlers' ways of expressing themselves.

Teachers' actions and training

Three papers were presented in this session, by Erfjord, Carlsen and Hundeland, by Barkai, Levenson, Tsamir and Tirosh, and by Anantharajan. The questions discussed were based on the premise that teachers in early childhood education must make many choices related to mathematical activities, materials (books), drawings, etc. These choices may be influenced by their beliefs or their curiosity to learn more about children's thinking. So, we discussed the factors that affect these beliefs and these choices and whether professional development can affect them. We concluded that teachers in early childhood education often have quite stable beliefs, much dependent on their subject-specific knowledge, e.g., beliefs about teaching and learning arithmetic may be different from those about geometry. Similarly, teachers with a problem-solving view on mathematics may highlight different counting strategies, while teachers with an instrumentalist view encourage fixed algorithms. Moreover, teachers' mathematics anxiety may occasionally lead them to resources with immediate use, e.g., a textbook or a worksheet.

Curriculum and inclusion

Three papers were presented in this session, by Henriksen, by Højsted and Rasmussen, and by Lange, Lembrér and Meaney. There was also a poster presented by Walla. The questions discussed were focused on the challenges and the benefits of comparing mathematics curricula from different countries, but also on achieving inclusion. We agreed that considerable differences concerning the role of curriculum in early years mathematics exist among different countries. Therefore, comparing curricula may be quite challenging. The same is the case with the use of textbooks, since most countries do not use textbooks in kindergarten. This, combined with a non-explicit curriculum on mathematics, may in turn lead to a reduction of mathematical activities in kindergarten. We also agreed that the issue of equity is not solely an issue for the mathematics classroom but affects all school subjects.

Analyses of children's activities

Three papers were presented in this session, by Baumanns, Pitta-Pantazi, Demosthenous, Christou, Lilienthal and Schindler, by Conceição and Rodrigues, and by Vogler, Henschen and Teschner.

There was also a poster presented by Pettersen, Volden and Justnes. The questions discussed were based on the premise that often, a narrow focus is needed when we analyze very young students' mathematical activity. So, we discussed the possibilities for triangulation and for generalization of our results. In our discussion, we acknowledged the context-bounded nature of most research on early years mathematics, which in turn leads to qualitative approaches. We also noted that the culture of assessment in the early years varies across different countries.

Measurement in early years mathematics

Three papers were presented in this session, by Robotti, by Pytlak and Maj-Tatsis, and by Hoth and Fricke. We discussed whether a more general measurement sense is useful, e.g., one that incorporates comparisons of objects, informal measurements, etc. We also considered whether the introduction of standard units or other measurement tools support children's first approach to measurement or their intuitions about measure sense. It was noted that in many countries there are learning trajectories beginning with comparisons and by using non-standard units, while there are also activities on measuring with meters or rulers. We agreed that the use of standard units relies on the ability to identify numbers, while the use of non-standard units relies on an experience which supports the discovery of measuring and contributes to construct the measurement sense. Thus, children's experiences of length comparison as a daily activity and the use of non-standard units give rise to the necessity of standard units.

Literature overview

Three papers were presented in this session, by Flaten, by Maffia and Silva, and by Tzekaki. We discussed what can be learned by literature reviews or syntheses. We also discussed the necessity of reaching consensus on our terminology, e.g., on playful learning, board games, and patterning. We agreed that the field of early years mathematics education is blurred, as is mathematics education in general. This is mainly due to the concepts and methodologies adopted from different research areas, such as psychology or sociology. Therefore, literature reviews and other systematic reviews are very helpful in planning and conducting research, since they allow us to systematize our own approach and to look at a given issue from different points of view. We also agreed on the necessity of consensus on our terminology, regardless of cultural differences. At the same time, we need to be aware that sometimes we observe the same terminology used to describe different aspects of mathematics education, but also different terminology used for the same aspects of mathematics learning. This makes it sometimes difficult to synthesize the results of empirical studies.

Prospective teachers' education

Two papers were presented in this session, by Sabo Junger, Ferme and Lipovec, and by Sala-Sebastià, Breda and Farsani. Given the fact that most, if not all, early years teachers do not have a solid mathematical background, and do not necessarily believe in the importance of problem solving, we discussed two issues concerning teacher training, namely how can prospective teachers' problem solving skills be enhanced and how can their attitudes towards problem solving be changed. In our discussion, we agreed that the terms 'problem', 'word problem', 'task' and 'modeling' should be clarified, in order to assist prospective teachers. We also agreed on the necessity of incorporating problem solving as part of prospective teachers' training, including the use of various heuristics. This in turn, may assist them in identifying and utilizing opportunities for spontaneous problem solving, which will appear during their teaching. Additionally, it can make them cautious before providing children with too much help during problem solving.

Future directions

From the summaries of each session above, we observe that, besides the growing interest in early mathematics education, there are a number of issues that need attention, also in the future. The various curricula, kindergarten systems, and school systems differ significantly across the European countries. These cultural issues cause difficulties for researchers when comparing, aligning, and reflecting on the research results from different countries. However, the discussions that took place in CERME12 were promising in the sense that a growing understanding and acknowledgement of these cultural differences were emerging. Related to the issue of cultural differences, researchers might focus more on early mathematics education in the family, by asking questions such as: How may parents and other responsible adults provide opportunities for young children to engage in mathematics at an early age? Another issue that emerged was the need for developing systematic research in our field, as a way to cope with the multiplicity of approaches and methodologies. Finally, few studies were concerned with affective issues related to early years mathematics. This might be another avenue for future research.

We look forward to further scientific discussions in future CERME conferences. Several of the participants of CERME12 have attended the Early Years Mathematics group several times in the past. We thus see the contours of a research community across country boarders.

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Teachers’ curiosity about their own student’s counting over time

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Using the novel construct of teacher-curiosity, this study extends the idea that teaching in early mathematics is effective when based on students’ mathematical thinking. The study examines how six early childhood mathematics teachers’ curiosity towards one of their own students’ thinking changes over five weeks, alongside professional learning (PL) to support teacher-curiosity. Findings suggest that teachers can stay curious and continue learning about students with whom they are familiar.

Keywords: Teacher-curiosity, Early childhood mathematics, Teacher-learning.

Introduction and literature review of teacher-curiosity

Attending to children’s mathematical thinking is a vital component of teaching early mathematics (Carpenter et al, 2017; Franke et al., 2001). A potential way to help teachers attend to children’s thinking is to support teachers’ curiosity about how students think. This study uses the framework of teacher-curiosity to examine changes of early childhood mathematics educators’ curiosity about one of their own students’ mathematical thinking. Teacher-curiosity is defined as an instance where “teachers recognize something as unknown, unfamiliar, puzzling, uncertain, or new in the context of teaching and learning, and feel motivated to initiate inquiry into that instance” (Anantharajan, 2020. p.31). This definition is based on a framework of teacher-curiosity (Figure 1) developed from literature in teacher-learning and mathematics education; philosophical work on curiosity; and studies of curiosity in psychology.

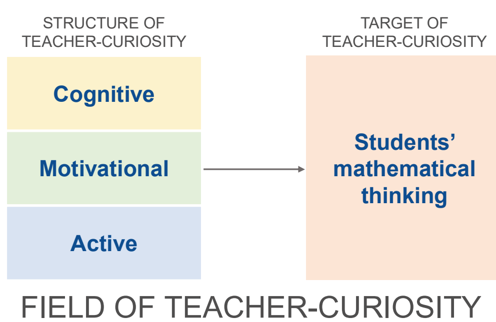

Figure 1: Framework of teacher-curiosity

The framework of teacher-curiosity comprises of cognitive, motivational, and active aspects (Anantharajan, 2020). Psychological studies describe the cognitive aspect of curiosity as experiences of surprise, ambiguity, puzzlement, or novelty (Berlyne, 1966; Kashdan, 2004; Lowenstein, 1994). In the current study, this involved teachers identifying something as surprising, ambiguous, puzzling, or new in student-thinking. The motivational aspect of teacher-curiosity involves what the teacher wants to learn about, from among elements that are surprising, ambiguous, puzzling, or new. Finally, the active aspect of teacher-curiosity involves the steps that teachers take to pursue their curiosity and learn more. The framework also relates curiosity to its target: knowledge. The target of teacher-curiosity in the current study is teachers’ knowledge of students’ mathematical thinking. This study focused specifically on children's understanding of counting: principles of counting like one-to-one-correspondence, cardinality, and number-sequence, and strategies such as grouping, skip counting and so on (Carpenter et al., 2017).

Approaching children's thinking with curiosity may help teachers identify what they wish to learn about, and teaching moves they can implement, which is often a challenge (Jacobs et al., 2010, van Es et al., 2008). The broader study in which the current paper is situated aimed to support teachers' curiosity about student-thinking through a PL. The current paper analyzes some potential transfer of teacher-learning from the PL context to the participants' classroom practice. The research question addressed in this study is: How does participants' curiosity about their individual focal student's mathematical thinking (FST) change over five weeks during which the participants take part in a professional learning module to support teacher-curiosity?

Research Design and Methods

The study was grounded in the early mathematics activity of Counting Collections. Children count collections of objects and teachers observe them count, not telling them what to do but asking students to explain their work. These interactions allow teachers to make sense of students' counting. The activity was a common element across the PL and participants' practice. The six participants taught grades TK[1] to 1 in three schools in the US, in California. Four participants taught multi-grade classrooms of K-1 and the participants' teaching experience ranged from one to twenty years. Participants implemented Counting Collections prior to the PL and were familiar with teaching based on children's thinking. They were paid a nominal honorarium to compensate them for their time in the study and the professional learning module.

The PL was informed by the research on teacher-learning (Borko et al., 2014; van Es & Sherin, 2008) and supporting curiosity (Kashdan & Fincham, 2004), and facilitated by the author. Participants were individually interviewed before and after the PL, during which they also responded to a video of a child counting. The PL included six weekly, after-school sessions of 90-minutes. One session was held after a two-week gap. Sessions included video-based discussions based on the teacher-curiosity framework, opportunities to engage in mathematics, and reflect on curiosity in practice. Participants also shared video from their own classrooms, recorded with a wearable camera during interactions with a student. Participants presented and posed questions about aspects of the clip they were curious about and received feedback from other participants. Of these multiple forms of data about the participants' curiosity, the current paper analyzes one specific data set described below.

Focal student documentation

At the start of the PL, each participant chose a focal student from their classrooms whose thinking they wished to understand better. Each week, the participant independently interacted with and documented this student's thinking during Counting Collections. The researcher did not observe this interaction. Participants used the focal student documentation tool (Table 1) each week to document their curiosity about the focal student. The tool progresses from noticing to the cognitive,

1 Transitional kindergarten (TK) is a grade level in the California public school system for 4-year-olds who turn five after September 2 of the school year. TK is followed by kindergarten.

motivational, and performative aspects of teacher-curiosity. Participants sometimes collected the student's work or took photographs of the student's counting, but only the weekly documentation tool was collected as data for this study. Participants did not share any identifiable details about the focal student. The effects of participants' curiosity on FST were not studied. During the last session of the PL, the participants also completed a focal student reflection tool, where they reflected on the previous six weeks of their own curiosity about the focal student. The data for the current analysis consists of the weekly documentation tool. Thus, each of six participants completed five instruments.

During the last part of the weekly PL meeting participants shared with a discussion partner what they learned about their focal student, what they still wished to learn, and asked for suggestions and feedback from their partners on next steps. Participants were paired with different partners each week, to enable a variety of feedback and suggestions based on their partners' knowledge and experience.

Table 1: Focal student documentation tool

Question/ prompt in documentation tool	Dimension of teacher-curiosity
What did you notice while watching this student count?	Target
What did you think this student understands or is able to do in terms of counting?	
List all the things that felt new, surprising, puzzling, unclear or unknown to you about the student's thinking.	Cognitive
Among these things, what interested or mattered to you to find out more about? Why?	Motivational
What next steps can, or did you take, to find out more?	Active
What can or did you do to record or keep track of the student's thinking?	
How can you confirm your understanding of the student's thinking?	

Analysis

The data was analyzed using a coding framework of *a priori* and emergent codes, in Nvivo (Table 2). Code 1 captured the *structure of teachers' curiosity about student-thinking*. The subcodes represent the dimensions of the teacher-curiosity framework, namely (1a) cognitive, (1b) motivational and (1c) active aspects of teacher-curiosity. The motivational aspect included subcodes to capture whether the participant wished to learn about something related to student-thinking (1bi) or not (1bii). Similarly, the active aspect of curiosity included subcodes to capture whether the participant's response to the student is meant to elicit student-thinking (1ci) or direct it (1cii). Code 2 captured the *mathematical ideas that participants noticed in student-thinking* based on principles and strategies of counting identified in the early mathematics research. Each of these sub codes was coded as a strong understanding or a partial understanding, and the participant's use of evaluative language to describe student-thinking. Inter-rater agreement was calculated for about 30% of interview data, as a combination of Kappa statistics at the parent code level, and discussion and consensus at the grandchild code level. The Kappa values ranged from 0.7-1.0. After reaching agreement on the interview data the two coders discussed the focal documentation data to confirm that the codebook could be applied to these data. The researcher coded the focal student data. The second coder independently proposed codes. Disagreements were discussed to reach consensus.

Table 2: Samples of coded data

Code	Sample of coded data
1a. Cognitive	I was happily surprised that she knew to count by tens this time. (Beth, wk 3)
1b. Motivational	
1bi. To learn about student-thinking	Would he be able to add any of his groupings together or would he revert back to counting by 1s [?] (Lillian, wk 2)
1bii. For other reason	Because she wasn't using counting by 2s to her advantage to help her double check. (Hannah, wk 4)
1c. Active	
1ci. Elicit student-thinking	"I asked her to show me how she made her rows again with more items from the collection. She continued to make uneven rows. I said I wondered why they had different amounts. She said they needed to get "bigger."" (Rita, wk 1)
1cii. Direct student-thinking	I asked her what 2 100 charts would make to see if she had the vocab & she did. So I guided her using the 100 charts as reference on how to count to 100. (Stacey, wk 5)
2. Mathematical ideas in student-thinking	
2a. Strong understanding	She has pretty good one-to-one correspondence. (Bella, wk 1)
2b. Partial understanding	"She counted by 10s and stacked the 10s into 10s (for the second time) But when counting she couldn't go past 100 by 10s." (Stacey, wk 5)

Participants' responses were organized into a table with responses to each question, each week. As each participant documented a different student no analyses was done across participants. Each participant's data from week 1 was a starting point. This initial data was compared with the combined data from weeks 2-5 for that participant. By comparing the week 1 data to the cumulative data over the subsequent four weeks, the analysis illustrates the participants' knowledge and curiosity about FST at the beginning of the PL (week 1), with what they subsequently learned and wanted to learn about the focal student over the next four weeks. The data from week 2-5 were combined because the iterative weekly changes for each participant were small and happened at different points. However, the cumulative change over the subsequent four weeks is clearer to perceive and acknowledges the overall magnitude of learning and effort by the participants. Combining the data from these weeks did not inflate what participants learned over that time but simply illustrates the change in what they learned or wanted to learn about their focal student from week 1.

Findings

Participants identified more mathematical understandings in FST

All participants noticed more strong and partial understandings from weeks 2-5, compared to what they started with in week 1 (Table 3). All participants also noticed the change in students' understanding of at least one mathematical idea over time. Participants were not prompted to follow the student's understanding of one idea. However, all participants did so, alongside other ideas. None of the teachers used evaluative language when documenting their curiosity about their own focal student during the five weeks of documentation.

Table 3: Changes in teacher-curiosity about FST

	Bella		Beth		Hannah		Lillian		Rita		Stacey	
	Wk 1	Wk 2-5	Wk 1	Wk 2-5	Wk 1	Wk 2-5	Wk 1	Wk 2-5	Wk 1	Wk 2-5	Wk 1	Wk 2-5
Mathematical understandings	3	12	3	13	3	17	4	15	3	17	3	9
What teacher found surprising etc.	2	7	1	4	1	4	2	4	2	10	1	7
What teacher wanted to learn more about	0	3	1	4	1	2	2	3	1	6	1	4

Participants identified more surprising, puzzling, ambiguous or new aspects of FST

Compared to the number of instances that participants found surprising, puzzling etc. about their respective focal students in week 1, all teachers record between 4 and 10 additional sources of surprise, puzzlement etc. about the focal student's counting, from weeks 2 to 5. (Table 3) As an example, Table 4 illustrates data from Rita's documentation.

Participants posed more questions about FST

At the end of five weeks, all participants identified more things about FST that they wanted to learn about, compared to week 1. From week 1 to 5, the number of things the teachers wanted to know about their students' thinking increased by between 2 and 6 (Table 3).

Participants implemented/ proposed multiple responses intended to elicit FST over five weeks

All participants proposed or enacted between 3 and 12 types of responses to elicit specific details of FST (Table 5). All participants mention certain types of responses: specific counting tasks to elicit

Table 4: Change in cognitive aspect of Rita's curiosity about her focal student

Week 1	Weeks 2-5
It was surprising that she would make rows with different amounts. I also wondered why she chose the large collection she did when she hadn't before.	I was surprised she began numbering the second row of objects at 1 again. I have seen her write her numbers 1-20, so I wondered why she couldn't in this instance. New: she matched color of beads to bowls Unclear: her plan for counting all Puzzling: why she thought matching colors would help. Why is she successful at grouping and adding when she uses one-color collection? How does she see this collection differently? Why did she work better by herself? I wondered how she made the connection to start grouping. I wondered if using the days-in-school chart supported this learning. I wondered what attracted her to use the same collection and how that helped her.

mathematical ideas, questioning and asking students to explain their thinking, and documenting or note taking about the focal student. Four participants indicated directive moves as part of their response. Of these, three participants indicated directive moves in the first week and not after that. One participant did not indicate directive moves till the fourth week. The documentation did not indicate a clear shift from directive to eliciting moves by the participants. Over the duration of the PL participants may have felt a conflicting need to learn about student thinking and have the student meet certain learning goals. This conflict may represent a challenge that teacher-curiosity poses in practice.

Discussion

Analysis of the focal student documentation provided a brief glimpse of how participants' learning in the PL transferred to their own practice over time. The results indicate that over five weeks, participants identified more understandings in their respective FST compared to week one, more aspects of FST that they found surprising puzzling, ambiguous or new, and more questions about FST. Finally, participants implemented or proposed multiple responses intended to elicit FST over the course of five weeks. Thus, the findings suggest that the PL and the teacher-curiosity protocol may help teachers perceive changes in, and develop a deeper understanding of, their students' thinking. The findings also suggest that, with consistent support, the participants' familiarity with their own students did not prevent them from identifying new, surprising, or puzzling things about the student. Rather, the results align with Loewenstein's (1994) notion that overcoming "manageable gaps" in knowledge and learning increases, rather than decreases a person's curiosity.

Participants likely drew on a wealth of implicit knowledge about the focal student and their class contexts, which informed what they noticed, found surprising, wished to know about, and how they responded. For example, knowing what mathematics the class had engaged with up to that point of time may have led participants to notice if the focal student struggled with something when others in

Table 5: Types of responses to focal student proposed by participants from weeks 1-5

Types of responses	No. of participants
Give tasks to reveal specific understanding	6
Question student or listened to student explain	6
Make notes or document student work	6
Have student record their counting	5
Observe student	5
Provide or suggest tools and other physical supports	4
Demonstrate to or direct student	4
Give group work and whole class activities	1
Give open-ended counting task	1
Give tasks to extend student thinking	1
Compare student work over time	1

the class did not. When focusing on the whole class, participants could likely only manage brief observations and conversations with the focal student, limiting the detail of what they were able to document. However, they had the opportunity to repeat their observations, which may have shaped participants' curiosity about their focal students. The fact that all participants followed their focal student's understanding of one mathematical idea over time suggests that the documentation process may have allowed them to consistently focus on a single idea. All teachers happened to observe a change in their student's understanding of grouping to count. This is likely a common mathematical idea at this grade level that the participants saw as an important part of curriculum. It is also possible that having more opportunities to interact with the focal student allowed participants to propose or try different responses with the student. The documentation tool included detailed questions regarding how they documented FST and how they might confirm what they learn. These questions suggested that responding to FST may involve multiple moves. The participants' suggestions to each other during PL meetings may have also motivated them to try different responses.

One concern may be that the comparison between week 1 and the combined responses of weeks 2-5 may misrepresent the participants' curiosity. However, rather than obscure details of weekly changes, this approach makes explicit that by actively engaging with FST over five weeks, participants were able to identify several aspects of their students' thinking that they did not fully understand and wanted to learn about. These additional questions may not have emerged for the participants if they stopped engaging with the student's thinking after one interaction. Overall, the findings suggest that actively supporting teachers' curiosity about their students can help them use their knowledge of their students to discover details of their students' mathematical thinking over time, rather than view their students as "known" entities. Classroom-based research on teacher-curiosity over a longer period is necessary to better understand how teacher-curiosity can be supported in PL and motivate practice.

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The effects of an intervention on adults' beliefs and self-efficacy for implementing numerical tasks with young children

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This paper describes a course attended by 30 graduate students in mathematics education, non-preschool teachers, that aimed to promote participants' knowledge of young children's development of numerical competencies. Prior to the course, participants held high positive beliefs towards children's engagement with numerical activities, but their self-efficacy was low. Findings indicated that mastery experience, mostly by repeatedly analyzing videos, and repeatedly designing tasks, afforded participants a chance to see how they were progressing and increased their self-efficacy.

Keywords: Adults' beliefs, adults' self-efficacy, mastery experience, numerical competencies.

Introduction

In recent years, there has been a growing emphasis on early childhood mathematics education and the professional development of preschool teachers (e.g., Tirosh et al., 2019). While this is commendable, young children spend much of their time at home with parents, grandparents, and other responsible adults. As such, increasing adults' awareness of their roles in promoting early numerical skills is important. In a previous study (Levenson et al., 2021) we investigated 91 adults' beliefs (none were early childhood educators) regarding supporting children's engagement with various numerical activities. Findings indicated that in general, participants had positive beliefs towards early mathematics.

The current study builds on that previous study by investigating a small group of adults who participated in an intervention aimed at promoting their awareness to and knowledge for engaging children with numerical activities. In this paper, we describe the intervention and present findings related to adults' beliefs before and after attending the intervention. In addition, we investigate their self-efficacy for engaging young children with numerical activities. Even if adults believe in the importance of their involvement and the importance of engaging children with numerical activities, they might not believe they are capable of doing so, and thus might avoid such activities.

Background

Studies of adults' beliefs regarding early mathematics have mostly focused on parents. Those studies found that while parents tend to agree that mathematics can be and should be promoted in the years before first grade (e.g., Missall et al., 2015), they also believe that engaging young children with mathematical activities is less important than engaging children with literacy activities (Blevins-Knabe et al., 2000). Not surprisingly, in that same study, parents also reported reading to their children more often than engaging them with mathematics activities. Whether during everyday contexts, such as carrying out household chores, or during more structured contexts, such as direct teaching, parents reported helping their children learn language skills more than mathematics (Cannon & Ginsburg, 2008). Thus, it may not be enough to have positive attitudes and beliefs towards early mathematics.

In addition to a person's beliefs, studies have found that self-efficacy can also impact on a person's behavior (Bandura, 1986). Bandura defined self-efficacy as "people's judgments of their capabilities to organize and execute a course of action required to attain designated types of performances" (1986, p. 391). Self-efficacy was found to be related to the effort, persistence, and commitment one places in their actions (Allinder, 1994). Parents' self-efficacy to parent successfully was found to impact on their involvement in their children's academic domain (Hoover-Dempsey et al., 2001), even at the preschool level (Pelletier & Brent, 2002). According to Bandura (1997), there are four main sources for a person's self-efficacy: mastery experience, vicarious experiences, verbal and social persuasions, and emotional and physiological states. The most powerful source is mastery experience; when a person completes a task successfully and deems the effort to complete that task worthwhile, that person's confidence to complete similar tasks will rise. In studies of teacher self-efficacy, this source has been found to be very influential (Arslan & Bulut, 2015). After mastery experience, vicarious experiences also impact on one's self-efficacy. A person gauges their success at a task by comparing themselves to others with similar abilities or attributes, who have engaged in the same task. The third source, verbal and social feedback, is considered weaker than the first two, although people do tend to see themselves through the eyes of others. Finally, the last source of self-efficacy is based on how people interpret affective states such as anxiety and mood.

Related to this study is how interventions can impact on self-efficacy. In one study, parents' self-efficacy increased after participating in an intervention that helped them focus on their children's strengths, as well as their own parenting strengths, while practicing new parenting skills (Waters & Sun, 2016). In another study, parents and their preschool children attended a school-readiness program where parents interacted with their children and observed teachers interact with their children (Pelletier & Brent, 2002). Findings indicated that parents' self-efficacy related to their own ability to teach and motivate their child increased. Parents reported that having the opportunity to interact with their child along with teacher support, impacted on their self-efficacy. Thus, mastery experiences and social feedback were sources for self-efficacy. For pre-service mathematics teachers, a methods course that involved group work when learning how to implement challenging tasks, served as an intervention that raised participants' self-efficacy through vicarious experiences (Yurekli et al., 2015).

The current study and intervention were based on the Cognitive Affective Mathematics Adult Education (CAMAE) Framework (Levenson et al., 2021). The framework considers knowledge adults need for engaging children with playful mathematics, including knowledge of content and children (e.g., knowing that children aged three may not yet have acquired the cardinality principle of counting) and knowledge of activities that can promote numerical thinking. The framework also considers accompanying beliefs. In our previous study, we found that adults believed in the importance of their own intervention for promoting children's numerical knowledge, yet significantly less participants believed that they needed guidance to do so (Levenson et al., 2021). Furthermore, no differences were found between parents, adults who have some other connection with young children, and those who claimed to have no connection with young children. In that study, self-efficacy was not investigated. In the current study, we investigate the beliefs and self-efficacy of adults prior to and after participating in an intervention aimed at promoting their awareness and knowledge for engaging children with numerical activities. Our research questions are: (1a) To what degree do adults hold positive beliefs towards promoting numerical skills in early childhood and (1b) believe that adults' interactions with young children are important? (2a) What are adults' self-efficacy beliefs regarding knowing children's numerical skills and (2b) what are their self-efficacy beliefs regarding designing numerical activities for young children? (3) How might a course that aims to promote adults' awareness and knowledge for promoting young children's numerical skills impact on adults' beliefs and self-efficacy?

Method

Participants and setting

The setting for this study was an elective course entitled *Early childhood numerical thinking: Theory and research,* attended by students working towards their master's degree in mathematics education. There were 30 participants (not the same as in our previous studies), of which twelve were parents of children between the ages of three and six, 13 had some other relationship with children of that age, and five claimed to not have any significant relationship with children of that age. None were early childhood educators. We chose this context for our study, wishing to include at this point in our research only adults who we hypothesized would have a general positive disposition towards mathematics. The aim of the course was to raise participants' awareness of number competencies developed prior to first grade, and in accordance with the framework, to increase their knowledge of children's development of those competencies, as well as the tasks that might promote early number knowledge and competencies.

The course included 13, 90-minute sessions, and focused on three major numerical competencies: counting and enumerating, comparing sets, and number composition and decomposition. During the first and last session, participants watched and then individually analyzed a video of Omer, a three-year old boy and his grandmother Esther (the second author of this paper), engaging in various counting activities while baking cookies. Analysis of this video, as well as other video clips viewed during the course, focused first on the child's ability to carry out the activity (e.g., Could the child count the cookies on the tray, and what were his difficulties?), as well as the adult's role in the activity (e.g., What exactly did the grandmother ask her grandson to do? How were the cookies arranged on the tray?). We will relate to this video and its analysis as the first activity (1) of the course. The rest of the course consisted of a series of activities repeated for each competency: (2) read and discuss related research (e.g., Gelman & Galistel, 1978; Tsamir et al., 2015), (3) view and analyze together YouTube videos of preschool children practicing that competency, (4) individually design a task to implement with a preschool child aimed at promoting that competency, (5) discuss together proposed tasks and agree upon one common task that each participant would carry out, (6) implement the agreed upon task with a young child while videoing the activity, (7) individually analyze the child's knowledge and competency when carrying out the task, (8) view and analyze together participants' videos. In accordance with the aims of the course and the framework, for each element of the course, we discussed children's ability to carry out a particular skill, as well as how a task may be designed and implemented to focus on a specific skill. For example, when discussing Tsamir et al.'s (2015) study of children's strategies for decomposing numbers, we discussed which strategies were more advanced and appropriate and how the task challenged children or supported a specific strategy.

Tools

A two-part questionnaire was handed out during the first and last sessions of the course, the first part focusing on beliefs and the second part focusing on adults' self-efficacy. A six-point Likert scale (1 being the lowest degree of agreement) was used to rate participants' agreements with the belief and self-efficacy statements. There were six belief statements related to promoting early numerical skills (see Table 1) and twelve self-efficacy statements, of which six related to knowing children's conceptions (e.g., I am capable of identifying different arrangements of 8 items which children find difficult to count) and six related to building appropriate numerical activities (e.g., I am capable of building activities that will promote children's skill in counting eight objects). Each group of six self-efficacy statements was combined to give an average self-efficacy score (see Table 2). Cronbach's alphas for each of the two categories was .924 and .965 respectively.

About a month after the course ended, participants were contacted and asked if they would agree to be interviewed. Twelve participants agreed, five parents of young children, four who had some other relationship with children, and three who had no significant relationship with young children. A semi-structured interview was conducted with the aim of identifying elements of the course that impacted on participants' self-efficacy for engaging young children with numerical activities. Participants were asked: The course focused on young children's numerical competencies. Which of the following elements of the course (here, participants were shown the eight course activities listed above) impacted the most on your self-efficacy for knowing about children and their ability to successfully carry out various numerical tasks? Which elements of the course impacted on your self-efficacy for designing tasks to promote early numerical competencies? Interviews were recorded and transcribed.

Findings

As seen in Table 1, and as we hypothesized, participants' beliefs regarding young children's engagement with number activities were positive from the start. At the end of the course, participants were even more positive. The only significant difference was noted for question 5. At the beginning and at the end, not all participants believed that every activity has the potential to invite children to engage with numbers. Yet, participants' belief in this statement on the posttest was significantly greater than on the pretest, perhaps due to the focus on activities in the course.

Table 1: Beliefs regarding engaging young children with numerical activities

Belief questions	Pre M(SD)	Post M(SD)	t(29)	p
1. Children enjoy activities/games that deal with number aspects.	5.23(1.006)	5.47(.819	1.191	.243
2. Children's number knowledge can be promoted.	5.43(1.006)	5.77(.430)	1.836	.077
3. It is worthwhile to engage children with activities/games that deal with number aspects	5.70(.596)	5.87(.346)	1.306	.202
4. Activities/games that deal with number aspects can increase children's knowledge of number concepts.	5.67(.661)	5.70(.535)	.226	.823
5. Almost every activity/game can invite children to engage with aspects of number.	4.17(1.341)	4.70(1.264)	2.570	.016
6. Interaction between a child and an adult while engaging in an activity/game can increase the child's knowledge of number.	5.43(.774)	5.67(.547)	1.756	.090

While participants held positive beliefs towards children's engagement with numerical activities, their self-efficacy at the beginning of the course was less positive, although not definitively low (see Table 2). As Michelle (all names are pseudonyms) said during her interview, "Before the course I thought I was aware and had a way (of doing mathematics with my child) and it was important to me." By the end of the course, self-efficacy beliefs were significantly higher. Interestingly, their self-efficacy related to building numerical tasks was greater (although not significantly) than their self-efficacy for knowing children's competencies.

Table 2: Self-efficacy before and after the course

Self-efficacy	Pre M(SD)	Post M(SD)	t(27)	p
…related to knowing children's competencies	3.869(1.23)	4.869(1.008)	3.831	.001
… building numerical tasks	3.887(.802)	5.125(.802)	4.862	.000

Regarding elements of the course that impacted on participants' self-efficacy, it was difficult for many participants to separate their self-efficacy for knowing children's competencies and their self-efficacy for building numerical tasks. As they talked with the researcher, they intertwined both. This is not surprising as no significant differences were found between these two self-efficacy beliefs. Furthermore, building appropriate numerical tasks and knowing about students' competencies go hand in hand. As Shur stated, "When I choose a task, like counting to 30, I need to know the age of the child, if he is 3, 4 or 5, and then I need to make a task for that child, for what that child knows, at his age." Thus, below, we report in general how the various elements of the course seemed to impact on the participants' self-efficacy for engaging young children with numerical activities.

Nearly all the participants found the video of Omer and Esther interesting. However, interesting does not impact on self-efficacy. What did impact on their self-efficacy was viewing the video twice, once during the first lesson and once during the last lesson. As Levy stated, "In the beginning, the story of Omer offered a taste, but did not help me know about what might be easy or difficult for children. But at the end of the course, … then I could see how much I learned, that is, I could name the different competencies and say where he got stuck and where he didn't." Dahesh stated, "watching the video of Omer and Esther helped me see that maybe I could do those activities with cookies, and that I could build all kinds of tasks with different things." In Levy's case, we could say that the analysis of the video of Omer afforded him a mastery experience, seeing how much he had progressed at analyzing what a child can and cannot yet do. For Dahesh, it was a vicarious experience, seeing that someone else was able to do something led her to believe that she could do it as well.

There were several other activities where participants watched and analyzed videos – watching YouTube videos in class, watching videos they had made of themselves engaging children with numerical activities, and watching videos other participants had made of their activities with children. Regarding the YouTube videos, either participants did not mention this activity, or they specifically said that it made less of an impression than watching videos of themselves or of others in the course. Regarding the other two video activities, watching videos made of themselves was carried out individually as a home task, and watching others' videos was carried out in a group situation. Interestingly, it was the context of the activity, doing it alone or as a group, which seemed to make a difference to some participants. For example, Dave remarked,

> The main thing for me was the individual work, writing the summary of the activity carried out with the child…when you sit at home and watch your recordings, you concentrate much more… To me, the (numerical) activity seemed at first boring, and then you see that the child is having fun… I was always surprised that the child was having a good time.

For Dave, it was necessary to have the time and quiet to focus on a recording. Dave's feeling of surprise when he realized that the child was actually enjoying the numerical activities, hints at an affective impact. Salin emphasized the importance of the group setting, that it increased the opportunities to view different children engaging in numerical activities, "When you see each child that was videoed and the child's age, then I learned more what is easy or difficult for children." Michelle related to the discussion of the videos, and not just watching others' videos, "discussing the videos lets you see things from others' perspectives." All are relating to mastery experience, that is, their experiences in analyzing several children's engagement with numerical activities.

Another aspect of analyzing one's own videos was the fact that participants completed this task three times, one for each numerical content. Like Levy's comment that he was able to see the difference between his first and last analysis of Omer, Dave remarked on his analysis of his own recordings, "I could see my progress from the first analysis I did till the last. You can see it in the length of each analysis, that it increased…Each time I viewed the same recording and saw it again, that was real learning for me." Dahesh also referred to her progress when she analyzed each of her videos, "With each summary and analysis of the activity carried out with the child, there was improvement. The first summary was totally different than the last."

In addition to watching and analyzing videos, participants noted the activity of making up numerical tasks for children as an influential activity. This activity also had two parts, writing the task at home, and then discussing everyone's tasks in class and deciding together what to implement. Both parts seemed to impact on the participants. For Levy, building tasks and discussing with others their tasks, made him notice the specific components of various numerical competencies, which enabled him to address those components when building a numerical task, "In the beginning, I didn't know… that counting and enumerating are different competencies and they are different from composing and decomposing numbers, … [Now] it's easier to focus each task I build and think in which direction I want to take the task." For Neta, discussing different tasks also had an affective impact, "The discussion [of different tasks] … opened my mind to other ideas… and made me feel, wow, there are lots of important things [to consider]." Her exclamation of "wow," signifies an emotional response.

Less participants mentioned the importance of implementing the tasks with children. However, those that did stated that they were able to deal better with unexpected circumstances, such as asking a question in a different way when it seemed that the child did not understand directions. Neta stated, "such experiences increased my self-confidence to guide the child I was sitting with." She also stated that this part was "fun." The element least noted was reading research papers. Two participants asked to be reminded of which research papers were read during the course. It could be that as graduate students, participants had read so many papers and theories, that additional reading of research papers left less of an impression on them than other activities.

Discussion

As found in previous studies (e.g., Missal et al., 2015), participants in this study believed in the importance of promoting early number knowledge and believed that adults have an important role in this effort. Yet, before the intervention, adults' self-efficacy regarding knowing children's conceptions and building appropriate numerical activities was rather low. Perhaps this is one reason that previous studies (Cannon & Ginsburg, 2008) found adults engaged their children with more literary activities than number activities. That self-efficacy was rather low, contradicts finding from a previous study (Levenson et al., 2021), where adults believed their involvement was important, but did not necessarily feel they needed guidance. It might be that because all participants in this study were graduate mathematics education students, they realized, more than other adults might, that they did not have adequate knowledge for promoting young children's' mathematics knowledge. Further study is needed to investigate the self-efficacy of adults who do not have a mathematical background.

Regarding how elements of the course impacted on adults' beliefs and self-efficacy, in accordance with Bandura's (1997) theory, we found that mastery experience, mostly by repeatedly analyzing videos, and repeatedly designing tasks, afforded participants a chance to see how they were progressing. This finding leads us to conclude that short-term workshops may not be enough. One needs to accumulate positive experiences to feel that they have mastered a task and are able to carry out a task. We also found that it was not just one activity, but a combination of several activities, along with positive feelings of surprise and enjoyment, that resulted in higher self-efficacy. Previous studies have reported the benefits of supplying parents with mathematical activities to be done at home (Vandermaas-Peeler et al., 2012); this study has shown the benefits of having adults practice designing their own tasks to carry out with children. Furthermore, supplying activities might not be enough if adults do not have the self-efficacy to implement them. Watching videos of other participants' implementations and discussing those activities with one's peers, can lead to helpful vicarious experiences that may also boost self-efficacy. Further research is needed to investigate the impact of similar courses for adults who do not have a mathematical background.

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Identification of first-grade students at risk of developing mathematical difficulties through online measures in arithmetic and pattern tasks: A study using error rates and response times

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For researchers and practitioners, it is important to identify students at risk of developing mathematical difficulties. The aim of this pilot study was to investigate whether it is possible to identify first-grade students who are at risk of developing mathematical difficulties (RMD) through online measures in arithmetic and pattern tasks. In our study, 54 first-grade students worked on 75 tasks in twelve sets on a computer screen. We also carried out a standardized mathematics test to identify students as RMD. We then investigated if error rates and response times as online measures allow to replicate the identification of students as RMD. Using a logistic regression model, we found that the error rates and response times allow identifying students as RMD with acceptable accuracy. We also found that tasks on symbolic number comparison, completing color patterns, and enumeration of small sets were particularly informative to identify students as RMD.

Keywords: mathematical difficulties, early identification, digital tools.

Introduction

Mathematical difficulties often begin early, before primary education and can—due to insufficient support—cascade into severe mathematical problems (Geary, 2013; Moser Opitz, 2013; Sasanguie et al., 2012). Longitudinal studies confirm that students who enter school with mathematical difficulties do generally not overcome these during primary school (e.g., Viesel-Nordmeyer et al., 2019). It is significant for teachers to be aware of such difficulties, be able to identify them, and to provide adequate support for students. Yet, identifying student difficulties at an early age is challenging—among other reasons, because young students often lack the ability to report their difficulties (Klein et al., 2010). There are numerous tests for identifying students at risk of developing mathematical difficulties at an early age (Hellstrand et al., 2020). These tests, some of which are conducted individually, require a considerable amount of time for both conducting the tests and evaluating results. Also, they are often not feasible to use by practitioners at school. A digital screening offers the possibility to identify, also in practice, a large number of children who are at risk to develop mathematical difficulties. Digital tools are particularly suitable as both the collection of data and its evaluation can be conducted on the same digital device. The Erasmus+ project DIDUNAS builds on this idea. It aims to develop an app, which incorporates tasks in the domain of early arithmetic and pattern tasks, that identifies first-grade students in need of support. For this identification, the app uses data such as error rates and response times.

This pilot study aims to investigate whether it is possible to identify first graders who are at risk of developing mathematical difficulties through online measures (response times and error rates) of students' work on tasks in the domain of early arithmetic and pattern tasks.

Related work

Early identification of students with mathematical difficulties

A variety of reliable and valid standardized tests are commonly used in identifying mathematics difficulties. To develop these tests, researchers focus on identifying variables that appear to be good predictors. Some tests assess speed and accuracy with which students can identify object sets (e.g., Geary et al., 2009). Early numeracy skills, such as quantity comparison, number identification, and counting, appear to have much predictive power for mathematical difficulties (Gersten et al., 2005). Most of these tests include tasks on: Object counting, number comparison, sequencing, connecting numbers to quantities, number recognition, and counting back and forth. Fewer tests include number calculations such as additions and subtractions with symbols or word problems. Even fewer tests include patterning tasks such as copying or extending a pattern or identifying the pattern unit. However, a growing number of researchers (e.g., Verschaffel, et al., 2017) argue that that for investigating mathematical abilities of young students, patterning tasks also need to be considered. Students' number sense and awareness of structure and patterns contribute substantially to students' later success in mathematics (Wijns et al., 2019). Pittalis et al. (2018) in a longitudinal study with first-grade students suggested that that the growth rate of algebraic arithmetic has a direct effect on the growth rate of conventional arithmetic, and subsequently the growth rate of conventional arithmetic predicts the growth rate of elementary arithmetic.

Error rates and response times as measures to identify mathematics difficulties

Several studies investigated differences in error rates and response times of students with and without mathematical difficulties. Some of these studies addressed students' enumeration ability and indicated that students with mathematical difficulties have longer response times within the subitizing range as compared to students without such difficulties (Moeller et al., 2009; Schindler et al., 2020; Schleifer & Landerl, 2011). Other studies addressed students' comparison of symbolic and non-symbolic numbers and found no significant differences in error rates and response times between children with or without mathematical difficulties (Mussolin et al., 2010). However, the slope of response times was significantly steeper for students with mathematical difficulties. Other studies found that response times and error rates for non-symbolic number line estimation are significant predictors of mathematical achievements (Sasanguie et al., 2012). These studies indicate that error rates and response times for suitable tasks can be a predictor of mathematical difficulties. Thus, we intend to investigate: *To what extent can error rates and response times of early arithmetic and pattern tasks be used to identify students that may be at risk of developing mathematical difficulties?*

Methods

Participants

The study was conducted with 54 first-grade students (age: M = 7.38; SD = 0.55) from two primary schools in Germany. In the German federal state that the study took place in, a social index classifies schools into levels from 1 to 9 which is based on factors such as child and youth poverty, family language, and special educational needs of students. Index 1 represents the most favorable conditions. The two participating schools had an index of 7 and 6 which means they tended to have a higher number of students in need of support. Of the students, 34.6% had German as their mother tongue.

Procedure and tasks

For the study we used two tests: (1) the standardized ZAREKI-K test to identify students at risk of mathematical difficulties and (2) a self-developed computer screen test with twelve sets of arithmetic and pattern tasks (75 tasks in total).

Standardized mathematics test: ZAREKI-K is a standardized test for identifying children at the transition from kindergarten to primary school of being at risk of developing mathematical difficulties (von Aster et al., 2009). The test battery is constructed as an individual procedure and consists of 18 subtests. For the present study, an adaptation of ZAREKI-K was used, which requires only six subtests: (a) Counting up to 30, (b) Numbers that precede or follow, (c) Word problems, (d) Visual calculation, (e) Number conservation, and (f) Writing numbers. This adaptation has been shown to yield excellent prediction rates for identifying students at risk of developing mathematical difficulties (Walter, 2020). The students took an average of 14.6 minutes to complete the ZAREKI-K.

Early arithmetic and pattern tasks: The students worked on twelve sets presented on a computer screen (Fig. 1). Every set had an example task to get the students acquainted with it. For sets (1), (4), and (5), students were to determine the number of objects. For set (2), students were asked to determine a number on a number line. Set (3) asked for the number behind the sun. Set (6) asked how many dots needed to be added or subtracted to make it equal to the number shown on the right. In set (7), the largest number was to be determined. In set (8), the number of persons' legs behind the wall was to be determined. In set (9), students were to determine the number of bricks of the tower behind the white blob. In set (10), the result of an addition/subtraction problem was to be determined. In set (11), the students were to compare quantities. For set (12), a color pattern was to be completed. Students could skip a task if they found it difficult by saying "next". There are no identical tasks between these sets and ZAREKI-K. However, both include cardinal and ordinal aspects of numbers. Additionally, set (12) is a pattern task from early algebra, which is not included in ZAREKI-K.

Students answered by tapping on the computer screen. The answers to each task were given with a single tap on the screen. For sets (1)–(10), a number field with the buttons labeled with 1 to 20 was shown at the bottom of the screen. Set (11) had a yellow, a blue, and an equal ("=") button for answering. Set (12) had a yellow, a blue, and a red button for answering. It took students an average of 20.8 minutes to complete these tasks (including all instructions, explanations, and trial tasks).

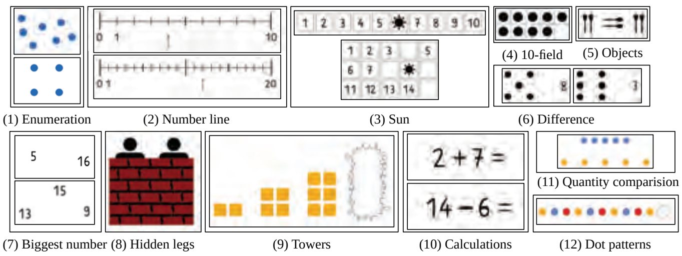


Figure 1: Example tasks of the twelve sets

Measures

We use the following data sets:

(1) **Identification of mathematical difficulties at risk:** ZAREKI-K identifies whether a student is at risk (RMD) or not at risk (¬RMD) of developing mathematical difficulties. We used an Excel spreadsheet provided by Walter (2020) for entering the students' individual points achieved in each subtest, which then calculated the risk of students to develop mathematical difficulties. Of the 54 students, ZAREKI-K identified 18 as RMD and 36 as ¬RMD.

(2.a) **Error rates:** Mean error rates were calculated for all 75 tasks in total as well as each of the twelve sets separately. We considered tasks, where the students answered wrongly or did not answer at all, as being not solved correctly. We considered all tasks, which were not being solved correctly, as error.

(2.b) **Response times:** For each task, the time from when the stimulus was first shown to when the student typed the response on the computer screen was measured. Only response times of correctly answered tasks were taken into account, since tasks that were not understood by the students sometimes were quickly skipped and since students partially rashly guessed wrong answers. Mean response times were calculated over all tasks that were answered correctly as well as for each of the twelve sets separately.

Statistical analysis

We followed the guidelines of logistic regression analysis and reporting by Peng et al. (2002). Logistic regression was performed using SPSS 27 in order to calculate a probability value between 0 and 1 for each student using mean error rate and mean response time over all tasks. For different cut-off values p between 0 and 1, students are identified as RMD or ¬RMD. A ROC (receiver operating characteristic) curve was then plotted, which indicates the sensitivity (true positive rate) and specificity (true negative rate) for all cut-off values p as an indicator of the overall classification accuracy. The area under this curve (AUC) is a measure of the classification quality.

Next, we ask which of the twelve mean error rates or the twelve mean response times are most informative for identifying students as RMD, according to ZAREKI-K. We thus carried out a backwards selection, subsequently for both mean error rate and mean response time for all twelve task sets in early arithmetic and patterns using our multiple-logistic regression model.

Results

We conducted a t-test to compare mean differences of error rates and response times between students identified as RMD and students identified as ¬RMD. Using the Shapiro-Wilk test, the normal distribution of mean error rates ($W(54) = .974$, $p > .05$) and mean response times ($W(54) = .967$, $p > .05$) was checked. Using the Levene test, the homogeneity of variances of mean error rates ($p > .05$) and mean response times ($p > .05$) were checked. Thus, variance homogeneity exists between the groups. With regard to the mean error rate, the 18 students identified as RMD had a significantly higher mean error rate ($M = .29$, $SD = .12$) as compared to the 36 students identified as ¬RMD ($M = .19$, $SD = .09$; $t(52) = –3.48$, $p < .05$). With an effect size of $r = .43$, this is a medium effect. With regard to the mean response time, the 18 students identified as RMD did not have a significantly higher mean response time (M = 7.04s, SD = 1.63s) as compared to the 36 students identified as ¬RMD (M = 6.65s, SD = 1.29s; $t(52) = -.957$, $p = .34$, $r = .13$).

The Likelihood ratio test indicates that the logistic regression model is significantly more effective than the null model (constant only) ($\chi^2(2) = 11.67$, $p < .05$). Goodness-of-fit was assessed using the Hosmer-Lemeshow test, indicating a fit of the logistic model ($\chi^2(8) = 5.15$, $p > .05$). Wald test indicates that mean error rate of the 75 tasks is a significant classifier of RMD ($\chi^2(1) = 8.53$, $p < .05$). The mean response time is not a significant classifier in this regard ($\chi^2(1) = .987$, $p > .05$).

The logistic model calculates a probability value between 0 and 1 for each student based on the error rates and response times. The cut-off value p then defines at which probability value a student is identified to have RMD or ¬RMD based on the logistic regression model. Choosing cut-off values of p thus means to trade off sensitivity (true positive rate) and specificity (true negative rate) as they change diametrically. Table 1 shows the sensitivity and specificity for different cut-off values p. The total accuracy is computed as the number of all correctly identified results in relation to all results.

Table 1: Sensitivity, specificity, and total accuracy of the model for different cut-off values *p*

Cut-off value *p*	.05	.1	.15	.2	.25	.3	.307	.35	.4	.5	.6
Sensitivity (%)	100.0	100.0	88.9	83.3	72.2	72.2	72.2	61.1	61.1	55.6	33.3
Specificity (%)	5.6	13.9	25.0	44.4	61.1	69.4	72.2	77.8	80.6	91.7	94.4
Total accuracy (%)	37.0	42.6	46.3	57.4	64.8	70.4	72.2	72.2	74.1	79.6	74.1

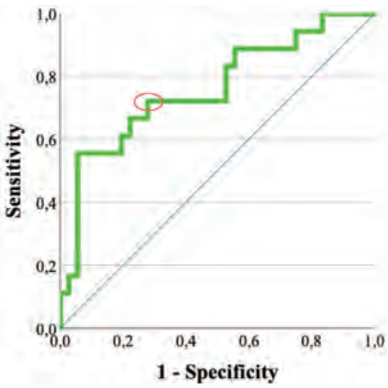
For the identification of students at risk of developing mathematical difficulties at an early age, a high sensitivity is often desirable even at the expense of a decreased specificity. A high sensitivity would ensure that only a few students with mathematical difficulties at risk are missed. However, this has the consequence that the specificity decreases and students with mathematical difficulties at risk are not detected. For the cut-off value $p = .307$, a reasonably high sensitivity of 72.2% is achieved at a still high specificity of 72.2%. Table 2 displays the classification of the students identified to be RMD and to be ¬RMD through the participants' error rates and response times compared to the students identified as RMD and ¬RMD from the standardized ZAREKI-K for cut-off value $p = .307$.

Table 2: Classification table[a]

		Identification		Percentage correct	
		¬RMD	RMD		
ZAREKI-K	¬RMD	26	10	72.2%	specificity
	RMD	5	13	72.2%	sensitivity
Overall percentage				72.2%	

[a]cut-off value $p = .307$

The ROC curve (see Figure 2) is the generalization of a single classification table (see Table 2). Each point of the ROC curve indicates sensitivity and (1–specificity) for a given cut-off value p. The drawn diagonal would be expected if the classification was purely random. A measure of the classification quality of the model is the area under the ROC curve (AUC). Following Hosmer et al. (2013), the classification accuracy can be considered "acceptable" with an AUC = .761.



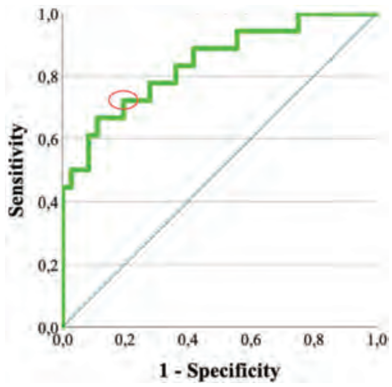


Figure 2: ROC curve of the general model (left; AUC = .761; ellipse marks cut-off value p = .307) and the reduced model (right; AUC = .841; ellipse marks cut-off value p = .351)

To identify those sets whose error rates and/or response times are particularly good for the identification of students at the risk of developing mathematical difficulties, logistic regression was performed through backwards selection. This backwards selection is first done with all twelve mean error rates of the sets. Step by step, the twelve mean error rates are removed from the model, starting with the one that has the lowest significance for predicting the ZAREKI-K outcome. All variables that are significant to replicate the classification based on the ZAREKI-K outcome at the $p < .1$ level remain included according to the Wald test. At the same time, the Likelihood ratio statistic is used to check whether the model would improve by adding another variable. After eleven steps, the mean error rates of sets 7 (symbolic number comparison) and 12 (completing color patterns) (see Figure 1) remained. For the response times, the mean values of set 9 (completing growing number patterns) could not be included, since the number of incorrect answers were too high for which response times were not considered. After eleven steps, only the response time of set 1 (enumeration of small sets) remained. Applying logistics regression onto these three variables identified through backwards selection, Likelihood ratio test indicates that the logistic regression model is significantly more effective than the null model (constant only) ($\chi^2(3) = 22.99$, $p < .05$). Goodness-of-fit was assessed using the Hosmer-Lemeshow test, indicating a good model fit; $\chi^2(8) = 3.58$, $p > .89$. Furthermore, Wald test indicates that the mean error rate of set (12) ($\chi^2(1) = 6.88$, $p < .05$) and the mean response time of set (1) ($\chi^2(1) = 6.91$, $p < .05$) are significant classifiers of developing mathematical difficulties. The mean error rate of set (7) is not a significant classifier in this regard ($\chi^2(1) = 3.256$, $p > .05$), but since $p < .1$ it remained in the model. Following Hosmer et al. (2013), the classification accuracy can be considered "excellent" with AUC = .841. With a cut-off value of p = .351, this model has a higher specificity of 80.6% compared to the previous model at a sensitivity of 72.2%. The total accuracy of this model is 77.8%.

Discussion

The results of our study should be viewed and interpreted against the backdrop of the following limitations: Logistic regression requires sufficiently many training samples, i.e., RMD and ¬RMD cases. Having only 18 students identified as RMD and 36 students identified as ¬RMD limits the certainty of the learned logistics regression model. A larger sample could provide further certainty. In addition, we optimized the classification threshold for the logistic regression model on a single data set and did not evaluate the classification accuracy on an independent test set. In practice, the classification threshold needs to be learned on a training set, which would likely decrease the classification accuracy on independent data.

This pilot study addressed the question to what extent error rates and response times of correctly solved tasks as online measures in early arithmetic and pattern tasks can identify students that may be at risk of developing mathematical difficulties (RMD). Using logistics regression, we found that the mean error rate across all 75 tasks is a strong classifier of RMD, whereas the mean response time was a weaker classifier. Combining error rates and response times in our study yielded an acceptable discrimination of the model of AUC = .761. Furthermore, we investigated to what extent the error rates and response times of twelve sets can be used separately to identify students' RMD. We found that the error rates of two sets (symbolic number comparison, completing color patterns) and the response time of one set (enumeration of small sets) appeared to be particularly informative. The influence of the set about complementing color patterns is especially noteworthy since the standardized mathematics test focused on early arithmetic, not patterns. Combining error rates of symbolic number comparison and completing color patterns with response times of small set enumeration yielded a discrimination of the model of AUC = .841.

Since the DIDUNAS project aims to develop an app that identifies first-grade students in need of support, what can we learn from these results regarding the app development? In our pilot study, we found that it is possible to use online measurements of certain tasks or a set of them (here: symbolic number comparison, completing color patterns, enumeration of small sets) to achieve a reasonably reliable identification of students at risk of developing mathematical difficulties in grade 1. This is promising for future developments. In the future, we will build on these results for developing the DIDUNAS app, which then can be used by teachers around the world to help identify students with risks of mathematical difficulties early on. As the app will require less effort in the conduction and evaluation, compared with some standardized mathematics tests, the app will enable a resource-saving use for teachers. The study presented in this paper is an important first step in that direction.

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First grade students' decomposition of 3D shapes: the interconnection between spatial reasoning and spatial structuring

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Assuming 3D decomposition as a form of spatial reasoning, in this paper, it is our aim to understand how 1st grade students decompose 3D shapes and how these decompositions are related to spatial structuring. Data were collected during a learning experiment where students were asked to decompose three 3D shapes, in two different ways, without disassembling the models, and draw each part. We categorize 21 first graders strategies and analyze more deeply the records from three students that are representative of the different strategies. Results show that students decompose shapes by establishing local relationships, when they recognize, for instance, composites, but cannot yet coordinate them to form the whole; or global relationships, when students coordinate different composites, relating them with the whole, and apparently, have a previous mental model that is coherent with the structure of a shape.

Keywords: Spatial reasoning, spatial structuring, decomposing 3D, early grades.

Introduction

Recent studies show the importance of spatial reasoning (e.g. Mulligan, 2015; Sinclair & Bruce, 2015), especially its relationship with mathematical achievement and with success in STEM disciplines, being these of great demand in the 21st century. Jones (2001) defines spatial reasoning as "the process of forming ideas through the spatial relationships between objects" (p.55). Also, to Davis et al. (2015), spatial reasoning involves two interconnected parts, (mental) *understanding* and (physical) *transforming*, in which composing/decomposing shapes constitute a form of spatial reasoning associated to *understanding*. In early grades, decomposing shapes contributes to a better understanding about shapes' structures. This is also important in more advanced grades, where students have shown difficulties (Duval, 1995; Duval, 1998; Spiegel & Ginat, 2017). Spiegel and Ginat (2017) describe some aspects where 7th and 8th grade students struggle concerning decomposing and recomposing shapes, in geometry problem solving. These researchers relate these difficulties to lack of fluency and flexibility, within this type of spatial reasoning, since "decomposition requires careful fluency analysis, of possibly overlapping elements. Recomposition requires flexible manipulations with elements and resources" (p. 215). This points to students understanding of how to coordinate different parts of the shapes by spatially structuring the shape. That means that students need to understand how the shape is composed, which relationships can be established within the shape and how these work together. In what concerns early grades, Hallowell et al. (2015) describe 1st grade students' difficulties in relating 2D shapes with 3D solids, through composing and decomposing, suggesting the importance of providing adequate experiences. Previous learning experiences seem to influence students' performance in tasks involving decomposition and recomposition (Spiegel & Ginat, 2017). As Sinclair and Bruce (2015) argue, even though 3D shapes have a strong presence in early grades' classes, there is still little research about the way students learn about these shapes.

Clements and Sarama (2014) propose two learning trajectories for composition of 2D and 3D shapes, based on relationships between components and composites and among components, composites and the whole, similar to Battista and Clement's (1996) construct of spatial structuring. Considering composing and decomposing shapes as a type of spatial reasoning, it seems to be relevant to understand and characterize the relationships 1st graders establish while decomposing 3D shapes, that is, how do they structure 3D shapes.

For that purpose, we have designed a learning experiment where we seek to understand how students spatially structure 2D and 3D shapes, through composing and decomposing shapes. This paper aims to answer the following questions: How do 1st grade students decompose 3D shapes? How the decompositions are related to the way students spatially structure those shapes?

Spatial reasoning and spatial structuring

Spatial reasoning is often associated to the process (Jones, 2001) or the ability (Battista, 2007) to create spatial images, and manipulate those images to generate new information. In order to create images, students need to understand objects' structures and by understanding their structures they will be able to manipulate them, physically or mentally.

Spatial structuring is defined by Battista and Clements (1996) as the mental act of constructing a mental representation for an object or set of objects. It consists in identifying units of composition (components), establishing relationships among those components forming composites and establishing relationships among components, composites and the whole. In 3D shapes, built with cubes, components are the cubes while composites are organized sets of cubes. Through this process, students go from seeing a shape as a whole to identifying its parts and establishing relationships among them, into a deeper understanding of shapes' structures. Decomposing and composing shapes, as a form of spatial reasoning, (Davis et al., 2015) seem to be straightly connected to understanding the shape's structure, as it contributes to establishing different relationships among the parts and between these parts and the whole. As students decompose or compose shapes, they establish relationships that allow them to build a mental model for that shape, to find new paths to construct the shapes, to find more sophisticated relationships within the shape. Also, working with different representations for the same shape, as it happens when students draw a 3D shape or when they interpret a 2D depiction of a 3D shape, fosters students understanding of the structures and its codes of representation. As such, we assume a interconnectedness between spatial reasoning and spatial structuring in composing and decomposing 3D shapes.

Spatial structuring shapes can be local or global (Battista & Clements, 1996). Local structuring is related to recognizing components or to establishing relationships among components into composites without yet establishing relationships among components, composites and the whole. Global structuring, in its turn, is evident when students relate components, composites and the whole in a way that makes a previous mental model correspond to a shapes' structure. For global structuring, the operations of coordination and integration are fundamental.

Coordination operation involves establishing interrelationships between different parts considering spatial relationships between them (Battista & Clements, 1996) and eventually with the whole. The operation of integration is linked to a previous existence of a mental model that represents a shape in a way in that each part of this mental model corresponds to a part of the shape, considering its relationship with the whole. The coordination operation is of great importance to the integration operation since it allows to interrelate the different parts of that mental model and also to relate different forms of representing the shape. Both operations are also involved in spatial reasoning processes like creating mental images and models.

Composing and decomposing shapes is important in understanding shapes' properties, but it extends to other domains in mathematics, such as part-whole relationships and fractions, among others (Clements & Sarama, 2014).

Method

The research presented in this paper is part of a teaching experiment focused on first-graders' spatial structuring processes. The teaching experiment was organized in three sequences of tasks where students are asked to explore relationships in 2D and 3D shapes and also the relationships between 2D and 3D representations for the same shape. Each task was implemented in a 60 minutes weekly session.

In the task reported here (12^{th} task), students were asked to decompose three 3D shapes in two different ways (Figure 1) and to draw these forms of decomposition (2D). 3D multilink cube models were available for each shape. Students could manipulate the models without disassembling them. Some of the students' solutions for decomposing 3D shapes were presented and discussed in a whole-class discussion, after they solved the task.

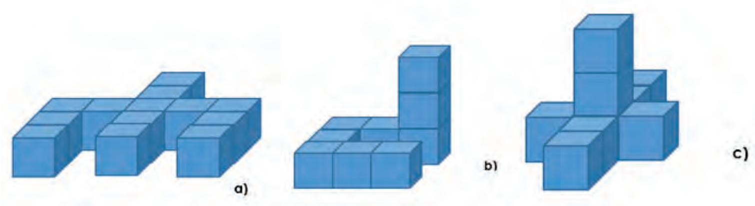

Figure 1 – Shapes presented to students

All the three shapes require the students to coordinate different parts. Shape a) has only one layer, as it can be fully placed on a plane base. Shapes b) and c) have different layers and require the coordination in three dimensions. Additionally, shape c) has a hidden cube that students might not be aware of, when the shape is in the position depicted in Figure 1.

Data were collected in a class with 21 students and their teacher, with great experience, of a public school in Lisbon. All students' names are pseudonyms. We have collected and analyzed all students' drawings. In this task, we have found five different strategies. For this paper, we present drawings from three students, Bruno, Carolina, and Duarte, which are representative of the whole range of strategies.

Data analysis

We started by noticing and categorizing students' strategies as presented in Table 1:

Table 1 – Categorization of students' strategies in both forms of decomposition (*n*=42)

	Decomposes into a smaller piece and a bigger piece	Decomposes by separating into shapes' composites	Decomposes using symmetry	Decomposes into equal or quasi-equal composites	Decomposes into components	No answer or incomplete
Shape a)	12 %	43 %	17 %	14 %	9 %	5 %
Shape b)	12 %	35 %	5 %	25 %	12 %	12 %
Shape c)	9 %	24 %	0 %	29 %	14 %	24 %

Since our focus was on spatial structuring, we built an analytical framework based on local and global structuring (Battista & Clements, 1996) for analyzing the strategies above, being these two progression levels for structuring. The strategies presented above point to different types of relationships established that can be associated to local or global structuring, depending on the ability to coordinate components and composites. Thus, a thoroughly analysis of these strategies led us to include progression sublevels for each level, as presented in Table 2. For each sublevel, we include a brief description of the strategies as aspects emerging from data analysis that we call Indicators/strategies. "No answer or incomplete" was especially due to a matter of time, which might explain, at least partly, its increasing. The categorization we propose here is hierarchic, thus each level includes the previous. We considered the children's difficulty of representing 3D shapes in a sheet of paper. However we assumed the duplication or omission of cubes as essential indicators of lack of coordination.

Table 2 – Analytical framework for decomposing 3D shapes

Level	Sublevel	Indicator/strategy
Local structuring	**E1-** Identifying components	Decomposes shapes into components (cubes).
	E2- Establishing relationships among components	Decomposes into two composites, being one of them almost the totally of the shape and the other a small part. Relates components when representing each of the composites.
	E3- Establishing relationships among composites	Decomposes, establishing relationships among composites (e.g., through symmetry).
Global structuring	**E4-** Establishing relationships between components, composites, and the whole by coordination.	Decomposes shapes into equal or quasi-equal composites. Coordinates position and orientation of components and composites with the whole, recognizing cubes that are part of more than one view.
	E5- Establishing relationships between components, composites, and the whole by integration.	Decomposes shapes, recognizing hidden cubes

Results

In the previous section, we have presented a distribution of student's strategies (Table 1). We use our framework to analyse these strategies, assuming that the same strategy may be allocated either to local structuring or global structuring, depending on the relationships students establish.

We start by analyzing Bruno's drawings (Figure 2).

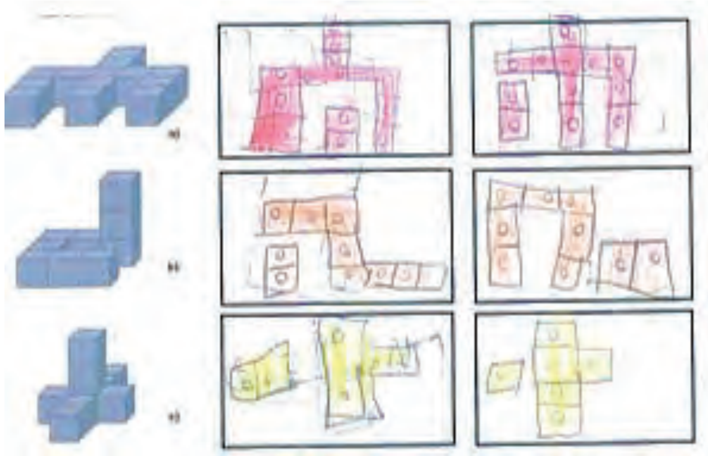

Figure 2 – Bruno's drawings

Bruno decomposes all three shapes into two composites, separating a small composite made of one or two cubes from the rest (level E2). Bruno presents two different decompositions for each shape. These forms of decomposing seem to be anchored in visual aspects. We can also observe that Bruno duplicates cubes that serve more than one composite instead of coordinating both composites. This was the case of the first decomposition for shape a) and b). In shape c), Bruno seems to miscount cubes. We infer that this could be related to the fact that he may not recognize the hidden cube, placed on the bottom of the central composite, and to the difficulty of representing, in a 2D sheet of paper, cubes from the second and third levels, while observing from above.

Carolina's drawings, presented in Figure 3, seem to show that the student establishes equal or quasi-equal composites which are used as complex units for decomposing the shapes.

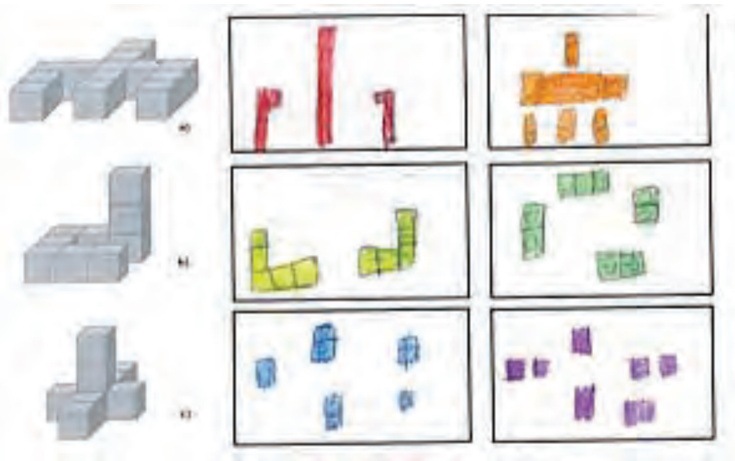

Figure 3 – Carolina's drawings

For the first shape, Carolina seems to consider a larger composite and several smaller composites, in both decompositions. However, for the first decomposition, Carolina apparently establishes a relationship of symmetry within the shape and uses it to decompose the shape, duplicating cubes. For shape b), Carolina presents two different decompositions. In the first, she decomposes the shape into two quasi-equal parts, apparently recognizing two complex symmetrized composites. Nevertheless, Carolina duplicates one of the cubes that is part of both composites, drawing 5 cubes for each. Thus, this student seems not to be able to coordinate the composites properly, while drawing. In this way, both composites seem to be drawn, in the worksheet, as being the reflection of each other. This could also be the way Carolina found to represent the cubes from the levels above, in a 2D paper. For the second decomposition, in shape b), apparently the student chooses to decompose using equal or quasi-equal composites, being able to coordinate them, almost like iterating the smaller one. For construction c), Carolina decomposes it into small parts, apparently without considering the hidden cube, since she draws only eight cubes, in both forms of decomposing. Therefore, Carolina's work could be included in level E3, since the student seems to relate several composites, without yet coordinate them consistently.

Carolina´s forms of decomposition seem to show that the student uses composites that are, apparently symmetrical, in some cases, and, in other cases, are quasi-equal. Hence, this form of structuring is more complex than those showed by Bruno.

Duarte's drawings, presented in Figure 4, seem to indicate that this student decomposes the shapes using equal or quasi-equal composites made of two or three cubes, with exception of the last where he uses a larger composite.

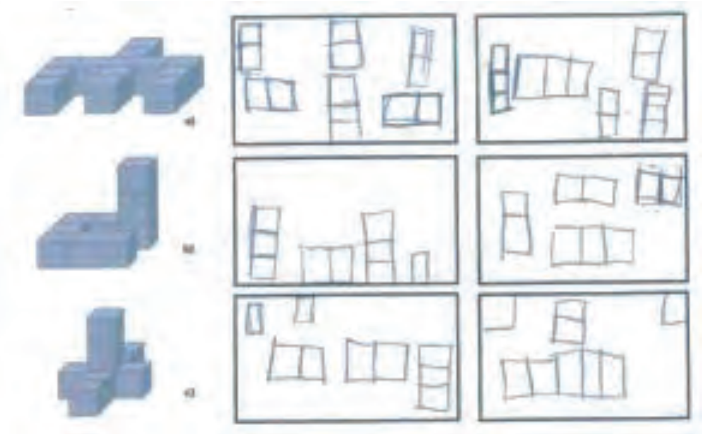

Figure 4 – Duarte's drawings

The horizontal or vertical orientation the student uses to draw each composite seems to be close to its orientation in the 3D shape. Duarte's drawings seem to show that he unitizes shapes using composites as complex units, repeatedly, to decompose shapes. He seems to coordinate the composites adjusting the number of cubes in each composite so that their size fits the shape. Duarte also shows to be able to coordinate different composites that he uses do decompose shapes, not duplicating cubes that are part of more than one view. Moreover, Duarte is the only student, from this group of three, that recognizes the hidden cube in shape c), by drawing it and mentioning it while presenting his strategy in whole-class discussion: "We made the top part with the square [sic] below". This seems to show that his mental model is coherent with the shape. Hence, we consider that Duarte’s work could be included in sublevel E5 from global structuring level.

Conclusions

This paper aimed to understand how 1st grade students decomposed 3D shapes and how these decompositions are related to spatial structuring. As suggested by Battista and Clements (1996), we observed two levels for spatial structuring in students’ work while decomposing three different 3D shapes: local structuring and global structuring. For the former, local structuring, students seemed to maintain a bigger composite and take apart one or several smaller composites. Another strategy for decomposing was establishing symmetrical composites within the shape’s structure and drawing these symmetrical composites as complex units. In most of the strategies, where structuring seems to be local, students showed a tendency to duplicate cubes, while drawing, meaning this that they struggled in coordinating composites when they had overlapping cubes in the corners (Spiegel & Ginat, 2017). Also, for local structuring, students seem not to recognize hidden cubes within the structure. Therefore, local structuring comprehends different levels of sophistication according to the complexity of the relationships that are established. That is, students recognize different kinds of complex units for structuring a shape, using different kinds of relationships, like repeating or symmetrizing composites. In local structuring, students show difficulties in coordinating composites.

For the latter, global structuring, one student showed to be able to coordinate properly different composites, as well as to recognize hidden cubes. This student used consistently quasi-equal composites, as units, to decompose all three different shapes, but he also showed ability to adjust the number of cubes in each composite to fit the shape. Global structuring implies that students have previous mental models that represent accurately the shape, its components and composites and that these parts are properly coordinated.

Either in local structuring or in global structuring, students were capable of suggesting two different forms of decomposition for each shape. However, it is only in global structuring that we can admit that students are able to coordinate in properly all the elements, considering overlapping elements, as mentioned by Spiegel and Ginat (2017), revealing a deeper understanding of shape’s structures. By establishing different relationships, students had the opportunity to deepen spatial relationships. However, we must also consider as a limitation the fact that the students did not have any previous lessons about how to draw a 3D shape.

Through this work, we expect to contribute to a deeper understanding of spatial structuring in geometric contexts, where spatial reasoning processes such as decomposition of 3D shapes play an important role. Also, we seek to reinforce the importance of learning experiences focused on decomposition, since early grades, as a fundamental process for developing spatial structuring and fostering different ways of thinking as Duval (1998) purposes.

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Identifying mathematics in the actions of very young preverbal children when playing with trains

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This paper will report on the mathematics that may be visible when very young preverbal children play with trains. The paper analyses video obtained using a 360-degree camera in an early learning setting in Sweden. The video involves two preverbal children playing with train carriages and train track at a table. The work of Barad (2008), Franzén (2015), and Meaney (2016) underpin the use of the preverbal child's actions to identify the potential mathematics that may be evident from these actions. The iterative process used to analyse the video is described. The analysis of the children's actions is linked to the six mathematical activities identified by Bishop (1991). A discussion of the effectiveness of using the 360-degree video and the iterative process is followed by implications for practice. Limitations of the research are considered.

Keywords: Very young preverbal children, actions, mathematical activities, 360-degree video.

Introduction

Research into early childhood mathematics is now considering children of younger ages, even from birth (Alsina & Berciano, 2018). Very young children, even preverbal children, demonstrate mathematics in their behaviours and actions. Franzén (2015) states that there are many instances where very young children engage with mathematics during self-selected activities in early learning settings. Further, that very young children can express mathematics through their use of their body (Franzén). Meaney (2016) proposes that researchers working with very young children need to recognise how the child's uses their body to convey their understandings. She describes her investigation of how two very young children (referred to in the paper as toddlers) use their bodies to when solving problems, connecting the children's actions to their mathematical understandings, specifically Bishop's (as cited in Meaney, 2015, p. 11) mathematical activity of locating. Children's actions are communicative of their mathematical ideas, with Flottorp (2011) linking these to Bishop's mathematical activity of explaining. Although the children were verbal, Flottorp stated minimal language was used during the experiences.

Bishop's (1988, 1991) mathematical activities

Bishop (1988, 1991) presents six mathematical activities that he describes as universal, due to their presence in all cultural groups and the way they support how mathematical knowledge develops. Counting involves ordering or comparing discrete phenomena systematically. Locating is the exploration and conceptualization of the spatial environment. Measuring comprises the comparison or ordering through quantifying qualities (in contrast to counting, which involves discrete phenomena). Designing incorporates creating designs or representations of the spatial environment and objects. Playing recognises creating, using, or adhering to rules (both formal and informal), whether in games or general activities. Explaining is the accounting of phenomena, whether physical or non-physical.

Although Bishop (1991) states his mathematical activities were created by examining language, their descriptions can also be applied to actions. Cooke and Jay (2021) described how Bishop's mathematical activities could be reframed to focus on the actions of very young children. Counting was reframed to consider matching one-to-one (such as using one hand to grasp one item), changes in quantity, and recognizing patterns in actions, music, or items; the provided example was a child using one hand to select one object from a collection. Locating reframed focuses on children using their bodies within their environment and to engage with their environment, such as moving objects or positioning themselves; the example was a preverbal child moving objects in a purposeful way to make them useful. Measuring reframed involves preverbal children investigating attributes within their environment, identifying how much there is, and cataloging these features to enable them to compare other objects and environments; a provided example was trying to reach an object and, when not successful, moving their body closer to the object. Designing reframed occurs when the young child can identify the function of objects and consider this in terms of the attributes of the object; the example provided involved placing toys with wheels on the ground with the wheels touching the ground (and not upside down). Playing reframed focuses on the joy children display or the process of replicating everyday happenings; the provided example involved a child picking up a book and turning it so they can open it and look inside. Explaining reframed is evident when children's engagement shows they are using previous understandings in current environments or with objects and situations; the example given involved a child holding a bookcase to pull themselves up to standing.

Using video to collect data

Lynch and Stanley (2018) describe how video can be a valuable data collection tool in qualitative research as it enables a holistic way collect events as they occur – capturing the physical, temporal, and social contexts that exist beyond language. They state how this can be invaluable when working with children. In addition, the capacity of the researcher to revisit and review the captured video enables researchers to be consistent in their analysis of the video. Björklund (2008) also highlights the capacity of video to enable researchers to meticulously observe the actions that have been recorded, and de Freitas (2016) links this to the opportunities that video provides to make the understandings of mathematical concepts visible. The camera for collecting video captures 360-degree audio video. This enables all of the room to be seen and heard. The video format provides the capacity to focus in on a particular part of the video.

Theoretical Framework

Several elements that Barad (2007) proposes underpin the research. The first is the de-prioritising of language. Barad states that language has been elevated to a position of power, gifting it the role of most trustworthy in the representations of reality. She proposes that language not be preferenced and that attention also be given to matter – comprised of human and nonhuman – as it is not immutable and passive. This is connected to the second element – intra-action. Barad considers everything – all matter – to be intra-active in its becoming. Specifically, "intra-actions are nonarbitrary, nondeterministic causal enactments through which matter-in-the-process-of-becoming is iteratively enfolded onto its ongoing differential materialization" (p. 179). Reality is conceived as continuous and dynamic, where people are part of the world and their intra-action contributes as both cause and effect to the world's becoming.

In their research with very young children, Franzén (2015) and Meaney (2016) both use the work of Barad (2007) to depower language and focus on intra-actions, enabling the child's actions with their bodies to be interpreted as demonstrating mathematics. Franzén (2015) uses Barad's work to widen how young children's learning is perceived. Specifically, Franzén incorporates all ways that the child intra-acts with the matter within the world, such as through their body, and uses this to identify the mathematics that may be evident. She recounts the intra-action of a very young child (aged one year) with a large climb-on toy car and connects this to the mathematics that may support the child's engagement. In a similar way, Meaney (2016) also focuses on how young children learn through using their bodies to interact with their environment. Meaney's focus is on how the very young children use their bodies to demonstrate understandings of locating, which is viewed as the understanding of spatial relationships, in their world.

The aspects of Barad (2007), Franzén (2015), Meaney (2016) that are critical to this research are the emphasis placed on the child's actions, the recognition of the intra-active nature of the child's engagement with their world, the reduced focus on language to convey mathematical ideas, and the deconstruction of the child's actions to identify the mathematics that may support their intra-actions. These aspects enable the actions of the very young preverbal children, who are the focus of this research, to be analysed to identify the mathematics the children use when engaging with their world. In doing so, the child's agency is recognized. That is, they are able to act as they wish to act and to have the freedom to make choices (Lange, 2010).

Research question

As indicated by the work of Franzén (2015) and Meaney (2016), very young children, even preverbal children, can demonstrate mathematics through their actions. This paper considers what mathematics very young preverbal children might demonstrate through their actions when playing with train carriages on a train track without educator engagement. Specifically, the research question is – What might the actions of very young preverbal children show in terms of Bishop's (1988, 1991) six mathematical activities when they are engaging in self-selected play with toy train carriages and a train track?

Methodology

Before entering the learning centre, information about the project was provided to the manager, the educators, and the parents of the children and consent to participate was requested. A camera capable of capturing 360-degree video was attached to the ceiling of the room, making it unobtrusive to those in the room. The camera recorded whilst the children and educators were in the room. A section of video is analysed from a video that is 34 minutes and 19 seconds long.

Context and participants

Two educators and three children were in the room. All three children were female and one year in age. Two of the three children did not use verbal language and the third has minimal verbal language. The two educators interact with the children, but the children had choice in what they did – that is, the activity was self-selected by the children. The educators' interactions involve speaking with the children, asking the children questions (related to what the children were doing), and responding to the children when they approached the educator, for example, handing the educator toys.

The section of the video that is analysed is 31 seconds in length. The two preverbal children had chosen to play at the train table and, although the educator was present, the educator did not engage with or interact with the children during this section of the video. The educator is watching the children and her only involvements are placing a train carriage on the train track after it had fallen to the floor (Child 1 had picked up two train carriages joined together, holding only one of the carriages, with the second train carriage falling to the floor); helping one child walk between her and the train table (holding the child steady as she passes by placing her hands on the child's sides); and moving the table and keeping a hand on it to hold it in place when one of the children tries to pass between the other child (sitting on the floor) and the table. The section starts when the educator places two trains on the train track then sat back to watch the two preverbal children play. Initially, Child 1 plays with the two train carriages the educator has placed on the train track and Child 2 plays with three train carriages on the train track. The video section ends when Child 2 walks towards the educator and hands her two train carriages.

Analysis

An iterative process is used to identify the actions of the child and then to identify which of Bishop's (1988) mathematical activities may have been evident through the children's engagement with their environment. The section of video was initially viewed in its entirety and a broad description of what occurred in the video was created as a frame. Then, each child was the focus for the subsequent viewings. The second viewing enabled details of the child's actions to be gathered. During this second viewing, the video was stopped frequently and reviewed to enable the detailed notes to be written. After the detailed notes were written, the video was viewed a third time, again in a stop-start manner, to enable the detailed notes to be checked against the video. A fourth viewing also followed the stop-start manner, but longer sections of notes were compared. A final viewing occurred to compare the full notes with the video, this time without stopping during the viewing. Narratives were created from the notes for each child that could be synchronized. The process of placing the narratives side-by-side to be synchronized with each other involved identifying actions of each child that occurred at the same time. Spacing was then used to clearly indicate which of the actions described for each child occurred at the same time.

Once the iterative process to create the detailed notes was complete, the actions in the notes were compared to the detailed descriptions Bishop (1991) provides for each of the mathematical activities and the ideas provided in Bishop (1988) to identify when the actions reflect the mathematical activities. As Bishop (1991) notes, it is possible that there will be instances where more than one mathematical activity is evident. The author has used this process with videos of young children for seven years with preservice early childhood educators as part of their work in early childhood mathematics education units. Work with colleagues interpreting videos of preverbal children has occurred over four years.

Results

The synchronized narratives for the two children are presented side-by-side in Figure 1.

Child 1 picks up the train carriages (as they are joined together), with the white train carriage in her right hand then drops the red train carriage to the floor (potentially just a result of only holding on to the white train and swinging her arm as she walks) and walks around the table to place the white train carriage on the train track curve at the end. Child 1 doesn't release the train, and takes it with her as she moves away from the train track to walk around the table.	Child 2 has three connected train carriages in front of her, placed on the corner of the table and not the train track – she has her left hand on the left-most train carriage and her right hand on the middle train carriage and is turning the carriage in the middle trying to reposition it (as it is upside down). The right-most train carriage detaches as Child 2 twists the middle train carriage – she looks briefly at the carriage that has detached then back at the two train carriages that are still joined (with her left hand on the left-most one and her right hand on the right most one).
Child 1 walks around Child 2 to the other end of the table. She places her left hand on the table with her fingers splayed. The train carriage is still in her right hand. Child 1 places the white train on the track. It is positioned sideways (the front and back off the track) and starts pushing it around the train track, even though it is not placed correctly.	With her left hand on the first train and her right hand on the second train carriage, Child 2 drives the two train carriages on the table corner in front of her. Child 2 stops and manipulates the second train carriage with her right hand, then starts driving them on the table again, following the curve of the train track (but driving on the table, not the train track). When Child 2 gets to the end of the table, the two train carriages tumble to the floor (separating as they do so).
Child 1 squeezes between the educator and the table, taking the train carriage off the track, and then walks to the other curved end of the train track. Child 1 places the white train carriage on the table next to the train track but slides it partially on the track (at right angles again) and proceeds to drive it around the curve of the train track. Child 2 is in the way (sitting on the floor), so Child 1 partially steps over and partially sits on her to get past while keeping the train carriage on the train track and her left hand on the table (the educator also moves the table slightly to enable the child to fit through).	Child 2 looks down at the train carriages on the floor then reaches simultaneously with each hand to pick up the two train carriages (the right hand goes to the train slightly under the table and the left to the train further away), crouching and bending to reach them. Child 2 fumbles then drops the train carriage in her left hand when it gets closer to her body, so she uses both hands on the other train carriage (she is still crouching). Child 2 briefly looks at Child 1 who is driving a train carriage around the train track and getting closer to her, then passes the train carriage to her left hand and uses her right hand to pick up the train carriage from the floor. Child 1 slightly knocks her as she is picking up the train carriages. Child 2 ends up with one train carriage in each hand. Child 1 bumps her again and Child 2 falls forward and to her left.
Child 1 takes the train carriage off the train track, placing her left hand on the table as she goes around the corner and down the long side. When Child 1 gets to the hill in the train track (roughly the middle of the track/table), she puts the train carriage on the train track and holds the train track with her left hand. Child 1 swings both hands, so the train carriage is going forward and back on the train track in her right hand and her left hand is going forward and back, with the train carriage and her left hand hitting together several times. While Child 1 is doing this, she is leaning her stomach against the table, which makes it move slightly away from her. Child 1 places the train carriage on the table on the other side of the train track and removes her right hand and raises it, while her left hand is on the table. Child 1 reaches forward again with her right hand and drives the train carriage on the table, then closer to the grey tunnel.	Child 2 uses her hands to help push herself up to stand, with the train carriages still in her hands. Child 2 lets go of the train carriage in her left hand and it drops to the floor. She reaches for the train carriage, looking like she is trying to reposition her grip. She picks up the train carriage with her left hand and straightens to stand. Child 2 walks back towards the train table and moves past it to where the educator is at the other end of the train table. She still has one train carriage in each hand. As Child 2 gets closer to the educator, she stretches out her hands to give the train carriages to the educator (one in each of her hands).

Figure 1: The narratives for the two children side-by-side

Bishop's (1988, 1991) mathematical activities evident in the children's actions

Each of Bishop's (1988, 1991) mathematical activities were evident in the children's actions. There are actions where more than one mathematical activity is evident (Bishop, 1991, p. 108). Example actions for each mathematical activity are listed below:

- Counting

Child 1 demonstrates one-to-one correspondence by using one hand to hold one train carriage. Likewise, Child 2 demonstrates one-to-one correspondence when she holds one train carriage in her left hand and one train carriage in her right hand. Child 2 also shows ideas around quantity when the third train carriage is detached from the other two – she looks at the one that had detached then places one hand on each of the remaining train carriages.

- Locating
  Both children demonstrate their capacity to orient themselves in space. Child 1 walks around the table, places the white train carriage on the track, and drives the train carriage held in her right hand forwards and back to meet her left hand engaging in the mirror action. Child 2 positions her body lower to pick up the two train carriages that have fallen on the floor. She also uses her right hand to push herself up to standing.
- Measuring
  Child 2 demonstrates measuring when she reaches her hands out to pick up the two train carriages after having crouched down to do so. This indicates that she recognizes that her hands were too far away from the train carriages initially, and that she has to change position and reach to be close enough to them. Child 1 demonstrates measuring when she tries to lift her leg and foot high enough to step over Child 2 when she is walking around the table.
- Designing
  In rolling the train carriages on their wheels to perform a 'driving' action, both children are showing their understanding of the properties of the train carriage (that it can be 'driven' on the surface). Child 2 shows this when she is trying to manipulate the middle train carriage to turn it over onto its wheels as it is the wheels that 'roll'. Child 1 demonstrates designing through the templating of the table to ensure she can move around it while 'driving' the train carriage on the train track.
- Playing
  Both children follow the procedures for placing the train carriages on the train track by putting them wheels down. Although Child 1 places the train carriage incorrectly on the train track several times, she 'drives' it around the track, walking around the table as she does so (removing the train carriage from the train track when she cannot reach). Child 2 uses her past experience in how to hold the train carriages to change her grip to increase her chance of picking the train carriage up from the floor.
- Explaining
  The children show their understanding of the conventions of train carriages by placing them wheels-down when they are 'driving' them on the table or on the train track. Likewise, although there are other objects on the train table, the children only 'drive' the train carriages on the train track, showing they know which items are classified in the category of 'driving on the train track'. Child 2 uses gestures to explain that she would like the educator to take the two train carriages by holding them out towards the educator.

Discussion

The 360-degree video makes it possible to identify opportunities where the very young preverbal children were engaged in self-selected activities (Lynch & Stanley, 2018). The positioning of the camera on the ceiling enables the collection of video that shows the full room, making visible a range of activities and levels of children's agency – their choices (Lange, 2010) – to be viewed. The format of the video from the 360-camera also provided opportunities to focus in on specific activities. The captured video enabled the use of an iterative process through which to analyse the actions of the children (Björklund, 2008). Without this iterative process, it would have been more difficult to observe both children and to note down sufficient detail to compare to Bishop's (1988, 1991) mathematical activities and, in effect, make the mathematics visible (de Freitas, 2016).

As Meaney (2016) noted, even though very young children do not reflect on or share their learning using language, they are capable of showing what they know through their actions. The actions of the two very young preverbal children in self-selected play with train carriages and a train track can be used to identify mathematics in terms of Bishop's (1988, 1991) mathematical activities. This is made possible through the deprioritizing of language and the focus on intra-actions (Barad, 2007) to enable

Conclusion

Even though these very young children are preverbal, they both demonstrate each of Bishop's (1988, 1991) mathematical activities and do so without language. This underscores the importance of not preferencing language when looking for ways that very young children may engage with mathematics, much as Franzén (2015) demonstrates with her identification of the mathematics the young child's engagement with the climb-on toy car. Likewise, the children's spatial awareness, both in terms of the positioning of the physical objects and in their use of their bodies to engage with the environment, reflects the capacities Meaney (2016) shared.

This research has limitations. The interpretation of the children's actions and the identification of the mathematics behind their actions cannot be checked with the participants themselves. Franzén (2015) states that researchers need to use children's actions to try to identify children's intentions, but that this may be limited due to the differences in perspective between the researcher and the child. This can occur in all research but is very much present with preverbal children – those who are the focus of this paper. In some ways, it is similar to the issue Bishop (1988, p. 184) expresses regarding his work, that "there is no real prospect of my being able to test whether or not this 'universal' structure will be adequate for describing the mathematical ideas of other cultural groups" – however, he did not consider this a weakness. Bishop (1988) hopes that his analysis would encourage others to engage in similar analysis, an aim that this research would do well to emulate.

Franzén (2015) discusses how the researcher who interprets the actions of young children holds much power. As part of this power, the researcher needs to recognize they are interpreting what they are seeing from a perspective that includes both their own and the child's. Franzén suggests that the researcher needs to accept that there will be ambiguity in interpreting the preverbal child's actions, but in providing an interpretation of the preverbal child's actions, the researcher may be giving voice to these very young preverbal children. However, it is the researcher's interpretations of the actions and, even with vast experience, there is no opportunity to confirm the interpretations are correct. Further investigations with more children may assist in the veracity of these interpretations and the connections of the actions in terms of Bishop's (1988, 1991) mathematical activities.

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An analysis of a kindergarten teacher's choices and justifications regarding teaching of a mathematical activity

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This paper aims to increase the insight into kindergarten teachers' justification of their teaching. The study was done in a context with mathematical activities developed by the researchers in collaboration with the teachers. This is a context we consider as rich to reveal their choices and justifications made in advance of and during the teaching of a mathematical activity. Data from a semi-structured, post-teaching stimulated recall interview based on an activity with 5-year-old children was analysed employing the Knowledge Quartet proposed by Rowland et al. (2005). We identify seven choices made by the kindergarten teacher that reveal aspects of the kindergarten teacher's knowledge-in-action and knowledge-in-interaction. She adapts the activity through questions, comments, particularities within the activities, and use of artefacts. Thus, she nurtures the children in their mathematical learning process.

Keywords: Knowledge Quartet, Mathematics teaching in kindergarten, Teacher's choices, Teacher's justifications.

Introduction

The aim of the current study is to reveal insights regarding the choices kindergarten teachers (KTs) make during the teaching of mathematical activities in kindergarten and the justifications they provide with respect to these choices. Research has shown that high quality early childhood programs carry the potential of improving children's early learning. Moreover, intervention programs have shown to positively effect children's learning of mathematics (Clements & Sarama, 2011; Stehler et al., 2013). The current study draws on data collected as part of the research conducted within the Agder Project[1] (AP). The AP is a research and development project, designed as a randomized control trial with an experimental group and a control group, with the intention to even out differences between children as they enter school. The AP thus bares characteristics of an early childhood intervention program. In the project focus was on nurturing 5-year-old children in their development within four competence areas (social skills, self-regulation, literacy, and mathematics) that research has shown to significantly contribute to children's school readiness. In the AP, researchers designed mathematical activities in close collaboration with participating kindergarten teachers (KTs); activities that the KTs taught with the group of 5-year-olds from their own kindergarten.

The mathematical activities were designed based on two main principles, playful learning (Hirsh-Pasek et al., 2009) and inquiry (Wells, 1999; see also Breive et al., 2018 for further details with

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respect to the design). The current study thus takes as point of departure these two principles when researching the teaching of mathematical activities in the kindergarten context. Research has documented that emphasis on playfulness in engaging with mathematics in the early years is particularly important with respect to long-lasting effects (Marcon, 2002; Singer et al., 2009). Research has also documented that adopting an inquiry approach to mathematics learning is promising regarding children's learning of mathematics (Breive, 2019; Wells, 1999)

A huge amount of research has been conducted with respect to mathematics teachers' knowledge regarding mathematics and mathematics teaching. These studies are drawing on well-known frameworks such as Mathematical Knowledge for Teaching (cf. Ball et al., 2008), the Teaching Triad (cf. Jaworski, 1994), and the Knowledge Quartet (cf. Rowland et al., 2003; 2005). However, the number of studies analysing kindergarten teachers' practice is more scarce. Mosvold et al. (2011), drawing on the framework of Ball et al. (2008), analysed mathematics teaching in Norwegian kindergartens. They found that they needed to adjust the theoretical framework slightly to make it fit with the kindergarten setting. Breive (2019) also analysed mathematics teaching in a Norwegian kindergarten setting, and she revealed insights into the subtleties of KTs' practice. However, these studies drew on observational data to reveal insights into the KTs' mathematics teaching. We, on the other hand, draw on interview data associated with a taught mathematical activity to reveal glimpses into one KT's choices and justifications for her choices in the activity. Through this approach, we seek a more personal view upon mathematics teaching in kindergarten. Our approach is to some extent in line with the study of Sæbbe (2019), as he, building on the work of Ball et al. (2008), also drew on interview data immediately following observations of mathematics teaching in kindergarten. Nevertheless, our foci on choices and justifications for choices deviate from Sæbbe's, as he discussed whether KTs' practice may be characterised as teaching, whether it is mathematical, and the challenges and demands for KTs' competence. Moreover, our study deviates from Sæbbe's as we adopt a complementary, theoretical framework in our study, the Knowledge Quartet (Rowland et al., 2003; 2005). As such, our study is theoretically in line with Hundeland et al. (2017), who also drew on this quartet in their analysis of a KT's practice. To our knowledge, the Knowledge Quartet has not been used as an analytical framework to analyse interviews before. Thus, our study represents a novel attempt to scrutinise a KT's choices and justifications for her choices in the activity.

Previous studies have shown that substantial insights into KTs' accounts of their teaching practice may be revealed through interview data (Erfjord et al., 2012; Hundeland et al., 2011). The current study draws on this experience as interview data from one volunteering KT from the AP has been analysed. This KT was selected based on the following procedure: The 42 participating KTs in the experimental group were, during the intervention year, invited to accept an invitation from the team of researchers who are the authors of this paper as well as teachers in the professional in-service program the KTs were part of. 12 of the KTs, randomly selected from the 42, accepted the invitation to be observed during their implementation of designed activities as well as a post-teaching interview. The particular KT in this paper was selected from one of these 12, at no other basis than an initial judgment that the activity was conducted as planned (that nothing unfortunate happened like sickness, external need for change in plans etc.). The interview of this selected KT has been analysed in order to address the following two research questions:

1. What choices does a kindergarten teacher make in her teaching of a mathematical activity?
2. What justifications does the kindergarten teacher provide regarding her choices in teaching the mathematical activity?

The KT taught an activity called 'The secret bag'. An analysis of the mathematical discourse emerging in this activity is communicated elsewhere (Hundeland et al., 2020). The activity encompasses reasoning with respect to two-dimensional and three-dimensional geometrical shapes, firstly displayed for all and secondly located in an in-transparent fabric bag. The KT and the children first discussed features of and connections between these shapes. Then, having the shapes into the bag, one child at a time tactilely felt and reasoned what shape the picked one was.

The Knowledge Quartet (KQ)

Rowland et al. (2005) describe mathematics and mathematics teaching and the mathematics teacher's subject matter knowledge regarding these areas through four dimensions. These dimensions are called Foundation, Transformation, Connection, and Contingency, hence the label 'The Knowledge Quartet'. This label not only signifies that there are four dimensions of mathematics teachers' knowing, but also that these dimensions are intertwined – as a quartet. These dimensions emphasize the situations during mathematics teaching in which one may observe the mathematics-related knowing of the teacher.

Foundation is the dimension directed towards the use of propositional knowledge. That is, in the current study in what ways the KT, in her interview responses, reveals relevant knowledge of mathematics and mathematics education as well as her view regarding the goals of mathematics education and ways children appropriate mathematics. The analytical contributory codes used were: "awareness of purpose; identifying errors; overt subject knowledge; theoretical underpinning of pedagogy; use of terminology; use of textbook; reliance on procedures" (Rowland et al., 2005, p. 265).

Within the KQ there are two dimensions addressing knowledge-in-action. *Transformation* refers in the current study to the KT's choices of representations, demonstrations, and use of examples, e.g. geometrical shapes, and how these are revealed in interview responses. Basically, this dimension concerns in what ways the KT shows evidence of appropriately transforming the mathematics for the children's learning. The analytical contributory codes used were: "choice of representation; teacher demonstration; choice of examples" (Rowland et al., 2005, p. 265). *Connection* refers in the current study to how the KT makes connections between the activity involved mathematical concepts and procedures, i.e. features of geometrical shapes and their relations, as well as ways to challenge the meaning of these concepts and ideas. The analytical contributory codes used were: "making connections between procedures; making connections between concepts; anticipation of complexity" (Rowland et al., 2005, p. 265).

The fourth dimension is called *Contingency* - a dimension of knowledge-in-interaction. This dimension encompasses the nature of adult-child interactions and responses, and in the current study as how the KT responds appropriately, take advantage of emerging opportunities for learning, make the activity her own, and to what degree she deviates from her set foci and goals. The analytical contributory codes used were: “responding to children’s ideas; use of opportunities; deviation from agenda” (Rowland et al., 2005, p. 266).

According to Rowland et al. (2003), “the quartet is comprehensive as a tool for thinking about the ways that subject knowledge comes into play in the classroom” (p. 97). Nevertheless, we will analyse the KT’s answers in a post-teaching stimulated recall interview and align her responses with the relevant dimension(s) and associated codes.

Semi-structured interviews

Semi-structured interviewing is a commonly used method of data collection (Bryman, 2016; Kvale, 1996). Characterising features of semi-structured interviews are (non-exhaustive): 1) A list of questions is prepared; 2) The respondent may elaborate the response; 3) There is no strict order of the questions; 4) New, non-prepared questions may be asked; and 5) All prepared questions are asked.

We are aware of the deficits of qualitative interviews when it comes to trustworthiness, as people tend to rationalise on their thinking rather than trying to tell what they are/were thinking in interview settings. However, one of the authors both observed the taught activity as well as conducted the interview immediately after the activity. Extensive collaboration between the researcher(s) and the KTs was also established in the AP. These aspects add to the trustworthiness of the KT’s answers with respect to choices made and the justifications for these choices.

In the current study we particularly analyse a post-teaching stimulated recall interview. Designing our interview in this way is in line with how Rowland et al. (2005) designed their interview when developing the framework of the Knowledge Quartet. Furthermore, we designed our interview paying close attention to the codes behind the four dimensions of the Knowledge Quartet. Thus, we were formulating our questions aiming to address the particular codes in order to implicitly address the four dimensions. For example, we asked the question “What was the intention behind the activity?”, to address the code ‘awareness of purpose’.

Context and participant

The current study analyses a semi-structured interview with the KT, immediately following her teaching of a mathematical activity, ‘The secret bag’. The activity was designed based on the principles of playful learning and inquiry. The written description of the activity encompassed an intention of the activity with respect to participating children’s possibilities for making mathematical experiences, suggestions for preparations, and suggestions for how to implement that activity including mathematical questions and prompts to use.

Results

Despite the collaborative design of the mathematical activities, including explicit intentions and suggestions for implementation of the activity, the KT was left with a lot of freedom regarding how to teach the activity in detail. Each interview was analysed based on the following procedure: Initially, as mentioned above, the interview guide was collaboratively developed by the three researchers adaptive to the codes behind the four dimensions of the KQ. The conducted interviews were transcribed, and individually we analysed the transcripts, marked the statements and selected illustrative statements for the choices made. After the individual analysis, we met, shared and discussed our outcomes and found that we strongly agreed on the outcomes and the respective statements. Finally, the statements were translated to English and added to our analysis section. From this analysis of the post-teaching stimulated recall interview, we identified seven novel choices that the KT made prior to and during her teaching. We have grouped them according to the dimensions of the KQ.

Choices and justifications associated with Foundation and Connection

With respect to these dimensions, we asked questions about the intentions behind her activity, whether the intentions were achieved, her emphases on making connections between involved mathematical concepts, decisions about sequencing. The KT communicated the following choices:

1. She wanted to promote a particular mathematical focus and mathematical learning purpose of the activity.
2. She was conscious of the design and order of elements engaged with in the activity.
3. She was conscious of what mathematical questions to ask each of the children.

Even though the written description of the mathematical activity encompassed an intention, the KT explained that she made explicit choices in line with her intention. With respect to 1): "I wanted to emphasize a mathematical discussion concerning the features of the geometrical shapes"; "I wanted to let the children come up with answers, to give them time to think, to let them philosophize"; "I have twisted the activity a bit, made it my own". With respect to 2): "I wanted the children first to engage with the geometrical shapes that I brought, and then for them to recognize geometrical shapes in their environment"; I wanted the children to experience the shapes in two different learning arenas". With respect to 3): "There are substantial differences amongst the children. Thus, I do not ask them questions with equal difficulty"; "If I am to include mathematics into the children's play, it is important to ask the good questions". These choices testify that 1) the KT consciously decides and is aware of the mathematical learning purposes of the activity; 2) the KT consciously decides on the organisation of the activity; 3) the KT consciously decides on the mathematical foci through her questioning; and 4) the KT anticipates the mathematical complexity involved in the activity. Thus, we argue that these choices exemplify an operationalisation of the dimensions of Foundation and Connection. The KT draws on her propositional knowledge both with respect to mathematics and mathematics education and establishes connections between the purpose of the activities and the children by adapting the inherent mathematical difficulty.

With respect to the first choice above, the KT argued that she wanted to nurture a discussion amongst her children concerning the characteristics of two-dimensional and three-dimensional shapes. Furthermore, she wanted her children to use visual, auditive and tactile senses to make mathematical experiences from the activity. Concerning the second choice, the KT argued that she followed the written activity quite closely, but also that she had prepared extra material for the children to engage with. She argued that her activity was flexible in its own right, offering her possibilities to both make it easier and to develop the activity further. Justifications for the third choice were as follows. The KT argued that the mathematics questions ought to be adapted to each child's level of competency and that it is important to know each child's competency in order to ask appropriate questions.

Appropriate questions are furthermore needed to make the activity successful in terms of participation and mathematical learning opportunities.

Choices and justifications associated with Transformation

With respect to this dimension, we asked questions about how she adapted the activity to suit the children's mathematical experience, her reasoning with respect to how successful she was in adapting the mathematics, explanations, and use of manipulatives. The KT communicated the following choices:

4. She was conscious of how to engage each of the children in the mathematical activity.
5. She was conscious of how to use manipulatives and other mediating artefacts.

Both these choices signal that the KT is consciously aware of how to best communicate with her children. The KT is careful in how she mathematically approaches each of the children, and she adapts her questions and prompts to each child based on her particular insights regarding each child. "I wanted all children to be able to participate in the mathematical activity"; "There are many ways to engage the children in the mathematics". Furthermore, the KT reveals that she made conscious choices with respect to the various manipulatives she used in the activities, both particularly mathematical manipulatives and other mediating artefacts such as what shapes and number of shapes to include. The KT was also attentive to how she used mathematical language, use of pointing gestures, and displaying artefacts. "I used the fabric bag to introduce some mystery, something exiting, and I used a balloon and some pictures". These choices demonstrate, we argue, that the KT's taught activity was revealing the dimension of Transformation.

Regarding the fourth and fifth choice, the KT argued that all children were actively engaged in the activity. She sought to make them curious about the shapes and used the shapes of artefacts in the room, a brought artefact, and the small plastic shapes to let the children "experience the shapes in two different learning arenas".

Choices and justifications associated with Contingency

With respect to this dimension, we asked questions about how the KT responded to and took advantage of the children's various contributions, whether and why she deviated from her plans for the activity. The KT communicated the following choices:

6. She made choices regarding how to communicate with each of the children.
7. She was attentive to children's comments and made deviations from plans and acted in the moment.

These two choices demonstrate that the KT is particularly aware of how she communicates with the children, especially to what extent and how she responds to the children's questions and initiatives during the activities. "My plans are rarely executed fully. It's about being spontaneous. The children steadily discover new things". "I wanted to let the children come up with answers, to give them time to think, to let them philosophize". The KT was also explicit about the various deviations she made due to the children's contributions, and she explained that she had to act in the moment according to the emerging issues. "I planned to use a game with shapes, made copies for each child. But then I experienced that the mathematical conversation was running smoothly. Introducing the game would then just disturb them"; "I had to make one the children sit on my lap"; "I saw that the children were engaged in the activity. But after 30 minutes I realized that it was time to finalize".

With respect to the sixth choice, the KT emphasized to let the children come up with the answers, to let the children philosophise. She also gave the children time to think for themselves for a while, for them to come up with answers. As regards the seventh choice, the KT argued that it was important to her to follow up on the children's philosophical-mathematical questions. She wanted to discuss what the children were mathematically occupied with, to be spontaneous and flexible and react to the mathematically unexpected. However, she also took actions on the spot with respect to her leadership of the activity.

Discussion

This paper aims to increase the insight into KTs' justification of their teaching. The study was done in a context with mathematical activities developed by the researchers in collaboration with the KTs, a context we consider as rich to reveal their choices and justifications. We set out to come up with answers to the following two research questions: What choices does a kindergarten teacher make in her teaching of a mathematical activity? and What justifications does the kindergarten teacher provide regarding her choices in teaching the mathematical activity? Findings suggest that the revealed justifications for the teaching choices made, give insights into this KT's knowledge-in-action and knowledge-in-interaction.

From the analyses of the post-teaching stimulated recall interviews, we argue that the KT made deliberate choices with respect to her teaching of the mathematical activity as well as justifications for these choices. She chose the particular mathematical learning goals for the taught activity, she chose to ask particular mathematical questions and raise mathematical issues for the children to wonder about. She chose various mathematical artefacts and talked attentively with each child in a deliberate way. All these choices were thoroughly justified.

Based on the KT's revealed justifications for the teaching choices made we argue that the KT made the activities her own, in a way featured by the four dimensions in the Knowledge Quartet (Rowland et al., 2003; 2005). She adapts the activity through questions, comments, particularities within the activities, and use of artefacts. Thus, she nurtures the children in their mathematical learning process.

These insights reveal that the Knowledge Quartet is applicable as an analytical framework for analysing post-teaching stimulated recall interviews of kindergarten teachers. Furthermore, through our elaborations of the KQ dimensions and associated codes in our analyses, significant insights are implicitly revealed regarding a kindergarten teacher's teaching of mathematics and the issues she has to handle in situ to make the participation in the activity a nurturing mathematical learning process for the involved children.

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Opinions of prospective elementary school teachers on word problems in mathematics

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Word problems are an important part of learning and teaching mathematics. However, they differ, among other things, in the extent to which they are related to reality. From this point of view, we distinguish intra-mathematical word problems, "dressed up" word problems, and modelling problems. Our research revealed that prospective teachers of lower elementary grades (N = 81) consider intra-mathematical problems and "dressed up" problems more suitable for use in mathematics instructions than mathematical modelling problems. In addition, prospective teachers would like to include the mentioned two types of word problems to a greater extent in their teaching of mathematics than modelling problems. Additionally, we will show what advantages and disadvantages they identify for each of the presented word problems.

Keywords: "Dressed up" problem, intra-mathematical problem, mathematical modelling, prospective elementary teacher, word problem.

Theoretical framework

Word problems help prospective teachers understand the role of mathematics in reality and prepare them to apply mathematical skills in everyday situations. We know that there is a wide range of mathematical tasks used to apply mathematical models and procedures to reality, ranging from simple word problems to complex modeling problems (Dröse, 2019). In addition, word problems can deepen and broaden prospective teachers' knowledge, improve their logical and critical thinking skills, and enhance their creativity. Moreover, they can provide prospective teachers with the opportunity to apply their mathematical knowledge to complex real-world problems, as they can work at their own pace and decide for themselves how to approach the problem (see, e.g. Van de Walle et al., 2016; Verschaffel et al., 2020).

One of the best-known characteristics of word problems is how they are connected to reality (Krawitz & Schukajlow, 2018; Rellensmann & Schukajlow, 2017). From this perspective, there are generally two types of problems: problems connected to reality, real-world problems, and problems without such a connection, also called intra-mathematical problems.

Real-word problems describe a reality-based problem situation involving objects from the reality, such as objects from nature and everyday life. Many authors distinguish two types of real-world problems: modelling problems and "dressed up" word problems (Rellensmann & Schukajlow, 2017; Krawitz & Schukajlow, 2018).

The basis of modelling problems is a demanding translation process between reality and mathematics (Blum et al., 2007; Krawitz & Schukajlow, 2018). Solving modelling problems is often described as a cycle of activities that begins and ends with a real-world situation (Galbraith & Stillman, 2006; Blum & Leiß, 2007). According to Blum and Leiß (2007), this process consists of the following seven steps: (1) understanding the problem and constructing an individual ”situation model”; (2) simplifying and structuring the situation model and thus constructing a ”real model”; (3) mathematising (translating the real model into a mathematical model); (4) applying mathematical procedures to derive a result; (5) interpreting mathematical result in terms of reality and attaining a real result; (6) validating the obtained result concerning the original situation (if the result is unsatisfactory, the process may start again with step 2); and (7) exposing the whole solution process. Note that prospective teachers' processes in mathematical modelling are usually not linear but jump back and forth between mathematics and reality several times (see, e.g. Borromeo Ferri, 2007).

“Dressed up” word problems are also connected to reality, but in these problems, the reality-related mental activities are much simpler than in modelling problems. This is because a real model, which is simplified, is already included in the description of the problem. Actually, “dressed up” word problems are just mathematical problems to which we add a figurative context related to reality. Consequently, prospective teachers do not have to structure and idealize the given information, and interpret and validate the mathematical results according to the real-life situation. In addition, “dressed up” word problems do not contain redundant or missing data. This means that prospective teachers do not have to make assumptions about missing data and separate important from unimportant information, which is a challenging property of modelling problems. Finally, from the teachers' point of view, validating the result of a “dressed-up” problem is much easier than in the case of modelling problems since it is mainly limited to checking the mathematical part (Krawitz et al., 2016; Schukajlow et al., 2012; Krawitz & Schukajlow, 2018).

Intra-mathematical problems are those which have no relation to reality. They are, therefore, pure mathematical word problems. Consequently, these problems do not require any reality-related mental activities (Krawitz & Schukajlow, 2018).

Note that the differences between these problems arise from the cognitive processes required to solve these problems (Blum & Leiß, 2007; Galbraith & Stillman, 2006; Schukajlow et al., 2012). All three types of problems have their own advantages and disadvantages, depending on their purpose. All types of problems require mathematical-technical skills, whereas modeling problems and “dressed up” word problems (but at a very low level) also require cognitively demanding translation processes (Krawitz & Schukajlow, 2018). Consequently, all three problem types have their characteristics and of course, they are all essential for learning mathematics.

Since word problems have many other characteristics, for example, linguistic characteristics, numerical characteristics, and the interaction between linguistic and numerical characteristics (Daroczy et al., 2015), it is not easy to determine whether or not each word problem is appropriate in a particular situation.

Particularly for teachers with little experience, it is often difficult to select or create a suitable word problem (Luo, 2009; Simon, 1993; Lee & Kim, 2005). For example, Luo (2009) found out (N=127) that a significant percentage of the prospective teachers from the United States could not construct appropriate word problems for the given symbolic expressions of fraction multiplication. Simon (1993) also found that 70% of the prospective elementary teachers in his study could not form an appropriate word problem for fraction division. In addition, Chapman (2012) has highlighted that studies of prospective elementary mathematics teachers have raised issues about their problem-solving knowledge, suggesting potentially matters related to their problem-posing knowledge.

Leading educators and researchers have proposed some guidelines to determine the level of appropriateness of a problem in a given situation (e.g. Cathcart et al., 2000; Hyde & Hyde, 1991; NCTM, 2000, as cited in Lee & Kim, 2005; Van de Walle, 2001) that can help teachers in selecting and creating word problems. Appropriate problems should be taken out of reality; should be recognised by the prospective teachers as meaningful and valuable; includes significant mathematics; integrates multiple topics and makes connections between different mathematical fields; requires justifications and explanations for answers and methods; is solvable in many ways; generates student interest in working on it; sometimes contains missing, redundant, or contradictory information; the problematic or exacting aspects of the problem must be related to the mathematics the prospective teachers are supposed to learn (e.g. Cathcart et al., 2000; Hyde & Hyde, 1991; NCTM, 2000, as cited in Lee & Kim, 2005; Van de Walle, 2001).

Purpose of a research

This study aims to investigate prospective elementary school teachers' views on different mathematical word problems. We were particularly interested in their opinions about the various word problems depending on their connection to reality. In addition, we want to examine what are, in their view, the advantages and disadvantages of a given word problem.

Methodology

The quantitative empirical pedagogical research methods we used are descriptive and causal non-experimental methods.

The study has been conducted based on a survey of 81 prospective elementary school teachers, more precisely 4th and 5th-year prospective teachers, since the study for an elementary school teacher lasts five years at the University of Zagreb, Croatia, after which they receive a master degree. Printed surveys were distributed to prospective teachers after the lecture, and all 4th and 5th year prospective teachers completed the surveys. The survey was conducted in June 2020. The data obtained from the surveys were analyzed using IBM SPSS Statistics 23. Here we list only a part of the results from the described study.

The anonymous survey included questions about the prospective teachers' year of study and attitude towards mathematics and some questions about word problems. The latter refers to word problems in general and to specific word problems proposed by us (intra-math problems, “dressed up” problems, and modelling problems). In the survey, we presented prospective teachers with six word problems, and in this paper, we will show the results for three of them. For each word problem, prospective teachers had to list some advantages and disadvantages and answer whether they thought the problem was suitable for lower grades of elementary school. In the last part of the survey, prospective teachers had to rank the given word problems according to the level of difficulty and according to what extent they would include a particular problem in their math classes. We also asked them if they felt competent enough to create word problems.

The word problems, which were given to the prospective teachers, were the following.

Word problem 1 (intra-mathematical word problem): *The addend is 14 400, and the augend is 16 500. What is the sum?*

Word problem 2 ("dressed up" word problem): *Carlo wants to buy a car that costs 18 600 euros. Can he buy that car if he only has 16 000 euros in his bank account?*

Word problem 3 (modelling problem): *Carlo and his mother were shopping for a car. Carlo wants a car that will be fun to drive, does not consume a lot of gasoline and is not too expensive. On the other hand, Carlo's mother, who will help pay for the car, wants the car to be reliable and safe. Your job is to make a list for Carlo and a list for his mother that shows which cars are best for them. Then they will have to decide which one to buy! Car information is given in the table below.*

Table 1: Car information

Car	Year	Price	Colour	Mileage	Fuel consumption per 100km	Additional equipment
Nissan Juke	2015	11 000	Red	112 000	9	navigation, automatic air conditioning, radio, front fog lights, cruise control
Ford Mondeo	2017	16 000	White	83 400	10	navigation, automatic air conditioning, radio, front fog lights, cruise control, parking sensors
Audi A4	2018	21 000	Black	91 600	11	navigation, automatic air conditioning, radio, front fog lights, cruise control, parking sensors, leather seats, rain sensors
Ford Fiesta	2016	8 500	Red	60 400	8	automatic air conditioning, radio
Hyundai Tuscon	2017	18 600	Blue	40 900	11	automatic air conditioning, radio, front fog lights, cruise control, parking sensors
BMW X2	2018	35 000	Silver	38 600	11	navigation, automatic air conditioning, radio, front fog lights, cruise control, parking sensors, leather seats, rain sensors, LED lights, sports seats

Reanult Captur	2018	11 600	Blue	111 400	9	automatic air conditioning, radio, front fog lights, cruise control
Opel Astra Karavan	2017	10 400	Silver	112 300	10	automatic air conditioning, radio, front fog lights
VW Golf	2016	12 000	White	70 000	9	automatic air conditioning, radio

Results and discussion

We asked prospective teachers if they thought the given word problems were appropriate for lower elementary grades. This was followed by two open-ended questions in which they were asked to state the advantages and disadvantages of the given problems. The last question related to the word problems was whether they would do such a problem in their mathematics classes. The results for all problems can be found in Table 2 and are described below.

Table 2: Questions about given word problems (WP)

	WP 1		WP 2		WP 3	
	f	f %	f	f %	f	f %
The word problem is suitable for lower grades of elementary school.						
Yes	77	95.1	75	92.6	25	30.9
No	4	4.9	6	7.4	56	69.1
I would do such word problems in my math classes.						
completely disagree	3	3.7	5	6.2	36	44.5
partially disagree	7	8.6	5	6.2	15	18.5
neither agree nor disagree	36	44.5	19	23.5	18	22.2
partially agree	20	24.7	24	29.6	8	9.9
completely agree	15	18.5	28	34.5	4	4.9
∑	81	100.0	81	100.0	81	100.0

Table 2 shows that prospective teachers consider intra-mathematical problems and “dressed up” problems to be more suitable for mathematics classes than mathematical modelling problems. Moreover, the mentioned two types of word problems would also be included to a greater extent in their teaching of mathematics. These responses were to be expected since students consider it less important to be able to solve modelling problems than “dressed” up word or intra-mathematical problems (Krawitz & Schukajlow, 2018).

We asked prospective teachers to write in an open-ended question what they considered to be advantages or disadvantages of a particular word problem. For the first word problem, 33 prospective teachers, i.e. 41% of them, mentioned the simplicity of the problem, and 23 of them, i.e. 28%, noted the repetition of terms in addition and the practice of adding numbers as an advantage. As for the disadvantages of the first word problem, 14% of the prospective teachers stated that the numbers are too large for lower elementary grades, and the vast majority, 48% of the prospective teachers, stated monotony and lack of context as the disadvantage of the word problem. In other words, prospective teachers indicated that word problem 1 seemed boring and uninteresting to them.

Among the advantages of the second presented word problem, more than half of the prospective teachers (57%) stated that this problem requires prospective teachers to think, and that the problem is good because it has a context from everyday life, and therefore could be more attractive to prospective teachers than the first presented problem. Other benefits prospective teachers mentioned were comparison and evaluation. Prospective teachers disagreed on what was a disadvantage of the second word problem. To some, it seemed too simple, while others thought the problem was too complicated. Some even noted that the problem focused too much on material things and that it would be better to have something closer to the prospective teachers’ experiences in the problem, i.e., their age and not cars.

For the third word problem from mathematical modelling, 17% of the prospective teachers mentioned a table presentation as an advantage. Another advantage was that the problem was realistic, excellent for inquiry, and engaging. For the third word problem, 40 prospective teachers (49%) did not state any advantage. Therefore, we can conclude that they do not see the advantages and benefits of this type of word problem at all in the lower grades of elementary school mathematics. As for the disadvantages of the third problem, prospective teachers were almost unanimous. Those who mentioned a disadvantage (84%) stated that the problem was complex/complicated and would take too much time. Additionally, they stated that such a word problem would be more suitable for prospective teachers who are better at math.

Looking at the results overall, we see that prospective teachers are much more inclined to standard word problems. By this, we mean intra-mathematical word problems and “dressed up” word problems, which they encountered much more frequently during their education than word problems involving mathematical modelling. More than 90% of prospective teachers responded that problem 1 and problem 2 are suitable for lower elementary grades, while nearly 70% of prospective teachers answered that problem 3 has no place in lower grades of elementary school. The vast majority of prospective teachers thought problem 3 was too complicated, with too much data and would take up too much of their mathematics class time. Other studies have come to similar conclusions. For example, Lee and Kim (2005) conducted a survey (N=22) that revealed that most teacher candidates found typical routine problems to be good and showed strong resistance to some non-routine problems with atypical features. In addition, Krawitz and Schukajlow (2018) found that there are significant differences in prospective teachers’ task values depending on the type of problem (intra-mathematical problems, “dressed up” problems, and modelling problems). Namely, in their study, prospective teachers reported the lowest task values for modelling problems compared to the other types of problems.

In this paper, we have only presented a part of the survey and the questions from the survey questionnaire. The challenge for all teachers, particularly prospective elementary school teachers, is to constantly upgrade their knowledge and experience in their professional development, including knowledge of word problems and their quality creation.

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The construct of playful learning in primary mathematics: A literature overview

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This paper scrutinises the main scientific journals and books concerning early years mathematics education for the learning of playful mathematics in primary school. The search process resulted in 2633 studies which were then screened according to title and abstract before reading 61 studies in more detail. The resulting 13 studies were further examined to explore how the different mathematics education researchers characterised playful learning in mathematics. Based on these examinations, the paper provides a working definition of playful learning in primary mathematics education.

Keywords: Literature overview, mathematics teaching in primary school, play, playful learning.

Introduction

This literature overview aims to define playful learning (PL) in primary mathematics education (ME). When reading previous research literature, I discovered that early years ME researchers often do not define PL but view PL situations regarding teaching opportunities by stating what children learn, such as dealing with counting, operations on numbers, shape, and measuring (Ginsburg, 2006), geometrical thinking (Clements & Sarama, 2014), classification, seriation, conservation, one-to-one correspondence, estimating, quantitative concepts, number words, space-time orientations (van Oers, 1996) to name a few. Of course, what pupils may learn through play is highly important, as education has a learning perspective. However, I argue that PL does not depend on the specific mathematical content. In this literature overview, a total of 2633 studies were screened before reading 61 studies in more detail. When scrutinising the resulting 13 studies and identifying the ME researchers' common features of PL, my argumentation contrasts with the argument of Brooker et al. (2014), who concludes that a consensus on the definition of PL in early childhood never will be reached. Also, I argue that researchers studying the effectiveness of a PL approach and what mathematical content pupils learn when participating in PL situations could benefit from a definition of what constitutes PL in mathematics in the first place. Therefore, next, I draw on previous research providing insights into key concepts and a further rationale for conducting this literature overview, deliberately labelled an overview, rather than a review, because the interest is not in the studies' research findings. The aim is purely theoretical; to explore how ME researchers characterise PL to define PL in primary ME.

Background

As a pedagogical approach, PL is a broad construct capturing the interrelationship between play and learning (Hirsh-Pasek et al., 2009). It encompasses learning through free play, guided play and games (Fisher et al., 2012; Ginsburg, 2006). Free play is child-initiated and child-directed. Guided play is adult-initiated and child-directed. In both, the child is active and in the lead. The difference is the adult's passive role in free play, compared to an active role in initiating the activity in guided play (Fisher et al., 2012; Weisberg et al., 2013). Thus, the adult can create more learning opportunities by enhancing the children's engagement in the activities (for a review, see Fisher et al., 2010). The pupils are unlikely to get the full benefit from PL without teachers' engagement (Ginsburg, 2006). However, balancing adult and child participation can be challenging (Breive, 2019), with a risk of the activity becoming of the instructional type. Compared to free and guided play, direct instruction is adult-initiated and adult-directed (Fisher et al., 2012; Weisberg et al., 2013). Thus, guided play lies between free play and direct instruction, involving adult guidance while allowing children to direct the activity (Weisberg et al., 2015). However, there exist various perspectives and differences in opinions of play and learning. Some might even view the two as incompatible (Fisher et al., 2010). What defines PL is unclear (Samuelsson & Carlsson, 2008), as is the distinction between the three approaches (free play, guided play, and direct instruction) it overarches. Especially according to the degree of adult guidance where guided play falls on a continuum where the adults' involvement "varies according to the adults' curricular goals and the child's developmental level and needs" (Fisher et al., 2010, p. 343). In general, it is essential to differentiate between child-initiated (play-based) and adult-initiated activities (instruction or more school-like tasks). However, mathematics instruction can involve various instructional approaches (Sarama & Clements, 2009). It does not have to be direct instruction, and PL can also include different instructional approaches. The integration of play in the learning process is precisely why play in teaching has such great importance (Wood & Attfield, 2005), a potentially valuable educational tool also in primary school mathematics teaching and learning.

The relationship between mathematics and play can be seen as either "mathematics made playful" or "mathematising elements of play" (van Oers, 1996). Mathematics is made playful when it is the primary activity, e.g., games where counting or sorting activities are transformed into playful activities. Elements of play are mathematised when play is the primary activity, e.g., when the teacher tries to be responsive to the children's actions and introduce mathematical concepts to the activity. As such, in both conceptions of the relationship, the teacher may provide opportunities for further mathematics learning. Teachers' ability to respond to the opportunities during play is critical to enhance the children's mathematical thinking (van Oers, 1996), in line with Fisher et al. (2012) and Ginsburg (2006) regarding the adults' role in guided play. However, as play is challenging to define, it is also challenging to assess its quality (Samuelsson & Carlsson, 2008). Thus, it is difficult to draw clear lines between different types of play and between play and instruction. These demarcation difficulties may explain why existing research on play often has focused on mathematical content. Thus, a literature overview is needed to provide a working definition of PL in primary ME.

Methods

The literature overview, conducted in June 2021, was limited to searching six resources for studies of pupils aged 5-12, published in 2010-2021, with no limitations regarding research methods. The resources and the respective number of studies screened were: Educational Studies in Mathematics (ESM, 610), Journal for Research in Mathematics Education (JRME, 177), Journal of Mathematical Behaviour (JMB, 366), (The) Journal of Mathematics Teacher Education (JMTE, 248), The International Journal on Mathematics Education (ZDM, 784), European Early Childhood Education Research Journal (EECERJ, 127), Early Childhood Education Journal (ECEJ, 160), Nordic Studies in Mathematics Education (NOMAD, 87) and four conference proceedings from A Mathematics Education Perspective on Early Mathematics Learning between the Poles of Instruction and

Construction (POEM, 74). Reasons for choosing these journals and proceedings were: The first five journals are ranked A* and A in ME (Törner & Arzarello, 2012). EECERJ and ECEJ are dedicated to early childhood education in psychology and sociology. NOMAD captures the social pedagogical tradition in Scandinavia, relevant to my future research on PL in primary mathematics in Norway. PL has also been a reoccurring topic at the POEM conferences. The keywords were limited to *play* and *playful* but combined with *mathematics* for EECERJ and ECEJ. Both keywords proved influential as three papers only containing *play* were eventually included. The keyword play was expected to capture studies on games, which was investigated in four of the included papers. NOMAD was also searched for the Scandinavian countries' word for *play* ("*lek*"). Table 1 provides the collective screening based on *one* reason for each study's exclusion, with descriptions exemplifying the criteria. The search process resulted in 2633 studies that were screened by reading the title and abstract in phase one, with italicised numbers of excluded studies. In phase two, no papers were excluded based on criterion 3 as non-empirical studies were identified and excluded based on title and abstract. However, when in the slightest doubt of exclusion, the paper was read in more detail, e.g., when *play* appeared in the abstract, only to reveal in phase two that it was used without providing any features (criterion 2). Therefore, phase two included 61 studies, with bold numbers of excluded and included studies. For example, the search of JMB provided 366 studies (*157* + *181* + **9** + *16* + *2* + **1**), excluding *157*, *181*, *16* and *2* studies in phase one according to criterion 1 – 4, and reading ten studies in phase two of which **9** was excluded according to criterion 2 and **1** was included. The collective screening provided 13 studies ($n = 13$) scrutinised for features of PL. However, due to the mentioned limitations, there might be research that this overview does not capture.

Table 1: Results of the first (numbers in italic) and second (numbers in bold) screening phases

Reason for exclusion. Description exemplifying each criterion.	**EECERJ**	**ECEJ**	**ESM**	**JRME**	**JMB**	**JMTC**	**ZDM**	**NOMAD**	**POEM**
1) The school level. The pupils' age or school level was not stated or was not relevant.	*22* **3**	*22*	*113* **3**	*40*	*157*	*46*	*113* **3**	*27* **2**	*34*
2) The use of play or playful. E.g. used without providing features, in a different context (like the theatre) or words like dis*play* or inter*play*.	*25* **4**	*45* **6**	*412* **4**	*80*	*181* **9**	*178*	*632* **9**	*40* **1**	*24* **4**
3) The paper is not empirical. E.g., editorial, a review etc.	*17*	*6*	*71*	*53*	*16*	*24*	*25*	*17*	*7*
4) The subject. The study was not specific to mathematics.	*64*	*81*	*5*	3	*2*	*0*	*0*	*0*	*0*
Studies included ($n = 13$)	**2**	**0**	**2**	**1**	**1**	**0**	**2**	**0**	**5**

Results

The resulting 13 studies were all written in English and conducted in Norway (2), Sweden (2), England (1), Germany (1), Italy (1), Canada (2), the Netherlands (3), and Switzerland (1). I will now give an account of the 13 studies' perspectives when italicising the features of PL that they provide.

Gejard and Melander (2018) studied five-year-old children's geometrical learning and multimodal resources during block play. No clear definition of play was found to be provided in the article. However, they underline the importance of *balancing* the adult's and pupils' *control* in the activity. Further, they emphasised *active participation* in *collective* and *social* activities within a cultural setting where the pupils *negotiate* when displaying their understanding of the encountered geometry.

The role of teaching in a game setting was emphasised by De Simone and Sabena (2020) when investigating five-year-old children playing strategy games in a guided play setting where the teacher initiated the activity and supported the children in the reasoning processes. Thus, involving *interactive participation* where the "attention is on participating in the game (possibly on winning), and feeling pleasure and enjoyment are essential parts of the game" (De Simone & Sabena, 2020, p. 157). As such, it appears the researchers emphasised *interaction*, *communication* (of strategies) and *participation* and the children's perceptions like feeling *pleasure* and *enjoyment* in the PL situation.

McFeetors and Palfy (2017) investigated pupils' reasoning and strategies playing commercial games in a multi-aged grades five and six class. By implementing games dependent on logical reasoning, the authors aimed to "value reasoning as an integral part of thinking mathematically" (McFeetors & Palfy, 2017, p. 536). Overall, they emphasised providing an *engaging*, *authentic*, *collaborative*, and *social* context. Also, the teacher posed questions verbally and in writing to encourage pupils to express their reasoning and explore more sophisticated reasoning, which was recognised by the pupils as helpful and by the authors as vital for the advancement of pupils' reasoning in the play context.

In the following study, McFeetors and Palfy (2018) emphasised the participants' activity when 5th and 6th graders *interacted* while playing in pairs, prompted by adults' questions to emphasise *conversation* about strategic moves and strategies. The pupils were *encouraged to reflect* and build on their previous strategies. Games thought to foster *discussion* and which the pupils would find *appealing* was chosen. By being commercial games, they were perceived as *authentic*. Thus, *interaction*, *reflection* and *communication* in *authentic* game-playing contexts found *appealing* by the participants are features emphasised by McFeetors and Palfy (2017, 2018) in their two studies.

The participants' experiences were also emphasised by Vogt et al. (2018), indicating higher learning gains for pupils experiencing a PL approach. Activities that "are *fun*, *voluntary*, *flexible*, involve *active engagement*, have no extrinsic goals, involve active engagement of the child, and often have an *element of make-believe*" (Weisberg et al., 2013, in Vogt et al., 2018, p. 592, own italicisation).

Van den Heuvel-Panhuizen et al. (2013) investigated the role of a dynamic online game in 10–12-year-olds' early algebra problem-solving. They considered mathematical play in a game context as "that part of the process used to solve mathematical problems, which involves both *experimentation* and *creativity* to generate ideas, and using the formal *rules* of mathematics to follow any ideas to some sort of conclusion" (Holton et al., 2010, in van den Heuvel-Panhuizen et al., 2013, p. 285, own italicisation). Also, PL was aligned with mathematical processes by contributing to non-threatening environments making it safe for pupils to present incorrect solutions and confront misconceptions.

Helenius et al. (2016) identified features that linked mathematics to play, the interacting components being *creative*, *participatory*, and *rule negotiation*. The *creative* aspect involved the 6-year-olds' modelling of a situation, where they *incorporated some elements of reality and altered others* when posing and solving problems they encountered. Furthermore, playful mathematics activities were dependent on *participation* and contributions from others in a *collaborative, social* context "both at the local level of the immediate situation and also at the societal level which determines the rules and values that affect immersion in reality" (Helenius et al., 2016, p. 146). Thus, the participants engaged in the free play situation and a more comprehensive societal reality, excluding individual play as mathematical. In these situations, the participants abided by *rules* which could be changed and *negotiated*, thus "forming the boundaries of the play situation" (Helenius et al., 2016, p. 147). The criteria were independent of the mathematics content and identified as interactional.

Two studies by van Oers (2010, 2014) were included in the overview. Building on the study from 2010, van Oers (2014) considered mathematising as "the activity of producing structured objects that allow further elaborations in mathematical terms through problem solving and (collective) reasoning/argumentation" (p. 112). Productive mathematising was defined as a "playful activity that has its roots in young children's playful participation in cultural practices" (van Oers, 2014, p. 112). Thus, productive mathematising could be interpreted as PL activities when contrasting productive mathematising to re-productive activities or instruction. The characteristics of the play activity were that the activity was *rule-driven* with a high level of *involvement* and some *degree of freedom* given to the pupils. According to van Oers (2014) the activity could contain elements of instruction if it was meaningful, contributing to the children's participation, and balancing "creative construction and sensitive instruction" (p. 121). Thus, the degree of freedom might vary "as long as the activity as a whole remains a playful activity, i.e. is based on personally acknowledged rules, is engaging, and preserves some degree of freedom" (van Oers, 2014, p. 121). The level of involvement included the motivation to keep the activity going, to *engage*, *collaborate* and be *creative*.

Black et al. (2019) built on the characteristics by van Oers (2010) when they investigated a six-year-old boy's expression of his emotion-cognition experience, who described the playful activity as "fun" and the school mathematics experience making him "tired".

Also, Tubach and Nührenbörger (2016) adopted the characteristics of van Oers (2014). They investigated play as a promising approach to link the informal with the more formal mathematics learning in the transition from kindergarten to primary school.

Hundeland et al. (2020) studied the quality of a kindergarten teacher and five-year-olds' mathematical discourse, emphasising *active* children *in the lead* of the PL activity. They referred to Hirsh-Pasek et al. (2009), who stated that "playful learning, and not drill-and-practice, *engages* and *motivates* children in ways that enhance developmental outcomes and lifelong learning" (p. 4, own italicisation).

Incorporating inquiry and playfulness studying five-year-olds engaging in PL activities in kindergarten, Breive et al. (2018) stated that playfulness "has to be founded in *rules* acknowledged between the players, the activity has to be *engaging* and the activity has to emphasise the *player's* *possibilities to deliberately play in his/her own way*" (p. 185, own italicisation). Furthermore, adult guidance provided children with the needed will to ask questions and construct mathematical ideas.

Discussion

Even though not explicitly revealed, several of the excluded papers incorporated PL as an approach to mathematics learning without providing features of PL or clarifying what it constitutes, which is a finding in agreement with other research (e.g., Helenius et al., 2016; Samuelsson & Carlsson, 2008).

Researchers providing features of PL in mathematics do so frequently in terms of *interactional*, *participatory*, and *social* situations. These situations are characterised by *involvement* (van Oers, 2014) and *participation* (Helenius et al., 2016), allowing pupils to *engage*, be *creative*, and *collaborate* when *negotiating* and *discussing* the encountered mathematics. To keep the activity going by *engaging*, *collaborating* and being *creative* are included in the level of involvement by van Oers (2010, 2014), whereas *creativity* was singled out as a separate criterion by Helenius et al. (2016). Several of the researchers provided features of PL independent of the specific mathematical content and more related to mathematical processes in guided play (e.g., Breive et al., 2018; van Oers, 2010, 2014), free play (Helenius et al., 2016) and games activities (e.g., De Simone & Sabena, 2020; McFeetors & Palfy, 2017, 2018; van den Heuvel-Panhuizen et al., 2013).

Further, the PL situations are characterised by researchers as involving *authentic* (McFeetors & Palfy, 2017, 2018), *cultural* activities (van Oers, 2014) with an imaginative *element of make-believe* (Vogt et al., 2018) or *incorporating and altering elements of reality* (Helenius et al., 2016). Also, PL activities are rule-driven (van Oers, 2014), potentially involving negotiation of implicitly or explicitly expressed *rules* (Breive et al., 2018; Helenius et al., 2016; van den Heuvel-Panhuizen et al., 2013).

Several researchers also argue for a need for a mutual understanding and coordination of participants' perspectives of what is engaged in, talked about, experienced, and learned (e.g., Breive et al., 2018; Gejard & Melander, 2018; McFeetors & Palfy, 2017, 2018). It is especially crucial regarding the adults' role in PL situations, which should provide the pupils with the opportunity to be in the lead (Hundeland et al., 2020) and to play in their own way (Breive et al., 2018). Thus, PL situations are characterised by balancing the adult's and children's control (Gejard & Melander, 2018) in activities where creative construction and sensitive instruction provide a degree of freedom to the children (van Oers, 2014). This feature, mentioned by several researchers, could collectively be termed as participants *right of co-determination*, an element allowing pupils a degree of freedom to be *creative* and influence the activity, which may also contribute to the pupils' *feeling of enjoyment*.

Based on this literature overview, common features of PL among ME researchers are identified. Following the identified features, I thus define PL in primary school mathematics as *situations where participants with a right of co-determination actively participate in a rule-driven, imaginative, cultural mathematics activity while discussing the encountered mathematics*. Since the 13 studies included all three approaches, the definition applies to PL as an overarching construct of learning mathematics through free and guided play and games (Fisher et al., 2012). The features of *collaboration*, *interaction*, *creativity*, *emotions* and *authenticity* are not mentioned explicitly. However, following the previous argumentation, the definition encapsulates these features. There are aspects of *collaboration*, *interaction* and *creativity* encompassed when pupils are given a *right to co-*

determination when engaging in PL activities *discussing* the encountered mathematics. Also, *creativity* is encompassed by the *imaginative* feature, allowing participants to influence the activity, which may also lead to feelings of *enjoyment* and *pleasure*. Thus, the features align PL with mathematical processes rather than with mathematical content. The participants coordinate their perspectives when posing their suggestions and developing the activity while trying to solve the encountered mathematics tasks. However, emotions are highly subjective and can vary within the same activity. By intending to provide a working definition applicable for teachers and researchers assessing or investigating the quality of play (Samuelsson & Carlsson, 2008), emotions are not mentioned explicitly. Also, the *authentic* feature (of games) is encompassed by the *cultural* feature in the definition. Notably, there can also be a varying degree of fulfilment of the different features, as in the scrutinised studies. As such, the definition includes a familiar resemblance of features of PL, without necessarily each situation exercising all features to the same extent. Also, since the taken approach has its limitations, it will be interesting to test and, if needed, refine the definition when researching primary mathematics teaching claimed to be playful.

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Didactic transposition of natural numbers in the first year of compulsory schooling: a case of comparative curricula analysis

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The first year of school has recently become mandatory in Denmark, and for the 5 to 7 age group, a national curriculum for mathematics has been designed. This study aims to present an analysis of the didactic transposition of natural numbers in the first year of school to foster reflection on the national curriculum design. The focus is on the didactic transposition from scholarly knowledge into knowledge to be taught occurring in the national curricula of three countries: Denmark, Sweden, and Australia. The methodological approach is curriculum mapping using a table. The results show that the main elements of scholarly mathematics related to natural numbers on a sector level are identified in all three curricula; still, analysis on a theme level shows significant differences.

Keywords: Didactic transposition, the first year of compulsory schooling, comparative curricula, natural numbers.

Introduction

Guidelines have existed for many years in early education, but curricula for preschool on a national level in OECD countries are a newer creation (Samuelsson et al., 2006). In Denmark, the first year of school, which forms a transition year, changed from voluntary to mandatory in 2009. In 2014, the Danish Ministry of Education presented a national curriculum for compulsory schooling (Retsinformation, 2014). One subject/competence area was mathematics. Play constitutes a critical element in teaching, and teaching should be based on several subjects/competence areas simultaneously, e.g., mathematics and commitment and community. Comparing this national curriculum with other international curricula may provide a broader insight into international perspectives on the knowledge to be taught designed for early years mathematics. This study aims to analyse the didactic transposition of natural numbers from scholarly knowledge into knowledge to be taught occurring in three national curricula covering the first year of compulsory schooling. On this background, the research question is: How can scholarly mathematics be used to analyse aims related to natural numbers for the first year of compulsory schooling in the national curricula from three different countries: Denmark, Sweden, and Australia?

Theoretical framework

The framework is built on the theory of didactic transposition and central theories of young children's development of different aspects of natural numbers.

Didactic transposition

The theory of didactic transposition is anthropological as it considers knowledge as "a changing reality embodied in human practices taking place in social institutions" (Chevallard & Bosch, 2014, p. 173). A systematic epistemological and institutional approach is employed to study these knowledge activities as they undergo a process from produced, designed to be taught, actually taught in school, and learned by students. When bodies of knowledge developed in one social institution are transposed to another social institution, they undergo a transformation, deconstruction, and reconstruction to adapt to their new social institution (Chevallard & Bosch, 2014). Chevallard and Bosch (2014) list four social institutions involved in the didactic transposition: Scholarly knowledge, knowledge to be taught, taught knowledge and learned knowledge. The scholarly knowledge, e.g., aspects of natural numbers, is the starting point of the process. Chevallard and Bosch describe scholarly knowledge as "…generally produced at universities and other scholarly institutions, also integrating elements taken from a variety of related social practices" (2014, p. 171). The transposed work from scholarly knowledge to knowledge to be taught is chosen and decided by actors who belong to the "noosphere," meaning "the sphere of those who "think" about teaching" (Chevallard & Bosch, 2014, p. 170). These actors can, for example, be researchers, employees in the ministry of education, and teachers/educators. A national or local curriculum and school textbooks governed the knowledge to be taught. The teacher's task is to transform the knowledge from national/local curriculum and textbooks into taught knowledge. Finally, the last part of the transposition is learned knowledge acquired by students. This study focuses only on the transposition of the scholarly knowledge into knowledge to be taught, also named the external didactic transposition (Bosch & Winsløw, 2020), and more specifically, the transposition of knowledge concerning natural numbers.

Natural numbers

Natural numbers constitute a broad research field in mathematics education; therefore, the total content elements on natural numbers of the three chosen curricula for this study have given a direction of a relevant theoretical framework. Scholarly knowledge on young children's development of natural numbers carried out by Fuson (1998) and Gelman and Gallistel (1978) is considered to be an essential part of the theoretical framework.

Historically, the Hindu-Arabic system of numerals, widely used today, has developed over many years from the tally system as a unary system to the positional numerical systems built around the base 10 (Sun et al., 2018). Conceptual development progress of the numeral systems can be grouped: "the tally system, additive system, multiplicative-additive system, and decimal place value system" (Sun et al., 2018, p. 96). This study concentrates primarily on the first group: the tally system connected with young children's learning of the system. A central origin in scholarly mathematic knowledge for young children, considering natural numbers, can be found in the notions of cardinality and ordinality.

> Early number consists of a network of inter-related skills and knowledge, broadly divided into cardinal and ordinal aspects, that is, those concerned with number as a representation of quantity, and those concerned with number as a representation of position in a sequence. (Bruce & Threlfall, 2004, p. 3)

Concerning ordinal aspects, Siegler (2009) states that the most basic ordinal concepts are more and less. The cardinal aspects of number include determining quantity by counting or by subitising. Subitising is the ability to instantly recognise the cardinal value of a small set of objects without counting them (Bruce & Threlfall, 2004; Fuson, 1988; Siegler, 2007). When children are 3-4 years old, they become proficient at establishing the cardinal value of a set by counting (Siegler, 2007). However, children saying a number sequence or string is not necessarily bound to cardinality or ordinality. Fuson (1988) states that children can repeat number sequences only as a verbal activity, independent of cardinality and ordinality. There must be a shift from counting to numerical meaning for children counting to be cardinal or ordinal. Five developmental stages in young children's ways of counting are outlined: The number sequence: 1) as a string 2) as an unbreakable list (one, two, three …) 3) as a chain that can be broken 4) as a numerable chain 5) as a bidirectional chain where the forward- and backwards- counting is related to each other.

Decomposition and composition or part-whole number knowledge are stressed to be fundamental to developing a deep understanding of arithmetic. Both cardinal and ordinal situations are related to numerical sequence, i.e., when a child knows that there are n-1 entities in the cardinal set, which precedes the n^{th} ordinal entity (Fuson, 1988)

Gelman and Gallistel (1978) identified a set of principles that preschool children at age 5 possess when counting. The first three principles, one-one, stable order, and cardinality, focus on how the process of counting is carried out. The last two principles, abstraction and order irrelevance, focus on what to count and define what can be counted.

To communicate effectively using numerals, children need to know three numerical representations, i.e., number words, numerals, and nonsymbolic quantities and understand each representation and how a representation maps to other representations (Hurst et al., 2017).

Method

Curriculum mapping

Curriculum mapping refers to the method for developing and employing a curriculum map "A curriculum map is a visualisation of relationships within and between a curriculum or curricula" (Greatorex et al., 2019, p. 3). This study follows the 6 key stages of the curriculum mapping method for comparability research: "1. Define study aims and use; 2. Decide which curricula will be considered; 3. Determine the curriculum features that will be the basis of comparison; 4. Collect relevant documentation and sources of data; 5. Extract data and input it into the standard instrument; 6. Consolidate findings through visual representation" (Greatorex et al., 2019, p. 5). However, changes concerning key stage 3 are made as the features that will be the basis for this study are not solely the curricula but scholarly mathematics/theoretical framework. As stated in the theoretical framework section considering natural numbers, the content of the table is not designed independently from the three curricula to be analysed. In this study, the 6 key stages are covered in the paper sections: key stage 1, introduction; key stages 2 and 3, selection and context; key stages 4, 5 and 6, results. Based on the theoretical framework, a table has been designed.

Selection and context

This study analyses aims related to natural numbers learning for young children in the national curricula from three different countries Denmark, Sweden, and Australia. The selection of those three countries' curricula is based on similarities and differences in their early childhood education. The school system in Denmark and Sweden is similar about a transition year from kindergarten to school and a strong cultural emphasis on play. Australia differs from Denmark and Sweden as the country follows a British school tradition focusing on formal learning and less on play (OECD, 2006). The data collection comprises what three curricula say on natural numbers for age 5-7 years, from Denmark, Sweden, and Australia. Natural numbers related to other content areas of a curriculum, e.g., algebra/pattern and measurement, are not included in the analysis of the study. Mathematical processes such as, e.g. proficiency strands of problem-solving are included if they involve natural numbers. The Danish national curriculum includes learning objectives and teaching instructions (Danish Ministry of Education, 2019). The Swedish national curriculum consists of core content and competencies (Skolverket, 2018). The Australian national curriculum includes content descriptions and proficiency strands related to natural numbers (Australian Curriculum, Assessment and Reporting Authority, 2016). Children must have begun compulsory schooling in all three countries by turning six years of age. In Denmark and Sweden, the children are five-and-a-half to six years of age when they start compulsory schooling. However, some children start compulsory schooling in Australia when they are four and a half years of age. In the three countries, the children attend the class for one year. The terminology used to show the first year of compulsory schooling in Australia is "foundation year," in Denmark "kindergarten class" and in Sweden "preschool class". In the three countries, the teachers or the educators below first grade are generalists with 3½-4 years of education at bachelor a level.

Terminology

Following Bosch and Winsløv (2020), terminology of levels has been applied to organise this table appropriately concerning the data collection (curricula). The levels are domain, sector, and theme. The mathematical domain is natural numbers; the sectors are cardinal aspects; ordinal aspects; numerals as names. The themes are subsectors, e.g. the theme one-one correspondence is a subsector of cardinal aspects. The table thus also forms a reference model with an explicit description of relevant scholarly knowledge of the domain natural numbers, which is used as a reference for analysing the presence or absence concerning the content of natural numbers on sector and theme levels. Explicit presence is marked in the table with a code, and an empty box shows absence. Based on the table and code extracts, the three curricula are shown. Finally, a comparison regarding similarities and differences is made.

Results

Table 1 shows the mapping of the analysis of the three curricula with a code: a letter and a number indicating the presence of scholarly knowledge/theory in the form of theme and the belonging sector in the current domain.

Table 1: Comparative curricula analyses – presence and absence of sectors and themes

Domain: Natural numbers				
Sector	Theme	Danish curriculum	Swedish curriculum	Australian curriculum
Cardinal aspects	The one-one correspondence	D1	S1	A1
	The stable order	D1	S1	A1
	The cardinality		S1	A1
	Order irrelevance			
	Abstraction			
	Subitising			A2
	Composing and decomposing (whole-part)		S1	
	Early arithmetic	D2		A3
Ordinal aspects	The ordinal property "more numerous" (more and less)	D3		A4
	The relative magnitudes of numbers	D4	S2	A4
Numerals as names	Saying the number names	D5	S1	A1
	Connect number names, numerals, and quantities	D6		A5

Below are content extracts from the three curricula, informing the coding decision shown and supplemented with an explanation, if necessary. The content extracts are listed in the native language and translated into English where necessary. The native content extracts are stated in italics.

D1) *Har viden om metoder til antalsbestemmelse*/knows methods for counting. Methods for quantification are not explicit in the curriculum, but the first two principles of counting, as one-one correspondence and stable order, are fundamental. Regarding Fuson (1988), children may count but do not make a cardinal integration; therefore, the third principle (cardinality) is not coded.

D2) *Regnehistorier, der rummer et problem, som eleven skal regne på*/problem solving with early arithmetic in a real-world context.

D3)*Bruge forklaringer med ord som større og mindre, når der arbejdes med antalsbestemmelser*/uses explanations, which include terms as more and less when working with number sets.

D4) *Har viden om talsymbolerne og deres ordning*/knows the number symbols and their order.

D5) *Kan læse etcifrede naturlige tal*/can read single-digit natural numbers.

D6) *Forstå sammenhængen mellem mængde, antal, talord og talsymbol*/understands the relationship between quantity, number, number word and number symbol.

S1) *Naturliga tal och deras egenskaper och hur de kan användas för att ange antal and ordning*/ Natural numbers and their characteristics and how they can be used for counting and order. Concerning natural numbers, the Swedish curriculum only contains 24 words. The sentence of cardinality is interpreted to include the three principles of how to count one-one correspondence; stable order; cardinality (Gelman & Gallistel, 1978).

S2) *Del av helhet och del av antal*/part of a whole and part of a quantity.

A1) *Establish understanding of the language and processes of counting by naming numbers in sequences, initially to and from 20, moving from any starting point. A point marks a position but has no size. Understanding that numbers are said in a particular order, and there are patterns in the way we say them.*

A2) *Subitise small collections of objects. Using subitising as the basis for ordering and comparing collections of numbers*.

A3) *Represent practical situations to model addition and sharing. Estimating and calculating with whole numbers.*

A4) *Compare, order and make correspondences between collections, initially to 20, and explain reasoning. Comparing and ordering items of like and unlike characteristics using the words "more," "less"… and giving reasons for these answers. Understanding and using terms such as "first" and "second" to indicate ordinal position in a sequence.*

A5) *Connect number names, numerals and quantities, including zero, initially up to 19 and then beyond. Estimating and calculating with whole numbers. Understand and use number in context.*

A comparison of the three curricula regarding central similarities and significant differences is outlined. Regarding similarities, all three sector levels (cardinal aspects, ordinal aspects, and numerals as names) were identified in the three curricula. This means that the main elements of natural numbers concerning scholarly mathematics are preserved in all three curricula or the knowledge to be taught. None of the curricula has explicitly included scholarly mathematics of order irrelevance and abstraction on the theme level. Only the Swedish curriculum has explicitly included the theme composing and decomposing. Only the Australian curriculum has explicitly included the theme subitising. The Australian curriculum has explicitly included more themes (9) than the Danish curriculum (7 themes) and the Swedish curriculum (6 themes). The themes indicate that the

Australian curriculum is more detailed and precise and goes closer to practice than the Danish and Swedish curricula. Concerning the levels of didactic transposition regarding the content, the Danish and Swedish national curricula tend to be organised at the sector level. In contrast, the Australian national curriculum tends to be organised at the theme level.

Discussion and conclusion

The present study sought to answer the research question on how scholarly mathematics can be used to analyse aims related to natural numbers for the age 5-7 years in the national curricula from three different countries. The study shows that the main elements of scholarly mathematics related to natural numbers on the sector level are identified in all three curricula. Analysing and identifying the aims of the curricula elements related to natural numbers on theme level call for some validity considerations because the designs of the three curricula differ. A significant difference is the detailed design of the Australian curriculum versus the compact design of the Swedish curriculum, while the design of the Danish curriculum is in between. According to Greatorex et al. (2019), the level of detail of curricula data varies, which can influence the usability of data. In the present study, the varying degree of detail makes the selection criteria in the analysis "explicit presence" challenging. The challenge is addressed by informing the coding decision, e.g., the Swedish curriculum where the sentence of cardinality is interpreted to include the three principles of how to count (S1). Furthermore, this study is constrained to the national curricula, while other documents of the knowledge to be taught, e.g., school textbooks, may elaborate on these content areas.

When the scholarly knowledge of natural numbers is transposed into three national curricula, the transformation, deconstruction, and reconstruction (Chevallard & Bosch, 2014) of each curriculum are linked to the educational culture and tradition of the current country (Greatorex et al., 2019). Australia has a longer curriculum tradition within early childhood education than Denmark and Sweden have. The contrasting can foster reflection on the two relatively compact Scandinavian curricula, especially considering the next step of the didactic transposition from knowledge to be taught into taught knowledge. It is the teacher's task to transform the knowledge from the national curriculum into taught knowledge, and it is an open question to what extent she needs scaffolding.

The present study has made some of the cultural differences of the didactic transposition of natural numbers on the national curricula level transparent. Thus, the curricula analyses have provided reflections on different ways to select and design knowledge of natural numbers to be taught in early childhood. The study could be supplied with an analysis of school textbooks considering natural numbers. Also, the study of the national curricula could be extended to include other content areas and mathematical processes. In a questionnaire for educators in the Danish kindergarten class, I will follow up on how often different mathematics skills, including natural numbers, are taught in the classes.

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Content and organization of grade 0 mathematics education in the Faroe Islands

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In this paper we study the content and organization of preschool mathematics for grade 0 in the Faroe Islands from a formal and praxis point of view. We elaborate on motives for the genesis of grade 0 and unveil formal documents. We present data from two cases in which we interview two grade 0 mathematics teachers. The data is analyzed in relation to Bishop's six fundamental activities for mathematical enculturation and we report on the educational resources that are used in these activities. Our results show that the content and organization from a formal point of view is limited, while data from our cases indicate that mainly two of Bishop's activities are employed, and that the content is heavily influenced by the chosen mathematics textbook system. We conclude that the lack of formal organization is problematic from an educational point of view and deliberate future perspectives.

Keywords: Preschool grade 0 mathematics, policy and curriculum, Bishop's activities for mathematical enculturation, educational resources.

Introduction

During the last three decades, national level preschool curricula have become more commonplace in early year mathematics education in OECD countries (Henriksen, 2021a; Samuelsson et al., 2006). Members of the CERME Early Years Mathematics thematic working group have previously noted that there are similarities and differences between countries in terms of the curriculum and organization of preschool and express that there is a "need to know more about the organization of preschool in each country" (Bartolini Bussi et al., 2015, p. 1887). For example, in Italy, the preschool curriculum was developed based on Bishop´s (1988) six universal activities for mathematical enculturation: counting, locating, measuring, designing, playing and explaining (Bartolini Bussi et al., 2015), whereas Levenson and Barkai (2013) report that Israeli curriculum documents describe quite extensively the competencies and concepts that children should grasp in preschool by listing "explicity and separately which of those concepts may be promoted and which skills should be enhanced for children ages 3-4, 4-5 and 5-6 years old." (Levenson & Barkai, 2013, p. 2158).

In Nordic countries such as Denmark, Sweden, Finland, a one-year participation at grade 0 for children at the age of 5-7 (usually called "preschool") is now prevalent with the purpose of supporting the transition from kindergarten (age 0-5) to compulsory primary school (grade 1). Since the introduction of the grade 0 in Denmark in 1912, attendance has been voluntary, meaning that some children rather remained at kindergarten one more year, before entering primary school grade 1. However, in 2009, attending grade 0 became mandatory, and new foci on playful learning and cognitive development were introduced (Henriksen, 2021b, Vejleskov, 2017). The shift was soon followed by the implementation of national curricula for grade 0 (Børne- og

Undervisningsministeriet, 2014), which came during a time when results from PISA showed that a higher percentage among weak readers did not attend preschool for one or more years compared to all students (Fredrikson, 2012).

In the Faroe Islands[1], the grade 0 tradition is less established historically. Beginning in 1963, a single primary school introduced grade 0 (students aged 5-7), which was physically located at the school, and it remained the only school with this structure until 2010 (Matras et al., 2014). Since then, several of the larger public primary schools of the Faroe Islands have introduced this system, and as of September 2020, a third of Faroese public school students attend grade 0, even though attendance is still voluntary.

In this paper, we attempt to address the call of Bartolini Bussi et al. (2015) of the need to investigate the organization of preschool mathematics in different countries, in our case focusing on the context of preschool in the Faroe Islands. In particular, we center our attention on grade 0 (students aged 5-7), in which attendance has grown extensively in recent years, and on which no previous research has been conducted.

Our study includes, perhaps obviously, the elaboration of formal curricular documents, however, since the unfolding of curriculum in praxis is not necessarily a one-to-one correspondence, we also seek to collect perspectives anchored in praxis. In this paper, we therefore put forward the following research question:

What is the content and organization of preschool mathematics for grade 0 in the Faroe Islands from a formal and praxis point of view, respectively?

To shed light on our question, we investigate the historical motives and intentions behind the genesis of grade 0 in the Faroe Islands and unveil formal documents that describe the organization and content in preschool mathematics. To gain qualitative insights from a praxis point of view, we interview two mathematics teachers working in grade 0 and investigate which resources they use.

In the next section, we present Bishop's (1986, 1988) six fundamental activities for mathematical enculturation, which is the conceptual background that is used in our analysis. Afterwards, we present the methodological choices underlying our study, which is followed by an elaboration of pertinent policy documents as well as an historical account of Faroese grade 0. We then present interview data and ensuing analysis. Finally, we conclude on our study by referring back to our research question and deliberating future perspectives.

Conceptual background

As mentioned previously, in Italy, the preschool curriculum was developed following Bishop's (1986, 1988) six fundamental activities for mathematical enculturation (Bartolini Bussi et al., 2015). The activities are:

[1] The Faroe Islands is a self-governing country within the Kingdom of Denmark with a population of approximately 53.000.

"*Counting*. The use of a systematic way to compare and order discrete phenomena. It may involve tallying, or using objects or string to record, or special number words or names. […]

Locating. Exploring one's spatial environment and conceptualising and symbolising that environment, with models, diagrams, drawings, words or other means. […]

Measuring. Quantifying qualities for the purposes of comparison and ordering, using objects or tokens as measuring devices with associated units or 'measure-words'. […]

Designing. Creating a shape or design for an object or for any part of one's spatial environment. It may involve making the object, as a 'mental template', or symbolising it in some conventionalized way. […]

Playing. Devising, and engaging in, games and pastimes, with more or less formalised rules that all players must abide by. […]

Explaining. Finding ways to account for the existence of phenomena, be they religious, animistic or scientific." (Bishop, 1988, pp. 182-183)

According to Bishop (1988), these fundamental activities are, on the one hand, universal, because "they appear to be carried out by every cultural group ever studied" (p. 182) and, on the other hand, they are "necessary and sufficient for the development of mathematical knowledge." (p. 182).

Method

To investigate our research question from a formal point of view, we searched for and studied regulative frameworks, executive orders, ministerial reports (including PISA reports), and curricular documents published by the Ministry of Education as well as the Faroese primary and lower secondary school council.

In addition to our studying of policy and curricular documents, we visited two teachers in order to interview them and to see their facilities. This could provide us with some qualitative insights of the content and organization of grade 0 from a praxis point of view, although only as reported by the teachers. Our method comprised a mixture of closed and open questions in a semi-structured interview approach, which is characterized by the interviewer preparing a guide that can serve as a starting point for a conversation, but where the interviewer can improvise if there is a need to ask for examples or elaborations (Arksey & Knight, 1999; Tanggaard & Brinkmann, 2015). The interview guide contained our research questions, followed by interview questions that were formulated in everyday language (Hansen & Andersen, 2009), which we used in the interview situation. Besides asking broadly about what the mathematics practice in grade 0 comprised of, we asked directly if the teaching practice involved each of Bishop's (1988) six fundamental activities, and also which types of resources were used.

The collected data was transcribed and analyzed in order to unveil the constituents of the grade 0 mathematics practice in our cases, investigating from a praxis point of view to what degree Bishop's (1988) six fundamental activities were part of this practice, and which resources are used.

Our findings are presented in the next two sections, beginning with an elaboration of pertinent policy documents followed by interview data analysis.

From PISA to grade 0 – policy and curricular documents analysis

When the Faroe Islands participated in the PISA survey for the first time in 2005/2006, the academic results sent shockwaves through the Faroese education system and society in general. The students' results in mathematics, reading and science were by far worse than every other Nordic country, in fact, also below Mexico, which was the worst performing country in the OECD at the time (Egelund, 2006). In the aftermath, comparative analysis highlighted differences between the Faroese education system and education systems in other Nordic countries, and one detail that gained traction was that almost all Faroese students at the end of 9th grade had received only 9 years of education since there was only one school that included grade 0 in the Faroe Islands, whereas >99% of Danish 9th grade students had spent 10 years at school (Egelund, 2006). In the subsequent years, politicians and other stakeholders voiced their opinions on the matter (Nielsen & Joensen, 2013) including the Faroese primary and lower secondary school council, which recommended that children enter grade 0 (Matras et al., 2011).

As an increasing number of schools began to offer grade 0, a regulative framework, “Executive order on organization and learning in prechool” (Uttanríkis- og mentamálaráðið, 2013) was developed by the Ministry of Education, which outlined the aim, content and organization of grade 0. The two-page document mentions mathematics once: “The content of the education should at least include the foundations of […] mathematics and nature” (Uttanríkis- og mentamálaráðið, 2013). In the already existing regulative framework for grades 1-10, a paragraph was added concerning grade 0, which also mentions mathematics once “The children should learn the foundations for reading, writing and mathematics…” (Uttanríkis- og mentamálaráðið, 2019). Curiously, no curriculum has yet been developed for grade 0 in the Faroe Islands.

Interview data and ensuing analysis

In table 1, below, we have collected some of the interview excerpts that pertain to each of Bishop’s fundamental activities.

Bishop’s activities	Classroom practice
Counting	Teacher 1: “We use many activities of counting […] count how many days they have been to school. We use straws […] when we get to 10, then we bundle the straws with an elastic and call it ‘one tens’ […] centicubes […] jars with different colors to put the straws in ones, tens and hundreds […] We have the abacus, which is old and actually brilliant, but actually it is not easy for them to understand…”
Locating	Teacher 1: “exactly that is in the book […] They see a tall building where there are different characters […] then I ask ‘what is to the side of the man?’ […]” Teacher 2: “the teacher guidance book is very good in that aspect (it says) ‘use the students to explain prepositions, e.g. can you, Linda, stand in front of Magnus’”

Measuring	Teacher 1: "it's one chapter [in the book]. how long is the table… how many pencils is the table… and if we have clips, then ask, how many clips long is the book. Or they throw a paper ball on the wall and measure how many clips the ball fell from the wall – then it becomes a tournament, then they cheat and the clips become shorter, but that is also fun, it is a process. Teacher 2: "and we measure heads, hands, noses etc. using strings with clamps" Teacher 1: "yes, there is a page about that"
Designing	Teacher 1: "yes that is a really good chapter. There is a lot concerning patterns. It is further ahead in the book. You see a picture from an Asian city, where there are towers with different beautiful shapes […] The shapes are typically triangles, quadrilaterals, both rectangles and squares, and then we have circles. Circles we call '0-edges' in grade 0 […] a student said it this year, and I like it so much I will use it going forward"
Playing	Teacher 1: "There is a lot playing, we have something called 'free play' where they can play what ever they want […] And they play many games such as dice games" Teacher 2: "Cards and dice games […] board games, memory and some of them can play beginners chess"
Explaining	Teacher 1: "explanations is mostly… so we try to explain extremely well when we are explaining, but to make them explain… we do try to ask them questions to make them explain, some of them explain in great details […] in each chapter there is a page that is about describing" Teacher 2 (with the book in his hands): "now I took a random page – they have to explain how many apples there are […] I am not sure if this is what you are asking"

Table 1 – Bishops' fundamental activities in the classroom practice

From the teachers' utterances, we can identify that each of the fundamental activities are present to some extent. However, the frequency of the activities, and the reasons that they are present, differ across the categories. On the one hand, **counting** and **playing** are frequent activities that the teachers consciously and in a goal-oriented manner activate in many different contexts. On the other hand, the activities of **locating**, **measuring**, and **designing** are present, yet to a lesser degree, and seemingly only because there are certain pages of the chosen mathematics textbook that encourage this type of activity. **Explaining** is the lone activity that the teachers are hesitant about, which indicates that it is not a particular focal point, however, they explain that it is partly included in the classroom discussions, in which the teachers ask the students questions that stem from the textbook, hence requiring explanations from the students.

When asked directly and indirectly about the type of resources that the teachers used in relation to each of the fundamental activities, the following resources were described: Counting (Straws, jars, centicubes and Abacus); Locating (Textbook, teacher guidance book, and chairs); Measuring (Textbook, pencils, clips, amd strings with clamps); Designing (Textbook, and the outdoors); Playing (Cards, dice games, board games, memory, and chess); Explaining (Textbook).

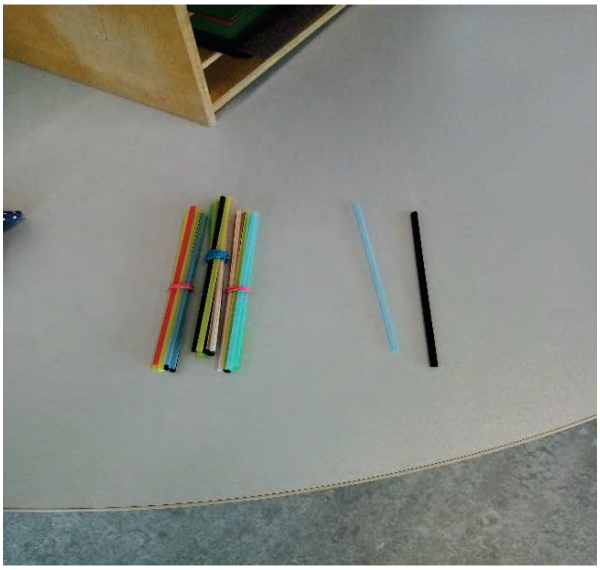

Figure 1 – Straws as ones and bundles of ten

Figure 2 – Cards, dices and other games

Most of the resources are oriented towards **counting** and **playing**, and a few that are used in relation to **measuring**, while the other fundamental activities are based on pages from the mathematics textbook.

In figure 1 we see some of the straws that are used in different teaching activities related to counting, while figure 2 shows a few of the educational resources that are used in playing activities.

Concluding discussion

Referring back to our research question, we can conclude from a formal point of view, firstly, that the content and organization of preschool mathematics for grade 0 in the Faroe Islands is scarcely developed, with only vague peripheral policy documents available. Secondly, from a praxis point of view, our cases indicates that Bishop's six fundamental activities are to a varying degree a part of the classroom practice, with **counting** and **playing** being the main activities that the teachers in our cases emphasize. That is supported by the fact that many of the educational resources that they employ are oriented towards counting and playing. Conversely, the mathematics textbook seems to be the main reason that **locating**, **measuring**, and **designing** are employed. We should stress that our insights into praxis are limited in this case study that consists of interviews with only two teachers. In addition, our data is teacher reported and not observed from the classroom. Therefore, the data may not depict the actual praxis for various reasons.

As mentioned previously, there are many possible approaches to organizing preschool, e.g., the Danish focus on playful learning; the Italian focus on Bishop´s (1988) six universal activities for mathematical enculturation; or the extensive Israeli description of learning goals in terms of competencies and concepts that children should grasp. Deciding which approach is suitable is not a trivial task from an educational point of view, however, we would argue that **some** organization is needed.

Hence, we find it problematic that there is no curriculum for grade 0 in the Faroe Islands. The consequence being that the content and organization is altogether in the hands of the individual teacher, and possibly quite different between schools. A conceivable consequence, which is somewhat evident in our cases, is that the content will be heavily based on the chosen mathematics textbook system.

Looking forward, we argue that the important goal of supporting Faroese students' mathematical development demands a coherent goal-oriented organization, which includes at least a curriculum that describes goals for these critical years in children's development.

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Preschool children estimating lengths – the role of standard units as relevant prior knowledge

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Length estimation is a relevant ability in our everyday life. Especially during the Corona Pandemic, estimating a distance of 1.50 m is crucial for our health and social life. Standard units play a central role in many length estimation processes that occur in our everyday surrounding. However, standard units as well as approaches to length estimation are generally discussed with children in elementary school while children already face situations already in their preschool age. This research addresses the question of how preschool children deal with length estimation. In a study with 189 preschool children who were about to start elementary school, we assessed their length estimates and analyzed the role of standard units in this context. It shows that standard units are used by the majority of kids to estimate lengths. However, their estimates are far from precise but the relation between the measures they name is mostly correct.

Keywords: Length estimation, estimation strategies, preschool education, prior knowledge.

Introduction

Estimating lengths as precise as possible is one main ability that becomes relevant in various situations of our everyday life. Especially in traffic there are many situations that require length estimation – the GPS system gives directions in standardized measures or there are zones with reduced speed for a given length. Children get in touch with length estimation in several situations as well. Reading books about dinosaurs holds several situations about length estimation as well as length comparison: the tooth of a Tyrannosaurus Rex was about 15 cm long which is about as long as a banana while the dinosaur itself could reach a length up to 12 m. In these books there are several pictures showing the relation of lengths between the dinosaurs and objects that are known from today. Another example situation where children are faced with length estimation is when they are baking cookies with their parents who have to place the cookie dough in a certain distance from one another or when they are travelling and they ask their parents how much longer they have to drive, they might get a response about the remaining kilometers. These examples are just a small selection of all the situations where children meet length estimation requirements. The majority of these situations also hold standardized units of lengths. Many of these situations already occur before the children start their school career meaning that they necessarily start school with prior knowledge about how to estimate lengths and a variation of intuitive approaches. As the examples already show, there are different estimation situations that require different approaches: Estimating a length in a standardized measure – such as: how many centimeters is this sheet of paper long? – requires knowledge about the standard unit centimeter, about scaling, conceptions about this specific measure as well as knowledge about the measuring process. In contrast, when asking children about what is as long as a Tyrannosaurus Rex, they might just provide a reference object (e. g. a truck) while they do not necessarily need prior knowledge about standard units but rely on direct comparisons instead. If precise concepts about standard units are still missing, this will affect the length estimation's approach as well as its success. It is an open question, to what extend preschool children use standard units for length estimation and how their estimation results are affected if their concepts of these units are still imprecise. With this research we want to contribute information about this point of children's early years' mathematics in this field.

Measurement estimation

Measurement estimation is defined by Bright (1976, p.89) as "the process of arriving at a measurement or a measure without the aid of measuring tools. It is a mental process though there are often visual or manipulative aspects to it". In the process of estimating measures such as lengths, there is specific knowledge and certain skills (such as strategies) that become relevant in order to reach a precise estimation (D'Aniello et al., 2015; Sowder, 1992; Joram et al., 1998). This is supposed to be knowledge about physical measurement (Joram et al., 1998), types of measures or basic conceptual knowledge that can be used as a reference to estimate (Sowder, 1992). Coming back to the dinosaur example from the beginning, children have to know the length of an adult man in order to estimate the height of a dinosaur that is pictured right next to him. In this case, the knowledge about a man's height functions as a reference in the estimation process. Alternatively, they might have a precise concept about a specific standard unit (such as meter) and use this knowledge to estimate the length of a man by counting how many units fit the length of the person. In these examples, we described the two main strategies that are differentiated in the literature for length estimation (1) benchmark comparison and (2) unit iteration (Joram et al., 1998)[1]. Empirical research shows that students from higher grades estimate lengths and areas more precisely than students from lower grades (Huang, 2014; Desli & Giakoumi, 2017).

This theoretical part about the process of length estimation shows the relevance of knowledge about standard units (e. g. the strategy unit iteration can only be used if standard units are known and concepts about their lengths available). In addition, if children encounter length estimation in their everyday life, the estimate will generally be given in a standard unit. Therefore, we put a focus on the role of prior knowledge about standard units for length estimation in this research.

(Prior) knowledge about standard units in the process of length estimation

Measurement generally bridges two mathematic fields: geometry / spatial relations and real numbers (Clements, 1999). Geometrical objects are assigned with a size indication. This consists of a number and a unit. Understanding the various concepts belonging to length measurement (including the meaning of size indication and unit) is a central part of mathematics education. Referring to the developmental theory of measurement by Piaget et al. (1960), it is generally taught in several steps, starting with direct comparisons followed by indirect comparison with a non-standardized unit and, finally, with a standardized unit. This approach is generally chosen in order to reveal children the need for standard units (Clements & Stephan, 2004). However, there is critique about this approach

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[1] There is a third approach classified in the literature – the decomposition / recomposition approach – which is also based on either unit iteration or benchmark comparison (e. g. Joram et al., 1998).

(Clements, 1999) because children (already in preschool) prefer to use standard units comparing objects and they are even more successful using standard units to measure as compared to measuring with non-standard units (Boulton-Lewis et al., 1996, Kotsopoulos et al., 2015). Empirical studies show that preschool children are already able to use standard units. Boulton-Lewis et al. (ibid.) observed that they used it even if they do not understand it. In this regard, Barret et al. (2011, p. 638) summarize that “students’ unit concepts are rarely well formed. Although students may use unit labels to name a quantity, they often do so without being able to show the meaning of the relevant unit.” Therefore, we find some evidence that preschool children already have prior knowledge about standard units although – as Boulton-Lewis et al. (ibid.) point out – they do not understand them and may not develop conceptual understanding about their lengths and relations to one another yet. However, this understanding may become relevant in the process of length estimation. As Desli and Giakoumi (2017) analyze, third as well as fifth grade students estimated more successful if they used nonstandard units instead of standard units. Therefore, we aim to analyze how this (however incomplete) prior knowledge about standard units may affect their length estimation efforts.

Bringing the ideas about length estimation and unit concepts together with regard to preschool children and their previous experiences, it can be hypothesized that they are missing relevant knowledge in certain situations. Overall, neither length estimation nor standard units were systematically discussed at that point of their education, meaning that they might choose intuitive approaches that they either observed in their surrounding or developed during that situation. For example, due to the fact that they generally did not yet discuss the different standard units and their relations to one another, providing length estimations in a standard unit may not be possible. Therefore, this study focuses on the question of how preschool children deal with different situations of length estimation and what role standard units play in these approaches.

Research questions

As the literature review indicates, preschool children prefer to use standard units for measurement purposes. However, there is some evidence that their unit concepts as well as their conceptual knowledge about these standard units are still incomplete but may be relevant in order to estimate lengths in a standard unit. Therefore, we focus the following two research questions:

1. To what extend do preschool children use standardized units to estimate lengths?

In order to gain information about how the preschool children’s understanding of standard units affects their length estimates, we additionally focus on the estimations’ accuracy. If children only estimate a number and add a standardized unit that they know about, their estimates will be somewhat imprecise. Therefore, the second research question focuses this aspect.

2. How precise are the preschool children’s length estimates?

Methodological approach

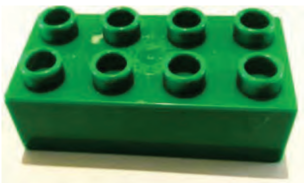
In order to analyze the role of standard units in preschool children’s length estimates, we conducted short interviews with standardized questions in Kindergarten with those children who were about to start elementary school only a couple of months later (school enrollment in summer 2021).

Interview design and administration

Four trained student teachers who interviewed the children individually conducted the interviews. Children's responses were assessed on a documentary sheet. The student teachers worked with a standardized interview script that held text about what the students should say as well as specifications about what kind of gestures they should use and what kind of material is given. The interview contained a couple of objects whose lengths should be estimated by the children. The children were asked in each of the five cases to estimate how long the object might be. The questions were intentionally held as open as in order to not limit children's responses and / or strategies. Of course, all the to-be-estimated-objects were standardized in their lengths as well.

In total, the children were asked to estimate the lengths of five objects. For item construction, we followed the suggestions by Heinze et al. (2018) and varied the different characteristics regarding the To-Be-Estimated-Object (TBEO)[2], e. g. its physical presence, accessibility, size and need to construct a representation. Table1 shows one example item of the interview.

Table 1: Task example and excerpt from the interview guide

	Material	**Material treatment**	**Question / verbal instruction**
T2[3]	Green brick (see picture) 	Place the green brick in front of the child. Move your finger along the longest side of the brick.	What do you think: How long is this brick? *Wait for the response.*

The other four items were constructed similarly. In T1, children were asked to estimate the length of a rope, T3 asked the children to estimate the height of a carton of milk, children were asked to estimate the height of the door in their kindergarten classroom in T4 and, finally, were asked to draw a line as long as a piece of toilet paper in T5.

[2] Benchmarks were intentionally not given (dimensions 5 & 6 by Heinze et al., 2018) and it was not prescribed whether children should estimate in a standard or non-standard unit (dimension 7) in order to leave the preschool children the choice what kind of approach they want to choose (they might provide an estimate in a standard unit but might as well just name an object that is about as long as the TBEO). However, the dimensions regarding the TBEO were varied in the five questions.

[3] T2 means second task. In tables 2 and 3, all entries in the line called 'T2' refer to this task.

Sample size and characteristics

In total, 189 preschool children participated in the study. All of these children were about to start school only about two months after the interview. The children attended 14 different Kindergartens in the north of Germany – more precisely in the areas of Mecklenburg-Vorpommern and Schleswig-Holstein.78 of the children are girls (41.3 %) and 111 boys (58.7 %). The children were averagely 5.99 years old – there were 171 children in the age of 6 years, only nine kids were five years old and seven children were seven years old[4].

Data analysis

In order to evaluate the amount of children who use standardized unit in length estimation (research question 1), we determined the absolute and relative frequency of children in our sample that gave an answer holding a standardized unit. In order to analyze the role of standard units in children's length estimation processes (research question 2), and how precise children's' concepts about standard units are, respectively, we analyzed the accuracy of children's estimates. For this second analysis, we only used the sub-sample of those children who estimated in a standardized unit because we cannot evaluate whether estimates such as 'the brick is as long as my hamster at home' is true without measuring the length of the hamster. In order to analyze the estimates' accuracy, we used the deviation of the child's estimate and the actual length of the TBEO. We clustered the children's results in the intervals that are generally chosen in research to code the accuracy of length estimates: if the estimate deviates equal or less than 10 % from the actual length, we consider the estimate as very precise. A deviation of $10\ \% < x \leq 25\ \%$ is still somewhat precise while a deviation between $25\ \% < x \leq 50\ \%$ is acceptable. If the estimate deviates between $50\ \% < x \leq$. 100% from the object's actual length, it can be considered imprecise. Finally, if the estimate deviates more than 100 %, the estimate is very imprecise. In order to gain additional information about the children's approaches and estimation accuracy, we analyzed whether the numerical values fit the right relation to one another. In this regard, we converted all given estimates into meters and evaluated whether the children gave adequate order relations between the measures.

Results

Regarding the four questions that required the children to estimate a length[5], it shows that a majority of the preschool kids use standard units for their length estimates. Due to the fact that a systematic introduction to length units occurs generally in second grade in Germany (Franke & Ruwisch, 2010), these results give another indication that children start with a great variety of prior knowledge in this field. Table 2 shows the amount of children using standard units for their length estimates, and what kind of unit they chose, respectively.

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[4] For two children, the age is missing.

[5] In T5, children were asked to draw a line that is about as long as a piece of toilet paper. Therefore, standard units were not named here.

Table 2: Amount of children using standardized units for their length estimates

	Amount of estimates given in mm	Amount of estimates given in cm	Amount of estimates given in m	Amount of estimates given in km	Total amount of estimates given in a standard unit
T1 (length of TBEO: 1.3 m)	1 (0.5 %)	18 (9.5 %)	119 (63 %)	6 (3.2 %)	144 (76.2 %)
T2 (length of TBEO: 6.2 cm)	4 (2.1 %)	43 (22.8 %)	90 (47.6 %)	5 (2.6 %)	142 (75.1 %)
T3 (length of TBEO: 20 cm)	4 (2.1 %)	41 (21.7 %)	87 (46 %)	6 (3.2 %)	138 (73 %)
T4 (length of TBEO: 2 m)	2 (1.1 %)	17 (9.0 %)	118 (62.4 %)	6 (3.2 %)	143 (75.7 %)

Table 3: Accuracy of preschool children's length estimates in T1 – T5

	Actual length of the TBEO	Range of the children's responses	Children with a difference < 10 %	Children with a difference 10-25 %	Children with a difference 25-50 %	Children with a difference 50-100 %	Children with difference > 100 %
T1	1.3 m	*Min*=10 cm *Max*=100 km	0	44 (30.6%)	0	32 (22.2 %)	68 (47.2%)
T2	6.2 cm	*Min*= 0.5 cm *Max*= 14 km	3 (2.1%)	6 (4.2%)	14 (9.9%)	18 (12.7 %)	99 (69.7%)
T3	20 cm	*Min*=0.8 mm *Max*=10000 km	6 (4.3%)	6 (4.3%)	6 (4.3%)	31 (22.5 %)	89 (64.5%)
T4	2 m	*Min*=10 cm *Max*=100 km	15 (10.5%)	3 (2.1%)	15 (10.5%)	25 (17.5 %)	85 (59.4%)
T5	12 cm	*Min*=2.5 cm *Max*=25 cm	7 (3.7%)	26 (13.8%)	29 (15.3%)	8 (4.2%)	70 (37%)

However, if we focus on the accuracy of the estimates (table 3), it shows that only a small amount of children reached an estimate that differs less than 50 % from the actual length of the object. It shows that the estimate of the majority of the children (generally about half of the sample) varies more than 100 % of the actual length while only some children estimated very precisely (discrepancy < 10%).

These results show that preschool children are aware of standard units and use them in length estimation situations to provide the lengths of different objects. However – as table 2 shows – they may not choose the different length units most appropriately and their length estimates are not very precise (table 3). This may indicate that they do not yet have precise conceptions about the standard units. The analysis about the order relation showed that 161 kids (79.3 %) gave all four estimates in the correct order (however we did not check for proportional differences). This indicates that preschool children are already able to assign objects with numbers that are given in the correct order relation. However, they may just add the name of a standard unit without really understanding their meaning.

Discussion

Since children get in touch with lengths as well as situations of length estimation in their everyday life, it can be assumed that children bring a lot of prior knowledge and skills in this field when they start elementary school. It is crucial for teachers to be aware of these preconditions in order to start efficient learning processes. The results of this study give hints about these starting points. It becomes apparent that the majority of preschool children provides an estimate using standard units, however, the estimates are often not very precise. This is in line with previous research about students' estimation accuracy and its connection to students' age (e. g. Huang, 2014; Desli & Giakoumi, 2017). However, if we focus on the relation between the estimated values, almost all children's estimates are in a correct order relation to one another. This may indicate that children know about standard units and use them in length estimation contexts but their understanding of the units is still incomplete (as indicated by the imprecise estimation). Therefore, the majority of children uses standard units to estimate lengths even though their understanding is not yet sufficient. These results closely connect to the findings of Boulton-Lewis et al. (1996). Supposedly, children merely estimate a real number and simply add a standard unit that they have heard of before.

Of course, there are limitations to this study. From our results we cannot reconstruct what exactly the children thought in their process of length estimation. We can only hypothesize their approaches from our results. In addition, the Kindergarten teachers may have talked with the kids about standard units before we arrived in their classes (even though we asked them not to do that) because the amount of kids using standard units is amazingly high.

Overall, we suggest that students' prior knowledge should be closely considered as a starting point of learning processes. Further research should provide evidence about how different starting points combined with different learning approaches lead to specific learning outcomes in the field of length estimation.

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A framework for analysing drawings as tools for mathematical reasoning

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The current paper presents a way of explaining and understanding children's drawings as tools for mathematical reasoning. The paper has two aims; to present a framework for analysing drawings as tools for problem-solving and describe how different drawings support children's mathematical reasoning in problem-solving. The study shows that drawings can offer different support for children's reasoning. Some drawings offer explicit tools to be manipulated for solving a problem. Some drawings are structuring tools, supporting systematic checking and re-checking of solutions, while others do not help solve a problem, but serve as tools for communicating with others.

Keywords: reasoning, collaboration, drawing, representation, early years mathematics

Introduction

Reasoning has acquired a more prominent role in several nations' curriculum, including Norway (Utdanningsdirektoratet, 2019). The Norwegian curriculum states that children should learn to 'follow, assess and understand mathematical chains of thought' and 'formulate their own reasoning to understand and to solve problems' (Utdanningsdirektoratet, 2019). Even though teachers are expected to facilitate reasoning on all school levels, some find it challenging to identify reasoning in young pupils' actions or words and, therefore, challenging to facilitate (Bragg et al., 2016). They also claim that the development of mathematical reasoning requires 'appropriate encouragement and feedback from [their] teacher who can only do this if they recognise mathematical reasoning in children's actions and words' (p. 523). According to Battista (2016), mathematical reasoning is about making conclusions based on evidence or assumptions after having manipulated and analysed objects, representations, and statements (p.1), and Bragg et al. (2016) claim mathematical reasoning consists of 'following a line of enquiry, conjecturing relationships and generalisations, and developing an argument, justification or proof using mathematical language'. In this study, reasoning is understood as children making conclusions or assumptions regarding problems after manipulating the task using drawing, and using drawing to follow lines of inquiry while solving tasks.

One way of developing reasoning skills is through paying attention to the different ways children create meaning and their representations. Mathematical representations capture the process behind mathematical concepts or relationships (Woleck, 2001) and play a critical role in children's conceptual and mathematical development (MacDonald, 2013). Mathematical representations can be symbols, physical objects, verbal language, or drawings used to understand mathematical concepts and relations and are a part of construction knowledge (Bobis & Way, 2018). Despite a large body of research on representations, little research focuses on one of the most important representations for the youngest mathematicians; drawings (Woleck, 2001). Drawings can serve as mediation or re-presentation of meaning that allows children's inner pictures and reasoning to become available to others. Thus, children's drawings play a role in meaning-making, problem-solving and early symbolism (Thom & McGarvey, 2015). Drawing is one of children's earliest (mathematical) representations (Papandreou, 2014; Woleck, 2001). Drawings and other written signs have a degree of permanence, allowing them to be re-examined, revised and argued for (Papandreou, 2019), which is why some claim that drawing play a central role in fostering mathematical reasoning for children (Saundry & Nicol, 2006).

By utilising children's representations of their understandings, like drawings or narratives, we are offered a window into their minds (MacDonald, 2013; Woleck, 2001). Children use drawings to bring their ideas or thoughts to the surface and to communicate them with others (Papandreou, 2014; Woleck, 2001). Learning to draw and drawing naturally occurs for most children, but the difference between drawing for fun and drawing with a purpose is big, and the transition from drawing out-of-school and drawing in-school contexts is a topic where several researchers claim teachers lack knowledge (Bakar, 2017; Woleck, 2001). Given the lack of research on drawings as representations (Woleck, 2001) and their central role in fostering children's mathematical reasoning (Saundry & Nicol, 2006), drawings should be offered more attention when researching reasoning and young children.

The project applies a sociocultural perspective on learning and development and considers knowledge to be developed and shared between people (Vygotsky, 1978). Although Vygotsky is mainly known for the relation between speech and reasoning, we can also relate his work to the drawing and reasoning. According to Vygotskian sociocultural theory, higher mental functions and human actions, like communication, are mediated by tools and signs (e.g. language and drawings) (Dahl et al., 2017). Tools are, within sociocultural theory, defined as the resources – both language-based and physical, available to us that we use to understand the world around us (Säljö, 2009, p. 21, my translation). Applying this perspective, viewing drawings as tools used to communicate and to understand the world around us, we can see the connection between drawing and reasoning.

This paper has two aims; the first is to present a theoretically anchored framework for analysing children's mathematical drawings as tools for mathematical reasoning. Which can be used for further research on drawing as a representation, both for researchers wanting to contribute or further develop the framework and teachers wanting to develop their pupils' competencies of using drawing as a meaningful representation and tool for reasoning. The second aim is to explain how different drawings support children's mathematical reasoning through some illustrative examples from two first-grade classrooms.

Framework for analysing drawings as tools in mathematical problem solving

The framework that is the starting point for what is presented below was initially developed by me for my master thesis (Kleven, 2019). It consideres two aspects of drawings in mathematics: the use of the drawing, and visual and multimodal aspects of drawings, and is a synthesis of other frameworks for examining drawings as a mathematical representation.

The first aspect concerns if and how drawings are used. It includes categories like used or did not use drawing, and different ways drawings that can be used for and of problem-solving (Saundry & Nicol, 2006). If used *for* problem-solving, the drawing is used for solving the problem, and drawing is both a process and a product (p. 57). In drawing for problem-solving, the drawing and problem-solving happen simultaneously. The drawing represents thoughts or internal pictures linked to children's mathematical reasoning. A drawing *of* problem-solving is produced after the task is solved, representing a solution (rather than a process). It functions as a tool for communicating with others (rather than solving the problem). Even though drawings of problem-solving are considered to be produced after solving the problem, Saundry and Nicol (2006) claim that pupils' artefacts when working on mathematical problems are a part of their reasoning and cannot be separated from it.

The second included approach to analysing drawings are the *visual and multimodal aspects* of drawings. The distinction between pictographic and iconic drawings is commonly used to describe visual aspects of drawings which could be useful for identifying and describing mathematical reasoning. Both pictographic and iconic drawings can support children's reasoning and problem-solving in mathematics, but at different times and in different ways. A pictographic (situational) drawing can, according to Rellensmann et al. (2017), help pupils better understand a problem by providing a way of organising the information provided in the task. However, simultaneously, a pictographic drawing can also include irrelevant details, getting in the way of an effective problem-solving process. In contrast, an iconic, mathematical drawing often only includes relevant elements, but given its abstraction, it often requires a higher level of mathematical skills to utilise. Further categories are based on Papandreou (2009), and concern whether the pupils include numbers, letters or words in their solution, and if the children use gestures or verbal language as a supplementary mode of communication. Although the table below (table 1) presents the whole framework, this short paper will only focus on a few categories and aspects. The colour-coding in the table is as follows: use of drawings in blue, and visual and multimodal aspects of drawings in purple.

Table 1 – A framework for analysing drawings as tools for mathematical problem solving

Category	Description and indicators of use
Used drawing	The pupil found one or several solutions using drawing.
Did not use drawing	The pupil only used numbers, letters or did not draw at all. Including the use of other concrete manipulatives (like counting on the fingers or using counters). Both correct and incorrect answers included.
Manipulative	Movement, like circles or lines, represent calculations or operations, similarly to physical manipulatives (Saundry & Nicol, 2006; Woleck, 2001). Drawings are used to organise and count the elements needed to solve the problem (Woleck, 2001) and serves as a placeholder for thoughts.
System support	A passive drawing without movement. Drawings are used in an elimination process by systematically sorting elements and creating a structure (Saundry & Nicol, 2006). The drawing is crucial for solving the task, and pupils often use it to count and re-count their solution.
Narrative	Pupils create a story with elements in the task or their surrounding life (Soundy & Drucker, 2009), then act out the story. Creativity, previous knowledge, and the ability to differentiate relevant and irrelevant information become explicit.
Dramatic representation	The pupil draws themselves as part of the problem (Woleck, 2001). E.g., a pupil draws himself pointing at a number line, showing the physical drama of solving the problem.

Imagery/ Visualisation	The problem is solved internally. The process is played out in their mind, and then they draw to *communicate* a solution (Saundry & Nicol, 2006). Information is still processed visually, even though it is not always visible on paper.
Pictographic	Is recognised by its realism compared to the elements in the task. E.g., If the task asks the pupils to put flowers into vases, the pupils will draw either flowers, vases, or both. Also called situational drawings (Rellensmann et al., 2017). Depict the surface of the problem have a low level of abstraction.
Iconic	Simple lines and shapes created to imitate the elements in the problem. It lacks the realism or affiliation to reality found in pictographic drawings. Also called mathematical drawings (Rellensmann et al., 2017). It depicts a mathematical structure and has a high level of abstraction.
Symbolic	Pupils include numbers or letters/words. Defined in this framework as the conventional numerals 0,1,2,3,4,5,6,7,8,9, all letters/words and non-conventional number symbols (Papandreou, 2009).
Gestures and verbal language	The pupils supplement their solution with gestures or use verbal language to support their communication.

The framework above shows different aspects concerning drawing, that one might consider when researching childrens mathematical drawings. Several other aspects were also considered in putting together the framework, and there is a possibility to build further on the framework, including categories and aspects relevant for the particular study one is conducting. The framework is meant to be a flexible framework, where one can make adaptions based on context and needs.

The examples below are from an Educational Design Research study investigating how teachers can facilitate productive conversations, where young pupils are provided with opportunities to develop their reasoning competencies. Video recordings of twenty-seven first-grade pupils (age 5-7) and three teachers have been collected to understand how pupils use drawings to reason and communicate in mathematics. Pupils worked in groups (of 2-4) collaboratively solving problem-solving tasks in mathematics. Participation was based on informed consent from pupils and parents, and the names included below are pseudonyms. For analysing how drawings were used, collected pupil work and video recordings were used. Video recordings were helpful in addition to the drawings because video recordings could help determine whether the children drew while solving the problem or afterwards. All video recordings were transcribed and analysed to supplement the analysis of the drawings. After categorising the ways of using drawings, I aimed to identify reasoning in the drawings, using the video recordings and transcriptions as supplementary information about the process. The examples below are from different sessions, where pupils work on the following problems: (1) 8 children share 12 cookies equally. How many cookies does each child get? And (2) A farmer has some animals that altogether have 14 feet. How many animals, and what animals does the farmer have?

Reasoning in drawings for problem-solving

The examples illustrate drawings used as manipulative and system support. In Figure 1a, drawing is used as manipulative to give one cookie to each child physically. We can see lines representing the process of sharing twelve cookies equally between eight children, resulting in each child getting one and a half cookies each. Based on the drawing, we can see that the pupils gave each child one cookie and then split the leftover cookies to give the same amount. Based on the drawing and the video recording of the two pupils collaborating, the drawing appears to be closely linked to their reasoning process of giving as many whole cookies as possible first and then splitting the rest equally as well.

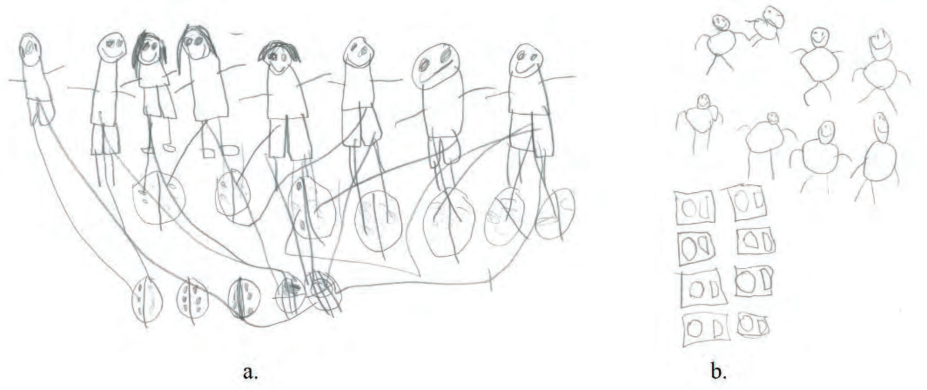

Figure 1a - Manipulative and 1b - System support

In comparison, Figure 1b shows a drawing used as system support for the same problem. The drawing provides an overview and organisation of the elements, but no operations or calculations are visible. In this drawing, we could need to turn to the video recording alone to identify whether the pupil put one and a half cookies in the lunch boxes at once, or if they put one cookie in each of the lunch boxes first, and then put half in each when it was not possible to give a whole cookie more to each child. Identifying the children's reasoning processes can be more difficult in drawings as system support, but the drawing still depicts some of the children's mental processes of sharing cookies equally.

Reasoning in drawings of problem-solving

For identifying imagery, one needs to be present during the process or have recordings (audio or video); this is because one requirement is that the problem is already *solved* when the drawing is produced. As an example of imagery, I would like to highlight Leo and Sam working on problem 2, the farmer problem (figure 2a below). Initially, Sam suggests five sheep, five cows and four hens while holding up five, five and then four fingers. This adds up to a total of 14 animals. After a while, Sam realised that his answer was wrong because it was 14 animals and not legs. Sam starts counting using his fingers and stops while holding up eight fingers. He says, "Two cows", and then Leo draws. Sam continues using his fingers to count and ends up at one sheep and one hen. The boys count all the feet and end up at 14 feet and are satisfied. Sam solved the problem before they made the drawing, and the drawing was used to communicate with others, not to solve the problem. The boys did, however, use the drawing to check their solution.

Figure 2a - Imagery and 2b – Visual aspects

Visual aspects and multimodality

As a short example of an analysis of visual aspects of a drawing, I would like to present a drawing made by a group of four pupils working on problem one, the cookie problem. Malin, Leo, Eric and Martin made an iconic drawing of cookies and children (figure 2b), in addition to a number line of the even numbers from 2-12. The video recordings show the four pupils discussing and trying to agree on which of the circles are children and which ones are cookies because they used the twelve circles to count both children and cookies. This sparks a discussion about if a circle is a cookie or a child, making the equal sharing of the cookies difficult. They are not able to solve the problem using drawing. In addition to using an iconic drawing, we can see that the group used both conventional number symbols and the words for children and cookies in Norwegian. The abstract nature of the drawing made it difficult for the children to reason both individually and in collaboration. A more pictographic drawing with a clear difference between children and cookies could have made this easier.

Discussion

This study aimed at presenting a theoretical framework for describing and analysing how children use drawings in mathematical problem-solving and show some examples of how different drawings can support mathematical reasoning. The framework was presented as a whole, although I only focused on small aspects of the framework in this study. However, the framework is a synthesis of different theoretical frameworks for describing different aspects of drawing in mathematics, enabling research on many different areas or aspects connected to mathematical drawings.

This study further shows that different drawings can support differently in reasoning processes. In drawings as manipulatives (Figure 1a), we can see that a manipulative-drawing allows children to use the drawing to solve the problem physically. A drawing as a manipulative can support the pupils reasoning process by giving the pupil something concrete (external) to solve the problem, thereby allowing the internal processes to become external. All problem-solving processes are visible on paper and are useful for teachers to identify how the pupils solved the problem without being present during the process. Looking back at Battista (2016), defining reasoning to concern manipulating a problem to make decisions or conclusions, a drawing used as a manipulative explicitly enables the use of a drawing to solve the problem, making the drawing a tool for the child's reasoning. As we can see in figure 1a, the lines enable the children to give one and one cookie, keeping tabs on the division happening in the problem. In drawings as system support, more extensive parts of the mathematical reasoning happen internally and therefore "beyond reach" of the drawing or those witnessing the process. Drawings as system support can support children's reasoning by providing them with a tool for organising elements systematically, making it easier to check multiple solutions, and keeping tabs of all elements of the problem without being overwhelmed by the problem itself (Woleck, 2001). As figure 1b shows, the lunch boxes with cookies allow the pupil to count and check his solution, both whilst solving the problem, and afterwards. Reasoning can also be identified in the process and products of pupils using drawings as *imagery*. A requirement for imagery is that the problem is already solved when drawing occurs. The drawing is then used to communicate or show the process or product to others. The drawing serves as mediation or re-presentation of meaning, allowing pupils inner pictures and reasoning to become available to others. Applying a sociocultural perspective allows us to think about the process (and product) of drawing as a direct re-presentation of mental pictures, and therefore as reasoning made available to others. To identify reasoning in drawings of problem-solving, we need to turn to children's verbal utterances and descriptions of their drawings, as illustrated above in the discussion in figure 2a.

Regardless of how the children use drawings, whether as manipulatives, system support, narratives or dramatic representations, all drawings have a degree of permanence to them, allowing them to be discussed, revised and argued for in collaboration with others or for oneself (Papandreou, 2019). By gaining knowledge on children's drawings as mathematical representations, we are offered a window into their minds and understanding of mathematical problems and concepts (MacDonald, 2013; Woleck, 2001). Though examining the children's drawings in the cookie-sharing context, one can gain insight into how young children treat the concept of dividing equally, which again can tell us something about their understanding of division more generally. Facilitating reasoning and sense-making should be the primary goal of mathematics instruction (Battista, 2016). It is, therefore, interesting to investigate how teachers can achieve this, particularly in the early years, where the amount of empirical research is limited.

A possible further analysis of data is needed, in order to see if there is possible to identify a connection between the different ways of using drawing (as a manipulative, system support and so on) and whether the students are able to present understandable chains of reasoning and arguments for their statements when communicating with others. Further studies into the use of more than one modality simultaneously and the quality of young children reasoning is another aspect which it will be both exiting and relevant to look at in the future.

The study had its limitations. What is presented above are empirical examples from one study, and should for that reason, not be generalised without considering all contributing factors. In addition, the framework could benefit from studies in other contexts with more participants with other prerequisites. Given the limitations in scope, several of the framework categories were not discussed, and more studies utilising and further developing the framework is both welcomed and needed to improve the quality of the proposed framework.

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I didn’t notice that there was mathematics in kindergarten: Polish parents’ views about Norwegian kindergartens

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Immigrant parents’ views about mathematics education are rarely investigated, yet these are likely to affect their relationship to early childhood education and care (ECEC) and potentially their adopted country. In this study, Polish parents, who had immigrated to Norway, were surveyed about their views about mathematics education for young children, including what was made available in ECEC institutions in Poland and in Norway. We investigated whether the responses showed that parents’ views about what is mathematics in early childhood and how it should be taught or learnt could be related to parents’ considerations of the power and authority linked to their position as immigrants. The results have implications for multicultural ECEC and policy makers.

Keywords: Immigrant parents, counting, playing, power and authority

Introduction

Immigration in Europe is often described as a recent phenomenon and as such requires policies to support integration. Dustmann et al. (2012) described how different European countries implemented integration policies, including requirements for language fluency, as part of the immigration process, and how the contributions that immigrants bring to a society are frequently considered in relationship to the economic benefits for the receiving country. Education is considered important for ensuring that economic benefits are achieved from children of immigrants gaining the necessary skills to be employable (Sadownik, 2018). As stated by Bratsberg et al. (2012), “the convergence of educational attainment across generations [of immigrants] to that of natives is commonly seen as a key indicator of successful integration and several analysts emphasize education as the key pathway for integration of immigrants and their descendants” (p. 212). In integration policies, the knowledge about education that immigrant parents have is rarely acknowledged. Instead the focus is generally on how immigrant parents’ values about education could/should be brought in line with those of the adopted country (see for example, Sadownik, 2018). This can situate existing educational culture, such as in early childhood institutions, as being static and non-changing. In order to consider what immigrant parents’ views on mathematics education for young children could contribute to discussion about the role of education, we describe Polish parents’ views about mathematics education in Norwegian early childhood centres, known as *barnehage/r* (kindergarten/s). In Norway, there are more Polish immigrants each year than any other nationality (Østby, 2016). We particularly consider how Polish parents’ views about what mathematics is and how it should be learnt are related to considerations of power and authority to have those views heard by Norwegian barnehager.

On the whole, very little research has investigated parents’ views about ECEC. Nevertheless, Van Laere and Vandenbroeck (2017) show that in the Netherlands parents had a variety of views, such as a tension between the importance of developing children’s social competence so that children are not excluded and the importance of readying children for school. However, Sadownik (2021) highlighted that when parents considered their own children’s needs then they could move away from their value positions. In Sweden, Lembrér (2018) found that Polish parents' views about mathematics and how learning possibilities should be provided to children were aligned to those in the Swedish curriculum for early childhood, such as valuing learning mathematics through play. Similarly, Takeuchi (2018) found that immigrant Filipina mothers only taught their children multiplication strategies used in Japanese schools rather than the strategies they themselves had been taught and knew to work. Takeuchi (2018) described "how identity and power are strongly tied with appropriation of cultural tools" (p. 41). In a review of the literature about partnerships between early childhood professionals and immigrant parents, Norheim and Moser (2020) highlighted that asymmetrical power relationships often reduced the possibilities for parental engagement. The results from the studies of Lembrér (2018) and Takeuchi (2018) suggest that these power relationships may affect parents' views about what is valued as mathematics and how it should be learnt. This could have an impact on their engagement with educational institutions such as ECEC and on their integration.

To gain insights into this complexity, we investigate survey responses from 36 Polish parents on mathematics education in ECEC in Poland and in Norway. The research questions are: How are Polish immigrant parents' views of mathematics education for young children connected to authority and power relationships? How are their views likely to affect possibilities for integration in Norway?

Theoretical perspectives

For this study, we use three frameworks: Bishop's (1988) six mathematical activities to identify the mathematics that was in focus in the parents' comments; Walkerdine's (1988) distinction between pedagogic and instrumental interactions; and Wertsch's (1998) discussion of power and authority in mediated action. These frameworks we combine to be able to identify different aspects of the parents' responses. In doing so, we take licence from their perspective on networking strategies of Prediger et al. (2008), who state that "combining theoretical approaches does not necessitate the complementarity or even the complete coherence of the theoretical approaches in view" (p. 173), We then look at how these aspects, what is mathematics, how should it be learnt, and the ways that parents express their power and authority through their views about how barnehager work with mathematics, provide insights into the complexity of parents' views. We investigated whether the responses connected to the different aspects were related.

To determine the kind of mathematics that parents valued, we used Bishop's (1988) six mathematical activities. These are the basis for mathematics in the Norwegian barnehage curriculum (Reikerås, 2008) and have been used extensively in research on ECEC in Scandinavia (see for example, Fosse et al., 2020) . The six mathematical activities are: playing; explaining; designing; locating; measuring; and counting. Of these, counting is often emphasised by Norwegian parents (see Lembrér, 2020).

To identify parents' views on how mathematics should be learnt, we used Walkerdine's (1998) distinction between tasks where the focus was on the child learning specific number understandings (e.g., counting buttons in a cardigan), which she labelled pedagogic, and tasks where the child was engaged in solving an actual problem (e.g., finding the right amount of ingredients for a recipe), which she labelled instrumental. In earlier research (see for example, Helenius et al., 2016), this distinction has been used to determine how children are expected to learn mathematics in ECEC, either by being directly taught (pedagogic) or in the course of solving a problem that the child is invested in answering (instrumental). Sadownik (2018) highlighted that whereas in Poland early childhood education is focused on introduction to academic subjects, Norway's approach emphasised play and democratic participation. This can be considered a distinction between valuing a pedagogical and an instrumental approach, respectively, for engaging in mathematical learning situations.

To better understand how power relationships affected parents' views on the importance of different mathematical activities and the ways that mathematics should be learnt, we built on Takeuchi (2018) research by adopting Wertsch's (1998) ideas about the valuing of knowledge. In a discussion about mediated action, Wertsch (1998) described how "the acceptance of a particular utterance by an individual agent is not simply a matter of dispassionate, reflective choice" (Wertsch, 1998, p. 66), because the society provides input on what should/can be valued, and this can act as a source of authority within a situation. Agents can either reject, accept or be somewhere in the middle, the valued, societal views, depending on the power they see themselves as having within a situation, because "cultural tools are not always facilitators of mediated action, and agents do not invariably accept and use them; rather, an agent's stance toward a mediational means is characterized by resistance or even outright rejection" (Wertsch, 1998, p. 144). When parents with one set of values about engaging young children in mathematical activity from their home country, meet a different set of values in their country of residence, they can either appropriate (more or less), resist or reject the new set of values. Therefore, it is important to understand what immigrant parents value as mathematics and how it should be learnt or taught, and whether they consider this as being in conflict with the values of their home or adopted country.

Methodology

The data are responses to an online survey consisting of 14 questions answered between May and September 2017. The survey was provided in Polish and the link was made available on the website of the Polish organisation "Moja Norwegia" ("My Norway"). The survey asked about the parents' memories of mathematics in Polish ECEC, their children's experiences of mathematics learning in Polish and in Norwegian ECEC, and what would they tell barnehager about their children learning mathematics. These open-ended questions asked for examples of memories or experiences, which we considered were likely to provide insights into the complexity of parents' views. The second author, who is a native Polish speaker, translated the answers into English.

In this paper, we focus on responses from 36 parents who made an explicit or implicit reference to early childhood education in Norway and/or in Poland. All these parents had experiences of their children attending barnehager, some had themselves – or had children who had – attended ECEC in Poland. The responses do not describe what the situation is in either Norway or Poland but rather relate what the parents valued in these situations.

References to mathematics in the parents' responses were categorised according to Bishop's (1988) six mathematical activities. For example, when the parent described using number words to identify the total amount of things, the response was classified as being about counting. To classify what was valued as the ways that children should learn, we categorised the responses on whether the parents valued children learning through everyday situations and solving problems they were interested in (instrumental), or whether it was implied that an adult would teach the children (pedagogic) (Walkerdine, 1988). Finally, we looked for explicit (de-)valuing of what occurs in Norwegian or Polish ECEC to gain insights about whether there was an acceptance, resistance or rejection of a specific set of values (Wertsch, 1998). The combination of these results gave us insights into the role of power relations and authority that may have influenced parents' views.

Results and discussion

The results from the three analyses indicated that they were related and so in presenting them, we placed each parent's responses on a continuum, with specific aspects of the responses clustered at each end. The continuum was one way of showing the complexity of the relationship between the different aspects which enabled us to answer the research questions. At End 1 of the continuum, we placed the five parent responses, which identified several mathematical activities, including counting, described children as learning through everyday problem-solving situations, and depicted barnehager as providing good mathematical learning possibilities. At End 2, we situated the six parent responses in which counting was valued exclusively, children were expected to be taught mathematics, and approaches to mathematics in Polish ECEC were valued. The remaining 25 parents' responses were placed in between these two ends.

Parents' responses at End 1

The parents who valued the approach to mathematics education in barnehager tended to value a range of mathematical activities as being important for children and considered that learning should happen through everyday situations, play and problem solving. For example, Parent3 (P3), whose child(ren) had only attended barnehager, described their mathematical learning as involving, "sorting objects into collections, large/small/medium comparisons, measuring e.g., volume of liquids, playing with shapes, learning how to count". This indicated that more than just learning to count was valued. In responding to a question about why they thought these tasks could help children learning mathematics, P3 wrote:

> Mathematics is the science of abstract thinking. It's good that the beginnings were based on specific examples. Playing with water and in the sandbox is associated with measuring, that is, it familiarizes the child with the concept of volume. Puzzles and blocks are practical science of geometry. Counting and nursery rhymes are an introduction to algebra.

In this response, P3 makes connections to mathematics children will learn at school but indicates that learning happens through play as part of everyday interactions. This was reinforced in the response to the question about what parents would like to tell the barnehage:

> I am very happy with the way children are taught mathematics in our (Norwegian) kindergarten. Children learn it casually, on specific examples. The road to abstract thinking goes gradually, starting with things that children know, that they can touch. Thanks to this they get used to mathematics as a natural part of life. I am very happy about this approach.

Here, how barnehager support children to learn in everyday situations was considered valuable. This acceptance of the adopted country's values is very similar to what Lembrér (2018) found in Polish parents' views of Swedish ECEC.

Sometimes parents at this end of the continuum made explicit comparison with Polish ECEC, "At the Norwegian kindergarten, he focuses on learning logical thinking and associations. In the Polish ECEC, you go directly to memorising activities such as addition and subtraction" (P1). For P1, the barnehage approach was more valued than the Polish one, which they considered was based on memorisation. At this end of the continuum, the parents seemed to have appropriated cultural tools (Wertsch, 1998) from barnehager about mathematics and how it should be learnt so that they had adopted the views as their own, with no indication of resistance to these ideas.

Parents' responses at End 2

At the other end of the continuum, parents emphasised the importance of counting alongside children needing to be taught, and that Polish ECEC provided a better approach to mathematics education or that Norway lacked a good approach. For example, in the question about their own experiences of ECEC in Poland, P6, whose children had also attended ECEC in Poland, stated, "many mathematical play activities, counting fruit, objects, furniture etc, addition, subtraction, division, multiplication through play activities. Very well-prepared staff, nice atmosphere during play activities". When answering a question about why they thought situations would help develop children's mathematics learning, P6 reinforced the importance of children learning counting, "Because children are beginning to understand the connection between numbers and objects". Although P6 described situations in the Polish ECEC as play, the focus on counting things and the need for well-prepared teachers indicated that teachers were expected to be in control.

In response to the question that asked parents to describe a situation where children learn mathematics in barnehager, P6 stated, "instead of splashing around in the mud outside, they could be doing something useful". Other parents also commented on how children being outdoors so much, a valued activity in Norway (Sadownik, 2021), restricted the possibilities to learn mathematics in more appropriate ways, such as inside and being taught, "he learns outdoors when counting cones, stones, there are no activities" (P21). At this end of the continuum, the strong valuing of counting and the need for teachers to explicitly teach, often appeared in conjunction with a devaluing of barnehager:

> There is a tragic level of education in Norwegian kindergarten compared to any preschool in Poland. The Norwegian kindergarten is a children's storage room until parents take them home. I am very disappointed; I plan to return to Poland because I see that children do not learn anything here and only at school from the first Grade, do they start learning anything. (P6)

The language in these responses was often more emotive than the responses placed at End 1. Similar to P6, P10 responded to the question about what they would tell barnehager about their children learning mathematics by stating "a disaster". P19 highlighted the impact on future schooling, "it is a very important but neglected field of science, especially in kindergartens, which unfortunately later negatively affects school". These responses suggest that resistance and rejection of cultural values (Wertsch, 1998) of the adopted country may have led to these parents using stronger forms of language to highlight both that they have an alternative set of values and that these values would provide better possibilities for their children's future. Norheim and Moser (2020) found that immigrant parents faced difficulties in expressing disagreements with early childhood professionals, because of a lack of language and knowledge of how to approach professionals, exacerbated by an asymmetry in power relations. The survey may have provided a safe place for the parents to express their dissatisfaction with what they saw as restricted opportunities for their children to gain the academic preparation they felt their children needed for school.

For P6, whose dissatisfaction with mathematics education in barnehager had resulted in her considering taking her children back to Poland, integration as envisioned in the policy documents could be seen as failing, with the possibilities for economic gains for the adopted country from the parents and their children reduced. This was not because the parents did not value education but because they consider barnehage education as limiting their children's possibilities. As Wertsch (1998) stated, "a focus on resistance and rejection leads one to consider a host of issues that do not arise when one assumes that cultural tools are friendly helpers" (p.145). In the responses at End 2, these issues included a concern for their children's future school achievement.

Parents' responses along the continuum

The parents whose responses fell between the two ends of the continuum presented different views about how they viewed mathematics and how it should be taught/learnt. For example, P47 provided a range of learning situations, connected to different mathematical activities, but in reflecting on what should be said to barnehager, raised both positive and negative concerns.

> I am happy that mathematics is an integrated part of children's play, and that they meet with it on their own terms. Certainly, it is possible to nurture a child's interest in mathematics by supporting play. Another issue is whether this freedom of access to games that stimulate mathematics can increase the differences between children. So that children who are not interested in mathematics can avoid it [play activities which include mathematics] and not be noticed.

Although the barnehage approach to mathematics education seemed to be appreciated, P47 was concerned that teachers needed to notice who might be excluded or exclude themselves in engaging with mathematical activities. Several parents raised the need to improve the quality of barnehage staff. For example, when describing the benefits of learning through play, P12 wrote, "I think that teachers, especially those without education, should be involved in courses that make them more aware about it". The discussion of barnehage staff qualifications has regularly been in the news (see for example, Kunnskapsdepartementet, 2015). This may have provided parents with cultural tools (Wertsch, 1998) to support stating that staff needed to be better educated, which may have helped them to resist appropriation of the values connected to mathematics education in barnehager.

Other parents were less happy with what happened at barnehager, but, implicitly or explicitly, did not completely agree with the approach in Polish ECEC. P38 stated that children in Polish ECEC learnt mathematics through everyday activities, but in a class, suggesting set tasks. In barnehager, P38 stated that they did not learn mathematics, "only picture counting". However, when asked about what they would tell barnehager about their children learning mathematics, P38 wrote, "counting, for example, to 20 is OK, but not adding or subtracting as they teach in Polish preschools". This indicates that P38 considered that barnehager should do something, but they did should not adopt the approach of Polish ECEC. Similarly, P46 expressed disappointment that "unfortunately" mathematics was not taught/learnt in barnehager but offered the suggestion that this could be done in a way which echoed the valuing of play, "it is worth starting the adventure with mathematics through play". Hence, P46 criticised the lack of mathematics education in barnehager, while also suggesting that they should adopt an instrumental, rather than a pedagogical approach to mathematics.

The views that lay between the two ends suggest that the majority of parents evaluated what they knew about mathematics education in Polish ECEC and in barnehager and resisted appropriating some aspects of both. (Wertsch, 1998) described how actors can use alternative knowledge, such as in our case about mathematics education in Polish ECEC, to invoke an authority structure in order to be seen as knowledgeable. Given that immigrant parents struggled to have their views recognised and valued in barnehager (Norheim & Moser, 2020), their evaluating of cultural tools to do with mathematics and how it should be taught could be seen as a way to have their knowledge recognised as valuable and hence as having authority.

Conclusion

In this paper, we have analysed Polish parents' views about their children engaging in mathematical learning opportunities in barnehager. Barnehager are required to make available pedagogical plans and to provide mathematical learning experiences (Kunnskapsdepartementet, 2017). Yet, parents, such as P6, did not consider that there was any mathematics education in barnehager, similarly P8 wrote "I didn't notice that there was mathematics in kindergarten". As pointed out by Takeuchi (2018), cultural values can come into conflicts when there are no opportunities to discuss alternative approaches to mathematics education. The Polish parents came with expectations about how their children would engage in mathematical learning situations. Although some parents were able to adapt to the new cultural values in Norway, they did so by denouncing the approaches in Polish ECEC suggesting that integration is only achievable if they give up some of their Polish values. However, rejection of the barnehage approach to mathematics education of learning through play caused conflicts with Norwegian values.

Wertsch (1998) described appropriation, resistance and rejection of cultural tools. Present immigration policies with their emphasis on the role of education seem to assume the acceptance and thus the appropriation, of the adopted country's values about what should be learnt and how. However, the responses in the survey suggest that this is not necessarily the case. Many parents resisted or even rejected the acceptance of barnehage approaches to mathematics education, even if they had master knowledge of them. This is likely to have an impact of the integration of Polish immigrants into Norwegian society and the potential economic outcomes.

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On the use of boardgames to develop young children's number sense

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With the aim of nourishing the discussion that has taken place during CERMEs in TWG13, we present a review of results presented in previous CERMEs about the possible roles of boardgames in developing young children's number sense. Several contributions in CERMEs have shown that difficulties related to teacher education and curriculum design emerge when analyzing the boardgames that are proposed and their management in classrooms. However, the discussion about the contribution of boardgames to preverbal number sense is still underdeveloped when compared to literature from research journals. Also, the potential of boardgames in providing challenging mathematical tasks seems still unresearched. Hence, possible future paths of research for the use of boardgames in early years mathematics emerge.

Keywords: Boardgames, number sense, literature reviews.

Introduction

Play can be considered as the main learning experience for young children in general, and for early years mathematics in particular (Schuler, 2011). Among the many possible playful activities in which children may engage, we can roughly distinguish between free play and guided play, the latter consisting of adults structuring of the play environment but leaving control to the children within the environment (Weisberg et al., 2013). In this contribution we will focus on guided playful activities, to which we will refer as games. There is a large body of research about the use of games in learning (game-based learning), but not all kinds of games have been studied with the same attention. While literature about videogames is getting larger and larger (e.g. Yong et al., 2021), boardgames were rarely researched (Ramani & Siegler., 2008). We consider as boardgames all those games that are played on a printed surface by one or more people usually sitting around a table (Parlett, 2018); they might include the use of cards or dice. As testified by the available literature, boardgames can play an important role in the teaching and learning of mathematics (see next sections). Such relevance might be even higher at early grades, as stated in some contributions from previous CERMEs (e.g. Schuler & Wittmann, 2009; Schuler, 2011; Tubach, 2015).

The aim of this paper is to nourish the discussion that has taken place during CERMEs in TWG13 by summarizing results presented in past proceedings. Such review of the literature is intended to suggest further limits and opportunities in using boardgames in early mathematics education. Our discussion is limited to number sense; the next session is devoted to explaining what we mean by that.

Number sense

The construct of 'number sense' is present since many years in the discourse of researchers in mathematics education in general, and in the ERME community in particular (Rezat & Ejersbo, 2018). Several definitions have been proposed, mostly in the form of a list of abilities that a child should develop in order to show number sense. Such lists are often not equivalent. In this contribution we adopt the classification provided by Andrews and Sayers (e.g. Andrews & Sayers, 2015) who distinguish three categories of number sense.

Preverbal number sense includes the innate abilities necessary for quantitative understanding. Research literature (within and outside mathematics education) shows that very young children can discriminate small quantities (e.g. subitizing) and arrange them as a linear representation, called 'mental number line' (e.g. Deahene, 2001).

Applied number sense refers to competences that prepare the learners for the adult life. Applied number sense should enable a person to

> look at a problem holistically before confronting details, look for relationships among numbers and operations and […] consider the context in which a question is posed; choose or invent a method that takes advantage of his or her own understanding of the relationships between numbers or between numbers and operations and […] seek the most efficient representation for the given task; use benchmarks to judge number magnitude; and recognize unreasonable results for calculations in the normal process of reflecting on answers (Reys 1994, p. 115, as quoted in Andrews & Sayers, 2015).

Finally, there is a set of intermediate abilities that are developed thanks to instruction, usually between the end of preschool and the beginning of primary school; then they are particularly relevant for early years mathematics. This set of eight abilities, called *foundational number sense*, is listed in Table 1.

The distinction between preverbal, foundational, and applied number sense will serve as a framework for organizing our presentation of the reviewed literature. The next sections are devoted to describing if and how each of these three types of number sense was considered in past contributions to TWG13 about boardgames. Furthermore, we will list limits evidenced by researchers.

Selection of papers

The papers selected for the presented review are taken from proceedings of the last six editions of CERME. A mapping review of all the abstracts of the paper presented in the working group about 'Early Years Mathematics' (TWG14 in CERME6, when it was founded; TWG13 in the following edition) served for a first phase of selection. Among the 124 contributions, we selected those explicitly referring to numerical abilities and to a playful/game context. We then realized a second phase of selection by reading the full text; only papers specifically including boardgames were left. We ended up with the four papers listed in Table 2.

The fact that a low number of contributions were selected sustains the claim that research about boardgames is still underdeveloped. Surely, TWG13 is not the only TWG interested in boardgames and number sense (e.g. Sensevy et al., 2001), however as we will discuss later, we believe that this topic is particularly relevant for TWG13. We can notice that papers about these topics appeared in this working group during several editions of CERME; we may then affirm that a discussion has started, and there is still room to widen it. The four selected papers have been categorized referring to the framework introduced above. Results are presented in the following sections.

Boardgames and preverbal number sense

Research on preverbal number sense usually focuses on children aged 0–6 and then this stream of research should be considered as part of research in early years mathematics education. Indeed, TWG13 has collected contributions about subitizing (e.g., Schöner & Benz, 2017), but none of them included the use of boardgames. One of the selected papers refers to subitizing (Schuler, 2011), but it is not the main focus of research. Surely this fact does not depend on the impossibility of using boardgames to assess or develop preverbal numbers sense. For instance, we may notice that many traditional boardgames (like Shut the Box) include the use of dice. Traditionally, cubic dice show quantities through dots arranged in a canonic way. Research shows how subitizing gets easier when children recognize patterns in the arrangement of dots; furthermore, developmental dyscalculia can cause a deficit in the estimation of canonically arranged dots (Ashkenazi et al., 2013). This fact suggests that playing boardgames can help in recognizing specific difficulties that might depend on inexperience with canonical arrangement of dots or on learning disabilities.

Table 1: Components of foundational number sense (Andrews & Sayers, 2015)

(1) Number recognition	Recognition of number symbols; their vocabulary and meaning. Ability to identify a particular number symbol from a collection and name a number when shown.
(2) Systematic counting	Counting systematically to twenty and back or count upwards and backwards from arbitrary starting points; knowing each number's position in the sequence of all numbers.
(3) Awareness of the relationship between number and quantity	Not only understanding the one-to-one correspondence between a number's name and the quantity it represents, but also that the last number in a count represents the total number of objects.
(4) Quantity discrimination	Understanding magnitude and comparing of magnitudes. Use of language like bigger than or smaller than.
(5) Understanding of different representations of number	Understanding that numbers can be represented differently, including the number line, different partitions, various manipulatives and fingers.
(6) Estimation	Estimation, whether the size of a set or an object. Moving between representations of number; for example, placing a number on an empty number line.
(7) Simple arithmetic competence	Performing simple arithmetical operations, transformation of small sets through addition and subtraction.
(8) Awareness of number patterns	Extending simple number sequences and identification of a missing number in simple number sequences.

Another clear example is given by research conducted on the development of the mental number line using linear boardgames (boardgames with linearly arranged, consecutively numbered, equal-size spaces, e.g. Chutes and Ladders, Game of the Goose) – we are here referring to research that was not presented in CERME proceedings, but in research journals. According to Siegler and Booth (2004) and to Ramani and Siegler (2008):

> In such games, the greater the number in a square, the greater (a) the distance the child has moved the token, (b) the number of discrete moves the child makes, (c) the number of number names the child has spoken, (d) the number of number names the child has heard, and (e) the amount of time since the game began. The linear relations between numerical magnitudes and these visuospatial, kinesthetic, auditory, and temporal cues provide a broadly based, multimodal foundation for a linear representation of numerical magnitudes. (Ramani & Siegler, 2008, pp. 376–377)

Research shows that playing these boardgames strengthen preschoolers' number line estimation, magnitude comparison, numeral recognition, and counting skills (Siegler & Booth, 2004; Ramani & Siegler, 2008). Also, evidence shows that playing boardgames correlate positively with numerical knowledge, while this is not the case for videogames and card-games. Children from middle-income backgrounds reported playing more boardgames, and fewer videogames, than age peers from low-income backgrounds. This is considered as one of the possible explanations for differences in the development of preverbal number sense in relation to socio-economic status (Ramani & Siegler, 2008). Whyte and Bull (2008) have compared linear boardgames with card-games and found that card-games can improve some aspects of children's number sense, but not numerical estimation (evaluated as positioning on a number line). Playing with linear boardgames helps children to shift from a logarithmic to a linear representation of numerical magnitudes (Whyte & Bull, 2008).

Table 2: Selected papers from past CERME proceedings

Author(s)	Title	Edition
Dorier & Maréchal	Didactical analysis of a dice game.	CERME6 (TWG14)
Schuler & Wittmann	How can games contribute to early mathematics education? A video-based study	CERME6 (TWG14)
Schuler	Playing and learning in early mathematics education–modelling a complex relationship.	CERME7 (TWG13)
Tubach	"If she had rolled five, she'd have two more": Children focusing on differences between numbers in the context of a playing environment.	CERME9 (TWG13)

Boardgames and foundational number sense

All the contribution selected for our review focused on foundational number sense. The quality of the game and the role of the educators (parents or teachers) appear to be central for all the authors.

Foundational number sense needs explicit teaching (per definition) and thus the choice of a boardgame should be guided by the aim of a teaching intervention. Following Brousseau's Theory of didactical situations, an appropriate game is selected when it allows "bringing together a 'milieu' and a 'player', with this game being such that a given piece of knowledge will appear as the means of producing winning strategies" (Brousseau, 1998, p.57). The equilibrium between the game and the mathematical content seems hard to achieve, as has been shown by contributions to previous CERMEs. On one side, the game should be mathematically productive, meaning that the game materials should be reinterpreted as representations of mathematical relationships (Tubach, 2015).

On the other side, it is important that the game is an actual game to children for exploiting the game's idea and affordance (Schuler & Wittmann, 2009). However, there is also the risk that a game remains just a game. As reported by Schuler (2011, p. 1920) while referring to a boardgame about counting and comparison of numbers:

> Mathematical potential develops through the educator's comments on the game's course, through questions that stimulate explanations, reflections on actions and thoughts, and reasoning. She has to communicate individually challenging rules through stimuli, comments, questions and requests what requires a sensitivity for possibilities and variations in the games course.

Dorier and Maréchal (2009) refers about teachers selecting games based on the pleasure they are supposed to give to students, while the mathematical content remains secondary. They analyze a game called 'Turn the Dice' proposed for first grade in the official curricular material of the French-speaking Switzerland. By their analysis, to play correctly, students should know how to make sums correctly. However, if they do not, they may play anyway, because nothing in the managing of the game is organized to provide any feedback. In their words "nothing is organized didactically for them to learn sums, they have to know, but they can make errors without being corrected, except if the other player knows better or the teacher is there to correct" (Dorier & Maréchal, 2009, p. 2580). They also observed a teacher conducting the game in one of her classes and found how she had probably underestimated the difficulty of the game.

While research elsewhere as shown how boardgames may provide a good opportunity for the development of foundational number sense (Peters, 1999; Stebler et al., 2013), the discussion going on in TWG13 has pointed out possible limits for the exploitation of such potential.

Boardgames and applied number sense

Potentialities of boardgames for developing applied number sense are as many as the different representations of numbers, calculation algorithms, and so on. In the context of TWG13, the interest could be in understanding how boardgames may help, in the context of early years mathematics, to foster the development from foundational to applied number sense. In one of the selected papers, Dorier and Maréchal (2009) note that even simple games can hide interesting opportunities for introducing more complex mathematics. For instance, they show how, by slightly changing the rules of the game, 'Turn the Dice' shows strong similarities with the famous game 'Race to 20' which offers interesting opportunities to introduce the Euclidean division (Sensevy et al., 2001).

This example helps us in introducing an opportunity provided by boardgames that has been studied scarcely. In their paper from the *Journal für Mathematik-Didaktik*, after analyzing children playing the game Shut the Box, Stebler and colleagues conclude that:

> This first exploratory analysis leads us to the hypothesis that boardgames like Shut the Box can provide a high-quality teaching and learning arrangement, offering cognitively activating and challenging learning tasks, adaptive for different levels of mathematical competencies and allowing for diverse strategies, embedded in a collaborative setting (2013, p. 172)

Such hypothesis needs confirmation, but the possibility of using boardgames as context for mathematical activity that is adaptive for different levels of competencies is proposed by other authors in research literature (e.g. Vogt et al., 2018). Some authors notice that the challenges that are posed by a boardgame are particularly effective for children with higher mathematical competences, while more traditional training program could lead to better results for low achievers (Vogt et al., 2018); other authors suggest using boardgames even in the case of disabled children since they prove to be successful with students of different ability levels (McConkey & McEvoy, 1986).

Conclusion

General education literature converges on attesting that game-based education can foster learning providing students with a high motivational context, and this is also confirmed in the case of mathematics (Yong et al., 2021). Research about motivation and games is widespread in journals and conferences, but we can recognize that the reviewed research focuses more on the cognitive and relational aspects that are specific to the development of number sense using boardgames. The focus on the different didactical variables, including the teacher's role, appears peculiar to TWG13.

Drawing on literature from research journals, we have noticed that linear boardgames and playing with dice can help in fostering preverbal number sense (specifically, the mental number line and the subitizing ability) and in assessing deficits, however research about boardgames and preverbal number sense is still missing in CERME. Contributions in CERMEs have shown that difficulties related to teacher education and curriculum design emerge when analyzing the boardgames that are proposed in mathematics classes. However, we believe that these difficulties do not constitute good enough reasons to avoid the use of boardgames in preschool or primary school, since there is evidence that these games can provide the context for challenging tasks. Downton and Sullivan (2017) have documented (in the context of word problems) how challenging tasks may prompt the use of more sophisticated calculation strategies and then flexibility. This could be particularly true in the case of strategic games. Further research is needed to prove this kind of conjectures. If any, positive results may serve as basis for a larger implementation of boardgames in mathematics classes.

Research literature testifies that teachers participating in experimental interventions involving boardgames are particularly enthusiast (Vogt et al., 2018), and such involvement is considered one of the possible causes of good results of these experiments (Ramani & Siegler, 2008). Enjoyment may lead teachers to propose tasks that are more challenging than those they are used to and then prompts an explorative approach that is often prevented by teachers' insecurities (Peters. 1999). However, as noted above, teachers may misjudge the mathematics involved in a boardgame (Dorier

& Maréchal, 2009). The design of teacher professional development programs appears as needed and may constitute an interesting context of research (Maffia & Silva, 2021).

Concluding, we can claim that boardgames show several potentialities in relation to the development of young children's number sense, however in TWG13's discussion there is still room for more research aimed at: (1) understanding the features of boardgames for developing and assess preverbal and foundational number sense; (2) developing teachers' education to help them in analyzing games and exploit their potential.

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Toddlers’ ways of experiencing the meaning embedded in the question “How many years are you?”

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This paper focuses on toddlers’ (1- to 3-year-olds) ways of responding to the question “How many years are you?” and in what respect their responses may connect to a mathematical meaning. As a part of a longitudinal study on toddlers’ numerical development, the question was posed to 15 toddlers five times at approximately six-month intervals. At the beginning of the study, no toddler responded to the question with a mathematical representation. The results show how the toddlers started to discern the question as one to be answered with a mathematical representation, either number words or a finger pattern. The results also show how some of the toddlers started to connect these two modes of representations. In the paper, we also explore the different ways the toddlers used finger patterns to answer the question.

Keywords: early mathematics, finger pattern, numbers, representations, toddlers

Introduction

This paper focuses on toddlers’ (1- to 3-year-olds) ways of responding to the question “How many years are you?” (direct translation from the Swedish expression “Hur många år är du?”). The starting point for this interest was a Swedish combined research-development project on toddlers’ numerical development. Even though there is a growing consensus in research and practice that children’s numerical competencies start to develop at a very early age, few tools have been developed to make their progress in learning visible and researchable. Thus, one part of the project was to develop a research tool to make toddlers’ ways of experiencing numbers researchable (Björklund & Palmér, 2021). In this paper, we take a closer look at one of the questions used in that tool.

There are challenges in exploring young children’s numerical competencies, since verbal utterances cannot be taken as the primary source for analysis of their understanding. Furthermore, it is crucial to establish joint attention between child and researcher in a conversation or task to maintain the focus on a common objective (Siraj-Blatchford, 2010). When developing the research tool, we thus wanted to make it possible for the young children to relate to and reason about the content based on their previous experiences as well as enable them to express different ways of experiencing numbers. Based on methodological reviews on studies involving children, we knew that, especially in research with very young children, multiple-method activity approaches had proven successful (Aubrey & Dahl, 2005). Thus, the decision was made to design a task-based interview that included a variety of tasks and manipulatives making it possible for the toddlers to express themselves in different ways. This task-based interview was to be held in a naturalistic setting, placing the children at the centre of focus, recognizing their social and cultural experiences and understanding. The context of the interview became the birthday of Kitty the cat as even toddlers have experienced birthdays. Also, the use of a toy cat was intended to engage the children's interest and promote joint attention.

As an introduction to the interview, the children got to see and, if they wanted to, hold Kitty the cat. The children were told that it was the cat's birthday. Then the question "How many years are you?" was posed to the children. The question had a twofold purpose: partly to arouse curiosity and interest by inviting the children to a joint conversation and partly to see how the children responded to this question, which they had most likely heard before. At one point, when we presented the study at a national research conference, a researcher in mathematics education asked whether the question "How many years are you?" is a mathematical question. Inspired by that question, we here analyse how toddlers responded to the question. Our approach is that it is not up to us to decide whether the question is mathematical or not; it depends on the toddlers' ways of experiencing the meaning embedded in the question. Whether or not the question "How many years are you?" is mathematical is thus an empirical question, which this paper will focus on by answering : How do toddlers respond to the question "How many years are you?" and in what respect do their answers connect to a mathematical meaning?

Four fundamental aspects of numbers

We cannot empirically investigate whether children experience the question as a mathematical question; we can only investigate their responses to the question and whether their responses include mathematical representations. Together with ordinality, cardinality and part-whole relations, representations are fundamental aspects of the complex construct of numbers that are known to be necessary for children to learn about in order to develop their numerical competence (Björklund et al., 2021; Baroody & Purpura, 2017). The meaning of ordinality implies that every object or number word has an exclusive position in a sequence and relates to the others in the same sequence (Fuson, 1988). "How many" questions mostly refer to the meaning of cardinality in numbers, that is, number words are used to describe a coherent set. This is often observed in counting acts where the last uttered number word includes all the counted items (Gelman & Gallistel, 1978). Knowing that number words represent a set or composed unit of items is, further, a requisite for understanding numbers' part-whole relations, which allows children to compare sets, add, subtract and in different ways operate with numbers to solve numerical problems (Venkat et al., 2019). Thus, number words include several aspects that become significant for making use of them, and their meaning is also connected to the context in which the words are used (see Fuson, 1992). As numbers are abstract by nature they have to be *represented* in some way in all communicative situations, for example, by number words as exemplified above. Lesh (1981) emphasizes five modes of representation that are important in mathematics education, where the learning (for example, of numbers) is reflected in the ability to make connections *between* and *within* these modes of representation: Real-World Situation, Pictures, Verbal Symbols, Written Symbols and Manipulatives. But, even if toddlers respond to the "how many" questions, for example, with finger patterns or number words, it is not self-evident that the representation used mediates a mathematical meaning. The meaning of representations has to be learnt through communication with others whereby representations assigned to children's actions mediate mathematical meaning (Van Oers, 2010).

Theoretical framework

The theoretical framework in this study is the variation theory of learning (Marton, 2015) which brings to the fore the child's perspective, or more precisely, children's ways of experiencing a phenomenon (see also Sommer et al., 2010). In every situation, several aspects of a certain phenomenon can be discerned and the aspects that are discerned are decisive for how the phenomenon is experienced by the child. The theory directs attention towards children's learning and explains learning in terms of changes in the child's ways of experiencing a phenomenon. In the analysis, how children respond to the question "How many years are you?" and changes in these responses are interpreted as expressions of discerned and undiscerned aspects. As we in the study focus on the emergence of numerical awareness among toddlers, non-numerical meaning is just as important to observe.

Method

The project was conducted in close collaboration between researchers and three preschool teachers in Sweden over two years. The project, including development of tools and methods, has been approved by the Swedish ethical review authority (Dnr: 2019-01037). During the longitudinal study, the play-based interview was conducted five times at approximately six-month intervals. The interviews were all video-recorded, with the written consent of the toddlers' legal guardians. The interviews were conducted by the preschool teachers. This was important as it is not reasonable to expect that a researcher who does not know the toddlers would be able to provide the children with the best conditions to show their knowledge. As toddlers' expressions are very subtle, the interviewer needs to have extensive knowledge of the individual children's ways of expressing themselves.

Altogether, 110 video-recordings focusing on the question "How many years are you?" were transcribed. Also, how the children responded to the follow-up question, "Can you show with your fingers?", was noted. This follow-up question was asked to the children who did not use finger representation in relation to the first question. The results presented in this paper are based on analysis of 75 of these video-recordings from those children who were between 12 and 18 months when the project started and who took part in all five interviews. This was in total 15 children born between January and September 2018. The children's responses were first categorized based on correct or incorrect answers and then inductive based on the representations they used. Then, each inductive category of representation was analysed focusing on discerned and undiscerned aspects in line with variation theory principles (Marton, 2015). This way of analysing the data provides information on toddlers' ways of experiencing the phenomenon that in this study is the question "How many years are you?".

Results

First, the results from the inductive analysis on the representations used will be presented. Each category (C) of representation will be elaborated on separately, then followed by an elaboration of the categories related to the longitudinal perspective constituting the five interviews. Finally, the toddlers' use of finger patterns in all the interviews are focused on.

Table 1: The children's responses to Q1 "How many years are you?" and Q2 "Can you show with your fingers?" (asked to children who did not use finger representation in answer to the first question)

		Int. 1	Int. 2	Int. 3	Int. 4	Int. 5
C1	Q1 No mathematical response	15	13	8	2	3
C2a	Q1 incorrect number word		1		1	
C2b	Q1 correct number word			6	5	
C3a	Q1 incorrect fingers				2	1
C3b	Q1 correct fingers			1	1	1
C4a	Q1 number word and fingers, but one of them incorrect				1	1
C4b	Q1 number word and fingers, correct		1		3	9
C5a	Q2 incorrect fingers		3	2	4	2
C5b	Q2 correct fingers		3	4	2	
C5c	Q3 no response	15	8	8	2	1

Table 1 presents an overview of the 15 toddlers' responses to the questions "How many years are you?" (Q1) and "Can you show with your fingers?" (Q2) in the five interviews (Int. 1-5). In C1, the children gave no apparent mathematical response to Q1. However, many of them did interact with the interviewer, for example, by talking about Kitty the cat. Some children did use numbers and/or fingers as responses in other tasks but did not seem to discern any numerical aspects in this question. For example, one boy did not respond to this question, but afterwards when the interviewer showed him two fingers asking "Is this three?", he answered "No, it's two." Thus, this category includes both children who did not use mathematical representations at all in the interview and children who did not experience this specific question as one to be answered with a mathematical representation. The nature of the question here comes to the foreground since it is likely to be more challenging for the child to present a number word as an answer to a "how many" question (regardless of the items asked for) than to determine the quantity of a (re)presented number (for example shown finger pattern).

In C2a and C2b, the children responded by saying either an incorrect (2a) or a correct (2b) number word. Both responses indicate that these children experience this question as one to be answered with a number words. However, based on only this question it is not possible to determine whether the number words used by the children have a mathematical meaning, for example ordinal or cardinal meaning, or not.

In C3a and C3b, the children responded by showing either an incorrect (3a) or a correct (3b) finger pattern. Both responses indicate that these children experience this question as a question possible to be answered with a mathematical representation. What distinguishes this way of responding to the question from the verbal number words is that the question was posed in one mode of representation (verbal) but the children responded in another mode of representation. However, based on only this question it is not possible to determine whether the finger patterns used by the children have a mathematical meaning for them, it may be a learnt procedure.

In C4a and C4b, the children responded by saying a number word *and* showing a finger pattern at the same time. The number word and/or finger pattern may be incorrect (4a) or correct (4b). Both responses indicate that these children experience this question as a question possible to be answered with mathematical representations and that they have discerned a connection between modes of representation. However, as previously mentioned it is not possible to determine whether the number words and/or finger patterns used have a mathematical meaning for the children.

As mentioned, the second question (Q2) was only asked to the children who did not use fingers in response to the first question (thus C1 and C2ab). Some children gave no response (5c) while others responded by showing either an incorrect (5a) or a correct finger pattern (5b). Some children showed a finger pattern consistent with the number word used to answer the first question. Thus, they may have been connecting these two modes of representation. Some children who had not answered the first question now showed the correct finger pattern, which may be a representation with mathematical meaning but could also indicate that the question "Can you show with your fingers?" induced a procedure (showing a finger pattern) not grounded in any discerned mathematical meaning.

A longitudinal perspective

In the first interview, the age of youngest child was 12 months and the oldest was 18 months. As the intention in the project was to initiate the investigations before the toddlers had begun to use numbers, the toddlers' limited responses that may be connected to mathematics in interview 1 were expected. In the second interview (age span 18-24 months), 13 children gave *no mathematical response*. Two children responded to the first question, one with an incorrect and one with a correct number word, "one", in combination with showing one finger. To Q2 (asked to the 14 children that did not use fingers in Q1) six children responded with finger patterns, three of them correctly. As mentioned, this representation may or may not be connected to a mathematical meaning.

In the third interview (age span 21-28 months), eight children gave *no mathematical response*. Seven children responded correctly to Q1, six used number words and one used a finger pattern. The use of number words and finger patterns indicates that these children had an expectation to answer with a (to the observer) mathematical representation. The children aged one answered "one", and the children aged two answered "two". Thus, they may not have known "how many" one or two is, but they experienced that these representations can be used to answer a "how many" question. To Q2 (asked to the 14 children that did not use fingers in Q1) six children responded with finger patterns, four of them correctly. As mentioned, this representation may or may not be connected to a mathematical meaning.

In the fourth interview (age span 27-34 months), two children gave *no mathematical response.* Six children responded with a number word on Q1, five of them correctly. Three children responded with finger patterns, two correctly. Four children responded with number words in combination with finger patterns, thus coordinating two representations. Three of them did this correctly. Similar to the above, these children may not have known "how many" one or two is regardless of representation, but they seemed to know that this was a way to answer this question. On Q2 (asked to the eight children that did not use fingers in Q1), six children used finger patterns, two of them correctly.

In the fifth interview (age span 33-40 months), three children gave *no mathematical response.* Two children responded to Q1 with finger patterns, one correctly and one incorrectly. 10 children responded to Q1 with number words in combination with finger patterns, thus coordinating two representations. Nine of them did this correctly. Similar to the above, these children may not have known "how many" one or two is regardless of representation, but they seemed to know that this was an answer to this question. On Q2 (asked to the three children that did not use fingers in Q1), two children used finger patterns, incorrect.

How do toddlers use finger patterns?

In the five interviews, three different ways to use finger patterns as a response to Q1 and Q2 emerged:

1) Respond by showing all fingers (three examples in Figure 1). This way of using finger patterns indicates that the children have discerned a representational aspect, but this is not yet connected to any numerical aspect. Nevertheless, the questions can be answered by showing finger patterns as representation, although the finger patterns used are not yet *mathematical* representations.

Figure 1: Different ways of showing all fingers

2) Respond by showing a finger pattern with a selected number of fingers, but not the correct number. Some children instantly selected a number of fingers, others worked to fold down a number of fingers with the help of the other hand. This way of using fingers indicates that they have discerned that different finger patterns mean different things and it is therefore necessary to be precise when creating the finger patterns. It is possible that these finger patterns are experienced as representations of a "unit". But because of the lack of connection to the number asked for, it is likely that the finger patterns are still undifferentiated in an exact cardinal sense.

3) Respond by showing a correct finger pattern (three examples in Figure 2). Some children used fingers on one hand and others used fingers from two hands, the last when showing two by

using the index fingers close together. Similar to the above, this way of using fingers indicates their discerning that finger patterns have a specific meaning and may also be based on an experience of finger patterns as representing a "unit". Some sense of cardinality may here come through in that "two" is not merely an act of creating a pattern but an intended action of bringing one finger on each hand together to compose a set.

Figure 2: Showing two with one or two hands

Discussion

The questions elaborated on in this paper are how do the toddlers respond to the question "How many years are you?" and in what respect do their answers connect to a mathematical meaning? In accordance with the variation theory of learning (Marton, 2015), how children respond to the question is interpreted as expressions of discerned and undiscerned aspects. Someone might argue that the toddlers, over the course of the five interviews, learnt which number words or finger patterns to use. However, one delicate issue is that the correct answer to this specific question changed with each birthday. Thus, the children changed their answers between the interviews (correct or incorrect), which indicated that they had not learnt one specific way to answer this question by taking part in the interviews. The question is quite difficult form a mathematical perspective based on the lack of visible set to connect to the question "how many". The results show that the toddlers initially did not discern aspects of the question that brought them to respond with any mathematical representations. One explanation may be that the toddlers had not yet discerned mathematical representations and therefore these were not part of their possible responses to the question. However, this is not true for all the toddlers as there were toddlers who did not respond to this question with a mathematical representation but used mathematical representations when responding to other questions in the same interview. Throughout the study the children on group level discern that the question can be answered by using what we interpret to be mathematical representations (number words and/or finger patterns). However, based on the analysis in this paper, we cannot know if the children do experience mathematical meaning of the representations that they use (Van Oers, 2010). To know the explicit meaning the representations have for the children, more situations in which they use representations must be analysed. Nevertheless, recent findings have shown that young children who make use of several representations, whether used in correct correspondence with a number of objects or not, are more attentive to learn cardinal meaning of numbers (Gibson et al., 2019). Lesh (1981) emphasizes five modes of representation, three of which are at play in this task, real-life situation provides context for the task while the toddlers respond by using number words and/or finger patterns. The children most often used one representation, then they started to use two representations in combination to a larger extent. However, the toddlers may not experience them as two different representations but as a whole, and they may not have known that the representations were mathematical; that definition is in the eyes of the observer.

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Spontaneous activities in kindergarten as inquiry-based mathematics

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Keywords: Inquiry-based mathematics, spontaneous activities, kindergarten teachers, maps

Finding the flowers – a spontaneous IBMT activity initiated by the teachers

This pilot study is a cooperation between the project SUM, organized by UiT The Arctic University of Norway, and the Norwegian Centre for Mathematics Education. The main focus in the SUM project is to investigate challenges in the transitions in the Norwegian educational system and how inquiry-based mathematical teaching (IBMT) can make these transitions more coherent. The participants in the SUM project, from both kindergarten and school, were introduced to a 3-phased didactical model for structuring IBMT (Haavold and Blomhøj, 2019). The model includes these elements: Phase 1: Setting the scene, phase 2: investigating and phase 3: shared reflections. This view on how to approach and learn mathematics resonates with how the Norwegian Framework Plan describes the mathematics subject area (Quantities, spaces and shapes) and how to work with it (Norwegian Directorate for Education and Training, 2017).

The participating kindergarten teachers (KTs) and school teachers problematized differences in, for instance, teachers' and kindergarten teachers' planning time and the distribution of spontaneous activities, free play and planned activities in the kindergarten context compared to the school context. In this pilot study we approach these differences with the following research questions:

How might the three phases in the didactical model for structuring IBMT appear in a spontaneous activity in a kindergarten context, and can spontaneous activities be regarded as inquiry-based mathematics?

The term "teaching" itself is not commonly used in the Norwegian kindergarten context although the work that kindergarten teachers do is considered as professional work. The Norwegian Framework plan is specific about what kindergarten teachers must do to achieve the content and tasks described in the plan. We therefore choose to use the term teaching to describe the professional work that both kindergarten and school teachers do. However, there are some differences in the work of teaching in Norwegian kindergarten contexts compared to the school context, which also is described and debated by the participants in SUM. Hence, we argue that a model suitable for teaching in a school context may need to be adapted to fit the kindergarten context.

We define spontaneous mathematical activities as activities that are not planned and can be initiated by either children or kindergarten staff. In the IBMT didactical model, mathematics is in focus. In daily life, however, children just want to solve the problem. Spontaneous activities therefore present some challenges along, due to the influence of the staff. Lossius and Lundhaug (2020) discuss the dilemma that arises when choosing between joining or observing an activity. Mason and Spence (1999) argue that this is knowledge about how to act in the moment.

This study is a pilot case study and a part of the SUM project. The data collection method is observation, field notes and interview. The researchers participated in the everyday life in one Norwegian kindergarten for one day. The researchers observed and interviewed two kindergarten teachers in daily situations and aimed to focus their observations on the mathematics involved in spontaneous situations. However, as Flottorp (2020) states, it is difficult to capture these situations by being present. Consequently, participating kindergarten teachers are co-researchers since they made additional observations and fieldnotes without the researchers present. We illustrate one of the spontaneous activities by the following vignette:

> The kindergarten teachers had planned to go looking for meadowsweet (mjødurt in Norwegian) which is used to make juice. The aim of first excursion was just to find the plants, not to pick them, so one of the kindergarten teachers asked the children how they would be able to find the plants again the next time, when they were going to pick them. After some discussion, they agreed upon making a map about the places.

In the vignette above, the KT made the activity more inquiry-based by posing a question. This question acted as a phase 1 element and set the scene for children to investigate the problem in phase 2. Making a map was not part of the original plan, but a spontaneous idea from one of the KTs. In the interview the KT told the researchers that the suggestion was based on her experiences from the SUM project. She decided in the moment that this idea had the potential to turn the "looking for plants" activity into an inquiry-based mathematics activity for the children.

Knowing how to act in the moment and to change your plan is an important part of kindergarten teacher's mathematical knowledge. We argue that the vignette is an example of an inquiry-based mathematics activity, that was not planned. It also illustrates our initial finding, that at least phases 1 and 2 can appear in spontaneous activities. Our next step is to further investigate how we can identify phase 3.

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Intuitions of 5-6-year-old children related to measure sense and measurement

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In this paper, we present the results of a study conducted among 5–6-year-old children, with a focus on their intuitions regarding aspects of measure sense and measurement. The results show that children at that age have different intuitions in terms of measure sense and measurement and they can use them while problem solving. Additionally, our results show that appropriate activities can assist children in developing their abilities of comparing different distances and applying measurement rules.

Keywords: Early years mathematics, measure sense, measurement, intuition.

Introduction and theoretical framework

One may claim that the two basic pillars of preschool education are arithmetic and geometry. In the field of arithmetic, counting and comparison are the competences which are mainly developed: children at such ages are quite good at counting and at providing the cardinality of a set. In the field of geometry, preschool education focuses primarily on recognising and measuring basic geometrical shapes (Clements & Sarama, 2000). According to Buys & de Moor (2008), there are three forms of measurement that are essential for young children:

- Measuring through comparing and ordering
- Measuring through pacing off, using a measurement unit (natural or standard measure)
- Measuring through the use of a measuring instrument. (p. 17)

The first experiences in measurement are related to comparisons. Children compare each other on an everyday basis: who is taller, who has a longer foot or who built a higher tower (Buys & de Moor, 2008). This is why van den Heuvel-Panhuizen & Buys (2008) claim that “Both measurement and geometry enable children to make connections with their daily environment” (p. 10), which in turn stresses the need for everyday contexts as the starting point for relevant activities (MacDonald & Lowrie, 2011). The use of informal units of measurement with young children is advocated in most studies (e.g., Haylock & Cockburn, 1989), while in other studies children are asked to use informal and standard units of measurements to solve some tasks (e.g., Boulton-Lewis, Wilss, & Mutch, 1996). In the process of developing measure sense and measurement, one cannot ignore the importance of intuition:

> Mathematizing involves reinventing, redescribing, reorganizing, quantifying, structuring, abstracting, and generalizing that which is first understood on an intuitive and informal level in the context of everyday activity. (Clements, 2004, p. 12)

The above considerations have led us to design a research project to investigate measure sense and measurement among young children, with a focus on children’s intuitions. Our research project was realised in two stages, which will be described in the next section, aiming to answer the following research questions:

1. What intuitions related to the concepts of measure sense and measurement do 5–6-year-old children have before their formal introduction to these concepts? (Stage 1)

2. How do children measure different objects, what units do they use if any? (Stages 1, 2)

3. Are children able to compare different objects by using their dimensions? Can they develop a universal unit and/or measurement rules? (Stages 1, 2)

Context of the study and methodology

The present study is part of a larger project related to the understanding of the concepts of measure sense and measurement by children at the preschool and early years levels. The research group consisted of twenty 5–6-year-old children; eighteen of them took part in the first stage and all of them in the second stage. They were attending a public kindergarten in Rzeszow, Poland.

According to the Polish curriculum for the preschool level, numbers and arithmetic are the most significant topics (Podstawa Programowa/ National Curriculum, 2017). Although many activities are related to counting and calculating, geometrical concepts appear less often, and are usually related to the recognition of figures and shapes; measurement contexts appear less frequently. Comparison skills usually refer to comparing the number of elements in sets, rather than in the context of measuring length and size.

The first stage of the study focused on children's intuitions regarding aspects of measure sense and measurement. Individual meetings took place with the children, which were designed to identify their intuitions on measuring. Our aim was to answer the question on what "to measure" means for them. We were also interested in how children compare the length of two objects (e.g., two trains made of blocks) and on what tools they use to measure the length (how they check which object is longer).

The second stage focused on analysing how children use their intuitions when solving measurement tasks. The children were divided into two groups of ten. Each group worked separately under the supervision of the researcher, performing the same tasks. During the group discussions, the researcher did not assess the answers of individual students. The two tasks were presented in the form of stories, depicted with physical props. They were as follows:

Task 1. In the world of toys, a doll, a ball, a cube and a teddy bear decided to visit each other (Figure 1). Who is the furthest away from the bear? And who is the closest to the ball? How can we check it?

In this task no way of solving was imposed, nor it was suggested that the distances between the toys should be measured. The idea was to provoke pupils to discuss the problem posed: how to compare the distances between objects? A string and paper feet were used as measurement means in this task.

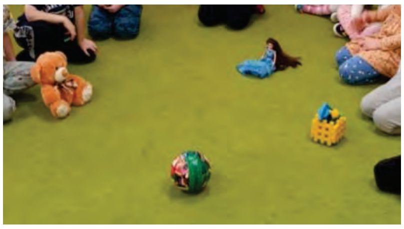

Figure 1: The initial situation of task 1

Task 2. a) The animals held a competition in throwing balls. The fox and the bunny were judging, and the deer, the owl and the beaver were the players and stood on the starting line (see Figure 2, from left to right we see the fox, the deer, the owl, the beaver and the bunny). Each of the players made one throw. Which of the animals won the competition? Why?

Figure 2: The initial situation of task 2

Figure 3: The measurement done by the fox (task 2b)

b) The fox was the first to judge. He used feet for the measurement (see Figure 3). The fox announced his verdict: the beaver won (the blue feet in the Figure 3). Do you agree with the verdict of the fox judge? Justify your opinion.

c) A bunny has joined the judging. With his feet, he marked each of the distances between the player and his ball. It looked like this (Figure 4):

Figure 4: The measurement done by the bunny (task 2c)

The bunny announced his verdict: there is a draw, everyone threw his ball on the same distance. Do you agree with the bunny judge's verdict? Justify your opinion.

At the beginning of this task, the children could propose their own solutions. This was to examine the children's intuitions about measuring, especially if they perceive distance as the shortest line (segment) between objects, which is perpendicular to the end line. Then, the solution proposed by the children was confronted with two other, erroneous solutions.

The research data consisted of video recordings of the individual interviews with the children as well as video recordings of the activities conducted with the two groups of children. Our analysis of the transcribed discussions mainly focused on the way the children were working, as well as their interactions with the researcher (the first author of the paper) and with other children. The collected data were analysed quantitatively (Stage 1) and qualitatively (Stages 1, 2). In this paper, we focus on presenting Stage 2 of the study, together with a discussion on the obtained results. Stage 1 of the study is presented in another paper (Pytlak & Maj-Tatsis, 2021), but we shortly describe its results in the next section, in order to assist the reader of the paper. The collected data of Stage 2 were analysed interpretatively, by categorising the children's utterances according to: measurement methods and rules, comparison methods, and measurement units.

Results

Results from stage 1

The analysis of our data showed that it is possible for children aged 5-6 to hold valid intuitions on measure sense and measurement. They were able to interpret the distance between objects as a straight segment connecting them. At the same time, their understanding of the shortest way was twofold. For some children, it was the 'straight' way (e.g., a straight line without loops and bends, which could be verified visually). Other children understood the shortest route as a segment perpendicular to the starting line, which is consistent with the mathematical understanding of the concept of distance.

During measuring, the children paid attention to both the units and the way the measurements were made. In order to compare two objects (trains made of blocks), they arranged them next to each other, from the same starting line. However, this way of comparison was important mostly in a situation where trains were made of blocks of different lengths. When both objects were made of the same blocks, the children did not feel the need for physical comparison. Then, it was enough for them to count the elements in both trains. Thus, in this situation, measuring was identified with the cardinality of a set (for details see Pytlak & Maj-Tatsis, 2021).

Results from stage 2

Task 1. In both groups, children agreed that individual distances should be somehow measured. Various proposals were made to make measurements, most often it was to determine the distance by using the spat of the arms. The teacher's proposal to use the string for distance measurements was accepted. They correctly marked the distances between individual toys with a string. They also proposed a way to investigate which distance is the longest. In order to do that they compared the strings with each other. The shortest of them was a 'scoop', according to which the length of the others was determined. One of the boys took the first of the strings and successively applied it to the second and the third. He commented on the results in the following way: "this one is longer", "this one is shorter".

Researcher (R):	To what toy does the bear have the furthest?
A:	To the doll.
F:	No, to the cube.
R:	A. says to the doll, and F. says to the cube. Who is right? How to check it?
F:	You need to measure the strings.
R:	So measure the strings.
F:	[takes the strings and lays them along each other, "centered"] This one is longer and this one is shorter.
A:	No. [corrects the strings so that they all start from a common point, from a teddy bear]
R:	Ooo, A. put all the strings together. Is that a good thing? Do you agree?

F: Here you can see that this one is longer, and these are shorter (Figure 5).

In the above excerpt, we can notice that when comparing objects (here: strings illustrating the distance between toys), children do not always pay attention to the appropriate position. The visual aspect is more important. If something is directly visible, certain properties can be visually evaluated, then additional aspects do not have to be taken into account.

Figure 5: Measuring and comparing the distances between the teddy bear and the other toys

In addition to measuring the distance with strings, the children also proposed the use of a ‘centimeter’, which could be a reference to experiences from everyday life. Furthermore, attempts were made to use hands (spreading arms over a measured distance), legs (distance measured with legs’ extension) or one’s own body for measurements. However, the children themselves noticed that in this way it is not always possible to measure something. The researcher suggested measuring the distance with her feet. Firstly, she asked one of the girls to measure the distance between the teddy bear and the ball with her feet, and then she measured the same distance herself. Two different results were obtained: 3 feet of a researcher and 6 feet of the girl. For the children, this situation did not cause any conflict, it was unanimously noted that the results differ due to different foot lengths. It seems that the children did not see the need to use the same measuring units in all measurements. All that mattered was the way it was measured.

R: L. measured and she got 6. I will also measure [measures with her feet the distance between the teddy bear and the ball, counting loudly] once, two, three. So, who is right? Is there a distance of 6 or 3?

F: Well, because L. has smaller feet.

R: Oh, so if there are smaller feet it will be differently, and if they are larger it will be differently, right? [children are nodding]

F: Because larger feet take up more space and smaller feet take up less.

R: If I wanted to measure the distance between all the toys, do I have to measure with the same feet or can I measure a bit and L. a bit?

Children: I don’t know [shaking their heads]

R: Well, listen, L. measured the distance from the teddy bear to the ball and it was 6 feet. I measured the distance from the teddy bear to the cube and it also was 6 feet. You said that the teddy bear is further to the cube than to the ball, right?

Children: [nodding]

R: But it was the same - 6. So, can you measure with different feet to compare distances?

Children: [silence]

The children unanimously approved the researcher's method of measuring distance with her own feet and began to use it by themselves. They noticed very quickly that the result of the measurement depends on the unit used. Therefore, obtaining different values for the same distance did not cause any cognitive conflict in the children. They were a bit surprised when the researcher pointed out two identical results obtained when measuring two different distances (from the teddy bear to the cube and from the teddy bear to the ball). They knew that these distances differed greatly, but they could not explain why the same numerical result came out in both cases.

Task 2. The children unanimously suggested that the middle player (the owl) won the competition (Figure 2). They justified it by the fact that the ball had the longest distance from the starting line. They indicated with their hands how the segments depicting individual distances were running. Here it was clear that the children appealed to the intuitive understanding of the distance between objects as the shortest distance between them.

When assessing the fox's judgement (Task 2b in Figure 3), the children referred to two aspects: the process of measuring the distance and the result obtained by individual players.

R: Who according to the fox threw the ball the farthest?

Children: The beaver.

R: And do you agree with such a measure by the fox?

Children: Yes... No…

R: Did the fox measure well?

D: Yes, because here is 8, and here is 9 and here it is so straight [in the air he outlines segments depicting the distance between the fox and individual balls] (...)

R: At the beginning you said that you thought the owl won. Why? How did you measure it?

W: From the owl [points his finger from the owl to the ball]

R: So, arrange it as you would count it.

W: [arranges – initially uses all the feet, creating an arc, but after a while he removes some of the feet, and arranges the rest in a straight line]

The way of measuring the distances by the fox did not arouse any protests in children. They treated it as a measure of three different segments that had a common beginning and different ends. This was consistent with the understanding of the distance between objects as the shortest segment connecting them. Only the researcher's reference to the original solution presented earlier caused them hesitation. Then they paid attention to which segments had to be compared with each other. They very quickly verified the solution and made changes in the arrangement of the paper feet (Figure 6). By measuring subsequent distances with the paper feet, the children tried to maintain the parallelism of the segments and start them from the same line. Counting the paper feet (treated as units of measurement) in the segments rather referred to the comparison of the sizes of three sets.

When the children gave the numerical value of the measured segments, the cardinal aspect came to the fore.

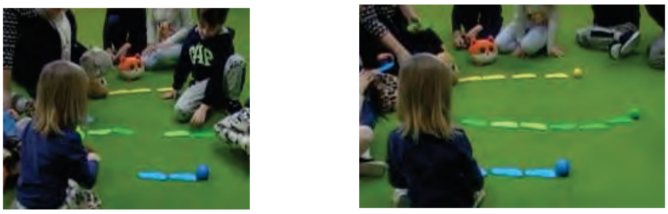

Figure 6: Rearranging the paper feet while measuring the distances between animals and balls

In the next part of the task (2c), it turned out that the judge (the bunny) measured distances in such a way that each of the competitors achieved the same result. The children immediately noticed the mistakes. First of all, they did not agree with the result itself, clearly indicating that the distances obtained in the ball throw are different. They primarily referred to the visual representation of the task. In addition, the children paid attention to the precise way of measuring: the feet were not laid in a straight line, but in an arc (Figure 4).

Discussion

Children enthusiastically and actively participated in solving the proposed tasks. They tried to justify their answers, strongly supporting themselves with gestures. The results obtained during the first stage of research allowed us to assume that that it is possible for children aged 5-6 to hold valid intuitions on measure sense and measurement. They were able to interpret the distance between objects as a segment perpendicular to both objects. Straightness was understood by them both as a straight line but also as a perpendicular line. This corresponds with the formal mathematical understanding of the concept of distance. The second stage of our research confirmed these assumptions. Moreover, the children have shown their intuitions about measuring. They understood the distance between two objects as the shortest segment connecting them, the end of which is perpendicular to the final object. This is consistent with the notion of distance that students encounter during mathematics education in primary school. The comparison of segments was made through visual verification: they arranged them parallel to each other and checked which one protrudes beyond the others.

The introduction of measuring with feet can be identified with the use of a measurement unit. Measuring with different feet (the researcher's and a child's) relates to the use of different units of measurement. Our results have shown that in making measurements of length, the unit of measurement was not so significant. The process of measurement by itself was more important: maintaining a straight line and accurately measuring units (putting foot by foot). The children were aware that by measuring the same segment with different feet (that is, different units) they would receive different results. However, they were not fully aware that in order to compare different segments, they had to use the same unit. We believe that this result goes in line with studies which suggest the use of informal units with young children (e.g., Haylock & Cockburn, 1989; Heuvel-Panhuizen & Buys, 2008).

The children in our study perceived measuring as an activity, a process, not as providing a numerical value. It was important for them how the measuring process takes place, whether the right direction and straightness of the line are maintained and whether the measuring unit is applied evenly. The numerical result of this process was of secondary importance. This was especially evident when working on the second task. When assessing the judgement of the fox and the bunny, the main consideration was whether the designated segment is straight and connects the starting point with the end point. Therefore, the arcs made by the bunny were immediately treated as an erroneous way of measuring. However, without hesitating much, the children agreed that the straight lines set by the fox are the correct way to measure the distance at which the players threw the balls. The results of our study show that length measurement should not be treated as a simple skill, but rather as a combination of concepts and skills that develop over time as part of a learning trajectory (Clements, 2004; Heuvel-Panhuizen & Buys, 2008). Moreover, informal activities based on everyday and playful contexts, which include object comparison can be useful in enhancing children's measurement and measure sense.

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Building the meaning of measurement in kindergarten: the case of length

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This paper deals with a teaching experiment concerning length measurement in kindergarten for children aged 4-5. Through a classroom-based intervention, designed according to the Semiotic Mediation theoretical framework and developed on a multimodal approach, children produce a definition of measure as product of the action "measuring", working with a chosen measurement unit. The qualitative analysis of data collected highlights the specific role of the teacher and of artifacts in supporting the process of semiotic mediation through which children transform the signs linked to artifacts (used to develop the meaning of measure) into mathematical signs (which present the results of measuring in geometrical way).

Keywords: Measurement, kindergarten, length, multimodality, semiotic mediation.

Introduction.

In the Italian National Guidelines for kindergarten (3-6 years) children are asked to acquire skill in measuring. This doesn't occur necessarily through standard tools of measuring (such as ruler, meter, balance), rather through any tools aiming to develop the meaning of measure, measurement unit and measure as product of the action "measuring". Accordingly with this perspective, I focus on the concept of length measurement, placing the teaching-learning act in the context of kindergarten. Research has shown that the understanding of space and its representations in mathematics has an intuitive experiential basis before primary school (Bryant, 2008) and that development of children's spatial skills are crucial nodes in the conceptualization of space and geometric space (Sinclair & Moss, 2012; Robotti 2018). Furthermore, bodily experiences and experiences mediated by the use of tools act as a bridge between the physical modelling of space and the conceptualization of a geometric space (Casey, et al., 2008). For this, I consider the multimodal approach crucial to pursue the educational aim. As far as measurement is concerned, research has long underlined that, already in pre-school age, children experience the comparison of quantities and, explicitly or implicitly, measurement (Buys & Veltman, 2008). Despite this, the action of measuring and the conceptualization of measurement are not without difficulties for children. Peter Bryant (2008) identified some difficulties in the measurement procedure for preschool children. Let's consider, for example, the act of measuring a length. Generally, children are perfectly able to define the sequence of actions needed to measure, as well as understanding the importance of one-to-one correspondence, which is an essential part of relating measurement unit with the length to be measured. However, measuring hides pitfalls and difficulties related to the iteration that is, in fact, at the basis of the action of measuring with an instrument. If we consider, a ruler, the measurement of a linear length consists of repeated actions (measurements), just as the ruler is made up of a set of iterated measurement units such as centimeters. The iteration is necessary when the length to be measured is longer than the measurement unit. Another difficulty identified by Bryant concerns the "asymmetric" one-to-one correspondence involved in measuring a line with a ruler: the measurement units (centimeters, inches) are visible and clear on the ruler but must be imagined on the length to be measured. Thus, through body experiences according to the multimodal approach (Arzarello & Robutti 2009; Radford, Edwards & Arzarello, 2009; Nemirovsky 2003; Radford 2014), the activities described in the following paragraphs will show how children developed the definition of measure as physical quantity, and also how they related measure with a chosen measurement unit. Moreover, I'll show how the teachers faced the difficulties highlighted by Bryant with appropriate didactic choices, designing activities through Semiotic Mediation framework (Bartolini Bussi & Mariotti, 2008). With these premises in mind, I argue it makes sense that geometry could be explored much earlier and more robustly than it is at the moment (at least in Italy), already during the kindergarten years (Robotti, 2018).

Theoretical perspectives.

The theory of Semiotic Mediation (TSM), developed to design and analyze educational activities, is one of the theoretical frameworks of reference in this research. It provides a powerful way for teachers and researchers to study the process by which activity with artifacts (in this research with Montessori sticks) can be turned into mathematical activity (geometrical activity allowing children to develop the definition of measure). Summing up the main elements of the TSM (for more details, see Bartolini Bussi & Mariotti 2008), the teacher takes in charge two main processes: the design of activities and the functioning of activities. In the former the teacher makes appropriate choices about the artifacts to be used, the tasks to be proposed and the mathematics knowledge at stake. In the latter, the teacher monitors and manages the students’ observable processes (semiotic traces), to decide how to interact with the students in order to transform signs linked to the use of artifact into mathematical signs. The students are in charge of the resolution of the task through the use of the artifact. Making this, they produce signs (objects, drawings, words, gestures, bodily movements, and so on), which are linked to the artifact and therefore they aren’t yet explicitly math signs. The teacher collects all these signs, in order to organize a path for their evolution towards mathematical signs that can be put in relationship with the aimed mathematical knowledge. In this research I also refer to Gallese and Lakoff’s (2005) theoretical approach, according to which mathematics teaching-learning processes are multimodal activities. Nemirovsky (2003) states that, understanding and thinking, included mathematical thinking, are perceptuo-motor activities, which become more or less active depending of the context. This means that, exploiting perceptual-motor components, the body becomes essential in the learning processes. In this perspective, the term multimodality is used here to underline the importance and mutual coexistence of a variety of cognitive, material and perceptive modalities or resources in teaching-learning processes and, more generally, in constructing of mathematical meanings (Radford, Edwards & Arzarello, 2009). According to these premises, this research aiming to investigate whether and how, in kindergarten, it is possible to transform signs related to the use of unconventional measuring instruments into geometric signs supporting the definition of measure as the result of the measurement action.

Methodology and teaching experiment.

The teaching experiment took place with 15 children of kindergarten aged 4–5. Two teachers were involved during the classroom activities. Seven sessions were carried out for 4 weeks (more or less twice times a week, 1 hour each) either in the classroom or outdoor, with a careful alternation of whole-class (with adult's guidance) and pair activity. The teachers and a researcher (the author) designed the educational activities. Each session was carefully observed by one of the teacher involved, with the collection of photos, graphical productions, videos and transcripts. Iterative qualitative analysis for data subsets and overall were carried out. Due to space constraints it is not possible to report all the activities, so I'll focus on the sessions in which the production of signs was been particularly meaningful and rich.

The teaching experiment.

The teaching experiment is part of a more articulated path, where Digital Sticks (Montessori material, see Figure 1) are used. In previous activities, sticks were used to introduce different aspects of mathematics including memorizing arithmetic facts as sums of numbers up to ten (in this educational activities children named them the "friends of ten"). Children, through the use of Montessori Digital Sticks, memorized the "friends of ten". The "friends of ten" act in a way that the length of the stick 10 can be obtained summing two lengths-sticks (9+1, 8+2, 7+3,...). The experience that I'll describe in this article arose spontaneously from the children's need to identify a not colored stick having the same length as a given colored stick (stick "five", as shown in Figure 2) in order to complete "the wall of friends of ten" (in that case with 5+5, as shown in Figure 1).

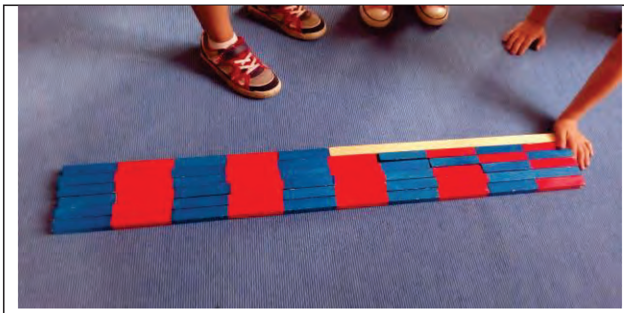	
Fig. 1: The not colored stick is placed next to the stick "five" to complete "the wall of friends of ten"	**Fig. 2: Children superpose non-colored stick on the colored one in order to verify they have the same length**

Children referred to "a stick as long as the colored stick five" by comparing two objects directly in length. This is a necessary skill to develop the meaning of measure, but teacher knows that it is not a form of measuring. Moreover, since children referred also to "a stick that measures like the stick five", using the word "measure" in context, teacher decides to define a new didactic objective: introducing the concept of measurement of a length. Therefore, the teacher designs the educational activity in two phases having the following aims:

Phase 1: Building and sharing among all children a definition of measure and of measuring. This allows children to construct a meaning of measurement in geometrical terms. In this phase the educational activity is performed in classroom using Montessori sticks.

Phase 2: Applying the definition of measure in situations different from those they generated that definition and defining the relationship between "measurement unit" and "measure". In this phase the educational activity is performed in outdoor, still with the use of sticks.

Phase 1.

In order to complete "the wall of friends of ten", children need another stick "five" from another series of sticks (5+5=10). Teacher provides children the "non-colored sticks" series, so they decide to superpose the non-colored stick "five" on the colored one, verifying that it is as the colored stick (Figure 2). The superposition of sticks (signs linked to artifact) allows children to access to "comparison of length" meaning (math signs) expressed in verbal sign "the stick is as long as the colored stick five". Teacher's aim is now to approach definition of length measurement as much as possible. For this, relaunching an issue introduced by Nicolò, she opens the discussion with a question: is this [non-colored stick] a stick "five" as Nicolò said?

While children begin to think of a strategy to answer question, Matilde puts the non-colored stick next to the colored one (Figure 3) and she begins to count the colored segments of the colored stick through deictic gestures.

Fig. 3: Matilde counts the colored segments of the stick next to the non-colored one

This strategy implements several counting skills: the one-to-one correspondence, with the labeling process and the splitting process correctly implemented, is clearly detectable in pointing finger step by step on blue and red color; also the cardinal principle and the stable order of the numerical sequence are detectable, due to the verbal label “five” and the stated numerical sequence (Gelman & Gallistel, 1978). This strategy puts in acts the definition of measurement as the action of counting how many measurement units (stick “one”) are in the length to be measured. At the same time, this strategy puts in evidence the result of this action: the measurement of a length as physical quantity, that is as a number with the measurement unit. Matilde said: “yes, its length is five”. In this way, the signs linked to the use of artifact (deictic signs and numerals) are linked to math signs (measurement as physical quantity).

Unlike Matilde, Davide answers the question through an inference, exploiting arithmetic facts (Figure 4):

Davide: Five plus five equals ten. There are ten here [pointing the stick "ten", he refers to the measurement units]. Because here there are five [he points the colored stick "five"] then here necessarily there have to be five [he points the non-colored stick] and five plus five should make 10!

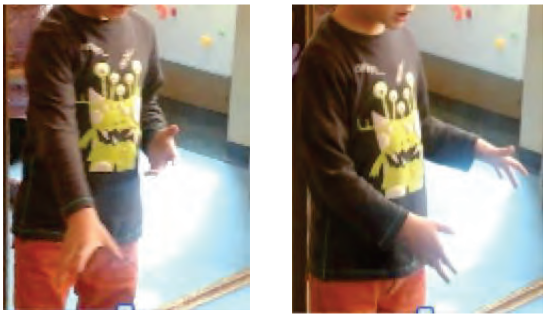

Fig. 4: Divide's gestures show the known part -the stick "five", that corresponds to 5 units- (Figure on the left) of a known whole -the stick "ten", that is 10 units- (Figure on the right)

Thus, if the stick "five" plus an non-colored stick are equal to 10 (stick "ten"), then the unknown measurement of non-colored stick has to be 5 (stick "five"): the equation $5 + x = 10$ is solved. Interestingly, here the measurement of the length is obtained without any action of measuring but through an arithmetical inference. The gestures evoke signs linked to artifact through which arithmetic fact 5+5=10 is recovered.

Afterwards, children are asked to measure non-colored sticks through colored sticks. This means teacher asks children to apply the definition of measurement shared by the class. Gabriele points the sticks "one" which compose two staircases, one of colored sticks and the other one of non-colored sticks, and says:

Gabriele: you see, the first two steps are the same
Alessandro: if they are the same, they start the same and they end the same [they have the same length]
Teacher: ok, but... do you also know their measurements?

In this excerpt the comparison of lengths is present yet as dominant concept. Therefore, teacher shifts the focus on the measurement. Alessandro and Ilaria take into account the non-colored stick "three" and they measure it (Figure 5) making explicit the meaning of "measurement of a length" and of "act of measuring" as follows:

Ilaria: I can put one after the other [the colored stick "one" as measurement unit] and I see that the non-colored one is like three of colored sticks [action of measuring].
Alessandro: so, we can see what the measurement [of the non-colored stick] is: it's 3 sticks "one" [measure].

Fig. 5: Alessandro measures the stick “three” with 3 sticks “one” as measurement unit

Note that Ilaria defines the action of measuring in terms of "seeing how many times the chosen measurement unit [stick “one”] is in the given length" and this corresponds to the shared definition by all children involved in the activity. Instead, Alessandro defines the measurement in terms of the result of this action, that is a physical quantity (3 measurement units, 3 stick “one”). Thus, the use of Montessori sticks promotes the measurement as repeated process in "posing measurement unit", giving sense to the measuring action and overcoming difficulties underlined by Bryant.

Phase 2.

In order the definition of measure becomes an "operational knowledge" and it falls within of the competences (that is a knowledge in act in other situations/contexts), the teacher asks children "measuring what they want" in the school garden, using Montessori sticks. This is why the children leave the classroom and chose to measure the side of an anti-shock mat about 3m long, placed under the swings in the school garden (Figure 6). Interestingly, children choose different measurement units.

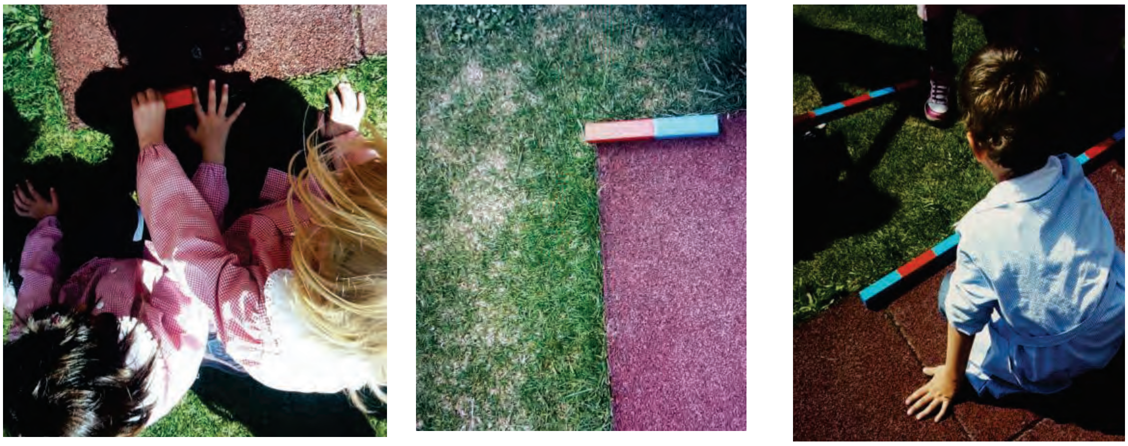

Fig. 6: Children measure the length of the side of the anti-shock mat through different measurement units (stick “one”, image on the left, stick “two”, central image, stick “ten” image on the right)

Children work in pairs: one child places one of the end of the unit-stick on the edge of the mat, the other child tracks, with his/her finger, the other end of the unit-stick, which will become the new starting point where placing the unit-stick. This repeated action allows children to cover the entire length of the side of the mat in *n* times the unit (iterated process). Note that, posing unit-stick one after another one is a repeated action that allows children to objectify the definition of measurement. In other words, this action, which is a sign linked to the use of the artifact, became mathematical sign in the verbalization of the obtained measurements. As matter of fact, the children find that the stick "one" is 29 times into the side of the mat (we consider a certain measurement error due to the approximation of measuring action), the stick "two" is 14 times into the side of the mat, while the stick "ten" is 3 times into the side of the mat. The following excerpt of discussion is representative of these results:

Teacher:	ok guys, how many did you had with the stick "one"?
Alessia:	29
Teacher:	ok, and with the stick "two"?
Alessandro:	14
Teacher:	and… with the big one, the stick "ten"?
Gabriele:	only 3, 3 times
Teacher:	but... why do the measures change so much?
Alessandro:	because the unit-stick change: one was bigger and the others smaller
[…]	
Gabriele:	[…] [the measure] it's larger if the stick unit [the measurement unit] is small, [the measure] it's smaller if the stick [the measurement unit] is large

It should be noted that teacher starts a discussion to highlight the relationship between the measurement unit and the measurement obtained in order to link the signs linked to the artifact (how many sticks) to math signs (measurements of the length). In fact, she asks why the measurements of the mat changes. Alessandro recognized a link between the measurement unit chosen and the measurement obtained, and Gabriele makes this relationship explicit in terms of inverse proportionality. The comparison among different measurements of same length reinforces the idea that measurement is a physical quantity (with a measurement unit of length).

Discussion and conclusion.

Different signs related to the use of Montessori sticks and the transformation of them into geometrical signs through the teacher's mediation, allowed children to develop the meaning of measure and the measure definition in relation with the chosen measurement unit. In order to reach this aim, and to overcome difficulties underlined by Bryant, perception and bodily experiences seem to play a key role in this teaching experiment. In particular, the use of Montessori sticks seems promoting the measurement as repeated process in "posing measurement unit", giving sense to the measuring action. As matter of fact, the physical, repetitive movement of positioning the measurement unit associated to the enumeration support the construction of the meaning of measurement as a physical quantity (i.e. *n* times the chosen measurement unit). This could reasonably facilitate the reading of the measurement on the ruler in the future (helping students to give meaning to the reiterated process that allows reading the length measurement on the ruler itself). At the same time, the one-to-one correspondence involved in measurement of a length with the Montessori sticks, overcomes the difficulty linked to the asymmetry of the ruler instrument: the measurement unit (the chosen unit-stick) must be physically superposed *n* times on the length to be measured. No ambiguity, therefore, linked to the need to imagine each measurement unit on the ruler (centimeters) on the length to be measured.

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Criteria of future early childhood teachers to design problem-solving activities

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This work aims to identify the criteria contemplated by early childhood education preservice teachers when they reflect on the design of problem-solving tasks. The Didactic Suitability Criteria (DSC) and their indicators were used to analyze one of the tasks performed by future teachers. As a result, it was identified when they design their classes in a consensual way, they are implicitly based on the DSC. Still, not all their indicators emerge since their reflection is not guided by an explicit guideline that serves them to show their didactic analysis in detail.

Keywords: Early childhood education, Didactic Suitability Criteria, Ontosemiotic approach, problem-solving, task design, teacher training,

Introduction

In the field of Mathematics Education, problem-solving is considered one of the optimal scenarios for teaching and learning skills in this discipline. Current trends in this area consider that problem solving encourages the development of mathematical processes, such as reasoning, the establishment of connections and representation, and communication, among others (NCTM, 2003).

Starting from a competence approach from an early age is crucial since it is a stage where the curriculum presents content and processes in an integrated way, which favors new views that emphasize the content, the process, and the relationships established between them (Alsina, 2016). In this sense, the tasks posed by the teachers to the students are the starting point of their activity, which, in turn, produces their learning as a result (Pochulu, et al., 2016). Teachers' teaching practice determines the students' learning, and, in that sense, designing, implementing, and evaluating tasks is a crucial aspect and competence that the future teacher must develop in their training.

This professional competence, characterized by knowing how to design, apply and assess learning sequences through didactic analysis techniques and suitability criteria to establish planning cycles, implementation, assessment, and propose improvement proposals, is called, within the framework of the Ontosemiotic Approach (OSA) (Godino et al., 2007) as competence in analysis and didactic intervention (Godino et al., 2017). It is a competence long studied by the OSA with mathematics teachers where the results showed the Didactic Suitability Criteria (DSC) were a valuable tool to organize their reflections and evaluations and improve their task designs (Breda et al., 2017). Although there is research with DSC in early childhood education (3-6 years old), in particular, the design of a proposal for evaluation teaching and learning methods (De Castro, 2007) and the analysis of a sequence of tasks for the development of spatial perception (Moreira et al., 2018), until now, no research has been carried out that focuses on the design and analysis of tasks, in particular, problem tasks, from the perspective of DSC in the training of future early childhood education teachers. In this work, we focus on the DSCs agreed by future early childhood education teachers when asked to reflect together to justify the design of their tasks.

The objectives of the paper are: to study what aspects future early childhood education teachers consider essential when planning and designing a sequence of arithmetic problems for the teaching and learning of numbering; which of these aspects can be identified as DSC and; to identify which elements of the DSC are not considered by the students.

Theoretical framework

Design of problem-solving tasks for the first ages

Concerning problem-solving-type tasks, according to NCTM (2003), four aspects should be worked on in the classroom: a) build new mathematical knowledge through problem-solving; b) solve problems that arise from mathematics and in other contexts; c) apply and adapt a variety of appropriate problem-solving strategies; and d) control and reflect on the process of solving mathematical problems. Along the same lines, there are some key aspects regarding the use and meaning of problems in the classroom and that, therefore, should be taken into consideration when designing the tasks. There is an agreement among researchers that a problem situation must be a new situation for which the resolution method is not known in advance. There is also agreement on how one learns to solve problems by manipulating, simulating, discussing, sharing, imagining, observing, visualizing, etc. Moreover, in this sense, in the resolution process, each child should be allowed to use the strategy that best suits his or her possibilities: a drawing, a diagram, mental calculation, the manipulation of a particular material, etc. (Alsina, 2016).

Focusing on the mathematical activity that students develop when solving tasks, there are different teaching approaches but, in the case of early childhood education, citing Baroody and Coslick (1998), the following can be distinguished: 1) the skills approach; 2) the conceptual approach; 3) the problem-solving approach; and 4) the investigative approach, which is a combination of conceptual and problem-solving approaches, the primary purpose of which is for students, with the mediation of teachers, to reach their conclusions through reflection, reasoning, representation, problem solving and research. In this study, students were asked to work assuming the latter approach.

Didactic Suitability Criteria in the design and management of tasks

The use and application of the DSC, proposed in the OSA, allow a teacher to guide the teaching and learning processes of mathematics, design tasks and assess their implementations. The OSA considers the following DSC: Epistemic suitability; Cognitive suitability; Interactional suitability; Mediational suitability; Affective suitability; Ecological suitability. To operationalize the DSC requires defining a set of observable indicators, which allow assessing the degree of suitability of each of these criteria (Breda et al., 2017). For example, from the epistemic aspect, there is consensus that it is necessary to implement "good" mathematics. DSC construct considered good mathematics those that are rich in mathematical processes (connection, argumentation, problem-solving, etc.) and contemplate certain representativeness of the complexity of the mathematical object to be taught (different meanings of the mathematical object, different representations and languages or a variety of problem typologies), etc. Based on the DSC, Gusmão and Font (2020) defined a set of observable indicators (included in

Table 1) to be able to assess, in an operational way, the degree of suitability of each of these criteria in the design and analysis of mathematical tasks.

Table 1: Task design indicators according to the Didactic Suitability Criteria (Gusmão & Font, 2020)

Epistemic suitability
1. Is the task's description in a clear, correct, and appropriate language for the level of education?
2. Are different languages and forms of mathematical expression used (verbal, graphic, symbolic, etc.)?
3. Is the selection of tasks representative and varied, and whether it includes tasks of a closed and open nature?
4. Are the tasks of different types?
5. Does the generation of hypotheses promote open thinking (reversible, flexible, decentralized thinking) and encourage the use of argumentation and justification processes?
Cognitive suitability
1. Is it based on the prior knowledge of the students?
2. Is knowledge expanded, reinforced, and systematized?
3. Is the level of cognitive development of the students respected?
4. Is the use of different, creative, and original resolution strategies encouraged?
5. Are different learning objectives met, and are students developing different cognitive and metacognitive skills?
Interactional suitability
1. Are there moments of dialogue and discussion between students or between teacher and students?
2. Is the resolution of tasks individually, in pairs, or in groups encouraged?
3. Does it allow the generation of cognitive conflict and the negotiation of meanings?
4. Do they promote responsibility for the study (exploration, formulation, and validation)?
Affective suitability
1. Does it promote interactivity, attraction, fun, raising self-esteem, the feeling of inclusion and a taste for mathematics?
2. Are the different types of reasoning and responses valued?
3. Is participation encouraged and interest generated?
4. Do they favor the perception of the usefulness of mathematics in life and at work?
5. Is student involvement promoted in solving tasks (return of learning in Brousseau's sense)?
6. Are there possible challenges to be achieved, triggering levels of thought, each one more complex?
7. Do they present the application and beauty of mathematics?
Mediational suitability
1. Are manipulative and/or technological materials provided, or is their use recommended?
2. Is sufficient time allowed for its completion and the maintenance of concentration and interest?
3. Are the times appropriate for each of the different types of tasks?
4. Are adequate spaces provided for its realization?
5. Are moments of hands-on experimentation provided to aid understanding of concepts and their applicability?
Ecological suitability
1. Are official curricular documents (national and local) considered?
2. Is the articulation between different contents of Mathematics and between different areas of knowledge sought?
3. Are the tasks contextualized with the social and cultural environment?
4. Are the contents of the tasks useful for social and work life?

Among other authors, Canals (2009) and Chamorro (2005) highlight some aspects to consider in the design of problems for early childhood education. These aspects coincide with some of the DSC indicators in Table 1. For example, starting from the children’s interest and curiosity and promoting playful situations (indicator 1 of Affective suitability); contemplating the children's different languages and forms of expression, such as oral, gestural, pictorial, musical, plastic, dramatic, corporal, etc. and promote communicative processes that favour the exchange and exploration of ideas, allowing children to advance in language and modes of representation (indicators 2 and 5 of Epistemic suitability and indicator 1 of Interactional suitability); favour situations that offer challenging experiences (indicators 5 and 6 of Cognitive suitability), that encourage children to explore, observe, compare, pose and test hypotheses, make decisions, propose and solve problems, and that consider the different fields of mathematics, articulating them whenever it is possible and necessary (indicators 3, 4 and 5 of Epistemic suitability).

Methodology

The participants were 76 students from two groups of the Mathematics Didactics subject that were carried out in the next-to-last (3rd) of the Early Childhood Degree of a Catalan public university (Spain), during the first semester of the 2020-2021 academic year. This degree is organized in four years with only two mathematics subjects: “Mathematics, Science and education” (in 2nd year) and “Mathematics Didactics” (in 3rd year). The course began with mixed teaching and, given the evolution of the health situation due to Covid-19, it became virtual and the BlackBoard Collaborate application was used in the course schedule which has made it possible to configure rooms for work in small groups when it was necessary. However, when students worked independently outside of class sessions, they could choose the platform that best suited their preferences. The teacher of these groups along with the first author had to redesign the programming of the Numerical Thinking content block to adapt it to the new teaching modality. For this, he prepared various tasks that are part of the context of a broader study than the one presented in this work (some parts of which we are still analyzing), with the general objective of studying how future early childhood education teachers build their didactic mathematic knowledge about arithmetic problems solving.

One of the tasks before the one analyzed in this work carried out by the participants (we will call it Task A) had the purpose that the students knew a didactic-mathematical source, as a practical example, on which to base the development of their next task we will call it Task B. This work only focuses on Task B because it is the part of the data currently being analyzed and responds to the specific objectives indicated in the introduction section.

To carry out Task A, the students organized into workgroups of 3-5 people (in total, 19 workgroups), carried out an analysis task outside class hours from De Castro and Escorial (2007) reading, "Solving verbal arithmetic problems in early childhood education: an experience with an investigative approach." This article explains, among other aspects, how a teacher raises a series of seven problems for her 5-year-old pupils and the development of the sessions where they solve them. The problems and the resolution strategies used by the children were classified based on a table prepared by the authors of the reading article and based on the typology of Carpenter et al. (1993).

First, each person in the group, individually, had to read the article and, later, share doubts and reflections on it with the rest of the members of their workgroup. Second, they had to choose by consensus at least three of the seven problems in the article and analyze them based on the following demands: identify the problem statement; identify and justify the type of problem involved (according to the table in the article) and what strategies the children use in solving, as explained in the narration of the article; explain if the strategies identified are the ones that would correspond according to those set out in the table of the article; identify the material resources used by children and how they use them for the resolutions presented; they also had to solve the chosen problems with material resources other than those in the article and reflect on whether, when changing resources, the resolution strategy also varies (comparing the resolution strategy itself with that used by the children in the article).

Description of Task B analyzed

This task, which was presented to the students in a virtual joint session, consisted of designing a minimum of three problem-solving sessions for a classroom of twenty 5-year-old children. This task had to be developed with the same groups as those in Task A, outside the hours of the virtual joint sessions. For this, the teacher recommended consulting the early childhood education curriculum, the article by De Castro and Escorial (2007) analyzed in Task A and the rest of the documents of the content block. It was also part of the task to present the recording of a video conference where the group members discussed the aspects of the design and the implementation methodology that they considered essential to incorporate and reach a consensus and/or seek solutions. Before the final delivery of the works, a 2-hour class session was dedicated to presenting their designs and obtaining feedback from their classmates and the teacher. Thus, they had time to incorporate improvements in the design of their sessions. Finally, they presented a file in PDF format with the design of the sessions and a file in MP4 format with the recording of their decision-making session (video conference). We analyzed the recordings of the video conferences and the document of the design of the sessions of the 19 working groups.

Analysis methods

This is a study of qualitative characteristics on the aspects that the participants consider essential to consider in their designs of a sequence of problems for the teaching and learning of numbering for 5-year-old pupils and analyze which of these aspects can be identified with DSC and which elements of the DSC are not considered. The phases of the thematic analysis proposed by Braun and Clarke (2012) have been adopted in this study. In the first phase, to familiarize ourselves with the data, the group's videoconference recordings were viewed concerning the planning of the work to be carried out, and aspects of the design and management of the problem-solving sessions addressed by the participants were identified. Then, it was compared with the written works sent as a final document to verify that these aspects were reflected in their final works and/or considering other aspects that emerged. After this, the presentations before the final submission have been reviewed to observe the improvements made in the final work because of the influence of the sharing session, but it is a part of the research that we will not deal with here due to the limitation of the space.

In the second phase, the first author systematically applied the a priori categories based on the indicators in Table 1 to the evidence (identified basically in their written works) of the aspects that the students considered necessary for the design and management of their resolution sessions of problems to analyze in a descriptive and interpretive way which criteria could be identified with the DSC and which aspects of these were not contemplated. For example, when students said their problem is contextualized with “a story that calls them [the pupils] the attention”, we considered it is evidence of Affective suitability, specifically of indicator 3 (see Table 1). The third phase was focused on the categorization revision by three authors. Also, in this phase, the identification, analysis, and interpretation of the pieces of evidence were triangulated with an expert in using the DSC. In the last phase, the three authors discussed the results.

Results

Although the participants were not given any training on DSC, all groups implicitly used DSC to justify decisions about the design of their tasks, although no group evidence is observed for all indicators. All designs have high *Affective suitability*. For example, in videoconferences, the 19 participating groups express the intention that the activities motivate children to participate in them and find them fun. To do this, seeking to activate empathy, characters such as the class mascot or the school cook are incorporated, who supposedly live in problematic situations and ask for help, games are created, stories, in their own words, “a story that calls them the attention,” etc.

Regarding *Mediational suitability*, all groups plan to offer material resources to children to solve their problems. However, all adhere to the resources seen in the training classes and add non-specific materials (pencils, pasta, stones, chickpeas, etc.) and others related to the context of the problem (reproductions of fish, drawings of trees, and apples, sports medals, etc.). Only one of the guideline designs offers five specific material resources and a program that rotates through the small groups of children formed for each session. However, it is not contemplated that each child can choose and experiment with the material that he or she sees fit, and it is not considered that some of the resources are better than others depending on the problem to be solved. In addition, in all the designs except one, the activities are planned to be carried out entirely in the classroom environment. Only one designer schedules sessions on the playground and in the psychomotor room.

Regarding *Epistemic suitability*, 14 of the 19 designs are concerned that the children’s organization and time includes a space for the explanation and argumentation of the process through which the answer has been found. However, only 2 designs specifically explain how to manage it, and the only one provides possible dialogues with good questions to promote the generation of hypotheses, argumentation, and justification. Although future teachers have studied problem-solving from an investigative approach and have analyzed an example (in Task A), none of the designs considers it. All the groups work from verbal statements, with clear and straightforward questions, and freedom is left for different proposals for resolutions to arise that will be put together (only 5 of the 19 designs do not foresee a pooling to explain the possible diverse resolutions of the children).

There is less evidence that students took the criteria of *Cognitive suitability* of their tasks into consideration, but all designs consider introducing problems with a gradual increase in their difficulty. In their dialogues, they start from what theoretically 5-year-old pupils should know according to the curriculum (for example, in their words: “they have to know how to count to 20”, “we have to consider the 5-year level”), but only one of the designs incorporates prior knowledge exploration activities. In their dialogues, future teachers do not debate about which concepts the teacher is going to reinforce or systematize; they only say that “she will have a guiding role,” “she will interact with the children”, “she will go around the groups to review the processes and results,” “at the end the children will stand in a circle so that the teacher can clarify doubts.” On the other hand, we have observed aspects related to the evaluation of pupils that, although implicit in cognitive (and ecological) suitability, are not explicitly contemplated with indicators in the DSC. The 19 works analyzed are concerned with evaluating pupils learning, with various proposals such as rubrics, lists of actions to observe, or lists of questions for children to answer.

Regarding the *Interactional suitability*, all the students' designs organize the activity to solve the problem in small groups, arguing: "so that they help each other." When they want to increase the difficulty, they propose that they be individually resolved. However, no design reflects how to get 5-year-old children to collaborate and, therefore, no working group considered that the activity implies that they must negotiate meanings and promote it. None of the groups designed activities self-validating: the process and the validation of the results always go through the teacher's approval.

Evidence of *Ecological suitability* is obtained since curricular objectives and competencies are incorporated for the second cycle of early childhood education, specifying them for each activity. However, none of the groups attempts to articulate different mathematical contents since, apart from the typology of problems studied, no other concepts are incorporated (such as logic, which is a block of knowledge dealt with in the course in the months before this work), or other knowledge areas. The statements of the problems contain elements that future teachers think are every day for children but are based on imaginary stories with children's characters, more in the sense of activating empathy and obtaining a high emotional fitness. Only 3 of the 19 designs are based on real experiences: athletics games at school, activities in the school garden, and a trip to a lake. However, they are only a motivating contextual excuse to put elements in the statements of the problems, and in no case are the activities proposed in an interdisciplinary way.

Discussion and Conclusions

It can be affirmed that, in the training of early childhood education teachers, it is essential to promote dialogue to make consensual decisions in the design of tasks for the teaching and learning of suitable mathematics, since group reflection enables the generation of proposals, opinions, and decisions taken as a group. Regarding the aspects that future preschool teachers consider essential when planning and designing a sequence of arithmetic problems and which of these are identified with the DSC, future teachers are implicitly based on criteria identified with the DSC. However, not all its indicators emerge since their reflection is not guided by specific DSC and they do not have an explicit guideline to guide their didactic analysis. This result agrees with the conclusion reached in recent works such as Breda et al. (2017) and Sánchez et al. (2019). In this sense, it is observed that DSC appears as a regularity when teachers or future teachers want to justify the criteria on which their decisions are based without being taught the use of this construct. The reason could be related to the fact that DSCs reflect consensus on what good mathematics teaching should be, widely assumed by the educational community. From a didactic point of view, this study offers indications that it would be convenient to offer early childhood education preservice teachers a tool such as DSC so that they have explicit criteria to guide the designs of their mathematical tasks. In this sense, a future line of research opens, much needed, to adapt the DSC to the singularities of this educational stage.

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Patterning in Early Years,

A Synthesis of Research

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This paper attempts to summarize the research related to patterning in early childhood (4-8 years). In doing so, the basic concepts, rationales, methods, and findings are systematized to provide a foundation for forthcoming design. The purpose of this presentation and the suggested discussion is not to present new data but to systematize existing as a guide for future research.

Keywords: patterning, definitions of pattern, pattern activities, pattern abilities, pattern strategies

Introduction

Systematic exploration of early patterning begins in nineties as an important element of early mathematics education. Researchers accept that children's familiarization with patterns and structures influences their future mathematical development (Clements & Sarama, 2009) and provides a foundation for understanding complex mathematical situations and making generalizations (Rivera, 2013).

To date, a large number of different studies have been conducted on early patterning in mathematics education. These studies approach the idea of pattern differently, use a variety of pattern forms, propose similar but also dissimilar pattern activities and understand the connection of pattern to mathematics in various ways. Therefore, we believe it is important and mature to attempt a synthetic presentation of this research in the following axes, covering aspects such as: (a) the *conceptual definitions* and types of patterns, (b) the *rationales* for linking patterns to mathematical development in different contexts, (c) the research related to patterning *skills and strategies*. These elements are considered important for planning future research. A synthesis of the research of this content would also require an outline of the main findings, but as we will see, due to the diversity of research, this synthesis could only be done on certain axes, therefore it is limit to presenting only findings on patterning skills and strategies.

Clarifications

Pattern definitions

Not all researchers define the meaning of pattern in the same way, since pattern is not a distinct mathematical concept. A first approach describes pattern as *a sequence of elements* of different nature (numbers, shapes, sounds, symbols) organized according to a rule. This definition implies regularity and therefore predictability (Papic, Mulligan & Mitchelmore, 2011; Radford, 2008).

However, according to Liljedahl (2004), the *nature of the elements* of a pattern is crucial. He argues that the concept of a pattern is a primary notion defined within the set of specific forms of regularities to which it belongs (e.g., sound, space, arithmetic, etc.) and has features within that set. Thus, he supports that, for example, a sound or a spatial pattern, even if they have the same e.g.

ABB structure as an arithmetic pattern, they are of different natures and can be approached differently.

Extending the above view, Papic et al. (2011) identify patterns in the context of mathematics as, a) a *regularity in an object* in which some of its elements are related in a systematic way, b) an *ordered set of objects* in which each element is connected to the next by a specific relation, and c) *two ordered sets of objects* whose elements correspond to each other. So, the above researchers extend the corresponding concept by including elements organized on the basis of a structure (such as a clock), endogenous or constructed or introduced by a mathematical system (Mulligan, Mitchelmore, English & Crevensten, 2013).

Summarizing the above, it is understood that different research approaches in pattern definition are recorded, but the *basic component of a pattern* - common to all views - is the relationship between the elements that constitute it, which either enforce its organization into a structure or allow the prediction of the subsequent terms of its sequence (Radford, 2008; Rivera & Becker, 2009; Warren & Cooper, 2008).

Pattern categories

There are also a variety of categories and terms regarding patterns that appear in relevant research. In general, researchers have identified and widely reported *three specific forms*: repetitive, evolving, or relational patterns (Lünken, 2018; Mulligan, Mitchelmore, & Prescott, 2006; Threlfall, 1999).

For repetitive patterns, a further distinction is made based on the type of *iteration unit* (e.g. ABAB, ABABA, Liljedahl, 2004; Papic et al., 2011; Threlfall, 1999), but also on the changing parameters (e.g. shape only, or shape and size, etc., Fyfe, Evans, Eisenband Matz, Hunt, & Alibali, 2017; Skoumpourdi, 2013). As for the developing patterns, Rivera and Becker (2009) group them into *additives* or *multiplicatives*. Rivera (2013) also distinguishes patterns in terms of their elements in arithmetic and geometric forms, while Warren (2005) includes regularities with elements in the form of shapes, colors, movements, sensations and sounds (especially for young children). The relevant findings are related to these categories.

To these forms Mulligan et al. (2006) add *algebraic* (triangular formations with numbers), *metric* (with repetition of unit elements) and *diagrams*, but also *two or three-dimensional* formations (Mulligan et al., 2013). Similarly, Ma (2009) refers to *iconographic patterns* (mainly geometric) that lead to the formulation of verbal rules.

In this category, Chua and Hoyles (2013), who study schematic or figurative forms that lead to arithmetic relations and refer to rules (function type), distinguish their categories on the basis of the formations, but also of *the type of the function type* to which each form refers (e.g. some lead to types of linear functions, e.g. $5 + 3ν$, while others lead to types of secondary functions, $ν^2 + 2ν$, etc.).

Composing the above, a variety of categories emerges in the relevant research, which can be organized in the *following axes*, with mainly mathematical content (i.e. without sound or kinetic patterns) and corresponding combinations: *content or type of elements* of the pattern (iconographic, schematic, geometric, metric, arithmetic, algebraic, etc.), *type of development* (repetitive, evolving, retrospectively evolving, etc.), *spatial dimensions of development* (e.g. one-dimensional, two-dimensional, three-dimensional), *structure of the pattern* (for repetitive: AB, ABB, ABBA, etc. – developing: additive, multiplicative, etc.), *material* (tangible, virtual and/or symbolic). The research has been conducted only in some of these categories.

Pattern activities

In this section, it is important to add the activities suggested by the researchers from the youngest to the older students.

At young ages, activities with patterns include *replication* with or without copying, *continuation* or extension of regularities, and *recognition* of structurally similar regularities. To these activities there is also research with *pattern creation* (Papic, Mulligan & Mitchelmore, 2011), *search of a missing element* (Warren, 2005), and *alteration of material* (Clements & Sarama, 2009).

Finding and describing the *iteration unit* or smallest unit of a pattern is recognized by many researchers to be a particularly important activity (Mulligan, Mitchelmore, English, & Crevensten, 2013; Clements & Sarama, 2009). Similarly, *predicting later terms* for generalization (near or far) (Ma, 2009). These two activities, distinguishing units and predicting terms, as well as linking them to a next position (e.g. what is the 10th term?) are considered very important as they lead to generalizations and symbolism (Michael, Elia, Gagatsis, Theoklitou, & Savva, 2006; Lannin, 2005). It is argued that pattern generalization but also verbal explanations (Lüken, Peter-Koop, & Kollhoff, 2014), drawings, or other forms of symbolism (Mulligan, et al., 2013) are an important introduction to both later numerical or developing patterns leading to formulas at this age.

In composing, research suggests the following activities, different in their treatment and difficulty: *copying*, *reproducing*, *extending*, *transferring to another material*, *recognizing the repetition unit*, *identifying the same units of repetition*, *finding the missing element* (interpolation), *finding the error* (Wijns, Torbeyns, De Smedt, & Verschaffel, 2019), *generalization* (finding a *formula*, *finding a term in a specific position*, Lannin, 2005).

Students' skills in pattern's development

The finding that young students (ages 4-8, kindergarten and elementary school) have an ability to develop patterns is a common thread throughout the literature. The results of research on children's performance on activities that involve patterns reveal the basic types of regularities and the types of activities that young children successfully master without a specific didactic approach. Thus, in general, even preschoolers are able to explore repetitive patterns, language patterns, and some linear patterns and perceive regularities during their leisure activities (Lüken, 2018; Fox, 2006; Waters, 2004).

Most research in early childhood focuses on examining performance on repetitive patterns, with the majority of children able to reproduce, continue, and create a repetitive pattern AB or ABC (see, e.g., Skoumpourdi, 2013; Papadopoulou, 2010; Warren & Cooper, 2008). More complex forms, including those of the ABB type, seem to cause greater difficulty, while the pattern material does not seem to have a significant impact.

However, analyzing the pattern in its structural elements and finding the iteration unit is becoming more demanding issue, which has led researchers to consider iteration unit location, complementing (finding the missing element) and identifying (which patterns are the same? is this a pattern?) at a higher developmental level of the respective ability (Wijns et al., 2019).

Developing patterns that is an early introduction to functional reasoning, appears to be more challenging, leading to less emphasis at younger ages. Although the performance of young children has not been systematically studied, the relevant research mainly concerns children in the last grades of elementary school and the first grades of high school, it seems that in this area the material and the structure play a crucial role (one-dimensional or two-dimensional growth). Several researchers point out that children reach high percentages in repetitive patterns by the 2nd grade, but less in increasing geometric or arithmetic patterns (Papadopoulou, 2010; Warren, 2005; etc.).

In relation to the building patterns of objects of different types (Mulligan et al., 2013), the developmental trajectory of young children has been studied in depth through a systematic program of activities from the prestructural to the structural level (intermediate levels: emerging and partially structural). This view brings the importance of curriculum and instruction to the first level, as a number of studies - (not examined in detail in this paper) - highlight the positive impact of appropriate educational interventions and programs.

In summary, children up to first grades succeed, even without educational help, to recognize patterns with simple structures or materials and identify a simple rule, but have difficulty with more complex forms as well as numerical, developing patterns. There is a lack of meaningful research on systematic and verbal finding of the unit of repetition and near or far generalization (e.g., what is the form at the next or next but one place - near generalization, or what is the form at the 10th or 20th place - far generalization).

Patterning strategies

By studying the activities of students with patterns, some researchers have also studied the strategies they use at an early age. A common assumption is that children make one-to-one correspondences to the proposed pattern activities that mainly involve copying, reproducing, extending, or transferring to other material, or that they take a rhythmic approach based on a sequence (which, however, according to Threlfall, 1999, cannot lead to generalization), such as finding the unit of repetition or 'breaking into pieces' (Mulligan et al, 2013).

Papic et al. (2011) categorize children's strategies into *direct matching*, *alternation*, *repetition* of basic units, and *integrated repetition* of units. Accordingly, Lüken (2018) summarizes relevant results and records five strategies of students in solving activities with patterns: in the 1st, there is *no reference* to the design, except for the holistic reproduction of the pattern; in the 2nd, the focus is *on some features*; in the 3rd, *on comparison* and classification; in the 4th, *on the sequence*; and finally, in the 5th, there is *a focus on the unit of repetition*.

In summary, young children successfully develop strategies to find the next element in a pattern or its rule, but according to some researchers (see Lüken (2018)), copying, imitating or reproducing without generalization and symbolism cannot be recognized as a mathematical activity, that is the finding of the iteration unit, its generalization and its possible symbolism. The next section gives some answers about the importance of developing patterning skills and strategies.

Relationship between pattern development and mathematics

Researchers engaged in studies with patterns argue that patterns are important because "... *Mathematics is the science and language of regularities*" (Steen, 1990, p. 5, in Fox, 2006) and familiarity with them also promotes mathematical thinking, exercises the ability to find rules and structures, both in everyday situations and in symbolic objects or symbols, such as the elements of algebra (Vogel, 2005).

In general, connections to Mathematics relate patterns to numbers (e.g. repetitive structure of the decimal system, cumulative and multiplicative structures), to geometry (e.g. regular shapes), to measurement (e.g. creating and repeating a unit of measurement), and to data processing (e.g. finding rules and relationships) (Lüken, 2018; Mulligan, Mitchelmore & Prescott, 2006).

The connection with algebraic reasoning (Fox, 2006) as well as with functions (Warren, 2005) is another axis that many researchers have worked on (see Ma, 2009; Papic et al., 2011). More generally, it is agreed that the formation of generalizations, subtractions and symbolizations enhanced by pattern activities as an indispensable element of Mathematics (Lüken et al., 2014).

Several studies examining the relationship between performance in patterns and performance in other mathematical domains (e.g. arithmetic) or other cognitive skills (e.g. reading, memory, etc., Fyfe et al., 2017) show important connections. Some others, also investigating whether high performance in patterning is a predictor of later mathematical achievement (Rittle-Johnson, Zippert, & Boice, 2016), reach different conclusions.

Combining the above, it becomes clear that different pattern forms and activities support different aspects of mathematical concepts. Thus, *repetitive patterns* (with the characteristic feature of circular character and repetition unit) and activities requiring *continuity, completion, finding a missing element, finding an error, finding a unit of repetition*, etc., can help students approach mathematical objects or processes that are systematically repeated phenomena, such as number systems, regularities in shapes, repetition of units of measurement, etc. (Liljedahl, 2004); *generalization, and symbolism* are important in mathematical domains such as problem solving (Rivera & Becker, 2009; Radford, 2008) or equations (McNeil & Alibali, 2005). Accordingly, *arithmetic regularities* and activities requiring *finding not only some next elements* or the type/function that generates them (e.g., a number sequence 1, 5, 8, ...), but also some elements of the pattern that are in a particular position (far generalisation), support functional thinking from an early age.

Discussion for future research

From the above review, we need to highlight the following points for future research: First, the number of studies that have been conducted on early patterning is large, but there is a variety of contents, aims, questions and approaches, thus the concepts and terms used in them are not always clearly or similarly defined. For this reason, systematic concern regarding terms, categories and activities are needed in any future research.

Then, the review presented show that regularities or patterns, as well as related activities, can be categorized in many dimensions. Therefore, it would be important to conduct synthetic comparisons and compare performances surveys but only in common approaches. Also it is also important to systematically investigate the extent to which certain pattern forms and activities are related to mathematical content and the mathematical skills they promote, as not all regularities lead to mathematical activity and thus to Mathematics.

Finally, we know that children in general can develop important patterning skills at a younger age, but not all of them are related to mathematics. Therefore, in research or classroom practice, the researcher/teacher needs to know which developmental pathway serves which content, material, structure, and/or action.

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Early childhood teachers' awareness of the characteristics of picture books for learning mathematics

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Keywords: early mathematics, picture books, teachers' awareness

Research topic and theoretical framework

There is growing research which provides evidence that the use of picture books in the early years of schooling can contribute to the learning of mathematics (e.g., Van den Heuvel-Panhuizen et al., 2016; Van den Heuvel-Panhuizen & Elia, 2013). However, it is also known that different books can generate different amounts and different kinds of mathematical talk (Anderson et al., 2000). Therefore, it is essential to know what picture books can offer to elicit mathematical thinking of children. In particular, this knowledge is important for early childhood (EC) teachers when they use picture books to create opportunities for children to gain informal mathematical experiences. The study presented in this poster is a part of a research project which is carried out to acquire cognizance of how to enhance teachers' competence in using picture books in early childhood mathematics education.

The study builds on our previous research in which we developed and researched a framework of learning-supportive characteristics of picture books for learning of mathematics (Van den Heuvel-Panhuizen & Elia, 2012). This framework consists of two parts. Part A incorporates the mathematical content that is offered in a picture book. Besides the usual content domains this also refers to mathematical processes and attitudes, and mathematics-related themes. Part B describes how the mathematics is presented. It includes both the way of presentation and the quality of it.

The goal of the present study is to investigate whether the teachers are aware of the learning-supportive characteristics of picture books, specifically whether they are aware what mathematics can be found in picture books and the way the mathematics is presented, and whether the framework is of use for recognizing opportunities of picture books to support children's mathematical development. The latter is the focus of the poster.

Method

To investigate whether the framework helps to improve teachers' awareness of the opportunities picture books offer for early childhood mathematics education, a qualitative study was conducted in Cyprus and Norway with respectively 6 and 4 in-service EC teachers, who had to evaluate two picture books with and without the framework by filling in for each book two times an open questionnaire specially developed for each book. The picture books are regular trade books of high literary quality, not purposely written for teaching mathematics, but they tell appealing stories which form a meaningful context for engaging children in mathematical thinking. In both countries we used the same wordless picture books. In this way teachers have also more freedom to recognize learning-supportive characteristics and can develop their own ideas of using them. In the data analysis the focus was on finding noteworthy changes in teachers' answers between the two rounds.

Results and discussion

Overall, we found that in both rounds teachers' answers involved to a greater extent learning-supportive characteristics included in the supply of mathematical content (Part A) than characteristics referring to the presentation of mathematical content (Part B). Moreover, the answers often were short and without explaining their thinking. They rarely referred to mathematical processes or dispositions. Possibly they are not accustomed to using picture books or the framework did not appeal to them. Regarding the mathematics that was found, the framework helped the teachers to see different or more mathematics and further specify the identified mathematics from the first round. The framework also had a positive contribution to teachers' evaluation of how the mathematics is presented. However, in some cases the framework did not have added value. In the second round a few teachers just used the wording of the framework, while their answers earlier were more detailed and relevant.

Our study has shown that the framework has potential to make teachers aware of the learning-supportive characteristics of picture books for supporting children's learning of mathematics. Our next step will be to use our findings to set up a professional development for EC teachers.

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Analysing early reasoning: A pilot study with prospective early childhood teachers

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This paper presents the design and construction of a professional task aimed at analysing experiences that foster reasoning in children aged 0 to 3 years old. The process served to organise and validate a task that mobilises future teachers' knowledge of how to foster reasoning in early childhood education. The task allows the recognition of metacognitive actions that show how, in the cases studied, relationships of cause and effect are produced as a prelude to strengthening mathematical reasoning. This permits a reflection on the way in which children at this stage intuitively construct their knowledge.

Keywords: Initial teacher training, professional task, reasoning, early childhood education

Introduction

Some research reflects on the capacity of young children to establish causal relations and their capacity to draw conclusions based on certain experiences and representations (Sobel & Legare, 2014). It is explained that educators can improve thinking skills in children aged 0 to 3 (Sturman et al., 2005). Recent research findings attest to the importance of early mathematical learning. They show that the understanding of complex mathematics and abstract reasoning develops much earlier than was once thought (Vukatana, 2013; Clements et al., 2019, among others). Although these authors presented their work some time ago, the amount of literature on early childhood mathematics education remains rather limited as regards logical reasoning in the early years. On the other hand, Ivars and Fernández (2018) and Moreno et al. (2021) point out that during their initial training it is important to encourage prospective teachers to gain experience from different contexts where they can observe teaching situations in a structured way. One of the possible contexts is the analysis of in-school experiences. Such experiences, when described as a narrative, can be a powerful instrument for the construction of professional tasks. Our interest in the initial training of early childhood education teachers has prompted us to consider, among other aspects, the different approaches and professional tasks faced by future teachers.

The purpose of this paper is to describe the process of designing and structuring a professional task focused on the analysis of experiences that promote reasoning in children aged 0 to 3 years old. This kind of task seeks to involve pre-service teachers in processes of interpretation and decision-making (in teaching situations) so that they begin to "notice" the mathematical activity in the classroom (Mason, 2021). This type of study is considered to play a key role in reformulating training programmes and responding, on the one hand, to the challenge of involving future teachers in reflective processes and, on the other hand, identifying experiences that make it possible to promote mathematical logical reasoning in young children. In our context, early childhood preservice teachers are trained to teach children aged from 4 months to 6 years old. In this study we will focus on the mathematical education of the youngest group, specifically children aged up to three years old.

Theoretical background

The logical mathematical reasoning of children aged 0 to 6 years old can be organised according to its complexity (from least to most) in three parts: a) identifying, defining and/or recognising qualities, b) relating qualities, and c) operating qualities (Canals, 2009). This organisation can not only help teachers to design activities but also to analyse children's level of understanding and the problems children face when they have to tackle a task that requires logical-mathematical reasoning. Some studies describe the emergence of category-based inductive reasoning during infancy, focusing on adherence to a fundamental induction principle, premise-conclusion similarity. It is shown that infants aged 13-22 months use both category information and perceptual similarity to guide their inductive inferences about non-obvious properties under various conditions (Graham et al., 2020).

One of the difficulties encountered by future early childhood education teachers consists of recognising the knowledge that underlies children's actions, especially when it is not verbalized, as in the case of children under 3. It is important that pre-service teachers are familiar with activities that foster reasoning at this stage. And that they are able to design and manage tasks that address processes such as attribution, designation and characterisation, which lead to the idea of classification (Peres, 1984). Prospective teachers must be provided with enough tools to be able to develop the specific skills needed for educational practice (Llinares, 2013). One of these competences is the so-called noticing, which involves the development of the prospective teachers' cognitive skills in order to identify and interpret the students' mathematical thinking and enable informed decision-making with regard to their teaching proposals (Jacobs et al., 2010). In recent years understanding how this teaching competence can be developed has become a research goal in the mathematics education (Dindyal et al., 2021).

Criswell and Krall (2017) argue that the noticing competence enables teachers to go beyond the more easily observed issues, such as students' behaviours and actions. It enables them to address issues that need to be meaningfully inferred, such as students' thinking about a given mathematical notion. Ginsburg (2016) states that teachers cannot teach well if they do not understand children, mathematics and the associated pedagogy. Therefore, it is crucial that teachers and future teachers learn to recognise and understand children's mathematical thinking.

Research on task design focuses on a number of aspects (Swan, 2007; Charalambous, 2010). One of them is the use of narratives in pre-service teacher training, which provide relevant starting points for the discussion of problems faced by teachers when making decisions about the various teaching and learning situations, such as how to apply certain curriculum requirements to school activities, classroom management techniques, and/or how to assess and track students' learning (Chapman, 2018). We use narratives to reflect on the thinking and actions of prospective mathematics teachers in the context of mathematics content and mathematics teaching and learning itself.

As pre-service teacher educators, we have to select, plan and design professional tasks that enable them to identify children's behaviour, thereby supporting the development of their thinking and consequently their ability to make justified professional decisions. In early childhood education, prospective teachers are not usually specialists in mathematics teaching and this means that the selection and design of professional tasks must be more carefully carried out if a satisfactory understanding of mathematical notions and processes and the associated didactics is to be achieved.

Methodology

This research features a qualitative approach of a descriptive nature (Cohen et al., 2018). It is based on intervention research that focuses on action, seeking to understand and explain its effects. This intervention study was carried out with students pursuing a degree in early childhood education at a Catalan university during the 2019-2020 academic year. The participants had worked on the content dealing with logical-mathematical reasoning and the mathematical problem-solving process. The professional task was designed to enable future teachers to observe, analyse and reflect on real experiences that promote logical-mathematical reasoning in the 0-3-year-old period. The professional task had five stages: 1) a review of studies on the teaching and learning of logical-mathematical reasoning in 0-3-year-olds; 2) a review of studies, materials and activities used to work on logical-mathematical reasoning in early childhood education; 3) a choice of narratives for future teachers to observe and analyse; 4) the implementation of the aforesaid professional task in a pilot group; and, 5) the definition of levels to characterise teaching competency in advance. The written work produced by each group of students provided the data for this study.

The design of the professional task (PT) presented in this study is based on a previous professional task (PPT). This task (PPT) focused on the design and implementation of activities for children (0-3-year-olds) based on the use of non-specific materials. The PPT was developed in the 2018-19 academic year by 80 students pursuing an early childhood education degree, organised into working groups of four to five and from two Catalan universities. Each working group designed and implemented an activity using non-specific materials, with the purpose of developing logical-mathematical reasoning in 0-3-year-old children. The goal was to provide opportunities for the children to discover and establish common (or distinct) characteristics by manipulating and experimenting with different objects. This is the perceptual basis of processes such as attribution, designation and classification. The PPT also included a reflection on the materials used, on its potential and relevance when promoting reasoning in this children's age group. As a final product the future teachers submitted a written report and a visual record of the task implementation.

In the following course (2019-2020), some of the proposals made by the pre-service teachers who developed the PPT were used as the basis for the design of the PT for a new group of prospective teachers. This new task focused on the study of narratives, with proposals specifically derived from the PPT. The aim of the new task was to strengthen other aspects of teaching competency in the new group of prospective teachers, particularly as regards the analysis of real classroom experiences.

The professional task (PT) was organised into three parts. Firstly, the prospective teachers were asked to examine the main characteristics of two activities (a sensory wall and heuristic play respectively). Identifying the type of objects used in each activity, indicating the attributes of each object or groups of objects, and describing the actions that could be done with each object. Secondly, the future teachers were invited to associate each of the children's actions with the logical thinking skills of identifying, relating, and operating, and also to evaluate the design and management of each of the activities. Finally, the prospective teachers were asked to give their reasons for whether or not they would again use the same or similar objects to those employed in the two activities.

The PT was initially carried out with 37 students pursuing an early childhood education degree, who were studying the course's introduction to mathematics didactics. These future teachers had studied content related to logical-mathematical reasoning and the problem-solving process. We proposed that they develop the task in small working groups (3-4 people) for purposes of a joint reflection that could promote the competence of didactic analysis of learning situations. As a result of their analysis, each group submitted a written report that provided the data for this study. In total there were 10 groups and 10 papers.

Results

Having completed the task, the answers provided by the different groups of future teachers were organised in order to refine the task for future use and also characterise the teaching competence of noticing. Different levels associated with the different teaching competences – identifying, interpreting, and making decisions – were established (Jacobs et al., 2010).

By way of an example, we present the levels associated with interpreting skills. In this research we found that prospective teachers recognised measurable attributes, spatial relationships associated with movements, and attributes associated with comparisons and order (e.g. large-small; thick-thin; high-low). They also identified different relationships, for example, when children showed unequivocal signs of being able to identify similar elements. Given the type of action performed, they distinguished causal relationships.

Regarding the interpreting competence, three levels were established previously:

L1 – Identification of actions of comparison and some relationships of cause and effect by indicating them descriptively

L2 – Metaphorical allusion to types of reasoning, indicating terms such as 'visual' and 'deductive', but without specifying what they mean. Describing actions associated with the different logical thinking skills (identifying, relating, operating), but without a theoretical justification.

L3 – Evaluation of children individually, appropriately associating types of abilities and possible types of mathematical reasoning. Discrimination of key aspects of the adequacy of the activities according to age.

Table 1: shows extracts from the answers provided by the different groups and the allocation to the different levels

Table 1: Examples of answers and the levels associated with the interpreting competence

G	Level	Answer Extract	Analysis
10	L1	"…They realise that when they take a *metal tube* and *hit* it, it makes a *noise*, and that the *harder they hit it*, the *louder the noise*"	The group of prospective teachers described actions that allude to relationships of cause and effect, but without any justification. They only refer to them.

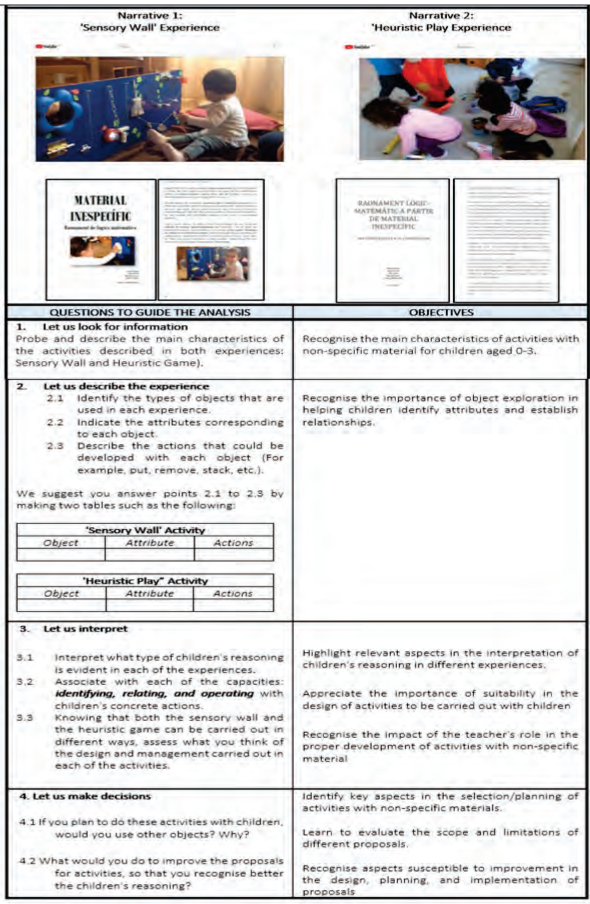
2	L2	“On the sensory wall, we saw that the girl can *identify different elements* through touch. It is a process of identification by touch.” “...They observe that when *they move some objects* in a *specific way* or when these *collide with each other*, they make a *noise*. They establish a *relationship* between the *action (hitting) and the noise.’*	This group interpreted different actions and associated them with different competences (identifying and relating).
4	L3	“…In other words, with this experience, children begin to *discover and get to know the different objects* in the classroom. *Then* they begin to *do actions* with them, for example, making a noise, and finally they *play the symbolic game*, where those objects stop being what they are to become other things such as necklaces, snails, telescope, etc.” “Perhaps if we used a treasure basket instead for this activity, *it would be more appropriate*. Given that Eva *cannot crawl yet*, the educator had *to help her to reach* the objects’	This group recognised key moments in the development of experiences where children move from observation to discovery. They also interpreted the value of heuristic play for the introduction of the symbolic game. They appreciated the relevance of the activities according to the age and characteristics of the children (for example, their degree of independent mobility and how this conditioned exploration).

Thus, in the case of the interpreting competence, we observe that the prospective teachers alluded to types of reasoning (inductive, abductive, etc.) metaphorically, erroneously referring to the visual and deductive without specifying how to verify it. They cited descriptive traits of actions associated with logical thinking capacities without offering any theoretical discussion.

After the first implementation and an initial analysis of the reports submitted by the sample of prospective teachers involved in this research, we decided to make some adjustments to the final structure of the professional task. One of the products generated by this study was a professional task that aims to mobilise the professional knowledge of prospective early childhood teachers with regard to experiences and activities that help promote logical-mathematical reasoning at the 0-3-year-old stage.

The final version of this task considered the skills that characterise professional noticing: identifying, interpreting and making decisions (Jacobs et al., 2010; Ivars & Fernández, 2018); the notion of narrative and its potential to encourage reflection on mathematics teaching and learning situations; two types of activities with non-specific material, which we consider suitable for encouragement of reasoning in children at this stage (Goldschmied & Jackson, 1994); and, lastly, specific aspects of reasoning in early childhood education such as the logical thinking skills of identifying, relating and operating (Canals, 2009). The final version of the professional task was structured in four parts (see Figure 1).

1. The future teachers were asked to explore the main characteristics of the two activities (sensory wall and heuristic play).
2. The future teachers were asked to identify the type of objects used in each activity, indicate the attributes corresponding to each object or groups of objects, and describe the actions that could be carried out with each object.
3. The future teachers were asked to associate each of the children's actions with the logical thinking skills (identifying, relating and operating) and assess the design and management of each activity studied.
4. The future teachers were asked to decide (and justify) whether or not they would again use objects similar to or the same as those employed in the two activities and invited to reflect on how they could improve the two activities in order to better understand the children's reasoning.



QUESTIONS TO GUIDE THE ANALYSIS	OBJECTIVES
1. Let us look for information Probe and describe the main characteristics of the activities described in both experiences: Sensory Wall and Heuristic Game).	Recognise the main characteristics of activities with non-specific material for children aged 0-3.
2. Let us describe the experience 2.1 Identify the types of objects that are used in each experience. 2.2 Indicate the attributes corresponding to each object. 2.3 Describe the actions that could be developed with each object (For example, put, remove, stack, etc.). We suggest you answer points 2.1 to 2.3 by making two tables such as the following: 'Sensory Wall' Activity: *Object* / *Attribute* / *Actions* 'Heuristic Play" Activity: *Object* / *Attribute* / *Actions*	Recognise the importance of object exploration in helping children identify attributes and establish relationships.
3. Let us interpret 3.1 Interpret what type of children's reasoning is evident in each of the experiences. 3.2 Associate with each of the capacities: ***identifying, relating, and operating*** with children's concrete actions. 3.3 Knowing that both the sensory wall and the heuristic game can be carried out in different ways, assess what you think of the design and management carried out in each of the activities.	Highlight relevant aspects in the interpretation of children's reasoning in different experiences. Appreciate the importance of suitability in the design of activities to be carried out with children Recognise the impact of the teacher's role in the proper development of activities with non-specific material
4. Let us make decisions 4.1 If you plan to do these activities with children, would you use other objects? Why? 4.2 What would you do to improve the proposals for activities, so that you recognise better the children's reasoning?	Identify key aspects in the selection/planning of activities with non-specific materials. Learn to evaluate the scope and limitations of different proposals. Recognise aspects susceptible to improvement in the design, planning, and implementation of proposals

Figure 1: Professional task of analysis of experiences that promote reasoning in 0-3 cycle

Final considerations

It is in the process of designing and redesigning tasks that the possibility of increasing teachers' knowledge arises. In this case, the professional tasks designed involved future teachers in what Serrazina (2010) calls the process of planning - action - reflection, which plays a fundamental role in the construction of prospective teachers' mathematical and didactic knowledge. Learning to evaluate the adequacy of activity planning and design though the analysis of other people's proposals is also a key competence that should be developed by prospective teachers. Being able to observe and study the implementation of these proposals and observe how an activity with children functions in different contexts allows the integration of both the knowledge developed during training and the observation of how certain processes actually develop in early childhood education.

Thanks to the study of the two activities (sensory wall and heuristic play), the future teachers were able to recognise different types of attributes (sensory and measurable characteristics) and interpret the link between children's actions and different types of relationships (equivalence, order, causal, spatial, etc.), as well as observing actions that indicate changes (shape, position and measurement, among others). These factors play a key role in the promotion of intuitive, informal mathematics in children under three years of age.

The professional tasks described made the prospective teachers more aware of what happens during different activities carried out with children under three. One of the tasks focused on the process of design, planning and implementation of activities to promote reasoning in the 0-3-year-old stage, while the other professional task invited the prospective teachers to evaluate the said activities in order to make decisions for their improvement. All this implies an analysis of the mathematical content, reflecting on how to organise this content in order to teach it, analysing and interpreting children's productions, and considering how this type of activity can be managed. In short, as proposed by Moreno et al. (2021), the goal is to help prospective teachers learn to notice and provide them with a wider range of resources to make effective teaching decisions.

We intend to continue developing this approach, designing and redesigning proposals to improve prospective teachers' professional knowledge and contributing material to the discussion around the type of education in mathematics and didactics of mathematics needed by prospective early childhood teachers.

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The multifaceted argumentative structuring of peer interactions in block play situations: Opportunities for early mathematical learning

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Since the 2000s, early co-constructive mathematical learning in kindergarten has been the focus of political discussion and mathematics didactic research. This is based on the fact that kindergarten is the first place for subject-specific mathematical learning, alongside socialization in the family. The focus of research on this first institutional learning in kindergarten is often on the interactions between children and preschool teachers, which are depicted as a key variable for learning. Beyond this, it is the interactions between peers that take up a large part of the time in everyday kindergarten life. In these interactions, children negotiate a variety of topics that are relevant to them. In the following article, these peer interactions in block play situations and especially collective argumentation processes in these interactions are the focus of the research interest.

Keywords: Preschool children, kindergarten, collective argumentation, interaction process analysis, peer groups.

Introduction

Early learning in different contexts is of especially high importance for children's learning biographies. This also applies to mathematics. Already in their first months and years, children experience mathematical contexts and develop a variety of skills in different mathematical areas (Sarama & Clements, 2009) by interacting with others and thus participating in different early mathematical discourses. A large number of studies deal with early mathematical learning in terms of enculturation (van Oers, 2019) in interactions with adults, family caregivers and professionals in daycare centers and kindergartens (cf. Vogler, 2019). In contrast to these studies, there are only a few studies, that investigate children's mathematical exploration while playing with peers for example Flottorp (2011) and Zippert and collegues (2019). Thereby, peer interactions account for a large part of the interaction time (in kindergarten). Following Vygotsky (1978), it can also be assumed that interactions between peers, in particular, can be especially beneficial for the learning process, as these are characterised by a high degree of interactional proximity and adaptivity. The following article, therefore, focuses on such peer interactions in block play situations in a German kindergarten, which Henschen (2020) has analyzed. A special focus is placed on collective argumentation processes between peers, which, according to Miller (1987), assume a special role in learning, since they are constitutive for the construction of new (mathematical) knowledge. In this context, the question to be empirically answered is how peer interactions lead to collective argumentations that creates conditions for the opportunity of mathematical learning.

Theoretical framework

Early mathematical learning in interactions

Mathematical learning already takes place in children´s everyday life before they enter school. This initial learning occurs primarily in a co-constructive manner, and children develop mathematical content actively and through discovery in interaction. Learning, from this co-constructivist perspective, is to be seen as a process of becoming increasingly autonomous in interactions of discourse (cf. Krummheuer, 2013; Vogler, 2019). In line with this, conditions for an opportunity of learning (Miller, 1987) of "new things" are created by enabling children to actively and productively participate in the negotiation processes of discourse, thus opening up different so-called "leeways of participation" (Krummheuer, 2013 p. 253). Bauersfeld (1988) describes that such increasing autonomy is possible when children "rebuild" or "expand" their so-called "Domains of Subjective Experiences" (hereafter: DSE). These DSEs are knowledge resources that are connected to interactively meaningful action in an individual context of experience. They are activated in interaction to cope with the demands of the process of negotiation of (mathematical) meaning - to recognise what is being negotiated and how contributions can be realised.

Collective argumentations in early mathematical interactions

According to Miller (1987), collective argumentation in interactive processes of negotiation is particularly conducive to the development of such new, mathematically rich meanings, or DSEs. This is mainly because, during productive or receptive participation in collective argumentation processes, the possibilities of one's constructions or individual contextualization (based on the DSEs) are systematically exceeded (Miller, 1987). The more comprehensible and convincing the argumentation is to the individual, the more effective the collective argumentation is concerning the individual construction of new knowledge. For mathematical learning, it is of significant importance that collective argumentation is not only generally conducive to the growth of 'new knowledge' (Miller, 1987), but also determines the "thematic shaping" of the (new) individual constructions or contextualization (Krummheuer, 2013). In this context, Krummheuer (2013) speaks of the convergence function of collective argumentations. The collective and argumentatively structured negotiation process consequently conditions and influences learning. Such processes of argumentation are therefore the focus of research interest in this paper.

The role of peers in interactions of early mathematical learning

Miller (1987) considers also the adults to have a special role in the co-constructive and argumentatively structured negotiation processes described above. In early (mathematical) learning, it is referred to primarily as guardians, parents and elementary teachers (cf. Vogler, 2020). They are considered to be the persons who, through their head start in knowledge and scope of (formally) mathematically oriented DSEs, "introduce" children to the culture of doing mathematics and, in negotiation, support children in building new mathematical DSEs. In addition to adults, however, peers can also play an important role as interaction partners in processes of negotiation or argumentation and learning (Youniss, 1980). While language acquisition research focuses on these peers as important instances of language socialization in interactions in the kindergarten context (Arendt, 2019), there has been limited work in mathematics education to address the important role of peer interactions in early mathematical learning in the kindergarten context, although Henschen (2020), among others, can show that diverse mathematical content emerges in peer interactions. This research lack is particularly surprising because it can be assumed that learning in childhood occurs to a large extent through interactions in groups of children.

Research questions

Given the importance of peer interactions as a catalyst for early mathematical learning, it seems essential to closely examine the argumentative negotiation processes of peers and their contribution to the formation and restructuring of mathematical DSEs. Consequently, the questions emerge: (1) How do peer interactions lead to collective argumentations that creates conditions for the opportunity of mathematical learning? (2) What kinds of new or existing DSEs of children, resulting from these argumentative negotiations, can be reconstructed in these argumentations?

Data and method

Data corpus – Block play situations with peers in daily life at the kindergarten

The video recordings of autonomous play situations with building materials, already examined by Henschen (2020) about their mathematical content, serve as a data basis. These so-called block play situations are particularly suitable for the ethnographic observations presented here, as they represent an interaction situation that frequently occurs in the kindergarten context. Especially at preschool age, actions that enable mathematical experiences also result from the play event and from the stimulus of the play itself, as well as from the emotional proximity to the playmates.

Method of analysis

Henschen (2020) identifies in her work on block play with the help of Qualitative Text Analysis sequences that are particularly characterised by the density of mathematics-related negotiation. Starting from Henschen´s (2020) results, an appropriate example of interaction sequences between the peers from this data corpus is applied here. These interaction sequences are examined micro-analytically in terms of their thematic development and argumentative structures. The micro-analytical reconstruction of the interactions and the process of negotiation of mathematical meaning are carried out with the help of interaction analysis (Krummheuer, 2013). The argumentative structures are realised with the help of the functional argumentation analysis according to Toulmin, which has been successfully adapted for mathematics didactic research by Krummheuer (2013) and others. Thereby, argumentations are analysed with regard to their functional components and it is worked out how a conclusion (C) is drawn from existing data (D) through argumentation and how this conclusion is substantiated using warrants (W) and backings (B) (see figure 3).

Empirical example of a block play situation with peers

The situation presented and analysed in the following takes place in the block area of a kindergarten in Germany. Two boys, Ron and Max, aged 5, are building with the building material Sonos. The analysed scenes follow a building play phase in which the two boys have placed a self-made “forklift truck” on top of a “building” (Figure 1). They then decide to build a ladder so that the forklift can be reached. The design of the ladder in terms of its stability, attachment (Figure 2) and flexibility in use is the central theme of the negotiation in the following scenes. These topics show that the children are confronted with certain "construction problems". Therefore, it could be stated that the children's examination of these problems does not have a mathematical but rather an engineering focus. However, it should be noted that the solution of such problems involves, stimulates, or even requires mathematical thinking as Helenius and colleagues (2016) or Henschen (2020) shows. The children's argumentations, therefore, seem relevant from a mathematics didactics perspective.

Figure 1: Joint building action on the ladder

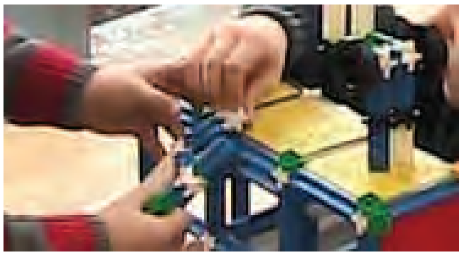

Figure 2: Fixing the ladder

Analysis of scene 1 – "And then how do they get up there?"

001	Max:	I have to fix there (incomprehensible).
002		*Actions on the building* (...)
004	Ron:	And then how do they get up there?

From the utterance in line 1, it can be interpreted that fixing the ladder is the focus for Max. The previous interaction, in which the children are looking for a solution to the problem of reaching the "forklift truck" on the building, "forces" the ladder to be fixed, from the boy's point of view. What is remarkable in the sequence is the obvious everyday world reference of Ron's utterance <4>. Ron raises the question of how anybody (for example humans) can get on the top of the building <4>. From a mathematical perspective, this construction problem focuses on the connection between two points or even two planes. Ron's question on this reveals mathematical experiences with the aspects of topography and spatial orientation (Henschen, 2020). The children seem to be in an "expert discussion" here about the design of the ladder. In this context, the question can be understood as a problem outline that requires argumentative discussion. In the sense of the Toulmin analysis, the fact that the forklift is on the roof of their construction is, therefore, the datum of the argumentation (D I), while building a ladder to get onto the roof can be interpreted as a conclusion (CI) (all argumentation elements are shown in Figure 3). It is possible that at this point Ron does not agree with Max's proposal for fastening or with the ladder and therefore asks for clarification, but this does not directly follow. In various subsequent scenes, however, it can be seen that the topic continues to be the subject of negotiation between the children: in the following turns, for example, different variants for dealing with the ladder are first considered, but without a taken as shared meaning being reached.

Analysis of scene 2 – "But they build the ladder together, you fool."

The main issue in the negotiation between the two children is whether the ladder is long enough for anybody to get onto the building and whether it is sufficient to place smaller pieces of the ladder between the 'floors' of the building. While Ron sees the solution as a construction task (to build a large ladder), Max shifts the solution to the narrative play action and pleads from line <53> that ladders can also be assembled by somebody if necessary.

052	Ron:	But otherwise they can’t get up there at all.
053	Max:	But they build the ladder together, you fool–

Here, Max’s statement can be interpreted as a narrative warrant (W I – Flexibility) for the conclusion (C I) that someone comes to the building through the ladder(s) because they assemble ladders in the course of the ‘story’. Together with the construction idea, which also becomes emergent in the following scenes (3 and 4), a non-hierarchical solution diversity is revealed here in the children’s contributions, since several warrants are identified for the implied argumentative conclusion that somebody gets onto the building with one or more ladders (C I). After some time, the two boys turn together to the construction of a large ladder from already existing sections and steadily lengthen it. In the following scene 3, the focus is on the further handling of the large ladder that has now been created in this way.

Analysis of scene 3 – “There must be a white button on it.”

080	Max:	We still have to be fixed up there.
081	Ron:	No.
082		*pulls the ladder a little in his direction*
083	Max:	So that it does not slip down. (...)
089	Ron:	No, we have to- I know. You have to put it away. Because otherwise you can’t curve it, you have to (incomprehensible).
(...)		*Construction work on the ladder* [Figure 1]
092	<Max:	No. Look at it!
093		*Runs his hand along a bar of the upper edge of the building*
094	<Ron:	There must be a white button on it. Otherwise you can’t put it here.
095		*Construction work on the ladder* [Figure 2] (...)
097	Max:	But look. Then it stretches the buttocks out backward. That’s stupid too, isn’t it?

In scene 3, the fixing of the ladder is negotiated. The need to use certain components <80>, to omit them <89> and <94>, or to pay attention to the design of the structure <97> is discussed. The different contributions can be understood as backings (B I and B II) of the warrant (W II) of the argumentative conclusion that someone can scale the building with a ladder, because this ladder is sufficiently fixed (C II) and reaches the bottom or can be fastened diagonally so that it is possible to climb up from the base of the building (W II - Anchorage). The negotiation subsequently intensifies and reaches its argumentative climax in the following scene 4.

Analysis of scene 4 – “And besides, I had the idea!”

117	Ron:	I HAVE TO FIX IT!
118	Max:	*constructs on the building*
119		No, you don't have to fix it at all.
120		*continues to construct the building*
121		Only this one has to be, simple.
122	Ron:	*stretches his hand with components towards Max and looks at Max*
123		But otherwise it can't hold.
124	Max:	It holds all-
125	Ron:	*kneels next to Max*
126		Please! (.) And besides, I had the idea!
127	Max	Look at this. Still holds.

Max statement in line <119> could be interpreted on the one hand as a support of the warrant ‘flexibility’ (W I) from line <53>; on the other hand, it could be interpreted as a further warrant of the conclusion that a ladder on which somebody can reach the upper floor must be stable (C II). This conclusion emerges in line <83> and can be interpreted as shared from line <123> onwards. With this, the children's statements in lines <119>, <124> and <127> can be interpreted, on the one hand, as a generally valid rule that even leaning ladders have this stability and can still be flexibly accommodated onto the building and, on the other hand, as justification for the above-mentioned conclusion (W III). Line <126> can also be interpreted as backing (B III) of the warrant 'anchoring' (W II). Social persuasion can be attributed to this statement by Ron. In the request, the rule can be recognized that in the building game, actions also receive their validity if they are demanded by a playmate. In a sense, it is a social warrant.

Empirical findings

Overall, the argumentation that emerges across scenes 1 to 4 can be summarised in the form of the following Toulmin scheme (figure 3).

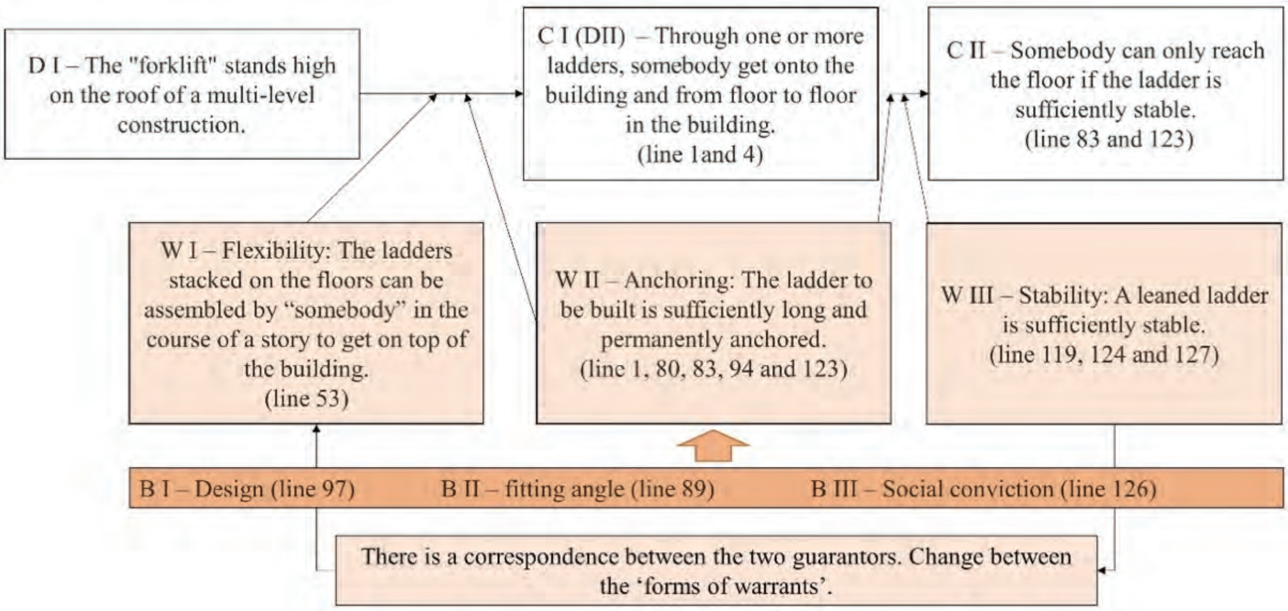

Figure 3: Toulminian argumentation scheme of scenes 1-4

In the reconstructive analyses, it becomes clear that the processes of negotiation in the block play situation with the children Ron and Max described here are structured argumentatively concerning mathematics-related content. A variety of warrants and backings which are collectively realized by the peers or negotiated over a longer time in the situation can be reconstructed. In connection with this, it becomes apparent, not least through the Toulmin schema presented, that the collective argumentation that emerges in the presented play situation has a special 'depth', as it is characterised not only by diverse warrants but also by the manifestation of supports or rules that support the validity of the securing warrants. According to Krummheuer (2013), it is precisely this 'depth' that constitutes a social persuasiveness of argumentation. The warrants and backings refer to multifaceted mathematics-oriented topics such as the stability and geometry of the structure 'ladder', with its various connections and angles of attachment, its length or its orientation in space, but also to informal, everyday considerations or domains of experience embedded in narratively structured play actions, such as the possibility that somebody can also assemble ladders to reach an elevated point.

Thus, what emerges here is not only a collective argumentation that leads to new interpretative perspectives for both boys, but there seems to be a networking of contextualised knowledge that is fostered by the varied argumentative structure. The children's reasoning processes, which are evident in the context of solving their construction problems, require and enable mathematics-related knowledge or DSEs about fixing angles, lengths, and the statics or geometry of buildings, even if the children do not (yet) negotiate this with formal or explicit mathematical expressions.

Concluding remarks

The paper aimed to work out how collective processes of argumentation emerge in peer interactions in block play situations, which create conditions for the possibility of learning "new" mathematical knowledge. Based on the theoretical approaches, analyses and empirical findings presented here, it can be assumed that it is precisely through the play situation, in which peers interact together, that many discursive degrees of freedom emerge that enable the children to construct different possible solutions and to pursue them over a long chain. Probably especially in such peer settings it is possible, for children to realise argumentative connections that are characterised by their "depth" and thus enable the development of sustainably networked knowledge. One reason why this is realised could be that the peers are interactively very coordinated because there is a symmetry in terms of power relations and a proximity in terms of their knowledge resources; both children can situationally slip into the role of the more competent other. In contrast, for example, analyses in Vogler (2020) show, in interactions in asymmetrical teaching-learning situations (e.g., with an authority figure, e.g. elementary education professionals) there is often a lack of argumentative depth. Children's participation is usually exhausted in one-word answers that appear in funnel pattern (Bauersfeld, 1988) and offer less opportunity for the sustained interconnection of knowledge. Following these remarks by Vogler (2020), it can be assumed that the adults (in contrast to the children) pursue a specific learning goal and therefore, in the case of hierarchically organised interaction roles, only anticipated responses are accepted by them in the interactional interplay. Consequently, the peer interactions seem to be less characterised by their "stringency" compared to the adult-child interactions, but they are particularly conducive to learning due to their argumentative multifacetedness, and the associated opportunity for participation and linking of informal and formal knowledge. From a pedagogical point of view, it would be advisable to systematically observe these peer interactions, to support them interactionally and to take up the negotiated attributions of meaning and argumentative structures with children afterward and develop them (professionally) further.

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An educational design research on equity in early mathematics education

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Keywords: early assessment, design research, equity

This poster presents an educational design research study focused on equity in the Swedish preschool class. In Sweden, six-year-old students attend preschool class for one year of schooling with the aim to create continuity in early education, by relating to surrounding institutions – preschool and compulsory school. In comparison with the compulsory school, students' free choice and play is more focused on in the preschool class. According to Espinoza (2007), equity can be described as educational justice, meaning that both individual circumstances and differences related to individual needs and requirements are to be taken into consideration. In the Nordic countries, equity is emphasized as important in the creation of a "fair and equal society supporting democracy, participation, welfare, and life-long learning" (Klette, 2018, p. 59). However, since the turn of the millennium, Sweden of all the Nordic countries has departed the most from the Nordic model with a "control regime overshadowing the learning, equity and democracy agenda that is still in the curriculum" (Blossing & Söderström, 2013, p. 32). The differences between low- and high-performing students and between schools have increased, as has the significance of students' socio-economic backgrounds in terms of their educational outcomes (National Agency for Education, 2010).

According to Gutierrez (2002), equity issues can be addressed from four different dimensions: access, achievement, identity, and power. Foregrounded in this study are the equity dimensions of access and identity. In the field of mathematics education, equity can be described as "the inability to predict mathematics achievement and participation based solely on student characteristics such as race, class, ethnicity, sex, beliefs, and proficiency in the dominant language" (Gutierrez, 2002, p. 153).

The aim of this study is to explore how to design an equal education in early mathematics in the sense of an education that affords young students access to as well as prosperous attitudes towards mathematics. The basis for this study is a previous study on assessment in the Swedish preschool class. In that study the teachers talked about an increased awareness of the different needs that students have when it comes to mathematics. They described a dilemma in providing all students an equal possibility to learn mathematics, this as the students are at different levels. In addition, the teachers described the challenge with teaching students in such a way that they all find mathematics to be interesting. In this new study, the following research questions will be addressed: What does an equal mathematics education where all students have access to mathematics imply? What does an equal mathematics education where all students can develop prosperous attitudes towards mathematics imply? How can a mathematics education, that affords young students access to as well as prosperous attitudes towards mathematics, be designed?

Method

This study is developed within the frame of educational design research (McKenney & Reeves, 2012). The research approach is a case study, located within two classrooms. Educational design research can be described by five common characteristics: "theoretically oriented, interventionist, collaborative, responsively grounded and iterative" (McKenney & Reeves, 2012, pp. 12-16). In 2019, mandatory assessment material was developed and put into use in Sweden as a way to improve equity within the Swedish preschool class. Through the assessment, students' knowledge is assessed with the aim to identify what adaptions are needed for each student to meet the knowledge requirements, and for them to "reach as far as possible" (National Agency for Education, 2019, p. 1). In this study, this assessment of all students serves as a starting point of the design.

The process of generating data is defined by a number of cycles of invention and revision. Each cycle will consist of the following elements: classroom observations, student interviews, and dialogs between the teachers and the researcher. In line with educational design research, in each cycle adjustments will be made based on the two research questions. This iterative process intends to produce practical solutions based on a theoretical understanding, where the results of the research are relevant for education practice (McKenney & Reeves, 2012). The research results will be formulated as design principles, thus not only statements like what to do and how to promote equity in early mathematics education, but also theoretical and empirical explanations to support these knowledge claims (Van den Akker, 2013).

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This poster presents an educational design research study focused on equity in the Swedish preschool class. In Sweden, six-year-old students attend preschool class for one year of schooling with the aim to create continuity in early education, by relating to surrounding institutions – preschool and compulsory school. In comparison with the compulsory school, students' free choice and play is more focused on in the preschool class. According to Espinoza (2007), equity can be described as educational justice, meaning that both individual circumstances and differences related to individual needs and requirements are to be taken into consideration. In the Nordic countries, equity is emphasized as important in the creation of a "fair and equal society supporting democracy, participation, welfare, and life-long learning" (Klette, 2018, p. 59). However, since the turn of the millennium, Sweden of all the Nordic countries has departed the most from the Nordic model with a "control regime overshadowing the learning, equity and democracy agenda that is still in the curriculum" (Blossing & Söderström, 2013, p. 32). The differences between low- and high-performing students and between schools have increased, as has the significance of students' socio-economic backgrounds in terms of their educational outcomes (National Agency for Education, 2010).

According to Gutierrez (2002), equity issues can be addressed from four different dimensions: access, achievement, identity, and power. Foregrounded in this study are the equity dimensions of access and identity. In the field of mathematics education, equity can be described as "the inability to predict mathematics achievement and participation based solely on student characteristics such as race, class, ethnicity, sex, beliefs, and proficiency in the dominant language" (Gutierrez, 2002, p. 153).

The aim of this study is to explore how to design an equal education in early mathematics in the sense of an education that affords young students access to as well as prosperous attitudes towards mathematics. The basis for this study is a previous study on assessment in the Swedish preschool class. In that study the teachers talked about an increased awareness of the different needs that students have when it comes to mathematics. They described a dilemma in providing all students an equal possibility to learn mathematics, this as the students are at different levels. In addition, the teachers described the challenge with teaching students in such a way that they all find mathematics to be interesting. In this new study, the following research questions will be addressed: What does an equal mathematics education where all students have access to mathematics imply? What does an equal mathematics education where all students can develop prosperous attitudes towards mathematics imply? How can a mathematics education, that affords young students access to as well as prosperous attitudes towards mathematics, be designed?

Method

This study is developed within the frame of educational design research (McKenney & Reeves, 2012). The research approach is a case study, located within two classrooms. Educational design research can be described by five common characteristics: “theoretically oriented, interventionist, collaborative, responsively grounded and iterative” (McKenney & Reeves, 2012, pp. 12-16). In 2019, mandatory assessment material was developed and put into use in Sweden as a way to improve equity within the Swedish preschool class. Through the assessment, students’ knowledge is assessed with the aim to identify what adaptions are needed for each student to meet the knowledge requirements, and for them to “reach as far as possible” (National Agency for Education, 2019, p. 1). In this study, this assessment of all students serves as a starting point of the design.

The process of generating data is defined by a number of cycles of invention and revision. Each cycle will consist of the following elements: classroom observations, student interviews, and dialogs between the teachers and the researcher. In line with educational design research, in each cycle adjustments will be made based on the two research questions. This iterative process intends to produce practical solutions based on a theoretical understanding, where the results of the research are relevant for education practice (McKenney & Reeves, 2012). The research results will be formulated as design principles, thus not only statements like what to do and how to promote equity in early mathematics education, but also theoretical and empirical explanations to support these knowledge claims (Van den Akker, 2013).

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Theoretical frameworks and the danger of relying on the one!

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Inspired by Sfard we investigate the stories of teaching and learning situations using two different theories. We look at how the stories that the two theories tell are the same or different in order to answer the question about what can be gained and what will be lost when using two different theories in this case variation theory and pedagogical strategies. This is done using a video from a mathematics lesson in preschool class setting, 6-year-old children as a base and then comparing the stories that the two different theories tell about this situation.

Keywords: Theory of variation, sociological theory, domains of action, stories

Background

In this paper we use the definition of research as presented by Sfard (2018), where she states that research "is the activity of telling stories about some aspects of reality" (p.220). Using the terminology of Sfard (2018) we use two different theories and thus two different routine actions (analysis). This approach can be either using a multi-theoretical approach (Johnson et al) or coordinating different theories (de Freitas, et al., 2019). The use of theory and its necessity is an ongoing discussion (Bikner-Ahsbahs et al., 2019). In their paper, Bikner-Ahsbahs et al. (2019) give different approaches to the use of theories depending on if the theory is in the foreground or background or even if they are seen as a tool or an object in order to use as a lens. We will continue to use the vocabulary presented by Sfard (2018), when we talk about the different theories and the stories that they tell. If a theory consists of a small set of stories or narratives, which story do variation theory and pedagogic strategies (a sociological perspective) tell. The rational for choosing these theories lies in our interest and deep knowledge about them and a curiosity on what story each of them will tell when using the same set of reality. Lerman (2010) used sociological theories to claim that the use of several theories is a necessity. On this note, we use the same reality. We tell two, in some sense, different stories but they are both grounded in a common interest in teaching and learning mathematics. In this case we use a teaching and learning situation in mathematics in a preschool class setting (6-year-old children). For us it is important to spell out the primary stories, where the base for our different stories is. The base for the stories is the part of the theories that helps us perform routine actions, or in other words analyze the data. We do that by presenting the subset of stories valuable for our analysis across both theories. In the sense of asking a question about differences, we need to also address the following question: Do the two different theories add any new stories or do they just use different words to describe reality or are they even mutually exclusive? We tell the stories, that is, we craft the narratives based on the data set at hand. In this case, the data is a video of a teaching and learning moment in a preschool class mathematics lesson.

Let us discuss the word different in this case, since different is the core of the paper. Can these stories be seen as different, or do they add aspects to one another? None the less, one has to remember that from our different perspectives we have to choose which aspects to focus on. Sfard (1998) argues that there are two metaphors for learning and there is a problem of relying on just one of them. Inspired by this idea we argue that to tell the stories of a learning situation we would benefit from using different theories to tell different stories about the situation at hand. Still, we know different theories make us ask different questions and hence tell different stories.

Theoretical frameworks

In order to answer our research question, what can be discerned in the narrative using two different analytical tools, we use Variation theory (Marton, 2015) and Pedagogic strategies (Dowling, 1998) (a sociological perspective) in our routines (method of analysis) of a videoclip from a preschool class.

Variation theory

Although variation theory was developed in school-settings about 20 years ago it has also been used as a theoretical tool in preschool-settings (e.g., Björklund et al, 2021; Wernberg, 2017). Marton (2015) describes how learning means a qualitative change and a more developed understanding of knowledge. The understanding is developed by distinguishing more and more aspects of phenomena in the outside world (e.g., mathematical and mathematical didactic concepts) and giving meaning to the one who distinguishes them. The distinction presupposes an experienced variation of these aspects. Here we have mainly adopted the notion of critical aspects and critical features, while also adhering to the assumptions of the overall theory. One theoretical assumption is that learning is always the learning of something, and the ability to learn presupposes an experience of variation. Critical aspects can be defined as necessary features to be discerned for learning the object of learning. To understand what it is that enables learning in one situation, one can illuminate what varies and what is invariant using contrast. This pattern of variation is created by patterns of variation of the same aspect (Wernberg, 2009). The difference between a critical feature and a critical aspect is that the latter refers to a dimension of variation and the former is a special value in this dimension of variation. To discern something, it is necessary to experience a variation. In Figure 1 we use one pattern of variation, contrast, making it possible to distinguish the difference between a triangle and a rectangle/quadrilateral. It opens up a dimension of variation, geometric shapes, where triangle is a value in this dimension of variation.

Figure 1: Critical aspects and features

“Values correspond to features, and dimension of variation corresponds to aspects… A critical aspect is thus a critical dimension of variation and critical features is critical value" (Marton, 2015, p.47).

Pedagogic strategies (a sociological perspective)

The following theoretical perspective is based on Dowlings (1998), analysis of high school textbook series and redefined by Johansson (2012) for use in a mathematics classroom and in the interaction between students and teacher. The redefined method of analysis by Johansson (2012) has also been used in preschool and preschool class settings by Helenius et al (2018). Dowling (1998) describes four domains of action and a set of strategies within those domains. The four domains where esoteric, descriptive, expressive and public. These four domains were distinguished in two different ways based on content (signifieds) and expression (signifiers) and whether they were strongly classified or weakly classified. The domains of practice are shown in Figure 2.

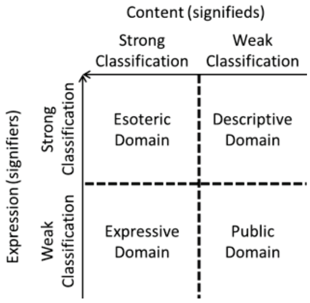

Figure 2. Domains of practices (Dowling, 1998, p. 135)

A mathematical activity in a teaching and learning setting uses all these four domains. However, the boundaries between these domains are not sharp and hence it is not always easy to separate them from one another. In order to clarify the impact of practices in these domains Dowling (1998) elaborated on a series of pedagogic strategies. These redefined pedagogic strategies are then used as a methodological tool for interpreting interactions (Johansson, 2012).

In a preschool class setting where the mathematical activities contain tasks that are not strongly classified regarding content or expression, the different strategies that come to use contribute to the development of more or less generalised mathematical knowledge or more or less localised knowledge.

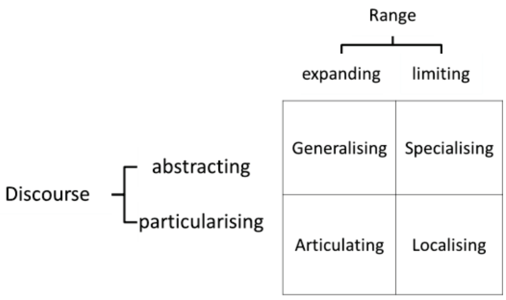

Figure 3: Distributive strategies (Dowling, 1998, p. 147)

As seen in figure 3, the four strategies are categorized in terms of range and discourse. When it comes to range of possible solutions, the strategies are categorized regarding whether they limit or expand the range and, in the same manner, regarding if they support an abstracting or particularizing discourse. The pedagogic strategies are used by both teacher and children and in one utterance the children and the teacher vary in their use of these strategies hence it is used in a microlevel analysis where each part of the dialog is categorized. The following description is based the redefined pedagogical strategies by Johansson (2012) and the examples is from the age group that is present in the situation at hand.

The *Specialising* strategy separates different cases or methods or concepts. In this strategy the children or teacher use specific ways when discussing the task or solution; this also means using specialised terminology. For example, using addition or subtraction. Since this strategy separates the methods or concepts, it reduces the range of possible solutions. However, it can contribute to the children's understanding of a concept or method. The *Generalising* strategy, on the other hand draws on different cases in relation to a common principle and so this strategy is expanding the range of solutions. Hence, it provides an understanding of how a common principle can be used in different settings and cases. These two strategies provide the children with opportunities for discussing and understanding the mathematical practice and the underlying principles. Specialising does this by providing opportunities to discuss differences in examples or cases and generalising does this by providing opportunities to discuss the commonality in examples or cases. Some examples of specialising are solving a word problem and modelling. Examples of generalising could be finding an appropriate drawing to illustrate an argument or applying a solution method to a range of different examples.

The *localising* strategy constructs one particular example but does not give the children any opportunities to engage in a mathematical discourse. The *articulating* strategy on the other hand highlights different examples, but these examples are not connected to any mathematical principle and so it is not possible to generalise from this in a mathematical discourse. These two strategies occur in preschool and preschool class settings and, by not making the connection to the mathematical discourse, the children must themselves determine what knowledge to be used in solving the tasks. If the children cannot determine that they should use generalising or specialising strategies, the mathematical discourse will not be visible.

Perceptual mediator - Data

The research was undertaken in a Swedish preschool class where most of the children were six-year-olds. Preschool class is the first place children have contact with formal school knowledge and ways of working. In the first half of 2013, video recordings were made on four separate occasions in one preschool class in Sweden. The preschool class had two different teachers and the video recordings captured a range of different interactions, set up by one or other of the teachers. We had asked to film problem solving tasks and later a free play situation. In this paper a part of one problem solving lesson is analysed. The teacher initially reminded the children about the activity previously done; They had asked different people if they preferred chips or candy and put the result in a bar chart (cubes on top of each other).

Figure 4a: The teacher coloring the cubes **Figure 4b: The teacher showing the newspaper bar chart**

The teacher colored the cubes to get the children's attention on the importance of making the cubes evenly high (figure 4a). Thereafter she showed a bar chart from a newspaper where you only could discern the height when comparing the different bar charts (figure 4b). Then again, she reminded the children about an earlier episode when she had asked the children to vote on a book to read in whole class. She took two books and put eight equally small blocks in front of one book and four bigger blocks in two different sizes in front of the other book (figure 5).

1 Teacher: Which pile became the largest, who received the most votes?

2 Children: That one. [Most of the children point at the pile with four cubes]

3 Teacher: Yes, is that the book that then won?

4 Children: No.

5 Teacher: Well, that pile is the biggest?

6 Theo: Well, it's not, check it's one, two, three, four blocks. One, two, three, four, eight, nine, no. Yes, but it's that one, that one has at least smaller blocks.

7 Klara: Eight

8 Teacher: Eight in that, and that one has only four. Well then, I say that is the biggest [the teacher points at the pile with four blocks] then I choose that book.

9 Children: No.

10 Teacher: But, why not?

11 Theo: Because it has four blocks, and four blocks are less than eight blocks.

Figure 5: Placing the blocks in front of the books

12 Teacher: Yes, that’s right, but if this were stacks then as it will be there later, then you do not see these boundaries [once again, the teacher pulls out the diagram from the newspaper, with no visible blocks, and compares it with the one they made for candy and chips, with visible blocks].

13 Teacher: That is why it is very important that we draw all the way up to the number. And here, all the way up to second. Do you remember that someone wanted to do a little less here so we had to sort of pull it up to the line? It is very important that it is right in both directions. It was chips and so it was up to the number. So, if we were to draw this, four, it would be there, right? And here is? How many were there now?

14 Children: Eight.

15 Teacher: Then it was eight here.

Routines – methods of analysis

In this part, we present the separate analysis for the teaching and learning situation in mathematics in the preschool class setting using variation theory and sociological theory. The analyses were made in parallel and were undertaken by one or the other of the authors (Wernberg VT, Johansson SP) and then discussed. The rationale for both the research question and the analysis is the authors different theoretical backgrounds which has been present in other common work where questions/curiosity have been raised due to this difference in their background.

Story - Variation theory perspective

In the introduction to the assignment, the teachers open up for a dimension of variation by contrasting the two diagrams: The one made in the classroom with the pupils and the one from the newspaper (figure 1a and 1b). Thereby, the teacher opens the ability for the children to discern one critical aspect, the importance of the "cube" being of the same size. The same size is, thus, a value in the aspect size with the values big, small and equal.

In the utterance (line 1) the teacher draws the children's attention to the two piles of blocks in front of her to, again, make them aware of the importance of the blocks being the same size. One pile with eight small books with the same size and the other pile with four bigger blocks with two different sizes. In line 8 she challenges the children's understanding by claiming that the children mostly voted on the book with four cubes in front of the book since that pile is the highest. Here, the teacher opens up for a dimension of variation by using blocks of different sizes in the highest pile and blocks of the same size in the lowest pile. Again, she opens up for the children to discern the critical aspect, same size, but in a different way. Here, she contrasts the height of the piles with the total number of the blocks in each pile.

In the end of this assignment (line 12), the teacher invites the children to look at the different diagrams, the one they made and that from the newspaper, to make them aware of the different appearances. Again, she opens up for the children to discern the critical aspect, same size, but in a third way. Here, she contrasts the bar chart from the newspaper with the data the children collected and put in a bar chart. It is when the two diagrams are being discussed at the same time through contrast it is made available for the children to discern.

Story - Sociological perspective

In the beginning of this utterance (line 1) we can see that the teacher is using a specialised language but is ending the sentence with a localising question. She is using the task at hand, namely asking: "Which pile became the largest, who received the most votes?". The children are specialising when most of them are replaying on the first part of the question: "which pile became the largest?". The teacher is then continuing with localising the discussion by asking: "Yes, is that the book that then won?" (line 3). The children then reply to the question with a "no" and the discussion continues by generalising and arguing why a large pile with few blocks is not the pile that has the most blocks (line 6-11). In this part the teacher is localising and specialising, and the children are specialising by focusing on the number of blocks at hand.

In line 8 the teacher is first responding to the students talk about the numbers of block: "Eight in that, and that one has only four". Then, she is localising by again taking the question back to the task of which book is most popular; she says: "Well then, I say that is the biggest then I choose that book". The children reply by generalising. By using two of the strategies, shifting between the discourses abstracting and particularizing, the teacher gets the students to generalise. Even though she does it on the limiting range, she still manages to get the children to generalise and see the point that she is trying to make. The interaction continues in this pattern. We can see that the teacher is shifting between localising and specialising and finally gets to the point where she can give the children an esoteric point with a specialised language (line 14). She is presenting how to make a bar chart.

Results

In this section we will highlight the findings from the analyses when exploiting the same data but two different theories. To be more precise, since research in mathematics education narrates the process of teaching and learning mathematics, the analyses of the same data (perceptual mediator) can be seen as two methods of analysis (routines) with its own keywords.

The most obvious differences between the stories are the depth of the mathematical discourses. The pedagogical strategy (sociological perspective) offers an insight to how the mathematical content is understood in a broader story in relationship to the discourse, while the theory of variation offers a more detailed description of the mathematical content. In the pedagogical strategy the routine reveals the vocabulary pattern, localising and specialising, the teacher is using to get the children to grasp the mathematical content being taught (how to make a bar chart) and she is given the opportunity to give the children an esoteric point with a specialised language (see figure 2 and line 14). If we look at the other analysis done with variation theory, the story comes closer to the mathematics being taught in the routine (method of analysis).

Maybe one could have predicted the findings because of the different possibilities the various routines offer, but as stated in the beginning of the paper we are not interested in discussing the possibilities of combining them but rather look at how the stories are the same or different to answer the question about what can be gained and what will be lost when using two different theories? In their paper Bikner-Ahsbahs et al., (2019) argue for the danger of becoming blind to aspects that a theory does not capture. We find that the pedagogical strategy revealing the discourse of language and how the teacher using questions make the children aware of how to construe a bar chart, while the theory of variation afforded opportunities to follow the mathematics being taught. So, the stories are different and give us different perspectives. However, by using both of them on the same situation it might be so that other questions can be asked and yet another story told.

Conclusion

We are aware that in general different theories ask different research questions to the same perceptual mediator (data) and thus get different stories. In this paper, we do not ask explicit research questions to the perceptual mediator, even if each analysis contains an implicit question. This means that we tell the story from two different perspectives, regardless of the research question. Hence our conclusion forms new research questions. More data needs to be analysed and the analysis compared in order to get answers and not only new questions. Johnson et al (2019) linked theory and method in their research design for an expansion of design possibilities. In this paper we used “an old” episode and analysed it, but could we like Johnson et al integrate theories, and thus their different epistemological roots?

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TWG14: University Mathematics Education

Introduction to the papers of TWG14: University mathematics education

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Keywords: University mathematics education, (digital) resources in university mathematics, university teachers' practices and knowledge, mathematics for non-mathematics professionals, university mathematics students' identity and experiences.

Introduction

TWG14, *University Mathematics Education* (UME), was launched in CERME7 (Nardi et al., 2011) acknowledging the fast growth of research in UME, as well as some specificities proper to UME. Some of these specificities are: the abstract, formal nature of a significant portion of the mathematical content; the absence of national curriculum guidelines, leading to great variations in organization and practices across institutions; the general lack of systematic preparation for teaching; and, the volume of content to learn in a short period of time and the degree of autonomy expected from students. The consolidation of this research area, both outside and inside CERME, is visible through the number of UME-related activities in the last years: the publication in 2014 of a special issue about theoretical frameworks being used in UME research; the creation in 2015 of the *International Journal of Research in Undergraduate Mathematics Education*; the creation in 2015 of the *International Network for Didactic Research in University Mathematics* (INDRUM), which since 2016 has launched the biannual INDRUM conferences (an ERME Topic Conference) with associated special issues in journals (IJRUME and IJMEST) and the book *Research and Development in University Mathematics Education* (Durand-Guerrier et al. 2021); and, the organization in 2019 of the first *Calculus in upper secondary and beginning university mathematics* conference, to cite just a few.

In 2021 and 2022 we celebrated important anniversaries for the UME field: in 2021, the 10th anniversary of the creation of TWG14, the 30th anniversary of the publication of *Advanced Mathematical Thinking* (Tall, 1991), and the 20th anniversary of the publication of the ICMI study *The Teaching and Learning of Mathematics at University Level* (Holton, 2001); in 2022, the 15th anniversary of the emblematic handbook chapter *Mathematics Thinking and Learning at Post-secondary Level* (Artigue et al., 2007) with the follow-up chapter 10 years later, *Post-Calculus*

Research in Undergraduate Mathematics Education (Rasmussen & Wawro, 2017). For all these reasons, CERME12 provided an opportunity to celebrate UME research and to discuss its future.

This year, we received 41 paper and 9 poster submissions, with 27 papers and 14 posters presented at the conference and published in the proceedings. This number of presentations led to the decision to have parallel session in two groups (TWG14A and TWG14B) addressing six themes (see below). We also held some common sessions to discuss transversal issues and recent achievements and challenges in UME research. This introductory paper summarizes the works presented in both groups organized according to the six themes, as well as the common discussions.

As in previous CERMEs, a significant number of papers focused on *students' learning of mathematical topics and practices* (9 papers), such as Calculus, reasoning, and proof. Compared to CERME11, we received less papers in total (27 vs. 35) with some variation in the distribution in each theme: the number of papers addressing *teaching and teachers* decreased from four to three; the number of papers addressing *students' identity and experience* went from three to five; the number of papers addressing the *use of mathematics by non-specialists* remained the same (two); and, the number of papers in *interventions* decreased from seven to two. The CERME11 theme about resources and curriculum (five papers) became the new theme on *teaching and learning with digital resources* (six papers). Finally, studies on transition (a theme in CERME11) were discussed within the six themes above. In the next section, we briefly present the six themes with examples from paper contributions. While many papers could fit in more than one theme, this classification helped structure our work at the conference and the presentation below.

Themes and contributions

Students' learning of mathematical topics and practices

Ten papers (Baldino & Cabral; Borji et al.; Karavi & Mali; Körtling & Eichler; Noah-Sella et al.; Rogovchenko & Rogovchenko; Spratte; Utsch; Wallach et al.) and six posters (Beran; Fuchs; Hanke; Oldenburg et al.; Piroi; Vincenzi) were classified under this theme.

These studies investigated different mathematical areas, with Calculus being the dominant (Baldino & Cabral; Fuchs; Körtling & Eichler; Noah-Sella et al.; Oldenburg et al.), including advanced Calculus topics such as multivariable functions (Borji et al.), convergence of sequences (Utsch) and differential equations (Rogovchenko & Rogovchenko). There were also studies focusing on Linear Algebra (Beran; Piroi; Wallach et al.), Complex Analysis (Hanke; Karavi & Mali), and Abstract Algebra (Beran). Finally, one paper was on students' proof reading (Spratte) and one poster on incommensurability in regular polygons (Vincenzi).

Some of the papers discussed students' learning (Körtling & Eichler; Spratte; Utsch) focusing on students' definitions (Körtling & Eichler), their intentions when reading proofs (Spratte), and students' connections between their concept images and definitions (Utsch). There were also studies that discussed teaching innovations (Baldino & Cabral; Beran; Borji et al.; Fuchs; Piroi) focusing on visualization in Linear Algebra (Piroi); the connection between Calculus and modeling techniques in Physics (Fuchs); the introduction of the Fundamental Theorem of Calculus using discrete graphs (COVID-19 graphs) and then moving to continuous graphs (Baldino & Cabral); the use of mathematical structures (Beran); and, the introduction of activities designed using APOS theory (Borji et al.). Other studies investigated experienced and less experienced learners (Noah-Sella et al.). Finally, there were studies focusing explicitly on teaching resources and lecturers' practices (Hanke; Karavi & Mali; Rogovchenko & Rogovchenko; Wallach et al.). These dealt with definitions in textbooks (Hanke), the potential of Linear Algebra tasks to assist in transitions between multiple discourses (Wallach et al.), an investigation on differential equations tasks in terms of assessing students' conceptual understanding (Rogovchenko & Rogovchenko), and the proving routines used by a lecturer in Complex Analysis (Karavi & Mali).

Teaching and learning with digital resources

Six papers (Albano; Broley et al.; Davies et al.; Donevska-Todorova & Turgut; Gueudet et al.; Przybilla et al.) and one poster (Thoma & Iannone) were discussed in this theme.

Some authors considered the use of *traditional* technologies: Broley et al. explored the issue of learning programming for mathematical investigations with the Anthropological Theory of the Didactic (ATD – Bosch et al., 2020). They established an epistemological reference model and investigated a student's praxeological equipment. Gueudet et al. studied a similar issue with the instrumental approach (Gueudet, 2017), focusing on the social aspects in the schemes developed by students and associated to the programming artefact.

Other authors considered relatively *unexplored* technologies: digital assessment and its use by university teachers (Davies et al.) or an automated theorem prover and its impact on students' reasoning (Thoma & Iannone). Digital maps also seem to be a promising new tool, with different intended uses, such as fostering students' collaborative work and their conceptualization processes in Linear Algebra by connecting different representations (Donevska-Todorova & Turgut). Experts can design digital mathematical maps to evidence connections between secondary school and university mathematics (more precisely Geometry, in the study by Przybilla et al.).

As we mentioned above, the theme of digital resources was new in TWG14. One of the reasons for the emergence of this new theme was the COVID-19 pandemic, which was evoked in several papers and played a central role in the study by Albano around the new orchestrations required in the context of hybrid teaching. The generalized use of digital platforms in university courses and its consequences were discussed in TWG14 and identified as directions requiring further research.

Students' identity and experience

Five papers (Gandell; Göller; Kontorovich & Greenwood; Mullen & Cronin; Rasmussen et al.) and one poster (Nardi) were presented and discussed in this theme.

Three of the contributions to this theme focused on students' in-class experiences. Gandell investigated students' spontaneous mathematical thinking in movement, illustrating how this approach offers new insights into students' mathematical knowing. Kontorovich and Greenwood investigated student experiences with proof in a Topology course, where students were provided with opportunities to prove the same mathematical statement in different social situations. Rasmussen et al. analyzed the individual and collective mathematical progress of one small group of four students in an inquiry-oriented differential equations classroom as they reinvented Euler's method.

The other three contributions focused on students' out of class experiences and identity. Göller investigated first-year mathematics students' everyday coping strategies while dealing with the challenges they face transitioning from secondary to university mathematics. Mullen and Cronin described a suite of online and in-person mathematics supports designed for in-coming first-year university students. Finally, Nardi took up the question of how well undergraduates understand the actual work that mathematicians do and argued that more needs to be done to make this work more visible and salient for undergraduate mathematics students.

Teaching and teachers

Three papers (Nseanpa & González-Martín; Tabchi & Sabra; Viirman) were in this theme.

The first two papers paid particular attention to the use of resources in teaching, building on the documentational approach (Gueudet, 2017). Tabchi and Sabra investigated the connection between the teaching practices of a lecturer at a Lebanese university and her activity as a researcher in Graph Theory. The lecturer herself perceived little connection between the teaching and researching aspects of her work, but analysis of her teaching practice and use of resources revealed such connections, for instance, regarding the use of generic examples. Meanwhile, Nseanpa and González-Martín, in a study of the teaching of derivatives at the pre-university level in Cameroon, focused on how strong institutional constraints shaped teachers' practices and use of resources. These constraints included a prescribed textbook, official teaching guidelines and a high-stake national examination. Findings indicate that national examinations strongly influenced the didactical choices of teachers concerning the teaching of the derivative, shaping, for instance, the way derivatives are introduced. Calculus was also the topic of the paper by Viirman, which otherwise has quite a different focus from the first two papers in this theme. Viirman analyzed a set of 14 national accounts, written by experts in the field, of the teaching of Calculus in secondary education, at university and in teacher education. Differences and similarities between the accounts were highlighted, and findings were used to discuss how Klein's second discontinuity plays out in different countries around the world.

Interventions

Two papers (Albano et al.; Markulin et al.) and three posters (Akrouti; Dreyfus et al.; Vourenpää et al.) were presented under this theme.

The research work presented by Albano et al. addressed the development of the problem-solving competence at the university level. In particular, they presented the design, implementation, and analysis of an activity in Topology. Markulin et al. discussed the use of Study and Research Paths (SRP) in statistics courses. We return to this paper in the next section.

Regarding the posters in this theme, they discussed activities designed for the teaching of integrals (Akrouti), the use of flipped classroom formats (Vourenpää et al.), and the development of a methodological approach for characterizing the interplay of mathematical progress across individuals, small groups, and the whole class (Dreyfus et al.).

The use of mathematics by non-specialists

Two papers (Florensa et al.; Hitier & González-Martín) and two posters (Feil & Strauer; Schmitz et al.) were presented under this theme.

The two papers of this theme and one in the previous theme (Markulin et al.) used ATD as theoretical framework to address issues related to mathematics for students who did not chose mathematics as their main subject. Markulin et al. presented an analysis of the conditions and constraints affecting the implementation of SRPs in statistics courses at university level. Florensa et al. drew on ATD to analyse the discontinuities of the mathematical education of engineers. Finally, Hitier and González-Martín conducted a praxeological analysis of the use of the derivative by students in post-secondary institutions concurrently following Calculus and Mechanics courses. The use of gaps in worked-out examples (Feil & Strauer) and of application examples (Schmitz et al.) were also discussed.

Transversal issues addressed in plenary discussions

Resources (including digital) and interventions

The focus of this discussion was on interventions and digital resources and was organized around four thematic areas: design and sustainability of interventions; nature and impact of interventions; design of digital resources and use by teachers; and, digital resources and students' learning.

In relation to the design and sustainability of interventions, collaboration of mathematics education researchers with teachers with mathematics or non-mathematics specialty (e.g., physicists, biologists, etc.) is pertinent and necessary. However, practice highlights tensions in such collaborations, which calls for systematic studies of what makes those collaborations work effectively. Additionally, more evidence is needed on the sustainability of interventions. In this sense, it seems that having a community already working together helps the initiation and stability of changes. Our discussions also suggested that institutional and socio-cultural perspectives have the potential to capture the development of such changes. Moreover, there is an overall agreement that the pandemic has triggered significant changes to UM teaching and learning practice. Digital resources obviously played a significant role in those changes; however, doubts were expressed about whether and which of those changes will remain. Other questions we discussed are: Have online practices changed practices and interaction with mathematical content? If writing mathematics by hand is important, how can online platforms support mathematical communication effectively?

In relation to the nature and the impact of interventions, it seems that there is a variation of models regarding design and expectations. Several interventions aim towards inquiry-based learning and more student-centred approaches. However, the nature of those interventions is influenced by institutional characteristics. Consideration of such institutional characteristics should be critical in future investigations. Furthermore, clarity on the aims of proposed interventions is essential. Such clarity can assist the evaluation of the impact of interventions. Such impact is discussed in recent studies also in relation to specific student demographic profiles (gender, socio-economic status, etc.), a discussion that opens new opportunities for research on equity and access issues at UM.

Regarding the design and use of digital resources by teachers, some papers addressed new types of technology, such as theorem provers or digital assessment by indicating the pertinence of research into innovative technologies, in particular artificial intelligence, for future works. Other studies that are necessary concern teachers' use of digital resources (and how we could support productive uses and orchestrations), theoretical approaches and methods that can be appropriate to analyse the design and use of technology at UM (and the possible differences with other educative levels), and the short- and long-term impact of the COVID-19 pandemic on practices related to the use of technology (e.g., teachers' use of flipped classroom approaches during the pandemic may continue).

Finally, regarding the use of digital resources and students' learning, we discussed the importance of identifying some consequences of technology use on learning, as well as on the epistemological level of the mathematics taught and learned. We also discussed the importance of theoretical and methodological approaches to address these issues, and of assessing the short- and long-term impact of the use of technology during the COVID-19 pandemic.

Students' identity, experience and learning (including non-specialists)

Regarding these issues, we saw some innovative contributions at the tertiary level: the analysis of data about movement and its role in mathematical activity, the analysis of the individual-collective dynamic, models to study different transitions for non-specialists (pre-university to university; university to workplace; between mathematics courses and non-mathematics courses), studies about advanced topics (such as Topology and Complex Analysis), and studies about students' appreciation of mathematics (for instance, in support centers, or with non-specialists). We also discussed the methodological challenges for some of these studies and how to upscale them.

Among the main challenges for the future, we discussed those related to remote teaching (for instance, how to assess, how to consider communication) and its impact on students' learning and experience of mathematics. Another issue of interest is that, to better understand the experience of students in programs for non-specialists, we need to better understand how mathematics is used (or not) in other disciplines. These studies can also open new perspectives about students' appreciation of mathematics, their identity and experiences. Such studies have the potential to move beyond a decontextualized investigation of students' learning of specific topics.

We also discussed some potential challenges on research findings dissemination, such as the communication of research data related to innovative topics (e.g., students' movement, or how to share the large amounts of data that can be collected during remote teaching). Another challenge concerns the replication of studies conducted in other education levels with consideration of the cultural and institutional characteristics of UME. Also, some of the issues discussed in our group are not UME specific. These observations call for more interaction with other CERME groups.

Supporting university mathematics teachers and teaching

Research on UM teachers and their teaching has been gaining attention in recent years. However, research related to UM teacher education and professional development is rather scarce (Winsløw et al., 2021). Overall, there is a variation of practices and approaches on the preparation of UM teachers and the support they need for their profession. With this in mind, we opened the discussion around two questions: "What support for teaching do mathematics teachers have access to at your institution?" and "What would you like to see in research on UM teachers' support in the next years?"

Regarding the availability of support, practices shared by participants in the group confirm the variability mentioned above. Very often teacher education and support is non-mathematics-specific, with attention to general skills such as use of digital resources, organising lessons, engaging with educational literature and developing pedagogical practices. Mathematics-specific support often relies on ‘local’ projects supported by individuals or small teams. The proximity between mathematics and mathematics education departments seems to influence the interaction between mathematics teaching and research on mathematics teaching. Sustainable models of mathematics-specific teaching support seem to be those that are institutionally embedded and maintain mutual participation of mathematicians and mathematics educators. Institutionalized acknowledgment of and support for teachers and teaching-developmental activities (e.g., release of time, accreditation, funding) is also mentioned as a factor facilitating interventions and other teaching-related projects.

Education and support for UM teachers is an emerging area of research with need for further studies. Suggestions for future research proposed in our group include the study of the role of institutional structures on the support for mathematics teaching; the preparation of new or graduate teachers; the investigation around support for non-mathematics specialists who teach mathematics; and, the identification of the characteristics of productive collaborations between mathematics education and mathematics communities (see a recent publication by Goméz-Chacón et al., 2021). Furthermore, developmental research projects in this area should be more attentive to designing, implementing and evaluating pedagogical interventions towards institutional change, including the development of appropriate resources (design principles and implementation) for UM teaching.

Reflections and ways forward

We can see that many notable contributions of this year, as well as challenges for the future, are in line with the overview on CERME research in UME identified by Winsløw et al. (2018): (a) *what is it?*, namely research into current practices of UME (with no direct intervention), such as: mathematical content; methods and resources; transition phenomena; student experiences; and, teaching non-mathematics specialists; and (b) *what could it be?*, namely developmental or experimental research, that includes an intervention design as part of the research project (e.g., research on, and for innovation in UME; i.e., interventions in specific courses or programs) and professionalization of UME practice (preparation of mathematics teachers).

In our discussions, the different impacts of the pandemic on many aspects – such as teaching and learning, conducting research or collecting data – appeared as an important point for the research agenda. This impact may lead to more studies considering technological issues in UME research in the coming years. Other important topics are in line with those identified in CERME11 (González-Martín et al., 2019): 1) the establishment of different types of collaborations and the development of theoretical tools to study them; 2) the study of complex phenomena, and the networking of theories (or the use of theories from other fields); 3) the need for large-scale studies and replications to consolidate results; 4) the development and testing of innovative research methods and data collection procedures; 5) the identification of the cultural, institutional, and local characteristics in some studies, and how changes in these factors may influence studies and results in other contexts; and, 6) studies on new or understudied topics, such as equity, access, and inclusion in UME.

We believe the first ten years of existence of TWG14 have brought many advances in our field, and we predict more important contributions in the years to come. We are confident that the coming CERME conferences will allow us to pursue research on the areas and questions discussed this year, to address and propose implementations for practice, and to open new areas for investigation.

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Students' process of learning integration in an adidactical situation at the first year of University

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Keywords: Integral, adidactical situation, approximation process

The integral is one of the most important topics in Calculus and is difficult to understand for many students. When solving definite integral application problems, previous research has emphasized that students found the antiderivative procedure more useful and easier than the approximation process or area (Akrouti, 2020). Kouropatov and Dreyfus (2014) argue that "students rarely acquire comprehension regarding the integral concept; rather, even in the best cases, students acquire no more than formal techniques for the solution of specific exercises and problems" (p. 533). Students have a great overreliance on algebraic representation. Some authors explain the difficulties by a lack of understanding of the multiplicative structure of integral (Orton, 1983; Sealy, 2014), a poor understanding of the rate of change (Thompson, 1994), and resistance toward accepting accumulation as a function (Kouropatov & Dreyfus, 2014).

To improve students' understanding of integration and deepen their knowledge, I propose an approach based in a physical phenomenon. My idea consists in that the formal definition should emerge from informal ideas that are intuitively clear. Students should use a form of intuitive knowledge through their interaction with the real world. In addition, these types of knowledge lead them to work with their personal experiences (Moreno-Armella, 2014). This approach consists of presenting a situation to introduce the definite integral where students are invited to implement an approximation process through which its underlying structure could progressively emerge (Fig. 1). My goal is to investigate how students understand the underlying structure of integral when working with an adidactical situation. The framework guiding the analysis of student understanding is based on Bloch and Gibel's (2019) Calculus Knowledge framework.

Data from this study is from a larger data set, which included audio recordings of two instructors teaching the Riemann integral. Students participating (18 students were present in these lessons) in this study were attending a first-semester course in calculus. In February 2020, I suggested the situation to a group of students (19/20 years old) enrolled in the first preparatory year of Mathematics-physics (MP) at IPEIT (Institut Préparatoire aux Ecoles d'Ingénieurs de Tunis). In preparatory classes, the courses are organized into lectures and tutorials that each lasts two hours. I developed a detailed analysis of the lectures on the Riemann integral. Observation of these lessons allowed me to see the interaction between the teacher and the students.

The analysis of the data collected emphasizes two obstacles encountered by students. The main one was how to consider the bar as a set of points. In fact, a few students viewed the bar as a continuously distributed line and then considered it as a collection of point masses. Therefore, they used the total mass divided by the total length of the bar. Then, they considered the length of the bar as a proxy for the number of points in order to implement an acceptable expression for dr.

The second obstacle for students was how to evaluate an infinite sum of products. In fact, it is a question of confusion between the limit of a sequence that converges to zero and the sum of an infinite number of terms of this sequence.

Through this research, I hope to be able to design activities for students about integration that develop their understanding of underlying integral structure and improve how integral can be used in physics phenomenon.

The proposed situation

Evaluate the force of gravitational attraction between a punctual mass m of 2 kg and a continuously distributed bar of 6 m long and mass of 18 kg in the position below:

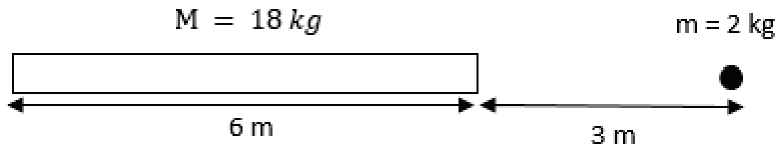

Figure 1: The bar situation

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Orchestrating affective, cognitive and metacognitive dimensions of undergraduate mathematics learning in digital environments

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This paper focuses on the notion of orchestration and explores its possible meaning in digital environment in the new distance/blended setting caused by pandemic. I review the literature about the concept of orchestration. Starting from the description of a learning scenario, the emergence of a new possible meaning of orchestration arises and some insights for further research are given.

Keywords: university, orchestration, digital environment, blended learning

Introduction and context of the study

This paper wants to contribute to the debate on the theoretical construct of orchestration. The metaphor of orchestration has been largely exploited in education, both in mathematics education research and in research about technology-enhanced learning environments. However, the recent new context of digital education created by the pandemic, involving distance or mixed forms of participation in lessons (all the students at home or some students in the classroom and others at home), seems to highlight a needed evolution of the concept of orchestration.

The study is embedded in the such a context of distance learning caused by the pandemic. University courses moved to digital platforms for synchronous lectures (in our case Microsoft Teams), supported by Learning Management Systems (in our case Moodle) for providing students with materials and individual or group learning activities. According to previous studies, our experience in teaching mathematics courses (especially Linear Algebra) for freshmen engineering students confirms that the didactic contract rules of the teacher and of the students are not aligned:

> At the mathematics (subject) level, both in the UK and in France, the lecturer expected that the text of the lecture would be used by the students, not only to learn and understand the concepts, but also as a model for certain mathematical practices, in particular mathematical proof. In France, we observed that the novice students did not adhere to this rule, and searched for worked examples (as models for such practices) in different kinds of resources such as their tutorial notes, textbooks and websites. […] students were searching for worked examples trying to reproduce techniques (Gueudet & Pepin, 2018, p. 69, 71)

This is even more true in degree courses (such as Engineering degree) where mathematics is considered as a service subject, which often leads students to make the equivalence between learning mathematics and learning (by rote) mathematical procedures. In order to fostering change in students' attitude towards mathematics learning, I started to be interested in designing and exploiting digital activities, which students can be engaged in. Before the pandemic, the Information Engineering students attended all together (almost 150 students) face-to-face Linear Algebra lectures, addressing the topic from both theoretical and procedural point of view. Then they were offered a 2-hours session per week in smaller group, consisting of 50 students, in order to work on and deepen the content of the week lectures, supported by a tutor. Due to my interest, I designed and provided students with digital activities (e.g. Albano & Pierri, 2014; Albano, 2017), already implemented before the pandemic, with the didactic purpose of helping them in constructing relational mathematics/conceptual understanding of mathematics, as opposite to instrumental mathematics/procedural understanding (Skemp, 1976; Hiebert, 1986). Such activities have been designed according to an asynchronous setting and thus engaged students in their personal study time/space outside the classroom, so that they could participate at their own pace working at their home or wherever. In the weekly tutoring sessions, the students could discuss the activities with the tutor, as well as the tutor, having access to the digital platform where the students worked on the designed activities, could focus her interventions on the points of greatest difficulty detected. The new distance setting forced us to rethink the structure of synchronous lectures, with particular reference to the interactions and the involvement of the students that no longer could be the same than in a traditional face-to-face lecture. So I was concerned with two interrelated issues: finding out which Moodle tools offered support for interactivity and designing their use in order to enable students to develop relational mathematics knowledge. On one hand, questioning and answering is recognized as one of the most effective practice for promoting interactivity. On the other hand, conceptual understanding is not just a cognitive issue, but it is affected by affective (beliefs, perceptions, attitudes) and metacognitive (learners' awareness and control of their own learning processes) factors. This means that any successful teaching/learning intervention should take into account all these three learning dimensions (cognitive, metacognitive and affective). I was interested in exploiting tools allowing to implement closed-ended questions/answers sessions and to collect and display right away the class-wide distribution of responses. Two tools offered by Moodle are Quiz and Feedback. The former allows to create (self-)assessment tasks. The latter allows to construct and submit a survey. Feedback falls into the category of so-called classroom response system (CRS). One of the early works on the use of such systems (Siau et al., 2006) points out some relevant pedagogical and curriculum issues, that can be valid also for the Quiz, including 'when to introduce the questions, what questions to ask, and how much class time to allocate' (ibid, p. 402). In this paper we state that the design issue should be focused on a learning scenario where each tool serves a particular purpose within a more general didactical objective. The metaphor of orchestration immediately comes to mind and it will be the underlying concept through the rest of the paper.

Theoretical framework

In this section we make a review of the concept of orchestration in technology-based educational research and mathematics education.

Dillenbourg (2013) refers to orchestration as a form a management (regulation process) of integrated pedagogical and technical scenarios, including on one hand activities, that can be face-to-face or online, and on the other hand tools enabling the implementation of the activities. In his view, orchestration is not concerned only with learning, but also with various extrinsic constraints (time, space, discipline, curriculum,…). This is one of the features that distinguishes orchestration from instructional design, in addition to the fact that it relates to a group of students rather than an individual and that the teacher's control prevails over that of the system. Dillenbourg states that 'orchestration' strengthens the teachers' potential in steering classroom activities and enables teachers to view things otherwise invisible. In response to Dillenbourg position, Kollar & Fischer

(2013) argue that the metaphor of music orchestra can be effective if it refers not only to the arrangement aspects (that is real-time management of activities and events) but also to whole process underlying the creation of music, which includes composing and conducting aspects too. They consider orchestration as “the process of creating, adapting and enacting a technology-enhanced learning scenario under complex classroom conditions” (ibid, p. 508). In their view, what Dillenbourg refers to as orchestrating actually means conducting. Composing consists of describing a scenario, constituted by resources and tools, specifying how they are combined and used by the teacher. Arranging is what the teacher does adapting the defined scenario to her classroom’s constraints. All the three processes are essential for technology-enhanced learning (TEL) being effective in classroom. Finally, they emphasise that the main objective of TEL is to facilitate student learning, which should always be taken into account. A further conceptualization of orchestration concerns the way in which the students are involved in the activities: individual, small groups, large groups. The design can foresee more than one mode of involvement or the real constraints can ask for changing the designed ones. Weinberger & Papadopoulos (2016) introduce the idea of orchestration of different social modes of learning. Students can learn individually or collaboratively, in small and large groups. Orchestrating social modes of learning means organizing learning choosing one of them or merging some of them. They argue that the transition from one social mode to another one should be carefully planned by the teacher taking into account how each of them help the students to reach the global learning objectives of the course. The teacher is recognized as the centre of a complex technology-enhanced environment, where technology both requires to be orchestrated and can facilitate orchestration.

In mathematics education, Trouche (2004) proposes the term ‘instrumental orchestration’ in a computerized learning environment (CLE). An instrument encompasses an artifact (i.e. a given object) together with utilizations schemes socially constructed by the subject. The process which gives rise to an instrument is called instrumental genesis. Trouche highlights that the complex artifacts present in CLE produce a set of instruments. The process of instrumental genesis as well as the articulation of instruments in CLE cannot be left to the students themselves but demands the guidance of the teacher, which can be done by means of instrumental orchestration. In this strand, Drijvers and colleagues (2009) propose a three layer model: *didactical configuration*, meaning the setting of the teaching environment equipped with artifacts (technological tools and tasks); *exploitation mode*, that is the way the teacher uses the didactical configuration in order to reach her didactical objectives; *didactical performance*, referring to ad hoc and run-time decisions taken by the teacher while teaching. Within the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008), two different meanings of orchestration come into play: one referred to Trouche in relation to the use of artifacts, and one related to mathematical discussion, intended as the coordination of various voices emerged by the students with the voice of the mathematicians.

In this paper we want to investigate the potential of the concept of orchestration as lens to analyse a learning scenario, implemented in a distance setting and based on the exploitation of technological tools offered by digital learning platforms. What is being orchestrated? The tools? The purposes for which the various tools are used? Something different?

A learning scenario

In this section we present an actual learning scenario, implemented in a Linear Algebra distance course for freshmen Information Engineering students, equipped with Teams and Moodle. The course provides the students with synchronous online lectures (7 hours per week), tutoring sessions (2 hours per week), didactical material (videos, books, notes from digital board, worked-out exercises, slides) and resources (weekly tasks, quizzes, FAQ forum, periodic workshops for reviewing macro-sections of course contents). Various questions arose: concerning the precise didactical purpose of promoting conceptual understanding, how can Quiz be used? and Feedback? And how to handle the use of both so that one can takes advantages of the other one?

The scenario and its implementation in class can be analysed in terms of orchestration. I choose to refer to Drijvers et al. (2009) orchestration model. I designed a didactical configuration, arranging an environment composed of three artifacts: a Moodle Feedback activity, investigating students' perception of their mastery on a given topic; a Moodle Quiz activity, investigating students' learning on the same topic; a Teams talk session for discussing the outcomes of the previous activities. The exploitation mode concerned the way I designed each artifact, described below, and the delivery timeline, which envisaged first the delivery of an affective activity, then a cognitive activity and finally a discussion that possibly moved to the metacognitive level. The design and the outcomes of the activities have been analysed both content-based and using the "in class" observation of the teacher-student interaction. The data have been collected by using Moodle reports related to Feedback and Quiz activities and recording the Teams talk session. I carried out a survey, by means of the Moodle Feedback tool, to collect students' opinions on their level of knowledge for the topic of linear systems. Thus I submitted the closed-ended question in Figure 1.

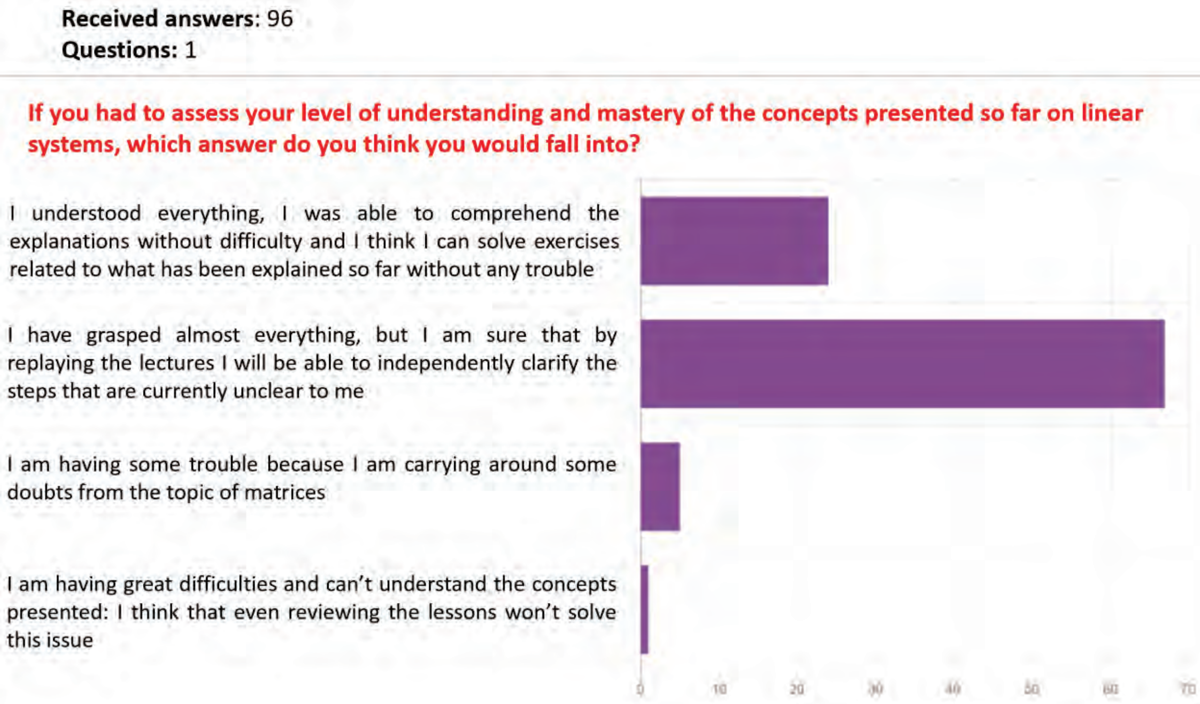

Figure 1: Results of the Feedback

I collected 96 answers, pointing that most of the students were satisfied with their comprehension of the topic (Figure 1). Indeed 25% of all the participants were completely convinced of their mastery, about 70% of the students admitted to having some doubts which they felt they could clarify by studying in more depth with the support of the recorded lectures, very few students reported having gaps from the preliminary topic or being unable to recover the gap. After showing the students and commenting on the graph in Figure 2, the mood in the class was very positive as it reflected the fact that the students felt very confident with the subject matter.

In the next lesson, I submitted a short quiz to the students: the first question shown in Figure 2 dealt with the notion of solution of a linear system, while the second question shown in Figure 3 dealt with the discussion of systems with echelon matrices. The former can be classified as an exercise, since establishing whether an item is correct or not requires direct application of previous knowledge, that is a definition. The latter can be classified as a problem, since it requires a certain reorganisation of the information given in the text and pieces of knowledge about linear systems and matrices in order to draw conclusions about the correctness or otherwise of the items at hand.

Let us consider the following linear system:

$$\begin{cases} x + 2y + z + t = 2, \\ 3x + 4y + z + 2t = 5, \\ 2x + 2y + t = 3. \end{cases}$$

Indicate which one(s) of the following statements is true.

- ☐ (2,1,1,-2) is a solution of the linear system
- ☐ $\forall y, z \in \mathbb{R}$ (z+1,y,z,-2y-2z+1) is a solution of the linear system
- ☐ none of the other items is true
- ☐ the system is incompatible
- ☐ the solutions of the linear system are (x,y,x-1,-2x-2y+3) $\forall x, y \in \mathbb{R}$

Figure 2: Question 1 of the Quiz

Let Ax=b a linear system of 4 equations in 4 variables, and let A be an echelon matrix.

Indicate which one(s) of the following item is true.

- ☐ If A has a zero row, the linear system is incompatible
- ☐ None of the other items is true
- ☐ If b=0, the linear system has only the zero solution
- ☐ If each non-zero row of A corresponds to a non-zero row of b, the linear system is compatible
- ☐ If A has two non zero rows, the system has ∞^2 solutions

Figure 3: Question 2 of the Quiz

Figure 4 shows the graph (produced by Moodle) of the marks (between 0 and 1,5 per question) received by the participants in the Quiz, with the total number of students distributed by grade range.

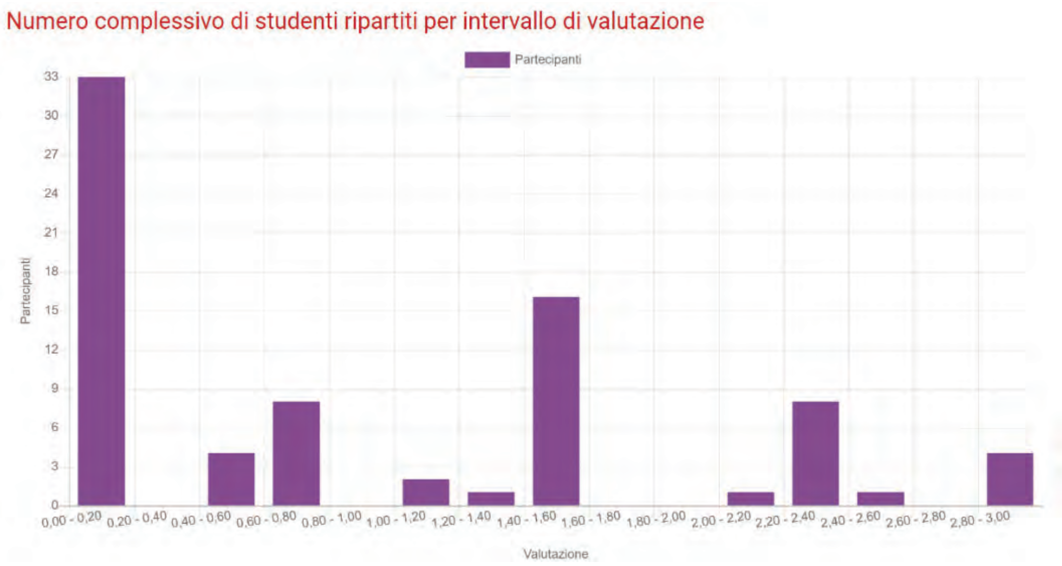

Figure 4: Results of the Quiz

After showing the results of the Quiz, everyone was astonished: they made evident a great gap between the students' perception of their mastery of the topic showed by Feedback and their actual mastery in solving the questions on the topic posed by Quiz. Thus the teacher started a collective discussion aimed at making sense of the gap. In terms of orchestration, the way in which the teacher led the discussion, the questions she chose to ask to guide the students' reflection, the mirroring of some of the students' interventions and the recapitulation of what emerged from the discussion concerns didactical performance. The discussion started with a question concerning the first question of the Quiz (Figure 2):

Teacher: How do you determine which of the items presented is correct? Tell me which strategy you used.

Student 1: Prof, I solved by using Gauss and got different solutions.

Various students agreed with Student 1, so the teacher asked for someone who acted differently.

Student 2: Yes, me, Prof. When he asked me if that 4-uple was a solution of the system I substituted it in and saw if it was equivalent.

Using this intervention, the teacher focused the students' attention on the definition of solution of a linear system and launched a collective discussion about its potential to investigate the items: it could be directly applied to items 1, 2 and 5, and depending on their value of truth some inferences could be done concerning the correctness of the remaining items. Then the students proceeded to apply the strategy come out from the discussion to answer to Question 1.

Once completed, the teacher opened a further strand of discussion concerning the difference between the approach used by most of the students and the one used by few, as Student 2, which turned out successful. The teacher highlighted that the request of the question was to establish the value of truth of the items, whilst many of them seemed to have acted as if the request had been 'solve the linear system'. Some students recognized that, taking this approach, they missed some correct items, since they were not able to recognize the equivalence between what they got solving the system and further description of the same solution set. From the discussion the need of a relational approach in contrast to a instrumental approach emerged. This let the teacher shift the focus to Feedback results and gave her interpretation of the results, based on the assumption of students' procedural learning, confirmed by Quiz outcomes and discussion:

Teacher: the perception of your mastery emerged from Feedback is not false because I am sure you know how to carry out exercises but you have to move a bit further ... a few days ago someone asked me if in the exam quiz there will be theory questions ... of course, this is a theory question.

The lecture proceeded discussing question 2 (see Figure 3). It was particularly suitable for activating relational knowledge, both because the question dealt with a generic linear system (therefore no solving procedures could be applied as in the previous case) and several items were "if... then..." propositions, which brought into play reasoning and argumentation competencies.

Insight for new research

The previous learning scenario does not remain confined to distance learning, but it is inherited by the face-to-face lectures. Indeed, most of the students attend lectures equipped with their personal mobile device (smartphones, iPads and notebooks), thus lectures can be redesigned according to BYOD approach. In my view, this scenario poses the issue of a further development of research. In mathematics education the concept of orchestration has been developed mainly with the perspective of guiding students' instrumental genesis. This is not the focus of the paper due to the general-purpose nature of the digital artefacts used by the students. But the instrumental genesis of the teacher using the digital tools could be a very interesting focus to be developed. For example, as we teach remotely, we develop many new schemes and sets of artefacts for teaching, and it can even influence how we teach in-person (providing evidence of the development of a scheme).Taking up Dillenbourg's idea of technology as something that makes visible what was invisible, we can look at what has been made visible by the orchestration performed in the above learning scenario. The use of feedback and quiz together, even in a specific order, made the students experience a discrepancy between their idea of mastery and that of the teacher that is realised in the exam paper (affective dimension), the subsequent (mathematical) discussion allowed the students to become aware of and reflect on their own learning processes (metacognitive dimension) through a cognitive analysis of the questions and answer items proposed (cognitive dimension). An hypothesis following this exploratory study could be that I started developing a scheme emerged by the presented scenario which allowed a successful integration of the three dimensions of learning that took place in the orchestration of the three activities feedback, quiz, discussion. This seminal study suggests new research to investigate the emerged hypothesis addressing the development of a renewed orchestration framework, taking into account the pedagogical aim of the teacher in using the digital tools, and a renewed teachers' instrumental genesis for reaching this pedagogical aim. Such a new framework should exploit elements from various theories, in mathematics education and in technology-based education, according to networking of theories (Prediger et al., 2008).

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Cognitive roles in cooperative problem solving at university level

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This paper addresses the didactic issue of promoting the construction of the problem-solving competency for undergraduate students. We report the design of a learning activity, engaging students in collaborative problem solving in topology, each of them playing as a cognitive function coming into play when a mathematician faces a problem. To shed light on the impact of the roles played by the students with respect to the solving-problem process, we analyze some student's protocols from which we carry out for each role some actions as well as some benefits recognized by the students.

Keywords: mathematics education, university, problem solving, cognitive functions, role-playing.

Introduction and conceptual background

Problem solving has a fundamental role in mathematics learning and its educational importance is stated at each school level and in every field of mathematics, as it provides experience of those key processes in mathematics education that are involved in exploring, conjecturing, proving, constructing examples and counterexamples, representing, monitoring etc. (Polya, 1945; Schoenfeld, 1992; Carlson & Bloom, 2005). An important problem-solving activity, particularly efficient for the construction of new concepts (Dahlberg & Housman, 1997) is the students' generation of examples satisfying particular constraints. In fact, the students' production of examples can be a very complex activity, also for university students, that promotes the activation of fundamental cognitive processes (Antonini, 2011). At university level, a rich and deep field of mathematics is topology, a basic working tool of mathematicians in a variety of fields that can offer the undergraduate mathematics students the experience of engagement in a real problem-solving process. In this paper we are interested in promoting the construction of problem solving competency for undergraduate students, attending a topology course within Bachelor of Mathematics. The study of introductory general topology topics requires activating the significant cognitive functions, that are the mental processes (Kiely, 2014) that come into play during the problem-solving process. Polya (1945) highlighted that often the teacher poses questions and suggestions useful to the problem solver and recognizes them as indicators of mental operations typically useful for the solution of problems. In this strand, Albano, Coppola and Dello Iacono (2021), looking at how mathematicians behave when solving a problem, individuated some mental processes they usually activate (e.g. looking for paths, questioning herself, organizing herself, systematizing the findings, …), identifying them as roles/cognitive functions that a problem-solver should activate. These roles are: 1) Boss: she manages the work from every point of view (organizes actions, calls to the task, requires participation); 2) Promoter: she gives insights to promote a path that, starting from and manipulating prior mathematical knowledge (concepts and propositions), leads to the construction of examples, conjectures and outlines a solving strategy. In case of trouble, she asks for the teacher's help; 3) Critical mind: she questions the truth of the arguments and the validity of the answers proposed by the group, with the aim of corroborating their findings; 4) Blogger: she collects and rearranges everything that emerges from the discussion, to draw up a document containing all the arguments, notes, doubts, questions, answers.

We face the issue of promoting the development of students' ability to solve problems by offering them structured opportunities which allow the students to imitate and practice (Polya, 1945) and to become aware of their cognitive processes, so being able to monitor and coordinate them (Schoenfeld, 1992). The assignment of roles expands the potential of students as it gives them the opportunity to improve their problem solving skills at various levels: cognitive, metacognitive, affective. Exploiting and extending the results of a national project[1], we used the metaphor of storytelling to characterize the problem solving process at two levels: the process itself is seen as a mathematical story and the cognitive functions coming into play during the process become characters of such a story. Students are engaged in collaborative problem solving, where each student is required to play the role of a character of the story, that is she personifies one of the envisaged cognitive functions (Albano, Coppola & Dello Iacono, 2021). During the problem solving process, the students are expected to develop a mathematical narrative, consisting of their co-constructed notes on a digital board and a final collective report on the problems' solution. In this paper we report the design of a learning activity (Podolskiy, 2012), engaging undergraduate students in collaborative problem solving tasks. We discuss the first outcomes of the analysis concerning the metacognitive aspects, that is the students' declarative knowledge and awareness of the cognitive functions they experienced by playing the corresponding roles during the activity.

Experimental design

The design of the learning activity foresees that the students face a problem-solving task, consisting of three problems. According to the engagement model of Albano, Coppola and Dello Iacono (2021), the students work in groups of four or five people, each of them playing the role of one cognitive function. The groups are engaged in problem solving at different levels. One group, called Solver group, is devoted to collectively solve the three problems, and each of the students in the group acts according to one of the cognitive roles described in the previous section. The remaining groups, called Onlooker groups, are required to observe how the Solver group is working. Each student is guided to reflect both on how a specific one Solver member acts with respect to her role and on how the mathematical process is carried out by the entire Solver group. Therefore, personifying an Onlooker role stimulates a critical reflection not only at the cognitive level, as it allows a student's engagement in the mathematical problem, but also at a metacognitive level, as it fosters student's monitoring skills related to a role to play in a subsequent activity. A further level of reflection is added with respect to the previously cited engagement model of Albano et al. Indeed, we assume an incremental goals' structure of the problem-solving task: each of the three problems aims at a specific sub-goal, going from routinary employing mathematical concepts, facts, procedures and reasoning to creating new

[1] PRIN "Digital Interactive Storytelling in Mathematics: a competence-based social approach", PRIN 2015, Prot. 20155NPRA5, national project funded by the Italian Ministry of Education, University and Research.

mathematical knowledge. We envisage as many Onlooker groups as there are problems, so that each of them is focused on one specific problem. Each student is required to draw up a personal Logbook, containing some guidelines to reflect on the role she assumed, both as Solver and as Onlooker. Furthermore, at the end of each activity, every group is required to draw up a collective Logbook, reporting the solution of the problem and the process made to reach it (detailing the experience, how the construction of the answers took place, paying particular attention to the arguments they adduced in solving the three sub-problems). In order to foster the awareness of all the cognitive functions, students change roles with each new activity, also changing Solver with an Onlooker group and permuting the Onlooker groups (so changing the problem to be focused on).

Methodology

The experience involved about fifty students attending the course of *Geometry III* at the second year of a degree course in mathematics. The course Geometry III aims to introduce students to the fundamental concepts of general topology and to stimulate them to be able to use ‘topological eyes’, often far from the Euclidean ones. Besides acquiring content knowledge, the main educational goal is to construct students’ mathematical reasoning capability, by means of analyzing and exploring problems, with an efficient use of topology concepts and results.

According to the design, each activity has an assignment on which students work in groups of 16-20, divided into 4 subgroups corresponding to the Solver group and the three Onlooker groups. In our experiment, all the participants have been split into three groups, named WG1, WG2, WG3, each of them consisting of four subgroups WSGi.1, WSGi.2, WSGi.3, WSGi.4 (i=1,2,3). Each student has been associated with a role-pair (subgroup role, individual role). Each subgroup acted as Solver or as Onlooker, so the corresponding value of the variable ‘subgroup role’ could be S or Oi, where Oi means that the subgroup acted as onlooker on the i-th sub-problem. The values assumed by ‘individual role’ corresponds to the cognitive functions played.

Along the course, students have been involved in three activities CWi (i=1,2,3). Both individual and collective roles changed as the activity CWi changed. More precisely, the assignment of roles in the passage from one activity to another can be described by a double permutation, one corresponding to the subgroup role and the other to the individual role. As an example, a student who has been assigned the role-pair (O1, Promoter) could take on the role-pair (S, Critical mind) in the next activity. This means that, while in the first activity the student belonged to the Onlooker group focused on the first problem and she was required to observe the work of the Promoter in the Solver group, with whom she confronted, during the second activity, she belonged to the Solver subgroup and acted a Critical Mind. In every activity CWi, the problem-solving task consisted of three problems: the first two concerned basic concepts introduced during the lectures and required the construction of examples of topological spaces or subspaces under given constraints; the third one was less routine (for instance, students may be asked to provide some characterization related to the property that is being investigated). Figure 1 gives a flavor of the kind of problems the students were asked to face.

PROBLEM n.1

1.a Construct a topological space (S, τ), where S is a non-empty set and τ a topology on S, different from those studied in class (invent it !!), in such that S contains two proper non-empty subsets X_1 e X_2 such that $Fr(X_1) = \emptyset$ e $Fr(X_2) \neq \emptyset$ in (S, τ), where $Fr(X_1)$ e $Fr(X_2)$ are the boundaries of X_1 and X_2 in (S, τ).

1.b Consider the set S introduced in point $1.a$. Denoted by τ_1 the topology τ and by τ_2, τ_3, τ_4, respectively, the trivial topology, the discrete topology, a topology of your choice distinct from the previous ones on S, determine

1.b.1 the frontier of X_1 in (S, τ_i), $Fr(X_1)^{\tau_i}$, for $i = 2, 3, 4$.

1.b.2 the frontier of X_2 in (S, τ_i), $Fr(X_2)^{\tau_i}$, for $i = 2, 3, 4$.

The boundary of a set can vary as the topology varies. Explain why.

PROBLEM n.2 Consider the topological space you defined in the previous problem (point 1.a). Construct, if they exist, three proper non-empty subsets of S, X_3, X_4, X_5, each distinct from X_1 and X_2, so that

2.a X_3 has an empty boundary with respect to all topologies.

2.b X_4 has a non-empty boundary with respect to all topologies.

2.c X_5 has empty boundary only with respect to some topology.

PROBLEM n.3

3.a Choose one of the topological spaces (S, τ_i) with $i \neq 2$ among those considered in problems 1 and 2 and investigate how a subset with an empty boundary can be constructed. In other words, characterize, if possible, the subsets with empty boundaries.

3.b Do the considerations made in the previous point or the statements you have reached also apply to the other topologies? Consider at least one of the two topologies τ_j on S with $j \notin \{i, 2\}$ and determine if the results found in (S, τ_i) continue to hold in (S, τ_j).

Figure 1: The task for the students

Each working group WGi had at their disposal a digital environment, consisting of various tools:
a) a collaborative board Miro, enabling to collectively brainstorm in order to solve the problems, by adding post-it, importing images, drawing and connecting ideas, exchanges comments by chat; b) a collaborative document (shared Google-doc), used to report the solution product and process; c) a personal document (private Google-doc), used to report the reflections concerning the roles played during the experience.

All the data concerning the learning activity have been digitally stored, by means of the used tools (Miro and Google-doc). At the end, a questionnaire was submitted for investigating the students' perceptions. Here, we focus the qualitative analysis of the roles, looking at the students' answers in:
- the personal Logbook: *Describe how you played your role (as Solver and Onlooker) and what your contribution was. Do you think that the interventions related to your role were useful to achieve the objective? Why? Would you have done something differently? Why?*
- the questionnaire: *In the activities you played a role as Solver or Onlooker. For each role, tell us from your point of view how it contributes to solving the problem. If you think about when you are solving a problem on your own, do you recognize any of these roles in what you do during the solving process? If so, which ones more frequently? Are there any other roles you identify? Tell us.*
In particular, we are looking for excerpts describing the characteristics of the roles and the benefits of playing such roles, as perceived by the students.

Preliminary findings and discussion

We analyzed the Logbook and the questionnaires of 24 students. The answers to the above questions shed light on the impact of the roles played by the students with respect to the problem-solving process and to their usual own experience of problem solving.

Concerning the role of the Boss, the student St1 says:

St1: I immediately noticed her attention to details. Thinking of doing something useful for those of us who might have some deficiency on the tools required to face the task, she wrote in a frame [on Miro] all the useful definitions, such as boundary, interior, closure [of a set]. This has created a sort of "safe-spot", a point where you can have everything you need ready for use. She acted as a leader.

This excerpt seems to highlight the Boss' characteristics of a leader, implemented as someone who makes the group "safe" with respect to the problem-solving process. This means that she takes charge of supplying the group with mathematical recall notes connected to the given problem. The student St6 reports a similar work of the Boss (recall definitions), highlighting that such work let all the students be more comfortable with the needed background mathematical knowledge.

The role of the Boss is a disputed one. On the one hand, it is considered as a key role:

St1: The only real distinction was therefore that of boss / rest of the solver group, but without this negatively affecting the results.

St4: The boss of the solver group is the main role because he coordinates the entire group and manages to make the onlookers understand the work done.

On the other hand, sometimes, it can be perceived/played as a rote-role, without any added value, as shown by the following excerpts.

St15: I think it is the least useful role, since its role is only to check if all the work is progressing correctly, but does not interact in any way with the other students.

Actually, it seems that the experience of student St15, playing as 'boss' or onlooking 'boss', leads her to perceive the role differently from the designers of the activity. In fact, the cognitive function corresponding to 'boss' envisages someone who coordinates and takes care that everyone participates in the solving process. It is not her business to check the correctness of the work.

The role of Critical Mind seems to have promoted the action of questioning, and it has been recognized how much such action fosters a broader view of the problem solving:

St7: This work allowed her to ask more questions about the topics useful for solving the problem, and to have a broader vision of how a given exercise could be solved with different methods and observations. The role of Critical mind and, at the same time, that of the onlooker of the critical mind, proved useful in stimulating new proposals and new points of view, questioning the choices that are made each time and rattling off the strategies adopted as much as possible.

It is worthwhile to note that the role of Critical Mind has stimulated the student to pose questions to herself and to promote a critical attitude:

St17: It helps me to understand the mistakes I make, saying to myself: "stop, think: why did you carry out this calculation like this? Would it be simpler in another way? Is there any theorem or statement that can help me solve this exercise more quickly? And I find all of this very useful for the smooth running of a problem".

The student S12 highlight the importance of this role at individual and group level:

St12: In addition to an individual utility, I believe that the role of critical mind was very important for the whole group because often in the resolution of an exercise it can happen to make mistakes or inaccuracies without realizing it and the critical mind, insinuating doubts, manages to bring to attention the steps on which we need to work better.

The student St5 points out the role of the Promoter as someone who sketches a working outline, to engage the whole group in developing a solution and activating a thinking process:

St5: My idea as a promoter is precisely this, to propose your ideas despite some small inaccuracies in order to encourage the intervention of others and reach the final solution.

St13: I think about how to develop a certain problem.

It is interesting to note that the Promoters' suggestions are constructed on previous knowledge:

St14: She manipulated known propositions and concepts to build examples and strategies that led to the resolution of the problems

However, the role of Promoter does not always find the same appreciation and this is probably due to different values and personal beliefs about the roles:

St22: I also appreciated the promoter but to a lesser extent than Boss and Critical mind only because I personally think that finding an idea is less interesting than stressing it.

Concerning the role of the Blogger, it seems to impact on the cognitive level (e.g. reasoning) as well as on the affective one (e.g. engagement of the group), as shown in the following:

St11: The student I observed, that is the one who played the role of Blogger, played his role well, collecting and tidying up the board, highlighting some definitions and observations useful for carrying out the exercises, through the use of arrows and schemes that made the key concepts clear. He exploited and used the comments made in the chats by the other members of the group as well.

St13: His interventions were therefore also useful for taking stock of the situation from time to time, reorganizing ideas and, asking to repeat some of the concepts, also giving the possibility to those who may have remained behind to pick up the thread of the discussion.

As for the Boss, also the Blogger is a disputed role. On the one hand, it is considered pivotal:

St12: I believe that the role of blogger was fundamental for managing space and order on the board. Working in an orderly environment, in my opinion, favors concentration, allows you to easily identify the elements necessary to conduct a certain reasoning.

On the other hand, it can be played at surface level, just looking at aesthetic aspects:

St15: It is a secondary role in my opinion, since in most cases he was only concerned with the stylistic point of view of the board, caring little or nothing about the content.

St22: …the blogger did not impress me as I found it marginal with respect to solving the problem.

Table 1 shows a synoptic picture of the performed analysis. The first column shows some excerpts from the personal Logbooks that the students filled along with the activity on the basis of the roles they played (steered by the questions reported in the 'methodology' section). The second column shows excerpts from the questionnaires, submitted at the end of all the activities.

It seems that the logbooks show descriptions of the roles according to the actions that are recognized as belonging to a certain role, while the questionnaires show the roles in terms of the characteristics that define them in a problem-solving process. Furthermore, the table shows that sometimes the roles are experienced by the students in a way that corresponds to the designers' model, while in other cases some students find them difficult or uninteresting or give them an interpretation different from that of the designers, as in the following examples.

Table 1: Students' excerpts

	Excerpts from the personal logbooks	Excerpts from questionnaires
Boss	– plans, arranges and coordinates the work – pays attention to certain details that can benefit all the group – briefly recalls the key starting concepts that could be useful for solving the problem – enforces the assigned roles – involves all members of his group in solving	– is a predominant role taking care of the group – puts all members of the group in the same starting conditions trying to eliminate all possible differences or prejudices about individual abilities – makes you break down shyness – keeps his group cohesive – is the least useful role
Critical mind	– raises questions, also concerning the validity of the arguments – allows questions to be asked – analyze the various procedures critically – stimulate new proposals and new points of view	– allows you to gain awareness and to understand your own mistakes – allows you to develop a broader view of how a problem can be solved – fosters reasoning and searching for alternative simpler ways of moving in the problem space
Promoter	– sketches ideas to be completed – manipulates known propositions and concepts to build examples and strategies – retrieves useful material (sources from books, lecture notes, ...) – communicates with the teacher for help	– allows you to start reasoning – allows you to repeat useful notions – allows you to think how to develop a problem – opens to change the way to solve the problem – promotes the engagement of peers – is less interesting than critical mind
Blogger	– rearranges the various information emerged from the discussion – clarifies the many ideas presented by peers – uses arrows and diagrams to make the key concepts clear – tidies up the board	– makes the solving process clear – promotes concentration – helps to keep the logical thread – helps to take stock of the situation from time to time – has a secondary role – it is marginal to the resolution of the problem

Sometimes roles didn't work out because students got 'too busy' with solving the problem. Here the roles functioned as a 'group', not fragmented:

St16: Personally, I didn't fully respect my role when I was in the solver group, as working all together, and taken up by problem solving, we didn't pay much attention to the roles we had to fill. .… I think this was the most difficult aspect to respect, everyone was aware of the roles, which were respected as a group but not by the individual.

What emerged and synthesized in the above table seems to confirm that the students actually grasped the characteristics of each cognitive function, recognizing their functional goal with respect to successful problem solving. Assigning a role serves to activate a specific cognitive function, to stimulate the activation of some cognitive processes, and the roles would seem to be tight especially to those students who usually activate all of them. Sometimes these kinds of students tried to cross over and join the group of solvers even if she was not part of it. We could compare the action of assigning a role with giving the student a piece of chalk, associating chalks of different colors to the different roles by which to write on the board Miro and to contribute to the construction of the story and the fabula. Thus, each group is engaged as in a *thinking classroom*, as "a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together and constructing knowledge and understanding through activity and discussion" (Liljedahl, 2016, p. 364). We could speak of “thinking groups” (Thinking Solver group and Thinking Onlooker group), where each member performs an active function to solve a problem, by means of an action that stimulates a cognitive thought process that could remain dormant and blocked.

We conclude by noting the educational importance of engaging students’ in experiencing all the roles, in order to recognize all of them as pivotal to be successful in problem solving, similarly to how a mathematician usually uses them all when facing a problem. However, it is not taken for granted that a student activates them all, and this can cause difficulties. There are problems in front of which some roles do not come out, maybe because students are not used to or are more comfortable only with some of them. The experiment allowed the students to share the profile of a mathematician and in some cases to recognize themselves as mathematicians. Generally speaking, the learning activity proposed aimed at promoting and developing processes involved in problem-solving competency and an attitude of a mathematician towards problems mathematics, not aimed at ‘re-producing’ theorems and proof learned in class, but to autonomously ‘produce’ something of their own, new and original. Further research will be needed to investigate the processes involved in playing specific roles in problem solving and in the personal development of such important competency.

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Dialectics of language and Plato's cave of mathematics

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This is a theoretical article hailing at classroom practice. It was originally intended to be a report on an ongoing study introducing the fundamental theorem of calculus in a STEM freshman course, but the necessary explanation of our epistemological position has taken up most of the available space. Based on Hegel and Lacan, we argue for the reality of language against the implicit realistic positions that dominate mathematics education research. We describe how Cauchy's "one says that" (on dit que) *created a bubble of semantic conventions. From inside this bubble, mathematicians only recognize the students by their shadows on the wall, like in Plato's cave. We argue that learning occurs outside the bubble. Instead of dragging the students in, to teach them ready-made notations and methods, our study exercises the dialectics of language in interaction with the students. We stick faithful to our lemma: one teaches by listening and learns by talking.*

Keywords: Hegel and Lacan, dialectics of language, Cauchy, fundamental theorem of calculus, infinitesimals.

Introduction

Initially, we intended to report on our experience with constrained remote teaching during two semesters as teacher-researchers (Tabach, 2006). We introduced the fundamental theorem of calculus (FTC) by working out the graphs of Covid-19 as displayed in the media. Once the students could meaningfully state the FTC about the discrete bar graphs, we moved on to the continuous case by a sudden change of scale (Ellis et al., 2020), thereby changing the referent of discourse. Ideas of limits and infinitesimals emerged from the students. This didactic strategy is justified in the ensuing theoretical development. Some of the worksheets used are available in https://cabraldinos.mat.br/

In the present analysis, in order to bring the dialectics of language and the subject of speaking to the foreground, we engage in a critical dialogue with a certain mathematicians' perspective about teaching (Tall, 2009; Tall & Katz, 2014; Thompson, 1994, 2019; Ely, 2017). We focus on three signifiers used by Cauchy (1821): "one says that" (*on dit que*), "becomes an infinitesimal" (*devient un infiniment petit),* and "the neighborhood" (*le voisinage*). Referring to Hegel (Hippolyte, 1977) and Lacan (1973), we show how these signifiers have created a bubble of semantic conventions that can be called *mathematics of the twentieth century* (M20); this bubble is the epistemological habitat of mathematicians. We argue that, due to the inherent semantic limitation, mathematicians are unable to leave the bubble to meet the students on their path to the bubble. They can only drag the students in and produce models of students' ways of thinking by looking at their shadows on the wall. Important concepts like *mathematics*, *numbe*r, and *infinity*, remain outside the semantic reach from the bubble. Mathematicians tend to mix up teaching, understanding and learning.

Language and the subject come first

To ground our study, we sought to discern events that could be placed upstream of the epistemological flux that led to the FTC. Marx's maxim advised us: the anatomy of man is the key to the anatomy of the ape. We asked ourselves: do we know what we are looking for? Yes, in our classes we expect the students to express in current language: *the integral of the derivative is the variation of the function,* and *the derivative of the integral is the function itself.* Guided by these current language statements, we were able to locate one of the origins of the FTC in Barrow's theorem (Barrow, 1976, p. 78, Figure 109). It is well-known that this theorem became the cornerstone of a discussion that lasted two hundred years. We will take this theorem as an example of a *signifier*, a core concept for us.

Barrow did not write the theorem for us; he was addressing his contemporaries that shared a certain common epistemological spirit. We say that Barrow *took that theorem as a signifier to represent himself as a subject to another signifier* (Lacan, 1973, p. 188). More than three centuries later, it happens that this second signifier is available to us, so that not only can we understand the theorem and its proof, but we can *represent ourselves as subjects* by this second signifier to a third signifier, belonging to the reader, etc. This movement of language and subject from one signifier to the next we call the *dialectics of language*.

> Language is not only a system of signs alien to the signified, it is also the existing universe of sense, and this universe is the interiorization of the world as well as the exteriorization of the "I". Language is a double movement that must be understood in its unity. (Hippolyte, 1977, p. 24)

Due to this subtle but ubiquitous double movement, words acquire their meaning and we become their speakers. We were able to locate Barrow's theorem because what we were looking for was already specified in language. *Insofar as we 'make sense of the world' it is the world that posits its sense across us*. A double movement constitutes both, *human subjects* and *language*. The movement is prior to the opposing poles that it generates. Accordingly, there is neither a universal "I", owner of an inner meaning to be expressed, nor a world out there of which one can speak and make sense by "using" language. There is no "dialectics between", no "dialectic relation" (Pais, 2016).

We apologize for demanding of the reader the effort to understand ideas so strange to commonsense. They are necessary to elicit the realistic[1] positions prevailing in mathematics education research. For instance, the belief in "underlying mechanisms that shape human thought, building from the fundamental level of human perception (…)" (Tall & Katz, 2014, p. 100) leads to the postulate that "the underlying brain activity is more fundamental" (p. 102). The next step in this line of reasoning is to state that "our brains make sense of the world by assembling neuronal information" (Tall, 2009, p. 482). The apex of this chain of psychological materialism is the search for the "number neuron" (Dehaene, 1997, p. 57). A criticism of this chain of thought may be found in Pais (2019) and Webel & Stigliano (2004), as well as in Baldino (2019), a parody about the number neuron.

[1] https://en.wikipedia.org/wiki/Scientific_realism

Dialectics vs understanding

By the time of Barrow, mathematics and philosophy had formed a single body of knowledge whose esthetic was modeled on Euclid. Newton followed this model, while Leibniz broke apart from it, and the two vied for being credited with the "discovery" of calculus. Berkely criticized them both. The story of this imbroglio is well known.

In the beginning of the nineteenth century, Hegel criticized the mathematicians for not been able to provide rational foundation for their science (1985, p. 236-309). By the end of the eighteenth century, Kant's philosophy had become prevalent. Hegel called it *understanding* (*Menschenverstand*) and referred to his own system as *speculative philosophy*. Hegel's went beyond Kant's philosophy. Understanding establishes clear cut distinctions between thought and being, subject and object, form and matter, discourse and referent, us and the world, etc. Due to these distinctions, understanding was prone to being well received in the imbroglio of philosophy and mathematics. Understanding was a reference for the movement for rigor that ran throughout the nineteenth century and is becoming stronger in STEM today. For instance, Kant's philosophy has been explicitly evoked to support the hegemony of mathematics over mathematics education; an "apprenticeship model" attributed to Dawkins and Weber (2017) is proposed by Rittberg et al. (2020) and criticized in Baldino and Cabral (2021).

Understanding postulates the existence of a beyond that cannot be reached by knowledge nor, consequently, by *language*: the so-called *thing in itself* is not cognoscible. This philosophical position favors the polysemy of the signifier "mathematics" discussed in Cabral and Baldino (2021): it favors inconsequent research in mathematics education, while the classroom resists. We argue that language must be brough to the fore. This endeavor naturally points to Lacan wo goes beyond Hegel and states plainly: "there is no being outside language" [2]. We cannot survey the meaning of the word "dialectics" from Plato to the present. We will retain the meaning it has in a formula that may condense the whole of Lacan's work: "dialectics of the subject and the Other" (Lacan, 1973, p. 205, 239). Accordingly, by "dialectics" we will refer to both, the dialectics of language, as in Lacan, and to Hegel's speculative philosophy which Žižek has elicited as the background of Lacan's work.

However, *dialectics is not a substitute for understanding*; this would be an assumption proper to understanding. *Dialectics exerts itself on understanding*, leading it to recognize the contradictions that unavoidably stem from its black-and-white divisions. Dialectics does not fight for victory; it is guided by the political necessity of the moment; it fights to continue fighting. If understanding dies, speculative philosophy dies too "and in this night of mere reflection and of the calculating intellect, in this night which is the noonday of life, commonsense and speculation can meet one another" (Hegel, 1977, p. 103).

The disentanglement of mathematics and philosophy is taking place today, under the arrogance of science over humanities. In his paper on quantum mechanics, Gauthier (2010, p, 2) aims to "make explicit the concept of probability in order to extract the mathematical content from its mystical

[2] " Il n'y a d'être que dans le langage." https://www.youtube.com/watch?v=njA-1a4N_iw

(philosophical) gangue [sic]". What is it that is being separated from its "philosophical gangue"? Strangely enough, this is left to philosophers to explain, since mathematicians do not say what mathematics is. Scott (2012) shows what happens when a physicist (Stephen Hawking) draws on his professional status to gather audience to speak about what he does not know. We must be aware of such antagonisms because understanding and dialectics face off when they meet the student in the common ground of the classroom, in the "noonday of life".

Cauchy and the bubble: *on dit que*

Concerned with the polysemy of the signifier, Cabral and Baldino (2021) characterize "mathematics" as special form of discourse that was born in Ancient Greece. This discourse, which they call *quilted speech*, aims to stop the slide of the signified under the signifier: each signifier would have a single precise meaning:

> The understanding would like "a fixity and an exactitude that is not found in existing language; the idea of creating a pure language, a system of symbols which remain absolutely invariant over the course of the diverse combinations they undergo comes from this. (Hippolyte, 1977, p. 46)

The whole imbroglio following Newton and Leibniz may be looked at as an attempt to quilt speeches; people refer to it as "rigor", "formalization", "definition of the concept" etc. In 1821, Cauchy introduced a precision into quilted speeches that eventually completed the separation of mathematics and philosophy. In his Cours d'analyse de l'École Royale Polytechnique, he wrote: "*One says that* a variable quantity *becomes* infinitely small when its numerical value decreases indefinitely so as to converge towards its limit zero" (Cauchy, 1821, p. 26, added emphasis).[3]

On the one hand, he cut the gordian knot of misunderstanding with a *semantic convention*: "one says that" (*on dit que*). However, on the other hand he conserved the movement of dialectics of language by saying "becomes" (*devient*).[4] Something ceases to be what it is and becomes something else. Hegel would appreciate this proposition as a perfectly dialectical one: its very utterance imposes a movement of language that preserves what it denies; this is expressed by the German verb "to transcend" (*aufheben*). Hegel (1985, p. 69) calls this movement of language "becoming" (*das Werden*).

Cauchy's semantic convention was followed by Weierstrass, Hilbert and many others. They created a language bubble that we call M20. This bubble became the habitat of a *community of speech* whose members are the mathematicians (Cabral & Baldino, 2021). This community decides what counts as a valid inference; for instance, it informally accepts the use of the axiom of choice and the continuous hypothesis.

The membrane that delimits the bubble is opaque for those inside it. Like in Plato's cave, mathematicians only see shadows of the external world. The infinite movement of the dialectics of

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[3] On dit qu'une quantité variable devient *infiniment petite*, lors que sa valeur numérique décroit indéfiniment de manière à converger vers la limite zéro.

[4] In Portuguese, *devenir*, to become, is a reflexive verb: *tornar-se*: to change oneself is a perfect expression for learning, a movement that occurs in language through which a subject changes herself. It is up to the native speakers to compare the relative weight of these verbs in their respective languages.

language is left outside, including important concepts, like “mathematics”, “number” and “infinity”, that find no “definition” inside the bubble. The “repetition of a process with an underlying pattern of successive states” (Tall, 2009, p. 483) is what Hegel calls the bad (*schlechten*) infinity. The passage from one member to the next in a sequence is an indefinite repetition of sameness, with no risk of producing changes either in the subject or in the object in the dialectics of language. Only this finitized infinity is admitted into the bubble; the apex of finitization of M20 is Weierstrass epsilontics.

When mathematicians face the necessity of teaching, they have an inkling that something is wrong. They do not realize that the student is outside the bubble, plunged into the infinite movement of language. They try to pull her in to teach her the concept by “defining” it. Looking at her shadow on the wall of the cave, they offer us this precious confirmation of our criticism: “Students’ mathematics is the mathematical reality they experience, which is wholly theirs and is unknowable to us in the same way dark matter is unknowable to us” (Thompson, 2019, p. 39).

It is transparent that Kant’s philosophy supports this ideology: the thing in itself is not cognoscible. Mathematicians are aware of the failure of their teaching methods. For instance, even after four semesters of calculus, students are not only unable to use the FTC, but they do not recognize its use when it is presented to them (Tompson, 1994, p. 256, ex. 7.10). Limited by the opacity of the membrane, mathematicians tend to explain this failure by the tautology of lacking: the difficulties with the FTC “stem from impoverished concepts of rate of change and from poorly developed and poorly coordinated images of functional covariation and multiplicatively-constructed quantities” (Thompson, 1994, p. 229). That is, the difficulty lies precisely in what was taught to these students.

Tall (2009, p. 484) refers to the “notion of a generic limit” to account for the difficulties of students with the limit of sequences and the concept of infinitesimal. However, as justification of this concept we only find the expedients: “a *natural human belief* that the limiting object is endowed with the same properties as the individual terms” and “infinitesimal concepts are *natural products* of human imagination” (p. 483, both emphases added). We finally collect the mathematicians’ recognition of the impossibility: “Like with dark matter, the best we can do is make models that fit observations and are consistent with other models” (Tompson, 2019, p. 39).

Cauchy and infinitesimals: *devient…le voisinage*

A full account of the polemics around Cauchy can be found in Katz and Katz, (2011) and Tall and Katz (2014); these articles include an impressive list of references only available in universities of the so-called First World. By requiring “mastery of the field” as an academic prerequisite for anyone who has something to say on the subject, one risks blocking out the new and foment cultural imperialism.

We read the second part of Cauchy’s “becomes”, together with this other excerpt: “Besides, one also says that the function $f(x)$ is, in *the neighborhood* of a *particular value* assigned to the variable *x*, a continuous function of this variable (…)” (Cauchy, 1821, p. 35, added emphasis).[5] The French

[5] On dit encore que la fonction *f*(*x*) est, dans le voisinage d’une valeur particulière attribué à la variable *x*, fonction continue de cette variable (…).

singular "*le*" stresses that a numerical "value", has only one "neighborhood". Consequently, when this value is zero, we may assume that this neighborhood consists of the "infinitely small variables" (*variables infiniment petites*, p. 65). These signifiers do not belong to the quilted speech of the bubble. We agree with Katz and Katz (2011, p. 426) that we should "jettison the automatic translation-to-limits" in reading Cauchy. Instead, we propose a reading-to-infinitesimals as in Sad et al. (2001). Cauchy is saying that a variable that converges to zero becomes an element of the *monad of zero*, o, a whole universe of infinitesimals that is incommensurable with ours.

From the dialectical perspective, Robinson is not a "consequence" of Cauchy. Marx's aphorism about the man and the primate must be evoked once more. Cauchy's discourse on infinitesimals has actually been quilted and finitized by Robinson. Only from this perspective can we look back and ask where this quilting came from and what its trajectory to the present has been. To the myriad things that have been said about Cauchy, we dare to add: the mature Cauchy of 1853 was *waiting for rigor to catch up to him* when he restated his 1821 theorem (Sad et al., 2001). This was the trajectory of the dialectics of language that led from Leibniz to Robinson.

The completion of $\mathbb{Q}$ with the monads stemming from his sequences was already an embryo in Cauchy's "intuition", with no need to invent classes of equivalence of such sequences to fill in the gaps of $\mathbb{Q}$. Indeed, consider sequences (a_k) of rational numbers that satisfy Cauchy's condition:

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists N \in \mathbb{N})(\forall m, n \in \mathbb{N})(m, n > N \rightarrow |a_n - a_m| < \varepsilon)$$

We can take adequate equivalence classes of such sequences and form what we now call the field of finite hyperrational numbers O. If we introduce the operations of addition and multiplication as well as the order relation elementwise among the monads, we get a complete ordered field (isomorphic to $\mathbb{R}$), namely, the quotient ring O/o (Stroyan & Luxemburg, 1976, p. 9, Katz & Katz, 2011, p. 448). We say that a single movement of the dialectics of language that is still taking place posits these signifiers under the form of an identity quilted speech inside the bubble and also *posits ourselves as subjects who utter these speeches.*

Consequences for mathematics education

The teaching experience that we initially intended to report consists in working on the COVID-19 graphs until the students are able to formally express and meaningfully utter that 1) the variation of the running total is the area under the daily deaths, 2) the moving average of the daily deaths is the variation rate of the running total and 3) the one-day-based moving average is the number of deaths of the day before. These, of course, are the statements of FTC. Next, we replaced the discrete graphs of a few days in the pandemic with continuous graphs of an endemic disease that lasted for many years (Ellis et al., 2020). We insisted upon the expression and formula for the one-day-based moving average. The whole discussion occurred in the realm of the *dialectics of language*; we only tried to collimate the students' discourses towards the signifiers in the bubble. For instance, only when they were trying to express one day in sixty years did we suggest *dt*, "a little bit of", according to Tompson (1914). For the areas under the graph, the students suggested $A(a, b)$; neither Leibniz notation for the integral nor Riemann sums were necessary.

We did not drag the students into the bubble to teach them how to interpret the M20 readymade symbols (Ely, 2017). We believe that the more one tries to do so, the greater the difficulty for the students. Asunder from the dialectics of language and modeling shadows of students with finitary processes, mathematicians can do no more than teach finite processes. Learning depends on the infinity of variables in the dialectics of language, insofar as the students start representing themselves as subjects in *current language* by new signifiers. In this respect, the possibility of open-camera, eye-to-eye communication and video recording was a blessing to our long-standing methodology (Cabral, Pais, & Baldino, 2019) based on the aphorism *one teaches by listening and learns by talking.*

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Exploring symmetries with first-year students: a gateway into mathematical structures

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Keywords: Mathematics education, mathematical structures, structural thinking, symmetry, university mathematics.

In the poster, I describe one part of my project of “structural thinking and its teaching” that focuses on the design and implementation of a seminar for first-year students at the Faculty of Mathematics and Physics. The aim of the seminar is to bridge the gap between secondary and university mathematics. This seminar is based on exploring symmetries of regular polygons and polyhedra, building on students’ previous knowledge and being a starting point to more abstract disciplines.

Mathematical structures constitute the core and simultaneously the image of modern mathematics (Corry, 2004). On the other hand, the attempt for the *modernisation of school mathematics*, promoted by the Bourbakist movement and culminating in the 1960s, ended rather unsuccessfully. This attempt focused on incorporating mathematical structures into secondary school mathematics via highly abstract university-style “definition-theorem-proof” led to formalism. Hence the gap (and so Klein’s *double discontinuity*) between “modern” university mathematics and “classical” school mathematics has remained wide open and still forms a barrier, especially for first-year university students of mathematics, including future teachers.

For this reason, the seminar at the Faculty of Mathematics and Physics was created aimed at trying to help students to accommodate different styles and also to inspire them for further studies. I have a repeated opportunity to make a few sessions with students where we are exploring the structure of symmetries of geometrical objects. My goal is to familiarize them with tools and concepts which they will meet in the following obligatory courses of linear algebra (in their first year) and abstract algebra with the basics of group theory (in their second year). I use a more constructivist and visual approach and relate new concepts to their current knowledge from secondary school.

Content

In this seminar, together with students, we first explore one particular, well understandable example: symmetries of a square. From secondary school, students are well acquainted with geometric transformations (from a synthetic viewpoint) and they can easily enumerate that square rests invariant under three rotations and four reflections. However, we realize that these transformations do not form only a set, but, as we can compose them – follow one by another –, they form some algebraic structure. Therefore, students are asked: How could we describe this structure, group of symmetries of a square?

Students’ first idea is usually to record it by a table of composition (*Cayley table*), as it is analogous to a well-known multiplication table. However, to deduce all combinations geometrically is exhausting. Moreover, students realize that some of the transformations do not commute. We would like to represent the symmetries in a way we can really *compute* with them – to represent them analytically. Here, I follow students' secondary school knowledge of analytical geometry and introduce linear transformations of a plane, represented by matrices. With a general form of a matrix of rotation and reflection, completion of the table is much easier. Another suitable representation is by permutation of vertices.

Then I raise a question: Are there any effective tools that enable us to understand such structure? In a similar manner, together with students we explore possible substructures and organize them as sets by inclusion – another already known concept for students – into partially ordered set (poset), more precisely *lattice of subgroups*. By this, students also recognise various properties of different subgroups: some are commutative, some are even cyclic, and we discuss how we could represent them by *additive and multiplicative groups of integers modulo n*. The next topic is the *presentation of a group* by generators and their relations; we visualize this with the *Cayley graph*.

The example of square symmetries is particularly suitable because we can illustrate enough interesting properties and also this group is small enough to make computations inspiring rather than tedious. In this elaborate example, I provide students with principal tools and concepts enabling them to grasp the structure of a given group. Subsequently, we can extend the scope and leave students to get familiarized with tools. Students have to describe the group of symmetries of an equilateral triangle and then generalize it on a dihedral group. After this, we move one dimension higher for symmetries of tetrahedron and cube and the most interested students can also explore the icosahedral group. Physical models of those regular polyhedra, a familiar topic from secondary school, are used to make the transformations more tangible. Later we can add more concepts (e.g., quotient groups) and also explore infinite groups: symmetries of friezes and wallpapers. Further connections to Galois theory can be added from Gray's textbook (2018).

Conclusion

The content rises from current students' secondary school knowledge: geometric transformations, analytical geometry, sets and their ordering by inclusion, regular polygons and polyhedra etc. It can be a starting point for more sophisticated concepts (matrix eigenvalues and eigenvectors, normality, group automorphisms) and abstract disciplines (not only linear algebra and group theory but also universal algebra, Lie groups etc.). This content is also used in a seminar for talented students at secondary school. I hope that these ideas may be inspiring for teachers both at the university and secondary school levels. Mathematical structures do not have to be taught as an axiomatically presented subject for itself but can be incorporated into secondary and university mathematics more implicitly, mainly as an effective tool for dealing with mathematical problems. In this way, they can also promote the development of students' general structural thinking.

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A replication study of university students' understanding of functions of two variables

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We used Action-Process-Object-Schema (APOS) theory to analyze the possibility of replication of results obtained in our previous research on student understanding of two-variable functions when using a similar teaching approach in a different institutional context. The experience was conducted at a university in a different country from those in previous studies. The experience consisted in comparing two groups of the same course, one taught through lectures and the other using collaborative work and activities designed with APOS theory. In this study, we show a summary of results obtained through a comparison of students' performance in both groups. Findings show the generalizability of results obtained in previous studies and the possible replication of didactic aspects across institutions. In particular, it was found that using APOS theory's didactical approach favors a deeper understanding of functions of two variables.

Keywords: Functions of two variables, APOS, reproducibility, understanding.

Introduction

Multivariable functions play an important role in many professional fields in modeling numerous phenomena which naturally depend on several variables. In this research, we discuss results obtained in a research study where the didactical approach developed by Martínez-Planell and Trigueros (2012, 2013, 2019) and Trigueros and Martínez-Planell (2010) through three cycles of research on the learning of two variable functions was used by a different teacher (the first author of this article) and in a different country. Our interest was to compare results obtained in a new context in terms of similarities with those obtained previously and critically analyze the possible similarities and differences obtained. Results of the first research cycle (Martínez-Planell & Trigueros, 2012; Trigueros & Martínez-Planell, 2010) stressed the importance of helping students construct an R^3 Schema including different subsets such as points and their movement in space, and fundamental planes and their intersection curves with surfaces (for a detailed description of an R^3 Schema see Martínez-Planell & Trigueros, 2019). The authors stressed the importance of conversions between different representations. They also underscored that generalization from the one-variable to the two-variable context is not easily done by students; the reconstruction of many basic ideas is needed. In their second research cycle, Martínez-Planell and Trigueros (2013) stressed the need to explicitly consider situations where the notion of free variable is needed to graph two-variable functions, cylinders, and make sense of other subsets of three-dimensional space. In the third research cycle, Martínez-Planell and Trigueros (2019) found that students in a section that had used their research-based activity sets and using the ACE cycle (activities, class discussion, and exercises) as didactical strategy outperformed students in a regular section (without the use of specially designed activities), and were more likely than students in the regular section to construct a conceptual understanding of function of two variables.

This study can be considered a replication study (Melhuish & Thanheiser, 2018) as it aims to confirm or refute previous results in the literature. “Replication study” has various possible meanings, none of which is generally accepted (Sanchez, 2020). In this article by “replication” we mean a test of a result of earlier research work that replicates most of the methodological features of the original study. Our study also satisfies conditions proposed by Star (2021), for a study to be a replication rather than a follow-up study, in that our study starts from results (rather than from a new idea) and it has a basic structure isomorphic to that of the original study, with methodological differences that do not alter that structure. Our replication study cannot be said to be either internal or external, in the sense that two of the researchers participated in the original study (internal) and one did not (external). The advantage of an internal replication is the knowledge of instruments, methods, and contributions of the original study, thus allowing a more faithful reproduction; the disadvantage is that it may be subject to some unconscious confirmation bias and that it might allow the incidence of “inside” knowledge that would make it difficult for others to conduct equivalent studies (Schoenfeld, 2018). Our study pairs internal and external elements and so has the experience advantage of internal replication, while the external member affords the needed independence to safeguard against confirmation bias and the incidence of “inside” knowledge. The merit of this type of study has been suggested by Melhuish and Thanheiser (2018), We consider that the inclusion of the external researcher helped in overcoming both the incidence of internal knowledge, with his independent teaching, and the subjective bias that the original authors may introduce in their results. Generalizability and confirmation bias (Schoenfeld, 2018) is further addressed by the negotiation of results in the independently obtained analyses. Mathematics education, as a field of study, strives to understand and describe findings but also to change and improve the way mathematics is taught. It is important to aim to go beyond basic research and propose and test research findings under the same conditions and also under new ones to describe the conditions, affordances, and constraints of the experience (Maass et al., 2019). Replication, as described above may validate research results, and help understand their possibilities and constrains under different conditions. Given the importance of multivariable calculus and the observed difficulties that students have learning this topic, it is important to study if pertinent research findings are applicable in different classrooms and to verify if this is the case in different institutional contexts. That is, it is important to question, could other researchers, in other types of institutions, with a different sample population, obtain comparable results?

Theoretical framework

In APOS (Arnon et al., 2014), an Action is a transformation of a mathematical Object that the individual perceives as external. An Action may be the rigid application of an explicitly available or memorized procedure. When an Action is repeated, and the individual reflects on the Action or on a chain of Actions, it might be interiorized into a Process. A Process is perceived as internal. The individual is able to justify the Process, to omit steps and anticipate results without explicitly performing the Process, and thus to generate dynamical imagery of the Process. When the individual is able to perceive a Process as an entity in itself, and is able to do or imagine doing Actions on it, the Process is encapsulated into an Object and new Actions can be performed on it to determine, for example, its properties. An Object can be de-encapsulated into the Process it came from when necessary. A Schema is a coherent collection of Actions, Processes, Objects, and other previously constructed Schemas having to do with a particular mathematical notion or topic. In APOS, mathematical knowledge with respect to a specific mathematical notion or topic, is defined as the general tendency of the students to perform Actions, Processes, Objects or Schemas in different problem situations related to the notion. Research in APOS typically starts by proposing a model, in terms of the structures and mechanisms of APOS, of how a student may construct a specific mathematical notion. This model is called a genetic decomposition (GD). It is used to design research instruments and didactic activities. After implementation of the GD-based activities, research is undertaken to find out what conjectured constructions students can do, which cause them difficulty, and what unconjectured constructions are done by students. Research results may suggest revising the GD and, consequently, the designed activities. The revised GD may then be tested through another research cycle. One may continue doing research cycles until the GD no longer needs revisions. At that moment the GD will model how students, in practice, do construct the mathematical notion of interest. The research questions in this study are: How do students' constructions when using research-based activity sets and the ACE cycle compare to those of students in a lecture-based section not using the activities? How do students' constructions compare with those obtained in the previous study?

Methodology

Two groups of an Iranian university participated in this study, one which will be called the APOS group, and the other the regular group. The APOS group worked collaboratively with GD-based activities designed for the third cycle of the Martínez-Planell and Trigueros (2019) study. Students in the regular section were taught mainly through lectures. Both groups used the same standard textbook (Stewart, 2012) and followed a very similar course syllabus (chapters 12 to 16 of Stewart), including the same assigned homework exercises. The main difference was the use of the additional GD-based activities (Martínez-Planell & Trigueros, 2017) for the APOS group and the teaching methodology. Eleven students from each group were chosen to be interviewed so that in each group they represented the spectrum from above average to below average students as determined by their one-variable calculus course grade. The participating students were chosen so that those course grades were as similar as possible. Both groups (APOS and regular) had the same professor in their previous one-variable calculus course; the professor that taught the multivariable calculus course to the regular group of the present study. The professor of multivariable calculus for the APOS group was one of the authors of this article. All of the interviews were conducted by the instructor of the APOS group. Each interview lasted about 1 h. Interviews were conducted in person (not online), recorded, transcribed, and translated. The transcripts were individually analyzed by the researchers and differences in opinion were negotiated. Students' response to the interview questions were graded for their mathematical correctness; this was used to identify general patterns. The instrument involved questions related to constructions of: 1) fundamental planes (planes of the form $x = c$, $y = c$, $z = c$, for c constant) and their intersections with surfaces, 2) free variables (i.e., variables that can take any value without affecting the values of the other variables, like y in $f(x,y) = x^2$, $z = x^2$, or in the intersection of the plane $x = 0$ with the surface $z = xsin(y)$), 3) graphing two-variable functions, and 4) domain and range.

Results

We compared the performance of students in the APOS and regular sections. Our comparison showed that the total percentage of correct answers in the graded interview questions obtained in the APOS section (65%) was more than twice that of the regular section (25%). Table 1 compares the average scores of students in the APOS and regular sections in problems dealing with the intersection of a surface with a fundamental plane, the notion of free variable, domain and range, and graphing. It also compares the results of this study with those of students of the original article (Martínez-Planell & Trigueros, 2019). The table suggests that the GD proposed by Martínez-Planell and Trigueros (2019) and the activity sets, designed to foster students' constructions described in the GD, seem to help students construct a deeper knowledge of basic and geometric notions of functions of two variables.

Table 1: Comparison of APOS and regular sections in the original and reproducibility studies, by problem categories (total points obtained/total possible points)

Problems dealing with:	% APOS section reproducibility study	% Regular section reproducibility study	% APOS section original study	% Regular section original study
Fundamental planes	70	15	85	25
Free variable	67	18	58	17
Domain & Range	64	45	72	21
Graphing	60	20	84	23
% entire instrument	65	25	77	34

Fundamental planes and their intersections

Eight of the eleven students in the APOS section showed to have constructed fundamental plane as a Process or were in transition to doing so. Student A6 demonstrated the construction of fundamental plane as a Process. When drawing in 3D space the collection of points in that satisfy the equation $y = 2$ and that are also in the graph of the function $g(x,y) = x^2 + x^3(y-2) + y^2$:

Student A6: I only need to know the graph of $g(x,y)$ when $y = 2$, umm the $y = 2$ is a plane. I consider $g(x,y)$ as z. I have to substitute $y = 2$ into $g(x,y)$ and x can be everything. So z will be umm $x^2 + 4$, its graph is something like a parabola which is placed on the plane $y = 2$. When x is 0 then z will be 4 so the minimum height of the parabola is 4 umm at the point $(0,2,4)$ [Figure 1].

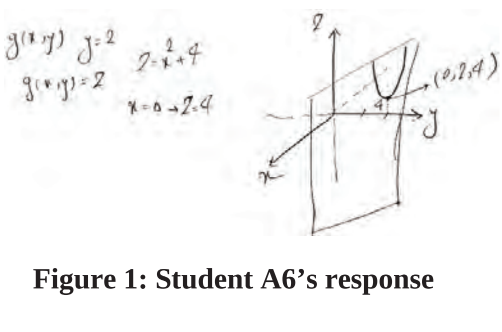

Figure 1: Student A6's response

Student A6's overall behavior gave evidence of construction of the Process of fundamental plane. He gave evidence of relating algebraic and graphical representations of fundamental plane, including its placement in space. Moreover, A6 also demonstrated to have encapsulated this Process into an Object by performing Actions upon it, in order to obtain the resulting curve and to place it in its appropriate place in space. In comparison to the APOS section's students, only three of the eleven students in the regular section showed the construction of fundamental plane as a Process or showed to be in transition to doing such a construction. To exemplify the understanding of fundamental plane constructed by most students in the regular section, we consider R4's response to the same problem:

Student R4: I know $y = 2$ is a line in 2D and it's a plane in 3D.

Interviewer: Okay, find what the question asks for.

Student R4: I can't draw $g(x, y) = x^2 + x^3(y - 2) + y^2$ in 3D.

Interviewer: Is it necessary to draw it?

Student R4: For solving this question yes, I need, so I can't solve this question.

Student R4's response showed that although he was aware that $y = 2$ is a plane, he believed that he needed to draw the surface in order to represent its intersection. He seemed not to be aware of the geometrical meaning of substituting a number for a variable in an equation; he showed a rigid understanding of its being a plane with algebraic representation as a variable equal to a constant but he was not able to use this information to do the needed Actions to find the intersection with the surface. R4's responses throughout the interview showed that R4's understanding of fundamental plane can be considered as consistent with an Action conception.

Free variables

We found that six students in the APOS section could construct a Process of free variable, meaning that they coordinated the Processes involved in relating the algebraic context of an equation (some with unnamed variables) and its solution set, with the verbally or symbolically given geometric context in which the equation and its solution set were to be interpreted. Two more students in the APOS section evidenced to be in transition to constructing such coordination. In contrast, two students in the regular section showed they had constructed or were in transition to constructing that coordination. We consider A5's response to one of the questions related to free variables. In the question, students were asked to draw the intersection of $S = \{(x, y, z): x^2 + 2x + y^2 = 3\}$ with the *x* axis:

Student A5: On the x-axis, we have $y = 0$ and $z = 0$, so I have to put 0 for y, I should solve the equation $x^2 + 2x = 3$, the values of x are 1 and -3, therefore the answer will be two points, (1,0,0) and $(-3,0,0)$.

A5 was able to interrelate the given equations, the unmentioned variable *z*, and the context of the *x* axis he was asked to presume. By contrast, only two students in the regular section (compared to eight in the APOS section) gave evidence of relating the given equation, unnamed variable, and presumed context for solving problems related to free variable, or were in transition to constructing such relations. There were seven students in this section who were not able to correctly or even partially solve any of these problems. R3 is an example of such a student. In the same problem as above:

Student R3: For $z = 0$ we have a circle umm it's $(x-1)^2 + y^2 = 2$, [sic] umm a circle with radius 2... Since we are in 3D, I think we have a sphere with center $x = 1, y = 0$, and $z = 0$. So, the intersection with the x axis will be a segment of the x axis.

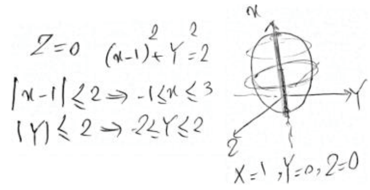

Figure 2: Student R3's work

Note, not taking into account her algebraic mistakes, that R3 attempted to graph the surface S rather than interpret algebraically the required context of the x axis, and she set the missing variable equal to zero ("for $z = 0$"), without making reference to the x axis, rather than treat it as a free variable.

Graphing

In general, the graphical representation of functions of two variables posed difficulties for students in all the sections. However, students in the APOS section were more likely to exhibit behavior consistent with a Process conception of function graphing than students in the regular sections, who seemed to rely more frequently on memorization or who showed not to have yet constructed a relation between fundamental planes and graphs. We consider A4 as an example of a student in the APOS section who evidenced construction of a graphing Process. When graphing $f(x, y) = x^2$:

Student A4: If my parabola $z = x^2$ that I drew it in the previous part moves in the y direction then I can imagine the graph $z = x^2$ in 3D, a surface umm it's like this [See Figure 3, left].

In her response, A4 showed to relate fundamental planes to graphing ("if my parabola $z = x^2$ that I drew it in the previous part") and gave evidence of dynamical imagery ("… moves in the y direction"). Showing connections between different representations and generating dynamical imagery is consistent with a Process conception. Eight of the eleven students in the regular section could do none or only one of the five problems dealing with the graphical representation of a function. We considered R3 as an example of these students. When graphing $f(x, y) = x^2$, R3 interpreted it as $y = x^2$ and directly generalized from the 2D to the 3D context to obtain a paraboloid (Figure 3, right). Moreover, she was not be able to justify, which is consistent with performing Actions.

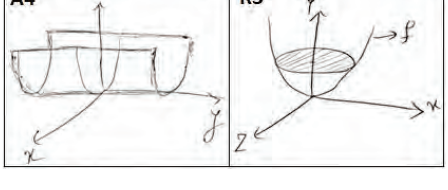

Figure 3: A4's work (left) and R3's work (right) in drawing $f(x, y) = x^2$

Domain and range

We will focus on two of the interview problems dealing with domain, where students had to represent the domain of $f(x, y) = x^2 + y^2$ restricted to the pairs that satisfy $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$ as a subset of 3D space, and also had to find the domain of $g(x, y) = x^2$.

Student A4: It explicitly tells us that the domain of $f(x, y) = x^2 + y^2$ umm is all the pairs (x, y) such that $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. Its figure I mean the domain in 3D is like this square in the xy plane [Figure 4, left].

Student A4: [in the next problem] The domain of function $g(x, y) = x^2$ is R^2 because I can put all the points (x, y) of the xy plane into $g(x, y) = x^2$.

The above response shows that A4 reconstructed her notion of function domain to deal with the new situation of functions of two variables. Domain elements are now ordered pairs of real numbers rather than real numbers. Further, she relates her verbal set-theoretic description to the graphical representation of domain in 3D. Now consider the case of student R4.

Student R4: The domain of f is umm set of points $(x, y, 0)$ such that $-1 \leq x \leq 1, -1 \leq y \leq 1$, and x, y belong to R^3. The domain is this part on the x axis and this part on the y axis [Figure 4, right].

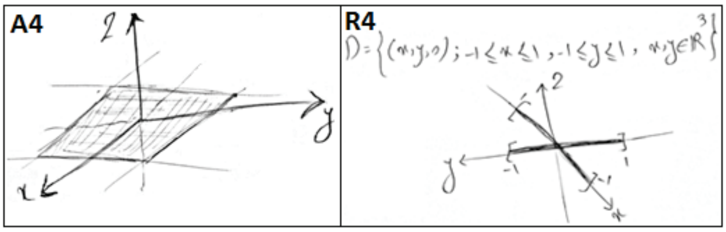

Figure 4. A4's work (left) and R4's work (right) in representing the domain

Student R4's drawing shows that he attempted to directly generalize his notion of one-variable function domain as an interval of real numbers to deal with the new two-variable function context. He did not show the reconstruction of domain as a set of ordered pairs.

Discussion and conclusions

In this study, as in the original, considerable differences in the constructions made by students in the APOS and regular sections were found. Observation shows that in both, the original and new studies, students who were taught using APOS theory's didactical approach and using the activities designed with a validated genetic decomposition showed the constructions of the expected Actions and some of them showed the construction of Processes demonstrating a deeper learning of topics related to functions of two variables. Discussing replicability of studies in the context of mathematics education research is difficult. Mathematics education is a social phenomenon and, as such, is also complex. We limit our attention to study if the use of a specific didactical approach based on a cognitive theoretical approach to teach a specific mathematics topic, functions of two variables in this case, results in similar learning in two different institutions. It is very interesting to observe that the mental constructions observed in students in the APOS and regular sections in both institutions seem, for the most part, to be independent of the country and the institution where they were studied. So, our results show that there may be cognitive factors that are somehow independent of social, cultural and institutional differences. Results show that the use of activities designed in terms of the constructions described in a validated GD, in this case for the teaching of two-variable functions, are useful in promoting a deeper learning of this topic in two different contexts when the teaching approach follows the ACE cycle. This may ratify the strength of the GD as a design model and also how research-based activities using this prediction model and collaborative work can result in students' learning with some independence of the teacher, the institution and the country where they are used.

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University students learning programming-based practices for mathematical inquiry: Contributions of an institutional approach

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In mathematics education research, the use of different theoretical lenses can lead to a deeper understanding of phenomena such as the teaching and learning of programming for mathematical inquiry at university. In light of our past work leveraging the instrumental approach, this paper seeks to explore the potential contributions of a different lens: the Anthropological Theory of the Didactic (ATD). Through a comparative analysis of "practices to be learned" in a mathematical inquiry project and "practices actually learned" by one student, we demonstrate the usefulness of the ATD's notion of praxeology. The complementarity of the analysis with our past work calls for further reflection on the networking of institutional and instrumental approaches.

Keywords: Institutional approach, praxeology, mathematical inquiry, computer programming.

Introduction

Researchers working on the networking of theories in mathematics education claim that using different theoretical lenses can lead to a deeper and more complex understanding of a phenomenon of interest (Bikner-Ahsbahs & Prediger, 2014). In our work, we are interested in the teaching and learning of programming for conducting mathematical inquiry at the university level. So far, our research team has utilized the instrumental approach and demonstrated the usefulness of several of its tools: e.g., the notion of scheme for understanding individual students' learning over time (e.g., Buteau et al., 2019) and the notion of instrumental orchestration for exploring how instructors create a learning environment to support students' learning (e.g., Buteau et al., 2020). In this paper, our aim is to see what additional understandings could be gained about our phenomenon of interest when using a different theoretical lens: namely, the Anthropological Theory of the Didactic (ATD). More specifically, the research question guiding our work is: How can the theoretical tools offered by the ATD contribute to our understanding of students learning to use programming for mathematical inquiry at university?

The selection of the ATD was inspired in part by the fact that it has been used by the first author in several past works (e.g., Broley et al., 2018, which reports on mathematicians' use of programming in research and teaching). The ATD is also becoming increasingly used by researchers of university mathematics education; however, as far as we know, it has not yet been used in research about students learning to use programming for mathematical inquiry. Moreover, overviews of the use of different theoretical lenses to investigate university mathematics education have highlighted the potential complementarity of the ATD and the instrumental approach (e.g., Gueudet et al., 2014; Winsløw et al., 2014). As such, we see our work as providing both an empirical and a theoretical contribution. In particular, this paper is a natural first step towards a deeper reflection on the networking of the ATD and the instrumental approach, for the purposes of better understanding how university students learn to use programming for mathematical inquiry.

Theoretical framework

The ATD was initiated by Yves Chevallard in the 1980s and has since grown to include a significant collection of theoretical tools for investigating mathematical and didactic activities. In this paper – the beginning of our work on the question posed above – we will consider only a subset of these tools.

Central to the collection of tools offered by the ATD is the notion of *praxeology* (Chevallard, 1999), which provides a way of describing the practices (i.e., regularized and purposeful human actions) that are involved in any human activity. According to the notion, every practice is composed of four interrelated, essential components: a type of task (generating the need for the practice), techniques (ways of doing the types of task), technologies (discourses producing, justifying and explaining the techniques) and theories (the rational discourses underlying the technologies). Hence, practices necessarily comprise both a practical part (the types of tasks and techniques, called the *praxis*) and a theoretical part (the technologies and theories, called the *logos*).

Another critical tool in the ATD is the notion of *institution* (Chevallard, 1991): a relatively stable structural element of a society that frames and promotes certain kinds of human actions (towards the achievement of certain aims). Indeed, a fundamental idea behind the ATD is that the praxeologies of an individual (e.g., a student) do not exist in a vacuum, but are shaped by the social institutions (e.g., universities) where they are developed. Artigue (2016) explains that with the ATD:

> The lens is no longer directed towards the student and her cognitive functioning or development, but towards the institutional practices that condition and constrain, both explicitly and implicitly, what she has the possibility to learn or not. (p. 17)

Bosch and Gascón (2014) add: “an ATD analysis therefore starts by approaching *institutional praxeologies* and then referring individual behavior to them, talking in terms of the ‘praxeological equipment’ of a given person” (p. 69).

A third critical tool in the ATD is the notion of *didactic transposition* (Chevallard, 1991), which highlights the institutional relativity of praxeologies with respect to three institutions that are pertinent to thinking about university mathematics education: professional communities (which produce and use “professional practices”), an education system (which, through programs, curricula, course outlines, textbooks, etc., determines “practices to be taught”) and a classroom (where interactions between teachers and students determine “practices to be learned” – e.g., through the assessments given in the course, and “practices actually taught and learned”).

In this paper, we propose to use the notion of praxeology to describe and analyze practices that are involved in the activity of “using programming for conducing mathematical inquiry”. Since we are interested in “students learning” to engage in this activity, we start by working at the level of a classroom institution, embedded in a particular education system (described in the next section). Following an institutional approach, we start by examining institutional praxeologies (in particular, practices to be learned, inferred from assignment guidelines and anticipated solution approaches), which may shape the practices that students have the possibility to learn. We then use this as a base to investigate the potential praxeological equipment of (or practices actually learned by) students.

Context and methods

The study we present is part of a larger 5-year (2017-21) iterative design, non-interventional research that uses as a context three programming-based math courses at Brock University: Mathematics Integrated with Computers and Applications (MICA) I, II and III. In these courses, students design, program and use computer environments to investigate mathematical concepts, conjectures and real-world applications (Buteau & Muller, 2010) – i.e., to engage in mathematical inquiry.

We focus on MICA II, the second course in the sequence. Certain features of the education system (Brock's MICA concentration) ensure that the structure and operation of MICA II is relatively stable. With MICA I as a prerequisite, MICA II students are expected to have developed some practices for using a programming language to solve some foundational types of tasks (e.g., produce the graph of a given function); in MICA II, the aim is for students to use and build on these practices to engage in mathematics inquiry projects. Course outlines specify the evaluations: 4 mini-projects (worth 12% each), 1 final project (worth 22%) and 2 midterms (worth 15% each). Each week during a semester, students participate in 2-hour lectures (where the professor mainly introduces math content related to the projects) and 2-hour labs (where the students can work on their projects among their peers and with access to teaching assistants and/or the professor). In a certain semester, the professor giving the MICA II course determines the topics of the projects and hence the particular practices to be learned.

The current study used data collected from one MICA II classroom, when the course was given in 2019. The data is of two types: (1) guidelines for the 4 mini-projects and the final project (i.e., 5 assignments), which can be found in Ralph (2020); and (2) semi-structured interviews that were conducted with volunteer MICA II students shortly after they completed each of their assignments, which aimed to guide the students in reliving and describing their actions. In alignment with these two data types, our study proceeded in two stages. First, we constructed a reference epistemological model (Bosch & Gascón, 2014) of practices to be learned in the first MICA II assignment: i.e., practices that students may (be expected to) learn when engaging in the assignment. We modelled types of tasks and techniques by looking at the formulation of the assignment questions, considering the kinds of objects involved, and thinking about anticipated solution approaches of students, based on our understanding of the MICA courses (as researchers, instructors, and/or past students) and the mathematics involved in the assignment. We modelled technologies and theories by thinking about mathematical justifications for the modelled techniques. Note that our model does not necessarily reflect the exact material presented in lectures (we did not have access to that data for the current study) or the intentions of the professor (they were not interviewed for our study). Moreover, we do not claim that our model is absolute or comprehensive: It contains elements that helped us as researchers begin to explore what students may learn when engaging in the assignment. Second, we explored the praxeological equipment exhibited by one MICA II student, Mark, during the interview that followed assignment 1. To accomplish this, Mark's interview was coded to find evidence of his perceptions in relation to the different components of our reference model. These perceptions were recorded in "praxeology tables", with evidence sorted in rows according to whether it corresponded to types of tasks, techniques or technologies (as is typical of analyses of students' praxeologies, we did not find evidence specific to the level of theory). It is important to note that the interviews were not designed for the purpose of probing into Mark's perceptions in relation to the praxeologies in our reference model; and yet, Mark spontaneously provided evidence of his perceptions. This is part of the reason he was selected for the current study, to allow us to explore potential new directions for our work in terms of theoretical lenses and data analysis. It is also important to note that Mark may not be representative of all MICA students. He is a computer science and mathematics co-major who, unlike many of his peers, had significant programming experience prior to the MICA I course.

Results

A reference epistemological model of the practices to be learned in assignment 1

In this section, we present our reference model of the practices to be learned in assignment 1, using the notion of praxeology. Assignment 1 contains 5 questions (Q1, Q2, Q3, Q4, Q5), which invite students to work on and explore the scope of a statistical computational technique ("Monte Carlo") for "estimating numerical values" (we note that this is a *genre* of task, which is more general than a type of task; Chevallard, 1999). All students are required to do Q1-3 and then can choose either Q4 or Q5. For the complete assignment, including complete question statements, see Ralph (2020). In this paper, we focus on Q1, Q3 and Q5, due to space constraints, and since these were the ones for which Mark presented explicit evidence of his related praxeological equipment in his interview.

In lectures prior to assignment 1, students are introduced to the Buffon needle problem, including an analytical solution (a derivation of a formula) and a computational solution (the writing of a code) for the task: find/estimate the probability that a needle touches a line if it has length $l = 1$ and it is dropped onto a plane of parallel lines that are $d = 1$ unit apart. Crucial to these solutions is an initial modelling of the situation, which transforms the original task into a new one: find/estimate the area in $[0,\pi]$ x $[0,0.5]$ such that $y \leq (1/2)\sin(x)$. In Q1, students modify the code (and model) given in class to create a program (Figure 1) to "find" the probability if the length of the needle is changed to 0.5. Generally speaking, the task solved in Q1 (like the task solved in class) belongs to the type of task T_1: Estimate the probability of an event in a random experiment. However, the modelling of the random experiment leads to a task of another type: T_2, Estimate the volume of a bounded k-dimensional subset A. In Q3, students create a program to solve another task of this type: i.e., estimate the hypervolume of the unit hypersphere in R^4 ($k = 4$, $A = \{(x,y,z,w)|\ x^2+y^2+z^2+w^2 \leq 1\}$).

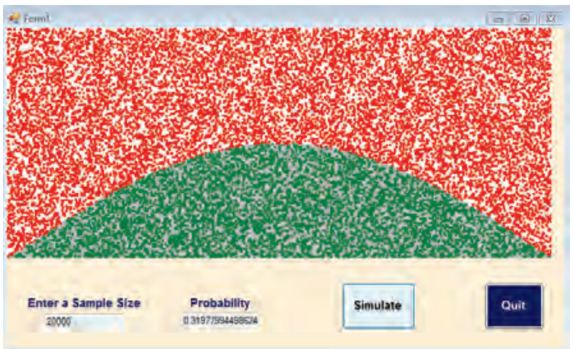

Figure 1: A student's program for solving Q1

We could model the Monte Carlo technique that can be used to solve T_2 as: τ_2, put A in a set B whose volume is known (e.g., B is the hypercube in Q3), choose n points at random in B (with n sufficiently large), keep track of the number of points m that are in A and calculate m/n*Area(B) (the estimate). One main technology underlying τ_2 is: θ_2, as the number of points increases, the estimation becomes more accurate. This is a particular instance of the "law of large numbers", which is supported by the theory: Θ, Probability and Statistics.

For MICA II students, a computational version of τ_2 is implemented in the vb.net programming language typically through modifying the original code given in class: studying the code and making relevant modifications may support students in abstracting the technique. In the tasks faced by students, B is provided (e.g., Q3 specifies to carry out the estimation by "choosing n points at random inside $[-1,1]^4$"). Moreover, the program students create enables them to vary n until it is judged to be sufficiently large to produce an accurate estimate; and in Q1 and Q3, this judgement can be made by comparing the output (an estimation) with a *known* actual value. Such a comparison may support students in developing θ_2 (though it would not be possible when trying to estimate an *unknown* value).

An additional part of Q3 introduces another way to judge the accuracy of an estimate. Students are invited to estimate the hypervolume accurate to one decimal place, which leads to a modification of $[T_2, \tau_2, \theta_2]$. To estimate the volume to a certain accuracy (T_3), students build on τ_2 to implement a new technique, τ_3: they create a program that carries out the estimation process described by τ_2 a specified number of times (w) and calculates the mean and standard deviation of the estimates (the mean is the new potential estimate), and then they vary n and w (increasing them) until a sufficiently small standard deviation is obtained (e.g., taking into consideration that 99.7% of all means would lie within three steps of the standard deviation from the mean). In this case, one main technology underlying the technique is: θ_3, a specified version of the "central limit theorem", which ensures an approximate normal distribution for a large number of independent, identically distributed variables.

Finally, Q5 introduces students to another praxeology based on a widened theory, including Calculus and Analysis. In particular, Q5 asks students to: suppose that two numbers a and b are chosen at random from $\{1, 2, \ldots, n\}$; let P_n be the probability that a and b are relatively prime; and answer: As n goes to infinity, does the limit of P_n exist? Can you guess the exact limit? The type of task explored here could be modelled as: T_4, find the limit of a sequence $(P_n)_n$, where the sequence values are the probability of an event of a random experiment. The technique τ_4 has two parts: students go back to engaging in a task of type T_1 to estimate a sequence value P_n for a given n using Monte Carlo techniques; then they vary the value of n, for larger and larger values, to see if P_n appears to approach a certain value. Underlying this technique are technologies, θ_4, related to the estimation of limits: e.g., if the values seem to get closer to a specific value as n increases, this may be the limit of the sequence.

Mark's praxeological equipment related to assignment 1

In this section, we use perceptions shared by Mark when describing his actions in completing assignment 1 to think about his praxeological equipment with respect to the reference model outlined above. Tables 1, 2 and 3 provide some selected quotes from Mark's interviews, which serve to exemplify his perceptions in relation to the three main praxeologies in our model.

Although there are some imprecisions in Mark's descriptions, they suggest a praxeological equipment that reflects well the techniques and technologies in our reference model. Consider, for example, the way Mark describes the technique for estimating the volume of the hypersphere (Table 1): he explains the existence of two embedded spaces, the dropping of random points and the required check for points that hit (or miss) the smaller space. He also justifies the technique by referring to the "large amount of points" he dropped, suggesting an awareness that the accuracy improves as the number of points increases. In a similar vein, Mark's perceptions of the practice for estimating the volume to a certain accuracy (Table 2) includes some key elements of the technique (the replication of the process of dropping points and the aim of "a really small" standard deviation) and the technology (the idea that the desired accuracy should be maintained several standard deviations from the mean since “over 99%” of means lie in that range). Finally, Mark’s perceptions of finding the limit in Q5 (Table 3) reflect a technique using “trial and error of inputs” with bigger and bigger numbers, supported by a technology of the sort: if the outputs get “closer” to a value, the limit exists (and that’s the limit).

Table 1: Mark’s perceptions in relation to [T_2, τ_2, θ_2]

<table>
<tr><td>T_2</td><td rowspan="2">“the code was similar to question one in the fact that you have um, a space, uh contained within a, a larger space, and you're dropping uh, random points and you're … simply checking uh if you're hitting or missing this smaller space, uh to estimate the volume of it”</td></tr>
<tr><td>τ_2</td></tr>
<tr><td>θ_2</td><td>“because I was able to … run a simulation where a large amount of points … are dropped, uh that gets a fairly high accuracy obviously because there's so many data points”
“because it's on computers you're able to do it so many times”</td></tr>
</table>

Table 2: Mark’s perceptions in relation to [T_3, τ_3, θ_3]

<table>
<tr><td>T_3</td><td>“once the program was done I had to switch over into math mode to … prove that the 4.9 was accurate”</td></tr>
<tr><td>τ_3</td><td rowspan="2">“I was able to run a simulation where a large amount of points … are dropped … and then that's also replicated … I was able to do I believe almost 100,000 or so points, um, nearly 200 times. Uh so because of this you're able to get a, um, fairly accurate mean value of uh 4.9 um, … the main thing was that you're able to get a really small standard deviation of 0.002 or whatever, um, so even three or four standard deviation points off you're still accurate to that 4.9 decimal place which um, obviously means that over 99% of your trials show up that it's going to be that 4.9 accuracy”</td></tr>
<tr><td>θ_3</td></tr>
</table>

Table 3: Mark’s perceptions in relation to [T_4, τ_4, θ_4]

<table>
<tr><td>T_4</td><td>“you're looking for basically a probability as Pn approaches infinity”</td></tr>
<tr><td>τ_4</td><td rowspan="2">“what I ended up … doing was … there's just a small table that kind of shows … the outcomes that um I did so that way I could show um as the numbers got bigger it did get closer to that probability.”
“it was more so … trial and error of inputs to uh, try and prove that uh, that the limit did exist, so um, it was uh trying to find a match between incredibly large numbers that you know, can quote ‘prove infinity’, uh versus finding numbers that would make you sit there through the entire lab waiting for an output”</td></tr>
<tr><td>θ_4</td></tr>
</table>

Interestingly, Mark does not refer to specific mathematical theoretical elements such as the “law of large numbers” or “central limit theorem”; in fact, he indicates that some technologies are “obvious” (Tables 1 and 2), which could be indicative of a “non-mathematical” theory (i.e., an explanation of a technology that is not based on relevant mathematical properties). Also, Mark’s perceptions highlight other kinds of technologies – specific to computational techniques – that were not included in our reference model and that relate to both the affordances of computers (e.g., they allow you to repeat mathematical processes many times; Table 1) and the constraints (e.g., even computers have a limit as to how many times they can repeat certain mathematical processes; Table 3).

Towards the end of his interview, Mark gives his view of the key idea behind assignment 1:

Mark: I would say this one the key kind of concept was finding exact, um, exact values through estimation … it was definitely a cool thing so, when you're building your program, and you're like "Alright I'm just going to throw in some really big numbers and we're going to get really close to this number that's mathematically correct".

Mark seems to perceive a genre of task that differs slightly from the one in our reference model, emphasizing the estimation of *known exact* numerical values (or numbers that are "mathematically correct"). In each of Q1, Q3 and Q5, Mark found an exact answer (e.g., using Google) to compare with the estimate produced by his computer program. It would be interesting to see how his praxeological equipment might differ when the numerical value being estimated is unknown.

Conclusions

In this paper, we sought to explore how the theoretical tools offered by the ATD can contribute to our understanding of university students learning to use programming for mathematical inquiry. We claim that our analysis demonstrates the potential usefulness of the notion of praxeology: e.g., it allows us to describe and reflect on specific mathematical practices students may have the possibility to learn while (and for) engaging in programming-based mathematical inquiry projects, and to investigate the degree to which certain students learn those practices or not.

We note that the praxeologies presented in this paper are not specific to a particular programming environment. Although we see the development of such general praxeologies as a pertinent aim of teaching students to use programming for mathematical inquiry, the question remains as to how the specificities of an environment could shape students' praxeological equipment. One limitation of our study is that it used existing data from interviews that were not framed by the ATD. Further investigation into students' praxeological equipment would require a revision of existing research tools and a reflection on the usefulness of other data sources (e.g., students' responses to midterms).

In relation to our past work using the instrumental approach, we see the analysis presented in this paper as complementary. For instance, using the notion of instrumented action schemes, we have so far focused on modeling operational knowledge that is developed and used across MICA inquiry projects, primarily when students program a computer environment for the purposes of their inquiry (e.g., Buteau et al., 2019). In comparison, using the notion of institutionalized praxeologies, we were brought to focus on modeling mathematical knowledge that is developed and used in particular MICA projects, primarily at stages outside of the programming: e.g., when students use their programmed computer environment to conduct the inquiry. The current study seems to open a window into other key parts of students' learning. Such complementarity warrants further reflection not only on the potential contributions of the ATD, but also on its networking with the instrumental approach.

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STACKification: automating assessments in tertiary mathematics

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In this paper, we report on four university lecturers' first-time experiences with computer-aided assessments. They were required to automate a significant proportion of the pre-existing weekly coursework for modules in first- or second-year undergraduate mathematics using STACK. We consider lecturers' perspectives on the role of computer-aided assessments in course design for undergraduate mathematics; the knowledge of technical aspects required to implement STACK-based assessments; and the perceived merits of automated assessment for different aspects of mathematical study. We conclude with a series of reflections upon our departmental practice and the process of enculturating mathematicians into the realm of automated assessment.

Keywords: Automated assessment, instructional design, mathematics coursework, thematic analysis.

Introduction.

In this paper, we focus on the introduction of STACK (a **S**ystem for **T**eaching and **A**ssessment using a **C**omputer algebra **K**ernal) to a Russell Group University in London. In particular, we set out to study a department-wide initiative where lecturers are expected to implement the majority of coursework using STACK. Students' weekly submission of handwritten solutions to problem sheets transitioned to the use of the STACK online environment which automatically assesses their answers and provides feedback.

The COVID-19 pandemic has dramatically increased the urgency and extent to which tertiary education has transitioned online. However, we understand this to be an acceleration of changes already underway in many parts of the tertiary sector. While we position our research as having general applications independent of the global health circumstances, we must acknowledge the environment in which this data was collected. Computer-Aided Assessment (CAA) has been on the agenda for the department from which we report for several years. However, the immediacy of the transition away from traditional handwritten assessments is, in large part, the result of the urgent need for remote, contactless instruction.

Given the urgency with which lecturers were required to automate their assessments, the default workflow for the majority of modules focused on the 'translation' or 'STACKification' of existing materials into CAAs. Some scholars may argue that this workflow is inherently flawed, and that effective CAAs should be generated in isolation, free from the restrictions of human graders (Sangwin, 2013). In the interests of space, we prefer to acknowledge the pragmatism of STACKification, and conjecture that many others using STACK for the first time are likely to follow a similar workflow. The process of STACKification warrants structured investigation, independent of scholarly arguments regarding the optimal origins of CAAs.

We report on semi-structured interviews with four lecturers and two postgraduate students, employed to support the design and implementation of STACK-based assessment across the department. All participants have been involved with the project for less than one year, and none had any prior experience with STACK (or any other CAA) prior to the project. The first author of this paper is also in the department and is responsible for co-leading the development of STACK-based resources.

Assessment in tertiary mathematics, and the increasing role of CAA.

Despite decades of innovation in assessment methods and tools, closed-book written examinations continue to dominate assessments for tertiary mathematics (Iannone & Simpson, 2011). Recent decades have seen an increase in the variety of assessment methods available to practitioners, but many of these innovations have struggled to gain popularity beyond the researcher communities in which they are developed. Researchers have highlighted the value of low-stakes formative assessments (Black & Wiliam, 2010), and called for greater assessment variety across undergraduate degrees. In this paper, we focus on Computer-Aided Assessment (CAA) and its role in a balanced 'assessment diet' (Iannnone and Simpson, 2011) alongside other modes including written and oral modes.

The last decade has seen significant growth in the availability of CAA technologies, from which STACK has emerged as a major player in the assessment of tertiary mathematics (Fahlgren et al., 2021). STACK uses a computer algebra system to evaluate students' responses against a wide array of mathematical properties. Unlike many of its predecessors that invoke little more than string matching or numerical equivalence, STACK uses a computer algebra system, based on open-source Maxima, to establish numeric and algebraic properties of students' answers. While STACK can be used for summative assessment, 'the actual potential lies in the possibilities for formative assessment; eliciting evidence of student understanding and providing feedback that moves learners forward' (Fahlgren, et al., p. 74). A detailed exposition of the affordance of STACK can be found in Sangwin (2013), and on stack-assessment.org. This software is currently 'used by universities, commercial [entities] and developers in over 15 countries' (www.stack-assessment.org, Sept 13, 2021) and can be integrated with a wide suite of Virtual Learning Environments including Moodle and ILIAS.

Recent developments with STACK have included a fully integrated online module in introductory university mathematics (Kinnear, 2019), and an exploration of task design for proof-based mathematics (Bickerton & Sangwin, 2021). Kinnear (2019) outlines an exemplary approach to embedding CAA in an introductory course for tertiary mathematics students. The author notes the time- and resource-intensive process required to fully integrate the technology, but from preliminary results, concludes that these investments were worthwhile for both instructor and student. Bickerton and Sangwin (2021), on the other hand, focused on higher level concepts associated with proof and argumentation. These authors provided a suite of design suggestions for proof comprehension tasks using STACK, including faded worked examples, reading comprehension activities and example generation tasks. Again, while time intensive to generate, such tasks appear to have the potential to contribute greatly to the varied assessment diet suggested by Iannone and Simpson (2011).

In this paper, we discuss the development of CAA in STACK by first-time users in one particular department of mathematics. While we did not set out to replicate Kinnear (2019), or to explicitly implement the design suggestions of Bickerton and Sangwin (2021), these works provide an important grounding against which to compare our own progress.

Aims, Research Questions and Methodology.

Consistent with the traditions of Design-based Research (Cobb et al., 2003), the aims of this research are two-fold; namely to develop our theoretical understanding of the assessments we design as a department, and to improve upon both our understanding of the design-process, and the assessment materials we offer our students in future iterations of the relevant modules.

Our research questions for the study reported in this paper are:

RQ1: What challenges are faced by first-time STACK users when implementing CAA assessments in tertiary mathematics?

RQ2: What are mathematicians' views and approaches to implementing CAA in tertiary mathematics?

Methods.

Participants.

Four lecturers (referred to as L1 – L4) and two postgraduate students (S1 and S2) participated in semi-structured interviews with two members of the research team (also the authors of this paper). S1 and S2 were members of a larger design team including two full-time faculty and six postgraduate students employed at different times throughout the year. Each lecturer was the leader of at least one undergraduate module and was responsible for overseeing the design of their own assessments. The extent to which lecturers engaged with the design team varied substantially.

Procedures and materials.

All interviews were conducted via Zoom, running between 35 and 45 minutes, and comprised two parts. First, participants were asked a series of questions about their experiences designing and implementing STACK-based assessments. The interviewers also asked about relationships between various members of the design team; the process of 'translating' existing items into CAAs in STACK; their level of satisfaction with their existing bank of STACK-based tasks; and what they would like to improve upon in future iterations of their STACK assessment. The second part of the interview was a stimulated reflection task. One week before their interview, participants were asked to select their favourite, and least favourite tasks to which they had contributed. Interviewers then asked a series of questions about each task, probing for information about the perceived strengths and weaknesses of CAA in general.

Data analysis.

In this first instance, a member of the research team watched each interview multiple times, tidying the imperfect automated transcripts in real-time. A series of latent themes were then identified, with supporting excerpts extracted iteratively through several passes through the data. A preliminary report was then produced, highlighting four themes with supporting excerpts and commentary for review by other members of the research team. This report forms the basis of the results section to follow.

Results.

Our thematic analysis (Braun & Clarke, 2006) identified three themes related to the design of STACK-based assessments by first-time users: *1) the process of STACKification, 2) technical challenges with coding in STACK,* and *3) the role of CAA in undergraduate math.*

Theme 1: The process for STACKification.

The course lecturer had for each of their modules a set of problem sheets that they used to provide homework and assessment tasks for their students. Despite variations in other aspects of their approach to CAA, all four lecturers adopted a surprisingly similar four-step workflow for translating their existing materials into automated assessments using STACK.

Phase one: Lecturers would parse their list of existing questions to identify which they believed would make suitable CAA items. This often involved identifying answers that required limited or simple input, and items that could be coded with relatively little technical expertise. L4 noted that "you cannot simply take an exercise sheet and immediately turn into STACK. It requires some effort [to identify appropriate items]". Only L3 focused on including items that were "most critical to capture the coverage of the material" when selecting items for CAA. All four lecturers worked largely independently on this phase, although two lecturers did consult the project leadership team for advice on which items were most suitable to automate.

In many cases, the mathematical content of existing questions would be preserved, but the response required from the student would be altered to suit the STACK environment. For example, with items from the Introductory Analysis course, the lecturer would choose a short series of proofs that were important for students to know and understand. Since STACK cannot currently facilitate the evaluation of student-produced proofs, the design team proposed a series of reading comprehension activities akin to those proposed by Bickerton and Sangwin (2021) that would still assess students' understanding of the proof in the absence of a 'prove that'-style task. In some cases, a series of multiple-choice items similar to those discussed in Mejía-Ramos et al. (2017) were also appropriate.

Phase two: In consultation with the design team, 'preSTACKed' documents were produced for most items. These were most frequently written in LaTeX, and resembled pseudocode outlining the design feature a future coder should implement. These included the types of inputs required from students, the scope and placement of random variables, and the specific question text to be shown to students. In some cases, this preSTACKing phase was a lot less structured, and simply comprised an itemized list of questions to be coded.

Phase three: These preSTACKed documents were then translated into functioning code. For three of the four lecturers, these preSTACKed documents were posted on a shared workflow tracker, to be picked up by the design team. By contrast, L1 did the majority of their own coding, consulting others only when "there was some finessing that I wasn't aware or didn't know how to do".

The design team collaborated frequently, checking each other's work, and coding additional question when the member responsible did not have time. This coding process worked well when the postgraduate student was familiar with the mathematical concepts and methods being developed in the module. However, S2 noted that "some of the hardest second year modules that I didn't take…I found hard, especially when the preSTACK document was vague [and] there was a lot of having to speak to the lecturer, [asking] how do you actually do this?”. We return to this back-and-forth dialogue between designers and lecturers later.

Phase four: After initial coding was completed, lecturers were invited to review each item and encouraged to check the code for the intend functionality. Given the inexperience of the design team, several items had early bugs. In some cases, variations on correct answers were marked incorrect (e.g. an answer such as 4/2 would be marked incorrect when the desired solution was the integer 2), and vice versa. As a tool, STACK gives tremendous control to the user regarding how to assess such variations and can facilitate the vast majority of desired responses in each case. However, given the inexperience of the coding team and the speed at which items needed to be produced, bugs of this nature were frequent in the early stages of the project and caused significant problems to lecturers and students.

Open communication between the lecturer and the coding team on checking how the STACK quizzes would be seen by the students was really important. L3 noted that “There were occasional things where the solutions that have been typed in weren't in the notation that I would teach and so I changed those. Little formatting things and a bit of debugging, so I would have a go at the questions and sometimes I came across errors and got them fixed before the students hit them, but other times, of course I didn't find them until the students found them and then we had to debug them live”.

We expect that these teething issues will reduce in future iterations of these modules. However, we note their significance here because of their impact on attitudes to the value of the technology, in particular with respect to (automated)-assessment, discussed later in this manuscript.

Theme 2: Challenges in early implementations of new STACK materials.

Lecturers tended to focus on assessing procedural tasks (in the sense of Sfard, 1991) in which a numeric or simple algebraic expression could be entered by the student. We, the research team, note that in theory, STACK has the capacity to implement a wide variety of question formats accessing a range of different understandings and approaches. However, anything beyond numeric or algebraic equivalence tests proved to be a significant challenge in many cases.

L1 noted that when the answer to the problem involved surds, STACK had no difficulty when the square root was in the numerator, but when it was in the denominator and the student rationalised the denominator, STACK “could not see that this was a correct answer”. Interestingly, this excerpt doesn’t draw a distinction between the capacity of the tool, and the capacity of a given implementation. While our data does not facilitate a more in-depth discussion on this point, we conjecture that this attitude may have been a barrier to higher quality design in some cases.

And L3 noted that when students were required to type in formulae “then one function of STACK that I hadn't really realized is, if you make one mistake, one small mistake [typing in a formula] which could be just a typo, it blanks all your answers”. S1 noted the importance of students needing to be shown how to input formulae correctly in STACK, for example how to input Greek letters such as lambda and theta, and how to input terms with subscripts such as x_0.

In contrast, L3 highlight one particularly successful episode, in which the coders initially “struggled because there's more than one right answer. So, in the end they worked out a really cunning way to work out whether the student’s [solution] was correct”. S1 also recognised the need for creativity with STACK, and appeared to understand that a solution should exist, even if it couldn’t be implemented in this case. STACK recognises algebraic equivalence, but students can write the solution of a differential equation in many different ways and “you kind of have to think a bit more about all the all the possible answers that the students could give you”.

L2 also highlighted problems associated with the inputting formulae: “One of the big challenges of STACK is making sure you get it right, because the system is only good as it is accurate, so if you have a mistake in your answer in STACK, then the whole, the whole thing is pointless”.

Further professional development for lecturers and coders, and in some cases of students (as pointed out by L3) will seek to minimize the problems raised in this section. However, we note that even experienced coders have difficulties in this regard. To readers considering using STACK in the future, we recommend having a robust system of peer-review in place before AND after implementation with a student cohort.

Theme 3: The role of CAA in different content domains.

All four lecturers started with problem sheets that had been used as homework and assessment activities in their previous teaching. They felt that STACK could handle examples that required a numerical or simple algebraic answers but were reticent to explore opportunities to assess more conceptual aspects of their module curriculum using STACK. For example, S1 noted the ease of assessing calculus: “[it] was fairly straightforward because it involved fairly straightforward kind of mathematical methods so we had weekly quizzes for that”. However, they asserted that the answers needed to be “well defined”. L4 felt similarly, claiming that problems requiring students to input formulae can lead to difficulties “because formula can be written in slightly different ways and sometimes it doesn't recognize these things as the same”.

While questions that required a numerical or algebraic answer could be easily STACKified in most cases, it was more difficult to test theoretical knowledge and proofs. L1 asserted that “when it comes to proofs one would use a normal Moodle (VLE) quiz and do some kind you know very smart multiple-choice type of question”. Similarly, L2 claimed that “Not all [examples] were suitable because some of the questions involve some theorem or some proving which possibly could be STACKed or, if you like, but I couldn't see a way to do that, so I concentrated on questions with numerical answers”.

Further, L4 questioned how a simple numerical or algebraic answer in STACK could show the students had understood the theory and methods they had been taught. “[In my course] it’s not a matter of manipulating formula like in school, right. It's a matter of showing that you understand what's going on and it's somehow difficult to transform it into computer-based assessment”. This lecturer went on to query the suitability of STACK “at a serious university... In a very good math department, you have to show that you understand, then you have to write, and explain”. L4 did concede that “STACK is more suitable for an ancillary course [for non-math majors], but still, it's somehow lame even for chemists”. In contrast, however, L2 felt that if you defined the question carefully then the student would have to understand the methods and the theory in order to get the right answer. This sentiment was also echoed by L3.

These excerpts suggest that the scope and merit of STACK vary greatly for different parts of the undergraduate maths curriculum. In this manuscript, we intentionally abstain from passing judgement on the commentary of L4 and others. Here, we prefer simply to report on the perspectives offered by our participants and reflect on ways in which we can improve our offering to students in future iterations of these courses. Bickerton and Sangwin (2021) propose a series of alternatives for STACK-based assessment of proof that seek to address many of the limitations raised by L4. We acknowledge that these alternatives are time consuming to implement and not applicable in all cases. However, we suspect that none of our four lecturers are aware of this recent work and intend to provide some professional development workshops in the future. In doing so, we seek to broaden the range of tasks offered to our students and the range of conceptual understanding accessed by our STACK-based assessments.

Discussion.

From our interviews with lecturers and postgraduate students, we identified three themes with consequences for the future development for the CAAs at our university and for the wider community planning to implement CAA for the first time.

First, we enumerate the process of ‘STACKification’, in which traditionally handwritten coursework tasks can be translated into CAAs using STACK. While the process has several possible refinements, it is interesting to note the relative uniformity with which this process was used by all four lecturers. Future iterations of these courses will involve cyclic redevelopments of many items, adding new features, resolving bugs, or adding more detailed feedback.

Second, we have identified a primary challenge for first-time users of STACK associated with evaluating algebraic equivalence in various forms. In several cases, lecturers and members of the design team were aware than an alternative coding solution should exist but could not execute a solution within the time constraints afforded. Again, these concerns will diminish with time, and as a department, we now have the opportunity to revisit those items that did not function as expected.

Finally, and perhaps most importantly, we considered lecturers’ perspectives on the role of CAA in tertiary mathematics more generally. All four lecturers acknowledged that STACK had the potential to assess at least some proportion of the undergraduate curriculum. However, these were heavily weighted toward applied mathematics, and to more procedural (rather than conceptual) tasks. Of particular note was L4’s belief in the inability to assess mathematical proof using STACK or other forms of CAA. It is unclear from the data available whether these perspectives would change with further professional development, focused on the potential for STACK to assess a wider array of question formats.

One final challenge not yet discussed lay in the design and implementation of feedback. The automation of personalised feedback proved to be a time intensive process, with most lecturers providing at most a correctness evaluation and a general solution for each question. We report on this feature of our data in more detail in future publications, focusing on routes for realising the potential for productive formative assessment discussed by Fahlgren et al. (2021).

Final remarks and future development.

At the time of writing, the STACK project at our university has been running for approximately 12 months. While we have now had a large bank of STACK-based items, the process of successfully integrating CAA into our curriculum is an on-going challenge. In the future, we will develop a greater variety of question forms, with a clearer focus on the learning outcomes for students and lecturers, and a more rigorous consideration of the formative and summative roles that these assessments play in our courses. This will feature a structured research programme intended to understand students' and lecturers' experiences with CAAs, and further iterations of the design-based research cycle we began with the data reported here.

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Epistemic Potentials and Challenges with Digital Collaborative Concept Maps in Undergraduate Linear Algebra

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This paper aims to investigate epistemological potentials and challenges of digital concept mapping in collaborative activities of pre-service teachers regarding conceptualization in undergraduate linear algebra. Design experiments were undertaken within a larger design-based project with pre-service mathematics teachers for upper secondary school in Germany to look at students' connections and translations between three modes of representations and thinking of concepts such as matrices and determinants. Besides testifying that concept maps have the potential to foster students' organization of the concepts, the results also show how collaborative digital mapping can support three kinds of transitions and students' experiences: (1) *within a digital CmapTools,* (2) *across a digital and a physical medium and* (3) *beyond a single digital resource by integrating DGS and CmapTools, which gained importance since the pandemic outrage.*

Keywords: Pre-service mathematics teachers, Concept mapping, Design research, Linear algebra, Digital tools.

Introduction

While undergraduate linear algebra for engineering and science students is based on the vector space definition and other axiomatic definitions of the concepts, courses about didactics of linear algebra for pre-service secondary school mathematics teachers usually treat the concepts in a non-axiomatic manner in Germany (Donevska-Todorova, 2018a, 2018b). Such approaches may result in the lack of *cognitive flexibility* (Alves Dias & Artigue, 1995) where students perceive many of the concepts disconnected from one to another and prevent them from getting a wider picture of what constitutes a typical axiomatic-structural undergraduate linear algebra curriculum. Hence, pre-service teachers may gain disorganized, fragmented knowledge with little connections between many of the concepts or insufficient conceptual understanding without recognizing the concepts' meanings (Donevska-Todorova, 2016, 2017).

Preventing such development has become even more challenging since the global COVID-19 pandemic. Remote teaching has challenged educators to search for adequate novel tools that may quickly and effectively transform traditional mathematics classrooms and simultaneously involve learners in resource designs that maximize their activities and engagement. It seems to us that collaborative concept mapping in the virtual space has the potential to respond to this challenge. Thus, the purpose of using collaborative digital concept maps in our teaching approach is to enable future teachers to systematically connect different modes of representation and thinking in linear algebra in an organized structure, which is in line with Papert's notion about digital tools being "vehicles" for development of "Mathematical Ways of Thinking" (Papert, 1983). We focus on the following research question: what are the epistemological potentials and challenges of collaborative digital mapping through algebraic, geometric, and axiomatic transitions and experiences with linear algebra concepts by pre-service mathematics teachers?

Theoretical framework

Regarding the representation of mathematical objects in linear algebra, Hillel (2000) addressed three modes (p. 192): the *abstract* mode, the *algebraic* mode, and the *geometric* mode. The abstract mode is referring to axiomatic definitions, structures and language of vector spaces, subspaces, linear combination, and linear independence of vectors in linear algebra. An algebraic mode refers to the use of algebra of (specifically) properties of $\mathbb{R}^n$, like algebra with *n*–tuples, the interplay between matrices and associated systems of linear equations. A geometric mode refers to the language of *2D* and *3D* geometry and geometric vectors, such as parallelogram rule with vectors, lines, planes, their intersection(s) etc. The interplay between these modes is often necessary for cognitive flexibility (Alves Dias & Artigue, 1995).

However, this is not an easy and trivial task. As addressed by Sierpinska (2000), students tend to think practically rather than theoretically. Practical thinking can be described as thinking *locally* and an effort to reason that is changed by the action itself, while theoretical thinking is about generalising and reflecting on the situation through linking the action and associated mathematical objects by using different procedures. In other words, theoretical thinking concerns concepts (i.e., definitions, the use of set theory etc.), while practical thinking is limited to the results and/or experiences in the involved action. From an epistemological point of view, Sierpinska (2000) characterizes three thinking modes referring to practical and theoretical thinking: synthetic-geometric thinking mode, analytic-arithmetic thinking mode and analytic-structural thinking mode. In the synthetic-geometric thinking mode, the learner refers to geometric properties of the action (possibly) based on 'practical' observations but does not refer to thinking on how observed mathematical objects are created. The analytic-arithmetic mode is associated with referring to algebraic features of the objects, for example, thinking and reasoning with *n*–tuples, coordinates of objects in Cartesian geometry, the system of linear equations and associated matrix algebra. The analytic-structural thinking mode requires a synthesis of progressive reasoning on different situations and mathematical objects and thinking of them as (a part of) a conceptual system. We note that the three thinking modes are in parallel to Hillel's (2000) three modes of representation.

Concept mapping in mathematics education

Several resources (e.g., Brinkmann, 2003) point out that concept maps were first introduced by Novak and Gowin (1984) as research tools for structuring an individual's knowledge. The initial intention was to use the concept maps as graphical representations of one's knowledge for research purposes in science. Other authors define the concept maps as advanced organizers with a meaningful and practical structured approach (Willerman & Mac Harg, 1991) or as an aid to instruction in science and mathematics, new teaching strategies that will enhance the understanding of those concepts which are common for both disciplines (Malone & Dekkers, 1984). The concept maps are often referred to as instruments, tools, techniques, methods, graphical displays, or networks with a very wide range of aims.

The historical development of concept mapping follows the order of their implementation in research, teaching and learning. The later tendencies lead to the implementation of concept mapping in investigations on how learners learn. Qualitative analysis of students' concept mapping has been in expansion for different purposes: suggesting teaching approaches that help students integrate new knowledge and build upon their existing naive concepts, learning by illustrating patterns of conceptual development (Kinchin, Hay & Adams, 2000), assessing conceptual understanding (Williams, 1998; Varghese, 2009). Concept maps can be used to organize information on a topic, to facilitate meaningful learning, to identify students' knowledge structures, especially misconceptions or alternative conceptions, to serve as a memory aid, to revise a topic and to design instructional materials (Brinkmann, 2003). The "theory underlying concept maps" (Novak & Cañas, 2008) points out two major important foundations of concept maps:

- *Psychological,* related to learning processes like discovery learning, rote learning, meaningful learning, etc., thus they serve as a kind of template or scaffold to help organization of knowledge and to structure it, and
- *Epistemological,* so serving new knowledge creation as a constructive process involving both previous knowledge and emotions or the drive to create new meanings and new ways to represent these meanings (Novak & Cañas, 2008, p. 9).

Related to the epistemological aspect, McGowen and Davis (2019) analysed a sequence of concept maps and corresponding schematic diagrams and together with quantitative and qualitative data found out that there are students with low gain in undergraduate mathematics who seem unable to productively integrate new knowledge into an existing structure and that they reveal radically different processes of knowledge construction and organization.

Digital concept maps as tools for dynamic synchronous systematic and structural organization of concepts through collaboration

Previous research (in the previous sub-Section) shows the variety of potential that concept mapping offers in an organization, structuring and consolidation of mathematical knowledge, yet what do their digital forms have to extend or promote differently? The following Table 1 offers answers to some insights to this question.

Table 1: Comparison of characteristics of physical and digital concept maps

Concept map	Individual's map	Collective/ Collaborative map
Physical (non-digital, e.g., paper-pencil, flipchart-marker, flipchart-stickers, board-chalk, board-stickers)	Single	Static
Digital (e g., Miro, CmapTools)	Multiple	Dynamic, synchronous, integrates e-content, DGS files, shareable, extensive, adaptable

Table 1 shows a comparison of characteristics of physical and digital concept maps including examples of digital maps such as Miro and CmapTools. Further aspects of the question are explained below in the section Results and Discussion.

Concept mapping in the teaching and learning of linear algebra

An example of a concept map for systems of linear equations in two unknowns showing lower stage algebraic methods and geometric method with lines can be found in Brinkmann (2003). Lapp, Nyman and Berry (2010) examined the connections of linear algebra concepts at the undergraduate level. They have developed two techniques for qualitative analysis of student-constructed concept maps and showed that eigenvalues and eigenvectors seem to be the most disconnected concepts from the concepts as basis and dimension in the conceptual network. Another research (Stewart, 2008) aimed to discover students' difficulties in understanding some linear algebra concepts and to suggest possible ways for their prevention, through students' involvements in tests, interviews, and concept maps. In the light of the obtained results, we hypothesized that (digital) concept maps would be a key tool to connect abstract notions belonging to linear algebra context, such as linear independence, matrix algebra and determinants etc. In other words, we focused on whether digital concept maps would coordinate geometric, algebraic, and abstract representation and associated thinking modes.

Methods

During the initial design experiments in a first design-based research (DBR) cycle with pre-service teachers at a large university in Germany, physical concept mapping was applied in a course about linear algebra and analytic geometry at the beginning and the end of the second semester. The collected data, scans of flip charts were stored, qualitatively analysed and the results suggested that the time distance of the mapping activities should be reduced, and their frequency should be increased to enable students to capture the vast number of new concepts completely and structurally.

In a second DBR cycle, in addition to physical maps, CmapTools were implemented according to the previous findings. Besides regular weekly exercises, a group of 15 students were asked to create digital concept maps with an opportunity to update them every week and finally submit them in three-time slots during the semester. They were also encouraged to collaborate in small groups of up to three students during the digital mapping, edit and advance their maps by linking a variety of resources at any time. The created maps were collected via the course in the Learning Management System Moodle where students reflected and engaged themselves in further discussions in a Forum.

Results and Discussion

Instead of reporting on quantitative data, this section represents a qualitative analysis of a case related to the research question. It offers a collection of exemplary concept maps of one group consisting of three students. The maps show students' connections of different modes of representations and thinking modes. For example, Figure 1 shows students' work regarding the link between 2×2 and 3×3 square matrices and determinants.

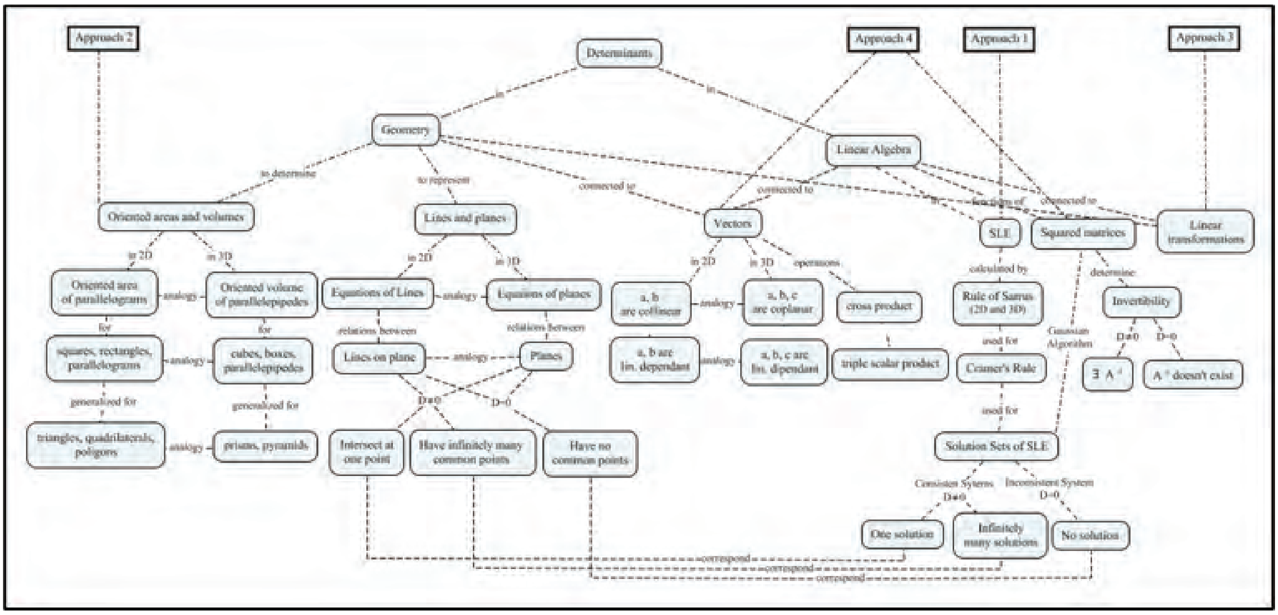

Figure 1: Students' concept map about determinants of square matrices in dimensions 2 and 3 with CmapTools

Related to the *psychological* foundation of the maps according to the "theory underlying concept maps" (Novak & Cañas, 2008) which was mentioned above, this map shows scaffolding and structuring knowledge hierarchically in seven layers of nodes-concepts and established arcs-processes of thinking. Further, related to the *epistemological* foundation, Figure 1 shows students' rich network of concepts related to determinants of matrices and their mathematical meanings. Moreover, it shows relations and transitions between Hillel's (2000) algebraic representations and modes of thinking of concepts such as systems of linear equations, linear transformations and geometric representations and modes of thinking of concepts as oriented areas of parallelograms in 2D and oriented volume of a parallelepiped in 3D geometry. Finally, this concept map represents a structured and dynamic network of epistemological connections of linear algebra concepts (1) *within a digital tool* (CmapTools).

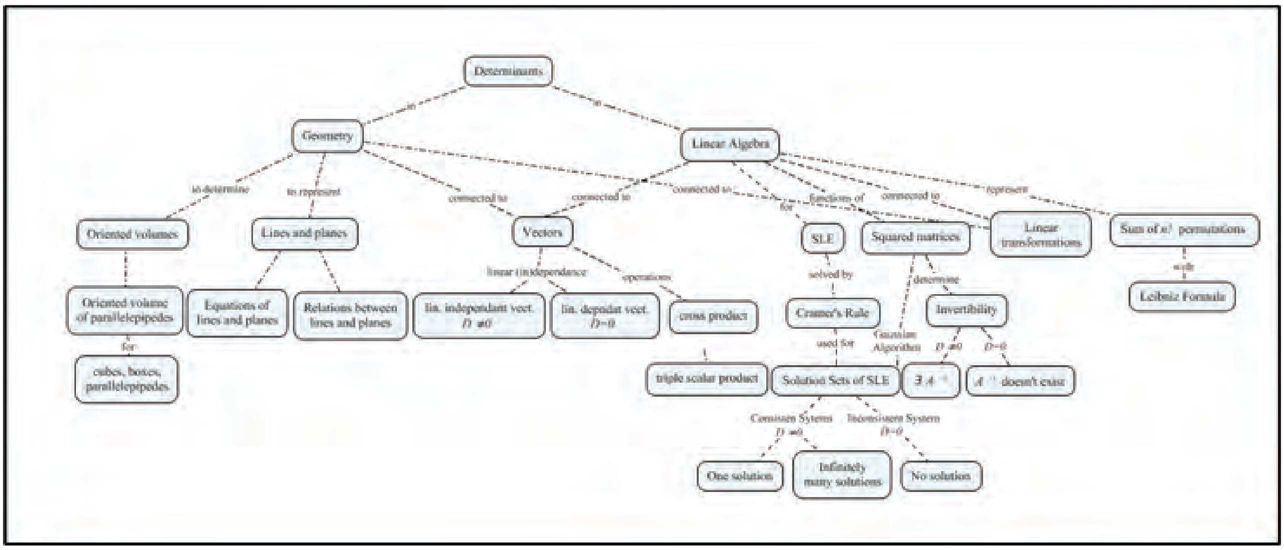

Figure 2: Students' concept map of determinants of matrices in dimension *n* with CmapTools

Likewise, the concept map in Figure 1, the one in Figure 2 created by the same group of students, shows the connections between the algebraic and the geometric representations of the concepts, yet in a generalized form for the dimension *n*. It enabled this group of students to complete their initial static flipchart-marker map, used as a sketch, with the missing concepts and relations in the digital map. The initial static map was later incorporated as a JPG file in the digital form offering possibilities to reflect on the advances and enrichment of the network. These are not only *epistemic* values of the collaborative digital concept map but also didactical and confirm the characteristics of the digital maps for easy adaptations and editing (given in Table 1).

Figure 3 describes exemplary cases regarding matrices and associated concepts, which illustrate Sierpinska's (2000) notion of students thinking of prototypes of concepts locally. This concept map is a specification and reduction of the concept in dimensions 2 and 3.

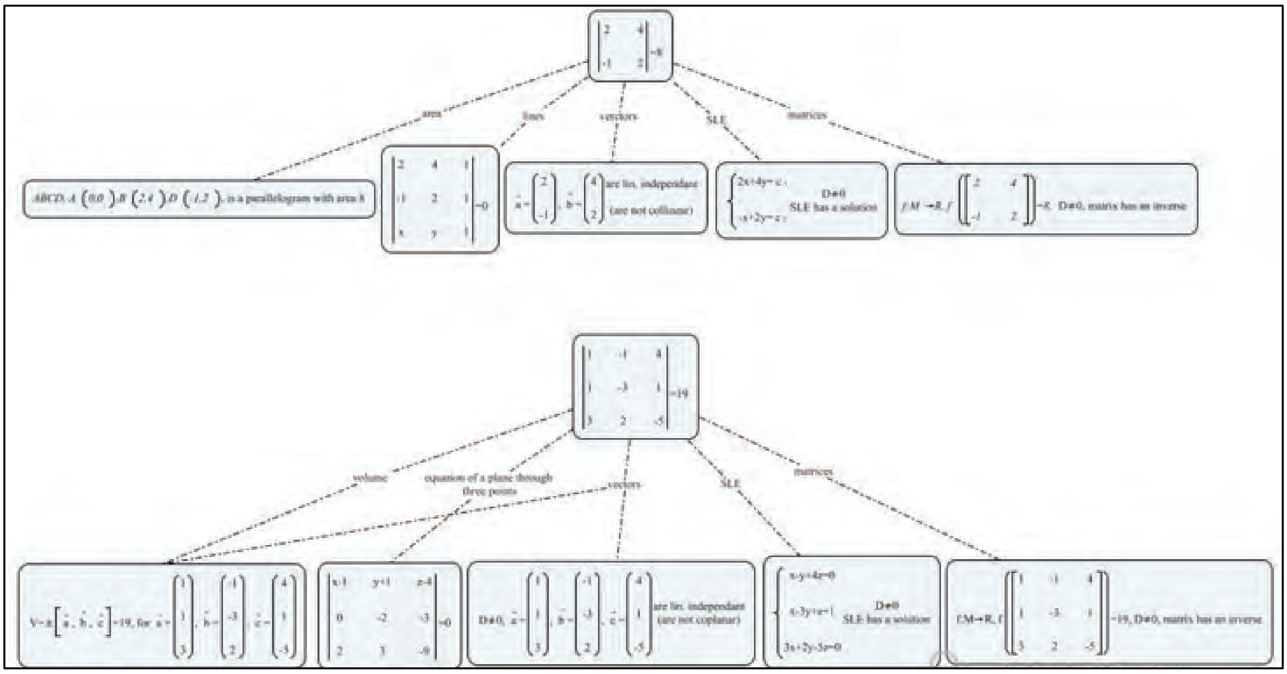

Figure 3: Students' concept maps with examples of determinants of square matrices in dimension 2 and 3 with CmapTools

These empowered students to create DGS GeoGebra files and directly link them with the map. These files enabled students' translations between geometric, algebraic, and axiomatic structural modes of representation and thinking of determinants of matrices. All three concept maps represent a collection of collaboratively created digital resources for the teaching and learning of linear algebra.

In reference to the characteristics of the physical and the digital maps presented in Table 1, some *challenges* are worth mentioning. Students' practical work showed how individual maps differ from collective ones. The synchronous engagement and the integration of the additional technology-enhanced resources were not constantly smooth to maintain. These affordances required more time and effort invested in *often adaptations*. We further consider that *'zoom in'* into the depth of each of the nodes-concepts (e.g., providing 'on click' definitions or dynamic visualizations of the concepts with other digital tools, e.g., a GeoGebra file ((3) *beyond a single digital resource of use*) in the concept maps may potentially bring one more dimension and meaningfulness of the maps.

Conclusions and further research

This paper tried to respond to the Call of the CERME12 TWG14 by tackling challenges when implementing novel approaches in the teaching and learning of undergraduate linear algebra with pre-service teachers. Based on the "theory underlying concept maps" (Novak & Cañas, 2008) we identified *psychological* and *epistemological* potentials and challenges of innovative collaborative digital concept maps in supporting conceptual understanding by connecting three modes of representations in linear algebra: algebraic, geometric, and abstract (Hillel, 2000) and the corresponding thinking modes analytic-arithmetic, synthetic-geometric and analytic-structural (Sierpinska, 2000). Concerning the research question, we have undertaken design experiments in two DBR cycles with physical concept maps and the digital tool CmapTools. Through a qualitative analysis by comparing the collected data we found a case of a group of students providing a collection of concept maps bringing into forth the *potentials* of the maps in showing connections (1) *within a digital tool* (Figure 1), (2) *across a digital and a physical medium* (Figure 2) and (3) *beyond a single digital resource of use* by integrating CmapTools and DGS (Figure 3) which enabled students to reflect, consolidate and reorganize their knowledge. To summarize, we have investigated how digital collaborative concept maps enable learners to construct, scaffold and consolidate an individual's knowledge and how they contribute to meaningful learning, effectiveness in conceptual understanding in linear algebra, negotiation of the meaning of mathematical concepts while establishing connections between them, assigning them appropriate placement in a structured and hierarchical network of concepts through its nodes and arcs. Often requiring adaptations of the maps and 'zoom in' options in the nodes leave room for further research.

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Collective and individual mathematical progress: Layering explanations

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Keywords: Individual, collective, mathematical progress, methodology, Sierpiński triangle.

Introduction and background

We report on the latest development of our efforts to coordinate analyses of individual and collective mathematical progress. We build on and extend a series of theoretical-methodological analyses aimed at networking Abstraction in Context (AiC, Dreyfus et al. 2015), and Documenting Collective Activity (DCA, Rasmussen & Stephan 2008). AiC is commonly used for the analysis of knowledge construction by individuals or small groups of students, and DCA, which accompanies the emergent perspective, is commonly used for analyzing the mathematical progress of the whole class or a small group of students (Hershkowitz et al. 2014; Rasmussen et al. 2015; Tabach et al. 2014; Tabach et al. 2020). Our research goal is to further develop a methodological approach for characterizing the interplay of mathematical progress across individuals, small groups, and the whole class. We refer to this approach as “collective and individual mathematical progress: Layering explanations” (CIMPLE, pronounced as “simple”). “Layering explanations” pertains to the use of both theories on the entire data set, and to transparently layering analysis upon analysis, unlike our previous efforts that leveraged AiC on small group work (SGW), and DCA on whole class discussions (WCD). The significance of this ongoing work lies in the identification of nuanced ways in which students’ knowledge progresses in inquiry-oriented classrooms.

The context for this study was a semester-long intact graduate level mathematics course on chaos and fractals at a State University in the USA. Ten of the eleven students were pursuing a master’s in mathematics education. The students worked in four stable groups: A (Carmen, Jen and Joy); B (Kevin, Elise and Mia); C (Soo, Kay and Shani); and D (Curtis and Sam). All names are pseudonyms. Groups A and B were video-recorded during SGW; the class was video-recorded during WCDs.

Analysis and results

In Lesson 9, students carried out the first few iterations of a recursive geometric process (given a triangle, connect its midpoints and remove – or color white – the middle triangle); if continued infinitely, this process produces the Sierpiński triangle (ST). Students were asked to imagine the ST and discuss what they could say about the area and the perimeter of the ST.

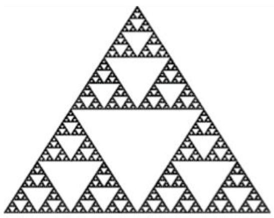

Figure 1: The ST

After 3 WCDs and 3 SGWs, the instructor (who had listened in on Group A) convened the class and asked Carmen and Joy to report on their opposite views of the perimeter. According to Carmen, as one keeps zooming in, the entire triangle is “going to be white, so there’s no area, so there’s nothing to … put a fence around; so there’d be no perimeter”. According to Joy, “if you zoom in…, there is more and more to fence; Until…" Joy was unsure whether the perimeter increases to a finite value or to infinity. Although in the preceding SGW and WCD episodes the students had made little progress constructing knowledge about the features or the length of the perimeter, Elise immediately reacted to Carmen and Joy by Connecting Area to Perimeter (CAP): "what you are coloring in *is* perimeter, to some extent". She was followed by Kevin, and later Curtis: "The perimeter of the white is also the perimeter of the black" (PW=PB); and in between Carmen: "The fence is guarding both properties". Our analysis shows that in these 2 minutes, several students constructed the CAP knowledge element (according to AiC) and PW=PB started functioning-as-if-shared (FAIS) in the class (according to DCA). In other words, the students constructed new (to them) knowledge within a WCD, and this new knowledge immediately began to FAIS in the class. This constitutes substantial mathematical progress, achieved in a pattern that is very different from the standard trajectory.

Earlier in the same lesson, the students made mathematical progress in two further patterns. In one, "Area goes to 0" passed from FAIS in Group A to FAIS in the whole class without any reaction or even question. In the other one, the everyday metaphor of "zooming in" which had earlier appeared in a movie about the fractal nature of the coast of Britain was appropriated by the students as a tool to deal with the infinite nature of the ST (as used by Joy and Carmen above).

There are certainly additional patterns of mathematical progress. We conclude that the standard trajectory is only one of many possible ones for mathematical progress in inquiry-oriented classrooms. The coordination of AiC and DCA is an efficient methodology to identify such patterns.

Acknowledgment

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Design principles for gaps in worked-out examples in mathematics for science students

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Keywords: Worked-out examples, modelling cycle, undergraduate students.

Learning how to apply mathematics is crucial for university level science students. Modelling tasks are especially challenging for undergraduates, since solving such tasks involves performing various actions. These actions can be described as transitions between successive states in modelling cycles (e.g., Blum & Leiß, 2006). Students often focus on calculations and neglect the actions of modelling and interpreting, even though these are important for their future studies and work. In order to support students in modelling tasks, we use worked-out examples (WOEs) with gaps and self-explanations to encourage active work with the problem at hand (Hilbert et al., 2008). Our poster shows design elements of our WOEs, in particular the different types of gaps, and the design of a pilot study with the idea of analysing how students work with these gaps (link to the poster).

Worked-out examples with self-explanations related to a modelling cycle

It is well known that beginners benefit from WOEs, especially when they include self-explanations (Atkinson et al., 2000). Some research reports the use of WOEs with gaps. For instance, Stark (1999) showed that gaps improved the students' performance in case of probability calculation, while Hilbert et al. (2008) found that gaps impaired the learning gain in the context of proving in geometry. In our opinion, the greatest advantage of WOEs is the possibility to provide a structure for a solution of a modelling task as well as to enable students to think deeply about the solution by filling in the gaps.

We have taken the modelling cycle from Schupp (1988) as a basis to structure the WOEs. Our WOEs usually start with a scientific problem in a real-world situation (see Figure 1) and contain various tasks of modelling, deducing, interpreting, and validating that correspond to transitions in the cycle. In addition, we extend the action of modelling by the translation between scientific, mathematical and daily languages as a bridge in order to better comprehend the scientific problem and the mathematical model as well. Moreover, we feel it is important to emphasize the underlying mathematics, which is, for instance, a crucial point in a mathematically deduced solution.

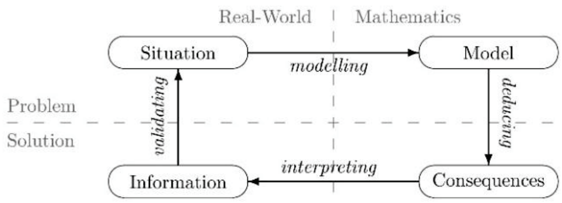

Figure 1: Modelling cycle (our representation, following Schupp, 1988)

Design principles for gaps in worked-out examples and research questions

Strauer et al. (2019) suggest four types of gaps; each of these types is used to foster the actions in the modelling cycle. *Gaps for translation:* These gaps activate the translation between scientific, mathematical and daily language to refer to the transition from real-world to mathematics and vice versa, i.e. for modelling and interpreting. *Gaps for calculation:* Students are asked to calculate some steps on their own. This type is always inserted for deducing on the mathematical side. *Gaps for mathematical terms and notions:* The students should practice the use of mathematical language, including mathematical symbols. These gaps are applied on the mathematical side of the cycle for both deducing and modelling. *Gaps at key points:* At these points, students should become aware of crucial points or general subgoals in modelling, deducing, and interpreting.

According to our experience and the students' comments during lectures and tutorials, students do not always estimate the difficulty of gaps in the same way as the designers of the WOEs.

1. What *types of gaps* in the WOEs do students find *easy/moderate/difficult*, and why?

2. To what extent do the estimates of *the difficulty of gaps differ* between the designers and students?

Design of a pilot study

We proposed the following interview study on the topic of discrete growth with first-year pharmacy and biomedical students. The designers of the WOEs set gaps of different types and estimate the difficulty of the gaps in advance. Between a lecture and the tutorial, student volunteers first complete a given WOE while thinking aloud. Second, they are asked to estimate the difficulty of the gaps and give reasons for their estimate if possible. Afterwards the designers' and the students' difficulty estimates are compared and analysed, particularly with respect to the types of the gaps.

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The triple discontinuity in engineering mathematics education

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This paper takes as object of study the discontinuities in mathematics education in engineering. In particular, the internal discontinuity between mathematics education and the role of mathematical activity in engineering courses, is characterised through diverse interviews with mathematics and engineering teachers of an engineering school in Barcelona (Spain). The results show that the internal discontinuity can be characterised by diverse elements that make it an institutional phenomenon further from the specific discontinuities found in previous punctual studies.

Keywords: Anthropological Theory of the Didactic, mathematics in engineering education, discontinuities.

Introduction: transition between institutions and discontinuities.

The beginning of the analysis of discontinuities in mathematics education started with the work of Felix Klein (1908) who announced the famous double discontinuity concerning the mathematical education of teachers. However, research in didactics of mathematics at university level is a relatively young field: the first works date back to 1970. From the beginning, this domain of research considered the didactic phenomena associated with discontinuities as a relevant study area.

The first discontinuity announced by Klein concerns the transition problems that students experience when they enter the university. This has been extensively studied in recent research in mathematics education (Gueudet, 2008). The second discontinuity concerns the disconnection that mathematics students experience when they join the school as teachers. At this point, they are faced with a process of transposing academic knowledge acquired at university into knowledge needed for teaching, which is far from being an easy process (Winsløw & Grønbæk, 2014).

Another field of study in didactics at university level is that of mathematics education for non-mathematicians in the field of engineering. Quéré (2019), in his doctoral dissertation, shows that the number of research works at the CERMEs in this field has been growing in the last editions.

This paper is framed within the Anthropological Theory of the Didactic (ATD, Chevallard 2000), and takes as its starting point the study of phenomena associated with discontinuities and mathematical education in engineering. The ATD approach on discontinuities considers that they are associated to a transition between institutions: discontinuities appear when the students move from one institution to another one. Analysing the existing work to date, we hypothesise that the path followed by people learning engineering experiences what we could call a triple discontinuity (see Figure 1), making an analogy with Klein's famous double discontinuity. In fact, the first and third discontinuities are homologous to those announced by Klein in mathematics teacher education. The second is the one that would be more specific to engineering education: it is an internal discontinuity in engineering education institutions due to the transition between the mathematical courses for engineers and courses in engineering courses, in line with what has been observed by Romo Vázquez (2009) and Quéré (2019).

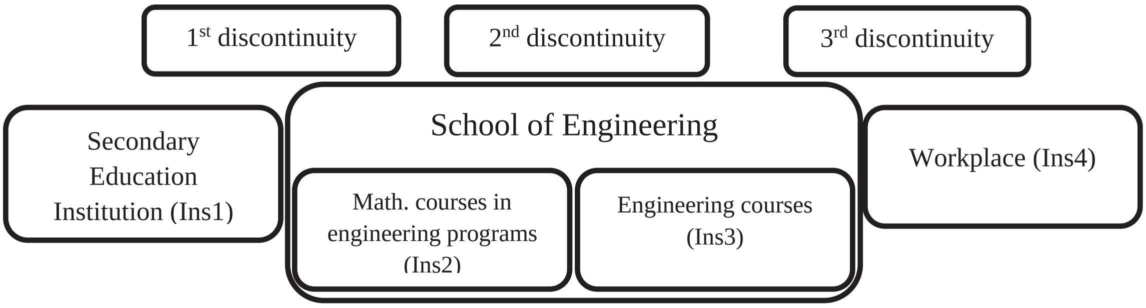

Figure 1 - Diagram of the triple discontinuity in engineering and institutions considered

Theoretical framework

Some studies referring to the ATD address the second discontinuity by analysing the different mathematical activity in the engineering courses (institution 3) and contrasting it with the activity promoted in mathematics subjects or in the workplace (institutions 2 and 4). We present in the following some of the most relevant works analysing the second discontinuity in engineering degrees as well as the theoretical tools mobilised in them.

One of the pioneer woks is Romo Vázquez's doctoral dissertation (2009) that studies the origins of mathematical education for engineers and highlights the differences in mathematical activity in mathematics courses, engineering courses and professional practices. Romo Vázquez (2009) and Castela and Romo Vázquez (2011) adopt as the main theoretical tool the praxeological model proposed by Chevallard (2000). Praxeologies allow researchers to model knowledge: they are entities formed by the inseparable combination of a praxis or know-how made of types of tasks and techniques, and of a logos or knowledge made of a discourse aiming at describing, explaining, and justifying the praxis. Romo Vázquez (2009) and Castela and Romo Vázquez (2011) identified the differences that appear in the logos block (especially in the technological environment) depending on whether the training institution or that of the practitioners is considered (mathematical courses or engineering courses).

In the ATD studies analysing the mathematical training of engineers, a key notion is that of institution and institutional position. According to Chevallard et al. (2015) an institution in the sense of ATD is just a more general conception than the general meaning: for example, a class with the students and teacher is an institution. In every institution, each person may occupy different institutional positions, for example, in the case of a class, that of student or that of teacher.

Another relevant work analysing the mathematical training in engineering is that of Barquero et al. (2014). They hypothesize that the applicationism is an essential component of the epistemological conception of mathematics in applied sciences degrees at university. Specifically, they characterise the applicationism by five indicators: (1) Mathematics are presented independently of other disciplines, (2) Mathematical tools are considered the same for all scientists, (3) Organisation of mathematical courses is based on logical concepts, (4) Applications are always presented after the basic mathematical courses and (5) Extra-mathematical systems can be constructed without any reference to mathematics. In their work, they observe, through interviews with university mathematics teachers in applied sciences, a clear applicationist conception of mathematics. In addition, the authors consider this conception as an element hindering the implementation of mathematics (and science) teaching based on modelling processes.

More recent works within the ATD framework are those of González-Martín and Hernandes-Gomes (2019) and Hochmuth and Peters (2020), both mobilising the praxeological model. Specifically, the first addresses the differences in the use of the integral in mathematics and in strength of materials textbooks while the second addresses the use of mathematics in signal theory courses.

Research questions

Considering the previous works in the ATD framework in the field of mathematics for engineers and the discontinuities as well as our hypothesis, our research question is:

What are the epistemological conceptions of these two institutional positions: first-year mathematics teachers and engineering courses' teachers? Do these conceptions reveal a discontinuity in the institution further from punctual issues already characterized?

Institutional context and interviews

The institution where the interview campaign was carried out is the Escola Universitària Salesiana de Sarrià (EUSS), a centre associated to the Autonomous University of Barcelona (Spain), where five engineering degrees are taught (electronics, mechanics, automotive, industrial organisation and renewable energies and energy efficiency). This is an institution with a tradition of implementing teaching innovation processes and where specific training courses in didactics for university teachers are offered to teachers, see for example Florensa et al. (2017).

Mathematics teaching in all EUSS degrees is concentrated in three courses. Mathematics course is offered in the first semester with 7 ECTS, Calculus in the second semester with 8 ECTS and Statistics in the fourth semester with 6 ECTS. The syllabuses of each degree can be consulted in detail on the centre's website: www.euss.cat. In order to characterise the possible discontinuity, two interview protocols were designed: one for teachers of mathematics courses and the other for teachers of engineering courses. Both interview protocols can be consulted in full at the following link (https://sites.google.com/euss.cat/cerme12).

Both interview protocols have the same structure with two common blocks and a third differentiated one. The first one characterises the academic and professional career of the person interviewed, as well as the degrees and courses in which they teach. The second block is dedicated to identify the conception of each person with respect to the applicationism described by Barquero et al. (2014). The third block aims to identify the opinions of the teaching staff on the most important elements in mathematics education in engineering in terms of techniques and type of tasks (accordingly to the praxeological model). In particular, teachers in the mathematics field are asked to identify the most important elements of the courses they teach, the most common difficulties detected in students, as well as the criteria used to design the contents of each course. In contrast, engineering teachers are asked to identify the types of mathematical tasks and techniques required in their subjects, and whether these are of a routine or algorithmic nature or whether they are related to modelling. These protocols were used to interview teachers of mathematics courses (Mathematics and Calculus) in the first year (MT, 5 in total), as well as teachers of engineering subjects (ET, 8 in total). In Table 1 the academic and professional backgrounds of the interviewees are summarized.

Table 1: Academic and professional backgrounds and teaching experience of interviewed teachers

	Academic background	**Professional background**	**Teaching experience**	**Courses**
MT1	PhD Mathematics	Academia	30 years	Mathematics, calculus, statistics
MT2	PhD Chemistry	Academia	9 years	Mathematics, engineering courses.
MT3	MSc Mathematics	Teaching (not researcher)	30 years	Mathematics, calculus, statistics
MT4	BSc Mathematics	Secondary education,	10 years	Mathematics, calculus, statistics
MT5	PhD Biochemical Engineering	Academia with some experience in industry	7 years	Mathematics, calculus, engineering courses
ET1	PhD Materials Eng.	Academia	20 years	Strength of materials, elasticity, and structures
ET2	PhD Telecommunication	Academia (1 year in telecommunication industry)	20 years	Industrial computing and communications
ET3	PhD Electronic Eng.	Academia	10 years	Automation and power electronics
ET4	PhD Material Science	Vocational training, laboratory technician and academia	10 years	Materials, manufacturing processes and strength of materials
ET5	PhD in Business Administration	Academia	4 years	Strategic management and Quantitative Methods
ET6	PhD Local Development	Consultant, in academia since 2009	12 years	Quality and Industrial plant location
ET7	MSc Electronic Eng.	Academia	12 years	Electronic systems, Project Management and Digital signal
ET8	PhD in civil engineering	Academia	7 years	Physics, Strength of Materials, Machines and Fluids

Results

Interviews of teachers of mathematics courses

The results regarding the applicationist conception of mathematics education in engineering show a strong consensus except in the first question that refers to the uniformity of mathematics education in any university degree. In this question, teachers do not have a clear position (see Figure 2).

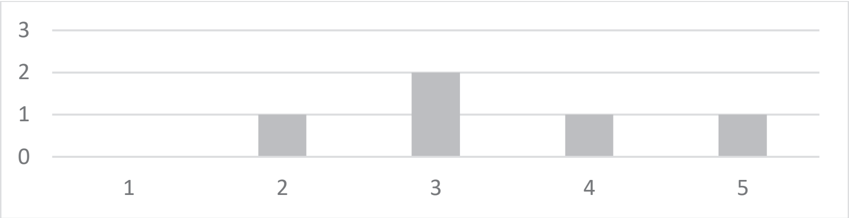

Figure 2- Answers of mathematics teachers to the statement "The mathematics taught in the first university courses of Engineering are the same as those taught in the other university degrees" with 1 being " I disagree" and 5 being "I strongly agree"

Concerning the answers to the criteria used to define the syllabus of the courses, these reveal the existence of a shared conception for the meaning of mathematics by mathematics teachers in the first years. For example, MT2 states that "the subject was already set up, but it is similar to what is offered in other centres" In the same direction, MT3 speaks of "a common profile for Engineering Mathematics" A second phenomenon that comes to light at the same time is the influence of previous curricula on the definition of the contents included in the courses. In this sense, MT4 states that "the contents of the old engineering program (3-year program before the adaptation to the European Higher Education Area standards) were adapted to the new engineering degrees" or MT1 states that "the syllabus was established with the technical industrial engineering, I suppose based on what was done in other universities" MT3 also states that "there has been continuity since the initial creation of the courses and the contents have been adapted...".

Related with the mathematical content, there are two types of answers. Firstly, MT1 and MT4 attach great importance to the ability of mathematical content to develop "abstract thinking, reasoning skills and the ability to adopt strategies" (in the words of MT1). MT4 formulates it as follows: "finding primitives is important, not because of the technique, but because of the associated mental process" Secondly, most of the interviewees propose a set of contents justifying its importance. These include finding primitives and derivatives (MT2, MT3, MT4), working with matrices (MT3, MT4, MT5) and solving differential equations (MT4 and MT5). It is important to note that the importance assigned to each content is heterogeneous and there is not much uniformity: some teachers emphasise the structuring role of algorithmic work, while others prioritise, in the words of MT4, "knowing and understanding how to solve, since calculation is mechanical and/or programmable"

Although the two groups of teachers do not prioritise the same contents on the list, there is a great consensus that the manipulation of algebraic expressions in the study of functions is the greatest difficulty that students face when learning mathematics. Other difficulties that appear are the calculation of integrals and derivatives and the confusion between vector and scalar magnitudes.

Table 2- Results of the question "Highlight from the following list of linear algebra contents the five most relevant to your course"

Contents of “Linear Algebra”	Number of votes MT	Number of votes ET
Matrix transposition	0	1
Matrix multiplication	1	4
Scaling matrices row / column, pivot, rank	5	0
Elementary matrices	0	2
Inverse matrix	5	2
Determinants, Sarrus, determinant properties	3	3
Systems of linear equations, Rouche, Gauss-Jordan	5	7
Inverse matrix - adjugated matrix	0	0
Eigenvectors and eigenvalues,	5	1

Interviews of teaching staff in engineering courses

As in the case of mathematics teachers, there is consensus in all the questions related to applicationism, except on the question whether the mathematics taught in the first years of engineering is the same as that taught in other degree courses where there is not a clear agreement on the answers.

Teachers of engineering courses attach particular importance to the interpretation of the results obtained, and decision-making based on these results, beyond the technique applied. ET1 expresses it as follows: "I am not so much interested in them finding integrals or derivatives, but they need to acquire the concept behind them and use it to solve engineering problems". ET2 also expresses in very close terms: "Mathematical elements are useful for modelling, the difficult thing is to set out and validate the model, the resolution itself is algorithmic...". ET3, speaking of the use of integrals and derivatives in a strength of materials course, states that "the bending moment that we obtain allows us to select a beam for a structure: this is what matters, beyond the algorithmic calculation" ET5 also emphasises this duality: "the convolution of functions can be very routine, but what is interesting is that it allows us to analyse, study and explain typical phenomena of electronic systems"

Characterising the phenomenon: applicationism and internal discontinuity

The results obtained on the applicationist conception in mathematics education in engineering, confirm those obtained by Barquero et al. (2014) in the field of applied sciences. Moreover, there are no significant differences between the opinions of mathematics and engineering teachers, which seems to indicate that the applicationist conception is proper to the institution.

With respect to the internal discontinuity, two points stand out. Firstly, the MTs' and ETs' interviews show some items with a very high level of consensus in the fields of linear algebra and analysis (see Table 2). The maximum consensus is obtained in solving systems of equations, finding the derivative and the integral. However, there is a significant disparity in matrix scaling and operations with matrices, as well as the determination of the graph of a given function. Another point of consensus detected in the open comments on the questions, is that training in mathematics should provide abstraction, reasoning and generalisation skills beyond working with specific content, what Chevallard et al. (2015) characterises as transcendent utility.

However, the main difference between the MTs' and ETs’ answers, lies in the role played by the solving technique as opposed to its justification. The ETs emphasise that mathematical education should have as its main goal the development of a well-defined logos around certain mathematical praxeologies rather than the work on technique. ET2 expressed it in the following way: "I am interested in the concepts of integral and derivative in contrast to them knowing integration by parts which I do not consider necessary at all". ET3 also states along the same lines: "They have to know and identify functional relationships, the variation of data, identify trends, beyond finding integrals or concrete derivatives by hand". ET4 states it as follows: "They don't have to know how to solve almost anything by hand, on the internet there are symbolic calculators: they have to know what it is to integrate and derive" ET5 puts it like this: "In practice, solving by hand is not used, there is technology available that solves it [...] On the other hand, it is more important that they know how to identify trends, analyse data" This type of discourse in which technical work is not a priority does not appear explicitly in the interviews with the MTs.

In conclusion, the interviews allowed us to identify an applicationist conception of mathematics in the institution under study. Moreover, the internal discontinuity that we hypothesised appears to be characterised by the contrast between a technicist teaching model (in the sense of Gascón, 2001), in which the raison d'être of mathematics education is mainly the resolution of exercises using concrete techniques and another model closer to the conception of mathematics as a modelling tool. These results confirm the local characterisation of the discontinuity described by Castela and Romo Vázquez (2011), González-Martín and Hernandes-Gomez (2019) and Hochmuth and Peters (2020).

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Physics for mathematicians. How physics can help to develop mathematics in higher education.

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Keywords: Higher education, mathematics, physics, anthropological theory of the didactic.

Different studies in the field of didactics have shown difficulties that arise in the teaching of mathematics in relationship with physics at university level. In her opening lecture for INDRUM 2016, Artigue (2016) recalls the work done in the 1980's by a group of didacticians, mathematicians and physicists, where no agreement could be made on the notion of differential and the way to teach it. In this study (*Groupe Maths- Physique Enst. Sup.*, 1989, page 15), the researchers show that mathematics students and teachers are very uncomfortable when they are asked about the validity of differential procedures. Recent research shows that the object 'integral' has different aspects and interpretations (sometimes common to physics and mathematics), between which the students have to juggle when they use integrals in applied contexts (Jones, 2013). This suggests that students who encounter the integral in many different applied contexts develop a deeper understanding of the whole object. Mathematical modeling in a physics context might allow to develop mathematical knowledge. The process of modeling is often described as an iterated process, where the modeling hypothesis is subject to modifications depending on the results obtained at the end of the mathematical work. This implies that the mathematical treatment has to be valid and thus legitimated by a mathematical theory.

Our global research question is: *How mathematical modeling techniques in physics can be related to classical calculus courses content?* We search how the physics modeling contexts can enrich calculus knowledge.

Modeling physics tasks involving integrals in polar coordinates to determine a magnitude sometimes makes use of a technique in which we consider surface elements delimited by two semi-lines going through the origin at an angle of $d\theta$ and two concentric circles distant of dr (where the smaller circle has radius r). This surface element is said to have approximately the area of a rectangle of side lengths $rd\theta$ and dr. The searched magnitude is then obtained if we "sum-up" these areas or a magnitude depending on these areas by the integral: $\iint m\,(r,\theta)rdrd\theta$. We tried to analyze this technique in terms of praxeology (Chevallard, 1985). The theorem that is often cited as a justifying technology to this technique is the change of variables theorem. However this theorem already needs the function to be integrated to be known, and thus it cannot be applied in such a physics modeling situation whose goal is precisely to find this function. Instead we searched if the error we get by this approximation of the area has or not an impact on the limit of the Riemann Sum. Indeed the real surface of such a surface element can be calculated using simple proportionality and the area formula for a disk: $\Delta S = r\Delta r\Delta\theta + \Delta\theta\,(\Delta r^2)/2$. Comparing this to the formula obtained by the approximation by a rectangle, we can determine the error term of the approximation $\Delta\theta\,(\Delta r^2)/2$, for which we can show that its sum can be made arbitrarily small if the partition is fine enough. This technology is embedded in the theory of Riemann integrals.

To further investigate if and how this link could be implemented at the student level, we tried to teach this missing brick to two second year mathematics students who were volunteers and interested in electrostatics. In four sessions of two hours each, we gave them an overview of the physics subject of electrostatics, divided the proof of the missing brick into subtasks in a purely mathematical context (integrate a function over a disk) so that they could do the proof and at the end we proposed them the following modeling task (standard in electrostatics courses): "Determine the force exerted by a uniformly charged disk onto a charged particle situated above its center". For the last session, only one of the students was present. He first wanted to repeat the end of the proof from the session before, to be sure to understand. After that, he followed the proof from the last time for this physics task. This modeling task contained new elements, making the task more complex. For instance, the magnitude to be calculated was a vector instead of a scalar. Here the student correctly explained that we could do the sum component by component, and thus passing to the limit do the integral component by component. He first calculated the vertical component, and then he was asked to formulate a conjecture about the horizontal components of the force. He was able to give a heuristic argument for his conjecture that they would vanish:

E2: We turn around the point […] the components er… in a horizontal direction, well it, it cancels out then.

In physics courses such simplifying techniques that use the symmetry of the objects are explicitly taught. Our student never had such courses, but he was still able to give a correct answer and an argument for this. This gives us new praxeologies to investigate.

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Moving mathematically

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Over the last twenty years, mathematics education research has become increasingly interested in how the body interacts with students' thinking and knowing. However, researchers in this field often theorize the body in diverse ways. From Ingold's post-humanist perspective, humans live in animate bodies which are inseparable from their thinking and knowing. Movement, for Ingold, is not a support for, or an expression of, thinking, rather human bodies think in movement. This paper studies a small group of students, as they engage with a mathematical task, to investigate students' spontaneous mathematical thinking in movement. As this study illustrates, students' spontaneous thinking in movement can offer new and valuable insights into students' mathematical knowing. These findings suggest mathematics educators may need to reevaluate what might be considered students' mathematical thinking in research and in the classroom.

Keywords: Thinking in movement, mathematics education, embodiment, post-humanism.

Within mathematics education, the links between the body and mathematical thinking are an increasing focus of research. Mathematical education researchers employ a variety of theoretical perspectives to investigate movement and thinking. Although these theories generally reject the historical western separation of mind and body in cognition, they often theorize the body from a variety of different perspectives. Recruiting Tim Ingold's (2013) post humanist theory of making, this paper investigates students' spontaneous movement and thinking. For Ingold, new things emerge from the correspondence of animate human and non-human material flows. Following Maxine Sheets-Johnstone (2011), Ingold argues thinking and movement are inseparable: animate human bodies "think in movement" (Sheets-Johnstone, p. 451).

This paper is part of research for a doctoral thesis which explores students' activities as they engage with a mathematical problem task. As part of the wider thesis, this paper investigates a different fragment from, but the same group and session as, Gandell & Maheux (2019). The focus for this paper is an exploration of students' spontaneous full body movement rather than investigating movement as a learning tool. In order to include all aspects of movement, space, body and dynamic qualities, Laban's (Moore & Yamamoto, 2012) movement elements are employed as a movement framework for this study. The aim of this paper is to investigate students' mathematical thinking in movement during a mathematical task. The paper begins with a brief background into the research and theories underpinning the research, describes the research design and movement framework, analyses a small fragment of student activity, and ends by discussing the significance of the findings.

Background

Over the last twenty years, embodiment research, investigating the role of students' and lecturers' bodies in and as their knowing, has gained traction in mathematics education (Abrahamson et al., 2020; Roth, 2016). Although challenging the long-held paradigm, separating mind and body in cognition, embodiment research encompasses a variety of, sometimes conflicting, theories (Abrahamson et al., 2020; De Freitas & Sinclair, 2014; Maheux & Proulx, 2015; Roth, 2016). In addition, many of these theories, which include sensuous cognition, cognitive psychological frameworks (for example, grounded blends, conceptual metaphors, and gesture research), enactivism, and inclusive materialism, hold a variety of diverse perspectives on the body.

As Ingold (2013) and Sheets-Johnstone (2011) explain, embodiment research often appears to continue the divide between mind and body. For example, in some embodiment research, students' movement may be dictated by specified tools (often digital tools), or students may be required to use preplanned movements, to produce a predetermined output (Abrahamson et al., 2020). In this way movement may be conceived as a demonstration of concepts held in the mind with the body positioned as an instrument of that intellectual knowing mind (Roth, 2011). For Ingold and Sheets-Johnstone, however, humans are tactile-kinaesthetic beings, not reified minds enclosed in a body package. As primarily tactile-kinaesthetic beings, humans use their bodies, from before birth and before language, to explore and come to know the world (Sheets-Johnstone). From Ingold's post-humanist perspective, humans come to know as human and non-human material flows correspond in an ongoing, ever changing, "dance of animacy" (p.101). In this way knowledge emerges from the flows of animate materials answering to each other, rather than by building representations of the world in ordered steps. So that, as Sheets-Johnstone explains, humans experience "thinking in movement" (p. 451), not by having thoughts in the mind expressed as movement, nor by having movements creating thinking in the mind, but the movement itself is the thinking.

Research design

To investigate students' spontaneous mathematical thinking in movement, this paper follows the flows of materials forward, as Ingold (2013) suggests. Working together, students offer their actions, including their movements and verbalisations, to each other as indications of their knowing (Roth, 2016). Any actions, made available to others, can then be used by researchers as a representation of the students' knowing, without resorting to presupposing students' intentions or thoughts (Roth). By micro-analyzing students' verbalizations and movements, this analysis is concerned only with the actions the students provide for each other, as they engage with a mathematical task. Thus, students performed actions are taken as their knowing, rather than guessing at students' intentions.

Movement framework

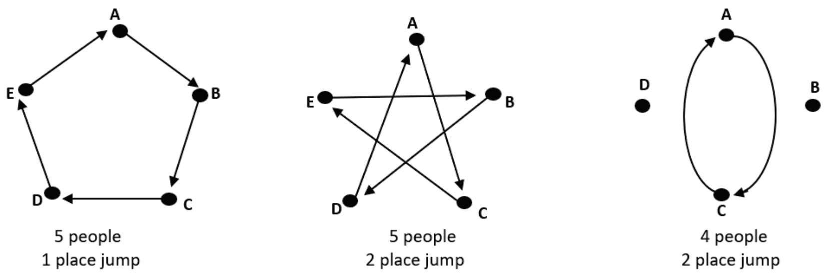
Humans, with similar bodies, share an understanding and recognition of how bodies move and how they feel as they move (Sheets-Johnstone, 2011). This shared social understanding of movement is essential for survival and reproduction in any animate social species. Typically, mathematics education research usually considers how the body moves through space. Inherent in any movement, however, are dynamic qualities which evoke sensations for both the performer and for the observer (Laban & Ullman, 2011/1966, Sheets-Johnstone). Consider how stomping heavily feels different to running lightly and how these movements might feel different for an observer. During the 1930's Laban developed detailed framework of movement elements describing both quantitative (body and space) and dynamic qualitative (effort actions) of movement. Laban's elements are now used in a variety of fields including the arts, industry, psychology research and computer interface technology. Over time Laban's framework has been adapted, however, three elements are generally described: body (parts and actions); space (reach and direction); and dynamic qualities (force and timing)

(Moore & Yamamoto, 2012). Although Sheets-Johnstone's movement descriptions are similar to Laban's elements, Laban's framework provides more detail and is more widely used in movement analysis. Thus, this paper analyzes students' movements using Laban's movement elements.

One difficulty for movement research is how to record dynamic movement in a static paper. A variety of methods have been used including Laban Notation (Laban & Ullman, 2011/ 1966), developed prior to easily available video recording, a series of still images of progressing movement and annotated still images. This paper uses still images with annotated arrows to provide some animation to the still images as Ingold (2013) describes in drawing lines.

The mathematical task

Nic watches a game where a ball is being thrown around a group of people in a clockwise direction. The number of people in the group is called the *people number*. Each time the ball is thrown in a game it is thrown in equal size *place jumps*. Each person throws the ball to the person on their left the same number of *place jumps* away. When the ball gets back to the first person the game ends.



In some games (like the 5-people 1-place jump game and the 5-people 2-place jump game) Nic notices that all the people throw the ball. In other games (like the 4- people 2-place jump game) only some people throw the ball. Nic wonders whether everyone gets to throw the ball in a 4-people 3-place jump game and a 6-people 3- place jump game.

Nic wants to make a dance using this game with people moving between each of the positions instead of the ball being thrown. Nic wants to know if everyone gets to move for different size *people number* and *place jumps*. Create and explain a shortcut that Nic could use for any size of *people number* and *place jump* size. Present this shortcut in the last 5 minutes of the session.

Figure 1: The task

As a modular arithmetic task, the modulus, n, is the people number and the place jump number is repeated addition (multiples of), m. For example, in the five-person three-place-jump game: the multiples of 3 mod 5 are (3, 1, 4, 2, 0/5). As this is the set of numbers in modulus 5, everyone gets to throw the ball (or move/swap places in the dance). A possible shortcut could be written as: if the people number and place jump number don't share a common factor everyone gets to throw the ball (move/swap places).

The setting

This paper is part of a larger thesis studying students' movement, as they explore a mathematical task and follows on from the analysis of students' mathematical problematizing reported in Gandell and Maheux (2019). For Maheux & Proulx (2015) is the posing and solving of smaller self-generated problems in response to a mathematical task. The environment and task were intentionally designed to elicit students spontaneous full body movement.

The participants, four non-maths major students aged 18 and 22 years, were recruited from a six-month tertiary bridging programme which provides entry into degree and diploma programmes. During an hour-long session, the students engaged with a mathematical task in an open room with no tables and chairs. The task (Figure 1), printed on A3 paper, was attached to one of several vertical whiteboards positioned at the edges of the room with whiteboard markers and magnetic counters. All student activity was captured by three video cameras positioned at the edges of the room. The fragment transcribed below, occurs thirteen and a half minutes from the beginning of the session, and six and a half minutes after the first fragment transcribed in Gandell & Maheux (2019). The session began with a movement warm up, led by the researcher, who then invited the students to move freely around the room. The students then alternately used the vertical whiteboard and task sheet and acted out two games from the task (a four-person three-place-jump game and a six-person three-place-jump game) in the open area of the room. In the first enactment the students used a counter as a ball, in the second enactment the students changed the game into a dance, as requested by the task (Figure 1).

Mathematical thinking in movement

At the beginning of this fragment the students have just completed acting out a six-person, three-place-jump game, which they call a six-three. As 3 modulus 6 has only two multiples (3, 0/6) only two people move, during the enactment of the game, which the students verbalize as swapping places. Returning from the open area of the room to the white board and task sheet they utter "that one doesn't work". The group, Chas, Kit, Ala and Paige, stand quietly for a few minutes. Kit first problematizes finding a formula, and then how to change a six-three game so that everyone can move.

1	Kit:	A six-two for everyone to move I think (suddenly dabs three discrete positions with this right index finger, inscribing a circular path in front and to his right side with this right arm, and gazing first between Chas and Ala then towards Ala and Paige, Figures 2 a, b and c).

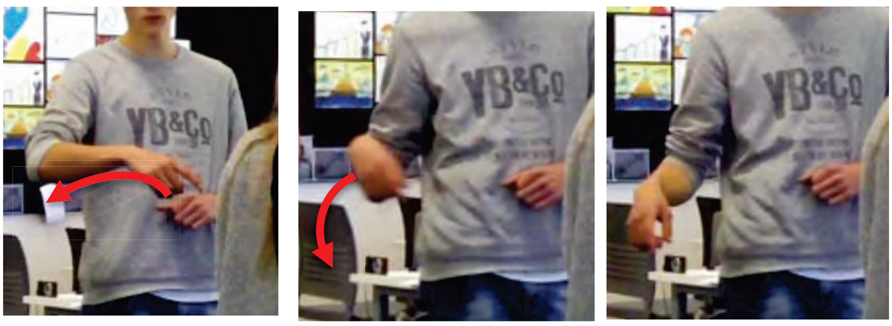


Figure 2: a) Kit dabs position one b) position two c) and position three

For Sheets-Johnstone (2011) “thinking in movement is not that the flow of thought is kinetic, but that the thought itself is” (p. 421). Thinking in movement does not require students to learn which movements will produce specified answers, or to pre-plan movements to express ideas: thinking in movement is the spontaneous activity of a dynamic thinking body. In the fragment above Kit demonstrates spontaneous, self-generated, mathematical thinking in movement providing a mathematical solution for his problematization of how many people move in a six-person two-place jump game (line 1).

In Line 2 Kit verbalizes that everyone will move, for a six-person two-place-jump game and performs a movement indicating three positions in a rough circle with his right index finger (Figure 2). Kit performs this movement very quickly, with the three positions distinguishable by the variations in dynamic qualities. The curved paths between each of the positions have a light, gliding quality, like a bounce. Kit pauses, at the end of each bounce, with a heavier and more bound quality, like a dab. Although these bounces and dabs serve to indicate different three positions to his right front and side (Figure 2), the sudden and indirect qualities of this movement give Kit’s performance the feeling of a sketch. With his movement in line 2 Kit appears to be trying out, rather than defining, the solution of three positions for the game.

Kit clearly provides a solution for the six-person two-place-jump game with his movement: a solution not available from his verbalisation that everyone moves. No previous movements in this session have sketched a solution using Kit’s bounce and dab movement, so Kit is not reproducing a movement. Thus, a new mathematical movement has emerged spontaneously from a moving dynamic body. As Kit’s movement (Figure 2) does not replicate his verbalized solution the movement cannot be a pre-planned or pre-thought embodiment of that verbalization. In line 2, then, Kit performs a mathematical solution in movement: Kit is thinking in movement as Sheets-Johnstone (2011) describes.

Evolving thinking in movement

2	Chas:	The trouble is instead of swapping (elbows bent index fingers touching, right finger traces horizontal curve forwards, left hand traces straight line backwards and up, Figure 3a) you go around (spiral trace with right arm across left, up and forwards, Figure 3b).

Figure 3: a) “instead of swapping” b) “you go around”

3	Paige:	(lifts hands to hips) yeah that’s what I was thinking
4	Chas:	(touches left and right index fingers in midline) it goes around to the next person (spiral right arm trace left, forward and up across his body) instead of swapping (index fingers touching, traces two lines right hand forward, left hand backwards and up then rolls right hand backwards under left hand going forwards, then rolls right hand forwards under left hand going back, Figure 4a). So that’s the swap (Rotates right and left hands around each

other tracing horizontal circles Figure 4b. Holds final position index fingers pointed upwards for 2 seconds, Figure 4c).

Figure 4: a) over and under rolls b) spiral around vertically c) final held position

5 Chas: From there (points right arm to left side, rotating torso and head to left, Figure 5a) goes around to (traces horizontal circle around body with right arm, rotating torso, to point to right side. Holds arm extended to right and looks back to Kit, Figure 5b) … the next person

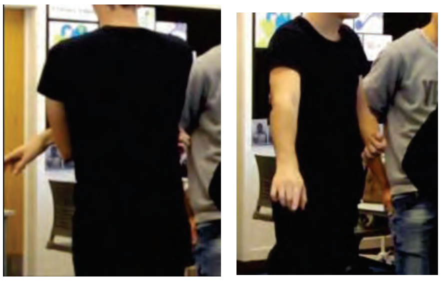


Figure 5: a) "from there" b) "to the next person"

Thinking in movement arises from a body that resonates with the world (Ingold, 2013; Sheets-Johnstone, 2011). Rather than reified minds creating symbols and representations of the world, new things, including mathematical things, emerge as tactile-kinaesthetic bodies correspond with animate material flows. Thus, thinking in movement is emergent, evolving and adapting to an ever-changing environment, and may take many different forms (Sheets-Johnstone). Chas, in lines 2, 4, and 5, demonstrates an evolving thinking in movement which contrasts with his almost unchanging verbalizations.

In line 2, Chas verbalizes and performs two problematizations which he differentiates both verbally and by using different space, body and dynamic movement qualities. For the first problematization Chas traces a line with each hand (Figure 3a) as he utters "instead of swapping back". In the second problematization Chas performs a spiral trace with his right arm (Figure 3b) uttering "you go around". The small, direct, straight line "swap" movements are performed with two hands, tracing a mainly horizontal pathway. In contrast, for the larger spiral "around" movement, Chas's right arm traces a path to the left, across his midline, then up and forward, through horizontal, vertical and transverse planes, with a sustained, indirect, light, floating, quality. Although Chas uses different words for his verbalizations, how "swap" might be different to "around" is not clear without his movements. Chas' verbalizations, then, appear to be labels supporting his movement problematizations.

In line 4, Chas begins by repeating the swap and around problematizations, with little change to his movement or verbalizations, but in reverse order. Immediately after he repeats the 'swap' movement from line 2, Chas begins a new movement (Figure 4a) rolling his hands over and under each other in vertical semi-circles. Chas then verbalizes “so that’s the swap”, as he spirals his hands around each other in a horizontal plane (Figure 4b), finally stopping and holding the position in Figure 4c. Thus, in line 4 Chas performs three distinct “swap” movements.

These “swap” movements, which are performed in quick succession, are differentiated by transforming dynamic qualities from sharp, bound and direct to more continuous, freer and indirect. In this way the “swap” movement appears to enfold the dynamic qualities of both the “swap” and “around” movements, initially performed in line 2. In addition, the movement pathways change from almost straight lines to semi-circles and finally to spirals. Although performed with two hands the spiral pathway of this final swap movement (Figure 3b) reflects the spiral of the initial “around’ movement from line 2. However, while Chas performs ever-evolving “swap” movements in line 4, he continues to verbalize these movements as a swap.

Finally, in line 5, Chas places himself in the centre of the movement and performs a large continuous, sustained horizontal curved pathway with his right arm. By combining a curved path, a sustained floating dynamic quality, and the use of one arm this final movement performs some elements of the “around” movement from line 2. However, in this movement Chas also includes elements of the line 2 swap movement, performing a more horizontal, single direction, pathway and a more direct dynamic quality. Thus, Chas’s movements (lines 2, 4 and 5) evolve and, by merging elements of both the “swap’ and “around” movements, seem to resolve the problematization of how counting around becomes swapping places in the game as a dance. Although Chas’s movements transform, in lines 2, 4 and 5, his verbalizations remain very similar, with his final verbalization indicating positions rather than referencing any resolution to his problematizations. By adapting and evolving his movements to merge his two problematizations, Chas shows how thinking in movement may evolve and change as a dynamic body resonates with an ever-changing environment.

Discussion and conclusion

For Sheets-Johnstone (2011) humans move and know the world through their tactile kinaesthetic bodies. Humans cannot remove themselves from their bodies and think in some disembodied mind. As this paper demonstrates, students’ movement is not only integral to their thinking and knowing, students think mathematically in movement. Kit’s movement illustrates how students animate bodies perform mathematical thinking: thinking that may not necessarily be articulated, expressed or made available in any other way except in movement. Similarly, Chas demonstrates how thinking in movement can emerge and evolve, even while accompanying verbalizations remain static. Thinking in movement is not about making decisions about how to move or where to move, rather spontaneous movements emerge and evolve in correspondence with an animate world (Ingold, 2013; Sheets-Johnstone).

For a long time, western mathematics education has de-privileged movement and the body in mathematical thinking. By ignoring students’ spontaneous thinking in movement, mathematics educators and researchers may be missing valuable instances of students’ mathematical thinking and knowing. As Sheets-Johnstone (2011) explains

> “thinking in movement is a way of being in the world, of wondering or exploring the world directly, taking it up moment by moment and living it in movement, kinetically. Thinking in

movement is clearly not the work of a symbol making body, a body that mediates its way about the world by language, for example, it is the work of an existentially resonant body" (p. 425).

Although many educators approach movement in the classroom in many different ways, movement is not simply a learning tool. Students' spontaneous movements provide access to mathematical thinking and knowing that may not otherwise be available. To fully understand students' mathematical thinking, mathematics educators need to develop approaches that recognize and support students' thinking in movement, rather than considering movement as an adjunct to verbalization or an expression of concepts held in a mind. In research, and in the classroom, mathematics educators need to consider movement not just as another resource for learning, providing space and support for students to move, mathematics educators also need to pay closer attention to students' mathematical thinking in their spontaneously performed movements.

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Coping strategies: A rather neglected perspective of research on first year university mathematics students' goals and strategies

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Research on undergraduate mathematics students' goals and strategies has mainly focused on students' learning and performance goals and their learning and problem-solving strategies that correspond to these goals. This paper presents results from an interview study on self-regulated learning with 18 first-semester mathematics students, which indicate that, in addition to those, well-being goals and coping strategies may provide an important contribution to understanding students' engagement with mathematics at university. These results provide approaches to a more holistic view of the everyday learning of mathematics students at universities with possible implications for research and practice, which are discussed.

Keywords: Mathematics Education, Higher Education, Learning Strategies, Coping

Introduction and theory: Students' goals and strategies

Research on undergraduate mathematics has grown steadily in recent years, particularly with regard to the transition from school to university mathematics and revealed many difficulties of students with the mathematical contents at tertiary level (e.g., Biza et al., 2016; Gueudet & Thomas, 2020). Reasons for these difficulties are considered to be the differences and ruptures between the mathematics taught at school and the mathematics taught at university, but also the institutional settings (e.g., Gueudet, 2008). For example, the proportion of time, where students have to shape and regulate their own way of learning is significantly higher at universities than at schools and, according to the study regulations of German universities, amounts to about two thirds of the learning time for the mathematics modules of the first academic year. This increases the importance of self-regulated learning at universities which will be examined in the present study with a focus on first-year mathematics students' goals and strategies.

There are many different models of self-regulated learning with varying definitions and foci (e.g., Panadero, 2017; Zimmerman & Schunk, 2011). Consistent with most of these models, self-regulated learning can be defined as *"an active, constructive process whereby learners set goals for their learning and then attempt to monitor, regulate, and control their cognition, motivation, and behavior, guided and constrained by their goals and the contextual features in the environment"* (Pintrich, 2000, p. 453). This definition shows the importance of goals in models of self-regulated learning. A distinction is often made between *learning goals* which focus on increasing competence and *performance goals* which focus on the attainment of positive judgments, e.g., good grades (Pintrich, 2000). Additionally, the dual processing self-regulation model of Boekaerts (2011) takes *well-being goals* into account that focus on preventing threat and harm to the self. This model theorizes that when tasks are appraised to be a threat to the self, *coping strategies* such as seeking social support, join with people who have the same concern, wishful thinking or working hard, are used to deal with the stressful situation (Boekaerts, 2011). Generally, we define *strategies* as goal-oriented behaviors and thoughts, which can be distinguished by the goals they aim to achieve: *Learning strategies* aim at learning, *problem-solving strategies* aim at solving a problem, *coping strategies* aim at preventing threat and harm to the self. This is in line with Weinstein and Mayer's definition of learning strategies which *"can be defined as behaviors and thoughts that a learner engages in during learning and that are intended to influence the learner's encoding process"* (Weinstein & Mayer, 1986, p. 315).

Research in university mathematics education has provided important insights into mathematics students' general learning strategies surveyed via questionnaires (e.g., Griese, 2017; Lahdenperä et al., 2019; Liebendörfer et al., 2021, 2022), students' problem-solving strategies (e.g., Pólya, 1945; Schoenfeld, 1985; Selden & Selden, 2013; Sommerhoff et al., 2021), their activities learning new mathematics (Wilkerson-Jerde & Wilensky, 2011), types of students' engagement with proofs (Selden & Selden, 2017) as well as their proof-reading strategies (Panse et al., 2018; Weber, 2015). As with any research, the explanatory power of such studies for students' actual day-to-day mathematics learning is framed by the questions asked and, possibly, limited by the study design of the particular studies. For example, when participants *"were asked to read the paper and try to understand it such that they would be able to teach it to a colleague"* (Wilkerson-Jerde & Wilensky, 2011, p. 27) the learning goal "to understand the paper" and the strategy "to read the paper" is given by study design. However, it is not clear in which situations of everyday engagement with mathematics students pursue such goals and use such strategies, or whether other goals and strategies might occupy a more prominent position.

Studies which rather try to capture students' usual everyday strategies suggest that, when working with textbooks, students' rather spend their time on searching worked out examples or for surface similarities with a procedure and rather not to read the (closed) text (Lithner, 2003, 2004; Weinberg et al., 2012). On a broader picture, it seems that first year mathematics students spend more time on searching for worked-out examples in books, their lecture notes, or on the internet or on working in peer groups trying to solve mathematics exercises (and seem to consider this their main responsibility) than trying to recall their lecture notes to understand mathematical theory (Göller, 2020, 2021; Gueudet & Pepin, 2017). This is in opposition to expectations of lecturers, who rather anticipate that students work on their lecture notes, especially on the proofs of the theorems, and then are able to work on exercises by themselves (Gueudet & Pepin, 2018). Of course, such expectations of lecturers vary and, in addition to personal conceptions, depend on the institutional setting in which the lecture takes place. Therefore, in the following, the organization of typical first semester mathematics courses at German universities will be briefly described.

Goals implied by the institutional settings of first year courses at German universities

First year mathematics modules at German universities often consist of lectures and related exercises. The lectures introduce mathematical theory, i.e., definitions, examples, theorems and their proofs are presented. The exercises are handed out weekly and worked on by students in self-study. Students submit their solutions which are corrected, graded, and discussed in a separate lesson. To pass such a module, usually a certain number of exercises (often 50% of all exercises) must be solved correctly and a written exam has to be passed.

These formal requirements imply at least two types of performance goals which are necessary to pass such mathematics modules: *Generating solutions to the exercises* and *passing the exam*. In addition, the division of the courses into a lecture and exercises may imply certain learning goals: For example, the *comprehension or understanding of the lecture contents* and the *development of problem-solving competencies* to be able to solve the exercises.

Research questions

This paper aims at investigating the everyday self-regulated learning of first-year university mathematics students within the described institutional setting based on the following questions:

RQ 1: Which study-related goals do mathematics students report in their first year of study?

RQ 2: Which study-related strategies do mathematics students report in their first year of study?

RQ 3: How can these strategies be explained as the result of a balancing of goals?

Research methods

In order to investigate these questions, problem-centered interviews (Witzel, 2000) were conducted with a total of 18 students (13 of whom were female) at up to four interview times in their first year of study. Ten interviewees (9 female: T1-T9, 1 male: T10) were enrolled in a degree program for *mathematics teachers at upper secondary level*, six (3 female: M1-M3, 3 male: M4-M6) in the degree program *Mathematics B.Sc*. and two (1 female: B1, 1 male B2) in the degree program *Business Education*. The respective interviews had a duration of about 45 minutes, were audio-recorded, completely transcribed, and analyzed using methods from Grounded Theory Methodology (Strauss & Corbin, 1990). To pre-structure the data material, passages in which the interviewees talked about their goals and strategies were identified first and then these were coded inductively regarding further subcategories (cf. Göller, 2020 for a more detailed description). The interview excerpts presented below were translated from German by the author.

Results

With regard to RQ 1, learning and problem-solving goals as well as well-being goals were reported (see Table 1) and could be identified as influential for students' strategies.

Table 1: Categories of reported goals

Category	Quote example
Performance goal: Pass exam	I said from the beginning, even when I started studying, the main goal is to pass math, nothing else matters. (T1)
Performance goal: Solve 50%	My goals are to get at least 50 percent of the points on the exercise sheets. (T2)
Learning goal: Understand lecture notes	To actually understand it. That is my goal. To actually understand what I wrote down in the lectures. Maybe not everything, that will be impossible, I think. (T3)
Learning goal: Solving ability	You also have to be able to solve exercises. (M1)
Well-being goals	Because I'd also like to enjoy the time a little and don't want to drown in stress. (T4)

Among these reported goals, the interviewees' overall focus was primarily on achieving the institutionally defined performance goals: Pass the exam and solve at least 50% of the exercises. It could be observed that especially when the achievement of these goals is in question, learning goals were inhibited or postponed to a later point in time.

The strategies were dominated by the work on the exercises (RQ 2). Thus, work with the lecture notes was comparatively rarely reported and, in many cases, even explicitly negated. A frequent reason given is that the exercises take so much time that there is no time left for working on the lecture notes:

> T3: I really don't have time to look at the lecture notes again. Well, I'm kind of geared up to try to finish this exercise sheet somehow within the week. And most of the time I sit there at ten o'clock on Wednesday evening and write down these exercises. That's basically all I'm trying to do in math, because time just isn't there.

At the same time, experiences of success when working on the exercises often occurred in the context of working through the lecture notes. But even for students who had realized this for themselves, working through the lecture notes was a question of time resources:

> M1: Sometimes I even manage to learn the topic before handing in the exercise. And that's when I notice that it really makes a difference. Because if you have learned the topic beforehand and really understood it, then the exercise doesn't take ten hours. Then it takes maybe two or so.

Table 2: Categories of strategies for exercise solutions with subcategories for "work with others"

Category	Quote example
Work with lecture notes	I start with the exercise where I think you could make good progress, to get a positive feeling. Yes, then I look in the lecture notes to see if there's anything I can use. This semester I have already been able to use the lecture notes surprisingly often. (B2)
Problem solving strategies	I always write it down: What do I have? What do I want? What do I have to show? (T6)
Use other materials	I occasionally also look on the Internet. (M4)
	Sometimes you can find something in books that helps a lot. (M2)
Work with others	And analysis, well, we do that in a group, that's much faster than doing it alone. (T7)
- Compare solutions	For example, today we also meet and do the exercises together and then talk together about who has solved and how we solved which exercise and so on. (M2)
- Teamwork	We complement each other. Sometimes he has things that I don't have, and vice versa. (B2)
- Ask for hints	And if I get stuck, I write to someone and ask if he can give me a hint. (B2)
- Comprehend solutions of others	So, when other people already solved it, that I then look at it and try to understand it for myself and write it down again. (T4)
- Copy solutions of others	Exactly, that's why we just bluntly copied some things without comprehending what was being done. (T5)

Regarding the work on the exercises, different strategies can be distinguished, which are listed in Table 2. The subcategories of *work with others* given in Table 2 can be seen as arranged in descending order for the achievement of learning goals (also from the students' perspective) and thus describe a gradual transition toward coping strategies. Regarding RQ 3, the respective situationally used strategy then depends on a balancing of situational goals and self-assessed internal (time, ability, effort) and external (materials, other people) resources. Although these strategies varied situationally, there were students who preferred to work on their own and rated related strategies rather high,

M4: I usually work completely alone, and I refuse to copy solutions anywhere, which might mean that other people then have more points, but I think I've built up more understanding by the end of the semester. That's how I would evaluate it now.

while others rather worked in groups to generate exercise solutions in teamwork:

T5: So, the other girl, who is with me then, that's more the one who also looks up a bit what we've done or something. She also has ideas from time to time about what to do about it. I think she also looks at things at home from time to time to understand. Then we have another guy. He also does some calculations on the weekend. So, he works on the things himself. He also wrote the best exam of all four of us. Then there's the guy who failed everything, he's just not the brain, but he, I don't know how he manages it, but every now and then he listens to the lecture, and then he can summarize the things that the professor explains in a totally complicated way, he can simply apply them to school knowledge in a way that every eighth grader will understand. And he always makes these simpler connections. And so, he can derive these proofs. But apparently that didn't work in the exam. And I'm the one who always looks at all the theorems. I do not do anything before or something like that, but we really sit down on Tuesday and do the exercises. Then I look at the exercises. Or get solutions or something. From someone else. Or some ideas. And I'm also pretty good at searching for definitions. So, I find definitions and find parts: Oh, this could be important. And then the girl takes that and forwards that. So, it's a good team, I would say.

One main reason for working in groups seemed to be that many students did not feel able to solve the exercises on their own within a reasonable amount of time (from their point of view):

M5: First, because of the motivation, because otherwise I would have to work on it alone. But it also helps to have the others because the ideas come from several people. You could also say that it might not be so good if you don't have all the ideas yourself, but then we wouldn't be able to finish in time. So, if I had to do it alone, then I would need much more time, much more.

Working in groups with all its sub-strategies (cf. Table 2) thus did not only address students' learning and performance goals, but also their motivation and well-being goals, i.e., it is (also) a coping strategy. The following interview excerpt provides a more detailed insight into this balancing and negotiation of different goals with regard to available resources:

T3: And with the exercises, it's pretty much the only thing I do. But even there, I have to admit, I don't really engage with it. Because we have such an exercise group now, and somebody always has the solution. And most of the time I just write it down to get my permission [50%], because there is no time for it. […]

Interviewer: But I mean, it was a little different last semester, right?

T3: Yes, but that was simply because we had no solutions [from others]. There was no other option for us. That was simply the fact. But it was actually better that way. I

> must say quite honestly. Because you don't get anything out of it. Everything just passes by. And that's exactly the problem. And actually, I have to say, somehow, I mean, I got my admission in Ana[lysis] I, too. And somehow, I understood more. So, it could have been done in Ana[lysis] II as well. And, I mean, I somehow found time to make these exercises at some time. I just have to say that at that time I sat there until two o'clock in the morning, sometimes on Wednesdays, and did math. Fortunately, I don't have to do that anymore and I'm actually glad that I don't have that stress anymore. But, as I said, I don't get anything out of it.

Discussion

This paper tries to give a qualitative insight into first year mathematics students' everyday strategies while dealing with the challenges of their study. The results indicate that a closer look at coping strategies can contribute to a better understanding of students' everyday engagement with mathematics at university. Boekaerts' (2011) theory predicts such coping strategies, as the exercises pose a potential threat to well-being because they often frustrate self-confidence on the one hand and can actually determine the success or failure of the module on the other. The results of this study highlight the importance of students' balancing and negotiation of different goals, considering their respective resources, to understand their use of learning, problem-solving, and coping strategies.

Implications for research and practice

The categories of goals and strategies presented here are based on different theoretical perspectives and research practices. Questionnaires usually operationalize theoretical considerations on general learning strategies (e.g., Griese, 2017; Lahdenperä et al., 2019; Liebendörfer et al., 2021, 2022), observational studies from research in undergraduate mathematics education often examine problem-solving strategies or proof-reading strategies in a more clinical setting (Panse et al., 2018; Weber, 2015; Wilkerson-Jerde & Wilensky, 2011). This research is undoubtedly important but seems not to represent the totality of the everyday strategies used by students. Well-being goals and coping strategies are rarely represented in common questionnaires and are rather prevented in observational studies by study design. However, based on the data presented here, they seem to offer an important perspective for a deeper understanding of mathematics learning at universities regarding the respective institutional settings and their influence on students' learning.

The requirement, e.g., to achieve at least 50% of the possible points for the weekly exercises entails that most of the time spent on mathematics is spent working on exercises. Students invest a lot of time and effort on the exercises, while working on the lecture notes is often neglected. In addition, this external pressure provokes coping strategies when students feel unable to meet the requirements on their own. From the data of this study, however, it is unclear how an omission of the 50% barrier would affect students' goals and strategies. On the one hand it seems that the emphasis on such extrinsic incentives inhibits or postpones learning goals, on the other hand it seems that at least for some students the absence of these extrinsic incentives (e.g., because of the possibility to copy solutions of others) leads to a situation in which they hardly engage with the mathematical content at all. Accordingly, further research in this direction is desirable in various institutional settings.

In the institutional setting described, exercises can be seen as an important institutional tool to regulate students' strategies. Accordingly, goals and strategies considered essential should be represented as explicitly as possible in the exercises. For example, special task formats can be considered which support the work on lecture notes, in particular the reading of proofs, if these are identified as important institutional goals. In terms of coping strategies, attention must also be paid to the difficulties involved in the exercises, to enable as many students as possible to participate in working on the exercises in a way that promotes learning and at the same time challenge all students.

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Understanding how students learn programming for mathematical inquiry at university: Schemes and social-individual dialectics

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The instrumental approach, and in particular the concept of scheme, can contribute to our understanding of how students learn programming for mathematical inquiry at university. While most studies consider only individual schemes, in this paper we propose to investigate schemes' social aspects, focusing on the scheme of "Validating the programmed mathematics". Through a questionnaire conducted with students over 2 years, and interview data, we identify shared rules-of-action developing over time. Deepening the analysis for the case of two contrasted students, we observe that common rules-of-action can be associated with different theorems-in-action.

Keywords: Social aspects of schemes, Programming, Instrumental approach, Mathematical inquiry.

1. Introduction

The study presented here is part of a larger 5-year project, whose aim is to better understand how students learn programming for mathematical inquiry at the university level. In our previous work (Buteau, Gueudet, et al., 2020; Gueudet et al., 2020) we have shown that the instrumental approach (Rabardel & Béguin, 2005; Trouche, 2003) and the concept of scheme (Vergnaud, 2009) can contribute to this aim. This concept allowed us in particular to investigate how students develop operational knowledge (knowledge that provides means to do and succeed, Vergnaud, 2009), through instrumental geneses. We focused only on the individual level: the schemes developed by a given student. Nevertheless, at university, students learn in a social context, and most probably their schemes also comprise common elements. Some aspects of the operational form of knowledge are shared; identifying these aspects is essential in particular to inform teachers' orchestrations (Trouche 2003) in this context. The question we investigate here is: Can we identify patterns in the culture of a community of students learning to use programming for mathematical inquiry?

In terms of research, we claim that studying this issue is an important theoretical and methodological contribution, because the instrumental approach has been used mostly for studying individual learning processes. In terms of teaching, we have observed in a previous study (Buteau, Muller, et al., 2020) that insights into students' schemes can be very helpful for teachers orchestrating their learning of programming for mathematical inquiry.

Our study takes place in the context of a sequence of three university mathematics courses called *Mathematics Integrated with Computers and Applications* (MICA I-II-III) taught at Brock University since 2001 (Buteau & Muller, 2010). In these project-based courses, mathematics majors and future mathematics teachers learn to use programming for mathematical inquiry (e.g., using programming to simulate a battle between two armies; see section 5).

2. Theory and research questions

We draw from Vergnaud's (2009) theory of conceptualization, distinguishing between an operational form of knowledge (knowledge that provides means to do and succeed) and a predicative form of knowledge (knowledge that consists of means to express ideas in words or symbols). We argue that Vergnaud's theory is relevant for our study because operational knowledge is very important when learning to use programming for mathematical inquiry.

Operational knowledge's development and evolution are captured by Vergnaud (2009) through the concept of scheme, which is central in our study. A scheme is an invariant organization of activity for a certain goal and consists of four components: the goal and sub-goals; rules-of-action (RoAs), to generate action, information seeking, and control; operational invariants (concepts-in-action, which are concepts considered as relevant, and theorems-in-action (TiAs), which are propositions considered as true); and possibilities of inferences.

We also use the instrumental approach, particularly its theory of instrumental genesis, which conceptualizes the process of how users (learners) transform an artefact (a human product, designed for a goal-directed activity) into an instrument for a specific goal and situation (Rabardel & Béguin, 2005). For Rabardel and Béguin (2005), an instrument is a hybrid entity: partly an artefact and partly scheme(s).

According to Rabardel and Béguin (2005), schemes have both a private and a social dimension. The private dimension is specific to each individual, while the social dimension reflects the fact that schemes may be shared by members of social groups. The social dimension also may be a consequence of schemes developing during a process involving individuals who are not isolated; for example, classmates working on assignments in a shared space such as a computer lab. Schemes may be shared informally or prompted/promoted formally by training, such as in teaching and learning situations (e.g., through assignment guidelines, lectures, etc.).

This points to the possibility of the development of social dimensions of schemes in teaching situations. In fact, Trouche (2003) claims that through instrumental orchestration, one aim of the teacher is to reduce the variability of the individual schemes students develop, in order to strengthen the social dimensions of schemes; for instance, reflecting those shared by an intended community of practice, such as that of mathematicians (Lave & Wenger, 1991).

The above theoretical elements lead us to refine our initial question as follows: Can we identify common elements in the schemes developed by students learning programming for mathematical inquiry? For a common goal, do students share common RoAs, and if so, are these rules associated with the same operational invariants? How do these common elements develop over time, in a course, and over several years of courses?

3. Methods

In our past work, we developed a general model of students engaging in learning to use programming for mathematical inquiry (Buteau & Muller, 2010), which enabled us to identify common goals in the schemes developed by students. For this study, we focus on one such goal, namely "Validating the programmed mathematics", considered as particularly important because it associates mathematics and programming. For this goal, we further focus on RoAs because most operational invariants and inferences cannot be made explicit; RoAs are more explicitly identifiable.

We mainly draw from two types of data: a questionnaire and individual interviews. The questionnaire was given to students at the end of each MICA course over 3 years (2019–2021), and included questions on demographics, confidence levels in programming (mathematics), et cetera. Since 2020, it also included two questions seeking insights about two specific schemes. Each of these questions presented a list of RoAs in a matrix format and a 5-point Likert scale for students to answer if they used the RoA always, often, sometimes, rarely, or never. Most of these RoAs were identified in our previous work analyzing a student's activity (Buteau, Gueudet et al., 2020), and some more were added based on our experience as mathematicians researching with programming or teaching programming for mathematics investigations. The questionnaire was answered by 30 anonymous volunteer students, in addition to another 13 volunteer students whom we have been following closely throughout their MICA courses (e.g., using also individual interviews).

We used relative frequency bar graphs for responses to each of the questions and for responses regrouped into three categories (always/often, sometimes, rarely/never) in order to identify RoAs that could be considered as social. To do so, we used an arbitrary threshold of 70% of the regrouped always/often responses. We note that the sample of 43 participants is not necessarily representative of the entire MICA community but can nevertheless give insights into potential social dimensions of schemes. For triangulation purposes, we also called upon interview data. For this paper, we selected data collected in Year 2 (2019) of our research since we had the opportunity to collect data both from an instructor (Bill) and some of his students. In one of the interviews with this MICA II instructor, we showed him the list of RoAs and invited him to comment on whether, according to him, students enact the RoAs, and to elaborate about his related guidance.

Finally, we went to individual student cases. Two among eight student participants enrolled in Bill's MICA II class were selected due to their more reflective answers in interviews and for their differing profiles: Kassie is a female student enrolled in the mathematics and education program who had no programming background prior to starting her university studies and Mark is a male computer science and mathematics co-major student who had a significant programming background prior to MICA I. After each of the programming-based mathematics inquiry projects (five assignments), individual semi-structured interviews were conducted. We used guiding questions incorporating "explicitation" techniques (Vermersch, 2006) to help students relive their actions during the development of their investigation projects. Kassie and Mark's interview data were coded to identify potential elements of schemes (in particular, the "Validating" scheme) according to their RoAs and TiAs (Gueudet et al., 2020), which were then confronted with their assignment reports. The outcomes of this analysis were organized in terms of RoAs and summarized for each project in a common Excel table, to which we added Kassie and Mark's own questionnaire responses as well as the relative frequencies of the overall questionnaire responses. Using these tables, we identified individual elements of the "Validating" scheme for Kassie and Mark, respectively.

4. Identifying social aspects of a scheme

In this section, we present the results of our analysis identifying the social RoAs for the "Validating" scheme (Buteau, Gueudet, et al., 2020), as part of the process of creating a program for conducting a mathematical inquiry. We consider an aspect of a scheme to be "social" when it is shared by a social community. In our analysis, we realized that RoAs may be identified as social (or not) depending on whether the community in which they are shared comprises the students in MICA I or the students in upper MICA courses (i.e., MICA II and III). These two communities are not independent: We might even see the MICA I student community as in the process of "becoming" or "developing" into an upper year MICA community. This relates to the instrumental lens, whereby the study of students' instrumental geneses is considered as progressing over time. Hence, we will not consider as our reference community all MICA students. Note that we grouped 2nd- and 3rd-year students (into upper MICA) to have more appropriate sample sizes for comparison purposes (N=18 for upper MICA, N=25 for MICA I). This also reflects the implementation model in these courses: MICA II and III invite students to use programming skills to engage in mathematics inquiry, while MICA I is also focused on developing students' programming skills (due to their lack of background in programming).

RoA	Upper MICA	MICA I
1. I check a few times as I program by compiling with a few inputs.	94, 6, 0	72, 28, 0
2. Once I have programmed it all, I run the program with a few different inputs and compare the output with my hand calculations.	89, 0, 11	92, 8, 0
3. I compare the output of my program with that of a peer or with examples from the internet.	83, 6, 11	40, 24, 36
4. I compare my program with that of a peer.	72, 17, 11	32, 24, 44
5. I ask someone (a peer, a TA, the instructor, etc.)	61, 22, 17	52, 20, 28
6. I trust that I translate in vb.net/python what I do on paper. († This is the original wording in the questionnaire. However, we acknowledge that this formulation rather points to an operational invariant and a 'no RoA'.)	56, 22, 22	28, 32, 40
7. I use other technology (Maple, Desmos, etc.) to generate an example and compare it with the output of my program.	28, 50, 22	64, 24, 12
* I don't really know if it works. (* We label this statement differently because it is not a rule-of-action. It is more an indication that students do not have a fully developed scheme for validating the programmed mathematics.)	11, 22, 67	8, 24, 68

Figure 1: Percentages of participants who say they do the RoA always/often, sometimes, rarely/never, in response to the question "When I program a mathematics concept, I know that it works because…"

Figure 1 depicts findings from the online questionnaire, listing the RoAs in the order of highest to lowest percentage of upper MICA participants stating that they use it always or often when validating a programmed mathematics process.

While these RoAs can seem to concern programming in general, we note that the question invites the students to answer only about programming a mathematics concept. Some of the rules are directly linked with mathematics: calculating by hand (RoA 2), or with Maple (RoA 7). Moreover, when the scheme is mobilized in a specific mathematical situation, these rules are specified in relation to the mathematical contents, as we illustrate in the following section.

The top four RoAs are used by more than 70% of the upper MICA participants, suggesting that those RoAs may be social components of the validating scheme for the upper MICA community.

RoAs 1 and 2 appear to be social also in MICA I. For RoA 2, the proportions are similar: 89% and 92%, respectively. One reason for this may be that the lab and assignment guidelines explicitly encourage students to check their output with hand calculations, starting in MICA I. In contrast, for RoA 1 the proportions have a greater difference: from 72% in MICA I to 94% in upper MICA. One way to explain this is that RoA 1 goes together with the practice of coding incrementally. The MICA II instructor from our study, Bill, explains:

> Bill: They will often test. … They develop programs incrementally, so they'll write … just a couple of … a loop or something and they just [say]… see: does that make sense?

Such a skill may require time to develop. Also, the mathematics problems explored in upper MICA lead to more elaborated programs, which can encourage students to code incrementally. This is in contrast with RoA 2, where students check after the program is completed.

RoAs 3 and 4 appear to become social only in the upper MICA community. For RoA 3, the proportion increases from 40% in MICA I to 83% in upper MICA. This could be related to upper MICA students having become more confident in their own skills and more comfortable with their peers; hence they may be more willing to interact and share with their peers. We note that interaction and sharing among MICA students, as well as comparing output with examples from the internet, is also encouraged by instructors. Bill confirms that students do RoA 3 and says:

> Bill: There are places where the answers are on the internet and I encourage them to check that their program gives that output.

Similarly for RoA 4, the proportion increases from 32% in MICA I to 72% in upper MICA. Bill again confirms that students do this constantly and are encouraged to do so:

> Bill: It's a public forum. Chat, talk, discuss. … We are collaborating and I want to foster that atmosphere.

The last four RoAs do not appear to be social in the upper MICA community. Bill confirmed this in some cases: E.g. for RoA 6, Bill noted that it is not common for students to trust their translation of what they do on paper into the programming language. Finally, we note the low proportions (in *I don't really know if it works) that may be interpreted as indicating that all students already have started to develop their "Validating" scheme to a certain degree by the end of MICA I.

5. Social and individual aspects of schemes

In this section, we further analyze the "Validating" scheme for two students, Mark and Kassie. Considering this individual level allows us to observe similarities and differences when a common RoA is mobilized by different students, and to deepen our analysis of the intertwined mathematics and programming knowledge involved in this process. For the sake of brevity, we focus on assignment 4 (question 1) from the MICA II course taken by Mark and Kassie, and we select only one of the social RoAs. We also evoke other MICA II assignments that confirm the stability of the organization of students' activity and illustrate variability depending on the mathematics.

Presentation of assignment 4

In this assignment, students investigated simulations of a battle between two opposing armies using discrete equations. The assignment contained three questions that became progressively more complex. In the first question, students were told to create a program to output the day-by-day evolution of the battle represented by the Lanchester equations: $X_{n+1} = X_n - a*Y_n$ and $Y_{n+1} = Y_n - b*X_n$, where a and b are fixed parameters and X_n and Y_n represent the number of soldiers in the two armies on day n (the battle ends when X_n or Y_n is less than 1).

RoA 2: A common mobilization, different TiAs

For question 1 of assignment 4, both Kassie and Mark mobilized RoA 2: "Once I have programmed it all, I run the program with a few different inputs and compare the output with my hand calculations", which we found to be social among both MICA I and upper MICA communities. In their individual interviews, they declare for example:

Kassie: I did like the equations myself, just after like the first day of battle and kind of like, compared them with what I got with my program.

Mark: You're able to, again, at least begin on paper to kind of understand and write it out yourself of what the expected results are going to be. So, because of that, I kind of just tried to do the first few questions by hand.

We found evidence of the mobilization of RoA 2 by Kassie and Mark for other assignments as well, confirming that it is part of a stable organization of their activity. We infer that both Kassie and Mark developed a corresponding TiA, such as: "When the result I compute by hand and the output of my program coincide, my program is correct". Both this general TiA and RoA 2 take different forms depending on the specific mathematical content. In assignment 4, once the parameters a and b and the initial values X_0 and Y_0 are chosen, simple computations using the recurrence relations can provide the successive values X_n and Y_n. Students look for an exact match between the output and their hand calculations; however, they are only able to do this for a finite number of days in the simulated battle and eventually trust that the further iterations will be calculated properly too. In comparison, in assignment 1, where students adapted a program (presented in class) simulating the Buffon needle random experiment, they could compute by hand the exact probability (applying a formula given in class). They knew (law of large numbers) that the frequency computed by their program should converge towards this exact probability and used this mathematical result to check their program (in particular, Mark described his actions in this way).

In the questionnaire, while Kassie said that she "Always" uses RoA 2, Mark answered "Rarely" (nevertheless we found evidence that he used this rule on several occasions when completing his MICA II assignments). One possible explanation for this difference is that Mark is aware that a hand calculation is not always possible, depending on the mathematics involved. For example, about assignment 2, where students had to use the daily return percentages of the Dow Jones from 1950 to 2020 to compute the probability of a 2% drop, Mark said:

Mark: If you're given, uh, a mean … or a standard deviation or anything like that, um, you can't just verify your answer because you'd have, what over 50 years of data there.

It seems Mark developed another TiA associated with RoA 2: “The validation through hand calculation is only possible in some particular cases”. Thus we observe in this case that in the “Validating” scheme, even if RoA 2 can be considered as social among the upper MICA community because it seems to be shared by a majority of students, the associated TiAs are not necessarily shared: Mark has developed a TiA about the possibility, or not, to apply this rule, but we did not observe the development of this TiA by Kassie.

6. Conclusion

In our study we attempted to identify patterns in the culture of a community of students learning programming for mathematics inquiry, and more precisely to identify common RoAs in the schemes students develop. We also have investigated the development over time of these shared RoAs, and the TiAs associated with them. In this paper we focused on the scheme developed for the goal of “Validating the programmed mathematics”, which associates mathematical and programming knowledge, and RoAs for this goal that were identified in an earlier study.

Students’ answers to a questionnaire confirm that some of these rules are shared by more than 70% of the students—we consider these rules as social aspects of the “Validating” scheme. A further study of schemes developed individually by two students evidences that the same rule can be associated to different operational invariants, perhaps depending on the profile and experience of the student. We also observed evolutions between MICA I and upper MICA students. We claim that the social aspect of the teaching (the orchestration by the teacher, collective students’ work, etc.) contributed to create a community, with its shared patterns. MICA I students progressively entered the upper MICA community and aligned with its practice (Lave & Wenger, 1991) in terms of using programming for mathematical inquiry. In this university context, the horizon is given by the practice of mathematicians using programming for their own research.

The patterns of this practice were not explicitly stated in the MICA curriculum; they can be considered as operational knowledge, which is often not explicit. The teachers might be aware of some of these patterns and try to support students in addressing a goal in “proper” ways—using certain RoAs, based on certain TiAs. Nevertheless, they might also ignore some of these patterns; the identification of patterns by a research study can be helpful for them.

Most research using the concept of schemes has focused on individual schemes. However, as emphasized by Vergnaud (2009), the scheme–situation pair is a powerful tool in mathematics education research. Studying the aspects of schemes shared or not by students in a situation can also help refine our understanding of the situation. Focusing on social aspects of schemes is important to better understand what the situation is, for a group of students. We claim that using schemes not only as a theoretical tool for research but also as a lever producing interesting results for teachers requires a consideration of the social aspects of schemes, and their social-individual dialectics. In our future research we will further investigate social schemes by networking the theoretical frames of the instrumental approach (Trouche, 2003) and communities of practice (Lave & Wenger, 1991).

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Aspects of complex path integrals

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Keywords: Aspects, complex analysis, complex path integrals, definitions, partial aspects.

Introduction

Complex path integrals appear in various undergraduate mathematics curricula (e.g., for pure mathematics students, pre-service teachers, or engineers) and have many applications in pure and applied mathematics (e.g., computing Riemann integrals). Yet, there is only very little research on educational issues of complex analysis education so far (e.g., Oehrtman et al., 2019). Regarding the heterogeneity of students attending courses in complex analysis and deep curricular connections of complex to real analysis, it is relevant to analyze various ways to define complex path integrals, and how real analytic notions, which are usually foreseen earlier in mathematics curricula, can be used for that purpose. Moreover, Oehrtman et al. (2019) indicate that even experts in complex analysis struggle interpreting complex path integrals, so the epistemological endeavor undertaken here is likely to contribute to the teaching and learning of complex analysis. Hence, I address the following question: *"How can complex path integrals be defined and how are these definitions substantiated?"*

Theoretical frame and methodology

Greefrath et al. (2016) introduce the notion of mathematical "aspects": "An *Aspect* of a mathematical concept is a subdomain of the concept that can be used to characterize it on the basis of mathematical content" (Greefrath et al., 2016, p. 101; emph. and capital. orig.). For example, the *product sum aspect* characterizes Riemann integrals as limits of "product sums" and the *anti-derivative aspect* as the difference of the values of a primitive function at the upper and lower bound of integration. The last aspect actually requires an additional constraint (continuous instead of integrable integrands), and so do several characterizations of complex path integrals. Therefore, I follow Roos (2020) in distinguishing aspects from "partial aspects", which are only valid under additional constraints.

Definitions from approximately 50 textbooks from Cauchy's time to present and articles on interpretations of complex path integrals (see Hanke (2020) for some references) were analyzed with regard to the constraints imposed on the integrands (e.g., continuity or holomorphicity) and the paths of integration. In addition, I analyzed connections to real analysis as well as authors' substantiations for the definitions in order to abstract them into aspects and partial aspects of complex path integrals.

Exemplary results

Whereas the poster presents all (partial) aspects, I can only indicate four aspects (Table 1) and one partial aspect here. Assume that $f = u + iv$ is a complex function defined on the trace of a simple continuously differentiable path $\gamma: [a, b] \to \mathbb{C}$. The *product sum aspect* resembles that for Riemann integrals. Making use of the *substitution aspect*, the complex path integral is defined by symbolically substituting $z = \gamma(t)$ into $\int_\gamma f(z)\,\mathrm{d}z$. The *mean value aspect* characterizes $\frac{1}{L(\gamma)} \int_\gamma f(z)\,\mathrm{d}z$ as the average of the values obtained by rotating and dilating $f(z)$ by the unit tangential vector at z on the path of integration, not the values of f itself. The *vector analysis* aspect expresses $\int_\gamma f(z)\,\mathrm{d}z$ in terms of two real path integrals, which can be interpreted as the flow and flux of the so-called Pólya vector field $\boldsymbol{w} = (u, -v)^T$ associated to f along and across the path of integration. Therefore, there is a risk of confusion in the last two aspects in terms of what is averaged or which vector field occurs. One of the partial aspects is the *residue partial aspect*, which characterizes the complex path integral for closed paths and holomorphic integrands as a finite sum of residues.

Table 1: Four aspects of complex path integrals

Product sum aspect: $\int_\gamma f(z)\,\mathrm{d}z$ is the limit of complex Riemann sums $\sum_{k=1}^{n} f\left(\gamma(\xi_{n,k})\right)(\Delta\gamma)_k$, where $a = t_0 < \cdots < t_n = b$ ranges over the partitions of $[a,b]$, $(\Delta\gamma)_k \coloneqq \gamma(t_{k-1}) - \gamma(t_k)$, and $\xi_{n,k} \in [t_{k-1}, t_k]$ are tags.	**Mean value aspect:** $\frac{1}{L(\gamma)}\int_\gamma f(z)\,\mathrm{d}z$ is mean value of $(f \circ \gamma) \cdot (T \circ \gamma)$ on the trace of γ, where T is the unit tangent vector at the path of integration
Substitution aspect: $\int_\gamma f(z)\,\mathrm{d}z$ is the complex number $\int_\gamma f(\gamma(t))\gamma'(t)\,\mathrm{d}t$.	**Vector analysis aspect:** $\int_\gamma f(z)\,\mathrm{d}z$ is equal to $\int_\gamma \boldsymbol{w}\,\mathrm{d}\mathbf{T} + i\int_\gamma \boldsymbol{w}\,\mathrm{d}\mathbf{N}$, where $\boldsymbol{w} = (u, -v)^T$.

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"It all depends on the sign of the derivative": A praxeological analysis of the use of the derivative in similar tasks in mathematics and mechanics.

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Calculus and mechanics are closely connected disciplines. However, students are not always able to establish links between them or use knowledge from one to solve problems in the other. In this paper, we study students' solutions of two tasks: a familiar mechanics task and a similar yet unfamiliar calculus task. Our results indicate that while students had acquired praxeologies from mechanics suitable to the mechanics task, they had difficulties transferring these praxeologies to the task presented in a calculus context.

Keywords: Calculus, mechanics, anthropological theory of the didactics (ATD), derivative task, college mathematics.

Introduction and research problem

Calculus and physics have been closely linked since the birth of the former: the emergence of the notion of derivative is tied to the formalisation of velocity and acceleration as concepts. Taşar (2010) points out that "in the educational literature [velocity and acceleration] emerge as nodes of difficulties", but also that "students experience great difficulty when they need to transfer knowledge between these two domains" (p. 142).

International literature has identified a number of difficulties that students face in grasping the notion of derivative (e.g., Montoya Delgadillo et al., 2018; Orton, 1983). Taking into account different representations of the derivative (graphical, verbal, symbolic, paradigmatic physical, and others), Zandieh (2000) observes in particular that if a student understands the derivative in one context, this does not mean that they can solve tasks in another.

The notion of "transfer" has been studied in the literature, initially from a cognitive perspective examining the transfer of knowledge from one situation to another, and more recently from the perspective of the *Actor-Oriented Transfer* (AOT) (e.g., Lobato, 2012; Roorda et al., 2015). With AOT, transfer is defined as "the influence of a learner's prior activities on his activity in novel situations" (Lobato, 2012, p. 233). From a cognitive perspective, the fact that students fail to intuitively transfer knowledge from mathematics to physics has been highlighted in the physics education literature (e.g., Christensen & Thompson, 2012; Planinic et al., 2019). As for mathematics education, our literature review of the main international journals identified only two studies that focus explicitly on the links students make between each domain. In the first study, Marrongelle (2004) provides evidence that students can use physics to solve calculus problems. In the second one, Roorda et al. (2015) use AOT to study the transfer of procedures between mathematics and physics in both directions (from the traditional cognitive perspective, transfer is "usually expected to occur from mathematics to physics" [Planinic et al., 2019, p. 235]). In Roorda et al. (2015), the participants did not tend to explicitly mention links between the formulae learned in physics and the results obtained through symbolic differentiation in mathematics. However, they were able to establish further relationships between the procedures learned in both courses over the long term.

Hitt and González-Martín (2016) and Rasmussen et al. (2014) state that research on the teaching and learning of the derivative in relation to physics is still scarce, calling for more studies in that area. In particular, they highlight a lack of research in mathematics education examining teachers' and students' practices at the intersection of both disciplines. To address this issue, in her PhD, the first author of this paper examines practices related to the use of derivatives in calculus and mechanics courses at the college level. In this paper, we present some preliminary results of this ongoing work. Our research is divided into three main stages: (1) a praxeological analysis of calculus and mechanics textbooks (Hitier & González-Martín, 2021); (2) an analysis of teaching practices, through interviews with mathematics and mechanics teachers and in-class observations of differential calculus and mechanics courses; (3) an analysis of student practices using a questionnaire and task-based interviews. In order to shed light on the complex relationship between the two disciplines, our research aim is to study the similarities and differences between the way the notion of derivative is used in mathematics and mechanics courses, and the resulting impact on students' learning.

In the first stage, we observed that while the different representations of the derivative considered by Zandieh (2000) appear in the mechanics and calculus textbooks analysed, the practices studied do not seem to consider the difficulties inherent in shifting among these representations. We also observed that the practices employed in the context of one-dimensional motion were different in calculus and mechanics, relying on the limit definition and differentiation formulae in calculus, while applying given formulae in mechanics (Hitier & González-Martín, 2021). In this paper, we provide preliminary results from the third stage of our research, focusing on two tasks included in the students' questionnaire (see Methodology). We aim to answer the following research question: How do the praxeologies that students use when solving an unfamiliar calculus task relate to the praxeologies they use when solving a similar but familiar task in a mechanics context?

We note that in the international literature, certain studies remove the physics context from mechanics tasks, using a graphical representation to create calculus tasks (e.g., Christensen & Thompson, 2012). Their results "suggest that students have difficulties conceptualizing mathematics tasks that are common to the ways in which [they] ask questions in physics courses" (p. 5). However, we have found no example where essentially the same task is presented both with its physics context and without it. The tasks discussed in this paper do exactly this, and it is in that sense that we consider them to be "similar".

Theoretical framework

As stated above, we are interested in analysing practices related to the use of derivatives in two different courses (calculus and mechanics), in particular when solving a similar task in each discipline. We draw on the Anthropological Theory of the Didactic (ATD—Bosch et al., 2020), which considers human activities as institutionally situated. In ATD, knowledge is seen as embedded in practices which are conceptualised through the key notion of *praxeology*. Praxeologies are formed by a quadruplet $[T/\tau/\theta/\Theta]$, where T refers to a type of task to perform, τ to a technique that allows the completion of the task, θ to a rationale (called "technology") that explains and justifies the technique, and Θ to a theory that includes the rationale. These four components form two blocks: [T/τ] is the practical block that describes tasks and ways to solve them (*know-how*), and [θ/Θ] is the knowledge or *logos* block that describes, explains, and justifies what is done.

Additionally, ATD distinguishes between the *knowledge to be taught* (i.e., predetermined praxeologies that appear, for instance, in textbooks), the *knowledge actually taught*, and the *knowledge actually learned* (Bosch et al., 2020). In our analyses, we refer to our study of calculus and physics textbooks (identified as *knowledge to be taught*) and connect it with the students' responses (as evidence of *knowledge actually learned*). Finally, we consider a task to be familiar when it is part of the *knowledge actually taught* (it is also generally part of the *knowledge to be taught*).

Methodology

Our research is taking place at a large Canadian college (College *A* hereinafter). In Québec, students attend colleges after finishing high school and before entering university. Pre-university science programs are four-term, two-year programs. At College *A*, science students usually take their differential calculus and mechanics courses in their first term. On top of the *Regular Science* program (R), College *A* offers an *Enriched Science* program (E) featuring extracurricular activities, such as weekly seminars. Enrolment in the *Enriched Science* program is primarily based on motivation and interest. College *A* also offers two other programs (O): an *Explorations Science* program (consisting mostly of remedial courses) and continuing education classes at night or on the weekend.

During the fall of 2020, all courses were taught online due to the Covid-19 pandemic. That term, the college established two cohorts of students: one Enriched (EP, 37 students) and the other Regular (RP, 35 students). Each cohort attended calculus and mechanics courses together. A number of other students were not assigned to specific cohorts. The cohorted students followed the same curriculum as the non-cohorted students, with the calculus and mechanics teachers working in collaboration and sometimes attending each other's classes. At the end of the term, between the final classes and the beginning of the examination period, we sent an online questionnaire to all science students at College A (approximately 1,200 students). We chose to use an online questionnaire despite its limitations, due to the pandemic and the fact that students were not physically present at the College throughout the 2020–2021 academic year.

The questionnaire was divided into three main sections. The first included questions concerning the student's profile (the name of their program, whether they were attending a calculus and/or a mechanics course, etc.). The second section consisted of three contrasting questions addressing the students' views on calculus and mechanics, inspired by Halloun (2004)'s Inventory of Basic Disposition. The third section included seven questions containing tasked to be solved, focusing on either calculus or mechanics. Among them were two pairs of similar questions, with each pair containing one calculus and one mechanics question. In this paper, we focus on one of these pairs (Question 4 and Question 7, see Figure 1).

Question 4 is considered a familiar question in mechanics. As our textbook analysis (Stage 1 of our research project) revealed, such tasks are part of the praxeologies developed in mechanics courses.

Question 7 is similar to Question 4: the velocity v is replaced by a function f, so that the acceleration a becomes the derivative f'. However, this "translation" comes with a few adjustments. We considered the absolute value of f in Question 7 for two reasons. First, *speeding up* and *slowing down* refer to the variation of speed, which, in one-dimensional mechanics (the content considered in our study), is the absolute value of the velocity. Second, without the absolute value, Question 7 could become a familiar task in differential calculus, likely prompting students to use learned calculus praxeologies. Another important difference: to avoid an explicit connection between both questions, we used different terms: "not moving" for Question 4 and "constant" for Question 7. We are also aware that "not moving" in mechanics means $v = 0$, whereas f constant allows for more possibilities for f. We did not deem this difference to be crucial in the resolution of the task.

Question 4	Question 7
An object, with velocity v_x and acceleration a_x is moving on a straight line. In each of the situations, state whether the object is *speeding up, slowing down, not moving*, or there is *not enough information to decide*. $v_x > 0$ and $a_x > 0$. | $v_x > 0$ and $a_x < 0$. | $v_x < 0$ and $a_x < 0$ | $v_x < 0$ and $a_x > 0$. For each of the situations, explain your reasoning.	In each of the situations, state whether $\lvert f(x) \rvert$ is *increasing, decreasing, constant*, or there is *not enough information to decide.* $f > 0$ and $f' > 0$. | $f > 0$ and $f' < 0$. | $f < 0$ and $f' < 0$. | $f < 0$ and $f' > 0$. For each of the situations, explain your reasoning.

Figure 1: Question 4 and Question 7 of the online questionnaire

Of the students who received the invitation to take the online questionnaire, 179 accessed it and 62 answered at least one of the questions in Section 3. Question 4 was answered by 27 students; of these 27 students, 23 also answered Question 7, while four students skipped Question 7 but completed the rest of the questionnaire. To preserve anonymity, we identified the participants with numbers, preceded by letters corresponding to their academic profile ("N" for students who had already passed differential calculus and mechanics, "O" for Explorations Science or continuing education, "R" for students in the Regular program, "E" for students in the Enriched stream, adding "P for students in a paired group). We conducted a thematic analysis of the students' answers to each question, identifying the main elements of their praxeologies (mainly, their techniques and the rationales provided). We then cross-referenced the categories that emerged from our analysis of each question.

Data analysis

Question 4

Only a few participants had difficulties answering this question correctly (see Figure 2 or Figure S1 for a more detailed distribution of the answers). Consistent with our analysis of textbooks, the data indicates that this content is part of the *knowledge to be taught*: the students' praxeologies match the textbook praxeology, and we observe that the students use rationales present in their textbooks. For instance, one of the textbooks used in mechanics at College *A* (Serway & Jewett Jr, 2014), invites the reader to "think about the signs of velocity and acceleration by imagining a force applied to an object and causing it to accelerate" (pp. 32–33). Some examples of rationales are:

> when a and v are the same sign, the object is speeding up. If they are opposite signs, the acceleration is against the movement, which means that the object is slowing down. (EP6)

> when acceleration and velocity have the same sign then the object is speeding up and when they

are opposite signs (or their vectors are in the opposite direction) the object is slowing down. (N1)

We note that these rationales were also present in the mechanics teaching practices we observed in class. In this case, the *knowledge to be taught* also seems to be the *knowledge actually learned* by most of the participants.

Question 7

In contrast with the results of Question 4, the results of Question 7 (see Figure 2 or Figure S2 for a more detailed distribution of the answers) confirm that this is an unfamiliar question in calculus, with participants displaying greater difficulty in answering correctly. With the exception of one student who mentioned the first derivative test (RP13), the rationales used to support correct answers are common to rationales for Question 4 (referring to the sign or using the idea of (co)variation). Many participants develop techniques related to the use of rules based on the signs of the derivative, but only a quarter of them do so successfully. This could be because they have memorised the rules without truly understanding their meaning (which could in fact be understood using knowledge from mechanics). The majority of the students who use rules based on the signs of the derivative refer to the relationship between the sign of the derivative and the variation of the function (part of the *knowledge to be learned* in calculus), without clearly considering the absolute value of the function *f*. For example, RP5 states:

> If $f'(x)$ is bigger than 0, then it is increasing. If f' is smaller than 0, it is decreasing. The sign of f does not matter.

R3 goes further in her explanation, linking the sign of the derivative with the slope of the graphical representation:

> […] This is because a function will have a positive slope when it is increasing, and therefore a positive derivative. The opposite applies for decreasing.

Here also, the sign of the function, and therefore its absolute value, is not taken into account in the student's reasoning.

Cross-referenced results of Questions 4 and 7

Figure 2 provides an overview of the answers to Questions 4 and 7. To simplify the coding, each response was coded in only one category. We can see that of the 24 students who answered Question 4 correctly, only seven answered Question 7 correctly. Additionally, none of the students who failed to answer Question 4 correctly was able to answer Question 7 correctly.

In both questions, when students refer only to the sign(s) or to vector direction (e.g., EP5, O2), the rationale is relatively poor, as it comes down to the technique itself. For instance, EP3 wrote: "In mechanics, we saw that whenever the acceleration is of opposite sign to the velocity, the object was slowing down". We observe that this type of rationale, typical of mechanics courses, allows students to be successful in the familiar task (Q4), but it mainly leads to failure in the unfamiliar task (Q7). These findings are consistent with those of our textbook analysis (Hitier & González-Martín, 2021), as well as with other studies on the use of calculus in other disciplines, such as engineering. Many results that depend on calculus are proved once and then taken for granted, indicating that students may learn the explanations by heart without fully understanding them (e.g., Faulkner et al., 2020; González-Martín, 2021; Hitier & González-Martín, 2021).

Question 7 \ Question 4	Idea of variation	Idea of acting on movement	Referring to vector directions	Referring only to the **signs** of v and a	"Not enough information" in the case $v<0$ & $a>0$*	Depending only on the **sign of** a	"Not enough information" for all cases*	Total	
Idea of (co)variation, taking the absolute value into account	R2	O3						2	7
First derivative test, taking the absolute value into account			RP13					1	
Referring only to the **signs** of f and f' (correct)			N1	EP3 R4 RP7				4	
Referring only to the **sign of** f' (correct-inconsistent)						E1		1	3
Referring only to the **sign of** f' (incorrect-inconsistent)				EP5				1	
Constant in the case $f<0$ and $f'>0$			N2					1	
Referring only to the **sign of** f' (incorrect)		RP5 RP6 EP6	O2 RP8 RP14	N3 EP7				8	13
Graphical argument		EP2	R3 RP11					3	
Concavity mixed with variations				O1				1	
Unclear explanation		RP10						1	
-- No answer				RP9 RP12	EP4		R1	4	4
Total	1	5	9	9	1	1	1	27	
	24				1	1	1		

* No rationale given in the response.

Figure 2: Summary of responses for Question 4 and Question 7

We also note that although the students were not asked to make explicit connections between Questions 4 and 7, two students (R2 and, to a lesser extent, RP7) did so. For Question 4, R2 and O3 provided the most detailed rationales. Remarkably, R2 does not use the vocabulary *speeding up* or *slowing down* but provides explanations in terms of the variation of the absolute value of the velocity (the only participant to do so). Below is an excerpt of her rationale:

> for i and iii: since both v_x and a_x are in the same direction, there is no change in direction, so there is no decrease of the absolute value of velocity. […]
>
> iv: the absolute value of the velocity will decrease until zero, then the velocity will increase in the positive direction. (R2)

Her ability to rephrase Question 4 in terms of the variation of the absolute value of the velocity might have allowed her to identify the link between the two questions. Regarding Question 7, her rationales are:

> i and iii: since f and f' have the same sign, the abs value of f will continue to grow; f' will not subtract from the absolute value of f.
>
> ii and iv: since f and f' have different signs, the abs value of f will decrease until it reaches zero. After that point, f will increase in the direction of f'. (R2)

In her explanation, we see the expression "f' will not subtract from the absolute value of f" as hinting at the "idea of acting on movement" that we found in the other students' rationales for Question 4.

We believe that this provides evidence, as observed by Roorda et al. (2015), that students are able to establish certain links between the knowledge learned in the calculus and the physics courses.

Final remarks

In this paper, we analyse students’ solutions to two tasks: a familiar mechanics task and a similar yet unfamiliar calculus task. As in the work of Marrongelle (2004), we observe certain students using praxeologies from mechanics to solve the calculus task. However, those students make up only a small fraction of the study’s participants. Even if we consider that the students who provided correct answers to Question 7 (*referring only to the sign of f and f'*) might also have seen the link between the two questions without mentioning it explicitly, they still represent less than a quarter of our participants. In fact, our analyses tend to indicate that, for the calculus task, most students develop techniques based on results from calculus without giving any real sense to them. These techniques allow students to solve familiar calculus tasks but seem to lead them to failure when faced with unfamiliar tasks. Connecting these results with our textbooks analyses (Hitier & González-Martín, 2021), we see important implications for the teaching of calculus and mechanics. Praxeologies in calculus seem to foster the development of practices whereby results and formulae are learned without necessarily connecting them with a physical meaning or an interpretation. On the other hand, although derivatives are used in mechanics, the praxeologies rely heavily on given formulae and the link with calculus is obfuscated. Figure 2 shows that the small number of successful rationales in mechanics does not translate to successful rationales when solving a similar calculus task. This sheds some light on students’ difficulties in transferring knowledge between mathematics and physics, as discussed in the introduction. These difficulties may be related to praxeologies in both disciplines, which do not establish explicit connections between the two. This is consistent with Planinic et al.’s (2019) view that “students’ almost exclusive reliance on formulas in physics presents […] an important obstacle for the development of students’ deeper reasoning in physics and sometimes even an obstacle for the application of their already existing knowledge and reasoning developed in other domains” (p. 243). In this vein, Taşar (2010) also noted that the transfer of knowledge between the two domains seems problematic. We believe our results help to reveal institutional reasons for these difficulties in transferring knowledge between the two disciplines, as a consequence of institutional choices to organise separate praxeologies in each one.

In the previous paragraph, we mentioned the possibility that students may have seen the link between Questions 4 and 7 without mentioning it. The fact that the questionnaire was completed online could have affected the students’ ability to develop their rationales. We acknowledge this limitation and we hope to gain further insights through the analysis of the students’ interviews. This analysis, as well as other parts of our study, will be the focus of future publications.

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Investigation of the connections within proof in complex analysis lecturing

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Studying teaching in lectures at the university level is a topic of particular interest in mathematics education as lectures are still the predominant mode of teaching at the advanced level. In this study, the aim is to examine the teaching of mathematical proofs in a complex analysis course offered to second-year bachelor students of mathematics and science, using the commognitive framework. We focus on the mathematical discourse for proof teaching and specifically on the proving routines for the connections among the arguments within the proofs. In particular, three sub-routines were identified: setting the proof, applying, and computing. The lecturer performed these routines for the students intending for meaningful connections within the proof. We conclude by providing implications for education and future research.

Keywords: Proof teaching, lecturing, complex analysis, commognition, routines

The commognitive framework (Sfard, 2008) highlights, among other issues, the fine aspects of teaching in lectures that concern the communication of proof in lecturers' mathematical discourse for teaching. Routines have an important role in both teaching and learning mathematics; they are the patterns that appear repeatedly within a discourse as results of rules, which indicate the *when* and *how* one should perform the pattern (Lavie et al., 2019). While starting engaging with mathematical discourse, the routines of a learner (namely rituals) are process-oriented and relate to performance. Lavie et al. (2019) asserted that one of the teaching goals might be to help learners move toward outcome-oriented routines (namely explorations). A shift of the focus of the learners from the process to the outcome could be identified by addressing the *changes* in their performance (Lavie et al., 2019). When it comes to teaching for the facilitation of the changes in the performance of the learners, the researchers could seek for the characteristics of routines in the lecturer's mathematical discourse for teaching that aid the learner's move toward explorations.

In this study, we focus on the teaching that aims at the facilitation of changes related to proof production. In doing so, we seek for characteristics in the proving routines of the lecturer while teaching in lectures. The proving routines are the patterns that appear in the discourse of the lecturer while proving. In our case, the changes are intended for the students but are performed by the lecturer and identified as characteristics of the routines. The focal point of this study is on one of the changes, the *change of bondedness* (Lavie et al., 2019). The change of bondedness, in this study, is the move from a list of unrelated arguments within a proof to a sequence of arguments related to each other. Thus, bondedness, as a characteristic of the proving routines of the lecturer's discourse, is concerned with the connections between the different arguments of a proof. As the lecturer of this study stated at the beginning of a lecture:

> I urge you to take some notes and carefully look at how the various statements I'm going to make relate to one another, which one implies which one, what are the essential assumptions in each theorem.

The connections among the different arguments within the proof are an important aspect of proof production because the formal proof consists of sequences of arguments where the last is the theorem, which is aimed to be proven (Tall et al., 2012). Within the sequence, axioms and other results (theorems, lemmas, definitions) are used. Tall and his colleagues presented a macro level description of the connections, whereas with commognition we are seeking for the identification of the connections at the micro level by focusing on the routines governed by bondedness within a proof. Also at a macro level, Lew et al. (2016) identified the key points in the presentation of a proof by an exemplary lecturer and it appeared that the students did not recognize them. One explanation was that the students mainly focused on what was written on the blackboard and not on the verbal explanation of the key points by the lecturer.

The use of the commognitive framework seems promising to highlight a communicational aspect of university teaching even if there is a lack of dialogue in this context. For example, Kontorovich (2021) highlighted the communicational aspects of the lecturer’s feedback and the importance of didactical discourse on proof (DDP) for the reflection on the given feedback not only for mathematicians but also for all the communities related to proof and proving. Thus, studying the teaching of proof by using commognition adds to the discussion and understanding of the attempts of the lecturers to communicate mathematical proof with the student in university lectures. Not directly focused on a proof-oriented course, Viirman (2014) studied the teaching practices of lecturers during the teaching of functions in introductory university courses. The researcher identified the different routines in the mathematical discourse of the lecturers and provided a categorization of construction routines (which aim at the creation of new endorsed narratives) and substantiation routines (which focus on the decision to endorse an earlier presented narrative). He distinguished the emerging routines into types; for example, the different types of substantiation routines included definition verification, proof, and claim contradiction. In the follow-up study (Viirman, 2021), the courses analyzed were introductory, so proof and proving were not central. However, the study of proving routines is fruitful in analyzing mathematical activity at university, because proof is core of many introductory courses at the university level. In this study, we focus on the proving routines of a lecturer to get insights into making connections within a proof.

Methodology

This study is part of a wider ongoing doctoral project that aims to characterize proof teaching in proof-oriented university courses. For this study, we investigated proof teaching in online lectures of a second-year complex analysis course. Formal proof raises difficulties for the students. It is important to highlight how the lecturer attends to the connection among the different arguments for the facilitation of students to endorse a proof. The connections in our case indicate the intended change to bondedness. Thus, our research question is “How does bondedness within proof emerge in the proving routines of the mathematical discourse of proof teaching of the lecturer?”

The lecturer in this case study was a theoretical mathematician with less than five years of teaching experience. The course concerned a group of bachelor students enrolled in mathematics or science (physics, astronomy) degree programs. In total, 242 students were registered, while between 30 and 80 students attended the live (online) lectures, which were recorded and students had the possibility to watch anytime. In this course, complex valued functions were defined in the complex plane and their properties were explored. The lectures were based on the textbook while the lecturer referred to the chapters and the topics that were covered in each lecture. The data were generated from the transcripts of five (out of 16) live lectures. The data analysis had two stages and we used the commognitive framework (Sfard, 2008) for the identification of the proving routines in the mathematical discourse for teaching of the lecturer.

In the first stage of the analysis, firstly, we identified the mathematical narratives, sequences of utterances that describe objects, processes, or relations among them, which the lecturer presented in the lectures. These narratives concerned definitions, the statement of the theorem, lemmas, proofs, examples, exercises – problems, remarks (comments of the lecturer on the statement of the theorem which were used in the proof), or notes (comments of the lecturer after the proof). The identification of the narratives assisted in the selection of the episodes. For this study we analysed ten episodes. Each episode had a duration of five to 45 minutes and included the narratives relevant to the teaching of proof. For example, the most common episodes included the narratives: theorem, proof, remarks. Then, we highlighted keywords and phrases of lecturer's discourse, relevant to proving. They were related to the mathematical aspects of proving (e.g., "the first thing we're going to do is replace this simple closed contour"), and the pedagogical aspects of proving (e.g., "let's see how you go about proving this"), and lecturer's comments about proving (e.g., "to make things formal and nice"). Also, the visual mediators, visual objects that are employed for communicational purposes (graphs, notations, interaction with the online environment), were identified. The visual mediators were of two kinds, either pre-existed in the slides or were created on the spot by the lecturer. The first stage ended with overarching proving routines. Some of the routines were "making connections within the arguments of the proof", "presenting a proving technique" (i.e., proving by contradiction), and "generalising to the complex plane".

In the second stage of analysis, we focused directly on the overarching routines. Doing thematic analysis, we used open coding for analyzing the episodes in Atlas.ti software because previous literature is limited on this topic. The coding was necessary for the rectification of the routines and the characterization of each routine. Within the routines of stage one, we identified, through constant comparisons, sub-routines. In this study, we focus on the proving routine "making connections within the arguments of the proof" which is closely related to the change of bondedness. Within this overarching routine, we identified three sub-routines namely: *setting the proof*, *applying*, and *computing*.

Results

The first sub-routine, *setting the proof*, was used to connect the statement of the theorem with the intended outcome of the proving process and the proving strategy (i.e., an outline of the arguments that will lead to the desired conclusion). In this paper, we focus on the former. The lecturer, when performing this sub-routine, either drew analogies with the real case which was familiar to the students or discussed a proving strategy. For example, an analogy with the real case appears in the following excerpt where the lecturer introduced the fundamental theorem of calculus for the complex plane to the students:

> So, if you recall from calculus [writes the formula on the right corner of the slide], you know, the fundamental theorem says that integral of f of a continuous function on an interval a b [meaning: $\int_a^b f(x)dx$], equals $F(b) - F(a)$ where F is any anti derivative of f on this interval. So, we have basically the same statement here… The proof is essentially a simple application of the fundamental theorem of calculus in the real case.

The lecturer, in this excerpt, reminded the students of the theorem in the real plane after presenting the statement of the fundamental theorem in the complex plane. The proof that followed after the excerpt, was described as an application of the theorem that the students were familiar with. The lecturer in that case did not explain a new proving strategy but connected the proof with a familiar theorem to students and placed this theorem within a familiar proving strategy to them.

The next example comes from the Laurent series theorem, which statement appears in Figure 1.

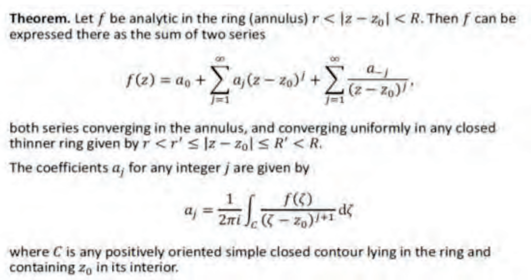

Figure 1: Slide presented for the Laurent series theorem

The excerpt of the lecturer's discourse highlights the case when a new proving strategy is needed:

> So, we start with a little theorem whereby we consider functions that are, well, we already know a name for them, analytic functions, that are expressible as a power series in some domain, right? And we said that if we have an open disk, then an analytic function on it, and its Taylor series are in sort of one-one correspondence… that's what we talked about in the previous lecture. So now, we will be interested in considering something slightly different that would be any annuli… So, we will be interested in these and functions which are expressible on them as power series, because it will let us slightly expand the family of functions that interests us, because you know, functions can have singularities.

In the excerpt, the lecturer discussed the statement of the theorem by bringing results from the previous lectures; the students were familiar with the case of the open disc. Now the problem changes to the case of any annuli and to general functions with singularities. Then, to describe the annuli, he made the visualization on the top right corner of the slide [Figure 2].

Also, the lecturer added examples about the different types of functions that related with the case that was described in the statement:

So, easy example would be something like $1/z$ at the origin, and so forth.

Next, the lecturer discussed a proving strategy to tackle the case of the annuli:

So, we will be interested in functions with that kind of property that there is some bad point. And then, in a sense, the natural thing to look at is, well, you remove a small disk, okay, and then the function will be analytic, that's the idea right.

Within this sub-routine, the lecturer discussed the statement of the theorem and used visualizations for its exemplification, formulated the problem that the proof will answer, and sometimes gave some examples. We showcased the lecturer's intention of focusing on connections within a proof in the introduction of this paper. In particular, the lecturer's excerpt showcased that one of his intentions was to "carefully look" at the relations among the statements he made. Given this intention, it seems reasonable that the sub-routine *setting the proof* happened mainly at the beginning of the proving process, and the lecturer set the scene for the proof that followed. Thus, the lecturer connected the statement of the theorem with the strategy they would follow to prove the theorem, building on familiar or new arguments. Within this sub-routine, the discussion and the visualizations around the statement of the theorem were connected with the other intention of the lecturer to "carefully look" at "the essential assumptions in each theorem". Indeed the sub-routine *setting the proof* placed the theorem within known theorems and results from that or previous courses and made connections between the results that appeared within the proof and the intended outcome of the theorem.

Next, is the sub-routine *applying*, which had three aspects. The first aspect concerned the introduction of a result (definition, theorem, lemma, preposition, axiom, remark, note, example). The second was the connection of this result to an argument, not shaped yet, with the form of this result that was suitable for the specific case in order to proceed with the argument. The third aspect was related with the development of the result. In this sub-routine, the lecturer elaborated on a visualization, described and applied results. The following excerpt is an application in the Laurent series theorem. The lecturer introduced the result, Cauchy's integral formula in that case, which he intended to apply (first aspect of this sub-routine). Next, he elaborated on that with the visualization [Figure 3] (second aspect of this sub-routine). By the end of the excerpt, the lecturer described the outcome of the application that $f(z)$ can be expressed in terms of values on the boundary as was already written in the slide [Figure 3] (third aspect of the sub-routine).

So, for any z, which is inside the annulus, Cauchy's integral formula applies. Now, mind you, you can think of these if we say that $C1$, so this can be a little bit confusing, perhaps, when we say a positively oriented circle, in just in in a vacuum that's just counterclockwise oriented circle. But there is also orientation of a boundary where we said the positive orientation means that if you reverse it, you're going if you reverse the boundary, the set is on your left side. Well, of course, for the outer circle, its counterclockwise orientation on the annulus indeed corresponds to that orientation consistent orientation that the set is on the left side right, but if you reverse the inner circle, counterclockwise, the set the annulus is on the right side. So, it is also possible to think of this minus sign as coming from the sort of wrong orientation on the smaller circle not consistent with the annulus. Well, at any rate, reserved which is properly inside, you can use Cauchy's integral formula to express $f(z)$ in terms of values on the boundary.

In this excerpt, the lecturer used a result, Cauchy's integral formula, to proceed to an argument related to an intended outcome within the proof, $f(z)$, as an expression of terms of values on the boundary. The statement of the theorem had an important role in this sub-routine as it facilitated the emergence of the results. Indeed the lecturer used and explained the orientation of C described in the statement of the theorem. Taking into account the statement of the theorem and the ideas formulated at the beginning of the proof, he then brought and applied the aforementioned result to have a specific outcome. In this excerpt, the visualization was the same with the one in Figure 2 but was of different nature. At this sub-routine the visualization assisted the lecturer to describe how the use of Cauchy's integral formula can be applied. However, in the sub-routine *setting the problem* the visualization was used to describe the statement and set the picture of the statement and the problem that was needed to be solved with the proof.

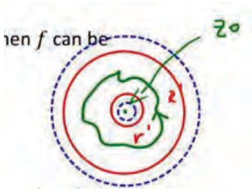

Figure 2: Visualization for the exemplification of the theorem's statement

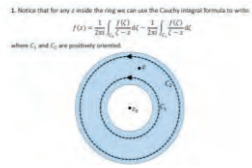

Figure 3: Slide used for the presentation of the theorem's proof

The applications within the sub-routine *applying* happened mostly after the verbal representation of the proving process and the explanation of the reasons why the result can be applied. Looking at the lecturer's excerpt in the introduction of this paper, his intention to "carefully look" on how the statements related and implied one another explains his focus on the verbal representation of the reasons why a result can be applied and related with the arguments that needed to be shaped for the proof. Moreover, the students could see the application from the beginning of his verbal representation because it was also written in the pre-written slides. We interpret that this choice of the lecturer is connected to his intention to relate the statements, as the attention is moving from an application to the reasons why an application (here Cauchy's integrals formula) is valid. In a nutshell, the sub-routine *applying* indicated how one uses a result for a specific proof.

The last sub-routine we present is *computing*. Computing relates to either algebraic or logical manipulations and the connections appeared within and among the arguments. The algebraic manipulations had to do with substitution or division of quantities. The logical manipulations indicated the equivalences among the arguments. These connections happened when all results within the proof were set, so in the final version. An example of the lecturer doing the algebraic or logical manipulations (i.e., rearrangement of terms, subtraction of terms) is the following from the fundamental theorem of calculus for the complex plane:

> So, if said zero is the initial point of γ_1 or the initial point of the whole contour, this integral equals $F(z_1) - F(z_0)$, that's the first integral. $+F(z_2) - F(z_1)$ is the second integral, this is just the antiderivative at the endpoints, the endpoints z_1 and z_2 of this contour γ_2 and so on.

$$\int_{\Gamma} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z + \cdots + \int_{\gamma_n} f(z)\mathrm{d}z$$
$$= (\cancel{F(z_1)} - F(z_0)) + (\cancel{F(z_2)} - \cancel{F(z_1)}) + \cancel{\cdots} + (F(z_n) - \cancel{F(z_{n-1})})$$
$$= F(z_n) - F(z_0)$$

Figure 4: Fundamental theorem of calculus computations in lecture slide

In the excerpt, the lecturer proceeded to the computations that led to the intended outcome of the theorem. To support his verbal explanation, he crossed the terms in the pre-written slides and circled the remaining terms [Figure 4]. An example of the lecturer doing the logical manipulations is the following from the Laurent series theorem. The explanation was verbal and in the slides, only the inequality appeared:

> And, of course, the opposite inequality holds in that case, namely $\zeta - z_0$ is less than $z - z_0$ in absolute value, which is just to say that if you take ζ on the boundary, $C1$ which is closer to the center, that's going to be, well, ζ is going to be closer than an arbitrary to the center than an arbitrary point, which is inside the annulus. So the inverse hold.

The lecturer in this excerpt explained why an inequality was true. The logical manipulation appeared in the colloquial of arguments that are connected with logical laws; equivalence in this case.

The connections in the sub-routine *computing* are within arguments and between arguments and the statement of the theorem. The lecturer's intention to relate arguments, found in the excerpt in the introduction of this paper, is at a different level in this sub-routine compared to the previous sub-routines, because the arguments are connected with each other or connected within. The computations led to the argument that was expected to be proved and as such they are connected with another intention of the lecturer to bring the "essential assumptions in each theorem" that can give the intended result.

Discussion

In this study, we discussed the proving routines of a lecturer's mathematical discourse for proof teaching. The proving routines indicated connections within a proof and related to an intended change, for the students, from a list of unrelated arguments to bondedness within a proof. We identified three sub-routines, *setting the proof, applying,* and *computing*. The first connects the statement of the theorem with the intended outcome of the proving process and the proving strategy (i.e., an outline of the arguments that lead to the desired conclusion). It appeared mainly at the beginning of the proving process. The second connects the argument that is about to be developed with the results (definition, theorem, lemma, preposition, axiom, remark, note, example) that are needed to be applied in order to proceed with the argument. The third one is the connection between the different arguments of the proving process and the statement of the theorem. With the use of commognition we shed light on the relations between the different arguments of the proving process and the lecturer's associated routines. In our search for bondedness as a characteristic of proving routines, we identified sub-routines, which gave a micro-level analysis of the connections within a proof in the discourse of the lecturer. These sub-routines may facilitate students' understanding of the connection between the statement of the theorem, the arguments and the intended outcome of the proof.

In previous studies, Lew et al. (2016) interviewed students and found that they did not attend to the key ideas of a proof when those were delivered verbally by the lecturer. Viirman (2021) criticised such an approach of a one-time interview and suggested an investigation for a longer period, because students need time to accumulate the key ideas of a proof and change their performance. In the current study, which is part of an ongoing project, taking into consideration that a prolonged period of data collection is needed, we focused on one lecturer for a period of five lectures. We were able to identify in a micro level some connections within proof that were present in the discourse of teaching of the lecturer. In contrast, Tall et al. (2012) discussed proving in a macro level as a linear process. In our micro-level analysis, we found that formal proofs, as appeared in lectures, were more than a sequence of arguments. Within the arguments, we found the three sub-routines that intended to connect the arguments with the statement of the theorem (sub-routine: *setting the proof*), with previously presented results, which were also shaped for the development of arguments (sub-routine: *applying*) and with each other via algebraic or logical manipulations (sub-routine: *computing)*. Thus, we conclude that commognition offered a rich description of the teaching of proof and contributed to understanding communication of proving in lectures. A follow-up study is about the characteristics in the proving routines of the students during a semester. Such studies could only but help lecturers take a closer look at their teaching and raise their awareness about their proving routines, which sometimes they perform tacitly.

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Mathematics learning through a progressive transformation of a proof: A case from a topology classroom

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We report on an ongoing project in a cross-level topology course, where students have been provided with opportunities to prove the same mathematical statement in different social situations. This paper focuses on a pair of students who proved a statement collaboratively before one of them volunteered to re-prove it at the board for the whole class to observe. We offer a commognitive analysis of students' discursive activity in each situation and trace the transformations of their proof throughout the process. This process is discussed with a focus on students' mathematics learning.

Keywords: Commognition, graduate students, proof and proving, topology, university mathematics.

Introduction

In their comprehensive overview of the mathematics education literature, Stylianides et al. (2017) identify three broad perspectives in the area of proof: the cognitive – proving as problem solving, the constructivist – proving as convincing, and the social – proving as an activity that is embedded in communities. Within the latter, typical proof-related tasks (e.g., constructing or presenting a proof) are not viewed in isolation but as constituents of a broader mathematical activity. Stylianides et al. (2017) further explain that "If a student or teacher produces a proof, research in this perspective would frequently place emphasis on the meaning of this artifact and how that individual and members of his or her community could subsequently use it" (p. 247).

Stylianides et al. (2017) describe the social perspective as less developed, not yet coherent, and lacking "common, widely used concepts" (p. 248). These descriptors seem especially appropriate in the university mathematics education literature where the cognitive and the constructivist perspectives dominate the area of proof. This situation opens the space for socially-oriented studies on how proof is practiced in university classrooms. Stylianides et al. (2017) propose considering these practices in relation to the mathematics community.

Much has been written about proof in the mathematics community. For instance, Rav (1999) stresses that "the intricate role of proofs [is] in generating mathematical knowledge and understanding, [that goes] way beyond their purely logical-deductive function" (p. 6). He associates this role with "inventing methods, tools, strategies and concepts" (Rav, 1999, p. 6) for solving problems.

We suggest that proof often actualizes its intricate role when mathematicians prove collaboratively, participate in research seminars where proofs are communicated, engage with published proofs, et cetera. These are structured situations with particular rules of the game that shape the course of a mathematical activity and yield proof transformation. In the case of a new proof, the transformations often occur through interactions between the proof constructors and outsiders to the construction process. For instance, as a response to the feedback from reviewers and editors, the constructors can revise the proof. Alternatively, a familiar proof can be restructured in a way that makes it more readily available “to build on it”. Morgan (1998) highlights the importance of the media through which mathematics is conveyed, which draws attention to instances where a proof transfuses from one communicational channel to another (e.g., from oral to written). Accordingly, we propose that proof transformation is a multifaceted process that can unfold in various social situations.

This paper comes from our ongoing project that unfolds in a proof-based course in topology (Kontorovich, 2021). This relatively advanced mathematical context has received limited attention in TWG14 (for an exception, see Stewart et al., 2017). Our project features a sequence of classroom situations where students develop proofs individually or in small groups, share them with the whole class at the board, and receive feedback from their peers and the course teacher. The project’s overarching aim is to explore opportunities for mathematics learning that emerge when students engage in progressive transformations of a proof. In this paper, we analyze an interaction between two students as they collaboratively constructed a proof, and the subsequent public re-proving of the same statement by one of them at the classroom board.

Theoretical framework

Our project is grounded in the *commognitive framework* (Sfard, 2008). This framework has been acknowledged for its capability to account for the complexity of university mathematics education and for offering tools to analyze learning and teaching in fine grain (e.g., Nardi et al., 2014). Commognition posits that mathematics as a whole and its particular disciplines (e.g., topology) can be construed as a *discourse*. Discourses are distinguishable through *keywords* (e.g., “Hausdorff space”) and their use, *visual mediators* (e.g., diagrams) and their use, endorsed narratives (e.g., a proof), and *routines* (e.g., proving). A person’s participation in a discourse is viewed as a patterned activity, when features of one’s public communication that remain relatively stable across interactions with different interlocutors constitutes a *personal discourse*.

Lavie et al. (2019) introduce the notion of a *task situation* to refer to “any setting in which a person considers herself bound to act—to do something” (p. 159). Then, they define “a routine performed in a given task situation by a given person is the *task*, as seen by the performer, together with the *procedure* she executed to perform the task” (p. 161, our italics). A procedure can be constructed through abstracting the commonalities of steps that a performer undertakes in similar task situations. Having no access to the performer’s interpretation of a task situation, one approach to deducing their task is attending to what the implemented procedure achieved.

The task situations at the heart of our project invite students to prove mathematical statements. We propose that in university mathematics classrooms, the keyword “proof” is often used to refer to a narrative that is targeted at endorsing a statement. This substantiating narrative is expected to unfold as a sequence of utterances (or sub-narratives), each either an “accepted fact”, or derived according to a well-defined set of rules (e.g., deduction, induction). Sfard (2008) comments that “routines of substantiation are probably the least uniform aspect of mathematical discourses. The very term *endorsement* may be interpreted differently by different people” (p. 231–232, italics in the original). These intrapersonal differences may emerge in students’ interpretation of which elements of their substantiating narratives constitutes a “classroom fact” and when a substantiation is required. We associate these interpretations with students’ tasks (cf. Lavie et al., 2019). We also note that the same dilemmas feature in the mathematics community, when different resolutions were offered in different historical periods. Even today the approaches to proof vary greatly across mathematical communities.

Whether the narrative "truly" endorses the statement or not, is a matter of social sanctioning. Manin (1977) writes that "a proof becomes a proof after the social act of 'accepting it as a proof'" (p. 48). In a similar vein, we use "proof" as a discursive label that is allocated by a particular community to a substantiating narrative (Kontorovich, 2021). This community can be as small as the person who constructed the narrative in the first place.

Lastly, Sfard (2008) defines *learning* as a lasting change in one's discourse. This change can be triggered by *learning opportunities* – "circumstances that call for, and support, *a change* in the learner's participation in a discourse, a transformation that would bring him or her closer to the discourse required by school curricula" (Chan & Sfard, 2020, p. 3, italics in the original). Chan and Sfard distinguish between opportunities for a change in *the learner's command of the discourse* and for a change in the *discourse* itself. Within the former, the learner becomes more fluent in the target discourse by realizing the opportunity to mathematize according to its rules. In the latter, the learner enriches their discursive repertoire of endorsed narratives and routines. Students' discourses are expected to get closer to the university version of a topological discourse in our project.

The case of Grace and Jonah

The project data comes from a semester-long course in a large New Zealand university. The course cohort consisted of six students: four were studying towards post-graduate degrees in mathematics, and two undergraduates were in their final year of a mathematics major. This was the only course in topology offered by the university's mathematics department, and it covered standard topics in point-set and algebraic topologies (e.g., continuity, convergence, homology). For illustrative purposes, we selected a case where transformations in students' proof were evident.

The data is extracted from the lesson on Hausdorff spaces, which took place towards the end of the first third of the semester. At the beginning of the lesson, the teacher defined Hausdorff spaces as those where every two elements can be separated by open sets (i.e., for each $x \neq y$ in X there are open sets $U, V \subset X$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$). After discussing this definition and specific examples, the students self-divided in pairs, and the teacher invited them "to have a go at proving" that if $f: X \to Y$ is a continuous one-to-one function and Y is Hausdorff, then X is also Hausdorff. The protagonists of our case are a doctoral student, Grace, and an undergraduate, Jonah. They collaborated for nearly 4 minutes before the teacher asked "who is ready to present?"; then, Jonah volunteered to prove the statement at the classroom board.

The data corpus consisted of video-recordings of students' activity and written work. We embarked on the analysis with two questions: "*what routines did the students implement in each task situation?*" and "*how did their proof transform in the progression from one task situation to another?*" After transcribing the data, we examined students' activity to identify routines implemented in each task situation. We scrutinized the utterances to delineate substantiating narratives and routines before characterizing discursive similarities and differences between them. To present the findings, we begin with analyzing students' collaboration, and then turn to Jonah's mathematizing at the board.

Collaborative work

1	Jonah:	Okay, so… [sketches two ovals for the sets X and Y in his notebook]
2	Grace:	So we want to show that in X, yeah
		[Jonah completes the diagram reproduced in Figure 1]
3	Jonah:	That’s basically it.
4	Grace:	[a] Yeah, that is kind of it, right? [b] Well, if they weren’t disjoint…
5	Jonah:	[a] Oh, that’s true. [b] Sounds really simple.
6	Grace:	It’s almost too simple. [pause of 5 seconds]
7	Jonah:	I feel like, I feel like something’s missing.
8	Grace:	Yeah, I feel like something is missing as well. [pause of 15 seconds]
9	Grace:	So the function is from X to Y. You have two points here [in X], let’s call them little x, little y. [notates the points on the sketch]
10	Jonah:	Oh, doesn’t it imply that these two actually are in the same… You have an intersection.
11	Grace:	Yeah, that’s seems wrong because then it [the statement] is true for all functions. Oh, but the fact that it is a continues function…
12	Grace:	But why do we need one-to-one? I feel that we got to use that.
13	Grace:	So, so… Let’s actually do this super logically. We start with two points.
14	Jonah:	Yeah.
15	Grace:	We want to put an open set around each.
16	Jonah:	Yeah, yeah.
17	Gemma:	We go to $f(x)$ and $f(y)$.
18	Jonah:	Yeah, yeah.
19	Grace:	We put an open set here and here, which we can do because it is Hausdorff.
20	Jonah:	Yeah.
21	Gemma:	We can pull these back and get two open sets here [pre-images in X].
22	Jonah:	Yeah.
23	Grace:	If there was a point in this intersection but it can’t get mapped to two points.
24	Jonah:	Yeah.
25	Grace:	One-to-one means that these two… Oh!!! These two can’t get mapped to the same point. Because if they got mapped to the same point, this argument wouldn’t work. It has to be two different open sets. That’s why [1:1].
26	Jonah:	Oh, wait, what? I still don’t see where the one-to-one.
27	Grace:	So our argument would fail. If f wasn’t … because of if f wasn’t one-to-one, then you could have $f(x)$ equals $f(y)$.
28	Jonah:	Oh, oh, oh [as if realizing this]… Yeah.
29	Grace:	[a] And then you definitely couldn’t do this picture. [b] So I think that’s where it happened.
30	Jonah:	Okay. Right, right. That’s kind of subtle.

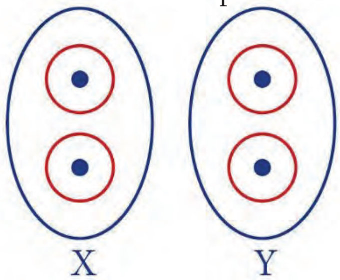

Figure 1: Jonah's diagram

The presented transcript features three rounds, differing in task situations and students’ routines. In the first round in [1-5], Jonah sketches a diagram that both students treat as a visual mediator of a proof of the assigned statement. Note that in [2] Grace appears to commence the construction of the substantiating narrative, but once Jonah completes the diagram the construction is relinquished. Accordingly, we suggest that generating a verbal version of the proof was not within the students’ task (in this round). This suggestion explains why the mathematically experienced students endorsed a diagram as “basically it” and “kind of it” in a proof-requiring task situation.

In [3-8], the pair implements what we term as a *proof-monitoring* routine: a procedure of “looking back” at the previous discursive activity with the task of assessing whether or not it can be sanctioned as a proof of the assigned statement. Herein, the students monitor the diagram visually and only the routine outcomes are articulated: Grace and Jonah both do not identify issues with their never verbalized proof. Notwithstanding, both agree that “it” (the diagram, the proof, or their construction) was “too simple”. In tune with our approach to proof, we interpret the appearing tension between not identifying an issue with their work and being not satisfied with it as students monitoring not only their previous activity but also how it appears to them in the broader context. For instance, they could recall that the task situation was set up by a research mathematician in a cross-level course. Within this view, it may seem unlikely that the dyad could generate a proof just in seconds.

In [7-8], the wholistic “it” turns into a focused “something’s missing”, and the identification of a potentially problematic element turns into a task for the second round. Pursuing this task seems impossible without narrating the proof, and this is what happens in [9-11]. We refer to these students’ utterances and actions as the implementation of a *proof-growing* routine: a procedure through which a substantiating narrative is not constructed “from scratch” but becomes more extensive, elaborate and detailed based on previously conducted work. In this case, Grace uses Jonah’s diagram to name the sets and points, and in [10] Jonah appears to rephrase Grace’s utterance from [4b].

This round’s task is completed in [11-12] with Grace identifying that their (still partially narrated) proof does not capitalize on f being one-to-one. This identification is not unlike the one in the previous round where the problematic spot was also not detected in proof-monitoring. The progress is in students delineating an element that they expect to feature in the substantiating narrative, and that it is currently not there. This recognition illustrates that a proof-growing routine can impact the substantiating narrative, not only by broadening its previously recognized constituents with new details, but also through its expansion to elements that were not addressed beforehand.

But why do Grace and Jonah expect the function’s injectivity to play a role in their emerging proof? Both appear to agree that their work substantiates the assigned statement “for all functions”, and injective functions are a subset of “all” – so why wouldn’t the pair see their proving mission as accomplished? As before, we propose that the students’ proof-monitoring went beyond *their* discursive activity to account for how this activity may appear in a broader context. For instance, drawing on their previous experiences, they could be driven by such considerations as “a teacher would not provide a redundant condition” or “we would be asked to prove a stronger version of the statement if it was possible”.

In tune with the above, delineating the role of the function’s injectivity becomes the task for the third round. The [13-30]-section features proof-growing and proof-monitoring, but the students’ interaction changes: Grace leads the implementation of both, narrating one proof element at a time, while Jonah endorses her statements. In [25], this interaction bears fruit: Grace realizes that their diagram had highlighted the function’s injectivity at the start, by depicting $f(x)$ and $f(y)$ as distinct points. Thus, in this round, their substantiating narrative grew by generating a verbal utterance about a visual element of the diagram. To appreciate what comes next, note that other parts of the diagram, especially those substantiating the Hausdorff-ness of X, did not feature in students’ discussion.

Jonah’s mathematizing at the board

Due to space limitations, we present an abbreviation of Jonah’s work at the board (for the full transcript see Kontorovich et al., in press). Jonah approached the board leaving the notebook with his much-discussed diagram on the desk. He stood facing the board and with his back to the class throughout the process, often blocking the board with his body. Figure 2 shows a snapshot of Jonah's board when he finished.

Jonah began by articulating every word in the sentences that he wrote on the board. The first three lines in Figure 2 were generated in this way. The fourth line emerged in silence. Then, Jonah appeared hesitant: he stopped writing and his gaze oscillated between the target statement and what he had written to that point. He took a step aside and sketched a diagram with two ovals, points x and y, and dotted circles U and V. After a few seconds, he giggled and smiled as if embarrassed, went back to his desk and returned with his notebook featuring the original diagram presented in Figure 1. After a quick glance at it, Jonah exclaimed "oh yeah!" and generated lines (5-6). Then he returned to the diagram on the board and completed it (see Figure 2). He wrote the lines (7-8) in silence, and instantly went back to his seat while cracking a smile to the video-camera.

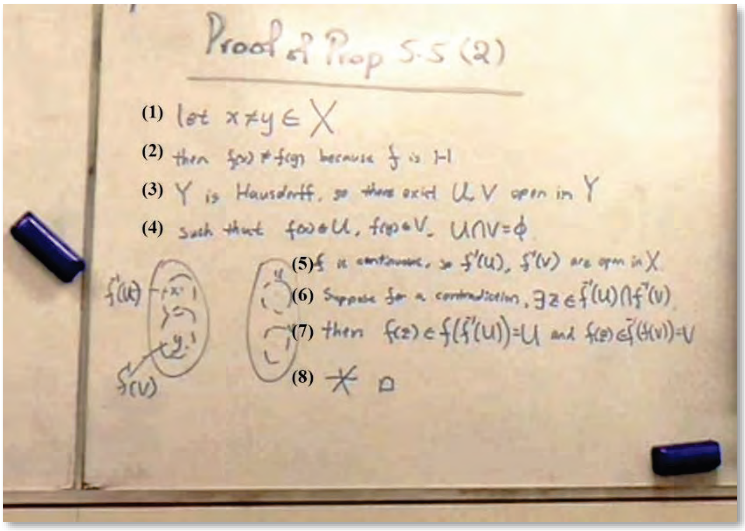

Figure 1: Snapshot of Jonah's board (numbering added)

In terms of routines, there is a visible change in how Jonah proceeded with his proof in this task situation: from *utterance-duplicating* in the first three lines, where he articulated what he put on the board as he wrote, to *silent writing* towards the end. We propose that while proving publicly for peers and the teacher to observe, Jonah mathematized for himself. This explains him investing almost no effort in elaboration on his text and eventually "turning off" the oral component – when one is communicating with themselves, the talk is loud even when all others hear is silence. Furthermore, his task appears to generate a written and self-contained proof, only parts of which were earlier discussed with Grace. Given that Grace did most of the "heavy lifting" in the earlier generation of the substantiating narrative, Jonah's proving appears as a self-imposed challenge of constructing the board narrative on his own. This is in tune with him volunteering to prove at the board, initially leaving his notebook behind, and using it half-heartedly when getting stuck.

The transition from collaborative work to the board entailed a transformation of the proof in terms of *restructuring*, *formalization*, and *growth*. Indeed, recall the insight on the function's injectivity in [25-29]. The dyad clarified it in the third interactive round and referred to it as a matter of contradiction (see [25,29a]). On the board, Jonah transformed this element into the symbolic "$f(x) \neq f(y)$", eloquently substantiating it with f being "1-1" (see line (2)). The formalization aspect can be captured through what Sfard (2008) terms as *reification*: a discursive process in which humanized formulations about processes turn into a talk about mathematical objects (e.g., [19] vs line (2), [21] vs line (5)). Perhaps the most visible instance of proof growth is evident in the transition from "if there was a point in this interaction […] but it can't" in [23] to the deductive sequence in lines (6-7), where Jonah declares his intention to obtain a contradiction and explicates the steps that do so.

Summary and Concluding Remarks

Focusing on a single mathematical statement, we analyzed the progression of students' proof between two task situations. Collaborative work on a statement "from scratch" and its consequent proving at the board are distinct social circumstances, and thus, it is barely surprising that different substantiating narratives emerged in each of them. What seems less obvious is what transformations a narrative can go through when "rolling over" such task situations. In the presented case, the proof was restructured, it grew previously not articulated elements and became more formal and elaborate. We acknowledge that these developments relate to the shift from the oral to written communicational medium, at least partially. Indeed, academic mathematical texts are renowned for being dense with terminology and symbols, modest in their use of "grammatical words", and having impersonal and authoritative formulations (e.g., Morgan, 1998). Then, it may be expected that such an experienced mathematics student as Jonah would write in this way in the presence of a mathematically mature audience.

Can these transformations count as evidence of Jonah's mathematics learning? The answer depends on whether Jonah's discourse underwent a lasting change, which is out of the research scope in this paper. Our analysis captures the short-term developments in Jonah's communication: from his limited contribution to the generation of a substantiating narrative in the first task situation to a fully-fledged narrative in the second. Some may argue with our claim about Jonah's discursive development, noting that he was the one to generate the original diagram and declare "that's basically it". He delivered on this declaration at the board after a short glance at that diagram, something that may be explained by him as "holding the proof in his mind" all along. We remind the skeptics that commognition operates with communication that rests in a publicly accessible space and it recognizes the effort that is often needed to switch from inner dialogue with oneself (i.e., thinking) to conversing with others (e.g., recall how Jonah got stuck at his board proof). In the minimal case, by volunteering to mathematize at the board, Jonah realized a learning opportunity to change his command of an academic topological discourse. Indeed, he not only mathematized through using conventional keywords, symbols, narratives, and routines, but he did that at the board, which is characteristic to research mathematics.

A somewhat similar argument can be made regarding Grace's discursive development. Throughout the interaction, Grace led the proof growth to broaden its components and expand it to new elements. Specifically, Grace's discourse was enriched by a narrative about the role of the function's injectivity in the assigned statement, a condition that initially appeared redundant.

Two interrelated aspects of the students' interaction are noteworthy. First, the case illustrates how a substantiating narrative can develop through the alternation of proof-monitoring and proof-growing. Students' monitoring is interesting since its first two implementations did not remove a blind spot in the proof, even though the dyad was convinced that this spot existed. We accounted for this conviction by proposing that the pair monitored not only their activity but also how it may appear in a broader context. The inaptness of their activity within this broader context explains why the students kept growing their proof further. This account illuminates how social considerations can permeate what could be expected to be a "purely logical-deductive function" (cf. Rav, 1999) of proof-monitoring.

Second, the students knew that in their topology course and beyond, a request for a proof is tantamount to generation of a written self-contained narrative. Yet, the dyad referred to a wordless diagram as “basically it” and “kind of it”. This illustrates that students’ familiarity with “a certain kind of action” (Lavie et al., 2019, p. 159) that the task setters expect does not always prevent students from pursuing a different task. Lavie et al. refer to “task” as “a person’s interpretation of a given task situation” (p. 161), but this case shows that it can be a deliberate choice. Janah’s proof at the board offers a colorful example that the choice of an alternative action in one task situation is not necessarily evidence of one’s incapability to undertake an expected action in different circumstances.

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Students' development of mathematical language regarding definitions

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Research showed that university students struggle with the mathematical language, particularly when beginning mathematics studies. Therefore, we investigate a group of 15 mathematics students from a German university that were interviewed at seven different times during their first year of study to describe the process of their language development. Focusing on the aspect of definitions as a specific part of mathematical language, our results imply three types of students with respect to their development of mathematical language. Students of the first type seem to have an adequate understanding of mathematical language from the beginning. By contrast, students of the third type have severe problems and show no real development, whereas students of the second type start also with severe problems, but show improvement during their first year of study. The main aim of this paper is to describe and analyze the development of a student belonging to the second type.

Keywords: Language, definitions, transition, university students.

Introduction

The transition from school to university is strongly influenced by a transition from school-mathematical language to university-mathematical language. According to Gueudet (2008, p. 244), the mathematical language at universities can be seen as a gatekeeper that ensures or impedes access to the mathematical community: "It is the language of advanced mathematics, required to enter the mathematical community, and to communicate inside this community". Furthermore, Seaman and Szydlik (2007) describe the development of the mathematical language as part of mathematical sophistication, by which they mean the "enculturation into the community of practicing mathematics" (p. 170). Mathematical language at universities is the basis of stating and using mathematical definitions, understanding mathematical concepts and developing mathematical argumentations and proofs (Moore, 1994). Schleppegrell (2007, p. 140) stated: "the language [of a discipline] and learning [a discipline] cannot be separated". This deep dependency of language and mathematical learning is widely accepted in mathematics education research (e.g., Morgan, 2005; Pimm, 1987; Tabach & Nachlieli, 2015). However, research showed that university students struggle with the mathematical language, especially when beginning mathematics studies (e.g., Gueudet, 2008). The students' difficulties with mathematical language could be understood as a potential reason for the high drop-out rate in mathematics studies (e.g., Heublein, 2014).

Whereas the topic of language and mathematics has been studied from various perspectives, especially in recent years, research concerning the development of mathematical language as a main part of students' mathematical sophistication, is sparse. Therefore, we contribute to this line of research with a project conducted within the Competence Center for Higher Education in Mathematics (khdm, www.khdm.de) that has the aim to investigate the process of mathematics students’ development of mathematical language within their first year of studies. In this paper, we focus on definitions as a specific part of mathematical language that seem to provide a specific obstacle for mathematics students (Alcock & Simpson, 2002; Moore, 1994). For example, Edwards and Ward (2008) have shown, that many freshmen in mathematics are unable to understand and apply mathematical definitions correctly, which can lead to further problems, e.g. with proofs (cf. Moore, 1994). Other studies showed similar results for prospective mathematics teachers, who were not able to formulate precise definitions (Özyildirim & Sahiner, 2017; Zazkis & Leikin, 2008). To conclude, our main question for this paper is the following:

> What development of the mathematical language do students show within their first year of study?

Construct and content of mathematical language

Characterizing the mathematical language and its key features, one usually finds a subdivision into word, sentence and text level (e.g., Prediger et al., 2019). Mathematical symbols can be understood as specific linguistic entities and therefore they can be assigned on the word level (Meyer & Tiedemann, 2017). Symbols in mathematics are used as abbreviations for mathematical objects or technical terms and without them, many abstract objects, constructs and thus meanings could not be described and grasped (Schleppegrell, 2008). Besides symbols, the mathematical vocabulary (technical terms) is to be classified on the word level. Compared with everyday language, mathematical vocabulary can be divided into three categories (cf. Maier & Schweiger 1999; Monaghan, 1999): The first category includes words that are not used in everyday language (e.g. “bijection”), the second words that are used in everyday language with largely similar meanings, for example “completeness”. Words that occur in both registers but have different meanings are assigned to the third category (e.g. “field” or “series”).

On the sentence level, there are certain norms and rules about how mathematical symbols may be linked with each other and may be linked to words. For this, it is possible to establish references between different concepts and objects and to express mathematical relationships (Maier & Schweiger, 1999). Furthermore, special patterns of grammar and syntactical structures like using long and dense noun phrases, passive voice and the verbs “be” and “have” are typical for mathematical language (e.g. Schleppegrell, 2008; Maier & Schweiger, 1999). Definitions as an important element of mathematical language are to be located at the sentence level. Mathematical definitions – in contrary to definitions in general - do “have the property that everything satisfying it belongs to the corresponding category and that everything belonging to the category satisfies the definition.” (Alcock & Simpson 2002, p. 28). Definitions in mathematics do not include more properties than necessary to describe the relationship or the object that should be defined. Further, usually only previously defined terms are used. Grammatically, in the (conditional) subordinate clause the condition or premise is presented and in the main clause the consequence resulting from the premise, the conclusion, is described (Meyer & Tiedemann, 2017). Regarding the transition from school mathematics to university mathematics there are changes concerning mathematical definitions (cf. Chesler, 2021): In school mathematics, objects and terms mostly have an experimental or descriptive basis, using less formalism and a more informal language instead of the exact terminology (Reiss &

Nagel, 2017), whereas “they are specified by formal definitions and their properties reconstructed through logical deductions” (Tall 1992, p. 495) in university mathematics.

Mathematical theorems are another important element of mathematical language, which often consist of more than one sentence and are therefore located at the text level (Meyer & Tiedemann, 2017). Besides theorems, proofs are of course assigned to the text level. Generally, properties of technical jargon such as completeness or precision are also characteristically for mathematical language.

Method

The students were analyzed concerning their language development in a longitudinal setting at seven different times in their first year of study and were interviewed at these seven times. The group of students we interviewed comprises 15 students (of which ten were female and five male), all beginning with their mathematics studies (prospective mathematics teachers (upper secondary school) and mathematics students). We obtained them through convenient sampling and they were paid for making the interviews in the same way student workers are paid. The interviews lasted about 30 to 60 minutes and were transcribed completely. In the interviews, among other aspects, the students were asked to read specific mathematical sentences providing theorems or definitions, to formulate specific theorems, to describe the meaning of mathematical objects or to give definitions. One of the tasks was to give a definition of the term function (other terms that should be defined were for example linear combination and divisibility). The interviewer asked the participants to write down the definition and to read it out loud afterwards. If necessary, the participants were asked, how a particular phrase or term in their definition should be understood. If the interviewer noticed that the participants had difficulties in defining the term, he asked them to try to explain these difficulties.

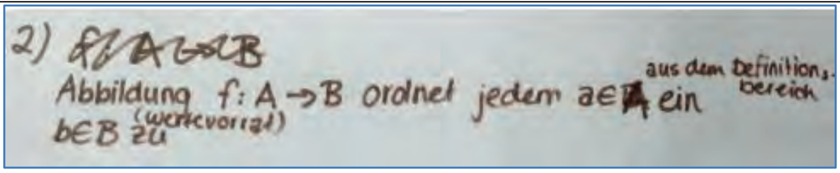
To evaluate the collected data, we developed a coding scheme on the basis of preliminary interviews for analyzing the definitions provided by the students (cf. Kuckartz, 2016). These codes were used to describe differences among the students and to identify developments in the students’ mathematical language use. We coded the students’ use of symbols and technical terms with the following codes:

- the number of symbols, quantifiers, variables and arithmetic operators
- the number of sets and symbols for set operations
- the number of technical terms

In addition, we coded failures that are apparent in the students’ written answers. As failures we coded

- the number of undefined variables,
- missing parts of a definition,
- erroneous terms

After coding the students written answers, we analyzed and compared, what the missing parts and erroneous terms in the students’ definitions exactly were. An example for our coding process referring to a student’s solution concerning the definition of the term function is given in Figure 1: The student used eleven symbols overall in her definition. In detail, she used two (different) variables that are defined ($a \in A$ and $b \in B$), six symbols for sets or set operations ($A, B, \in$) and one calculus symbol ($\rightarrow$). She further used four technical terms and there are no erroneous terms in her definition. Finally, the aspect of the uniqueness of the assignment is not mentioned explicitly (the student wrote “ein” which translates to “a” or “one” but not “genau ein” which translates to “exactly one”), so based on the written answer we coded that as a missing part of the definition. For each excerpt of students' expression, we show the original expression and a translation.



2) Image $f: A \rightarrow B$ assigns every $a\epsilon A$ of the definiton set a $b\epsilon B$ of the value set

Figure 1: Example of the definition of the term function from a student (first type, fourth interview)

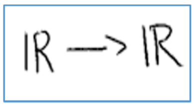
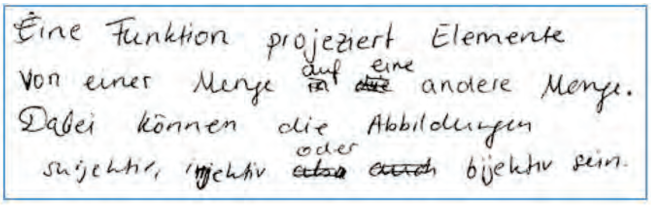
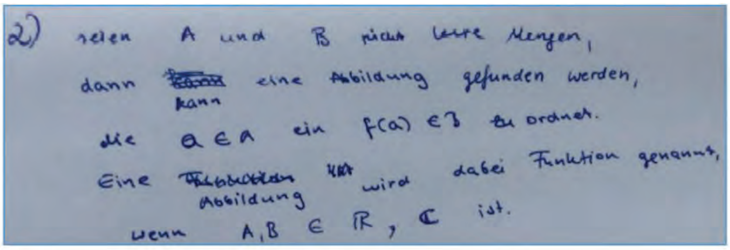
Besides being asked to give and write down definitions of specific terms, the students were asked by the interviewer to explain what characterizes mathematical definitions in general and what differences they see between definitions and theorems. The interview transcripts were analyzed and coded following Mayring's procedure of inductive category development (cf. Mayring, 2014).

Results

From the perspective of qualitative analysis on the basis of the codes described above, we identified three types of students with respect to the development of mathematical language regarding the questions and tasks in the interviews about definitions. Students who can be summarized to a first type (three out of the 15 participants) use mathematical language from the beginning with little or no errors. They use technical terms correctly, the variables that they used are defined and they rarely use irrelevant terms in their definitions. Still, a development of mathematical language can be observed among these students. In the course of their first year of study, their definitions become more precise by avoiding ambiguous terms and an adjustment of word orders. On the other hand, students who were unable to formulate the required definitions in the first interview, or at most incomplete and incorrect definitions, can be summarized to a third type (seven out of the 15 participants). These students start at a low level and hardly any development is seen in their use of mathematical language when trying to define the requested terms over the course of the interviews. Instead of generally valid definitions, students of this type can usually only give examples, so most of their definitions are no sentences in a linguistical sense. Besides that, they often do not define the variables they use and use fewer technical terms than students of the other two types. The second type is represented by students, who have difficulties at the beginning with the use of mathematical language and, therefore, especially with defining mathematical objects. Still, a positive development can be observed during their first year of study. Out of the 15 participants, five belong to this type.

As mentioned before, we focus on a student representing the second type who showed heavy difficulties referring to her mathematical language at the beginning of mathematics studies and analyze her development. Table 1 shows the definitions of the term function from the first, fourth and last interview of a student of the mentioned type. In the first interview, she only managed to write down three symbols (two symbols for sets and one calculus symbol) and besides the fact that not all functions are from $\mathbb{R}$ to $\mathbb{R}$, it remains unclear how this even exemplifies a function. The student seems not to be able to connect her expression of an assignment with more information that it could become a meaningful mathematical sentence. Being asked what characterizes definitions in mathematics and what differences between theorems and definitions are, the student answered the following in the first interview: "A definition [...] that is, that one, […] has an assumption and that one somehow tries to explain by a solution in a mathematical way. [...] I would say a theorem, um, a theorem is basically a formula perhaps. [...] The theorem was rather that one writes down this formula and with defining, it was rather that one deduces something.”. A definition for her at that point was an “(explained) assumption” and “something deduced” – an ambiguous and vague description that she could not explain further.

Table 1: Examples of a student’s definition of a function (first, fourth and last interview)

Interview 1:  IR —> IR	
Interview 4:  Eine Funktion projeziert Elemente von einer Menge ~~in~~ auf ~~die~~ eine andere Menge. Dabei können die Abbildungen surjektiv, injektiv ~~also~~ oder ~~auch~~ bijektiv sein.	A function projects elements of one set to another set. The mappings can be surjective, injective or bijective.
Interview 7:  2) seien A und B nicht leere Mengen, dann ~~kann~~ kann eine Abbildung gefunden werden, die a ∈ A ein f(a) ∈ B zu ordnet. Eine ~~Abbildung~~ Abbildung ~~wird~~ wird dabei Funktion genannt, wenn A, B ∈ IR, C ist.	2) be A and B non-empty sets, then a mapping can be found that assigns $a \in A$ to a $f(a) \in B$. Thereby, a mapping is called function if $A, B \in \mathbb{R}, \mathbb{C}$.

Her definition from the fourth interview is at least a complete sentence, but she used no symbols this time. We coded six different technical terms instead, but three of them are irrelevant for the definition of the term function (e.g. surjective). There are no erroneous terms in the definition, but she used the ambiguous formulation “to project” and furthermore, the aspect of uniqueness is missing. In her fourth interview, she describes definitions first as “basis for proofs” and then as “basis for theorems”, so it seems that at least the deductive character of mathematics has become clearer to her and she can better distinguish definitions and theorems from each other: “A definition is there to give a mathematical proof. [...] Well, a theorem is, so to speak, when I (...) want to prove something and for this proof I then apply various corollaries, I think they are called, and lemma and perhaps also definitions. [...] And the definition is so to speak what I use for the theorem.”.

She managed to produce a more elaborated definition in her last interview (Table 1). In that definition, she used fifteen symbols (the variable she used is defined) and three different technical terms, but we coded again the aspect of the uniqueness of the assignment is missing explicitly. She explains the following about definitions and theorems in general in her last interview: “[...] when you have a term or an expression, that the definition is simply a more detailed explanation of that term or expression. And a mathematical theorem [...] basically just, um, a statement or rule. [...] If you have previously defined something, a term, the theorem simply serves to explain how you calculate with this definition". She describes more concretely what she thinks a definition is ("definitions explain terms") and expresses more clearly, that for her theorems are based on definitions.

The question of what characterizes a mathematical definition could not be answered correctly or only vaguely by almost all participants at the beginning. However, differences between the three types can be observed here as well, and these differences continue to manifest themselves over the course of the first year of study. While students of the first type at least usually answer in their first interview that definitions, unlike theorems, do not need to be proved and must be unambiguous, students of the second and third type are usually not aware of the difference between theorems and definitions, cannot answer the question at all or give a nonsense response. Nevertheless, a positive development can be observed among those students who can be assigned to the second type as described below, whereas for students of the third type it seems not to be clear, even in later interviews, what characterizes definitions in mathematics.

Discussion

We investigated students' development of a mathematical language as a crucial part of students' mathematical sophistication in our study and focused in this paper on the students' ability to formulate mathematical definitions, for what a precise and elaborated mathematical language is necessary (cf. Moore, 1994; Zazkis & Leikin, 2008; Chesler, 2021). As mentioned above, our results indicate a division into three types of students. On the one hand, the students we assigned to the first type already seem to have a good knowledge of the mathematical language at the beginning of their studies. They are able to define familiar terms linguistically correctly and almost without errors and the mathematical language does not seem to be a barrier when they access new content. They quickly adapt to the conventional mathematics language and the deductive structures of university mathematics experienced in their courses and lectures. Therefore, it is reasonable to assume that they do not need specific language support to master their mathematics studies successfully. By contrast, students of the third type are unable to formulate adequate definitions in most cases and have serious problems with using mathematical language. They make errors in linking symbols to each other and use more colloquial paraphrases. It should be noted that during their first year of study the majority of students we assigned to the third type dropped out of their mathematics studies (five of seven). For those in this group who participated in all interviews and did not drop out of their studies, we assume that they will have low chances of success in their studies, particularly because of their difficulties with mathematical language.

The students of the second type are to be classified between these two groups. Here, a positive development can be observed in the course of the interviews as we presented before in more detail, but even at the end of their first year of study, they are still far from being able to use mathematical language adequately and reach the goal of mathematical sophistication. Especially at the beginning of their studies, mathematical language seems to be a barrier for them. Together with the higher pace in university lectures compared to school classes and the new content, a cumulation of different transition problems can occur (cf. Gueudet, 2008). Another noticeable aspect is that students in this group sometimes try to use new mathematical symbols quite quickly and they succeed in doing so to some extent. However, whereas students of the first type seem to have grasped the meaning of the symbols they use, it is at least debatable whether students of the second type actually already use all symbols meaningfully. Following Berger's (2004) hypotheses, this form of functional use and imitation could help students to learn using mathematical language correctly - but it is not clear whether these students will succeed by the end of their studies and achieve the linguistic sophistication that is necessary without specific opportunities for support or whether they remain on the level of imitating phrases based on school mathematical knowledge. Therefore, we assume that these students seem to need substantial support for developing their mathematical language. One possible option could be to discuss failures in using the mathematical language appropriately so that their failures could be used in a productive sense (Loibl & Rummel, 2014). To reveal the relevance of a sophisticated mathematical language, it could also be helpful to confront these students with examples where an ambiguous school-related language is not appropriate. Whereas students of the first type at least have the right idea of what characterizes definitions in mathematics, it turned out that this is not clearly the case for students of the second type. Thus, another supportive measure could be a learning unit that points out the importance of definitions in mathematics and in which the students practice defining.

Due to our small sample and the qualitative approach, we cannot generalize our results or quantify how many students belong to one of the types. However, the results of our study imply that a larger group of students has problems with developing a sophisticated mathematical language.

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A link between a second course in calculus and engineering applications: what the students think about.

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Keywords: Mathematical Modelling, Problem Posing, Engineering Mathematics, Thematic Analysis

Theoretical background and aims

One of the problems in the early years in an engineering curriculum is that mathematics is taught as a separate subject and, in many cases, the students do not recognize the mathematics they use (Kent & Noss, 2003). Modelling activities are usually recommended to bridge the gap between mathematical content and engineering practices (González-Martín et al., 2021). Problem posing offers opportunities to connect mathematics to students' interests. Moreover, within a classroom community, students could be encouraged to pose problems that others in the class might find interesting or novel. Students' beliefs about the relevance of mathematics are considered as one of the factors that can play an important role in their attitude and motivation (Hannula et al., 2016).

To help student to recognize the relevance of mathematics and make it more interesting, as volunteer supplementary problem posing work, assigning as incentive of 3 over 30 extra bonus points to add to their final grade, to Civil and Environmental Engineering (University of Udine) students were asked to find possible applications of the topics of the course (i.e. differential and integral multivariable calculus and differential equations and systems), to engineering and to present them to the class as completion and support of the standard lessons (see also Lepellere, 2021). At the end of the academic year, we also asked to students who participated to the project to write a report about their impressions of the work done. Thematic analysis technique (Braun & Clarke, 2021) was carried out to identify benefits do students report participating in the project. An anonymous survey on the Moodle platform was also administered. The research questions were: what kinds of application will the students choose? Does participating in the project affect students' belief that knowledge in analysis is relevant to engineering? What benefits do students report participating in the project gave them?

Results

Twenty-five students, out of seventy-five, prepared and presented their project: 46% chose a modelling of real problem; 38% a topic connected to other courses (rational mechanics, construction sciences and hydraulics); 21% presented a simplified extract of a research article. Twelve students wrote a report. From its analysis, three important themes emerged: enjoyment; to challenge understanding; to improve connections of mathematical concepts in other contests. Eleven students expressed positive enjoyment, as: *It is a good idea to motivate students to go beyond the classic lesson study…* Just one student said: *However, I cannot claim to be satisfied*. Eleven students said that it challenges students' understanding: *This forced us to understand the proposed arguments more fully… from a previous study of the subject I did not understand its essence and its applicability in real contexts…* Just one student said: *it did not help me in the study of the subject*. It helps to make connections of mathematical concepts in other contests: *It allowed us to broaden our knowledge and get closer to aspects of engineering that perhaps in other cases would not have been of personal* *interest… It has allowed an easier understanding of the same topics seen in other courses allowing a considerable simplification of the learning phase…* Seven students spontaneously highlighted their beliefs about mathematics: *abstract topics…, a very theoretical subject in itself…, many concepts are delicate to understand…* There was no lack of critical issues: *It took me a long time to find something… it is not always easy to find precise, certain, and secure information… it is easier for students who have already attended more "technical" courses… I had a hard time developing the idea and find a conclusion…* About the survey, to the multiple-choice question: What do you think of the supplementary work on applications proposed during the course? Only 5% (2 out of 40) wrote the project is not useful, 85% find it useful, but just 7.5% would make it mandatory, 10% would extend the project even at the end of the course given the excessive workload during the semester.

Conclusions

Despite the efforts that teachers do to make calculus more attractive for future engineers, by offering application examples, often related to physics, the students continue to show little interest in the discipline, deeming it “too abstract”. Furthermore, the students are not always able to connect the topics covered in class with those of other courses, sometimes the simple use of a different “language” confuses them. However, it is difficult to propose alternative activities, as evidenced by the fact that while they consider the proposed project useful, they prefer it to remain optional. But if the students are stimulated, they are able to pose and solve problems using mathematical modelling too. From the analysis of the data, we can conclude that the activity had a positive impact on students’ beliefs that mathematics is essential for the continuation of their studies and career as engineer.

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Study and research paths in statistics: an ecological analysis

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We present a study about statistics university education focusing on study and research paths. An SRP is a didactic proposal based on the inquiry of questions, introduced as a research and teaching tool within the Anthropological Theory of the Didactic. Our research addresses the problem of the SRPs' ecology in the first courses of statistics considering the conditions and constraints enabling or hindering the proposal to be implemented in a sustainable way. We perform a qualitative analysis of consecutive experimentations of the same type of SRP using the methodology of didactic engineering. In this paper we highlight the last stage of didactic engineering, the a posteriori analysis, and develop the analysis of the SRPs ecology at four levels: epistemological, didactic, pedagogical and school. In the results we emphasise the impact of the SRP in the course organisation and point out its strength of being both a research and teaching tool.

Keywords: Study and research path, anthropological theory of the didactic, project-based learning, statistics, ecology.

Introduction

Statistics has rapidly evolved during the past two decades, primarily thanks to technological developments that brought out the possibility and the need for collecting and effectively analysing massive amount of data. Such a change in the profession requires adaptations of the teaching of Statistics that keeps up with its application. A similar cause in architecture resulted in the beginning of a reform of its teaching back in Italy in the 16th century (Knoll, 2014). Knoll locates the origins of the Project Method as an instructional approach for "the academisation of a profession" and not the result of "abstract philosophical deliberations" (Ibid.). Throughout centuries, the method evolved, penetrating various branches at all educational levels and is continuing to adapt, grow and being disseminated. Nowadays, the most common expression for the method's successor is "project-based learning" (PBL) (Batanero et al., 2013; Harmer, 2014).

The research works assuming the PBL pedagogical model in statistics teaching has grown steadily these past decades. These research works address two main aspects: first, its founding principles and second, the description of diverse teaching experiences (Markulin et al., 2021a). In addition, Batanero et al. (2013) propose implementing projects and investigations, bridging the relationship between the mathematical concepts and alive statistics environment, as a method to develop a "statistical sense" of the students. They show that the project fosters not only technical knowledge but a strategical one too (to know when to use a specific content or analysis tool).

As a contribution to the field of statistics education, our research follows the line of research within the Anthropological Theory of the Didactic (ATD) on the study of the ecological conditions in the design, implementation, analysis and development of a new type of instructional format – study and research paths (SRPs) – based on the continued inquiry of problematic questions (Chevallard, 2015).

In a way, SRPs include aspects of project-based learning (PBL). However, they also provide new perspectives and methodologies currently not elaborated in the PBL literature (Markulin et al., 2021a). The question addressed in an SRP has a main goal to generate an answer, regardless of the knowledge mobilised during the inquiry, while in PBL the central aim is meeting a specific piece of knowledge. This communication forms part of the first author's doctoral work on the advancement in the dissemination and impact of the SRPs in university education of statistics for non-mathematics students. We particularly analyse the didactic ecology of SRPs, that is, the conditions that allow their implementation in the classroom, and the constraints of all kinds (epistemological, pedagogical, school and social) that hinder their generalised development.

Research framework

During the past 15 years, a line of research has been developed within the ATD to study the conditions needed for a change of the prevailing pedagogical paradigm. We are now in the paradigm of visiting works, where syllabi are usually a list of themes, topics or disciplines to learn, without necessarily knowing why it is important to learn them. The new paradigm considers knowledge as a tool to question the world and elaborate answers to the questions raised.

To analyse the conditions for transitioning to the new paradigm, the ATD proposes to design, implement, analyse and develop a new type of didactic device: SRPs. This proposal is based on organising teaching and learning processes taking the study of open questions as the central activity. These *generating questions* are posed to students in order to develop answers under the direction of a teacher or team of teachers. The process of elaborating the answers will give raise to research activities (such as: searching for information, collecting data, comparing the information collected, producing partial answers, etc.) and study activities to understand the information collected, acquire and be able to mobilise new tools of analysis. In that sense, the paradigm of visiting works is being replaced by a new paradigm: questioning the world. An important aspect to highlight is that in SRPs, the study of new knowledge or know-how is integrated into the research process (Bosch, 2018; Chevallard, 2015) and that they do not oppose transmission and inquiry.

The research follows the line of implementations of SRPs in chemistry, business administration (BA), medical sciences and mechanical engineering and continues in the case of statistics in BA degrees. These investigations show different modalities of SRP's integration in the courses and their management using different tools originated in didactic research, such as question and answer maps, research and study dialectics and the methodology of didactic engineering (Barquero et al., 2020).

Research questions

The emphasis of this communication, and the doctoral work it forms part of, is to address the problems associated with teaching statistics at the university level in a moment where knowledge in this field is evolving rapidly. We investigate actions that can and do appear in the process of the change from the traditional teaching of statistics towards the acknowledgment of the evolution of the profession and its application in the teaching of it. Our research questions are:

- What conditions can be established in a current degree of business administration (in Spain) to organise a first course of Statistics in the transition between paradigms (where the students' work is led by the need to answer some questions instead of the need to visit some works)?
- What constraints appear at the epistemological, didactic, pedagogical and school level?

To answer these questions, we rely on the literature about similar teaching proposals, their design and implementation, as well as on our own research experiences.

Methodology

As said before, this work is framed in a wider research project intending to study the ecology of SRPs at higher education institutions. More specifically, our research group has been implementing SRPs in experimental sciences (Barquero et al., 2011), engineering (Florensa et al., 2018; Bartolomé et al., 2018), and business administration (Markulin et al., 2021b) among others. The first approach to study the conditions and constraints governing the implementation of SRPs at higher level takes a macro-didactic point of view and assumes as an object of study the collaborative work needed between researchers and lecturers to design and implement SRPs at higher education. Previous research showed that the ecological fragility of the implementation of SRPs is due to the fact that in most experiences the researcher and the teacher are the same person (Florensa et al., 2019). Our work intends to explore how this collaborative work, materialised in teams of researchers and teachers, can address this issue and to what extent this collaborative setting may enhance the long-term viability of SRPs at higher education. Our proposal is not far from the practitioner-researcher collaboration as described by action research method that faces similar challenges concerning this situation where the researcher and the teacher are close collaborators or even the same person (Townsend, 2014).

The second approach is the study of the ecology of SRPs by designing, implementing and analysing SRPs at higher education level, in particular in the fields of mathematics and engineering (STEM). We follow Florensa's (2018) proposal on the use of Didactic Engineering (DE) to do so. In fact, one of the main particularities of SRPs compared to other PBL proposals (Markulin et al., 2021a) is the existence of this four-phase methodology (Barquero & Bosch, 2015). The first phase is the *preliminary analysis* where the institutional conception of knowledge is explicitly characterised in order to identify the foundations of the didactic phenomena addressed. The second phase is the *a priori* analysis of the SRP that will include not only the selection of the generating question but also a new proposal of the knowledge at stake considering it as a consequence of the inquiry process. The third phase is the *in vivo* analysis concerning the implementation and observation of the SRP as well as the data collection. Finally, the fourth phase is the *a posteriori* analysis. The main work is to analyse collected data, to compare the a priori analysis with the actual activity experienced, to describe the different roles played by teachers and students and to highlight the constraints and conditions identified during the implementation.

In this work we adopt this second approach, and we analyse the implementation of three SRPs in a second-year course on statistics in Business Administration degree at IQS School of Management (Barcelona, Spain). We present here the last phase of the DE methodology, this is the analysis of empirical material generated during the implementations (further details on the design and the implementation of the SRPs can be found in Markulin et al., 2021b; Bosch et al., in press). The first implementation was done during the year 2019-20, the second during the year 2020-21 and the third implementation is ongoing (fall semester of year 2021-22). We have collected and analysed different students' productions (weekly reports, final reports), we have also conducted naturalistic teachers' observations, and students have filled in surveys. The survey had been previously piloted in other SRPs implementation (Florensa et al., 2018) and can be found here (https://sites.google.com/view/cerme12-statistics-srp-ecology). To complete this information, we have conducted semi-structured interviews with students focusing on students' experience regarding: the aim of the study, data gathering, teamwork, assessment, etc.

Results and discussion

In this section we present and discuss the results obtained during the last phase of DE methodology. We propose an overview of the a posteriori analysis structured in four levels of didactic co-determinacy: epistemological, didactic, pedagogical and school (Chevallard, 2015). For each level we present the conditions and constraints detected during the implementations of the SRPs. An overview of the results is also shown in Table 1.

Epistemological level

The different implementations of the SRPs have caused important modifications at the epistemological level. Changes at this level are very challenging as they tend to modify the curriculum, or at least to heavily modify the previous conception of knowledge. This modification contrasts with the prevailing epistemology at the institution: that is the conception on what statistics is and the attributed raison-d'être. Even if the adoption of the European Higher Education Area standards led to competence-based curricula, the institutional conception, often implicit and ill-defined, remains in terms of conceptual organisations and pieces of labelled knowledge.

For the case of statistics, we consider it as an area where its epistemological modification towards a knowledge conception in line with the paradigm of questioning the world is favourable. First, because scholarly knowledge is evolving rapidly, especially in the past 20 years with significant technological developments and a shift in the way of collecting and dealing with data. Secondly, because statistics is placing the focus of the course more on the interpretation of the results than on the mathematical procedures that are at the foundation of the calculations, and this fact enables a better fit with the degree of BA and nurtures the motivation for the field. Nowadays, there is no shortage of questions and problems to pose when proposing any activity or project in the teaching of statistics.

However, this favourable situation is not without a challenge: a transposition work needs to be done to reorganise the praxeological components of the statistics organisations. This didactic transposition process should consider the selection and adjustment of the scholarly knowledge to become the knowledge to be taught. In the process, new technologies in the ATD sense will get elaborated to adapt into better correspondence with the statistics software as well. Epistemologically, the area is compliant with the creation of new exercises and teaching proposals. Nevertheless, teachers can have a lack of legitimacy to "invent" new teaching materials and knowledge organisations. In our case, such a constraint is not an issue thanks to the freedom that the teacher-researcher has for transforming the knowledge to be taught according to some didactic and epistemological principles.

Interviews with the students revealed their views on the project and the area of statistics in general: "I think I learned more in the project than in the rest of the course", "I learned that you could know something about subjective things about people and that, if you could do it with vegetable diets, you could also do it with other subjects, such as politics."

Didactic level

On the didactic level, we consider a crucial point that the community of study takes the SRP's generating question seriously. The need to produce an answer is, in fact, the generating force of the whole inquiry process and points to the direction for the final answer to be presented. However, the variables governing this process are difficult to identify and of diverse natures.

The first variable identified is the origin of the dataset, this is if the dataset and the survey are given (external) or produced by the community of study (internal). In the first two SRP implementations, the survey was composed by the marketing experts and the students were in charge of collecting the data. We considered that this external origin would engage students because of its closeness to a real setting. However, in the interviews we found out that the external origin of the survey led to students' detachment mainly because of the difficulties to understand its structure and questions: "At the beginning the questions seemed strange. But when we started analysing with graphs and summaries, we started understanding it." Consequently, for the third implementation, we plan to build the project survey together with the students, making it more a product of internal descent to encourage the involvement, making the structure more understandable than in the previous implementations.

On the other hand, the receiver and validator of the question and its answer can be inside the course, the institution or from outside the education setting. We administer a combination of internal and external receiver/validator, depending on the step of the project. Derived questions and the feedback provided based on the intermediate students' reports are validated by the course teachers, while the final answers are presented for the external jury (members of the institution and the external client). The validation of the results is announced beforehand using an evaluation rubric which forms part of the established didactic contract. Furthermore, the case studies during the course, before the project, help introduce (or train) didactic strategies and devices that are later vastly used in the project activity. An example of such a strategy is a practice of always starting with a Q_0 that is open and needs to be studied in several sessions, meaning it is presented as an activity that requires deep attention and investigation by the students and cannot be solved within 20 minutes or even in one session. The teacher is then a guide in the process and not an instruction giver.

Pedagogical level

As the business requirements for our students require collaborative work, most of the students' activities during the statistics course we present are carried out as teamwork. On top of that, the idea of an SRP by the ATD proposes raising above the individual work. It encourages connecting, interfering, brainstorming and advancing in a problem setting by interaction within and between groups. Case studies that form the course before the project, apart from being managed in teams, cherish investigative environment and set pace to the statistical activity. They have a fixed duration of two weeks and always end with a submission of a team report that is afterwards corrected by the teachers. The culmination of the project, on the other hand, occurs in a session of posters or presentations, depending on the conditions the course is going under (pre-pandemic project finished with poster sessions in the classroom, during the pandemic project finished in online presentations). Although the project is being organised and driven by the problems that are considered relevant for the students' degree, statistics remains a topic from a group of courses based on quantitative methods. On average, this brings a pre-established aversion of the students towards the course and a lack of dedication in its components. However, the inclusion of a project is perceived as a valuable stimulation for some of the low-motivated students, as noted in the interviews: "It makes a huge difference; what we will encounter tomorrow, the fact that our work is going to have an impact on the work of different people".

Table 1: An overview of the SRPs conditions and constraint and their evolution

	CONDITIONS	CONSTRAINTS	EVOLUTION
Epistemology	° Flourishing area of research ° Area of knowledge in fast evolution ° Importance in BA ° Facility to modify the knowledge to be taught	° Prevailing epistemology: difficulty to accept new organisation of knowledge ° Didactic transposition work needed to adapt new areas of knowledge	From SRP1 to SRP2: ° Re-scheduled detaching from traditional organisation (kept during SRP1) → Planning SRP3: continue the praxis from SRP2
Didactics	° External Q_0 proposed by real companies - partners ° Report writing ° Final assessment	° Data are given ° Difficulty to incorporate new tools by students ° Didactic contract based on external validation of answers	From SRP1 to SRP2: ° Incorporation of intermediate reports allowing to give feedback to students ° The project was initiated earlier in the course → Planning SRP3: continue the praxis from SRP2
Pedagogy	° Teamwork ° Calendar of the course: case studies ° Oral presentation	° Low degree of students' engagement ° Lack of creativity	From SRP1 to SRP2: ° COVID pandemic heavily affected the development of SRP2 ° Explicit management of the process of study in the second edition.
School	° Assessment ° Computer availability ° Use of free software R ° Moodle, MS Teams ° Teacher's accessibility ° Small groups	° Private university: complaints on changing the terms of didactic contract	The school stays the same

School level

At school level, we present our institutional setting, a private university, shows no or few changes from one SRP implementation to another. A direct consequence of the COVID19 pandemics was the implementation of the MS Teams for all the teachers and students. There were also vast investments in the technological equipment to ensure the quality and synchronous teaching and interaction, but no other changes. Teachers have the autonomy to arrange assessment of the course (two to three teachers depending on the implementation, well connected and easy to agree between themselves) and to adapt it from one year to another. They also have freedom with the calendar organisation of the course (no need to be in accordance with teachers of other courses or with the instructional organisation). The students' attendance is compulsory, and they are organised in relatively small groups (big teacher-student ratio compared to usual university groups). Each student owns a personal computer and is required to always carry it. In Statistics we introduce R software, more precisely its user interface R Commander, and use it in every session. For the provision of the materials, we use a learning platform Moodle. Implementing the SRP in a private university modified its initial schedule due to the students' requirements for extending the pre-SRP sessions devoted to statistical inference. Conditions are certainly different in this respect in Spanish public universities.

Conclusion

The consecutive implementations of SRPs in statistics for BA students allow us to transform the Statistics course in several aspects. The didactic phenomenon at stake becomes questioned before and after every implementation, providing new perspectives both from the epistemological (the profession) and the didactic (reorganisation and enrichment of didactic devices) point of view. In a way, already by seizing to detect the phenomena in the foundation of the professional area, occurs the questioning of the educational setting and puts the basis for the design of the SRP. Furthermore, according to the teachers' observations, every implementation that was experienced suffered from different or repeated obstacles. Such constraints bring to consciousness the impact of different didactic strategies or influences originating in other levels, whether epistemological, pedagogical or the school. For example, in the statistics course presented, the case studies facilitated certain skills that appear in the SRP as well. However, this situation can be a limitation, for instance, when a certain activity appears in the project but is not previously encountered during the case studies. The analyses of each implementation support the common findings and stimulate the design of the following implementations of the SRP and the course in its entirety. The long-term viability of the SRP implementations to this Statistics course is still under study. In addition, a major ecological change will take place in 2022-23 when the researchers will not be involved in the course anymore.

In the line of the SRPs developments, we cannot avoid stating that an SRP is a transformative and questioning instrument, not only for the students but also for the teachers who design the course. We observe that the implementations of SRPs progressively influence the organisation of the program. If it was first included as a complement to a course typically structured according to the logic of the contents, it gradually gained more prominence and made the content subordinated to the anticipated needs of the research. This goes in favour of extending the unit of analysis of the SRPs' methodology to the entire course of Statistics.

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A university mathematics transition programme designed to increase student engagement with mathematics support

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This paper describes preliminary findings from the ongoing project known as MathsFit, a suite of online and in-person mathematics supports designed for in-coming first-year university students of service courses. We describe the rationale of the project, some engagement metrics, and present results from the first year of the programme. Early results indicate that MathsFit does improve student engagement with the mathematics support services available. Plans for the next iteration of MathsFit, based on these results, will also be discussed.

Keywords: Undergraduate students, diagnostic teaching, sector transition, mathematics support.

Introduction

Diagnostic testing of incoming university students has been occurring for at least 30 years (Lawson et al., 2002; Hyland & O’Shea, 2021) and often leads to the identification of students at-risk of underachieving in mathematics courses with the objective of supporting these students as early as possible in their transition to university mathematics (Gillard et al., 2010; Ní Fhloinn et al., 2013). However there have been no attempts, that the authors know of, at developing best practice in terms of how and why such testing should be conducted and what follow-on supports are most effective given varying institutional, socio-cultural and didactical contexts. In this on-going project MathsFit, a blend of in-person and digital supports, we employ an action research methodology of cyclical critical reflection using learning analytics, student interviews, and student survey response data.

Literature Review

The growing ‘Mathematics problem’ (Hawkes & Savage, 2000), i.e., the difficulty students, especially service students, experience in the transition from secondary to university mathematics has led to a number of initiatives in universities worldwide to support such students. Common measures include the expansion of mathematics supports such as drop-in centres and extra organised support tutorials (Lawson et al., 2019), the creation of bridging or preliminary courses, some of which have been discussed in previous CERME conferences (Kürten & Greefrath, 2015; Biehler et al., 2011), and diagnostic testing which has been researched extensively in the UK and Ireland (Gillard et al. 2010; Hyland & O’Shea, 2021). Diagnostic testing can allow for the identification of students who would most benefit from additional mathematics support during their initial time at university and beyond, commonly known as ‘at-risk’ students. Lee et al. (2008) used diagnostic test results among other variables to predict the performance of students in later mathematical exams. Other uses for diagnostic testing found in a survey of UK institutions by Gillard et al. (2010) include tracking changes in student profiles (e.g. Faulkner et al., 2011), informing students and staff about the students’ mathematical competencies and guiding the development of additional support services.

There are two parts of diagnostic testing that necessitate sound learning design principles – the test itself and the follow up support (Savage et al, 2000; Lawson et al. 2002). In choosing the mathematical topics to include in a diagnostic test, the mathematical expectations for students at university are communicated (Hyland & O'Shea, 2021) and a connection to their previous mathematical learning is made. Other design decisions, including whether the test is paper-based or electronic, and if questions are multiple-choice or not, affect the efficiency of the marking process and thus the speed of follow-on support. Follow up support can vary, with most including recommendations to engage with the mathematics support available (Hyland & O'Shea, 2021; Gillard et al. 2011). Personalised and/or monitored support has a significant positive effect on students' engagement with offered support (Gallimore & Stewart, 2014; Burke et al. 2012). Large online systems with automatically marked multiple choice diagnostic tests followed by a personalised, gamified online remediation course that allows immediate feedback and increased student choice have also been successful (Sharma et al., 2019). Ensuring diagnostic test results provoke action by students and university staff is the key theme across the literature on how diagnostic testing can aid students' transition to university mathematics.

Rationale

The rationale for implementing this project at this time were numerous. Firstly, quantitative research conducted on visits to the University College Dublin's Maths Support Centre from 2015-2020 showed that the majority of students attending the centre achieve high grades in the courses they sought support for (Mullen et al., 2021a). Conversely those students who failed mathematics courses, particularly in the first semester of first year typically did not attend the MSC or if they did it was often on only one occasion and late in the semester. Thus, it was conceived that new strategies to engage and support these under-performing students were needed beyond existing measures of advertising and promotion of the service. Secondly, given the significant disruption caused by the COVID-19 pandemic on second-level students' mathematics education from March 2020 there was a concern among MSC management that the mathematical preparedness of incoming university students may not be as strong compared to pre-pandemic times. Thirdly, given the disruption to state examinations and the delay to the collation and awarding of teacher awarded grades, university students started later than usual. This meant a reduction in instruction from 12 to 10 weeks. Combined with the fact that first-year final university examinations could not proceed in the traditional fashion, these students needed to be assessed within this same 10-week period. These factors persisted for the 2021/22 student cohorts also.

Methodology

In year one, MathsFit targeted two of the largest service modules taught by the UCD School of Mathematics and Statistics, Mathematics for Business (515 students) and Mathematics for Agriculture (371 students). These student cohorts come into university with very similar mathematical prior learning standards and university entry requirements, with Business requiring slightly higher matriculation points. A 30-question diagnostic test, referred to as 'Proficiency quiz' in communications with students, assessing fundamental concepts was devised by the authors. Existing diagnostic tests from Ireland, the UK, and Australia were consulted. The motivation was for students to reach ~~a~~ mastery level in the basics of mathematics fundamentals; Arithmetic and Trigonometry, Algebra, and Functions and Calculus. Questions were chosen on the basis that (i) they were taught in the Ordinary Level Leaving Certificate course (the state terminal second-level school examination sat by all school leavers in Ireland), (ii) they would be required in the forthcoming university courses, and (iii) they were a mixture of both procedural and conceptual questions. Figure 1 shows the first question of the Algebra and seventh question of the Functions and Calculus sections. All but two of the questions were multiple choice with five options: the correct answer, three distractors (based on common misconceptions), and 'I don't know'. Non-multiple-choice questions (such as Q1 shown below) were included to avoid 'process of elimination' type responses via the multiple choices offered. We included 'I don't know' as an option throughout to discourage random guessing and to differentiate between students who made errors in their solution and those who could not approach the question.

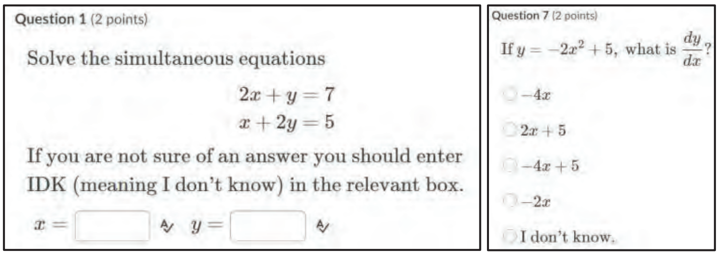

Figure 1: Question 1 of the Algebra and question 7 of the Functions and Calculus sections of the MathsFit Quiz

The quiz was delivered online through the Quiz feature of the university's VLE Brightspace and students had 45 minutes (15 minutes for each section) to complete the quiz. In advance of the quiz students were shown an introductory video welcoming them to the university and explaining the rationale of MathsFit. The quiz counted for 3% of the continuous assessment grade for the Business module and did not count for credit in the Agriculture module. Demographic information including students' entry route to university and feedback on the students' experience of the quiz were collected through pre and post quiz surveys.

Utilising Intelligent Agents and Release Conditions features of the VLE, each student was sent a personalised email within 24 hours of their first attempt which included their results for each of the three sections, their overall result and a set of feedback instructions detailing how they may like to remediate. The email wording of these remediation factors was pre-programmed and dependent on their quiz score, which had five bands of classification. For example, those students who did most poorly were invited to book a one-on-one maths support session with a tutor to discuss their quiz and work through the areas they scored poorly on. Those students who scored the highest grade possible were invited to lead a study group of their peers under the facilitation of a senior MSC tutor. All students who did not score in the highest band were invited to retake the MathsFit quiz the week after their first attempt so that they may gain the extra academic credit, in the Business case, or refresh their mathematics, in the Agriculture case. This allowed us and the students to measure their improvement and further identify students who still score poorly after being offered initial support.

Results

446 Business and 254 Agriculture students gave their consent to participate in the study. There were 164 Business (37%) and 185 Agriculture (73%) students scoring in the lowest band. Only 11 Business (2.5%) students and no Agriculture students scored in the highest band. In terms of the specific mathematical areas tested students scored best on Arithmetic and Trigonometry and worst on Functions and Calculus. Figure 2 below displays the results by topic area for both attempts for the Business student cohort and for the first attempt of the Agriculture students. The second attempt results of the Agriculture students were not included due to low participation (n=13).

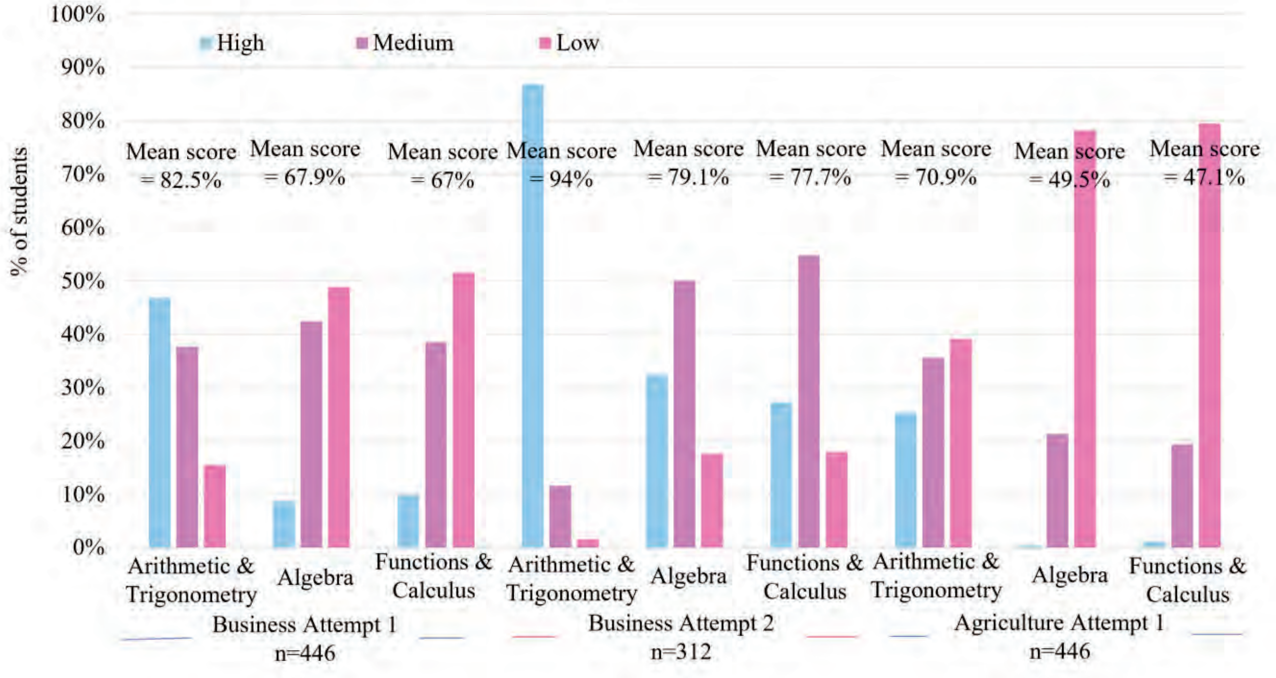

Figure 2: MathsFit Business and Agriculture Results by Section

The pre-quiz survey included a question on how well students felt they had covered their second-level mathematics curriculum and what topics they may not have covered sufficiently or at all. 38% of Business students and 34% of Agriculture students were unsure if they had covered the curriculum while 30% and 38% of Business and Agriculture students respectively were certain they had not. The post-MathsFit quiz survey results for Business students indicated that 11% did not have enough time for the Arithmetic and Trigonometry section, 46% did not have enough time for Algebra, and 27% did not have enough time for Functions and Calculus. The corresponding percentages for Agricultural students were 18%, 36%, and 31% respectively. Answers to these questions may explain in part, why the majority of students fell into the 'Low' category in the Algebra and, Functions and Calculus sections, calculus especially being a topic only covered in the Leaving Certificate curriculum, while other topics would have been covered prior to that. The module lecturers were informed about these results including students' particular struggles with exponents and features of functions (e.g. range) to aid their conceptions of students' prior knowledge and thus their teaching.

To assess the predictive nature of MathsFit, linear regression analysis was performed on the Business students' MathsFit data and other continuous assessment data from their Mathematics course. Both attempts at MathsFit were positively correlated with all the assessment components (six quizzes and two exams, all taken online). Students' first attempt at the MathsFit quiz had a correlation coefficient of 0.48 with their final percentage grade (see Figure 3). Their second attempt had a correlation coefficient of 0.53 with their final percentage grade (see Figure 4). Stepwise linear regression found the best model to predict students' final percentage using MathsFit data from both the quiz and the survey contained ten highly significant terms including their MathsFit second attempt result (denoted MF2 in the regression equation below), their gender, their degree, their Leaving Certificate result (H1 is the highest possible grade achievable), whether they thought they had covered the curriculum, and whether they had enough time in the Algebra section.

Final Percentage = 22.6533 + (0.61 × MF2) + (9.02 × H1) + (4.2 × H2) + (3.93 × Female) − (3.36 × NotSure, Covered Curriculum) + (0.3 × Yes, Covered Curriculum) + (6.5 × Other Degree) + (3.65 × H5) + (2.28 × A Enough Time).

The model had an adjusted R-squared value of 0.3816, which means 38.16% of the change in Final Percentages can be explained using this model, similar to Lee et al.'s (2008) model. In particular, for every 1% increase a student scored on MathsFit, the model predicted a 0.6063 increase on the Final Percentage. Of 446 students, the model correctly predicted 433 of the students passing. It incorrectly predicted 11 students would pass, however they failed (false positive rate of 2.5%). It also incorrectly predicted 2 students would fail, but they passed (false negative rate of 0.45%). This model shows that MathsFit data can be used to identify at-risk students; however, linear regression may not be the best model to capture the trend. The quartile-quartile plot (Q-Q plot) for this model reveals a straight line for the middle values, but deviation at either end, particularly the lower end which means the residuals are not normally distributed. It implies that the model is working well for students in the middle range, but not well for students with lower marks, unsurprising as the model assumes normal distribution and the data is left skewed. Despite not being the best possible fit, this linear regression model shows that the data gathered through MathsFit, like other data from diagnostic tests (Lee et al., 2008, Gilliard et al., 2010) can be used to identify students at risk of failing and hence those in need of support.

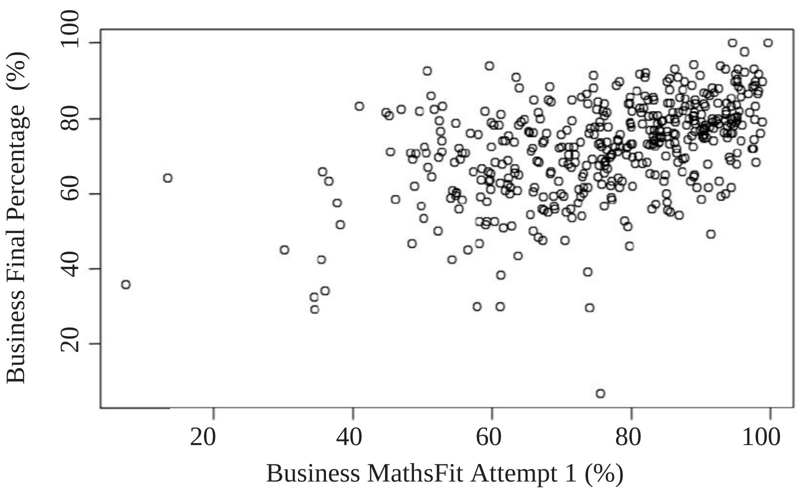

Figure 3: Correlation between Business students' MathsFit Attempt 1 results and final percentage

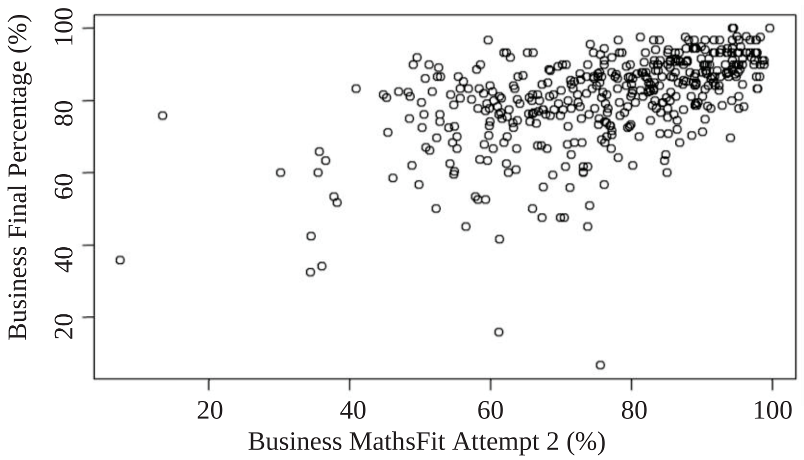

Figure 4: Correlation between Business students' MathsFit Attempt 2 results and final percentage

Analysis of visits to the university's Mathematics Support Centre (MSC) by Mathematics for Business and Mathematics for Agriculture students was completed to investigate the potential effect of MathsFit. Some 17 of the 51 Maths for Business students identified as needing support most (scoring 'Low' in all three quiz sections) visited the MSC, a positive beginning but engaging this cohort still requires attention. Comparing the percentage change in number of visits from the academic year 2019/20 to 2020/21 for those two courses to the 200 other courses using the MSC reveals a positive impact of MathsFit. The average percentage change in number of visits per course from 2019/20 to 2020/21 was -60% due to the sharp drop in visitors caused by Covid-19 and the move to online support provision (Mullen et al., 2021b). The 14% drop in visits for Mathematics for Agriculture and the 166% increase in visits for Mathematics for Business are thus above average. In particular, Mathematics for Business was one of only 20 courses whose number of visits increased in 2020/21 in comparison to 2019/20. Some of this increase in visits can probably be attributed to the impact of MathsFit, but other factors also may have played a role, such as, the course lecturer's increased advertising of the Mathematics Support Centre when lecturing online in 2020/21 in comparison to lecturing in person in 2019/20. MSC visits have been proven to positively impact students' grades both in UCD (Mullen et al. 2021a) and in other international contexts (e.g. Lee et al. 2008) so this increase in MSC visits by Business students due to MathsFit is a positive outcome.

Next steps

In September 2021, MathsFit was extended to incoming first-year students of five service courses including Calculus for Science, Introduction to Calculus for Engineers and the Business and Agriculture modules targeted in 2020. This involved approximately 1,600 students. Based on the authors' analysis of the first iteration of MathsFit, four of the five course coordinators agreed to award 3% of their continuous assessment for MathsFit. For comparison purposes a core set of 15 questions was common across all five MathsFit student cohorts, though the Engineering and Science quizzes were assessed at a higher cognitive level than the Business and Agriculture quizzes due to the mathematical requirements of these courses.

Analysis of the survey (39 respondents - 29 Business / 10 Agriculture) conducted in June 2021 found that students felt being asked to take a quiz in the first week of university without preparing for it was not student-centred. Hence, this year participating students had the option to engage in a refresher course before (and after if necessary) attempting the quiz in their first tutorial. Both the refresher course and quizzes were delivered using the open-source e-assessment tool Numbas (https://www.numbas.org.uk/). This allows for corrective feedback and multiple attempts (in the refresher course context) and a randomization of question~~s~~ variables. Hence students get multiple opportunities for practice and do not see the same quiz if a second attempt is necessary as was the case with the static quiz in Brightspace. Due to public health restrictions in Ireland, lecture and tutorial slots must be capped at 45 minutes and so it was necessary to shorten the quizzes to 24 questions. Discriminant analysis and correlation analysis was conducted on the 2020 answers so that the questions with the lowest discriminant indices and/or highly correlated with another question were removed from the 2021 version of MathsFit.

Conclusion

MathsFit has reached one objective in that it had a positive impact on the number of student visits for the Business course involved in the study. It has also alerted lecturers to the gaps that some of their students have in terms of mathematical preparedness for university mathematics, in particular exponents and the range of a function. Finally, it has targeted the students who are most in need of timely mathematics support and introduced them early to the services on offer. Clearly the carrot of offering even a small percentage of credit for mastery within MathsFit had a significant impact on engagement with the second attempt and subsequent mathematics support engagement among the Business students.

Acknowledgments

We would like to acknowledge the Maths Support Centre tutors who helped test the MathsFit Proficiency quiz questions and helped design the refresher courses in both years of the study.

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TWG15: Teaching Mathematics with Technology and Other Resources

Introduction to the papers of TWG 15

Teaching mathematics with technology and resources

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Introduction

The two technology groups at CERME (TWG15 and TWG16) adopt a broad view of technology resources in mathematics education to include tangible devices (hardware), and the associated applications (software), within the context of learning and teaching. TWG15 addresses the related issues that mostly concern teachers and teacher education. Previous discourse at ERME conferences embraced a wide variety of research topics, theoretical and methodological approaches. Most recently, this has focused on: teachers' uses of students' (digital) productions; sorting and organising digital content such as: simulations, applets and Open Educational Resources (OERs); the teaching of computational thinking in, and through, mathematics; teachers' choices and beliefs concerning technology use; and the ever-challenging process to develop useful theories and new pedagogies. The group is keen to learn more about actual uses of technology in classrooms (and beyond) to understand both the prevailing classroom practices and the implications of this on policy, practice and theory. The Covid-19 pandemic offered such a context and two papers (Barlovits, Kolokytha, Ludwig & Fessakis and Ramirez) and one CERME12 award-winning poster (Bini) reported such research from Chile, Germany and Italy respectively.

The group concluded the following perspectives and suggestions for future research. Continued effort is needed to enable wider understanding of the theories and methods applicable to this domain. More specifically, a greater awareness of the epistemologies that underpin these theories and methods, particularly when adopting more than one frame and/or attempting to network them in some way. The prevalence of quantitative studies led the group to conclude that increased attention is needed to the sampling of research participants and their associated characteristics, which can greatly impact the research claims that can be made when reporting findings. In addition, the roles and activities of participants within research studies is also an area of growing interest for the field. Finally, the nature of innovation within the field of technology in mathematics education results in many future research opportunities that have both pedagogical and mathematical implications, and make this a vibrant area for study.

Organisation of TWG15 and its sub-themes

TWG15 accepted 22 papers and 6 posters for presentation at the conference, which reported research from 20 countries (Argentina, Austria, Canada, Chile, China, Denmark, England, France, Germany, Ghana, Greece, Ireland, Israel, Italy, Norway, Portugal, Spain, Sweden, Turkey & United States). Each paper was allocated to one of six sub-themes and all authors of papers recorded 5-minute presentations of their papers, which were made available for asynchronous viewing by all CERME participants. In each TWG session, the allocated video presentations were broadcast prior to the TWG discussions. Each Sub-theme chair structured the whole group and breakout group discussions around key ideas that emerged from the related papers. In the text that follows, we summarize these ideas and suggest implications for our field of research. We conclude with 'hot topics' for the TWG15 community, which emerged from our final collaborative discussions.

Sub-theme 1: Theoretical frames for investigating the teaching of mathematics with technologies (Chair: Melih Turgut)

Tools and technologies for teaching and learning mathematics continually change. Accordingly, it naturally takes time for teachers to adapt the use of technologies in classroom practice, a process that is known to be challenging. Alongside, pedagogies also evolve and emerging theoretical/conceptual perspectives are needed to provide insights on how to integrate both new technologies alongside new functionalities of well-adopted tools. Many theoretical and conceptual frameworks have been developed to help understand the inevitable gaps between teachers' existing content and technological knowledge, and the processes through which new technologies transform classroom practice. The TWG15 papers provided a broad spectrum of emerging theoretical perspectives for designing and understanding teaching practice. We underline two main applications of theories: (i) as more deductive research frames, such as Ruthven's *Structuring Features of Classroom Practices* (SFCP) (Ruthven, 2009) (Simsek, Bretscher, Clark-Wilson & Hoyles), Gueudet and Trouche's (2009) *Documentational Approach to Didactics* (Basturk-Sahin & Tapan-Broutin), and Koehler and Mishra's *TPACK* (2009) and TPACK developments (i.e. Bray & Oldham; Lyublinskaya & Du; Meier & Oliveira); and (ii) as multiple frameworks that are merged to offer new theoretical/conceptual lenses, such as: Verillon and Rabardel's (1995) *Instrumental Genesis* and Robert and Rogalski's (2002) *Double Instrumental Approach* (Haspekian & Fluckiger), and *TPACK* and Valsiner's (1997) *Zone Theory* (Lindenbauer, Lavicza & Weinhandl).

Barlovits et al. propose a set of "design requirements" for the development of mobile environments for teaching distance mathematics, where they combine a community of inquiry lens, with e-pedagogy and mobile learning models. Gonscherowski and Rott conduct an interview study to explore pre-service and in-service teachers' argumentation and justification regarding digital tool use in mathematics education. Gonscherowski and Rott consider a combined analysis tool; teaching phases and classification of the use of digital tools (Clark-Wilson, Robutti & Thomas, 2019) and they further elaborate different levels of *decision-*

making competencies about appropriation of the digital technologies. Haspekian and Fluckiger combine the five components of the Double Approach (Robert & Rogalski, 2002) with notions of instrumental distance and didactic landmarks to analyse teachers' instrumental geneses when integrating programming in their practices. Their combined analysis addresses teachers' difficulties, which mainly concern '... the changes that ICT introduce[s] at cognitive and meditative levels'.

Sub-theme 2: Methodologies and methods for investigating the teaching of mathematics with technologies (Chair: Alison Clark-Wilson)

The TWG15 papers and posters predominantly report empirical research that spans a diversity of methodologies. Each study is underpinned by the respective researchers' epistemologies, which frame their thinking on what it is possible to know and conclude from their research findings. Consequently, this TWG session encouraged participants to think critically about if, and how, the application of research methods in different studies did (or did not) shed light on the chosen phenomena, and in accordance with the selected theoretical frame.

The majority of studies adopted qualitative research designs that sought to reveal aspects of the different phenomena at hand: (i.e., Engelhardt & Roth; Meier & Oliveira; Vilchez & Lemmo; Speer & Eichler; Tortoriello & Veronesi). Common qualitative methods included questionnaires, surveys, observations, interviews (individual, focus group), and document scrutiny/analysis. In some cases, data is analysed to present findings quantitatively. For example, in the study by Lyublinskaya and Du, a novel data visualization method reveals interesting characteristics of pre-service teachers' development of TPACK over time.

The second most common method used in the studies (i.e. Gavor, Clark-Wilson & Hoyles; Kristinsdóttir, Hreinsdóttir & Lavicza; and Lindenbauer, Lavicza & Weinhandl) is the increasingly used *design (or design-based) research approach* (Bakker, 2018). At the heart of such an approach is the aim to research innovations (i.e. design new materials or approaches) for which existing theories and methods might not be appropriate. Instead, more 'humble' theories inform iterative cycles or research that involve different methods to enable the theoretical, methodological and practical knowledge to emerge.

Two mixed method studies were presented. Fan and colleagues propose a mixed method approach to answer the question: How do mathematics secondary school teachers in China use digital resources in their teaching? They design a questionnaire for teachers about their uses of digital resources, before, during and after lessons, according to the three dimensions (content, function and infrastructure). The results are analysed quantitatively, alongside a sample of follow-up interviews, which are analysed qualitatively. The quantitative results show that more than one half of the teachers use digital resources often or always, and especially during lessons. However, they less frequently use resources after lessons, and the variety of resources used before, during and after is still limited. Interviews are used to ask the reasons for the use of specific resources. Segal and Biton's study, conducted in Israel, research teachers' perceptions of their work in the Whatsapp environment, also using questionnaires and interviews, and additionally Whatsapp messages and observations of four teaching groups.

Results of this study are at two levels: for students, in feeling free to make mistakes without fear, and for teachers, who gained in their professional technological knowledge.

The three studies by Thurm, Geraniou and Jankvist, Müller and Wachte, and Sharkia and Kohen all adopted quantitative designs.

Finally, TWG15 discussed how well we paid attention to, and made explicit in our research, our methods for sampling the participants of our studies. The prevalence for 'opportunity sampling' of pre- and in-service teachers, was acknowledged as a concern when set alongside claims for wider application or generalisation of research findings.

Sub-theme 3: Teachers' different roles in diverse educational settings (Chair: Ornella Robutti)

There is a recent increase in research concerning the role of teachers within in educational settings and processes. From being recognized as a "dimension" in studies on mathematics teachers (Robutti et al., 2016), to becoming one of the four fundamental themes of ICMI Study 25 *Tools and resources used/designed for teacher collaboration and resulting from teacher collaboration* (Robutti et al., in press). TWG15 interpreted teachers' roles within the different institutions as: prospective teachers (e.g., undergraduates in mathematics courses with a clear aim in preparing future teachers); pre-service teachers (e.g., participants of university-led teacher education program); in-service teachers (e.g., within their professional work context, or in formal projects); teacher educators (e.g., participating in formal or informal teacher education settings). The papers presented below relate to teachers in the aforementioned roles, focusing on the different types of involvement for teachers as participants in the research.

One role is teachers as *designers* of tasks using technology (Guerrero-Ortiz & Camacho-Machín; Speer & Eichler; Fahlgren, Szabo, & Vinerean): teachers are protagonists and have an active role in the design of tasks. Another role for teachers is that which is presented by Lyublinskaya and Du. In their study mathematics pre-service teachers are *learners* of technologies to be used in teaching mathematics. Their learning trajectories are analysed using the TPACK frame within the institutional context of a non-discipline-specific online educational technology summer course. The findings show that the pre-service teachers grew in their discipline-specific TPACK, under the influence of personal and contextual factors. The study demonstrates an innovative use of TPACK, combined with digital timelining analysis, to describe teachers' growth in professionalism as dynamic processes.

Jacinto and Carreira show another kind of teachers' involvement in the research: as *problem solvers*, with the use of technology. Contextualising the teacher's approach to technology in the general lens of *humans-with-media*, the study questions the role of technology within the mathematics teachers' problem-solving processes. The study reports the gain in the teacher's competences according to the MPST model, facilitated through the different micro-cycles in the overall process.

A community of inquiry comprising teachers as *participants* constitutes another kind of involvement: active participation is a distinctive feature of the community, intended not only in presence but also at distance. The ASYMPTOTE project (Barlovits et al.) is presented as an evolution and adaptation of the MathCityMap system in the COVID-19 pandemic era. The particular setting of the technology, which uses the idea of decision trees, enables teachers to design differentiated tasks for students in a flexible way, directing students to different learning paths, according to their performances.

Sub-theme 4: Innovations in technology for teaching mathematics (Chair: Daniel Thurm)

Technologies in mathematics education are constantly evolving. “New” innovative technologies emerge and become over time more established technologies. For example, dynamic geometry systems, computer-algebra systems, or function plotters - technologies which were considered “new” in the 1990 – have by now a strong research base. However, even for those technologies there is still much to be understood with respect to teaching and learning which, is evidenced for example by the papers of Fahlgren, Vinerean, and Szabopaper (task design for dynamic geometry environments) or Jacinto and Carreira (problem solving using spreadsheets). At the same time some TWG15 papers focused on more emerging technologies, e.g., videogames (Vilchez & Lemmo), WhatsApp-messenger (Segal & Biton), videos in a flipped classroom setting (Sharkia & Kohen) and multitouch environments (Bakos).

Vilchez and Lemmo presented research on a game-based approach, concluding that the selected videogame has potential to support the teacher to scaffold students’ relational thinking during whole-class discussions. In particular, the authors hypothesize that observing students play the videogame might implicitly help teachers to orchestrate classroom discussions.

Another emerging technology not so widely investigated from the perspective of the teacher is WhatsApp. Segal and Biton’s research focuses on opportunities for learning and teaching that can be created using WhatsApp as a social network. While teachers report positively on many aspects of using WhatsApp some teachers felt limited to generate collaboration between students, as they would in a regular class. From the researcher’s point of view, decentralized learning made it difficult for the teachers to capture the learning processes of the students.

Sharkia and Kohen investigate how online videos in a flipped classroom setting can effectively utilize an Inquiry-Based Learning (IBL) approach. For this they analyze seven filmed lectures from an advanced mathematics course. The discussions during the TWG15 session suggested extending the investigation to include the students’ views and experiences, since IBL is fundamentally a student-centered practice.

Bakos focuses on teaching and learning with the app TouchTimes, which is a multi-touch iPad application. By using the notion of double instrumental genesis, Bakos examines how teachers experience TouchTimes as learners, alongside their subsequent transitions to adopting it as a didactical instrument. The paper highlights that the notion of instrumental distance can be a promising way to examine the impact of integrating emerging digital technology on mathematics teachers’ practices.

The TWG15 papers evidence how the constant emergence of technological innovations brings some recurring challenges for the educational research community. For example, new technologies might require new pedagogies, for which existing theories may neither be sufficient, nor sensitive enough, to detect new phenomena. Conversely, there is the challenge to relate research on emerging technologies to research on established technologies and the "old" theories (Jankvist & Misfeldt, 2021). The TWG discussed how the research community can respond to the accelerated emergence of "new" technologies and in what way "new" technologies demand and/or inspire "new" pedagogies. It was suggested that looking back at the characteristics of technologies which remain constant might help to us to better connect our research to existing knowledge and practices. For example, focusing on dynamic mathematical representations could be a common theme across different technologies. Finally, it was highlighted that use of (and research on) digital technologies is often inherently linked to a particular mathematical concept or topic. Hence, for researchers and teachers alike, the challenge to grapple with both the mathematics and the tool is very real, as clearly described in the paper authored by Bakos.

Sub-theme 5: Pedagogical approaches and mathematical content (Chair: Gülay Bozkurt)

As indicated in the previous theme, technology use demands different pedagogical approaches, which are also likely to be affected by the aspects of mathematics that is being taught. Hence, in this theme, we focused on elaborating the foci of *pedagogical approaches* and *mathematical content* in our studies. The studies discussed in the TWG15 indicated a great variety in both their pedagogical framing and the particular aspect(s) of the mathematics curriculum such as: developing problem solving skills in algebra (Jacinto & Carreira); integrating computer science in mathematics education (Haspekian & Fluckiger); encouraging modelling activity in proportions and areas (Guerrero & Camacho); and improving distance education (Barlovits, Kolokytha, Ludwig & Fesakis). Within the TWG15 discussions on this theme, we particularly focused on perspectives pointed out in the three papers by: Abu Raya and Olsher; Bretscher; and Simsek, Bretscher, Clark-Wilson and Hoyles.

The paper of Abu Raya and Olsher explores the potential of a technological environment (STEP) on formative assessment as a process of teachers' pedagogical approach. They particularly examine the effect of accessible learning analytics on teachers' formative assessment practices, by providing them with means to respond to student submissions. The FaSMEd framework is adopted as an analytical lens through which to focus on teachers' interactions with the STEP environment's learning analytics. The researchers explore the impact of this on both class discussion, and the content for teaching functions. Findings indicate that teachers use all the technology's functionalities, enabling them to advance learners' understanding through a class discussion that deployed the five key strategies of formative assessment interactions.

Bretscher examines one teacher's knowledge about a particular aspect of geometry, circle theorems, focusing on his transition between dynamic and static technologies through which he would support students' related conceptual understanding. The researcher uses the TPACK framework to explore such knowledge with a particular focus on comparisons between angle definition and measurement within GeoGebra, and a paper-and-pencil environment. The author's analysis of a clinical interview with the teacher, reveals an example of this teacher's dilemma on defining angles in GeoGebra indicating his technological content knowledge

(TCK). Bretscher concludes “mathematical knowledge for teaching using technology is always situated, since the technological context in which it is being applied is central to its meaning”.

Simsek, Bretscher, Clark-Wilson and Hoyles apply Ruthven’s (2009) SFCP framework into teachers’ domain specific practices. They select the construct of curriculum script in their aim to characterize teachers' knowledge regarding key aspects of their practice. Focusing on three different teachers’ classroom practices and interview accounts in the context of English secondary schools, this study highlights differences in both the quantity of teachers’ anticipated or identified misconceptions about geometric similarity in their curriculum scripts, and the ways they use the dynamic mathematical technology in addressing such misconceptions.

Emerging “hot topics” relevant to the TWG15 community

During the final session, the TWG15 divided into self-identified breakout groups of early, mid and late career researchers/practitioners for the purpose of reflecting on the implications of the TWG15 session inputs (and associated discussions) for our field. Each group generated and discussed a long list of personal ‘hot topics’, prior to agreeing those that are summarized below.

Early career researchers want to get to know and deepen knowledge on theories that were introduced within the different papers’ frameworks and the group discussions, e.g., on TPACK and (Double) Instrumental Genesis as major theories; or specific theories like FASMEd as possible components of theoretical frameworks. Also, this group is keen to learn more about how these frameworks are operationalized within the respective research methodologies. The group also want to explore more the social, political and economic aspects of the use of technology within mathematics teaching, from both teachers’ and students’ perspectives (i.e. the availability of technology, the learning opportunities offered, etc.).

Mid-career researchers identified a methodological hot topic, which concerns the design of research instruments that capture mathematical and digital competences for teaching and/or the associated knowledge and beliefs/orientations. This seems important both for conceptualizing what the competences are, and for supporting teachers’ professional learning. Associated with this, how to develop methods to help them understand whether such knowledge is associated with improved pupil outcomes? On a theoretical level, the selection of theoretical framework(s), how frameworks are put to use (operationalized) within studies and, in the case when multiple frameworks are used, how they are synthesized (e.g., examining the resulting data from different perspectives) is also an ongoing hot topic. Finally, the group acknowledged the need for research designs that include strong connection, communication and collaboration with pre-service and in-service teachers to enable both a deeper understanding of teacher characteristics and practices, and to be able to develop more impactful teacher education and development programmes/initiatives.

Late career researchers identified the hot topic of how best to identify and connect theories, frameworks and methodologies. The group poses three questions, “What are the considerations that guide us as researchers in choosing a theoretical framework?”; “Are the existing theoretical frameworks suitable for the purpose of analyzing the research data?”; and “How to merge different theoretical frameworks for the analysis and characterization of findings, including mergers between theories on the integration of technologies in teaching and research and theories from other fields?”. For example, Segal and Biton’s study, which explores teachers' perceptions of the contribution of teaching and learning in the WhatsApp environment, implies that theoretical frameworks from the field of social sciences might be profitable. A second hot topic concerns the need for methodologies that take account of the evolution of teachers’ practices over time, a perspective that demands a mixed (quantitative/qualitative) study design. Finally, the group was most concerned about the need for studies that conclude findings that can be applied directly to teachers practices and/or support the design of teacher education/preparation programmes.

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Teachers’ formative assessment practices when interacting with learning analytics

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Conducting a student-centered discussion in a mathematics classroom is not a trivial task. Teachers must follow their students’ work, then use the relevant information to conduct a meaningful discussion. This challenge is greater when digital environments are involved, or when students work remotely and submit their work only online. A possible solution to this challenge may come in the form of accessible learning analytics (LA) that can assist teachers to gain insight about their students’ work. In the presented case study, which is part of an extended research project we focus on the interaction of teachers with LA and the formative assessment practices of teachers when dealing with students' answers to example-eliciting tasks in a digital environment.

Keywords: Teacher dashboard, topic-specific learning analytics, online formative assessment

Rationale and Background

Conducting mathematics lessons based on students’ ideas and on analysis of their answers is a good example of meaningful student-centered teaching. Using this analysis, teachers are required to conduct class discussions that promote formative assessment (FA) interactions. We focus on formative assessment, conceptualized as “all those activities that are undertaken by teachers, and/or by their students, which provide information to be used as feedback to modify the teaching and learning activities in which they are engaged” (Black &Wiliam, 1998, p.7-8). In technology-enhanced learning settings, teachers are required to assess their students’ work in multi-participant classes and direct their teaching accordingly, often in real time. Studies show that teachers who integrate digital systems in their classes choose to do so based on their usual habits and views on teaching mathematics in general (Drijvers et al., 2010). At the same time, the technological learning environments can provide an immediate picture of students' work for teachers. Each environment offers different learning analytics (LA) in the form of a single display that aggregates various indicators about learners, learning processes, and learning contexts into one or multiple visualizations (Schwendimann et al., 2016). In the meantime, it remains a challenge to provide a meaningful LA to teachers in real time. Numerous technological learning environments have been developed to provide teachers with an immediate snapshot of students' work for example, DESMOS and TI-Nspire (Clark -Wilson, 2010). Another example is STEP, a formative assessment platform that uses example-eliciting tasks (EETs), with more than one correct answer, to support conceptual learning. Students are asked to construct an example that meets or contradicts certain conditions, after which the environment analyzes the answers according to mathematical characteristics (Olsher, Yerushalmy, & Chazan, 2016). STEP provides teachers with accessible statistical data and helps them shift from focusing mostly on errors to making decisions based on the analyzed data (Olsher & Abu Raya, 2019).

In this study, we explore the effect of accessible LA on teachers' formative assessment practices, by providing them with means to respond to student submissions based on the LA. We explored the potential of a technological environment on teachers' formative assessment using the FaSMEd framework evolved to describe the use of technology in formative assessment. The three-dimensional FaSMEd framework extends Black and Wiliam’s (2009) model, which includes five FA processes: (a) clarifying and sharing learning intentions and criteria for success; (b) engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding; (c) providing feedback that moves learners forward; (d) activating students as instructional resources for one another; and (e) activating students as the owners of their own learning. The FaSMEd model takes into account three main dimensions: the five key FA strategies, three main agents (teacher, student, peers/group), and functionalities by which technology can support the three agents in developing the FA strategies. The three categories of the functionalities of technology are: sending and displaying (SD) when technology is used to support communication among the agents of FA processes, processing and analyzing (PA) includes all the functionalities that support the processing and the analysis of the data collected during the lessons, and providing an interactive environment (PIE) are functionalities of technology that enable to create a shared interactive learning environment where mathematical/scientific contents could be explored (Aldon, Cusi, Morselli, Panero, & Sabena, 2017). Aldon and Panero (2021) use the FaSMEd framework to demonstrate that different FA practices of the different participants (teacher, students) are not necessarily similar during different parts of a learning activity, and for different ages levels. In this study we hypothesize that teachers' practice is affected by the type and design of data that LA offers teachers, which means that the functionalities of each report affects teachers` formative assessment practices.

Methodology

This study examines teachers’ interactions with LA as well as its effect on class discussion and on changes in the contents of teaching the topic of functions using STEP. The lessons were conducted in Israeli middle schools (grades 8 and 9) during the 2020-21 school year. The research question is: What formative assessment processes do teachers use when they interact with LA when evaluating student answers to EETs in a digital environment?

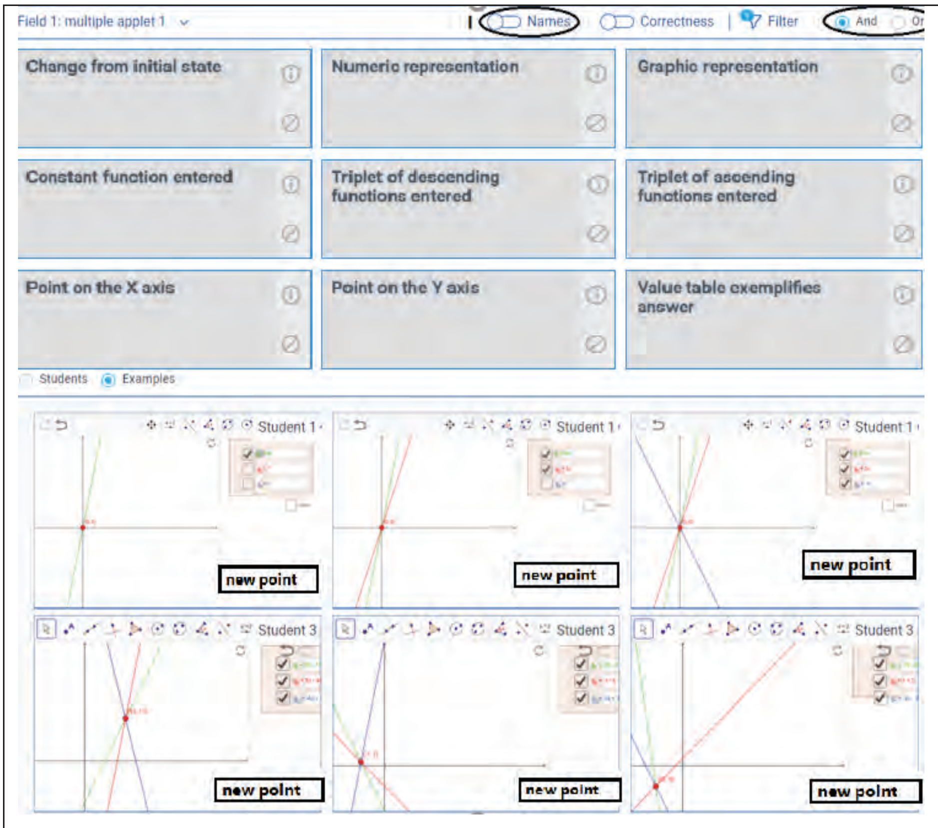
Sample

Three 8th and 9th grade teachers, experienced in teaching using the technological platform, participated in the study. The teachers had attended a professional development program and used the platform in their classes, which made them appropriate candidates for the study.

Research tools

The research tools included STEP reports and video records of the teachers using the reports. Teachers used STEP in their classrooms as part of their routine teaching sequence and conducted a discussion after the students completed the activity based on the STEP reports. The lessons were video-recorded, and the teachers were interviewed to clarify various decisions and actions that were noted during the lesson. The STEP teacher dashboard produces six types of reports: tables, grids, histograms, Venn diagrams, perceptual landscapes, and bubble reports (Abu Raya & Olsher, 2021). Each interactive report was designed to address specific pedagogical needs. Mathematics teachers were introduced to the design principles of STEP and its usage ideas in professional development programs. In this report we focus on four of the reports: table, grid, histogram, and Venn diagram, and present separately the goals and mechanisms of each report.

The grid report presents a snapshot of student submissions in a collage (Figure 1) that resembles those of other platforms (e.g., TI-Nspire). This report makes available to the teacher the work of the students in the classroom, allowing the teacher to filter student work based on predefined characteristics. The filtering mechanism is similar to that of e-commerce websites. For example, users looking for a small black purse can check each one of the characteristics for the relevant meta-data category (size, color). The grid report allows teachers to choose one or several characteristics as either present or absent and to create "and" and "or" relations between them (Figure 1), making possible in-depth analysis of submissions based on their characterization. The interactive report also enables teachers to choose a picture by clicking on it and to display the interactive diagram that the student submitted. Filtering helps the teachers detect various phenomena in the students' work, as opposed to merely scanning the snapshots to identify characteristics relevant for the ensuing classroom discussion.



1. presents a snapshot of student submissions in a collage

2. allows teachers to filter student work based on predefined characteristics.

3. allows teachers to choose one or several characteristics as either present or absent.

4. enables teachers to choose a picture by clicking on it and to display the interactive diagram that the student submitted.

Figure 1: Grid report with filtering activated, name display button (I), and/or buttons, (II).

The table report displays a row for each student, and a column for each characteristic, indicating which characteristics are present in each student submission. Presentation is similar to that of conventional spreadsheets. The goal of the report is to provide information about submissions of student in class, especially about characteristics that are prominent in each submission.

The histogram report presents a distribution of the characteristics across submissions. Each characteristic is represented as a bar whose height corresponds to the number of submissions having that characteristic (Figure 2a). The report provides a visual representation of the frequency of each characteristic of a task in students' submissions. Clicking a column brings up a filtered grid report below the histogram, allowing the teacher to further analyze student work directly.

The Venn diagram shows the relations between characteristics in the students' submissions (Figure 2b). It displays up to three characteristics simultaneously, each color representing a different characteristic. The Venn diagrams show the distribution of more complex phenomena in student submissions, which could not be captured by a single characteristic but only in the relations between several. The diagram provides an indication whether certain characteristics coexist. It also displays numeric values that enable determining the phenomena that are more prominent in students' work. Similarly to the histogram report, clicking on a part of the diagram displays a filtered grid report according to the selected region, for example, the intersection of three characteristics.

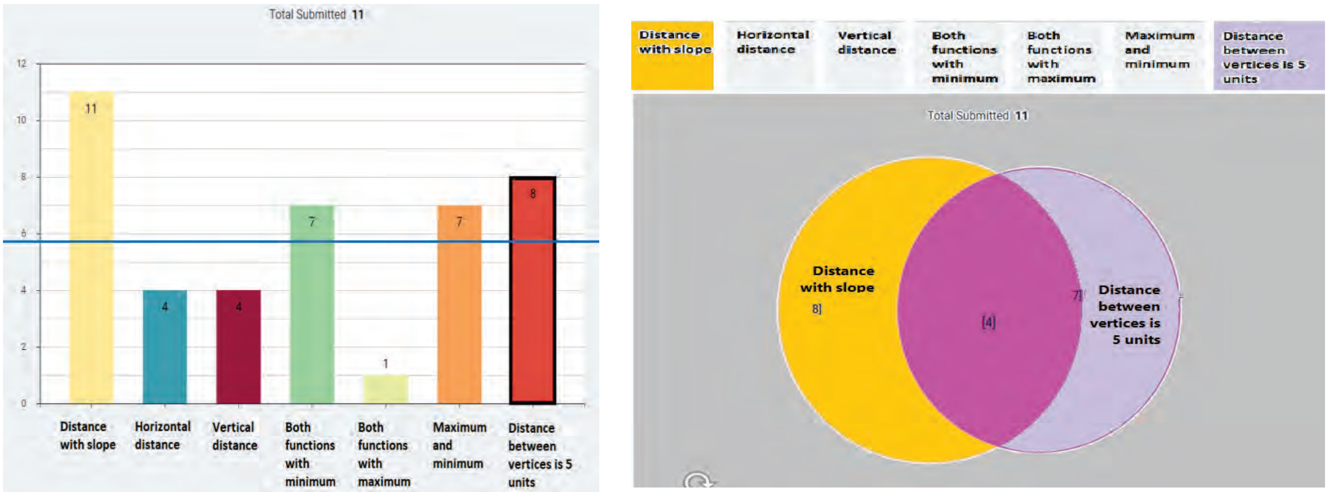


Figure 2: Histogram report (a) and Venn diagram (b)

Data sources and analysis

Data sources include student submissions for the tasks, interviews with each of the teachers, conducted after the class discussion that followed the students' submissions, and classroom observations. Each teacher assigned five tasks in five separate lessons. We used the FaSMEd analytical framework to describe the findings.

Findings

The findings show how teachers conducted formative assessment in their teaching using the data they obtained from the different reports. Below we describe the teachers' use of each report, focusing on the different report functionalities, indicating the role of these affordances in describing the FA properties that teachers focused on in the ensuing discussions.

Grid report

The use case for this report concerns a task asking students to find a linear function equation from two given points, which was assigned in the course of an 8th grade mathematics lesson. The students were asked to "Construct a linear function whose graph passes through the two given points. If you believe this cannot be done, explain why." The interactive diagram that was provided included multiple linked representations of the function, and the students could choose the different points by pressing a "New points" button that generated semi-randomized pairs of points (Bagdadi, 2019).

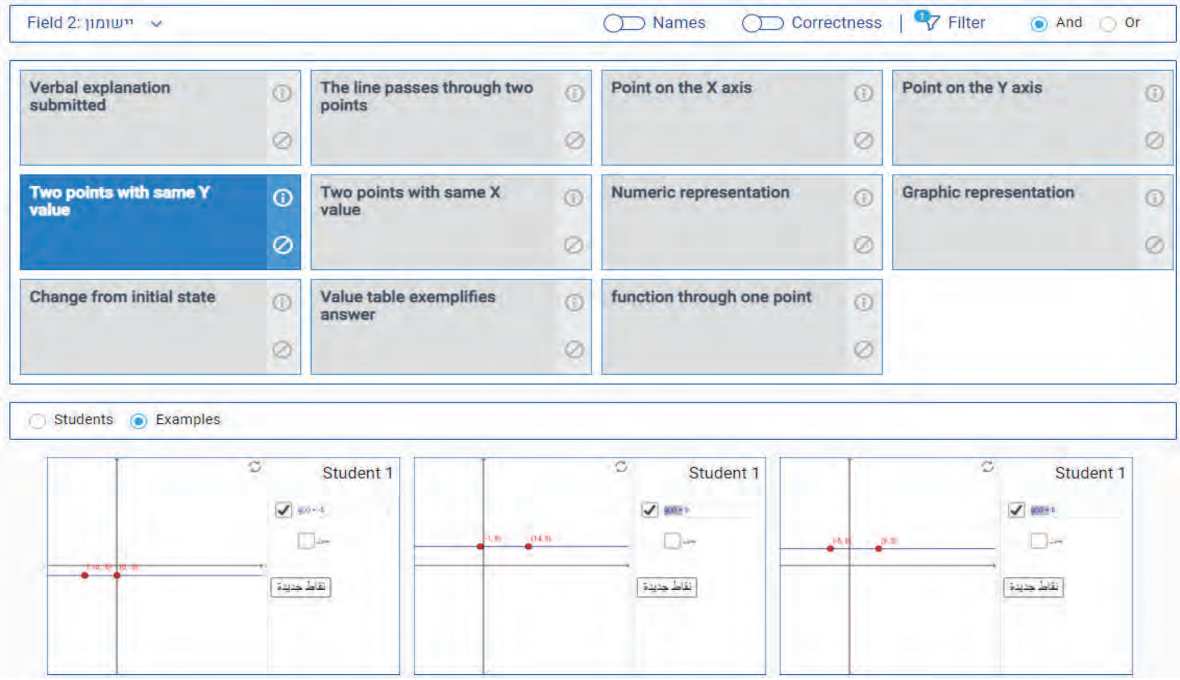


Figure 3: Filtered grid report presented during classroom discussion

The teacher chose to address the characteristic "Two points with same Y value," based on the analysis (PA) provided by the platform. The teacher *clarified and shared learning intentions and criteria for success.* She indicated that "this characteristic describes a constant function and helps students solve this type of a task efficiently, saving them time" (Asala, questionnaire). Analysis of the classwork reveals that the teacher presented examples of student submissions with this characteristic (Figure 3). In the process, the teacher used an interactive component of the report (PIE) to choose a filter based on the automatic analysis and displayed students' work (SD). Next, the teacher decided to give students tasks for discussion to *elicit evidence of their understanding*, *organizing an effective classroom discussion* based on the data she collected from the report. Lesson observations reveal that she asked students questions like "What did these students do?" "What type of function do you get in this situation?" "Do you think it [constant function] is easier [than a function with a slope]?"

"How would you find the equation?" The teacher also addressed student submissions in which the two points coincide, which enabled students to submit non-constant functions. Generally, it was apparent that the teacher *activated students as instructional resources for one another and as the owners of their own learning.* The teacher indicated that this choice of points helped students understand that an equation can be written without calculating the slope and plugging in the X and Y values of the points. The teacher used the different functionalities of the report to *provide feedback that takes learners forward*. In general, the grid report offered the three functionalities of technology that enabled teachers to relate to all the five FA strategies (Figure 4a).

Table report

The use case for this report concerns a task asking students to find several linear functions that intersect a given point and also intersect the positive part of the Y axis. The task was part of an 8th grade mathematics lesson. The students were asked to "choose a point and construct several expressions of linear functions whose graph passes through the given point and intersects the positive part of the y-axis." Based on the report (PA), the teacher found that none of the students paid any attention to the table of values provided in this task ("numeric representation" characteristic in the table). She later said: "I chose to pay attention to this characteristic because none of the students used the table. In addition, it was important for me to emphasize that the table helps illustrate characteristics of the function" (Asala, questionnaire). Observation of the lesson revealed that the teacher used the report to *clarify and share student learning intentions* and to *orchestrate the discussion*. The teacher continued to *provide feedback that takes learners forward* by showing the students how the table of values could assist them in verifying their solutions by displaying an additional report, the grid report (SD). This report is limited to two functions of the technology, *sending and displaying* and *processing and analyzing* (Figure 4b). Although teachers can use the analysis provided by this report to activate the students, our current findings do not indicate that they did so.

Histogram report and Venn diagram

The use case for these reports concerns a task involving quadratic functions, specifically, identifying the extremum and calculating the distance between two extrema of two different quadratic functions. The task was formulated as follows: "Functions f(x), g(x) are from the family y=a(x-p)^2+k. Claim: there is only one situation for the functions f(x) and g(x) in which the distance between the extremum points of each function is 5 units. If you think this claim is true, provide the algebraic expression of each function. If not, use the interactive diagram to create five examples of different functions." The teacher used the histogram to *clarify learning intentions and criteria for success*: "I presented the grid for each characteristic separately and we discussed the effect of the parameters for each characteristic" (Ranya, interview). She noticed that most of the students gave correct answers, and that most of the students provided a distance that was not a vertical one, something she did not expect (PA). Following this insight, she presented (SD) the filtered grid report for the characteristic "vertical distance" and *conducted a discussion around it*. Next, she used the filtered grid report of a distance that has a slope (i.e., not vertical/horizontal) to *share the students' work* (PIE) (Figure 2a).

The Venn diagram was used for the same task. Indeed, it was the histogram report that encouraged the teacher to use another representation of the data from her classroom. In the histogram report, the teacher noticed that most of the student submissions were correct, and that most of the submissions had distance with a slope (PA): "The histogram showed that most of the students gave examples with a slope, which was strange for me since they didn’t study the distance formula yet. So I turned to the Venn diagram to see the intersection between the characteristics” (Ranya, interview). On the Venn diagram (PA), the teacher checked the interrelations between the two characteristics (PIE) (Figure 2b) and noticed that only 4 out of 11 submissions were both correct and with a slope. She said: “I chose to look into this characteristic. Even though there were many submissions with a slope, they were not correct, since they calculated the distance only according to the X values” (Ranya, interview). One student used the Pythagorean theorem and calculated the distance based on the length of the hypotenuse. Following this insight, the teacher asked this student to reveal her solution (SD) and used it to explain to the other students why it was not correct to calculate the distance the way they did, *activating students as instructional resources for one another and as the owners of their own learning*. Based on the analysis and on the interactive reports, the teacher gathered valuable information about the students’ learning that assisted her *to manage an interesting discussion* about a rare correct solution of one of the students. In the course of this discussion, she provided feedback to the whole class about the common mistake. In general, these two reports offer the three functionalities: sending and displaying (SD), processing and analyzing (PA), and providing an interactive environment (PIE), of the technology to teachers. By using these functionalities, the teachers can apply the five strategies of the FA (Figure 4c). By clicking on any characteristics in these reports, the platform provides additional information about the student's works, which helps the teacher *give effective feedback and serves as a powerful resource for activating the students*.

Summary and discussion

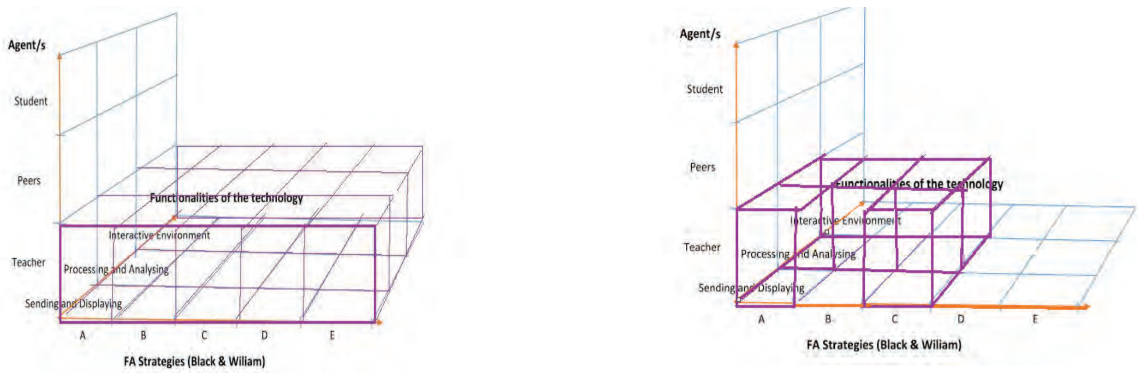


Figure 4: Grid report and Venn diagram interactions (a) and table report interactions (b)

In this report, we used the FasMed framework to describe the FA practices of teachers interacting with LA based on students' answers to EETs in a digital environment. LA were accessible by means of different interactive reports. The use cases presented show that in using the reports, teachers resorted to different strategies, ranging from locating work with specific characteristics to discovering meaningful complicated logical conjunctions of characteristics. In the grid report, students’ work is accessible in a single report, and teachers could filter student work according to pre-designed characteristics. Teachers used all the functionalities of the technology, which enabled them to advance learners in the course of a class discussion that deployed the five key strategies of FA interactions (Figure 4a). By contrast, the table report provided only the functionalities of sending and displaying, and of processing and analyzing. Teachers used these functions to clarify, guide the discussion, and provide feedback. (Figure 4b). The histogram and Venn reports offer correlations between the different characteristics, and grids of students answers for the correlation they chose. In these reports, findings show that teachers used all three technological functionalities for effective FA that also included all five strategies (Figure 4a). The examples show how the design principles of each report served teachers in different ways and demonstrate the role of interactive environment in the teacher's FA. At the same time, other facets of the interaction and of FA characteristics remain to be further investigated. By enabling the teacher to display the students work (SD), then to click on one of the columns in the histogram report (PIE) then get the filtered grid (PA), STEP demonstrates a wide range of opportunities to use technology in teacher's formative assessment practices.

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Primary teachers implementing *TouchTimes*

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After integrating TouchTimes (hereafter, TT) into their teaching practice, four primary school teachers (K–5) in British Columbia, Canada share their experiences, both as learners of this relatively new technology, and as teachers utilising TT as a tool to guide and support student learning. Using the notion of double instrumental genesis, I examine how these teachers experienced this digital technology from 2018–2021, both as learners themselves, as well as their subsequent transitions to thinking about and adopting it as a didactical instrument for teaching mathematics. The aim of this research was to identify and highlight specific ways in which these teachers adopted technology-enhanced mathematical learning, managed obstacles they experienced during personal instrumental genesis and how instrumental distance affected their professional instrumental genesis.

Keywords: Touchscreen technology, TouchTimes, instrumental distance, double instrumental genesis.

There are a wide variety of resource options available for the teaching and learning of primary school mathematics. Physical (hands-on) objects have been used to support the development of mathematical understanding for many years and, with the emergence of touchscreen devices which are better suited to the as-yet developing fine motor skills of younger students, digital technology is becoming a more viable resource to primary school classrooms (e.g. Sinclair & Baccaglini-Frank, 2015). Integrating technology into their teaching repertoire and becoming adept at leveraging the opportunities that technology can offer for teaching and learning remains challenging for many teachers (Trigueros et al., 2014), and this is also the case when implementing TT in primary school classrooms (Sinclair et al., 2020). The notions of *instrumental distance* and *double instrumental genesis* (Haspekian, 2014) are useful for examining the impact of integrating digital technology on mathematics teachers' practice.

TouchTimes (Jackiw & Sinclair, 2019), a multi-touch iPad application, enables primary school children to experience relational and functional aspects of multiplication through engagement with two different microworlds, *Grasplify* and *Zaplify*. While using their fingertips to create and transform pictorial representations of multiplicative situations on an iPad screen, the children receive immediate visual, tactile and symbolic feedback from TT in response to their actions.[1]

The design of TT focuses on the two quantifying dimensions that comprise the multiplicative relationship; the first being the unit of measurement (the multiplicand), and the second involving the quantity of that unit (the multiplier). Multiplication from this perspective involves "a count of a [larger] unit for which a relationship to another, smaller unit, is already established" (Davydov, 1992, p. 12). The design of Grasplify is influenced by Davidov's (1992) double change-in-unit approach to multiplication that is grounded in measurement. The first unitising occurs when the multiplicand is established (in Grasplify, this is the creation of the pips) and the second occurs when determining the number of units to be used (the number of pods). Rather than repeated addition, both Grasplify and

1 For further information on *TouchTimes*, view this short video demonstration. https://youtu.be/JkznPdu8RkA

Zaplify focus on doubling, tripling, etc., a design choice influenced by the relational and functional aspects of Vergnaud's (1983) work on the conceptual field of multiplication.

Theoretical framing

The instrumental approach extends Vérillon and Rabardel's (1995) theory of instrumentation of human tool-use into the domain of mathematics education, utilising its focus on instrumental genesis for analysis of technology-mediated teaching/learning (Artigue, 2002). There is a two-way process during instrumental genesis, in which a physical object or tool, defined as an artefact, influences the user (*instrumentation*) whilst the user adjusts to the tool (*instrumentalisation*). It is during this process that the artefact develops into a functional instrument for the user. This progression is even more complex in the case of teachers, who must engage in what Haspekian (2014) terms *double instrumental genesis* when adopting unfamiliar technology for teaching. Initially, a personal instrumental genesis occurs when teachers first engage with an artefact as learners themselves. As the artefact becomes an instrument for mathematics, teachers then engage in a professional instrumental genesis as they appropriate and construct the technology into a didactical instrument for use with students. Haspekian declares, "The teacher's professional genesis with the tool is much more complicated as it includes the pupils' instrumental genesis" (p. 254).

In examining the sustained integration of technology into mathematics teaching, Haspekian also refers to the notion of *instrumental distance* between the digital technology and the mathematics. This relates to the gap between 'current school habits' and the didactical experiences offered by the technology, and can include the computer transposition (how the computer mediates the mathematical concepts in question, as per Balacheff, 1996), institutional, didactical or epistemological changes that occur when a tool is introduced into mathematics teaching. The gap must be large enough to make the benefits of adopting the technology apparent, but not so large as to discourage teachers from its integration.

Given that TT was developed specifically for mathematics teaching, and that none of the teachers had any prior experience using it, double instrumental genesis provides a way to examine how these teachers experienced this digital technology as learners themselves, as well as their subsequent transition to thinking about it as a didactical tool for teaching mathematics and the effects of instrumental distance during the process of integrating TT. The research questions specifically relate to these ideas. (1) During their personal instrumental genesis of TT, were there specific problems or obstacles related to either the technology or the mathematics it represents, encountered by the primary school teachers interviewed? (2) How did the instrumental distance between prior ways of teaching multiplication and using TT affect the evolution of these teachers' professional instrumental genesis of TT? In order to respond to these questions, I draw on data gathered during interviews with these four teachers about their experiences implementing TT as a teaching tool in primary classrooms.

Method

Three of the teachers are primary school generalists, and one is a former secondary mathematics teacher, who is now a mentor teacher that works with K–12 teachers in her school district to improve mathematics teaching. Their teaching experience ranged from 9–24 years, each had a master's degree, and all were working in grade 3 or 3–4 classrooms in British Columbia, Canada when utilising TT. Each teacher had volunteered to provide feedback for a larger, multi-phase project, in which the author is part of, involving the implementation of TT in primary school classrooms, and to contribute to the development of tasks and assessments to be used with it.

The teachers were initially introduced to TT during the first meeting of this larger project, which was videorecorded. With the exception of Leah's first reaction to TT (which is used as a comparison to her later thoughts about TT that emerged during the interviews), the data in this paper has been taken from 60–80–minute semi-structured interviews with each of the teachers individually (one for each teacher), which were conducted via Zoom in June through August of 2021. Having observed that teacher responses in the larger research group meetings would often build from the ideas shared with each other, it was hoped that this may also occur by interviewing pairs of teachers together. Therefore, two additional interviews were conducted where the four teachers were interviewed in pairs (one interview for each pair). In each interview, the teachers were asked about their initial experiences with TT as learners, their thoughts about how TT presents multiplication, how they used it as an instructional tool and their observations related either to TT or to its mathematical representations, as well as what they noticed about student learning during the implementation of this digital technology. Each interview was transcribed in its entirety and the resulting transcripts were then analysed for common themes that emerged based on the experiences shared by the four teachers.

Data analysis and results

With my research questions in mind, the data was analysed with two specific aims. The first was to identify instances of obstacles or challenges related to the personal instrumental genesis of TT shared by each teacher, while the second was to look for specific examples of instrumental distance that influenced the evolution of each teacher's professional instrumental genesis. I first categorised the experiences of instrumental genesis of TT as either personal or professional, while noting any obstacles or challenges shared prior to examining the examples of the latter more closely to determine if instrumental distance was apparent and, if so, whether the resulting gap was related to computer transposition, institutional, didactical or epistemological changes. I wanted to understand better the challenges these teachers experienced with TT as learners themselves, as well as what factors influenced the integration and use of TT into each teacher's mathematical teaching practice. I will now discuss five specific challenges in the teachers' double instrumental genesis.

(a) Leah: "Grasplify is backwards". A member of the research group, Leah's initial encounter with TT occurred during our first teacher–researcher team meeting. While using the app, she noticed the multiplicand × multiplier = product ordering displayed by TT and stated to the group that this was "backwards". Leah shared how she would refer to the textbook to guide her teaching. The textbooks that she referred to all showed 4 × 3 as four groups-of three, and were always in the multiplier × multiplicand = product ordering. Grasplify however, displays the equation in the opposite order, where 4 × 3 is four, three times (see Figure 1).

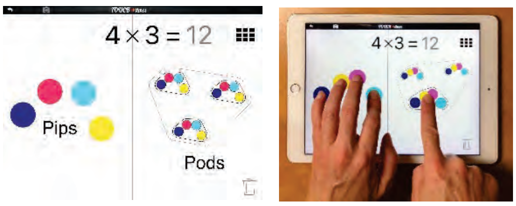

Figure 1: (a) Pips and pods; (b) Grasplify display of 4 × 3 = 12

Leah’s reaction to this ordering occurred during her first exposure to Grasplify, and was immediate. Though she was still learning how TT functions, and in the very first stage of personal instrumental genesis, this obstacle intertwined with her professional instrumental genesis and how she could use Grasplify with her students. The instrumental distance between the computer transposition and the epistemological aspect of Leah’s personal representation of multiplication and how she taught it was significant from her first use of the Grasplify world and continued to be problematic for her.

When I interviewed her three years later, Leah mentioned again how she “couldn’t get past the groups-of thing and it was so huge for me”, but explained that, “if I really believe in […] how this [app] works and what multiplicative thinking means, it doesn’t matter what happens next. It’s what happens in their [her third grade students’] thinking.” She went on to say, “I was so stuck on this groups-of thing and then I started thinking about, well, what does multiplication mean? So, it really changed my thinking about what it [multiplication] means.” It was through discussion in the TT teacher–researcher group, and during Leah’s use of Grasplify in her classroom, that the instrumental distance began to narrow. She explained that the commutative property makes the order of the factors irrelevant, so the product will be the same. Observing her own students using TT was what was most convincing for Leah, who noted that students, “didn’t know any different, and so they were understanding it [the ordering] the way it was, and it didn’t matter”. The most significant growth in Leah’s personal instrumental genesis occurred as a result of her professional instrumental genesis, rather than preceding it, as often occurs when implementing new digital technology in the classroom.

Another challenge that Leah described involved the impermanence of the pips and pods on the iPad screen. When first using Grasplify with her students, she was projecting it onto the wall for all to see. At that point, she only had access to one iPad with TT on it, and was engaging in teacher-led prediction tasks. For example, she would create pips and pods on the screen and asked the students to predict how she would double the product by only changing the pips (see Figure 2). Leah shared her frustration when, each time she removed her pip-making fingers, it would reset the screen. This left her ‘stuck’ at the front of the classroom and unable to view the predictions students were making on their mini-whiteboards. However, once multiple iPads were available for student use, she noted how the impermanent nature of the pips and pods forced her students to think more carefully about what they were doing, adding an element of concentration and a bit of planning that resulted in a more “metacognitive aspect to it. It’s not just playing […], it’s thoughtful play”. This aspect of the technology and how it affected Leah’s ability to use it and monitor student progress, created an obstacle for her professional instrumental genesis. It was not until she observed how this lack of permanence affected student engagement with the tasks, that the instrumental distance grew smaller.

Figure 2: Doubling task progression

(b) Rachel: It was not intuitive. When first using TT, Rachel did not find the app to be intuitive. She described it as hard to use and admitted to having difficulty thinking of ways to use it with her students. As was observed with Leah, Rachel’s personal and professional instrumental geneses were closely intertwined and the instrumental distance was initially large, making TT difficult for her to adopt and implement without assistance. It was through the shared experiences of other teachers within the teacher–researcher group, the teacher discussion about what they had difficulties with or found valuable when using TT with their students, as well as the provision of task ideas to be used with students, that helped Rachel overcome these initial obstacles and begin using Grasplify in her classroom. She explained that she needed teacher tasks that were already developed to support her initial implementation of TT and also suggested the creation of short videos to help other teachers better understand the app and the tasks that can be used with it. Rachel described the benefits of being part of a cohort of teachers who were talking more deeply about multiplicative thinking, their experiences teaching with TT and its accompanying tasks and how this influenced her thinking about the properties of multiplication and being more purposeful about this in her own teaching. The instrumental distance narrowed over time, and although TT was not initially intuitive to use as a teaching tool, its use began to change the way she thought about and taught multiplication.

(c) Amy & Rachel: Using Grasplify in the "opposite way". As these two teachers continued to use Grasplify with their students over the course of two school years, the ways they described using it were becoming more personalised, reflecting on-going professional instrumental genesis. When discussing Grasplify as a teaching tool, the differences between traditional methods of teaching multiplication and those afforded by TT were described as beneficial by both teachers. When interviewed together, the pair would often elaborate on and extend each other's ideas. They explicitly stated that TT is not the only model that they use with students when teaching multiplication, that it is simply one model. Going further, both described teaching multiplication using other models in comparison with how they used TT. Amy commonly begins by writing a multiplication equation on the board for all to see. After writing 3×2, for example, she would then proceed to use manipulatives or drawings to create three groups-of two, or the applicable array or area model or a number line drawing for skip counting. She was starting from the symbolic mathematics and then working to create either physical or visual representations that explained what the symbolic mathematics meant. Whereas when using Grasplify, Rachel described using it in the "opposite way". When asking her students to skip count to twenty, she was, "not necessarily looking to show a model for that equation, […] the equation is there, but you're trying to get at concepts that might be harder to get by just drawing something"[2].

Amy pointed out that Grasplify provided a visual representation of multiplication for students that is difficult to demonstrate using physical manipulatives or drawings and that, because Grasplify is constantly changing and moving, it encourages more open-ended thinking and discussion. For her, "the emphasis is on the exploration because the answer is already provided by TT and therefore that isn't where the focus is". She used other models when she wanted the focus to be on the answer. Rachel agreed and reiterated that she wanted her students to notice what happens when they add or remove fingers, how that relates to what is happening within the pods and then how this influences the numbers within the equation and the product itself. She was trying to enhance her students understanding of multiplication in a different way, through the growth of pips spreading across the pods. Rachel used these types of tactile experiences with immediate visual feedback, to "enhance

[2] For a more detailed description of Leah and Rachel's classroom implementation of Grasplify, see Bakos (in press).

students’ comprehension of multiplication in ways that are different, you know harder to get at through pencil and paper or even manipulatives”.

Teaching with TT begins with students exploring the app, noticing the effects of their fingers on the pips and pods, then later, using intentional questioning, she directs student attention towards the mathematical symbols that are also visible on the screen. Student experience with the multiplicative models takes place first and then she builds her teaching on connecting such experience with symbolic mathematics. The more operational approach of starting with an equation and explaining it through a multiplicative model and using Grasplify to provide dynamic and relational experiences with multiplication was a complete inversion of their approach to teaching multiplication and yet this was what Amy and Leah both welcomed about the digital technology. It was not seen to be detrimental: rather it was considered advantageous for student learning.

(d) Kate: Transitioning across multiplicative models. Of the four teachers I interviewed, Kate was the only one who had utilised both of the TT microworlds with her students. The use of the two different multiplicative models represented by Grasplify and Zaplify allows for examining what I term *intra-instrumental distance*, in that there are two related tools and a possible distance between them.

Kate was very purposeful in taking advantage of this *intra-instrumental distance*, wanting students to learn what multiplication is and for them to understand the different representations and how to go between them, while also recognising “that multiplication is the common theme” in the different multiplicative models embodied through Grasplify and Zaplify. She engaged students in activities that explicitly directed attention towards comparing and contrasting both worlds. For example, after sharing screenshots of the same multiplication sentence represented by Grasplify and Zaplify (see Figure 3), students were asked to describe how these were the same and how they were different. Kate would sometimes provide screenshots of a multiplication equation in one microworld and ask students to draw what that equation would look like in the other. Kate’s goals were for students, “to make connections between the two different worlds, make connections between the symbols, the equations, the representations, because at the end of the day, I wanted them to know how to multiply and what multiplication was, so it kept coming back to that one idea”. Kate’s professional instrumental genesis involved her prioritisation of the symbolic mathematics and the instrumental distance between TT and her epistemological beliefs of what was important for students to learn mathematically, she very intentionally kept narrow. The learning activities that she designed for students prioritised the symbolic and representational models of multiplication and how TT could be used as a vehicle to drive students towards those goals.

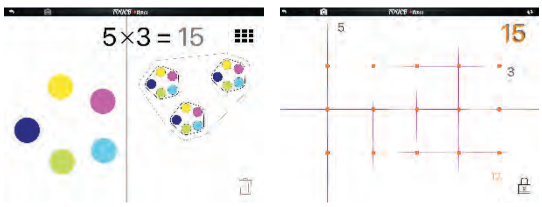

Figure 3: (a) Grasplify multiplicative model; (b) Zaplify multiplicative model

After engaging the children with various TT tasks for a few weeks, Kate described how she projected some examples of different representations of multiplication onto the board and found it “really powerful” when the students could easily identify Zaplify or Grasplify in the models, even though the models projected were taken from another teacher resource. “Switching back and forth between the models connected to the different representations. So, when they see an array model, they’re connecting it to Zaplify or when they’re seeing the groups-of, they’re seeing it as pips and pods.” For Kate, this would be useful when transitioning into what she referred to as “the more formal symbolic type of math” that most teachers teach. She explained that this is beneficial to students because when they move on to another teacher, they can carry their TT experiences into other contexts and that multiplication would still make sense.

(e) Amy, Leah & Rachel: How Grasplify shaped their teaching. Both Amy and Rachel would sometimes have students create drawings to depict what they had learned after completing a task using Grasplify. Amy used these drawings as a formative assessment tool to “see what they noticed. Like did they notice the colours? What were they able to pick up on? What did they attend to?” Of importance to Rachel was determining what her students were seeing and what they understood from TT, so that she could use this information to plan what experiences she needed to provide during the next class. Her goal was not on students transferring this knowledge to an equation: rather, her goal was to know more about what her students, both individually and collectively, understood from that day’s task and to try to glimpse what it was that *they* were seeing.

As her students used Grasplify to “play with” and learn about multiplication, Leah would watch what they were doing. She explained how, even if students were not always going in the direction she had hoped for, that she was better able to understand where they were coming from and that she could redirect with a different question to get them thinking about the relevant mathematical concept that was emerging from their explorations. Leah found that, “TT allowed me to actually see how kids were thinking about multiplication”, in comparison with her traditional teaching where she would show students what to do and expected them to mimic this.

Discussion and conclusion

Throughout the process of double instrumental genesis, all four teachers became increasingly responsive to the teaching opportunities that emerged from student experiences with TT. For Leah in particular, the reaction of her students to Grasplify significantly influenced how the instrumental gap continued to narrow. The mathematics learning that her students were engaged in influenced her comfort with using digital technology with which she was initially very uncomfortable. She explicitly shared that, “one thing that affected me is the conversations I had with the kids as a teacher”.

For Rachel, she was not focused on memorising facts or writing equations, and therefore TT meets her where she is concerned, which is about providing new meanings for multiplication. The use of Grasplify allowed her to provide learning experiences with visual representations of multiplication that were dynamic and with which students could interact in a relational way. The limits of static drawings and the difficulty for children to build multiplicative situations out of physical manipulatives without error was very visible for Rachel.

The process of double instrumental genesis was not straightforward for these teachers during their integration of TT. Although there was an initial experience of learning to use the digital technology personally, for these teachers, it was difficult to differentiate between their personal and professional instrumental geneses. Their reactions to TT and the manner in which it presents multiplication were clearly related to how they would use it as teachers to promote student learning. The instrumental gap narrowed significantly as these teachers used TT with their students. When Leah's students were not having difficulty with the multiplicand × multiplier ordering, she began to re-evaluate her own thinking. When Kate's students were able to identify the multiplicative models experienced in TT to static drawings of multiplicative models, the *intra-instrumental distance* narrowed for her.

Although obstacles were encountered by these teachers during their personal instrumental genesis of TT, the instrumental distance between previously used ways of teaching multiplication and the relational experiences with multiplication offered by TT were either embraced as positive differences or narrowed as the teachers' professional instrumental genesis of TT advanced.

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Designing mobile environments for mathematics distance education: The theory-driven development of the ASYMPTOTE system

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Distance teaching and learning due to the Covid-19 pandemic was perceived as a major challenge by both, teachers and students. With new ways of instruction emerging in a matter of days, the resulting instruction formats – summarized in the term Emergency Remote Teaching (ERT) - could not fully meet the needs of online teaching and learning in Spring 2020. With a particular focus on the reported problems of mathematics distance education during the Covid-19 pandemic, this paper firstly aims at providing a theoretical framework for the development of mobile environments for distance math class. In doing so, we refer to the Community of Inquiry (CoI) model and e-pedagogy to deduce design requirements for online learning environments. Secondly, the ASYMPTOTE system is introduced. Based on the theoretical considerations and the discussed Covid-19 experiences, necessary further developments and requirements for the project are identified.

Keywords: Covid-19, design requirements, distance education, mobile learning, online systems.

In the following, a theoretical framework for mobile learning environments aiming at mathematics distance education is provided. Hereby, we refer to the Community of Inquiry, online learning and practices of online instruction as well as mobile learning. The theoretical framework is used to evaluate the present status of the ASYMPTOTE system and to identify needed further developments.

Theoretical Framework

Community of Inquiry

A well-known approach for the design of digital educational environments is the Community of Inquiry model (CoI), which is based on Dewey's (1897) assumptions made on the value of social interaction and collaboration to the learning process. The constructivist model describes learning in online communities – composed of teacher and students – by the interplay of three core elements: cognitive presence, social presence and teaching presence (Garrison et al., 2000).

Cognitive presence in online environments is considered as a holistic multi-phased process initiated by a triggering event. According to the practical inquiry model, it is followed by an individual exploration which result in a reflection process – and thus in the creation of new concepts. Afterwards, the skills and knowledge acquired are practiced in real situations (ibid.). Thus, cognitive presence is defined "as the exploration, construction, resolution and confirmation of understanding through collaboration and reflection in a community of inquiry" (Garrison, 2007, p. 65).

Social presence describes the ability to position oneself in the online environment as real person (Garrison et al., 2000) and to collaborate in the digital world by establishing personal and purposeful relationships (Garrison, 2007). The main aspects of social presence are the expression of emotions, the open communication and group cohesion in the online format (Garrison et al., 2000).

Teaching presence – mainly referred to the instructor's role – includes firstly the design of educational experiences, i.e. organization of the course as well as design of learning activities and assessment formats. Secondly, teacher presence involves a facilitation function during the lesson (ibid.). Thirdly, direct instruction is considered to be a part of the teaching presence (Garrison, 2007). All three functions of teacher presence focus on the goal "to support and enhance social and cognitive presence for the purpose of realizing educational outcomes" (Garrison et al., 2000, p. 90).

Viewing educational experiences as the result of successful interplay between all three elements, CoI assumes that knowledge construction in online settings depends on the ability of instructors and learners to create forms of perceived presence despite spatial separation (Shea & Bidjerano, 2009).

Online Learning

According to Allen and Seaman (2013), online learning can be defined as a course that is mostly or entirely delivered online, so that typically no face-to-face meetings are conducted. From an e-pedagogical perspective, three different instructor roles can be taken in online environments: Teachers can firstly act as excessively active leaders of students' learning progress or secondly be in a reactive position as facilitators which only respond to students' questions. Thirdly, and preferably, they take the position of a reasonably active mediator, i.e., instructors engage and interact without directly guiding students' learning progress (Serdyukov, 2015).

Even though online learning is from its origin an independent and self-determined process, the need of critical support for the online learner is highlighted in e-pedagogy. To address the demand of social interaction, four forms of organization can be identified: interaction can be enabled by both, text-based and video-based communication, either in synchronous or in an asynchronous setting (ibid.).

Principles for Online Instruction

The value of social interaction for online teaching is also highlighted by Sorensen and Baylen (2009) who adapted the *principles for good practice* of Chickering and Gamson (1987) to online instruction settings. The seven principles for teaching in online environments can be summarized as follows:

1 Enabling student-teacher interaction
2 Facilitating cooperation among students
3 Empowering active learning
4 Providing prompt feedback
5 Managing time on task
6 Communicating high expectations
7 Respecting diverse ways of learning

As shown by Fiock`s (2020) literature review on empirical studies about instructional strategies in online communities, the three core elements of CoI can be related to each of the seven principles. Consequently, the seven principles can be seen as a starting point for implementing instructional activities which consider the CoI model in school praxis (ibid.). In this paper, we understand the seven principles of online instruction as design principles for the creation of educational online environments in line with CoI.

Mobile Learning

Mobile learning can be defined as learning progress facilitated by the use of mobile devices, such as smartphones. Characteristics are the portability of devices as well as their possibilities for communication and interactive representations (Kearney et al., 2020). In a meta-study, Sung et al. (2015) report a positive mean effect of using mobile device on students' learning. The authors emphasize learning environments that offer broad functionalities including authoring tools for teachers to flexibly adapt the system to their own instruction (ibid.).

Mobile apps can promote students' learning progress in numerous mathematical areas, e.g., arithmetic and geometry. Further, the use of apps in math class can foster problem solving strategies or mathematical programming skills. Additionally, the embedding of mobile learning in math education tends to have a positive influence on students' motivation and enjoyment (Drigas & Pappas, 2015).

State of the Art: Mathematics Education in Covid-19 Pandemic

Due to the Covid-19 pandemic, abrupt changes of instruction were needed to deal with the crises circumstances. Within only a few days, new teaching practices were rapidly established (Hodges et al., 2020) mostly by the help of digital media and internet-based communication (Crompton et al., 2021). This temporary shift without appropriate time for designing online teaching courses is described as Emergency Remote Teaching (ERT) (Hodges et al., 2020). Hereby, several challenges concerning mathematics education are reported from a teacher's point of view:

Empirical cross-national research highlights the issue of assessment. Concerning the formative assessment, teachers reported problems in diagnosis as well as individual feedback and support (Aldon et al., 2021; Barlovits et al., 2021; Drijvers et al., in press). During distance education in Spring 2020, instructors mainly focused on procedural skills rather than conceptual understanding (Drijvers et al., in press). In addition, teachers reported difficulties in finding adequate forms of summative assessment (Aldon et al., 2021).

Furthermore, it arouses the question of how to implement inclusive distance education in which students can participate at their individual achievement level (Aldon et al., 2021), especially underachieving students (Barlovits et al., 2021). In addition, teachers reported a lowering of the course level through an increased rate of standard and reproduction math tasks (ibid., Aldon et al., 2021) and an infrequent use of modeling or reasoning tasks (Drijvers et al., in press).

Also, social factors are emphasized in research on mathematical distance education during the ERT phase. Here, the interaction between teacher and student (Aldon et al., 2021; Barlovits et al., 2021) as well as among students, e.g., in form of peer instruction (Drijvers et al., in press), is perceived as challenging. The general ability of fruitful content-related social interaction in math class seems to be dependent on the ratio of synchronous distance education (ibid.)

Moreover, with regard to the finding that teachers paid more attention to the use of general tools, such as video conferencing systems, than on the use of tools designed for math education (Barlovits et al., 2021; Drijvers et al., in press), the availability of technical equipment and the handling of accessible tools must be taken into consideration in the context of distance education (Barlovits et al., 2021).

From the outlined empirical studies, we derive four requirements for digital tools in math distance education during Covid-19 pandemic. An appropriate digital tool in math education has to (i) enable appropriate formative assessment, (ii) promote inclusive forms of teaching and foster different performance levels, (iii) address the issue of social relatedness and (iv) follow a low-barrier approach concerning technical requirements. Being aware that only a few challenges that arose due to crisis situations are presented here, these four design requirements are considered in the following analysis.

Theory-based Design of a Mathematics Online Environment

Mathematical learning in fully online learning environments differs substantially from traditional classroom instruction since it involves a greater amount of asynchronous and self-directed instruction and a lower degree of collaborative work (Choi & Walters, 2018). However, following CoI and the seven principles for online instruction, the opportunity for a teacher-learner interaction and the student collaboration is clearly emphasized. Following the assumption that ability of successful social interaction in math class depends on the number of synchronous lessons (Drijvers et al., in press), online environments should enable *synchronous distance learning*, *direct interaction* and *collaboration* in mathematics distance learning.

Both, the seven design principles as well as the challenges of Covid-19 math distance education, underline the importance of *immediate feedback* and *adaptive design* of learning environments to support and challenge learners at their individual level. Concerning the lack of technical equipment or challenges in its handling (Barlovits et al., 2021), a *mobile learning* approach seems promising due to the high rate of smartphone ownership in Europe and worldwide. Regarding the instructor role, online environments should enable the teacher to act as a *mediator* (Serdyukov, 2015).

The ASYMPTOTE Concept

Based on these considerations, the ASYMPTOTE[1] project is presented as one example for the development of a synchronous mobile environment in the context of the Covid-19 pandemic.

The project aims at the development of a cost-free, open-accessible and low-barrier tool for mobile distance learning in mathematics education. Due to availability and handling of the environment, it is designed for smartphones and tablets. In order to set up this tool for distance mathematics education in a reasonable time, the already existing MathCityMap system (Ludwig & Jablonski, 2019) will be adapted to the needs of online education. The stand-alone system ASYMPTOTE will be available at the beginning of 2023. In the following, the features of MathCityMap are evaluated in light of the theoretical considerations for online instruction. Subsequently, based on the theoretical framework, steps for the further development of ASYMPTOTE are identified.

The MathCityMap System

The MathCityMap system – developed for mathematics outdoor education using the math trail method – consists of two components, namely a web portal for teachers and smartphone app as

[1] Adaptive Synchronous Mathematics Learning Paths for Online Teaching in Europe. Erasmus+ Strategic Partnership; co-funded by the European Union from 03/2021 to 02/2023 (grant no. 2020-1-DE01-KA226-HE-005738).

working space for students (Gurjanow et al., 2019). In the MathCityMap app, students can easily download samples of tasks via code. The app displays the tasks (Figure 1, left; here: task for distance education), shows up to three hints during and a sample solution after the task solving process on demand. Further, an immediate answer validation in form of a systemic feedback is implemented.

Besides these asynchronous features, the web portal offers teachers within the Digital Classroom the possibility for a real-time monitoring of the students' working progress on both, class and individual level. In addition, a chat function is implemented for a one-to-one synchronous communication between instructor and student (Figure 1, center right and right).

By using MathCityMap in the context of distance education, students work on digital learning paths, i.e., internet-based task sequences to be completed independently at their own performance level with the help of hints and result checks (Roth, 2015). Different answer formats such as values or vectors, cloze texts, or multiple-choice selection are available for task creation. For a detailed description of how the MathCityMap system has been adapted for distance education, see Larmann et al. (2021).

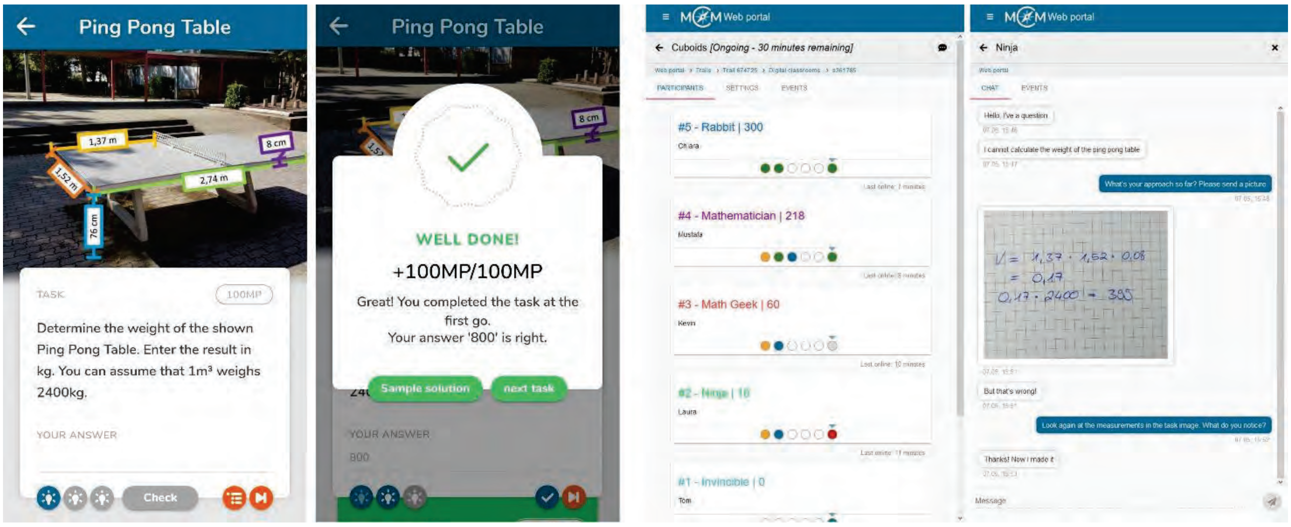

Figure 1: MathCityMap used for distance education from the student's perspective (app; left) and the teacher's view (Digital Classroom in the web portal; right)

From MathCityMap to ASYMPTOTE

Following the considerations for math education online environments, we evaluate the present status of MathCityMap and derive steps of further developments for ASYMPTOTE.

Features of MathCityMap based in line with the identified design requirements: As for *student-teacher interaction*, the chat function of the Digital Classroom provides a possibility for a dialogue between the instruction and the student. Therefore, shown by Larmann et al. (2021), students often start the communication with the teacher to receive feedback on their solution process. Further, teachers can send messages to the whole learning group.

The system provides *prompt feedback*: the student's numerical or text input is immediately validated by the system (Figure 1, center left). In addition, the possibility to send images and audio messages via chat allows teachers to provide individualized support to the student in the sense of formative assessment. Since the systemic feedback distinguishes between correct and incorrect solutions, instructors gain more time to analyze solutions. Thus, the interplay of systemic and individual feedback follows the idea of semi-automated feedback (Fest, 2011).

For student's *time on tasks*, Sorensen and Baylen (2009) highlight well-structured and -planned lessons as well as the value of teacher's time management. Within MathCityMap, a structured setting by using digital learning paths is provided on which students work in a predefined time slot synchronously. Within the Digital Classroom, instructors can monitor student's individual working progress, track student's engagement and directly contact students via chat if needed. With the monitoring function of the Digital Classroom and the semi-automated feedback, the MathCityMap system seems to address the need of mathematical distance education for formative assessment.

To further encourage the student's *time on task* and to *respect mutual ways of learning*, the system offers the ability for teachers to select available digital learning paths or to create own digital learning paths related to the individual needs of their class. Thus, according to Sung et al. (2015), it provides teachers the ability to make adaptations needed for their own classes. Additionally, MathCityMap enables the structuring of complex tasks in manageable subtasks. While upper-performing students can work on the task at a whole, under-archiving students can be guided through the solution process by the subtasks. Also, the online learning setting itself (Serdyukov, 2015) as well as the concept of digital learning paths (Roth, 2015) respect different paces of learning and levels of performance.

Further developments for ASYMPTOTE based on identified design requirements: Within the ASYMPTOTE project, the possibility to create adaptive learning paths will be developed. Following the idea of decision trees, teachers will be able to define so-called status indicator tasks. Depending on the quality of learner's solution, the app will allocate the next task according to the student's performance. Through the definition of several indicator tasks, the content level of the digital learning path can be more and more adapted to the individual level of the student. Moreover, to facilitate learning for students with disability, a read-out-loud mode and a zoom function will be implemented. Tasks, which are formulated in easy language, will be specially marked. Also, tasks will be translatable in various language which might help to bridge language barriers.

The *high expectations* in online education can be addressed through the use of challenging mathematical tasks. These addresses in particular the focus on standard and reproduction math tasks (Aldon et al., 2021; Barlovits et al., 2021) and the infrequent use of modeling or reasoning tasks (Drijvers et al., in press). Within the ASYMPTOTE project, a broad database of open-accessible tasks and learning paths from lower secondary to university level will be created with a special focus on modeling, problem solving and reasoning tasks.

For enabling *communication* and *collaboration* between students in line with CoI and addressing the issue of students' social relatedness in math class, a teamwork mode will be developed. This involves setting up a group chat for synchronous and text-based interaction (Serdyukov, 2015) to support collaborative task processing, discussion and reflection. Whether this type of interaction is sufficient, of course, will depend on the embedding of the lesson conducted with the ASYMPTOTE system. The authors suggest a joint preparation phase and a post-discussion via video call.

Active learning will be fostered by the ASYMPTOTE system by the provision of complex tasks on a variety of topics from lower secondary to university level as well as the teamwork mode. Since

ASYMPTOTE can be considered as an adaptive tutoring system, however, active learning is not additionally promoted by manipulative and interactive tasks: the system does not provide an integrated dynamic geometry software or computer algebra functionalities.

Conclusion

This paper presents a theoretical framework for mobile learning environments aiming at distance mathematics education. It is referred to the Community of Inquiry model, e-learning and online teaching practices as well as mobile learning. Based on the theoretical considerations, design requirements for distance learning systems for mathematics education are developed – also in line with reported challenges of distance mathematics education during Covid-19 pandemic.

Furthermore, the ASYMPTOTE system is introduced, with technical development scheduled to be completed by the beginning of 2022. The system is characterized by an adaptive, synchronous and mobile learning setting. Based on the theoretical framework, we evaluate the present status of ASYMPTOTE and identify needed demands of further developments. Due to the theory-based development of ASYMPTOTE, the mathematics online environment can be considered as a promising tool for mathematics distance education. Thus, we expect that the system will make a substantial contribution to the improvement of online instruction in math class. To which extend the system addresses the issue of social presence in online environments (CoI) and how it affects teachers' and students' perception of the distance situation, has to be investigated by future research.

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Investigation of the usage of digital and non-digital resources in the transition from being teacher candidate to teacher

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This study aims to investigate the resource usage of teachers from the beginning of their careers. For this reason, first, we examined the digital and non-digital resources of the teacher candidates. After two years, we studied with the same group to examine their resources while they were teachers. In this phase, we utilized Documentational Approach to Didactics (Gueudet & Trouche, 2009) as the framework and we designed the study as a case study with two mathematics teachers that were mathematics teacher candidates. We used the reflective investigation method. As a result, teacher candidates used digital resources in a more personalized way. When they became teachers, it was observed that they tended to use interactive e-books. Also, teacher candidates tended to use university textbooks and the resources that their advisors suggested. When they became teachers, they tended to use the resources that the ones accurate for the national exam system.

Keywords: Documentational approach to didactics, resource usage, teacher candidates, teachers.

Introduction

Teachers utilize both digital and non-digital resources in their teaching. Also, their usage of those resources can switch according to certain variables, such as classroom conditions, student attention, etc. While using those resources and selecting and organizing them, teachers are actually designing both their lessons and their resources at the same time (Brown, 2009). However, not only teacher affects the lesson and the resources, but also, the lesson and the resources can affect the teacher. This is an interrelationship between the teacher and the tools (Trouche et al., 2020).

Some of the resources diffuse the teacher's teaching day by day, but others can fade away. This happens with or without the consciousness of the teacher, but there are certain factors affecting teachers' choices. In this study, we aim to investigate the teachers' non-digital resources as well as digital resources and we try to examine the factors affecting usage of them in transition from being teacher candidate to teacher. The problems that guide this research are in the transition from being teacher candidate to teacher:

- What are the resources the use of which has ended and continues?
- What are the resources whose usage schemes remain the same and change?

Theoretical Framework

Considering that the resources used by teachers can be both digital and non-digital, it is important to use a theoretical framework that allows examining all kinds of resources in such a study that examines the resources used by teachers in the transition from being teacher candidate to teacher. For this reason, Documentational Approach of Didactics (DAD) (Gueudet & Trouche, 2009) framework was used to guide this study.

For a better understanding of the framework, one would need to clear all the concepts special to the framework. The first of those concepts is a resource. A resource can be thought such as a tool (Adler, 2000), but it is more than a tool according to DAD. For example, teachers' drawings on the blackboard can be a resource (a tool) and also students' responses can be a resource, too. Teachers use several resources in their daily work for teaching. This work can include selecting, modifying and creating new resources. According to DAD, this work is called documentation work and the outcome of it are teacher documentation. In their documentation work, teachers utilize both the text resources (e.g. textbooks, teacher guiding books, workbooks) and digital curriculum resources (e.g. digital interactive books, technological tools).

One of the concepts of DAD is a document and it consists of resources and their usage schemes (Vergnaud, 1998). A scheme is a teacher's stable organization of their activities, for a given class of situations. A scheme has four components: (i) the aim of the activity, (ii) operational invariants, (iii) rules of action, (iv) possibilities of inferences. In DAD, schemes are considered as usage schemes of resources. All the resources of a teacher, form their resource system (Ruthven, 2007). When the resource system is associated with the usage schemes, they form the document system together.

In this study, DAD was chosen as the framework because it allows us to evaluate a larger collection, considering not only digital resources but all kinds of resources available to teachers. This holistic approach is thought to lead to a broader view of the relationship between the teacher and resources (see Figure 1).

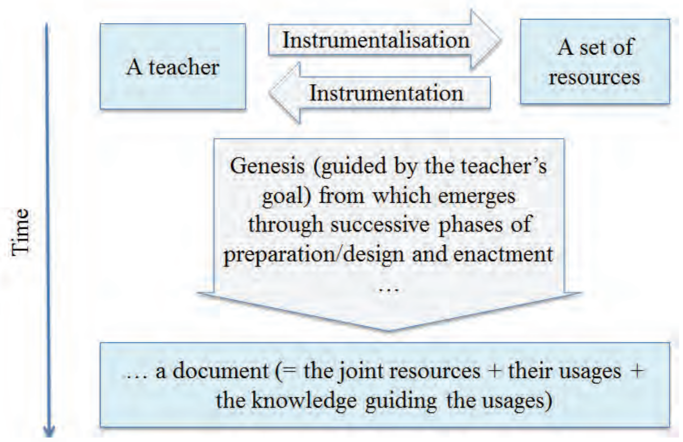


Figure 1: Documentational genesis process in DAD (Trouche et al., 2020, p.3)

As seen in Figure 1, DAD focuses on the relationship between teachers and their resource system. As well as special concepts, there are special names of the processes for teacher's relationship with resources. When the requirements of the resources influence teachers' practice, the process is called instrumentation. When the teachers' aims and knowledge lead to their choices and changes on their resources, the process is called instrumentalization. And the DAD suggests that these processes work both ways. In time with the ongoing instrumentation and instrumentalization processes and the usage schemes of the resources, teachers create documents which is called documentational genesis.

Research Design

In this study, we utilized qualitative research methods to be able to investigate the usage of the resources in detail. The study was designed as a case study. And reflective investigation approach was adopted as a data collection method.

Participants of the case study

In this study, we studied with two teacher candidates and followed them when they started teaching. The teacher candidates, Mary and Keira (pseudonyms) were not the best technology users or with the most resources. Mary and Keira were willing to share their work with resources and were able to introduce their usages in detail. Also, they were volunteering for the lesson preparation sessions as an extra activity. That’s why we chose them for this study to investigate their resources. So the sampling method is criterion sampling method and the criterion are: (i) Being in the last year of undergraduate education in the 2016-2017 academic year, (ii) those who have taken the courses "Field Papers in Mathematics Education" and "Teaching Practice", (iii) to be appointed as a teacher to public schools affiliated to the Ministry of National Education (MoNE) in 2018, (iv) being volunteer for the study.

Mary

Mary is a teacher who has drawn the attention of researchers with her studies even before the research. She came to the fore during her student years in terms of both the scores she got from the exams of the courses related to mathematics teaching and the emphasis she made on the notes she took during the lessons. Another remarkable point about Mary is that she attends classes like a teacher. She has become a student who feels like she will tell her students what she has learned after leaving the class. Mary has been a student who has attracted the attention of the instructors, especially while teaching lessons that require exemplary lectures. During the example lecture, she had no problems while controlling the class and giving appropriate answers to the questions asked. She was also able to use applications such as the Cabri Geometry on the smartboard in the teaching practice as needed and gained the appreciation of her mentor teachers. It is noteworthy that there is great diversity in her teaching practice file in terms of resource use. Mary generally accepted teachers as resources, even contacted teachers who own resources shared on websites, tried to learn how they used these resources, and sought alternative teaching methods for different courses.

Keira

Keira is a teacher who has come to the fore with her desire to use different materials in teaching lessons during her student years. Keira, like Mary, has shown outstanding success in teaching lessons, especially drawing attention to designing and using teaching materials that can be used in lessons. The teaching materials designed by Keira drew attention as they were suitable and durable for student use as they would be used in the classroom, and she stated that he wanted to use these materials when she became a teacher. She showed outstanding success both in teaching courses and in pure mathematics courses. She was able to successfully use smartboard applications in her teaching practice course. In addition, Keira's resource system is quite extensive, often containing resources for different teaching materials. In addition, Karen also used the resources recommended by her mentor teacher in her teaching practice and added them to the resource system.

Method of data collection

We utilized the reflective investigation method which is suggested by Gueudet and Trouche (2009) for those to use DAD. There are some principles of the method (Trouche et al., 2020, p. 6):

- Broad collection of the resources,
- Long-term follow-up,
- In- and- out-of-class follow up,
- Reflective follow up,
- Confronting teachers' views on the documentation work and the materiality of it.

According to these principles we utilized semi-structured interviews, schematic representations of resource systems (SRRS), teaching practice files, interviews on the lesson preparation, lesson observations, and recall sessions. Semi-structured interviews and SRRS diagrams were handled together, because we used the semi-structured interview to understand the resource system better. All the interviews were audio-recorded. The SRRS diagram is a data collection tool that is "not structured for the researcher but it is structured for the participant" (Trouche et al. 2020). In other words, we did not structure the diagrams, they did draw their own SRRS diagrams to explain their resource system.

Table 1: Data Collection Methods according to the Reflective Investigation Method

Reflective Investigation Steps	Data Collection Methods
Broad collection of the resources	SRRS diagrams
Long-term follow-up	Teacher candidate and teacher observations
In- and- out-of-class follow up	Teaching practice files
Reflective follow-up	Interviews and Observations of the lessons and lesson preparations
Confronting teachers' views on the documentation work and the materiality of it	Recall Session

Teaching practice files were used as a written interview about what they realized in their teaching practice classes. This data collection tool did not use when the time they were appointed as teachers. Semi-structured interviews were used instead.

Both before their teaching practices and teaching sessions, we interviewed them about how they planned their lessons, then we observed them in the lessons. In the observation, one of the researchers took field notes and the lesson was video-recorded, too. After the lesson observations, we planned a recall session to get the interpretations of their own lessons.

Data collection procedure

Data collection of this study started in the 2016-2017 education term. In 2016-2017, teacher candidates were in their teaching practice year and the first data was collected. Afterward, a two-year break was given for the appointment of teacher candidates and for obtaining official permissions from their schools. And in the 2019-2020 term, data collection has completed while they were teaching in their own classes (see in Figure 2).

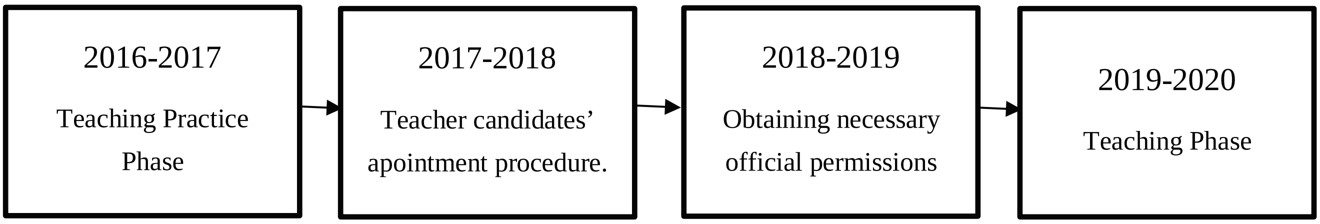


Figure 2: Data collection procedure

Teaching practice files of pre-service teachers who took the course were collected and analyzed. At this stage, the files of all teacher candidates who took the course were examined. However, after the participants were determined, the files of the selected participants were included in the study. Each participant's lesson was observed for 4 hours before the teaching period and 25 hours after being actual teachers, during the lesson observation, the researcher sat in the back rows like a student and took notes and did not interfere with the process. In addition, the lessons were recorded in order to prevent data loss and to be able to re-examine during the analysis. After all the lesson preparations, observations, and interviews, a recall session was organized in order to realize reflective remarks.

Data analysis

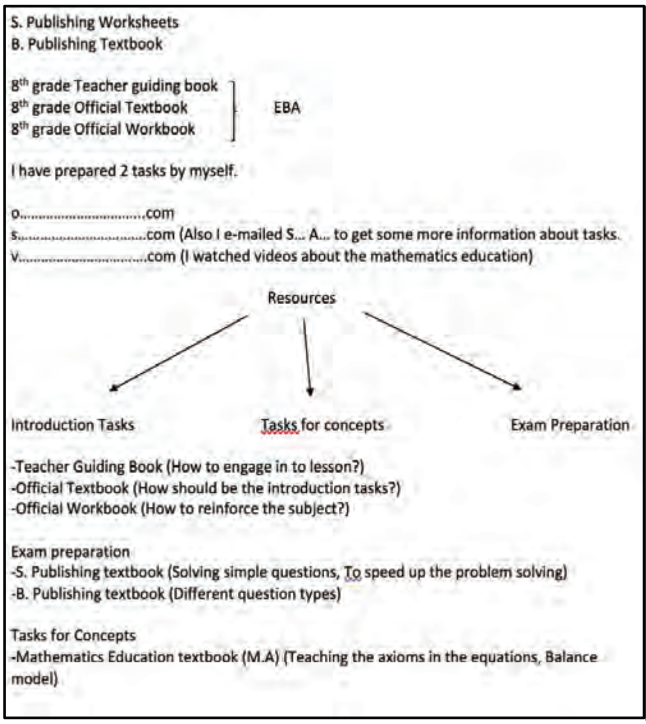
The audio-recordings and video-recordings of the interviews and lessons were transcribed. Initial coding was based on the resources and their usages of the participants. We focused on DAD's key processes- instrumentation and instrumentalization. When we realize that there could be such a process, we coded it accordingly and checked it in the recall session. Because there were a lot of data collection tools, we followed the instrumentation and instrumentalization processes during the coding. This is in line with Yin's (2009) approach about using a framework to be able to do systematic coding.

The use of various data collection tools was in order to enhance the trustworthiness of the findings. Triangulation was employed to cross-check the conclusions. The triangulation was between the interviews, observations, and recall sessions.

Findings

Teacher candidate phase

When Mary and Keira were teacher candidates, they were using smartboards in their teaching practice lessons. However, they both did not mention it in their SRRS diagrams (see Figure 3 and Figure 4).



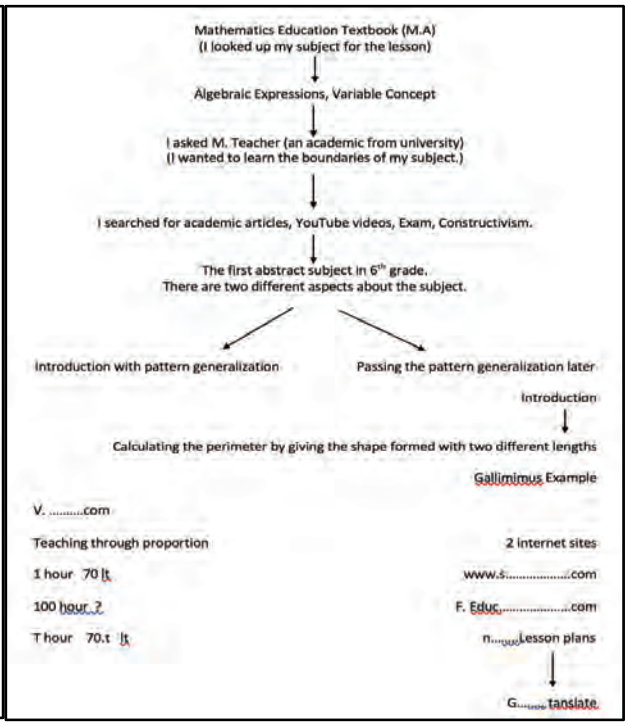


Figure 3: Mary's SRRS diagram **Figure 4: Keira's SRRS diagram**

In Mary's lesson in her teaching practice, she used a balance system to emphasize the inequality concept. Even if she utilized a special program in the smartboard, she used only the pictures to demonstrate the systems she would like to exemplify.

In Keira's lesson in her teaching practice, she used a picture and brought concrete material in the class. She utilized a special program in the smartboard, but she generally used the smartboard just for demonstration. Additionally, she used a concrete material of the picture she shared on the smartboard.

In the interviews, they both mentioned that they used a specific smartboard program to be able to teach an effective lesson. And they emphasized that both of them designed their sessions all by themselves. They mentioned only the resources they used while designing.

Teaching phase

Mary and Keira utilized the smartboard in their teaching, too. But this time, they both mentioned smartboard in their SRRS diagrams. In the teaching phase, according to lesson observations, they used interactive e-books in their lessons. But, while Mary's students have also textbook of the same e-book, Keira used the e-book only on the smartboard. Mary stated her situation as follows:

Researcher:	Mary, you use an e-book on the smartboard. But, as I see, the students follow the lesson from their own books. Did you arrange it?
Mary:	Yes, I asked them to buy this book. I also get the same book's teacher book version and e-book. Thankfully my students were able to get the book. Time is so important for us. And thanks to this e-book, I can complete all the aims in the curriculum. I also use another textbook with this e-book. When I want to give different examples about the national exams, I use that book. At that time, I close the smartboard and write the questions on the blackboard.

As it can be understood from the transcript, Mary combines the e-book with another book. But she used them for different aims. When she wants to solve harder questions for national exams, she closes the smartboard and the e-book for a while. In this situation, we can see the instrumentation process of DAD.

Also, Keira mentioned that her students could not get the students' book. That's why she explained that she could not use the students' book in the class. But she used the e-book for demonstration.

Researcher:	Keira, you use an e-book on the smartboard. But the students have another book, not the same as the e-book?
Keira:	Yes, I asked them to buy the students' book. But some of my students get and others were not able to buy the book. So, I cannot use the students' book in the class, but I use the e-book. At least I don't have to write the questions to solve on the board. I am already late in the curriculum.

It can be understood from the transcripts that they use the smartboard to gain time for the goals of the curriculum. When examining the transcripts, we can see that Keira limited her own usage of the e-book, because the students' book is not available for her students. This points out to instrumentation process of DAD. There were some notes on the teachers' book of Keira, which shows some definitions of mathematical concepts. In this case, we were able to see an example of the instrumentalization process, that Keira used her own definitions for the mathematical concepts in the teacher's book of her e-book.

Discussion and Conclusion

In this study, we aimed to investigate the digital and non-digital resources in the transition from being teacher candidate to teacher. As a result, one of the resources, the smartboard, remained the same for both of the participants. However, the usage of it differentiated when started teaching. They used interactive e-books in their teaching, even if they did not use any e-books before. Also, the way both participants use these e-books is different from each other. Mary, for example, used the smartboard for speeding up the problem-solving. However, Keira used it both for the e-book and as a music and video player. They basically explained that their initial aim is to be able to catch the curriculum (Şahin, 2010). In their opinion, the curriculum is congested and they have to catch the curriculum as well as they have to catch their colleagues.

In this study, instrumentation and instrumentalization processes were also detected. According to their own aims and knowledge, they were able to revise their resources (instrumentalization process). For example, Keira was noted her own definitions for some mathematical concepts on the e-book she used. Also, she noted some rhymes about divisibility rules to remember them quickly. And Mary noted her own draft exam questions on her teachers' book. And she presented them using the smartboard.

Also, they were able to revise their own lesson practice to use that specific resource they aimed to use (instrumentation process). For example, Mary used another textbook for the national exams, and she needed to close the smartboard for this usage. Since she was not able to use the other book on the smartboard, she closed it and continued with the blackboard. Even if she stuck to the e-book most of the time, she changed her own practice.

Therefore, it can be said that they used the smartboard just for demonstration while they were in the teaching phase. They did not use its qualifications as a smartboard. So, we can say that they used the smartboard as a resource when they were teacher candidates, but they just used it as an educational technology when they were teachers (Adler, 2000).

Another result is that teacher candidates tended to use university textbooks and the resources that their advisors suggested. When they became teachers, they tended to use the resources that their colleagues used and the ones accurate for the national exam system (Gueudet et al., 2013).

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MathMemeThon: how mathematical memes bring teachers and students together during Italy's pandemic lockdown

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Keywords: Mathematical memes, boundary objects, learning across contexts, distance learning.

Introduction: a tale of two cultures

The global switch to distance education imposed by the Covid-19 pandemic in 2020 and 2021 dramatically exposed teachers' difficulties to engage students in a distance-learning setting, as acknowledged by recent studies (Bakker & Wagner, 2020; Fondazione Agnelli, 2021). Indeed, these difficulties are more related to the existing technological and cultural discontinuities between 21st-century learners and teacher-centred educators (Bronkhorst & Akkerman, 2016) than to the distance-learning setting enforced by the pandemic. Feedback collected from Italian teachers posting in social media groups on Facebook and from personal acquaintances evidenced that the majority of teachers simply moved their usual teacher-centred lessons from in-person to distance mode, resulting in one-way lessons with teachers lecturing passive students (often with no camera on), which dramatically failed in engaging learners. In other words, the pandemic just amplified malfunctions that were already there, making them clearly observable. It acted as the PCR, the Polymerase Chain Reaction used in molecular biology to amplify DNA samples: now these malfunctions are evident and we can observe them and act on them. One possible way to act on them, bridging the discontinuities and engaging students both in-person and in distance-mode learning settings, is by challenging the dichotomy between the two cultures represented by in-school formal learning and out-of-school informal learning. This can be done, among others, by importing a product of the out-of-school digital culture, such as mathematical Internet memes, into the in-school formal learning environment.

What are mathematical Internet memes?

Internet memes are digital objects pervasive on the Web (221 million occurrences of the hashtag #memes on Instagram in February 2022) created by Internet users adding original humorous captions to existing popular images. Mathematical Internet memes are mathematical mutations of Internet memes: they combine mathematical and memetic elements to produce hybrid representations of mathematical statements, endowed with an epistemic power to initiate argumentation processes among users inside dedicated online communities (Bini et al., 2020). Despite these evident potentialities, mathematical memes are still widely understudied in mathematics education.

The activity: theoretical framework, research question, methodology and results

In 2020, Bini and Robutti conducted an exploratory study on mathematical memes as boundary objects (Star & Griesemer, 1989) between the communities of students and teachers during in-person school activities. The purpose of this work is to move forward along this line of research, investigating if mathematical memes can "fulfil a bridging function" (Akkerman & Bakker, 2011, pp. 133) between students and teachers also in distance-mode settings. The study is guided by the research question: Can mathematical memes act as boundary objects between students' informal out-of-school culture and teachers' formal in-school mathematical culture in distance-mode activities? Mathematical memes' boundary-crossing nature has been taken into account in two different ways in designing the activity: (1) the task design requested students to create a composite object, i.e. a mathematical meme *and* a presentation providing a brief insight into the mathematical content, and (2) during the activity the author acted as a boundary broker, facilitating the "processes of translation, coordination and alignment between perspectives" (Wenger,1998, p. 109). The result is MathMemeThon, a distance learning activity with mathematical memes, structured as a team competition inspired by a computer hackathon. The activity took place in the second quarter of the 2020/21 school year, involving 7 class groups of 9th grade students (15yo) and 2 class groups of 10th grade students (16yo), for a total of about 180 students and 6 teachers from 3 different institutes located in Piedmont, in the north of Italy. It developed in three online meetings of 2 hours each, where students teamed and competed creating mathematical memes and presentations, and sharing them online on Padlet walls. Students' productions were then presented remotely via WebEx to a jury of experts made up of teachers, Master and PhD students of mathematics, who judged the productions evaluating the mathematical and memetic content and the quality of the presentation. Memes were then shared on the project's Instagram page (https://www.instagram.com/lifeonmath/).

Observations of the interactions and feedback from teachers and students showed that, even in the distance learning setting, the hybrid language of mathematical memes succeeded in connecting the two communities: teachers appreciated the idea of communicating mathematics through an object "very close to the world of students", and students valued the fact "it was necessary to give importance to the mathematical content but at the same time to find the right idea to create the meme".

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An exploration of the use of structured reflection tasks to surface prospective teachers' awareness of TPACK

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Keywords: Reflection, TPACK, prospective teachers.

Introduction

This study focuses on undergraduate mathematics students in an Irish university who choose to take a module on mathematics education – as part of their academic course, not in an initial teacher education (ITE) programme; thus, the students can be considered as *prospective teachers*. Among the assessment elements for the module is a small component consisting of tutorial exercises. One exercise sets out some structured, technology-facilitated mathematical tasks, involving use of a spreadsheet (Excel), a function graphing and dynamic geometry package (GeoGebra), and a simple programming language (Scratch); the students do the tasks and then post written reactions and reflections to the module's discussion board. The tasks are framed for the students by the overarching idea that – appropriate – use of technology can support, enhance or change teaching and learning, and also by Taylor's (1980) simple but enduring *tutor, tool, tutee* classification (with the tutorial exercise involving the tool and tutee roles). However, the students have not been introduced to the Technological, Pedagogical and Content Knowledge (TPACK) framework (Herring et al., 2016).

Although the section of the module is short (two or three lectures and one tutorial exercise), it has produced insightful reflections from participants over the years. This prompted the research question: Do the tasks involving structured activities and reflections enable prospective teachers, in particular those who took the module in 2019-2020, to surface tacit awareness of components of TPACK?

Theoretical framework

The theoretical framework for the study is founded in the various technological components of TPACK (Herring et al., 2016): Technological Knowledge (TK), Technological Content Knowledge (TCK), Technological Pedagogical Knowledge (TPK), and Technological Pedagogical Content Knowledge (TPACK). The latter denotes the knowledge required by teachers for the effective integration of technology into their teaching in any content area. Teachers who display TPACK demonstrate a good understanding of the complex interplay between technological, pedagogical and content knowledge, by teaching content using appropriate pedagogical methods and technologies.

Methodology

Ethical clearance was granted, and signed consent forms were returned by 18 of the 25 students taking the module in 2019-2020. Fourteen of them contributed to the discussion board for this task.

Directed Content Analysis (Hsieh & Shannon, 2005) was used on the texts of the (n=14) posts, using the definitions of the four technological components of TPACK as outlined on www.tpack.org. Thus, the analysis aimed to identify instances where the students displayed TK, TCK, TPK, or TPACK.

Results

In total, 48 sections of text were coded in relation to the TPACK framework. Of the 48 codes, 15 showed evidence of students' TPACK, for example (from student 7):

> These graphing programmes certainly have a role to play in the teaching of school mathematics. Such educational tools help students to visualise mathematics in a more relatable manner, hence strengthening the understandings of mathematical concepts. For instance, first-year groups were introduced to Scratch and encouraged to learn about geometric shapes and their properties by using the tool, thus making the learning of abstract concepts more meaningful.

Here the student is clearly considering the role of technology in relation to different representations of content, as well as showing an understanding of how it might change the way the students learn. Regarding TCK, there were 16 instances, such as the following, in which student 12 reveals an understanding of how GeoGebra can be employed to create new representations of content:

> …very valuable as a tool in teaching maths, especially in terms of coordinate geometry and the links between algebraic expressions and their geometric realisations. For example, it could […] be used to explore the relationship between the coefficients of a function and the shape [of] its graph.

A further 12 sections of text were coded as TPK, illustrating both a knowledge of how technology can be used in teaching and an understanding of how its use may change teaching and learning. Student 5 posted:

> … the students being able to visualize it easier because of the spreadsheet would help and it might also help students that are not very good at computations in maths gain confidence if they could see that they could understand what's happening even if they couldn't do the calculations by hand.

Only five instances of TK were identified, such as (from student 6): "When creating worksheets, GeoGebra is a very accessible tool which makes diagrams clear and easy to paste into a document." This is indicative of an appreciation of how technology can be used for a simple educational goal.

Conclusions and implications

This research has identified the potential of using structured tasks, followed by reflection and analysis, to surface prospective teachers' awareness of TPACK concepts. We propose that the process could be effectively integrated into ITE programmes to expose students to the framework in a way that makes it accessible and personal. Having completed the structured tasks and reflections, the students would be introduced to the TPACK framework and asked to analyse their own reflections. We aim to conduct further research using this model with post-graduate students enrolled in ITE.

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Teachers' knowledge for supporting transitions between dynamic digital technologies and static environments

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This paper explores teachers' knowledge for supporting students to transition between technologies, for example, moving between using dynamic digital technologies and using static environments for mathematics. The TPACK framework is used to explore such knowledge through an examination of one teacher's frustration with the 'rigidity' of angle definition and measurement in GeoGebra, expressed in a task-based interview on circle theorems, compared with the relative 'flexibility' of a static environment. The nature of the central TPCK construct is discussed and implications for teacher education are identified.

Keywords: teacher knowledge, dynamic technology, circle theorems

Introduction

This study investigates the nature and content of teachers' mathematical knowledge for teaching circle theorems using dynamic digital technologies, such as GeoGebra. The specific knowledge required for teaching mathematics with technology has been a long-term interest in TWG15 (Clark-Wilson et al., 2019). Less commonly, research has focussed on teachers' knowledge for supporting students to bridge or transition (Geraniou & Mavrikis, 2015) between technologies, for example, moving between using dynamic digital technologies and using static environments for mathematics. Hence this study focuses on investigating what knowledge enables teachers to support transitions between dynamic and static environments in their teaching of circle theorems. The main research question is: what is the nature and content of teachers' mathematical knowledge for teaching circle theorems using dynamic digital technologies? The sub-question is: what knowledge enables teachers to support transitions between dynamic and static environments?

Theoretical background

Mathematical knowledge for teaching, as defined in this study, is when tacit knowledge-in-action (Ruthven, 2014) is underpinned by and coincides with a teacher's articulated knowledge that provides for "a rational, reasoned approach to decision-making" (Rowland et al., 2005, p. 260). The *Technological Pedagogical and Content Knowledge* (TPACK) framework is suitable for this study because the framework enables a focus on teachers' mathematical knowledge for teaching *situated* in a technological context (Mishra & Koehler, 2006). In this study, the central TPCK construct is viewed as a new domain of synthesised knowledge, that is, a transformation (Rowland et al., 2005) of mathematical knowledge for the purpose of teaching using technology. This paper focuses on the dyadic construct TCK (technological content knowledge) as a means of exploring, by comparison, the nature and content of the central TPCK construct. Mishra & Koehler (2006) define TCK as knowledge about how technology and content influence and constrain one another; that is, how the mathematical content can be changed by the application of particular technologies. For example, dynamic digital technologies, such as GeoGebra, embed mathematical rules in their design, constraining the user to obey these rules e.g. by imposing an explicit order in constructing geometric figures (Jones, 2000). This mathematical rigidity may be fruitful in supporting the user to appreciate and explore the rules embedded. By contrast, a static environment, such as paper-and-pencil, is relatively flexible, allowing the user to draw mathematical 'sketches' without the necessity of obeying a specific set of mathematical rules. An appreciation of how different environments provide contrasting representations of mathematics exemplifies the TCK construct. An appreciation of how to capitalise on such contrasting representations for the purposes of teaching, e.g. to support pupils' transitions between environments, would exemplify the TPCK construct. Adler's (1999) dilemma of transparency provides a means of explaining why teachers' knowledge might enable them or not to support transitions between dynamic and static environments.

Methodology

As part of a larger doctoral study (Bretscher, 2015), four case study teachers were selected, as self-described technology enthusiasts confident in the use of technology, to take part in semi-structured interviews based around a GeoGebra file on circle theorems. As enthusiasts, the case study teachers were likely to display mathematical knowledge for teaching using technology. The semi-structured interviews provided a common situation across which the case study teachers' mathematical knowledge for teaching using technology could be contrasted. Examples of TCK were identified where a teacher's articulated knowledge and knowledge-in-action coincided to place emphasis on technology and mathematical content when addressing a situation involving a synthesis of mathematical, pedagogical and technology knowledge. The case study teachers were prompted to show (knowledge-in-action) and discuss (articulated knowledge) how they would use diagram D1 presented in the GeoGebra file (see Figure 1) to demonstrate that the angle at the centre of the circle, subtended by an arc, is double the angle at the circumference subtended by the same arc.

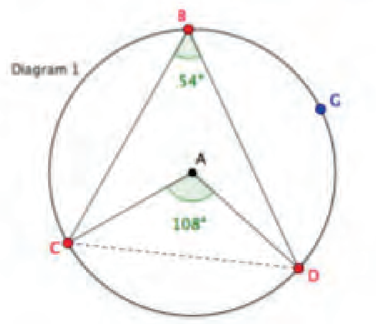

Figure 1: Diagram D1 in the GeoGebra interview file on circle theorems

Diagram D1 was designed to be similar to resources found on a web-search. Circle theorems were chosen since it is a topic, in the English mathematics curriculum, which is commonly identified with the use of dynamic geometry software (Ruthven et al., 2008). It was therefore reasonable to assume that the case study teachers would be familiar with technological resources similar to D1 and might even have previously used such resources in their own teaching. Thus, they would be likely to have some mathematical knowledge for teaching circle theorems using the GeoGebra file, even if they were unfamiliar with the particular software. In addition, the topic of circle theorems is at the apex of geometry in the compulsory English mathematics curriculum, hence it provided a potentially challenging context even for experienced teachers who were both mathematically and technologically confident.The semi-structuring of the interview allowed some flexibility to respond to events during the interview, whilst maintaining an overall structure that would allow for and facilitate comparison. Both the visual and audio aspects of the GeoGebra interviews were recorded and analysed. This paper focuses on one of these case study teachers, Edward, whose expression of frustration at the way angles are measured in GeoGebra provided a particularly illuminating example of TCK.

Defining angles in GeoGebra: Edward's dilemma

The analysis presented in this section focuses on an indicative example of TCK, from Edward's interview, as a means of exploring the nature of the central TPCK construct.

Edward was prompted to question how angles are defined for the purposes of measurement in GeoGebra by unexpected configurations of D1 appearing during dragging, displaying the 'incorrect' angle at the centre (see Figure 2). After experimenting by dragging points C and D, Edward concluded the angle measured at the centre was dependent on the relative position of points C and D. More specifically, in GeoGebra the angle measured at the centre in D1 is defined by specifying the ordered triad of points CAD and measured anticlockwise from the line segment AC to the line segment AD. Thus, when the relative positions of C and D are reversed, as in Figure 2, the angle appears to 'flip' between being less than 180 degrees and being reflex.

D1 had been designed so that, whilst the angle at the centre could become reflex, the angle measured at the circumference was constrained to be less than 180 degrees whatever the relative position of points C and D. Hence the 'correct' angle at the circumference in relation to the circle theorem was always displayed, however some configurations of D1 displayed the 'incorrect' angle at the centre. Edward's questioning of how the software defines and measures angles and his realisation of the angle at the centre's dependence on the relative positions of C and D is an example of TCK because it shows a developing understanding of how the GeoGebra software models geometric concepts and relations.

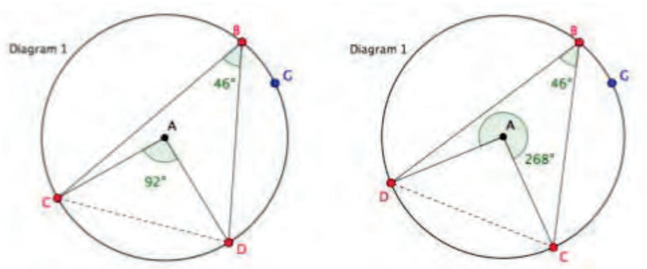

Figure 2: Angle measurement and reversing the relative positions of C and D

For Edward, the software's definition and measurement of angles was a source of frustration, appearing idiosyncratic in the way D1 'flipped' between displaying the correct and the incorrect angle at the centre. He argued:

E: ... this is sort of a function of how the software works isn't it, rather than a ... is that bringing out anything useful mathematically that ... that's just a bit annoying the way it does that, isn't it?

His frustration with angle definition in the software led him to suggest that, for proof, he would prefer a static environment: “I’d project this on the whiteboard […] and then mark on the angles that I want”. Implicitly, Edward compared the difficulties he faced understanding how angles are defined in GeoGebra to the flexibility of being able to mark the angles that he wants in a static environment. Diagrams presented in software such as GeoGebra are constrained to follow the rules for defining angles that have been programmed into that piece of software. One of the affordances of drawing diagrams without digital technologies is that the relevant angles of the circle theorem may simply be marked on a diagram with a brief stroke of a pen or pencil, without needing to consider how they are defined precisely. It is not that a precise definition of the angles does not exist or is not necessary in a paper-and-pencil environment, of course, but that often it does not appear necessary to give it explicit consideration.

A case where it might be necessary to give explicit consideration to a precise definition of the angles, even in a static environment, would be when giving a full statement of the circle theorem, rather than a commonly-used, abbreviated form such as ‘the angle at the centre is double the angle at the circumference’. For example, a full statement of the circle theorem is ‘the angle subtended at the centre by an arc is double the angle subtended at the circumference by the same arc’. The difference between the abbreviated form and the full statement is in the specification that the two angles must be subtended from the same arc. More specifically, using the full statement of the theorem clarifies which is the ‘correct’ and ‘incorrect’ angle at the centre.

In his initial discussion of D1, Edward assumes the angles are defined as being subtended by the chord CD:

E: ... so what it shows is the angle subtended at the circumference by chord CD is always twice the angle at the centre, irrespective of where B is.

Defining the two angles as subtended from the chord is unproblematic as long as the two angles remain in the same segment; however, when they are in opposing segments the theorem appears to break down (see Figure 3 a and b).

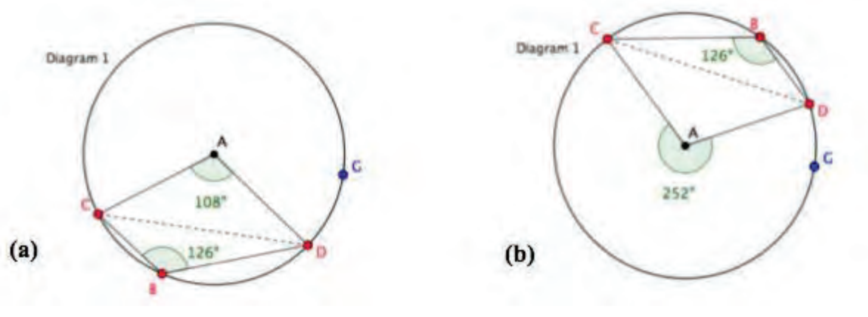

Figure 3: Angles in the opposite segments with (a) the ‘incorrect’ angle at the centre displayed and (b) the ‘correct’ angle at the centre displayed.

The situation where the two angles appear in opposing segments occurred twice during Edward’s GeoGebra interview. Firstly, as depicted in Figure 3 (a), it occurred where the ‘incorrect’ angle at the centre is shown, assuming the angles in the circle theorem are defined as being subtended from the same arc. He had anticipated this case to some extent. Thus, for Edward, this case was not unduly problematic and did not disrupt his statement of the circle theorem defining the angles as subtended from the chord CD, as the quote below suggests:

E: And then if you drag B this side [onto the minor arc CD], then suddenly it goes from 54 to 126. So ... uh ... what's happening there? So ... uh ... what's happening there is the angle on the other side of the 108 is now double the angle at the centre, the angle at the circumference ... but it's not showing on the diagram, the computer's not showing that other angle ... but you can calculate it as 360 – 108, so 252. And 252 is double 126. Yeah.

Instead, he called this case a "complication", suggests "ignoring" it at least initially with pupils, and refers to the 'correct' angle at the centre, measuring 252 degrees, as "the reflex angle". His treatment of the case in Figure 3 (a) as a sort of deviant example or extension of his statement of the circle theorem, where the angle at the centre is reflex, avoided a mathematical critique of his definition of the angles being subtended from the chord. However, the situation arose for a second time, similar to Figure 3 (b), where the 'correct' angle at the centre is shown, assuming the angles in the circle theorem are defined as being subtended from the same arc. This time, the situation was unexpected and troubling for Edward. In particular, it led him to question his previous definition of the central angle as being subtended by the chord CD. The following quote indicates his struggles as he attempted to find a correct mathematical interpretation of this configuration of D1, see Figure 4 for the numerical example he discusses at the start:

E: Um ... so ... let's take an example ... so 94 doubled is 188, so it's still true that ... so that angle is twice that angle. But uh ... how do you know it was that angle ... so the computer is kind of showing you the right angle for what it's working for isn't it? But in words, how do you explain what that angle is, it's not really the angle that chord CD is subtending at the centre is it? Because it's that ... chord CD is subtending that angle at the centre, so suddenly you have to say it's the other angle, the reflex angle at the centre that's subtending. So... so CD is subtending 99 at the circumference and, ... er ... the reflex angle is 198 yeah. Uh ... which is not a very good explanation. [E laughs]

At the end of this quote, Edward tries to re-state the theorem using a particular numerical example, taking into account his realisation that the 'correct' angle at the centre was not, as he previously assumed, the angle subtended by the chord CD. He struggles, eventually settling for "the reflex angle", whilst acknowledging this seemed inadequate.

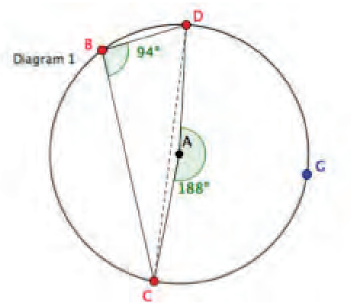

Figure 4: Edward's angle definition dilemma

Returning to Edward’s frustration at the apparently idiosyncratic way GeoGebra defined and measured the angles in D1, the discussion above shows that instead of being “just a bit annoying”, the way GeoGebra defines and measures angles *does* bring out something mathematically useful. The variation in whether the ‘correct’ or ‘incorrect’ angle is displayed in D1 provides a means of discussing how angles are defined in other contexts and, in particular, how the angles referred to in the (abbreviated) ‘angle at the centre is double the angle at the circumference’ circle theorem are defined precisely in a full statement of the theorem. Articulating a strategy to use the way GeoGebra defines angles to raise these issues for the purposes of teaching circle theorems would be an example of TPCK. Such a strategy would not appear to depend on integrating pedagogic knowledge with TCK. Instead, it requires mathematical knowledge regarding the precise definition of the angles in a full statement of the angle at the centre circle theorem. Hence, TPCK appears to be mathematical knowledge, abstract in the sense that it generalises across particular technological contexts and mathematical topics. For example, a precise definition of the angles in a full statement of the angle at the centre circle theorem should hold both in the context of using GeoGebra or a paper-and-pencil environment.

However, TPCK also appears simultaneously to be mathematical knowledge *situated* in the context of teaching using technology. The issue of how angles are defined appears more salient and even surprising – for Edward at least – in the context of GeoGebra. In addition, at the time, the high-stakes GCSE (General Certificate of Secondary Education) examinations in England only required pupils to state an abbreviated form of the circle theorems. As a result, it is possible that the case study teachers were unaware of a precise definition of the angles in a full statement of the angle at the centre circle theorem. Hence, an individual teacher’s TPCK may also be seen as situated in the examination system and national curriculum of the country in that teacher is working. This argument suggests that TPCK is a synthesis of mathematical, pedagogical and technological knowledge, highlighting its situated nature as a transformation of mathematical knowledge for the purposes of teaching using technology.

In summary, the central TPCK construct is exemplified as having teaching strategies for exploiting the opportunities that arise from contrasting and complementing the affordances and constraints of different technologies, in this case, the mathematical rigidity of angle measurement in GeoGebra relative to the flexibility of paper-and-pencil environment. Using such teaching strategies and making affordances and constraints of technologies explicit to pupils should support them in transitioning between different technologies.

Discussion

In this section, I apply Adler’s (1999) dilemma of transparency to explain Edward’s frustration further and so identify implications for teacher education. Adler (1999) uses Lave and Wenger’s (1991) notion of transparency to describe teachers’ dilemmas in negotiating the dual visibility and invisibility of talk as a resource in the practice of school mathematics. In this paper, the notion of transparency is applied to the use of technology as a resource in the practice of school mathematics. Adler (1999) describes Lave and Wenger’s use of the metaphor of a window to explain their notion of transparency:

> Lave and Wenger (1991) used the metaphor of a window to clarify their concept of transparency. A window's invisibility is what makes it a window. It is an object through which the outside world

becomes visible. However, set in a wall, the window is simultaneously highly visible. In other words, that one can see through it is precisely what also makes it highly visible.

Thus, technology as a teaching resource for mathematics needs to be simultaneously both visible, so that it can be noticed and used in the practice of school mathematics, and invisible so that attention is focused on the subject matter of mathematics and not solely on the technicalities of the environment. The particularities of using a specific technology to teach mathematics influences the mathematics that can be taught. For example, sketch diagrams in paper-and-pencil environments are relatively flexible in that they do not have to obey fixed rules in relation to defining and measuring angles. The flexibility of the paper-and-pencil environment affords the user the freedom to imagine they are working in an ideal mathematical world, where geometric relationships embedded in figures can be imagined without being weighed down by rigid rules of construction and where perfect circles, exact angle measurement, circle theorems and proof 'exist'. Diagrams in GeoGebra appear more mathematically rigid in this respect, hence Edward's irritation with the software. However, this rigidity can be useful in forcing attention to mathematical details, such as defining angles, which the sketch diagram in a paper-and-pencil environment allows the user to elide. Similarly, the window frame, its shape and positioning on the wall, influences which part of the outside world can be seen. Thus, teachers need to understand the significance of the particular technology for the mathematics they are teaching: the technology requires teachers' explicit attention, it needs to be visible. In this sense, mathematical knowledge for teaching using technology is always situated, since the technological context in which it is being applied is central to its meaning. Simultaneously, technology should enable the teaching of mathematics, in this case the GeoGebra software should enable the teaching of circle theorems and should thus be invisible. It is the window through which mathematical knowledge can be seen: the GeoGebra software is a means of controlling numerical and geometric variation so that pupils are exposed systematically to examples of the circle theorem. Here, mathematical knowledge for teaching using technology appears more abstract, allowing teachers to make mathematical connections across technological contexts.

Adler's description of a dilemma of transparency where the teacher manages talk as a classroom resource, so that it is neither too visible for pupils, obscuring the mathematical subject matter, nor too invisible so that they are unable to access it, has some explanatory value for this study. However, here, the dilemma is managing technology so that it does not become too visible for teachers, obscuring mathematical knowledge for teaching using technology, nor too invisible, so that teachers assume that technology can be used unproblematically for teaching mathematics. For example, on the one hand, Edward's irritation with the definition and measurement of angles in GeoGebra indicated that the software was too visible for him. In this case the GeoGebra software obscured his access to mathematical knowledge for teaching using technology, leading him towards rejection or restriction of technology use. On the other hand, the case study teachers' lack of awareness of how dragging imposes a particular order on how different configurations of the circle theorem arise (reported elsewhere, e.g. in Bretscher, 2015) provide an instance where technology seems too invisible. Here, the unintentional pedagogic structuring of mathematics suggests that the technology has become too invisible, with an assumption that technology provides unproblematic access to mathematical knowledge for teaching. The implication for teacher education is that they need to help teachers manage the dilemma of transparency. That is, just as teachers need to making affordances and constraints of technologies explicit to pupils, teacher educators need to make affordances and constraints of technologies explicit or ‘visible’ to teachers. In addition, teacher educators need to support teachers in identifying teaching strategies that exploit the contrasts and complementarities of different technologies, such as dynamic digital technologies and static paper-and-pencil environments, so that the technology becomes ‘invisible’ and just another resource for teaching mathematics.

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How do pre-service teachers evaluate dynamic worksheets for learning functional relationships?

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The appropriate choice of media is a key task in lesson planning. In the past years, there have been several studies which indicated potential in the use of digital media in the classroom. Nevertheless, these technologies are rarely used in everyday teaching. While one initially thought that this was due to the lack of infrastructure, it seems to be more due to a lack of skills. For this reason, the present study investigates what skills teachers need in order to teach functional relationships with dynamic worksheets, what skills pre-service teachers already possess and how these can be fostered.

Keywords: Dynamic worksheet, (technological) pedagogical content knowledge, pre-service teacher training, functional relationships, qualitative research

Theoretical background

Lesson planning is a daily and important part of teachers' work. Here, the choice of appropriate media is a key task, especially when the aim is on achieving mathematical learning goals. From a theoretical point of view, dynamic worksheets have high potential for developing functional thinking as seen with Lichti & Roth (2018). A dynamic worksheet consists of an applet (dynamic figure constructed with GeoGebra which is often embedded in a web page) and accompanying tasks and/or explanations (Hohenwarter & Preiner, 2008, p. 318).

The ability of evaluating dynamic worksheets for learning functional relationships

With the TPACK model, Mishra and Koehler (2006) have created a framework for what teachers need to know in order to use technology in a meaningful way. In order to adapt this framework to the ability of evaluating dynamic worksheets for learning functional relationships, the special features of developing functional thinking were brought into connection with the features of dynamic mathematics software like GeoGebra. In addition, e-learning principles according to Mayer (2009) were considered, as the extraneous cognitive load in dynamic worksheets "should be small in order to foster more effective learning of mathematical concepts" (Hohenwarter & Preiner, 2008, p. 314). As a result, five main aspects are identified: Learning goals, representations, interactivity, tasks and multimedia principles. These five main aspects and their sub-aspects have been validated by experts (N=14) for adequacy and completeness.

Research questions

The study aims to describe how pre-service teachers engage in evaluating dynamic worksheets and enhance pre-service teachers' ability of evaluating dynamic worksheets. Resulting research questions are:

1. How do pre-service teachers engage in the evaluation of dynamic worksheets?
2. How does training influence the way pre-service teachers engage in the evaluation?

3. To what extend can the ability of evaluating dynamic worksheet be promoted?

Method

Data collection runs in two parts. During the entire duration, the students' screen is recorded. In the first part, the students are encouraged to think aloud, as this method enables the reconstruction of thought processes (Wallach & Wolf, 2001) and thus resembles a simulated lesson preparation.

Specifically, the students are asked to evaluate a dynamic worksheet in relation to a given learning goal for its use in the classroom. In the second part, a pre-structured guide is used to give students a reflection scheme as a scaffold. In this way, differences within a data collection can be investigated and the data collection itself can serve as a learning opportunity. Data will be collected at three different times during a teaching-and-learning-lab course in which students are trained to evaluate and develop dynamic worksheets.

Data will be analyzed with qualitative content analysis (Kuckartz, 2018), with the aim of a potential type-building of how students proceed when evaluating dynamic worksheets. In addition to the interview and think aloud data, students' topic-specific PCK and CK, previously GeoGebra experience and teaching experience are collected as secondary features.

Expected Results

A pilot study indicated that students tend to address only superficial features of applets, especially in earlier stages of the course. Deepener argumentations and suggestions for improvement were more frequently found in later stages of the course. Furthermore, it seems that in later stages of the course students find relations between different aspects.

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Prospective teachers designing tasks for dynamic geometry environments

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The paper examines the quality of digitized tasks designed by 10 (small) groups of prospective upper secondary school teachers as part of a geometry course assignment. The results indicate that a small instructional intervention, addressing the planning and implementation of tasks in digitized task environments as well as how to stimulate students to make mathematical generalizations, led to a relatively high proportion (8 out of 10) of high-quality tasks designed by the prospective teachers.

Keywords: Task design, dynamic geometry environment, prospective mathematics teacher.

Background

Dynamic Geometry Environments (DGEs) have been used as educational tools for several decades (Sinclair et al., 2016). Mainly, it is the dragging function that is regarded as the defining feature of a DGE. By dragging points linked to geometrical objects, students can interact with these objects to search for regularities and invariances and to generate conjectures (e.g. Leung, 2011).

However, to utilize the potentials provided by DGEs, there is a need for carefully designed tasks. Indeed, designing DGE tasks or even evaluating existing tasks is not easy for teachers (Trocki & Hollebrands, 2018). To address this issue, researchers suggest models or principles for designing tasks that take advantage of DGEs as tools for exploration that might lead to conjectures, explanations and proofs (e.g. Fahlgren & Brunström, 2014; Leung, 2011). For example, Leung (2011) suggests a task design model composed of three epistemic modes that resemble different phases of the proving process: exploration, re-construction and explanation. These three modes are sequentially nested in the sense that one mode is a precursor for the next mode, which in turn, is a cognitive extension of the previous one. In this way, "…this task design model can be seen as a vehicle to carry the acquisition of mathematics knowledge." (Leung, 2011, p. 328).

In recent years, there has been an increasing interest in how to support teachers in their process of designing DGE tasks (e.g. Komatsu & Jones, 2019; Trocki & Hollebrands, 2018). While the study by Komatsu and Jones concerns specific task design principles to engage students in heuristic refutation, Trocki and Hollebrands provide a more generic framework with the intention to serve as guidance for teachers "…both for identifying and for writing high-quality tasks for DGSs [i.e. DGEs]." (p. 111). This framework, entitled the Dynamic Geometry Task Analysis Framework, is inspired by Smith and Stein's classification of tasks based on the level of cognitive demand that they require (Smith & Stein, 1998) as well as theories linked to various technological action linked to DGEs. Although Trocki and Hollebrands demonstrate the effect of the framework on teacher knowledge for recognizing and designing DGE tasks, they argue that this is only the beginning because there is a need for more research on investigating the usefulness of the framework (Trocki & Hollebrands, 2018).

Bozkurt and Koyunkaya (2020) address this request by investigating how prospective mathematics teachers (PMTs) developed their task design skills in DGE during a period of 14 weeks. Their study involved three cycles: (a) seminars on task design, followed by design of DGE tasks, (b) implementation of peer micro-teaching, followed by task revision, and (c) implementation in classrooms. Besides using Trocki and Hollebrands' (2018) framework as instructional material to develop the PMTs' skills in designing DGE tasks, the framework was used as a research tool to analyse task prompts as well as the questions posed and responses made by the PMTs during their teaching practices (Bozkurt & Koyunkaya, 2020). The micro-teaching cycles revealed that the PMTs were unable to reach neither the mathematical depth nor the technological actions that they planned for. However, Bozkurt and Koyunkaya found an improved development in PMTs' classroom practices after the micro-teaching. Having participated in each other's micro-teaching lessons, including follow-up discussions, the PMTs revised and developed their DGE tasks. Accordingly, Bozkurt and Koyunkaya suggest micro-teaching as an important component in teacher education courses aiming to develop PMTs' technology integration skills (2020). Moreover, they confirm the usefulness of the framework by Trocki and Hollebrands, both as instructional material and as a research tool.

In a similar study, Gulkilik (2020) examined DGE tasks designed by PMTs. The focus of this study was to examine in detail how PMTs' DGE tasks supported students' "…acquisition of mathematics knowledge"…(p. 2). To enable this, Gulkilik used Leung's (2011) model for task design. The PMTs were introduced to Leung's model and asked to analyse sample DGE tasks to examine their potential of engaging students in activities such as exploration, re-construction, and explanation, which relate to the three epistemic modes in Leung's model. In line with Bozkurt and Koyunkaya (2020), the PMTs implemented their designed tasks in micro-teaching with peers acting as students. To analyse the PMTs' tasks, Gulkilik developed a coding manual with descriptors related to each of the three epistemic modes as well as the transition between them, which enabled "…a continuous description of how PMTs guided students to mathematical understanding in DGE tasks." (2020, p. 13). One prominent finding was that the focus of the PMT tasks was on the construction of geometrical objects, i.e. without using pre-constructed sketches. Instead, the tasks included step by step instructions to build robust constructions, i.e. constructions were the properties are perceived under dragging. In this way, Gulkilik argues, the focus of the PMTs' tasks were limited to observe and explain invariants in the first constructed object, and "…did not utilize the potential of DGE to engage students in terms of exploration, re-construction, predicting, conjecturing, or proving…" (2020, p.13), i.e. activities for knowledge acquisition in DGE according to Leung's model (2011). Overall, the literature highlights the need for more research on task design within DGE to utilize the potential of the technology to reach a deeper mathematical understanding (Sinclair et al., 2016).

Dynamic Geometry Task Analysis Framework

Trocki and Hollebrands' (2018) framework consists of two components: mathematical depth and technological affordances (see Table 1). Central in the framework are the prompts, i.e. questions or directions that require written (or oral) responses and/or technological actions. Besides the prompts, a DGE task most often includes a pre-constructed or partially constructed sketch of a geometrical object (Trocki & Hollebrands, 2018).

Table 1: Dynamic Geometry Task Analysis Framework (Trocki & Hollebrands, 2018, p. 123)

Allowance for Mathematical Depth		Types of Technological Action	
Levels	**Descriptions**	**Afford -ances**	**Descriptions**
N/A	Prompt requires a technology task with no focus on mathematics.	N/A	Prompt requires no drawing, construction, measurement, or manipulation of current sketch.
0	Prompt refers to a sketch that does not have mathematical fidelity.	A	Prompt requires drawing within current sketch.
1	Prompt requires student to recall a math fact, rule, formula, or definition.	B	Prompt requires measurement within current sketch.
2	Prompt requires student to report information from the sketch. The student is not expected to provide an explanation.	C	Prompt requires construction within current sketch.
3	Prompt requires student to consider the mathematical concepts, processes, or relationships in the current sketch.	D	Prompt requires dragging or use of other dynamic aspects of the sketch.
4	Prompt requires student to explain the mathematical concepts, processes, or relationships in the current sketch.	E	Prompt requires a manipulation of the sketch that allows for recognition of emergent invariant relationship(s) or pattern(s) among or within geometrical object(s).
5	Prompt requires student to go beyond the current construction and generalize mathematical concepts, processes, or relationships.	F	Prompt requires manipulation of the sketch that may surprise one exploring the relationships represented or cause one to refine thinking based on themes within the surprise (adapted from Sinclair (2003, p. 312).

While the levels (0 to 5) of mathematical depth that a prompt allows reflect the progression of cognitive demand, the descriptors for technological actions (A to F) reflect the potential of using a DGE. It is the degree of coordination of mathematical depth and technological actions that indicate the quality (low, medium, high) of a DGE task (Trocki & Hollebrands, 2018).

Trocki (2015) reports findings from case studies involving three in-service teachers and three prospective teachers. Mainly, the research setup involved three parts: (a) design of a DGE task (with a given learning goal), (b) a tutorial session introducing the framework, and (c) redesign of the DGE task. By using the framework, Trocki found that all tasks redesigned by the participants increased their ranking to the highest level. Among the initial versions of the task, all (except for one) ranked medium in quality. Based on these findings, Trocki and Hollebrands suggest that the framework can serve as a useful tool in teacher education programs (Trocki & Hollebrands, 2018).

We had occasion to investigate this issue as part of a geometry course for prospective secondary and upper-secondary mathematics teachers. In contrast to the studies by Bozkurt and Koyunkaya (2020) and Gulkilik (2020), we only had a very limited amount of time at our disposal; one and a half week (over 6 weeks). Accordingly, we designed a small intervention, where the PMTs were asked to design DGE tasks as part of the course assignment. This paper aims to gain insight into what impact a small intervention might have on PMTs' abilities to design DGE tasks by exploring the following research question: *What quality of DGE tasks designed by PMTs can we expect of the instructional intervention?* As Bozkurt and Koyunkay's (2020) study, this paper is also guided by Trocki and Hollebrands' (2018) framework.

Method

The study was conducted in spring 2020 in the context of a geometry course for PMTs in secondary and upper secondary school (ages 14–18) in Sweden. In total, 24 PMTs were enrolled in the course. Although the main aim of the course was to develop PMTs' content knowledge, in this case, both Classical Euclidean and Non-Euclidean geometries, there were some seminars on geometry teaching embedded in the course. Particularly, these seminars intended to develop PMTs' skills in planning and implementation of tasks in digitized task environments (such as DGE), as well as their abilities to stimulate each students' learning in the ordinary classroom, including those students who easily reach the knowledge requirements. The instructional intervention addressed both these issues by offering a seminar that introduced geometrical tasks whose numerical solutions could be developed into general results on giftedness and one homework with a follow-up seminar on task design in DGE. After the intervention, as part of the course assignment, the PMTs were asked to design (in pairs or small groups) DGE tasks for (upper) secondary school for all students at different levels of knowledge. Preliminary versions of the tasks were trialled by peers, who provided both oral (at a small seminar) and written responses. The PMTs were then expected to revise and provide a final version of the DGE tasks. Although the participating PMTs were familiar with dynamic mathematics environments as learners of mathematics, the role as task designers were new to them.

The intervention

The mentioned seminar, based on a systematic review (Szabo, 2017) was performed by the second author of this paper, and highlighted the importance of designing tasks that offer opportunities for students to reach general solutions, thereby addressing students performing at higher qualitative levels. To achieve this, we suggested using DGE tasks, in which the participants were encouraged to explore mathematical relationships, to make and verify conjectures, to generalize (if possible) and eventually to construct a proof (Fahlgren & Brunström, 2014). To introduce the ideas behind Trocki and Hollebrands' (2018) framework, the PMTs were encouraged to perform a homework as a preparation for a follow-up seminar. As a basis for the homework, we used three versions of a sample task, provided by Trocki and Hollebrands, to demonstrate various levels of DGE task qualities. These tasks address "…the same two learning goals: 1) justify that opposite angles of parallelograms are congruent; 2) justify that the diagonals of parallelograms bisect each other." (p.127). Each of the tasks consists of a combination of a sketch of a parallelogram and some associated prompts for students to achieve the learning goals. The homework included a brief introduction to the task, of which three versions, A, B and C were provided, and the two learning goals (as described above), followed by some prompts (see Figure 1). At the follow-up seminar, the PMTs discussed the homework in small groups before a whole-class discussion. The focus of these discussions was on to what extent the three versions of the task took advantage of the DGE. For example, which of the versions (A, B, and C), if any, encourage students to explore and discover mathematical relationships.

(a)	Start by constructing a parallelogram in GeoGebra. Make sure that your construction is robust, i.e. that the properties of the parallelogram are perceived even when one of its vertices is dragged.
(b)	Perform the three versions (A, B and C) of the task. Reflect on possible constraints and opportunities that each version entails for a student to achieve the learning objectives.
(c)	Reflect on the quality of the different versions of the tasks by considering the following questions: • How is the potential of the DGE utilized? • What is it that makes one task of higher quality than another?
(d)	How can the task be adapted for students who easily reaches the knowledge requirements? Give suggestions.

Figure 1: The homework prompts

Data collection and analysis

The unit of analysis was the DGE tasks (both the preliminary and the final version) designed by 10 groups (A to J) of PMTs, and the written responses from peers. Each task included a number of prompts for potential students. Some tasks also included pre-constructed (manipulable) sketches. For each task, all prompts were coded with Trocki and Hollebrands' (2018) framework. The coding process was done independently by two of the authors of this paper and then comparisons were made followed by discussions (between all authors) until full agreement was reached. Although the framework was straightforward to use, some subtleties emerged, which are also recognized by Trocki and Hollebrands (2018). First, the distinction between the mathematical depth codes 4 and 5. According to Trocki and Hollebrands, the

> [c]hoice of the word *explain*, as opposed to *justify* or *prove*, was deliberate, in that it serves expose the student to the need *for* explanation as opposed to a particular type *of* explanation (e.g. deductive proof). The code is also based on research that emphasizes a need for students to explain what they notice when using a DGS. (p. 124)

Another subtlety concerns the technological action codes E and F. To sort this out, we needed reexamine Trocki´s original work (2015). A prompt is considered a code E when it "…requires manipulation and directs the student on what to notice." (p.173), while a code F was used if the manipulation is based on a student conjecture, i.e. "…not on a preconceived conclusion on behalf of the task writer."(p.174) .

When all prompts related to a task have been coded, they are assessed holistically to define the quality of the task ackording to the three levels described by Trocki and Hollebrands:

> **Low:** The task *does not* contain a collection of prompts that co-ordinate mathematical depth and technological actions in such a way as to *require* the student to make generalized conclusions based on emergent invariant relationships that go beyond a static sketch.
>
> **Medium:** The task contains a collection of prompts that co-ordinate mathematical depth and technological actions in such a way that *may encourage but does not necessitate* that the student make generalized conclusions based on emergent invariant relationships that go beyond a static sketch. (2018, p.126)
>
> **High:** The task contains a collection of prompts that co-ordinate mathematical depth and technological actions in such a way that *requires* the student to make generalized conclusions based on emergent invariant relationships that go beyond a static sketch. (2018, p.125)

For each of the 10 DGE tasks, all associated prompts were coded by indicating the level(s) of mathematical depth and the type(s) of technological action (see Table 1). This coding generated 10 individual summary tables, which formed the basis for ranking the task quality. To illustrate the coding process, we use one of the tasks, designed by Group D (see Figure 2).

Prompt	**Code (within brackets) and Explanation**
1. Create an arbitrary quadrilateral (convex) with the tool "Polygon". Remove the labels on the sides of the quadrilateral. Then mark the midpoints on each side of the arbitrary quadrilateral. Use the "Polygon" tool to construct the inscribed quadrilateral.	**(1,A,C)** To "create an arbitrary quadrilateral" (coded A) and to "mark the midpoints" (coded C) in Prompt 1, the students need to recall the definition of a (convex) quadrilateral (coded 1).
2. Formulate a hypothesis for the type of geometric figure that is created when the midpoints are connected. Write down your hypothesis on paper. Also, try to drag the corners of the original quadrilateral, before formulating your hypothesis.	**(2,3,D)** Students are asked to drag (coded D) the figure created to formulate a hypothesis (coded 3) about the type of geometric figure it represents (coded 2).
3.After formulating your hypothesis, read the length of the sides and measure the angles of the inscribed quadrilateral. Also, try to drag the corners of the original quadrilateral to see any relationships. Does this result agree with your hypothesis? What type of geometric figure did you get and what characterizes one? If your result is incorrect, justify why and state what assumptions you made that were incorrect and what should have been your correct conclusion. Write down all conclusions on paper.	**(3,4,5,B,D,E,F)** The codes of technological action emerged due to the requirement of measuring the angles of the inscribed quadrilateral (coded B), and then to drag (coded D) the corners to obtain multiple examples (coded E) from which one can generalize to "…see any relationships" (coded F). Concerning the mathematical depth, the students are encouraged to consider relationships in the current sketch (coded 3), and to justify (coded 4) the hypothesis from Prompt 2. Since the prompt requires the student to go beyond the current construction and generalize the mathematical relationships, it receives a code 5.
4. Formulate a hypothesis about the relationship between the area of the inscribed quadrilateral and the area of the original quadrilateral. Write down your hypothesis on paper. Also, try to drag the corners of the original quadrilateral to try to see connections before formulating the hypothesis.	**(2,3,D)** Coded in the same way as Prompt 2.
5. Measure the area of the inscribed quadrilateral and the original quadrilateral. Drag the corners of the original quadrilateral to discover interesting relationships. Does the result agree with the hypothesis that you formulated in point 5? Write down your conclusions on paper.	**(3,5,B,D,E,F)** Coded in the same way as Prompt 3, except for code 4. In contrast to Prompt 3, Prompt 5 does not require the student to justify the conclusion (code 4).

Figure 2: Analysis of one of the DGE tasks (Group D) designed by the PMTs

Since the task includes prompts that co-ordinate mathematical and technological actions in ways that requires students to draw generalized conclusions (code 5) based on emergent invariant relationships that go beyond a static sketch, we ranked the quality of the task as 'high'. During the quality ranking process, we compared and contrasted our interpretations with those made by Trocki (2015).

Results and discussion

Table 2 shows our task ranking of the DGE tasks designed by the 10 groups (A-J). Notably, there was no difference in terms of task ranking between the preliminary and final versions of the DGE tasks. The reason for this might be that the written feedback provided by peers foremost concerned clarification of the DGE tool instructions and/or formulations of questions. We also (in Table 2) indicate whether the tasks provide students with manipulable pre-constructed sketches or step-by-step guidance for constructing geometrical figures.

Table 2: Overview of the results

Group	A	B	C	D	E	F	G	H	I	J
Task rankning	High	High	High	High	High	High	Medium	High	High	Medium
Pre-constructed sketch?	No	Yes	No	No	Yes	Yes	No	No	No	No

As seen in Table 2, 8 (out of 10) tasks ranked high, which indicates that the instructional intervention worked well. However, these results should be interpreted with caution. Besides the subtleties concerning the coding of prompts indicated by Trocki and Hollebrands (2018), the subjective nature of the task ranking method must be taken into consideration. In contrast to Trocki's (2015) study, the tasks designed in this study aimed to address different learning goals, which made the holistic analyses of the prompts associated with a specific task challenging due to fewer comparison opportunities between the tasks. So, for example, to distinguish between 'may encourage but does not necessitate' (medium') and 'requires' (high), was not straightforward. Consequently, we argue that this perspective may affect the validity of our study. Therefore, the relatively high proportion of high-quality tasks designed by the PMTs can be questioned. Still, the tasks ranked as high quality according to the definition (Trocki & Hollebrands, 2018), indeed include prompts that coordinate mathematical depth and technological actions. For example, in this study, the tasks ranked 'high' all offered opportunities for students to reach a generalization beyond the DGE sketch (Code 5) by directions for technological actions such as dragging to recognize invariants in the sketch (Code D and E). A possible explanation for the relatively high proportion of tasks including generalization-making prompts might be the first seminar in the intervention. This seminar highlighted, among other things, the importance of encouraging mathematical generalizations as a way to challenge high-achieving students. Nevertheless, reconsidering the comparatively small size of the instructional intervention, we argue that it was successful in that most of the PMTs designed DGE tasks were ranked at high quality, at least according to Trocki and Hollebrands' (2018) framework.

Moreover, Table 2 shows that several groups did not provide pre-constructed sketches in their tasks. This is in accordance with Gulkilik's (2020) finding that PMTs' tasks provided students instructions to make (robust) constructions on their own. There are several possible explanations for this result. In previous courses, the participating PMTs experienced, as learners, tasks designed for dynamic environments (although not DGE) that offer construction guidance rather than providing pre-constructed sketches. Further, the homework (see Figure 1) prompted the PMTs to make a robust construction before examining the sample tasks, which might have influenced their task design. Moreover, in contrast to previous studies utilizing Trocki and Hollebrands' framework (e.g. Bozkurt & Koyunkaya, 2020; Trocki, 2015), the PMTs in this study were not introduced to the framework itself. Instead, they were asked to examine the quality of three sample tasks, as the PMTs in Gulkilik's (2020) study. Since Trocki and Hollebrands' framework is strongly influenced by the work of Sinclair (2003), who provides guidance for designing tasks utilizing pre-constructed DGE sketches, the introduction of the framework to the PMTs might affect their choice of providing pre-constructed sketches, which was the case in Bozkurt and Koyunkaya's (2020) study.

To sum up, despite its limitations (e.g. no data was collected during the intervention), this study adds some different perspectives to the emergent research field of DGE task design (Sinclair et al., 2016), particularly in teacher education programs (Trocki & Hollebrands, 2018). This study confirms the usefulness of Trocki and Hollebrands' (2018) framework as instructional material, although not necessarily by presenting the framework itself but by asking teachers to evaluate the quality of sample tasks. As a suggestion for further research, we propose deepening this study by analysing all steps of the intervention, not only its outcome (in this case the designed DGE task and associated written responses from PMTs). We also suggest comparing the usefulness of this framework with the suggested operationalization of Leung's model by Gulkilik (2020) to analyse the educational potential provided by DGE tasks.

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Investigating Chinese mathematics teachers' use of digital resources before, during and after mathematics lessons

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Abstract: This paper aims to investigate mathematics teachers' use of digital resources (DRs) before, during and after mathematics lessons. The data were collected from a questionnaire administered to 146 secondary mathematics teachers from 11 schools in China and follow-up interviews with 19 of them. The results revealed that the majority of teachers often or always use DRs in their teaching, especially during lessons. In particular, teachers frequently searched online for rich exercise problems before lessons, displayed PowerPoint slides during lessons, and used online test makers after lessons. There still existed a digital divide between secondary schools in rural and urban areas of China in teachers' use of DRs. Teaching experience had a significant impact on teachers' use of DRs before lessons.

Keywords: Chinese mathematics education, digital resources, mathematics teachers, ICT.

Introduction

With the fast development of information and communication technology (ICT), growing attention has been paid to promoting educational modernization in mathematics education. Given the growing availability and variety of ICT, in particular, digital resources for mathematics teaching and learning, along with relatively little research available on teachers' use of digital resources, there is a clear need for a greater understanding of how digital resources impact mathematics teaching and learning, especially from teachers' perspectives and practice (Pepin et al., 2017; Remillard et al., 2021). Drawing mainly on Remillard et al.'s (2021) work, we use the term *digital resources* (DRs) to refer to a broad array of digital applications, tools, media, systems and platforms that provide teachers with digital textbooks, exercises, Dynamic Geometry Software (DGS) files, videos and other resources to empower mathematics teaching before, during and after lessons.

In this study, we aim to examine Chinese mathematics teachers' use of various DRs and the factors that influence their use. The research questions are: (1) How do mathematics teachers from secondary schools in China use digital resources in their mathematics teaching? (2) Are there differences in the use of various digital resources by different groups of mathematics teachers?

Literature Review and Conceptual Framework

Prior research showed that teachers' use of DRs varied in different educational contexts. For example, Ibieta et al. (2017) surveyed 6932 secondary teachers in Chile and found that 91% of them always used DRs for "search and preparation of resources." A study with 514 Chinese teachers conducted by Shi and Li (2015) reported that only 0.8% of the participating teachers never used computers in class. In addition, researchers have examined mathematics teachers' usage of particular DRs, some

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of which were content-based (Vermeulen et al., 2017), function-based (Engelbrecht et al., 2020) and DR infrastructure, e.g., devices and Internet (Trouche & Drijvers, 2010).

In terms of the process of teachers' teaching, earlier studies have revealed that mathematics teachers use DRs before, during and after lessons with different instructional purposes. Before lessons, DRs play a crucial role in teachers' lesson preparation. Teachers nowadays have access to various resources for lesson planning, such as videos (Oechsler & Borba, 2020) and DGS files (Bozkurt & Uygan, 2020). Forgasz (2003) found that 40 of 96 Australian secondary mathematics teachers used DRs (mathematics software) for lesson planning. In China, Wang et al. (2019) reported that teachers in rural schools commonly used DRs for lesson planning. During lessons, teachers utilize different functions of DRs to present instructional materials. Devices used by teachers are not limited to computers, Interactive Whiteboards (IWB), projectors and tablets (Trouche & Drijvers, 2010). Bretscher (2013) reported that 85% of 188 English secondary mathematics teachers used IWB and 63% of them used digital projectors in almost every lesson. In contrast, a similar study with Spanish mathematics teachers revealed less frequent use of IWB (Gómez-García et al., 2020). After lessons, teachers mainly use DRs for assessment, e.g., automatic assessment (Joubert, 2013). However, the use of DRs outside the classroom, while more frequent, has been less studied (Ibieta et al., 2017).

In Chinese educational contexts, researchers have reported that there has been little empirical work about the types of DRs available for use in education and which DRs are used by teachers in China, especially those in rural areas (Wang et al., 2019). In a large-scale survey of teachers in 2168 schools in China, Lu et al. (2015) pointed out that there were significant differences in uses of multimedia courseware between teachers from urban and rural areas. These studies revealed that some digital divides existed between urban and rural schools in China. However, researchers' knowledge about mathematics teachers' use of DRs remains vague, particularly about the difference between rural and urban schools. In this regard, we hope that the current study, with a more holistic view of mathematics teachers' use of DRs in their mathematics teaching, can contribute to filling the gap and deepen our understanding of what and how DRs are actually used and what are the associated influencing factors.

Table 1: A conceptual framework on teachers' DR usage before, during and after lessons

Dimension	Before lesson	During lesson	After lesson
Content	PowerPoint slide, micro-lesson video, digital teacher manual, etc.	N/A	Problems
Function	Individual lesson planning, collective lesson planning, etc.	Displaying, screenshotting, drawing, virtual teaching aid, etc.	Assigning homework, teaching reflection, etc.
Infrastructure	Devices: computer, tablet, smartphone.	Devices: computer, tablet, IWB, etc.; Internet: online, offline, mixed	Devices: computer, tablet, smartphone.

In order to obtain a relatively comprehensive picture of teachers' use of DRs, we conducted a survey of the literature concerning DRs, e.g., digital textbook, DGS file, flash file, short video, online test maker (OTM), automatic grading, and devices like computer, tablet, IWB, smartphone, in education and more particularly, mathematics education. We also identified other related DRs, e.g., micro-lesson video, digital teacher manual, screenshot, digital handwriting note and online assessment, through our experience and observations. Partly based on a multi-layer model for analyzing "eSchoolbag," a personal learning tablet containing DRs for teaching and learning school subjects (Zhu & Yu, 2011), we categorized DRs in three dimensions: content-based, function-based and DR infrastructure, and investigated specific DR usage in the process of teachers' mathematics teaching, that is, before, during and after lessons. Table 1 lists examples of DRs in the conceptual framework that we established for this study.

Methods

This study adopted a mixed-methods approach. Following the conceptual framework, we designed a questionnaire to collect the data about teachers' uses of DRs. The first part captured the profile of the participants, and the following three parts questioned the use of DRs before, during and after lessons, respectively, with items corresponding to the DRs mentioned in Table 1. Each item was measured by a 4-point Likert scale, with 4 for "always; more than 4 lessons per week", 3 for "usually; 3 to 4 lessons per week", 2 for "sometimes; 1 to 2 lessons per week" and 1 for "never". Open-ended questions were used to collect more information about the most frequently used DRs, the difficulties encountered when using DRs and the specific functions of DRs that they were most satisfied or unsatisfied with.

Additionally, interviews captured the reasons or examples for teachers' use of specific DRs. Questions were asked such as: "For DRs use before lessons, you rated 4 for item 1, while 3 or 2 for the rest. Are there any reasons why you use this more frequently? How do you usually use it?" We also conducted a pilot test (n= 8 teachers) to refine the instruments and ensure reliability and validity.

Table 2: Profile of participating teachers (*n*=146)

Characteristic	School Location	Gender	Educational Background	Title[1]	Teaching Experience (Year)[2]
Participants	Urban: 105 (71.9%); Rural: 41 (28.1%)	M: 56 (38.4%); F: 90 (61.6%)	Bachelor: 116 (79.5%); Master & Doctor: 30 (20.6%)	Junior: 59 (42.1%); Intermediate: 57 (40.7%); Senior: 24 (17.1%)	(0,5][3]: 43 (29.7%); (5, 20][4]: 58 (40.0%); >20: 44 (30.3%)

Note: All percentages are based on valid responses. [1]*n*=140. [2]*n*=145.[3]5 years or less. [4]More than 5 years and less than 20 years. M=Male; F=Female.

Overall, teachers from 11 secondary schools, 7 urban and 4 rural, in 6 cities of China: Shanghai, Hangzhou, Nanjing, Jinan, Yiyang and Changde, participated in this study. We received 146 questionnaires back (response rate: 94.1%; see Table 2 for the profile of participants) in June, 2021 and conducted interviews with 19 teachers (referred to T1 to T19) from June to August, 2021.

Findings

Use of DRs before lessons, during lessons, and after lessons

Table 3 shows how often teachers used various DRs for lesson planning. 65.7% of 102 participants always or often used DRs before lessons (*M*=2.82). Digital exercises (including problems from national/regional exams) received the highest rating (*M*=3.25), and 80.4% of them always or often used exercises. This result was not surprising since rich problems could be downloaded online by teachers. For example, T17 mentioned that when she prepares lessons, she first downloads exercises online and then designs teaching activities accordingly. In comparison, flash files received the lowest rating (*M*=2.08). Only 4 teachers always used flash files, and 24 teachers never used flash files. During the interview, T11 reported that it was difficult to obtain suitable flash files. Regarding functions, teachers used individual lesson planning (*M*=2.89) most frequently, followed by collective lesson planning (*M*=2.75) and analysis of students' prior learning (*M*=2.51). T10 and T12 explained that individual lesson planning helped them build their own lesson plan database, which was beneficial to their long-term professional growth. Computers (*M*=3.37) were the most frequently used devices before lessons, followed by smartphones (*M*=2.36) and tablets (*M*=2.14).

Table 3: Frequency of using DRs before lessons and functions used during lessons

Rank	Content before lessons	Mean	SD	Function during lessons	Mean	SD
1	Digital exercise (*n*=112)	3.25	0.77	Displaying* (*n*=105)	3.39	0.74
2	Digital textbook (*n*=107)	3.08	1.07	Touching, writing and erasing (*n*=103)	3.20	0.87
3	Digital teacher manual (*n*=109)	3.06	1.00	Drawing (*n*=103)	2.87	0.87
4	PowerPoint slide (*n*=111)	2.87	0.82	Supporting interactions (*n*=99)	2.67	0.97
5	DGS file (*n*=110)	2.35	0.81	Saving handwriting notes (*n*=101)	2.50	0.98
6	Micro-lesson & short video (*n*=110)	2.32	0.77	Screenshotting (*n*=97)	2.43	0.90
7	Picture and GIF (*n*=110)	2.17	0.76	Virtual teaching aids (*n*=99)	2.34	0.93
8	Flash file (*n*=104)	2.08	0.78			

Note: *The "displaying" function combines five items, including displaying PowerPoint slides, DGS files, or using an IWB, projector or students' tablets/computers to display teaching content.

During lessons, 78.2% of 96 participants always or often used DRs during lessons (*M*=3.14). 53.3% and 44.7% of the teachers always used displaying contents (*M*=3.39) and touching, writing and erasing (*M*=3.20), respectively. In contrast, only 12.1% of them always used virtual teaching aids (*M*=2.34). During the interview, 12 teachers indicated that displaying DGS files was a good way to teach geometry and functions. Some teachers said that they were not quite proficient to use many functions of DGS. In terms of devices, teachers used computers (*M*=3.16) and IWBs (*M*=2.93) more often than smartphones (*M*=2.04). Challenges like (1) pop-ups and advertisements, (2) switching displays, (3) equipment and network error and (4) deviation issues of devices received similar ratings, between 2.15 to 2.41. For the internet connection, teachers used offline (*M*=2.43), mixed (*M*=2.38), online (*M*=2.34), in a similar frequency. 19 teachers mentioned in the open-ended questions that the instability of the Internet caused great inconvenience.

The results revealed that 55.1% of 78 participants always or often used DRs after lessons (*M*=2.56). It is highly related to recent policies as teachers were not allowed to assign digital homework after class (Ministry of Education, 2021), as mentioned by five teachers during the interview.

Regarding various functions, teachers used online test maker (OTM, e.g., ProProfs Quiz Maker) most frequently (M=2.48). T5 said she would download exam papers and adapt them for exercises; T10 said that OTM allowed him to organize and produce high-quality exam papers. In contrast, the automatic grading (M=1.82) was rarely used, probably due to the low accuracy (stated by three interviewees) and limited types of problems (mainly multiple-choice), as T2 said: “It was not meaningful to me as students can check the answers of multiple-choice questions on their own.” Furthermore, OTM is rather well-developed, while other functions were not user-friendly and helpful to teachers. For content, online problems from previous High School Entrance Examinations (HSEEs) (M=2.64) were used most frequently, while original and self-design problems were used least frequently (M=2.29). As T10 commented, problems from HSEEs were regarded as “well-designed” and “of high quality,” while the quality of original or self-designed problems was doubted. Problems from digital textbooks, teacher manuals, exercise books and resource books were used almost equally frequently (M=2.46~2.53). For problem sorting criteria, knowledge (M=2.90) and chapter and section (M=2.88) were used most frequently. Overall, teachers used tablets (M=1.82) less frequently than computers (M=2.79) and smartphones (M=2.13) after lessons.

Differences in the use of DRs by different teachers

Table 4 lists items with statistically significant differences from the chi-square tests. Before lessons, digital textbooks and exercises were used significantly less frequently by teachers with experienced teachers (more than 20 years) than less experienced teachers. In the interview, T2 pointed out that he didn’t search exercises online as he had accumulated enough exercises over the past 30 years, which was “what makes us (experienced teachers) different from novice teachers.” T13 said that experienced teachers relied on printed textbooks more heavily than novice teachers. It can be inferred that experienced teachers are more familiar with printed resources and reluctant to use digital ones. Besides, female teachers used digital exercises significantly more frequently than male teachers; teachers with graduate degrees used digital exercises significantly more frequently than those with bachelor degrees. Teachers from rural schools used tablets significantly less frequently than their counterparts from urban schools, which may be related to the availability of tablets. T16 from a rural school said in the interview that there was no tablet available in his school. During lessons, female teachers adopted the online mode significantly more frequently than male teachers. Teachers from urban schools used DRs offline in the whole class at a significantly higher frequency than those from rural schools. After lessons, there were statistically significant differences between teachers from urban and rural schools in many aspects, especially in OTM. T14 and T16 from rural schools both mentioned the lack of devices in their students’ homes, implying a relatively low frequency of DR usage. Gender differences were found in manual grading and sorting problems by the level of difficulty, in which female teachers used them more frequently. Teachers with graduate degrees selected problems by the year of exams more frequently than those with bachelor degrees.

Table 4: Items showing statistically significant differences in frequencies of DRs use between teachers

Dimension	Before lessons	During lessons	After lessons
Content		-	Digital textbook: 8.335*(SL); Digital teacher manual: 18.109***(SL)

	Digital textbook: 13.564*(TE); Digital exercise: 6.381*(G), 8.479*(EB), 9.655*(TE)		*Sorting mathematics problems by:* 1. Mathematics knowledge: 15.646**(SL); 2. Level of difficulty: 12.980**(SL), 8.551*(G); 3. Chapter and section: 15.122**(SL); 4. Upload time: 17.274***(SL); 5. Year of exams: 18.831***(SL), 11.054*(EB)
Function	-	-	Online test maker: 33.864***(SL); Manual grading: 10.409*(SL), 9.237*(G); Teaching reflection: 11.683**(SL)
Infrastructure	Tablet: 9.220*(SL)	ON: 11.233*(G); OFF: 11.005*(SL)	Computer: 18.405***(SL)

Note: SL=school location; G=gender; EB=educational background; TE=teaching experiences; ON=online; OFF=offline. *p<.05, **p<.01, ***p<.001.

Conclusions

Overall, more than one-half of mathematics teachers in our study often or always used DRs, especially during lessons. However, teachers' use of DRs is still limited in variety. Mathematics teachers used DRs at a relatively low frequency after lessons, compared to their use of DRs before and during lessons. This result was consistent with the finding reported by Ibieta et al. (2017) that teachers in Chile used DRs more frequently outside the classroom for class preparation before lessons.

Before lessons, digital exercises, individual lesson planning and computers were the most frequently used DRs by mathematics teachers. From the content-based perspective, digital exercises were the most frequently used DRs, which was partly due to the influence of high-stake examinations (see also Leung, 1995). Moreover, the finding that teachers used DRs with complex procedures (e.g., DGS, GIF and videos) at a relatively low frequency was in line with an ecological metaphor in which simpler technologies requiring little adjustment to existing practices are more frequently used (Zhao & Frank, 2003). Thus, teachers' limited proficiency in designing DGS files and other dynamic resources leads to the low frequency of using these types of DRs. Also, it was consistent with the finding that teachers mainly used DRs for presentations during lessons, as well as the findings of Bretscher's (2013) study where presentation-oriented software dominated English mathematics teachers' IWB use. The frequency of teachers' use of a particular function varies, which is affected by their proficiency in that function. Thus, more professional development programs regarding how to use DRs efficiently are needed to support teachers in this aspect. And future research on how DRs are used in mathematics classrooms in various contexts may produce new possibilities of integrating DRs effectively. After lessons, the relatively low frequency of teachers' use of DRs was largely related to policies, as well as limited and helpful functions of DRs. The well-developed and user-friendly exam/test production was used most frequently, possibly due to the high-stake examinations.

Overall, experienced teachers tended to use digital exercises and textbooks less frequently than less experienced teachers before lessons. There were statistically significant differences between urban and rural school teachers in tablet use before lessons, adopting the offline mode during the whole lessons, and in most of the aspects of using DRs after lessons, which indicated some digital divides between urban and rural schools in China. More support in terms of providing devices professional training programs for effective use of DRs are needed, especially for rural schools in China. Meanwhile, the flexibility of professional training programs should be maintained among different regions, as teachers may evolve in different ways in the same training (Prodromou et al., 2018).

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Integrating dynamic mathematics technology in pre-service teacher education: the case of Ghana.

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Keywords: Ghana, Dynamic mathematics technology, Teacher education curriculum, Mathematics teacher education, Professional development.

Background

In 2018, the Ghana Tertiary Education Commission revised the teacher education curriculum to a 4-year Bachelor of Education, which specifically identified the use of dynamic mathematics technology (DMT) as a pedagogical tool in the preservice teacher mathematics curriculum to ensure college graduates are well-equipped to leverage digital technology for the teaching of mathematics and science (UEW, 2018). DMT for the purpose of this research refers to technology offering different linked mathematical representations (geometric shapes, graphs, tables, algebraic expressions) that teachers and pupils can manipulate and by doing so, engage with the underlying mathematical concepts and relationships.

Problem Statement

Research studies have reported the many ways that dynamic mathematics technologies can support deep and lasting learning by offering learners connected multiple representations -graphs, tables of values and equations (Clark-Wilson & Hoyles, 2017) and by providing tangible mathematical objects to explore shape enlargement or angle changes (Dick & Burrill, 2016). However, due to the absence in Ghana of courses on teaching mathematics with technology, coupled with mathematics teacher educators' lack of training, knowledge and confidence, preservice secondary mathematics teachers leave college without the adequate training and exposure to dynamic mathematics technology. Hence, they are mostly unable to use dynamic mathematics technology for teaching mathematics both to improve the reasoning and creative problem-solving skills of secondary school students and to expose secondary students to mathematical opportunities in the digital age.

Research Aim

My research aim is to investigate how to foster the integration of dynamic mathematics technology within secondary mathematics teacher education in Ghana by improving the mathematical pedagogical technological knowledge (MPTK) (Thomas & Hong, 2013) of mathematics teacher educators in the colleges of education.

Research Questions

My research will investigate mathematics teacher educators MPTK within the context of known underuse of Dynamic Mathematics Technology in mathematics classrooms across the forty-six (46) colleges of education. The questions below will guide the study.

- RQ1 How do the current Ghanaian mathematics teacher education curriculum conceive the integration of dynamic mathematics technology?

- RQ2 What are the obstacles to the integration of DMT for Ghanaian mathematics teacher educators?
- RQ3 What are the design features of professional development course for mathematics teacher educators that support them to use DMT in ways that become embedded in their practice and lead to effective teaching?
- RQ4 What is the impact of the professional development course on the mathematics teacher educators' MPTK?

Theoretical Framework

This design-based research study will adapt the *Conversational Framework* pedagogic theory (Laurillard, 2002) to create an online asynchronous professional learning course for mathematics teacher educators to improve their mathematical pedagogical technological knowledge (MPTK). Laurillard's framework argues that teaching is a dialogue and shows what it takes to learn using the ideas of instructionism, social learning, constructionism, and collaborative learning.

Methodology

The study will be an exploratory mixed methods design that would adopt a design-based research approach. Semi-structured interviews will be administered first to the curriculum designers and policy makers followed by a survey questionnaire for mathematics teacher educators. The analysed data from the interviews and questionnaires will inform the design of the intervention - an online asynchronous professional learning course on a particular DMT- the *GeoGebra* application (GeoGebra, 2021) for the teaching of geometry. In the final phase, the impacts of the course on mathematics teacher educators' MPTK will be evidenced.

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Digital competencies of pre-/in-service teachers—an interview study

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Since the role and the availability of digital technology in society is growing, educators need to increasingly more often decide when and what digital technology to integrate into their teaching. Thus, those decision-making competencies need to be developed and measured especially for pre-service mathematics teachers. Therefore, we conducted an explorative interview study to understand the variety of argumentation and justification of pre- and in-service mathematics teachers on whether to use or not use digital technology in the teaching phases of Prediger et al. (2013), and what criteria they use in making those decisions. The analysis shows that (1) based on the arguments and justifications different level of decision-making competencies can be distinguished and (2) the recorded criteria on whether to use or not use digital technology are suitable for reflective practicing of such decisions.

Keywords: Digital competencies, Pre-service teachers, Student teacher evaluation, Technological advancement, Decision making skills.

Introduction

Given the growing role of technology in society and education, as well as the growing number of digital technologies available to teachers (Ally, 2019; Clark-Wilson et al., 2020), it is important that the technology-related competencies of educators constantly evolve and valid instruments to evaluate such competencies are necessary. Educators have to decide increasingly more often when and what technology to integrate into their teaching practice to be effective and meet the demands of society and their learners — not only because of curricular guidelines/standards — but also because the aim to prepare their students for a work environment in the future which will be even more digitalized and driven by technology. In summary, the skill and the knowledge of educators to make appropriate decisions on when and what digital technology to use in teaching is driven by the increased digitalization of society and an increasing number of digital technology available and thus of high importance to educators. Not surprisingly the skill and the knowledge of selecting suitable digital resources have been added to educator competency frameworks like ISTE Standards for Educators (International Society for Technology in Education, 2000) or the DigCompEdu (Redecker, 2017). Even though the DigCompEdu framework lacks references to the educator's orientation towards implementing technology in classrooms, it has been cited as one of the most compressive frameworks in its suitability of educators' need to integrate digital technology (Tabach & Trgalová, 2020) and therefore is used as a basis for our research. DigCompEdu entails twenty-two competencies and one of them is "Selecting digital resources," which is defined as "To identify, assess and select digital resources for teaching and learning. To consider the specific learning objective, context, pedagogical approach, and learner group, when selecting digital resources and planning their use." (Redecker, 2017, p. 20). It should be noted that although increasing the efficiency of an educator because of the use of digital technology is not included in the DigCompEdu definition, it has been cited as a significant factor in the selection process of educators (McCulloch et al. 2018). Given the importance of the competence "Selecting digital resources," it needs to be fostered as part of the educational program of pre-service teachers; to evaluate such processes, the competency needs measure instruments and items which reflect that importance. With that in mind, we aim to develop open-end items for an external assessment and conducted this interview study to better understand the decision processes of pre- and in-service teachers on the use or not use of digital technology in teaching. In the following sections, we first describe the theoretical models, we used in preparation for this interview study. Subsequently, we present the results of the study with pre-/in-service mathematics teachers in Germany.

Theory

In the planning of the interview study, it became apparent that (1) a clear definition of digital technology within education and (2) a model to describe the situational context of teaching settings is required. Thus, we briefly introduce the digital technology definitions given by Clark-Wilson et al. (2020) and the model of teaching phases of Prediger et al. (2013).

Definition of digital technology in education

Clark-Wilson et al. (2020, pp. 1225–1226) group digital technologies by function. The function of the technology inherently determines its use within a teaching situation and are termed as follows.

a) Organizing: "As a support for the organization of the teacher's work (producing worksheets, keeping grades)"

b) Representation: "As support for new ways of doing and representing mathematics"

c) Collaboration: "As a support for connecting, organizing in communities, communicating and sharing materials"

d) Independent: "…a commercial and industry driven function, which supports students' more independent work and focuses on practicing and assessing previously taught mathematical knowledge and skills in a range of online formats."

We use these technology categories since they align with the definition of the competency "Selecting digital resources" and with aspects of the competencies in the area of "Teaching and Learning" of the DigCompEdu framework.

Core teaching phases by Prediger et al. (2013)

Prediger et al. (2013) describe a model of teaching that can be used to design teaching arrangements and to make situational teaching decisions. In the model, four core phases are differentiated*: Connecting (C), Exploring (E), Systemizing (S), and Practicing (P)*. These core phases are defined by their didactic function, cognitive activity, and epistemological quality. Whereby the didactic function designates the perspective of the educator and the cognitive activity encompasses the perspective of the learner. The authors state, that other aspects of teaching e.g., creativity and collaborative exchange can be developed within each phase by providing those stimulations. This aligns with the digital technology categories of *Collaboration* and *Independent* as outlined in the previous section. In the context of this paper, we use the model of Prediger et al. (2013) to evaluate the decision-making processes regarding whether to use or not to use digital technology within each phase. Digital technology can be used in all four teaching phases to either support a didactic function, the cognitive activity or both. (Büchter et al., 2019; Hillmayr et al., 2020; Peschek & Schneider, 2002; Reinhold et al., 2020; Swanson, 2020). While some of the studies are directly attributed to a particular phase, others are applicable to all or multiple phases.

Research questions/evaluating educators' competency of "Selecting digital resources"

To understand the selection and decision process on whether to use or not to use digital technology in teaching lessons of (prospective) teachers, we conducted exploratory interviews with in-service and pre-service teachers to answer two research questions (RQ).

> RQ1: How do pre-service and in-service teacher reason for or against the use of digital technology in the context of a specific learning subject, learner age group, and teaching phase?

Regarding this question, we let the interviewees choose their preferred content and learning group to reason on familiar grounds. All four phases of the model of Prediger et al. (2013) are addressed by the respondents.

> RQ2: What criteria do pre- and in-service teachers use when deciding on the use of digital technology in teaching?

With this question, we specifically inquired what role the age group of the learner and the learning content have in the decision process and on the learner/educator perspective.

Methodology

We have used semi-structured interviews in the explorative study. Next to some demographic questions, the respondents were first asked to explain along with the four phases of teaching on a specific mathematics subject and student group whether to use or not use digital technology. In the second part of the interview, we enquired about the use of digital technology in teaching—not restricted to a particular subject and student group. This two-prong approach was taken to gain an understanding of how participants would reason within a specific setting as well as in more general terms of teaching with digital technology. We interviewed five pre- and five in-service mathematics teachers in Germany. Regarding the former, two interviewees were at the beginning (1st and 2nd semester) and three at the end of their studies (6th, 8th, and 10th semester); they studied with an emphasis on special or on lower secondary education. Regarding the latter, teaching experiences varied between four and thirty years in lower and upper secondary education. The broad diversity of participants was chosen to get insight into the maximum variability of answers and reasoning. All interviews were conducted in German and transcripts were coded using qualitative content analysis (Mayring, 2000).

Qualitative analysis of the interview study

In the following, the results of the coding are presented. Citations are abbreviated and designated as either pre-/in-service teacher indicated by the prefix "Pre-T-"/ "In-T-" followed by a number indicating the semesters of study or the years of teaching as well as the point in time in the interview. Only the English translations of the interview coding by the authors are presented to save space.

Results regarding research question 1

Two different approaches within the first section of the interview were taken by the participants. One participant, a pre-service teacher in the sixth semester, stated that the selection of digital technology cannot be made by the teaching phases and that the decision is rather based on the type of technology. For the analysis, this interview was excluded as although the approach is valid it does not answer the research question at hand. The responses of the remaining participants were used as the basis for answering the imminent research question using a two-layered approach. In the first layer, we differentiated the type of arguments and justifications and in the second layer, we reviewed the types of digital technology groups used within each phase.

For the first layer of analysis, we differentiated argumentations with (i) no argument, (ii) arguments(s) which were not substantiated or overly generalized, and (iii) argument(s) substantiated by either a didactic function, cognitive activity, or own application. First, we provide some examples of the applied coding system and later a summary of the types of argumentations clustered by participant groups. The following response is an example of "no argument," indicating to not use digital technology in the *Connecting* phase because of no knowledge of technology to be useful in this phase.

[06:17] – Pre-T-2: *Connecting. Well. I would say no…At the moment I cannot think of a use for software to capture one's previous experience. I would do that I think rather in a conversation ...*

An instance of a response with an overly generalized argument and justification reads like this.

[09:38] – Pre-T-2: *So first of all, I think in general it makes sense to use it [digital technology], because I just think that it totally focuses the attention of students.*

In addition of being an overly generalizing argument, it could be debated if this line of argumentation is correct, as studies suggest that prolong use of digital technology or media can have adverse effects on learners' cognitive ability and attention span (Lodge & Harrison, 2019). A response with an argument substantiated by a didactic and cognitive function is the following explaining the use of a quiz app [Biparcours] versus paper & pen worksheets in the *Connecting* and *Exploring* phase.

[11:16] – In-T-10: *So now I use a digital tool [Biparcours], which is not explicitly mentioned in the curriculum... I found this methodologically useful here for motivation and to promote self-directed learning. The [students] can work through the questions in the [Bipar] course at their own pace and receive automated feedback and continue…*

The app is used to reactivate the learner's prior knowledge of a related subject and letting them explore the new content. The decision is explicitly supported by the didactic goal in those phases and fostering self-learning. Also the motivational aspect of technology on the cognitive activities of the learners has been mentioned in the reasoning. The response also implies the aspect of automated feedback and leveraging aggregate instead of individual results to assess the outcomes of the phases. This speaks for the aspect of educators' efficiency because of technology. A response using in the argumentation a cognitive perspective is given next.

[07:45] – In-T-4: *I think that when you're exploring and discovering new mathematical facts, the effort is very high... and if you can relieve this high cognitive hurdle that's involved in the discovery, in some form by having the technology take away certain repetitive sequences of actions, like the construction, which is done identically over and over again, we can outsource that…*

Following the outlined coding the responses of the participants were categorized by (1) the four teaching phases, (2) the type of argumentation and substantiation of the arguments and (3) the types of participants. We differentiated three groups, Pre-T-2, Pre-T-8/10, and In-T-x.

The analysis yielded the following results: First, the arguments and the level of justification progresses from "Pre-T-2" who provide only arguments in the *Exploring* phase which were substantiated by either their own experience in using digital technology or by providing examples of the use of digital technology, to Pre-T-8/10 who provided arguments supported by cognitive perspective for the *Connecting* phase and arguments supported by cognitive as well as didactic perspective for the *Exploring* phase. And then to the in-service teachers who provided arguments for all four phases supported by didactic, cognitive aspects or own application of technology and are most comprehensive. Second, regardless of the participant group a justification is given by the respondents for the *Exploring* phase. Thirdly, only the in-service teachers provided substantiated arguments for the *Practice* phase. In summary, based on the coded argumentation and justification pre- and in-service teachers can clearly be distinguished.

The second layer of the coding reviews the technology groups *Organizing*, *Representing*, *Collaboration*, and *Independent* in the context of the teaching phases. The results by the three participants groups Pre-T-2, Pre-T-8/10, and In-T-x are shown in Table 1. Pre-T-2 named only technologies in the *Representation* group (and here also only GeoGebra) but were able to argue for their use in the *Exploring* phase like Pre-T-8/10 and In-T-x. The latter however did not only provide a more comprehensive list of technologies in the group (e.g., GeoGebra, Excel, CAS) they also were able to identify and argue for the use of digital technologies regarding the *Independent* and *Organizing* group in this phase. The * in the table denotes phases in which participants argued not to use digital technology or did not see the value of it. Their level of argumentation was assessed in the first layer of the coding.

Table 1: Digital technology groups by phase and participants group

	Pre-T-2	Pre-T-8/10	In-T-x
Connecting	None*	None*	Independent, Organizing
Exploring	Representation	Representation	Representation, Independent, Organizing
Systemizing	None	Collaboration	Collaboration
Practice	None	None	None*

Peculiar is that the In-T-x see the value of the technology grouping *Independent* in the *Connecting* phase, but do not see the same benefit in the *Practice* phase, in particular they do not see the aspect of using digital technology for assessments. A possible explanation is that the use of digital technology for assessments is not widespread and not necessarily the main didactic objective of the phase. It also becomes apparent in this layer of coding, that the decision of using digital technology in a particular teaching phase is a decision on how a digital technology supports the didactic function and a cognitive activity. Since the didactic function and the cognitive activity in the four phases are discriminative, different technology groups are highlighted for a phase. The aspect of "independent working" supported by the technologies in that grouping, although potentially applicable to all four phases, is more relevant in the *Connecting*, *Exploring* and *Practice* and to a lesser extend relevant in the *Systemizing* phase. The same applies to technology in the *Representation* grouping, which enables to present content in multiple and new forms and thus is more relevant in the *Exploring* phase and not or to a lesser extend relevant in the other phases. *Collaboration* technology, which enables the sharing of information -- educator to learner and learner to learner – was assessed to be more relevant in the *Systemizing* phase, where the didactic goal is to link the learning of the individual to the group. Inductive the hypotheses can be made that the *Representation* technology used in the *Exploring* phase is more dependent on the teaching domain and that the technology in the grouping *Independent, Organizing* and *Collaboration* are neutral to the teaching domain. Based on the two layers of analysis we can summarize (1) the competency, meaning the argumentation and reasoning skills as well as the knowledge about digital technology in education distinctly differs and progresses from Pre-T-2 to In-T-x. (2) The technology groups differ regarding the teaching phase and distinct technology groups can be assigned to a single phase respectively.

Results regarding research question 2

In second part of the interviews, we investigated the selection criteria the participants use in their decision processes. The cumulative view of the results is organized by (i) learner age and abilities, (ii) learning content, and (iii) educator perspective. The criteria are formulated either as continuum statements (younger to older learners, less to more) or dichotomous yes/no statements. Criteria which are limited in their scope because of the context they were provided in or the participating teaching group being either special-, lower- or higher education are marked accordingly.

Learner age and abilities (In-T-x)

- The younger the learners, the lower the cognitive demand of the technology can be and the lesser own personal devices are available. Also the younger the learners, the more oversight is required to ensure the digital technology is used responsibly and in the intended didactic manner.
- Motoric abilities (Pre-T-2, Pre-T-8/10)
    - No: Do not use technology, if it is likely that it gets damaged.
    - Yes: If it enables the inclusion of learners [overcomes impairments of learners].

Learning Content (In-T-x)

-Yes: If the curriculum demands the use of a particular technology.
-Yes: If the content can only be taught using a particular technology.
-Digital technology should enable or support dynamic and different forms of representations and the ability to outsource repetitive activities which are not the focus of the didactic goal.

Educator perspective (In-T-x)

- Technology by commercial providers generally puts a higher burden on the user either because of financial costs, login and registration requirements or the data collection by the provider.

From the list it becomes apparent that the participants use no hard rules at what learner age to start with using digital technology. It should also be noted that some of the participants stated that they weigh in their decision and selection of technology different criteria and are making tradeoffs when deciding. Otherwise the list of criteria has been deduced from the participants responses and should be seen as an accumulation of the teaching practice of the in-service teacher in combination with the arguments and reasoning given in the previous section.

Discussion and outlook

In the interview study, we were first able to show that by the argumentation and justification of whether to use or not use digital technology that we can differentiate between three groups (pre-service teachers at the beginning and pre-service teachers at the end of their studies as well as in-service teachers) within all four phases of the model by Prediger et al. (2013). In the *Exploring* phase, the differences are seen regarding the justification, whereas in the other phases, the differences are seen regarding argument and justification. Additionally, whereas the in-service teachers provided multiple digital technologies for each technology group by Clark-Wilson et al. (2020), the pre-service teachers especially at the beginning of their studies were only able to provide a few examples in the *Representation* group. Either because they are not aware of digital technologies and/or they did not bring them in context with the teaching phases, which needs to be considered in the development and measurement of the digital competency "Selecting digital resources." To verify the reliability of the findings a quantitative study should be conducted.

Secondly, there are indications that the role of the technology groups, as described by Clark-Wilson et al. (2020), differs regarding the teaching phase and that the technology groups can be assigned to a teaching phase in the model by Prediger et al. (2013) respectively. As such, the decision process on the use of digital technology in a teaching phase can be seen as a decision regarding digital technology and not necessarily regarding a specific teaching phase.

Thirdly, although the technology grouping of Clark-Wilson et al. (2020) and the model of the core phases of teaching of Prediger et al. (2013) were created within the context of mathematics education, the findings of the interview study suggest that they are transferable to other teaching subjects/domains. Digital-technology grouped as *Independent, Collaboration and Organizing* potentially is common across teaching subjects and the technology in the group *Representation* is teaching-subject specific. In order to validate these findings, we are planning a quantitative pre/post survey with participants outside of mathematical contexts using the offered approach in this interview study.

The study is limited to pre-service teachers at one university and in-service teachers of one region in Germany, all in context of teaching mathematics. Nevertheless, we were able to provide insights into the decision-making process on the use of digital technology. The accumulated list of technology decision criteria will require further review with regard to its application in the development process of pre-service teachers. Especially the scope of the decisions they will have to make in their future teaching practice needs to be considered versus the decisions they are subjected to by others namely the school leadership or curricular guidelines.

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Design modelling tasks in digital environments

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This research explores the relationships that future mathematics teachers establish between modelling, mathematical content and technology when designing tasks for teaching mathematics. By means of a qualitative analysis, aspects were revealed in the design of tasks that provide evidence for the transversality of mathematical and extra-mathematical content. In the design process characteristics related to the modelling process were identified. Also observed was how the use of Dynamic Geometry Software (DGS), associated with the construction of static and dynamic simulations, can expand and transform the mathematical work initially planned in a task. The findings of this work show the potential of the design of modelling tasks to the development of knowledge for training future teachers.

Keywords: Modelling, Dynamic Geometry Software, tasks, pre-service teacher

Introduction and conceptual framework

The role of the teacher in selecting, designing or modifying tasks has been discussed in different environments, where two elements that frequently stand out are the intended task and the enacted task. This research explores the work that future mathematics teachers carry out during the process of designing the intended task. In particular, in keeping with the work of Sullivan and collaborators, we study the “interactions among aspects of task design: design elements of tasks, the nature of the mathematics that is the focus of the tasks, and the task design processes” (Sullivan, Knott & Yang, 2015, pp. 84). Tasks for learning mathematics can be classified in different ways, depending on the activities they promote. In this work, a modelling task is defined as one that encourages the modelling activity, where modelling is understood as a transition between reality and mathematics in order to address a problem that occurs in real life.

Various elements of teacher knowledge are brought to bear in the process of designing a task (Hill, Ball & Schilling, 2008). Moreover, when the design considers the use of technology and modeling, the range of knowledge required of the teacher is expanded. We have found results that indicate how participating in the design of modelling tasks in digital environments impacts the Technological Pedagogical Content Knowledge (TPACK) of future mathematics teachers (Guerrero-Ortiz, 2021). Different models have also been explored to describe the skills that a teacher should master to teach modelling, and ways have been suggested to operationalize development of these skills (Borromeo, 2018). The review of the literature reveals that designing tasks for learning is in itself a challenge for teachers and, by involving technology and modelling, the situation becomes even more complex as more knowledge is needed.

Based on the fact that the specific introduction of a type of tool promotes concrete changes in the activities that individuals perform (Jacinto & Carreira, 2017), the types of activities that emerge when modelling in digital environments reflect, to a certain extent, the potentials of the tool that individuals recognize, in our case, future mathematics teachers. By this way, we argue that in the modelling process, the actions of individuals reflect part of the mathematical thinking that is brought to bear when they explore ways of approaching the design of a modelling task. While technology itself does not have an objective, in this work we seek to answer the question: What are the characteristics of the modelling tasks designed by future mathematics teachers with the use of technology?

Methodology

By means of a qualitative analysis, we studied the design process and the modelling tasks for teaching mathematics developed by two groups of future mathematics teachers (G1 and G2). Each group consisted of three participants who had to design, implement and analyze a mathematics teaching task whose content and grade level were chosen by them. Before they designed the tasks, the participants had been introduced to problem solving and modelling in a Dynamic Geometric System (DGS) environment. The design of the task thus reflects in part their knowledge of modelling and their knowledge of teaching mathematics in digital environments. The work of these participants was chosen because it highlights the difficulties encountered by those who design modelling tasks, and it accounts of two modes of exploiting the DGS (Swidan & Faggiano, 2021) in teaching tasks. The information gathering instruments consisted of a document on which the participants reported the modelling process employed during the design of the task, and an electronic file containing the construction in GeoGebra. The written document was analyzed using the content analysis method, while the electronic file was analyzed using the “construction protocol” tool, which allows looking back and redoing every step of the geometrical configuration.

For data analysis, the task design process is first described, consisting of two major stages: the research and the pre-service teachers’ modelling process. Each of these stages in turn considers some of the actions shown in table 1. These stages highlight crucial aspects of the modelling process undertaken by the participants that have an impact on the task designed. Subsequently, based on how the participants exploited the technology, ways to expand or improve the task are explored.

Research		**Modelling**
- Choose the context - Explore mathematical and extra-mathematical knowledge -Focus on mathematical content	→	-Simplifying and idealizing - Construct models - Working with models - Obtaining mathematical results - Interpreting and validating

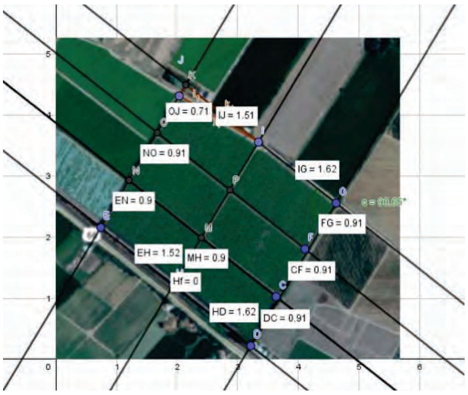
Table 1: Stages of the design process, compiled by authors

Data analysis

This section presents the analysis of the tasks designed by each of the groups. A general description is first shown according to the points listed in Figure 1, and the role of technology in each task is then analyzed, along with the ways in which the potential of DGS can be more fully exploited.

Analysis of design made by Group 1

Based on the school level of the course they were teaching during their professional training (12-13-year-olds), the participants in group G1 decided to address the *teaching of proportions and applications of areas*. Based on this choice of content, they studied the case of a crop farm. They began by exploring in Google Maps (Fig 1) the areas of some tomato, chard and potato farms, which are widely consumed in Chile, from where they obtained relevant data, such as the production yield per hectare, types of soils, climates, and recommended seasons for sowing (extra-mathematical knowledge).

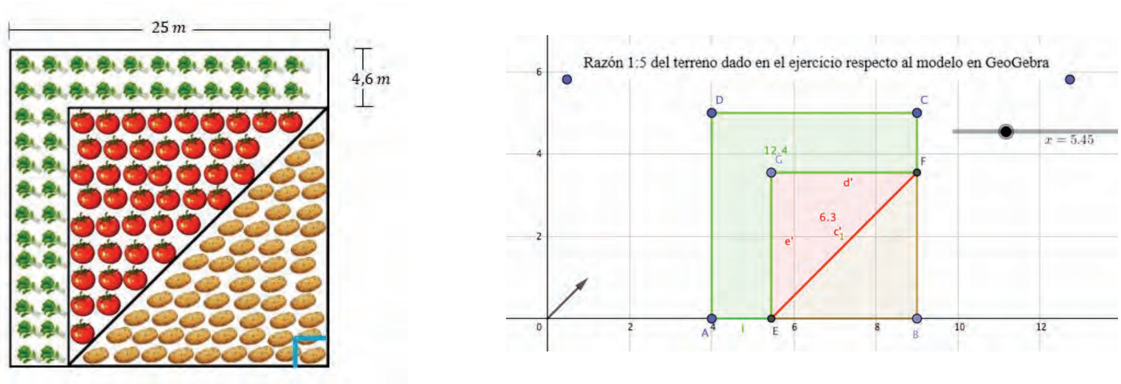


Product	Average yield per hectare	Distance between plants
Chard	16 294	0.4 to 0.5 m
Tomato	25 196	0.5 to 0.7 m
Potato	24 000	0.5 to 0.7 m

Figure 1. Initial exploration of the situation

After exploring the situation, an *idealization* process took place to determine the relevant information to be used in the teaching task, assuming a production yield per square meter. They also considered a square plot measuring 25 m x 25 m, with the condition that each field should have the same area (Fig 2). They also set the values by specifying the production yield per square meter: chard 15 kg/m^2, tomato 8kg/m^2 and potato 13kg/m^2. At this time, the study of the situation lost any semblance with reality, since when calculating the final production, they obtained quantities that differ considerably from the average values. This is explained by the *oversimplification* of the situation, which assumes that each plot of land produces the same amount of product and ignores the density of plants in a given area. This aspect is also related to the omission of the *validation* phase associated with modelling, so the initial approach to this task stands out for the participants' lack of control over their procedures and reflections. This is associated with a metacognitive ability for the development of the modelling process (Czocher, 2018).

In this case, the design of the task depends on the mathematical content for teaching (areas and proportions), an aspect that determined the study context selected and the restrictions in the exploration of a situation that should comprise a modelling task. Moreover, the questions that are finally posed to the students (Fig 2) denote that the activity could well be carried out by omitting the context. In other words, the context is not relevant to achieving the desired objective learning, nor is it a source of reflection. Regarding the DGS *exploitation mode*, since it was initially designed to have the students carry out the construction, it considers the use of tools, such as a polygon, a vector to define the diagonal that divides one of the plots of land, intersections and calculation of distances.



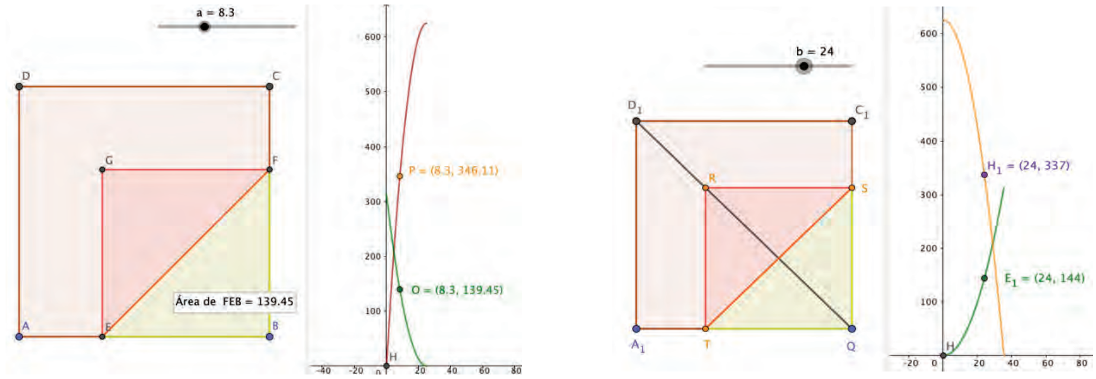
1. Calculate the area of each zone.
2. Calculate the length of the sides of each zone, since you will want to fence them in soon.
3. What is the approximate yield he can expect knowing that [...]
4. If he expands the plot with chard, knowing that the tomato and potato zones have to be the same, how will the proportion change […]

Figure 2. Task designed by a group of future teachers. This figure shows part of task 1 designed by G1 (the introduction has been omitted, which is of the type "Help Mr. Ramón to…").

The mathematical content present in the task, in coordination with the use of technology, can give rise to the exploration of mathematical concepts of another order, for example to introduce the image of simultaneous covariance of two quantities, an element that is essential to understanding the concept of function (Carlson, et al 2002). Based on the idea initially presented in the task, and in view of the fact that the context was irrelevant to the mathematical reflection, we next explore ways, from a purely *mathematical context*, in which the task could be exploited to encourage engagement by the students.

Figure 2 (right side) shows the construction made by the participants, where slider *a* controls the length of side AE, which is congruent to side FC, such that when it is moved, it changes the area of each figure in the geometric configuration. The triangles EFG and EFB are congruent. Graphing the area of one of the triangles and the area of figure AEGFCD yields the right graph in Figure 3a. This can be used to introduce the study of variation, with segment length AE as the independent variable and the area as the dependent variable. The point where the curves intersect represents the case in which all three areas are equal.

Another interesting case that allows us to explore the DGS is the construction where the independent variable is given by the values of segment QR, resulting in the graphical representation shown in Figure 3b. Of note in this case are the different growth rates.



(a) Graph of side AE vs area of triangle FEB and AEGFCD

(b) Graph of side QR vs area of triangle and $C_1D_1A_1TRS$

Figure 3. Exploring different ways to approach the task

These ways of approaching the task, shown in Figures 3 a and b, supported by the dynamism offered by the software, can be used to explore the covariance between the input value and the output value (area) in the same situation. The construction of the dynamic configuration and the use of the slider to control the independent variable are essential to determine the dependency between the variables. This paves the way to introducing the concept of rate of change, domain and range as a preliminary step to studying functions, aspects that had not been initially considered in the design of the task. And improving the context of the task can lead to the study of a crop yield optimization problem. Note that in the previous cases, the sides of the triangle vary similarly, at the same time, giving rise to quadratic behavior. If we vary only one leg of the triangle while keeping the other constant, this results in linear behavior, which introduces a modification to the task. The table 2 below summarizes the main characteristics of this task.

DGS exploitation modes	Predicted mathematical knowledge	Mathematical knowledge emerging from the construction	Technical knowledge
-Construction of a geometric configuration, considering equal areas and different shapes. - Exploration of changes in areas and perimeters, and changes in yield - Static simulation	- Direct and inverse proportions. - Graphic and tabular representation - Lines, perpendicular, parallel, bisector - Triangles, their dimensions - Midpoint of a segment - Congruence - Calculating areas	- Vector - polygon - variable - straight line (vector definition) - point and segment - intersection - symmetry - distance - ratio	- slider to control the characteristics of the configuration. - Point in - Straight line passing through - Vector - Segment - Intersection - Axial symmetry - Distance between points

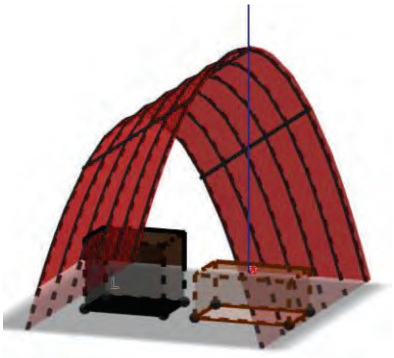
Table 2. Summary of the characteristics

Analysis of design made by Group 2

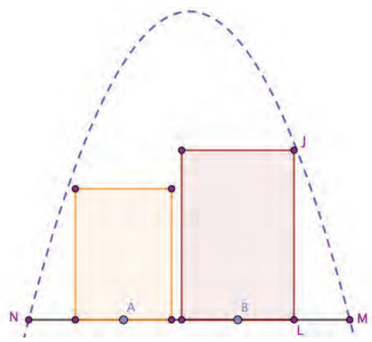
The design of task 2 is based on a study of the dimensions of vehicles that go through a tunnel. In this case, the participants looked for information online about the dimensions and shapes of tunnels, and about accidents where the vehicle impacts the structure due to exceeding the allowed dimensions. These aspects defined the *choice of context*. Based on the above, the situation was *simplified and idealized* assuming, primarily, that the vehicle moves in a straight line, and the shape of the tunnel was approximated by a parabolic arch with height and base width fixed (Figure 4a). From here, the participants determined the mathematical *learning objective* of the lesson (modelling situations using the quadratic equation, recognizing representations of the parabola and solving problems involving intersections) for high school students (16-17 years old). In this case, as in task 1, there was a certain disconnect with reality due to considering unrealistic data for the dimensions of the vehicles.

The simplification of the situation in this case leads to a study of different representations of the parabola in the context of DGS. On the one hand, by representing the shape of the tunnel with a parabola, the first element that emerges is its construction, which can be carried out in at least three different ways: *a conic given five points*, by means of its *algebraic expression*, and *using the parabola tool given the focus and directrix*. The participants chose the second option. The cars are then represented by means of rectangular prisms.

Task 2 asks, given the width, to determine the maximum height that vehicles can have that cross a tunnel shaped like a parabolic arch. This task requires broad mastery of DGS to construct the simulation that represents the situation in the 3D graphic view where the objects are moving (Fig 4a). The analysis is subsequently done in a 2D graphic view. The mathematical activity is then steered to determine the point where the height of the quadrilateral intersects the parabola (Figure 4b). This implies a mathematization process where the shape of the tunnel is represented with a parabola, with its axis on the y-axis, the maximum height of the tunnel (h meters) and its width at the bottom (*a* meters). These parameters are associated with the vertex of the parabola (0, *k*) and points (-*a/2*, 0) and (*a/2*, 0) respectively. This information is used to determine its equation $y = -\frac{x^2}{4p} + k$ and graph the parabola and quadrilaterals (Figure 4b).



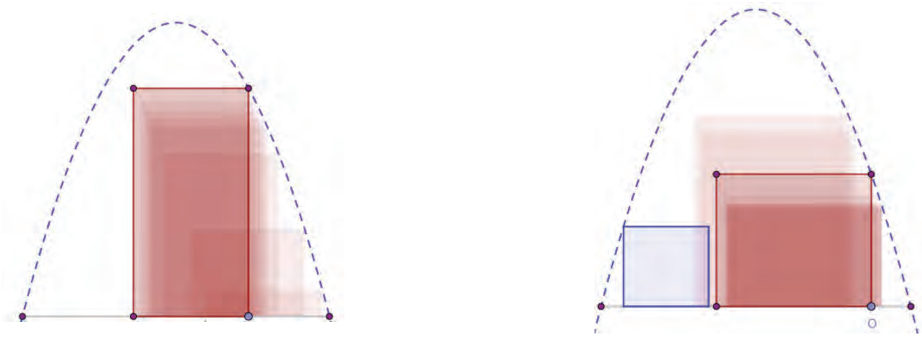
(a) Representation of the 3D situation



(b) Representation of the 2D situation

Figure 4. Students' representation of the situation

To explore the solution, if distance LM is fixed in figure 5b, the problem is reduced to determining the equation of the straight line that passes through points J and L, and solving the system of equations formed by the equation of this line and the equation of the parabola. But if this distance is not fixed, another DGS mode of *exploitation emerges* where the quadrilateral has horizontal movement, allowing for exploration to obtain different heights (Figure 5a).



(a) constant base, different heights **(b) allowed dimensions of two quadrilaterals**

Figure 5. Exploring the dimensions of the quadrilateral

Now, since there are two quadrilaterals, exploring the movement allows us to observe how the dimensions of one quadrilateral affect the dimensions of the other such that both can be inscribed in the area between the parabola and the line y = 0. Thus, another form to exploit DGS emerges when modifying the task to allow for exploration of the maximum dimensions that one or both quadrilaterals inscribed in the figure can have (Figure 5b). The characteristics of this task are summarized in Table 3.

DGS exploitation modes	**Predicted mathematical knowledge**	**Mathematical knowledge emerging from the construction**	**Technical knowledge**
-Exploration the intersections using DGS, considering the width and height of the quadrilaterals - Static simulation	- Midpoint - Elements of the parabola: concavity, vertex, endpoints, algebraic expression. - Intersections - Areas	- Line through two points - Segment, distances - Intersection. - Multivariate function with domain restrictions - Operations with functions - Parabola and its elements	- Slider to control the characteristics of the configuration. - Point in - Segment - Intersection - Distance between points - 3D graphic

Table 3. Summary of the characteristics of the task designed by group 2.

Discussion and Conclusions

In the above cases, we can differentiate between two elements associated with the design of the tasks. The first is the modelling process undertaken by the future teachers, which involves the following phases: selection of a real-life problem, simplification/idealization, recognition of the mathematical learning objective, and mathematization. When designing a modelling task, it was hoped that the participants would also model it; however, the analysis shows that their process does not span a complete modelling cycle, as defined in some of the commonly recognized modelling cycles (Doerr, Ärlebäck and Misfeldt, 2017). This is explained by the fact that the simplification was done with the goal of designing a task with a specific mathematical learning objective in mind, and not with the goal of teaching modelling.

The second element involves the potential of the software, as exploited by the participants. In both cases, they exhibit a general domain of DGS, their use of which is limited to fulfilling the objective of the tasks. Moreover, representing the situation or part of it in the DGS environment results in a mathematical exercise that, without adequate teacher intervention, could detract from the modelling objective. In the tasks analyzed, the software's role in mediating and enhancing the mathematical activity is evident in the mathematical work phase, a fact that should be taken into account in the pedagogical training of future teachers.

We also observed some tension between the choice of the mathematical learning objective and the choice of an interesting context for the students, in the sense that the choice of the mathematical objective precedes the exploration of a situation to model (task 1), or how the participants sought to determine the mathematical objects in a situation (task 2). This seems to give rise to a context dilemma, in which the solution to the task could either be found without involving a particular context, or the context may be irrelevant or be a source of difficulties in achieving the desired learning (Sullivan, Knott & Yang, 2015).

Acknowledgment

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Investigating the processes of Mathematical Problem Solving with Technology of experienced mathematics teachers: The case of Sofia

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This paper aims to characterize the processes in which mathematics teachers engage while solving non-routine mathematical problems and express their reasoning with technology. The descriptive model Mathematical Problem Solving with Technology was used to analyse an experienced teacher's utterances and actions while solving a mathematical problem with a spreadsheet. Our findings reveal the complexity of expert problem solving with technology, through regulation processes and several micro-cycles involving the processes integrate and explore. The teachers' techno-mathematical fluency seems crucial to solving the problem and expressing the reasoning with technology.

Keywords: Mathematical problem-solving, techno-mathematical fluency, experienced mathematics teachers, expert problem solving.

Introduction

Digital technologies provide significant opportunities for enhancing mathematical thinking processes. Yet, the literature shows that most of the technology uses in mathematics learning consist of forms of replicating traditional classroom approaches with some improvements (Bray & Tangney, 2017).

Mathematical problem solving (PS) has been over decades a fertile field of research (Santos-Trigo, 2020). However, the role and impact of digital tools in the processes of solving mathematical problems remains an underexplored topic. Some studies address the strategies and ways of reasoning developed by means of digital tools (Santos-Trigo & Reyes-Martínez, 2019; Silva et *al.*, 2021). Rott et *al.* (2021) propose a model of PS processes that can be used to characterize students' processes with dynamic geometry. Still, it is strongly influenced by Schoenfeld's (1985) model, which was based on paper-and-pencil work and did not account for the affordances of digital tools nor does it provide ways of explaining their role in the PS processes. In this paper we report on an exploratory case study developed to answer the following research question: what is the role of technology on mathematics teachers' processes of PS and how does it support their mathematical thinking?

Theoretical framework

Cognition in digital settings has been conceptualized as stemming from the interactions between individuals, technology, and the media; hence, *humans-with-media* entails the transformational and reorganizational power of the digital tools with which one thinks and acts (Borba & Villarreal, 2005). Technology plays a significant role in the development of mathematical thinking; it allows innovative ways of accessing information and affords new styles of thinking and knowing, producing a reorganization of cognitive activity, namely, in the PS processes.

Our research has been focusing on non-routine mathematical problems, i.e., challenging situations to which the solver does not have a straightforward mathematical process that leads to the solution. Solving these problems by means of digital tools requires the engagement in a mathematisation activity that leads to a productive way of dealing with the challenging situation (Lesh & Zawojewski, 2007), entailing both mathematical and technological knowledge. The development of a conceptual model is consistent with a progressive mathematisation activity, where a *model of* a particular situation evolves into a *model for* explaining or justifying the solution (Gravemeijer, 2005). As the conceptual model portrays the mathematical reasoning developed, it becomes difficult to establish a clear boundary between the solving activity and the explanation of the reasoning. Often, they are so entangled that *solving-and-expressing* summarizes the synchronous processes of mathematisation and expression of mathematical thinking (Jacinto & Carreira, 2017).

Regarding teachers' use of technology in PS, Silva et *al.* (2021) discuss the ways of thinking-with-technology developed by collectives of teachers-with-media, concluding that the tools brought to the fore, GeoGebra and a Spreadsheet, influenced the exploration of the problem, not only visually, but also numerically and experimentally. Hernández et *al.* (2020) analysed pre-service teachers' mathematical understanding when dealing with problem solving in GeoGebra. They found that GeoGebra played a fundamental role in articulating different approaches and in the effective use of control strategies (such as, evaluating the solution or finding support for their conjectures). This leads to consider teachers' proficiency in using digital tools in their problem-solving-and-expressing activity.

The ability to articulate mathematical and technological skills, such as 'techno-mathematical literacies' (Hoyles et *al.*, 2010), is seen as relevant to efficiently solve a problem from a mathematical perspective and to communicate its solution. The term 'fluency', adopted from Papert and Resnick (1995), seems appropriate to describe the ability of articulating a complex idea by means of a tool, being able to do or construct relevant things with it. Thus *techno-mathematical fluency* (TmF) refers to the ability to combine mathematical and technological knowledge for solving-and-expressing non-routine problems (Jacinto & Carreira, 2017). It entails bringing together digital tools and mathematics to create new understandings of the situations, develop techno-mathematical thinking and express it effectively. As with digital fluency, TmF involves to be able to select a useful technological tool from a pool of possibilities, the recognition of particular affordances in the tool, and knowing how it can be used to reach a mathematical outcome.

Research Method

This study investigated the processes of an experienced mathematics teacher, Sofia (pseudonym). A qualitative approach was used in collecting and analysing data. Data collection took place online, through Zoom. At first, in a semi-structured scoping interview the teacher was invited to talk about the role of PS and technology in mathematics teaching and learning. Then, she was asked to select one non-routine problem among four possibilities provided, and to solve it using the digital tools of her preference. She was asked to verbalize every thought and to make explicit every action, and to share her computer screen which enabled the video recording of actions and utterances while solving the problem. Data includes the video recording and the files produced by the teacher.

The teacher's processes were analysed, using NVivo, based on the transcript (utterances) and the video recording (actions), aiming to identify critical events (Powel et *al.*, 2003) that would allow to segment the activity. Deductive coding was performed on the segmented data, based on the ten processes of the descriptive model of Mathematical Problem Solving with Technology (MPST) developed earlier (Carreira & Jacinto, 2019; Jacinto & Carreira, 2017).

A keystone of the MPST model is the inseparability between the subject and the digital tool in solving the problems and expressing the solver's techno-mathematical thinking. The MPST model results from the combination of two theoretical lenses: Martin and Grudziecki's (2006) model for solving a technological problem, and Schoenfeld's (1985) mathematical problem-solving model. The MPST model includes ten processes: i) *Grasp* refers to the first encounter with the problem, either by reading or stating it, to an appropriation of the situation and early ideas involved; ii) *Notice* entails an initial attempt to understand what is at stake, the mathematics and the digital tools that may be useful; iii) *Interpret* is about placing affordances in the digital resources to ponder mathematical ways of approaching the solution; iv) *Integrate* refers to the combination of technological and mathematical resources within an exploratory approach; v) *Explore* entails the use of technological and mathematical resources to explore and analyse conceptual models that may enable the solution; vi) *Plan* involves the outlining of an approach to achieve the solution based on the analysis of the conjectures previously explored; vii) *Create* refers to carry out the outlined approach, recombining resources in new ways to enable the solution and synthesise new knowledge objects that will contribute to solve-and-express the problem; viii) *Verify* involves engaging in activities to explain and justify the solution based on the mathematical and technological resources available; ix) *Disseminate* refers to presenting the solution or outputs to relevant others and pondering on the success of the PS process; and x) *Communicate* comprises the interactions with relevant others while dealing with the problem.

The processes were used to code the segments and their analysis supported the writing of the case of the teacher Sofia solving-and-expressing a problem with the spreadsheet. In the next section, we describe the segments of her problem solving activity by summarizing, in the form of tables, the processes that she carried out at each stage.

Results and discussion

Sofia is a secondary mathematics teacher, with over 20 years of experience. She is highly enthusiastic about the use of technology, as she uses and promotes her students' use of digital tools (e.g. the calculator, GeoGebra, Kahoot). She thinks some tools are suited to particular kinds of problems and that the teachers' familiarity with such tools is fundamental to their successful integration. A detailed and clear explanation of the reasoning process is essential, and Sofia urges her students to do it.

Selecting the problem and grasping the conditions

Sofia started by reading the given problems but spent more time in some parts of the one she would choose (Figure 1), by reading out loud fragments of the statement.

Leonor borrowed the video camera from her mother to film the general rehearsal of the play she is preparing with her colleagues at the Theatre Club. She knows that the camera's battery lasts 2 hours if it is in recording mode and lasts 3 hours in playback mode. Leonor wants to record the rehearsal and immediately watch that video with her colleagues, but cannot re-charge the battery. What is the maximum amount of time of the rehearsal that she can record, in minutes, to be able to view everything she recorded, right after?

Don't forget to explain your problem-solving process!

Figure 1: The problem chosen by Sofia: "How long does the battery last?"

It became clear that it was a new problem for her: "I'm not seeing a way to solve it. At first sight, I would say least common multiple… no, the greatest common divider… I don't know… but it's interesting. I might try this one out!" (*grasp*) (Table 1). Sofia notices that the spreadsheet is an appropriate tool to deal with the problem. Realizing that, if the recording lasts 2h, it will not be possible to play it (*interpret*) she begins to consider recording only 1h, and creates a table with Excel inserting titles and colouring cells A1 and B1, resizing the columns, and formatting boundaries (*integrate*). Then she tests the previous hypothesis, filling 1 in the recording column and 1 in the playing column, although she thinks this experiment will not lead to the solution (*interpret*).

Table 1: Utterances and actions of Sofia during the initial approach

What Sofia did or said	MPST processes
- reads the problems to appropriate the notions that may be involved, identifies a situation as familiar, chooses the problem to solve: "*I like this one… it looks more challenging*"	Grasp
- "*I'm opening an Excel*", says, while overlapping that window with the page that contains the problems - reads the problem again: the battery lasts 2h when in recording and 3h when in playing mode	Notice
- hypothesis 1: "*let's see, a possibility… if she records 2h, she's not able to see it. If she only records 1h…*"	Interpret
- starts by organizing the information on a table: types "recording" in cell A1, and "playing" in cell B1 - formats the table: adjusts the cells dimensions, colours them in orange, formats table's boundaries - inputs 1 in A2 and 1 in B2	Integrate
- "*this is so basic… but I think this way won't take me there*"	Interpret

Testing with an erroneous approach

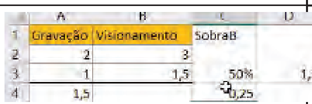
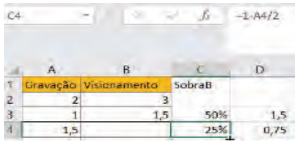
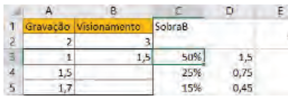
Sofia's subsequent activity is characterized by a micro-cycle between the processes *integrate-interpret* (occasionally, *explore*) which entails introducing formulas in the spreadsheet using its syntax, testing concrete cases and analysing the results obtained in light of what she was expecting.

She assumes that 5h is the total battery duration and writes 2/5 and 3/5 in her spreadsheet model, meaning the ratio of the recording time and of the playing time, respectively. After some attempts, she decides to test a familiar case: if the recording takes 2h, she knows that the battery will be empty. But the result obtained with her model in Excel (0.8h of playing time) is not what she expected (0h). The tests, based on the initial erroneous assumption, disregard that the battery fully charged lasts 2h in recording mode and the same full charge lasts 3h in playing mode. These experiences support a perspective that will become crucial in the development of the solution: the *time left* after a certain recording. However, another difficulty seems to persist which is related to the perception of the existence of two variables of the same nature – the "*amount of time* spent using the camera" and the "*amount of time* that the battery lasts", entailing an inverse proportion.

Testing and developing the conceptual model: "the percentage of the battery left"

The *exploratory* phase of Sofia's activity (excerpts in Table 2) entails the test of a familiar case (1h in recording mode) and a new way of looking at the problem: the percentage of the battery left. When recording 1 hour, half the battery is spent and the remaining half allows watching a 1h30 video (*explore*). Realizing the potential of this approach, she adds a column - "LeftB" - where she considers the percentages of battery left after a certain recording time. That column is a new techno-mathematical object that reveals how she is conceiving the path towards the solution (*planning*).

Table 2: Utterances and actions of Sofia during the development of the conceptual model

What Sofia did or said	MPST processes
- *"hum... 2h… I record 2h, its over, I can't, I can't watch it. (…) but if I have something previously recorded during some other time, I can play it for 3h. I can watch 3h of recording"*	Interpret
- *"If I record for instance 1h, let's see, I recorded 1h then I spent half of the battery. I recorded 1h, so 50% of the battery is left… 100% of the battery would allow to record 3h, 50% will allow for an hour and a half. There!"* - *"I'm in another line of thought now"*	Explore
- Inputs 1.5 in cell B3 (testing) and explains *"there will be 1h left of recording… the battery is left… if in here I record 1h, there will be 1h left"*	Interpret
- Inserts 'LeftB' (what is left of the battery) in C1 and in C3 inputs '1' justifying *"just to see if this allows me to generalize something faster"*	Plan
- *"there is 1h left, no, I have half of the battery left"*	Interpret
- In the cell C3 she records 50% and in D3 she computes the percentage of the battery that is left to play the video, that is $D3 = C3 * B2$ 	Integrate
- *"I have 50% left of the battery… so I recorded 1h, half is left, I kept half of the battery and that half battery will give, will be 50% of the 3h (…) I will have an hour and a half. An hour and a half to play the video"*	Interpret
- Inputs 1.5 on A4, which corresponds to 1.5h in recording mode	Integrate
- *"If I record 1h30 it is left…* [sighs] *I get a quarter of the battery left"* - *"I'm computing without generalizing and this should be generalized completely… so… I'm left with 25%..."* 	Interpret
…	…
- *"is it enough to play? I don't know, I'll see… I don't want to make one by one"* - Replaces the percentages of battery left after the recording with a formula, that is, C3=50% is replaced by $C3 = 1 - A3/2$ - *"I have 50% left. This is what is left and here* [C4], *it will have to be 25%"* - Drags the fill handle from C3 to C4 and confirms that the result is 25%. 	Explore
- *"I have 25% left, now what do I have to do? I have to calculate 25% of the 3h"* [using the spreadsheet]	Interpret
- *"let's see if it's enough, 1.7... 1.7"*, says, inputting 1.7 in cell A5 - inspects the formula, asking *"What percentage is left?"* and by dragging the fill handle from C4 to C5 concludes *"15%"* 	Integrate
- *"at the first sight there is still a little left over. Oh, but it's not enough to see, because then I need to play... I have to see what I recorded* [laughs]. *I was forgetting that detail of the problem, wasn't I? Then it's not working there* [1.5 in recording mode]*… right?"* - *Look, I think I've moved ahead but now I'm finding myself stuck somehow...*	Interpret
- engages in reviewing the steps taken so far and her reasoning, explaining the meaning of lines 1 and 2 in the spreadsheet and going through the several experiments made with particular cases	Verify

Sofia's conceptual model (a *model of*) is being developed as she tests other values even if, at this point, they are worked out mentally and manually inserted on Excel. The processes integrate, interpret and explore follow each other in a cyclical way (Table 2), while Sofia keeps aiming to find a more robust approach that takes advantage of the spreadsheet affordances: "now how do I put it this here in a formula? (…) this should be completely generalized". The process interpret includes observations regarding the testing of particular cases, whilst the process explore is related to the use of those experiences in the refinement of the conceptual model.

The exploratory activity continues until she obtains a formula for the case corresponding to the recording of 1.5h, in which the battery is left with 25% of its capacity, and uses it to test 1.7h. She concludes that the solution must be somewhere between 1h and 1.5h but, as she fills stuck, she decides to review her reasoning and processes (*verify*).

Finding and expressing the solution

After concluding that the overall reasoning is correct and confirming that the formula used throughout column C (to compute what is left of the battery after a certain recording time, “LeftB”) is correct, Sofia realizes that she is looking for two equal values in different columns (A and D). She then colours these two columns in orange for her own “guidance” which is a critical action that sets the *creation* of the solution (Table 3), that is, she will continue her approach by carrying out the plan using the spreadsheet model to test between 1h and 1.5h. She finds the solution in the second attempt: the camera may record a video of 1.2h and its battery will allow to play the whole film.

Table 3: Utterances and actions of Sofia finding and expressing the solution

What Sofia did or said	MPST processes
- about to finish the review, she inserts “play” in D1 to become clearer what the values in the column refer to - *“now, hold on… let me do just one thing for guidance”*, says, formatting the colour of cells A2 to A8 to orange - *“to organize my ideas… here and here…”* and formats cells D3 to D7 with the same colour - tests 1.1 in A5 and obtains 1.35 in the column B (view) and claims *“they must both be equal”* referring to the orange shaded cells - tests 1.2 in A6 and drags the fill handle from C5 to C6 and from D5 to D6 while saying: *“hold on… technology could still be of more assistance to me… I could take further advantage if I worked will with Excel”* - *“Look, 1.2… 1.2… I think that the maximum is 1.2… She could record 1h, 1h, and the 0.2 times 60… hum…”* - She starts to insert a formula in G4, but computes mentally first: *“12 minutes!” Oh, this is it! So… I think it is 1h and 12 minutes”*	Create
- *“Now I have to recapitulate everything to see if this makes sense again”*	Verify
…	…
- *“You have all the reasoning recorded, so it’s not necessary to explain it all”* - going back to statement, describes that she started by testing a concrete case because she already knew the solution, and then moves to explain how she obtained the formulas - Corrects the heading in the cell C1 changing it from “LeftB” to “%battery” to better adhere to the content - *“I think it’s fine now! I’m convinced!”* [laughs] - opens a text editor file and writes “Problem” as the title - uses the Snipping Tool to take a snapshot of the spreadsheet table and pastes it on the text file - starts typing and reading out loud a description of the processes followed, disregarding the initial erroneous path - presents the solution as 1.2 hours	Verify
- *“now a confirmation is needed, oh, not a confirmation, the calculations confirm... a process, a mathematical formula to reach this value”* [the solution] - sends the spreadsheet and the text editor files to the researcher via e-mail	Disseminate

As requested, Sofia engaged in explaining her problem solving processes by recapitulating, again, her thoughts and actions with Excel (*verify*). While doing so she revises the table, changing C1 heading from “LeftB” to “%battery” as she finds it to be more explicit. Even though she is continually seeking for a “mathematical formula”, she includes in the written explanation that she was “computing the percentage of the battery left after the recording (1- recording time/2)”, which worked as a *model for* explaining how the solution was achieved. Her reflections on the success of the activity and the files sent to the researcher containing the solution, characterize the process disseminate.

Conclusion

The Mathematical Problem Solving with Technology model allowed to analyse the role of the spreadsheet in the teacher’s processes. It also accounts for the complexity of expert successful problem-solving activity by revealing that metacognitive skills, namely control and regulation strategies are of paramount importance to progress (Schoenfeld, 1985; Hanin & Van Nieuwenhoven, 2020).

PS with technology takes place trough micro-cycles of several processes, as others suggest (Carlson & Bloom, 2005; Jacinto & Carreira, 2021). Initially, within an erroneous approach, the processes *integrate-interpret* support experimentations that will disclose the basis of the conceptual model: the battery time that is left. Then, other cycles comprise the processes integrate-explore-interpret, as she perceives a different approach in using the spreadsheet to organize the testing of particular cases. The conceptual model evolves through cycles of integrate-explore, from testing with cases (model of) to a confirmation that the approach works and the spreadsheet supports a general solution (model for).

The teacher used the spreadsheet output to create the final answer, as a solving and an expressing tool. Even though the exploratory activity has induced a plan based on the percentage of battery left, the solution emerges from a retrospective analysis of her reasoning, that lead her to look for equal values of the recording time (column A) and the playing time (column D). This reinforces the idea that the 'solving' and the 'expressing' are simultaneous activities of mathematisation.

Sofia's techno-mathematical fluency is revealed by her familiarity with a diversity of digital tools useful in mathematics teaching and learning. In this case, the spreadsheet was chosen because she is familiar with its syntax, recognizes several of its affordances (tabular representations, formulas, automatic fill) in organizing and developing a numerical approach. Later on, a more robust conceptual model emerges as she is constantly seeking a generalization, a formula. The spreadsheet's numerical feedback encouraged conjecture generation and exploration, by easily testing the effects of changing values or relations. By incorporating the formatted table on the text file, she created a techno-mathematical answer to the problem that represents her conceptual model of the solution. Her techno-mathematical fluency includes the recognition of affordances in the digital tools used with several purposes: to interpret the situation from a techno-mathematical point of view, to explore a conceptual model, and to produce the techno-mathematical solution.

Technology plays a paramount role throughout the mathematical problem solving and expressing activity, which suggests that techno-mathematically fluency is an essential skill for mathematics teachers to engage in successful problem solving with technology.

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Development of silent video tasks as a tool for formative assessment

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Keywords: Task design, formative assessment, silent video tasks

Introduction

This poster reports on results from a design-based research project (e.g. Bakker, 2018) focusing on the definition, design, implementation, and development of silent video tasks–tasks were students watch and add their voice-over (in the form of explanations, descriptions, narrative) to short, silent animated mathematics films. Experiences made with teachers in the first phase of the research project formed the basis of the first drafts of a definition of silent video tasks and a description of their instructional sequence that were presented in the form of comics on a poster at CERME11 in Utrecht (Kristinsdóttir, 2019). Based on reactions from participants in TWG15 at CERME and at ICTMT-14 in Essen, a second phase of the design-based research project was prepared and conducted in fall 2019 with the aim to develop further the process of assigning silent video tasks. The second phase of the research project–which could be viewed as a case study–was conducted in collaboration with three Icelandic upper secondary school mathematics teachers, who had some previous experiences with the use of formative assessment. This poster presents the refined instructional sequence of silent video tasks, in the form of comics, based on results from the case study.

Methodology

Prior to the case study, some potentials of silent video tasks to be used as a tool for formative assessment had been identified, using a list of technology-based formative assessment strategies by Wright, Clark, and Tiplady (2018). Hence, teachers with experience of formative assessment practices were purposefully selected to take part in the case study. It so happened that all three teachers, that accepted participation, taught a course for low-achieving students entering grade 11 (age 16-17) in upper secondary school. Based on teachers' suggestions and in accordance with the course curricula, I (the first author) created three one-minute-long silent videos on the topics of coordinate geometry and linear equations. Teachers planned to implement all three videos within the course of one semester. They were encouraged to suggest refinements to the tasks' instructional sequence already in the first interview prior to the silent video task implementation.

Data collection was mainly in the form of teacher interviews that were conducted before and after each implementation of a silent video task and classroom observation notes written during each implementation of a silent video task. Furthermore, during three of the interviews, the teacher who tried out all three silent video tasks was asked to do some think-aloud exercises to prepare and reflect on his task implementation. These think-aloud exercises proved to be helpful to understand the reasons behind his actions and decisions regarding the task implementation.

All interviews were transcribed verbatim in Icelandic. Analysis started immediately after the first interview, focusing on the instructional sequence design and development. After transcribing the last interview, an iterative process of open coding and detailed notes-writing was used to gain an overview of how teacher's ideas, experiences, and expectations developed over time. Thus, creating a base for forming the design principles of silent video tasks, using a format suggested by van den Akker et al. (2013, p. 67) that included characteristics, procedures, and underlying theoretical and empirical arguments for the design.

Results

In the end, only one of the participating teachers implemented all three videos, and the other two teachers both implemented one of the videos in their teaching. Based on teacher interviews, think-aloud reflections, and classroom observations, the instructional sequence of silent video tasks was further developed. Prior to the case study, the silent video tasks' instructional sequence involved the selection of a silent video; whole class viewing of the video; students working in pairs to prepare and record their voice-over for the video in a first lesson, teacher preparation (listening to all students' responses) before the follow-up lesson, and a group discussion based on a part of students' task responses in the follow-up lesson. Teachers in the case study, however, wanted feedback to be immediate and instead of only listening to selected task responses, they wanted a whole group listening and discussion as a reaction to all students' responses to the task. This proved to be challenging for teachers. Nevertheless, the teacher who implemented all three silent video tasks developed his ways of conducting the group discussion and during the third implementation, he was observed to make connections between students' discourse (as presented in students' responses to the task) and the mathematical discourse (as had been presented in classes before the silent video task implementation). This might be connected to the process of reification (Sfard, 2008, p. 44), which involves a transition from describing processes towards talking about objects.

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Initiating the development of a pre-service teacher training course based on research on students' digital resource and teaching designs

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*This paper reports an exploratory qualitative study conducted within the FLINK (*German acronym for *supporting learners for developing sustainable mathematics skills by using interactive materials) project in Austria. We present an overview of the study, its theoretical framework, initial findings from FLINK-based resource designs and the development of our TPACK-inspired professional development course focusing on assessing, designing, and implementing digital materials. Starting in the school year of 2021/22, all Austrian pupils in 5th and 6th grades will be equipped with digital devices. FLINK aims to support mathematics teachers in this digital transformation. In the pilot project, seven pre-service teachers participated in developing interactive, open-access learning and teaching resources for Austrian lower secondary mathematics courses, and initiated a design-based research project aiming to design a pre-service teacher training course within a masters program.*

Keywords: Professional development, digital materials, technology integration.

Introduction

In Austria, digitalization in teaching and learning is becoming increasingly important especially due to impacts of the Covid-19 pandemic. The ministry of education reacted by providing a digital work environment for (lower) secondary schools (BMBWF, 2018). Since autumn 2021, most 5th and 6th grade students will be equipped with digital devices (laptops or tablets with a digital pen and keyboard). According to Clark-Wilson et al. (2020), one key issue for the successful integration of technology in mathematics classes is teacher training (TT). For example, in Uruguay *Plan Ceibal* (a government policy to provide a laptop for each student and teacher) revealed the need for TT programs to support teachers in the integration of digital devices for teaching and learning (Vitabar et al., 2019). However, identifying key factors for a sustainable TT, which have an impact on teachers' everyday lessons, is an ongoing research topic (Zehetmeier & Krainer, 2011).

The joint mathematics TT program at Johannes Kepler University (JKU) and colleges of education includes two courses on the use of technology in mathematics classes. However, these courses mainly focus on software skills, and they do not include the intense didactical feedback necessary to enable students' professional development in this field (Zehetmeier & Krainer, 2011). Research also suggests that it takes more time than often anticipated for teachers to gain enough confidence to use digital materials in teaching (Clark-Wilson & Hoyles, 2017). Supporting Austrian teachers' longer-term integration of digital materials into classroom teaching includes several phases of work. First, we aim to develop a pre-service TT course in our master program that focuses on designing and/or creating digital materials and the accompanying task design, which should enable students to select digital resources and plan for their use. The second important part of implementation into regular teaching will have to be addressed in a later phase as part of our long-term research plans.

Theoretical considerations

Technologies could support teachers' work in various ways, such as assisting the organization of teachers' work or offering new ways of communicating, connecting, sharing, and doing mathematics (Clark-Wilson et al., 2020). In this study, we focus on the role of technology as a new way to represent and learn mathematics and in particular on the digital materials to be used. With digital materials, we mainly utilize dynamic objects in GeoGebra that enable pupils to engage in dynamic interactions with the material and/or provide automated feedback. These objects are embedded in a GeoGebra-based online worksheet additionally comprising of further tasks or explanations for pupils. We chose this software as it: is an open-source mathematics software for educational purposes; allows an interactive combination of several semiotic representations of mathematical objects; and is widely employed in Austrian schools. Due to the age and grade of the pupils first equipped with technology and their limited experience with GeoGebra, we focus on material with pre-designed configurations.

After reviewing several possible alternative models, we selected Koehler and Mishra's (2009) TPACK model as the most appropriate framework. According to this model, teachers need to develop skills not only concerning technological, pedagogical, and mathematical knowledge but especially on the respective interactions of these themes. TPACK encompasses teachers' knowledge of challenges and changes in teaching when using technology, factors that make mathematical concepts easy or difficult to learn, ways to overcome learning difficulties by using technology, and the basis for a sound, meaningful teaching with technology (Koehler & Mishra, 2009). In the following, we interpret professional knowledge in the sense of the TPACK framework.

The TPACK model covers a wide variety of factors relevant for integrating technologies successfully into teaching, its latest version also includes teachers' knowledge *of* their sociocultural environments as contextual knowledge (Mishra, 2019). However, for designing pre-service TT, also sociocultural environments *in which* these skills are to be acquired should be considered. In terms of networking theories, we additionally utilize Goos' (2005) adaption of Valsiner's (1997) Zone theory to cover a broader range of possibly relevant factors to complement the TPACK model. Goos adapted the framework suitable for examining factors influencing teachers' use of technology in secondary mathematics classes. The zone of proximal development (ZPD) can be interpreted as the zone where (future) teachers can develop skills guided by more experienced persons, and the zone of free movement (ZFM) refers to environmental issues constraining a teacher's development (e.g., access to hardware). While the ZFM implies possible actions of future teachers, the zone of promoted action (ZPA) represents professional development (e.g., pre-service TT) which promotes their teaching practices. Considering the interplay of these zones, novice teachers' ZPA has to be within their ZFM, and promoted actions must be within a novice teacher's reach, that is within their ZPD (Goos, 2005). In sum, Zone theory and TPACK together provide a more holistic view on content-related, contextual, sociocultural, and individual factors relevant for implementing and researching a pre-service TT course focusing on technology integration in mathematics teaching.

FLINK – Digital materials design

Our exploratory study is situated within the FLINK project at JKU, which aims to provide mathematics teachers with digital materials to be used alongside mathematics textbooks. The pre-designed interactive materials aim to support pupils' learning of mathematics and therefore focus on the following functionalities: (i) developing concepts and (ii) practicing skills (Drijvers, 2019). Teachers can implement specifically selected materials either during their lessons, or provide them for pupils' homework. A guiding objective in the design process is that each digital material offers a technology-added value compared to paper-and-pencil tasks, which means exploiting functionalities offered by technology such as: automated feedback; dynamic visualizations; or task randomization. For future materials, we also plan to examine the potential of technology for implementing more open-ended tasks, for instance tasks focusing on modelling or problem-solving.

Four mathematics teacher educators selected 18 high-achieving students (6^{th} to 10^{th} semester) enrolled in their university courses, seven of whom worked within FLINK project in July 2021 for an average of 28 hrs/week). These students designed digital materials for topics of grade 5 with a strong focus on the technology-added value. Their workflow is structured as follows: Firstly, students choose a particular small topic from the 5^{th} grade curriculum. Then they work through a checklist (guided by an accompanying script), which requires students to summarize relevant information on their topic (e.g., literature research, learning goals, design ideas, and technology-added value). The script provides additional information for selecting and designing digital materials and for task design, in particular information about: the introduction of the mathematical topic in a learning sequence; the structure of finished materials; the use of language; and references to curriculum and mathematics educational literature. Students are required to analyse the chosen topic from a content-specific pedagogical perspective. Afterwards, they (re-)design digital materials and review their ideas together with an experienced teacher and teacher educator. This review and redesign process takes place two to three times. In parallel, the students discuss their ideas with an 'authoring team', which comprises those who implement their ideas in GeoGebra. Finally, the finished materials are reviewed again by another experienced teacher.

All materials currently completed can be found at www.geogebra.org/flink. One exemplary GeoGebra Book is *Fractions as part of a whole* (https://www.geogebra.org/m/pge8d4x3). Each book consists of digital materials for exploring mathematical concepts and practicing skills. The first part ('Entdecken') consists of interactive tasks about new concepts accompanied by guiding questions. The second part ('Üben') provides self-checking tasks with immediate responses, hints, prepared solutions, or new tasks at the push of a button. Figure 1 presents one task, where pupils must decide if a given fraction is coloured blue. They receive feedback (correct, false) and can choose the next task by pushing 'Neue Frage'. Furthermore, a counter displays the number of already correctly solved tasks.

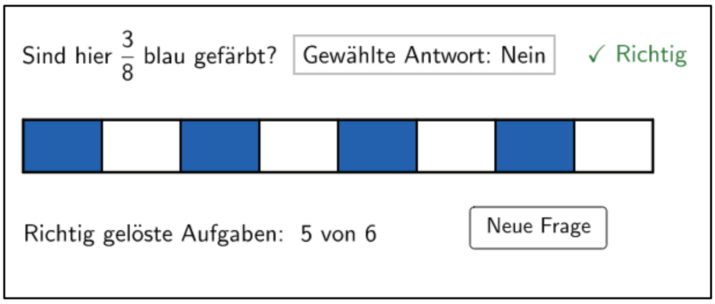

Figure 1: Task 'Can this be correct? – Level 2'

The project is framed by the following research questions: How can a pre-service TT course within a master program focusing on designing digital materials for lower secondary education support future teachers in (i) assessing the quality of digital materials, (ii) designing or (iii) implementing design ideas with technology, and (iv) planning lessons integrating digital instructional materials? We concentrate on the master program because it offers opportunities for students to integrate their prior mathematical knowledge, mathematical content knowledge for teaching, pedagogical knowledge, and knowledge about technology integration acquired in their university courses and school internships. The above-described structure will be the starting point for the design of the TT course. Hereby, we currently have two major priorities: First, we consider it important to provide students with relevant literature-based information about technology-integration in school and to guide them in conducting an intensive literature research about the chosen topic in terms of relevant content knowledge for teaching (e.g., typical errors of pupils, options for visualizations). This should result in an awareness of relevant quality issues in the design and use of digital materials as for example outlined by Trgalova and Jahn (2013). Second, we want to implement a structural collaboration between students and more expert practitioners from mathematics education and technology implementation (Clark-Wilson & Hoyles, 2017).

Research design

From a long-term perspective, this study focuses on developing pre-service TT, thus integrating theoretical findings into practice. According to Cobb et al. (2003), design-based research (DBR) aims at developing domain-specific theories about learning with a designed learning environment and improving this environment by conducting an experiment in an iterative, hypothesis-driven way. DBR involves multiple cycles of hypothesis-based design of learning environments and aims at bridging the gap between practice and research. Therefore, DBR is suitable for our research questions.

For our first cycle, we selected an explorative qualitative approach (data collection by means of students' learning diaries and working notes). It aims to develop inductively a local theory about the development of the professional knowledge of the participating students, which should inform our deeper understanding of the processes involved, and relevant criteria for the design of a pre-service TT course - our *local theory* (Teppo, 2015). This first exploratory study of DBR intends to examine how the participants' self-assessed professional knowledge evolved according to the TPACK model. In addition, we plan to identify crucial – and missing – factors that are relevant for supporting students in assessing, designing, and creating digital materials with technology and thus important for designing a TT course. In this phase, we focus on the first three sub-questions of our research aim.

Preliminary results

In this section, we present exemplary analyses of data collected from FLINK project students' learning diaries over a period of approximately five weeks. Students were encouraged to self-assess the evolution of their TPACK knowledge over time, which factors were helpful and what additional kind of support they needed. Our data analysis followed thematic analysis coding procedures (Braun & Clarke, 2012), which revealed several themes. In this paper we concentrate on: evolving students' professional knowledge; supporting and constraining factors, and the changing beliefs of students.

Students' professional knowledge

The emerging themes span the TPACK model; however, the participating students highlighted especially their evolving knowledge in two fields: pedagogical content knowledge (PCK) and the interaction between technological and pedagogical content knowledge (TPACK). This result also gives an account of the project structure, which focuses on students' intense work with mathematics educational literature.

Students' PCK includes: deepening understanding of PCK issues; how to enhance mathematical understanding for pupils; introducing mathematical topics in teaching; conceptualization in teaching, enhanced knowledge about typical pupils' errors and learning obstacles; topic-specific options for supporting pupils; and further topic-related themes. Most of the themes concentrate on a certain topic (e.g., typical errors when multiplying decimal numbers) but others cover more general aspects. Laura (all names are pseudonyms), for instance, noted:

Laura: Engaging with it [literature] opens new forms of conveying content. [21-08-16]

As the students' ideas are implemented in GeoGebra, one specific theme considers representation and visualization issues. Students discovered additional forms of representations (e.g., how to visualize rational numbers) and remembered rules for visualization (e.g., using various colours for visualizing a certain geometric object). Especially, they started to consider how related representations with technology and paper-and-pencil could interact; for example, Barbara's notes about possible representations of points or Flora's comment about the different representations of commas (in German as comma, in GeoGebra as dot).

Barbara: [Using] cross or point when constructing? Advantage of a cross: a compass can pierce it very accurately. Disadvantage of [using a] point: since it is quite large in GeoGebra, it could happen, that pupils draw thick 'dumplings' on paper. Therefore, … points will be represented as a cross. [21-08-23]

Flora: GeoGebra does not recognize a comma as such, but one must enter a dot … on paper, never dots but commas are made. [21-08-19]

Two further categories concerning PCK emerged from students' notes: language use and task design. We do not have space to expand on the theme task design, which obviously is an expected issue when designing materials. However, several comments revealed issues about language use, indicating the importance of this category, which included: the importance of formulating mathematically correct statements; using language precisely and sensibly; enhancing readability; and using inclusive wording (e.g., according to pupils' gender or cultural background). In addition, supported by an experienced teacher, students also evaluated inexact wording that occurs in schoolbooks. Here, one exemplary student comments about multiplication of decimal numbers, which reflects an expert's well-thought considerations:

Flora: Not the decimal point but the digits shift – [something] often described 'incorrectly' in schoolbooks. [21-08-06]

Finally, we discuss two technology-related knowledge categories: technology from a technical perspective and technology-related views on learning and teaching. The first category contains codes mainly concerned directly with the chosen software, its tools, and technical possibilities to implement students' materials ideas, which can be summed up as technology knowledge (TK) and technological content knowledge (TCK). The second theme relates to TPACK and mainly focuses on dynamic features and visualizations and especially on technology-added value (or limitations) of digital materials, which the following comments show:

Barbara: Dynamic visualization directly shows the effect of varying parameters. This can be very helpful for representations of solids or oblique projections. [21-08-05]

Julia: There are many tasks … that can be carried out particularly well with digital technology because they contain direct feedback, solution paths, and new tasks. Also, the aspect of dynamic variation can be particularly helpful ... [21-08-03]

Furthermore, there are some tasks with more value if they are solved with non-digital means. For example, construction tasks in which children should use ruler and compass. [21-07-28]

Additionally, results reveal various themes representing quality issues for technology integration similar to those outlined by Trgalova and Jahn (2013) (e.g., design and presentation or technical aspects, didactical implementations), showing an increasing awareness of students towards quality aspects. To summarise, the data analysis results in a broad range of issues especially concerning students' PCK and TPACK, which indicates that the structure of FLINK project supports students' professional development and thus provides a profound starting point for organizing a TT course.

Supporting and constraining factors

Supporting factors most frequently mentioned are discussions with colleagues and reflecting and discussing their ideas with experienced teachers and teacher educators. Laura, for instance, wrote:

Laura: … exchanging ideas with each other, being able to ask short questions. Sometimes, a short thought-provoking remark from a colleague is enough. [21-07-15]

Like Ida, all participating students highlighted the importance of discussing their ideas with experts:

Ida: I always find discussions with teachers … particularly useful. It is helpful to discuss other thoughts, perspectives, and experiences on various topics. [21-07-29]

Further supporting factors are: studying schoolbooks and literature about mathematics education; researching existing digital materials; the provided accompanying script and checklist; and the students' prior knowledge. One student in particular mentioned that after reviewing content-specific literature, structuring her knowledge and ideas, proved to be helpful. This can be interpreted as one step of 'decompressing' mathematics to make it explicit for learners as mentioned by Clark-Wilson et al. (2020). These factors support elements of students' ZFM. As most of these factors – except student discussion – are already implemented in the structure of FLINK project, they also are part of their ZPA. A finding is that student discussions should be integrated in the planned pre-service course.

Finally, we discuss findings concerning the main constraints. As mentioned before, the participants mainly design digital materials and tasks and other students (authoring team) implement their ideas with GeoGebra (authors). While students find it supportive to discuss their ideas with the authors as suggested in the project structure, they highlight the need for earlier and more regular discussions:

Christina: To start planning digital materials, it would have been helpful to learn … which ideas are implementable and which not, or what is particularly cumbersome and should be avoided, if possible, since I do not have so much experience with GeoGebra apart from basic knowledge. [21-08-04]

This theme indicates participants' missing TK and TCK, which only partly evolved in this project. Furthermore, it represents a constraining issue on students' ZFM, and there is a tension between students' ZPA and ZPD because designing the digital materials (promoted action) is not fully within their reach (ZPD). Therefore, a further aim for designing a university course must be to provide a setting and structure to enlarge students' knowledge about implementing design ideas.

Beliefs and attitudes

Interestingly, before the project we did not consider how students' beliefs and attitudes may change in this project. Ida's comment about the value of intensive literature already indicates such change.

> Ida: Before, I was not aware of how important it is to deal with mathematics educational literature to enable a topic for teaching to be prepared. [21-07-16]

In addition, Laura recorded after only two weeks:

> Laura: The intensive study of digital implementation possibilities, its technology-added value, and, above all, the already implemented materials are constantly changing my mind and my attitude towards it. I can very well imagine using some of these materials in teaching myself. [21-07-15]

Data analysis revealed a change in: students' attitudes about how to approach a new topic for teaching; the value of the literature review to inform their PCK; and their beliefs about added value of technology integration in teaching. Furthermore, one student commented that she felt more secure about using digital materials in the classroom, even though she was yet to use them. From the perspective of Zone theory, beliefs and attitudes are part of the individual student's ZPD; therefore, such change is one factor for a possible successful technology integration in teaching and learning.

Discussion and conclusions

The results indicate that various factors implemented in the FLINK project's structure support students to develop their TPACK and thus provide a basis for a pre-service TT course. Supporting factors are among others: providing students with a checklist and script as guideline in the beginning; emphasizing literature review on the topic involved; and – most important – enabling regular discussions with experts (experienced teachers, teacher educators, and software experts). Based on these experiences, we could recommend common supervision by experienced teachers and mathematics educators, as they bring different perspectives into discussions: While students adopt the role of learners of technology used in teaching, the involved experts act as guides in the process of designing digital tasks. Furthermore, for each topic students should be encouraged to structure their ideas and knowledge about teaching mathematics (e.g., mathematical concepts, introductions, typical mistakes) and to formulate precise learning goals; the latter is something we introduced during the project as it helped students to focus on the intention of each digital material. One further major issue for us to consider in our study is how to deepen students' TK/TCK, possibly through collaboration with software experts. Finally, it seems crucial that the course structure enhances peer discussions, and it will be important how to embed these factors in a university course's time frame, which differs from this project in terms of structure and especially in the amount of time required.

Results indicate that students' knowledge mostly evolved concerning PCK and TPACK. Hence, it may be advisable to use the MPTK framework (Clark-Wilson et al., 2020) in our next research cycles as it takes a specific stance towards mathematics and also includes teachers' beliefs and attitudes. In addition, students' self-assessment of skills may not always correspond to their actual knowledge, a fact to be considered in the next phase. Furthermore, quality issues emerged during data analysis implying that we already identified some crucial factors for a course design that supports students in assessing and designing digital materials and thus provides first answers to our research questions. Next we will concentrate on how to integrate the findings of this study into our master's program.

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Preservice mathematics teacher's learning trajectories of Technological Pedagogical Content Knowledge (TPACK) in an online educational technology course

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This case study examined mathematics preservice teachers' (PSTs') learning trajectories of discipline-specific Technological Pedagogical Content Knowledge (TPACK) in a non-discipline-specific online educational technology summer course. The study showed that all three PSTs followed a similar pattern of continuous TPACK growth with some variations. Annotated digital timelining of TPACK trajectories suggested that growth of PSTs' discipline-specific TPACK could be attributed to the course design, while variations could be explained by a variety of personal and contextual factors moderating the effect of the course on their TPACK development.

Keywords: Learning trajectories, Mathematics preservice education, Online learning, Technological Pedagogical Content Knowledge, Timelining.

Introduction

The need to prepare preservice teachers (PSTs) for teaching with technology, specifically in virtual environments, is widely recognized by the teacher education community. However, according to OECD (2020) on average, fewer than half of all teachers feel well prepared to use technology in their classrooms. The discussions about teacher preparation for technology integration started about two decades ago and led to the development of the Technological Pedagogical Content Knowledge (TPACK) framework, first proposed for mathematics education (Niess, 2005) and then extended to different subjects and contexts (Mishra & Koehler, 2006). The central component in this framework, an integrated TPACK, is the knowledge that relies on the interaction of content, pedagogy, and technology and is specific to the teaching context. Further, Niess et al. (2010) proposed the developmental model of integrated TPACK through five progressive levels: Level 1 (Recognizing) where teachers recognize alignment of the capabilities of technology with subject content; Level 2 (Accepting) where teachers form a favorable or unfavorable attitude toward teaching and learning a subject with technology; Level 3 (Adapting) where teachers adopt or reject teaching and learning a subject with technology; Level 4 (Exploring) where teachers actively integrate teaching and learning of a subject with appropriate technology; and Level 5 (Advancing) where teachers evaluate the results of integrating appropriate technology into teaching and learning a subject. This model is based on Roger's (1983) diffusion of innovation theory, which also served as a theoretical basis for the Stages of Adoption of Technology (SOA) model that was developed to understand the trends of technology adoption by preservice and in-service teachers (Christensen & Knezek, 1999). SOA model includes six stages in which teachers rate themselves: Stage 1 where teachers mostly avoid technology, Stage 2 where teachers are frustrated and lack confidence when using technology, Stage 3 whet teachers begin to understand the process of using technology, Stage 4 when teachers become self-confident with technology, Stage 5 when teachers start using technology as an instructional tool, and Stage 6 when teachers become creative in using technology as an instructional tool. This model provides an affective lens into teachers' trajectories towards technology integration that could provide additional insights into understanding of individual TPACK trajectories.

Growing body of research indicates that development of TPACK with new technologies is a complex and multifaceted process (Niess & Gillow-Wiles, 2021), but little is known about the ways PSTs' TPACK learning trajectories are shaped by external factors, such as practices and experiences in their teacher preparation courses or by internal factors, such as attitudes toward and confidence with technology integration. Therefore, this study combined the learning trajectories approach with developmental model of TPACK (Niess et al., 2005) and Stages of Adoption of Technology model (Christensen & Knezek, 1999) to analyze TPACK learning trajectories of mathematics PSTs participating in an online educational technology summer course. Specifically, the purpose of this study was to analyze how various contextual and personal factors influence TPACK development of mathematics PSTs.

Literature review

Studies indicate that TPACK framework-based courses support mathematics PSTs' TPACK development (Açıkgül & Aslaner, 2019); however, there is no consensus on what is the most effective approach to TPACK development. While many TPACK studies focus on factors that affect overall TPACK development, little is known about PSTs' daily experiences in programs that aim to develop their TPACK, specifically in and for online environment. Research shows that individual teacher learning trajectories are non-linear, complex processes developed over time through the active engagement with a set of ideas (Shavelson, 2012). The studies that focus on TPACK learning trajectories acknowledge the diversity of patterns in mathematics PSTs' TPACK development. For example, Niess & Gillow-Wiles (2021) showed that mathematics PSTs' learning trajectories of TPACK for teaching online are multifaceted in building knowledge for teaching and learning with technology. Özgün-Koca et al. (2010) demonstrated that engaging secondary mathematics PSTs in iterative design and implementation of technology-rich inquiry-based materials led to their shifts in identity from learners to teachers and their TPACK development. However, PSTs' TPACK learning trajectories were also influenced by their beliefs about the role of digital technologies in mathematics.

More studies are needed to understand mathematics PSTs' development of TPACK *in* and *for* online environments. Therefore, this study was guided by the following research questions: 1) In what ways online course practices influenced mathematics PSTs' individual learning trajectories of TPACK? 2) What factors contributed to variations in mathematics PST's learning trajectories of TPACK?

Methods

Study context

The educational technology course in the study was non-discipline-specific pedagogy course and therefore the whole-class instruction focused on the purpose and the role of instructional technology in non-discipline-specific teaching and learning, while tasks for small groups were subject-specific. The course content was delivered in a synchronous mode via Zoom video-conferencing and Nearpod platform (nearpod.com). The course met for seven weeks with two 1.5-hr sessions per week, one led by the STEM education faculty and another one – by a doctoral student-mentor with degree in politics, literacy and instructional technology. During faculty-led sessions PSTs experienced pedagogical practices with technology through both, student and teacher perspectives. Each immersion experience engaged PSTs in high cognitive demand tasks and guided or open inquiry to illustrate both the role of specific type of technology and the specific teaching strategy. Then this experience was analyzed from a teacher's perspective based on theoretical considerations. During mentor sessions PSTs learned how to use technology and engaged in developing their own technology-based activities while the mentor provided necessary scaffolding.

Each assignment aimed to assess PSTs' ability to use technology tools to facilitate specific high-impact instructional strategy for teaching a topic of their choice in online settings. During the 2nd week, PSTs developed a blog task for student research and communication. During the 3rd week, they developed a multimedia concept map task. In the 4th week, PSTs developed a small group exploration using Google Apps to support student discourse and collaboration. For the final project PSTs developed a full online lesson on a topic of their choice using Nearpod. Each assignment required PSTs to include subject-specific technology tools in their tasks.

An earlier study (Lyublinskaya & Du, 2021) demonstrated that the increase in PSTs mean TPACK scores in this course was statistically significant and with large effect size. The study suggested that the PSTs' overall TPACK growth could be attributed to the program design, specifically, inclusion of immersion - theory - analysis - digital content development cycles and focus on high-impact teaching strategies. The study also suggested that PSTs' individual learning trajectories fall into a limited set of distinct patterns. However, question remained how this non-discipline-specific educational technology course supports the PSTs' discipline-specific TPACK development. Thus, this study attempts to gain a more in-depth understanding of the factors affecting mathematics PSTs' learning trajectories of TPACK in such course.

Out of 25 PSTs' enrolled in the course, three PSTs were preparing to teach mathematics, and therefore were selected for this case study. PSTs A and B had no prior instructional technology or teaching experience, PST C had some K-12 teaching experience including experience with technology.

Data collection and analysis

The TPACK levels rubric (Lyublinskaya & Tournaki, 2012) was used to score PST-generated digital artifacts that included weeks 2-4 assignments and a final online lesson. The rubric measures the level of TPACK in four specific components: overarching conception, knowledge of curriculum, knowledge of instructional strategies, and knowledge of student understanding (Niess et al., 2010) and the range of possible scores for each component is from 0 to 5. PSTs' individual scores across four components of TPACK were averaged and plotted against time to visualize TPACK learning trajectories. Nearpod reports were collected and analyzed for completion of activities and understanding of the material.

The TPACK Likert-scale survey (Chai & Koh, 2017) was modified to collect PSTs' beliefs about their TPACK and design beliefs, that include four constructs: new culture of learning (NCL), attitudes towards technology (ATT), design dispositions (DD), and views about themselves as teacher-designers (TAD). The survey was administered in weeks 1 and 7. The Stages of Adoption (SOA) survey was administered during weeks 2-7 to collect self-assessment of stages. Three open-ended questions about challenges in using specific technology tools were added to the survey. Open-ended responses were coded in relation to SOA.

The study used annotated digital timelining to explore contextual and personal factors that affected mathematics PSTs' TPACK learning trajectories (Figure 2).

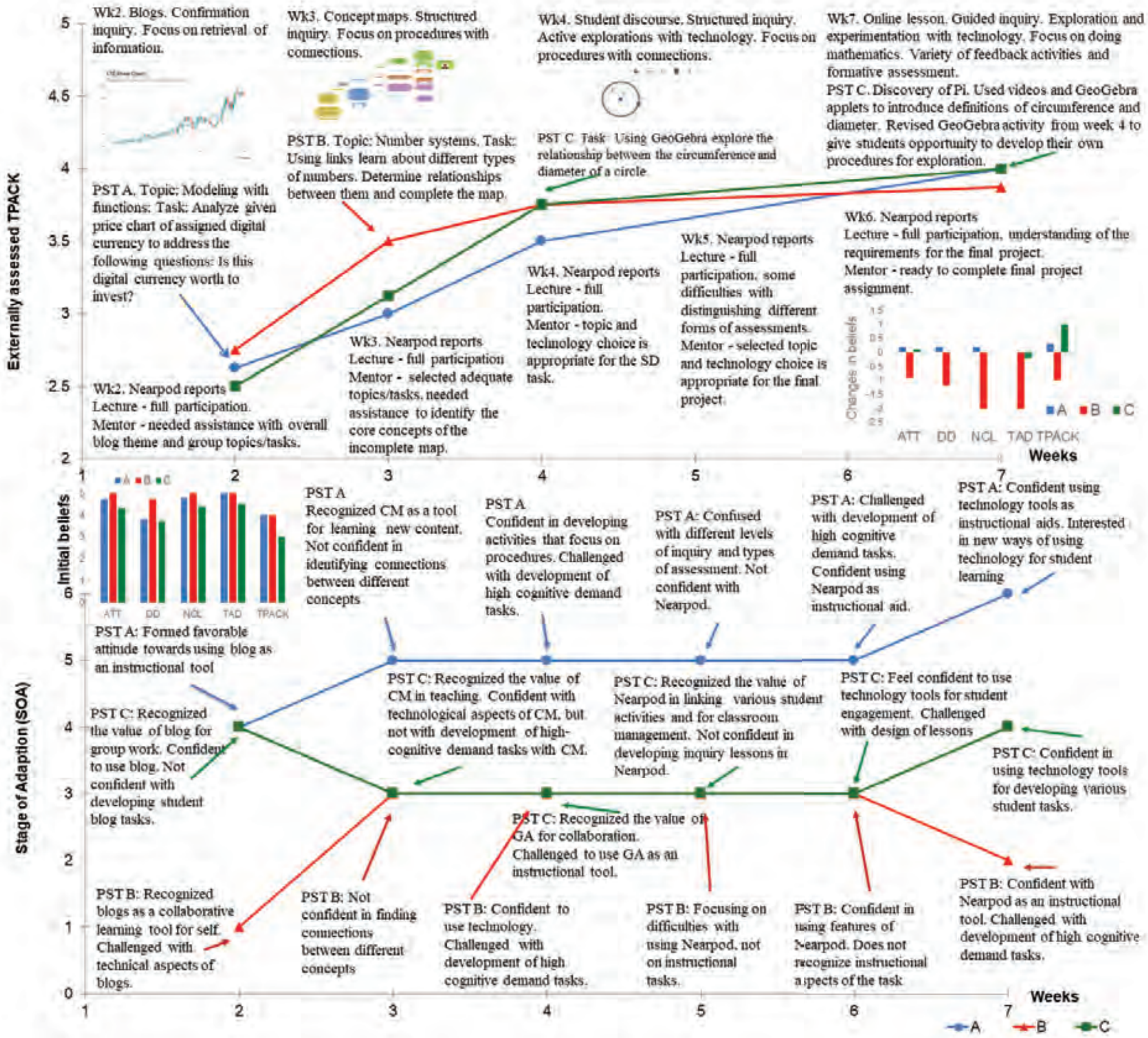

Figure 1. Timelining analysis of PSTs' trajectories of TPACK and SOA

Timelining is a visual research method that focuses on a time-based representation of one or more constructs of interest. The power of a timeline as an analytic resource is in the way that it allows to use multiple type of data from multiple sources on the same display that extends and deepens understanding of the observed changes in the constructs of interest over time (Chamberlain & McGuigan, 2019). In this study, the TPACK and SOA trajectories annotated with the summaries of the analysis of collected data. In order to do that, TPACK and SOA scores were first plotted for each PST to create simple timelines with aligned horizontal axes. Digital artifacts were analyzed for features reflecting the TPACK score. One artifact per assignment was selected to illustrate these features on the TPACK timeline. Nearpod reports provided records of participation and collected PSTs' questions for the mentor. The SOA timeline was annotated with summaries of individual responses to the survey open-ended questions. In addition, two bar graphs were added to the timelines: the first week's TPACK and design beliefs scores calculated from the survey and changes in these scores from the week 1 to the week 7.

Results/Discussion

All three mathematics PSTs demonstrated TPACK growth by one (PST B) or two (PSTs A and C) TPACK levels, which indicates that this non-discipline-specific educational technology course supported development of subject-specific TPACK. Comparison of their TPACK learning trajectories show that all three PSTs followed a similar pattern of continuous TPACK growth with some variations in actual scores and rates of change. In a previous study (Lyublinskaya & Du, 2021), this pattern was found for only 25% of the PSTs enrolled in this course. This continuous growth by mathematics PSTs could be attributed to their full participation in all course sessions as evident from the analysis of Nearpod reports. The analysis of mentor session reports indicated that PSTs recognized their challenges and reached out for support as they were developing their artifacts. That also could have contributed to the positive shifts in TPACK demonstrated by PSTs.

Analysis of PST-generated blog tasks (week 2) showed that all of them were at the Accepting level of TPACK. This level is characterized by confirmation inquiry with low level of cognitive demand tasks that focused mostly on retrieval of information and basic facts (Stein & Smith, 1998). Illustration of this level is a blog developed by PST A with the intent to address the topic of modeling with functions by using online price charts of different digital currency. However, the actual student tasks did not include modeling, trend lines were already included on the price charts that students accessed on the Internet, and questions did not ask for analysis of trend lines or identification of functions used as trendlines, mostly focusing on reporting information found on the websites. This level of TPACK was observed for a large majority of PSTs in the course, therefore course sessions put more emphasis on discussions about levels of inquiry and cognitive demand of tasks.

The following week all three PSTs reached the Adapting level of TPACK as evident from the analysis of concept map tasks. This level is characterized by structured inquiry with higher cognitive demand tasks that include procedures with connections. As an example, consider PST B's task for students to develop a map illustrating the real number system. She provided students with incomplete map that had 5 nodes – natural numbers, whole numbers, integers, rational numbers, and irrational numbers – each linked to an internet source. Students were expected to use provided Internet resources to explore the properties of different number sets in order to establish their hierarchy and complete the concept map. However, her instructions were very prescriptive and did not provide students with opportunities to explore their own questions about the numbers. These results were representative of the large majority of PSTs in the course, thus course sessions put more emphasis on developing guided and open inquiry tasks with technology.

By week 4 all three mathematics PSTs shifted towards using technology for more active explorations, focusing on student thinking and development of conceptual knowledge. However, student tasks with technology were still at the structured inquiry level and often just replaced non-technology-based tasks. As an example, consider a student activity to explore the relationship between the circle's diameter and circumference developed by PST C. She created a Google Document handout linked to an online GeoGebra App. Students were to draw five circles, measure their diameters and circumferences, calculate ratios of circumference to diameter, and arrive at conclusion about the relationship between the two measurements. Asking students to draw five circles similar to a pencil and paper activity indicated that PST C did not understand the dynamic nature of GeoGebra. Students could have varied the radius of a single circle to observe changes in the diameter and circumference. Moreover, instructions to calculate the ratio of the circumference and diameter took away student opportunity for discovery.

In the following weeks, while PSTs were working on their online lessons, the course sessions focused on the various models for technology-infused lessons, with continued emphasis on inquiry and cognitive demand tasks. Analysis of mathematics lessons indicated that all three PSTs incorporated guided inquiry by engaging students in experimentation with technology through high cognitive demand tasks. By week 7, PSTs A and C achieved the Exploring level of TPACK with PST B falling short of that level by a small margin. As an illustration, consider the lesson "Discovery of Pi" developed by PST C. The lesson started with interactive review of the concepts of radius, diameter, and circumference of a circle using GeoGebra applets followed by a matching activity and multiple-choice quiz to check student understanding of definitions. The next part of the lesson was a revised GeoGebra activity from the previous assignment. At the beginning of the activity students were asked to make a prediction about the relationship between the diameter and the circumference of a circle. They were then instructed on how to construct a circle, measure its circumference and diameter, and change its size. Students were to develop their own procedure to determine the relationship between these two measurements and record their work in the group's Google Document. The lesson concluded with student presentations of their discoveries and teacher's summary.

While the overall pattern of the TPACK learning trajectories was similar for the mathematics PSTs, there were some variations in TPACK scores over the course of the study. Therefore, the study analyzed PSTs' responses to surveys. As can be seen from the bar graph on Figure 2, PSTs had different initial beliefs about technology in teaching and learning mathematics and the role of the teacher as designer. PST B had the most positive attitudes, while PST C, the only one who had prior teaching experience, was the least positive. Starting week 2, PSTs self-assessed their SOA of technology and reflected on their experiences in the course. The summaries of qualitative analysis of these responses shown on SOA timeline (Figure 2) clearly indicate that PSTs had different challenges throughout the course.

PST A had the highest SOA self-assessment, mostly indicating his confidence in using technology for teaching. He consistently explained his lower performance by the lack of time rather than understanding: "Sometimes I don't have enough time to prepare for the lecture or finish my assignment perfectly." (PST A, week 3). However, his reflections about his struggles with instructional tasks suggest that he might have been overconfident during the first half of the course. During the second half of the course, he had the largest TPACK growth and by the end of the course, his attitudes slightly improved across all five categories, including self-assessed TPACK. Even though PST B had the most positive attitudes at the beginning of the course, she was the least confident in the group about her ability to use technology in teaching, even though her TPACK scores were highest during the first half of the course. This lack of confidence is evident from her comments: “It's hard for me to brainstorm a topic [for student tasks with technology].” (week 3), “Coming up with high cognitive [demand] activity ideas is challenging.” (week 4), “the workload is a bit heavy” (week 7). By the end of the course she had the lowest self-assessed SOA and her attitudes towards technology worsened (see the changes in beliefs bar graph on Figure 2). The lack of confidence and feelings of being overwhelmed might have affected her overall performance. As a result, her final TPACK only slightly improved, remaining at the Adapting level.

PST C started with relatively high SOA self-assessment, and throughout the course she consistently expressed confidence with using technology. At the same time, she was regularly challenged with pedagogical aspects of the tasks, as can be seen from her survey comments: “I'm confident in setting up and publishing a blog, but I’m not sure about the content of my blog” (week 2); “I think the main challenge at the moment is the [lesson] design” (week 6). Thus, her self-assessed SOA decreased early in the course and did not come back until the very end of the course, when she finally gained confidence in her abilities to design technology-infused lessons. Her increased confidence is reflected in highest growth in self-paced TPACK among the three PSTs as well as her reaching the Exploring level of TPACK by the end of the course.

These results suggest that level of confidence, attitudes towards technology, and design dispositions could have affected the variations in PSTs’ learning trajectories of TPACK.

Conclusions

The study results suggest that a non-disciplinary-specific educational technology course could support the development of PSTs’ subject-specific TPACK by organizing PSTs’ learning as the cycles of immersion – theory – analysis – digital content development, with the first and the last stages being subject-specific, while the theory and analysis aspects were discussed across different subjects. The study findings also indicated that mathematics PSTs’ learning trajectories were influenced by the focus on high-impact teaching strategies. Variations in the individual PSTs’ learning trajectories could be attributed to their attitudes towards technology, beliefs about the role of the teacher as a designer, self-regulating skills, and the level of confidence with technology and pedagogy.

This study contributes to the field in two ways. First, the study extends an innovative visual research method, timelining analysis, commonly used in forensics to an education mixed-method study. The use of timelining extends and deepens understanding of the observed changes in the PSTs’ TPACK over time by visualizing the links between multiple data. Second, the study contributes to the field by examining individual learning trajectories of PSTs’ TPACK *for* online teaching of mathematics and conceptualizing the variations in the paths taken by PSTs. More studies are needed to understand the complexity of developing TPACK for teaching mathematics online.

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Interactive Whiteboard: View of prospective primary and secondary math teachers

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Many schools possess interactive boards, which can be especially beneficial for teaching mathematics. This paper examines the attitudes of prospective primary and secondary math teachers towards the use of interactive whiteboards. It also shows the requirements they perceive to be important for the school, teachers, and students, and the different areas they would use interactive whiteboards for their lessons. The results showed that many prospective teachers, either for primary as well as for secondary school, have a vague intention of how they can benefit from interactive boards in their classroom. However, it can also be seen that these prospective teachers in most cases do not know about specific applications and overall (dis-)advantages of this technology.

Keywords: interactive whiteboard, prospective teachers, mathematics education, technology

Introduction

Over the past ten years, interactive whiteboards (IWB) have become increasingly popular in German classrooms. This trend spiked after a decision in 2019 to actively finance the introduction of new technology in German schools. Similar developments can be seen internationally as well, such as in North America, East Asia, Australia, and several European countries. For example, Denmark has already introduced digital whiteboards in about 40% of the classrooms by 2015 (Balta & Duran, 2015). Most likely, the number has increased even more in recent years and in the future, IWB will be used almost exclusively in schools. Since the preparation of teachers for future challenges is essential, such as the use of digital media and especially the IWB, this study aims to research on prospective teachers' attitudes towards IWBs and how they are implemented into the education at a German university. To gain an insight into the current situation in Germany, we decide to ask prospective teachers for their own opinion of experience.

Theoretical framework

IWBs are touch-sensitive boards, that can be connected to a computer and projector (Nejem & Muhanna, 2014). More recent models combine both of these functions integrated into the IWB (Saltan, 2019).

Along with tablets and computers, IWBs play a central role in the digitalization of school education. The effect of IWBs have been widely researched by several authors. There are also positive effects of IWBs on all levels of education (e.g. pre-kindergarten: Cozby, 2018; secondary education: Cabus, Haelermans & Franken, 2017; school and university: Jacob, 2014) have been repeatedly shown. In particular the motivational effects of technology, which might only be an illusion (Knaus, 2013), are highly regarded. Studies such as Türel's (2012), which focused on teaching with IWBs, show, that the majority of the 140 Turkish middle school teachers examined lacked the technical and pedagogical knowledge to teach with this technology. Knaus (2013) found similar results in a survey of German teachers. Regarding the usage of IWBs, The IWB is often used merely as a digital notepad (Cabus et al., 2017); a projection surface (Knaus, 2013); a communication, demonstration and presentation device or as an instrument for teacher organisation (Northcote et al., 2010) in schools.

Ellis (2010) researched the effects of technology in the classroom with the SMART board as a prime example and stated that "the impact and implementation of the technology is only as strong as the teachers' understanding of the materials being used" (Ellis, 2010, p. 12). This highlights the need to educate teachers in how to effectively use technology in the classroom. Teachers do not necessarily share this view however. As shown in Knaus' (2013) research on the use of IWBs in German schools and universities, some teachers believe that the use of technologies by one colleague is sufficient for its introduction in a new school. It is clear, that every teacher will have to deal with IWBs – which undeniably hold potential for improvement – in the near future, even though most of them were not trained on how to use them in their university courses (Jacob 2014). Therefore, it is not quite clear how (prospective) teachers are *using* these installments for their lessons. Teachers need to be educated such that they can use technologies like the IWB in a way from which pupils actually benefit.

With that in mind and because of the rapid development of digital technology, it is important that we check, and not just assume, if prospective teachers know the application and limitations of digital technologies like the IWB, especially in the context of mathematics education, and whether such technologies are being utilized in university education.

A desideratum exists regarding the education of prospective mathematics teachers concerning their experiences with IWBs and the usage of this technology in mathematics classes. This inspired the following research questions, that should be answered via a survey within the body of mathematic students at Saarland University – also having possible implications for teaching students in mind:

1. To what extent were students able to gather experience with IWBs?
2. Which digital technologies, particularly with regards to IWBs, would students use in mathematics lessons and how would they use them?
3. To what extent have these experiences influenced students' ideas for using IWBs in school?

Methodology

The survey consisted of an open questionnaire to find out if the prospective teachers are adept at using IWBs and how they would employ them in teaching mathematics. Therefore, prospective teachers enrolled in mathematics courses for primary and secondary schools were asked to participate voluntarily in an online survey. No specific selection was made by age or semester attended. The online survey was open for responses for three weeks.

The first section of the questionnaire asked for sociodemographic data. This was followed by open-ended questions on the subject of digital technologies, in particular the use of IWBs. For this, students were asked which digital media in general they had become familiar with during their studies, whether they have already gained experience with the IWBs, and about their attitude toward this technology. Furthermore, they were asked to name prerequisites for the use of digital technology in school, both at a teacher and at a student level.

After the survey period had expired, both authors independently performed a qualitative content analysis according to Mayring (2015) to identify categories based on the answers given by the students. Next, these categories were compared by both authors. Then, the answers of the prospective teachers were revisited again to ensure the reliability of the categories. Finally, the two authors' classifications of the answers were compared again and their agreement verified. Only data from prospective teachers who indicated mathematics as a studied subject and completed at least thirty percent of the questionnaire were included in the analysis – see next section. In the following section, we will present our results: the main- and subcategories we found as well as the number of students who mentioned each of these categories.

Results

A total of 48 respondents participated in the publicly available survey during the three-week period, whereof 31 met the aforementioned criteria and were thus included in the analysis. Slightly more primary education students (n = 17) than secondary education students (n = 14) participated. The percentage of women was noticeably higher in both groups. Most students in both groups participated in an internship at school. However, only one participant in the primary education group reported having worked as a homework assistant, while three students in the secondary education group have already taught a class, and one of them was a language instructor.

To answer the first research question, the prospective teachers were asked if they had come in contact with IWBs and could gather experience with IWBs and their applications. Table 1 shows that more than 80 percent of respondents have gained some type of experience. Most of this experience was reportedly gained in practical settings such as during the students' own school years or during an internship. Less than half of the students reported having encountered the technology in seminars or lectures at university and only one person during their free time. Almost equal numbers of students indicated that the IWB was used as a projection screen or as a blackboard replacement. The blackboard replacement is usually used at school, while the projection surface, such as for presentations, at university. Only a few students mentioned specific applications or operational scenarios (see Table 1).

Table 1: Experience with IWBs of primary and secondary prospective teachers

No experience (5)	
Experience (26)	- in practise (school) (18)
	- university (13)
	- free time (1)
Insert format (11)	- projection surface
	- blackboard replacement (8)

	- others (6)	- internet - DGS - presenting results - oncoo - writing in graphics - videos

After respondents were inquired about their accumulated experience, they were asked to indicate how often they would use an IWB for teaching. The majority of both prospective primary and secondary school teachers indicated that they would always or almost always use IWBs. Next, participants were asked (to predict) how often they would use IWBs. Students with positive attitudes toward IWBs, were asked how they would use them. Those who did not, were asked why they do not want to use it. Table 2 details how they planned to use this technology in their mathematics class, divided into three categories: *representation medium*, *communication medium,* and *working tools.* As previously shown in the results on this topic, there was a consensus that most participants would use it as a blackboard replacement or to present content. Both groups stated that geometry is a topic that could highly benefit from using IWBs. The respondents from the secondary school group also explicitly mentioned functions as an additional topic where they can find advantages in the use of the technology. More detailed insights into teaching content or scenarios were not mentioned.

Table 2: How prospective teachers would use IWBs

	Use of interactive whiteboards in school			
	never	sometimes	almost always	always
primary	1	1	8	7
secondary	0	1	6	7

Applications in school		
Representation medium (11)	Communication medium (6)	Working tools (6)
- geometry (4)	- board replacement (3)	- apps (Geogebra) (1)
- functions (3)	- teamworking (1)	- online elements (1)
- 3D-models (2)	- interexchange (1)	- presenting results (1)
- complex learning (1)		
- drawing (1)		
- pictures, movies (1)		

Finally, the survey asked about the requirements that must be met at the school level, student level and teacher level to enable the usage of this technology. In Figure 1, the blue side represents the categories of the primary prospective teachers, while the orange side represents the categories of secondary prospective teachers. Subcategories were formed first, which were then grouped into main categories. For this reason, the number of main categories and subcategories are not equal, since some people named several subcategories but were only included once in the main category, or only named the main category. In most cases, the categories from both groups are similar and differ only in a few points.

In total, three main categories could be identified. Almost all respondents indicated that financial resources must be available. Under the category of *technical precondition*, both *hardware*, such as equipment, and *infrastructure* were also frequently mentioned. However, it can be seen that the prospective primary school teachers consider both to be somewhat more necessary than the prospective secondary school teachers. Other subcategories include *teacher training*, *anchoring in curriculum,* and *development of teaching materials*, which can be combined into the main category *educational aspects*. Especially for primary education students, further training of teachers was an important requirement. *Supervisor*, *university cooperation* and *time for lesson planning were* subcategories mentioned by one person and categorized as *other*.

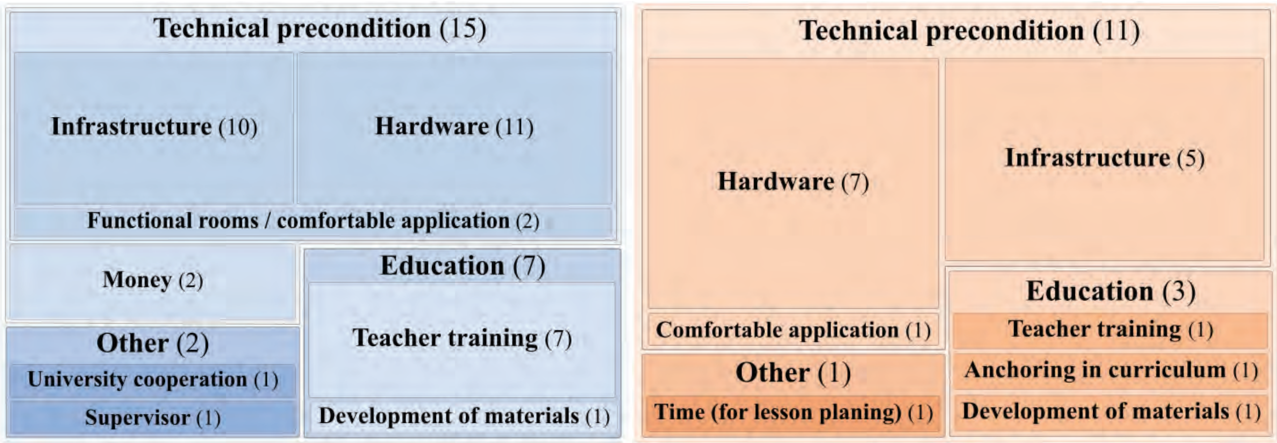

Figure 1: Requirements at school level of primary (blue) and secondary (orange) prospective teachers

Figure 2 shows the requirement at the teacher level. The three main categories *digital media knowledge*, *positive attitude* and *technology enhanced teaching skills* (TET skills) were established. While both groups of prospective teachers surveyed frequently mentioned digital media knowledge in particular. Primary school group focused on it even more frequently than secondary school students. Another important criterion mentioned is attitudes toward digital media, which we refer to as positive attitude. Many respondents expressed the belief that open-mindedness and a willingness to educate oneself further on digital technologies are necessary for their use them in classrooms. TET skills could be identified as another category, which include the subcategories teaching flexibility and application ideas. However, this was only mentioned by a few respondents.

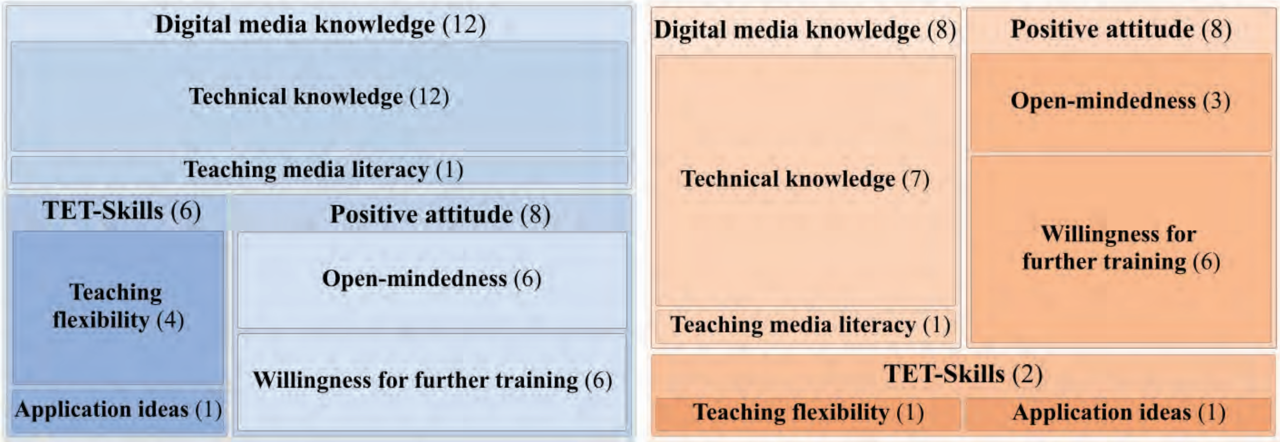

Figure 2: Requirements at the teacher level of primary and secondary prospective teachers

The results of the students' reported requirements for using the IWB are shown in Figure 3. Two main categories can be identified. In the primary school group, the results show disagreement between prospective teachers as to whether or not students need to have prior experience and knowledge of digital media. On the other hand, the majority in the secondary school group agreed that students need previous knowledge and experience. Two respondents in both groups commented that the students already have previous knowledge and experience to build on. Furthermore, the social skills of the students are also considered important, with prospective teachers from the primary school group citing motivation most frequently.

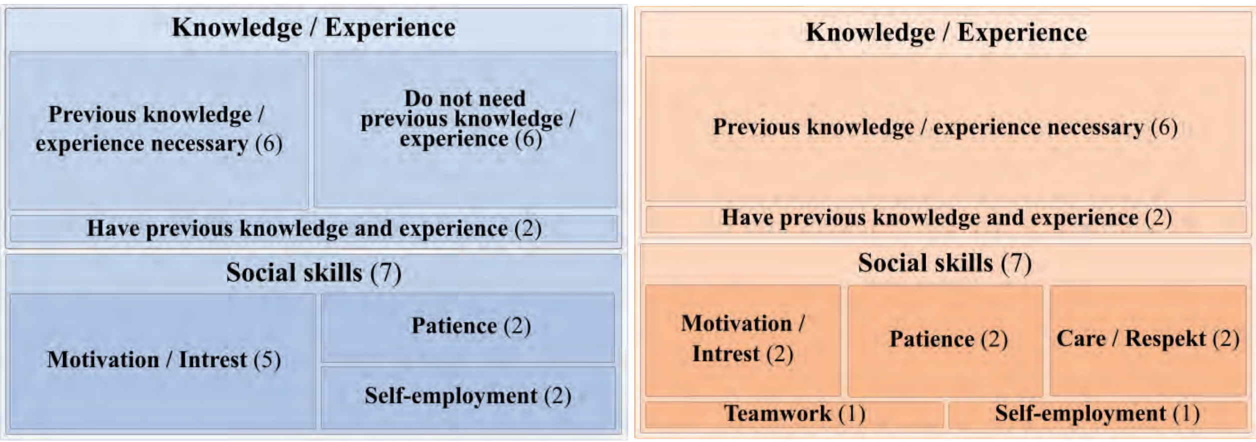

Figure 3: Requirements at the student level of primary and secondary prospective teacher

Overall, we can see that most of the interviewed respondents have some idea of what is needed (by the teachers, students and authorities) to succeed at using IWBs in the classroom, even if the number of explicitly named ideas for use-cases is pretty low. These results and their implications will be discussed in the following section.

Discussion

The results show that almost all surveyed prospective teachers have a positive attitude toward IWBs. This matches the findings of other authors (e. g. Warwick & Kershner, 2008; Littleton, Twiner & Gillen, 2010). While the majority of the respondents would use the IWB for teaching, they are only familiar with a few applications. Interestingly, although many participants chose infrastructure as a requirement for utilizing IWBs, only two respondents said they will use them for *online elements* and *movie / pictures,* where internet is most likely needed. However, it should be noted that there are also negative sides to the ability to live-research things during class, such as misinformation among various websites. Furthermore, teachers can show videos in the classroom. Online mathematics tutorials, which claim to provide quick and easy explanations for certain topics, also often contain oversimplifications and mathematical errors. This shows that prospective teachers have to learn how to correctly handle IWBs and the consequences of their application, such as quick access to online information just as students have to do at school. Given this fact, it is good to see, in this study a lot of respondents know the relevance of advance training. This result also underlines the importance of further training, because the teachers are uncertain about whether the students need previous knowledge and experience, which was probably not addressed in their previous study at the university.

Although we have to consider the rapid development of technologies like IWBs, a short comparison of the categories found in this paper and the research of Northcote and colleagues (2010) shows that the main ideas regarding working with IWBs are closely related to each other and seemingly stable over time. It follows that universities should properly integrate the technical knowledge and competencies needed to work with IWBs in the classroom into prospective teachers' training, just like didactic knowledge. This includes, for instance, discussing their advantages and disadvantages in certain situations, giving examples on topics that could be adapted for this technology, and also teaching about how the IWB is operated so that technical errors during classes can be reduced to a minimum. Especially in mathematics education, diverse applications exist to utilise the IWB as more than just a surface for presentations or as a blackboard replacement. In maths teacher training, there are benefits to using simple applications, which are more comfortable and intuitive. The majority of all surveyed students already mentioned presenting and operating with 3D-Models in the context of teaching geometry, e. g. using the software GeoGebra. One key advantage of this software is that it allows the user to operate on the models, e. g. to rotate them or zoom in and out. This can be done *directly* on the board in front of the students, in contrast to the software found on laptops, beamers or tablets (e. g. Kohls, 2012) and does not require any additional instruments.

Applications exist for other topics of mathematics as well. For example, in primary education, the students learn to read the clock. Showing an analogue clock on an IWB enables the teacher to hide and unhide certain elements of the clock and change its appearance and alter other functions without having to deal with the downsides of a homemade clock (not as precise, fragile, etc.). In the area of probability, an IWB can be useful to show a wheel of fortune with adjustable sections. Details can be hidden or revealed and sections can be replaced. In addition, the graphics, together with dynamic calculations related to them, can be integrated directly into the 'classic' chalkboard painting. Furthermore, not only for special mathematics applications, the IWB has the advantage of saving these panels and chalkboard presentations so to use and modify them repeatedly.

This study shows that there is a need for further debate on digital media such as the IWB. Therefore, more exploration into this topic is necessary, which might result in the cooperation with schools or universities to develop further ideas. It should be kept in mind that this is a relatively new topic for many fields and institutes, even at the university level and a knowledge of such research should be transferred to other digital technologies and media. The rapid development of digital technologies and their applications constitutes a major, novel challenge for everyone. Because of this, it is also a central purpose of the university and further research to find fields of applications and examine them carefully. Only in this way can prospective teachers of mathematics, but also of other subjects, be prepared for their further school life and accompanying tasks.

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What pre-service teachers observed from their virtual mathematics teaching practice

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This paper presents the preliminary findings of an ongoing project regarding what pre-service teachers and facilitators learned while teaching in an online environment during the COVID-19 pandemic in 2020. Using narrative inquiry as a methodology, this initial analysis suggests a need to examine their general descriptions in detail to recognise which actions pre-services teachers specifically identified in their own practice and what they noticed about their mathematics teaching.

Keywords: Observations, pre-service teacher, noticing, facilitator, online lesson.

Introduction

For pre-service mathematics teachers, one of the main components of their education is teaching practice, which enables them to gain experience, practice their professional skills and develop knowledge of how to conduct mathematics lessons at the school level. Teaching practice thus plays a crucial role in the learning of pre-service teachers (Guyton & McIntyre, 1990; Le Cornu, 2012) by enhancing their professional development. Pre-service mathematics teachers are supported during their teaching practice by experienced mathematics teachers and other academics, who function as 'mentors', 'supervisors' or 'facilitators'. However, little attention has been paid to facilitation (Even, 2008; Kelchtermans et al., 2017). By contrast, rich and varied research has considered pre-service teachers' professional identities; their influences during lessons (Zhao & Zhang, 2017), and their teaching values (Mergler, 2012). Moreover, it has been recognised that experienced teachers must improve and update their knowledge (Genç, 2016) to support pre-service teachers effectively.

Research has shown that concerning teaching practice, "pre-service teachers adopt survival strategies during this stage, avoiding risks or taking the initiative so as not to reveal their weak points; consequently, they do not develop the real professional competence that is expected to develop at this stage" (Correa Molina, 2011, p. 78, translated from Spanish). In particular, in the Chilean context, where this study is ongoing, one of the weaknesses noted about pre-service teachers is their failure to link practice and theory (Comisión Nacional de Acreditación (CNA), 2018). This points to a lack of knowledge about what is learned by pre-service teachers during teaching practice and specially in online environments. Even more, recent research shows that "pre-service candidates did not have the opportunity to demonstrate mastery of specific teacher performance expectations within the distance learning format" (Hill, 2021, p.1); therefore, knowing that there is an "evolution and transformation of the classroom with the growing integration of the internet and interactive digital devices into mathematics teaching and teacher mathematics teacher education" (Engelbrecht et al., 2020, p. 825) there is an urgent need to understand in greater detail what is learned through teaching practice, to include virtual mathematics teaching practice. One way of doing so is for the facilitator to note the details of pre-service teachers' actions when teaching and supporting pupils. The aim of this paper is to present the preliminary analysis of a study on what pre-service teachers and facilitators learned while teaching in an online environment during the COVID-19 pandemic in 2020. Our research question is: When pre-service teachers conduct their mathematical practice virtually in a school, what characterises their learning? Using narrative enquiry as a methodology, we examine what pre-service teachers noticed about their own practices, including whether and how these observations were linked to the discipline of mathematics. The three participants were pre-service teachers who were doing their final teaching practice in a school virtually, and the author of this paper was the facilitator of these participants.

Theoretical Framework

To consider the teaching practice of pre-service teachers as a situated professional experience that leads to a lack of articulation of the link between practice and theory, I examined what the pre-service teachers. and the facilitator. learned about what they learned about their own skills in their teaching practice and what this revealed about their learning during interactions with others (experienced teachers, students, and facilitators). As a theoretical framework, I referred to the discipline of noticing (Mason, 2002, 2011), which has been considered a means to promote professional development (i.e. Barnes & Solomon, 2013) and an essential part of pre-service mathematics teaching practice (Llinares, 2013).

Studies of noticing are usually carried out from a first-person perspective, as this way of approaching learning involves looking at one's own professional practices, thus allowing for the observation and investigation of professional pedagogical practice. This enables pre-service teachers in their professional practice to identify more feasible ways of working, consider incidents they experienced during their pedagogical practice, and discuss what they have noted professionally, as well as which characteristics others have recognised (Mason, 1989). Such an approach validates the observed actions that differ between pre-service teachers' own practices and those of others and allows teachers to become mindful of their own actions so they can act differently in the future and make effective changes (Mason, 2011). For example, if a student asks what 'definite integral' means, an 'automatic' answer is often 'the area under the curve', but noticing and applying the professional gaze involves "arranging to alert oneself in the future so as to act freshly rather than automatically out of habit" (Mason, 2011, p. 37); thus, providing a different type of answer becomes possible. Learning to notice professionally allows pre-service teachers, through the observation of their teaching practice, to distinguish the professional components of actions and avoid using generalities, such as 'it was an adequate class'. This is done by paying attention to the details of mathematics-related actions, which in turn helps them to give professional answers to mathematics questions from students.

Finally, although it is expected that future mathematics teachers will professionally consider their mathematical knowledge and while there are analytical models allowing this knowledge to be studied (e.g. the mathematics teacher specialised knowledge [MTSK]; Carrillo et al., 2013), the aim of this ongoing research was not to examine pre-service teachers' knowledge; rather, by working with the discipline of professional noticing, the study considered the actions that promoted learning during mathematical teaching practice, enabling pre-service teachers to act consciously.

Methodology

I used narrative inquiry (Clandinin & Huber, in press) as a methodology, which means the researcher sought to understand the experiences of the participants during their mathematics teaching practice in an online environment, as expressed by their stories. In this paper, I present their experiences through stories told by Sue, Jonah and Harry, considering the meaning of their experiences through the stories they told and the actions they observed.

To conduct the analysis, I used a modified version of Mishler's typology, which is a method of narrative analysis, which means "focusing on making meaning of events and experiences through the researcher's tellings" (Kim, 2016, p. 198), "imposing a told on the telling: identifying a story pattern" (Kim, 2016, p. 205) looking for emerging patterns in stories, chosen it without predetermined themes for the story to tell in advance.

Methods

The participants of this ongoing research were pre-service teachers doing their final teaching practice in two different Chilean schools during 2020. The criteria for their selection were that the teacher conducted their practice in an online teaching–learning environment using tools that involve access to live chat, audio and video conferencing, shared whiteboard, virtual "hand raising" and a break room; and the teaching took place in "real time" via the internet. For pre-service teachers in Chile, this teaching practice usually occurs in the last years of their study towards becoming a mathematics teacher.

The lessons taught by the participants took 45 minutes each. Two of the three participants were in a school that received only students with special needs, so the number of pupils that they have per lesson varied between three and four. The third participant was in a school where the number of pupils per lesson was 30 on average. In addition, the participants oversaw designing and teaching their mathematics lesson to their students over a period of three months and they attended to presentation on the discipline of noticing in mathematics classes.

I observed three lessons per pre-service teacher within the online teaching–learning environment and although it is recognised that one of the practices to support pre-services teacher on their lesson is through the use of observing a video of their own practice (i.e., Segal et al., 2018), I gave to them the observations of each lesson in a written manuscript to be read by them. The written transcripts consisted of a detailed dialogue of what the teacher and students were saying to each other). The reason for given those transcripts to the participants is because "transcripts can help reveal details that are easily missed when watching a video" (Reid et al., 2014, p. 373)

Having given the observation manuscript of the lesson to the participants, bearing in mind the discipline of noticing, I asked to them read it previous to collect research data through three separate conversational interviews about what the participants had observed during each online classroom, the length of each interview was on average one hour, and each was video-recorded within the video conferencing platform. After that, each participants wrote reports of what they had noticed in their own mathematical teaching practice and they submitted this document to an online platform. Following Mishler's typology the method of finding common patterns in the transcript from the interview as well on the written report was highlighting and coding the common characteristics that constituted a pattern of noticing on the data collected.

Results

The following analysis constitutes the preliminary findings regarding the common patterns of noticing among the three students—Sue, Jonah and Harry—based on the interviews and the written reports about what they observed about their own practices as pre-service teachers in charge of online classrooms.

The tension of teaching as a pre-service teacher

Sue was sitting at her desk waiting to start her class. She had to wait until the teacher stopped speaking before she could start her lesson; afterwards, she told me in the interview, '*I realised that when I read, according to my observations, Mary* (referring to the teacher responsible for the class in which she was doing her practice) *interrupted me sometimes. I waited to speak because she was saying something, and I thought I would make my point later, but by then she had already made it*'. I noted Sue's concern about the tension involved in being a pre-service teacher with another teacher close to her.

This tension of being a pre-service mathematics teacher was also experienced by Harry, but in a different way. His concern was, '*I always feel that students don't look at me as a friend…I know when, someday, I'm in a school, they will want to speak about their problems with me*'. He had the idea that a teacher should be a friendly person who can speak about things other than mathematics, but he noted that he was not yet a teacher; he identified himself as only a pre-service teacher who will become a teacher in the future, but for me, he is a teacher now, because he is teaching.

The importance of reflecting on details

By asking '*What do you think about the observation? I don't know what type of observations you have experienced before but…*', I was trying to provide the context for beginning to speak about the mathematics observations, bearing in mind the discipline of noticing and how to give sensible and professional mathematics answers. Sue said, '*I like that you generated a dialogue because, in general, when other teachers observed me previously, they only noted key things; that's all*'. She added, '*You gave us every detail*'. Harry commented, '*I was expecting a critique of my work, more judgments*'. His experiences of other instances of practice feedback were structured observations that seemed to him to be criticisms.

Going back to the details given in the transcript, Sue explained something about 'c'. Not knowing to what she was referring, I asked her to go to the relevant page of the transcript and explain a little more about 'c'. She found the point and exclaimed, '*Here*!' and relayed the following dialogue about Pythagoras' theorem in her class:

> Sue: We use a little formula to calculate c. Who is c?
> Students: The results.
> Sue: Hypotenuse or cathetus?

She added, '*I realised that I was asking about c as a person*. *I should have asked "What is c?"*' Sue was concerned about her mathematics expression and what she could do about it. How could she examine her own speech while being aware of the mathematics happening in that moment? Were there enough questions? I asked: '*What draws your attention in your observation?*' Her first observation was that although it was a story relating to mathematics, she was identifying the long side of the triangle, apparently working from her memory as a student and seeing that the ‘c’ in the ‘little formula’ (as she referred to Pythagoras’ theorem) was the result of an operation. Despite the richness of the details given by Sue to observe her action, I recognised that she was perhaps only noting the specific mathematics around the ‘c’ but not providing more information about a possible action that could be undertaken concerning the ‘c’ idea.

Jonah stated, ‘*I would like to extract some part of the observations of my facilitator*’:

Jonah: D (referring to the problem level); It looks a bit complex.
Student: Teacher, let me think about it, because I’m sure I have seen it before.
Jonah: It is interesting. I wanted to give you this [question] because…
Student: I’m sure we have seen it before.
Jonah: You bet. Let’s…

He noted Regarding this dialogue, we can see that *the student was interested in solving a complex problem*[1]*… At that moment, I took the decision to give him more time, but half of the class time could be lost, so I decided leave the problem to the end, but that didn’t happen … Looking back, I see this as an important chance that I lost.* Jonah wrote about his actions, reflecting on his own mathematical knowledge, his interruption of the student, and his concern when looking back on the class. He also observed, “*it looks like an error in the transcription* (shown below)*, but it’s not. I´m speaking about the whole problem, answering my own questions, and this type of action will never be favourable for promoting the autonomy* (of my students). *It has started to become something automatic’* (referring to probability lessons).

Jonah: That’s the thing about probabilities; three can come out many times on one day, but not on another.
Jonah: What are the favourable cases?
Jonah: Let’s look here. For every number that I get here, I have six chances of getting it. That’s 36
Jonah: So the probability that I get Tom …
Jonah: You understand? It wasn’t an easy problem to solve, but if you want to solve it, you must pick numbers close to seven.

He observes the transcript and from there, he took account that in his lesson, he was speaking all the time (which he acknowledges). Reflecting on his actions, as prompted by the transcript, he was able to understand more deeply his own learning without me saying, ‘You have been speaking all the time in your lesson’. I wanted them to learn from their own observations, avoiding compulsory descriptors, such as, ‘Does the teacher give the goal of the class? Are the students participating in the lesson’? In a similar way, Harry said, *I realised there is a lot of Harry speaking (in my transcript). I literally take the microphone…and I don’t allow the others to speak. I said to her* (referring to a student), “*We are seeing ratio. It’s a ratio,” and I didn´t give to her the opportunity to speak*. After going back and

[1] In the Chilean context, it is very common to encourage students to writing in the third person. Although, in this study, the participants were allowed to write in the first person, some of them wrote in the third person or used a hybrid of the first and third person.

reading his transcript, he noted what he had done on their teaching, seeing its importance in the way as he teaches his mathematics lesson.

Decisions and mathematical consequences

The students spent time speaking about their actions but without directly mentioning mathematics, so I started trying to direct the questions towards actions relating to mathematics in the transcript. Sue told me: *I knew the difference between eight to the power of two and eight times two, and I think they understood that, but I think they forgot it because, when I asked them, they still said that eight to the power of two was eight times two*. She observed she did not have the mathematics resources to correct the errors. How should she start giving professional answers about mathematics? Why did she give the same answer to the students if they were still making the same error? I was concerned with this situation, so I asked her, '*How can we give a different mathematical answer while considering the idea of doubling or the power of two?*' Sue replied, '*Developing the power*', detailing her own knowledge of the power and what she brings to her lesson.

Harry, in a similar way, seemed to decide to stop talking about mathematics because '*I didn't want to add more concepts to what they already had…I know what I said about the addition rule, but it is not as I said*'. This showed the paradox of deciding 'the best' mathematics for students, but what were the consequences of this decision? Is there a 'best' mathematics to be learned? Who decides this? Jonah said, '*I was thinking a change of measurement was trivial and the students must know this, because it is usual for me to know this as a Chilean citizen, but I failed on this activity*'; thus, he took responsibility for his mathematics actions. *I asked*, '*Thinking about the change of measurement and ratio that were part of your lesson, how would you take account of them?*' Jonah replied, '*In my next lesson, I will start with the change of measurement and ratio, presenting some double entry tables so that they can see they are ratios*'.

Discussion

In an on online mathematics teaching context, what the pre-service teachers noticed in these initial findings reflects the importance of discussing the details of their practice, recognising the tension between becoming, but not yet being THE teacher, and I ask 'who defines being a mathematics teacher?' 'are we considering what type of pedagogical identity is happening on the pre-services teachers?' Is it enough to teach mathematics to be considered a mathematics teacher? As van Gogh (2017, p. 45) asserted, 'One becomes a painter by painting'; likewise, one becomes a teacher by teaching".

This initial analysis suggests a need to move from general descriptions towards giving specific details of pre-service teachers' actions to encourage them to reflect on their own actions and what they notice during mathematical teaching practice, such as seeing the mathematical consequences of their decisions during teaching practice, as the stories in this narrative analysis have shown.

When presenting observations of the practices of pre-service teachers, moving to uncontrolled (but no less effective) noticing by each participant regarding what the pre-service teachers observed about their own practices in their reading of the transcript, allows them to link their own mathematics with their pedagogical practice and perhaps foster effective learning in their students.

Finally, the use of technological tools that involve video conferencing and recording in a research context facilitated the collection of the data. However, providing the transcript to the pre-service teachers allowed them to detail their own mathematical practices and their own learning. The pre-service teachers' awareness of the use of this type of technology in their lesson seems low, perhaps because they are building a windmill instead of a wall concerning technology in their own teaching practice. That is, as Brunetto et al. (2021) argue, a teacher can build a wall or a windmill when the wind changes the usual class environment into a virtual one, such as with the current pandemic.

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A videogame for supporting teachers' scaffolding in whole-class discussions

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Videogames are becoming a topic of interest in mathematics education. However, research in the field does not seem to clearly highlight the role that a videogame environment has in supporting teachers in promoting teaching-learning processes. The aim of this paper is to analyse how a videogame for learning can support a teacher in scaffolding relational thinking during whole-class discussions. We observed that the videogame does not explicitly seem a benchmark for offering scaffolding interventions, but the teacher states that observing students playing is relevant for orchestrating discussions, especially in anticipating and monitoring students' processes during the classroom activities.

Keywords: Game based learning, relational thinking, scaffolding.

Introduction

In recent years, the widespread use of digital games for learning has opened new frontiers for research in education, whereas Connolly, Boyle, Hainey, McArthur and Boyle (2012) show that this interest is frequently speculative and a lack of empirical evidence about the effectiveness of games. Concerning mathematics education, several research studies address the potential, promises, and pitfalls of digital games for mathematics learning by measuring, monitoring, and analysing the development of students' sense-making as they engage in games technologies, both in and out of school (Lowrie & Jorgensen, 2005).

In literature there are different and independent definitions of videogames for learning. For our purposes, we choose to consider the one provided by Perrotta and colleagues (2013), who describe game-based learning (GBL) as "[…] the use of video games to support teaching and learning." GBL could become a tool that supports teachers specific teaching and learning targets. However, it is important to select a specific area to investigate from both a mathematical and educational perspective.

In this study, we are interested in exploring the role of a GBL in supporting teachers' scaffolding during classroom discussions on relational thinking.

Theoretical framework

To achieve our goal, it is necessary to clarify what relational thinking is and what types of scaffolding teachers could carry out during classroom discussions.

Carpenter, Franke and Levi (2003) describe *relational thinking* as examining two or more mathematical ideas or objects, looking for connections between them and analysing or using those relationships to solve a problem, to decide, or to learn more about the situation or concepts involved. Carpenter, Levi, Franke and Zeringue (2005, p. 54) define relational thinking as "looking at expressions and equations in their entirety rather than as a process to be carried out step by step". In other words, relational thinking regards the use of fundamental properties of numbers and operations to manipulate numerical expressions rather than following sequences of procedures for reaching a result.

Carpenter and colleagues (2003; 2005) mentioned many times the importance of teachers' effort in designing suitable teaching and learning environments. The authors suggest engaging students in the solution and subsequent discussion of specific tasks; in particular, solving true/false and open number sentences could provide a flexible context for representing relations among numbers and among operations. However, involving students in well-designed tasks is not enough (see for example Lampert, 2001); one of the goals of research on relational thinking is how teachers might foster its development and its use to learn arithmetic. Therefore, the role of the teacher is crucial, and it could be defined as scaffolding, that is the "[...] support given by a teacher to a student when performing a task that the student might otherwise not be able to accomplish" (Van de Pol, Volman & Beishuizen, 2010, p. 274)

Carpenter et al. (2003; 2005) highlight the relevance of scaffolding, but it seems that the role of teachers and how they could scaffold students are not clearly defined. However, in most of their papers the authors present examples of interviews with students and transcriptions from discussions with their teachers. Such examples allow us to identify four kinds of strategies that teachers could use to provide scaffolding: (1) choosing suitable task to administer and then discuss with students; (2) inviting selected students to intervene during the discussion; (3) inventing or introducing new tasks to clarify or share the emerging mathematical ideas; (4) encouraging students to invent new tasks to clarify or share the emerging mathematical ideas.

In this scenario, a GBL could offer several scaffolding opportunities; for example, it could present a large number of suitable tasks in line with the ones proposed by Carpenter and colleagues (2003) (1). This large body of tasks could also allow teachers (3) and students (4) to invent new examples of similar tasks based on reasoning by analogy. Finally, the log files collected by the digital environment could permit teachers to select students for discussion considering their achievement in the game (2).

Methodology

To understand if and how a game-based learning could help teachers scaffold activities on relational thinking, we structured a field trial as follows.

1. We planned a teacher training focused on how to use the videogame, on relational thinking and the role of the teacher during whole-class discussions. Furthermore, we provided the teacher with some theoretical and methodological guidelines.
2. We designed the videogame tasks to be equivalent to those proposed by Carpenter and colleagues (2005).
3. We provided the teacher with worksheets to highlight the solution processes, which are not visible from the videogame log files. Moreover, she could use a web interface, where scores, access and play time and other useful information are reported.
4. We asked the teacher to present the videogame and the worksheets to her students and then to orchestrate a classroom discussion. Finally, we interviewed the teacher.

SuperFlat Math

"Matematica Superpiatta" (SuperFlat Math) is a game-based learning (GBL) about Mathematics developed by Prof. Leonardo Guidoni, from the Department of Physical and Chemical Sciences of the University of L'Aquila. It consists of a sandbox videogame that enables primary and lower secondary school students to explore a blocky, procedurally generated 3D world. This videogame is divided in mathematical activities, which are composed of different "minigames", that are short puzzles at increasing levels of difficulty.

In our trial we asked students to play two activities: "Parkour" and "Swimming Pools". The first one (Fig. 1) consists of a perilous uphill path, which in some points presents a number sentence or an expression with two possible solutions. Players should choose the correct one in order to advance in the path. The first half of Parkour minigames presents number sentences in which students should find the correct solution, whereas the second half contains an equivalence between two expressions.

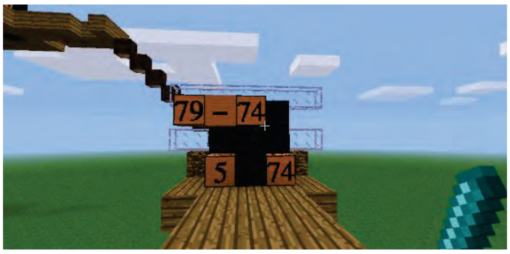

Figure 1: A Parkour minigame

The minigames in Swimming Pool (Fig. 2) consist of a pool full of block numbers from 0 to 100 and an open number sentence. The goal is to find the correct block number within the pool and place it in the sentence. The first half of Swimming Pool minigames contains an open number sentence with two operations and one missing number, whereas the second half presents expressions with parenthesis and two or more different operations.

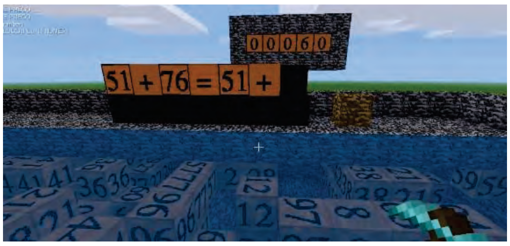

Figure 2: A Swimming Pool minigame

An additional feature of SuperFlat Math is the message given to players on the correctness of their answers. If the answer is correct, players gain points that could be converted in rewards.

A web interface has been developed, called "CLARAS" (CLAssroom Report And Supervision), which enables teachers to monitor students' achievements and to collect useful information about access and play time, scores, number of correct answers, number and type of wrong answers, number of attempts, and so on. This interface helps teachers and researchers identify students' misconceptions and solutions to the given tasks.

Worksheets

According to Carpenter, Franke, and Levi (2003), the easiest way to start a classroom discussion on relational thinking is to assign students true/false and open number sentences. For this trial we structured two worksheets, which were focused on specific number properties and ways of thinking about number operations: the first one was composed of 17 true/false number sentences, while the second one was composed of 13 open number sentences. We asked students to justify their answers for each task so as the teacher could understand the process carried out to solve it.

The true/false tasks concern three main topics: conception of equal sign, properties of addition of natural numbers, and properties of multiplication of natural numbers (Table 1).

Table 1: True/False tasks in worksheet1

Conception of the equal sign	Properties of addition	Properties of multiplication
1. $3+5=8$ 2. $8=3+5$ 3. $3+5=3+5$ 4. $8=8$ 5. $4\times 2=0+8$ 6. $9+5=14+5$ 7. $8\times 2+5=8\times 2=16+5=21$	8. $3+5=5+3$ 9. $10+2+8=10+10$ 10. $3+5=2+1+5$ 11. $20+7+33=37+40$	12. $8\times 6=8\times 5+1$ 13. $10\times 2=8\times 2\times 2$ 14. $15\times 2=3\times 10$ 15. $6\times 5+6\times 3=6\times(5+2)$ 16. $3\times 11+7\times 3=18\times 3$ 17. $7+14=7\times 4$

The open number sentences regard three main topics: properties of addition/subtraction of natural numbers, properties of multiplication of natural numbers, and more complex expressions with the four operations (Table 2).

Table 2: Open number sentences in worksheet2

properties of addition/subtraction	properties of multiplication	more complex expressions
1. $25+16=25+\cdots$ 2. $25+32=27+\cdots$ 3. $\ldots+60=57+83$ 4. $30-25=20-\cdots$	5. $7\times 3=\cdots+7$ 6. $8\times 3+8=8\times \ldots$ 7. $2\times 3\times \ldots=6\times 5$ 8. $2\times \ldots\times 7=14\times 5$ 9. $25+20=5\times \ldots$	10. $32+(20-7)=32+\cdots$ 11. $25+75=25+(30+\cdots)$ 12. $(\ldots+\cdots)-17=28-15$ 13. $25+\cdots=25+36:3$

Some of the tasks proposed in the worksheets were also added in Swimming Pool and Parkour minigames.

Teacher training

We structured the teacher training in three meetings. In the first one we asked the teacher to play SuperFlat Math as to familiarise with the game environment, the instructions, and the kind of proposed tasks. In the second meeting we presented to the teacher some examples about relational thinking taken from the textbook by Carpenter and colleagues (2003). Finally, in the third meeting we showed her CLARAS and its main features. We also provided the teacher with guidelines, where the fundamental topics of each meeting were summarised.

Sample and data collection

The sample is a fifth-grade classroom of a primary school located in a region of central Italy. The classroom is composed of 22 students, 14 males and 8 females. The whole-class discussion was orchestrated by their mathematics and science teacher.

The teacher lets students play for about 2 hours, then she assigns them the worksheets to be solved in small groups (composed of three students) and finally she orchestrates a whole-class discussion.

We asked the teacher to observe students play, collect, and read all the worksheets and record the whole-class discussion.

Results

The teacher conducted several classroom discussions involving all the topics proposed in the worksheets. In this result section we focus on the description and analysis of the classroom discussion about the conception of the equal sign. The discussion lasted 1 hour and 23 minutes and almost all present students got involved.

In the following we report and describe some discussion transcripts. We selected those that highlight the key role of teacher's scaffolding. We present a first example in which the teacher picks out some of the 7 tasks related to the conception of the equal sign, a second one in which she calls some students out to intervene in the discussion and a last one in which she introduces new tasks in order to clarify the emerging mathematical ideas. In the first example, we show an excerpt in which the teacher effectively selects only few tasks from the worksheet1 in the following order: $3+5=8$; $8=3+5$; $8=8$ respectively the task 1, 2 and 4 in the worksheet1. She starts by pointing out the difference between the first two sentences.

Teacher:	Student A, how do you read that [referred to $8=3+5$]?
Student A:	Eight equals three plus five.
Teacher:	Ok, Student A. And how do you read the first one [referred to $3+5=8$]?
Student A:	Three plus five equals eight.
Student B:	They [referred to the numbers] have been changed.
Teacher:	What have been changed?
Student B:	The results... Because in the first one it was in the first place, in the second one it was in the last one.
Teacher:	Do you agree? We all agree, do you all think that between the two [referred to $8=3+5$ and $3+5=8$] the results have been changed?
Student C:	To me, it is easier to decompose the second one [referred to $3+5=8$], 8 in $3+5$ and $3+5$ is equal to three plus 5 because we know that one can put the equal sign when either the first or the second part give the same result, so it is like $3+5$ is equal to 8 ...

The discussion goes on, and all at once the teacher asks some students if they have already met sentences of the form $8 = 3 + 5$ and if they consider it as equivalent to $5 + 3 = 8$. Thus, she encourages the class to invent new equalities equivalent to $5 + 3 = 8$ and $8 = 3 + 5$. In this case, almost all students answer autonomously with different examples like $2 + 2 + 2 + 2 = 1 + 1 + \cdots + 1 = 0{,}9 + 7{,}1 = 13 - 5 = \cdots$ During this part of the discussion, we noted that the teacher exhorts students to continue suggesting equivalent sentences. Then, we report a second transcript example in which the teacher selects certain students based on their answers in the worksheets. In the following, the classroom is discussing about the sentence $9 + 5 = 14 + 5$. The teacher calls Student E out because her strategy is the same as that of another one.

Teacher:	Since we are dealing with small numbers, we know which is the result of both sides of the equal sign. But before, when I asked if the sentence [referred to $9 + 5 = 14 + 5$] was true or false, Student D explained why it was false without telling the result. Has someone reasoned in the same way? Student E?
Student E:	I did not compute $9 + 5$ either. I saw that five was in both the additions, only 9 and 14 changed. Since 14 is greater than 9, the results could not be the same.
Teacher:	Ok. Student F.
Student F:	I saw that five was in other positions, it was the number that was always present in both operations, so I look at the first two addend of both operations and I realize that, since 14 is greater than 9, the result [in the right side] should be greater...

After discussing with Student E, the teacher explicitly calls Student F out, because she remembers she used a relational strategy to solve the task in the worksheet1: she compared the addends 14 and 9 without any computation.

Finally, in the subsequent example the teacher does not look for a task in the worksheet or among those proposed in the videogame, but she invents a new one, in order to consolidate the reasoning explained in the previous transcript by Student F.

Teacher:	According to you, this reasoning... Now we have small numbers... But let's try to think with bigger ones. According to you, could this reasoning help us in solving: $3527 + 1528 = 3682 + 1528$? Are they the same?
Student F:	No!
Teacher:	$3682 + 1528$. Are they [referred to the sentences] the same?
Chorus:	No!
Student G:	No, no, no. It is the same... It is the same...
Student H:	It is the same.
Student G:	Because... 3527 is smaller than 3682, but the other addend is the same.

At first, students do not easily understand teacher's example, but afterwards one of them realises that there exists a similarity between the invented task and the one from the worksheet1. Then, the teacher carries on the discussion emphasizing the potentiality of the student's observation: $a + b = a + c$ if and only if $b = c$. Finally, a student concludes with the following remark: if $b > c$, then $a + b > a + c$.

Discussions

In this section, we analyse the results described previously by exploring the role of a GBL in supporting teacher's scaffolding during classroom discussions on relational thinking.

In the three examples, we highlight when and how the teacher uses the four scaffolding strategies described in the theoretical framework. Therefore, in all the three examples, the teacher chooses the tasks to discuss with students without following the order of the tasks in the worksheets (1) and she invites selected students to intervene during the discussion considering their responses in the worksheets (2). In the first excerpt, the teacher encourages students to invent new tasks to clarify or share the emerging mathematical ideas (4), while in the last example she invents a new task to clarify the emerging mathematical ideas (3).

We assumed that a GBL could have offered several scaffolding opportunities. Thus, it could be interesting to observe the role of the videogame in this trial. During the interview the teacher reports she did not rely on the web interface, because it only displays correct and wrong answers. Indeed, she calls students out during the discussion considering their answers in the worksheets and not referring to game scores. We also supposed that the large body of tasks could have allowed teachers and students to invent new examples of similar tasks based on reasoning by analogy. In fact, the examples provided by students are not like the ones presented in SuperFlat Math ($2+2+2+2=1+1+\cdots+1=0{,}9+7{,}1=13-5=\cdots$) but probably they seem to have been invented considering other contexts. However, the teacher invents a task very similar to the ones proposed in the videogame. In line with this data, it seems that GBL offered to the teacher only one of the four scaffolding strategies, but we asked her if the videogame was useful in this trial. She answers affirmatively: in particular, she states it was remarkably interesting to observe students while they were playing in order to discover their strengths and weaknesses on equalities. Furthermore, the teacher maintains the videogame was an essential feature in the trial because of its motivational aspects, such as its power to captivate students' attention or the goals and rewards within the game. Finally, the teacher declares that the discussion would not have been the same without the videogame.

Teacher:	To us, as primary school teachers, the videogame is very important. We should intervene more during play time so that the following activity with worksheets enables students to consolidate all concepts emerging from the game. In this way, with the worksheets we can verify if all our work was profitable. For me, the videogame is essential, and it should become a pleasant practice.

Conclusions

The aim of this paper is to analyse whether and how SuperFlat Math could support teachers in scaffolding activities that involve relational thinking. To do so, we organized a field trial which involved three phases: playing the videogame, written activities with worksheets and a whole class discussion. Before the trial, we structured a three meetings teacher training in which we proposed some theoretical references on relational thinking, and we presented the videogame and the web interface. From the discussion analysis we can determine that in several scenarios the teacher performed all kinds of expected scaffolding. The excerpts reported in the above section are unambiguous evidence of what we have already described. However, the presented results seem to show that the videogame was used by the teacher as a tool for scaffolding only in the case of inventing new tasks (3). For the other strategies, she preferred to use the worksheets to select specific tasks, students, and students' answers to be discussed. In addition, students did not refer to videogame tasks when they invented new ones. For our analysis we refer only to the four strategies presented by Carpenter, Franke, and Levi (2003; 2005); probably, a more general framework on teachers' scaffolding (e.g., Van de Pol, Volman & Beishuizen, 2010) could be more useful. The teacher states that the motivational aspect of the videogame plays a particularly significant role in promoting classroom discussions; it could be considered as two of the six scaffolding intentions (Van de Pol, Volman & Beishuizen, 2010): Recruitment and Contingency. In addition, SuperFlat Math seems a remarkably interesting tool in monitoring students' activities for designing the discussion. Indeed, the teacher reported that observing students play allowed her to conduct informal observation of students' arithmetic knowledge and skills. According to the theoretical framework by Stein and colleagues (2015), the act of observing students' behaviour could be included in both "anticipating" and "monitoring" practices, which are the first two phases of the model for orchestrating productive discussions. For this reason, we propose that observing students play might implicitly help teachers orchestrate the discussion, but we should investigate this hypothesis further, for example by conducting a new trial using the thinking aloud method (Fonteyn, Kuipers & Grobe, 1993). Finally, there are some aspects that could be interesting for future research. For instance, we could develop sequences of adaptive tasks within the videogame to emulate teacher's scaffolding. Moreover, the videogame could propose more structured feedbacks that partially help teachers and researchers discover students' processes. For this purpose, we could ask students to keep a "diary" during play time, where they could write down the strategies used to solve the proposed tasks.

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Teachers' perceptions of teaching and learning mathematics in the WhatsApp environment through the "Bagroup" project

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In this study, we investigated what opportunities for learning and teaching could be created using WhatsApp as a social network to help students prepare for the final secondary-school Bagrut (matriculation) exam in mathematics. Launched by the Center for Educational Technology three months before the Bagrut examination, we initiated WhatsApp groups meant to provide an online review project during which teachers integrated blended learning, and students presented problems with which they were having difficulties. During this initiative, we applied a quantitative and qualitative research model to analyze the teacher's perceptions about what learning and teaching opportunities were created. In this particular report, we focus only on the quantitative.

Keywords: Social network, communication, mathematics teachers' knowledge

Introduction

Communication is a necessary and essential component of learning and teaching processes. Plentiful, diverse communication allows students to organize their mathematical thinking; analyze, evaluate, and enhance their ability to express their mathematical thinking consistently and clearly alongside those of others; and make proper use of mathematical language to accurately express mathematical ideas NCTM (2000). During lessons, it is important that mathematics teachers initiate a discourse to promote communication between students and use diverse tools for explaining, making connections, solving problems, and raising persuasive arguments (NCTM, 2019).

"Orchestrating such discourse presents a unique challenge in online settings where discourse usually takes the form of discussions about shared readings or experiences rather than collaborative problem-solving of a mathematical task" (Morge et al. 2020, p. 216).

Teaching and learning mathematics with social networks

Learning via social media can take place in many settings. It invites learners to collaborate in meeting their learning goals by communicating with colleagues over the Web, learning in any environment that social network learning allows, expedite learning as a result of exposure to many, varied learning interactions between the group’s partners, anonymous learning, and active or passive learning, among other things (Naidoo & Kopung, 2020; Moodley, 2019). Social networks can also be integrated into student learning processes, and the interactions that occur between teacher and student and between the students themselves allow immediate feedback, improvement of thinking and reasoning skills, more focus on addressing learners' difficulties in learning and understanding processes, and exposure to the mathematical ideas of their peers (Biton & Segal, 2021; Freeman et al., 2016; Greenhow & Askari, 2017). Thus, it is important for mathematics teachers to acquire appropriate technological knowledge about teaching over the social networks, options offered for teaching in this environment, and methods for integrating appropriate pedagogy for optimal mathematics instruction.

Technological pedagogical and mathematical content knowledge

Based on Shulman’s work (1986), Ball and Bass (2003) defined the term “Mathematical Knowledge for Teaching” (MKT) as the knowledge that encompasses areas and levels of school mathematics, supports mathematics teachers’ connected ideas, and emphasizes their ability to plan, evaluate, integrate, and manage appropriate mathematical content for teaching. Subsequently, Ball et al. (2008) proposed six different components of MKT, one of which – Knowledge of Content and Students (KCS) – combines knowledge about students and about mathematics. It entails knowing how students think, what mathematical tasks may be easy/ difficult for them, and so on. This type of knowledge requires interaction between specific mathematical understanding and familiarity with students’ mathematical thinking. Koehler and Mishra (2009) coined the term “Technological, Pedagogical, and Content Knowledge” (TPACK), which is an amalgamation of the three types of knowledge. The concept of TPACK implies that these bodies of knowledge intersect at various levels of complexity, and it encompasses the knowledge teachers require to effectively integrate technology into their teaching (Schmidt et al., 2009). One component of TPACK that is relevant to this particular study is TPK−Technological Pedagogical Knowledge. This is familiarity with the range of technologies that can be integrated into teaching and understanding how their use can affect teaching methods.

Clearly, making use of any web-based social platform requires the teacher to have TPACK: the technological capacity to operate within the platform, the pedagogical skills to use it effectively, and (as goes without saying) the content knowledge they aim to impart to their students.

The "Bagroup" project

The Bagroup project was initiated by the Learning Center for Education Technology for the Ministry of Education to offer the use of the WhatsApp application to high school students preparing for their math matriculation exams (“Bagrut”) at all levels. WhatsApp groups of approximately 100 students from across the country (who were not necessarily acquainted with each other) were set up and overseen by a professional, experienced teacher. The project ran approximately two to three months before the date of the exam, during which hundreds of thousands of messages were sent that included scholastic content, questions and solutions, and explanations. The WhatApp-based project offered the advantages of immediacy (students could get an immediate, professional response from a teacher or peer), equality (the entire student body had access), mobile learning (the smartphone supplemented classroom instruction anywhere and anytime), encouragement (strengthened understanding, feelings of capability, and self-confidence) and variety (exposure to a myriad of learning and problem-solving methods).

The study’s aim was to determine teachers' and students' perceptions of the project. For this purpose, five research questions were posed, but in the present framework, we focus only on the question: What are teachers’ perceptions of the WhatsApp “Bagroup” project learning environment?

Method

Setting

Learning in each of the WhatsApp “Bagroup” groups was based on the “blended learning” format, which includes synchronous and non-synchronous sessions (Schwartz et al., 2017; Tella, 2014). The group was overseen by a professional teacher to ensure consistency, but most of the learning was peer-to-peer. A learning program was designed specifically for the project, and every day, questions in one of the pre-defined topics were raised and solved within the group. For each subject, two or three 45-minute-long WhatsApp lessons were held in which the teacher presented the topic and emphasized significant points of knowledge. Learning was continuous (24/7) and facilitated in a variety of ways: text messages, voice messages, photos, video, questions, presentations, and more.

Sampling method

A nonprobability sampling method was used based on the availability and willingness of participants to reply to the questionnaire.

Research tools

Four data collection tools were used: (1) Teacher questionnaire distributed at the end of the three-month learning period that included 11 six-point Likert-type statements and open-ended questions. The questionnaires were constructed together with and validated by a team of mathematics education experts to monitor and evaluate the actions of the participants, particularly because they were strangers to each other at the outset. Out of the 40 participating teachers, 24 responded; (2) informal semi-structured interviews at the end of the learning period (friendly conversations via video chat with three teachers and two project managers); (3) the content of the WhatsApp messages; and (4) observations of four groups of students (chosen at random).

Data analysis

Four-stage data analysis was both quantitative and qualitative and included (Corbin & Strauss, 2014; Creswell, 2014): (1) quantitative analysis of closed statements in questionnaires; (2) qualitative analysis of open questions in questionnaires and interviews; (3) cross-matching information obtained from quantitative and qualitative analyses; and (4) searching for evidence/episodes from the students' and teachers' messages to corroborate the findings obtained in the previous stages.

Findings

Our quantitative findings of the responses to the teachers’ questionnaire are presented in Table 1 (list of statements, and average score and standard deviation of each; (internal reliability, α =0.785).

Table 1: Averages and standard deviations of the components and statements that relate to teachers' experience in teaching mathematics via the WhatsApp “Bagroup” project

Statement	Av.	SD
Category 1. Contribution of the WhatsApp environment to learner's emotional needs	**3.56**	**0.89**

I managed to track the students’ progress via the WhatsApp discourse.	3.83	1.17
I managed to explain the material just as much as in a regular class.	3.33	1.05
WhatsApp allowed students to participate without fear of committing mistakes.	4.14	1.39
Written discourse in WhatsApp is preferable over verbal discourse in regular classes.	2.50	1.41
Teaching through WhatsApp allowed me to meet the specific needs of each student.	4.04	1.20
Category 2: Factors that promote learning in the WhatsApp environment	**4.20**	**0.86**
The lesson setup with WhatsApp differs from that in a regular lesson.	4.92	1.10
The WhatsApp method made me more familiar with available math-teaching math technologies.	3.96	1.27
Students seem to invest more time and effort in learning compared to in a regular classroom.	3.38	1.47
I believe that WhatsApp will become an integral tool for teaching math in the future.	4.61	1.34
Category 3: Factors that inhibit learning in the WhatsApp environment	**3.10**	**0.97**
Some mathematical content cannot be explained via WhatsApp.	3.29	1.43
I could not manage to generate students' collaboration like I do in a regular class.	2.92	1.35

Figure 1 illustrates the distribution of answers. For ease of presentation, the Likert scores were divided into three levels (agree, somewhat agree, disagree) and the respective percentages are indicated.

In the first category, “Contribution to learner’s emotional needs,” all the teachers agreed or somewhat agreed that it reduced students’ fear of making mistakes, the majority agreed or somewhat agreed that it allowed them to meet the specific needs of each student, and they all agreed or somewhat agreed that they were able to successfully track their students’ progress. However, the matter of written vs. verbal explanation did trouble the teachers as only about a quarter believed that written discourse is preferable over verbal discourse (a little over a third somewhat agreed).

In the second category, "Factors that promote learning in the WhatsApp environment," all the teachers agreed or somewhat agreed that the lesson setup with WhatsApp was quite different than a regular lesson, that WhatsApp would become an integral tool for future mathematics instruction, and that teaching through WhatsApp improved their familiarity with the technologies available for teaching math. However, there was some difference of opinion regarding whether students invested more time and effort in learning compared to students in a regular classroom": although over four-fifths agreed or somewhat agreed, almost a fifth disagreed.

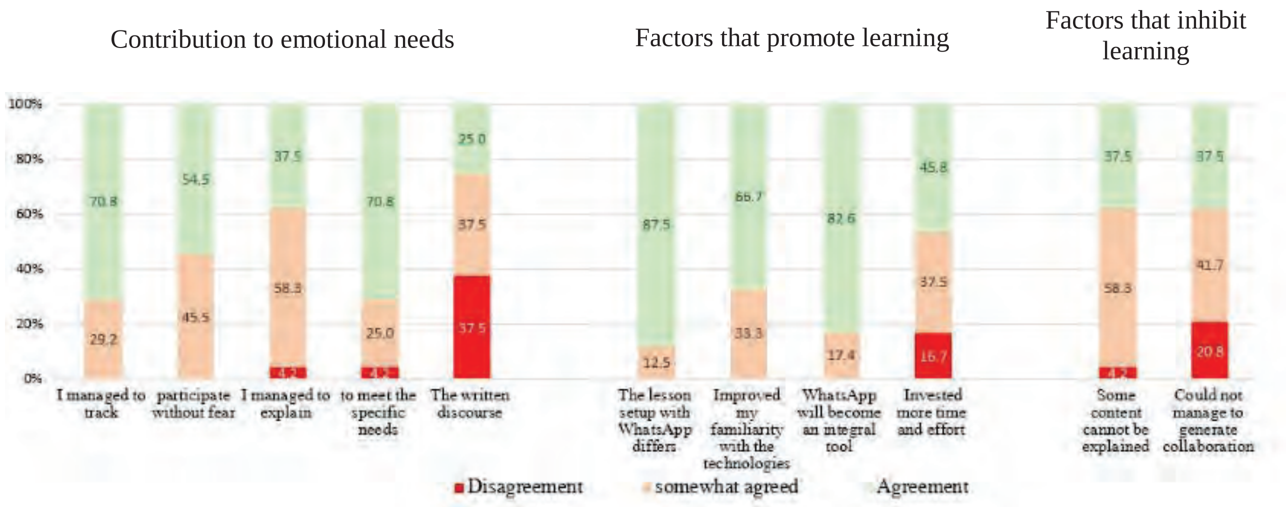

Figure 1: Levels of teachers' agreement with statements concerning their experience teaching mathematics via WhatsApp

In the third category, "Factors that *inhibit* learning in the WhatsApp environment," about a third of the teachers agreed and over half somewhat agreed that some mathematical content **cannot** be explained via WhatsApp." With respect to the last statement, "I could not manage to generate collaboration between students like I do in a regular class," note that this is a "negative statement," meaning that agreement illustrates a disadvantage. Almost 80% agreed or somewhat agreed with the statement meaning that only about a fifth did not find encouraging collaboration to be a problem.

Discussion

Because learning opportunities are created through the interactions of (at least) tasks, teaching, and students, measures of learning opportunities will need to develop through analyzing interactions among these factors and describing how some interactions help students achieve a specified learning goal more than others (Cai et al., 2020. p. 19).

The aim of this study was to investigate teachers' perceptions of the WhatsApp Bagroup program environment as a lever for learning and teaching opportunities. Analysis of the teacher questionnaires gives the overall impression that the teachers feel the program does indeed address students' emotional and scholastic needs perceived by them.

Emotional needs

The program involved interaction between students and teachers and between students and their peers through respectful communication. Each student's ideas were taken seriously and everyone had the opportunity to ask questions, make statements, and express their ideas. These have been shown to be necessary for a successful learning atmosphere (Chapin et al., 2013).

The first category of statements was to determine whether the teachers believed this platform met the learner's emotional needs. Overall, the response was favorable. Teachers believed that it allowed them to meet the specific needs of each student (4.04): most (70.8%) agreed; 25% somewhat agreed with the relevant statements. Only 4.2% disagreed. Similarly, fully 94.8% felt that the platform allowed them to successfully explain concepts or understand their students' difficulties (37.5%:

agree; 58.3%: somewhat agree) and only 4.2% did not. Results also showed that 100% of the teachers managed – either fully (70.8%) or somewhat (29.2%) – to properly track their student's progress through the WhatsApp discourse (3.83), a factor that would help meet students' emotional needs.

All the teachers either fully (54.5%) or somewhat (45.5%) agreed that the platform freed students from the fear of making errors (average 4.14), clearly a factor in meeting students' emotional needs. However, there seemed to be some hesitancy regarding whether the written discourse was efficient enough to meet student needs, as over a third (37.5%) disagreed with the statement "The written discourse in WhatsApp is preferable over the verbal discourse in a regular class" (average 2.5). Only 25% of the teachers fully agreed that it was preferable and 37.5% somewhat agreed.

Overall, it seems that the teachers' personal approach to the students, meeting needs, and providing a feeling that any question is valid encouraged learning and provided emotional support.

Scholastic needs

Most of the teachers expressed satisfaction with their ability to track students' responses and explain mathematical concepts over the platform, thus meeting their students' scholastic needs. However, many did not agree that written discourse was preferable (see above).

Regarding how much students invested in learning in this platform, most of the teachers agreed (45.8%) or somewhat agreed (37.5%) that the students in this project invested more time and effort in learning compared to students in a regular classroom" (3.38); about a fifth (16.7%) disagreed.

Challenges to teachers

It is not surprising that 87.5% of teachers fully agreed with "The lesson setup with WhatsApp differs from that of a regular lesson" (average 4.92), as an internet chat platform is clearly different from frontal teaching (even over the internet, such as with ZOOM), since all answers must be composed in writing. Regarding their success (or not) in explaining the content over the platform, it seems that the majority of teachers had difficulty, as a third of teachers agreed (37.5%) and over half of them somewhat agreed (58.3%) with "Some mathematical content cannot be explained via WhatsApp" (average 3.29). Only 4.2% seemed to have no problem explaining content over the platform.

Finally comes the issue of successful student collaboration (average 2.92). Only 20.8% of the teachers responded that they had no problem getting their students to collaborate, 41.7% seemed to have some problems with this, and over a third (37.5%) reported that they could not manage to generate collaboration between students like they do in a regular class.

Nevertheless, despite these challenges, all teachers believed (82.6%) or somewhat agreed (17.4) that WhatsApp (or similar) will become an integral tool for teaching math in the future (average 4.61). Furthermore, all the teachers responded that teaching through WhatsApp improved their mastery with technologies available for teaching math (66.7% agreed, 33% somewhat agreed, average 3.96).

Conclusion

Naidoo and Kopung (2020) emphasized the contribution that collaborative learning via WhatsApp made to pre-service mathematics teachers: it encourages ubiquitous, prompt, and anonymous mathematical learning. In the present study, in-service mathematics teachers are teaching students whom they have not met before, and the teachers identified a similar contribution to the students alongside its contribution to them as mathematics teachers. The majority of teachers reported satisfaction with the program: it allowed them to track student progress, assisted them inefficiently in solving problems, and helped them identify where students were having difficulty. They also felt that it helped meet students' emotional needs because students were not afraid to make mistakes, and they received an appropriate and immediate response to any difficulties according to their specific needs.

The teachers seemed to feel that the program increased their TPK (Schmidt et al., 2009) because they were forced to become familiar with a variety of innovative teaching-learning processes in technological environments and gain experience in how to teach, explain, and present mathematical processes and concepts. Nevertheless, the environment posed some challenges: for example, not all mathematical content is appropriate for this environment and written discourse is not always an alternative for verbal discourse to fully explain some concepts.

These and similar programs may help predict learning or teaching responses that have not yet been identified. Similar studies may assist in shaping future research into technological integration into formal education, leading to the introduction of solutions to ongoing educational problems or altering prevailing educational norms and methods towards more beneficial applications.

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Implementing the 5E inquiry model in an online platform of a flipped classroom environment

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In light of COVID-19 outbreak and the subsequent closures of educational institutions, the need for online environments has significantly increased. This study sheds light on the online component of a mathematics flipped classroom (FC) and aims to investigate if, and to what extent, it can effectively utilize inquiry-based learning (IBL). Based on the 5E inquiry model, the study focuses on an advanced mathematics course for high-school students taught in an FC environment. Analysis of seven filmed lectures on the subject of complex numbers was conducted using a validated 5E inquiry model scoring instrument. Results indicate promising findings for a wisely planned virtual platform that answers the 5E model requirements and may successfully demonstrate an IBL-supported environment.

Keywords: Inquiry-based learning, flipped classroom, 5E Model, mathematics education

Introduction

The difficult period resulting from the COVID-19 outbreak, which includes, among other things, the closure of educational institutions, has highlighted an urgent need for a worthy alternative to the traditional teaching method (Dhawan, 2020). Most recently, and due to COVID-19 pandemic, a huge variety of online learning environments have emerged. A potential alternative approach is the Flipped Classroom (FC), with the goal to provide students with a supportive environment alongside a deep and meaningful learning experience (Sharkia & Kohen, 2021). The research environment that served this study is a MOOC called Campus IL, an Israeli national digital learning venture, which offers the users the opportunity to experience an advanced and individualized learning process, by providing a huge variety of online content. Particularly, the FC method can be an ideal venue for transforming traditional learning into an engaging, inquiry-based learning (IBL) environment (Love et al., 2015). The FC approach includes two phases, of which the first phase occurs out of class, where students are expected to undergo an independent learning process, while the second phase takes place inside the classroom and involves extensive practice exercises of the materials learned outside of class (Bergmann & Sams, 2012). Previous studies in the field have explored the application of IBL in the physical classes, as the FC approach frees up class time for IBL-type activities (Sharkia & Kohen, 2021). Yet, limited research had been done on the application of IBL in the online component of the FC environment (Love et al., 2015). Thus, the current study aims to investigate the application of IBL in the online platform of an FC learning environment. We aim to explore whether, how, and to what extent an online platform for a mathematics FC utilizes IBL in the most effective way.

Theoretical Background

The FC Approach

Two components comprise the FC approach; the first is the independent learning process that takes place outside of the classroom, and the second is the active and collaborative lessons inside the classroom (Bergmann & Sams, 2012). The FC approach utilizes various technological means to provide students with instructional materials and related exercises. These technological resources allow students to learn the content outside of class (Dori, Kohen, & Rizowy, 2020), hence offer teachers considerable time in class to provide deeper explanations and practice alongside the students (Bergmann & Sams, 2012). The online component of the FC is where teachers provide filmed lectures, presentations, online assignments, and more, and expect students to independently learn the provided content before arriving to the class. With that, and according to Lo and Hew (2017), this approach promotes student-centered learning experience. Additionally, it allows a wiser management and exploitation of class time, by enabling teachers to roam around the classroom, identifying various individual difficulties, detecting different challenges and misconceptions among the students, and finally react accordingly by giving these students the support and encouragement they need (Lo & Hew, 2017). A main advantage of the class component of the FC approach is that it provides students with adequate amount of class time to extensively and collaboratively work on exercises alongside their teacher, which is considered an essential condition to master mathematical skills and achieve comprehensive understanding of the material (Kaiser & Vollstedt, 2007). A main advantage for the online component of the FC approach is that the instructional content is available for students out of class any time and place, hence they can access it countless times until they accomplish full comprehension (Lo & Hew, 2017). As a result of the COVID-19 pandemic, the "traditional" flipped classroom, in which students meet with the teacher in person, was almost impossible to adopt. Instead many institutions have integrated online teaching with FC. Research has shown that this combination resulted in positive effects on students, including increased learning, comprehension and attention, as well as positive evaluations of a variety of taught courses (Tang et al., 2020).

The 5E Inquiry Model

According to the 5E model (Bybee et al., 2006; Bybee, 2009), learning through inquiry follows five phases, which are identified as follows: (i) engagement, (ii) exploration, (iii) explanation, (iv) elaboration, and (v) evaluation. These phases can be implemented to various levels when planning and constructing different curriculum materials, lesson plans, and instructional strategies (Bybee et al., 2006). Classroom inquiry is composed of five essential characteristics (National Research Council, 2000), which can be addressed by applying the 5E model during the learning process (Schallert et al., 2020). The first characteristic is engaging the learner in scientific questions (National Research Council, 2000, p. 29). This feature can be found in the first phase of the 5E model, namely, engagement, which enables students to engage in the learning activities. In this phase, teachers need to motivate and engage students by presenting a certain problem that requires students' attention. The next phase is exploration. As soon as students are engaged and motivated in a certain activity, they should be able to explore their thoughts and abilities. In this phase, classroom inquiry is characterized by prioritising the evidence in response to questions, and formulating explanations based on evidence (National Research Council, 2000, p. 29). The next phase of the 5E model is explanation, in which concepts, ideas, procedures, and skills become clear and understandable. This phase relates to another inquiry characteristic that comprises explanations connected to scientific knowledge and communication and justification of explanations (National Research Council, 2000, p. 29). The explanation phase describes these features, offering students the opportunity to create connections between their explanations to scientific knowledge and further justify them. The next phase is the elaboration of the problem and its solution(s). In this phase teachers provide students with further experiences, tasks and challenges aiming to expose them to new but similar situations. The final phase of the 5E model is the evaluation, in which teachers are expected to provide their students with the appropriate feedback on the quality of their performance. Figure 1 presents visually the 5E inquiry model.

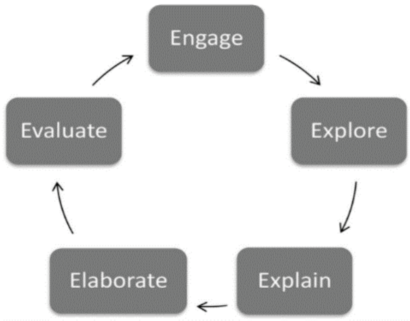

Figure 1: The 5E inquiry model (Schallert et al. 2020, based on Bybee and colleagues' model, 2006)

The research question of this study is: How, and to what extent, the online platform of the FC can effectively utilize the IBL approach.

Method

Research Environment - Campus IL

The current study focuses on one course that is held in Campus IL. Campus IL is a joint venture from Digital Israel and the Council for Higher Education that aims to provide all Israeli citizens the opportunity to pursue education and engage in intellectual development. This environment includes a huge variety of courses and resources from highly respected colleges, universities and other academic organizations. Having gained access to Campus IL, the user can benefit from a personalized and unique learning experience. The current study utilizes an advanced mathematics course in Campus IL. This course is designed for high school students who study advanced mathematics, and aims not only to prepare them for the final matriculation exam in mathematics that takes place at the end of 11^{th} and 12^{th} grades, but also to expand their horizons and nurture their mathematical thinking and prepare them for first year college mathematics.

All the content of this course is conveyed through short, filmed lectures that were filmed and produced in a specially designed studio at the Technion institution, under the supervision and direction of a professional photography and editing team. The average length of each video is roughly 5 minutes, and it is mostly followed by short assessment tests and quizzes that aim to assess students' comprehension of the mathematical content they have recently watched. Dr. Aviv Censor, one of the outstanding lecturers in the Technion institution (according to national and institutional instructional surveys) is the teacher who delivers the content in all the filmed lectures. In addition, an academic team under his supervision has developed, created, and written all the evaluation tests in a manner that suits the level taught in the videos. Figure 2 presents a screenshot taken of a filmed lecture about the subject of complex numbers.

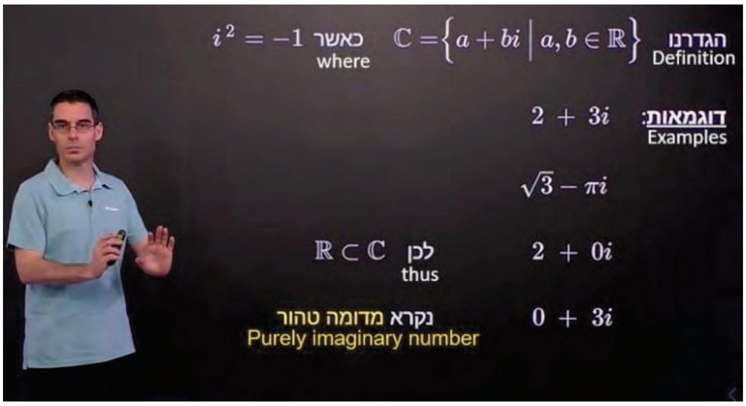

Figure 2: The teacher in a filmed lecture about complex numbers

Research Tools and Analysis

Research tools include a scoring instrument that was developed and validated in a research study conducted by Goldston and colleagues (2013). This scoring rubric was designed to evaluate IBL lesson plans according to the 5E model by Bybee et al. (2006) and is considered a reliable tool for its total reliability score that reached 0.98. It consists of several items for each one of the 5E model phases. Each item is given a score that ranges from 0 to 4, based on a 5-point Likert scale, where 0 stands for unacceptable, 1 is poor, 2 refers to average, 3 means good, and 4 represents excellent. This rubric was also found to help teachers and educators to revise their strategies of how to design a 5E-based lesson. The current study mainly used several sections of this rubric to assist in evaluating filmed lectures, in an attempt to investigate if and to what extent these lectures employ the IBL method. The elements of all phases were considered except for the explanation phase, which is basically based on assessing student perspective, therefore could not be taken into account in the scoring process which evaluates the teacher perspective in the filmed lectures. In the present study, we chose to present the analysis for the first seven filmed lectures which provides introduction to the subject of complex numbers. This subject is part of the advanced mathematics curriculum, that is required in the second matriculation test conducted at the end of 12th grade. These lectures started with three introductory videos, the next two videos explained the emergence of the groups of numbers and provided a definition of a complex number, and the last two videos illustrated the algebraic and geometric representations of a complex number. The videos were transcribed and evaluated quantitively according to the scoring rubric described above. We also present qualitative analysis of the videos, for examining the narratives of IBL that were captured in the videos.

Findings

In order to reveal the exemplification of IBL in the online platform of the mathematics FC Campus IL, a thorough and comprehensive analysis of several filmed lectures was completed. A score was given to each one of the different elements that are characterizing the various phases of the 5E model. Figure 3 below illustrates the means and standard deviations of the scores obtained to the seven filmed lectures. Scores were calculated separately for each one of the four considered phases.

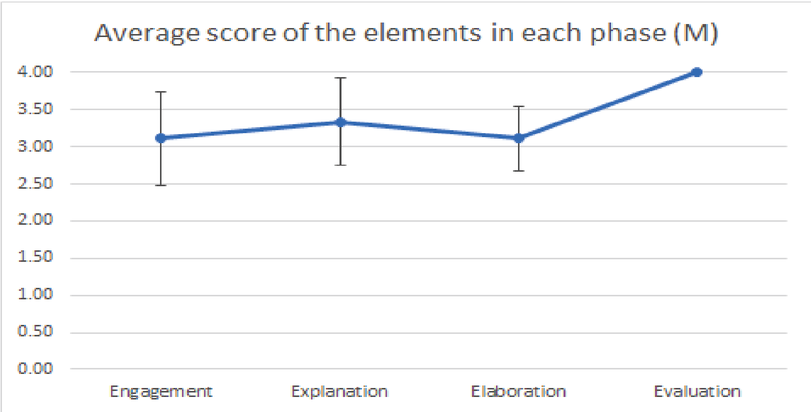

Figure 3: Average scores of the elements composing each phase of the 5E model

Figure 3 shows that all the examined elements were evident in the lectures and reached a score that is higher than 3, meaning that these lectures highly fulfill the requirements for implementing IBL during a mathematics lesson. Besides means and standard deviations, frequency distribution of scores that were given for each component was calculated. See Table 1 reflects the frequency distribution of scores that were given for each component, for all the seven explored filmed lectures.

Table 1: Frequency distribution of scores that were given for each component

Phase	Score (2)	Score (3)	Score (4)
Engagement	22.22%	44.44%	33.33%
Explanation	16.67%	33.33%	50.00%
Elaboration	11.11%	66.67%	22.22%
Evaluation	0.00%	0.00%	100.00%

It can be seen that score (4) has the highest frequency in both explanation and evaluation components while they have different means. Similarly, score (3) has the highest frequency in both engagement and elaboration components. These differences in the distribution can explain the differences in standard deviation in each component. Specifically, when comparing all four phases, it can be seen that the explanation and evaluation components were the most prominent in the examined lectures. This indicates that the explanation phase that is adopted in the videos largely promotes and contributes to inquiry in the classroom. Particularly, aiming to encourage and promote profound understanding, throughout this phase it was evident that the teacher was determined to introduce concepts that students were unfamiliar with or not aware of. Presenting a great variety of examples and strategies in the filmed lectures, the teacher was able explain these concepts and theories. Another interesting finding that is revealed is that classroom inquiry can be more efficient when the teacher performs evaluation and assessment activities at the end of the lecture.

From a qualitative perspective, we now describe some utterances that were observed in the filmed lectures, which illustrate the appearance of the inquiry components in part of the videos that were analyzed. In the introduction lecture (#1), the teacher opens the lecture with an interesting starting point in an attempt to gain students' engagement. He asked the students the following: "Did you know that bears can count to 4?!”, then continues: “When walking in bear-inhabited areas, the instructions recommend staying in groups of at least 5 people, as bears tend to avoid groups of more than four”. Following the intriguing introduction about counting natural numbers, the teacher refers to negative numbers, and asks: “what is -3?” and immediately tries to bring real life examples to illustrate the meaning of negative numbers. He then asks students an interesting question that stimulates students’ knowledge from their early years of study: “Back in 6th grade, you were taught that multiplying two negative numbers results in a positive number. Why is this true? I would like you to pause the video for a moment and try answering this question yourself”. The teacher tries to raise simple yet challenging questions to motivate the students and attain their attention and engagement, this is crucial to assure students will be intrigued to continue watching the next lectures seeking for logical answers. Following, in the second introduction lecture (#2), the teacher enters the explanation phase, in which he intends to answer the previously raised question, and thoroughly explain it to the students. At this stage, the teacher chooses a creative explanation of the answer, and uses a filmed example attempting to convey the concept of “the product of two negative numbers is a positive number”. He gradually poses a variety of questions to help students develop their understanding and skills, such as: “In this video, you can see Maya walking at a speed of 3 km/h. What if we run it twice the original speed? What happens if we run the video backwards? what if we run it backwards twice the original speed? etc.”. After the explanation phase the teacher moves to the next phase of elaboration revealed in lecture (#3) of the introductory films, in which he presents new problems, yet related to the previous parts of the introduction. This lecture aimed to elaborate the concept of square root of a negative number. He uses simple explanations, examples, and graphs to illustrate that no square root exists for a negative number. And from this point, he mentions the main subject of complex numbers by saying: “There were some who believed differently and raised the idea of imaginary numbers”. He concludes this video by adding that: “Let us consider complex numbers, without which there would be no Einstein relativity, no quantum theory, no autonomous cars, and no noise-cancelling headphones”. Doing so, he emphasizes the importance of mathematics in general and the significance of the complex numbers subject to students’ real life. All the above examples of this teacher’s lectures demonstrate the actual use of IBL in the online platform of the FC, through a variety of wise questions, certain metaphors, and unique delivery methods of the mathematical content. These examples supplement the quantitative findings which indicate a high degree of IBL applied by this teacher in the various online filmed lectures.

Discussion

According to the Principles and Standards for School Mathematics (National Council of Teachers of Mathematics, NCTM, 2000), an IBL approach should be integral to and at the core of good instructional practices. Yet, to promote rigorous and meaningful IBL, teachers should constantly strive to improve the quality of inquiry instructional practice (Marshall et al., 2006). In the present study, we provide insights into implementing IBL in mathematics flipped classrooms. It investigates if, how, and to what extent the online platform of the flipped classroom can effectively utilize the IBL approach. Using a validated scoring rubric (Goldston et al., 2013), analysis and scoring of several filmed lectures were accomplished. The findings indicate that the instructional process in these examined lectures exemplify the characteristics of an IBL method. Specifically, findings reveal that the explanation phase was prominently observed. This finding aligns with several characteristics for IBL defined by the National Research Council (2000). It was evident in the lectures that the teacher tried to raise questions that require the students' exploration, and subsequently offered them explanations that are science based. Thereby, teaching them to link their own explanation to scientific knowledge which leads to further comprehensible justifications. Another significant finding is that once evaluation process was observed, it indicated an excellence implementation of IBL. According to Bybee (2006), this is a critical phase, which enables not only teachers but also students to evaluate and assess their comprehension. Through applying short quizzes and a variety of tests, teachers may be able to evaluate the improvement of students' understanding and abilities (Schallert et al., 2020).

From a theoretical perspective, the study contributes to the limited literature about the interrelationship between IBL and FC, particularly referring to the online component of FC. From a practical perspective, we present promising findings indicating that when designed wisely according to the 5E inquiry model, the virtual component of the flipped classroom can represent an IBL environment which students will benefit from, especially in difficult periods such as the current COVID-19 period.

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Teachers’ classroom use of dynamic mathematical technology to address misconceptions about geometric similarity

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This paper examines what misconceptions about geometric similarity (GS) teachers can attend in their curriculum scripts and how they use dynamic mathematical technology (DMT) in the classroom to promote students to encounter, reflect upon and address their misconceptions. The Structuring Features of Classroom Practice (SFCP) construct of ‘curriculum script’ guided the collection and analysis of the data presented in the paper. The research worked with three teachers with different levels of experience and expertise in using DMT. Data collection involved video-recorded observations, audio-recorded, semi-structured post-lesson teacher interviews and lesson resources. The findings suggested there were differences in the quantity of misconceptions the teachers anticipated and/or identified and in the ways in which they used the DMT to help students confront, reflect upon and address their misconceptions.

Keywords: Geometric similarity, misconception, dynamic mathematical technology, teacher.

Introduction

Researchers have underlined that there is still little understanding of the phenomenon of the integration of dynamic mathematical technology (DMT) into classroom practice and we need more case studies guided by the available different theoretical lenses to explore teachers’ DMT-enriched practices within ‘real’ classroom settings (e.g. Ruthven, 2014). In our research, we aimed to contribute to addressing this gap by investigating secondary teachers’ actual classroom practices as they teach the key mathematical domain of geometric similarity (GS) with a carefully designed DMT. We focused on GS since it is a key but also difficult concept for students (Clark-Wilson & Hoyles, 2017). For instance, research has documented that the improper use of additive reasoning is one of the most common student misconceptions about GS (Noss & Hoyles, 1996). However, despite its importance in school mathematics and the difficulties that it poses to students, GS has been neglected in research on the integration of DMT into practice. This is surprising as the research evidence suggests that students’ engagement with DMT could help them overcome their (possible) misconception(s) with GS (Clark-Wilson & Hoyles, 2017).

The current paper presents part of a broader PhD study (Simsek, 2021), conducted by the first author and supervised by the co-authors. The paper concerns particularly teachers’ use of DMT in their practices to support students to confront, reflect upon and address their (possible) misconception(s) about GS. We formed our research question for this paper as follows: *What student misconceptions about the topic of geometric similarity can teachers attend in their curriculum scripts and how do they use dynamic mathematical technology in their classroom practices to support students to encounter, reflect upon and address them?*

The 'Curriculum Script' Construct of the SFCP Framework

We selected and used the Structuring Features of Classroom Practice (SFCP) framework to guide the entire research process of the broader study including both data collection and analysis. The lenses of this framework specifically concentrate on the five key structuring features of classroom practice that shape how teachers incorporate (new) technologies into their practices, namely *curriculum script*, *resource system*, *activity format*, *working environment* and *time economy*. The framework underlines how these features pertain to teachers' integration of technologies into the classroom and identifies the examples of the associated teacher craft knowledge in relation to integration of technologies (the full description of the framework can be found in Ruthven (2014)). Ruthven (2014) discusses teachers' craft knowledge is mainly tacit, illustrating why many teachers find it difficult to articulate or might not even be aware of. Teachers develop this type of knowledge, which is dynamic and evolving, primarily through "their own practical experience of learning and teaching the topic" or from "available curriculum materials" (Ruthven, 2009, p. 138). The SFCP framework therefore serves to identify and analyse teachers' craft knowledge that underpins successful classroom practice with (new) technologies.

Due to the focus of the present paper, we briefly describe *only* the construct of *curriculum script* as it also involves student misconceptions about a topic teachers anticipate and/or identify and how teachers address them using different pedagogical tools including DMT(s). The concept of *curriculum script* can be defined, in the psychological sense, as an event-structured organisation of knowledge incorporating well-defined sets of learning objectives, expectations and actions in a loosely logical sequence for a mathematical domain, likely misconceptions related to the domain, together with resources, activities, associated language and questions, and pedagogic strategies (Ruthven, 2009). When teachers implement (new) technologies in their practices, they need to draw on their knowledge to develop and adapt their curriculum script for teaching a topic. This script guides the ways teachers create the overall structure for a lesson (i.e. the dynamic plan or set of goals and actions for a particular lesson) and enact it in a flexible and responsive way (Ruthven, 2014). It is important to highlight a curriculum script goes beyond a lesson plan created by teachers in the written form in advance a lesson. In this sense, while a lesson plan seems to be fixed and contain details that teachers are able to articulate, a script can be considered as dynamic and evolving in light of what happens in the classroom and encompass content of teachers' craft knowledge that many of them are not be able to articulate or might not be aware of.

Design and Methods of the Study

This research adopted a multiple case study approach. We were able to identify and work with three teachers who had been previously involved in the original Cornerstone Maths (CM) project in England (see the details of the project in Clark-Wilson and Hoyles (2017)) and expressed an interest in being part of the research. We recruited them from the community of the project as they had engaged with a particular DMT tool, the CM software, and they were committed to use the DMT in their own classroom practice to teach CM curriculum unit on GS (including the CM software, the student booklet and the associated teacher guide) to students aged 13-14 years. The participant teachers were from the two different London-based co-educational secondary schools. While Jack (pseudonym) taught in a school in the east London, Alex and Lara (pseudonyms) were colleagues and taught in the north-west London. We expected Jack to exemplify characteristics of an *expert teacher* in the use of DMT for teaching and learning of mathematics as he had high-level of involvement in the original CM project and considerable experience and expertise in the use of DMTs. Whereas, we expected Alex and Lara to exemplify characteristics of an *advanced beginner teacher* as they had mid-level of involvement in the original CM project and some experience and expertise in using DMTs (Berliner, 2004). We assumed such selection would be helpful to elucidate differences in the characteristics of teachers' practices with DMT in relation to GS.

The CM software is a web-based DMT designed to support students' understanding of the set of three CM curriculum units for lower secondary mathematics in England. The units include GS, algebraic patterns and expressions, and linear functions. The software was designed by exploiting the dynamic, visual and multi-representational potential of digital technology. It contains several carefully designed dynamic tasks related to each of CM units and promotes the use of the 'predict-check-explain' pedagogical approach. These tasks were created on the basis of realistic contexts in a learning environment in which there are dynamically linked multiple mathematical representations such as geometric shapes, ratio checker and measurement tables. The CM software is therefore intended to offer the potential for students to make conjectures and test them by manipulating such representations and to explore the underlying mathematical concepts and relationships in a realistic context. In terms of GS, the tasks allow students to engage with, for example, shapes, ratio checker and measurement table by using several key features (e.g. scale factor slider, angle slider, dragging) in the dynamic learning environment. Therefore, they could potentially recognise and explore the embedded variant and invariant properties of mathematically similar shapes and how the properties are used to determine similarity. The CM software is also intended to help students encounter, reflect upon and overcome their (possible) misconceptions about GS.

Data collection involved video-recorded lesson observations, audio-recorded, semi-structured post-lesson teacher interviews and lesson resources. The first author observed eight each of both Lara's and Jack's lessons and seven of Alex's lessons. In the case of Jack, he conducted his lessons in his normal classroom with laptops. In the cases of Lara and Alex, they both taught their lessons either in a pre-booked computer room with desktop computers or in an ordinary traditional classroom with iPad computers. Moreover, the first author interviewed Alex and Jack six times and Lara eight times, which lasted between 35-45 minutes. Lastly, we collected teachers' interactive whiteboard or PowerPoint slides, examples of their task sheets, photographs of students' DMT screens and their associated written work in the workbook in response to the DMT-enriched tasks. For data analysis, a *within*-case analysis was carried out first to create an individual detailed description of each case in written form that helped examine each case in depth. The within-case analysis led to the identification of what student misconceptions about GS each case study teacher attended in her/his curriculum script and how (s)he used the DMT to help students address them. Then, using spreadsheets, a *cross-case analysis* was conducted by comparing and contrasting the cases based on the results of the within case analysis. The cross-case analysis resulted in the identification of the key differences between the teachers in terms of their anticipation and identification of student misconceptions about GS and their exploitation of the DMT to tackle them in the classroom.

Findings

In this paper, we present findings emanated from our data analysis that concern the case study teachers' anticipation and identification of student misconceptions regarding GS and of the ways through which to enable students to confront, reflect upon and address their (possible) misconceptions using the DMT. The within-case-analysis revealed the following student misconceptions about the topic of GS evident in the teachers' curriculum scripts.

1. If two shapes look similar in appearance (e.g. rectangles), they are mathematically similar;
2. An enlargement results in the creation of a mathematically similar shape which is *always* larger than the original;
3. A scale factor makes a shape only doubled and/or halved;
4. The use of additive strategies within GS is appropriate when determining whether shapes are mathematically similar;
5. If the lengths of a shape are multiplied by a scale factor of k, then the area of this shape is multiplied by a scale factor of k, too;
6. The proportionality of all corresponding pairs of sides is by itself sufficient to prove the similarity for shapes; and
7. The angles of a shape are multiplied by a scale factor along with its side lengths.

The cross-case analysis indicated that compared to Alex and Lara, Jack, as an expert teacher, showed more awareness of the likely misconceptions about GS and of the potential teaching strategies incorporating the use of the DMT in a dynamic mode to tackle them. While Jack anticipated and identified a total of six of the above stated seven key misconceptions and made dynamic use of the DMT to provide opportunities for students to address them, Alex and Lara predicted only three and two different misconceptions of them, respectively, and did not necessarily use the DMT in the same way as Jack did. In the following, we present results for each case to provide an insight into what misconceptions out of the above given ones each teacher attended in their curriculum script and how each used the DMT in their classroom practice to support students to encounter, reflect upon and address them.

Case Study 1: Lara

In her curriculum script, Lara identified the first and second misconceptions underlined above that her students faced when engaging with the DMT-enriched tasks. However, she did not make dynamic use of the DMT herself in the observed lessons to assist students in addressing them. Below, we present analysis of the second misconception to provide an insight into how she addressed a misconception with (and without) the DMT.

Promoting students' understanding that an enlargement does not only result in the creation of a similar shape larger than the original

In her post-lesson interview, Lara expressed students struggled to understand an enlargement may also result in a shape being smaller than the original as they were inclined to think an enlargement is a mathematically similar shape that should be *always* larger than the original. She outlined she recognised this misconception when interacting with a pair of students as they asked her to confirm what she had said in the previous lesson regarding the possible impact of an enlargement on the sides of shapes. She also stated students' engagement with one of the DMT-enriched tasks, where students predicted, tested and explored if the rectangles were mathematically similar to the original on the basis of visual and numerical cues, helped them realise an enlargement may create also a smaller similar shape than the original.

> They [students] cannot connect that an enlargement also makes a shape smaller because, in their mind, an enlargement is something getting bigger. They told me [that] "You said yesterday an enlargement can make the shapes smaller as well". So, for them, by themselves, they are not connecting it until we have done this [the DMT-enriched task]. So, from the original to copy 2 [the names of the shapes being enlarged in the task], they can see the bigger shape [the original shape] is getting smaller. Before that, they could not, that is how I assumed or observed.

However, Lara herself did not use the DMT dynamically to help students realise and explore that an enlargement is a similar shape which may be also smaller than the original. Instead, she only allowed students to use the DMT during their independent work with the DMT to confront and reflect upon this misconception by themselves.

Case Study 2: Alex

In his curriculum script, Alex anticipated the second, fourth and sixth misconceptions given above and addressed them with (and without) the DMT in the classroom. We present how he addressed the third misconception with the DMT to offer an insight into this aspect of his curriculum script.

Promoting students' understanding that the proportionality of all corresponding pairs of sides is not by itself sufficient to prove the similarity of shapes

Alex was aware students need to understand the need to consider both the corresponding angles and sides when determining if shapes are mathematically similar.

> If they [students] do not bring corresponding angles and corresponding sides together, they can go into an error. When they just focus on the [corresponding] sides, they might miss the fact that the corresponding angles [in mathematically similar shapes] need to be the same. They [students] need to be aware of both.

In the lesson that featured Task 5.2 (see Figure 1), during the subsequent whole-class discussion, Alex himself used the DMT dynamically to promote students' emergent understandings that inequivalent corresponding angles do not result in a similar shape to the original. From his computer desktop, he used the DMT to demonstrate and share students' responses with the class and to stimulate a discussion. For example, in the dynamic learning environment, Alex first measured the corresponding angles of the original and the parallelograms named Copy 1 and Copy 2 and then dragged the angle slider to change the angles of the original and Copy 1, followed by two questions he posed to the class: "What did you notice about all the angles?" and "What is going on with the angles at the moment?".

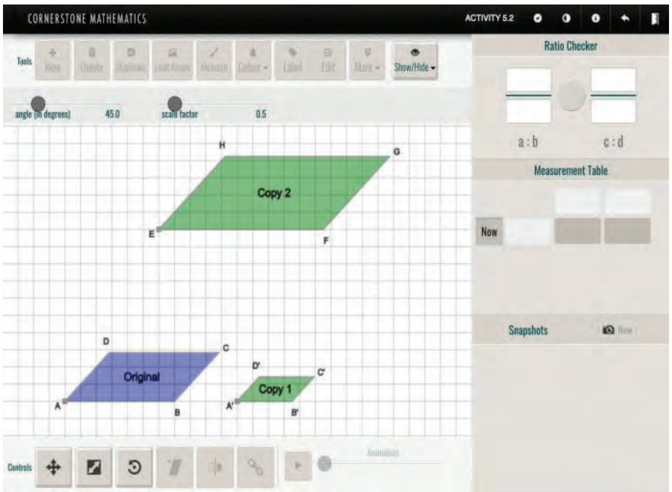

Figure 1: Task 5.2 in the DMT where students drag the angle and scale factor sliders and compare the three parallelograms as they change

Alex also used the types of geometric transformations (e.g. translation, rotation, enlargement) to superimpose Copy 1 on the top of Copy 2 and then dragged the angle slider. His aim was to show and confirm that since the corresponding angles were the same initially, Copy 1 and Copy 2 were mathematically similar, as the angle slider was dragged, the corresponding angles did not remain the same, which made Copy 1 and Copy 2 not mathematically similar. In his post-lesson interview, Alex outlined the DMT enabled students to identify and examine both the corresponding angles and sides, especially through the use of the types of transformations, which would not be possible with an exam-style question in a paper-and-pencil environment.

> If you get a shape where the shape is oriented, and you can use the software [the DMT], for example, you can use this software to orientate the shapes differently to see where corresponding sides and angles are. I think those types of things are useful in the software because you can use the software to rotate the shape to find, as I said, where these angles go. If you use an exam question [in a paper and pencil environment], it is hard for students to understand what they are focusing on the angles and the sides.

Case Study 3: Jack

In his curriculum script, Jack anticipated and identified all of the seven misconceptions provided above (except for the first one) and aimed to address them through the use of the DMT in a dynamic mode during the observed lessons. We particularly focus on the fourth misconception in detail below to give an insight into this aspect of his curriculum script. This misconception was spontaneously spotted and identified by him during a lesson and resulted in him making *ad hoc* decisions to address it using the DMT dynamically.

Promoting students' understanding that if the lengths of a shape are multiplied by a scale factor of k, then the area of the shape is multiplied by a scale factor of k^2

Jack identified this misconception in a lesson when interacting with students during their independent work with the DMT to check their written explanations in the workbook. He instructed students to engage with Task 2.1 (see Figure 2) in which they were instructed to play an animation, watch and decide which quadrilaterals (if any) were always mathematically similar to the original and then produce their written justifications in CM workbook. He noticed one pair of students used the concepts of perimeter and area in their written explanation to justify why the orange and red rectangles were always mathematically similar. The students had concluded the orange and red rectangles remained mathematically similar to the original as the area and perimeter of the rectangles increased by a scale factor of 3 when the animation slider came to the end.

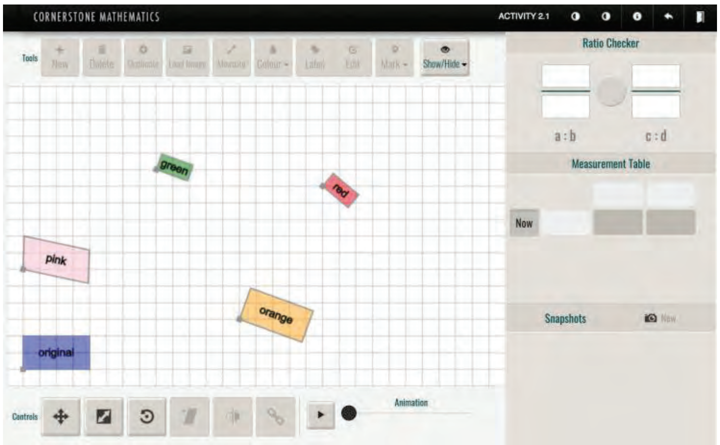

Figure 2: Task 2.1 in the DMT where students play the animation and identify the similar shapes

Jack, however, recognized a misconception as the students had concluded the area scale factor of the orange and original rectangles was the same as the length scale factor (a value of 3). To enable the students to address their misconception, he posed them the question "You see that the perimeter increases by 3 times, does the area increase by 3 times, too?" and then invited them to drag the animation slider such that the lengths of the orange rectangle were twice as large as those of the original. Having drawn the students' attention to that the side lengths of the orange rectangle were now double the side lengths of the original as it had been enlarged by a scale factor of 2, Jack asked them to examine what happened to the area of the orange rectangle in the dynamic learning environment. By looking at the original and orange rectangles on the screen, the students assumed the area of the orange rectangle had doubled. Jack then encouraged them to figure out "How many of them [the original rectangle] could sit into the orange one [rectangle]?" using the grid as a measurement tool available in the DMT, leading them to realise the area of the orange rectangle was 4 times that of the original.

In his post-lesson interview, Jack stressed when the students came up with the mathematical ideas concerning the area and perimeter of similar shapes, which actually went beyond his lesson agenda, he wanted to exploit this learning opportunity to promote students' understanding of the mathematics a stake using the DMT.

> What they [a pair of students] said the perimeter and the area has doubled, so it was a nice teachable moment to say, you know, that was not really part of my learning intention at all, but I felt like it was just a nice opportunity to talk about the behaviour of the area and the behaviour of the side lengths, and the software [the DMT] let them see really nicely what had happened to the area […], so it was just a nice teachable moment where the software gives some nice visual and dynamic pictures that let them notice what was happening.

Conclusion

The focus of this paper is what student misconceptions about the topic of GS teachers predict and/or identify as part of their curriculum scripts and how they exploit the affordances of DMT in the classroom to promote students to confront, reflect upon and address them. The findings suggested all three teachers' curriculum scripts contained some anticipated (and/or identified) misconceptions about GS to be addressed with (and without) the DMT in the classroom. During their post-lesson interviews, all three teachers acknowledged the key role the DMT played in addressing students' misconceptions about GS. However, there were differences in the quantity of misconceptions the teachers anticipated and/or identified and in the ways in which they used the DMT dynamically to help students address their misconceptions. The analysis suggested Jack, the expert teacher, could anticipate several misconceptions with GS and develop alternative approaches to address them using the DMT. He was also, perhaps more importantly, open to employ *ad hoc* strategies to spot and identify misconceptions about GS students encountered in the lessons and use the DMT dynamically to address them. This implies the expert teacher had both confidence and proficient skill in the use of the DMT (even in unplanned use) to respond to perceived students' emerging misconceptions and to develop the underlying mathematics at stake (Ruthven, 2014). However, Lara and Alex, the advanced beginner teachers, showed a limited but growing awareness of their need to anticipate student misconceptions about GS and to consider alternative ways to address them. Lastly, this research was conducted in the context of English secondary schools that limits the generalisability of the findings. Further research is necessary to determine the extent to which the findings of this study might generalise beyond the English context.

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Digital tasks with feedback as core of a mathematical learning concept for prospective teachers

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Due to the increasing importance of digitalization in our society, there are new demands on school, teaching and thus on teachers. Nowadays digital competences belong to the key competences of teachers. The acquisition of digital competences should already be developed in university education of teachers. In this regard, we refer to a university seminar aiming to support professional competences of prospective teachers in digital mathematical teaching and learning contexts. The seminar is based on learning mathematics with the digital assessment system STACK, reflecting the use of STACK, designing learning environments with STACK and, finally, reflecting school students' use of STACK. We collected data of prospective teachers' development of digital competences through interviews. First results show that prospective teachers' competences including knowledge, beliefs and motivation can be developed and differentiated.

Keywords: Electronic learning, computer uses in education, feedback, teacher education.

Introduction

"In a digital world, knowing how to use ICT and having access to such technologies are proving increasingly important for participating effectively in society" (Fraillon et al., 2020, p. 5). For this reason, gaining digital competences is a main issue for school students. However, research showed that in Germany digital competences of school students are partly poorly developed (Fraillon et al., 2020). One reason for the poor development of school student's digital competences seems to be caused by teachers (Hegedus et al., 2017; Misfeldt et al., 2016). In an international comparison, teachers in Germany use digital tools (computer, calculators, mobile devices etc.) less frequently in the classroom and also assess the potential of these less than teachers in other countries (Fraillon et al., 2020). For this reason, a main goal of university education and professional development of prospective teachers is to improve their digital competences as part of mathematics teachers' professional competences including knowledge, beliefs and motivation (Baumert & Kunter, 2013).

A contribution to this line of research will be made by this PhD-project called "Learning, Reflecting and Designing: Digital tasks with feedback as core of a mathematical learning concept", in which we investigate in a qualitative study how prospective teachers' knowledge, beliefs and motivation could be increased in a specific seminar in the teacher education program. In this seminar, the prospective teachers experience the importance of learning mathematics with the digital assessment system STACK by learning with digital tasks themselves. Afterwards they reflect on the potential of STACK such as the opportunity of the STACK system to give individualized feedback to learners and after that, the prospective teachers design their own mathematical tasks with STACK and use them with school students. Finally, there is a renewed reflection on the STACK system, the task, the feedback and the practical application. Our main research question (RQ) is as follows:

RQ: How does learning with and designing and reflecting on one's own digital (STACK) tasks in a university mathematics education seminar affect the knowledge, beliefs and motivation of prospective teachers?

In this paper, we primarily focus on prospective teachers' beliefs referring to the potential of digital tasks with feedback including partly also aspects of knowledge and motivation.

Knowledge, beliefs and motivation

According to Baumert and Kunter (2013), professional competences comprise knowledge, beliefs and motivation. With regard to digital competences Koehler et al. (2013) provide the TPACK-model which differentiates knowledge into three main components (Figure 1): content knowledge (CK), pedagogical knowledge (PK) and technological knowledge (TK). Content knowledge comprises factual knowledge and subject-specific ways of thinking and working (Koehler et al., 2013). The pedagogical knowledge contains the knowledge about different concepts for designing learning environments, such as teaching methods, classroom management but also motivational aspects (Koehler et al., 2013). Knowledge of how to use (digital) technologies, such as hardware and software, is called technological knowledge (Koehler et al., 2013). The three main components of teachers' knowledge are set in relation to each other. Research showed that the positive influence on student performance is greater if the teacher has previously done further training on the use of digital tools and has thus acquired digital competences (Hillmayr et al., 2020). Although an influence of the teacher on the design of lessons with digital tools has already been proven, offers for the acquisition of digital competences in university teacher education in Germany are only available to a small extent (Vogelsang et al., 2019).

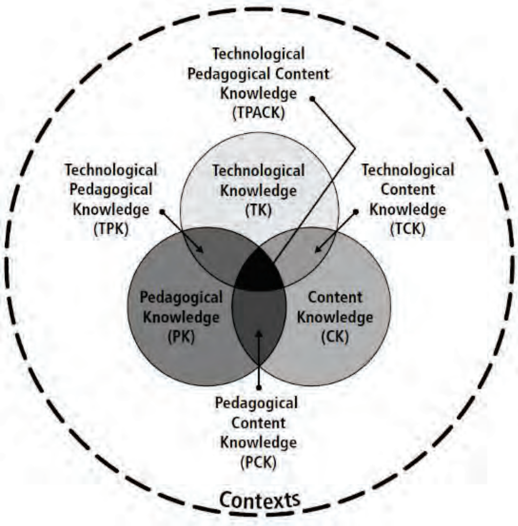

Figure 1: TPACK-model (Koehler et al., 2013)

Another essential competence facet of teachers are beliefs (Baumert & Kunter, 2013). We understand the term beliefs "as an individual's personal conviction concerning a specific subject, which shapes an individual's ways of both receiving information about a subject and acting in a specific situation" (Erens & Eichler, 2015, p. 136). Different dimensions of teachers' beliefs can be identified: beliefs concerning teaching, learning and a specific subject. (Fives & Buehl, 2012). Research showed that beliefs are difficult to change and that a belief change needs a substantial situational impact (Liljedahl et al., 2012). Thus, also to change prospective teachers' beliefs in a university seminar potentially needs to include a strong impact on their beliefs. Beliefs also shape the way, teachers understand teaching with digital tools (Erens & Eichler, 2015; Misfeldt et al., 2016). Specifically for the use of digital tools, Thurm et al. (2017) developed categories for teachers' beliefs about advantages, disadvantages and general issues of digital tools. In the category advantages are those teachers' beliefs that are positively disposed towards the use of digital tools in the classroom, for example, that digital tools can be used to support a change of representation (visual, symbolic). One of the disadvantages of digital tools mentioned by Thurm et al. (2017) is the high amount of time required for the introduction of digital tools in classroom. The general category includes beliefs about the timing and thus the question at what point in the lesson a teacher should use digital tools.

Besides knowledge and beliefs, motivation is another essential competence facet of teachers (Baumert & Kunter, 2013). According to Eccles and Wigfield's (2002) expectancy-value model, motivational orientation is influenced by expectation and value variables. The expectation variable can be defined as individuals' view "about how well they will do on upcoming tasks, either in the immediate or longer-term future" (Eccles & Wigfield, 2002, p. 119). The value variable can be divided into four facets: "attainment value, intrinsic value, utility value and cost" (Eccles & Wigfield, 2002, p. 119). The value variables comprise intrinsic components (intrinsic value), rather extrinsic components (utility value) and any perceived negative aspects (costs) of an action (Eccles & Wigfield, 2002). The motivation to use digital tools in one's own lessons is positively influenced by knowledge about the use of digital tools as well as by corresponding beliefs (Ertmer & Ottenbreit-Leftwich, 2010).

Design and materials

The seminar for improving prospective teachers' digital competences

We developed a seminar concept for prospective high school and vocational school teachers focused on digital tasks with feedback within this PhD-project (Figure 2). The seminar, which is one of the elective modules in the teacher education program, is divided into four parts. In the first part, prospective teachers learn with digital tasks (STACK) themselves. For this purpose, digital tasks are provided for them to deal and to learn with. After learning with the digital tasks, the prospective teachers change their role from learner to teacher and test, evaluate and assess the given digital tasks and feedback included in the digital tasks. They also reflect the potential of the digital assessment system STACK. In the second step, the participants independently design their own digital tasks with feedback in the system STACK. In the third part, school students selected by the prospective teachers work with these tasks and comment on them, the digital format of the task and the feedback given within working on the task. In the fourth part based on students' feedback, the task, the feedback, the system STACK and the practical application are reflected by the prospective teachers. The seminar has already been conducted, evaluated and optimized in three cycles.

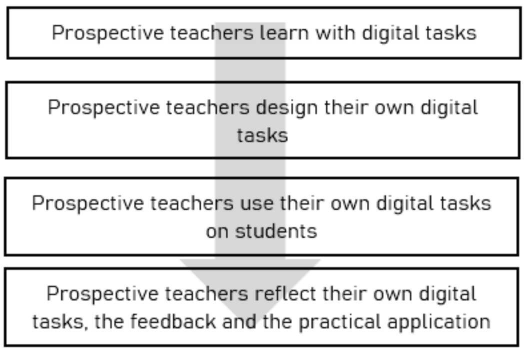

Figure 2: Seminar concept developed within this PhD-project

The digital assessment system STACK

STACK (System for Teaching and Assessment using Computer Algebra Kernel, Sangwin, 2013) is a digital assessment system in which digital mathematical tasks can be designed procedurally and conceptually (Rittle-Johnson & Schneider, 2014) and at different levels of representation (symbolic, graphic, interactive). The STACK system uses the computer algebra system Maxima (Sangwin, 2013), which makes it possible not only matching user input with stored sample solution, but also checking it for mathematical properties. Task developers are able to create a potential response tree (PRT, Sangwin, 2013) in STACK. The potential response trees, in which the input made can be examined for a specific mathematical property at each node, make individualized feedback for each user input possible (Figure 3).

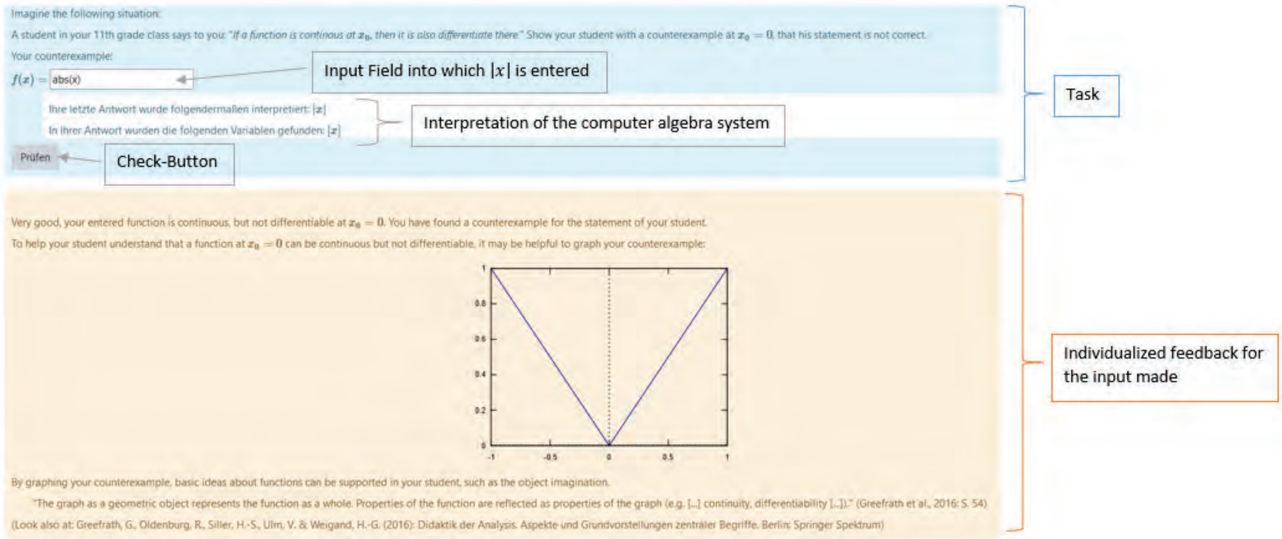

Figure 3: An exemplary STACK task with individualized feedback

Particularly, the possibility of STACK to provide individualized feedback is crucial since feedback is considered information that focuses on aspects of performance and understanding (Hattie & Timperley, 2007). Feedback is an effective intervention to support and optimize learning processes (Goldin et al., 2017). Feedback can increase cognitive performance, motivation and the feedback-recipients' willingness to make an effort. Furthermore, feedback can support learner individually and according to their potential. Within digital learning environments there are numerous possibilities to give feedback on learning processes (Goldin et al., 2017). The individualized, differentiated feedback for each user input seems to be crucial but often there is given rather simple and evaluative feedback (categorization into right or wrong) (Fraillon et al., 2020).

Design of the study

The sample for the data collection consists of three prospective teachers from each of the three seminar cycles. A main method for collecting data of the prospective teachers were semi-structured interviews which took place two times in each seminar. The semi-structured interview guide contains three domains (Figure 4). The first domain refers to beliefs and motivation from the learner's perspective. Therefore, this domain is about the prospective teachers' university education. This contains the use of digital tools in university studies and the attitude towards digital tools. The second and the third domain ask for beliefs and motivation from the teachers' perspective. In the second domain, we ask for the motivation to use digital tools and digital tasks in the classroom in the four facets that Eccles and Wigfield (2002) describe (attainment, intrinsic and utility value as well as cost). The third domain concerns school and teaching in particular. There we ask for beliefs towards digital tools and digital tasks with feedback while teaching and self-efficacy to design lessons with digital tasks. In this context we also collect beliefs about feedback in school context.

Furthermore, we work with written comments from the prospective teachers on different digital tasks. Within the written comments, the prospective teachers should vary the given tasks, design new feedback to different inputs and name advantages and disadvantages of the digital format and the digital feedback. With this method, we want to collect professional knowledge which contains variation of tasks and designing feedback, but also beliefs towards digital tasks and digital feedback.

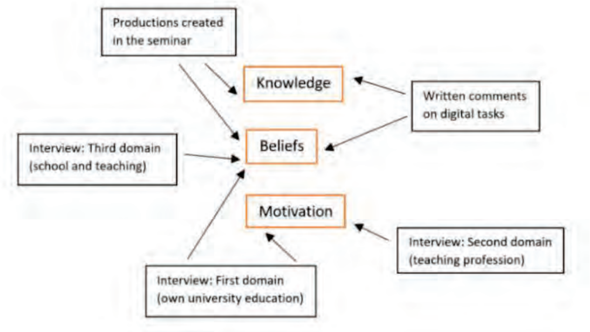

Figure 4: Methods and which competence facets we ask for within these methods

Moreover, we analyze productions from the prospective teachers, which are created in the seminar. This includes their own digital tasks, the written reflection on their own digital task, the designed feedback and their final term paper. With this, we want to survey the professional knowledge as well as the beliefs about the use of digital tasks in lessons (Figure 4).

We analyzed the data using content analysis. For this purpose, the competence facets knowledge, beliefs and motivation are subdivided into different categories, such as learning with digital feedback, advantages of the digital format for teachers and negative aspects concerning the STACK system.

Findings

In this paper we exemplarily show interview excerpts. In the first example, Adrian names the possibility to get feedback directly while working on digital tasks:

Adrian: [The system gives] feedback directly. This means that the student works on a task and during the work he already gets feedback on what is correct, what is wrong and why he does something wrong. This gives a great potential to directly address and counteract the problems a student has while working on the mathematical task.

The expressed belief relates to learning with digital tasks with feedback and applies more to an advantage of feedback than of the digital task itself. Adrian only mentions superficially the structure of feedback and the containing components. He rather focuses on one function of direct feedback which is the early prevention of misconceptions. This is not only a belief, but also concerns the knowledge facet TPACK (technological pedagogical and content knowledge, Koehler et al., 2013). TPACK describes the knowledge about mathematical digital learning environments. In this context, it is necessary for a teacher to know about the functions of the STACK system as well as to recognize misconceptions and problems of students while working on a task and to be able to design individualized feedback.

In the next example, Jacob's statement relates to teaching with digital tasks with feedback from the teacher's perspective and applies to an advantage of the digital format of the task:

Jacob: For me as a teacher it is important that I can see what the students' learning level is. The students' inputs are listed in a table. There I can also see how many attempts [to solve the task] were needed, what misconceptions existed among the students and what feedback the students received.

He explains that the digital format gives the teacher an accurate overview of students' missed attempts and misconceptions. This can form the basis for a better assessment of the students' performance on the one hand and for planning the following lessons on the other hand. This statement also refers to the utility value of motivation (Eccles & Wigfield, 2002). Digital tasks and the associated better overview on the students' learning level are useful for a teacher to plan further lessons. This is a rather extrinsic reason for using digital tasks in one's own lesson. The facet of knowledge addressed here is technological knowledge (TK, Koehler et al., 2013). The teacher needs to know about the function associated with digital tasks and how to handle the STACK system.

Patrick's statement concerns teaching and learning with digital tasks and applies to an advantage of the digital format and the digital task:

Patrick: The digital format of the task gets students motivated to start working. I believe that the students have more fun working digitally than calculating a task from the textbook. Especially students who are not mathematically inclined are more likely to want to work on digital tasks and try them out.

The first part of Patrick's statement is about teaching with digital tasks. He mentions the positive attitude of the students towards digital tools. Therefore, it will be possible to get students into action. The second part of the statement concerns learning with digital tasks. It explicitly refers to students who are not mathematically inclined. The inhibition to deal with mathematical content and to do something wrong can decrease through the digital format of the task. This also concerns the utility value of motivation (Eccles & Wigfield, 2002). The digital format of the tasks is useful to get students motivated to work and therefore the use of digital tasks in one's own lessons is influenced rather extrinsically. This statement also concerns TPACK (Koehler et al., 2013). Patrick focuses on cognitive activation within mathematics teaching which can be supported by digital tasks.

Not all beliefs towards digital tasks with feedback are positive. In addition to the advantage explained above, Jacob also mentions a critical point towards working on digital tasks. He discusses whether the process of working on the task is really digital:

Jacob: In case of a complex digital task that cannot be solved in head but, for example, requires five to ten intermediate steps before solution can be found, these steps must be written on a sheet of paper in analogue form. This means that the process finding a solution is partly not digital, but only the input and evaluation of the solution are digital.

Jacob explains that students need an analogue medium like a sheet of paper to solve a complex digital task. Therefore, the task as well as the evaluation of the input can be described as digital but not the process of solving. This belief concerns learning with digital tasks with feedback and the process of working on a digital task respectively and applies to a disadvantage of digital tasks because not the entire process of working can be considered digital. Jacobs' statement also relates to the facet cost of motivation (Eccles & Wigfield, 2002). Within this facet, all negative aspects of the use of digital tasks with feedback in the classroom are summarized.

Discussion and conclusion

The presented study investigates how prospective teachers' knowledge, beliefs and motivation towards digital tasks with feedback can be changed through a specific seminar in the teacher education program. First results show that prospective teachers' beliefs are varied. Some of the beliefs relate to feedback as a part of a digital task. Other beliefs concern teaching with digital tasks and associated with a better overview of student performance. Beliefs about learning with digital tasks were also expressed such as that students are more motivated to engage with a mathematical topic because of the digital task format. We can thus see that the prospective teachers' beliefs relate either to teaching and learning with digital tasks as well as to the specific subject of feedback, what has already been described by Fives and Buehl (2012). The fact that the prospective teachers adopt multiple perspectives could be due to the seminar concept presented. The prospective teachers first learn with digital tasks themselves, taking the learner's perspective. In the seminar, they then change to the designer's perspective when designing their own digital tasks and finally to the teacher's perspective when using them with school students. Furthermore, the beliefs can be classified into the categories advantages, disadvantages and general issues found by Thurm et al. (2017). It also becomes visible that the competence facets knowledge, beliefs and motivation are not disjoint. Many of the prospective teachers' statements can be assigned to categories of beliefs as well as categories of knowledge and/or motivation.

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Preservice teachers' beliefs about mathematical digital competency – a "hidden variable" in teaching mathematics with digital technology?

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Recently the construct of mathematical digital competency (MDC) was put forward in which mathematical competency and digital competency are seen as a connected whole. This entails that student understanding of mathematical concepts may be almost inseparable from digital tools. We report on a quantitative study with n=238 preservice teachers (PSTs) from Germany that investigates PSTs' beliefs about such a "connected position" of MDC. Results show that a large group of PSTs believe in the potential of digital technology but at the same time opposes the notion of MDC and rather believe that mathematical and digital competency should be separated. Furthermore, PSTs' beliefs about MDC are largely independent from epistemological beliefs. We hypothesize, that beliefs about MDC may be an overlooked variable which may influence how teacher think about and use digital technology in the mathematics classroom.

Keywords: Beliefs, digital technology, teacher education, graphing calculators, self-efficacy

Introduction

Teacher beliefs are an important factor for implementing digital technology (DT) in the mathematics classroom (Thurm & Barzel, 2021). The question arises which facets of teacher beliefs are relevant for teaching mathematics with (DT). In this respect, previous research has shown that teachers' beliefs about the potentials of DT use, teacher epistemological beliefs (i.e., beliefs about the nature of mathematics and teaching and learning mathematics) and self-efficacy are important dimensions of teacher beliefs (Thurm & Barzel, 2020). In this paper we investigate a somewhat novel dimension of teacher beliefs in the context of teaching mathematics with DT. Starting point for this research study was the work of Geraniou and Jankvist (2019) who argue that mathematical competencies and digital competencies are rarely seen as a connected whole even though students will have to simultaneously activate and use these competencies. Therefore, they conceptualize the construct of "mathematical digital competency" (MDC), which describes an amalgam of mathematical and digital competencies. In particular, they use the theories of conceptual fields (Vergnaud, 2009) and instrumental genesis (Guin & Trouche, 1999) to show that such an amalgam entails that a student's understanding of a mathematical concept may almost inseparably be connected to digital tools and the student's instrumented techniques. In this paper we investigate how PSTs think about such an amalgam and how their beliefs are related to other belief facets like epistemological beliefs and beliefs about affordances and risk of DT use. We start by elaborating in more detail on the two main theoretical frameworks for our study, namely MDC and teacher beliefs. Throughout the paper the term "digital technology" (DT) refers to mathematic-specific digital technologies like function plotters, dynamic geometry systems, computer algebra systems and multi-representational tools.

Theoretical background

The notion of competency has gradually gained momentum and is nowadays a key construct in the educational paradigm (Geraniou & Jankvist, 2019). Mathematical competency can be defined as "someone's insightful readiness to act appropriately in response to a specific sort of mathematical challenge in given situations" (Niss & Højgaard, 2019, p.14) while digital competency has been conceptualized as "the set of knowledge, skills and attitudes […] that are required when using ICT and digital media to perform tasks; solve problems; communicate; manage information; collaborate; create and share content; and build knowledge" (Ferrari, 2012, p.43). Geraniou and Jankvist (2019) linked mathematical and digital competencies by using the theory of instrumental genesis (TIG) and the theory of conceptual fields (TCF). TIG (Guin & Trouche, 1999) describes the process of transforming a digital tool (an artefact) into a mathematical instrument which is a psychological construct that combines (parts of) the artefact and cognitive schemes in which technical knowledge about the artefact and the domain-specific mathematical knowledge are intertwined. TCF (Vergnaud, 2009) takes, similar to TIG, a developmental point of view. TCF highlights that a concept is not only referring to explicit objects of thought, but comprises a set of schemes, a set of situations and a set of linguistic and symbolic tools of representation. In particular, Vergnaud (2009) stresses that different concepts and situations are interconnected forming conceptual fields, which he defines as "a set of situations and a set of concepts tied together" (Vergnaud, 2009, p.86). The set of situations gives meaning to the concept and acts as a point of reference.

Geraniou and Jankvist (2019) show that both TIG and TCF can serve as a lens to investigate the simultaneous activation and development of mathematical and digital competency which they call "mathematical digital competency" (MDC). In particular they highlight, that the situations that make up students' conceptual fields "*may be embedded so deeply in a techno-mathematical discourse that, potentially, also their understanding of the mathematical concepts involved is almost inseparable from the digital tools and the students' instrumented techniques*" (p. 42). This could for example mean that the set of situations that students use as points of reference to give meaning to a concept will largely comprise situations involving DT. Moreover, a student may only be able to do (some) mathematically activities within a digital environment. Hence students might only be able to think about, explain and do mathematics with reference to a digital tool. Starting from this notion of a close amalgam of student's mathematical competencies and digital competencies, the question arises what teachers believe about such a potentially close interwovenness.

According to the broadly accepted definition proposed by Philipp (2007), teacher beliefs can be defined as "psychologically held understandings, premises, or propositions about the world that are thought to be true" (p.259). Beliefs are part of teachers' conception - a general notion or mental structure encompassing beliefs, meanings, concepts, propositions, rules, mental images, and preferences (Philipp, 2007). Teacher beliefs are an important factor for teaching mathematics with DT since they act as a bridge between knowledge and action (Thurm & Barzel 2020; 2021). However, teacher beliefs are not an unidimensional construct but can be differentiated into various dimensions (clusters) and sub-dimensions which form a differentiated belief system in which (clusters of) beliefs can be logically connected and some (clusters of) beliefs are more important than others (Philipp, 2007; Leder et al. 2002; Thurm & Barzel, 2020; 2021). With respect to teaching with DT, three dimensions of teacher beliefs have so far been identified as particularly important: (i) beliefs about teaching and learning with DT, (ii) self-efficacy beliefs and (iii) epistemological beliefs (Thurm & Barzel, 2020). These dimensions of teacher beliefs can be further differentiated into various sub-dimensions (Thurm & Barzel, 2020; see figure 1).

However, teacher *beliefs about MDC* have not yet been investigated, which is not surprising given that the notion of MDC has just recently emerged with the work of Geraniou and Jankvist in 2019. Clearly, a teacher can have different belief positions with respect to the relation of mathematical and digital competencies. On one extreme one can fully embrace the "*connected position*" of MDC (e.g., that students understanding of the mathematical concepts involved may be almost inseparable from DT). On the other extreme, someone could strongly favor a "*independent position*" believing that mathematical and digital competencies should not be closely interwoven but rather clearly separated (i.e., a student should be able to think about, explain and do mathematics without a DT). Clearly if teachers oppose a close interwovenness ("*connected position*") as conceptualized by MDC this will be problematic if the goal is to develop students MDC or if students learn mathematical concepts with the support of DT. In the following, we will refer to beliefs about the interwovenness of mathematical competencies and digital competencies simply as "beliefs about MDC".

Research questions and methodology

In our exploratory research study, we investigate two distinct but interconnected research questions:

RQ1: What are preservice teachers' beliefs about MDC? (Belief position)

RQ2: How are preservice teachers' beliefs about MDC related to epistemological beliefs, beliefs about teaching and learning with DT and self-efficacy beliefs? (Belief system)

To answer the research questions, we used quantitative instruments to measure PSTs beliefs about MDC, about the nature of mathematics and mathematical learning, about teaching and learning with DT and PSTs self-efficacy beliefs. The reason why we use questionnaires to catch teacher beliefs is that questionnaires allow to simultaneously measure different belief dimensions and relate them to each other by statistical analysis. Furthermore, questionnaires also provide opportunity to investigate the discriminant validity of the dimensions (Fives & Gill, 2015). In the following we briefly elaborate how the different dimensions were assessed through multi-item-scales (figure 1 gives an overview of all scales).

Since there were no items/scales available to measure PSTs beliefs about MDC, we started to construct a scale following the recommendations for scale and item construction of Simms (2008). The item design was guided by the goal to write items that capture PSTs believe whether or not it is acceptable if a student's understanding of a mathematical concept is almost inseparable from digital tools. If it is inseparable, this would mean for example that a student is only able to think about, explain and do mathematics with reference to a digital tool. We set up an initial pool of eight items (sample items are given in table 1) which were further refined with PSTs and experts. Response format for all items was a 6-point Likert scale ranging from "1=strongly disagree" to "6=strongly agree" (hence higher values indicate a more "independent position"). After administering the scale to the PSTs, we conducted an exploratory factor analysis (EFA). Based on the results four of the eight items were dropped and the final MDC-scale was constructed using the mean of the items MDC1-MDC4 displayed in table 1. Hence high values on the MDC-scale reflect that PSTs oppose the notion of a close link between mathematical competencies and digital competencies (which we call “independent position”) while low values of the MDC-score reflect a more “connected position” in line with the MDC concept. Reliability (Cronbach’s alpha) of the scale was 0.75 indicating an acceptable reliability score.

Table 1: Sample items for the scale to measure PSTs beliefs about MDC

MDC1	A student should be able to explain a mathematical concept or relationship without referring to a digital mathematical tool.
MDC2	A student should be able to solve a mathematical problem without a digital mathematical tool.
MDC3	A student’s understanding of a mathematical concept should be independent from digital mathematical tools.
MDC4	A student should be able to give examples of a mathematical concept or relationship without referring to a digital mathematical tool.

- To measure PSTs’ epistemological beliefs about the nature of mathematics we used three shortened multi-item-scales from the international TEDS-M-Study (Blömeke & Kaiser, 2014). One scale captured to what extend PSTs believe that mathematics is a static collection of rules and procedures (Scale: “Rules and Procedures”, E1). The second scale captured to what extend teachers view mathematics as a dynamic science which consists of problem-solving processes and the discovery of mathematical structures and regularities (Scale: “Inquiry”, E2). The third scale captured to what extend PSTs believe that mathematics is discovered rather than invented (Scale: “platonic conception”, E3).

  Epistemological beliefs about the learning of mathematics were captured with two scales. A shortened scale from the COACTIV-Study (Kunter & Baumert, 2013) measured to what extent students believe that learning mathematics is best achieved by receptive learning (Scale: “Instructivist”, E4). We also asked PSTs to rank the four conceptions of learning mathematics put forward by Kuhs and Ball (1986) by preference (classroom-focused; content-focused with an emphasis on performance; content-focused with an emphasis on conceptual understanding and learner-focused). By taking the mean of the ranks of the two instructivist conceptions (classroom-focus, content-focused with an emphasis on performance) we derived a scale (E5) where higher values indicate a more constructivist approach to teaching.
- To measure PSTs’ beliefs about teaching and learning with DT we used five established multi-item-scales (all items can be found in Thurm (2020)). Beliefs about the potentials of teaching with DT were captured by the scales “Supports discovery learning”" (T1) and “Support of multiple representation” (T2), whereas negative beliefs were captured by the scales “loss of computational / by-hand-skills” (T3) and “Mindless working” (T4). The scale “Prior mastery of mathematics by hand” (T5) captured whether students believe that DT should only be used when the mathematics is thoroughly understood without DT. Response format for all items of the scales was a 6-point Likert scale ranging from “1=strongly disagree” to “6=strongly agree”.
- Self-efficacy was measured using two established multi-item-scales (all items can be found in Thurm (2020) capturing PSTs’ self-efficacy for task design and selection (S1) and lesson design

and implementation (S2). Following Bandura (2006) response was given on a scale ranging from 0 to 100, where higher values indicate higher self-efficacy.

The online-questionnaire which comprised all previous described scales was administered in 2021 to n=238 PSTs from two German universities in the state of North Rhine-Westphalia. Participation was voluntarily and anonymous. To ensure a certain level of basic expertise in mathematics education and a basic exposure to digital mathematical tools during university studies, the questionnaire was only administered to PSTs who had completed at least four semesters of teacher education. 35% of the participants were male while 65% were female. The PSTs indicated that they had used digital mathematical tools for their own learning approximately once a week during their university studies.

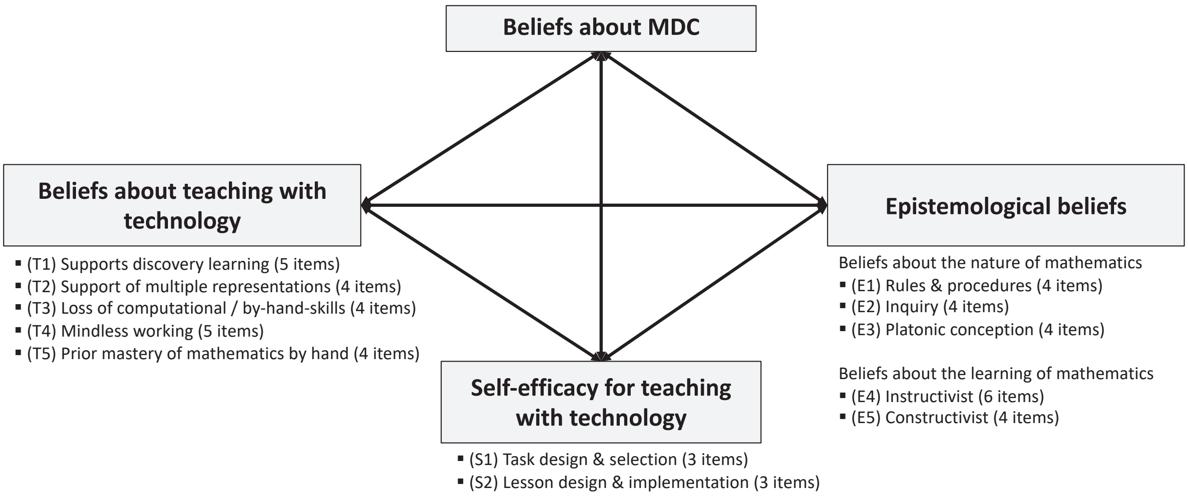

Figure 1: Overview of the different dimensions of teacher beliefs that were investigated

Results

RQ1: What are preservice teachers' beliefs about MDC? (Belief position)

Figure 2 shows the distribution of the MDC-scale (average of the items MDC1-MDC4). The mean of the MDC-scale is 4.51 indicating that PSTs on average clearly agreed more with an "*independent position*". In particular, 55.5% of the PSTs show an MDC-score ≥ 4.5 and thus strongly identified with the "*independent position*". A group of 40.3% of the PSTs was somewhat undecided/moderate (MDC-score between 2.5 and 4.5), while only 4.2% identified strongly with a *"connected position"* (MDC-score ≤2.5).

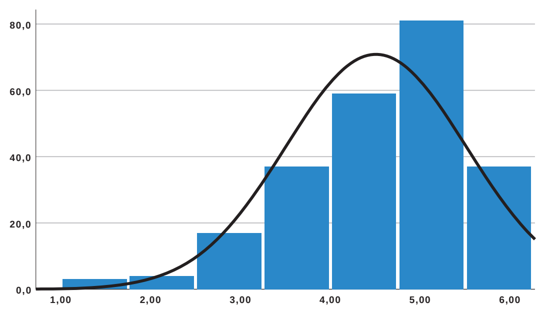

Figure 2: Frequency histogram of the scale measuring PSTs beliefs about MDC (mean of the items MDC1-MDC4)

RQ2: How are preservice teachers' beliefs about MDC related to epistemological beliefs, beliefs about teaching and learning with DT and self-efficacy beliefs? (*Belief system*)

Table 2 shows the means, standard deviation and reliability of all scales used in this study. Table 3 gives the correlation among the different constructs. First, none of the correlations are extremely high which indicates that the MDC-scale is indeed measuring a distinct construct different from the other constructs. The correlations in table 3 also show that a stronger "*independent position*" (higher values on the MDC-scale) is significantly associated with more negative beliefs about teaching and learning with DT (T1-discovery learning: ρ=-0.31**, T3-loss of skills: ρ=0.34**, T4-mindless working: ρ=0.39***). However, a closer look at the subgroup of PSTs that holds a particularly strong "*independent position*" (MDC-score ≥ 4.5, see table 2) reveals that this subgroup has still very positive beliefs about the potentials of DT for teaching and learning (DT supports discovery learning=4.65 on a 6-point scale, DT supports multiple representations=5.36 on a 6-point-scale). Remarkably, there were almost no significant correlations between MDC and epistemological beliefs (E1-E4) or self-efficacy beliefs (S1, S2). Only scales E2 ("Inquiry") and E5 ("Constructivist") were slightly negatively correlated with a more "*independent position*".

Table 2: Mean (M), standard deviation (SD) and reliability (Cronbach's alpha) for the total sample (Mean total) and the subgroup with strong "independent position" (Mean MDC≥4.5)

Scale	α	Mean total (SD)	Mean MDC≥ 4.5	Scale	α	Mean total (SD)	Mean MDC ≥4.5
MDC	.75	4.51 (1.00)	5.23	(S1) Task design & selection	.84	64.63 (20.12)	64.64
(T1) Supports discovery learning	.86	4.76 (0.86)	4.65	(S2) Lesson design & implementation	.87	62.47 (21.11)	62.64
(T2) Support of multiple representations	.70	5.44 (0.58)	5.36	(E1) Rules & procedures	.65	3.33 (0.94)	3.34
(T3) Loss of comp. / by-hand-skills	.87	4.03 (1.12)	4.28	(E2) Inquiry	.70	5.21 (0.66)	5.15
(T4) Mindless working	.90	4.23 (1.12)	4.44	(E3) Platonic conception	.77	4.25 (1.08)	4.38
(T5) Prior mastery of mathematics by hand	.94	4.29 (1.36)	4.69	(E4) Instructivist	.80	3.54 (0.91)	3.55
				(E5) Constructivist		3.03 (0.63)	3.05

Table 3: Correlation between the MDC-scale and scales measuring teacher beliefs about teaching and learning with DT (T1-T5), epistemological beliefs (E1-E5) and self-efficacy beliefs (S1-S2). (* < .05, **< .01, *< .001)**

	T1	T2	T3	T4	T5	E1	E2	E3	E4	E5	S1	S2
MDC	-0.31 **	-0.12	0.34 ***	0.39 ***	0.58 ***	0.12	-0.2 *	-0.04	0.07	-0.14 *	0.04	0.01

Discussion and conclusion

In this study, we took a first step to investigate PSTs' beliefs about the relation between mathematical and digital competency. We found that many PSTs in the sample agreed with an *"independent position"* meaning that they believe that a student should be able to think about, explain and do mathematics independently from digital tools. This clearly opposes a more "*connected position*" as conceptualized in the concept of MDC by Geraniou and Jankvist (2019).

Remarkably, beliefs about MDC were only barely associated with epistemological beliefs by the PSTs. Moreover, we found, that even the PSTs who were very strongly favoring an *"independent position"* at the same time strongly believed in the potentials of teaching and learning with DT (e.g., for discovery and investigation). These are interesting findings since they may point to some contradiction in the PSTs' belief system. If a PST highly values the potential of DT (e.g., to support discovery learning) but at the same time believes that mathematical understanding should be independent from DT, it will not be easy to balance these two views. In fact, if DT is used in a constructivist way for discovery learning of a new concept, the conceptual fields and the set of situations that students use as points of reference to give meaning to a concept will not be independent of the DT. Rather students' conceptual fields will be constructed around and therefore interwoven with DT. Hence, if PSTs hold an *"independent position"* this may limit the use of DT to understand and learn mathematics, or even reduces the use of? DT to "do mathematics". We hypothesize that PSTs may not be aware of this contradiction in their beliefs system. The observation that most PSTs agreed strongly that DT should only be used if the mathematics is thoroughly understood, may be a consequence of managing the tension of using DT (acting in line with their positive beliefs about DT) and at the same time maintaining an independence between mathematical understanding and DT (acting in line with their "*independent position*").

In total, the results of this study indicate that beliefs about MDC might be an overlooked variable—especially if the goal is to support students to develop MDC. In particular, the results of the study point to a discrepancy: If students learn with DT, they may develop some form of MDC (Geraniou & Jankvist, 2019). Yet, teachers expect students' understanding to be independent of DT and to be able to think about, explain and to do mathematics without DT. This discrepancy will be problematic and likely impact how teachers use DT. Consequently, it might be fruitful to engage PSTs in reflection about their beliefs on MDC and how these influence / conflict other beliefs in their beliefs system (e.g., beliefs about the potentials of DT for learning). Finally, we would like to mention one main limitation of the study – namely, that the belief position measured by the MDC-scale was not triangulated with qualitative data. Currently we are conducting qualitative interviews with PSTs after they have answered the MDC-scale to validate whether the MDC-scale is indeed measuring teachers' beliefs about MDC, and to uncover the PSTs' belief argumentation for their position.

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An interdisciplinary educational path enhanced with technologies: Cloud Computing in Education

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Keywords: Interdisciplinarity, cloud computing, education, technology, mathematical language.

This poster describes an interdisciplinary educational path developed in cooperation among different schools participating in Mathematical High School Project (MHS) and implemented thanks to Cloud Computing. The activities lead us to think that Cloud Computing in teaching can enrich curricular activities conducted with a constructivist methodological-didactic approach in a wide vision of knowledge. We present the case study, a laboratorial path of analysis and deepening of mathematical books that led to the production of an educational game.

Modern educational theories subline that students must be the protagonists of all educational activities, playing an active role in all training moments. The sharing of good educational practices allows the enrichment of the whole school communities thanks to the potentials of new technologies that enable remote access to tools, programs and didactic models. Sharing and collaboration become more difficult to manage when institutions are placed in distant locations. In these cases, the technologies are powerful and effective tools for cooperative work and the Cloud is a widespread methodology for sharing in educational activities is much less widespread even if it could become extremely powerful. Generic software, offers online productivity applications such as word processing, spreadsheets and functions that can be used in the classroom. This provides to work and share progresses or results in real time.

The project was implemented in the classroom in the year 2017/18 with the participation of researchers from Department of Mathematics of the University of Salerno, teachers (Italian literature, Math and Computer Science) and four middle classes (about eighty students) from different MHS Schools. In our activity we used the Cloud platform as an enhancement of a teaching-learning practice oriented towards metacognitive teaching that allows the enhancement of higher mental functions. Social interaction between students and teachers is fundamental for cognitive development as a result of interactions between peers and with experts in different contexts (Vygotsky L.S., 1934).

The transformation of mathematical narrative books into a video game required a careful analysis of different languages for different contexts. Students mediated the various communicative languages bringing their analytical and synthetic thinking to a higher level of sophistication. It is consistent with MHS activities oriented to insert mathematics as a transversal language between humanistic and scientific culture (Rogora, E., Tortoriello, F.S. 2021, Tortoriello, F.S., Veronesi 2021-1) to integrate curricular paths by exploiting the tools that technology offers (Tortoriello, F.S., Veronesi 2021-2).

The purpose of the activities carried out in cloud mode was to reconstruct the plot of popular books of mathematical interest in the form of a video game with the Skretch software. One group of students designed the scenes from the books, another group developed the backgrounds and game environments, the last group built the actions on the characters drawn by other groups.

Asset structure of the project:
Step 1: brainstorming - all students read the entire books, some of them are “What is the name of this book?” (Raymond Smullyan), “The Parrot’s theorem” (Denis Guedj), “A tangled tale” (Lewis Carroll). Their teachers solicited the exchange of opinions and suggestions to define the most engaging moments of the books and rewrite them by turning them into videogame scenes.
Step 2: creating scenes - students wrote the storyboard, organized the story sequences and transcribed the selected scenes storing them into a repository on a shared drive.
Step 3: creating the digital story - students organized into groups, used the scenes saved in the shared Drive folder and used the Scratch software to create the scripts with the collaboration of the teachers. Images, videos and sounds have been added to the game in which the protagonist (the player) faces challenges of different skills, logical reasoning, spatial observation, memory skills, speed and dexterity, historical and scientific cultural competences.
Step 4: presentation- a first sharing activity was carried out by each group of participants in their own school. A second dissemination meeting was the National Seminar of Mathematical High Schools that took place at the University of Salerno in a workshop.

The use of a constructivist approach enhanced with technologies has shown that the participation of students, the quality of the video game as the final product of the course were qualitatively excellent and this activity allowed students to learn by doing, enriching the path with active participation

In the analysis of the didactic impact of this laboratory activity, all the students were very interested and involved because each student found the most congenial area among the various ones of the project. The methodological choice of group-works and the active collaboration between different schools and students developed social skills as communication, respect for the other and allowed them to share their knowledge by experimenting the work of a design team. Students interacted with knowledge mediated by an artifact (software Scratch, congenial to them) that lightened the weight of the study of curricular disciplines. The processing of students’ reports to experiential stimuli highlighted the production of skills that are individual cognitive potentials enriched by the exchange in different school contexts and shows the high degree of satisfaction and the willingness to participate in similar activities in the future.

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TWG16: Learning Mathematics with Technology and Other Resources

Introduction to the papers of TWG 16: Learning mathematics with technology and other resources

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Abstract: This short contribution summarizes the results of TWG16 according to the five themes described in the Call for papers. As an overall conclusion, we notice that the somewhat general theme of good practices in technology-rich mathematics education attracts much attention, whereas the other themes apparently are addressed to a lesser extent in our research community, with the impact of digital technology on research methods as the most striking underrepresented example.

Keywords: Digital technology, embodiment, good practice, mathematics education.

Overview of themes and contributions

The scope of thematic working group TWG16 was to address opportunities and constraints of digital technology and other resources for students' learning of mathematics; a topic that is all the more relevant in times of an immense increase of distant learning and teaching practices. As a targeted outcome, we wanted to establish an overview of the current state of the art. We also aimed to suggest important trends for technology-rich mathematics education in the future, including a research agenda. TWG15 addresses a similar global topic but focuses on teaching rather than on learning.

Table 1: TWG16 themes and contributions

Theme	# accepted papers	# accepted posters
1. Theoretical advances on using digital technology in mathematics education	5	1
2. Embodiment and the use of digital technology in mathematics education	4	0
3. New roles for new tools (e.g., augmented and virtual reality, 3D printers)	5	1
4. Good practices in technology-rich design, learning and assessment in mathematics education	15	5

5. The impact of digital technology on research methods	1	0

The TWG16 call for papers identified five main themes to be addressed. Table 1 provides an overview of these themes and of the number of contributions per theme. Altogether, we had 30 accepted papers, 7 accepted posters. Some 60 participants from many countries within and outside Europe took part in the sessions, and contributed to having a positive and productive atmosphere.

Main results per theme

Theoretical advances on using digital technology in mathematics education

The crucial role of theory was an important issue within various discussions, not only limited to the contributions specifically addressing this theme. To a great extent this is due to the interconnectivity of the different themes. For example, innovative contributions focusing on embodiment or implementing new tools suggest a need for suitable theoretical frameworks to design and reconstruct mathematical learning activities. The contributions specifically addressing theoretical advances took particular aspects into account. On the one hand, some of the contributions addressed the value of theoretical frameworks when working with digital tools. In terms of extending a theoretical framework with regards to the use of digital tools, the potential of Cultural Historical Activity Theory (Engeström, 2000) was discussed in the context of programming activities. In terms of methodological considerations, the use of an instrumental approach in a quantitative study on functional thinking was explored. The role of an epistemological approach to inclusive settings using digital tools provided another theoretical lens. On the other hand, some contributions showed the need to develop new theoretical frameworks in the context of using digital technology. For example, a framework for creating heuristic videos to enhance students' modeling competencies was presented, as well as a model for learning about black boxes. Both the contributions and the discussions show that the key role of theory as well as theoretical advances keeps being a central theme for TWG16.

Embodiment and the use of digital technology in mathematics education

This theme included a range of papers, with some focusing specifically on embodiment—in terms of theorizing it—and others only evoking the role of the body in mathematical meaning-making. In the latter case, one study considered the dynamic animations created by students using geometric transformations. Although the conceptual focus here was on creativity, the analysis of the expressive and aesthetic dimensions of the students' work could certainly relate to theories of embodiment that account for the role of the senses in mathematical thinking. In the former category was a contribution that focused explicitly on embodied design. It explores a possible alignment between Abrahamson's embodied design (Abrahamson et al., 2020) and Sfard's (2008) commognitive approach, proposing the perception-actions that could correspond to saming, encapsulating, and reifying. A second example taking an explicit theoretical stance was on enactivism. The researchers compared the finger gnosis deployed in two digital technology settings, TouchCounts and Rakin, studying the fingers movements of a child as they relate to number sense. Finally, a systematic literature review was presented on the embodied approaches to functional thinking involving the use of digital technology. This paper leverages a particular approach to embodiment arising again from the work of Abrahamson (e.g., see Abrahamson et al., 2020).

Overall, it was remarked that the theories of embodiment used tended to be focused on the individual learner and their intact bodies, conceptualized as ontologically distinct from the environments (including the digital tools). This contrasts with approaches to embodiment that are more distributed, socially situated and politically inflected.

New roles for new tools

The thematic working group discussed the use of innovative digital tools in mathematics classrooms. The participants discussed specifically the use of augmented reality, robots, and 3D printers. The contributions evidence how such digital tools affected students' learning and teachers' instruction. In particular, the researchers presented studies that showed how such tools shape the students' language, formulation of mathematical concepts, and shed light on robust teaching. In addition, participants discussed how these new tools may invite negotiation of the mathematical meanings embedded in the tools and the learning environment. A comparison of physical objects, common-use digital technologies, and new digital tools was also presented. Most of the contributions to this theme were qualitative studies conducted with few participants. This fact required the participants to consider the feasibility of implementing these new digital tools in regular classrooms. Without any doubt, scaling up the use of such digital tools to regular classrooms will bring new opportunities and challenges for the learning and teaching of mathematics, and little is known about them yet. This may open a window toward a new research trend that will include questions on how to implement new tools in teaching practice. How should students engage and interact with such new tools? How will teachers' practices change through using such digital tools? This theme, of course, is far from having final answers, and seems suitable to revisiting during next CERME conferences.

Good practices in technology-rich design, learning and assessment in mathematics education

Developing good teaching practices that foster mathematics learning through technology-rich student activity is a prominent issue, all the more in times of emergency remote teaching due to the pandemic. Contributions describing good practices focused on topics such as the flipped classroom and distance learning environments, more established tools such as GeoGebra, tools for computer programming. Also, more innovative tools were used, such as a digital spirograph and the online application GeoGebra classroom. From the discussions we conclude that new pedagogies and practices have emerged in practices of using various digital tools, such as practices of digital-collaborative learning. It has become clear that the teachers' instrumental genesis of teachers is a prerequisite for good practice. Also, the availability of advanced digital technology used in the learning process questions the regular curricular goals. A main idea that emerged from the discussions is that good practices should use digital technologies to deepen mathematics learning through in-depth meaningful learning trajectories (teaching sequences), rather than through superficial tool use. To do so, new theoretical approaches might be needed. Another important conclusion was that social-media norms may change to enable new pedagogies to be put into practice. Finally, as our experiences with and knowledge of emergency remote learning with digital tools has drastically increased during the pandemic, we wonder how to capitalize on this in the future.

Many contributions in this theme presented examples of well-designed good practices. However, describing and investigating good practices also resulted in raising the issue of design principles. It was pointed out that design is group work, with different types of expertise involved, and that it is usually done in cycles. There is a need to identify overarching design principles, and theories that can underpin them, both for the design of tools and environments and for the design of learning activities. Moreover, the sustainability of such design principles was recognized as a key point, as they need to be still applicable—maybe in adapted form—when new, more advanced digital tools will emerge.

The impact of digital technology on research methods

Many types of digital technology impact on our work as educational researchers. We have eye tracking, video labs, data logging, learning analytics, machine learning, software for qualitative data analysis, and artificial intelligence, to mention just some. Even if the TWG team felt the need to address the question of how these tools affect our research methodologies, we hardly received any contributions to this theme. Apparently, this is not (yet?) a ‘living’ topic in our community. We wonder whether this will change in two years, if the new TWG team takes up this theme once more.

Conclusion

A first goal of this TWG was to provide an overview of the current state of the art in the domain of technology-rich mathematics learning. Based on the contributions and discussions, we conclude that the field is moving quickly, and that much attention is paid to the design, implementation and evaluation of good practices for mathematics learning and teaching using digital technology. The need for further foundations, in terms of theoretical frameworks—being new or adaptations of existing ones—and design heuristics is widely recognized. However, the suggested themes that aim at such foundations received not many contributions. This may be partly caused by the impressions that the TWG attracted many relatively early-career researchers, with seniors being somewhat underrepresented. As a practical feed-forward for CERME-13, particular attention might be paid to ensuring a balanced group of participants in terms of seniority.

A second goal was to identify a research agenda. Even if this was touched upon only implicitly, the above theme reports provide an interesting picture: whereas the need for theoretical foundations is widely acknowledged, the themes on theoretical approaches, new tools, embodiment, and new research methods were not popular in terms of numbers of contributions. One of the challenges for the next edition, therefore, is to reiterate these themes and ensure a more explicit place for them in the call for papers. The number of participants in the CERME-12 group shows that the field is alive and the topic is attracting much attention by researchers. The distribution of contributions over themes shows that there is a clear research agenda: let’s work on these theoretical and innovative foundations to ensure progress in these directions for the next edition.

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"Doing well" in the Teaching for Robust Understanding approach revealed by the lens of the semiotic potential of tasks with the GGBot

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In this paper we report on experiences we have provided students in a grade-3 classroom with the aim of introducing concepts in plane geometry through interaction with the GGBot, a drawing robot that can be programmed with SNAP! Blocks. We do this to explore the effectiveness of the use of a new theoretical approach that combines the Teaching for Robust Understanding Framework with the Theory of Semiotic Mediation. We claim that the articulation between the two frames is not only feasible but also insightful, providing an a priori analysis of two tasks, and a detailed analysis of a short excerpt from the corresponding Didactic Cycle.

Keywords: Coding, GGBot, geometry, personal meanings, situated signs, STEM education, Theory of Semiotic Mediation (TSM).

A STEM scenario: the position of mathematics when coding with the GGBot.

More attention is being placed on educational experiences in the context of STEM, as shown, for example, by the increasing importance and number of international STEM education journals (Li et al., 2020). However, STEM disciplines are not yet treated in a coherent way in educational settings; moreover, the role of mathematics in this panorama has been problematized by Li and Schoenfeld (2019) who advance an interesting and constructive proposal about the positioning of mathematics within STEM contexts within the Teaching for Robust Understanding (TRU) Framework (Shoenfeld, 2013). Presenting the framework, thew authors exemplify how it can be applied to mathematics, in order to help students "develop into powerful thinkers" (2019, p. 8).

In this paper we are interested in exploring the appropriateness of certain mathematical experiences we have offered primary school students using the GGBot, a drawing robot, in terms of STEM education and of the overarching goal of fostering their development into powerful thinkers. Specifically, we conceptualized the GGBot and students' experiences with it with the aim of fostering sense-making in a STEM perspective. Coherently with the TRU Framework, we followed Schoenfeld's suggestion "to operationalize what was meant by “doing well” in each of the five dimensions, and to see how “doing well” related to student outcomes." (Schoenfeld, p. 492, 2018).

However, given the breadth and generality of such a framework, in order to design and implement experiences with the GGBot, we felt the need of a more specific, though coherent, framework that describes the didactical potential of the GGBot, allowing a reformulation of chosen dimensions of the TRU Framework upon which we will focus our analyses. The specific framework we choose for this description is the Theory of Semiotic Mediation. Now we introduce our conceptual framework.

Teaching for Robust Understanding (TRU) and the Theory of Semiotic Mediation (TSM).

As explained by Li and Schoenfeld (2019), the Teaching for Robust Understanding (TRU) Framework is comprised of the following five dimensions, that can be introduced through guiding questions: 1) *content*: is it conceptualized as something rich and connected that can be experienced and codified in meaningful ways? 2) *cognitive demand*: what opportunities do students have to do that kind of sense-making and codification? 3) *equitable access to content*: who has such opportunities: is there equitable access to the core ideas? 4) *agency, ownership, and identity*: do students encounter the discipline in ways that enable them to see themselves as sense makers, building both agency and positive disciplinary identities? 5) *formative assessment*: does instruction routinely use formative assessment, allowing student thinking to become public so that instruction can be adjusted accordingly?

In presenting the framework as an alternative to viewing mathematics as "given" or "fixed" and arguing how such a framework also applies to STEM education, Li and Schoenfeld propose a conception of mathematics as "empirical". That is, mathematics can be (and should be) seen as products created through experience (as opposed to pre-existing). This perspective focusses on students' experience, in which a shift needs to take place from "instruction conceived as "what should the teacher do" to instruction conceived as "what mathematical experiences should students have in order for them to develop into powerful thinkers?"" (Li & Schoenfeld, 2019, p. 8). For mathematical experiences to accomplish this, the authors argue that they need to provide not only opportunities for making sense of the mathematics at stake, but for *sense-making* processes (McCallum, 2018), highlighting "the importance for students to experience mathematics through creating, designing, developing, and connecting mathematical ideas" (Li & Schoenfeld, 2019, p. 6).

While the TRU seems quite appealing for framing mathematical experiences that strive to find a place and identity in STEM scenarios, we find it too broad and general to actually help gain insight into how specific mathematical experiences are (or not) well-designed and, when carried out, whether they help students become powerful mathematical thinkers. Moreover, the context we are interested in studying – the learning of geometry through coding of a drawing robot – involves an artefact. Therefore, an appropriate framework that seems to compensate for the generality of the TRU Framework and allow for more detailed analyses in this setting is the Theory of Semiotic Mediation (TSM). Indeed, the TSM has its roots in Vygotskian socio-constructivism, according to which students are guided by their teacher to construct mathematical knowledge by solving appropriately designed tasks with the use of appropriately designed and chosen artefacts (Bartolini Bussi & Mariotti, 2008). Students' activity with artefacts allows the unfolding of their *semiotic potential* leading to students to produce personal signs that are closely related to the task and to the artefact, and that gradually, through mathematical discussions orchestrated by the teacher, are transformed into shared signs. The shared signs are generalizations of the situated personal signs, and they are more closely related to the mathematical signs belonging to the concepts being taught.

Focusing especially on three dimensions of the TRU Framework, we need to draw connections between such a broad framework and the TSM. The notion of semiotic potential expresses the relationship between the personal meanings emerging from the experience of acting with the artefact and the mathematical meanings recognizable by the expert in such actions; its strict dependency on the task to be accomplished by the students makes it the key tool for designing appropriate tasks. In doing so, it operationalizes a link between dimension 1 (Mathematical content) and dimension 2 (Cognitive demand) of the TRU Framework.

The structure of the iteration of the Didactic Cycle (work on a task with an artefact, individual production of signs, collective production of signs through mathematical discussion) organizes the implementation of teaching sequence in the classroom, and it is in line with dimension 4 of the TRU: the potential of agency ownership, and identity. For instance, typically the first cycle is characterized by a "discovery task" that foresees guided exploration of the artefact, in which the teacher's intervention is expected in order to prepare students to the following tasks, where their autonomous work is encouraged, with the aim of making personal meanings emerge and the semiotic potential unfold. Students' personal meanings can emerge in their conversation with peers during the solution of the task, providing a base for the following collective discussion (e.g., Mariotti, 2009) during which the semiotic production can (and should) foster the evolution of the expected mathematical meanings.

Objective and Methodology.

The broad objective of our work with robotic toys (e.g., Bartolini & Baccaglini-Frank, 2015; Baccaglini-Frank et al., 2020) is to study their educational potential with respect to specific geometrical concepts and with respect to significant reasoning processes in a STEM educational perspective. As for this paper, we explore the effectiveness of the use of a new theoretical approach that combines the TRU Framework and the TSM both for the design and the analysis of classroom activities. In order to show that the articulation between the two frames is not only feasible but also insightful, we provide an a priori analysis of two tasks to be carried out in a grade-3 classroom, and a detailed analysis of a short excerpt from the corresponding Didactic Cycle.

We will use a fine-grained analysis of a short excerpt, paying particular attention to the unfolding of the semiotic potential and to the teacher's actions, to show how the activity placed the students in a meaningful (to them) situation, in which they felt the need to explore and express mathematical ideas.

Presentation of the GGBot.

The GGBot (short for "GREATGeometryBot") builds on the convergence of physical and digital affordances, combining the well-known strengths and opportunities offered by Papert's original robotic drawing-turtle (more recently developed into robotic toys like the "Bee-bot") and LOGO programming with those of the block-based programming language SNAP!. The GGBot can hold a marker between its wheels (Figure 1a) that draws out its path as it moves on a sheet of paper on the floor, as well as a marker at the front, on its "nose", to highlight its movement when it changes direction (Figure 1b). Such traces provide situated signs that can be elaborated into geometrical notions – such as segment, vertex, angle, rotation, polygon – while still carrying the situatedness given by the real movement of the physical artefact. Commands are given to the GGbot through an SNAP! interface that was customarily designed, and they can be gradually added based on the teacher's needs. Other than the possibility of holding two markers, the way in which commands are given to the GGBot is quite different from other robotic toys like the Bee-bot, because the blocks represent commands (in the machine's language) that can be given to the GGbot by putting them together into sequences or *codes* (figure 2c) that are transmitted to the GGbot via a wifi module. Although these blocks are virtual objects that "live" on a screen (touch-screen of an interactive white board, tablet, or computer screen), they are concrete enough to be accessible to and shared by the whole class, and by each student–consistently with respect to TRU dim. 4, on agency, ownership, and identity–when engaged in tasks such as the following.

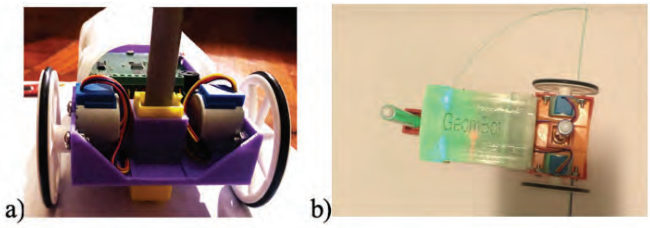


Figure 1*:* a) back view of the GGBot; b) top view of the GGBot

Though the GGBot also has a completely digital version (https://sprintingkiwi.github.io/virtual-geombot-snap), that is more similar to the LOGO turtle (though still in the block-based SNAP! environment), the physical artefact offers (especially younger) students a very different experience.

The semiotic potential of the figure-to-code and code-to-figure tasks.

A *figure-to-code task* consists in giving students the name of a figure and asking them to work in pairs and use the blocks to produce a code, so that when sent, the GGbot draws the required figure. The task we will be considering is: "Make a code so that when it is sent, the GGBot draws a square".

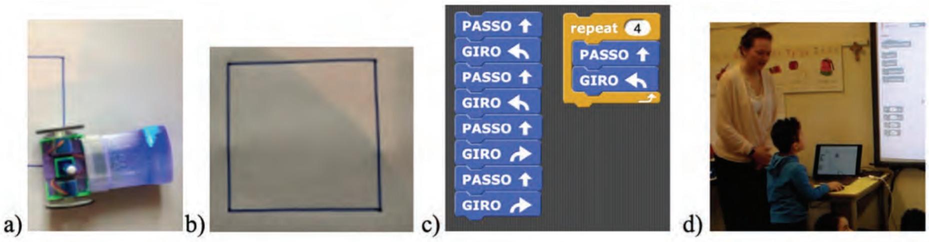


Figure 2: a) GGBot's moves; b) GGBot's trace; c) examples of codes; d) collective discussion of codes

In order to accomplish the task, it is necessary for the students to engage in a process that relates the GGBot's affordances with their conceptualization of "square shape". The shape needs to become an envisioned contour (e.g., figure 2b), a path along its border that corresponds to the GGBot's trace mark as it moves along such a path. Then, through the lens of the available commands (blocks), such a contour/path must be seen as a sequence of steps, leading to the realization of a code (such as those in figure 2c) to be communicated to GGBot. A spontaneous approach (e.g., Clements & Battista, 2001) consists in imagining to walk along the border of the figure and, based on such a simulation, planning the code to send the robot. While it is straightforward to identify the four *segments* constituting the sides of the square, and relate them to the step ↑ commands, it might be more challenging to decide how to connect these four steps. Indeed, it requires mastering the complex meaning of the Turn ↱ command (whether it is to the right or to the left) in order to relate it to the interpretation of each connecting point as "turn on the spot". Such points correspond to the thick dots left by the GGBot as it executes the turn ↱ commands, making the robot change direction before it takes another step forward to continue moving along the path. Moreover, a consistent interpretation, leading to complete the drawing, needs putting these points in relation with one another, as *the vertices of the square*, and as centers of the *rotations of the external angles of that polygon.* So, an essential feature of the semiotic potential of this artefact is its building on the relationship between the global movement and its breaking up into steps and turning points and the geometrical meaning of a polygon at a global and an analytical level. From a cognitive point of view (and this relates to TRU dim. 2, on cognitive demand), the task consists in breaking down a path that is imagined to be generated through physical continuous motion, into geometrical elements (TRU dim. 1) of a different nature: they are static and discrete.

Producing the code involves the use of signs corresponding to the required commands. Such signs refer both to the GGBot's movement and to the trace mark left, and their efficiency depends on the relationship between the drawing produced and the breaking up of the movement into its constituent elements and the corresponding commands. Comparing different codes produced in response to a figure-to-code task opens the way to the dual type of task: *code-to-figure*, that consists in focusing on a given code and asking the students to predict the trace mark that will be left on the paper by the GGbot when it executes such a code. This type of task asks to invert the relationship as the one described above between blocks in a code, the movements of the GGBot and the trace mark left. A potential challenge for students (TRU dim. 2) lies in finding a cognitive harmony between specific movements induced by a single command in the code and the continuous movement leading to the global trace mark imagined.

Design of the experiment and data collection.

We focus on a grade-3 class that worked with the GGBot over 3 sessions of one period (45 min) each. The students had been introduced to the bee-bot in earlier grades, but they had not previously worked with the GGBot. They had learned about plane geometry figures (though not specifically the notion of angle), but not in the context of coding with drawing robots. First, the students discovered and explored the GGBot, through the questions: What is it? What is it for? Why does is do it? How does it do it? The sessions were conducted by the first author in collaboration with the classroom teacher. After the initial discovery of the artefact (first cycle), the first author was prepared to propose both figure-to-code and code-to-figure tasks in the next cycles, based on students' responses. No rigid sequence of tasks (after the first exploration) was decided a priori. This design choice was made to increase the potential of agency, ownership, and identity (dim. 4), because the flow of the experience could better match students' actions, curiosity and involvement by building on the situated signs actually produced. Consistently with this purpose, the tasks proposed were intended to foster the unfolding of the semiotic potential: from the situated signs carrying the students' personal meanings the teacher would foster the development of mathematical signs. In the following section, we present an episode to illustrate a key moment of the unfolding of the semiotic potential, when the teacher changed the task from a figure-to-code to a code-to-figure task. In the analysis presented in the next section, we will consider a code-to-figure task.

"Doing well" in the TRU approach revealed by the lens of the semiotic potential.

After considering some of the codes produced in response to the figure-to-code task "Draw a square", the teacher selected the following: "step, left turn, step, left turn, step, right turn, step, right turn"; the teacher shared it on the main screen (figure 2a) and asked: "What will the GGBot draw if we give it this code?", posing a *code-to-figure task*. She then called on various students, asking them to share their predictions with the class. Below we present an excerpt from a student's response (among the six other ones provided by other students contributing to the discussion) concerning the interpretation of this code. We chose this excerpt because of how it shows the sprouting of an apparent cognitive conflict that triggers the unfolding of the semiotic potential of the GGBot. We use the example to show how the evolution of the semiotic mediation process is consistent with the TRU approach.

Silvia's prediction: the unfolding of the semiotic potential through a cognitive conflict.

Table 1 shows Silvia's prediction, including both words and gestures.

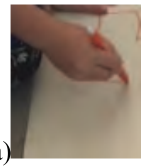
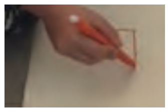
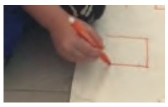
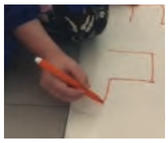
L.	S.'s words	S.'s gestures
1 2	So, he took a step forward (a), no? a rotation (b)	(a)  (b) 
3	then another, a ...another step forward, so (without touching the paper with the tip of the marker) the rotation...(c)	 [oscillating over the second segment] (c)
4	yes well, the, the step forward (c)	oscillating the marker over the two segments drawn
5	a rotation [(c) to (d)]	draws over the third short segment
6 7 8	and then after the rotation again a step forward (d) then he put the rotation the opposite way, and so like this (e) and like this (f)	   (d) (e) (f)

Table 1: Silvia's prediction expressed through words, gestures, and drawing

The task appears to be demanding for Silvia: she seems to be struggling as she thinks aloud (TRU dim. 2). Even though she had already envisaged a figure, she still seems hesitant about the "rotation", to which she returns again and again (lines 2, 3, 5). Silvia keeps on moving the marker on the paper as she says "rotation" (it. "rotazione") the first time (line 2), which suggests that she has not yet conceptualized a turn/rotation as a change in direction without continuing the movement in the new direction. Indeed, in real life, one does not rotate without continuing to walk in the new direction! However, there seems to be a seed of conflict (line 5), that might bloom into the mathematical notion of angle as rotation (dim. 1), by engaging in productive struggle (TRU dim. 2). The conflict remains unresolved, as Silvia then continues to draw a longer segment as she mentions again "rotation" and "step forward" (line 6). We interpret this as a global perception that seems to dominate over Silvia's analytic prediction: she starts off looking at each block on the screen and looking for a correspondence with the trace mark, but then her global perception takes over, making her lose control of the analytic relationship between blocks and the parts of the predicted trace mark.

We interpret Silvia's conflict as being (at least in part) due to her relying on personal meanings that are based on a physical everyday experience of "turning" during a walk, that is never accomplished as a rotation without moving along a straight trajectory "before" and "after" the turn". In this sense, the seed of the mathematical concept of *angle* is "empirical" in nature, and full of aspects that gradually will be "weeded out" as this (and other) mathematical discussion(s) move away from situated signs and towards mathematical signs, pointing to more abstract mathematical meanings. On the one hand, designing tasks that foster sense making through the production of personal meanings that live in a sort of "empirical" mathematics, seems to be very much in line with what Li and Schoenfeld (2019) argue for. On the other hand, being aware of the hybrid nature of the personal meanings produced by students like Silvia can help the teacher better exploit the semiotic potential of the tasks accomplished with the GGBot.

In terms of signs produced, we notice that Silvia produces distinct signs (word and gesture) only for the step ↑ block, which is a straight segment of a fixed length. While her drawing shows, in a global way, the "change in direction" in the form of perpendicularity between consecutive segments. However, though her struggle in trying to elaborate a sign for interpreting the Turn ↱ block remains unsolved, she successfully contributes to the conversation about mathematical ideas (dim. 4).

Discussion and conclusions.

The dual set of tasks proposed by the teacher led the students (not only Silvia, but all her classmates, too) to producing a very rich set of situated signs corresponding to personal meanings, that helped the students gain insight into the relationship between the single blocks, the robot's movement and its trace marks. Beyond the opportunity of reconceptualizing the square to be drawn as a contour, many students were led to conceiving such a contour as being made up of segments and points/turns (some spoke of "turning points") where the idea of *rotation* is associated to a specific point around which the robot stops and turns/rotates without moving forward.

The excerpt analyzed certainly shows that tasks such as those described have a high potential for mediating geometrical concepts. However, experiences such as those presented with the GGBot (and more in general with an artefact), may not be purely mathematical. Indeed, the cognitive complexity of tasks with a physical artefact such as the GGBot calls into play personal meanings that are indeed seeds of formal mathematical meanings, but initially they are hybrid, with an important "empirical" component. Such a component can be insightfully woven into the didactical cycle by the teacher, who can elaborate on it helping the students construct more formal mathematical meanings related to certain mathematical notions. Moreover, the way in which such an empirical component contributed to Silvia's cognitive conflict, fuels the notion of semiotic potential at a theoretical level. Indeed, according to its definition the notion of semiotic potential alone describes the possible coherence of the meanings emerging from the use of an artefact with the expected mathematical meanings. However, not always does such a coherence occur, and conflicts may arise; studying connections between the semiotic potential and cognitive conflicts that can emerge as students carry out the tasks, opens new theoretical and practical avenues.

Overall, this contribution shows that the articulation between the TRU Framework and the TSM, combined as we have suggested, is not only feasible but also insightful; and it operationalizes "doing well" in (3 of) the 5 dimensions in the context of coding with the GGBot.

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How do students describe and understand properties of special quadrilaterals with digital tools? – An epistemological perspective on mathematical interaction in inclusive settings

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Both digital learning and inclusive learning become of increasing relevance in mathematics education research and practice. Still, the opportunities the first brings for the latter are underexplored, especially in secondary education. This paper presents a digital learning environment on quadrilaterals as considering design principles for inclusive learning, and its implementation. Through an epistemological analysis of a collaborative activity involving students with different learning abilities, the paper reconstructs how mathematical meaning evolves in signs, focusing in particular on the role of the digital tool in use.

Keywords: digital learning environment, geometry, inclusive settings, epistemological analysis

Introduction and theoretical foundations

Inclusive education and equity play an important role within mathematics education research and practice (e.g., Scherer et al., 2016). One approach addressing the heterogeneity in such settings is geared towards allowing students to work on the same basic mathematical idea in different ways and through different, student-led, approaches, following the idea of natural differentiation (Scherer & Krauthausen, 2010). However, little is known about how a mutual mathematical understanding of the idea actually develops in interaction among students with different learning abilities, especially in secondary mathematics classrooms. This paper presents a case study that adopts an epistemological perspective to better understand such a process of meaning-making between two students with and without special educational support. In particular, the case explores the role of a digital learning opportunity designed to facilitate the exploration and negotiation of concave quadrilaterals and their properties. Hence, it draws attention to the potential of digital learning tools in inclusive settings as they become suggested in this specific case.

We will first elaborate on the notion of inclusive education and present an epistemological perspective for reconstructing students' processes of constructing meaning as developing in interaction through and with signs, together with the epistemological triangle as its analytical tool (Steinbring, 2006). We will then describe the methodological background of the study, including the mathematical content in focus, the design of the digital learning opportunity, its empirical implementation, and methods of data collection and analysis. From the then following case analysis, we will draw tentative conclusions on the two research questions:

1. How can mathematical concepts evolve in interactions among students with different learning abilities through the interpretation and the use of signs?
2. Which role can the digital tool play within such an interaction process?

Inclusive learning

Inclusive learning follows the idea of fostering all students with and without different learning conditions, taking into account the vast spectrum of abilities and needs (e.g., difficulties or special talents) encountered in the mathematics classroom. This understanding follows Article 24 of the Convention on the Rights of Persons with Disabilities, which claims equal opportunities for all people to participate in quality education and to develop their potential, regardless of special learning needs, gender, social and economic conditions (cf. United Nations, 2006). In inclusive learning, children should be supported individually and also learn collectively through collaborations and interactions (cf. Häsel-Weide & Nührenbörger, 2013; Scherer et al., 2016; Swidan & Daher, 2019). All students should get the same learning opportunities, but with individual learning aims. When implementing inclusive learning, enabling social participation and the explicit demand for social involvement should be taken into account. One challenge is to deal with heterogeneity and to develop learning environments accessible to all, while at the same time requiring social participation in collaborative learning phases. There is considerable need for research and development, particularly at secondary level. Didactic concepts suitable for inclusive teaching already exist for the primary level, such as *natural differentiation* through substantial learning environments (Scherer & Krauthausen, 2010). Although there are experiences of using digital tools in special education (Moyer-Packenham & Suh, 2012; Peltenburg et al., 2013), their use is often limited to learning and practice software for targeted training. In contrast, we understand digital tools more broadly as appropriate learning tools in the sense of enabling substantial mathematical experiences (Drijvers et al., 2016), acknowledging their potential for inclusive mathematics education. We therefore adapt the idea of 'natural differentiation' for the design of learning environments at the secondary level, taking into account the opportunities offered by digital tools.

Epistemological perspective

Children's mathematical knowledge is situational and linked to certain learning contexts and experiences. These learning and experiential contexts can be extended through interactions by constantly negotiating and interpreting mathematical understandings (Steinbring, 2006). The epistemological triangle (see Figure 1) can be used for analyzing such processes of meaning-making in interaction as processes of learning mathematics. It is "used for modeling the nature of the (invisible) mathematical knowledge by means of representing relations and structures the learner constructs during the interaction" (p. 136). Central to this model is the interplay between *signs* that are interpreted, the *concept* about which the learner constructs knowledge as understood to be represented in some way in the sign, and the *reference context* corresponding to the meaning of the concept through the interpretation of the signs. Figure 1 shows a pictorial rectangle as a sign, which doesn't have a meaning of its own. The meaning has to be produced by the learner by means of establishing a mediation to suitable reference contexts.

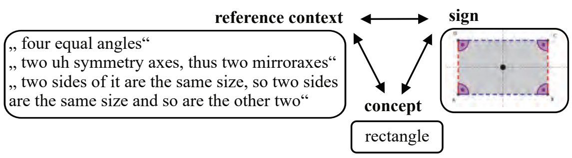

Figure 1: Exemplary epistemological triangle (adapted from Steinbring, 2006, p. 135)

In this case, the student describes the properties of the rectangle that represent the students' reference context, thereby establishing the meaning of the concept. This analytical tool allows a fine-grained reconstruction of evolving mathematical meaning in interaction processes. Therefore, the development of sign-use, the underlying reference contexts and the concepts involved can be analyzed in their synergy (cf. Steinbring, 2006).

Methods and methodology – from design to analysis

Mathematical background and didactical approaches

Quadrilaterals are characterized by their four sides and four angles and can be classified through hierarchy (De Villiers, 1994). Sub-concepts have the characteristics of the general concept and additional differentiating characteristics. A parallelogram has a center of rotational symmetry and opposite sides are parallel to each other, with sub-concepts (rectangle, rhombus, square) sharing these characteristics, thereby being considered special parallelograms. Conceptual understanding of quadrilaterals contains the understanding of a conceptual hierarchy in the sense of realizing ideas and sub-concepts *included* in the same concept (extensional definition of the concept), while also understanding the entirety of properties characterizing the concept and its sub-concepts (intensional definition of the concept) in order to *exclude* hierarchical relationships. Also, ideas like the operational principle consider actions crucial for conceptual development, with these being performed with real, visual and imagined objects, and described verbally. Designing a learning environment then means to offer all learners opportunities to openly approach and manipulate, such that properties can be explored on individually selected quadrilaterals.

The digital learning environment, the activities on it, and underlying design principles

The purpose of the digital learning environment 'dynamic quadrilateral' is to provide opportunities to explore and classify the properties of quadrilaterals (Bebernik & Schacht, 2019; https://www.geogebra.org/m/x7syhqny). It mimics activities manually carried out on GeoBoards, realized digitally through dragging corners or sides with the finger on a touchscreen or with a mouse. In this specific design, properties of the quadrilaterals - such as right angles and axis of reflection - are highlighted by specific markers. It thereby allows students to explore and order quadrilaterals by means of their properties locally through operative manipulations and visual feedback. The tasks focus on convex quadrilaterals, which will appear grey in the digital learning environment. Functioning as an example generator, the digital tool allows for exploring quadrilateral properties, to make discoveries, and to order different types of quadrilaterals (e.g., square and rectangle) locally by identifying similarities and differences. To do this, it is first necessary to ascribe meaning to the markings, such as interpreting equally colored angles as angles of equal size. It is about understanding logical structures of special quadrilaterals rather than naming the quadrilateral. The digital learning environment is first worked on individually, along with prompts to facilitate systematization of what is found out. Later, students work in pairs on a reconstruction task (Wollring, 2012) in which each of them has a specific role. In the setting shown in Figure 2, one person (sender) manipulates a particular quadrilateral with the digital tool and then describes its properties. The partner (receiver) now has the task of reconstructing the quadrilateral with the digital tool. The pair changes roles several times.

• Assemble in pairs so that you can't see each other.

• Person 1: Create with the file 'dynamic quadrilateral' a particularly beautiful quadrilateral and describe ist properties.

• Person 2: Create with the file 'dynamic quadrilateral' the described quadrilateral.

• Compare your quadrilaterals. Which one is it? Was the description good?

• Change the roles!

Figure 2: Reconstruction task (Bebernik & Schacht, 2019; Wollring, 2012)

The design of the digital learning environment and the accompanying tasks is informed by three main design principles specifically aligned to inclusive education. First, the already presented ***principle of natural differentiation:*** In the digital learning environment, the conceptual understanding of individual quadrilaterals can be worked on at different levels – some children are able to interpret specific properties, other children can investigate similarities and differences between individual quadrilaterals. Free exploration and allowing different approaches to deal and engage with the content enable different learning objectives (from pre-formal explorations to formalizations). Second, the ***principle of representation networking:*** In the sense of Duval (2006, p. 125), this involves operational variations in connection with different representations, considering *treatment* as transformation within one register of representation (e.g., within the geometric representation) or *conversion* from one register of representation to another one (e.g., the change from the geometric to the symbolic representation), whereby Duval considers the latter as being crucial for mathematical activity and learning (p. 125). With regard to the designed learning environment, the children should perform both treatment and conversion. For example, through the reconstruction tasks, the transition to symbolic representation or verbal description can be made by describing a quadrilateral. A third design principle concerns the relevance of ***integrating individual and cooperative learning phases:*** Conceptual understanding is built through interactions with others, as interpretations of concepts are negotiated and constructed together. This is realized through the reconstruction task that stimulated exchange about quadrilateral properties, thereby enabling social participation, but also demanding for social involvement.

Data collection and analysis

The data presented in this contribution has been drawn from a larger design research project on developing and researching substantial learning environments for inclusive classrooms. Following individual explorations with the digital learning environment, seven pairs of students (age 13-16) have been videographed while carrying out reconstruction tasks, stimulating the negotiation of mathematical understanding of the quadrilaterals and initiating social learning. The students had different learning conditions (with/without special educational needs) and attended different grades (5 to 8). In each pair, both students attended the same class. The students' interactions and the development of mathematical meaning are reconstructed using the epistemological triangle model (Steinbring, 2006). This paper provides an illustrative case study that allows to explore the research questions, providing rich conditions for reconstructing students' reference contexts through their use of signs while exploiting the affordances of the digital learning environment.

Results: Reconstruction of the learning processes

The results presented in this section especially address the reconstruction of an interaction process emerging while using a digital tool. Here, the students **S**am (sender) and **R**on (receiver) (7th grade, age 13-14) work on the reconstruction task as described above. Ron gets special educational support at school, he has been diagnosed with learning difficulties that encompass both math and written language. Writing and reading are difficult for him. In contrast, Sam has good mathematical and linguistic skills. This combination mirrors a prototypical inclusive learning setting. In the following transcript (translated from German), the two students use the dynamic geometry software GeoGebra to carry out the reconstruction task. Due to space limitations, the transcript only includes verbal interaction without a description of gestures and actions with the DGS.

1	S.:	So, my, my shape has four equal angles. (...) And (...) two uh symmetry axes, thus two mirror axes.
2	R.:	Wait, four large angles, two mirror axes.
3	S.:	But only two sides of it are the same size, so two sides are the same size and so are the other two.
4	R.:	Ah, I think, is it a kite?
5	S.:	No.
6	R.:	(...) Ok. It has two mirror axes and one rotation axis, right? Or doesn't it have a rotation axis?
7	S.:	No, um, just two mirror axes and, well, two symmetry axes and a centre of symmetry.
8	R.:	Ok, so yes. (.) That's what I mean by the rotation axis. Um, that should be the same *(manipulates the quadrilateral)*. Trapezium?
9	S.:	No.
10	R.:	(...) So (...) hmm (14 sec.). Is that a, a (...), a, what's it called? (...) Uh a question, may I please have my two sheets back where I uh, or where the things are on (...). So where here, where you can read the names, of the angles, of the different quadrilateral patterns. (...) I can't remember the name.
11	I.:	That's a rectangle. What you see there.
12	R.:	Oh, ok. Is it a rectangle?
13	S.:	Yes. (...) not like that, but it's also a rectangle.

The following analysis (1) first describes the specific structure of this interaction between the two students. In this context, we especially focus on the role of the digital tool. Secondly (2), an example of the development of reference contexts will be discussed. This will give insights into the conceptual development within a certain learning period based on the activities within the reconstruction tasks. A third analysis (3) will show the important distinction between sign-vehicle (*signifier*) and the underlying concept (*signified*) within a situation, in which a student uses the properties of a certain concept in order to determine the corresponding sign. Hence, the results presented in this section not only give insights into conceptual processes within the interaction, they also describe a process that starts with conceptual geometric activities with digital tools and leads to ways of signifying the underlying concept through verbal description.

(1) Evolvement of meaning during interaction

The following analysis shows how mathematical meaning can evolve during an interaction when working on reconstruction tasks. In a first step, Sam describes (to Ron) certain properties of the quadrilateral he has configured with his DGS (Figure 3 (left), [1] & [3]). For Ron these descriptions are signs to be interpreted first (Figure 3, in the middle is the ep. triangle of Ron being an interpreting agent). He partly repeats the described properties ([2]) and manipulates a quadrilateral with his DGS in order to find a quadrilateral that fits to the given signs by Sam. After a while, he has (supposedly) determined the corresponding configuration (∆, in process) and he asks: “is it a kite?” ([4]). Sam receives this question, which is now his sign, that he has to interpret (Figure 3 (right)). He compares the given sign (kite) with the properties of his initial quadrilateral. Because, for Sam, the underlying concept is the rectangle, he says “No” ([5]).

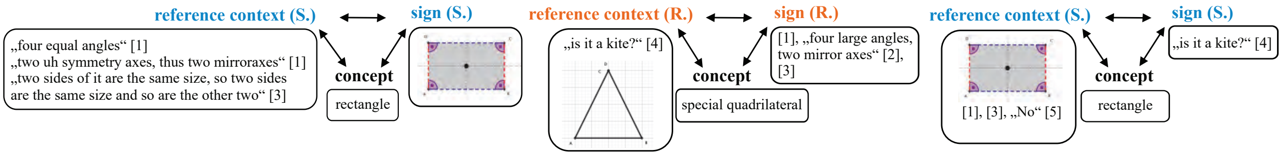

Figure 3: Epistemological triangles - Evolvement of meaning during interaction

The following interaction pattern can be identified here: First, a reference context (the description) is produced by Sam in the light of the sign (the geometric configuration) and the underlying concept. The description formulated by Sam is – in the next step – the sign to be interpreted by Ron. Ron produces a reference context and tries to find the matching quadrilateral with their own geometric configuration by using DGS. In the following reaction by Ron (in this case the question: “Is it a kite?”), Sam compares this sign to his initial situation (the geometric configuration and his properties) and formulates an answer (in this case: “No”). This pattern shows how the reference contexts produced by one of the two partners have to be interpreted by the second partner as a sign. It also shows that within this process of interpreting signs, producing reference context (descriptions) and integrating signs into one’s referential context, conceptual development can be observed in which properties are progressively made explicit by the students in the course of interaction. In this situation, Ron uses the possibilities of the tool in order to produce certain quadrilaterals which relate to the sign being given by Sam and his reference context.

(2) Development of reference contexts

Figure 4 shows the development of Ron’s reference contexts during the interactive situation when working with the reconstruction task with Sam. Note that the first reference context in Figure 4 (left) equals Ron’s first reference context in Figure 3 (middle). The development of Ron’s reference contexts shows that he cumulatively integrates questions to it in order to interpret the underlying signs (the described properties by Sam) ([4], [6], [8]). He uses the information received to manipulate the geometric configuration with his DGS. Note also that “new” reactions in the next step of the interaction are in black font color whereas former reactions are in grey font color.

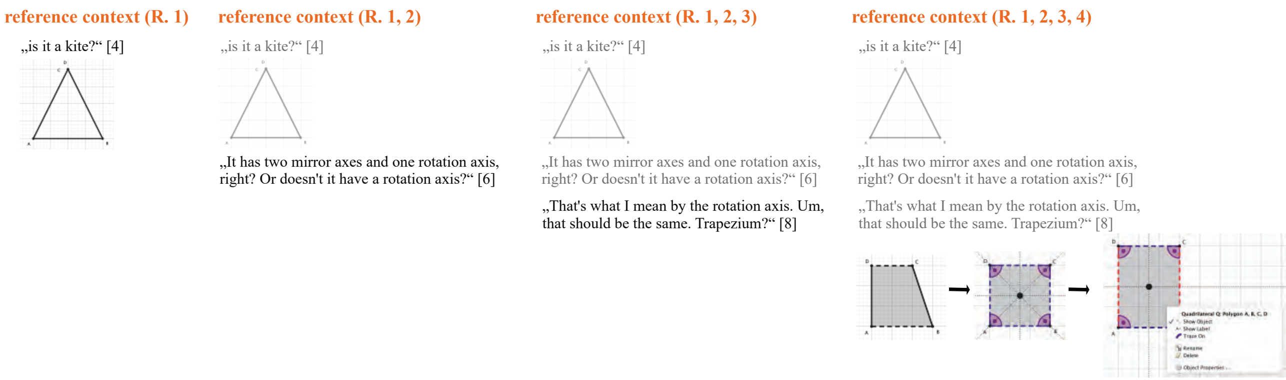

Figure 4: Development of reference contexts from the receiver (Ron)

Through the information Ron received in response to his questions, and the simultaneous tool use, the sign in demand from Sam (rectangle) becomes more and more meaningful. The conceptual development is manifested in this example by excluding properties and thus types of quadrilaterals. In this process the digital tool serves as an aid. Ron uses the digital tool as an example generator.

(3) From conceptual content to the corresponding signifier (sign vehicle)

The last example in Figure 5 shows Ron's epistemological triangle at the end of this episode. Ron and Sam have exchanged a variety of properties, questions and assumptions. Ron has found a geometric configuration with the DGS that seems not to contradict the received information. Indeed, the picture in Figure 5 shows that he constructed a rectangle as Sam did in the first step (Figure 3 (left)). In this situation, Ron asks if he can get back a previously edited sheet with different quadrilaterals and their properties to determine the signifier ([10]). He doesn't know what the correct mathematical term is of this specific quadrilateral with the properties exchanged. The interviewer provides the conventional terminology ([11]). This example shows a process of the evolvement of mathematical meaning by working on a reconstruction task. Signs are interpreted through reference contexts and thus the meaning of the concept evolves. The students determine and use the properties of the concepts before and with the aim of determining the corresponding signifier (rectangle).

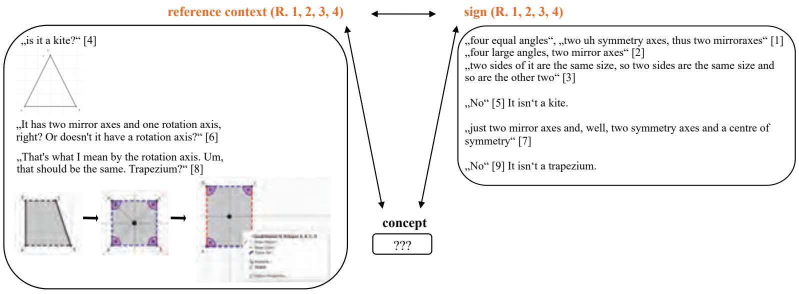

Figure 5: From conceptual content to the corresponding signifier

Conclusion and outlook

The analyses provide first insights into how mathematical concepts – here specifically quadrilaterals – can evolve in interactions among students with different learning abilities through the interpretation and the use of signs. In this process, the evolvement of mathematical concepts is enhanced by the use of digital learning environments. Related to our example, a mutual mathematical idea develops through negotiation of quadrilateral properties during interaction. The analysis of the interaction processes of the students engaging with the reconstruction task shows how mathematical concepts develop by focusing on sign, reference context, and concept. Patterns of interaction emerge that reveal the interplay between signs and reference contexts. A crucial characteristic of the specific type of the reconstruction task is that given signs (information/description) have to be interpreted by producing reference contexts. Within this cascade-like process, a conceptual network is established by sending and receiving information based on the interpretation of given information. The digital tool plays important roles in this context: On the one hand, it highlights certain properties of the quadrilaterals, thereby inviting the students to interpret and communicate these properties represented in signs in the course of interaction and by doing so, extends their conceptual network. On the other hand, it serves as an example generator that has potential to constantly check and re-evaluate information and interpretations. Future work might also explore design-modifications against an embodiment framework to integrate how conceptual understanding can be fostered through the development of sensorimotor schemes (Abrahamson et al., 2020). Together with the epistemic analyses of the interaction processes, this might further enlighten the role of digital tools for conceptual development as encompassing the individual and the social, also in inclusive settings.

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Students with special needs solve word problems in their mother tongue with an app

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Keywords: Special needs learning, word problem, mother tongue, digital media.

Description of the research topic

How can digital media support students with special needs by understanding word problems? One focus of the project presented here are word problems which are presented to students with a non-German mother tongue in their mother tongue. First, a tablet app was designed. Second, the app was tested in interviews with pupils with special needs learning.

Theoretical framework

Solving word problems is a complex process that is especially challenging for students with special needs learning (Häsel-Weide, 2016; Scherer & Moser Opitz, 2010; Verschaffel et al., 2000). Understanding the word problem is often a hurdle which affects subsequent processes of the process of modeling. The wording of a word problem is important, especially for non-native speaking students. Digital media can support mathematical teaching and learning processes (e.g. Krauthausen, 2012; Ladel, 2009; Walter, 2018). There are different possibilities to simplify the understanding of a word problem with digital media (Bierbrauer, 2021), e.g. the problem can be presented simply in different mother tongues. Based on the cognitive load theory, digital media offer possibilities to reduce cognitive loads when working on word problems (Paas & Sweller, 2014).

App Design

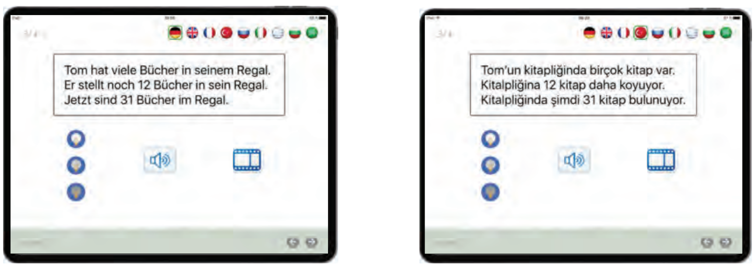

Figure 1: App "Understanding Word Problems"

Within the course of this research project, a tablet app was designed which is intended to facilitate the understanding of word problems. The design of the app is based on a theoretical background of mathematics, special education and multimedia learning. The app offers tools which reduce the cognitive load when trying to understand the problem, so that students can focus on dealing with the mathematical model. One feature is that students with a non-German mother tongue can chose simply their mother tongue. The word problem and other informations are then presented in this language. In Figure 1, the screen of the app is presented. The same problem is presented in German and in Turkish. The app offers further tools: hints for text interpretation, a speech recording and a video.

Testing the app by students

The app was tested in guided individual interviews with pupils with special needs learning who are taught in German schools and who are 9 to 12 years. Altogether 76 word problems are solved with the app, 28 word problems are solved by students with a non-German mother tongue. The collected data was evaluated using qualitative content analysis (Mayring, 2015) with the software MAXQDA.

Results and implications

The poster presents selected results of the project. No student whose mother tongue was not German selected the whole time their mother tongue during solving a word problem. The problems were mostly solved in German even if the knowledge of languages of the mother tongue was better. This result could be due to different reasons, e.g. German is the language of teaching. When the text was presented in a non-German mother tongue the word problem was not solved better. The students reported that a text in their mother tongue does not support them in solving the problem because they cannot read and write the language. Two design principles for the conception and evaluation of digital applications which intend to facilitate the understanding of word problems could be formulated by the findings: Information that is presented in the mother tongue is always accessible in the language of teaching. Text that is presented in a non-German mother tongue is always accessible audibly.

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Action-property duality in embodied design

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Action-property duality in embodied design

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For those working on embodied design it is a challenge to create tasks that enable students to develop abstract mathematical concepts. We approach this issue from the perspective of Sfard's notions of saming, encapsulation, and reification. We discuss a duality of properties and actions, and use this duality to review saming, encapsulation and reification from an action- and perception-based perspective. To illustrate the power of this theoretical contribution we discuss one new embodied task design and two from literature: MIT-P for proportion and a design for the gradient of a plane using the Augmented Reality Sandbox.

Keywords: Embodied design, Operational-structural duality, Action-property duality.

Introduction

In her 1991 paper on operational and structural conceptions in the formation of mathematical concepts Sfard writes that "we have good reasons to expect that in the process of concept formation, operational conceptions would precede the structural" (1991, p.10). Research into action-based embodied design seems to support and exploit this view that operations – in the form of goal-oriented actions that develop in the context of motor problems – form a ground for developing mathematical concepts (Abrahamson et al., 2020). Operational-structural theory and Abrahamson's embodied design theories share a central role for the transitions from process to object, and the aim of this paper is to study in more depth how a further application of ideas of operational-structural theory can inform embodied design. In particular, we are interested to see how the terminology of object formation—saming, encapsulation, and reification—as it evolved in Sfard's later work (2008), apply to the context of embodied design. To this purpose we write about *action-property duality*, a duality we believe to be at the heart of students' discovery and development of new mathematics in embodied designs. Whereas in Sfard's later work emphasis lies on how the development of saming, encapsulation, and reification take place in communication—in the introduction of new discourse, through signifiers, like nouns—we would like to draw attention to how these developments take place and are observable in students' non-communicative *actions* in embodied learning environments, in particular their interaction with artifacts (Shvarts et al., 2021).

In the next section we present a theoretical perspective on operational-structural theory and embodied design, immediately adding our view on action-property duality. In this section these theoretical ideas are illustrated a new embodied design for studying quadrilaterals. The next section illustrates how the theory applies to embodied designs in two studies, a well-known example from design for the concept of *proportion*, the MIT-P (Abrahamson & Bakker, 2016), and a more recent embodied design, using augmented reality, for the equation of a plane in a three-dimensional coordinate system and its relation to the gradient vector (Bos et al., 2022).

Theoretical contribution and background

Action-property duality

Let us postulate what we mean by a property in relation to actions:

A property is an invariant under constrained actions

The invariance means a property is perceived to be the same before and after (and possibly during) the constrained action. Trivial but crucial to us, a property is only invariant under actions that maintain the invariance. For example, an angle is only invariant under angle-preserving actions. There is an interesting duality in this formulation: properties and invariant-preserving actions cannot exist without each other. A naïve perception of sameness underlies the ability to discern a property. Our main claim is that such a naïve sense of sameness co-develops with an ability to maintain the property. To be able to compare the sameness across instances of the property in different objects, transformations of one object to the other need to be performed or imagined. In short, action-property duality refers to the phenomenon that the sameness with respect to a property across objects can only be perceived by transforming those objects into each other while maintaining the property either physically or in the mind's eye.

To make sense of this, let us look at the example of the angles of polygons. Suppose a student is invited to manipulate a polygon by dragging the corners. A naïve perception of *angle* depends on a naïve perception of the *sameness* of angles, since different instances of, e.g., straight angles need to be recognized as the same. As a consequence, a naïve perception of the sameness of angles must co-develop with a naïve ability to maintain an angle while transforming one polygon into another.

Operational-structural development

A theory of operational-structural development within mathematics education developed in the 90s within the framework of traditional cognitive psychology. Sfard argues that operational understanding precedes structural understanding of mathematical concepts (1991). She describes three stages in object/concept development: interiorization, condensation, and reification. Later, Sfard elaborated her perspective on operational-structural issues within the commognitive framework (2008). In the first two columns of Table 1 we present this later view on object formation (cf. Sfard, 2008, p.170). We do not intend to embrace or study the whole theory of commognition here, but find its description of saming, encapsulation and reification most suited to apply in the embodied design context.

Table 1: Operational-structural development from a commognitive and an embodied/action-based perspective

	Commognitive (Sfard, 2008)	Embodied: Perception-Action
Saming	Creating a subsuming discourse on hitherto unrelated objects with the help of a single signifier. Example: "This is a square and	Hitherto unrelated objects are perceived as similar and acted upon in similar ways. Example: two different squares are manipulated

	that is a square too".	with similar dragging schemes.
Encapsulation	Assigning a signifier to a set of objects and using this signifier in the singular when talking about a property of all objects together. Example: "A square has right angles."	A set of objects is perceived and/or acted upon as instances of a more abstract object. Objects in the set can be transformed into each other, if one perceives the defining properties to stay invariant. Example: dragging one square top of a congruent one, while maintaining equal sides and right angles.
Reification	Introducing a noun or pronoun with the help of which narratives about processes on objects can now be told as 'timeless' stories about relations between objects. Example: "These squares are similar through *rotation* and *translation*."	A series of actions on objects is perceived and performed as part of a single process. Example: rotating a square 90 degrees clockwise and then rotating it back to the original position.

A role of the operational-structural perspective in embodied design

Over the last 20 years, ideas from the psychological theory of embodied cognition have gained currency in mathematics education research, see (Abrahamson & Lindgren, 2014) and references therein. Embodied designs allow students to develop mathematical concepts from naïve perceptions (perception-based designs) or actions (action-based designs) in embodied learning environments (Abrahamson et al., 2020). In perception-based design students are challenged to use their innate perceptive qualities to observe certain events with potential mathematical meaning. Similarly, in action-based designs students are challenged to solve a problem of motor control with potential mathematical meaning. These naïve perceptions and actions are then developed into more robust mathematical concepts with the guidance of a tutor, thus grounding the meaning of the concepts in embodied (perceptive and motoric) experiences (Flood et al., 2020).

Returning to operational-structural development, as presented in the second column of Table 1, we argue, firstly, that saming, encapsulation, and reification are not exclusively revealed through communicative acts, but additionally through non-communicative actions: the way artifacts are handled in an embodied learning environment. This point of view is elaborated in the third column in Table 1. Below we present support for the idea that students' actions evidence stages of saming, encapsulation, and reification, before those stages are communicated through speech or gesture.

Secondly, from an action-property duality perspective, we argue that saming, encapsulation, and reification can be interwoven in embodied design. Motoric fluency in action-based design indicates that the series of necessary actions to solve the motor problem is perceived as part of a single process. These transformations contribute to the discovery of a property (of a new object) and hence contribute to a process of saming and encapsulation. As a consequence, development towards saming/encapsulation and reification are made simultaneously; this rephrases the idea of action-property duality within operational-structural perspective. This way action-based design offers an opportunity for the simultaneous development of a new object and the associated constrained actions (transformations).

To illustrate this let us look again at quadrilaterals. We developed a task series in which a student is invited to move similar quadrilaterals on a multi-touch screen on top of each other by dragging the corners with their fingers (see Figure 2). The corners must be moved independently but simultaneously by four fingers. Moreover, the similarity of the quadrilaterals must be maintained while moving – this is supported by color feedback as in MIT-P. Recognizing similarity of types of quadrilaterals relies on recognizing the similarity of angles and proportions of side lengths. This, in turn, co-develops with the ability to mentally or physically transform one quadrilateral onto/into the other while maintaining those properties. Movements that maintain similarity are turning, dragging, and mirroring. While students try to "same" similar quadrilaterals, they inevitably stumble upon those transformations as naïve actions. Naturally adaptive motor control might lead students to develop distinguishable fluent transformations that could be developed into more rigorously mathematical concepts of rotation, translation, and reflection. This illustrates the main point of how new objects and the associated transformations potentially codevelop in a saming-task.

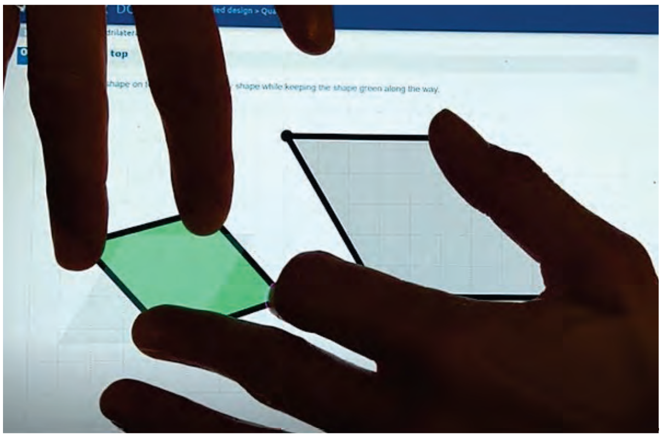

Figure 2. Action-based task: saming flexible quadrilateral by dragging four corners

Examples of embodied designs: the role of transformations

In this section we present two examples of the role of the action-property duality and the operational-structural development in embodied design. We emphasize how fluent motor-action could be reified into mathematical transformations (seen as objects).

Example 1: embodied design for proportion

A well-studied example of embodied design is the action-based task for proportion based on dragging two vertical bars (Abrahamson & Bakker, 2016). The student is encouraged to find positions where the heights of the bars are in a fixed proportion (e.g. $2:3$) by receiving green feedback when such a position is achieved, changing to red if the heights are not according to this proportion. Once a position has been achieved the student is invited to move the bars in a way that maintains the green feedback. In our interpretation from a procedural-structural perspective the student is hence invited to perform constrained transformations on the system of two vertical bars. The outcome is not only a naïve conception of proportion, but also a naïve conception of those transformations that leave a proportion invariant. Flood et al. (2020) report on a student, Ben, who arrived at a solution where the bars move with constant, but different speeds. We infer that Ben not only explored the invariant proportion, but also the actions that maintain the invariant. The latter could be reified into the notion of geometric vertical multiplication, i.e. the transformation that leaves the proportion of height invariant (see Figure 3). In particular, Ben established how (what we call) vertical multiplication has properties different from vertical translation, a transformation that does not leave the proportion of heights invariant. This transformation alludes to the property of proportional variables that, increasing one variable with a factor, the other must increase with the same factor. In general, this again illustrates how the process of saming situations of two bars' heights, the defining property of proportion codevelops with an ability to perform fluent vertical transformation, which could be reified into a mathematical notion of vertical multiplication.

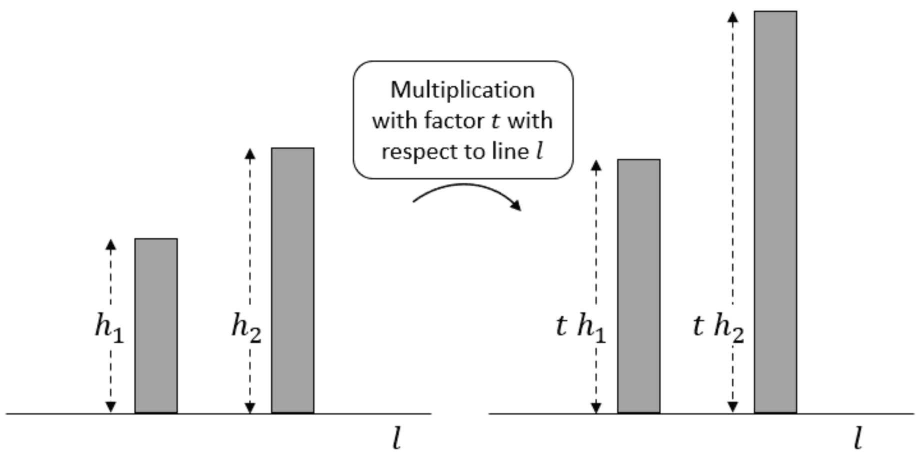

Figure 3. Vertical multiplication as a transformation that leaves the proportion of heights invariant

A transformation can also be associated with the eye movements from one bar to the other. For the task to make any sense the bars need to be considered 'the same' by the student: There must be reason to compare the heights. Some students tend to focus on an imaginary diagonal line between the tips of their two hand dragging the tops of the bars, a so-called attentional anchor (Abrahamson & Bakker, 2016). We argue that this diagonal line is a naïve conception that could (or even should) be reified into a more formal mathematical notion: the transformation of one bar into the other through enlargement with respect to a point (see Figure 4). The diagonal line is the essential ingredient for this transformation. This transformation alludes to the property of proportional variables $H_1 \sim H_2$ that one is a multiple of the other: $H_1 = c\ H_2$. So, yet another process of saming leads to a potential transformation to be reified.

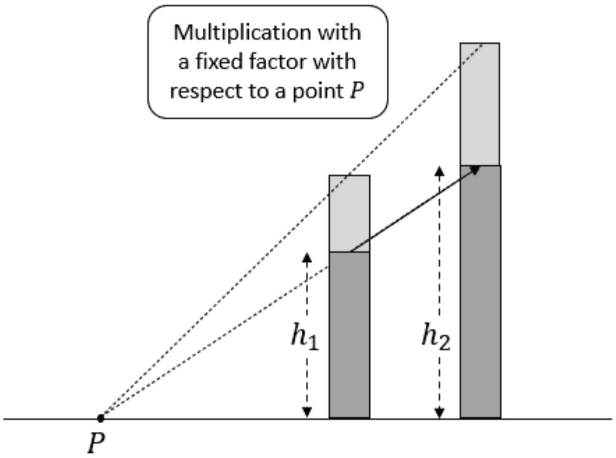

Figure 4. The diagonal line (attentional anchor) can be extended to form an essential ingredient for multiplication with respect to point P: a transformation from bar to the other (ignoring the width).

Example 2: embodied design for the gradient of a plane

In a recent study the author and collaborators investigated embodied design tasks for plane equations ($z = a\,x + b\,y + c$) and the relation to the gradient vector (a, b) (Bos et al., 2022). The embodied learning environment consisted of an Augmented Reality Sandbox (ARSB) – cf. https://web.cs.ucdavis.edu/~okreylos/ResDev/SARndbox/ – together with some plastic planes and bamboo sticks (see Figure 5). The ARSB consists of a box of sand of area roughly 0,6m^2, a stereo camera and a projector. The stereo camera captures images of the height of the sand and objects in the box, and the projector projects height lines and color feedback onto the sand and objects accordingly.

The first few tasks of the teaching sequence aim for students to discover the relations between the parameters a, b and c in the equation $z = a\,x + b\,y + c$ and the transformations rotation parallel to the y-axis, rotation parallel to the x-axis, and translations. In this paper we would like to highlight the next task in the sequence. This task aims to support development of the meaning of the gradient vector in a qualitative way. A vector is defined by its two properties: direction and length. In the case of the gradient vector these properties correspond to direction of the steepest ascent, and to the steepness, respectively. Applying the action-property perspective, each property has associated transformations that leave the property invariant. In the case of direction these are translations and rotations around an axis perpendicular to the direction; in case of steepness these are translations and rotations around a vertical axis. The task is divided into two sub-tasks accordingly: (1) move a circular plane in a way that the direction of the height lines stays the same, but the distance varies; meanwhile, roll a marble down the plane in several positions and try to explain the rolling direction; (2) do the same, but keep the distance between the height lines the same, and explain the speed of the marble. Observing the marble adds a perception-based element to these action-based tasks. Below is an excerpt from a dialogue between tutor Rogier and student Tiago, the case student in the study presented in (Bos et al., in press). The fragment begins while Tiago is working on sub-task 2.

1	Tiago	Like this you get [it] too, and then you can make all those movements again” [Tiago refers to horizontal and vertical translations] but then, in any

		case, it will remain just as steep. [Then Tiago begins to rotate the plane round the place where it touches the sand (see Figure 3)]
2	Tiago	Whether you hold it like this or this.
3	Rogier	Ah.. What movement are you making?
4	Tiago	I rotate it. I rotate it round an axis. That is actually what I'm doing. [Tiago keeps rotating the plane. He chooses a correct position, then waits for the feedback to update, thus clearly using the affordance of the ARSB for establishing a new action.]
5	Rogier	Do you pay attention to anything in particular while rotating?
6	Tiago	No, nothing special. Yes, that I keep it equally steep. Otherwise, it doesn't matter whether you rotate around the point at the bottom [where it touches the sand] or at the top [where Tiago holds it]

Tiago fluently performs a series of actions he calls rotation (line 4), "saming" the plane positions with equal steepness. In line 1 Tiago mentions that during his actions the plane "remain just as steep". This is the first time he associates this adjective "steep" to the plane as a property: one of the two which define the gradient vector. Moreover, Tiago's explanation in line 6 and remarks earlier in course of the experiment suggest he perceives an imagined triangle as depicted in Figure 4 as an attentional anchor, where the angle between the diagonal on plane and the horizontal line on the sand is a measure of steepness. In the light of Table 1 it is important to stress how the introduction of the noun "steep" (commognition perspective, column 2) is preceded and accompanied by the development of fluent action to solve the motor problem and a perception (attentional anchor) that facilitates this action (action-based perspective, column 3).

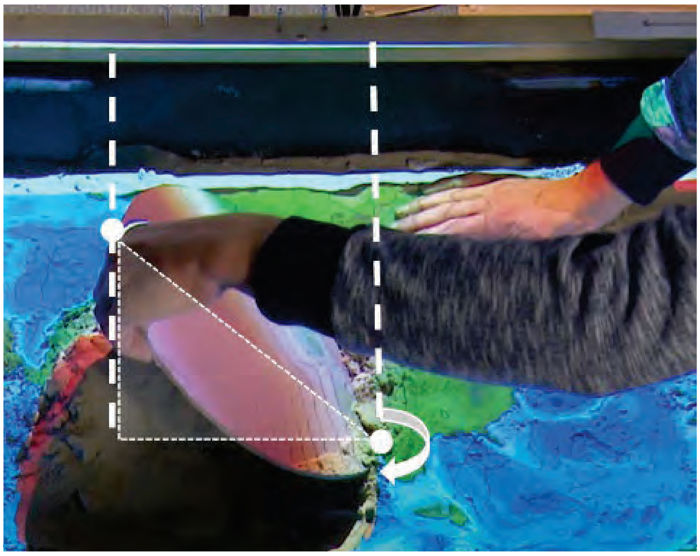

Figure 5. Tiago manipulating a plane while keeping the distance between height lines invariant

Table 2 summarizes the action-property duality for the two properties that constitute the gradient vector. The table highlights how each property is closely connected to an action that can be reified into a transformation as an object. We observed how the actions described in the first column are performed as single process with a clear goal, e.g. in line 6 during rotation "I keep it equally steep". The reification of the object "gradient vector" goes hand in hand with the reification of rotation round a vertical axis as a "keeping equally steep"-action and rotating round a horizontal axis as a "keeping same direction"-action.

Acknowledgment

I would like to thank Anna Shvarts for many discussions and the organization of the Embodied Instrumentation Group meetings at the Freudenthal Institute. The ideas in this paper originate from the inspiring conversations on embodiment with all participants.

Table 2. Action-property duality for two properties that constitute the gradient vector qualitatively

Action	Invariant (property)	Property gradient	Changing
Rotation round vertical axis	Steepness, angle of imagined triangle	Length	Direction of steepest ascent, direction of rolling marble
Rotation round horizontal axis	Direction of steepest ascent, direction of rolling marble	Direction	Steepness, angle of imagined triangle, speed of marble

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FALEDIA – Design and Research of a Digital Case-Based Learning Platform for Primary Pre-Service Teachers

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This paper illustrates the current status of the design and research of a digital, case-based learning platform (FALEDIA), which is being iteratively (re-)designed and researched to increase the diagnostic skills of pre-service teachers. In an interdisciplinary design team consisting of researchers from mathematics education and computer science, digital learning modules on central topics of arithmetic in primary school are designed and implemented in university courses. The learning modules are characterized in particular by the consideration of potentials of digital media from the mathematics education's point of view, such as multiple external linked representations and informative subject-related feedback systems. In this article, the design approach of FALEDIA learning platform is presented by outlining selected learning modules and insights into first results are given.

Keywords: Digital learning platform, case-based, diagnostic skills, place value understanding, understanding of operations.

Introduction

International assessment studies such as TIMSS (Mullis et al., 2020) indicate that pupils are not sufficiently supported in many countries. This particularly holds true for pupils with difficulties in learning arithmetic. Following these recurring empirical findings, it can be assumed that teaching should be consciously oriented towards the learners' individual learning levels and that diagnosis-based support for children should be practiced (Phelps-Gregory & Spitzer, 2018). Therefore, teachers necessarily need diagnostic skills. In order to support pre-service teachers in increasing their diagnostic skills, digital learning platforms can offer new learning opportunities by including multiple representations and intelligent feedback systems.

First, the theoretical background relevant to the design and research of the learning platform is outlined by addressing diagnostic skills, working with cases, and learning with digital learning platforms. The following two sections address the design of the FALEDIA learning platform and the design of the research study. Finally, the research questions and first results are presented.

Theoretical background

Diagnostic skills

‘Diagnostic skills’ are considered a key skill which all teachers should have in order to be able to provide individual support in (mathematics) lessons (Schulz, 2014). Although there is consensus that diagnostic skills are central to any teaching-learning process, the construct is often conceptualized in different ways. On the one hand, diagnostic skills are understood as adequately assessing pupil characteristics and learning as well as task requirements (*accuracy of assessment*, Karing et al., 2011). On the other hand, they are also understood as validly recording the learning status, difficulties and possible backgrounds on the basis of learners' statements (*diagnostic depth of focus*, Prediger et al., 2013). The latter conceptualization of diagnostic skills is followed in the FALEDIA study, since accuracy of judgement has a focus on performance assessments, whereas diagnostic depth of focus also takes potential causal factors of observable performance into account.

FALEDIA aims at contributing to raising diagnostic skills in central arithmetic topics such as *understanding basic arithmetic operations* (UO) and *place value understanding* (PVU).

Case-based learning

In order to prepare pre-service teachers to diagnose the learning levels of children and, based on this, to take appropriate support measures, case-based learning is of decisive importance (Syring et al., 2016). Cases can be in the form of vignettes – as a video, transcript or teaching/learning related document – and can be an occasion for linking theory and practice by establishing relationships between the general and the specific (Markowitz & Smith, 2008). When analyzing the cases, the pre-service teachers – unlike in classroom practice – are not exposed to any immediate pressure to act. Accordingly, it is possible to repeatedly work through a case and, thus, adopt different perspectives (Krammer et al., 2012). Furthermore, the analysis of cases can help to better cope with the diversity of individual cases without getting lost in the multitude of individual approaches.

Based on the promising experiences with the use of cases outlined above, the FALEDIA learning platform is characterized by a consistent integration of school practice cases.

Learning with digital platforms

Web-based, subject-specific learning with digital learning platforms has become increasingly important. With regard to the degree of student activity within learning systems, two approaches can be distinguished:

- *Worked-examples* (Renkl, 2017): Well-structured examples are presented, largely without learners' self-activity.
- *Problem-based learning* (Koedinger et al., 1997): The learners' own activity is encouraged and accompanied, for example, by intelligent tutorial systems.

Design of the FALEDIA learning platform

To emphasize this distinction, the first FALEDIA learning platform version includes two separate variants of the same content. One variant presents the content with informative elements (*worked-*

examples) only, while the other one includes elements to stimulate exploration (*problem-based learning)*. Each one of these will be briefly presented in the following.

FALEDIA conceptional design

Currently, the FALEDIA platform offers learning opportunities for two different topics – *UO* and *PVU*. The website content for each topic consists of three parts. It is considered fundamental for the pre-service teachers to gain (1) the necessary background knowledge. This forms the sound basis for (2) diagnosis on the one hand and (3) accordingly fostering pupils' learning on the other.

FALEDIA provides the necessary content concerning the three most prominent aspects of each topic at first. For *UO* the three aspects of basic mental model, linking representations and using numerical relations have been identified as the most prominent to know. Examples of competency expectations pre-service teachers should be able to acquire are given in Table 1.

Table 1: Competency expectations towards pre-service teachers

Basic mental models	Linking representations	Using numerical relations
Background knowledge – Pre-service teachers …		
can explain which basic image is addressed by given everyday situations or subjects.	can change and describe representation forms – even in more complex contexts.	can describe which task patterns do (not) suggest usage of a specific mathematical law.
Diagnosis – Pre-service teachers …		
can state in how far a basic image is shown in pupils' documents and name difficulties.	can state in how far a child is able to link representation forms and name difficulties.	can state in how far a child uses task relations und mathematical laws and name difficulties.

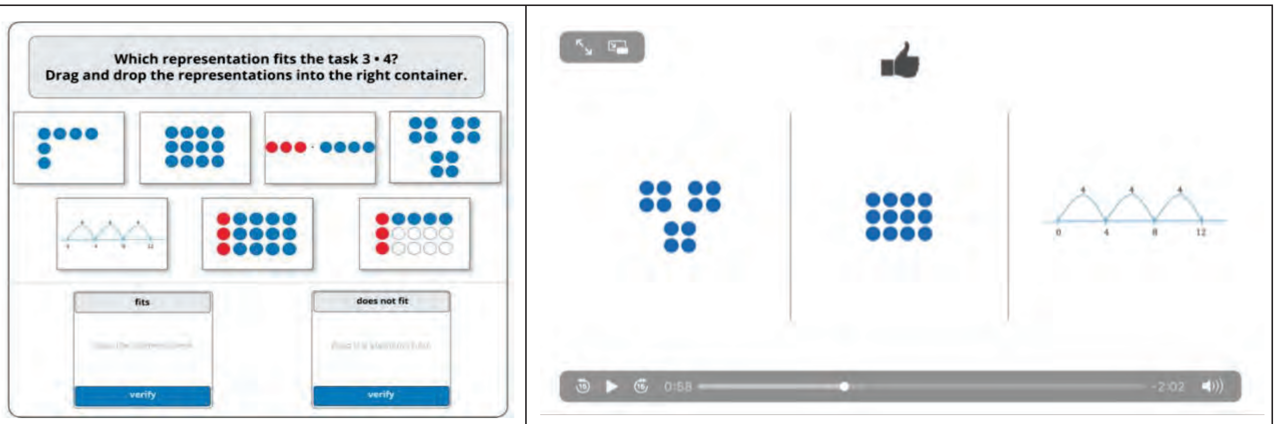
Based on these competencies FALEDIA focusses on background knowledge for pre-service teachers and exemplarily diagnosis of primary school pupils' documents. The two different variants are described in the next section.

Current FALEDIA variants

The current FALEDIA learning platform includes two variants – one with informative elements only and another one mainly with explorative elements. In order to make a comparison between the two variants, either worked-examples or elements of problem-based learning are offered at the same location of the learning content. The way the learning content is offered should contain the same information and be equally attractive to users. The accompanying text, in both variants, is the same.

In the following figures, an example of the implementation of the same content from the two different design perspectives is given. Figure 1a shows a problem-based learning element, where users actively group elements by ordering visual representations according to whether or not they fit the multiplication task 3 x 4. The German conventions dictate that multiplicator x multiplicand = product; the multiplicator expresses how many groups there are while the multiplicand defines the quantity of these groups. A figurative representation shows the quantity of the groups in rows. Users can drag and drop the representations into "fits" and "does not fit" containers. They can click "verify" at any time and receive solution-based feedback.

Figure 1b shows an excerpt from the implementation of the work-example. In this case, the user *is informed* by a video – not as in the example above actively interacting with the platform – about whether or not a representation resembles the task 3 x 4. In each case, a short explanation is given why the respective element was grouped accordingly. This comparison of the examples of the two variants demonstrates that both variants present the same content, but the access to the information differs.



(a) Problem-based learning element

(b) Worked-example

Figure 1: FALEDIA learning modules

The FALEDIA variant with elements of problem-based learning contains various so-called "learning modules"; one of which is the module for "grouping" shown in Figure 1a. To give only a few examples, there is the "slider" in which sequential processes are worked on. At certain important decision points, the pre-service teachers are supposed to choose between three different possible sequels, only one of which is correct. Another activity is "sorting", which is about linear ordering of elements.

The variant with solely worked-examples contains textual and tabular elements; informative videos and audios are integrated. All modules and informative elements are used for various content-related activities and information, both in the background and in the diagnosis.

The following Table 2 illustrates how some contents are realized differently, depending on which variant of the platform they are presented on. Each realization offers learning opportunities – depending on the specific content and its complexity, different accesses might be considered more supportive.

Table 2: Included learning and work-examples only models in comparison

Including interactive modules	Worked examples only

Sorting into groups whether the representation matches a certain task	Explanatory video concerning the connection of different representations
Sorting different contexts to the multiplicative basic mental model which were unknown before	Tabular presentation of the multiplicative basic mental model giving exemplary contexts
Linkage of single-choice tasks concerning an audio file of a pupil explaining her train of thoughts while using mathematical laws	Explanatory audio file including an audio file of a pupil explaining her train of thoughts while using mathematical laws

Design of the FALEDIA research study

The project is designed in a mixed methods design. In the current state, users are offered access to only one of the two pages for each of two topics (PVU and OU). Hence, the pre-service teachers once get access to the variant with worked-examples and once to the one with the problem-based learning, so both variants of access, based on a different topic, are offered. For example, if you see the variant with worked-examples for understanding place value, you see the problem-based variant for operation understanding.

In a next step, the alternative form of access is released for all users in each case, so that access to the information can be chosen freely as required. These multiple representations of the FALEDIA platform are used to gain insights into the perception of the participating pre-service teachers as well as their change in diagnostic tasks.

Research questions

The two following research questions will be addressed in what follows below:

1. Which diagnostic skills do pre-service teachers show before and after using the FALEDIA platform?
2. Which conceptional features and design elements prove to be conductive to learning and in how far are they applicable to other concepts and/or platforms?

Design of the study

To gain insights into the diagnostic skills of pre-service teachers all participating students of a course (N = 188) in their third or fifth semester of studying to become a primary school teacher are obligated to participate in two written surveys and use one of the FALEDIA platform versions for self-study purposes to complete another written task. The students were divided into two groups. One group had the variant with the problem-based learning elements and the other group had the worked-examples to PVU. For the OU, access to the other variant was later enabled for the respective group. The two surveys are designed in a pre-post design. The first test was performed. Then the background knowledge of PVU as well as exemplary diagnosis and fostering was elaborated with the help of the FALEDIA platform. Finally, the posttest was conducted. Among other tasks, the pre-service teachers are given a child’s solution which reveals weaknesses in understanding basic arithmetic operations, and are supposed to describe the errors and specify possible causes. Additionally, guideline-based interviews with randomly chosen pre-service teachers (N = 21) were conducted while they used the FALEDIA platform to provide more detailed information which elements of the two different variants pre-service teachers prefer over the other.

Data evaluation

A system of categories – based on a quantitative study by Brandt (in press) – has been developed to quantify the results of the written survey and get an overview of possible differences in results depending on which variant of FALEDIA was used by the participants. In the evaluation, the diagnostic subskills of describing mistakes, analyzing the causes of mistakes, and assessing diagnostic tasks are looked at. In each case, the three central content aspects of the respective topic are included in the evaluation. The correlations and significances were calculated using Anovas (RQ 1).

As important as that, these results are used to merge the current two variants of the FALEDIA platform into one variant that is tailored to the needs of pre-service teachers. For this purpose, the qualitative content analysis according to Mayring (2019) is applied (RQ 2).

Empirical findings

The first results of the surveys are available for both the diagnostic skills and the design elements. In this paper, a subset of the results on understanding basic arithmetic operations, looking at analysis causes of mistakes is focused on. Amongst others, a before-and-after comparison is used to examine whether the pre-service teachers were able to improve their score after working with FALEDIA and whether there are differences depending on which variant of the learning platform they worked with – the one with the worked-examples or the one with the problem-based learning elements.

Diagnostic skills (RQ 1)

For the average value of the scores achieved in the pre- and post-tests, it can be observed that the pre-service teachers improved both in the worked-examples variant and in the problem-based learning variant (Table 3).

Table 3: achieved scores (UO) averages out of max. 6 points

	score before FALEDIA	score after FALEDIA	score difference
worked-examples (N = 94)	1.33	1.60	+ 0,27
problem-based learning (N = 94)	1.32	1.64	+ 0,32

The increase of points for the analysis of causes of mistakes in the *UO* is statistically significant. Which variant of the learning platform the pre-service teachers had, however, is not significant. This means that the FALEDIA learning platform could help pre-service teachers to increase their diagnostic skills in the field of analyzing the causes of mistakes. However, there is no evidence that one variant of the platform can increase the learning success significantly better than the other variant in this area. As noted previously, prior research has found that a learning platform that includes both – work-examples or problem-based learning elements – proved particularly conducive to learning

(Saatz & Kienle, 2013). Qualitative interviews were conducted to determine which criteria should be used to decide whether a learning situation is better served with a worked-example or a problem-based learning element.

Design elements (RQ 2)

The qualitative interviews offer a more detailed insight into the subjectively perceived learning opportunities. In the interviews, various statements could be clustered into categories. Two particularly conclusive categories for the use of worked-examples and problem-based learning, which can be proven on the basis of the students' statements, are listed below.

Worked examples are preferred if…

… the mathematical content is considered rather difficult or new. ("For the videos the increase in value is when it explains an aspect that I've never noticed before." (Student 1))

… knowledge should be refreshed or presented in an overview. ("If there's something, where you have some kind of 'supercategories' and then you have examples in a comparison, where it's not about diagnosing something, but that you get a feeling for what it's based on, I think the table is more appealing." (Student 1))

Problem-based learning elements are preferred if…

… already acquired knowledge should be verified or deepened. ("This is a good exercise to get more confidence, because you also have to do something yourself […] because you simply have to think more yourself." (Student 2))

… practical diagnostic skills are to be fostered. ("It was good to work practically, like you'll do later as a teacher at school. You do it on your own, you can see how to implement [a diagnostic instrument] – not only theoretically." (Student 3))

These identified categorical differences will be used in the next step to elaborate on the merged site with elements of worked-examples and problem-based learning.

Conclusions

Looking at the insight into the initial survey results provided here, the following can be summarized: for RQ1, there is evidence that pre-service teachers increased their ability to analyze causes of mistakes through the FALEDIA platform. A correlation to the delivery method, i.e. which variant of the platform was used, could not be found significantly. Previous research has demonstrated learning effects for both approaches. However, learning platforms which combine elements of worked-examples and problem-based learning have proven to be particularly beneficial for learning (Saatz & Kienle, 2013). Students' comments from the interviews indicate that both variants are perceived as helpful in certain requirements (e. g. depending on the subject matter and phase in the learning process). Because of this, the final version of the FALEDIA learning platform should not and will not be strictly dedicated to one of the two approaches, but will contain elements of both. For this, with reference to RQ2, categories were worked out in interviews that serve as a basis for decision-making in order to determine which learning opportunity is offered in which variant on the merged platform. Research on the now emerging merged platform based on the initial results of the surveys will be established across sites in the same study design in the coming semester.

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Distance learning versus face-to-face classroom learning: student achievement in conditional probability when using a digital textbook with integrated digital tools

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The COVID-19 pandemic has had an impact on learning and teaching mathematics in many countries. In Germany, as in other countries, the measure of distance learning was taken. We consider the use of digital mathematics textbooks with integrated digital tools to be particularly suitable for remote or distance learning. The potentials of digital mathematics textbooks, such as assessment, interactivity or collaboration possibilities address the challenges teachers faced in distance learning. Whether mathematical learning processes can also be initiated among students during distance learning is investigated in this study using achievement tests. For this purpose, students in distance and classroom learning who have completed the same series of lessons on conditional probability and authentic contexts are compared. Results from the pretest-posttest design are presented.

Keywords: Achievement, digital textbook, digital tools, distance learning, influence of technology.

Introduction

In the project KomNetMath, teachers and their upper secondary mathematics courses in Germany are provided with a digital textbook with integrated digital tools for one school year. The use of the digital textbook is being empirically investigated. One of the main goals of the accompanying study is to investigate the influence on learning and achievement (Brnic & Greefrath, 2021). Of particular interest is the comparison of the use of the digital textbook to the use of analogue printed materials. This is done in the context of a series of lessons on conditional probability in a face-to-face classroom setting. As a result of the COVID-19 pandemic, a study was conducted in which students in a distance classroom setting performed the same series of lessons. Therefore, students who used the same digital materials in the face-to-face and distance classroom settings and received the same tasks and teacher instructions in synchronous learning can be compared. Using a pretest-posttest design, it is possible to measure the extent to which achievement differences occur depending on whether the students completed the series at school or at home using a digital textbook.

The COVID-19 pandemic and its impact on mathematics education

The COVID-19 pandemic has had a major impact on everyday life in many countries, as well as on educational institutions. Lockdowns and social distancing regulations have changed teaching and learning during this period (Engelbrecht et al., 2020). In Germany, for example, parts of the school year in 2020/21 took place in remote or distance learning. In such a teaching setting, the use of technology was crucial (Bakker et al., 2021). Technology could be used, among other purposes, to connect with students via videoconferencing or to work with digital materials, such as a digital textbook. At the same time, the school closures and the different access to technology are seen as possible causes for a deepening of social gaps (Engelbrecht et al., 2020). The long-term consequences for teaching mathematics and mathematical education are currently being discussed in particular in the editorials of various mathematics education journals (e.g., Cai et al., 2020; Engelbrecht et al., 2020).

First studies on the impact of the COVID-19 pandemic on mathematics teaching focus especially on the teachers' perspective (Aldon et al., 2021; Drijvers et al., 2021). In particular, interaction, communication, and opportunities for in-person feedback are reported to have changed in distance education compared to face-to-face instruction and are associated with greater hurdles (Aldon et al., 2021; Cai et al., 2020). Additionally, for some teachers in Germany, Flanders and the Netherlands the belief was found that distance teaching is more suitable for teaching procedures and algorithms than for teaching authentic and complex mathematical activities (Drijvers et al., 2021). Aldon et al. (2021) identified four challenges for teachers in particular:

> (a) managing distance learning to support students' learning through specific methodologies; (b) managing distance learning to develop assessment; (c) managing distance learning to support those students that face difficulties and/or are living a difficult situation/developing inclusive teaching; and (d) managing distance learning to exploit its potentialities for fostering typical mathematical processes. (Aldon et al., 2021, pp. 1, 6)

To what extent these challenges can be overcome in distance learning and how teaching-learning processes can be successfully implemented from the students' point of view, remains to be clarified.

The potential of digital mathematics textbooks

For many teachers, the textbook is the most important resource in the mathematics classroom (Fan et al., 2013; Rezat et al., 2021). Thus, the textbook has a significant influence on classroom instruction (Fan et al., 2013). Due to the importance of the textbook, it can be assumed that this resource was also relied upon in times of the COVID-19 pandemic. The use of digital textbooks exemplifies the use of technology in distance learning. Digital mathematics textbooks may differ significantly from analogue printed mathematics textbooks. The potential of digital mathematics textbooks is visible in various dimensions. Modern digital textbook offers opportunities to interact with the book and with individual tasks (*Interactivity*), but also with other people (*Socialization*). In addition, new adaptation and design options are available (*Customizing the Learning Experience*) and assessment is an integral part of a digital textbook (*Ongoing Assessment and Reporting of Student Progress)*. Furthermore, a digital textbook can be updated quicker and continuously (*Potential Economic Benefits of Digital Content)* (Choppin et al., 2014).

Pepin et al. (2015) distinguish three different types of digital mathematics textbooks, which differ in their design or structure as well as in their creation. The "integrative e-textbook" is a digitised version of a traditional, printed textbook. It therefore largely corresponds to the characteristics of analogue printed textbooks. The content is enriched by add-ons and links. In contrast to the integrative e-textbook, the "evolving or 'living' textbook" is not written by a classical authorship, but is continuously developed by a community. The third type corresponds to an "interactive e-textbook". This is a variant that mainly consists of learning objects. These learning objects include tasks and interactive elements. These objects can be linked and connected to each other (Pepin et al., 2015).

However, other types of digital textbooks or learning tools that do not fit into this categorisation are also conceivable (Pepin et al., 2017).

Learning with digital textbooks and its integrated digital tools

The structure and content of a (digital) mathematics textbook have an impact on the learning opportunities a textbook offers, and this can eventually result in an impact on learner performance (Pepin et al., 2015; Sievert et al., 2021). The textbook, as an artefact, is a mediator between the intended curriculum and the teachers' final actions (Fan et al., 2013; Valverde et al., 2002). According to the *didactic tetrahedron model*, the textbook influences the interactions between mathematics, students and teachers (Rezat & Sträßer, 2012).

So far, the use of textbooks and the influence on learners' achievement has only rarely been investigated (Fan et al., 2013). Research comparing digital and analogue materials has shown an initial positive effect of digital materials (Brnic & Greefrath, 2021; Reinhold et al., 2020). It became apparent that certain groups of people could particularly benefit from digital materials, specifically low-achieving students (Reinhold et al., 2020) or – in comparison of gender – female students (Brnic & Greefrath, 2021). With regard to the use of digital tools, meta-studies also found a positive effect on learning outcomes (Hillmayr et al., 2020).

Research Question

Initial study results indicate that digital mathematics textbooks with integrated tools have positive effects on learning compared to analogue materials (Brnic & Greefrath, 2021; Hillmayr et al., 2020; Reinhold et al., 2020). Digital textbooks are particularly characterised by their potentials and interactive features (Choppin et al., 2014; Pepin et al., 2015). These technological potentials precisely address the challenges that teachers face in mathematics lessons in distance learning, e.g. communicating with students or giving feedback (Aldon et al., 2021; Engelbrecht et al., 2020). However, the extent to which learning processes can actually be successfully initiated in a lesson with a digital textbook that also takes up authentic contexts is still a research desideratum. This results in the following research question:

What influence does the use of a digital textbook with integrated tools have on the students' achievement in the context of a series of lessons on conditional probability in distance learning?

This question will be investigated in a comparison with students who have completed the same series of lessons in a face-to-face classroom setting.

Method

The digital mathematics textbook Net-Mathebuch

The digital mathematics textbook Net-Mathebuch (www.m2.net-schulbuch.de) was used for this study. This interactive textbook was designed as a *digital* textbook and is not based on a printed textbook. The book can be used in the sixth form of the Gymnasium in Germany and covers the content relevant for the German Abitur. It is continuously revised and further developed by a group of mathematics teachers. Feedback from other teachers or students is incorporated. It therefore has the characteristics of an "evolving or `living` textbook" (Pepin et al., 2017). Besides, digital tools are constantly provided with integrated GeoGebra features, thus fulfilling the characteristics of an "interactive textbook" (Pepin et al., 2017). It takes up several of the theoretical potentials of a digital textbook (Choppin et al., 2014):

Potentials of digital textbooks	Examples from the Net-Mathebuch
Use of Multimedia	Learning videos; animations; simulations; pop-up windows and tool tips: gap texts; slide shows; …
Interactivity	Interactive tasks, e.g., through GeoGebra apps; tasks with selection options through checkboxes or radio buttons; drag and drop tasks; …
Socialization	Upload function for own materials; chat; …
Customizing the Learning Experience	Adaption to individual needs and learning styles, e.g., by setting bookmarks and links; unlocking of solutions by the teacher; …
Ongoing Assessment and Reporting of Student Progress	Integration of different types of feedback; print and save option; …
Potential Economic Benefits of Digital Content	Quick revision possibility of the content and features of a digital textbook by the editors; integration of current contexts; …

Figure 1: Potentials of digital textbooks with examples from the Net-Mathebuch

Sample and data collection

As part of the KomNetMath project, teachers and their students are provided with the digital mathematics textbook Net-Mathebuch free of charge for one school year. Teachers participating in the project receive regular in-service training to support the use of the textbook in the classroom. By participating in the project, the teachers agree that they and their students will take part in an accompanying study. In the study, different mathematics didactic questions are investigated, e.g. the influence of the use of a digital textbook on computer-specific self-efficacy expectations and the comparison between the use of a digital textbook and analogue printed materials (Brnic & Greefrath, 2021).

In the study presented here, 12 upper secondary mathematics courses (year 10 and 11) participated. The average age of the students was $M = 15.62$ ($SD = .69$). The five-hour series of lessons on conditional probability was conducted by 87 (39 female) students in September/October 2020 in face-to-face classroom settings and in April/May 2021 63 (28 female) students in distance learning. The distance learning students were home-schooled at this time due to the pandemic and a nationwide lockdown. The lessons were taught using the Microsoft Teams video conferencing system. The mathematics lessons took place regularly according to the timetable. In the video conferences, the students and teachers were online for the entire time. In the distance and face-to-face lessons, the pre- and posttest as well as the series of lessons were conducted synchronously in the respective courses. In both research groups the students always received the same instructions so that they worked on the same tasks in the same order in the series of lessons.

Design of the intervention and test instrument

A lesson series on conditional probability was developed for the KomNetMath project. In this five-hour lesson series students work on tasks from the Net-Mathebuch. Most of the tasks deal with authentic contexts, for example a home HIV test and DNA analysis. The lesson series is to be carried out according to predefined course plans for the individual lessons. In order to minimise the influence of the teacher, the students should mainly work on their own. The students can check their own solutions for some tasks with feedback options in the digital textbook independently (see Figure 2).

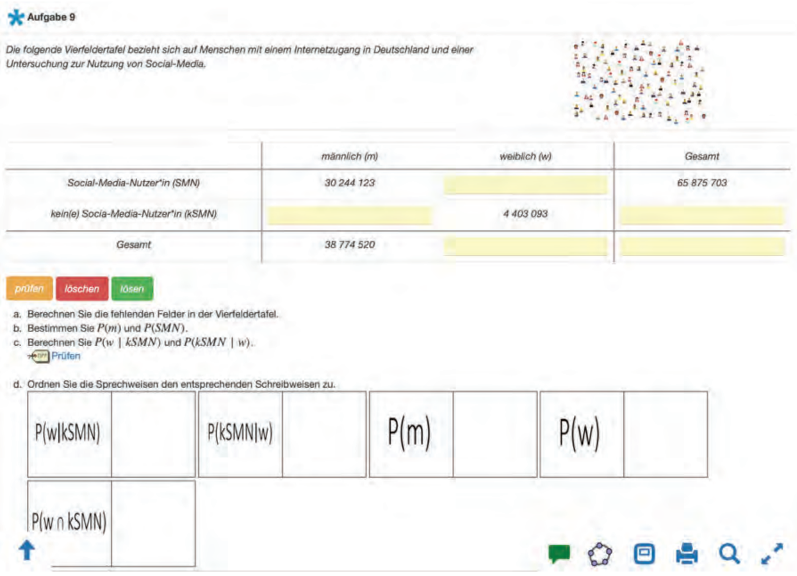

Figure 2: Sample task with feedback option for the solution

Before and after the lesson series, the students take a test on conditional probabilities. The test contains a total of 21 items. The pre- and posttest are linked with 5 anchor items so that the students also solve different tasks in both tests and no repetition effects occur. The test is originally designed as a paper-pencil test for face-to-face instruction so that students can draw their own tree diagrams or fourfold tables. For distance learning, the test format was slightly adapted and partially digitised so that the pre- and posttest could be conducted via video conference. In distance learning, the items were made available to the students digitally using the SociSurvey software. The answer options in multiple-choice format and semi-open response format could also be entered directly by the students digitally. Necessary calculations were carried out on a piece of paper as in a paper-pencil test and uploaded. Since the items were only coded dichotomously, a comparable evaluation of the tests was possible between the classroom and digital setting.

The data are scaled using the Conquest software and a Rasch model. Subsequently, person parameters are analysed with evaluation procedures of the classical test theory. A one-parameter and one-dimensional Rasch Model is used for the pretest and posttest (Rasch, 1960; Wu et al., 2007). Two items were excluded from the analysis because they had a discriminatory power of less than 0.2, which is too poor (OECD, 2012). To check the fit of item and model, the weighted mean square fit (WMNSQ) for each item is also considered. The values for the remaining items range between 0.89 and 1.07 and are thus in a suitable range for the discriminatory power (OECD, 2012). The EAP/PV reliability is 0.51, which is a sufficient value for group comparisons (Lienert & Raatz, 1998).

Results

Figure 3 shows the results from the pretest and posttest from both groups. A higher score corresponds to a better result in the pretest or posttest.

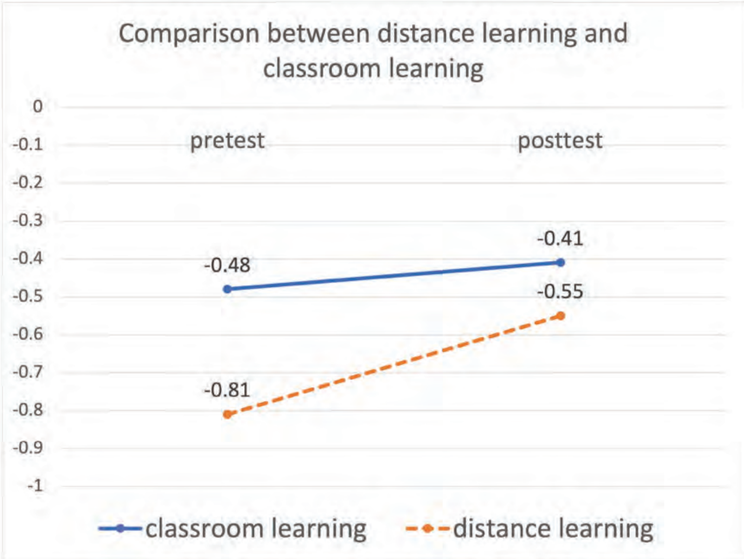

Figure 3: Results of the pretest and posttest on conditional probability

The pretest already portrays that both groups differ significantly with $t(147) = 2.23$, $p = .027$. The effect size of Cohen`s d = .37 corresponds to a small effect. A two-way repeated measures ANOVA shows a significant result for the main effect between the measurement time point with $F(1, 147) = 3.96$, $p = .048$, partial $\eta^2 = .03$. This is a small effect. The interaction between the measurement time points and the group reveals no significant effect ($F(1, 147) = 3.96$, $p = .254$, partial $\eta^2 = .01$). The within-subject factor group is not significant with $F(1,147) = 2.785$, $p = .097$, partial $\eta^2 = .02$. Analyses in both groups show that the face-to-face classroom setting group does not significantly increase with $t(85) = -.73$, $p = .229$. In contrast, the distance learning group increases significantly from pretest to posttest with $t(62) = -1.82$, $p = .038$ and a small effect of Cohen's d = .23.

Discussion

Although the groups already differ significantly at the first measurement point before the treatment, no interaction between the measurement points and the group can be found. Nor are differences in achievement attributable to the group membership.

The assumption that the students show a gap in distance learning to the students in face-to-face classroom learning seems to be confirmed by this sample. Although they manage to improve significantly, they are still at the level below the face-to-face group after the intervention. Whether these differences can be attributed to the challenges for teachers mentioned above (Aldon et al., 2021; Cai et al., 2020) cannot be answered by this study design. Differences between the groups can also be attributed to the fact that both reference groups were taught by different teachers.

The results give indications that the students in distance learning were even able to benefit comparatively more from the digital textbooks and its integrated tools than in face-to-face learning. The digital textbook seems to have particular potential in distance learning and the results indicate that especially low-achieving students from the pre-test benefit particularly from interactive materials (Reinhold et al., 2020). Which potentials and digital tools or in which areas of the tests the respective groups could particularly benefit from requires additional in-depth analyses. Further analyses can also examine whether multidimensional Rasch models can be used or whether a more differentiated analysis of student solutions is possible. Nevertheless, the study shows that students can also learn to deal with complex mathematical content in distance learning. A prerequisite in this study was that the lessons take place synchronously and all students have access to the technology. Nevertheless, it was also observed during the course of the lessons that technical problems can occur occasionally, making communication, for example, more difficult.

The results show that across the entire sample, students were able to improve their performance. This is consistent with other studies that have also found positive effects of digital tools (Hillmayr et al., 2020). In this project, we will further investigate to what extent achievement differs from the use of analogue materials and to what extent students benefit from the potentials of digital mathematics textbooks.

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Why introductory experiments on functional relationships should be qualitative to foster covariation

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The covariation aspect of functional thinking (FT) is difficult for students but at the same time central for the development of a concept of function. Looking at high school mathematics, covariation is not in focus. Prevalent approaches put the well accessible correspondence aspect in foreground and support this imbalance with numeric processing. Although student experiments have proven to be a beneficial introduction to functional relationships, the measurements collection could contribute to this problem. In our pre-post-test intervention study, we contrast a numerical and a qualitative introductory learning environment to functional relationships with a combination of hands-on and digital experiments. Results (N = 332) show significant increases in functional thinking in both settings, but significantly higher gains in the qualitative setting. Effects of the constraints cooperation level and school form indicate possibly relevant influential factors on this lead in line with theory.

Keywords: covariation, digital technology, student experiments, concept of function.

Developing a concept of function

The concept of functions is a major concept and at the same time a major hurdle in mathematics at school. Hence a considerable amount of research has been dedicated to the teaching and learning of functions. This study tries to bring together several branches of evidence to a coherent approach to the concept of functions. Breidenbach et al. (1992) used the Action-Process-Object-Scheme (APOS) theory for a developmental perspective on students' conceptualization of functions. The action concept on the lowest level is limited to the assignment of single output values to an input. With the more generalized process concept students consider a functional relationship over a continuum, enabling the reflection on output variation corresponding to input variation. Finally, functions conceptualized as objects can be transformed and operated on. Students with an elaborate concept of functions are supposed to be able to use the action, process or object conception depending on the mathematical situation (Dubinsky & Wilson, 2013).

Aspects of functional thinking

The developmental stages of APOS are in line with key elements of a function concept, that are described as aspects of functional thinking (FT) by Vollrath (1989) as follows: the *correspondence* of an element of the definition set to exactly one element of the set of values; the *covariation* of the dependent variable when the independent variable is varied and the final aspect, in which the function is considered as an *object*. Although with the APOS perspective one might deduce a teaching sequence with an initial focus on correspondence, then covariation and finally object, current research advocates for a major role of covariation. Thompson and Carlson (2017) argue that the correspondence aspect alone does not evoke an intellectual need for the new concept function and difficulties with functional relationships are mainly rooted in lacking ability and opportunity to reason covariationally. Johnson (2015) points out that correspondence induces a static view on a functional relationship, while a dynamic perspective is a prerequisite for covariation and a process concept. These arguments lead to the call for a qualitative approach to functional relationships in school.

Experimenting fosters functional thinking

Learning environments with experimentation activities have proven to be beneficial for functional thinking (Lichti & Roth, 2018). One possible explanation could be the proximity of functional thinking to scientific experiments as illustrated by Doorman et al. (2012): with a given variable as starting point, a dependent variable is generated in an experiment. Relating the output to the input clearly addresses the correspondence aspect and the action concept. Following manipulations of the input and concurrent observation of the output make the covariation of both variables tangible and enables a process view. Another benefit of student experiment is the inherit constructivist learning approach that leads to higher learning gains in combination with digital technologies (Drijvers, 2019).

Lichti and Roth (2018) implement the scientific experimentation process – preparation (generate hypotheses), experimentation (test the hypotheses) and post-process (reflect results) – in a comparative intervention study to foster functional thinking of sixth graders with either hands-on material or simulations and report learning gains for both approaches (ibid.), but a closer look reveals disparities that can be explained with the instrumental genesis.

Hands-on experiments and simulations in the light of instrumental genesis

The instrumental approach (Rabardel, 2002) and its distinction between artefact and instrument can be useful when interpreting these results: while the artefact is the object used as a tool, the instrument consists of the artefact and a corresponding utilization scheme that must be developed. This developmental process - the so-called instrumental genesis - depends on the subject, the artefact and the task in which the instrument is used. Hence, different artefacts lead to different schemes. Artefacts that are more suitable for the intended mathematical practice of a task appear to be more productive for the instrumental genesis and facilitate the learning process (Drijvers, 2019). When using simulations, schemes that develop are dynamic and concerned with variation as well as transition and hence support the covariation aspect (Lichti, 2019). Measurement procedures of the hands-on material induce static schemes for values and conditions, fostering the correspondence aspect (ibid.). While hands-on material stimulates basic modelling schemes, relating the situation to mathematical description, a simulation already contains a model of the situation. When used as multi-representational systems, the simulation illustrates connections between model and mathematical representations (e.g. graph and table) that evoke schemes for these representations and their transfer. The study presented here attempts to make use of both beneficial influences on the instrumental genesis through an appropriate combination of hands-on material and simulations in experimental activities to foster functional thinking.

Fostering the conceptual development

The measurement procedure is laborious, giving it a dominant role, which sets a focus on correspondence and induces static view on the relationship. As stated above, it would be desirable to shift to a dynamic view, a process concept and covariation. Thus, we explicitly developed a non-numerical approach for experimenting with immediate examination of covariation and compared this qualitative setting to a numerical one, following the implementation from Lichti and Roth (2018).

The learning environments

Both settings use a story of two friends preparing to build a treehouse and contain identical overarching tasks. The contexts are implemented with the same hands-on material and simulations (see Figure 1 and 2), but different components of the simulations are visible in the settings. The student activities are structured in six contexts (see below for details), each one laid out like a scientific experimentation process with preparation, experimentation and post-processing phase. The students work in pairs (A and B), each working on three contexts (see Figure 1). The contexts are chosen to represent a linear and a quadratic relationship and one with varying change rate.

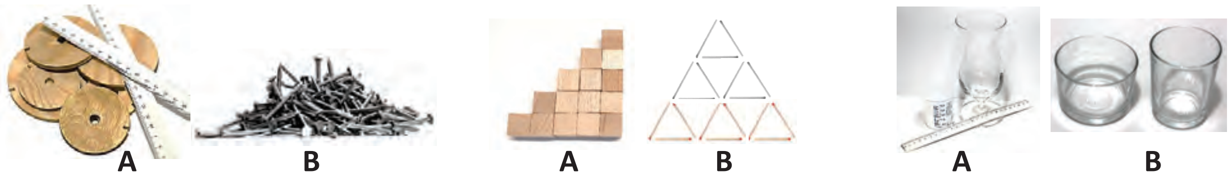

Figure 1: Hands-on material of the contexts for partner A and B

For partner A these are: the perimeter of a circular disc determined by its diameter, the number of cubes needed for a "staircase" determined by the number of steps and the fill height of a vessel determined by the volume of water filled into. Partner B examines the weight of a package of nails determined by the number of nails, the number of beams needed for a woodwork determined by the number of floors and the fill height of cylindric vessels with different diameters determined by the volume of water filled into. A bonus context for quick learning teams depicts the diameter of an unrolling tape determined by the length of tape that has been unrolled.

The **numerical setting** follows the scientific experimentation process: after initial hypotheses in the preparation phase, inspect hands-on material and estimate value pairs, students take a series of measurements and record their data in a table within a simulation (GeoGebra), which creates a graph from the data. The simulation also contains a model of the hands-on material, enabling systematic variation and parallel observation of the altering quantities in model and graph. In the post-processing phase the students verify their measurements, analyze the graph (interpreting and interpolating) and get back to the real material to check their estimations from preparation phase. The learners go through these phases for three contexts subsequently (see above), share their insight after each context with a partner and solve overarching tasks for each context as team.

In the **qualitative setting** the students also start off with hands-on material to activate modelling schemes and enable embodied experience. They are asked to make assumptions about a pattern and on that basis estimate subsequent values. With the aid of a simulation, where they can manipulate a model of the hands-on material, the students get a dynamic view of the relationship and are asked to identify the related quantities, which concludes the preparation phase. In the following experimentation phase students observe the variation and covariation of the quantities in the simulations and verbally describe the relationships discovered. Subsequently graphs are generated within the simulations to enable observing the covariation in multiple representations and in the post-processing phase students are asked to analyze the form of the graphs and connect their insights with the relationship described in the previous phase. The students then team up with their partner, compare both contexts and identify similarities in the relations. In an additional experimentation phase, they take measurements in the context of their partner, represent the covariation in the measurement table and compare this to the results reported by their partner. As a final task the partners are asked to group the contexts by the kind of covariation, i.e. build pairs of similar contexts based on their findings. Both settings can be accessed in digital classrooms[1].

Study Design

A comparative intervention study (pre-post design) is implemented both as in-classroom and as home schooling with seventh and eighth graders at grammar and comprehensive schools. It contrasts the qualitative and numerical settings and includes an additional control group with the simulation only implementation of Lichti and Roth (see above). In a subsample the settings are laid out as individual learning paths, i.e. without team phases. The intervention is designed for six lessons (split into three sessions). It is preceded and followed by a short test on functional thinking (FT-short[2]), to compare the learning outcomes in both settings. Students work in teams of two pairs (except the individual work subsample). A pilot study (ibid.) verified the comparability of the two settings in terms of processing time and difficulty. With this layout we aim to answer the following research questions:

RQ 1: Which setting is more beneficial for FT?

RQ 2: Is the combination of hands-on material and simulations more effective regarding FT than the setting with simulations only?

RQ 3: Do the systematic constraints cooperation level (individual/team) and school form (grammar/comprehensive) have an impact on the learning gains in the compared settings?

Method

Data analysis was conducted according to Item Response Theory. The dichotomous one-dimensional Rasch model and the virtual persons approach were used to estimate an item difficulty for every item of FT-short. The person ability was then estimated with fixed item difficulties. We applied mixed ANOVAs (between factors: setting, school form, teaching mode, cooperation level; within factor: time) after controlling data for normal distribution and homogeneity of variance. Pairwise t-tests were used to investigate differences of the settings. Due to the corona restrictions the distribution of the sample on the different constraints is somewhat imbalanced. For the mixed ANOVA of cooperation level, a subsample was selected out of the team sample and parallelized by pre-test (see values in brackets in Table 1 for team sample sizes).

A statistical power analysis (3 groups, 2 measurements, power .9, α =.05) for a medium effect (η_p^2 = .06) in a mixed ANOVA gave a desired sample size of 204.

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[1] www.geogebra.org/classroom numerical Setting: HQX7 UZRQ and covariational Setting: D3XM DDSB

[2] Rasch-scalable, 27 items, see Digel & Roth, 2020, online version of FT-short: www.geogebra.org/m/undht8rb

Results

Here we present quantitative results of the main study (N = 332, 121 female, 187 male, age M = 13.0, SD = 4.8). The distribution of the sample over the settings and constraints is shown in table 1.

Table 1: Data sample sizes of subgroups

	Numerical Setting	Qualitative Setting	Control Group
Total	125	114	93
Comprehensive / Grammar	52 / 73	39 / 75	27 / 66
Individual work / Team	20 / 20 (105)	18 / 18 (96)	16 / 16 (77)

The estimation of the Rasch-model, used to determine the person abilities for the total sample, showed good reliabilities in the pre- and post-test: $EAP\text{-}Rel_{pre}$ = .86 and $EAP\text{-}Rel_{post}$ = .80 as well as $WLE\text{-}Rel_{pre}$ = .85 and $WLE\text{-}Rel_{post}$ =.80.

Comparison of the settings in total

The mixed ANOVA (see Figure 2) resulted in two significant and one minor significant effects: first, there was a significant main effect for time $F(1, 329) = 188.17$, $p <.001$, $\eta_p^2 = .36$. The results in FT-short for the total sample (numerical, qualitative and control setting together) increased significantly with a large effect from $M = -.46$ logits (SD = 1.37) up to M = .26 logits (SD = 1.01). Second, there was a minor significant main effect for setting $F(1, 329) = 256.34$, $p <.01$, $\eta_p^2 = .04$. The subsamples of both treatment groups (numeric/qualitative) did not differ before the intervention ($t(198) = -.18$, p =.571), but they did afterwards ($t(198) = .26$, $p <.001$, d = .32) and both together did not differ from the control group before the intervention ($t(134) = -.78$, p =.219).

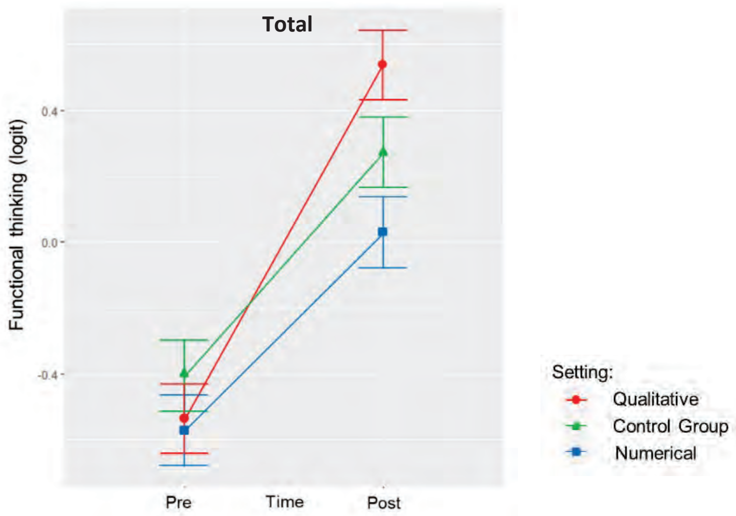

Figure 2: Increase in FT pre to post by setting

Results in all three settings increased significantly from pre- to post-test (see Table 2). The mixed ANOVA also showed a significant interaction between time and setting ($F(2, 329) = 5.33$, p =.005, η_p^2 =.03) with a small effect. Due to limited space, the results of the following ANOVAs are only reported briefly. If not stated otherwise, the remaining main and interaction effects were not significant.

Table 2: Learning Gains pre to post in subgroups per setting

Reported are effect sizes (Cohens' d) with significance level *** (p < .001) if not stated otherwise

	Numerical Setting	Qualitative Setting	Control Group
Total	.25	.51	.27
Grammar / Comprehensive	.27 / .32	.48 / .63	.28 / .34
Individual work / Team	.37** / .25*	.37** / .76	.26* / .23*

Comparisons of the settings under constraints

Regarding the **school form** (see Figure 3 left) the mixed ANOVA showed a significant main effect for time ($F(1, 326) = 197.34$, $p < .001$, $\eta_p^2 = .38$) and a significant effect of school form ($F(1, 326) = 87.82$, $p < .001$, $\eta_p^2 = .21$). Above, there are two significant interaction effects: between time and setting ($F(2, 326) = 5.92$, $p < .005$, $\eta_p^2 = .018$) and between time and school form ($F(2, 326) = 9.57$, $p < .005$, $\eta_p^2 = .029$). The grammar school students outperformed the comprehensive school students in the pretest significantly ($t(174) = 8.09$, $p < .001$, $d = .61$), but for both school forms students' ability increased significantly with a small to medium effect (grammar: $t(425) = 7.08$, $p < .001$, $d = .34$; comprehensive: $t(216) = 5.84$, $p < .001$, $d = .40$). In both school forms students in the qualitative settings showed the highest learning gains (see Table 2).

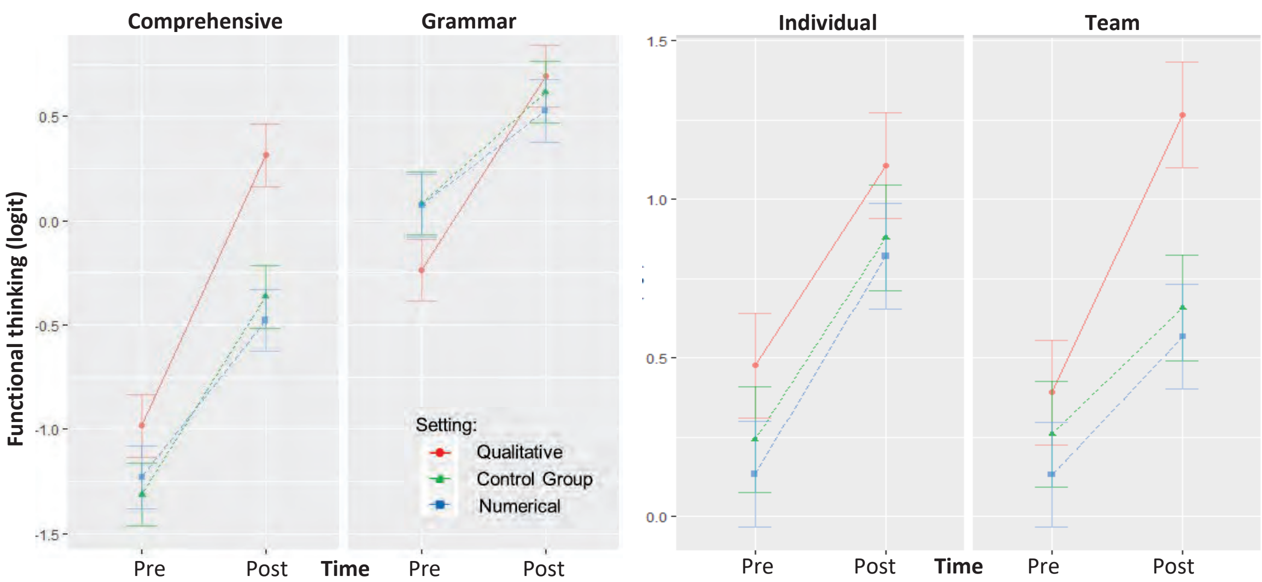

Figure 3: Increase in FT pre/post by setting & school form (left) / by setting & cooperation level (right)

The mixed ANOVA for **cooperation level** (see Figure 3 right) resulted in a significant main effect for time ($F(1, 102) = 79.38$, $p < .001$, $\eta_p^2 = .44$) only and no significant interaction effects. This subsample is part of the grammar school sample (high abilities in the pretest with $M = .36$ logits and $SD = .87$). The effect sizes of the learning gains are reported in Table 2.

Discussion

One of the major restrictions are the unbalanced subgroups, caused by requested flexibility towards the participating schools due to the pandemic restrictions. Above the results are not generalizable without reservation, since they depend on the concrete settings developed in the study. Nonetheless, the results show a significant increase of FT in the numerical (small effect d = .25) and the qualitative settings (medium effect d = .51), as well as the control group (small effect d = .27), from pre- to post-test. Hence all three approaches are suitable to foster functional thinking of seventh and eighth graders. The learning effect for FT in the qualitative setting is significantly higher (small interaction effect of time and setting η_p^2 =.03). Thus, we can conclude that the qualitative approach with a focus on covariation seems to be more beneficial for functional thinking than the other two (RQ1).

Since all three approaches in this study use identical (in case of the control group similar) simulations and contexts, it seems that the specific sequence and focusing of the tasks are decisive. Referring to our theoretical background, we consider two characteristics of the qualitative setting as influential aspects: first, the early focus on the dynamics of the observed variables in the qualitative approach provide opportunities to reason variationally and to develop a dynamic view on functions. Second, the shift of the measurement procedure to a very late step might also contribute to this view. We can assume that replacing early measurement with investigation and observation of the relationship initiates practice in covariational reasoning.

The learning effects in the numerical setting and the control group do not differ significantly as opposed to the qualitative setting. Regarding RQ2 we assert that the combination of hands-on material and simulations, as laid out in the qualitative and numerical setting, only lead to higher learning gains for FT (compared to the control group with simulations only), when the combination is embedded in a qualitative approach. From the perspective of instrumental genesis, we might conclude that the utilization schemes developed with hands-on material could have conflicting influences on FT. For instance, modelling schemes could be beneficial by facilitating the identification of independent and dependent variables, while schemes developed when investigating values and conditions of the hands-on material could hinder a dynamic view.

Regarding RQ3 the significantly different FT results of grammar and comprehensive school students in the pretest (d = 0.61) are in line with PISA results reported by Reinhold et al. (2019). But the medium learning effect in the qualitative setting for comprehensive school students indicate that the covariational focus is also accessible to lower levels of FT and not restricted to high achievers. Since the sample size does not match the power analysis, especially the results regarding the cooperation level must be handled with caution and need to be verified. The contrast of comparable learning gains for all three settings in the subgroup “Individual” and higher learning gains in the subgroup “Team” for the qualitative setting might allude to the importance of the team discussion phases, only present in the “Team” subgroup. They might represent the opportunities for co-/variational reasoning, Thompson and Carlson (2017) call for.

To sum up, a qualitative approach to the concept of function with experiments (1) attains higher learning gains across competence levels, (2) makes the covariational aspect accessible for high and low achievers and (3) benefits from the combination of hands-on material and simulations, when (4) opportunities to reason covariationally are included. In classroom practice, an approach to functions accommodating these aspects has the potential to enhance learning gains.

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Orchestrating the discovery of the Greatest Common Divisor and the Least Common Multiple hidden in a digital spirograph

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This paper focuses on good practices in technology-rich mathematics education, and in particular, on the principles and heuristics that may guide the teacher's orchestration of fruitful activities. We present a teaching activity, framed by the Method of Varied Inquiry (MVI), which involved a small group of 10th grade students, with the aim to let them discover the Greatest Common Divisor (GCD) and the Least Common Multiple (LCM) as objects that mathematize a digital spirograph. Results are analysed and discussed to show how students' learning can be supported by the teacher's orchestration of the activity.

Keywords: Digital spirograph, Greatest Common Divisor and Least Common Multiple, Instrumental Orchestration, Method of Varied Inquiry.

Introduction

Addressing opportunities and constraints of learning mathematics in a technology-rich environment has happened to be even more relevant in times of an immense increase of distance learning and of unexpected changes in teaching practices. In this paper we focus on the principles and heuristics that may guide the design and implementation of a didactic activity developed during the pandemic. More precisely, we are interested in reflecting on the way students' learning can be supported through appropriate design of the activity and teacher's behaviour. For this reason, we refer to the notion of teacher's orchestration developed within the research field of mathematics education, particularly in the case of the use of technological resources in classroom (Trouche, 2004; Drijvers et al., 2010).

The idea of the teaching activity we present in this paper comes from a study in which Ferrara, Ferrari and Savioli (2020) focussed on the mathematics "in movement" that can be addressed using a spirograph (a drawing tool used to create curves for recreational purposes): the movement is the key aspect used to give meaning to the mathematical relationships embedded in the spirograph, such as the concept of the Greatest Common Divisor (GCD) and the Least Common Multiple (LCM). In their work, Ferrara and colleagues suggest some reflections on the mathematical modelling of the spirograph, showing and analysing a teaching experiment developed with 5th grade students. Our attention was attracted by the idea that the characteristics of the spirograph fit with the purpose to offer students a situation in which variation gives power to the development of mathematical thinking, especially in terms of conjectures and argumentations. With this respect, the potential of the spirograph can be exploited not only at the 5th grade, but also at high school level. For this reason, we chose to use the spirograph as a didactical instrument with 10th grade students. The free online availability of a digital spirograph (https://nathanfriend.io/inspiral-web/) contributed then to design a teaching activity, to be developed in the distance context due to the pandemic. We wanted to exploit the opportunity given by the spirograph, in its digital version, to foster students' investigation of hidden mathematics. The functioning of the spirograph, with its related schemes of use, and the characteristics of the obtained curves do not depend on the version of the instrument (digital or not).

The reason why we have used the digital spirograph was due to the constraints of the pandemic. A comparison between the two versions of the spirograph was not an aim of this study but could be investigated in the future. According to this, our assumption was that, throughout the use of the spirograph, students can behave like researchers collecting data, formulating hypotheses and conjectures, and feeling the need to argue their own ideas and to discuss them with their peers. This requires, however, effective teacher orchestration in terms of task design and classroom interventions. To design and implement the teaching activity discussed in this paper we refer to the Methods of Varied Inquiry (MVI) (Arzarello, 2016). Results of this activity are analysed to answer the following research question: how can the students' discovery of the GCD and LCM hidden in a digital spirograph be supported by the teacher's orchestration of the activity? To answer to the research question, we describe and analyse the teacher's orchestration of the teaching activity, referring to the theory of variation developed by Marton and colleagues (2004).

Theoretical framework

Through the metaphor of the Instrumental Orchestration, Trouche (2004) offered a theoretical perspective to describe how the teacher can compose coherent sets of instruments within the classroom in order to guide students' instrumental genesis (Artigue, 2002) and its possible benefits for learning. In the teacher's intentional and systematic organisation and use of the various artefacts available in a learning environment in a given mathematical task situation, Drijvers and colleagues (2010) then distinguished three main elements: a didactical configuration, an exploitation mode and a didactical performance. In terms of the metaphor of the musical orchestrations: setting up the didactical configuration can be compared with choosing musical instruments to be included in the band, and arranging them in space so that the different sounds result in polyphonic music; setting up the exploitation mode can be compared with determining the partition for each of the musical instruments involved, bearing in mind the anticipated harmonies that will emerge; the didactical performance can be compared to a musical performance, in which the actual interplay between conductor and musicians reveals the feasibility of the intentions and the success of their realisation. In the next sections we will present the teacher's orchestration of the teaching activity describing the didactical configuration, the exploitation mode and the didactical performance.

This paper focuses on a teaching activity that was framed by the MVI method. It was introduced by Arzarello (2016) as a method to help students to consider a topic from different points of view and to understand it in a deeper way, fostering the transition from forms of "natural" argumentation to forms of mathematical reasoning. Its approach is near to that of a controlled experiment in science, in which the scientist can vary one variable and observe how another variable changes accordingly. Assuming that teaching with variations, in a controlled and systematic way, helps students to construct mathematical concepts, the MVI is based on the theory of variation, according to which in varying the didactical situations four schemes should be taken into account: to experience something, we must experience something else to compare with it (contrast); to fully understand what "three" is, we must also experience varying appearance of three (generalization); to experience a certain aspect of something, and to separate this aspect from other aspects, it must vary while other aspects remain invariant (separation); if there are several aspects that the learner has to take into consideration at the same time, they must all be experienced simultaneously (fusion) (Marton et al., 2004, p.16).

Methods

As said before, to identify the principles and heuristics that may guide the teachers to orchestrate the students' discovery of the GCD and LCM hidden in a digital spirograph, we analysed the development of the teaching activity in terms of didactical configuration, exploitation mode and didactical performance. In our study, the didactical configuration, characterised by the teaching setting and the artefacts involved in it, is defined by the digital spirograph and by the MVI. The exploitation mode, determined by the role of the spirograph, is described by the design of the teaching activity with its stages and its tasks. Finally, the didactical performance, in which the actual interplay between the teacher, the students and the spirograph reveals the students' discovery of the GCD and LCM, is the basis for the results' discussion.

The teaching activity, carried out in a distance context, was entirely video-recorded and transcribed. The transcripts were analysed according to the Marton's theory of variation, with the attempt to identify the variation schemes as they emerge in the students' intervention.

In this paper we briefly show and discuss results coming from the final collective discussion conducted by the teacher. They are representative of how the teacher's orchestration can guide the students' development and evolution of the variation schemes, thus fostering the discovery of the GCD and LCM hidden in the spirograph.

Orchestration of the teaching activity

In this section we describe the teaching activity focusing on the first two elements of the teacher's orchestration: the didactical configuration and the exploitation mode. The last element, the didactical performance, will be described in the results section.

Didactical configuration

As already underlined, in our study the didactical configuration is defined by the digital spirograph and by the MVI. The artefact, as it is explained below, was chosen for its potential to mediate the mathematical meanings of GCD and LCM. The MVI is used to design the tasks, because it well exploits the characteristics of the spirograph as an artefact in which everything is played on varying wheels and rings.

The spirograph is an artefact that can be used to draw a particular kind of geometrical curves (roulette) called hypotrochoids and epitrochoids. It is composed of a set of gear rings and wheels of different sizes which, by their combination, allow to create aesthetically fascinating curves. On each wheel there are some small holes positioned on a spiral at a different distance from their centre. To use this artefact, you must first choose a ring and a wheel, fix the ring on a sheet and position the wheel internally or externally to it. In our case the wheel was always put inside the ring and so the obtained curves are hypotrochoids. By inserting a pen inside one of the small holes in the wheel, the wheel is rotated around the ring and the curve begins to be drawn. After a complete rotation of the wheel inside the ring, a portion of the curve is traced, which could have cusps, or small roundings similar to flower petals. After a certain number of revolutions, the wheel returns to the starting position and then the curve closes, regardless of the choice of the hole on the wheel and the combination of the gears chosen (Figure 1).

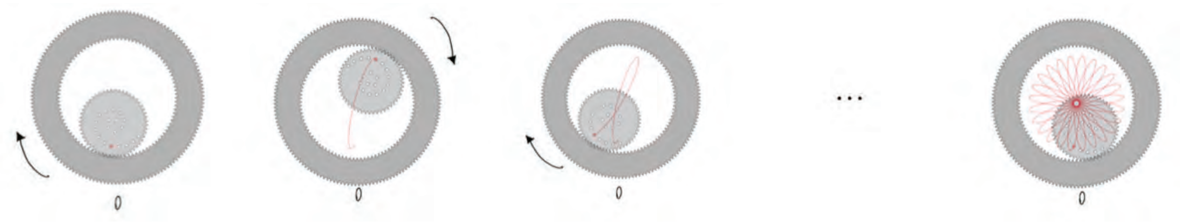


Figure 1: The drawing of the curve with the spirograph

Each curve that is drawn with the spirograph always has the following characteristics: a certain number of petals, a fixed distance between the petals (the number of ring teeth between two petals), a number of complete revolutions of the wheel inside the ring (up to the point where the wheel touches again the tooth of the ring from which it was started) necessary to complete the curve. Of these three characteristic elements, once the spirograph is removed, only the number of petals can be detected by the drawing.

What can be observed, using the spirograph, is that the number of petals, as well as the other characteristics, remains invariant if you change the hole, but instead varies if you change the choice of the ring-wheel combination. This is due to the mathematical relationships (involving the concepts of GCD and LCM) among the characteristics of the geometric curves and the numbers of teeth of the gears. Indeed, if R is the number of teeth of the ring, W is the number of teeth of the wheel, p is the number of petals, d is the distance between the petals and r is the number of revolutions of the wheel inside the ring, it can be seen that: $p \cdot d = R, r \cdot d = W, R \cdot r = W \cdot p$. Moreover, the number of petals corresponds to the number of times the wheel rotates around itself before returning to the starting point, while the curve closes when the repeated sum of the wheel teeth becomes a multiple of the number of the ring teeth. Hence, $R \cdot r = LCM(R, W)$. On the other hand, as $R \cdot r = W \cdot p$, it follows that $W \cdot p = LCM(R, W)$. So, $p = \frac{LCM(R,W)}{W}$ and $r = \frac{LCM(R,W)}{R}$. As a consequence, due to the relationship between GCD and LCM, it can be found that $d = GCD(R, W)$.

These considerations, which connect GCD and LCM to the hypotrochoids obtained with the spirograph, allow students to understand that the characteristics of the curves can be foreseen if the numbers of teeth of the chosen ring-wheel combination are known.

Exploitation mode

The goal of the activity is to use the digital spirograph as an instrument that allows students to untie the mathematical objects GCD and LCM from the idea of being useful objects only for the decomposition into prime numbers, but to see them as objects that mathematise the situation presented with the spirograph. The activity is divided into two phases, guided by appropriate tasks given to the students through an online word document to be filled and discussed: an initial phase of exploration of the instrument and a phase of discovery of the mathematics embedded in the spirograph. At the end of the two phases, to be done in small groups, a collective discussion is expected to be orchestrated by the teacher.

In the exploratory phase students are expected to answer to the questions "how is it done?" and "how does it work?" with the aim of making them familiar with the spirograph and discovering how it draws curves. The second phase of the activity can be divided into four sub-phases (described below), which takes into account the opportunity given by the spirograph to make variations and observe their consequences. To discover the mathematics of the spirograph, indeed, the tasks are designed to let students use different gears, observe and describe what was happening.

Variation of the small hole on the wheel - Students can be asked to choose a ring, a wheel and a small hole and to save the image obtained. Then keeping the same combination of the ring-wheel gears, students can be asked to choose another hole, to save the new image obtained and to compare the two images to describe what was happening. In this way, it is expected that the separation will emerge as a possible variation scheme of Marton's theory. Observing the two images, the students can also be pushed towards a first generalization: by changing the hole in the wheel, the characteristics of the curve obtained do not change, but that what changes is only its "size".

Variation of the wheel, keeping the ring fixed - Students can be asked to take the ring with 96 teeth, to choose a different wheel each, to draw the curve and to save the image. After completing the various drawings, students can fill a table in which, in addition to entering the choice of the wheel and the hole, they can also enter information relating to the characteristics of the curves obtained (the number of petals, the number of teeth between two petals and the number of complete revolutions of the wheel inside the ring) and to observe what happens. Varying the wheel while the ring is kept fixed can, again, boost the separation scheme to emerge. Moreover, as the characteristics of the curve vary when the wheel is changed, in this case, the students can experiment with the scheme of the contrast. The number of teeth of the wheel, in this way, can be seen as affecting the characteristics of the curve.

Variation of the ring, keeping the wheel fixed - Students can be asked to take the ring with 105 teeth and use the same wheels of the previous task. Again, students can fill a table entering the choice of the ring and the hole and the information relating to the characteristics of the curves. The comparison with the previous variation could allow the separation scheme to emerge again: the characteristics of the curve vary when the ring is changed while the wheel is keeping fixed, hence also the number of teeth of the ring affects the characteristic of the curve. This implies that the number of petals, the number of teeth between two petals and the number of complete revolutions of the wheel inside the ring, all depend on the number of the teeth of both the ring and the wheel.

Table 1: Summarising table of the characteristics of the curve according to the R-W choice

Ring (R)	Wheel (W)	Petals (p)	Distance (d)	Revolution (r)
96	36	8	12	3
105	36	35	3	12
96	52	24	4	13
105	52	105	1	52
96	63	32	3	21
105	63	5	21	3

Search for regularities - Students can be asked to take the rings with 96 and 105 teeth, use three wheels (such as those with 36, 52 and 63 teeth) and fill in a table to summarise the characteristics of the curves (see Table 1). With the questions, such as "Is it possible to find relationships between the numbers on each row? If so, which ones? Try to justify your answer", students can be helped to reflect and progress in the searching of regularities and in particular to find the dependence of the curve by both the ring and the wheel. The last sub-phase, thus, is meant to foster the fusion scheme to emerge, through the simultaneous experience with the critical and relevant aspects at stake.

The teacher, aware of the mathematics of the spirograph, can lead the collective discussion in order to guide the students in achieving their learning objective. Attention must be paid to the data shown in the table and the possibility of describing the regularities using mathematical formulas. For the collective discussion it is expected that the possible scheme of variation that can be reached is that of generalisation. From the discussion of experience, it will be possible to deduce: the independence of the characteristics of the curve from the choice of the hole; the dependence on both the number of teeth of the wheel and on the number of teeth of the ring; the mathematisation of the functioning of the spirograph, through the formulas that bring into play the GCD and LCM.

Results: the didactical performance

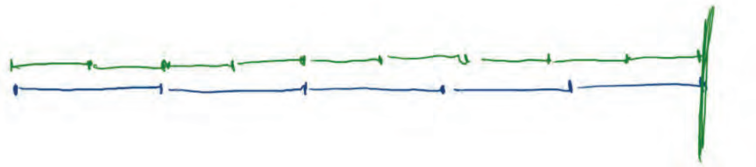
The teaching activity required three hours and involved 12 students. At the beginning of the lesson the teacher explained to the students that they would have been required to be divided into 3 sub-groups and accomplish two different tasks (related to the two phases described above) to be collectively discussed at end. The group work was conducted using the same online platform and sharing a unique document for each sub-group of students. Students were given 30 minutes to accomplish the first explorative phase and 60 minutes for the tasks of the second phase. We do not have enough space here to report the exact tasks given to the students, but the requests were made by the teacher according to what is described in the exploitation mode section. After a small break, the students were invited to leave the breakout rooms and to join the collective discussion guided by the teacher. In this section we focus on the final discussion concerning the second phase. In the first part of the discussion the teacher, showing and discussing a prepared picture, brought the students to share their considerations on what changes and what does not change if the combination of the ring-wheel gears remains fixed while the hole is changed. To guide the students to the shared conclusion that the petals size changes, but the number of the petals does not change, she did not re-voice those answers that are considered not pertinent to her aim. Afterwards, the discussion moved to the variation of the wheel and of the ring. The teacher boosted the students to experience the variation of one aspect at a time, keeping fixed the other, and the students arrived at the conclusion that the characteristics of the curves depend on both the ring and the wheel.

The excerpt below refers to the observations done when considering the obtained table (Table 1).

Teacher:	What observations did you make? Can you see any relationships between these numbers on each line?
Mattia:	8 times 12 gives 96, likewise 24 times 4 and 32 times 3 give 96
Teacher:	So, you are saying that: Petals ∗ Distance = Ring. Now we haven’t embedded the wheel in any relation yet. Do you see any other relations in which the wheel appears?
Michele:	Distance * Revolutions = Wheel, in fact 12∗3,4∗13...
Giulia:	I did something a little different: Ring/Wheel ∗ Revolutions = Petals
Teacher:	We could also write it as: Ring ∗ Revolutions = Petals ∗ Wheel. Indeed, we see that 96 ∗ 3 = 288; and if I do Wheel ∗ Petals, so 36 ∗ 8...
Michele:	I always get 288
Teacher:	What does 288 have to do with the wheel and the ring?

At this point, given the difficulty of the students, the teacher brought their attention to the fact that it is the revolution of the wheel around itself that draws the petals. In particular, in this case the curve is completed after 8 revolution of the wheel around itself and, at the end, the wheel has made 3 revolutions around the ring. The following excerpt starts with Michele's intervention after this last observation of the teacher.

Michele:	The Least Common Multiple! 288 is the LCM between 96 and 36
Teacher:	Why do you think it is the Least Common Multiple?
Michele:	Because we have to fill the same number of wheels and rings, i.e. we have to occupy with a certain number of revolutions the wheel and the ring and then return to the same tooth […] we put in sequence all the teeth of the wheel and of the ring, and we see that… it is the same segment. We must then get to a certain point where the two segments formed by all the ring teeth and all the wheel teeth are equal
Teacher:	That's like I have this kind of thing:



Michele:	Yes, we have to fill the same quantity with 2 different segments. And so that's 288
Teacher:	Now, the distance between one petal and another, if you were to read inside the ring, what will it represent?
Michele:	Perhaps the Greatest Common Divisor. […] Basically, it is the same reasoning, but we have to divide, not to multiply. That is... in this case we have to find a segment that divides both the ring and the wheel equally. And that segment is the distance.

Discussion and conclusion

Results were analysed using the lens of the Marton's theory. At the beginning of the final discussion, the students focussed on the effect of the change of the hole on the obtained curves. Thanks to the teacher's decisions (to show and discuss a prepared picture and to ignore students' answers that are not pertinent to her aim) the students moved from the separation scheme, fostered by the task itself, to the generalisation given by the recognition that the choice of the hole does not affect the number of petals. In the following part of the discussion, the students' focus moved from the observation of particular drawings to the understanding that the ring-wheel choice affects the characteristics of the curve. Also in this case, the students' development of the separation scheme was fostered by the teacher's orchestration, and especially her decision to let students experience the variation of one aspect at a time, keeping fixed the others. Students' awareness of the meanings of the variations among the aspects in the table is evident in the first excerpt: the teacher's questions, aimed at fostering the reading of the table with a researcher lens, allowed students to develop the fusion scheme and to recognise the complex relationships among the various aspects. However, the intervention of the teacher was fundamental to give meaning to those relationships. In the last excerpt, indeed, it can be seen how Michele, thanks to the teacher's suggestion, succeeded in connecting the spirograph movement to the numbers in the table, thus endowing the LCM with both the products $R \cdot r$ and $W \cdot p$, and the GCD with the distance between two consecutive petals.

To conclude we can say that the analysis of the teacher's orchestration showed how students can be supported in the discovery of the mathematics hidden in the digital spirograph, thus answering our research question. Indeed, the students' learning was supported through an appropriate task design of the activity and an aware teacher's behaviour during the classroom intervention. In our case study, the teacher's orchestration was based on her willingness to exploit the opportunity given by the digital spirograph to offer students a situation in which variation gives power to the development of mathematical thinking. Results showed how students behaved like researchers, feeling the need to argue their own ideas and to discuss them with their peers. This was fostered by the teacher's decision to use the MVI as a method to help students to consider a topic from different points of view and to understand it in a deeper way.

Although this study was developed by taking into consideration only one case, its results can give a contribution to the discussion on good practices in technology-rich mathematics education. Moreover, they can be considered as a starting point to develop further investigations focusing on the principles and heuristics that may guide the teacher's orchestration of fruitful activities.

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Perspectives and challenges on programming as a tool to learn mathematics

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This paper focuses on the terminology of computer programming as a tool for learning mathematics and the intersection between teaching programming and teaching mathematics. It shows different perspectives and challenges that might arise when combining these two disciplines. The foundation is the newly introduced Norwegian curriculum, where programming has been integrated into four school subjects in primary and secondary education, including mathematics. The aim is to take a closer look at the concept of 'computer programming' and 'computational thinking' and discuss in what ways programming as a tool in mathematics teaching and learning may be theorized.

Keywords: Programming, computational thinking, mathematics education.

Introduction

The development of digital tools in education used both inside and outside the classroom has increased in recent years. Digital tools have introduced new methods of communication, increased access to information, and streamlined daily tasks. The Ministry of Education and Research in Norway (Kunnskapsdepartementet, 2012) points out that in a digital society, it is important that pupils are given good opportunities to develop digital skills, skills that "involve being able to use digital tools, media, and resources efficiently and responsibly, to solve practical tasks, find and process information, design digital products and communicate content" (p.12).

To prepare pupils for this digital reality, many countries have changed the compulsory curriculum to facilitate the development of digital skills and programming to create a potential learning environment for future skills. The way programming was integrated to the curriculum differs from country to country. The United Kingdom had introduced a separate information technology subject called 'Computing', while the Nordic countries introduced programming and computational thinking as a separate subject or a part within the mathematics subject in primary and secondary education. In Sweden, programming is a part of 'algebra' in mathematics, while in Norway, programming has been introduced in subject's mathematics, science, music, and art and crafts (Bocconi et al., 2018).

In Norway, the new curriculum in mathematics states that: "Digital skills in mathematics refers to the ability to use graphing tools, spreadsheets, CAS, dynamic geometry software, and programming to explore and solve mathematical problems" (Utdanningsdirektoratet, 2019, p.5). This definition indicates that (1) children learning mathematics should also learn to use different digital tools in mathematics to develop their digital skills, and (2) the tools should provide opportunities to make deeper connections and understand mathematical processes when using the tool. By enlisting programming with tools like spreadsheets, graphic tools, or computer algebra systems (CAS), the curriculum indicates that programming can be a *tool* for teaching and learning mathematics. This definition states that programming should be used to *solve* mathematical problems and *explore* these problems as well. However, may programming be a tool to teach and learn mathematics? Is programming not a field in itself? May programming be used in a proper way to support mathematical thinking? This study aims to look at the concept of ‘programming’ and ‘computational thinking’ and discuss in what ways programming as a tool in mathematics teaching and learning may be theorized.

Digital skills and tools in mathematics

Fuglestad (2007) analyzed the use of digital tools by lower secondary pupils in Norway and argued that many tools can support pupils’ mathematical thinking if they are used through inquiry, exploration, experimentation and if the visual and dynamic properties of software are taken advantage of, allowing the pupils to be active and to investigate mathematical patterns. The research of Smeets (2005) also showed that digital tools give new possibilities and perspectives that can emphasize parts of mathematics that can be difficult to visualize. He concluded that the potential in digital tools lies with the pedagogical approach, and to maximize the pupils’ learning outcomes, the learning activity should highlight things that would otherwise be challenging. Thus, using digital tools as if they were not digital would eliminate any potential that the technology might bring, like visualization, dynamic interaction, or active participation that would differ from using pen and paper (Kim et al., 2013).

Trouche (2004) pointed the definition between the use of a digital tool. He commented that a tool can be used for many different purposes, while an *artifact* is a “tool before considering its users and its uses” (p.282), and an *instrument* can be considered when there is a meaningful relation between an artifact component and psychological component. This means that the instrument “involves the techniques and mental schemes that the user develops and applies while using the artifact” (Drijvers, 2010, p.108). This approach, the instrumental approach, emphasizes that the critical part is how “the user's conception of the instrument is formed through use” (Trouche, 2004, p.295). This suggests that not every use of a digital tool can benefit learning because it depends on how the instrument is used and how the users' thinking is changed through using the instrument.

Drijvers (2015) argued that there are factors that can influence the integration between mathematics and digital tools. He mentioned (1) the design of the lesson, the activity, and the digital tool, where the activity or the task could determine if the tool would become an instrument or not; (2) the role of the teacher and teacher's competencies, where the teacher's professional development of both digital and didactical understanding is crucial; and (3) the educational context, where he argued that mathematical practices need to be related to pedagogical opportunities. This can be interpreted as creating an environment for learning and using the potential of digital tools (Smeets, 2005).

Programming and computational thinking

Programming has been defined in different ways. When creating a stepwise description of an algorithm for solving mathematical problems in 1843, Ada Lovelace was called “the first programmer” (Computer History Museum, s.a.), but a precise description of what it does to ‘program’ was not given. A technical definition of the term ‘programming’ separated two working methods while writing a program. The first was the process of drawing up a plan of the sequences of decomposed problems and algorithms that the program would consist of (‘the programming’), and the second was to write the code in a programming language and implement the program (‘the coding’) (Hartree, 1950). Since then, there have been many new developments in programming, changing the programming process. Blackwell (2002) presented the development through a concept that started with “describing calculations”, then went on to “defining functions”, and finally developed further into “defining and treating objects” (p.205). The methods of programming have changed when new digital developments have been integrated into the structure of programming. Duncan et al. (2014) argued that programming has “changed and developed over time as software, hardware and usage of computers has changed” (p.62) and therefore, they suggest to “use the term ‘programming’ for the broader activity of analysing a problem and implementing a program that solves it” (p.62, emphasis in original).

This development has made programming and coding more available, creating new easy-to-use programming languages and environments that can both write and execute the code. In recent years there has been a development of programming/coding environments for children which are based on the idea of visual programming where a program “is a set of linear sequences of ‘jigsaw pizzle’ pieces representing commands” (Klassner & Anderson, 2003, p.15, quotes in original). As Duncan et al. (2014) stated, “Drag-and-drop environments on the other hand do not require users to manually enter programming expressions; instead they provide the user with a selection of ‘blocs’ that represent programming expressions (…) This prevents novices from encountering confusing error messages, which can be very discouraging to learners” (p.65).

The access to free software prepared to start to code without previous knowledge has made the terms ‘programming’ and ‘coding’ blurred, and there are people that present these terms as synonyms (Balanskat & Engelhardt, 2015; Resnick and Siegel, 2015). Duncan et al. (2014) explained that “[*c*]*ode* is a popular buzz word in today's technology driven world, and it also provides an element of mystery (there are hints of a secret code), and achievement (cracking the code)” (p.62, emphasis in original). This is clearly the opposite of Hartree's (1950) division of these two terms. Duncan et al. (2014) stated that “[i]n the context of programming, traditionally *coding* would only refer to the last stage of the process of programming, translating a designed program into programming expressions and typing/entering these into a computer” (p.62). Computer scientists still lean up to the technical definition and consider programming to be more than coding, but people from outside computer science often do not divide between the process of scheduling the sequences and writing the code in a certain programming language.

Papert (1980) argued that programming should be associated with the process of thinking when he wrote that by “teaching the computer how to think, children embark on an exploration about how they themselves think” (p.19), and then the children can use programming to construct their knowledge. He saw the possibilities of using programming as a way of thinking when solving problems. Wing (2006) built on this idea by reintroducing the concept of *computational thinking*, CT (first used by Papert in a different context in 1980). She defines it as: “an approach to solving problems, designing systems and understanding human behaviour that draws on concepts fundamental to computing” (Wing, 2008, p.3717). In her explanation, there is a clear link between the thought processes associated with processes of abstraction and decomposition and computing.

According to Wing's definition, computational thinking is a fundamental skill, which “means more than being able to program a computer” (2006, p.33). Computational thinking is dependent neither on technology nor programming languages (Bocconi et al., 2016). Denning and Tedre (2019) defined it as "(…) computational thinking, or CT– is not a set of concepts for programming. Instead, CT comprises ways of thinking and practicing that are sharpened and honed through practice" (p.6). The idea is to teach pupils how to think in structures so that they can gain knowledge about specifying and breaking a problem into several subproblems to find a systematic solution to the subproblems and evaluate whether the solution was useful and effective to solve the problem.

Lie et al. (2020) stated that Wing presented terminology and concepts as a computer scientist, and they argue that someone "outside that community might be prone to narrowly construe the idea of CT to direct connections with *number computation* or *computer*" (p.2, emphasis in original). Denning and Tedre (2019), also computer scientists, defined computational thinking as "the mental skills and practices for (i) *designing* computations that get computers to do jobs for us, and (ii) *explaining* and interpreting the world as a complex of information processes" (p.4, emphasis in original). Curzon et al. (2014) got into more detailed concepts of computational thinking (logic, algorithms, decomposition, patterns, abstraction, evaluation) and approaches (tinkering, creating, debugging, persevering, collaboration). Weintrop et al. (2016) presented a model of computational thinking that has four main categories: (i) Data practices, (ii) Modeling & simulation practices, (iii) Computational problem-solving practices, and (iv) Systems thinking practices (p.135). The interesting part of this model is relevant for this argumentation is that one of the sub-categories in the group (iii) is named 'Programming', which could mean that programming is considered a narrower approach than computational thinking. This is contradictory to the technical definition of Hartree (1950) and maybe has more in common with the perception of Resnick & Siegel (2015) and Balanskat & Engelhardt (2015).

The modern concept of computational thinking has in some way replaced the dual term of programming as presented by Hartree (1950), making the definition of the term 'programming' different than before. In addition, the programming environment that Hartree (1950), Papert (1980), and Blackwell (2002) considered were not the ones that are used in schools today. These scholars described a version of text-based programming languages. The languages of BASIC or LOGO were simplified and adapted for children, but they were still based on detailed syntax. Today's programming environments for children (for example, Scratch, Micro:bit, LEGO, Minecraft, etc.) are designed for the purpose of being easy to use, and many children meet programming through block-based programming languages, where the children are free to construct, modify and change codes with a simple push. Today, someone learning programming does not need to have a scheduled algorithm figured out before (s)he starts to code. One can code by trials and errors and continuously modify the program without stopping the software or creating bugs, but it is unclear if this can be considering 'programming'?

Challenges in introducing programming in mathematics teaching

Kilhamn and Bråting (2019) reminded that programming is a field with its own structures, rules, goals, processes, methods, and notations. The same symbol can be used both in mathematics and programming, but it does not necessarily have the same meaning. For example, '=' means equality in mathematics, but in most programming languages, it assigns a variable. This can be confusing when writing a variable like x=x+1. A common code for adding one to a variable in programming is mathematically wrong because there does not exist a value of x that could make this equation valid.

A few more challenges arise from teaching programming in mathematics. Firstly, the syntax in programming differs from one programming language to another, and mathematical symbols can have different meanings. How does it affect pupils learn to use the equal sign differently in programming and mathematics, and then change the programming language and use ‘:’ as equality. This would be confusing for the pupils that would need to be specified and clear in the syntax they use, as well as the terms and structures that each programming language represents. In Norway, there are not any national criteria for choosing a programming language, and each school, teacher or subject could use different software.

Secondly, programming, as mentioned before, is a field with its own rules and structures. The definition of programming differs between computer science and other disciplines, and many concepts are used to describe different approaches and actions. The terminology from computer science is not easy to translate to other situations and contexts, and much of the argumentation consists of explaining approaches from computer science used in other contexts. There are differences in syntax and in structure, different approaches in solving a mathematical problem with 'pen and paper' or solving it through the creation of new software. However, programming was implemented in four school subjects in Norway without explanation on how these disciplines could be combined. Currently, it is up to each teacher to choose how they want to implement programming in their subjects. It could be a part of computer science, or as simplified coding and gaming to motivate pupils in cases in different subjects, or as a tool for learning the subjects more in-depth. There is a lack of research on what would be most beneficial for the pupils, at their grade and in certain subjects, and how programming could be helping them to learn and develop skills.

Finally, the challenges lie within the teachers' competencies. The teachers need not only to learn how to program themselves but how to explain the program and support pupils in creating their programs as well. This could be solved by inviting computer scientists to teach programming parts at certain subjects. However, that may not give the educational and didactical results desired in the school curriculum. There could be some advantages if the pupils were taught to program by experts in their profession. Then the misunderstanding of structures, methods, terms, and syntax would not be problematic. The disadvantage of such an approach would be that the programming would be connected to computer science and used for the same purpose to create effective, friendly, and structural programs, but then there might be a lack of pedagogical or didactical approaches that children in primary and secondary school might need in their development.

Teachers' role in introducing programming in mathematics

Kaup (2019) has done a research review describing how in-service teachers and pre-service teachers understand the term computational thinking and what attitude they have towards that concept. Her results showed that many participants were not familiar with computational thinking, and even if they had noticed that concept before, their understanding of what it is was superficial and simplified. Also, the study of Misfeldt et al. (2019) showed that some teachers do not feel prepared to teach programming. Only 4,5% of the teachers participating in the survey answered that they feel ready “to a great extent”, even though almost 70% said that these disciplines are connected and can be combined within the same subject.

Bocconi et al. recognize the teacher's role when they admit that: “Evidence shows that the transfer of programming skills is more likely to happen when (i) transfer is addressed in the upskilling of all teachers involved and (ii) forms an integral part of the pedagogical approach adopted in the classroom” (2018, p.6). This claim emphasizes that upgrading mathematics teachers’ skills with programming may not give the wanted results if the knowledge is not adapted in a pedagogical approach. As Drijvers (2015) previously stated, teachers' development is crucial for integrating digital tools as instruments in mathematics teaching and learning. Yet, in many cases, the teachers today are confused about correct terms and methods (Kaup, 2019).

Conclusion

Can programming be a tool in mathematics? There are different definitions of 'programming', 'coding' and 'computational thinking', and these concepts are used in a variety of ways. The concepts themselves have changed in time, both because of further digital development (Blackwell, 2002; Duncan et al., 2014) and the need for more interdisciplinary connections (Wing, 2006; Weintrop et al., 2016). The integration of programming into mathematics education in Norway creates new possibilities for the pupils (Utdanningsdirektoratet, 2020), and the implementation of digital tools can support mathematical thinking (Fuglestad, 2007, Drijvers, 2015). Especially when the tool is used in a way that forms and influences the users' conception and becomes an instrument (Trouche, 2004). The remaining question is if programming can be such a tool?

The role of the teacher is significant in introducing programming with understanding and relevance (Drijvers, 2015, Bocconi et al., 2018), but the competence of today's in-service and pre-service teachers do not give them much confidence to teach programming in mathematics (Kaup, 2019, Misfeldt et al., 2019). Programming has some similar elements with mathematics (Papert, 1980, Kilham & Bråting, 2019), and studies of teachers' attitudes towards programming in mathematics show some connections between these two disciplinaries (Misfeldt et al., 2019, Kaup, 2019). Yet, there are many challenges that could arise when programming is introduced to pupils in teaching and learning mathematics, and the practical approaches of implementing and using programming as an instrument in mathematics have not been researched. There is a need for more knowledge in how programming can be used to solve and explore mathematical problems.

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Digital technology in mathematics education – more than a tool

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Digital technology in the mathematics classroom is often seen only as a tool. The purpose of this paper is to expand the discussion about the kind of roles digital technology can play by using programming activities as examples. We apply components from Engeström's Cultural Historical Activity Theory to discuss programming as the object of the activity; the tool in use; part of the division of labor; and as a part of a classroom community. Digital technology as a part of division labor and classroom community has the potential to provide rich classroom communications and learning processes in which the students, the teacher, and digital technology interact with each other.

Keywords: Digital technology, programming, roles, CHAT, mathematics education

Introduction

Digital technology has become an increasingly more important part of education in the last decades. In mathematics education, we see a development of concepts like ICT literacy (e.g. Dede, 2010) and digital competence, and research titles like Clark-Wilson et al.'s (2014) *The mathematics teacher in the digital era*. During the last decade, programming has been introduced as an important 21st-century skill. It is included in mathematics curriculums (and other subjects) in many countries (Balanskat & Engelhardt, 2015) and national policy documents (Bocconi et al., 2018).

Hoyles (2018) stated that the dominating view on mathematics is that "mathematics is simply a set of disparate rules for calculation and students attempt to master this 'mathematical machinery' without seeing its purpose" (p. 209). Teaching based on this view is dominated by students being given premade tasks to be solved with predefined mathematical tools and students have difficulties seeing the purpose of doing the tasks. Hoyles argued that the digital technology in such classrooms is used to speed up procedures, and calculations are largely only replicating doing mathematics with paper and pencil.

According to reviews by Batiibwe (2019) and Bray and Tangney (2017), digital technologies are often integrated into non-transformative ways in mathematics classrooms, and they are often regarded as tools only. Batiibwe (2019) reviewed articles on the mediating role of digital technology in mathematics education from a Cultural Historical Activity Theory (CHAT) perspective. All of the reviewed articles discussed the role of digital technology in classroom activities only as a mediating tool. Other roles, such as digital technology as an object, as the driving force in the activity, were not discussed. Bray and Tangney (2017) found that digital technology is primarily used traditionally in mathematics classrooms. Task assignments are not adapted to the integration of digital technology, and digital technology is used as a substitute for the teacher or to save time in calculations (e.g. Åberg-Bengtsson, 2006). Digital technology is often used because it simplifies things and releases students from tiresome calculations. Such use of technology generates to a little extent changes for the teaching and learning of mathematics.

However, digital technology can be used to explore mathematics in diverse ways. The integration of digital technology has the potential to transform activities in mathematics classrooms by engaging students and providing them possibilities to take ownership of their learning (Hoyles, 2018). According to Bray and Tangney (2017), programming activities were more transformative than the activities with other digital technologies used in mathematics classrooms. In most of the studies that discussed programming integration, the task assignments in programming activities were often collective, freer, and more student-centric. The teacher was often acting as a guide instead of just being a lecturer, and the students were able to use their ideas in the task design by negotiating with each other and the teacher and by interacting with digital technology.

To investigate the transformative possibilities of including programming in mathematics education, *the focus in this paper is on programming's potential to play roles that go beyond the role of being a tool*. We use examples from two earlier studies on mathematics and programming presented in Forsström and Afdal (2020) and Herheim and Johnsen-Høines (2020). In both studies, the students worked collectively with their programming activities. By taking a CHAT perspective, we exemplify and discuss programming's potential to be an object, to become part of the division of labor and community, as well as being a tool in the students' collective activities. The analysis is based on a micro-level approach of the activity system analysis in Engeström's (1987) version of CHAT, where social, multi-voiced interactions are part of the knowledge creation processes. Focusing on collective classroom activities instead of individual actions gives the possibility to get information about relational processes in the classroom. An activity system analysis enables a discussion of the potential roles of programming as a part of mathematics classroom activities in addition to being a tool. The different roles can be discussed in relation to other components in the activity system during the activity development. Taking a CHAT perspective makes it possible to see teaching and learning as dialectically intertwined processes (Engeström & Sannino, 2012).

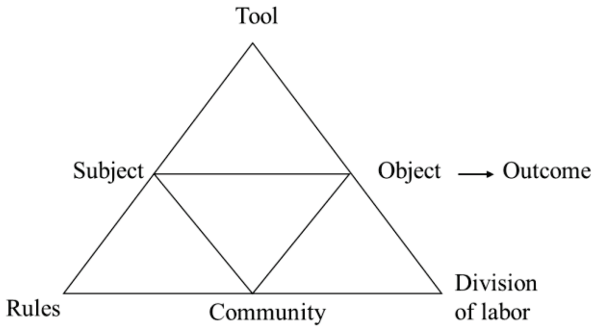

Figure 1: The activity system model from CHAT (Engeström, 1987, p. 78)

Cultural-Historical Activity Theory

The analytical approach is based on an activity system analysis, where the seven components: subject, object, tool, rules, community, division of labor, and outcome (see Figure 1 and Table 1) include the potential roles we argue that programming can play. According to Engeström (2005), the components are interrelated. For instance, in the uppermost sub-triangle, the tool mediates the subject's activity towards the object. With the help of tools, subjects interact with the object of the activity, which is the driving force in the activity. The activity is framed by collective components of rules, community, and division of labor, and the relationships between the components influence the activity development. Due to the interactions between subjects and dynamic relationships between the components, the collective activities are constantly transforming and developing. Knowledge is distributed between the different participants and components in the activity system, and learning is seen as a change in the components of collective activities; as an expansion of a collective object.

Table 1: Components in the activity systems analysis

Component	Definition/meaning	Examples from these studies
Subject	The individual/group of people who engage in the activity (Yamagata-Lynch, 2010)	The students and the teacher
Object	The driving force of the activity (motive and goal) (Engeström, 1987)	Fulfil a task by using programming
Tool	Instrument that mediates the activity (Engeström, 1987)	A robot, app, computer, and mathematical tools
Rules	The regulations that are relevant to the activity (Yamagata-Lynch, 2010)	Task assignment, the rules of the mathematics classroom
Community	The group the subject belongs to during the activity (Yamagata-Lynch, 2010)	The whole class of students and the teacher (or teachers)
Division of labor	How the tasks are shared during the activity (Yamagata-Lynch, 2010)	Collaboration between students, the role of the teacher and the programming
Outcome	The result of the activity (Yamagata-Lynch, 2010)	A robot drives a track, a shape is drawn etc. as they are programmed

When applying this theoretical perspective, we understand mathematics and digital technology such as programming, as components in an activity system. They can play different kinds of roles depending on the activity, the other components and the relationships between them, and how the activity develops. To understand the role of programming in relation to the role of mathematics, the activities described in Forsström and Afdal (2020) and Herheim and Johnsen-Høines (2020) were analyzed from a CHAT perspective, with a particular emphasis on the components object and tool, and on community and division of labor. To identify the object of the activities, we focused on the collective aim of the subjects (the students). We identified, for instance, what the students and the teacher were aiming to do, what the driving force in the activities was, such as to make a robot drive a track. The tools in the activity were determined by identifying what kind of tools the subjects used to achieve their object. The tools were distinguished from the objects by identifying the focus of the subjects. According to Engeström (2005), the focus can only temporarily be on tools. The activity components are dynamical and multilayered and the activities constantly transforming. The role of programming and mathematics developed in relation to other components in the activity system.

Expanding the discussion about the roles programming can play

In the following, we discuss how programming can play the role of an object and tool, be part of the division of labor and community, and how programming, when being more than a tool, can act as a resource in students' collective learning processes in mathematics.

Programming as an object and tool

In the programming activities discussed in this paper, the students are challenged to program a robot to drive a certain path (Forsström & Afdal, 2020) and to draw a particular geometrical shape with programming in Scratch (Herheim & Johnsen-Høines, 2020).

In Forsström and Afdal (2020), students aged 12–13 years old programmed Lego Mindstorm robots. The students were challenged to make the robot drive a circle, and the programming elements became their main object at the beginning. The students then used a trial-and-error strategy to achieve their object, but the mathematical tools were not used systematically (see Figure 2). The teacher negotiated with the students and suggested that they could program the robot to drive a circle with a radius of one meter. In that way, the teacher helped the students to mathematize their programming object and it developed a need for mathematical tools in the activity. The students focused on mathematics when they did the calculations. They used, for instance, the circle perimeter formula to find out the length of the route the robot had to drive, as well as proportions to uncover how much the robot had to turn. After the students obtained the needed results from their calculations, they used them in their programming to reach their object (see a more detailed discussion about the activity development in Forsström and Afdal (2020)). From a mathematics education perspective, such transformation of an object, from a programming object to a mathematical object, is often the intended purpose of including digital technologies in the teaching and learning of mathematics.

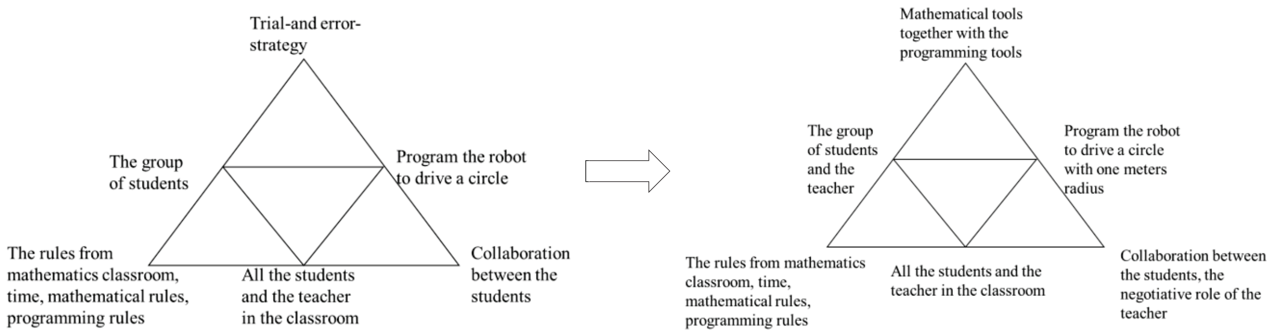

Figure 2: The activity development in Forsström and Afdal (2020)

Programming tools were in use together with mathematical tools. The students revised their codes according to their mathematical calculations to improve the programming of the robot. The programming tools provided an opportunity and a need to test mathematical tools. When the students did not remember the circle perimeter formula and used radius instead of diameter, they got immediate feedback because the robot only drove one half of a circle. Because of the visual feedback of an error, the students were able to return to their code and do corrections. Based on the feedback, the students successfully concluded that to get the robot to drive a whole circle, they needed to double their answer. The teacher encouraged, with his questions, the students to find out why they needed to double their answer. The students accepted the challenge and mathematics became the object of the activity.

Herheim and Johnsen-Høines (2020) investigated screen-based programming with Scratch where two 12–13 years old students collaborated to program a pentagon. This programming required both mathematical and programming considerations from the students. The students were unfamiliar with the geometrical properties of a pentagon, about the number of sides and the size of the interior angles. They also faced programming challenges such as how the turn block does not give the anticipated interior angle but the turning angle. The students struggled with both mathematical and programming aspects, but they used a systematic trial-and-error strategy to get closer and closer to programming a pentagon. The properties of the programming software allowed them to find out more about pentagons. They could test different angle sizes in their code and after each attempt, they received immediate feedback through the shapes drawn by the program: a hexagon shape that lacked one side; a pentagon where the first and last side intersected; and a pentagon with a tiny gap between the first and last side. This Scratch programming activity can be regarded as an example of how programming and mathematics can become an intertwined object of the activity – not only as of the overall object but for sub-objects during the process as well.

Being able to successfully program something can be the main driving force for students, and that makes programming an overarching object. However, during such work processes, students need not only to figure out what mathematics is needed and how to use it. They also need to figure out how a path or a shape can be programmed, what code blocks to include in the code and how to include them, and then they often need to try out different versions of a code. In several phases of their work, there are programming aspects that play the role of intermediate aims in the students' activities. This gives ground for saying that students can have several programming sub-objects as well.

Based on our discussion of the examples from Forsström and Afdal (2020) and Herheim and Johnsen-Høines (2020), we argue that programming can be an object as well as a tool. This can take place through a transformation between being a tool and an object, or as an intertwined object together with mathematics. Programming as an object in activities can enable the use of mathematics as a tool in students' activities. The programming tools enabled the testing of mathematical tools and by that, the programming tools brought a new dimension to the use of mathematical tools. As the activities unfolded, the programming provided feedback to the students about the mathematical tools they were using. When the students tested the codes, the robot and the visualizations on the computer screen gave them immediate feedback on the mathematical tools in use. The robot and the Scratch program acted as a part of the division of labor in students' activities. That will be discussed in more detail in the following.

Programming and robots as a part of the division of labor and as a part of the community

In some studies on digital technologies in mathematics education (e.g. Monaghan, 2005; Lavy & Leron, 2004), the technology is considered by students as a participant or quasi-human agent. The classroom activities can be seen as networks, which constitutes both human and non-human actors, such as the students, the teacher, and the robot. Students use screen images to express themselves and the other students use the same images to interpret the utterances – the technology provides language. Digital technologies can prompt, respond, and frame communication, but they have, unlike teachers and peers, infinite patience and do not have expectations and are not judgmental (Monaghan, 2005). Technologies can appear to act like subjects when they respond to inputs so that students get the feeling they must justify their responses, without feelings or expectations. Wegerif (2004) pointed out that this dual role of digital technologies can make them able to play a part in students' activities.

In the studies presented by Forsström and Afdal (2020) and Herheim and Johnsen-Høines (2020), knowledge was embedded in processes between robots and students and between computer screens and students. Students got information about their mathematical tools by getting feedback from nonhuman actors – the robots and the computer screens. With this feedback, the students got the opportunity to interact and negotiate with their mathematical object. The nonhuman actors worked as resources for students' understanding of their mathematical object and mathematical tools in use. The students interacted with their mathematical object through the mediation of tools but also through the mediation of division of labor. The programming, together with the students and the teacher, took part in the division of labor and mediated by that the relationship between the community and the object of the activity. In that way, programming did not act as a substitute for the teacher, but it acted to some extent as a participant together with the students and the teacher.

In Forsström and Afdal (2020), the development of the activity was constituted by the negotiations between the students and the teacher. The teacher encouraged the students to use mathematical tools with his questions. However, the teacher's questions would have been of little value without the participatory role of the robot. He referred to the robot's movements when discussing with the students, for instance when the robot drove only a half of a circle and he challenged them to find out why they needed to double their answer.

In Herheim and Johnsen-Høines (2020), the students discussed the geometrical properties of a pentagon. They based their discussions and revisions for the next attempt on the feedback from the Scratch program, on how the mathematics was represented by the code and in particular the drawings of the shapes. Through the collaboration with each other and with the programming, the students found out more about pentagons and programming. During this process, the programming played a role in the division of labor (see Figure 3).

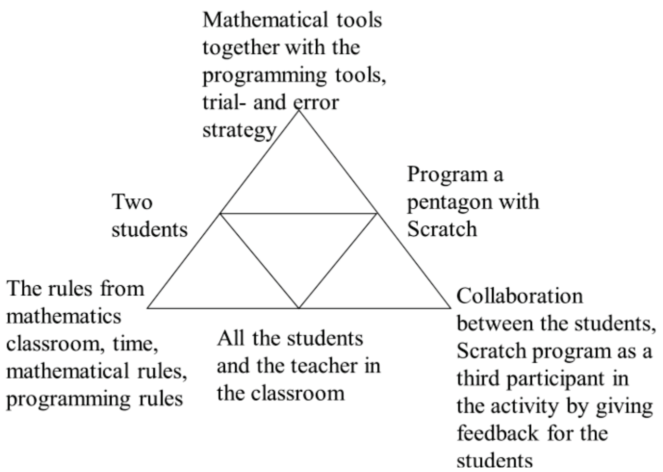

Figure 3: The activity system in Herheim and Johnsen-Høines (2020)

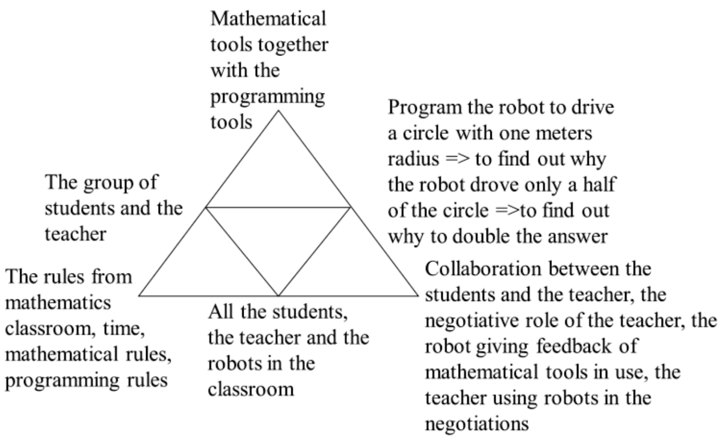

Figure 4: Robot as a part of the division of labor and community in Forsström and Afdal (2020)

When programming becomes a part of the division of labor and interacts with the students and the teacher by giving feedback, it also becomes part of the collective activities. The programming can be seen as a part of the social classroom group, as a part of the community in students' activity systems. Such communities can be regarded as digitalized classroom communities. In digitalized classroom communities, students and teachers interact with digital technology, and digital technology can be seen as a kind of a digital participant in the collective classroom activities (see Figure 4).

Concluding comments

As discussed in the introduction, digital technology is often regarded as a tool. Based on Bray and Tangney (2017), digital technology is mostly integrated into mathematics classrooms in non-transformative ways, used as a tool to save time in calculations, and as a substitute for the teacher or paper and pencil. We have discussed how digital technology can play more than the role of being a tool, how it can take different kinds of roles in classroom activities by introducing programming activities from earlier studies. The different roles discussed are programming as a tool, an object of the activity, a part of a division of labor in students' classroom activities, and as part of the community.

From a mathematics education perspective, the students' use of mathematics in programming activities is a priority, and the different roles of programming are discussed in connection to the role of mathematics in students' collective activities. We have argued that programming acted as part of the division of labor and as a part of the digital classroom communities in both the robot and the Scratch example and contributed to fruitful learning processes in mathematics. We called the classroom communities, where digital technology acts as a part of the community as digitalized classroom communities.

In the digitalized classroom community, digital technology can take different kinds of roles in students' collective learning processes in mathematics. The students, the teacher, and the digital technology can interact with each other, and programming and mathematics can have an active and transformative role. Digital technology can act as a resource for students' understanding of their mathematical object by giving students feedback and in that way play a participative role. In the examples described in this paper, it might appear as if the programming acted just like a paper and pencil by drawing a circle or a pentagon, but the programming acted more diverse than paper and pencil. The programming brought an extra dimension to the students' activities by acting together with them, the teacher, and mathematics. It gave confirmations of the use of mathematical tools by correcting students' mistakes and providing visualizations of key properties in mathematics and showed by that a potential to trigger and facilitate students' mathematical activity.

Furthermore, the teacher can use digital technology as a teaching partner. As in the examples discussed in this paper, the programming gave information to the teacher about students' understanding as well as their struggles and mistakes. The teacher and the students can base their comments, suggestions, and revisions on the movements of a robot or drawing of shapes on a screen. Digital technology as a part of the classroom community can play a valuable role in the collective activity with students and the teacher. In the digitalized classroom community, the students can work towards their object, together with the teacher, the digital technology, and the mathematics.

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Design considerations for facilitating mathematical learning online

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This paper presents a graduate student's reflections on the design of learning opportunities using the Desmos tool for carrying out mathematical activities and the online Zoom platform for facilitating mathematical learning. Using the theory of instrumental orchestration as our interpretative framework, we discuss the different types of orchestrations when a digital tool is used to support mathematical learning not in the familiar face-to-face classroom-based environment, but online instead. The contribution of this paper lies in the discussion of the design considerations and orchestrations to overcome the challenges of online learning and at the same time to capitalise on the opportunities it offers for mathematical learning.

Keywords: Online mathematical learning, distance learning environment, instrumental orchestration.

Introduction

There is a vast range of digital technologies (DTs), designed for and used in mathematics education, serving different purposes. Such technologies offer affordances for and constraints on students' learning, but also on teaching practices. DTs refer to digital tools and applications used "(a) as a support for the organisation of the teacher's work (producing worksheets, keeping grades) and (b) as a support for new ways of doing and representing mathematics" (Sinclair & Robutti, 2014, p. 598).

Using DTs for teaching mathematics at a distance has also been progressively researched, with many authors discussing the great opportunities of such a mode of teaching and learning, as well as the challenges teachers and students face (e.g., Silverman & Hoyos, 2018; Drijvers et al., 2021). Classroom-based strategies and activities, such as think-pair-share, role playing, group discussion, gesturing, modelling, assessing, etc., that are known to be engaging and effective for students' learning in the face-to-face classroom environment, and therefore widely used, present new challenges for teachers and students in online learning environments (Collison et al., 2000). During online synchronous teaching, the teacher-students interactions are certainly limited and the opportunities to observe individual students' work and intervene when necessary decrease (e.g. Silverman & Hoyos, 2018). In such online settings, assessing individual students' learning is challenged and the tools for doing mathematics need to be carefully chosen. In face-to-face classroom-based lessons, learning takes place in a social context, in which students and teachers use various strategies to communicate with one another, including verbal and non-verbal strategies. In online remote learning, certain aspects of such communications are more challenging to achieve, such as writing and sharing each other's written mathematical work, pointing at notations and symbols written on the board, and other gesturing used by students to support their explanations. In fact, while verbal communications are possible online (via video/audio/chat features), supporting such conversation with written notes is more challenging online, since such online spaces allow limited mathematical learning input (Aldon et al., 2021; Drijvers et al., 2021; Leventhall, 2004).

In this paper, we share how a graduate student on our MA in Mathematics Education master's course, Tania, capitalised on the opportunities DTs offer for remotely supporting the learning of mathematics. She designed learning opportunities to be used via the online Zoom platform that replaced the usual face-to-face classroom environment. Aware of the constraints of the online Zoom platform with regards to mathematical input and notation, she also planned for using Desmos, a digital tool that would 'make up' for constraints with respect to writing mathematics and sharing mathematical work in real time. The data presented in this paper is based on Tania's written assignment, which was submitted in September 2020 as part of the master's module on 'Digital Technologies for Mathematical Learning' (DTML). In her assignment, Tania shared her reflections on how best to support learners make links between different representations of quadratic functions using Desmos, and how face-to-face classroom-based activities were adapted to online learning via Zoom. We start by introducing the Theory of Instrumental Orchestration (TIO), which we used as our interpretative framework to present and analyse Tania's pedagogical considerations (or in other words orchestrations) when planning for mathematical learning with Desmos, and where the face-to-face classroom environment was replaced by the Zoom online learning space. We then give an overview of Tania's design considerations for online learning, followed by an analysis of Tania's reflections on how best to support online mathematical learning applying the TIO framework. We conclude with some reflections on broadening the TIO framework's orchestrations, as a result of overcoming the constraints, while capitalising on the opportunities when facilitating online mathematical learning.

Theoretical Background

The integration of DT into mathematics education has been an ongoing and non-trivial issue, mainly due to the complexity of the use of DTs. In order to describe how a teacher manages the use of DTs, steers students' instrumental genesis and orchestrates mathematical situations, the TIO was developed by Trouche (2004). Trouche (2004) introduced TIO by arguing that an instrumental orchestration describes the teacher's organisation and use of the different artefacts within a learning environment in a mathematical situation, so as to guide students' instrumental genesis (as cited in Drijvers et al., 2010, pp. 214–215). Trouche (2004) argued that an instrumental orchestration is defined by "didactical configurations (i.e. the layout of the artifact available in the environment) and by "exploitation modes of these configurations" (p. 296). A *didactical configuration* involves the teaching set-up and the artefacts available in the teaching environment and set-up. An *exploitation mode* involves the approach a teacher decides to take when exploiting a didactical configuration, aiming at supporting their didactical intentions (ibid). For example, how tasks are introduced to students and how they are solved, what roles the artefacts might play, or what schemes and techniques students may develop, are a teacher's decisions (ibid). Drijvers et al. (2010) added to Trouche's (2004) two elements: 'didactical configuration' and 'exploitation mode', a third element that of 'didactical performance'. *Didactical performance* involves the decisions a teacher takes while teaching, considering their chosen 'didactical configuration' and 'exploitation mode' (Drijvers et al., 2010). As Drijvers et al. (2014) add "what question to pose now, how to do justice to […] any particular student input, how to deal with an unexpected aspect of the mathematical task or the technological tool, or other emerging goals" (p. 191). Six orchestration types have been identified in the literature regarding whole class teaching and the seventh one involves students working on their own or in pairs with technology (Drijvers et al., 2014). These are:

> Technical-demo orchestration concerns the demonstration of tool techniques by the teacher […]. Explain-the-screen orchestration concerns whole-class explanation by the teacher, guided by what happens on the computer screen […]. In the Link-screen-board orchestration, the teacher stresses the relationship between what happens in the technological environment and how this is represented in conventional mathematics of paper, book, and blackboard […]. The Discuss-the-screen orchestration concerns a whole-class discussion about what happens on the computer screen […]. In the Spot-and-show orchestration, student reasoning is brought to the fore through the identification of interesting DME student work during preparation of the lesson, and its deliberate use in a classroom discussion […]. In the Sherpa-at-work orchestration, a so-called Sherpa-student uses the technology to present his or her work, or to carry out actions the teacher requests. (Trouche & Drijvers, 2010, pp. 219–220).
>
> In the Work-and-walk-by orchestration, the didactical configuration and the corresponding resources consist of the students sitting at their technological devices, and the teacher walking by in the classroom (Drijvers et al., 2014, p. 192).

Design Considerations for Online Learning

In this section, we present data from one graduate student's (Tania) reflections on designing activities to support eighteen 15-16 years old students' making links between different representations of quadratics functions using Desmos via Zoom. Tania was a student on the DTML masters' module and therefore a *learner* herself, who gained knowledge about technology-enriched practices in mathematics education. In this module, students learn about the affordances of various digital tools and critically discuss their value for mathematical learning. For their assignment they are expected to trial a digital tool with learners and critically reflect on the mathematical learning. Tania's assignment stood out, as she offered a particularly detailed account of and reflections on not just the mathematical learning with a DT, but also of doing so online. For this reason, we selected Tania's case study to illuminate how TIO can be applied to analyse orchestrations for online mathematical learning and how her design decisions showcase good practice in technology-rich learning.

The mathematics topic and choice of a tool

In her reflective writing, Tania justified her choice of a digital tool (Desmos – the technological artefact) for supporting eighteen Year 11 (15-16 years old) students' learning about 'quadratics'. In preparation for their mathematics examination (iGCSE), the students learned how to plot and recognise quadratic graphs; how to factorise, expand, complete the square of quadratic expressions; how to solve quadratic equations in various formats; and how to apply the 'rules' for graph transformations. By engaging with relevant mathematics education research (e.g. van der Meij & de Jong, 2006) as part of her the DTML module, Tania was aware that students master all of the above as separate knowledge and skills about quadratics, but may not necessarily develop a fuller, more holistic understanding of 'quadratics'. Inspired by her recent experience with graphware digital tools and knowledge of their potential for mathematics learning she gained in DTML, Tania thus planned for activities where Desmos would be used to support students to engage with the multiple representations of quadratics.

Planning for Online Learning

Tania trialed these activities with students via Zoom. In her written reflections, Tania was explicit about how much she learned from studying on the DTML and how it influenced her design decisions for mathematical activities which she ended up carrying out online. She became aware of the affordances and constraints of Zoom and Desmos, and there is evidence in her written reflections that such awareness affected how she orchestrated the learning of the students, as will be discussed next.

Tania's Instrumental Orchestrations

In this section we apply the TIO to describe and analyse Tania's own re-count of and reflection on her own practices to support online learning, where Desmos was used with the learning objective of supporting students make connections between different representations of quadratics. More precisely, we will be using Drijvers et al.'s (2014) interpretation of an instrumental orchestration consisting of three elements: a didactical configuration, an exploitation mode and a didactical performance. We will be using quotes from Tania's assignment, which we will indicate using single quotation marks '_'.

Tania wanted to find out about students' prior knowledge about the mathematics topic under scrutiny. She set a pre-task for the students to carry out on paper. Students then emailed her their scanned work in advance of trialing the online activities.

The *didactical configuration* for carrying out the online activity included the online platform (Zoom) as the online learning space which replaced the usual face-to-face classroom-based learning environment. Tania had to quickly become familiar with Zoom's functionalities, i.e. video camera (to make herself visible to the students), audio and chat features (to 'see' and talk with the students), breakout rooms for group activities, sharing screens and files (to share the resources she prepared for the online activities), Whiteboard, and certain functionalities that Zoom offered and which she referred to as the 'Pace', 'Pause' and 'Response' features; the Desmos tool; the mouse; and her computer. She referred to a 'Pace' feature of the Zoom platform to restrict students' access to specific screens. Tania also mentioned how she used the 'Pause' feature to remove students' ability to interact with the screen to capture everybody's attention and focus on the next activity; asking a student to provide an oral explanation; discuss a screen, etc. According to Tania, the 'Response' feature (how she referred to the chat box) allowed students to offer an answer for any questions or tasks that were posed.

Tania carefully thought about her *exploitation mode* regarding the online activity. For example, aware of the need for students to have plenty of time to explore the mathematics with the DT, Tania facilitated many such opportunities by organising students to work in pairs in breakout rooms. To better capture Tania's design considerations for the online activity, she created for the 18 students, we present these in Figure 1 below. In this figure, we exemplify the elements of mathematical learning Tania planned for (left column), the challenges she faced due to carrying out the activities online (middle column), and how Tania decided to exploit the artefacts mentioned above in her *didactical configuration orchestration* for the online activities (right column). Afterwards, we present her reflections on the elements involved with the *exploitation mode* regarding online learning and how she was able to overcome the challenges she was faced with, based on the affordances of the DTs she used.

Planning for…	Challenges for online teaching and learning	Tania's *Exploitation Mode* for online teaching and learning
Assessing prior knowledge	Writing mathematics in real time	Setting work for students to carry out on paper and then email their scanned work
Sharing work in lesson	Hand writing equations Hand sketching graphs	Using new symbolic representation x^2; Using Desmos for writing equations and sketching graphs; Treating computer screen like paper Sharing screens
Sketching graphs	Hand drawing graphs	Using Desmos to write equations and draw graphs, in parallel with verbal and written explanations
Managing the lesson	Presence and gesturing	Using the 'Pause class' Zoom feature
Monitoring and assessing students' progress	Walking through the classroom and checking on students' work	Joining break out rooms; Monitoring chats of break out rooms; Asking students to type explanations in the chat
Supporting collaboration - pair work; whole class	Students sharing working out in their books or on the board	Use of breakout groups; Students sharing screens
Gesturing	Using hand gestures to point at written work, to explain mathematics, to show own work on paper	Mouse used like a finger/pen to point to parts of the screen, while offering verbal explanation; 'Pause Class' for focus and joint attention
Sharing oral explanations	Facilitating rich conversations between teacher&students and among students	Teacher monitoring students' work by visiting breakout rooms, then asking the students to share their screens and explain their solutions.

Figure 1: Tania's design considerations for online learning

Tania's *didactical performance* involved the use of the 'Pause' and 'Pace' Zoom features in order to allow all students to see or work on the same screen together or have whole-group conversations and be provided with further support or introductions to new tasks, without many distractions. In her assignment, she referred to Godwin and Beswetherick (2003) by stating that 'teachers can be reluctant to use technology in the classroom if they are concerned that their students will lose focus and misbehave', and while she did not provide evidence that this did not happen online, she certainly hinted that she spent little time on managing students' transition to new activities. By asking the students to type their reasoning in the chat, Tania was pleased to be able to assess students' understanding, as she was able to quickly see their answers in the chat box. *Work-and-walk-by orchestration* and *Sherpa-at-work orchestration* were carried out with much ease, moving from one breakout room to another, checking on students' work, and hence monitoring students' work in less time than in a face-to-face classroom-based learning environment for example.

At the start of the online activities, Tania ensured that the students themselves had access to the Desmos application on their own devices, and they knew how to share screens. Her *technical-demo orchestration* included modeling to students how to input mathematics superscripts so that Desmos would recognise quadratic functions by using the '^' symbol for powers. Tania commented on how using Desmos to test out and modify their answers without judgement encouraged students to hypothesise about what would happen if variables changed. Tania noted that that was the case for all students, as she was able to quicky monitor their responses in the chat boxes, therefore gaining a window to every student's mathematical thinking. After sending the students to work in pairs in breakout rooms for the set activities, Tania would bring them back together and would always seek explanations from randomly selected students about how they (in pairs) found the solution to the mathematics question posed. *Spot-and-show orchestration* was used often in these online activities, due to the ease of sharing screens (hence students' work) and inviting students to explain their reasoning, followed by the *discuss-the-screen orchestration*, where their peers were invited to reflect and discuss the volunteer-student's contribution. Tania wrote explicitly about how she was able to check on students' progress through the 'Response' mode, being able to offer individual support, or lead a group discussion by *explaining-the-screen*, all of which were also highlighted by Tania as benefits of online learning. Tania's *didactical performance* was visible in her actions: 'In the 'Response' mode I could see students were happy to try out different equations. Only one pair were not inputting anything so I joined their breakout room' and while this aspect was clearly an advantage of online learning, Tania appreciated that it could be a limitation, as the rich mathematical conversations she had with pairs of students in breakout rooms were missed out by the rest of the students. She admitted '[All] students would also benefit from being able to listen in to each other's conversations' in a face-to-face learning situation.

For one of the activities, she designed (presented in Figure 2), Tania sent the students in breakout rooms for another opportunity to work with and consolidate their understanding of links between different representations. She particularly liked this activity as it asked the students to 'explain their thinking'. The justification box completed by students in the breakout rooms would instantly be visible to her, meaning that she was able to monitor students' understanding and support them when and if needed. At one point in the online activity, she noticed that one pair of students provided the answer as a quadratic expression in the expanded form, and not in the form that would evidence their understanding of the links between graphs as horizontal translations. She joined their breakout room to praise the students for finding the answer and asked them if their equation could be written in any other format. One student proceeded to re-write the equation of the parabola passing through the purple points on the graph as $y = (x + 4)^2$, while the other student immediately then said 'oh, because it's moved to the left four' having made the link to their knowledge of transformations of graphs. Tania brought everyone back together and asked this pair of students to explain what they had discovered by sharing screenshots and pictures of their graphs, their written work, and by actively interacting with Desmos while explaining their work. Using Drijvers et al.'s (2014) framework, there is evidence in Tania's reflections that the *link-screen-board orchestration* happened often in this online activity. In fact, throughout the activity, Desmos screens were shared via Zoom, which together with the written and verbal justifications, and pointing-at-screen by students and Tania, facilitated the *link-screenS* (Desmos and Zoom)-*board* orchestrations, supporting thus transitioning and making connections between the two DTs and conventional mathematical work.

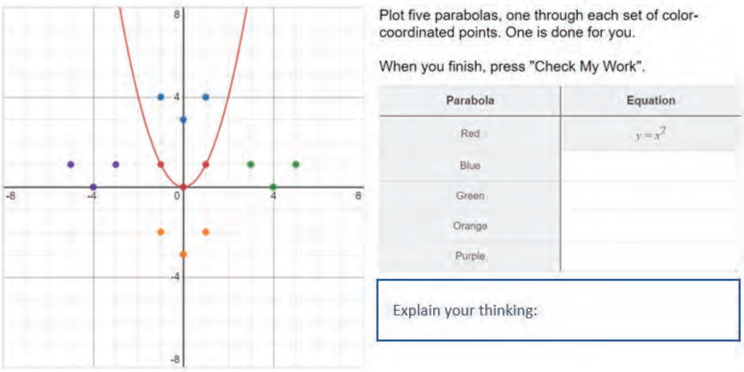

Figure 2: Mathematical Investigation with Desmos

Conclusion

The application of TIO to the data (graduate student's written reflections on her design considerations for online mathematical learning with a DT) we presented above indicates that the three elements of the TIO framework, namely *didactical configuration, exploitation mode* and *didactical performance*, which were developed for face-to-face classroom teaching and learning, provided us with a useful framework for describing the orchestrations necessary for designing and carrying out online learning activities. We investigated the reflections of a graduate student on practices involved when technology-rich learning activities take place online instead of a face-to-face learning situation. The nature of social interaction normally observed in face-to-face classroom-based mathematical learning activities, such as pair-work, students' and teachers' non-verbal gesturing, instant assessment of and feedback to students' learning, needed to be orchestrated for carrying out a mathematical activity online to ensure students' learning did take place. Breakout rooms, contributions to chat, freezing screens, pacing the learning, ease of use of mathematics specific DTs, were used to promote the same learning outcomes for students.

In many respects, such orchestrations facilitated learning in more productive ways according to Tania, as the ease of instant or timely access to each student's work and their contributions in the chat tool at the click of a button were pointed out as advantages of online learning. The ease of monitoring the students' work led to offering instant and timely individual support, and Tania reflected on these aspects as being more productive in an online environment. In other cases, Tania's orchestrations hindered certain pedagogical practices. For example, Tania had rich mathematical conversations with some pairs of students in the breakout rooms, but then the rest of the students missed these conversations as they were not in the 'same room'. This of course would not have necessarily been the case if Tania and the students were all in the same 'physical' room, where she could have easily initiated a whole group discussion.

Online input of mathematical writing is a well-known challenge (Leventhall, 2004), and Tania found a way to 'make up' for this limitation by designing the online activities around students' mathematical investigations using Desmos. Her pedagogical decision for this DT was taken in order to address both the learning objective of the mathematical activity (linking different representations of quadratics), but also to support students to do and share mathematical work through Desmos (graph drawings), and chat boxes (typing explanations). We argue that there are ways around such issues (e.g., Tania would take notes of the key messages in the breakout conversations she had and mention these to all students when they are back in the same virtual room), but this strategy, and the many others that Tania shared with us in her reflections, clearly indicate the need for careful considerations and investment of time in designing activities for online mathematical learning.

This paper's contribution lies in exemplifying how the TIO framework's orchestrations can be broadened as a result of overcoming the challenges and tapping into the opportunities of online learning, as supported by careful design considerations for technology enriched practices based on certain affordances of DTs. We discussed how a graduate student addressed the challenges of online learning and capitalized on the opportunities offered by DTs, while developing her own knowledge of and expertise with technology enriched mathematical learning. Tania had to quickly broaden her field of pedagogical expertise, by identifying how best learners interact online with their peers (in breakout rooms and the main Zoom room), and with technology (Desmos) to do mathematics and therefore accommodating their needs. These considerations should be central to any design decisions when orchestrating students' mathematical learning online.

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Learning about black-boxes: A mathematical-technological model

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Technology fastly becomes more relevant, both in real life and within mathematical and technological education. In everyday life, this technology frequently comes in form of a black-box, a system whose inner-workings are (partly) unknown to the user. In this paper, we argue that we should emphasize teachings about black-boxes in technological and mathematical education. This is because a black-box conception of technology is neglected by current teachings, but crucial for everyday life. To bridge this gap, we present a model that can be used as a basis for such teachings. We argue why the containing conception and analysis techniques are useful for mathematical and technological education. More precisely, we argue that such a model can be used as meta-informational knowledge to reflect upon technology, especially in cases with incomplete information.

Keywords: black-box conception, black-box analysis methods, mathematical-technological education, curriculum development, teaching models

Introduction

With the progression of the 21st century, technology becomes more relevant, both in real life and within mathematical and technological education. This is primarily because "[d]igital technology has the potential to open up new routes for students to construct […] new approaches to problem-solving" (Bray & Tangney, 2017). As such, technology is included more and more frequently in education.

This inclusion of technology can manifest itself in different ways. Firstly, one can differentiate between *tools* (used to automatically solve generic inner-mathematical problems) and *applications* (used to solve a specific real-world problem). Secondly, one can teach *with* and *about* technology. Teaching *with* technology focuses on teaching regular subject areas, but in a better way (whereas "better" can be understood in a huge variety of ways). Contrary to that, teaching *about* technology corresponds to an epistemic mediation of technology and focuses on goals internal to the user, e.g., his familiarity with or conception about technology (Rabardel & Bourmaud, 2003; Trouche, 2005).

Thus, teaching about technology can foster a reflective approach to the usage of technology (*critical reflection*). Notably, this critical reflection requires (some) insights into that technology. However, the mathematics of complex applications (like search engines, product recommendation, or self-driving cars) frequently exceeds the scope of typical school education. Additionally, many of these technologies hide their inner-workings from the user. Hence, at least parts of these applications must be treated as a black-box. Because of this, it is worth asking: "*What conceptions and techniques should be taught to students to foster their critical reflection of complex black-box applications?*"

In this paper, we contribute to this question with a normative approach. We first propose a theoretical model consisting of a black-box conception of technology and five analyzation techniques. Afterwards, we describe why we should teach this model to students. More precisely, we show how this model can be used as meta-informational knowledge to reflect about complex applications.

State of the Art of Teaching about Technology

Teachings about technology can be done both explicitly and implicitly: Students might get explicit instructions of how they should view and use technology, or they build their own model by their experience about their interaction with technology and its inclusion in their education.

Current Curricula Often Implicitly Build a White-Box Conception

As of right now, explicit teaching *about* technology is not part of most curricula. For example, in Germany, the central document guiding the creation of the state-local mathematics curricula states:

> "The inclusion of digital tools should foster the development of mathematical skills. The potential of such tools should support the exploration and understanding of mathematical relationships, the use of individual approaches to problem-solving, and reduce the focus of schematic activities in problem-solving." (KMK, 2015, p. 13, translated from German)

Thus, mathematical education focuses on mathematical *tools* and on teaching *with* technology; no explicit conception for technology is introduced. Hence, it is likely that students learn their conception implicitly, based on their learning and everyday experiences. Currently, the most prevalent theoretical model used to describe the inclusion of technology during learning is the technology enhanced modelling cycle (Greefrath et al., 2011). In this cycle, a mathematical model is translated into a technological model and the technological results are interpreted as mathematical results.

Thus, this model focuses on a white-box conception of technology: Students have full understanding of what the system does, how it is defined, why it is doing it, and know about all inputs and outputs.

Additionally, German computer science education complements this approach by teaching about how a system is doing something. This also includes lessons on how to implement such systems, e.g., using object-oriented programming. Notably, these courses also use a white-box conception where technology directly corresponds to a known model. Teaching about these models (e.g., a computer, the internet, databases, or an UML diagram) is an important part of the teaching (c.f. Röhner, 2016).

Strengths of the White-Box Conception of Technology

With this (implicit) white-box conception, students are able to understand a fundamental property of technology: In order to build or use it, explicit and correct translation between these representations is necessary. Using technology in this way can increase the scope of solvable problems with automaton. This might be because a single approach is executed faster, or multiple approaches can be executed simultaneously. However, it is not possible that technology solves problems a human (with infinite time) cannot solve, since technology is seen as a mere extension of a mathematical model. As such, technology can only be as valid as the models used to create it. More precisely: any output is an inherent property of the used model or algorithm, and not of the method of execution.

This coherence to a mathematical model can motivate the discussions of various concepts like determinism (the output of the model is defined only by its behavior and input values) and edge-cases (situations that defy the assumptions of the model). It can also be used to introduce the important difference between verification (the technology accurately represents the mathematical model) and validity (the model accurately represents the real-world).

Limitations of the White-Box Conception of Technology

While this current, implicit white-box conception accounts for many important properties of technology, it fails to account for black-boxes, in which (at least important parts of) the inner-workings of the technology are unknown to the user. Notably, this limitation is not the fault of the current teaching methods like the technology-enhanced modelling cycle: They were never meant to build a full conception of technology. Regardless of fault, the lack of an explicit black-box conception is undesirable.

Firstly, while technological white-boxes are prevalent in education, black-boxes are far more prevalent in everyday life. This is primarily because every technology not created by the user is, at least initially, a black-box. While it might be common that (in education) all applications or models within a tool are created by students, this is not true for the majority of applications used in everyday life.

Secondly, the impact of these black-box applications on our everyday life is significant. They might affect the information we see (Google), the peer group we interact with (Facebook), dates we arrange (Tinder), the safety of our commute (self-driving cars or trains), our fitness (smart watch and fitness apps), and even our health (medical technology like automatic insulin pumps).

And lastly, black-boxes are often used in the workplace to hide mathematics (Williams & Wake, 2007). As such, designing or working with these black-boxes is a necessity in many jobs. Hence, there is an additional incentive to teach about black-boxes as vocational preparation.

Overall, black-box applications are very prevalent both in everyday life, and in the workplace. As they might have a significant impact on both, a reflective approach to their usage is desirable. However, as the conception of a black-box is currently not taught, there is little students can actually do to pursue such a reflective approach.

Proposition: A Model for Teaching about Black-Boxes and their Analysis

As such, we propose to bridge this gap by explicitly teaching about black-box technology in mathematical and technological education. We describe a conception of black-boxes and five techniques that can be used as basis for such teachings. These techniques differentiate in the amount of resources necessary (from low to high) and the amount of insight gained by utilizing them (from high to low).

Proposed Conception: Black-Boxes

Firstly, we denote the proposed conception students should have about black-boxes:

> Frequently, (parts of) the inner-workings of a given application are unknown to the user. This occurs naturally if the creator and user are not the same person. In this case, a user typically has no influence on the quality of the software if applied in a given situation. Similarly, the creator typically has no influence on the situations a software is applied.

The most important aspect a user must know about a technological black-box is, that it still has all of the properties of a white-box. Most importantly, its behavior and results are deterministic. Notably, this includes the output of pseudo random number generators, as they solely rely on their input value (the seed). This is also true if the user (or, in the case of neuronal networks, even the creator) cannot explain the exact functionality. This also highlights the most important limitation of any technology:

It cannot (magically) adjust to the requirements of a situation and evaluating the quality of the result requires (some) knowledge about the (mathematical) models used to acquire that result.

Nevertheless, it importantly is often still possible to use the technology, even with this lack of understanding from the user or the lacking intention of the creator for this use-case. Because of this, it is crucial that the quality of the software and its applicability to a given situation were thoroughly evaluated. Otherwise, wrong or misleading results might follow, whose consequences can be dire.

Proposed Technique 1: Accepting the Black-Box

The first approach is *accepting* the black-box. In this case, the technology is used without examination and evaluation. We will call such a technology a *true* black-box.

When using a true black-box, one has knowledge about the most important input and output of that black-box: The most important output corresponds to why we use the black-box (e.g., for a self-driving car: “moving towards a destination”) and the most important input corresponds to how we use the black-box (“by inserting a destination into the user panel”). However, even this knowledge about the inputs and outputs might be incomplete or wrong.

At this point, it is important to note that true black-boxes both exist and have educational legitimacy.

On the one hand, accepting a black-box and using it as a true black-box is the default (and natural) behavior of many people. However, it is hard to discuss how desirable this is or to evaluate the consequences of that approach without explicit acknowledgement of the conception of a true black-box.

On the other hand, accepting a black-box can sometimes be the most reasonable choice: It might just be unfeasible or impossible to analyze a black-box at all. For example, many users will (likely) never have the ability (or feel the need) to analyze the algorithms used in self-driving cars themselves.

Proposed Technique 2: Testing the Black-Box

The first evaluation approach including interaction with the black-box is *testing* it, leading to a *tested* black-box. In this case, the technology is executed in a safe environment using specific inputs. If the observed output matches some expected output (to some precision), the technology is then applied to the real problem. Thus, if something fails, the resulting problems are limited to the safe environment.

For this technique, learners need to understand that it only works if the same inputs are used during testing and usage and if all relevant outputs are observed. It is important to fight the misconception “if all observed outputs are correct using one input, then all outputs are correct using any input”. Notably, this can be difficult, as the user does not necessarily know about all inputs and outputs.

However, a tested black-box already offers some amount of information to the user: If the test was successful and seems to represent the future use cases, we can infer that the most important inputs and outputs are likely recognized at this point and that the future outputs will suit the future use-cases.

Proposed Technique 3: Integrating the Black Box

A more comprehensive approach is *integrating* the black-box: First, the black-box is modeled as an unknown function from some inputs to some outputs. Notably, “some inputs and outputs” does not necessarily mean “all relevant inputs and outputs”. Then, outputs to selected inputs can be collected.

This systematic collection of inputs and outputs (rather than unsystematically collected tests), leads to more information as some generalizations can be made. For example, a user might notice that all outputs fall into some magnitude, regardless of input variation. Additionally, if we assume that the inputs are representative for any future use-case, we can infer that the product is likely safe to use.

Proposed Technique 4: Inferring the Black-Box

An even more comprehensive approach is *inferring* the behavior of the black-box. In this case, a re-modelling approach is taken: A mental model of the system is build, which is then verified by comparing coherence between this mental model and the actual behavior. As such, this techniques does not rely solely on some inputs and outputs of the black-box, but also takes the behavior into account.

The model-building process itself might use the observed dependency between inputs and outputs as basis for such a model and follow the steps of a modelling cycle. Regardless of approach, the resulting model can be represented in a variety of ways. This includes mathematical formulas ("speed of car = maximum allowed speed - 10"), verbal statements on different levels of abstraction ("The motor only starts if the battery is not empty", "The car follows the traffic laws"), algorithms ("if the cars sees a red light, then it breaks until standstill"), and meta-information ("This process takes 5s.").

The depth of both the mental model and the verification can vary significantly: The model can be a simple approximation (e.g., using a linear function) or be identical to the actual model. The verification can reach from a single execution in a common scenario to complex statistical tests using several carefully-constructed edge-cases (e.g., "how often does the motor starts if the battery is at 1%?").

Depending on the depths of the models and verification used, it is now *possible* to gain accurate knowledge about the exact lists of inputs and outputs, and the inner-workings of the technology. But this is no necessity: The model is only inferred and relies solely on a finite cutout of the observable behavior of the technology – contrary to any secured or proven knowledge of its inputs, inner-workings, or outputs. As Rice's theorem states that it is impossible to determine the exact functionality of a system without knowledge of its inner-working, this imples that this approach cannot lead to secured knowledge – even if the mental model used is actually the same as the real model (c.f. Rice, 1953).

Proposed Technique 5: Opening the Black Box

The fifth approach relies in *opening* the black-box. In this case, students not only examine its observable behavior, but also its inner-workings (*reverse engineering*). This approach can transform the black-box into a white-box. However, for comprehensive technology, accessing and understanding the implementation of a system is frequently outside the scope of education. For very comprehensive technologies, it might also be out of scope for any single person.

While it is frequently unpractical or impossible to open a black-box, understanding the limitation of the prior approaches only becomes possible if they are explicitly contrasted with this technique.

Additionally, some seemingly arbitrary aspects (like calling things that are treated as secured knowledge a "theory" in science, or arguing why mathematics can be seen as part of the humanities) only becomes comprehensible after understanding the epistemological difference between inferring and opening black-boxes.

Reasons for Teaching our Model

In this section, we show how the proposed model can be helpful and why to include them in curricula.

Reason 1: Intermediate Stages of Black-Boxes as Meaningful Conceptions

First and foremost, our model introduces "intermediate stages" of black-boxes (e.g., tested or integrated black-boxes) as meaningful conceptions. Without such an explicit conception, one might think about black-boxes as "incomplete white-boxes", i.e., a white-box that is left to be opened or inferred. Indeed, this conception was dominant in initial black-box research (e.g., Glanville, 1982) and remains relevant in many recent research projects. See (Krell & Hergert, 2019) for an overview of approaches.

However, this conception is not useful in practice: It is not always desirable to open or infer a black-box. Instead, depending on the time, skill, and jurisdiction of the student, the complexity of the technology under consideration and its available implementation, documentation, explanation, and licenses, it might be impossible, unfeasible, illegal, or uneconomic to apply some of these techniques.

This is quite apparent for self-driving cars or search engines: Access to their source code is limited because of intellectual property rights. Additionally, the complexity of their source code likely exceeds the capability for analysis for most (even trained) single individuals. Similarly, building and verifying an accurate mental model for such a technology also is both very hard and time-consuming.

As such, it is naïve to argue "one should always open or at least infer a black-box". In reality, opening or inferring certain black-boxes is not something most individuals will opt to do. But notably, this does not imply that one should not analyze a system at all. Instead, the intermediate stages show alternative and frequently valid courses of action or analysis one can take.

Reason 2: Stages of Black Boxes as Meta-Informational Knowledge

Secondly, our model highlights aspects of the trade-off between invested resources and gained insights. Thus, the techniques and their potential insights and requirements act as meta-informational knowledge: It can help to build an informed opinion on "whether one knowns enough about a black-box for a specific use-case, given the resources one is able or willing to invest".

As such, these stages can help to assess ones knowledge and identify limitations. Notably, this includes assessing ones knowledge about aspects where one has incomplete or uncertain information. Notably, the assessment of such information is an important step while "deciding what do believe or do", a phrase often used as definition of critical thinking (Ennis, 1987).

This is especially important for political participation: If one assesses that it not possible or feasible to analyze a black-box oneself, one has to ask about the implications. Most importantly, it might be desirable that somebody else analyses the black box. How to design such systems of evaluation can be an important part of a political debate, because different legitimate interests of stakeholders might collide: For example, a user has the desire to use a safe product even if he cannot verify its safety for himself. However, a company in a free market wants to keep certain implementation details as a company secret to keep their advantage over competitors and a government might try to reduce public expenses for institutional verification. Thus, a compromise between these legitimate interests has to be designed and evaluated. This requires insights into the technical aspects of potential solutions.

Illustration of the Benefits: Application of our Model in a Thought Process

To illustrate the prior benefits, we want to come back to the example of self-driving cars. More precisely, we want to exemplarily show how our model can guide the thought process of coming to a meaningful conclusion for the question “How likely is an emergency to occur while using this car?”

With our explicit model, we can first structure our knowledge: We can list available techniques for analysis and consider how they could be applied to the current application (the self-driving car):

True Black Box: Initially, one does not know anything about a self-driving car, expect for its basic functionality. It takes a destination and afterwards drives to this destination. Treating a self-driving car as a true black-box is easy, but reveals nothing about its safety.

Tested Black Box: A self-driving car can be tested in a specifically designed training course. If the course requires some representative situations (like interacting with different car or reading traffic signs), we can conclude that the car is safe to drive at least in some common situations.

Integrated Black-Box: To integrate the self-driving car, we can observe whether the car produces correct output (i.e., drives correctly and according to the traffic laws) for most anticipated inputs. If we chose a comprehensive set of inputs (e.g., a representative selection of all roads of the corresponding country and in a variety of different weather and traffic situations) and observe no (or few) invalid outputs, we can infer that the car is probably safe to use for most everyday cases we want to use it for

Inferred Black-Box: To infer more information about the self-driving car, we could build a set of models based on its behavior. For example, one can build the sub-model “If the car sees a red light, it breaks until standstill” and verify its correctness by building a representative amount of situations (including different distractors like weather and traffic). Afterwards, one can combine several sub-models (and verify this combination) to acquire an accurate mental model of the self-driving car. With this method, we can infer the safety of the car if the mental model indicates safe behavior.

As a second step, we can use this knowledge about the techniques to decide what to do: Using the system as a true black-box does not provide sufficient information about its safety. Since vehicle malfunction can be fatal, testing in a single environment also seems inadequate. However, both integration and inferring can give valuable information about its safety. Nevertheless, they require detailed knowledge and many resources (like verifying the behavior of the car in multiple real-world situations). As such, one might opt to not do this oneself, but rather vote for an obligation (by the producing company or by government regulators) to verify its safety using integration or inferring.

Overall, we used this model to progress from a simple “we don´t have any idea about its safety” to a more sophisticated “There should be an external entity that verifies its safety. An important aspect of this verification is the amount of hours driven in representative situations (rather than test scenarios)”.

Notably, this assessment benefits from knowledge about the techniques in the model. On the one hand, this is because they highlight the difference between behavior in a test vs. a representative environments (test vs. integration). On the other hand, the metric “amount of hours driven in representative situations” is a consequence following from the application of integration or inferring. Hence, it might be overlooked if only thinking in conceptions of “true” and “opened” black-boxes.

Summary and Conclusion

In this paper, we proposed a model of black-boxes in STEM education and argued why it accounts more for the special requirement of mathematical and technological education. This is primarily because technological system relevant for everyday life (especially those based on complex mathematical models like neuronal networks) are frequently too complex to open with typical school education.

Then, we argued why we should include teachings about black-boxes in mathematical and technological education. This is because, firstly, the conception of intermediate stages of black-boxes (rather than "incomplete white-boxes") are useful constructs if opening a black-box is impossible or unfeasible. As discussed before, this is often the case with complex mathematical technology. Secondly, our model can act as meta-informational knowledge to assess ones reflection about a system that is too complex for a full analysis. This also enables the generation of new knowledge. Thirdly, our models highlights that one does not necessarily know about all inputs and outputs of a black-box.

However, the proposition of this model can only be seen as a first step: How to design curricula and lessons based on this model is an important question for future research. Furthermore, it would be desirable to unite our technology-focused model with existing models in science education to achieve a more general "STEM approach to black-boxes" usable in interdisciplinary teaching.

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Communication of technology as a productive element in learning Mathematics in remote settings

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During the COVID-19 lockdowns, mathematics classes took place in distance learning settings in large parts with help of digital technology. Our research interest is to enhance digital face-to-face learning environments to stimulate and support mathematical interaction among learners with the help of digital media. In this article, we address the research question to what extent the consideration of the concepts of multiple external representations and communication of technology enable learning opportunities in online meeting tools. For this purpose, we present the learning environment developed based on these concepts and analyze a transcribed meeting with the help of an interpretative approach. By means of the analysis, identified potentials are presented.

Keywords: Communication of technology, Multiple external representations, Distance education

Introduction

During the school lockdowns caused by the worldwide COVID-19 pandemic, the way mathematic was taught and learned has changed. The traditional face-to-face interaction in classroom cannot take place. In Germany synchronous mathematics classes are held using online meeting tools like Zoom (https://zoom.us) or Big Blue Button (BBB; https://bigbluebutton.org). In our research project, we focus on diagnostic and support meetings held via online meeting tools. Various learning environments for online meeting tools have been constructed and implemented in our teaching-learning lab.

This article focusses on one developed learning environment: Interactive comparison of proportions using the rectangle model. In the theoretical background, the concepts of multiple external representations and communication of technology considered in the construction of the presented learning environments are introduced. Next, the developed learning environment is presented. In our empirical study we address the research question to what extent the consideration of the concepts of multiple external representations and communication of technology enable learning opportunities in online meeting tools. After presenting the selection of data as well as the method of analysis, empirical results from the implementation of the learning environment are presented and discussed.

Theoretical Background

Communication through and communication of technology

When learning with technology, communication is distinguished into two aspects: *Communication through technology* refers to the simple use of technology in mathematics classrooms (e.g., data projectors, displays, document cameras, tablets can be used to display and share the generated learning products). *Communication of technology* refers to the interaction evoked by the output of the technology (e.g., the discussion about the underlying mathematical concept while using a dynamic geometry software) (Drijvers et al, 2016). To realize a communication of technology via online meetings, for example GeoGebra worksheets can be used, which are programmed to show the learners a corresponding output. Depending on the conception of the task or the worksheet there is a potential that learners can generate numerous mathematical phenomena themselves with the help of digital technology (such as a GeoGebra applet or worksheet) to come to mathematical reasoning by observing them (Eichler, 2019). This can initiate fundamental learning processes (Nührenbörger & Schwarzkopf, 2016), which should be the overarching goal when creating a learning environment.

Multiple external representations

The concept of multiple external representations goes back to Bruner's remarks:

> Any domain of knowledge (or any problem within that domain of knowledge) can be represented in three ways: by a set of actions appropriate for achieving a certain result (enactive representation); by a set of summary images or graphics that stand for a concept without defining it fully (iconic representation); and by a set of symbolic or logical propositions drawn form a symbolic system that is governed by rules or laws for forming and transforming propositions (symbolic representation). (Bruner, 1966, pp. 44-45)

In (mathematics) learning environments, intermodal as well as intramodal transfers between and in these representation levels should be explicitly addressed so that learners link their knowledge from the respective areas (Bruner, 1966). One potential in the use of digital media is the synchronous and interconnected presentation of the different forms of representation via multiple external representations (MERs) (Ainsworth, 1999). When manipulating one level of representation, e.g., symbolic representation, the iconic representation changes at the same time. The use of MERs in learning environments can support learners in internalizing the interconnectedness of the levels of representation and in intermodal transfer between them (Moyer-Packenham & Bolyard, 2016).

The developed learning environment

Based on theoretical considerations above, the following learning environment was specifically designed for students attending the Teaching and Learning Lab to support their competencies in the area of numbers and operations, especially in fractions. To tap the potentials presented above as much as possible, tasks were designed with the help of a GeoGebra worksheet. The goal of the first task is to compare different fractions (see Fig. 1).

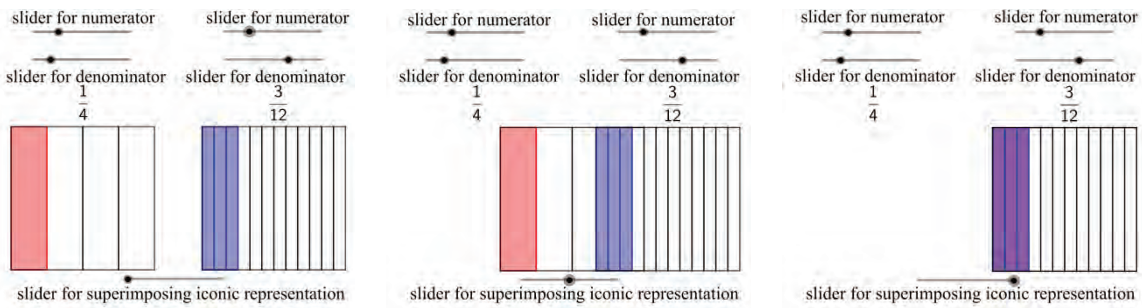


Figure 1: Designed tasks – numerator and denominator are manipulable via sliders

Besides symbolic representation, iconic and enactive approaches are offered. Digital enactivity is limited to the use of sliders to manipulate the iconic or symbolic representation. For example, the numerator and denominator of a fraction can be adjusted while changing the subdivision of the iconic representation. As an additional level of assistance, students can unlock another slider that allows them to superimpose the iconic representations of fractions.

The preliminary theoretical considerations have been taken up in the conception of the learning environment. Since the (enactive) change of the symbolic representation also adapts the iconic representation, the developed GeoGebra worksheet can be classified as MER (Ainsworth, 1999). The students can create different phenomena themselves (Eichler, 2019) by setting and comparing fractions and their iconic representation in quick succession using the sliders. Thereby, a Communication of Technology is initiated (Drijvers et al., 2016), as manipulation of the sliders produces an output in the form of change in the iconic representation, and the iconic representations can be manipulated so that they can be slid over each other to verify equality.

Methods

The collected data was gained in our virtual teaching and learning lab in Summer 2021. The learning environment presented here was conducted with three pairs of sixth graders attending a middle school in northwestern Germany. Partner schools were offered participation in the teaching and learning lab and teachers selected the participating students. The diagnostic and support meetings were held and recorded via the online meeting tool BBB.

In the following, the qualitative analysis of one of the three sessions is presented. The focus is on the pair of students that showed the greatest difficulties in the symbolic handling of fractions at the beginning of the session. Based on the interpretative analysis of the transcripts, the research question to what extent enables the consideration of the concepts of multiple external representations and communication of technology learning opportunities in online meeting tools will be addressed.

With the help of the interactionist approach (cf. Schreiber, 2004; Voigt, 1995), the interpretations of the participants are reconstructed in a turn-by-turn analysis of the transcript. Participants in interviews or classroom discussions interpret the actions of other participants themselves in these situations. These interpretations influence the further course of the social interaction. A detailed understanding of the course of interaction in a learning environment allows conclusions to be drawn about possible adaptations of the learning environment. These can be implemented in terms of the design research approach (van den Akker et al., 2006).

For the sake of clarity, this article only presents the results of the interpretations that have proven themselves within the framework (Krummheuer & Brand 2001). Starting from an observed phenomenon, an attempt is made to translate the cause into a general case with the help of a law. The resulting laws are always based on a hypothetical conclusion, so there is only the possibility of a plausible but not certain hypothesis (Meyer, 2009). The point is not putting the methodological background of the analyzes up for discussion. For the methodological discussion, the translation process of the original transcript would already be a strong interpretative intervention that is worth a discussion. Rather, exemplary episodes are presented that represent different ways of communication with the help of digital technologies.

In the following, the interpretative analysis of a scene from an online meeting with Angela and Susan is presented. Here, the empirical realization of the potentials is examined and, in particular, the question is pursued whether Angela and Susan can transfer the acquired knowledge in the digital-active and digital-iconic areas to the symbolic level with the help of the prepared environment.

Empirical Results

Before the present excerpt starts, the students were asked to use the sliders (Figure 1) to create the fractions $\frac{1}{4}$ and $\frac{3}{12}$. Andrea and Susan successfully solved this task. Simultaneously, the corresponding iconic representations were displayed to the students in the GeoGebra worksheet. The two students further used the possibility to superimpose the two representations with the help of the lower slider (Figure 1 and Figure 2).

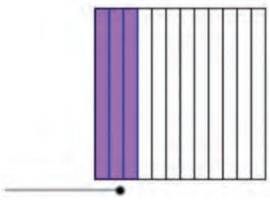


Figure 2: Superimposing the iconic representations with the help of the slider

The transcript (translated by the authors) starts at the moment when the students are asked by the worksheet to describe their observation:

12 S (Reads the task:) Tell what you have observed.. ehm well we have twelve parts three of them are marked and ehm on the quarter we had four parts but one was marked.. If you put them on top of each other three ehm three twelfths ehm the three are exactly as large as the ehm quarter

13 I Can you confirm your guess'

14 A Yes they belong to each other

The students describe the different subdivision of the two rectangles and emphasize that in one rectangle three of twelve parts were marked and in the other on of four. The equality of the given fraction representation assumed at the beginning is confirmed and justified by the students by taking advantage of the possibility to superimpose the two iconic representations with the help of the slider.

Afterwards, the interviewees are given the next task:

16 S (Reads the task:) Ohh interesting do you think that there are other fractions that belong to a quarter and three twelfths' enter your solution in the empty fields on the right side of the page

17 S Ehm (5 sec)

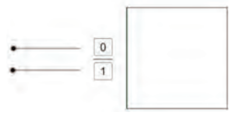
18 A I know ehm no (5 sec)

19 S There are only three twelfth ehm a quarter is slightly larger it therefore fits into the other one..

20 A So the marked part has to.. I don't know how to explain (5 sec.) ehm.. I don't know

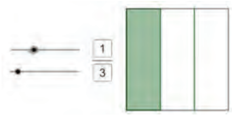
At the beginning of the task, the students can only enter their solutions using the symbolic notation. The task of specifying further fractions that are suitable for $\frac{1}{4}$ and $\frac{3}{12}$ cannot be processed by the students on the symbolic level. Susan focuses on the given iconic representation from the previous task without relating it to the new task in an explicit way. Angela seems to relate the given iconic representation to the task – unfortunately, she does not finish her thoughts.

21 I You can click on the help button (by clicking two sliders appear next to the upper rectangle)

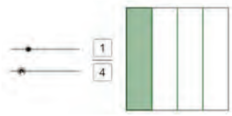


22 A Okay (starts to use the slider on the upper rectangle) ah okay

23 S Ahh one third



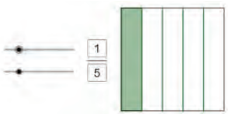
24 A But this is one more it has to fit in to the other one. We had a quarter



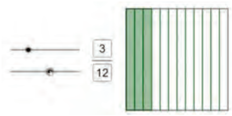
25 S This also does not work

26 A It does

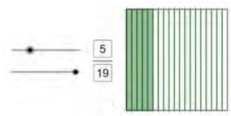
27 S (Keeps trying on the slider so that the rectangle is divided further) One fifth is too small yes that really does not work but I make it smaller.. wait three twelfth fit into three twelfth



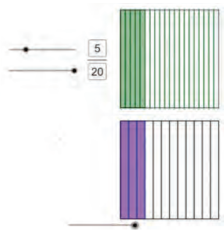
28 A That's right but now go on



29 S Wait five nineteenth no that does not fit



30 A What about five twentieth'.. that does fit



31 S We are ready

As soon as the sliders are provided as a support, Angela and Susan can take action. They try out different settings. Once the sliders are set, the corresponding symbolic notation and iconic representation is provided in the worksheet. The students compare the generated iconic representation with the permanently available representation of $\frac{3}{12}$ of the previous task (e.g. Turn 16). A mathematical systematic is not recognized, rather Angela and Susan persist on the comparison of the iconic representations. Nevertheless, Angela and Susan use the slider to make an increasing finer division and also verbalize this (Turn 27). Furthermore, it is noticeable that when the number of subdivisions is increased by the slider that influences the denominator, Angela and Susan also set the slider for the numerator to a larger setting to make the marked area of the rectangle as fitting as possible in relation to the given iconic representation of $\frac{3}{12}$. The central criterion for the two students is that the marked areas in the iconic representations take up the same proportion of the rectangle, so that they would fit on top of each other (Turns 29, 30). After finding the iconic solution, Susan formulates that they have finished working on the task.

32	I	Okay can you briefly explain how you arrived at your solution
33	A	We looked at how much ehm we needed and ehm how small the parts had to be so that they would fit into the three twelfths
34	S	So we have made the lower number always larger and thus made the lines shorter so that it fits

When asked by the interviewer to describe their solution procedure, Angela and Susan explicitly formulate the procedure interpretatively reconstructed above. By adjusting the sliders, the generated iconic representations were manipulated in such a way that a representation appearing congruent to the already given subdivision was produced. In Turn 34, Susan explicitly speaks of larger numbers without verbally associating them with the fraction representation as numerator and denominator.

35	I	Can you explain how you can get to the solution without using the sliders
36	S	Ehm
37	A	I can't do this
38	S	Five twentieths must have something to do with this.. I don't know

Explicitly asked for a possible solution without referencing the sliders, Angela and Susan are evasive. Susan names the generated fraction $\frac{5}{20}$ for the first time but does not relate it to $\frac{1}{4}$ or $\frac{3}{12}$.

Discussion

Angela and Susan generate a variety of mathematical phenomena (Eichler, 2019) in the form of iconic representations of fractions through the different settings of the sliders (Turns 21-30). They compare the generated fractions to the given iconic representation to find a solution to the task. By using the opportunity of the setting options via the sliders, Angela and Susan are enabled to indicate a solution using the technology. Through the settings of the sliders, the symbolic representation of the fraction is automatically specified, and the corresponding iconic representation is displayed. Angela and Susan argue with the help of the iconic representation via the offered GeoGebra worksheet and generate a solution for the task. However, although Susan and Angela develop findings on the iconic level, they are not sufficiently stimulated to transfer them independently to the symbolic level. An intermodal transfer does not take place. This might be explained by the interpretive hypothesis that from Angela's and Susan's point of view the argumentation on the iconic level is completely sufficient. There is no need for symbolic argumentation or backup of the iconically confirmed results. The apparent fit of the subdivision found through trial and error is seen as a complete solution. Reconstructively, there seems to be no need to give a symbolic argumentation for the iconically gained knowledge. Even though the digital medium offered a synchronicity of the presentation of iconic and symbolic levels, this does not necessarily mean the retrieval of the potential by the learners in form of a realization of an intermodal transfer.

Overall, it can be stated that Susan and Angela are enabled to solve the task on an iconic level through the tools offered by the worksheet which was constructed under consideration of the concepts of MERs and communication of technology. However, a transfer of the iconically obtained solution to the symbolic level does not take place and obviously requires a further impulse in the form of an authentic and precise anchoring in the given task. This adaptation of the learning environment with a view to forcing an authentic intermodal transfer stimulated by the worksheet will be focused on in the continuation of our Teaching and Learning Lab.

Based on the results presented and the discussions, the following possible adaptations of the learning environment emerge: The multiplicative link between numerator and denominator (for equivalent fractions), for example, could be focused more by corresponding buttons to double, triple or half the numerator and denominator. This highlights a more systematic change between numerator and denominator. In the presented form, the given rectangle for iconic representation is divided only horizontally. In a revision of the learning environment, both horizontal and vertical subdivision could be offered. Students might then experience a productive irritation (Nührenbörger & Schwarzkopf, 2016) since you get more examples with different structures for the same fraction. Likewise, the colored part is always the part to the left. In future, other parts might be colored to make clear that, for example, any three out of twelve parts (of equal size) form one quarter. Building on these constructive changes, learners should also be asked to make more arguments about whether the proportions presented are equivalent and why.

By considering the presented points, the potential arises to initiate the intended linking of iconic and symbolic levels and thus to take up the problems that could be reconstructed in the interpretative evaluation of the learners' use of the learning environment. Thereby, the given *model of fractions* could become *model for fractions* for the learners (e.g. Gravemeijer, 1999).

In addition to the reconstructed potentials regarding the argumentations, an organizational potential can also be identified: Due to the principal possibility of successfully conducting pure online teaching-learning arrangements via online meeting tools, the circle of learners who can be enabled to participate in our Teaching and Learning Lab can be enlarged insofar as learners from the entire catchment area in the northwest of Lower Saxony can be reached without hurdles of distance and travel.

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Characteristics of students’ covariational reasoning in an augmented reality environment: a language-oriented analysis

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This study reports preliminary results on the characteristics of covariational reasoning in an augmented reality-rich environment. In contrast to the research literature that has hitherto focused on levels of covariational reasoning, this study addresses its linguistic characteristics. Two groups of three 15 and 17-year-old students participated in this study. Each group carried out two activities (Hooke's law & Galileo) using AR headsets. The students' interactions were video-recorded. Based on inductive analysis of the students' covariational reasoning language, we identified five linguistic categories of covariational reasoning: direct-covariation, mediated-covariation, conditional covariation, multi-variable covariation, and second-order covariation.

Keywords: Augmented reality, covariational reasoning, linguistics, characteristics of covariational reasoning.

Introduction

Covariational reasoning is a central concept in mathematics and science education, which has drawn the attention of many researchers in these fields (Johnson et al., 2017). The research literature mainly focuses on the levels of covariational reasoning, as suggested by Thompson et al. (1994), Carlson (2002), and Thompson and Carlson (2017). In this context, several studies have focused on students' covariational reasonings as they learn in several learning environments, (e.g., Doorman et al., 2012). Among these studies, Swidan et al. (2019) is one of the few that focused on using AR technology to foster covariational reasoning. Swidan et al. (2019), like many others, focused on learners' covariational reasoning levels (see also e.g., Kertil, 2020; Sokolowski, 2020; Thompson and Carlson, 2017). However, little is known about linguistic aspects of students' engagement in covariational reasoning. By linguistic aspects, we mean the structure and the meanings of students' discourse about covariation.

The socio-cultural theoretical framework argues that language plays a central role in the development of human thinking. This suggests that identifying the linguistic characteristics of students’ covariational reasoning may shed light on that reasoning, and its development. Identifying these linguistic characteristics is therefore this study’s goal.

Theoretical framework

Covariational reasoning

Thompson and Carlson (2017) described understanding covariation as holding a sustained image in the mind of two quantities' values (magnitudes), which change simultaneously. In their study, they characterized six levels of covariational reasoning (1) No-coordination; (2) Pre-coordination of values; (3) Gross coordination of values; (4) Coordination of values; (5) Chunky continuous covariation; (6) Smooth continuous covariation. In the first level, the person has no image of variables varying together. In the second, the person can predict the change of each variable value separately but does not create pairs of values . In the third, the person perceives a loose link between the overall changes in the values of the two quantities. In the fourth, the person can match the values of one variable (x) to the values of another variable (y), thus creating a discrete set of pairs (x, y). In the fifth, the person may perceive that those changes in the two variables co-occur and vary smoothly, but only at separate domains. In the sixth and final level, the person can perceive an increase or decrease in the value of one variable as coinciding with changes in another variable value in its entire domain and see both variables as a smooth and continuous change.

Thompson and Carlson (2017) suggested descriptions and examples that illustrate each covariational reasoning level. We adopted their descriptions to identify the level of covariational reasoning and the linguistic structure of the utterances. Such identification of covariation reasoning may assist us in addressing the linguistic characteristics of each statement.

Augmented reality

Augmented reality (AR) is an innovative technology that combines layers of virtual objects or information about physical objects from the real world, such as texts, images, graphs, etc. This creates a kind of augmented reality, in which virtual objects and a real environment coexist to increase the learning experience (Arvanitis et al., 2009; Dunleavy et al., 2009). These virtual layers are created in real-time and layered on the physical objects in the real environment in 3D. Insofar as it augments mathematical representation with the real phenomenon, we assume that AR technology may play an essential role in engaging students in covariational reasoning, due to its ability to visually present information that is naturally invisible. It also simplifies objects' visual appearance and helps students think about their symbolic representations.

Fig.1 Students with AR headsets

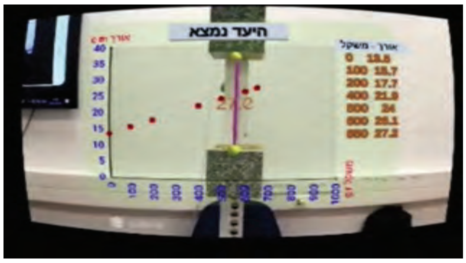

Fig.2 Virtual and physical objects as seen through AR headset

Methods

Research context and participants

Since this study seeks to understand the linguistic characteristics of covariational reasoning, it is based on the qualitative research method. Learning experiments were conducted with two groups of three 9th- and 11th-grade students. The students had already learned linear and quadratic functions. The learning experiments were held at Ben-Gurion University of the Negev. Each session lasted about

180 minutes, during which each group carried out two experiments: (1) the Hooke's law experiment, which examines the relationship between mass and the elongation of a spring (Figures 1 and 2), and (2) the Galileo experiment, which examines the relationship between time and distance that a cube travels as it slides down along an inclined plane. Each group worked on a task sheet corresponding to both physical experiments. The tasks were based on the inquiry-based learning approach. Each task included three inquiry phases: conjecturing, experimenting, and reflecting.

For this study, we designed a new AR prototype that utilizes augmented reality in an educational setting using a special headset, presenting a dynamic object in a real environment with virtual representations provided simultaneously. The prototype is designed to juxtapose mathematical representations (numbers, tables, graphs) with the dynamic real-world object to evoke engagement in covariational reasoning (Figure 2).

Research question

What are the linguistic characteristics of students' covariational reasoning when they learn in an AR environment?

Data collection and analysis

All the learning experiments were video recorded. Students' interactions and materials (written notes, files) were collected. Thus, a solid set of data was obtained. We used Thompson and Carlson's (2017) framework to identify covariational reasoning and determine covariational reasoning levels. We also used the inductive approach (Patton, 2002) to linguistically analyze (structurally and semantically) utterances that indicate covariational reasoning. Finally, we grouped the data into categories of linguistic characterization.

The semantic aspect of our analysis refers to the context in which covariational reasoning utterances emerge, while in the syntactic aspect we address the structure of the covariational reasoning utterances.

The inductive approach we adopted is based on the following phases:

- Looking for utterances indicating covariational reasoning.
- Identifying linguistic characteristics of each phrase.
- Identifying categories of characteristics (linguistic and content) in the statements.

Table 1 illustrates the ways we analyzed the covariation utterances:

Table 1: Illustrative analysis-table of covariational reasoning utterances

Utterance	Thompson-Carlson level lens	Linguistic lens	Category
Dennis: As time passes on, then the sliding cube gains acceleration, and so travels greater distance	**L3**: Dennis seems to perceive a loose link between the overall changes in the two quantities' values (time, distance)	Covariation between time and distance is ***mediated*** by another variable *(acceleration).* The linguistic structure is: As A, then B and so C	Mediated covariation

Alex: If the elasticity is weak, then, the more mass is added, the longer the spring length is.	**L3**: Alex seems to perceive a loose link between the overall changes in the two quantities values (mass, length)	Covariation is ***conditioned*** on a specific situation. The linguistic structure is: "If A then, the more B, the more C.	Conditional covariation
Shahar: At a weight of 200 grams, the length of the spring is 11.8 cm, and at 400 grams, it increased to 14.3cm.	**L4**: Shahar can match values of one variable (weight) to values of another one (length), creating a discrete set of pairs (weight, length).	Covariation is ***directly*** achieved between the variables' quantities (weight, length). The Linguistic structure is: at A=x, B=y.	Direct covariation

Results

Our data analysis produced five categories of linguistic characteristics that characterize students' engagement in covariational reasoning in an AR environment: Direct-covariation (n=130), mediated-covariation (n=12), conditional-covariation (n=7), multivariable-covariation (n=13), and second-order covariation - Cov2 (n=45).

Below, we present a series of excerpts that illustrate the linguistic characteristics of each category.

Direct covariation: In this excerpt, the students' coordination between variables, or variables' quantities, is achieved directly, without any mediation of other variables

Ex1. Lior: As *time* passes, then the cube travels a longer *distance*

Ex2. Shahar: At a *weight of 200* grams, the *length* of the spring is *11.8 cm*, and at 400 grams, it increased to 14.3cm.

In Ex1, Lior covaries time with the distance the cube travels. In Ex2, Shahar covaries the weight of the cube with the spring length. This kind of covariation is usually characterized as Level 3 of covariation. Here, the linguistic structure that the students use is: "as A then B" & "at A=x, B=y."

Mediated covariation: In this category, the covariation between two variables is mediated by another variables.

Ex3. Nir: As the plane's slope is greater, the cube gains more *acceleration* and the *distance* increases.

Ex4. Sagi: As the *elasticity* of the spring changes, so its <u>*length*</u> becomes greater and ..., so... The *graph slope* becomes sharper.

In Ex3., Nir covaries the plane's slope with the distance the cube passes; here, the covariation was mediated by the cube acceleration. In Ex4., Sagi covaries the elasticity of the spring with the graph slope. Also, here, the covariation was achieved through the mediation of the length of the spring.

The following images illustrate the students' engagement in the covariational reasoning that corresponds to Ex 4. As Sagi and Alex interact with the physical model and add mass to the spring (Fig 2a and 3b), they observe the mathematical representations through their AR headsets. They interact with virtual objects observed through the AR headsets (Fig3c). First, Sagi covaries the elasticity of the spring, which is a physical quantity, with the length of the spring, which is displayed on the AR headset; then, he varies the graph of the function shown in front of them via the headset. Hence, the covariation between the spring's elasticity and the function graph was mediated by the length of the spring (Fig 3c).

(a) (b) (c) (d)

Fig 2. (a)&(b) Alex and Sagi interact with the model by adding mass on the elastic spring. (c) Students interact with the virtual objects observed through the AR headsets. (d) The graph and spring's length as seen from the AR headset.

In these excerpts, students use the structure: “as A then B and so C” &, “As A so B and so C”

Conditional Covariation: In this category, the covariation is accompanied by the condition of a specific situation.

Ex5. Alex: *If the elasticity is weak*, then the more *mass* is added, the longer the spring's *length* is.

Ex6. Noam: *If the slope (of plane) is down*, then *as time* passes, so the cube travels a greater *distance*.

In these excerpts, the students use the structures: “If A, then: the more B, the more C” & “if A, then: as B, so C.”

Multi-variable covariation: In this category, the students' covariational reasoning is characterized by using a multiplicity of variables.

Ex 7. Sagi: In the graph, we simply see the change…that the distance increases ...That is, as *time* passes, then the *distance* increases, then you see that the *slope* (graph) simply increases from one second to another and ... and we also see this in the table of values in large jumps …that is the *acceleration* increased.

In this excerpt, Sagey covaries four variables: time, distance, the slope of the graph, and the acceleration of the cube. Here, students use the linguistic structure: “As A, then B, then C and D, so E.”

Second-order covariation: In this category, the students covary quantity and object, or two objects, but not two quantities as anticipated in covariation as defined in the literature.

Ex 8. Shaked: as the *spring* goes down, the *graph* grows up.

Ex 9. Shaked: The *graph shape* rises diagonally because the cube gains *acceleration* and time passes.

Ex 8. shows how Shaked covaries between two objects, spring and graph, while Ex 9. illustrates how she covaries between a quantity (cube acceleration) and an object (graph shape).

The following images illustrate the students' interaction and engagement in covariational reasoning corresponding to Ex 9. As Dennis slides down the cube, Shaked and Nir observe the phenomenon and mathematical representations through their AR headsets (Fig 3a). Shaked connects physical and virtual objects. She describes the graph (virtual) (Fig 3b) as growing diagonally because the cube (physical) is accelerating, and time is passing. Then, Shaked and Nir share their insights with Dennis (Fig3c).

(a) (b) (c)

Fig 3. (a) Dennis slides down the cube while Shaked and Nir observe the phenomenon through their AR headsets. (b)The graph as it seen through Shaked's AR headset. (c) Shaked and Nir share their insights with Dennis.

Final Remarks

We found five categories of linguistic characteristics that characterize students' engagement in covariational reasoning in AR environment: direct-covariation, mediated-covariation, conditional-covariation, multivariable-covariation, and second-order covariation.

With *direct covariation*, students find it easy to reveal the relationship between the two variables. Hence, they address the changes in variables' quantities directly. The AR environment, with its sensory and visual resources, seems to facilitate engagement in direct covariation. For example, in the Hooke's law experiment, the mass and length are visible in the real world and through the AR headset. Furthermore, AR provides numerical measurements augmented and juxtaposed to the observed phenomenon, facilitating students' direct engagement.

Mediated covariation may reveal the quantities or objects that relate to the phenomenon the students observed. The emergence of the mediated covariation might be attributed to the learners' need for such mediations to organize their reasoning, which may support their exploration. Such mediation seems to be an essential part of their deductive reasoning when exploring the relationships among variables.

Conditional covariation seems to be attributed to two factors: the task and the nature of the experiment. The students' utterances reveal that some of their conditional covariation statements seem to be inspired by the task questions. Since the task requested the students to explore the effect of the spring's elasticity on the mass-length relationship, the conditional covariation emerged as an answer to the task's question: "If the elasticity is weak, then the more mass is added, the longer the spring's length is."

The rest of the conditional covariation utterances arose in the context of task 1 of the Galileo experiment. In this task, the students were required to explore the time-distance relationship as a cube slides down along an inclined plane. Changing the plane inclination is not determined in the task questions. However, the students manipulated the plane inclination during their exploration. The features of the Galileo experiment evoke sensory interaction and invite available manipulations: adjusting inclination, sliding objects, free manipulations on the sliding object. Furthermore, this manipulation invites students to address other variables, such as inclination, even though the first task is not explicitly managed to require this.

During students' engagement in covariational reasoning, they resorted to other multiple variables. It seems that addressing such variables assists them in better understanding the observed phenomena and connecting with the variables determined in the task. This phenomenon of multi-variable covariation may be induced by the features of AR technology, which invites the interaction between the users and the real and virtual objects (Azuma et al., 2001). Utterances on multi-variable covariation mainly emerged in advanced levels of covariational reasoning. This may be attributed to students' attempts to utilize and organize the rich available variables' data resources to achieve deep insights into the observed phenomenon.

Although the students covaried between quantities along the learning process, we also identified situations in which students coordinate between a quantity and an object, or even two objects. This phenomenon, covarying two objects rather than two quantities, also appeared in Arzarello (2019), who investigated students' engagement in covariational reasoning when performing the Galileo experiment using computer simulation. This suggests that this type of covariation emerged due to the characteristics of the tasks, independently from the digital tools used in the experiments.

The linguistic characteristics provided in this study may shed light on the students' covariational reasoning processes. Furthermore, this categorization may help teachers foster the students' covariational reasoning and help them design dynamic technological environments. The findings of this study refer to a small sample of 10^{th} and 11^{th}-grade students. Therefore, additional research is required to further explore the linguistic characteristics of students' covariational reasoning as they learn in a dynamic and computer simulation technological environment, considering larger samples of high school and junior high school students.

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Swedish students' exploration of trigonometrical relationships: GeoGebra and protractors yield qualitatively different insights

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Trigonometry, an important pre-requisite for many advanced topics of school mathematics, links geometric, algebraic and graphical reasoning, but remains a difficult topic to teach and learn. The dynamic nature of many trigonometric functions is amenable to dynamic geometry software, which, in the form of GeoGebra, is the focus of this paper. However, both generally and in respect of trigonometry, research on GeoGebra's efficacy seems ambivalent. In this paper, we offer a case study of two groups of Swedish upper secondary students' solutions to the same tasks. One group was instructed to use GeoGebra and the other a protractor to investigate the sine and cosine functions in in the interval $0° \leq v \leq 180°$*. Analyses yielded qualitatively different outcomes; students using the protractor typically identified a geometrical relationship based on symmetry around the protractor's 90° line, while those using GeoGebra tended to identify only numerical relationships.*

Keywords: Trigonometry, GeoGebra, Protractor, Sweden, Upper secondary school.

Introduction

For at least half a century, technophiles have asserted the transformative impact of various forms of computer software on the teaching and learning of mathematics. An early example can be seen in Papert's (1972) enthusiastic promotion of Turtle Geometry as a way of engaging young children with mathematics in ways that generate rather than replicate mathematics. Later, spreadsheets were presented as a means of introducing various aspects of both school mathematics (Healy & Sutherland, 1990) and university mathematics (Steward, 1994). More recently, attention has been focused on, inter alia, dynamic geometry software (DGS), particularly GeoGebra, the focus of this paper. Unfortunately, despite the undoubted quality and availability of the software, much GeoGebra-related research seems to have been underpinned by the technophiles' desire to 'prove' its educational efficacy and is often little more than instructions or tips for teachers to develop their own applets for teaching purposes (e.g., Little, 2011; Phan-Yamada & Yamada, 2012). Even when attempts have been made to establish a baseline understanding of its impact on learning, research has typically been equivocal in its outcomes, methodologically problematic or both. In the following, before introducing our study, we summarise these methodological problems.

GeoGebra and the teaching and learning mathematics

Our reading of the literature indicates an important distinction between the use of preprepared applets, designed to facilitate the exploration of the mathematics under scrutiny, and expectations that students construct their own applets. In respect of the former, which is more frequently reported than the latter, studies have reported on, for example, undergraduates' investigations of the definite integral (Tatar

& Zengin, 2016), circle theorems (Sigler et al., 2017) and Fermat and Steiner points (Flores & Park, 2018). Such studies, while supportive of the software's potential efficacy, seem not only to lack details of the ways in which such applets support learning but assume that such learning could not be achieved via other means. When control groups have been employed, as with studies of high school students' learning of functions (Zulnaidi et al., 2020) and exponential functions (Birgin & Yazici, 2020), researchers typically report higher achievement and deeper learning in the groups exposed to GeoGebra applets than the control groups. However, these studies seem not to be without problems, particularly with respect to details pertaining to the students' experiences. For example, Zulnaidi et al. (2020p. 54) wrote only that "students in the control group were taught using the traditional method whilst their counterparts in the treatment group utilised GeoGebra", while Birgin and Yazici (2020, p.5) wrote only that "the control group was taught by using textbook-based direct instruction". In other words, such studies seem problematic in their unarticulated assumptions about traditional teaching and the experiences of control group students.

GeoGebra and trigonometry

Understanding trigonometric functions is a pre-requisite for understanding many other topics in science and engineering, and as "one of the earliest mathematics topics that links algebraic, geometric, and graphical reasoning, it can serve as an important precursor towards understanding pre-calculus and calculus" (Weber, 2005, p.91). However, it is a difficult topic to teach, with approaches based on the right-angled triangle stressing procedural skills at the expense of any conceptual understanding of either sine or cosine as functions (Kendal & Stacey, 1997). In attempts to address such matters, a number of studies have exploited GeoGebra in the teaching of different aspects of trigonometry.

Unfortunately, much of this research is prone to the problems as discussed above. For example, Kepceoğlu and Yavuz (2016) investigated the teaching of periodicity of trigonometric functions with grade ten students. In one classroom, the teacher demonstrated by means of an applet, while in the other, the same teacher adopted a traditional exposition. The post-test found the experimental group performing better than the control, although no tests of statistical significance were used, and little detail was offered with respect to what students were invited to do in either classroom. In similar vein, Zengin et al. (2012) developed a five-week trigonometry course for high school students. A post-test found greater improvements in the experimental group than the control. However, beyond the implied use of applets, nothing was said with respect to the interventions, other than to assert that "GeoGebra prepared activities aimed to make the subject more dynamic, concrete and visual" (p. 185). By way of contrast, Mosese, and Ogbonnaya (2021) undertook a controlled experiment in which one group experienced GeoGebra-based applet led instruction, while the control group experienced, inter alia, a model of a rotating arm to underpin a series of lessons focused on basic trigonometric functions. In both instances, some detail was offered with respect to what students experienced, with the experimental group performing better on a post-test than the control. Finally, Nordlander's (2021) study drew on observations of upper secondary students working on the limit of $\sin\theta/\theta$. Her hope was that students would "explore, compare, and connect items leading to discovering relationships and learning through their own reflections and self-explaining" (p. 3). She found students' conceptual and procedural knowledge to be enhanced by the visualisations afforded by the software.

In conclusion, much GeoGebra-related research seems problematic, mainly because the tasks students receive are rarely adequately described. Even the authors of review papers seem unaware of such problems. For example, Chan and Leung's (2014) meta-analysis, despite identifying 428 articles, evaluated just nine that satisfied their selection criteria and, while acknowledging factors like length of intervention, student age and teacher role, concluded that DGS-based instruction had a significant impact on achievement in relation to traditional instruction. However, no attention was paid to the tasks students received or, importantly, what is meant by traditional instruction.

This study and its methods

Acknowledging what seems to be a significant gap in the literature, this paper presents an investigation of the impact of different 'technologies' on students' learning of trigonometry. It is conceptualised as an exploratory instrumental case study in which two student groups, one working with GeoGebra and the other with protractors, solve the same tasks. Exploratory case studies aim to identify hypotheses for further inquiry (Woodside & Wilson, 2010), while instrumental case studies aim to advance knowledge of the issue under scrutiny (Garner & Kaplan, 2019). The study is framed by the question

> What differences emerge in upper secondary students' solutions to the same trigonometrical tasks when some students work with GeoGebra and others work with protractors?

The study involved 22 students in the second year of the Swedish upper secondary school's (17-18 years) natural science programme. Participants, who had previously encountered trigonometry only in relation to right-angled triangles, were randomly assigned to two groups, each split into pairs. Twelve students (6 pairs) worked with GeoGebra (G) and 10 (5 pairs) with the protractor (P). Each group completed the same tasks in, effectively, identical lessons of 60 minutes duration.

During the first part of each lesson, (ca 25-40 minutes), students undertook an investigative activity aimed at extending earlier trigonometric relationships beyond acute angles to arbitrary angles. During this time, both groups were allowed pocket calculators, paper and pencil, while group G uniquely used GeoGebra and group P uniquely used protractors and a unit circle drawn on paper. Next, participants in both groups worked through identical textbook-based tasks (ca 10-25 minutes), using whatever tools they preferred. Finally (ca 10 minutes), students solved a trigonometric equation, during which time they were allowed only a unit circle drawn on a paper, paper and pencil. At the beginning of the lesson, written instructions were distributed and afterwards the teacher clarified any misunderstandings for individual pairs. Throughout, students were asked to discuss their thinking, with each conversation being audio-recorded and every GeoGebra activity being digitally recorded.

In this paper, due to limitations of space, we focus on the results of students' work from the first and final phases of the lesson, which were structured by two tasks, presented in ways that would be amenable to either protractor or GeoGebra approaches.

Task 1, the introductory investigations, comprised several parts

- Draw a circle with radius 1 and its centre in the origin.
- Working from the centre of the circle O and the positive *x*-axis, draw an angle 125° and its corresponding radius OP.

- Note the coordinates for the point where the radius intersects the circle: the *x*-coordinate is the cosine value, the *y*-coordinate is the sine value.
- Find an angle for which $\sin v = 0.8$ in the interval

  a) $0° \leq v \leq 90°$ b) $90° \leq v \leq 180°$
- Find all angles in the interval $0° \leq v \leq 180°$ for which $\sin v = 0.5$.
- Examine the results of the previous tasks and make a note of the relationship between them.

The solutions to the first part are a) $v \approx 53°$ and b) $v \approx 127°$, while those of the second are 30° and 150°. Finally, it was hoped that students would have noticed that the angles are symmetrically placed either side of 90° and that the sum of the angles is 180°.

Task 2 comprised one part, namely, determine, without using a calculator or digital tool, which angles in the interval $0° \leq v \leq 180°$ are solutions to $\sin v = \sin 56°$. Its solutions are $v = 56°$ and $v = 124°$. During this time, it was hoped that students would exploit the relationship discovered during Task 1.

Data analysis

Analyses of qualitative data are typically either theory- or data-driven (Boyatzis, 1998). Our view is that the exploratory nature of this study is best served by the latter, as it privileges emergent insights that may be masked by the former (Andrews & Sayers, 2013). Consequently, participants´ utterances and actions, including the GeoGebra digital recordings, were interpreted, and coded in ways that would expose similarities and differences in how students approached their tasks. For example, during their work on the investigative task, Epsilon pair (P), offered the following:

Epsilon (P): But it is 90 degrees. Or is it 180 minus that angle, is the same … Look, it should be the same distance … 90 degrees plus ... or … Ok, check, ok, now, now I came up with it, check, if you have the angle here, you should add 90 degrees minus the angle you have here, because then you come here on the other side. So, if you have 90 degrees … So, it's 0 plus the angle ... is the same sine value as 180 eh ... minus ... yes ... 0 minus the angle *v*.

In this excerpt, the various utterances indicate that the two students had identified a symmetrical relationship around the protractor's 90° line and were attempting to articulate a geometrical relationship. By way of contrast, during their work on the same task, one of the GeoGebra pairs, Gamma, suggested:

Gamma (G): So, 53 divided by 127 might be the same thing as… 30 divided by 180.

Such an utterance, focused on division, indicates little, if any, awareness of symmetry and, we infer, reflects a loosely formed interpretation of a proportional relationship. Later, continuing to work with numerical values, the same pair identified a relationship, although there remained no explicit evidence that they had noticed the symmetry embedded in the situation:

Gamma (G): Now I have found another connection, 53 degrees plus 127 degrees is 180 degrees and 150 plus 30 is 180. It is also a small connection.

Interestingly, their description of the relationship as 'small' indicates a view that they thought an additive relationship of such a form was, possibly, too trivial to be correct.

Results

In the following, we summarise the results for each of the two tasks respectively, highlighting important similarities and differences.

Task 1

Overall, ten of the 11 pairs identified a connection between the angles in Task 1. Five of these, four protractor pairs and one GeoGebra pair, discovered the expected symmetrical relationship. The protractor groups typically drew on expressions like 'equally far' or 'mirror image' to highlight the physical location of the angles in relation to the vertical associated with the 90° line of the protractor. With respect to the former, in addition to the comments made by Epsilon (P) above, another of the protractor pairs, Lambda, was heard to say:

Lambda (P): They are on each side of... both are equally as far away from 90 degrees... the line. If you take 90 minus those in the ... 90 ... 0 to 90 range. Yes. Then it will be as much as if you take... 90 plus …Yes exactly. Between 180 and the 90 range. Yes... that is the connection. It is the same distance on both sides as well. Yes? The degrees are equally far from 90 degrees.

With respect to the latter, another protractor pair, Iota, seemed more explicit in their articulation of a symmetrical relationship. They said:

Iota (P): It looks like as if ... like a mirror image… Yes exactly. It's not plus 90 but ... but it is mirror-inverted, mirror-inverted ...Yes. Yes. So that it (the 90° line) is the line of symmetry.

In short, four of the five protractor pairs, drawing the physical characteristics of their given tool, were able to identify the expected symmetrical relationship.

By way of contrast, and in addition to the comments made by the Gamma pair discussed above, one of the Sigma (G), identified a numerical rather than geometrical relationship, was heard to say:

Sigma (G): Just that, if we add them, yes? Yes. Mm then it will be ... These, added to each other should always be 180. Yes. Is that the connection they want? I think that's the connection, it sounds like a nice connection.

Indeed, the utterance, "added to each other should always be 180", seems to confirm that this pair was thinking numerically rather than geometrically. Moreover, the closing comment that "it sounds like a nice connection" indicates that they were not only content with their conclusion but that they were not expecting to think in anything but numerical terms.

Finally, in this section, one of the GeoGebra pairs, Omega, initially struggled to make sense of their results, as seen in their initially confused and confusing comment that:

Omega (G): Is it the same, em, number of degrees between here it is 120 degrees between this … and this … and what were the angles? Ok, no it is not, because then it is … about 54 something, I think. And here it is ... no it is nothing, there is no connection, it seems…

The utterances "no it is not because then it is … about 54 something" and "no it is nothing, there is no connection, it seems" indicate that the students of this pair were struggling to identify any form of connection between the angles under scrutiny. However, after further thought, they shifted attention from the purely numerical to notice that:

Omega (G): I don't know if it's a connection but that they are just as far … from here to there and from there, em, from the *x*-axis, yes. Yes exactly, it's true. And from the *y*-axis.

In such an utterance, despite their obvious hesitancy, this pair had come close to identifying and articulating the expected symmetrical result.

Task 2

Overall, nine of the 11 pairs, four protractor and five GeoGebra, were able to solve Task 2 and find the values $v = 56^{\circ}$ and $v = 124^{\circ}$ respectively. In most cases, the successful pairs exploited the result they had identified earlier. For example, drawing on the symmetrical relationship they had found previously, Iota were heard to say:

Iota (P): Do we have *v* at all?... Can't we just do our ninety tactics?... If we have… like one answer, we take the other. Yes? …90 minus 56 is equal to 34. Yes? So, 34 plus 90, 124. Boom.

The somewhat rhetorical question, "Can't we just do our ninety tactics?" confirmed the pair's connection to their solution to Task 1. Afterwards, by performing "90 minus 56, is equal to 34... So, 34 plus 90, 124. Boom.", their task is solved efficiently and, it seems, with understanding.

However, not all protractor pairs made such an explicit connection between the two tasks. For example, uniquely among their colleagues, the protractor pair Epsilon solved Task 2 by drawing on a numerical interpretation of the symmetrical relationship they had discovered during Task 1. They began by repeating the task:

Epsilon (P): Determine without calculator which angles in the range are solutions to … Ok. This one we know already; it is 180 minus 56. Yeah!

The utterance, "This one we know already, it is 180 minus 56", came shortly after they had read the task and shows how they connected the two tasks swiftly. Interestingly, their conversation, due to its brevity, indicates that they transformed the relationship they found in Task 1 to the process used here.

GeoGebra pairs using numerical approaches that had not found the symmetrical relationship at Task 1, solved Task 2 by working once again with purely numerical values, and, as expected, did not used symmetry when solving it. For example, the Gamma pair noted that:

Gamma (G): … when it was sine, and it was 180 degrees, it was positive all the time…. So, sine 56 degrees … must be the same as … 180 minus 56 degrees what is … Yes, 130, 124 … it will be, because then the value is the same? Because it is always positive.

Finally, the Alfa pair connected the expectations of the task to their understanding that sine values were read-off the *y*-axis, before performing their calculation and commenting, with a degree of irony, that any connection had been obscured by the number of decimal places returned by the software.

Alfa (G): It's equal to 56 degrees? Yeah? Sine was the *y*-axis, right? Sine what *x*-axis ... *y*-axis. So, it should be 56 degrees.... Now, I do not understand anything. Wait, it's... So, first of all 180 minus 56. I was so confused, I just … how are we going to do this? … The connection is that there are unsatisfyingly many decimals.

Discussion

In this paper, acknowledging that trigonometry is a difficult topic to teach, we have examined the impact of two qualitatively different approaches, involving the same tasks, to the teaching of trigonometry to Swedish upper secondary students. The outcomes were unexpectedly different with, on the one hand, students using the protractor tending towards an awareness of a symmetrical relationship that allowed them to understand and exploit the sine of angles in the range 90-180 degrees, while, on the other hand, students using GeoGebra tending towards an awareness of a numerical relationship between such angles. Significantly, in contrast to Nordlander's (2021) study, there was little evidence of the G group's conceptual and procedural knowledge being enhanced by the visualisations afforded by the software. While further research would be necessary to establish the reasons for these differences, it is not improbable that the protractor scaffolds students' awareness of symmetry in ways that GeoGebra does not. Also, as noted by the Alfa pair of students, the decimal places offered by GeoGebra may interfere with students' interpretation and subsequent generalisation of their results, highlighting a need for teachers to ensure that the appropriate number of places is set.

In sum, the results indicate a need not only for further comparative studies but also for teachers to understand how different technologies may enable or hinder learning. If there are any generalisation to be inferred from a limited study such as this, they are, firstly, that teachers have a role in facilitating students' awareness of any teacher-expected outcomes and, secondly, if students are to make sense of their experiences with DGS, then the role of paper and pencil seems crucial (Komatsu & Jones, 2020); working without such tools may create unnecessary barriers.

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Mathematical creativity in coding dancing animations: a fallibilistic view

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In this paper we investigate student's creative mathematical activity, while engaging in coding digital animations. A logo-programming authoring system which integrates Turtle Geometry, 3D space and dynamic manipulation of figural models was used. Three students of 8th and 10th grade engaged in creating figural dancing animations and synchronizing them to the rhythm of a song, by constructing an artefact and manipulating its geometric transformations in space. We adjusted the Creative Mathematical Action Framework, that is based on a fallibilistic view on creativity, into a constructionist technological context. The results indicate that artistic values such as synchronicity, symmetry and periodicity were conceived and applied within the digital resource and acted as motivators for persistently tinkering with the artefact. This context provided fruitful ground for creative mathematical activity of forming, expressing, exploring and expanding mathematical ideas.

Keywords: Creative mathematical actions, fallibilism, programming, mathematics, art.

Introduction

During the last decades, creativity has been constantly revisited with research attempts for understanding, analyzing and cultivating it in mathematics education. Thus, a diversity of theoretical perspectives on mathematical creativity has been developed, mainly addressing it by means of inborn trait and giftedness (Mann, 2006; Sriraman, 2005) or problem solving-posing processes (Silver, 1997). However, Riling (2020, 2021) has recently made a critical review on the existing approaches, pointing out their limitations over how creativity takes place in mathematics classroom and who can be seen as mathematically creative. Riling proposed an alternative view where mathematical creativity is discussed in terms of students' actions and is open to cultural and social influences – in a parallel manner as in artistic domains such as painting or choreography. She developed a framework in which mathematical creative actions are defined in a concrete way, that enables their investigation in students' activity. In this context, creativity emerges as free, independent expression of mathematical ideas, that can be nurtured in all students. This kind of creativity dissociates mathematical meaning-making from taken-as-established mathematics and fixed curriculum structures (Kynigos & Diamantidis, 2021). In this study, we adopted this challenging view towards mathematical creativity within the context of middle school students using a digital resource called MaLT2, which provides means for developing, expressing and investigating mathematical ideas while modeling 3D figural objects (Kynigos, 2015). Our aim was to investigate how this kind of creativity can be cultivated in the context of open artistic creation. We employed Riling's framework as lens to identify and look into instances of student's creative actions while engaging in creating a digital animation in MaLT2.

Theoretical Framing

Fallibilism in Mathematics Education

The view of creativity we are examining has its epistemological roots into the fallibilist approach on mathematics nature, as opposed to the formalist one (Davis & Hersh, 1980; Ernest, 2003; Kynigos, 2015). Formalism is an epistemological paradigm that conceives mathematics as 'an objective, absolute, certain and incorrigible body of knowledge, which rests on the firm foundations of deductive logic' (Ernest, 2003, p. 2). Thus, mathematical systems are discharged from any human influence and are served as pure isolated knowledge that humans can only approach through discovery. On the contrary, fallibilism relies on the assumption that mathematics is originally created by human thought. Fallibilist mathematics is open to be modified or disproven according to a person's use of it. A main difference between these two paradigms is their educational purposes. Formalism aims at understanding the abstract products of the served mathematical knowledge through practice. On the other hand, fallibilism focuses on the activity of doing mathematics, rather than mathematics itself, aiming at enriching students' meaning-making process (Kynigos, 2015; Kynigos & Diamantidis, 2021). Traditional schooling consist of formalist practices, which are based on the assumption that every task has a unique right answer and any mistake is accompanied with disapproval and criticism. This approach leaves limited space for students to experience creative engagement in mathematics. Fallibilism, in contrast, is connected with constructionist practices in education (Kynigos, 2015). According to Constructionism, students' learning of mathematics occurs naturally while creating and sharing tangible artefacts (Papert & Harel, 1991). The design of constructionist activities within expressive digital media aims at providing fertile ground for powerful mathematical ideas to be constructed. Constructionist learning environments play a two-way role; that is an interplay of reinforcement between mathematics and artefacts. On one side, mathematics is used as tool for constructing or tinkering an artefact, taking a fallibilistic form. Conversely, creating an artefact provides dense opportunity for students to formulate and explore mathematical possibilities. Students are encouraged to engage in both directions and externalize their ideas by adopting the role of a designer, an engineer or an artist. The tangible outcome then becomes a public entity, accessible for reflection by its creator or others. In this fallibilist context, creativity becomes an integral part of mathematics learning in a direct way: mathematics is used for actual creation of personally meaningful content.

Creative Mathematical Action Framework

For our study, we adopted the Creative Mathematical Action Framework (CMAF) proposed by Riling (2020, 2021), which views creativity from a fallibilist perspective and places it into its community of action. We adjust this framework into a constructionist technological context. The CMAF offers a model for identifying creativity in mathematical contexts, such as a mathematics classroom providing a specific technological resource (Kynigos & Diamantidis, 2021). It puts emphasis on the process of *how* creativity takes place, instead of *what* final product is created. Creativity is conceived as a type of action connected to the emergence of new mathematical possibilities as experienced by oneself. Thus, CMAF links creativity to personal meaning-making process. We further connect new mathematical possibilities to the way they are communicated and represented through changes in a digital artefact. According to Riling, a creative mathematical action is defined “as one that transitions a given mathematical context into a new version of mathematics by creating ways of doing or thinking about mathematics that were previously not possible for a particular community of mathematicians” (2020, p. 17). We expand this consideration by adding the component of the way a technological resource can influence this transition in a constructionist learning context. Thus, we conceive a creative action within a technological resource as one that transitions a given mathematical context into a new or expanded form of mathematical meaning by using technological tools to generate, express or explore novel mathematical ideas for creating and tinkering digital artefacts. Two main criteria for an action to have creative potential are: a) to origin from students’ own account instead of others’ (i.e. their teacher) guidance and b) to refrain from the standard ways of doing mathematics in school, where actions are traditionally manipulated by superior established curriculum practices (Riling, 2020). She also emphasizes the importance of aesthetic experiences, such as fruitfulness, visual appeal, mystery and surprise, in activating creative actions. Riling (2021) listed six types of creative actions derived from her data analysis. These types were adjusted to our constructionist view as follows: 1. *setting out*: generating initial ideas and goals intuitively by using the digital tools spontaneously as means to explore unknown aspects of an artefact; 2. *imagining:* using the digital tools more systematically in order to imagine a plan of implementation of an idea and set more clear goals; 3. *manifesting*: taking concrete intended actions to make changes to the artefact by altering the existing mathematical context; 4. *familiarizing*: taking time to survey the current status of the artefact in a comprehensive or reflective way; 5. *recognizing*: reconsidering an aspect of the artefact or interpreting it in a new way; 6. *naming*: distinguishing an action pattern in the medium for a specific object or idea as a distinct entity. In this study, we consider creativity inside the CMAF and investigate students’ creative actions within the digital medium MaLT2 for the creation of a “dancing animation”.

Design of the research

The task in MaLT2

The digital medium used for this study was an online dynamic mathematical programming tool to tinker with 3D figural models called MaLT2 (http://etl.ppp.uoa.gr/malt2/), which is recognized by the Greek Ministry of Education. MaLT2 integrates a UCB-inspired Logo procedural language with affordances for dynamic manipulation of procedure variable values (Kynigos & Grizioti, 2018). Thus, MaLT2 provides three interconnected representations: programming, figural representations and dynamic manipulation of generalized values through their sliders. This last feature enables the dynamic behavior of a figural model that is constructed by a parametric procedure. Therefore, it provides a mathematical way to create an animation through geometrical transformations of figures. Dynamic change of figural models by manipulating variable values also enriches the opportunity for exploring mathematical properties, posing questions, formulating assumptions and getting instant feedback. In this way students can engage in a loop of interaction with the artefact. The task designed for this study was titled as “Dancing Animations”. We designed an artefact in MaLT2 that would be given to students to begin with, in order to make their own animation (Figure 1). The task was described in a worksheet as open as possible. The aim was to make an animation by manipulating only one variable (instead of six variables in the case of the given artefact) in the same rhythm as a song selected from a specific list. The final product would be a video of the animation (the middle representation in Figure 1) added by the song audio.

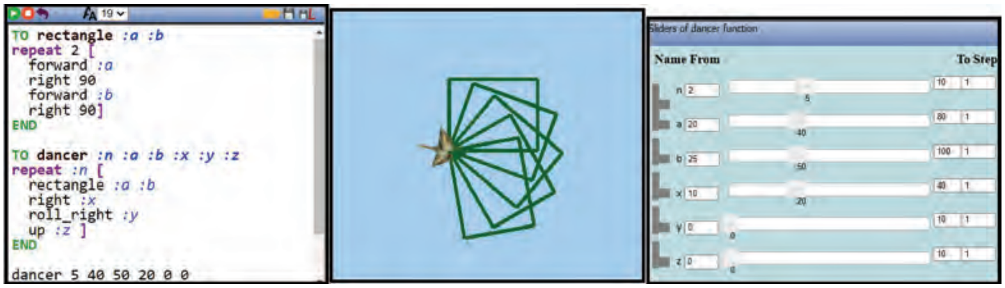

Figure 1. The initial artefact, in the three interconnected representations of MaLT2

Research Method

This study was implemented in a pilot level to three students; one girl of the 8th (Maria) and one boy and one girl of the 10th grade (Dimitris and Anna) of secondary school. They voluntarily participated all together in this two-hour-activity in an atypical after-school setting. They all had already participated in an introductory to MaLT2 activity. Each student had a laptop, a set of headphones and a notebook. A teacher-researcher was facilitating the activity by encouraging students to express their thoughts and ideas out loud. Data was gathered from screen-video, audio and voice recordings as well as students' notes. We analyzed data from students' discourse, notes and activity in MaLT2, following a grounded approach. We identified and analyzed instances of students' actions that demonstrated mathematical creativity in terms of the CMAF. We categorized them into actions of *setting out*, *imagining, manifesting, familiarizing, recognizing* and *naming*.

Results

We identified instances of creative actions for each student that led to the creation of three distinct "dancing animations" synchronized to the rhythm of a song. In Table 1 we briefly describe the sequence of eleven creative actions made by Dimitris and Anna, who collaborated during the process. The merged shells of the Table indicate a common, shared creative action.

Table 1: Description of Dimitris' and Anna's creative actions

Type of action	Description of Dimitri's action	Description of Anna's action
1. Setting out	They expressed the goal to create a dancing movement while dragging the sliders spontaneously and observing the changes on the artefact.	
2. Imagining	While focusing on dragging one slider, they set the goal of finding a way to create a complete rotation that simulates the spinning around oneself, in order to "create the sense of dancing".	
3. Manifesting	They selected the variable x that stands for the turn-command 'right' and tried different values for its upper limit.	
4. Familiari-zing	They ended up setting the limits from 0 to 360 and realized that 360 is the value for the whole rotating dancing move.	
5. Naming	They connected this value to the notions of "full angle", "full turn" and "full circle". Dimitris also referred to it as "the period of the animation".	

6. Recogni-zing	By dragging the slider of the variable x using the right arrow from the keyboard and simultaneously listening to the song each one selected, the expressed that their animation was "out of rhythm" and "too slow".	
7. Imagining	They set the aim to fix the period of dancing move to the period of the song and "find a way to make it move faster and fit".	
8. Manifesting	He considered the period of the song (Iggy Pop - The Passenger) to be 2 seconds and calculated that it corresponds to 60 units of the variable slider range. He used a multiplier to the command "right :x", which could control the speed of the animation. After trying different values, he ended up changing the command to "right 6*:x". He mentioned that "after I tried different values I realized 6 was the right one since 6 times 60 equals 360".	She considered the period of the song (Milky Chace – Stolen Dance) to be 9 seconds and calculated that it corresponds to 270 units of the variable slider range. She used a multiplier to the command "right :x", which could control the speed of the animation. After trying different values, she ended up changing the command to "up 1.3*:x". She mentioned that she found it "almost empirically".
9. Familiari-zing	They observed their animation and realized it was doing "a complete dancing rotation perfectly fitting to the period of the song".	
10. Recogni-zing	They realized their dancing move only lasts for a part of the song rather than its whole duration. They set the goal to make it last as long as the song extract.	
11. Manifesting	The song extract lasted 50 seconds. He calculated the right value of the upper limit of the variable slider for the animation to last 50 seconds, too. He counted the repetitive patterns (=25) and multiplied 25 by 60 (=1500), that is the number of degrees of each period of the animation.	The song extract lasted 34 seconds. She tried empirically values of the upper limit of the slider of the variable. She approached the exact value (=1020) by saying "I realized that each 3 beats it's almost 100 degrees so I was adding 100 each time to reach it.".

Dimitris and Anna followed a common sequence of creative actions, as a result of their collaboration and sharing of ideas. For example, they began with a shared instance of *setting out* and *imagining*, by manipulating the dynamic variation tools (sliders) of the procedure "dancer" (Figure 1). A dialog between them demonstrates their first instances of creative actions:

Dimitris: How can we make the shape dancing?

Anna: I imagine something like that... (She got up and made a complete rotation around herself.)

Dimitris: We need to find a way to make the shape spin. This (dragging the slider of the variable x) creates the sense of dancing. (...) As the variable x values change, the shapes turn continuously.

Anna: Yes, but we have to increase the limit value. Right? Because we want a whole rotation.

This investigation was the starting point of these two students' activity as a quest for answer to their own questions on how to create the sense of dancing. They posed the problem of "what value is suitable for the figure to complete a whole rotation" in order to simulate the dancing move of "spinning", formulating an example of creative mathematical action of *imagining*. During this action, they used the concept of variable and made sense on it. Afterwards, two shared instances of creative actions of *manifesting* and *familiarizing* took place, as the following dialog indicates:

Anna:	I tried changing the right limit of the variable x. I think 360 is the value for the whole rotation.
Researcher:	Why did you choose variable x instead of a or b?
Dimitris:	I thought that these variables (showing x, y and z) are for the turns. A turn means something that changes its angle. 360 is a whole angle. This is why it works for 360. It's 360 degrees, so it's a circle. I realized it right before it reached 360. A circle means a complete angle, so a full rotation.
Anna:	If you put more than 360, it starts repeating the same move again. We could use that to make it keep rotating for the whole song.
Dimitris:	Yes, you are right. (...) It makes another whole rotation when it reaches 720. And then 1080. So 360 is the period of the animation.

Dimitris connected the turn of 360 degrees to the concepts of a complete angle and a circle, noting that this realization was not possible before. In addition, Anna pointed out that values higher than 360 degrees make the animation repeat in the same way. After testing it, Dimitris realized that in every multiple of 360 the movement was making a whole rotation. They both imagined a choreography of repeated rotations fitting at the period of the song. The rest of their creative activity is captured at Table 1. They worked cooperatively, with actions of *imagining* and *recognizing* shared and discussed, but their actions of *manifesting* and *familiarizing* were independent, leading to two uniquely different animations.

Table 2: Description of Maria's creative actions

Type of action	Description of Maria's action
1. Setting out	While dragging the sliders freely, she set the goal to create a dancing movement by using them.
2. Imagining	While dragging back and forth the slider of some variables, she expressed the idea of finding a way to create the sense of moving "up and down around the horizontal state continuously".
3. Manife-sting	She selected the slider of the variable z that stands for the turn-command 'up' to move forth and back. She tried different values for both lower and upper limits. She ended up putting values that range equably from 360 which was the horizontal state: from 340 to 380; from 300 to 420; etc.
4. Familiari-zing and Naming	She observed it and expressed that "360 is the center of the move and I need to keep the same distance right and left". She called the value of 360^o as "the center of the animation" combining it to the notion of the center of symmetry.
5. Recogni-zing	She realized that her dancing movement "takes too long" and that she wants "to make it move faster" but "into the rhythm".
6. Imagining	She set the goal to find a way for the animation to "repeat the complete up and down move several times during the period of the song".
7. Manife-sting	She considered the period of the song (Milky Chace – Stolen Dance) to be 14 seconds. She selected the time period of 2 seconds for her dancing motion, in order to be repeated 7 times during the song period. She noted that "2 sec = 22 forth and 22 back". She ended up setting the value range from (360-11=) 349 to (360+11=) 371.
8. Recogni-zing	While observing the animation by dragging the slider of the variable z, she expressed that the motion was very slow and didn't fit in the 2-second-song period.
9. Manife-sting	She used a multiplier to the command "up :z", which could control the speed of the animation. She ended up changing the command to "up 1.5*:z".
10. Recogni-zing	She expressed that the speed of the animation seemed ok, but the shape does a different, less interesting movement. She said she needed to change the limits of the

	slider in order to make the previous symmetrical move in the selected speed.
11. Mani-festing	While trying different values she said: “when :z becomes 360, the shape is not horizontal like before. The new middle state is at 360/1,5=240. So the proper values are from (240-11=) 229 to (240+11=)251.
12. Familia-rizing	She observed the dancing animation and said that “Now it is even nicer, since the movement is faster and as a result its movement is longer, more extensive and even nicer!” (Figure 2).

Maria on the other hand followed a different path of actions. She did not participate to the discussion with the others and chose to work on her own. Her creative actions are shown at Table 2. Firstly, her idea of dancing animation included a periodic rising and falling motion and the problem of synchronization with the song emerged naturally:

Maria: Maybe I can create this dancing move. (She started moving her hands up and down periodically.)

Researcher: Great idea! Can you think of a way?

Maria: I would try moving the shape up and down by dragging it back and forth. I will write down the limits I like the most (referring to the right and left limit values for the slider of variable z). But, first I need to find the value for which the shape is horizontal and put it in the middle. (...) Well, it is 360. 360 is the center of the move and I need to keep the same distance right and left (...) But when I listen to the song, the movement is too slow. I need to make it move faster up and down.

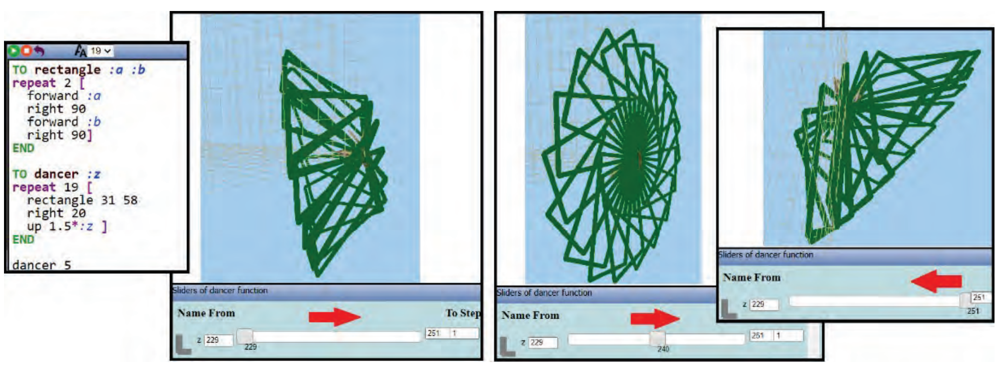

Figure 2. Maria’s final “dancing animation”

After Maria posed the question “how to create the dancing motion of moving up and down”, she generated answers through *imagining*, *manifesting* and *familiarizing* actions. As revealed by her words, she considered the value of 360 degrees to be the “center” of the dancing move, connecting it to the concept of the center of symmetry. Afterwards, she set the goal to synchronize the motion to the music rhythm. In order to achieve it, she listened to the song carefully and noted down its rhythmic period (14 seconds). She went back to her animation and tried various values for the variable z limits, all of them symmetrical to the value of 360. She timed the duration of her artefact’s period and noted that it takes 2 seconds for the slider to be dragged forth and then back when the value range is 22. Then she simulated the “up and down” motion by fixing the variable limits from 349(=360-11) to 371(=360+11). As it is shown in Table 2 (from 5th to 12th action) she went through three more circles of forming an idea, expressing it in the medium and reflecting on it, that led her to the creation of her final dancing animation (Figure 2).

Discussion

All three students participated in this study generated ideas while seeking ways for creating a dancing move and synchronizing it to the rhythm of a song. The CMAF provided an insightful scope for identifying and exploring their actions within MaLT2. We detected a sequence of creative mathematical actions, which formed circles of *imagining*, *manifesting*, *familiarizing* and *recognizing* actions, composing three unique paths of mathematical meaning-making processes. They approached mathematics in a fallibilist way, as it became the basic tool for applying their artistic ideas which were freely open for investigation and reformation. Conversely, their artistic ideas constituted of motivators for their creative way of thinking and doing mathematics. In the instances described above, they used and made meanings on the mathematical concepts of angle, variable, parameter, linear function, periodicity and central symmetry. MaLT2 was used as means for posing both mathematical and artistic questions and investigating their answers through testing and instant feedback. We conclude that both MaLT2 and the artistic context fostered naturally generated creative mathematical actions of all six types described in CMAF.

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SMART – online formative assessment: Professionalising teachers & enhancing students' understanding

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Keywords: Online formative assessment, student thinking, teacher professionalisation

This project of German Center for Teacher Education in Mathematics (DZLM) sets out to investigate the effects of SMART, an online formative assessment tool, on the professionalisation of teachers and on the development of students' understanding. Formative assessment can be conceptualised as "all those activities undertaken by teachers, and or by their students, which provide information to be used as feedback to modify the teaching and learning activities in which they are engaged" (Black & Wiliam, 1998, pp. 7–8). Core elements of formative assessment include eliciting evidence for student learning and understanding, for example, through appropriate tasks, as well as adapting classroom teaching based on the diagnostic information (Black & Wiliam, 1998). In this respect, it is important that diagnoses of students' learning do not remain on a superficial level such as the correctness of a task. Rather, diagnoses should focus on concept images and possible misconceptions of learners. Such deep diagnosis is the basis for teachers adapting their teaching in meaningful ways in order to foster their students' concept development. However, accurate and deep diagnoses usually require a considerable amount of time and effort on the side of teachers which might be the reason why formative assessment with a focus on understanding is often rarely implemented into classrooms. An expedient solution to this problem can be certain online tools such as the Australian SMART-tests ("**s**pecific **m**athematics **a**ssessments that **r**eveal **t**hinking"), which were specifically designed for this purpose by a group of researchers at the University of Melbourne (Stacey et al., 2018). SMART-tests provide not only understanding-oriented diagnoses within a few minutes, but also further teaching recommendations and information on common misconceptions. Hence, on the one hand, SMART delivers quick, directly usable results that can be used by teachers to enhance their students' understanding. However, its developers demand further research to scientifically investigate whether the system does improve student learning outcomes in general (Stacey et al., 2018). On the other hand, SMART can implicitly foster teachers' pedagogical content knowledge and thus their diagnostic skills. Nevertheless, "it also seems important to have professional development showing its advantages and distinctive features, and to provide teachers with advice on implementation." (Stacey et al., 2018, p. 20). Without such support, teachers may use the diagnosis from formative assessment environments more in the sense of a summative assessment instead of developing their teaching in an understanding-oriented way (Stacey et al., 2018).

Therefore, in addition to SMART-tests being adapted and translated for use in German-speaking countries, this project includes the design of an accompanying professional development (PD) programme to scale up the effects on teachers' diagnostic competencies and thereby students' understanding. The study will investigate to what extent teacher competencies, practices and students' understanding in the field of algebra develop through the use of the SMART system depending on whether teachers take part in an accompanying PD programme:

1.	How do teachers' diagnostic and support skills and teachers' self-efficacy beliefs about digital formative assessment develop through the use of the SMART tests?
2.	What kind of support do teachers implement in their lessons depending on the type of support teachers receive?
3.	How do teachers integrate SMART test results into their teaching?
4.	How do students' conceptual understanding and misconceptions develop through the use of SMART?

Figure 1: Research questions

These research questions are investigated by comparing three groups of teachers and their students (see Figure 2). While both first and second group will receive SMART diagnoses and teaching suggestions, only teachers from group 1 will be supported by an accompanying PD programme. Teachers in group 3 (control group) will only receive the corrected results of their students but neither a diagnosis in the form of stages of understanding and misconceptions nor any teaching suggestions.

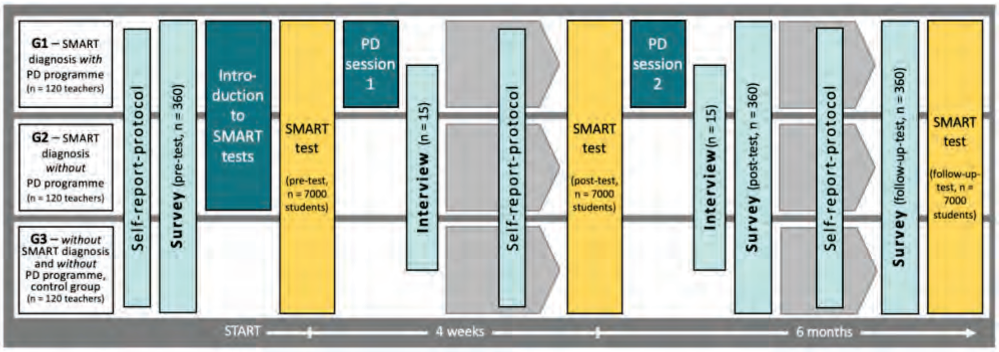

Figure 2: Research design

To answer the research questions, different types of data will be gathered: On the teacher level we assess teachers' diagnostic competencies and beliefs with questionnaires. Additionally, teachers' formative assessment practices in the classroom will be captured by pre-structured self-report-protocol. Furthermore, we conduct qualitative interviews with a subset of teachers from each group. On the student level, we gather data on student understanding by using the SMART-tests.

A first small pilot has shown, firstly, that German students indeed exhibit the misconceptions that were intended to be tested by the Australian developers. Secondly, although teachers acknowledge that knowledge gaps were filled by SMART information, their understanding and conclusions from the information differ considerably. This supports our presumption that additional PD might be helpful to further enhance teachers' formative assessment competencies.

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BYOD vs pool: Effects on competence development and cognitive load

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In this article, we investigated differences in the effects of student-owned smartphones (BYOD) and provided smartphones (pool) on mathematics modeling competence development and cognitive load. Therefore, we conducted an intervention study, used a Rasch model and applied a mixed model ANOVA to investigate the competence development (n = 176) as well as a Mann-Whitney U test to investigate the cognitive load (n = 186). We showed that there is no significant difference between the two treatment groups in terms of their competence development (p = .876). For cognitive load, we showed for one task that learners in the BYOD group were significantly less cognitively loaded than learners in the pool group (p = .047) and no significant differences for four other tasks (.08 ≤ p ≤ .579).

Keywords: Handheld devices, technology planning, influence of technology.

Introduction

An essential question when using digital media and tools in the classroom is with which device the students should access the digital materials. In addition to the question of the device type (e.g., desktop computer, laptop, tablet, smartphone, ...), it is also relevant whether students' own devices or school devices should be used—although it does not seem logistically sensible for all device types (e.g., desktop computers) to have students bring them to school. The question does not become clearer if one takes into account that different variants of the two possibilities mentioned exist, which differ, for example, in terms of the concrete design of the variables ownership, device management and device availability. Furthermore, differences arise from the device provisioning concepts, for example in the distraction potential of the devices or in the student motivation. In addition to the great uncertainty, this question also has great significance for school learning. In the study presented here, the two provisioning concepts BYOD ("bring your own device") and pool are compared with each other in terms of competence development and the directly connected cognitive load perceived by students when working either with their own devices or with provided devices.

Theoretical Background

Device Provisioning Concepts

Any use of digital media requires access to a suitable digital device. There is a wide range of concepts for the provision of mobile devices such as laptops, tablets or smartphones. In the school context, most commonly the concepts of *BYOD*, *GYOD*, *COPE* and *pool* are used (see Table 1). Under the

BYOD ("bring your own device") concept, students' private devices are used at school. Also, under the GYOD ("get your own device") concept, private devices are used at school, but the school limits more or less precisely which devices may be purchased by the students, thus reducing or eliminating device heterogeneity in the classroom. Under the COPE ("corporate owned, personally enabled") concept, learners are given school-owned devices to use on a long-term basis. Whereas in the concepts mentioned so far the devices are available to the students outside of class time and can thus also be used, for example, when working on homework, this is generally not possible under the pool concept. Within this concept, the school acquires a pool of devices—often several class sets—and usually only issues them to the students for short-term use during class time.

Table 1: Characteristics of Different Device Provisioning Concepts

Device provisioning concept	Device owner and cost bearer	Device selection	Device management	Device availability
BYOD [a]	Student	Student	Student	In and out of class
GYOD [b]	Student	Preselection or full selection by school	Student	In and out of class
COPE [c]	School	School	Student and school	In and out of class
Pool [d]	School	School	School	In class

Note. The most common realizations of the four device provisioning concepts are given. Some of the stated realizations are not unambiguous, but may vary slightly.

[a] Acronym for "bring your own device". [b] Acronym for "get your own device". [c] Acronym for "corporate owned, personally enabled". [d] Short for *device pool*.

The differing characteristics result in further potentially learning-relevant consequences between the concepts. On the one hand, digital devices have a certain potential for distraction (Karsenti & Fievez, 2013), which may also increase with greater possibilities for individualization and inhibits the growth of competencies. In contrast, however, students are more motivated when digital devices are used for learning (Burden et al., 2012). This motivation may vary depending on ownership or individualization opportunities.

Cognitive Load

According to the multi-store model of memory, a limiting factor for learning is the finite capacity of working memory (Atkinson & Shiffrin, 1968). However, this memory is stressed by different types of cognitive load during learning. Specifically, three types of cognitive load can be distinguished that affect learners: intrinsic cognitive load, extraneous cognitive load and germane cognitive load (Sweller, 1988). While intrinsic load is dependent on the complexity of the learning object, extraneous load is influenced by the external learning conditions. Whereas germane load is considered as positive cognitive load and is defined by the actual learning processes. Successful learning is achieved when the total cognitive load does not exceed the capacity of the working memory.

In a study of 520 smartphone users, Ward et al. (2017) showed that the mere presence of one's own smartphone causes cognitive load. Study participants who had their smartphone on the table during a corresponding test showed significantly worse results in the area of available cognitive capacity than participants who stored their smartphone outside the test room. Whether provided smartphones cause similar effects was not investigated in this study. In contrast, however, students' familiarity with their own device can be assumed to reduce the cognitive load in the BYOD concept (cf. Welsh et al., 2018).

Mathematical Modeling

Mathematical modeling is one of six general mathematical competencies of the German educational standards (see Sekretariat der Ständigen Konferenz der Kultusminister der Länder in der Bundesrepublik Deutschland, 2004) and allows for a diverse use of digital tools (Greefrath et al., 2018).

Research Questions and Hypotheses

Since in school currently different concepts for the provision of digital devices are practiced, the question arises whether these concepts have different effects on the competence development of the students. In view of the shown effects of the presence of one's own smartphone on cognitive load, a further question arises as to what extent the smartphone-caused cognitive load differs between different provisioning concepts when the smartphones are actively used. Specifically, the following questions are investigated and the related hypotheses are tested:

RQ1	Is there a difference in effects on mathematical modeling competence development between the BYOD and pool concept?
RQ2	To what extent does the cognitive load of a task differ between the BYOD and pool concept?
H1	The mathematical modeling competence develops differently between students taught under application of the BYOD concept and those taught under application of the pool concept.
H2	The cognitive load of a task differs between students taught under application of the BYOD concept and those taught under application of the pool concept.

Method

The comparison of the two device-type-independent provisioning concepts BYOD and pool has been carried out in this study using smartphones as an exemplarily device. The main reason for this is that the BYOD concept requires that students already have their own devices of the corresponding type. In order to not make the schools' participation in the study dependent on whether or not they have a device pool (of smartphones), the students in the pool group received university smartphones that were configured in a school-like manner. For example, students were not able to install apps on their own. Since it can be assumed that an established pool concept tends to cause fewer operating difficulties for the students than using an unfamiliar operating system, pool devices with the Android and iOS operating systems have been purchased and handed out according to the operating system used by the students privately.

The Project *smart for science*

This study is part of the research project *smart for science*, which investigates how student-owned smartphones (BYOD) can be successfully integrated into school classes. For this, the project offers workshops in the subjects of mathematics, physics and chemistry on the topic of electromobility for lower secondary school classes. During these workshops, qualitative data, namely videos from students' perspective, is recorded as well as a variety of variables is collected quantitatively. This data is complemented by further general data, collected at an additional date prior to the workshops.

Sample and Data Collection

In total, 234 students participated in the mathematics workshops between November 2020 and October 2021. The study sample is a convenience sample at the class level. The sample was formed through requests for participation made to school teachers or through self-selection by teachers. However, individual students were able to opt out of data collection and thus participation in the workshop. The assignment of the students to one of the two experimental groups was randomized at the individual level.

As for two classes the workshop realization differed from the manual, these classes had to be excluded from the analysis. Furthermore, some single students had to be excluded as well, as they showed up late, had to leave early or had to leave inbetween. Moreover, for analysis of the modeling competence development 10 further students had to be excluded from the analysis, as they did not participate in the posttest or swapped their assigned posttest with their seatmate. This leads to a sample size of 186 students ($M_{\text{age}} = 13.98$, $SD_{\text{age}} = 0.69$; 88 female, 70 male, 4 non-binary) for cognitive load analysis and 176 students ($M_{\text{age}} = 13.97$, $SD_{\text{age}} = 0.69$; 83 female, 67 male, 3 non-binary) for modeling competence development analysis. The students are distributed among nine classes of grades 8 and 9 in four secondary schools from North Rhine-Westphalia, Germany.

Quantitative Study Design

Prior to attending the workshops, students completed two general quantitative questionnaires once at a preceding meeting to collect background data. Study participants were then randomly assigned at the person level to one of the two study groups, BYOD or pool. The students then attended one, two or all three of the aforementioned workshops. The two study groups were each taught in parallel by a trained project staff member.

The data collection during the mathematics workshop days have been roughly structured as follows: First, a 15-minute organizational phase, a 10-minute questionnaire and a 15-minute competence-based performance test have taken place. Then, the intervention of approximately 3:00 hours has been conducted. During this intervention, students have completed a short questionnaire after completing tasks 1, 2, 3, 5 and 8 that measured, among other things, cognitive load. Further, during task 8, students have participated in a concentration test. Finally, again a 15-minute competence-based performance test has been conducted.

The restriction and selection of the tasks to measure cognitive load took place in order not to overload the students and at the same time to get an impression of possible differences between diverse tasks. For an insight in the tasks see the following section.

Intervention

During the learning time of the mathematics workshop, students completed eight mandatory and two optional tasks (no. 9 and 10) in the area of mathematical models describing the relationship between the mileage of an electric car and the emissions of climate-changing greenhouse gases it causes.

As an introduction to the workshop, in task 1 the students watched and worked through an interactive explanatory video with quiz questions embedded using H5P. These were multiple-choice questions or a drag-and-drop cloze. In task 2, students used a CO_{2eq} calculator to determine the related CO_{2eq} emissions caused in eight life domains based on a given person description. Then, in task 3, students used a bar chart applet to visualize the determined values, created a screenshot, and shared it with the class on a digital collaborative bulletin board. In task 4, only minimal smartphone use took place when writing down the determined values of the mobility life domain (cf. task 2) analogously. In task 5, the students first used the familiar CO_{2eq} calculator as a simulator to generate data pairs of distance driven and CO_{2eq} emissions caused, and then a multi-representation system to visualize the previously obtained table of values in a coordinate system and to determine a regression line using two sliders (one each for slope and y-axis intercept). In task 6, students interpreted the regression line in the factual context; little or no smartphone use occured here. In task 7, students used the smartphone as a function plotter to graphically determine intersections and potentially as a calculator beforehand. Task 8 allowed the students to use the smartphone in a variety of ways, especially for research on the World Wide Web and also as a calculator or function plotter.

Modeling Competence Instrument

To record the students' modeling competence before and after the workshop, a competence-based performance test was used. In total, there is a pool of 16 items, which are based on items of the Institut zur Qualitätsentwicklung im Bildungswesen (n.d.), developed for the German comparative test in grade 8 (VERA-8) and were used in prior years. All 16 items measure the competence *mathematical modeling* within the core theme *functional relations*. From these, four test versions with 8 items each were created. Since the data collected will be used to explore a possible effect of the digital device, the pre- and posttest were paper-based without the use of the smartphone.

Cognitive Load Instrument

To measure task-related cognitive load, the learners rated the item "How easy or difficult was it for you to complete the task?" on a 9-point Likert scale ranging from "very very easy" to "very very difficult" after completing the respective task. This instrument is adopted from Kalyuga et al. (1999).

Data Analysis

To answer research question RQ1, the underlying competence in mathematical modeling had to be determined for each student both before and after the intervention based on the responses in the pre- and posttest. Following Item Response Theory, a one-dimensional dichotomous Rasch model (1-PL model) was used for this purpose. Any tasks that were not answered were treated as if they had been solved incorrectly. Afterwards, a mixed model ANOVA was conducted to investigate differences within time between the two groups.

To answer research question RQ2, for each of the tasks 1, 2, 3, 5 and 8 a two-sided Mann-Whitney U test was applied to the data optained by the cognitive load instrument.

Results

Mathematical Modeling Competence

The Rasch model for the investigated data showed an EAP reliability of .723 for the pretest and .701 for the posttest. Thus, it can be stated that the data met the requirements for a Rasch model. The estimation of the Rasch model was run with the R package *TAM* version 3.7.16.

The mixed model ANOVA of the mathematical modeling competence development within time between the two device provisioning concepts BYOD and pool revealed no significant difference ($p = .876$). The R package used to run the ANOVA was *ez* version 4.4.0. Figure 1 presents the mean and standard deviation as well as the distribution of the students' mathematical modeling competence of the two groups BYOD and pool at both time points. This part of the study has already been presented in Krause (in press).

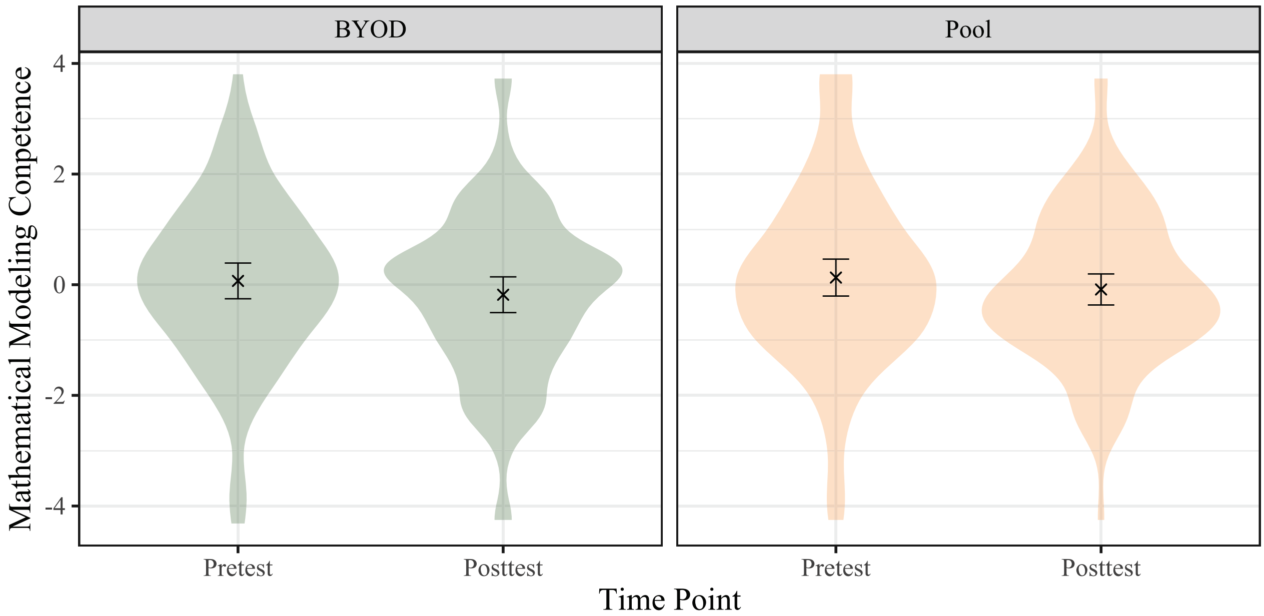

Figure 1: Mean With 95% Confidence Interval and Distribution of Students' Mathematical Modeling Competence

Cognitive Load

For task 8, students who used their own device to complete the task ($Mdn_{\text{BYOD}} = 7$, $M_{\text{BYOD}} = 6.64$) rated this task to be easier than students using a provided device ($Mdn_{\text{Pool}} = 7$, $M_{\text{Pool}} = 7.23$). A Mann-Whitney test indicated that this difference was statistically significant with small effect size, $W(N_{\text{BYOD}} = 83, N_{\text{Pool}} = 66) = 2228.5$, $z = -1.98$, $p = .047$, $r = .16$.

No significant differences were found between the two provisioning concepts for the remaining tasks 1, 2, 3 and 5 ($p = .423$; $p = .08$; $p = .579$; $p = .117$). The R package used to run the Mann-Whitney U test was *stats* version 3.6.2. Figure 2 presents the cognitive load distribution of the two study groups as box plots with whiskers with maximum 1.5 IQR for all five tasks.

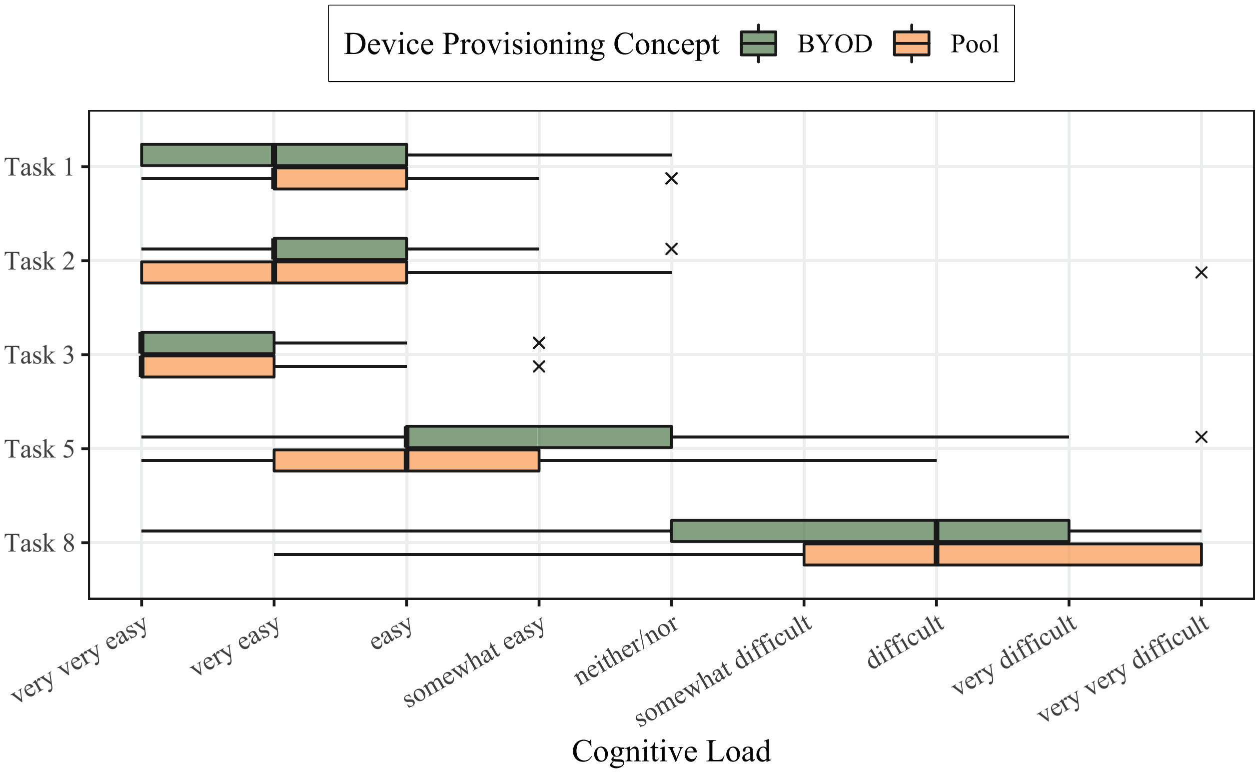

Figure 2: Cognitive Load of the Students Regarding Different Tasks

Discussion

As can be seen in Figure 1, there are only slight and non-significant differences in the students' mathematical modeling competence development within time and between the groups for this sample using the current analysis. To further investigate the research question, it is planned to repeat the analysis with a larger sample and also to consider other IRT models.

As can be seen in Figure 2, task 8 is rated as more difficult by the learners than the other tasks (in terms of 0.25-, 0.5- and 0.75-quantiles). In both experimental groups, only 25 percent of the students rated the cognitive load of task 8 as *neither/nor* or easier; whereas for all four other tasks, 75 percent of the students rated these tasks as *neither/nor* or easier. This raises the question of the extent to which the difference in cognitive load between the two groups depends on the task's cognitive load itself, possibly in particular on the cognitive load caused by the performed digital activities. Further, it is noticeable that exactly the last surveyed task shows a significant difference in cognitive load. This raises the question of the extent to which the duration of digital device use influences the observed effect. It is possible that the benefits of the BYOD concept will take effect after a (short) period of getting used to using one's own digital device at school.

Overall, we take the reported effects as indications that the advantages of using student-owned devices in the classroom may offset or exceed the disadvantages. Nevertheless, further analysis is needed to corroborate this interpretation. So far, it remains open what impact the single characteristics of the two device provisioning concepts (cf. Table 1) have on the results and thus also to what extent the results can be applied to other provisioning concepts (e.g., GYOD or COPE).

Acknowledgment

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Enriching videos with interactive questions to enhance students' cognitive activity: concept and implementation

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Keywords: Cognitive activation, interactive videos, H5P.

Cognitive activation in mathematics learning videos

Cognitive activation and individual learning support are key factors that explain students' learning in traditional classrooms (Baumert et al., 2010). Learning videos seem to hardly support both factors. However, activation does not mean a physical activity; instead, active participation includes active thinking. In mathematics, cognitive activation needs to be focused on mental activities that fit to the learning goals (see Leuders & Holzäpfel, 2011). Cognitive activation further means that learners engage with the subject matter at a cognitive level that is as high as possible and appropriate to their learning requirements. Cognitive activation can thus especially occur when videos are combined with tasks or interrupted for the learner's own activities. Even basic interactive elements like playback control, segmentation and table of contents improve both learning and learner satisfaction. As technology develops, there is an increasing number of possibilities to design videos in a cognitively activating way and to support individual ways of using them.

We aim to make mathematics learning videos more cognitively activating. In our poster, we present didactic scenarios taking account of the type of content during activation and individual use.

Technical background: the H5P software

The free and open-source software H5P is based on HTML5 and allows to create interactive learning content (https://h5p.org/). In particular, videos can be embedded in a container that offers predefined question formats, among others multiple-choice, drag-and-drop, and free text questions. In addition, a table of contents with navigation can be added to the video.

Didactic scenarios

Like in classroom settings, questions can serve different purposes in video: they can assess prior knowledge, support comprehension of the presented ideas, underline cognitive conflicts or stimulate further engagement with the subject (Lim & Wilson, 2018). The question formats mentioned above can be used to activate both prior knowledge and newly acquired knowledge while watching the video. The force to respond to the pop-up questions or to fill in the blanks may underline students' cognitive conflict.

Self-explanations have been found to improve both conceptual and procedural knowledge in the context of learning from worked-out examples (e.g., Berthold & Renkl, 2009). They can be supported with cloze texts. Renkl (1997) identified two types of successful self-explanations, anticipative and principle-based explanations. Implementing questions and cloze texts according to these types, we can encourage students to think ahead and establish links (anticipative explanations) as well as to reflect on the central principles underlying the presented content (principle-based explanations).

Practical Examples

In the project studiVEMINTvideos (Biehler et al., 2020), we develop videos embedded in digital learning material. The materials shall help students to autonomously review their mathematics knowledge when they transition to university.

To implement interactions in videos, we identify appropriate positions and question formats based on central problems and cognitive conflicts in the video. In the beginning of a video, we often opt for anticipative questions. In the end, we consider principle-based questions summarizing the main ideas. We adapt the number of interactions to the length of the video and try to vary the question formats within one video. In a video on the fundamental theorem of calculus, for example, we embedded three interactive questions. At the beginning, prior knowledge is activated by asking for the definition of the antiderivative with the help of a drag-and-drop question. After having explained the theorem, we explained how to generate an antiderivative. To consolidate this newly acquired knowledge and to transfer it to other functions, we embedded two single-choice questions to identify antiderivatives of given functions. Our poster will illustrate such examples. Further research will focus on students' learning with these enhanced videos.

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Affordances and constraints of the Dragonbox School teaching material

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The aim of this paper is to analyze the affordances and constraints of the way dynamic representations are used in the digital learning lab 'The set line' from the Dragonbox School teaching material. The learning lab is designed for second graders. The starting point of our analysis is the task 18 + 6. Our analysis shows that 'The set line' offers affordances for the users' learning of addition and other aspects of the number concept because the regrouping involved in the addition of 18 and 6 is explicitly displayed by the dynamic representations in the learning lab. 'The set line' may also cause constraints for students' opportunities to learn the inherent mathematical concepts and relations because several operations are carried out automatically by the learning lab.

Keywords: Addition, affordances, constraints, digital tool, dynamic representation.

Introduction

Digital tools may provide dynamic representations (Ainsworth & VanLabeke, 2004; Günster, 2019) which offer visualizations of mathematical concepts and relations. The digital tool under scrutiny here, is the Dragonbox School teaching material. When users, i.e., second grade students, interact with this tool, the dynamic representations are changed according to the input of the users. Such changes may make essential features of the mathematical structures more apparent, and this may facilitate understanding of the mathematical concepts and relations involved. If the students had been working with static representations, like concrete materials, these structures would in many cases not have been made apparent in the same way. For example, students may separate a digital rod representing the number 7 into a two rod and a five rod. This process demonstrates a decomposition of the number 7 not facilitated by physical rods in the same manner.

The reason for choosing the Dragonbox School teaching material, from now on called 'Dragonbox School', is that it contains digital learning labs which facilitate open-ended explorations and visualizations of mathematical concepts and relations using dynamic representations. These learning labs are designed in a way that enables the students to use them with little or no support from the teacher. The students are invited to explore on their own how the learning labs work and use these labs to explore mathematical concepts and relations. This approach to designing mathematical teaching material is rather novel. Because Dragonbox School was launched in 2018, very little research is carried out concerning this teaching material. Therefore, it is interesting to know more about the affordances offered by the learning labs, and the constraints they may cause in the learning process.

Dragonbox School is made by the company Kahoot Dragonbox AS and is used in Norway, Finland, and France. It consists of digital resources, textbooks, and concrete manipulatives for 1st to 4th grade. The digital resources consist of learning labs and quizzes developed for use on tablets or laptops with touchscreens. The quizzes are learning labs combined with mathematical tasks. The textbooks mainly consist of different types of tasks the students may work with. The concrete manipulatives, called *nooms*, are a development of the Cuisenaire rods (March, 1977). These concrete manipulatives also appear digitally in the learning labs and quizzes. Each of these manipulatives corresponds to a natural number between 1 and 10, see Figure 1.

Figure 1: The nooms in Dragonbox School

Dragonbox School is supposed to be used according to 'The Dragonbox method'. In the teacher manual (Dragonbox Kahoot, 2021), every lesson is structured according to this method. Each lesson consists of four phases: 1) exploration; 2) discussion; 3) practice; and 4) recapitulation. In the exploration phase, the students explore a digital learning lab individually, with a fellow student, or with the teacher in a whole class discussion. In the discussion phase, the students are given time to think, discuss with a learning partner, and share their reasoning with the teacher and the rest of the class. In the practice phase, it is recommended that the students start to work with the quizzes and then solve tasks in the textbook. In the recapitulation phase, the teacher facilitates a whole class discussion about the learning goal of the lesson and how the activities of the students are linked to this goal.

Theoretical framework

The work of Brissiaud (2016) has been central in the development of Dragonbox School (Uggerud, 2021). Brissiaud argues that solid understanding of a number means having access to its decompositions. This claim is supported by other researchers (Anghileri, 2000; Ma, 2010). With a developed conceptualization of the cardinality of number, e.g., the number eight, a student can easily think of different decompositions of this number: 8 = 7 + 1, 8 = 5 + 3, 8 = 4 + 4, etc. The idea of decomposition and composition led to the construction of the nooms, which play important roles in Dragonbox School. Using a tablet, one can split (decompose) one noom into two smaller nooms by 'slicing' it with the finger, and one can join two nooms (compose) by moving one on the top of the other. Then the bottom noom will 'eat' the upper noom and grow accordingly.

Duval’s (2006) theory of representations is central in our analysis of Dragonbox School. He defined the concept of representation as “something that stands for something else” (p. 103). He stressed that “the ability to change from one representation to another is very often the critical threshold for progress in learning and for problem solving” (Duval, 2006, p. 107). Dragonbox School offers several activities where students have to change from one representation to another. The concepts of treatments and conversions are core elements in Duval’s theory of representations. Treatments are “transformations of representations that happen within the same register” (Duval, 2006, p. 111), e.g., decomposing a 6-noom into a 2-noom and a 4-noom and composing a 5-noom by concatenating a 3-noom and a 2-noom. Conversions are “transformations of representations that consist of changing a register without changing the objects being denoted” (Duval, 2006, p. 112), e.g., transforming the number 6 into the 6-noom. When the students manipulate the dynamic representations in the learning labs and quizzes of Dragonbox School, both conversions and treatments are carried out. Skemp’s (1976) terminology of ‘relational understanding’ and ‘instrumental understanding’ of mathematics will also play an important role in our analysis of Dragonbox School. ‘Relational understanding’ refers to “knowing both what to do and why” (Skemp, 1976, p. 20), and ‘instrumental understanding’ refers to “rules without reason” (Skemp, 1976, p. 20). Relational understanding should, according to Skemp, be the goal of mathematical learning.

Affordances and constraints are two terms often used in mathematics education research in evaluation of teaching materials and learning activities (e.g., Carlsen et al., 2016; David & Watson, 2008). Gibson (1979) launched these terms. He defines *affordances* as relationships between the environment and the animal, in our case the digital environment of Dragonbox School and students. Gibson argues that these affordances exist independently of the user, but the user has to perceive these in order for them to be realized. Additionally, Norman (1988) emphasizes that affordances are linked to cultural conventions. We thus argue that in a digital environment, affordances denote action possibilities offered by a digital tool with respect to the capabilities of the user of that tool. The goals of actions and interactions with a digital tool, the user’s mathematical experience, and the mathematics classroom culture fundamentally informs the user’s perception of affordances. *Constraints* is a term that denotes factors delimiting the user’s actions and interactions with the environment. Such delimitations may support the user to focus at intended mathematical content.

In our study, inspired by Gibson (1979) and Norman (1988), we draw on these constructs and situate them to fit our purpose of analyzing Dragonbox School. Therefore, we use affordances to denote possibilities offered for the user’s actions and interactions with a digital tool that, if perceived, may nurture the development of relational understanding. We use constraints, despite the term’s denotations as to also support users’ attention, to denote the inherently emerging restrictions for the user’s actions and interactions with a digital tool that may delimit opportunities for developing relational understanding.

Based on these ideas and considerations, we will analyze a specific learning lab called ‘The set line’, which will be described below. We have thus formulated the following research question:

> Which affordances and constraints can be identified in ‘The set line’ with respect to how it facilitates treatments and conversions in addition of single- and two-digit numbers?

Presentation and analysis of the learning lab ‘The set line’

We immersed ourselves with the Dragonbox School teaching material and familiarized ourselves with its various resources. Particularly, we became interested in the learning labs due to their visualizations of natural numbers. In this study we analyze a learning lab called ‘The set line’ because it explicitly displays both treatments and conversions (cf. Duval, 2006). We inspected actions made possible by this learning lab, and further analyzed these actions’ affordances and constraints with respect to the task “Compute 18 + 6”. We have chosen this task because it involves the decomposition of 6 into 2 and 4. Such decompositions are a central theme both in the Norwegian curriculum (Utdanningsdirektoratet, 2020) and in Dragonbox School. Moreover, decompositions are fundamental to understanding addition and numbers in general (Brissiaud, 2016).

The insertion of 18 into the two first containers[1]

To carry out the calculation 18 + 6, the student must start by producing the number 18. To do this, the student has to press the grey area on the tube in the upper left corner of the tablet surface, causing three empty fields to appear (see Figure 2). Then, in the last two of these, the student must write 18 using a finger on the tablet surface. Then a 10-noom (black) and an 8-noom (pink) emerge from the tube displayed on top of each other (to the left in Figure 2). Using the terminology of Duval (2006), the digital tool conducts a hidden treatment of the number 18 into the numbers 10 and 8. Then, a conversion of the numbers 10 and 8 into the 10-noom and 8-noom is carried out and visualized. Therefore, this treatment-conversion may nurture the development of relational understanding of the number 18. This treatment-conversion may thus constitute an affordance in the student’s learning process.

To proceed with the calculation, the student has to press the 10-noom and the 8-noom using a finger. When this is done, a funnel appears at the left side of the first container, see Figure 2:

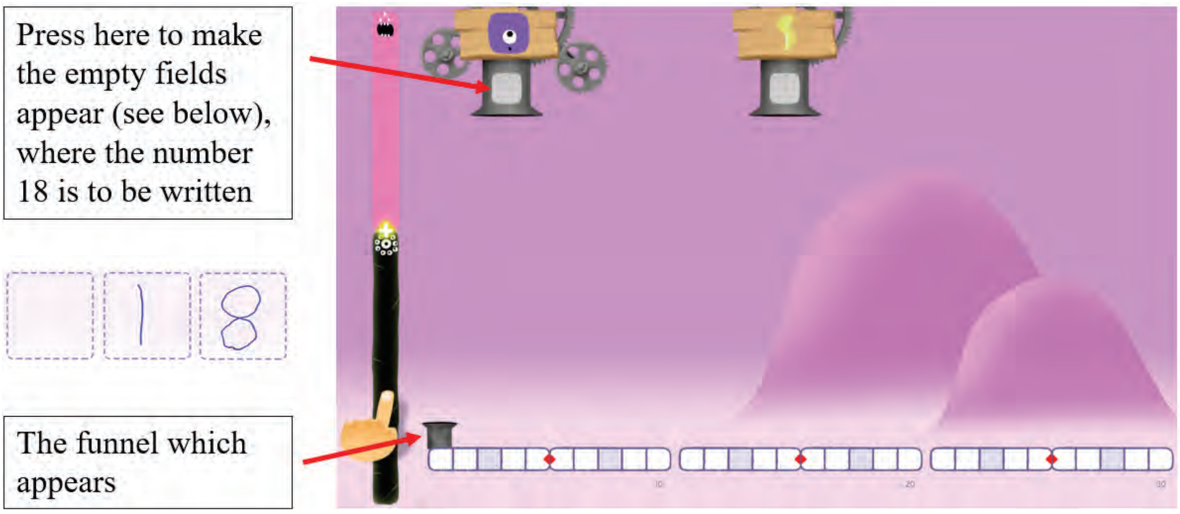

Figure 2: The funnel appears when the 10-noom and 8-noom are pressed. Text boxes and arrows are added by the authors

[1] By “container” we mean one tube with space for ten 1-nooms, as displayed at the bottom of the tool screen in Figure 2.

The funnel is automatically positioned at the correct place, namely at the first vacant cell in the first container. This informs the student about what to do next, that is, to insert the 10-noom and the 8-noom into the first container through the funnel. When the 10-noom and the 8-noom are inserted into the funnel, ten 1-nooms are fed one by one into the container, and a 'thumping' sound is heard as each 1-noom is fed into the container, see Figure 3. Then the feeding-process stops, and the remaining 8-noom is automatically moved to the second container. Using the terminology of Duval (2006), a treatment from one 10-noom to ten 1-nooms is carried out and explicitly displayed by the learning lab. This may nurture the development of relational understanding of this treatment and may thus constitute an affordance in the student's learning process.

When the remaining 8-noom is moved to the second container, a funnel automatically appears at the first empty cell in the second container, see Figure 3. In this way the student is informed about what to do next, namely, to insert the 8-noom into the first vacant cell in the second container. When pressing the appeared funnel, the 8-noom is fed into this container in a similar way as the 10-noom was fed into the first container. Nevertheless, the two automatic actions may constitute a constraint in the student's learning process. The only thing the student needs to do, is to insert the 10-noom and the 8-noom into the funnels. This may be mastered without relational understanding of the composition of the number 18. These two automatic actions may thus delimit the development of relational understanding.

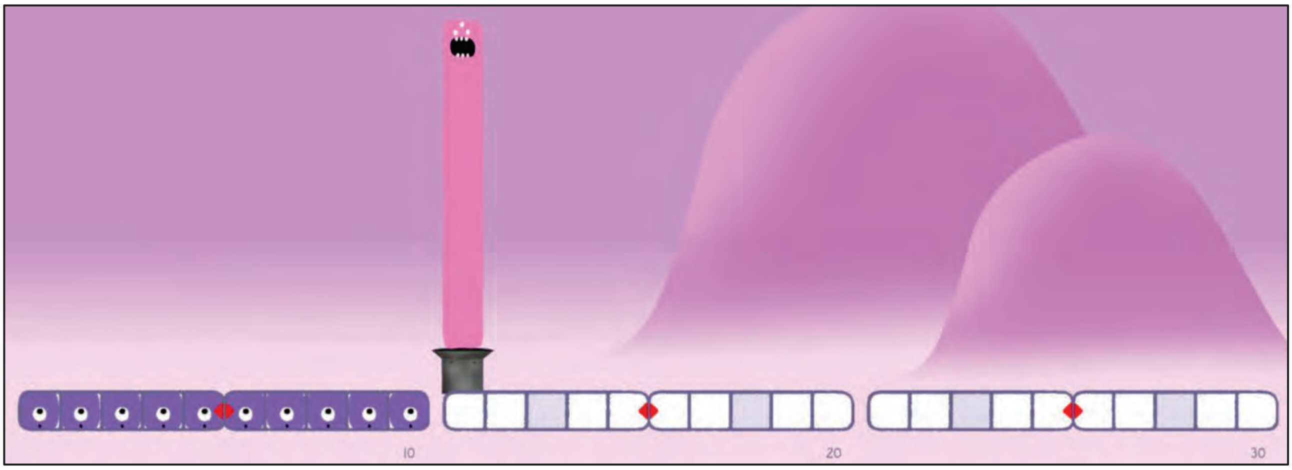

Figure 3: The remaining 8-noom is moved to the second container and a funnel appears

The insertion of 6 into the second and the third containers

To proceed with the calculation 18 + 6, the student needs to press the grey area on the tube once more, as when producing the number 18, and write 6 in the empty fields that appear to produce a 6-noom (orange). To complete the calculation, the 6-noom has to be decomposed into a 2-noom and a 4-noom. This decomposition can be carried out either automatically or manually. We begin with explaining how this can be done automatically. First, the student has to press the 6-noom. When this is done, a funnel with an addition sign automatically appears at the first vacant cell in the second container, see Figure 4.

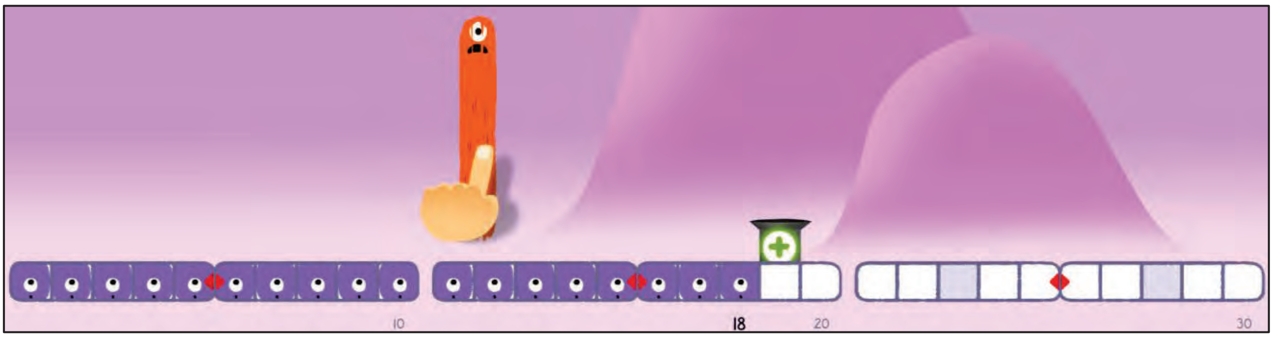

Figure 4: A funnel appears in the first vacant cell in the second container

The positioning of this funnel informs the student to insert the 6-noom into the funnel at the correct place, namely, the first vacant cell in the second container. This is indicating that the second container should be filled first. When the student moves the 6-noom to the funnel and presses the funnel, the 6-noom is starting to be fed into the container, but the process stops after the two first 1-nooms are fed into the container. Then the second container is full, and the remaining 4-noom (green) and the funnel are automatically moved to the first vacant cell in the third container, see Figure 5. Then the student has to press this funnel to insert the 4-noom into the third container.

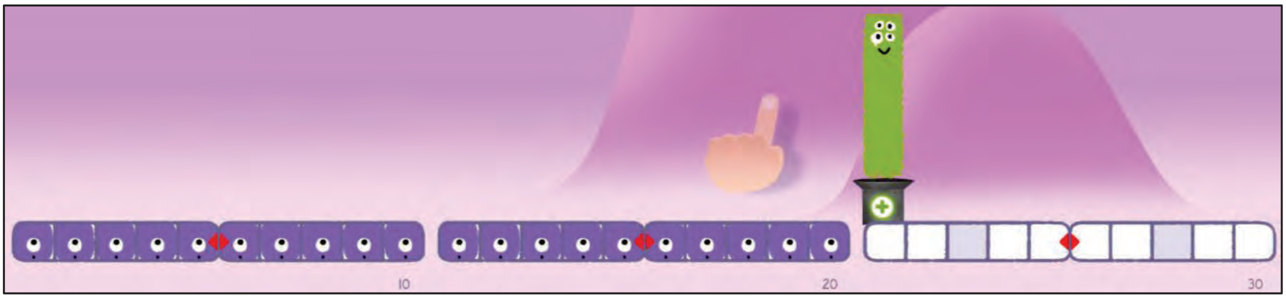

Figure 5: A funnel appears in the first vacant cell in the third container

In this way the student may notice the decomposition of the 6-noom into one 2-noom and one 4-noom. According to a Duvalian (2006) stance, this corresponds to a treatment from one 6-noom to one 2-noom and one 4-noom. This treatment is explicitly displayed, and this may nurture the development of relational understanding of the decomposition of 6 into 2 and 4. This treatment may thus constitute an affordance in the student's learning process. However, the automatic appearances of the two funnels at the correct places, may constitute constraints in the student's learning process. The only thing the student needs to do to proceed with the calculation is to feed the nooms into the funnels. This may be mastered without relational understanding of the decomposition of 6 into 2 and 4. The automatic appearances of the two funnels at the correct places may thus delimit the development of relational understanding.

When the student presses the funnel to insert the 4-noom into the third container, the number 24 automatically appears below the last 1-noom in the third container, see Figure 6. The appearance of this number corresponds to a hidden conversion from twenty-four 1-nooms into twenty-four 1s, which through a treatment become the number 24. This visualizes that the number 24 consists of two containers filled with ten 1-nooms each, and a container filled with four 1-nooms. This visualization may nurture the development of relational understanding of the number 24. Thus, the automatic conversion-treatment from twenty-four 1-nooms to the number 24, may constitute an affordance in the student's learning process.

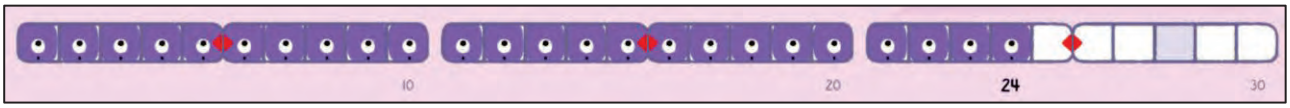

Figure 6: The conversion from twenty-four 1-nooms to the number 24

The decomposition of the 6-noom into the 2-noom and the 4-noom can also be carried out manually by 'slicing' the 6-noom with the finger. If this 'slicing' is done correctly, that is a bit above or below the middle of the 6-noom, the 6-noom will be decomposed into a 2-noom and a 4-noom. Then the 2-noom can be inserted in the second container, and the 4-noom in the third container. This manual decomposition may constitute an affordance in the students' learning process because it may nurture the development of relational understanding of the decomposition of 6 into 2 and 4.

Discussion and conclusion

Our analyses show that 'The set line' may constitute both affordances and constraints with respect to students' learning process. Firstly, we will address the affordances. These relate to how the learning lab may nurture the development of relational understanding by utilizing the dynamic potential of visualizing the addition process. With respect to conversions (Duval, 2006), the learning lab for instance transforms the numbers 10 and 8 into a 10-noom and an 8-noom. With respect to treatments (Duval, 2006), the learning lab visualizes transformations within the noom-setting: how the 10-noom is made up of ten 1-nooms, and how the 6-noom strategically can be decomposed into a 2-noom and a 4-noom in order to utilize the grouping of tens in our number system. Moreover, the learning lab visualizes how addition is executed by adding the second addend from where the first addend ends at the set line and then reading off the final endpoint. Thus, we claim that 'The set line' offers substantial affordances when it comes to visualizing basic number concepts and relations.

In our analyses, we have described operations that are automatically carried out by 'The set line'. These operations may facilitate the learning process because the students are guided regarding what to do next. Therefore, it is likely that the students may be able to operate this learning lab with little or no support from the teacher. This is an important point because one purpose of the learning labs is to enable users to explore mathematical concepts and relations on their own. However, the operations that are carried out automatically, may constrain students that would benefit from conducting these operations themselves. Moreover, these automatic operations may enable students to solve the tasks at hand without having relationally understood these operations. The students may write the numbers provided by the tasks without reflecting on the meaning of the numbers, and they may move the available noom(s) to the nearby funnel without reflecting on why this should be done and without paying attention to the ongoing visualizations. Furthermore, if the decomposition of the 6-noom into the 2-noom and 4-noom is carried out automatically, the students may be constrained from deciding on how the decomposition of the second addend is to be carried out. The operations that are automatically carried out may thus deprive the students of opportunities for learning.

We want to point out that if 'The set line' is used according to the Dragonbox method, the impact of the constraints we have described, may be significantly reduced. This method strongly recommends teachers to nurture student reasoning concerning the inherent mathematical content of the learning labs and quizzes. The Dragonbox method is explained in the teacher manual, and it is also taught in courses for the teachers who use Dragonbox School. Nevertheless, to inform and teach about the Dragonbox method do not necessarily prevent students from using the learning labs and quizzes without achieving relational understanding.

Our conclusion is that 'The set line' may offer both affordances and constraints in the users' learning process. The students' outcome is to a great extent dependent on teachers facilitating student reasoning about the dynamic representations and the inherent compositions and decompositions (cf. Brissiaud, 2016) of the learning lab. Further, empirical research is needed that investigate whether the identified affordances and constraints are experienced as such by students who engage with the learning labs.

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Instrumental orchestrations by a university teacher in an online linear algebra course

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In this article, we present an instrumental orchestration for online math education. The case study is of a university teacher in a linear algebra course with engineering students in a remote learning setting. We used the instrumental orchestration approach as a theoretical framework for planning and organizing the artifacts involved in the environment (didactic configuration) and the modalities of their implementation (mode of exploitation). The activities were designed using virtual manipulators, guided exploration sheets, and video recordings of individual or paired work by the students. The results of the observation of five sessions are presented and the orchestrations of a pedagogical sequence to introduce the concepts of value and eigenvector are briefly discussed. This work suggests new instrumental orchestrations for the teaching of mathematics online.

Keywords: Instrumental orchestration, linear algebra teaching and learning, university mathematics, online course.

Introduction.

In March 2020, the World Health Organization declared a global pandemic due to COVID-19 and recommended lockdowns and mobility restrictions, forcing educational institutions to migrate from their face-to-face teaching model to online education (Engelbrecht et al., 2020). Most institutions had to improvise online courses using various digital devices and internet platforms. Although remote education has been developing for years in various forms –distance learning, e-learning, and hybrid learning– to reach more students (Silverman & Hoyos, 2018), it had not fully permeated most educational institutions. Oktaç (2004) reports the implementation of a remote introductory linear algebra course in an asynchronous learning environment. Teaching linear algebra online is a challenge, as it is one of the first abstract math courses students encounter during their early years in college. At ICME13, there was a discussion group on teaching and learning linear algebra, which considered the ways a linear algebra course could be adapted to meet the needs of students from other disciplines, such as engineering, physics, and computer science. It also reflected on how technology should be used in teaching linear algebra (Stewart et al., 2018, p. ix). Gol-Tabaghi and Sinclair (2013) studied the geometry of eigenvalues and real eigenvectors of a 2×2 matrix using dynamic geometry software (DGS), in which by dragging the vector x onto the screen, the vector Ax moves accordingly. They concluded that dynamic geometric representations in the sketch enabled students to understand the concepts of eigenvector and eigenvalue by identifying their invariant geometric properties and helped them develop dynamic-synthetic-geometric thinking.

On the other hand, and perhaps in anticipation of the current circumstances, Trouche (2004) introduced the notion of *instrumental orchestration* for planning a lesson, defined as the didactic management of the different artifacts available in the classroom, the organization of space and time, and the way the artifacts are connected.

Theoretical background

An instrumental orchestration is “the intentional and systematic organization of the various artifacts available in a computerized learning environment by the teacher for a given mathematical situation, in order to guide students’ instrumental genesis” (Drijvers & Trouche, 2008, p. 377). An instrumental orchestration is defined by a didactical configuration and the way in which this configuration is exploited. The Sherpa-student orchestration, which involves a student whose computer or calculator is displayed to the rest of the class on a projector, was for several years an emblematic didactic configuration. For a given configuration, there are several possible operating modes. As an instrumental orchestration is partly prepared in advance and partly created "on the spot" during teaching, Drijvers and colleagues (2010) add a third level called *didactical performance* to account for the multiple adjustments that the teacher and students may make. Drijvers and colleagues (2010) suggest considering the triplets model, a sort of jazz ensemble composed of beginners, advanced musicians, and the teacher as a conductor who prepares joint participation, but who is open to the pupil’s improvisation and interpretation, on order to recognize the contributions of each of these levels. This metaphor became a reality in our experience because there was no precedent for online teaching of linear algebra, so we had to "improvise" ways and methods to teach both the mathematical task and the technological tool, for example, the forms of student-student and student-teacher through communication through the platform. Drijvers and his colleagues have extended the repertoire of instrumental orchestrations to include technical-demo, explain-the-screen, link-screen-board, discuss-the-screen, spot-and-show, Sherpa-at-work (Drijvers et al., 2010), board-instruction and guide-and-explain (Drijvers et al., 2013). In the first three orchestrations is it mainly the teacher leading the communication, while in next three students have more opportunities for participation. In the technical-demo orchestration, the teacher explains the basic techniques of a tool in order to familiarize students with it.

In the explain-the-screen orchestration the teacher explains what is happening on the computer screen, but unlike the previous one, this orchestration involves mathematical content. Technical-support orchestration refers to the process of helping the learner with technical issues with the tool, such as software and/or hardware issues. Discuss-the-screen orchestration involves a group discussion about what is happening on the computer screen. In the link-screen-board orchestration, the teacher refers to the relationship between what the tool displays and the mathematical concept. In spot-and-show orchestration, the teacher chooses a student's work before class, either for the student to explain in class or for discussion with the whole class. For the Sherpa-at-work orchestration, the teacher asks a student (referred to as the Sherpa) to demonstrate or perform a certain process in the technological environment (for details, see Drijvers et al., 2013). Thus, our research question is: What instrumental orchestrations are applied by a university professor using technology to teach the concept of eigenvalue and eigenvector in a distance education setting?

Method

The study was developed under a design-based research methodology, which consists of three research phases: preparation and design, teaching experiment and, retrospective analysis (Bakker, 2018). A first research cycle (Orozco-Santiago, 2020) was carried out in face-to-face teaching. For the design of our task sequence, we used a hypothetical learning trajectory (Simon, 1995) but, due to the global public health emergency caused by COVID-19, circumstances compelled them to use the Zoom platform for the first time for online teaching. However, the teacher has extensive experience in the use of digital resources. The trajectory consists of seven activities. During the second cycle of research, the teaching experiment was carried out with a small group of ten students (6 women and 4 men) in their second year of engineering studies at a Mexican public university, over a period of two weeks. When designing the didactic path of our work, we were able to detect a priori the use of the following orchestrations for the teacher (Figure 1), and each activity for the pupils was accompanied by a didactic interactive virtual environment (DIVE) and a guided exploration sheet.

	Session 1		Session 2		Session 3		Session 4		Session 5	
Technical-demo	✔		✔		✔	✔				
Guide-and-explain										
Link-screen-board,			✔							
Discuss-the-screen	✔									
Explain-the-screen	✔			✔	✔	✔		✔		✔
Spot-and-show,				✔	✔			✔		✔
Sherpa-at-work,				✔	✔			✔		✔
Board-instruction	✔		✔	✔	✔		✔			
Technical-demo individual,										
Guide-and-explain individual										
Link-screen-paper,					✔				✔	
Discuss-the-screen individual,										
Technical-support individual,										
Individual Work		✔	✔	✔		✔			✔	
Practicals in pairs						✔	✔			

Figure 1: Orchestration suggestions for teachers

The design of a configuration depends on the technological environment (Figure 2), and in this case the students had laptops or mobile devices (smartphones, tablets). However, the hardware and conditions available to them for remote study were less than ideal. Internet connections were quite weak, there was noise at home and outside, and due to pandemic conditions, students were prone to stress. Most students could not use their cameras or microphones during courses because these require more bandwidth, which many did not have. In the course, the following topics had already been taught: matrices, systems of linear equations, vector spaces, linear combination, linear independence, and linear transformations. The participants had not previously studied the concept of eigenvalue and eigenvector.

Throughout the course, activity data was collected by the following: a) e-mail; b) WhatsApp; c) Google Drive; d) the Zoom platform; e) free software for video recording OBS Studio; d) digital capture of students' notebooks; e) video recordings of the researcher's classroom activities on Zoom; and f) video recording of student work and teamwork, in the DIVE and guided exploration sheets. Five of the lectures on the concepts of eigenvalues and eigenvectors were recorded on video. To answer the research question, after the end of the teaching experiment, the five video recordings of the class were observed and the data analysis presented in this article focused on the identification of the orchestrations proposed by Drijvers and colleagues (2013), in the first three video recordings. In the video recording of session 1 (S1), 12 different episodes were identified. In S2 19 and in S3, 21.

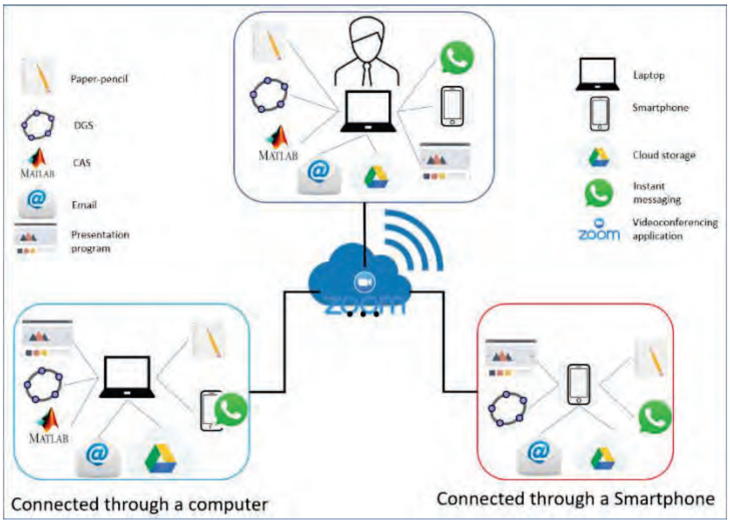

Figure 2: A remote configuration

Results

The instrumental orchestrations reported by Trouche (2004) and Drijvers et al. (2010, 2013), were developed within the framework of face-to-face teaching where the teacher and the students are in a classroom. But because of lockdown requirements during the pandemic, this teaching experience was developed synchronously online, which allowed for the use of some orchestrations and required the improvisation of others. For example, the technical-demo orchestration was used for the students to familiarize themselves with the Zoom software, offering instructions on how to use meeting control.

Since in the virtual model, the Work-and-walk-by orchestration was not possible, and we wanted to observe the individual instrumental genesis developed by the students and at the same time discuss as a group the correctness or not of the answers and the work of each student, the modality that we called *Student-shared-screen* appeared, which allowed the teacher to accomplish the task, it should be noted that the name, the peculiarity, and the characteristic of this orchestration are made in the analysis after the experiment with the data obtained described in Figures 1 and 2. As this orchestration was improvised, when we analyzed the data and using the taxonomy defined by Drijvers and colleagues, they mention that the didactic configuration for the Spot-and-show orchestration "includes access to the students' work in the technological environment during lesson preparation" and for Sherpa-at-work orchestration, "includes access to the technology and projecting facilities, preferably access to student work" (2013, p. 999), but to implement access to the pupil's screen, it is necessary to ask the student's permission. Not having found an orchestration for this type of activity which we believe is important in online education, we call it the new Student-shared-screen orchestration, which consists of a didactic configuration in which the student allows the whole class to access his work on his computer screen. As an exploitation mode, the teacher asks the student to share his screen and discuss his work with the whole group. As can be seen, this is an orchestration prior to Sherpa-at-work and spot-and-show.

To simulate chalk, the teacher chose to use the Zoom Annotation tool and use a blank PowerPoint slide or a DIVE as a whiteboard, on which the teacher and students could write in order to present, explain and solve the exercises. We will call this the *Share-board* orchestration, and it is a recurring didactical configuration very similar to traditional classroom teaching (see Figure 3).

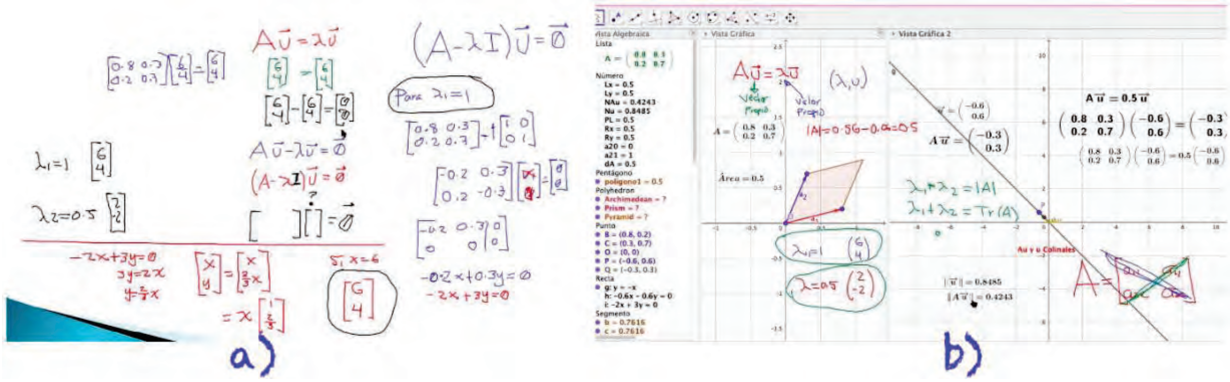

Figure 3: a) PowerPoint and b) DGE, both used as a whiteboard

In the third task, students had to drag two vectors onto $\mathbb{R}^2$ in the DGS and observe their relationship to the columns of a 2×2 matrix. The teacher identified an inappropriate response from the video recordings: some students did not drag the vectors, but directly changed the values of the matrix in the algebraic view. Through to the student-share-screen and spot-and-show orchestration, the teacher decides to ask the student to explain to the whole class how he modified the matrix,

Teacher:	The coordinates of the vector a: what relation does it have with the matrix A?
Student1:	None.
Teacher:	When you drag vector a1, what happens to matrix A?
Student1:	Nothing happens to it.
Student2:	Changes the values a11 and a22 of the matrix [this answer is provided by the chat].
Student3:	The values in the first column change.

Students who did not drag the vectors and directly modified the values of the matrix failed to identify the relationship between the vectors and the columns of matrix A. This didactical performance was observed thanks to the design of the DGS.

For the fourth task, the teacher used the explain-the-screen and share-board orchestration, using the work of one of the students as a mode of exploitation. The selected student's work was the criteria for the first work to be uploaded to Google Drive (see Figure 4).

For the next activity, construct matrix A.
Drag the point **P** (vector $\vec{u}$) and position (if possible) the vector $\vec{u}$ collinear with the vector $A\vec{u}$ and note its coordinates.

$\vec{u}$	$A\vec{u}$	λ Proportionality ratio between $A\vec{u}$ and $\vec{u}$
(6,4)	(6,4)	1
(−2,2)	(−1,1)	0.5
(−2.5122, −1.6682)	(−2.5102, −1.6702)	1
(4,−4)	(2,−2)	0.5
(2,−2)	(1,−1)	0.5
(−4,4)	(−2,2)	0.5

Figure 4: Search for collinearity of the u-vectors and the Au-vector

Thomas and Stewart (2011) point out that textbooks show $A\vec{x} = \lambda\vec{x}$, then $A\vec{x} = \lambda I\vec{x}$, but do not explain that it is $\lambda I\vec{x} - I$ is a matrix and not a scalar. Using the *board-instruction-tech* orchestration, the teacher explains how the equation $A\vec{x} = \lambda\vec{x}$ is transformed into $(A - \lambda I)\vec{x} = \vec{0}$ with student participation (see Figure 3a). In another activity, the teacher began with explain-the-screen orchestration, providing students with a DIVE that was not considered in the learning trajectory. After having carried out the calculation of the eigenvalues and the eigenvectors on the screen, the teacher explained to the students that in the CAS view of the DGS, they could observe step by step the calculation of the characteristic polynomial, of the roots of the characteristic polynomial (eigenvalues), and the solution of the equation $(A - \lambda I)\vec{x} = \vec{0}$ for each of the eigenvalues (if they exist). After each line of the CAS was explained, a didactical performance was presented. Here the students commented that on their computers, the IME given did not work like the one the teacher presented, and the CAS did not recognize the same commands as those of the teacher (see Figure 5a). Using student-shared-screen orchestration, a student split his screen to perform and follow the teacher's instructions, and the student's computer screen froze. The teacher had to ask another student to share his screen. To solve the command problem, the teacher was able to use his technological knowledge to search the GeoGebra Wiki and find another command which made the DIVE work correctly on the students' computers (see Figure 5b).

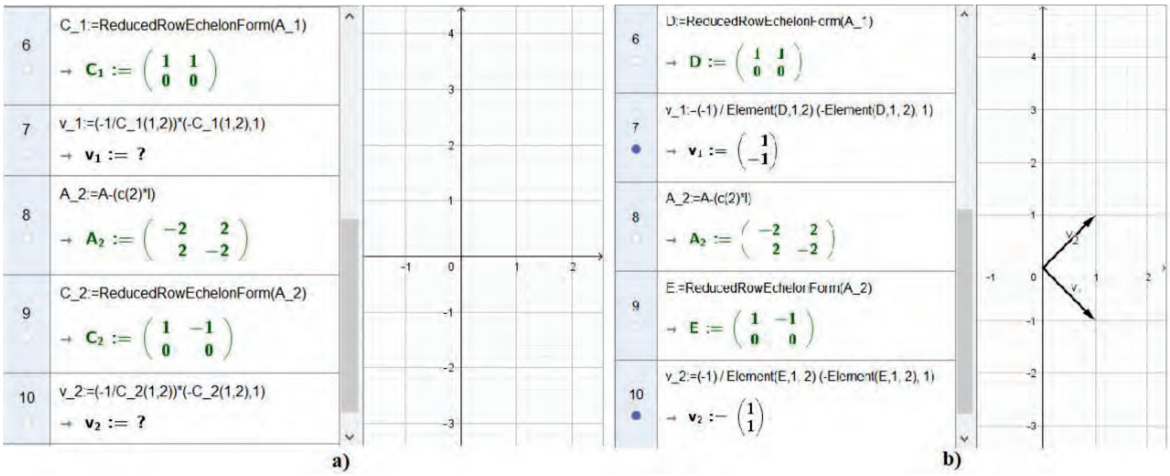

Figure 5: Calculation of eigenvalues and eigenvectors in the CAS view of the DGE

Discussion

For the student-share-screen orchestration, a didactical configuration for this orchestration is the sharing of the videoconference screen by a student, which allows the students to follow the discussion, using Sherpa-at-work or spot-and-show. As an exploitation mode, the teacher can ask the Sherpa-student to explain her work, or he can ask questions and ask the student-Sherpa to perform specific actions. In our case, when a student wanted to share their paper-and-pencil work, there were several exploitation modes. 1) A student with a weak internet connection could take a photo of their notebook using their smartphone, upload it to the WhatsApp group, their cloud folder, or conference chat, so that the teacher or another classmate could download and project their work while they explained it. This way we can define this orchestration as *Shared-resource*. But because of the weak internet connection, the gestures of the teacher and the students were overlooked.

The videoconferencing platform used allows for the creation of small work teams (Breakout Rooms) assigned either randomly using the app, or by the teacher. It also allows the teacher to move among the teams to observe their work and, on the same platform, the students in each teams can ask for help from the teacher (Ask for Help). A downside to this option is that if multiple teams ask for help with the same issue, only members of the team visited by the teacher will be able to hear and see the discussion.

Conclusion

Supporting mathematics teachers in their efforts to integrate technology into their teaching practice remains a challenge for the mathematics education community (Trouche, 2018). We believe that teachers need basic skills and knowledge in the use of technologies so that they can support their students' learning in these online environments.

Returning to our research question: What instrumental orchestrations are applied by a university professor using technology to teach the concept of eigenvalue and eigenvector in a distance education setting? As technology evolves, existing types of instrumental orchestrations (Drijvers et al., 2010, 2013; Trouche, 2004) need to be re-examined to determine how they might be modified and expanded. We examine the teaching practice of a university engineering professor during the onset of the COVID-19 pandemic through online education, using the theoretical framework of instrumental orchestration. Students contributed through video recordings of their work and ideas, which allowed the teacher to access this work and decide between Sherpa-student or spot-and-show orchestrations from this work. Some orchestrations from traditional education were also observed. In this observation, we identified four new types of instrumental orchestration: student-share-screen, share-board, board-instruction-tech and shared-resource.

In this course, being able to observe the personal development on paper-and-pencil by the whole class made the students more willing to share their incomplete attempts at problem-solving. Videoconferencing technologies offer means of communication that raise new questions regarding the organization of lessons. The pandemic has raised new questions about synchronous teaching: the use of computer screens and webcams instead of blackboards, the ability to carry out multiple activities such as Chat watching, student participation by raising their hand, muting and unmuting the microphone, solving technical problems on the fly. The activities developed in this study can be implemented in face-to-face classes, encouraging autonomous and collaborative work, and installed on a freely accessible server where students can learn at their own pace.

The role of the smartphone as a computer has changed the way we can experience mathematics (Borba, 2021). We believe that the impact of the pandemic has changed the traditional way of teaching and learning, which, together with the ever-increasing demand for study and the limitations of institutions, makes hybrid distance learning a necessity. It is, therefore, necessary to carry out studies on the design and implementation of educational software on digital devices and, of course, new orchestrations.

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The use of auditory media as mathematical language support in elementary classroom practice

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This paper focuses on the use of auditory learning material as specialized language support in elementary classroom practice. Radio features, with their relevance in everyday life, can serve as a natural auditory learning material for children. Thus, we investigated various applications of radio resources in teaching practice. The aim of this research is not only to investigate what a profitable use of auditory material would look like, but also to investigate the effects of such practice on the learning procedure in general and specifically on the development of the school register. First results show that auditory media can indeed be effectively implemented in teaching practice as a provided linguistic model and reveal just how these effects look like.

Educational media, aural learning, radio, language acquisition, visualization.

Cooperation with a Radio Station

In 2015, the Department of Mathematics Education at the University of Giessen started a project in cooperation with a regional radio station *hr2 – Hessen Radio for Culture*. Within this project, a series of radio broadcasts on mathematical topics for the primary school level was developed – calles *Kinderfunkkolleg Mathematik*[1], as well as accompanying material for use in elementary classrooms. More information about this can be found in the proceedings of CERME11 (Peters, 2019). Based on this project, research on innovative classroom practices was started that integrate auditory learning material as a central element to find out what a profitable use of such material could look like. Furthermore, the effects of auditory media as a specialized language support in mathematics education were investigated. In particular, the question was asked how auditory material can stimulate the development of the school register as a linguistic model.

The Potentials of Auditory Media in Mathematics Education

According to Mayer's memory model (2009; see also Baddley, 2007), visual and auditory information are processed in two different sensory channels. Using auditory media as learning material reduces the sensory impressions on the visual channel and focuses the sensory impressions on the auditory channel. Hence, the auditory channel becomes more tasked and can be trained more. Purposive use of auditory media can supply the listener with repeated and specific auditory input and thus foster the processing of information in the auditory channel. Based on this, the Purposive use of such media can furthermore foster the development of language (Niehbuhr-Siebert & Ritterfeld, 2012). Building up on the memory model, the Cognitive Load Theory by Sweller (1994) says that the capacities of the working memory are limited and should not be exhausted by extrinsic factors like too many animations or reading. Reading difficulties can exhaust the working memory and lead

[1] www.kinderfunkkolleg-mathematik.de

to not understanding mathematical content as well as not being able to solve mathematical tasks. Keeping mathematical concerns in the center of learning processes while reducing extrinsic factors is one important principle of auditory learning. For this reason, Rink and Walter (2019) are using text-to-speech technology (TTS) to support learners with difficulties in the written language while solving written tasks. Thus, it can be stated that auditory support can aid children with reading difficulties to understand mathematical contents and tasks without the need of having to read coherently and extract the meaning. The absence of a visual representation of the subject can be a learning potential as well. Children are cognitively challenged to develop their own visual representation of what they heard. When working with auditory media of mathematical content, students have to develop visual representations of e.g. mathematical concepts, geometrical shapes or figures (Peters, 2021).

In pandemic times, we also made first experiences of using auditory media in distance learning and developed accompanying material for homeschooling parents as well as for teachers that needed educational material suitable for distance teaching and learning.[2] There are a lot of good radio resources that can be found online and be used as educational material. So, good auditory learning material is very easy-to-access and all it takes for a teacher to get it to his students in distant learning is to send them a link. A very easy and useful tool - if implemented intently and well-thought-out i.e. if criteria for choosing and using auditory media are considered. Those criteria will be explained below.

Potentials of Auditory Media Concerning Language

One of the most obvious potentials of auditory media is the development of active listening skills. Such competence is a primary requirement for education, but it is seldom supported or even trained (Pimm, 1987). Another potential can be described by looking at the *Model of Orality and Writtenness* (Koch & Oesterreicher ,1985). In this model, auditory learning materials can foremost be categorized as being medial-oral. If they are designed for children, they are also more likely conceptual-oral because they utilize everyday language and explain in a child-orientated and situational manner. Still, they use mathematical terms and phrases of educational language. Thus, they are also characterized through conceptual-textual elements and – in the right learning environment – can be able to lead children from orality into writtenness. At best, this can possibly guide listeners to use mathematical terminology in orality as well as in writtenness. This makes them act in a more conceptual-textual way.

Prediger and Krägeloh (2016) are referring to a model of three registers relevant for mathematical learning (everyday register, school register and technical register). School register is an important factor for successful learning in mathematics, as it is a shared language basis and helps with explaining, describing and justifying (Götze, 2015). However, children do not bring this type of language to school with them. It must be learned. Thus, linguistic models are needed to develop educational language and to fill terms with representations. These linguistic models are scaffoldings

[2] https://www.kinderfunkkolleg-mathematik.de/unterrichtsmaterial

onto which children can lean (Gibbons, 2002). Here, auditory educational material could be a profitable addition.

Criteria for Choosing and Using Auditory Media

Exploiting the potentials of any media starts with choosing “good” media. Considering auditory media, conditions of success are necessary in order to support an active processing of what has been heard and then to achieve an output that is correct in terms of content and language. Thus, criteria for student-friendly audio offers must be established as well as methodological notes for good embedment in classroom practice. For example, listening to auditory resources should be combined with (listening) tasks to support selective and comprehensive listening. Also, opportunities to document results should be given. In accordance with the segmenting principle (Mayer, 2009), important sequences should be cut and listened to in segments. Those segments can then be repeated and talked about to make sure the brief acoustic representations are understood by every student. By these means, not only an effective use of auditory media in general but also specialized language support can be ensured.

As there is no visual representation of the subject given by auditory media, verbal explanations in such media need to be very accurate. While working with the production team of the radio station, the importance of a deep reflection and good structuring of the mathematical content became very clear. Verbal explanations in auditory media must also counter the fugacity of spoken language through e.g. linear representation, reputation or sound effects. With those criteria kept in mind, teachers will firstly be able to choose good auditory media for use in mathematics education. And secondly, they will be able to develop teaching concepts in which auditory media serves as transfer of mathematical knowledge, as stimulus in the sense of educational reduction and as language support for developing the academic register.

Methodological Approach

Following these concepts, my research can be focused on the evaluation of the use of auditory media for mathematics education in various settings – particularly regarding possible learning effects. Thus, my research issues can be stated as follows:

> RQ 1: How can auditory material support mathematical understanding as well as the development of school and technical register?
>
> RQ 2: What could a profitable use of such teaching concepts look like?

Building on the knowledge base described above, two teaching units consisting of four lessons on the topics *The House of Quadrilaterals* and *Probability and Random Experiments* were examined. In both units, auditory media was used as the central knowledge-imparting medium - the radio features *Wann ist ein Spiel fair?* (When is a game fair?)[3] and the radio feature *Wer wohnt im Haus der Vierecke?* (Who lives in the House of Quadrilaterals?)[4]. The teaching units were carried out in two

[3] https://www.kinderfunkkolleg-mathematik.de/themen/wann-ist-ein-spiel-fair

[4] https://www.kinderfunkkolleg-mathematik.de/themen/wer-wohnt-im-haus-der-vierecke

fourth-grade classrooms. The lessons were videographed, transcribed and categorized using the qualitative content analysis according to Mayring (2015). To develop the aspects of interpretation, the categories, as near as possible to the material, the procedure of inductive category development (Mayring 2015) was used to analyze the data with the software MAXQDA. After formulating a criterion of definition based on the theoretical background and the research questions, categories could be developed which determined the excerpts of the textual material considered for more detailed analysis. In this study those categories are the aspects of mathematical learning as well as the aspects of register development (RQ 1). All of the collected data was worked through and categories were deduced as well as revised in various steps. The detected categories were reduced to main categories and categories relevant to the research issues described above were determined. Subsequently, a selection of scenes for the detailed analysis was made from those relevant categories. Those scenes were interpreted in detail in the sense of interaction analysis (Krummheuer/ Naujok, 1999) to examine the specific research subject and to interpret not only the research itself, but also the specific situation of investigation – between the interaction partners and within their interaction.

All of the radio features from the *Kinderfunkkolleg Mathematik* are of 10-12 minutes length. The features are often built around a story line which includes a mathematical problem or situation and involve speakers as well as protagonists that use school register and mathematical terms to confront children with new terms in a playful way. The overall aim of the units is the development of concepts like *fairness* or *the house of the quadrilaterals,* while the linguistic aim is the understanding and the use of important terms that are presented by the radio feature. Those aims work together: language development can help to develop the concept. Working with the auditory media, the classes begin to listen to the first parts of the radio features. In class conversation, students repeat the content of what was heard and discuss the respective mathematical problem. Then, they imitate experiments or work on tasks and problems they heard about in the radio feature. In between those steps, the teachers present more parts of the radio feature and a lexical storage (posters with sentence phrases and guidelines for formulations as well as the necessary vocabulary) (Erath et al., 2021) is collectively developed based on the content of the radio feature. The units also involve tasks in which the students have to transfer their acquired knowledge whilst using the structures of language and reasoning they had been offered and trained while working with the radio feature.

Mathematics Register Acquisition (MRA)

To evaluate the teaching quality and the effectiveness of the use of radio resources in those teaching units, the *Model of Mathematics Register Acquisition* will be of help (Meaney et al., 2012, p.199). The MRA-model is divided into four stages: The first stage is *Noticing*. In this stage, teachers introduce new terms or expressions, use them frequently and then encourage students to using them as well. The second stage, *Intake*, describes the process of understanding. Students start to explore and work with the new terms. In the next stage titled *Integration*, testing, feedback and modification takes place. Students have a good understanding of the new term and are responsible for using it, but might be supported or reminded of their knowledge by the teacher. In the last stage, *Output*, there is a fluent use of the new terms. Teachers do not need to support, but should provide activities where the use of these terms would arise naturally. Auditory media can be of use in the first two stages. It can be used to introduce new terms and to repeat them frequently in the stage *Notice* and can serve as a linguistic model in the stage *Intake.*

First Findings

Initial observations of those interactions showed that students were highly concentrated while listening to the radio features. It can also be stated that nearly every student was able to correctly repeat the things that have been heard. If anything was unclear, it was easy for the teacher to repeat a certain part of the radio feature. This way, auditory material can counteract transitory learning and be a relief to the teacher as instructions and knowledge are proportioned between him and the auditory learning material.

The qualitative content analysis led to the identification of twelve categories which could be subsumed into three main categories:

- Mathematical-conceptual aspects
- Aspects of language acquisition
- Aspects of media didactics

Exanimating the relative frequency of occurrence of these categories showed that the aspects of language acquisition make up a significant proportion of the collected data. Thus, further in-depth examination of this area should be of main interest. On the basis of the quantified data and with a view to the research issues, a well-founded selection of transcript excerpts – mostly from this main category – was made for a subsequent detailed analysis in order to examine the influence of audio media more closely. The interaction analysis (Krummheuer & Naujok, 1999) then enabled a deeper insight into individual processes of technical language acquisition. Detailed analysis of interactions showed e.g. that the combination of listening, repetition of specific sequences as well as conversations and discussions of what was heard were of great support for teachers – especially whilst introducing new terms and developing the lexical storage. In terms of scaffolding, auditory materials served as linguistic models on which students were able to lean on. Furthermore, it has become evident that students demanded the visual level of what is heard (for detailed analysis see Peters, 2020). Although sometimes the teacher asked for something written, students developed a visual representation (e.g. by drawing) to support the visual work they were already doing in their head. Even if they didn't draw something (for example a geometric shape) they developed a mental representation without the help of a given image. This can be illustrated by the following example from the data:

01	Teacher:	So who remembers what they said in the audio about those special quadrilaterals?
02	Max:	Well, there was the convex quadrilateral where all the corners show to the outside… and then concave quadrilaterals…
03	Teacher:	Can someone help? What about the concave quadrilaterals?
04	Lea:	There the corners show to the inside… they are like inside the quadrilateral.
05	Teacher:	Okay… and how can you picture this?
06	Lea:	They said something like it's punched inside…
07	Teacher:	Yes.. Silas?
08	Silas:	Like an navigation arrow or boomerang.
09	Teacher:	Great that's correct. Can someone sketch how a concave quadrilateral could look like? Joella?

10 Joella:		Well… Maybe like the roof of a house but a roof that is a little bit thicker than normal.
11 Teacher.		Can you make a sketch on the blackboard?

After this Joella goes to the blackboard and draws the following sketch:

Figure 1: Joella's sketch of a concave quadrilateral

So even without ever having been confronted with the concept of convex and concave quadrilaterals and even with just the auditory (i.e. verbal) information and no image of such quadrilaterals, Joella was able to develop a visual representation of what she heard in the auditory resource. She was confronted with only the auditory information and therefore challenged to listen carefully, understand the explanation of the geometric shape – in form of technical register – and furthermore to make up here own visual representation. This cognitive challenge was clearly initiated by the need of visualization which emerged from the absence of visualization in the learning material. Thus, it can be stated that listening to auditory media without the direct provision of visual support led to a development of representations that was initiated by the learner himself.

Considering the language aspect, detailed analysis of interactions show that in the beginning of the teaching units most explanations of the students could be considered as a form of everyday register. However, during and after working with auditory media there was a visible change towards more school and technical register. At the end of both units (e.g. during the final presentations of their experiments, work or tasks), students did in fact use new mathematical terms and phrases that were introduced by the auditory material. Thus, referring to the MRA model (Meaney et al., 2012), it can be stated that there is a development from the first introductory stage *Noticing* to the second stage *Intake* (the process of understanding) in many of the analyzed sequences. Even the third stage *Integration* (testing and modification through feedback) is reached by many students.

Discussion

Despite those positive aspects of mathematical language development there are some limitations that need to be stated. To reach the stage of fluent *Output* students would still need more support and a longer period of use and training. Here a long-term study would be useful. However, the development from stage one to three – from *Noticing* to *Integration* – indicates an apparent improvement in the students' mathematical expression and understanding over the course of the units. The offer of professional language in (educational) auditory media combined with the absence of visuals, gestures and deictics (unlike YouTube videos etc.) might be a challenge – but a positive challenge – and represents an interesting opportunity for the development of mathematical language and thinking. In the next step of this study, the use and effects of auditory learning material is to be examined in grades two and six to get a broader view on the various aspects of mathematical learning and language development in different age groups.

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Fair games – A case study on a negotiation of meaning on the concept of fairness in probability in mathematics classes

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This paper describes a student's negotiation of meaning about the concept of fairness in a 10th grade mathematics class. Using the case study approach and the approach of empirical theories, key scenes in a learning unit about gambling are examined and knowledge structures are analyzed. A 3D printed dice tower plays a special role and is used to reflect on fairness in probability calculation.

Keywords: Empirical theories, knowledge development process, case study, empirical-oriented mathematics classes

Introduction and motivation.

Fairness is a much-discussed concept in various contexts. It often seems as if everyone has clear beliefs of fairness and "whether something is fair". Fairness for us people have quite different facets (Möbius, 2018). So we can say we want everyone to be rewarded equally. Or we can say that the person who has done more gets more. But we can also say we want those who have higher needs to get more (Möbius, 2018). A frequently made statement is "That is unfair!" and not "That is fair!" But how can we decide whether something is fair? Questions of fairness and justice (used synonymously in most contexts, e.g. Duden online, 2021) are dealt with by different disciplines. The concept of fairness thus has many facets, even beyond a simple mathematical distribution model (Möbius, 2018). In mathematics, the concept of fairness plays a crucial role. Looking at the work "An Introduction to Probability Theory and Its Applications" by Feller (1968), we can see that a "theory of 'fair' games" (Feller, 1968, p. 248) is treated here. Textbooks also address fairness for mathematics education. Often we can read: A game is said to be fair if the expected value for winning is 0. The expected value seems to be important in connection with fairness. A fair game is often discussed in connection with probability calculation. The Common Core for High School: Statistics & Probability. Using Probability to Make Decisions also states: "Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator)" (Common Core State Standards Initiative, 2021). This paper describes what beliefs of fairness can be analyzed in a student (10th grade of a high school) of our case study and what discussion impulses on "fair" and "unfair" can be gained from it. For this goal, the teaching unit "A fair game of chance?!" was developed (section: Material and data collection).[1]

Theoretical background – empirical-oriented mathematics classes.

If students acquire mathematical concepts by dealing with empirical objects (visual materials and physical objects, e.g., folded and drawn figures) in the constructivist sense, we assume according to the approach of empirical theories (Burscheid & Struve, 2020a; 2020b) that they develop a so-called empirical belief system (Schoenfeld, 1985). We describe the knowledge of students in empirical

[1] We note that parts of the results of this paper have also been published in the German-language volume "Neue Perspektiven auf mathematische Lehr-Lernprozesse mit digitalen Medien" by Springer.

theories which are constituted by dealing with these empirical. According to studies of Burscheid and Struve (2020a) and Witzke (2019), this "'non-abstract' point of view [...] is a reasonable one for the developing of mathematical knowledge" (Witzke, 2019, p. 199). Students construct their mathematical knowledge by themselves in interaction with their environment. With Hefendehl-Hebeker (2016) we can state that the concepts and contents of school mathematics have their phenomenological sources predominantly in our surrounding reality. For our analysis of the cognitive aspects in students' knowledge development processes, the concept of empirical-oriented mathematics class is crucial. An empirical-oriented mathematics class builds on, on the approach of empirical theories. Empirical-oriented mathematics classes are classes in which the teacher intendedly makes decisions to work with empirical objects as the mathematical objects of math classes. In mathematics classes, the empirical objects (e.g. dice or drawn figures) do not intend to illustrate mathematical concepts that are abstract in nature, but they are rather the subject of the lesson, where students acquire their knowledge on (Pielsticker, 2020). Empirical objects that play a role in this paper are 3D printed dice towers (Figure 1 and 6) and commercially available dice. Studies have shown that the use of 3D printing seems particularly useful for probability. For example, Pielsticker and Witzke (2022) argue that with a creation and use of 3D printed manipulated dice, profitable negotiation processes on the theoretical concept of probability can be initiated among learners. An empirical object such as the 3D printed dice tower described in this article (Figure 1 and 6) then serves in the classroom as an object of instruction (Pielsticker, 2020). 3D printing as a digital tool in a teaching conception of an empirical-oriented mathematics class creates student-oriented occasions to exchange ideas about mathematics. Students negotiate mathematics, which means that the digital tool 3D printing can be seen as a motor for the further development of student knowledge in the classroom. A self-construction of the dice tower in the CAD program in mathematics lessons is quite desirable and a crucial part also of the teaching unit "A fair game of chance?!" considered here. In this process, the learners negotiate the construction and thus also the functioning of the 3D printed dice tower. In the following, a teaching-learning process from a mathematics class that takes this conception into account is described.

Design and methodological decisions.

Research questions.

In this paper, we describe the beliefs of fairness that can be analyzed in a student in a 10th grade class at a high school. The research of Burscheid and Struve (2020a; 2020b) on questions of fairness of games of chance (Burscheid & Struve, 2020b) in historical reference is also guiding for this. There are many different games of chance. Often there was agreement on what the ratio of stake to winnings had to be in order for the game to be fair. In the "force majoure" problem, however, mathematicians pursued different solutions with respect to the question: How is the stake to be distributed fairly? Pascal and Fermat propose a solution to this in 1654, which corresponds to the modern probabilistic view (Burscheid & Struve, 2020b). Leibniz, knowing the solution of Fermat and Pascal, makes a different proposal for division in 1678. For normative problems there are just no right or wrong solutions, but only more or less appropriate proposed solutions (Burscheid & Struve, 2020a). In this context, probability theory nowadays plays an authoritative role for an assessment of the fairness of games of chance. This was not the case before, because probability theory has a root in the theory of the fairness of games of chance (Burscheid & Struve, 2020b). Thus, it is interesting to note that the concept of probability, which is fundamental to probability theory, has historically been developed within an application-related context: precisely the question of the fairness of games of chance (Burscheid & Struve, 2020a). For us, it is interesting to see how the student in our case study deals with the concept of fairness in the instructional context of probability. We pursue two research questions in this paper: 1. Which beliefs does the student of our case study develop about the concept of fairness? 2. To what extent can (students') theories about the concept of fairness be (further) developed when dealing with the 3D printed dice tower?

Material and data collection.

Data collection for the analysis of knowledge structures was based on student documents created before, during and after the implementation of the lesson "A fair game of chance?!". These include worksheets and a questionnaire. Both tasks of the worksheets and the questionnaire are part of the data collection and will be analyzed in this paper. The topic was introduced by the students' report about their experiences with gambling. Afterwards, there was a worksheet where cheating and fraud were discussed. This was intended to create a deeper awareness of the problem. In ancient times, for example, the use of a dice tower was considered a guarantee of fairness in dice games. The task of constructing a dice tower using 3D printing technology was intended to create an action-oriented approach in relation to a game situation. 3D printing is a quite new tool for mathematics education. It is an additive manufacturing method which allows digital models constructed with CAD-software to be transformed layer by layer into a real model (e.g. out of liquid plastic) (Gibson et al., 2014). This technology makes the development as well as the production of empirical objects in a variety of mathematical contexts quite easy and thus may facilitate embodied (Tall, 2013) approaches to mathematical content. In ancient times, a dice tower was seen as a way to prevent cheating. In the learning unit the students designed their dice towers on their own using the CAD software Tinkercad™ and 3D printed them out of plastic (Figure 4). To ensure that the tower had the conditions required for fair play, common criteria had to be defined within the student groups. In partner work, the students discussed the criteria. Subsequently, the students conducted test series with their dice towers and calculated the absolute and relative frequencies of their throws. In the worksheet with the sentence beginning "For me, fairness is ...", the students of the class summarized their view of the concept of fairness once again.

Methods.

In our instrumental case study (Stake, 1995), we describe the student Ardelin (a 10th grade student, 16 years old, name changed) in relation to our two research questions (see section: Research questions). „Case studies are undertaken to make the case understandable" (Stake, 1995, p. 85). With our case study we need to understand the case in order to understand our two research questions (Stake, 1995). We use the case study approach to identify our key scenes in terms of the research questions and to process the data material. We use the approach of empirical theories according to Burscheid and Struve (2020a) for informal description of the student's knowledge structures. We use the following terms (Table 1) as the basis for our analysis in terms of our research questions.

Terms	Explanation
intended applications	The phenomena of reality described and explained by an empirical theory.
empirical objects	In this study, empirical objects are understood as items and objects of reality that are immediately accessible to students, especially in a tactile or visual way.
theoretical concepts	Terms whose meaning can only be clarified by setting up or within a theory and which do not have an empirical reference object.

Table 1: Terms of empirical theories for analysis (Pielsticker, 2020; Burscheid & Struve, 2020a)

We use the terms from Table 1 as an analysis tool. In terms of our research concern, this is an informal description using the terminology of the empirical theories approach according to Burscheid and Struve (2020a).

Analysis and Results.

In order to achieve a deeper understanding of the case, key scenes have been selected, which we describe below. In the learning unit, the game of chance – dice game – was considered and knowledge about fairness was acquired. The question of when a game is fair should be addressed. In the learning unit, a link between ancient and digital tools – the dice tower – was used for this purpose. With reference to the construction of an (ancient) 3D printed dice tower (section: Material and data collection), the aim was to reflect on fairness. For the case of fair play, we will first discuss the results of the student Ardelin. To do this, we will start by looking at the student's answers in the questionnaire and then describe her answers on the worksheets. In the questionnaire, Ardelin is asked: "What does 'fairness' mean to you?". The student states, "That everyone is treated or valued equally, so that everyone is equal and no one is favored" (Figure 1).

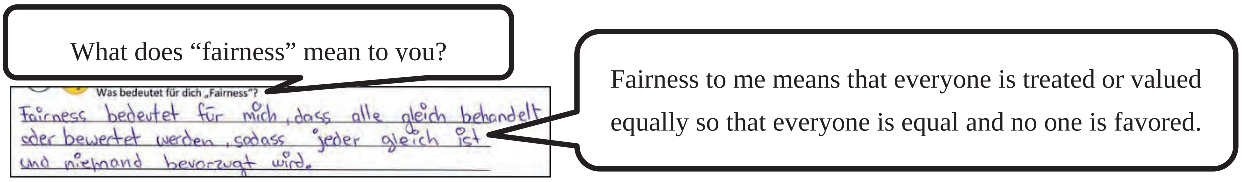


Figure 1: Answer to 2) from Ardelin in the questionnaire

The student bases her concept of fairness on dealing with people ("everyone" and "no one"). She thus refers to (her) fellow players. Ardelin is now asked to relate her concept of fairness to a game. She is asked, "When is a game fair for you?". The student states, "when everyone is treated equally and everyone gets the same chances and everyone gets the same rules" (Figure 2).

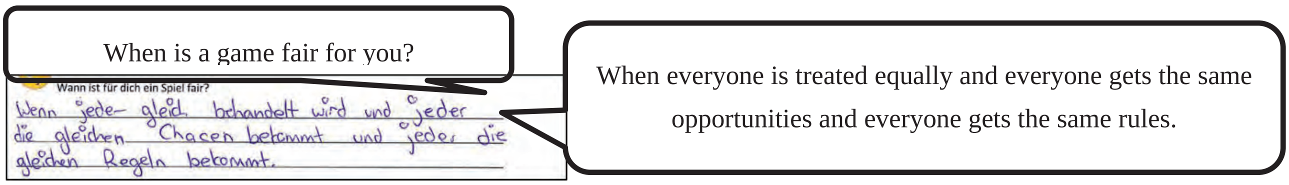
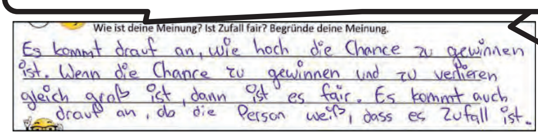


Figure 2: Answer to 3) from Ardelin in the questionnaire

Ardelin sticks to the reference ("everyone"), emphasizing that "everyone gets the same chances" and "everyone gets the same rules" (Figure 3). The student thus makes a fair game once to a game context in which "the same rules" apply, also she connects a fair game with "equal chances" and finally she names the players, "everyone", who should be "treated equally". When Ardelin is asked for her opinion in the 4th question of the questionnaire, "Is chance fair? Give reasons for your opinion.", the student first makes this point, about the chance of winning a game. "If the chance of winning and losing is the same, then it is fair" (Figure 3). At the same time, Ardelin also refers back to the players by noting, "It also depends on whether the person knows it's chance."

What is your opinion? Is coincidence fair? Give reasons for your opinion.



It depends on how high the chance of winning is. If the chance to win and to lose is equal then it is fair. It also depends on whether the person knows that it is chance.

Figure 3: Answer to 4) from Ardelin in the questionnaire

In her answers, the student thus distinguishes between the game, the people involved (players), and the chance of winning the game. At this point, we would like to discuss the results of the student's worksheets. Ardelin created a 3D printed dice tower and tried it out in the further course of the learning unit (Figure 4). Figure 4 shows the 3D printed cube tower. Ardelin has constructed some inclined levels (the student herself writes "steps") inside the cube tower and further a cube tower wall to be attached to the rest of the cube tower with tape afterwards. The student explained in the classroom situation that she only wanted to tape the cube tower wall because she wanted to control how the cube rolled along the inclined planes.

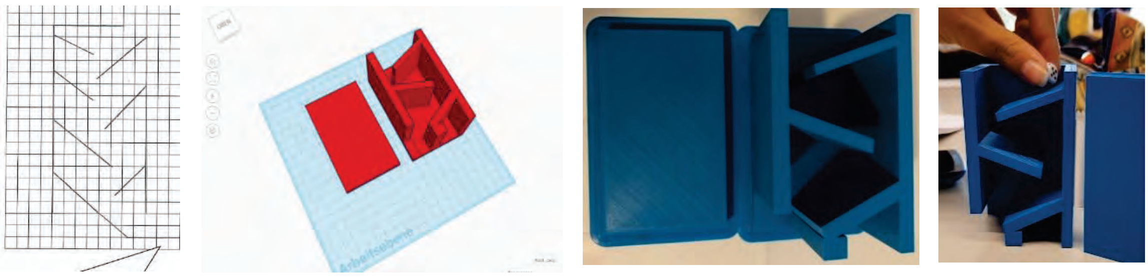
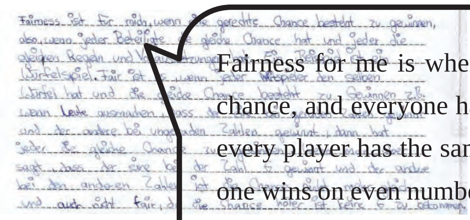


Figure 4: Adrelin's 3D printed dice tower

On her worksheet, the student describes regarding the criteria that should apply to her dice tower, "It must be fair" and "It must always be dropped with the “1” from the top and give more of these steps." The empirical object to which Ardelin refers at this point is the 3D printed dice tower. For the game with the dice tower, special rules should continue to be recorded, such as, "It [the dice] must always be dropped with the 1 from the top [into the dice tower]," there must be "more of these steps," which should be placed "at even intervals," and ultimately, "nothing should fall out of the sides." Using the example of the 3D printed dice tower, as an empirical object, Ardelin argues how it must be designed for fair play. To do this, she first states that the game must be fair, and then simultaneously formulates what rules must apply. Ardelin also describes her view once again on her worksheet (Figure 5). Here, Ardelin describes a "fair chance [...], that is, [...] equal chance" - she probably uses the two expressions synonymously - with "equal rules and conditions." She argues this with the help of the intentional applications "dice game" and "game of chance: even - odd" - as intentional applications of the student theory about games of chance. The dice can be regarded as the empirical object for the intended application dice game. Ardelin emphasizes here that "every player should have the same die" (Figure 5).



Fairness for me is when there is a fair chance to win. That is, when every player has the same chance, and everyone has the same rules and conditions. An example is a dice game. Fair is when every player has the same dice and the same chance to win. For example, if the players agree that one wins on even numbers and the other wins on odd numbers. Then everyone has the same chance to win. But if you say that one wins on the number 6 and the other wins on all other numbers, the chance is no longer equal. This is not fair, because the chance is higher not to get a 6.

Figure 5: Adrelin's 3D printed dice tower

In relation to the theory of the fairness of games of chance, the "game of chance: even - odd" is an intended application with even and odd numbers as failures of a random experiment. At this point, the student gives a counterexample, which she relates either to the intended application of the dice game or to another game of chance: "But if you say that one wins with the number 6 and the other with the other numbers, the chance is no longer equal and also not fair, because the chance is higher not to get a 6". If we assume the intended application of the dice game (where "one wins on the number 6 and the other on the other numbers"), the dice can be seen as an empirical object on which Ardelin explains her counterexample to "equal chance". It is also clear that she is oriented to the game context in her negotiation of meaning regarding the concept of fairness. The intended applications in relation to the student's theory of fairness of games of chance are games of chance such as "dice game" (with or without 3D printed dice tower) or "even - odd". In this context, Ardelin gives meaning to the concept of fairness in the context of gambling. Whether a game is fair is decided for the student by this game context, with the associated rules and "prerequisite" (Figure 5), such as, for example, that "every player has the same dice" (Figure 5). It should also be mentioned that Ardelin's examples probably tacitly assume that each player pays the same stake.

Discussion

1. Which beliefs does the student of our case study develop about the concept of fairness?

We can describe for Ardelin that she behaves as if she developed an empirical beliefs system, an empirical theory about the fairness of gambling. In doing so, the student repeatedly uses words like "chance" and "equal chance" (Figure 2 & Figure 5), with which a connection to probability theory can be assumed. This would not be surprising, because until today (not from the beginning!), however, probability theory has remained the authoritative theory for assessing the fairness of games of chance (Burscheid & Struve, 2020a). In relation to a fair game, the student – in relation to her theory on the fairness of games of chance – can describe the intended applications "dice game" and "game of chance: even - odd". Ardelin makes the "same rules and preconditions" (Figure 5), which apply to a fair game in the student's sense of the word, on empirical objects such as the 3D printed dice tower or dice. For a fair game, Ardelin emphasizes that "every player should have the same dice" (Figure 5). In her negotiation of the meaning of the concept of fairness, the student orients herself to the game context, and she gives her concept of fairness a meaning in this application-related game context. Whether a game is fair is determined for the student by this game context, with the associated rules and "prerequisite" (Figure 5), such as that "every player has the same dice" (Figure 5). The notion of fairness is seen in connection with the expectation value, as for example in Christian Huygens, who defines the notion of expectation in terms of fairness as equal expectation prevails in a fair game (Gigerenzer & Krüger, 1999). Burscheid and Struve (2020a) also make it clear with historical reference that the concept of expected value, which is fundamental for probability theory, did not develop within a formal theory, but within an application-related context (Burscheid & Struve, 2020a). In this context, expected value can be identified and described as a theoretical concept. Ardelin decides whether a game is fair along the chance of winning (and the expectation of winning) of the players in the game situation. In doing so, the concept of fairness takes on meaning for her in this context.

2. To what extent can (students') theories about the concept of fairness be (further) developed when dealing with the 3D printed dice tower?

Ardelin seems to add the dice tower as a (further) possibility of throwing the dice in her student theory about the fairness of gambling. The 3D printed dice tower is not crucial for the student to decide whether fair game prevails in a gambling situation (such as the "dice game"). The 3D printed dice tower is integrated by the student into her empirical theory as a further dice-rolling possibility and thus expands her knowledge of dice-rolling possibilities. The notion of fairness can be identified as an epistemological obstacle (Sierpinska, 1992) in probability theory and in connection with expected value (in relation to our case study). Here, parallels to the historical development of probability theory can also be identified on an epistemological level, in line with the remarks of Burscheid and Struve (2020b). With a development of meaning of the concept of fairness, as a theoretical concept in relation to a (school) theory of probability, special challenges are therefore associated in mathematics education. In the practical application (of probability theory in schools), time should not only be given for the development of the meaning of the concept of fairness, but it should also be addressed in game situations, e.g. in the sense of fair play and fair game. This impulse should also be followed up in further research and it should be considered whether and how both theories of fair game and fair behavior of persons in gambling are related.

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Digital support for proving arithmetic relationships - insights from the DigiMal.nrw project

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Keywords: Digital support, self-checks, explanatory videos, proof, pre-service teacher training.

Theoretical Background

The proof of arithmetical relationships is a mathematical content that causes difficulties especially for pre-service primary school teachers (Kempen & Biehler, 2019). They often encounter typical difficulties when carrying out proofs, formal transformation errors in terms as one example (Moore, 2016). In addition, pre-service teachers have problems when changing representations and showing completeness or generality (Kempen & Biehler, 2019).

To counter these difficulties targeted measures are needed to support pre-service teachers in constructing proofs. Besides analogue support, digital support may be helpful since it offers further specific potentials: For example, automated product-oriented feedback, as well as informative feedback, are possible (Hattie & Timperley, 2007). Furthermore, the synchronicity of presentations offers a high subject-specific didactic potential (Ainsworth, 1999). The production of explanatory videos may positively affect knowledge acquisition (Fiorella & Mayer, 2014), since they offer pre-service teachers to try out their future teacher role as an 'explainer', and their digital competencies can be promoted (Feurstein, 2017). As digital support in the context of proving is currently not widespread (Stylianides & Stylianides, 2017), this development project aims at creating and evaluating digital support materials for proving elementary statements on number theory. The findings are taken into account in the development of support options.

The project DigiMal.nrw and its subproject 'Arithmetics'

The project DigiMal.nrw (eng.: digital mathematics teacher training) aims at improving the quality of leading mathematics courses in pre-service teacher training (focus on primary schools and special education) with the help of digitally supported measures. In addition to teaching-related challenges, the project addresses the core contents of elementary school mathematics. As part of the DigiMal.nrw sub-project 'Arithmetics', corresponding teaching-learning opportunities in digital self-checks, and video-based learning environments for developing and working with explanatory videos are being developed and evaluated at the University of Münster and the TU Dortmund University.

The poster focuses on the following two questions: (1) How do pre-service teachers develop their mathematical and didactical knowledge of proof when working with video-based learning environments? (2) How do they perceive the subject-related feedback in the digital self-checks on the topic of proof?

Two first prototypes and their evaluation

On one hand, a video-based learning environment is being developed. Pre-service teachers prove the statement 'The sum of two consecutive square numbers is always odd' and deal with fictitious incomplete proofs. To deepen their knowledge, they create explanatory videos for 'fictitious pre-service teachers of the first semester'. The pre-service teachers analyse and reflect on their videos produced. They receive feedback from fellow students and lecturers. On the other hand, digital self-checks are being developed. They address the typical difficulties in proving. With the help of drag-and-drop tools, pre-service teachers deal with term transformations and their legitimation. Besides, pre-service teachers are given the task of deciding which formal and iconic representations fit a verbally given expression in different task formats. After completing the interactive elements, they receive automated feedback and have the opportunity to get further explanations via videos and texts.

The prototypes of the first survey cycle were evaluated with questionnaires due to their suitability for a large number of participants (Bortz & Döring, 2016). The evaluation of the video-based learning environment showed that pre-service teachers engaged more intensively with proofs and their explanations (RQ1). The evaluation data for the self-checks point to the effectiveness of the digital learning opportunities, especially since the majority of students rated their learning gains as 'high' or 'rather high' and indicated that the informative feedback supported their understanding of the content (RQ2). Based on the evaluation, the prototypes will be further developed.

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Students Undertaking an Elective Programming Course: Their Views on the Connection Between Programming and Mathematics

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In this study, we investigate how students undertaking an elective programming course experience the connection between programming and mathematics. Based on stimulated recall interviews with six Grade 8 students who have solved test items from PISA 2003, we identify various experiences among the students regarding how they draw on programming skills in mathematics and vice versa. We propose that three findings, in particular, are worthy of further discussion and investigation: the way some experience programming as a context for mathematics and mathematics as a context for programming; the challenges that follow from how some students assert that programming and mathematics complement each other while others struggle to see a connection; and, perhaps the most important, the way in which the students report that experiences from programming have equipped them with skills such as being systematic and exact, which they find they need in mathematics.

Keywords: mathematics education, problem solving, computational thinking, programming

Introduction

After recognising computational thinking (CT) as a vital 21st century skill (Bocconi et al., 2018), several countries have recently introduced CT to their curricula, either as a separate programming subject or as part of existing subjects. In Norway, CT is introduced in mathematics across grades (Directorate of Education, 2020), adding to being an elective programming subject in lower secondary school (Grades 8–10, ages 12–16). In the new Norwegian curricula, after Grade 4 in primary school, CT is exclusively addressed and talked about as *programming*. In programming, CT entails the ability to develop programmes to perform different algorithms and is considered a systematic description of how to solve a problem using a specific approach (Kaufmann & Stenseth, 2021, p. 1030).

As several studies have asserted that programming can induce improved understanding in mathematics (Moreno León et al., 2016), and even more precise – improved problem-solving skills in mathematics (Husain et al., 2017; Sinclair & Patterson, 2018) – it is easy to understand why many European countries introduce programming in mathematics (Balanskat & Engelhardt, 2015). However, Kaufmann and Stenseth (2021) warned against assuming that the programming's effect on problem solving is automatic. Exploring these findings while considering Kaufmann and Stenseth's (2021) warning, we seek to better understand whether those students undertaking an elective programming course in lower secondary school can communicate how mathematics and programming are interrelated. Hence, we give students a voice in the matter. It is imperative that the students themselves experience this connection. Hence, we ask the following research question: *How and to what extent do students undertaking an elective programming course experience a connection between programming and mathematics?*

Theoretical Framework

To investigate the interplay between programming and mathematics, we turned to Kilhamn et al. (2021) and Elicer and Tamborg (2021). Kilhamn et al. (2021) investigated the relationship between mathematics and programming in 32 mathematics lessons and found four 'categories' of relationships: *only programming* (lessons only focusing on programming); *mathematics as context for programming* (lessons where no new mathematical concepts are used, but programming can be used to repeat or confirm mathematical knowledge); *programming as a computational tool in mathematics* (lessons that clearly have mathematical content, and programming is used to carry out calculations efficiently); and *programming as a tool for exploring mathematical concepts* (lessons where programming is used to explore mathematical concepts and relations, where programming adds new insights and develops one's mathematical competence). Similarly, Elicer and Tamborg (2021) analysed activities from the Danish 'Texforsøget' and described the role of mathematics and what they refer to as programming and computational thinking (PCT) in different activities. They found six categories, where the first two are *no mathematics involved* and *no PCT involved*, implying either no mathematics or no PCT involved in the tasks, and that the focus of the task is either purely mathematical or purely programming. The two following categories, *mathematics as a context* and *PCT as a context*, are categories where mathematical concepts are explored using PCT operations or where PCT concepts are explored using mathematical operations. The next category is *conceptual integration*, which are tasks that are "not solved with mathematical or PCT actions but involve concepts in both math and PCT" (Elicer & Tamborg, 2021, p. 5). The last category is *operational integration*, where "mathematical and PCT competencies are interdependent" (Elicer & Tamborg, 2021, p. 6).

While exploring what these two studies revealed on the relationship between programming and mathematics, both at the level of a lesson and at the level of the tasks given, we sought to see how a discussion around some problem-solving tasks can work as a starting point for investigating how students see the relationship between programming and mathematics. Problem-solving tasks were a natural point of departure, as research seems to unanimously assert that problem solving is the most obvious overlap between mathematics and programming. For instance, in pursuing the definition of CT for mathematics and science classrooms, Weintrop et al. (2016) proposed a taxonomy comprising four practices, one of which focuses on computational problem-solving practices. In this practice, Weintrop et al. (2016) identified problem solving as a central practice for developing CT in mathematics (and science), which sometimes takes the form of conventional programming (p. 138). This made us take a set of mathematics problem-solving tasks as our point of departure when interviewing students on how and if they see a connection between programming and mathematics.

Methods

In Norway, students choose one elective subject in each of their three years in lower secondary school. This study used data collected from a larger project, where all participants (N = 247) completed a test comprising five different problem-solving tasks from the PISA 2003 test (these were the tasks on the library system, course design, transit system, irrigation and energy needs; see Faculty of Educational

Sciences UiO [2003] for details). Due to how the library system task was mentioned and discussed in most interviews, we give a brief account of it here to inform the reader:

> The task starts by giving a flowchart that shows a library lending system: If you are a teacher (yes), you are allowed to lend books for 28 days; if you are not (no), you are allowed to lend books for 7 days. Based on this system, the students are asked to build more complicated flowcharts.

Participants selected for this subproject are six Grade 8 students (ages 13–14) who chose programming as their elective subject (hereafter referred to as the *elective programming course*). The participants (five girls, one boy) came from four different schools (a mix of city-centre and rural schools) in four different counties in the western and southern parts of Norway and were chosen for interviews based on a) their choice of elective, b) their answers on the initial test, and c) how their methods on the initial test stand out from the rest of the students. Also, they were selected to ensure distribution across schools. The names presented later in the Results section are fictional.

Each of the six participants was interviewed by the first author of this paper on the same day as they had completed the initial PISA-2003-based test. The semi-structured interviews lasted for about 15 minutes (between 11–28 minutes) and were recorded. The overall intention of the semi-structured interviews was to explore the students' accounts of their solutions and to see if there were any cross-references between their experiences in mathematics and the elective programming course. The first author used stimulated recall when asking questions about what knowledge they drew on when solving the five selected problems. Additionally, the participants were asked if they had examples of when they had drawn on knowledge learned in their elective programming course in their mathematics class and vice versa.

The interviews were transcribed in full in their original language (Norwegian) and analysed in two steps. Step one involved a process of consistent coding (Mason, 2017) that entailed reading and re-reading the transcript to identify if and how the interviewees related programming and mathematics and their accounts of how they drew on their programming knowledge in mathematics and vice versa. Step two was to identify the central experiences and understanding of the students, which led to two main categories in the results: programming in mathematics and mathematics in programming. These two categories also build on the categories where mathematics is the context for programming and programming is the context for mathematics from Kilhamn et al. (2021) and Elicer and Tamborg (2021).

Results

This results section comprises two parts. In the first part, we will consider how the 8th grade students saw the possibility of using methods, knowledge and skills from their elective programming course when solving problems in mathematics, while in the second part, we will focus on how students used methods, knowledge and skills from mathematics in their elective programming course.

Drawing on Elective Programming Skills in Mathematics

When talking about the library system task, three of the students explained how they recognised the need to use a flowchart, which they had been introduced to in their programming elective course.

Peter immediately recognised the flowchart and explained how he had used it as part of his programming process:

> I recognised this one immediately [points at the flowchart] because we have had many flowcharts in the elective programming course when making programmes in different programming languages. The first thing we have to do is make pseudocodes and flowcharts.

We see how Peter drew on knowledge from his programming elective course when solving the Library system task. This also applies to two other students, Andy and Nick, who recognised the flowcharts as a method from the elective programming course.

While the section above provides examples of how students draw on methods from their programming elective course when solving problems in mathematics, we also found that the programming elective course provided the students with a new vocabulary that came in handy in their mathematical problem solving. During interviews, the students revealed how they tended to use programming concepts to solve and explain *how* they solved the library system task. Nick and Michael used TRUE/FALSE statements or IF-ELSE statements to explain how they solved the library system task and related this problem to earlier experiences:

> (…) is it a magazine? If yes, seven days. If no, go to the next step. If you do not return books or magazines, then you are not eligible to borrow (…) (Nick)

Peter, however, described how he solved other problems during regular mathematics classes using the programming concept variable and loops:

> One time, I developed a programme in Trinket or Python. Then I used a counting variable. I got a task where I could use the counting variable to solve it. This is a formula I can use in the programme because I have learnt to calculate this way using variables and 'x in range' [a command for loops in Python].

The third finding is connected to how the students identified whether or how the elective programming course changed their mathematics problem-solving approach. Due to how they had learnt to work in the elective programming course, Robbie revealed that he made more use of diverse strategies and focused more on finding multiple solutions to mathematical problems, and both Nick and Peter said they saw themselves as working more systematically when solving mathematical problems. Adding to this, Nick talked about how he felt more accurate in solving mathematical problems since this is vital in the elective programming course:

> My mathematical view has become slightly more systematic. (…) In programming, one must be more exact to make a programme work.

Overall, these findings indicate that students acquire general skills through the elective programming course that they see as positive for their mathematical problem solving. Andy even highlighted that he sees a clear connection between programming and mathematics – he sees that both subjects are about solving problems:

> Programming is much about solving different problems in a code, and some of the tasks in mathematics are about the same: solving a problem.

However, some students did not see a connection between the elective programming course and mathematics. Nora, like Michael, claimed that there is no connection between programming and mathematics, adding that there is much she has not understood in the elective programming course. As she finds the elective programming course challenging, kind of softenings her statement on the disconnect. When asked if the elective programming course had affected him in mathematics, Michael put it like this:

> That is a good question; I do not think so since programming does not involve, or it involves numbers, of course, but I am not sure if it is under [has to do with] mathematics. (…) It is not addition or subtraction (…)

Adding to this and despite his vague idea that the elective programming course might have changed his mathematical competency, Robbie revealed that he had never experienced any programming in mathematics and therefore saw no apparent connection. However, he said that one must know mathematics to programme, which we will return to in the second part of the results, which we turn to now.

Drawing on Mathematics Skills in the Elective Programming Course

Our data revealed three ways in which the students reported how they drew on mathematics skills in their elective programming course. The first entails how they report on the use of mathematical concepts in programming tasks, such as figure numbers and geometry. Five of the students had experience with mathematics in programming, one of which is Nora. She explained that she had experienced using rotation and angles when programming how to make different figures move. Likewise, Peter connected experiences from algebra to the concept of variables in the elective programming course:

> There is much mathematics in programming because there is much algebra in programming. One uses many variables to make calculations, and that is almost the same as in algebra.

The second entails that students report on how they find their experiences of using different digital tools in mathematics to better understand what is going on in the elective programming course. Andy described an experience with GeoGebra, where he talked about making and using different functions to solve problems in mathematics. Similarly, Nick drew on an experience of using Excel and claimed this to be programming when he was asked if he used programming to solve mathematical problems: "Excel is technically programming". Both Andy's and Nick's experiences come from well-known computational tools in mathematics, and they reported that their experiences with those tools can help them in the elective programming course.

Third, our analysis shows that all the students, in one way or another, agreed that one needs to understand mathematics to programme. For example, Robbie, who mentioned the four arithmetic operations earlier, used arithmetic to argue that one needs to know them to develop a programme. Nick adds to this, saying "… one has to know much mathematics to programme properly". From this, we can see that Nick sees how mathematics plays a role in the elective programming course.

Discussion

In answering the research question – How and to what extent do students undertaking an elective programming course experience a connection between programming and mathematics? – we decided to investigate how students tend to draw on programming in mathematics, and vice versa. Due to how, for instance, Weintrop et al. (2016) identified problem solving as a central practice for developing CT (and conventional programming) in mathematics, we decided to use the students' solutions on a set of five PISA 2003 problem-solving tasks for knowing their thoughts on the connection between programming and mathematics.

Our analysis resulted in three main findings, where the first two confirm, and considerably elaborate on, the findings of Kilhamn et al. (2021) and Elicer and Tamborg (2021). First, we saw that our students reported programming as a context for mathematics and mathematics as a context for programming. This became apparent in how some students argued that one needs to know mathematics to programme, and conversely, in how they revealed that they had experienced programming in mathematics. Andy, for instance, saw problem solving as a potential connection between programming and mathematics. We believe that this result argues for a potentially fruitful integration between mathematics and programming.

Second, we found differences in how the students talked about and experienced the relationship between mathematics and programming. If we lend the vocabulary of Kilhamn et al. (2021), we can say that some students tended to talk about how programming can work as a computational tool in mathematics. Peter, for instance, explained how he uses concepts from programming to solve problems in mathematics. Moreover, he asserted that programming tools made him approach and view mathematics differently. Robie added to this picture when revealing how he saw programming helping him find multiple solutions to a given problem. This adds insight into Kilhamn et al.'s (2021) category of "programming as a tool for exploring mathematical concepts", where they propose that programming can be used to explore mathematical concepts and add new insights. However, there are students, Nora and Michael in particular, who experienced the elective programming course and mathematics as two different subjects. We see how this adds insight into the categories of Elicer and Tamborg (2021) in the way Nora and Michael's experiences seem to reflect an understanding of "no PCT in mathematics" and "no mathematics in PCT". Hence, we see how some students enhance their understanding of mathematics when engaging in programming activities, while others struggle to see how they can leverage programming in mathematics and vice versa.

Our third finding adds to the "categories of interplay" between programming and mathematics set forward by Kilhamn et al. (2021) and Elicer and Tamborg (2021). Both Nick and Peter stated that the elective programming course provided them with new skills. They emphasised that their experiences with programming had made them see the added value of being *systematic* and *accurate* in mathematics. We propose that this is a new category, and more research is needed to elaborate on it.

Concluding Remarks

This paper contributes to the ongoing discussion on how (and if?) programming and mathematics should be integrated in an educational context. We report on the experiences and views of the students, which, in sum, speak for an integration of the two. In addition, we assert, that by comparing our results with those of Kilhamn et al. (2021) and Elicer and Tamborg (2021), we strengthen their findings regarding how we see that the students' utterances confirm most of their categories of interplay.

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Enriching learning of analytic geometry by augmented reality – development of an AR smartphone app

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Keywords: Augmented reality, digital tools, mathematics education, design-based research

Augmented reality (AR) combines reality with computer-generated representations in real time (Milgram & Kishino, 1994). Both are simultaneously present or continuously merged (Azuma, et al., 2001). Instead of replacing a real situation, e.g. by modelling with dynamic geometry software, AR supplements the real situation with objects or information (Avanitis et al., 2007; Martin-Gutierrez et al., 2010). Therefore, merging reality with additional objects is a unique way to make things tangible without switching to an artificial environment.

From a technical point of view, AR enables the mathematical situation to be represented in the familiar 3-dimensional environment around us and allows the user to navigate within this familiar situation in order to explore it. On a cognitive level, the theory of embodied cognition plays a role, which describes the mutual influence of physical interaction and human thinking (Tran et al., 2017). Through AR, learners can physically explore a mathematical situation given by a task instead of having to imagine it. Thus, it can be assumed that physical exploration reduces the "cognitive load", a measure of cognitive capacity, so that greater cognitive capacities are available to solve the task.

The AR smartphone app "MalAR"

The aim of the project MalAR (transl.: AR-supported mathematics learning) is to research the potential of AR-supported instruction on learning analytic geometry. In a first phase of the project, an AR smartphone app was developed, which is based on the needs of the learners and on the content of analytical geometry.

Due to the complex three-dimensional representations in analytic geometry, physical representations play a subordinate role. This favours a technical and not very descriptive view of these contents (Borneleitet al., 2001). In order to overcome this bias, Filler (2007) suggests a more experiential approach that involves visual perception. The developed AR smartphone app allows learners to place mathematical notions of analytic geometry such as planes (in parameter, coordinate and normal form), straight lines and points into a coordinate system that is embedded in their surrounding reality. The latter is a real novelty compared to other visualisation possibilities with e.g. dynamic geometry software. The input of mathematical objects can be done following the notation learners know from school, so that the barrier of input language, which is a problem with many current AR applications, is overcome. With the AR smartphone app MalAR, learners are able to physically explore the given mathematical situation as well as to evaluate their result in the context of the given situation.

Figure 1 (left) shows the AR implementation of the following task in the app. A plane $E_1: 2x - y - z = -1$ is given. A plane E_2 is sought, which contains point $A(3|1|2)$ and is orthogonal to E_1.

The learners have the possibility to explore the given situation with the help of the AR app by entering the coordinate shape of the plane and the point A in the app and placing them together with a coordinate system in their environment (Figure 1, left). In addition, learners can enter their calculated solution for plane E_2 (correct solution $E_2: x + 2y = 5$, purple plane in Figure 1, right). The required orthogonality as well as the position to point A can then be visually verified through changing the perspective by walking around the situation, i.e. by moving one's own body in the room.

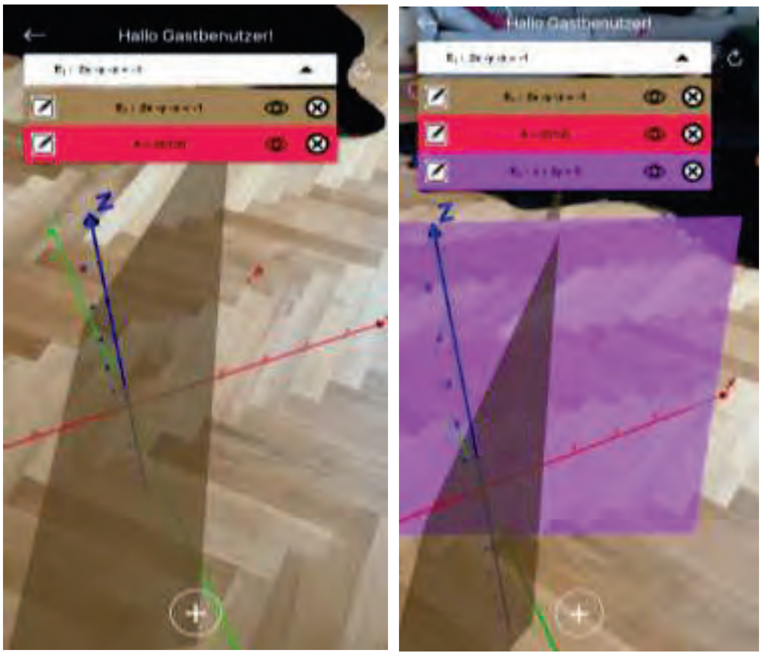

Figure 1: Implementation of the task in the AR app (left) together with the solution (right)

This representation of hitherto only 2-dimensionally represented 3-dimensional mathematical objects as part of the surrounding reality offers the potential to have a decisive influence on spatial perception and the fundamental understanding of those mathematical concepts.

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Near and far transfer in the flipped mathematics classroom: student's evaluation of learning activities

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In contrast to traditional forms of instruction, flipped learning enables students and teachers to focus on consolidating knowledge and fostering a deeper understanding of new mathematical content during class time. Both near-transfer problems and far-transfer problems can be the focus of such consolidation during a face-to-face lesson. In this study, we examine how students evaluate different learning activities in a flipped classroom setting when class time is dedicated to near transfer and far transfer, respectively. Results indicate, for example, a general preference for collaborative work, especially during face-to-face lessons that focus on far-transfer. To contextualize the results, we took the teacher's perspective into account. It adds a valuable point of view but is not always compatible with the students' perspective.

Keywords: Flipped classroom, Learning activities, Transfer of Learning.

Introduction

Flipping the classroom divides students' learning into two phases: the homework phase taking place before class and the face-to-face in-class phase. During the homework phase, students study new content at home, usually with the help of a video (Bishop & Verleger, 2013). Often, such videos focus on different forms of direct instruction (Bergmann & Sams, 2012), such as introductory videos, that explain new mathematical concepts, or illustrative videos, that present worked examples (Voigt et al., 2020). During the in-class phase, face-to-face time can then be used to focus on applying the content studied during the homework phase (Love et al., 2013). This can include various tasks and activities differing in their cognitive demand and the underlying educational objectives. One criterion to characterize different ways of content application in mathematics is the transfer of learning (Tüker, 2013). More specifically, near transfer and far transfer can be distinguished. Near transfer occurs between very similar contexts, whereas far transfer occurs between contexts that seem more remote to each other (Perkins & Salomon, 1994).

Near and far transfer in the flipped classroom

In a flipped learning environment, near transfer during the in-class phase can refer to tasks and activities in this phase being similar to the content of the video, especially worked examples previously presented in the video. For instance, in the context of teaching the Pythagorean theorem, near transfer occurs when students are given the task to calculate the length of the hypotenuse in a right-angled triangle in class after watching a video at home demonstrating such a calculation based on a different numerical example. Students can apply the algorithm displayed in the video directly in the context of the new task. In contrast, activities and tasks of far transfer in class go beyond the context presented in the video. For instance, far transfer occurs when students watch a video demonstrating how to apply the Pythagorean theorem to calculate lengths for one type of polyhedron, such as a pyramid, by looking for auxiliary right-angled triangles. In class, they are then asked to transfer this more general strategy to new types of polyhedrons, like cuboids, for example. While near transfer and far transfer are discussed in theory for designing activities for the in-class phase of a flipped classroom (Enfield, 2016), previous research on these forms of transfer in the flipped classroom is scarce or focusses on students' performance, e.g., Harrison et al. (2017). Hence, it has been suggested by Lo and Hwang (2018) that future research on course activities for flipped learning should also take into account near and far transfer as well as student perceptions of such activities. Therefore, we examine student perceptions of flipped learning activities in the context of near transfer and far transfer in this study.

Research Questions

The study aims to compare students' evaluations of learning activities in a flipped classroom for lessons that focus on near transfer and far transfer of the video content during the in-class time (hereafter referred to as near-transfer lessons and far-transfer lessons), respectively. To guide this comparison, we pose the following research questions:

1. How do students evaluate learning activities in the flipped classroom for near-transfer lessons?
2. How do students evaluate learning activities in the flipped classroom for far-transfer lessons?
3. How do students' evaluations of activities in the flipped classroom differ between near-transfer lessons and far-transfer lessons?

Methods

Participants

This study was conducted with students of four classes of the 9th grade of two German academic-track secondary schools (Gymnasium) in June 2021 (N = 79). These are schools granting the highest possible secondary school qualification in Germany. Convenience sampling was used to select the participants. The classes were taught by two different teachers. Their experiences in teaching mathematics in general as well as teaching the classes participating in the study were similar. Neither teacher had used the flipped classroom method regularly before this study. However, the teachers implemented flipped learning for several individual lessons before the start of the main study under the authors' guidance to accustom the students to the changed requirements in the flipped classroom (Lo et al., 2017). According to the teachers, students in all classes were familiar with cooperative forms of work, such as working with a partner or in small groups, before the study.

Design and Procedure

The students were taught two back-to-back flipped lessons on the Pythagorean theorem. The flipped lessons were conducted within a longer lesson sequence on the Pythagorean theorem, i.e., the students had prior knowledge of some aspects of the subject matter. However, each flipped lesson introduced new learning contents that had not been taught in previous lessons. The instructional material was designed and provided by the first author of this paper. During the homework phase of each lesson, students worked with an instructional video of nine to twelve minutes. These videos can be classified as illustrative videos (Voigt et al., 2020) since they contained worked examples or solutions for different problems related to the Pythagorean theorem, e.g., calculating the missing length of a side in a right-angled triangle or calculating the edge lengths of pyramids. Each video was accompanied by a short task relating to its content as an incentive for students to prepare for class (Kim et al., 2014). Completing these tasks, including watching the video, takes an estimated time of 30 minutes. The following in-class phase consisted of a face-to-face lesson of 90 minutes. It started with comparing students' solutions to the task related to the video and clarifying problems or questions. The remaining time of the lesson was devoted to exercising and consolidation. Here the two flipped lessons differed. Students' tasks for consolidation focused on near-transfer problems in one lesson and far-transfer problems in the other lesson. However, both lessons provided differentiated tasks according to students' level of prior knowledge and the opportunity for students to determine the pace of working through those tasks themselves. Furthermore, students could decide to work collaboratively with a partner or in small groups. For most tasks, students could also opt for working on their own if they preferred. During this phase, the teacher acted as a "guide on the side" (King, 1993), answering questions and providing scaffolding if necessary. At the end of each flipped lesson, students completed a survey evaluating the learning activities of the lesson. One teacher taught the near-transfer lesson before the far-transfer lesson. In classes of the other teacher, it was vice versa.

Instruments and Data Analysis

For the post-class survey, an adapted version of the Student Assessment of their Learning Gains (SALG) instrument (Seymour et al., 2000) was used after the near-transfer lesson and again after the far-transfer lesson. Students' evaluations of how much the learning activities in the flipped classroom helped their learning were gathered on a 5-point Likert scale ranging from 1 (no help) to 5 (great help). Since this does not constitute an interval scale, non-parametric tests were used for data analysis (Rasch et al., 2010). For students who did not engage in certain activities during a lesson, e.g., individual work, the option 'not applicable' was given in the questionnaire. In total, 75 students completed the survey after both flipped lessons. Four students, who only completed one of the two surveys, were excluded from the study. To determine whether students' evaluations for the individual activities during the near-transfer lesson and far-transfer lesson themselves differed, we applied the Friedman test for comparing multiple dependent samples (Janssen & Laatz, 2017). Since a comparison of all evaluated activities would not allow a meaningful interpretation, multiple Friedman tests were conducted for groups of related activities, e.g., comparing types of collaborative work. For the Friedman test, a post hoc analysis is necessary (Janssen & Laatz, 2017). Therefore, pairwise comparisons were conducted within each group using the Bonferroni correction. For further separate analysis of activities during the near-transfer lesson and the far-transfer lesson, the correlation between students' evaluation of the individual activities and their level of prior knowledge was examined through Spearman's rank correlation coefficient r_s for ordinal data (Janssen & Laatz, 2017). For this purpose, students were ranked into three groups according to their level of prior knowledge (low, average, and high) based on grade point average. Since students could choose not to do certain activities, missing data had to be taken into account. Therefore, a χ^2-test for the independence of nominal data (Rasch et al., 2010) was conducted to determine whether the decision against a specific type of activity depends on the students' level of prior knowledge for each form of transfer. Another χ^2-test was conducted to examine whether the decision against a specific type of activity during the near-transfer lesson coincides with such a decision in the far-transfer lesson. Finally, differences in students' evaluations of activities during the near-transfer and the far-transfer lesson were analyzed using the Wilcoxon signed-rank test for comparing two dependent samples (Rasch et al., 2010). All tests were applied at a significance level of 0.05 using SPSS. To contextualize the results of the SALG, we added an open-ended question to the surveys asking students to elaborate and explain their assessment. Furthermore, an interview with the teachers was conducted to gather an assessment on student learning from an observer perspective. The students' answers to the open-ended survey question and the teachers' statements were analyzed by the authors applying a thematic coding strategy (Kuckartz, 2010). When differences in the analysis occurred, the corresponding segments were discussed further until consensus was reached.

Results

Evaluation of learning activities for near-transfer lessons

Table 1 displays the evaluation results of the near-transfer lesson for all activities. The Friedman test for the different types of collaborative work, i.e., individual work, partner work, and group work, indicates significant differences (χ^2 (2, n = 47) = 38.38, p < .001). Post hoc analysis demonstrates that individual work was rated as significantly less helpful compared to partner work (z = -4.95, p < .001) and group work (z = -3.89, p < .001), respectively. However, the difference between partner work and group work evaluations was not statistically significant (z = 1.08, p = .836). Students' statements in the open-ended survey question yielded the same results with students describing both partner and group work mainly as a possibility to exchange ideas and thus to reach the goal faster.

Table 1: Evaluation results of individual learning activities for the near- and far-transfer lesson

	near-transfer lesson			**far-transfer lesson**		
Learning activity	n	M (SD)	Mdn	n	M (SD)	Mdn
Video-related activities						
Watching the instructional video at home	70	4.00 (0.87)	4.00	71	4.01 (0.80)	4.00
Completing tasks related to video at home	73	3.41 (1.05)	3.00	66	3.42 (0.96)	3.00
Comparing results of video-related tasks in class	62	4.02 (0.98)	4.00	71	4.13 (0.97)	4.00
Types of explanation						
Hearing explanations by the teacher in class	73	4.23 (0.84)	4.00	73	4.22 (0.97)	4.00
Participating in discussions in class	48	3.40 (1.22)	3.50	60	3.77 (1.06)	3.50
Hearing other students explain their work in class	51	3.49 (1.07)	4.00	65	3.23 (1.09)	4.00
Interacting with the teacher in class	65	4.34 (0.76)	4.00	65	4.63 (0.65)	4.00
Explaining my work to other students in class	47	3.19 (1.23)	3.00	64	3.13 (1.15)	3.00
Collaborative work						
Participating in group work in class	50	4.04 (0.95)	4.00	72	4.26 (0.89)	4.00
Participating in partner work in class	67	4.18 (0.82)	4.00	69	4.12 (1.02)	4.00
Studying on my own	69	3.32 (1.08)	3.00	65	3.08 (1.16)	3.00
Other						
Working at my own pace	75	4.32 (0.83)	5.00	75	4.39 (0.80)	5.00
The overall approach to teaching and learning	75	4.15 (0.73)	4.00	74	4.12 (0.78)	4.00

For the five different types of explanation (see table 1), the Friedman test also indicates significant differences (χ^2 (4, n = 37) = 45.59, p < .001). In this case, post-hoc pairwise comparisons revealed that interaction with the teacher was evaluated as significantly more helpful than participating in discussions (z = -3.01, p = .026), hearing other students explain their work (z = -3.57, p = .004) or explaining one's own work (z = -4.96, p < .001), respectively. Furthermore, hearing explanations from the teacher was rated significantly higher than hearing other students explain their work (z = 2.98, p = .029) or explaining one's own work (z = 4.38, p < .001). Between other pairings no significant differences were found. Using χ^2-tests for further examination of missing data, i.e., occurrences of the evaluation 'not applicable', did not reveal any significant relationships between the choice not to partake in a specific learning activity and the students' level of prior knowledge. Furthermore, the application of Spearman's rank correlation coefficient did not indicate a significant correlation of the students' level of prior knowledge and their evaluation of any learning activities during the near-transfer lesson.

Evaluation of learning activities for far-transfer lessons

The evaluation results of the far-transfer lesson for all activities are also displayed in table 1. Similar to the results for the near-transfer lesson, the Friedman test for the different types of collaborative work suggests significant differences (χ^2 (2, n = 60) = 49.69, p < .001). Again, post hoc pairwise comparisons indicate that individual work was rated significantly lower compared to partner work (z = -5.02, p < .001) and group work (z = -5.39, p < .001), whereas the difference between partner work and group work was not statistically significant (z = -0.37, p = 1.000). The explanations for this rating given in the open-ended survey question do not differ from those for the near-transfer lesson. The Friedman test for the five different types of explanation indicates significant differences (χ^2 (4, n = 50) = 84.79, p < .001) during the far-transfer lesson, too. Post hoc analysis is comparable to the near-transfer lesson revealing that interaction with the teacher was again evaluated significantly higher than participating in discussions (z = -3.92, p = .001), hearing other students explain their work (z = -6.19, p < .001) or explaining one's own work (z = -7.18, p < .001), respectively. Once again, hearing explanations from the teacher was rated as significantly more helpful than hearing other students explain their work (z = 4.05, p = .001) or explaining one's own work (z = 5.03, p < .001). However, unlike during the near-transfer lesson, the activity of participating in discussions was rated as significantly more helpful than explaining one's own work to others (z = 3.26, p = .011). No other significant differences were found. As was also the case for the near-transfer lesson, no significant relationships were found between students' level of prior knowledge and their evaluation of any of the learning activities of the far-transfer lesson or their choice not to partake in any specific activities during this lesson. This result is not consistent with the results gathered during the teacher interview. One teacher voiced the opinion that flipped learning as an overall approach is not as suitable for students with a low level of prior knowledge. The other teacher expressed the opposite view that activities of the homework phase are especially suitable for such students if they exhibit a certain level of diligence.

Comparison of the near-transfer lesson and the far-transfer lesson

The Wilcoxon signed-rank tests indicate significant differences in students' evaluations between both lessons only for some of the learning activities. Students' evaluations rated individual work ('Studying on my own') during the near-transfer lesson statistically significantly higher than during the far-transfer lesson ($z = -2.21$, $p = .027$, $n = 61$). In contrast, group work was rated significantly less helpful during the near-transfer lesson than during the far-transfer lesson ($z = -3.00$, $p = .003$, $n = 49$). A similar result was found for (individual) interaction with the teacher, which was also rated significantly less helpful during near-transfer lessons than during far-transfer lessons ($z = -3.05$, $p = .002$, $n = 62$). For certain types of explanations, one-sided hypotheses were formulated. In the case of the activity of 'hearing other students explain their work', it was assumed that during far-transfer lessons, students would be able to produce fewer mathematically correct explanations than during near-transfer lessons. Thus, it was expected that the helpfulness of this learning activity would be ranked higher during the near-transfer lesson than during the far-transfer lesson. This was confirmed by the Wilcoxon signed-rank test ($z = -1.83$, $p = .067$, $n = 48$). Since rank differences occurred in the predicted direction, this result can be interpreted as statistically significant in accordance with the one-sided hypothesis. For the other learning activities evaluated with the SALG, no significant differences between the near-transfer and far-transfer lessons were found. Comparing the occurrence of the evaluation 'not applicable' for the learning activities during both lessons, the χ^2-test revealed a significant relationship for the activities of explaining one's own work to others (χ^2 (1, n = 71) = 10.81, p = 0.001), hearing other students explain their work (χ^2 (1, n = 70) = 8.89, p = 0.003) as well as interaction with the teacher (χ^2 (1, n = 74) = 36.21, p < 0.001) during class. Students who did not partake in one of those activities during the near-transfer lesson were also unlikely to partake in the same activity during the far-transfer lesson.

Discussion

In the previous section, we have examined how students evaluate different learning activities in a flipped classroom setting when class time is dedicated to near transfer and far transfer, respectively. Most of the results were to be expected, and student comments and teacher interviews largely confirm the statistical results. However, the comments and interviews cast an interesting eye on the following aspects. As generally expected, partner and group work were both rated significantly more helpful than individual work. Also, the significantly higher evaluation of individual work in near-transfer lessons in contrast to far-transfer lessons is in line with our expectations, since here tasks and activities are similar to the content of the video. In contrast, this is hardly evident from the student comments, which focus almost exclusively on the partner and group work. However, this omission can be explained by observations of one of the teachers: the students frequently opted for partner or group work during the near-transfer lesson, but observing them, one got the impression that everyone completed the tasks on their own without any exchange in between them. During the far-transfer lesson, the teacher observed a much more vivid exchange during the group work. This comparison suggests creating opportunities for individual work phases in near-transfer lessons and offering more possibilities for group work in far-transfer lessons, a concept that needs further investigation. Nevertheless, the role of the teacher should not be underestimated during individual and collaborative work. In both the near-transfer lesson and the far-transfer lesson, students find explanations by their peers to be less helpful than explanations by the teacher. Especially interacting with the teacher, which can provide the opportunity to clarify individual problems, seems to be more important during far-transfer lessons than during near-transfer lessons. This was reflected in students‘ comments in the open-ended survey question which consistently emphasized the importance of interacting with others during group work, including the teacher. Another aspect worthy of discussion concerns the level of prior knowledge. The teachers assessed the suitability of flipped learning activities differently depending on students' level of prior knowledge. In contrast, the survey results suggest that students' preferences for any of the given types of activities do not correlate with their level of prior knowledge neither for near-transfer lessons nor for far-transfer lessons. This can also be seen from analyzing the occurrences of the answer 'not applicable' in the survey. A students' choice to not engage in certain types of activities at all (like interacting with a teacher) during a lesson cannot be explained by their level of prior knowledge in our study. Instead, a comparison of the near- and far-transfer lesson suggests that the more likely explanation for not choosing certain activities is that there are students who generally communicate less on an individual level with a teacher or other students during lessons. This effect seems to be independent of the teaching design.

Conclusion

In summary, our results show that students appreciate the possibility to work in groups during the in-class phase of a flipped classroom, both in near- and far-transfer lessons. The qualitative part of the study suggests focusing on individual work during near-transfer and group work during far-transfer lessons. A correlation between students' preference for certain types of activities and their level of prior knowledge could not be detected. This could be superimposed by other variables, especially aspects of behavioral engagement (Cevikbas & Kaiser, 2021). Traditionally, research on engagement in the flipped classroom focuses on the influence of flipped learning on student engagement (Bond, 2020). We suggest that further research also investigates the reversed case, i.e., the influence of student engagement on learning activities in a flipped classroom, particularly video-related activities.

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Fingers or rods? Using AI block play recognition to endorse children's mathematical re-worldings

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Keywords: Artificial intelligence, manipulative materials, mathematics education.

Research topic

What is a wooden cube? In the Kindergarten pedagogy of Friedrich Froebel (1782-1852), children and caregivers or teachers are encouraged to playfully transform a cube through storying and actions:

> for example, the cube can be now a table on which something is placed for the child. Again, it can be a stool on which the mother places her feet; again, a chair on which she sits with the child; again, the hearth on which something is to be cooked. (Froebel, 1895, p.99)

Similarly in mathematics classrooms children may be encouraged to transform wooden and plastic manipulatives, such as interlocking cubes and Cuisenaire rods, into 'real world' objects through storying and actions. For example, in enacting a word problem, cubes may become oranges or cupcakes etc, which are physically gathered by the child. In 'bar modelling' these cubes (aka 'oranges' etc) may be lined up in a row to facilitate acoustic counting, and eventually this row may be storied as having a 'length', then drawn as a rectangular 'bar' and labelled with a symbolic number.

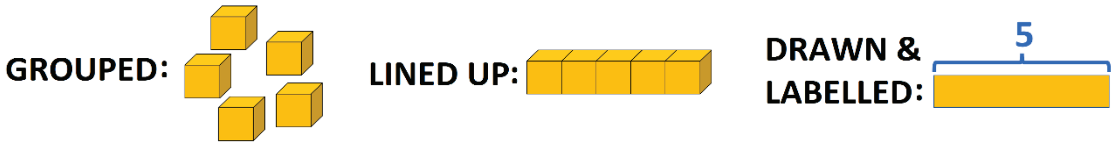

Figure 1: Typical bar modelling stages eg 5 oranges as cubes grouped, lined up and drawn as a length

In my doctoral study I am interested in studying these mathematical transformations, or 're-worldings', of wooden blocks, through children's storying and actions, often modelled and endorsed by teachers or caregivers, books, videos or apps. To support this research I was keen to explore technological tools, such as AI object detection to recognise and record children's positioning of blocks. In collaboration with a software developer, PySource, with seedcorn funding from the Bristol Digital Futures Institute, we developed an app in Python which uses artificial neural network (ANN) algorithms to recognise the placements of Cuisenaire rods on a tabletop, from a webcam, in near real time, and records the data in a spreadsheet for analysis. The rod recognition is triggered when the webcam detects no movement, ie when children take their hands away from the tabletop. In addition, the app can be programmed to respond to the placing of rods with sound files, such as speaking the colour, or length of the rod. It can also tell when two rods are placed end to end, or in parallel, and generate simple audible mathematical sentences accordingly, such as "two plus three equals five".

The app is still at a prototype stage, the ANN has only been trained on a small sample set of images so far, and requires further training and user-testing before it can be used as a reliable experimental research tool in the field. We hope as a tool it may eventually support a variety of experiments and methodologies. However, for the purposes of testing the app technically with users in a 'realistic' experimental scenario, we developed a theoretically ‘light’ methodology to inform task design, based on analysis of observed discourse as a highly reductive artificial Wittgensteinian ‘language game’, consisting of actions and interwoven language, where the actions are restricted to a child’s placements of Cuisenaire rods on a table, and the language is restricted to the app’s utterances, and questions, or ‘challenges’, posed spontaneously by the researcher, in the form of ‘Can you make it say…?’

Method

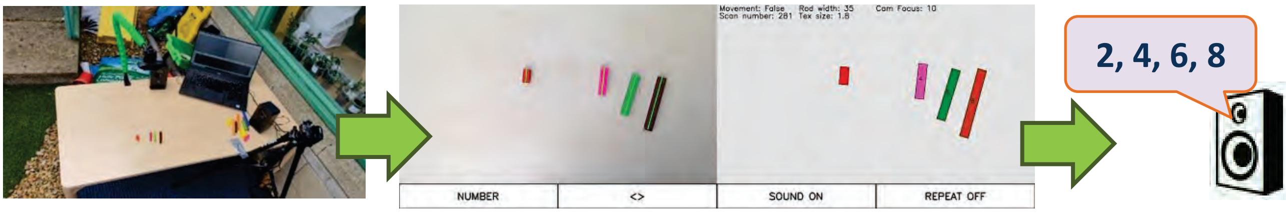

Figure 2: The app processes a webcam feed of rods placed on a tabletop and ‘speaks’ the lengths

In a local primary school in southwest England, Year 1 children (aged 5-6), who were not familiar with Cuisenaire rods, were invited individually to play a ‘game’ with rods, the app and a researcher, where the researcher challenged the children to ‘make it say’ certain numbers or sequences, such as the number five, or the two times table, or a sum, by placing rods on the tabletop. The children have to infer by experiment the ‘rules’ of which utterance is triggered by the placement of which rod. The tabletop block play was video recorded and the dialogue, and images grabbed by the app, analysed.

Results

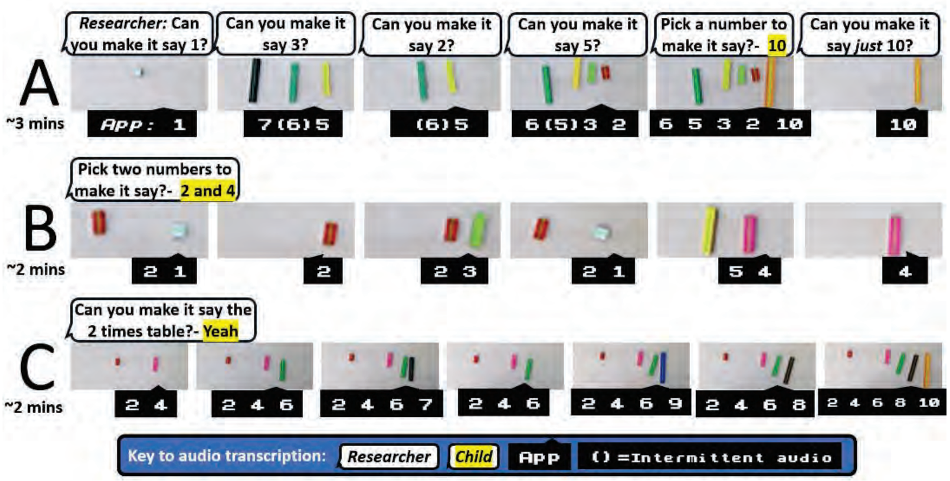

Figure 3: At first rods are placed as if ‘fingers’ eg 3 rods for ‘3’, then after experimenting , as lengths

As an early technical test we do not claim the tool was yet reliable enough for a rigorous experiment, with occasionally intermittent detection of some rods. However, on analysis of one 7-minute clip we observed an interesting shift in actions – of rod placements – consistent with a shift in inferred ‘rules’ from rods as ‘counters’ or ‘fingers’, eg placing 3 rods to (unsuccessfully) ‘make it say 3’, to rods as proportional ‘lengths’ eg placing 2, 4, 6, 8, 10 rods to (successfully) ‘make it say the 2 times table’.

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Design of digital teaching and learning materials for mathematics teacher education at the university level

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Keywords: Digital tools, OER, university mathematics education.

Motivation

In recent years there is a growing body of research concerning the use of digital tools at the university level. Whereas many studies focus mathematical programs like geometry or functional reasoning (cf. Albano et al., 2021), there has been less research concerning the use of digital tools in the context of mathematics teacher education, which addresses both, content-related as well as educational-related topics in mathematics and mathematics education. School-related research suggests, that digital technologies offer the possibility to help students autonomously acquiring mathematical knowledge (Drijvers et al., 2016), which is relevant for teacher training programs, too. Another advantage of digital learning materials is that they are not limited to one institution, as they can be distributed as Open Educational Resource (OER). Hence, OER has the potential to reshape and improve teaching in higher education (Camilleri et al., 2014). Not only the implementation of technological innovations but also their design is of great importance (OECD, 2015).

Goals and objectives of the explorative project 'DigiMal.nrw'

The development of digital learning materials for mathematics teacher education at the university level, and research on students' learning processes is the main purpose of the project DigiMal.nrw. The research activities in this project are oriented towards a design-based research approach. Developed materials address both, mathematical content and more didactical, practice-oriented content for learning situations (e.g. in seminars and lectures) and exam situations. The development and use of digital learning materials (e.g. interactive videos) and diagnostic tools (e.g. digital self-checks) shall enhance the individual student teacher's learning process to understand, reflect and use the content for learning independently of time and location. All digital materials developed will be available as Open Educational Resources (OER) on the (federal) state portal ORCA.nrw (Open Resources Campus NRW), a platform for exchange, communication and the use of digital materials.

Project group and sub-projects

DigiMal.nrw consists of five thematic sub-projects: The sub-projects 'arithmetic', 'geometry' and 'stochastics' refer to mathematical core contents. The sub-projects 'heterogeneity in mathematics classrooms' and 'subject-related language education' address didactical challenges. The project involves all eight universities in North Rhine-Westphalia (Germany) that provide teacher training for elementary school and special education. In this constellation, the project realizes a core idea of OER already in the process of the development: The material is developed and used in teams of different universities, and the material is exchanged within the project group. Hence, when distributing the material as OER on the platform, it is already exchanged and widely used in different locations so that the potential of further distribution will be increased.

First results

The developed digital learning materials vary in terms of their application and technical realization. The following types of materials have been created so far (most of them designed with the software H5P): 13 self-checks, 3 video based learning environments, in which students create videos, 27 interactive books, 1 digital introductory course, 1 digital learning environment, and several explanatory videos. Figure 1 gives an insight into different excerpts of an interactive book out of the sub-project 'geometry'. An interactive book consists of different pages. Every book page is a combination of e.g. interactive videos, GeoGebra-applets, tips, multiple choice questions, and drag and drop tasks, helping students to learn lecture contents autonomously.

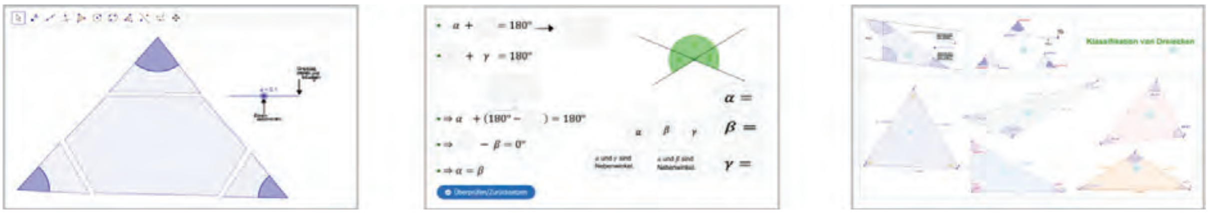

Figure 1: Extracts from interactive books (sub-project 'geometry')

The associated research projects especially address questions like the use of the digital resources. In the case of geometry, for example, first analyses of interviews and questionnaires suggest that the combination of videos, digital tasks and tips in an interactive book supports students investigating geometric proofs on their own and reflecting their work profoundly. Further evaluations will give hints for revising the material and derive implications for the design of digital learning materials.

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An impact of 3D computer and 3D printed models on the students' success in spatial ability and geometry testing

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The aim of the presented study was to measure whether 3D computer and 3D printed models could improve students' performance in testing their spatial ability and what brings the greater benefits. This study employed a quasi-experimental approach with a total of 25 secondary school students. A pre-test was employed to determine each student's level of spatial ability, namely the Mental Rotations Test. All students were taught the topic of three-dimensional geometry using 3D physical printed models, 3D virtual computer models or 2D drawings of 3D objects. A newly designed 3D geometry post-test was used in two groups of students - the first group was allowed to manipulate 3D printed models, the second group was allowed to use 3D computer models during the test. The group with 3D computer models outperformed peers with 3D printed models but a statistically significant difference was not found. Afterwards, the same students were tested again using the identical 3D geometry test without any visual aids. It was concluded that the 3D computer models provided statistically significant higher scores in comparison to the absence of any visual aid.

Keywords: Spatial ability, 3D printed models, 3D virtual models, geometry.

Introduction

Secondary school geometry is traditionally focused on studying geometric relationships in the two-dimensional space, i.e. planar geometry, and on studying solid geometry, i.e. spatial geometry. To study geometry is a big challenge and even if we live in the three-dimensional world it is difficult to work with three-dimensional objects especially when they are depicted on two-dimensional display. To be successful in the solving geometry problems spatial ability plays the main role. According to Lean and Clements (1981), *spatial ability* is the ability to formulate mental images and to manipulate these images in the mind. Very similar definition is provided by Linn & Petersen (1985) that is spatial ability generally refers to skill in representing, transforming, generating, and recalling symbolic, nonlinguistic information. Sorby (1999) distinguished *spatial ability* and *spatial skills*. Spatial ability is considered as the innate ability to visualize, whereas spatial skills are learned and acquired through training. Anyway, the both terms are very closely related and it is very difficult to distinguish them. In compliance with the literature, I will uniformly use the term spatial ability and will not examine the way in which the ability was acquired.

Spatial ability is a very essential part of intelligence and it is proved that continuous training has a great effect on its enhancement (Maresch & Posamentier, 2019, p. xv). Many studies undoubtedly proved that spatial ability of persons of different age can be trained (Maier, 1998). According to Sorby & Baartmans (2000) spatial ability can be improved through the targeted engineering graphics courses where diverse spatial activities are used ranging from manipulation of concrete models to computer visualization activities. Alias et al. (2002) state that spatial ability can be fostered trough activities predominantly consisting of free-hand sketching and object manipulation. Spatial ability is important for engineering, design, or other technology disciplines. It has been found to be very useful to a students' success in engineering related subjects as mathematics, engineering drawing, or computer-aided design (Alias et al., 2002). Moreover, spatial ability involves our mathematical, verbal, and logical capabilities and it is even crucial for our everyday life. We need the spatial ability to be able to orient ourselves in the environment, to understand the spatial relations among objects, to solve everyday tasks such as packing, moving, and many more.

The extensive research on spatial ability mainly based on factor analysis studies has resulted in the detailed categorization of spatial ability and its factors (Lohman, 1979). However, there was not a clear consistent model of different subcomponents of spatial ability. To illustrate this fact, let us mention that based on mental processes which are used for solving certain tasks McGee (1979) described two major components (factors) of spatial ability – *spatial visualization* and *spatial orientation*. On the other hand, the classification proposed by Lohman (1979) consists of three basic spatial ability factors – *spatial relation*, *spatial orientation*, and *visualization*. Five major factors of spatial ability – *visualization*, *spatial relations*, *closure speed*, *flexibility of closure*, *perceptual speed* were detected by Caroll (1993). Yilmaz (2009) provides a picture of a comprehensive model also with some another components of spatial ability. It seems that the number of underlying factors of spatial ability varies from study to study. However, *visualization*, *spatial relation*, *mental rotation*, and *spatial orientation* are subcomponents of spatial ability which are nowadays often designated as the relevant (Maresch & Posamentier, 2019). I will consider these subcomponents also in my research.

If we need to move or to alter in our mind some parts or the whole mentally presented objects, then it is considered as a spatial visualization task. Spatial visualization is the ability to imagine manipulating, moving, rotating, twisting, or inverting objects without reference to one's self. It means that the imagined object or its parts are moved or changed in our minds. Spatial relation works with the mental comparing of objects and with the identification of object parts which fits together. This subcomponent is not completely independent from the visualization subcomponent. The subcomponent mental rotation is the ability to imagine rotating a two and three-dimensional object or figure. Finally, the factor spatial orientation requires one's ability to imagine the appearance of an object from different perspectives. In other words, the imagined object does not mentally move, it remains the same and we mentally move ourselves to different viewpoints. Nowadays, the identification and description of the strategies for solving spatial problems emerge as a very interesting topic and it is brought into focus of the researchers. The classic research methods on factors of spatial ability assume that the spatial ability tasks of some category are solved using the same intended strategy. From the literature (Maresch & Posamentier, 2019) and also according to my own experiences with students, we know that geometric tasks are solved differently by different individuals.

In my research, I focus on the measurement and improvement of spatial ability of secondary school students. On the ground of the study, I include the optimal training and teaching methods into mathematics instructions to improve students' spatial ability. The objective of my research is to measure whether the 3D computer models and 3D printed models could improve students' performance in testing their spatial ability.

Spatial ability and the use of dynamic geometry and 3D printed models in mathematics education

In the presented research, let us focus on secondary school mathematics, namely on the topic of solid geometry. This traditional geometric topic covers the study of two-dimensional and three-dimensional Euclidean space. It includes the study of properties of and relationships between geometrical objects in the plane and in the three-dimensional space. It includes the measurements of volumes of various solid figures such as pyramids, prisms, polyhedrons, cylinders, cones, truncated cones, or spheres, cross-section of solids, transformations of two- and three-dimensional shapes in the plane and the three-dimensional space. Students determine the relative positions between two geometrical figures – lines and planes. Very typical tasks of solid geometry are to determine the size of an angle formed by two rays, to find the size of a dihedral angle, i.e. the angle between two intersecting plane, or to compute the size of an angle between a plane and intersecting straight line. Usually students solve such tasks depicted in two dimensional situations; so the proper visualization and its correct interpretation are crucial here. In the Czech mathematics textbook, usually the oblique projections are used. An oblique projection is a simple type of parallel projection which produces two-dimensional images with the specific properties (Carlbom & Paciorek, 1978). It holds that parallel lines are projected into parallel lines. If a polyhedron is projected, usually some its face or faces are parallel to the image plane (then these faces are projected in true shapes and sizes and are not distorted). I insist on working with an arbitrary position of projected solids, i.e. solids can be viewed from above from the right, from above from the left, from below from the left, and from below from the right. This is usually neglected in mathematics textbooks and only one position of the viewpoint is used.

I use dynamic software GeoGebra in my mathematics instructions. Indisputably, GeoGebra belongs among DGS (dynamic geometry systems) which are the most widespread all over the world among teachers and students. It is open-source software which is easy to use and understandable even for the absolute beginners. According to my experiences, GeoGebra software can be used with a potentially positive impact in teaching and learning process especially in such cases where its dynamic features can be used. It offers basic functions to model solid figures or more complex three-dimensional situations. Moreover, the modeled situation can be arbitrary rotated so it is viewed from different viewpoints; so it can substitute the real physical model to a certain extent.

Although, the computer-aided education is very modern and popular and brings indisputably advantages to the process of education, physical object manipulation plays the important role in the learning geometry and enhancing spatial ability. Especially action oriented training methods that work with real models have always shown good results in the improvement of spatial ability (Maier, 1998). This can be based on the approach of embodied cognition which emphasizes that cognition involves a motor behavior (Schneegans & Schöner, 2008).

In my mathematics instructions related to geometry, I use 3D computer models together with physical 3D printed models of solid figures and of three-dimensional situations as a visual aid. The properties of geometrical objects are demonstrated in my instructions using 3D virtual models and physical 3D printed models or students can use these models as a visual aid when they are solving the geometric tasks.

Methodology

As has been already pointed out, my aim is to measure students' success rate in geometric tasks; specifically whether and how much the 3D computer models and 3D printed models could improve their performance in testing their spatial ability. In order to explicate the effectiveness of visual aids, a quasi-experimental approach was chosen. 25 students (one class) in the third grade (17-18 years old) of a secondary school (grammar school) in the Czech Republic were involved in the experiment. The topic of solid geometry was taught for two months at the end of the school year 2020/2021. Partially it was a distance learning (3 weeks) which was caused by the worldwide pandemic situation; the rest was standard face-to-face education. All students were taught the topic of three-dimensional geometry (solid geometry) using 3D virtual computer models, 3D physical printed models (this visual aid was used only at school), or 2D drawings of 3D objects. The teaching method was the same for all students in the experiment realized by one teacher.

Firstly, the students took the *Mental Rotations Test* as a pre-test to determine their level of spatial ability. The students received the test approximately in the middle of the period when the teaching of solid geometry was realizing. The Mental Rotations Test is one of the most common instruments for measuring spatial ability. The original test was developed by Vandenberg & Kuse (1978) and it contains 20 items in five sets of four items. Each item consists of a criterion figure, two correct alternatives and two incorrect ones. The alternatives are always shown in a rotated position. Each item in the test is counted as correctly answered if both choices are correctly chosen. This eliminates the need to correct for guessing. The reliability of the test has been found satisfactory. In a sample of 3,268 adults and adolescents of age 14 year or older, was .88; in a similar sample of 336 subjects, the test-retest correlation was .83 after an interval of one year or more, and in an age corrected sample of 456 the test-retest reliability after a year or more was .70 (Vandenberg & Kuse, 1978). For the purposes of the research, I used a redrawn modified version of the Mental Rotations Test called MRT-A (Peters et al., 1995) with the official permission from the author Michael Peters. This modified version of the test consists of 24 items and the nature of the test is completely the same. The maximum score of the test was then 24 points.

Secondly, the students divided into two coherent groups took a newly designed post-test on solid geometry at the end of that two month period. According to the results from the Mental Rotations Test, it was shown that the both groups of students were close to equal. The first group of students (Group 1, n_1=13) was allowed to manipulate 3D printed models during the test as the visual aids; the second group of students (Group 2, n_2=12) was allowed to use 3D computer models during the test as the visual aids. Students of Group 1 are denoted by G1, students of Group 2 are denoted by G2. The 3D geometry post-test consists of five types of geometric problems in the three-dimensional space – the relative positions between two lines (12 individual tasks), the relative positions between a line and a plane (6 individual tasks), the size of an angle between two lines (3 individual tasks), the size of an angle between a line and a plane (3 individual tasks), and the size of an angle between two planes (3 individual tasks). All the geometric problems were depicted in two dimensional situations, i.e. projected to the plane; the oblique projection was used in all cases and different positions of the viewpoint were considered. In Figure 1 you can see three concrete tasks of the relative positions of two lines. Students were asked whether the depicted lines (drawn in the auxiliary cube) are parallel, intersecting, or skew. In Figure 2 you can see three concrete tasks of the size of an angle between a line and a plane (again drawn in the auxiliary cube). Students were asked to determine the size of an angle without any algebraic calculation. Every single task was for one point, so the maximum score of the test was then 27 points.

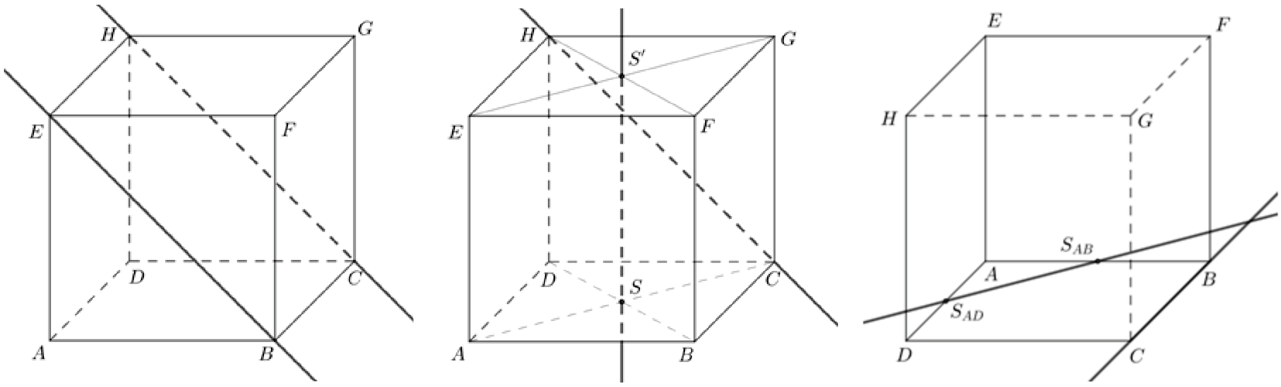

Figure 1: The relative positions of two lines – parallel, skew, intersecting, respectively

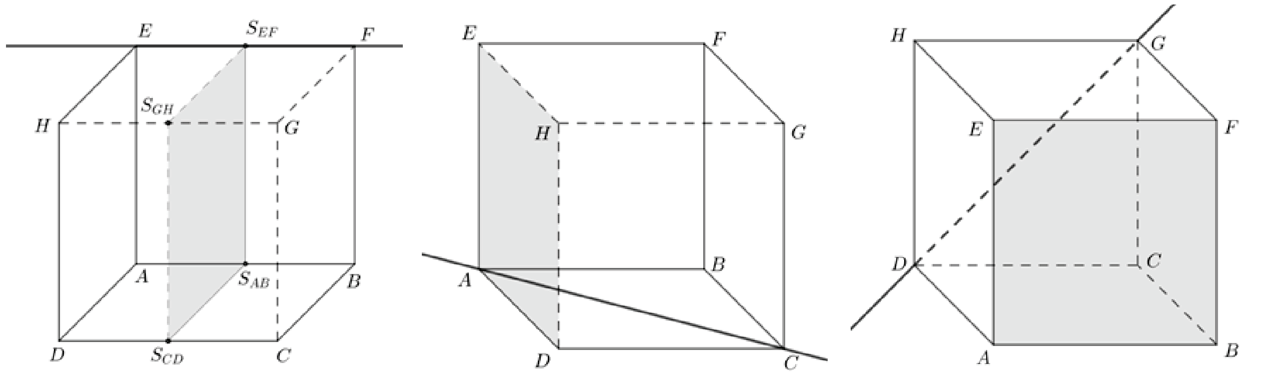

Figure 2: The size of an angle between a line and a plane – 90°, 45°, 0°, respectively

3D virtual computer models were created in GeoGebra software and 3D printed models were made on the 3D printer (Felix 3.0); the both by the author. The 3D virtual computer model is shown in Figure 3, on the left. Students were allowed to draw arbitrary lines into the prepared cube and they could rotate with the three-dimensional situation in GeoGebra software. 3D printed models are shown in Figure 3, on the right. Students were allowed to use them together with the sticks to model lines.

Thirdly, the same students were tested again using the identical 3D geometry test without any visual aids at the beginning of a new school year 2021/2022. For this final testing and for the interpretation of the results, the G1 is newly designated G1' and the G2 is newly designated G2'.

The relationship between the use of 3D computer models and 3D printed models and students' spatial ability was investigated in this research by focusing on the following research questions: What visual aid brings greater benefit to students when they solve geometric problems? Can 3D computer models and 3D printed models improve the students' performance in testing their spatial ability?

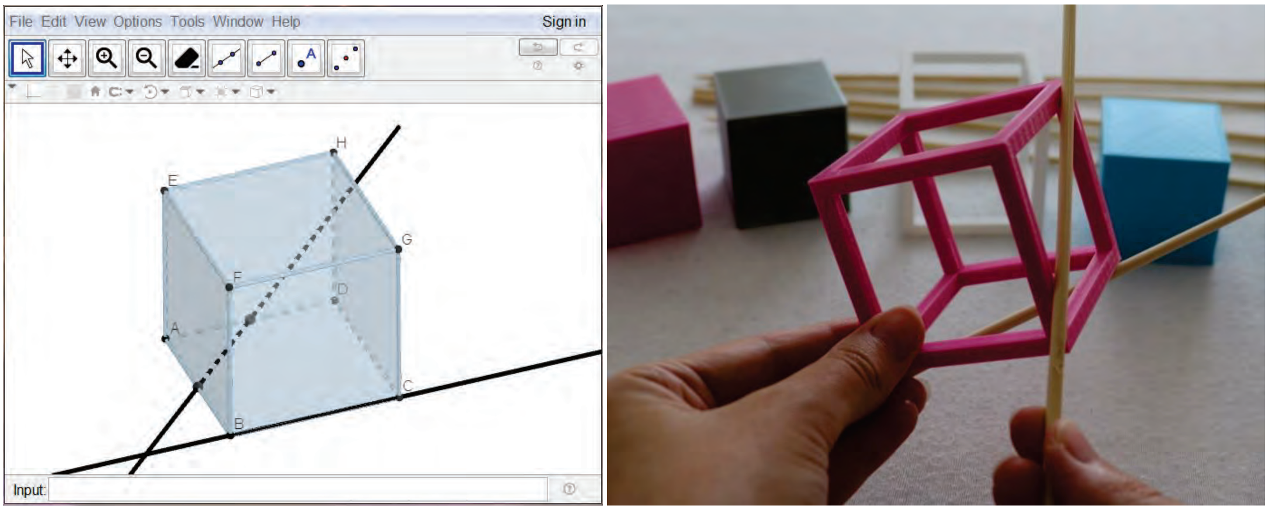

Figure 3: 3D virtual computer model in GeoGebra with allowed functions for drawing an object (a point, a line, a segment line, a midpoint) and functions for control (move graphics view, zoom in, zoom out, delete) on the left, 3D printed models with auxiliary sticks on the right

Students' scores in the Mental Rotations Test and in 3D geometry test were interpreted using the Mann Whitney U or the dependent *t*-test for paired samples. The significance level was chosen 0.05 in each test.

Results

There were no significant differences between the two groups (G1, n_1=13 and G2, n_2=12) of students as measured by the Mental Rotations Test. The G1 had a mean score of 16.92 and the G2 had a mean score of 16.25. The second method of data collection involved students' scores in 3D geometry test. The G1 that was allowed to manipulate 3D printed models during the test had a mean score of 21 and the G2 that was allowed to use 3D computer models during the test had a mean score of 22.92. The Mann-Whitney U test was used to compare differences in students' scores between the groups. The critical value of U at $p < 0.05$ is 41 which is lower than the test criterion of 52.5. Therefore, the result is not significant at $p < 0.05$. Students' performance can be observed in Table 1. Students are sorted according to their score (S) in the test and are assigned ranks (R). It can be visually observed that the group (G2) with 3D computer models outperformed peers (G1) with 3D printed models but a statistically significant difference between these two groups was not found.

Table 1: Students performance scores in 3D geometry test – Group 1 and Group 2

	G2	G1	G1	G1	G1	G1	G2	G2	G1	G2	G1	G1	G1	G2	G1	G1	G2	G2	G1	G2	G2	G1	G2	G2	G2
S	13	16	16	17	19	20	20	20	21	21	22	22	22	22	23	23	23	23	25	26	26	27	27	27	27
R	1	2,5	2,5	4	5	7	7	7	9,5	9,5	12,5	12,5	12,5	12,5	16,5	16,5	16,5	16,5	19	20,5	20,5	23,5	23,5	23,5	23,5

The third method of data collection involved students' scores in 3D geometry test again. This time any visual aid was not allowed. The first group (G1', n_1=13) had a mean score of 19.08, the second group (G2', n_2=12) had a mean score of 20.25. The dependent *t*-test for paired samples was used to compare the means between groups G1 and G1', and G2 and G2'. In the first comparison of groups G1 and G1', the p-value is 0.00043. The result of the test is significant at $p < 0.05$. In the second comparison of groups G2 and G2’, the p-value is 0.00185. The result of the test is significant at $p < 0.05$. So the both tests showed that there are statistically significant differences between G1 and G1’, and between G2 and G2’ too.

Discussion

I analyzed the effects of using 3D computer models and 3D printed models on students’ success rate in testing their spatial ability. Surprisingly, the group with 3D computer models performed better than the group with 3D printed models but a statistically significant difference between these two groups was not found. According to Katsioloudis et al. (2014), 3D printed models have greater positive effect on spatial ability than 3D computer models. This inconsistency could be probably caused by a small sample but still the difference was not significant. Using of 3D physical models is also supported from other researchers (Maier, 1998; Alias et al., 2002). The positive effects of using 3D virtual computer models on spatial ability over not using of any visual aid are consistent with earlier studies (Katsioloudis et al., 2014). Not to mention students’ reactions on visual aids; based on my observations students were much more motivated to solve geometric tasks if they were allowed to use some visual aid. Moreover, final testing showed that students’ results in the same 3D geometry test without any visual aids were worse even though they were solving the same geometric tasks.

Conclusion and future plans

This small quasi-experimental research showed that the use of 3D visual aids has a great potential in the process of mathematics education, namely in the topic of solid geometry. The study resulted in favor of 3D virtual computer models. It was demonstrated that 3D computer models can help students to better understand three-dimensional geometry. On the ground of this study, I include the optimal training and teaching methods into my mathematics instructions to improve students’ spatial ability. I plan to continue using the both types of visual aids. I also plan to repeat these experiments with bigger samples and with modified versions of the 3D geometry tests. The reliability and the validity of newly designed tests is also planned to be measured.

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Developing digital scaffolding modules to support oral description skills – Results of a teacher survey

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Oral descriptions of mathematic phenomena are indispensable for mathematic conversations in primary school but pose challenges for learners on a conceptual and linguistical level. Digital media offers potential to foster the oral descriptive skills, but the application of digital language scaffolding in mathematics has not been researched yet. Therefore, a digital scaffolding tool addressing primary school learners' oral descriptions of mathematical patterns and structures using the app 'Book Creator' is being developed in a design research project following the Integrative Learning Design Framework. This paper reports on teachers' evaluation of a prototype and its already implemented design principles, in order to identify implications and suggestions for the development in further micro-cycles. Results show that the current version is considered to be fruitful for primary school learners already but should be revised concerning the linguistic level and the cognitive activation.

Keywords: Digital scaffolding, Language in mathematics education, Design research.

Introduction

Oral descriptions are a very important part of communication in primary school mathematics as they lay the foundation for sharing different approaches to solve problems, explanatory statements and lines of reasoning, however, the necessary linguistic and conceptual requirements are challenging for many learners (Maisano, 2019). One possibility to promote language – including oral descriptions – in the mathematic classroom is by providing scaffolds according to the learners' needs. It has been shown that digital scaffolding that focusses on both the language and the mathematical content has positive effects on the students' math test scores and their math self-efficacy (Freeman, 2012). However, there are no findings yet whether digital scaffolding also has a positive impact on the students' language use in the mathematic classroom as seen from a commognitive perspective (Sfard, 2008). To fill this research gap, our design research project 'DeScriBe', following the Integrative Learning Design Framework (Bannan, 2013), aims at micro-cyclically developing and researching a digital scaffolding tool to promote the learners' descriptive skills with the app 'Book Creator' (https://bookcreator.com). This paper presents the development process of prototypical digital language supportive modules as well as results of a Local Impact Evaluation (Bannan, 2013) by teachers to assess the language supportive modules' technology-specific design principles' (DPs) fruitfulness for the learners' learning processes as well as the usability of the digital scaffolding tool.

Implications of oral descriptive skills for mathematical learning

Oral descriptions in the primary mathematics classroom can verbalize observable patterns and structures and therefore promote the children's understanding of the mathematic contents while simultaneously promoting the language skills themselves (cf. Lüken & Tiedemann, 2019). According to Sfard's (2008) commognition framework both thinking and talking about mathematics are communication processes, the former on an intrapersonal level, the latter on an interpersonal level.

This fusion of cognitive and communicative processes shows that language is required to both think and talk about mathematics and that communication is an integral part of doing mathematics. As such, communication is subject to determined routines in the process of mathematizing between the interlocutors (Sfard, 2008). A long-term-goal of mathematics education is to enable learners to explain mathematical patterns and structures. These explanations are based on the description of a mathematic phenomenon, the procedure of describing a phenomenon therefore needs to be established as a routine in the classroom. Since there exists a relation between the proficiency in the academic language and the access to knowledge and a successful education respectively, there is a mathematic educational interest in fostering the school or technical register (Tiedemann, 2015). Functional and generalized descriptions cannot be expressed in an everyday register (cf. Prediger & Wessel, 2013). The necessity to use a school or technical register to express these descriptions can however be challenging, especially for primary school learners. A promotion of oral descriptions is therefore necessary to lay the foundations for future explanations and argumentations. Teachers should make their context-specific linguistic norms explicit (Tiedemann, 2015) and include the student's progress related to the educational code in self-evaluations and feedback (Hammond & Gibbons, 2005). Besides the linguistic demands, oral descriptions also require a conceptual understanding of the phenomenon that is to be described, therefore language promotion should always go hand in hand with the mathematic content (Prediger & Wessel, 2013).

Fostering oral descriptive skills - analogue and digital

Scaffolding consists of specific support measures provided by more experienced persons that help learners reach their zone of proximal development (Hammond & Gibbons, 2005). Successful analogue measures that can also serve as the basis for digital language supportive modules in an adapted form are – among others – the inclusion of the learners' progress related to the educational code in self-evaluations and feedback, provision of transparency of the expectations as well as the use of different representational means that help the learners to structure their learning process (Erath et al., 2021). A checklist allows an integration of these facets as it is both a tool to work with for the learners as well as a synopsis of important aspects of a good oral description. In relation to specific tasks, it has been proven to be useful to offer sentence phrases and guidelines for formulations as well as posters with the necessary vocabulary and thus provide linguistic models (Erath et al., 2021). However, the limit of the analogue fostering of the oral descriptive skills is reached when learners cannot use a written linguistic support due to an uncompleted acquisition of the written language or insufficient reading comprehension. This problem is pervasive in primary schools (Mullis et al., 2017).

Digital scaffolding generally uses the same strategies as analogue scaffolding but offers additional opportunities by using text-to-speech technology (TTS) or linked scaffolds in hypertexts. Both features support learners with difficulties in the written language (Dalton & Strangman, 2006). Referring to Mayer's (2007) cognitive theory of multimedia learning, the use of digital media, which enables the combination of different media, such as pictures, videos, texts, TTS and audio samples can help students to engage in meaningful cognitive processes. Audio recordings with the voice memo function of a tablet can be played back multiple times and therefore avoid that the volatileness of oral utterances might hinder teachers and learners from precisely evaluating the descriptions.

The project ‘DeScriBe’

For the reasons given above, the project ‘DeScriBe – **D**igital**e** **S**pra**c**hfö**r**derbauste**i**ne für das mündliche **Be**schreiben mathematischer Strukturen’ (digital language supportive modules for oral descriptions of mathematical structures) aims at developing and researching a digital scaffolding tool to foster oral descriptions of mathematic structures. To develop the ‘digital scaffolding’, the project follows the design research approach of the Integrative Learning Design Framework (Bannan, 2013). This framework encompasses the four phases of Informed Exploration, Enactment, Local Impact Evaluation and Broad Impact Evaluation that are conducted in iterative micro-cycles to cover “the entire scope of research from initial conceptualization to diffusion and adoption” (Bannan, 2013, p.116). While the Informed Exploration phase built the theoretical base for the development, in the Enactment phase design principles and a first prototype of the language supportive modules were developed. The current Local Impact Evaluation phase puts to test, whether the developed design is usable, accessible, efficient and supports learning, by having different stakeholders rate the design which may then lead to the next micro-cyle for improvement (Bannan, 2013). This paper presents the theory-based development of prototypical digital language supportive modules as well as results of a Local Impact Evaluation by four teachers. As experts and stakeholders for the future adoption of the digital language supportive modules in the classroom, they test and review the digital scaffolding tool as well as the potential and challenges of the already implemented DPs.

The analysed potential of digital tools for scaffolding oral descriptions led to the formulation of the following language supportive modules and corresponding technology-specific DPs (Table 1):

Table 1: Language Supportive Modules and Design Principles

Language Supportive Modules	Design Principles (DPs)
Digitalisation of a Checklist for a Good Oral Description (Module 1)	• Interactive Feedback (DP 1.1) • Linking of Supportive Means (DP 1.1)
Digital Support for the Development of Meaning-Related Language (Module 2)	• Multimedia-Based Relation of Different Representations (DP 2.1) • Digitalisation of Task-Specific Scaffolding Means (DP 2.2) • Digital Differentiation (DP 2.3)

Enactment phase – Functionality of the digital scaffolding and implemented design principles

The digital language supportive modules are realised in the app ‘Book Creator’, a tool for effortlessly creating and sharing digital books. ‘Book Creator’ has been chosen due to its degree of popularity in German primary schools and its features (https://bookcreator.com/features/) that offer opportunities to overcome the limits of analogue scaffolding by integrating different media, TTS, linking pages to avoid difficult orientation to find the desired scaffolds and sending scaffolds via Airdrop or a QR-Code which allows a use in the classrooms as well as in distance learning settings or for homework. In the current version of the scaffolding tool, two DPs have already been implemented: **‘**Linking of Supportive Means**’** (DP 1.2) and ‘Multimedia-Based Relation of Different Representations’ (DP 2.1). In the following we will outline the central features of the prototypical tool and how it can be used to support learners’ competencies in orally describing mathematic patterns and structures.

The DP ‘Linking of supportive means’ has got the subject-specific aim of providing scaffolding elements according to the learners’ personal needs to allow content- and language-integrated mathematical learning (Prediger & Wessel, 2013). This aim should be reached by using the digital potential of hyperlinks as well as dialoguing and controlling interactivity (Hillmayr et al., 2020). It is implemented in the language supportive module ‘digital checklist’ and realised through the button “Tips” that leads to specific scaffolds of the respective aspect of the checklist (Figure 1).

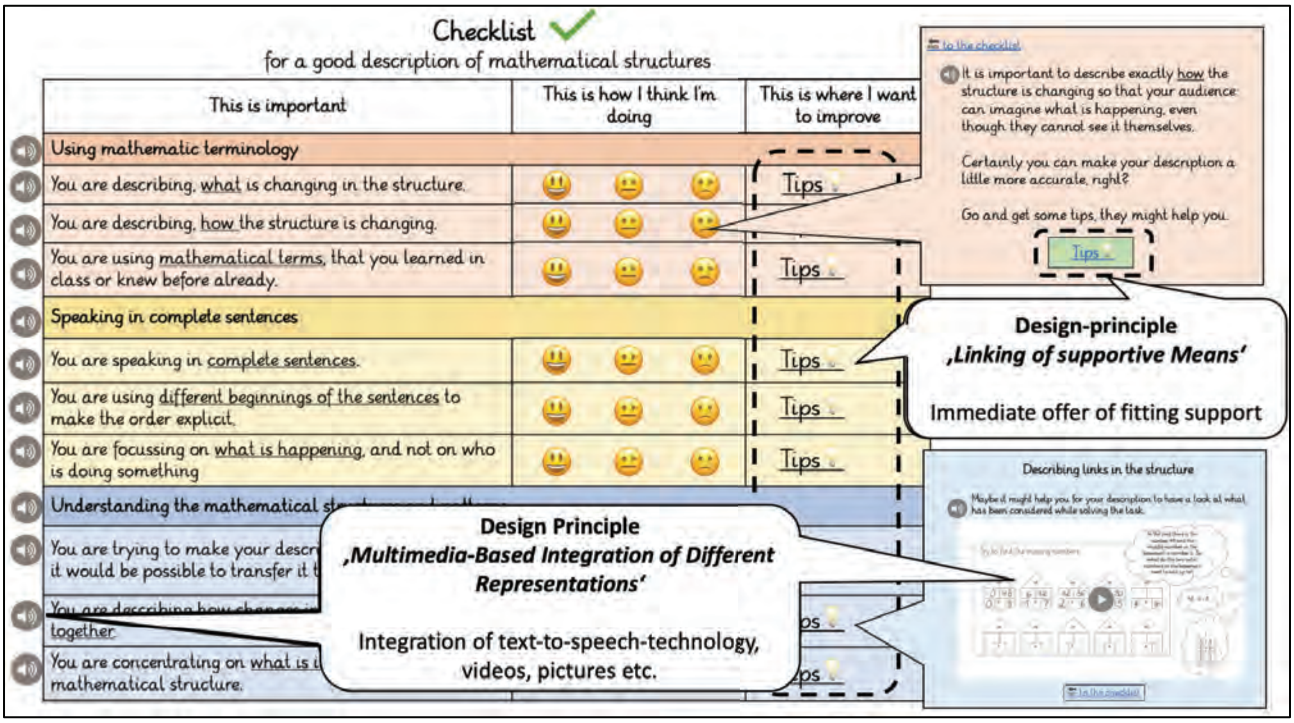

Figure 1: Implementation of DPs in the Digital Scaffolding Tool (translated screenshots)

The DP ‘Multimedia-Based Relation of Different Representations’ aims at the subject-specific goal of changing and relating registers (Prediger & Wessel, 2013). This aim should be reached by using the digital potential of the TTS (Dalton & Strangman, 2006), multiple external and linked representations (Walter, 2018) and by shifting learning processes to allow more capacities to focus on the mathematical content (cf. “cognitive load theory” in Chandler & Sweller, 1991). This DP is realised by enriching the digital checklist and scaffolds (that could generally also be used in an analogue form) via different related media, like audio samples, videos, TTS etc. (Figure 1).

Research questions and methods of the Local Impact Evaluation

The main goals of this Local Impact Evaluation were to assess the fruitfulness of the implemented DPs as well as to check the usability of the basic functions already available in the language supportive modules to compile aspects that should be considered when developing digital language scaffolds and to improve the design of the digital tool in the following micro-cycles. Teachers were interviewed as experts since they are competent in rating the helpfulness of (digital) scaffolding measures according to their learners’ needs. In addition, it was expected that as stakeholders they could think about possible extensions of the scaffolding tool on a meta level and therefore provide useful ideas for the future development. The following questions were leading: (1) What potential and challenges do teachers see in the implemented DPs? (2) How do teachers rate the usability of the basic functions in regards to their learners' experiences with digital tools? (3) Which additional functions are desired and suggested by teachers with regard to the support of oral descriptive skills?

The data collection was based on semi-structured guided interviews (Galletta, 2013) with a total of four teachers in which the teachers first used the scaffolding tool themselves and then answered open questions that asked whether the teachers would rate aspects of a respective DP, like TTS, videos or links as supportive for their learners and explain their assessment. Their answers were then analysed with a qualitative content analysis (Kuckartz, 2019). The resulting categories of the content analysis have been subdivided into positively and negatively connotated categories, named 'potential' and 'challenges'.

Results

(1) Potential and challenges of the digital scaffolding tool

The four teachers were asked to evaluate the usability and practicability of the digital language supportive modules in the scaffolding tool. A synopsis of the categories generated via qualitative content analysis, complemented with some exemplifications, can be found in the following Table 2.

Table 2: Assessment of the Digital Scaffolding Tool

Potential	Challenges
• Informative with regard to the aspects of an oral description • Intuitive handling • Very well arranged due to the use of different colours, underlined important words and the use of smileys for self-evaluation • Well suitable for primary school learners • Allows a versatile use in the mathematics classroom • Linguistic level well suitable for children with good knowledge of German	• Linguistic level might be too high for learners with little knowledge of German • Very extensive – might lead to an overload of weaker learners • Requires the skill to self-assess one's description • Requires previous experience with checklists • Some buttons do not function properly yet • 'Book Creator's' function to turn over pages might lead to difficulties in the classroom when learners accidently use this function

These categories indicate that in general, the teachers see opportunities for the use of the digital scaffolding tool in the primary mathematics classroom since they assess it to expatiate on the context-specific linguistic norms as claimed by (Tiedemann, 2015). While they rate the checklist as being well arranged and the digital aspects as being intuitive, it also has to be summarized that the teachers see challenges in including the student's progress related to the educational code in this form of self-evaluations that is based on Hammond and Gibbons (2005) claims, since self-evaluation requires additional skills of self-reflection that need to be acquired before working with this tool. However, all teachers claimed that this challenge can only be solved by a change of the socio- and sociomathematical norms and not by the digital language supportive modules themselves. When applying Sfard's (2008) idea of discursive routines on the 'rules' that are being set in the checklist, it might be concluded from the teachers' assessment that the routine that is supposed to be established with this tool might be not accessible for all learners yet due to the high linguistic level and the extent of the language supportive module. The technical issues reported as challenges will be revised.

(2) Potential and challenges of the implemented design principles

In addition to the assessment of the digital scaffolding tool, the teachers were also asked to evaluate the language supportive modules with a focus on the respective DPs' estimated fruitfulness for their students. The consolidated results of their assessment can be found in the following Table 3.

Table 3: Assessment of Potential and Challenges of the Implemented DPs

Design Principle	Potential	Challenges
Linking of supportive means	• Immediate reaction to the self-evaluation • Does not require posters and/or tip cards • Helps the learners to decide where support is needed • Use of links in the checklist before evaluation can help to get a better idea of the statement	• Risk of students' unfocussed use of all the linked elements due to easy accessibility
Multimedia-Based Relation of Different Representations	• Video that relates verbal, symbolic and (animated) iconic representations is very helpful for both the linguistic and conceptual understanding • TTS is particularly useful for weaker readers • Relation of different media caters for different types of learners • Related verbal and non-animated iconic representations in audio samples can be supportive for the learning process (mixed review)	• Related verbal and non-animated iconic representations in audio samples can be hard to follow (mixed review) • Headphones need to be used to make full use of the media • Risk of students' unfocussed use of TTS due to easy accessibility

The teachers' assessment of the DPs' current realization supports Dalton and Strangman's (2006) statement that TTS and linked scaffolds support learners with difficulties in the written language. Their positive evaluation of the implementation of different media second Erath et al.'s (2021) suggested support measure to use different representational means that help the learners to structure their learning process of language for mathematics. However, the mixed review of the relation between audio samples and non-animated representations points out that the use of different representations is not perceived as helpful in general, but only when fitted exactly to the different learners' needs. The teachers' ideas to improve this relation of representations are presented in the section 'Suggested additional functions'. The perceived challenges mostly consider the practical use of digital media in the classroom and can be traced back to the reported insufficient equipment of and experiences with digital media in primary schools.

(3) Suggested additional functions

To collect input for the future development in the next micro cycle of the integrative learning design, the teachers were asked to propose additional functions on both the level of the DP 'Multimedia-Based Relation of Different Representations' and the practicability for using the digital language supportive modules in their classrooms. Concerning the first aspect, they propose a stronger inclusion of the typical non-verbal illustrative and representational means like arrows, colours or magnifying glasses to help the students develop a conceptual understanding. Even though they already appreciated the video as a relation of different representations, they suggest to make even more use of the digital potential by linking the spoken text in the video and the visual stimuli closer together – e.g. while saying "the sum in the roof" the respective number should simultaneously be enlarged and accentuated. They also propose an interactive design of the video to include the students more and avoid passive watching. All these features could be summarized under the quality criteria of cognitive activation that is supported by the media design (Korntreff & Prediger, 2021).

Conclusion and Outlook

In this paper we discussed the Enactment and Local Impact Evaluation of a learning design's development addressing digital scaffolding using the app 'Book Creator'. The outcome of the Enactment phase is a prototype of the digital language supportive modules which can be used in order to support oral descriptions of mathematical structures. Given that language promotion is always supposed to go hand in hand with the mathematic content (Prediger & Wessel, 2013), it needs to be considered that this digital scaffolding tool can only be used to accompany the mathematics lessons and not to promote the mathematical content on its own. The results of the Local Impact Evaluation indicate that the general idea of a digital language scaffolding tool is well accepted by the teachers and their list of perceived potential (Tables 1 and 2) is long. In the following micro-cycles revisions of the current language supportive modules will be necessary to better align their linguistic standards with the primary school learners' competencies. To allow a digital differentiation that helps learners reach their zone of proximal development, it is necessary to analyse the competencies learners need in order to perform good descriptions of mathematic structures. This analysis will be undertaken in the PhD-project of Sophie Tittel. The teachers' suggestions for improving the videos by stronger linking the visual and auditive stimuli will be implemented in the future and may also be applied in other projects and explanatory videos. In addition, it is planned to further implement the DPs 'Interactive Feedback', 'Digitalisation of Task-Specific Scaffolding Means' and 'Digital Differentiation' on the basis of teachers' feedback in the next iterative micro cycle and evaluate their local and broader impact.

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Construction of Mathematical knowledge in digital-collaborative settings

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There are many different ways in which technology can enhance learning and teaching of mathematics. Therefore, a specific research focus is needed. One of the foci is to look at the role technology plays in learning mathematics through the lens of communication – especially the way in which it influences the construction of mathematical knowledge in interaction. To do this, student teachers were filmed while using Padlet to collaboratively summarise and discuss characteristics of different mathematical functions. First results hint at Padlet being a useful tool to induce and support meaningful mathematical discourse and mathematical thinking.

Keywords: Communication, technology, digital media, collaboration, knowledge building.

Introduction

Although the use of digital media for educational purposes has become an omnipresent topic – whether it is in research, politics or every-day-conversation – teaching with technology is often based on best-practice approaches or personal preference (Rink & Walter, 2020). Research also shows that while students use technology on a daily basis, they do not use it for academic purposes as often (Zawacki-Richter, 2021; Dolch & Zawacki-Richter, 2018). Although the academic use of technology in schools and universities has significantly increased over the last year, it often remains on a 'consuming' level (e.g. reading documents or watching videos) and rarely leads to subject-rich learning processes (Zawacki-Richter 2021; Biermann & Kommer, 2012). Considering that communication is fundamental for learning in general (e.g. Miller, 2002) and developing mathematical knowledge in particular (Steinbring, 2015), this problem is addressed by making communication the key focus when looking at digital media used in education (Ball & Barzel, 2018).

In this paper, we focus on the question of how technology influences communication and, thus, the construction of mathematical knowledge in collaborative settings. In what follows below, a brief theoretical background is given on *Construction of Mathematical Knowledge in Communication* and *Technology in Mathematics Education*. Afterwards the *Methodology* as well as some initial *Results* are presented.

Theoretical Background

Studies by Sung, Yang and Lee (2017) show evidence that digital-collaborative learning may lead to an increased learning performance and a more positive attitude towards learning in general. However, one should not assume that using technology will always be beneficial as "all [digital tools][…] come with affordances and limitations, with opportunities and constraints" (Drijvers, 2019, p. 9). With a view to university levels, research indicates that technology is in fact used a lot to share documents or to collect data (e.g. in online courses), but is only infrequently used to encourage collaborative learning (Zawacki-Richter 2021; Dolch &

Zawacki-Richter, 2018; Biermann & Kommer, 2012). This is problematic since working on tasks collaboratively in small groups is one of the best ways to induct productive mathematical thinking (Nührenbörger & Steinbring, 2009). Students working and learning together are forced to communicate and interact with each other. "Statements and suggestions are offered for joint consideration. These may be challenged and counterchallenged, but challenges are justified and alternative hypotheses are offered" (Howe & Mercer, 2007, p. 6). These situations are of uttermost importance, because – following enactivism – learning itself is manifested in communication (e.g. Miller, 2002).

Learning mathematics is somewhat unique since it is not accessible by senses (Ball & Barzel, 2018). In fact, *mathematical knowledge* is not a given set of definitions and theorems, but is only accessible "using signs, words or symbols, expressions or drawings" (Duval, 2000, p. 61). The meaning of those signs, symbols or expressions has to be constructed by the learners themselves during the act of communication (Steinbring, 2006). Therefore, communication is at the center of constructing any mathematical knowledge. Based on this underlying interpretation of learning, the construction of mathematical knowledge can be modeled using Steinbring's *epistemological triangle*:

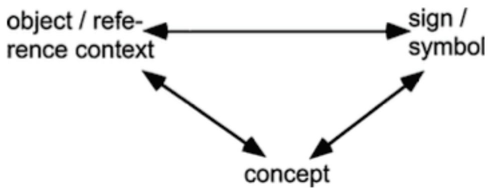

Figure 1: Epistemological Triangle

"Mathematics requires *certain sign or symbol systems* in order to keep a record of and code the knowledge. […] [The meaning of those signs] has to be produced by the learner by means of establishing a mediation to suitable *reference contexts*" (Steinbring, 2006, p. 135). A sequence of such triangles can be used to cover the whole learning process (Ball & Barzel, 2018) and to examine the impact of technology on those processes. "To use technology for effective collaboration and communication in mathematics classrooms it is necessary to consider the role of technology in the epistemological process" (Ball & Barzel, 2018, p. 229).

There is a great variety of digital tools that can be used in education. To gain some orientation it seems appropriate to characterise these tools with regards to how they are used in communication. Ball and Barzel (2018) distinguish between *communication through technology*, *communication with technology* and *communication of technology displays*. The first one involves use of technology to directly support online synchronous communication between people (e.g. Skype, Zoom). Communication *with* technology "considers the entry of syntax, selection of menu items, programming or any command that drives the technology to produce a display" (ibid.) (e.g. Apps like "digital-twenty-frame"). The third one describes a situation in which the technology display is a stimulus for discussion. "This discussion could occur in a range of contexts, for example, through two students' consideration of one shared screen or through public display of student work via technology such as an interactive whiteboard or data projector" (ibid.) (e.g. Padlet). While those categories are useful to distinguish and analyse the use of technology in communication, they may overlap and do not occur in isolation (ibid.). For example, students may discuss a technology display while at the same time interacting with the display itself. Thus, they may be even interacting with each other through the input itself. This shows how complex the connection between communication and technology is. Naujok (2012) and Knopf & Abraham (2016) mention that those discussions and interactions evolving while working with technology may be especially important and fruitful for deep learning processes.

Combining those three ideas – *constructing mathematical knowledge in communication*, *technology* and *collaboration* – it becomes evident that there are specific affordances and opportunities in teaching and learning mathematics in *digital-collaborative* settings. One such setting and initial results are presented on the following pages.

Methods

The research presented in this paper is part of a research project called K4D ('Collaboration for Digitisation'), funded by the German Federal Ministry of Education and Research. The project started in early 2020 at TU Dortmund University and aims at a better understanding of teaching and learning with technology in higher education.

During their first year at TU Dortmund University, students of mathematics education are obliged to attend the lectures '*Arithmetic and its Didactics I*' and '*Arithmetic, Functions and its Didactics II*'. Those lectures are accompanied by a mandatory seminar once a week in which students tackle mathematical tasks in small groups. Due to the Covid-situation, students worked remotely from home using Zoom. During some of those exercises, students were given tasks they had to collaboratively work on, while using Padlet, which is a tool considered a 'digital pinboard' enabling users to share, connect and sort documents (e.g. videos, recordings, pictures, text). Users can comment and react on those documents in real-time. Padlet can be categorized as a general educational technology and – in contrast to subject-specific technology – could be used for many different activities. The usage of Padlet to foster collaboration was planned before the pandemic, but within the usual seminar settings. Data was collected by screen-capturing and voice-recording those situations. The sessions were 30 to 60 minutes long and about 20 groups have been recorded. Additionally, questionnaires were given to the students to better understand how they themselves experienced the collaborative work with Padlet. Research is intended to continue in 2021 and 2022 to gain more data. At this stage of the research process, only exemplary insight into the data and findings can be given.

Tasks for working with Padlet

The topic of the recorded session is basic functions (linear, proportional, reciprocal) and the tasks (listed below) are supposed to engage the students in discussions about their specific content knowledge.

(1) Find a context/situation to those functions (linear function, proportional function, reciprocal function) and upload it to Padlet. Don't name the function in your context/situation.

[Additional information: (1) done by each student on their own the day before the group exercises took place. In each group, about 25 students uploaded their contexts in one Padlet.]

(2) Cluster the different contexts/situations and discuss which situation belongs to which type of function.

[Additional information: (2) done in small groups (3-5 students) during the exercises. The original Padlets were copied so that all groups could create and discuss their own cluster.]

(3) Name characteristics for each type of function and upload them to Padlet.

[Additional information: (3) done in small groups (3-5 students) during the exercises. One Padlet was created for each type of function. The students kept working in small groups (3-5 students), but those Padlets were not copied for each group, but filled by different groups of students simultaneously.]

Questionnaires

In order to gain additional insights into the processes, questionnaires were conducted to better understand how the students themselves experienced the collaborative work using Padlet. Items were given to the students (n = 220) which they could agree or disagree with on a 1 to 5 scaling. Some exemplary examples of those items are as follows: “Working with Padlet was very intuitive.”; “Other students’ postings confronted me with new ideas/approaches/ representations.”; “Other students’ postings led to more intense discussions within our group.”

Selected Results

The following transcript shows four students working on task (2) and (3). At the beginning, they discuss which type of function is represented in a given context-situation.
(Transcripts are translated)

1	Student 3:	[…] I don’t really get (…) get the difference. Between a reciprocal function (..) and the others.
2	Student 2:	[…] There is that mnemonic (*for reciprocal functions*): “the more, the less”, so#
3	Student 3:	#so it’s decreasing?
4	Student 2:	Exactly. Exactly.
5	Student 3:	[…] Okay, that makes sense.

The students proceed to cluster the given context-situations and decide – using the mnemonic: “the more, the less” – that the following is a reciprocal function:

“Peter spends a fixed amount of money each month. […] He got 3000€.
He withdraws 100€ each month.”

Later during the group-exercise, the students start working on task number (3). While doing so, they are confronted with other groups’ posts in Padlet. Two of those posts and the unfolding discussions can be summarised as below:

Post I:	Post II:
“a reciprocal function has no zero point and no intercept.”	“x * y is always the same for each coordinate.”

Confronted with those posts, a discussion emerges:

6	Student 3:	[…] But they wrote “it (*a reciprocal function*) has no zero point and no intercept” (..) but it could (..) it could start with an intercept, couldn’t it? (…) Or what do they mean?”
7	Student 1:	Yes.

8	Student 2:	Yes.
		…
9	Student 1:	[…] It's like with that money. If it starts at 3500€ (*3000€*), then that is on the y-axis."
10	Student 3:	Sure.
		…
11	Student 3:	[…] It's the same with those workers (*Referring to another context-situation: "For the construction of a new [building] a single worker needs 120 days. Two workers need 60 days. […]"*)
12	Student 1:	Sure.
13	Student 3:	Yes, and then you would still start with one, two, three, four workers on the y-axis.
14	Student 1:	But you have no zero point, like, like the intercept means that there is something like (..) zero, three thousand (*(0|3000))*. So x is always zero. And I think in that worker-context, that is not (..) it is not possible, because (..) one (..) zero workers, you can't say that, that they need twice as long as one worker. Because that makes no sense. Because if no worker is working, nothing ever happens.
15	Student 3:	Mhm.
16	Student 1:	So maybe the other context is wrong.
17	Student 3:	Maybe (*laughing*). But I don't get what they mean with "x * y is always the same for each coordinate".
		…
18	Student 3:	Especially "for each coordinate" (..) "for each coordinate". What does that even mean? (…) Maybe something like it's linear (..) so that (..) like it's (the function) increasing all the time (..) But#
19	Student 1:	#Ah, I think I know what they mean. They mean (..) like what you said before. That if one worker needs 120 days and two workers need 60 days, then it's still 120 days in total.
20	Student 3:	Ah okay, that's possible.
21	Student 1:	So for example one worker is your x and 120 is your y.
22	Student 3:	Yes, yes.
23	Student 1:	And 60 times two equals 120.
24	Student 3:	Mhm, okay. Nice.

The group decides that the first context (the one with the "money") does not represent a reciprocal function, but the latter (the one with the "workers") does.

Interpretation and discussion

The scene summarised above is analysed using an epistemological perspective. Since that analysis is open for discussion, we will try to re-construct the underlying knowledge and ideas, using Steinbrings epistemological triangle and focusing on how the process is shaped by the use of technology.

Clustering the different context-situations in Padlet proves to be a solid way to force students to interact. They engage in meaningful discussions about whether or not a given situation can be linked to a specific type of function. By doing so, they create a common ground for different *concepts* – for example, what they understand of a reciprocal function (1-5). The students link the situation ('*Peter and his money*') to a reciprocal function. This link acts as the *sign/symbol* the students try to interpret, using their *reference context*. In this specific case, the reference context is the mnemonic ("the more, the less" – meaning: for an increasing x value, the y value is decreasing) which S2 mentions to legitimate the situation describing a reciprocal relation (2) (Fig. 2).

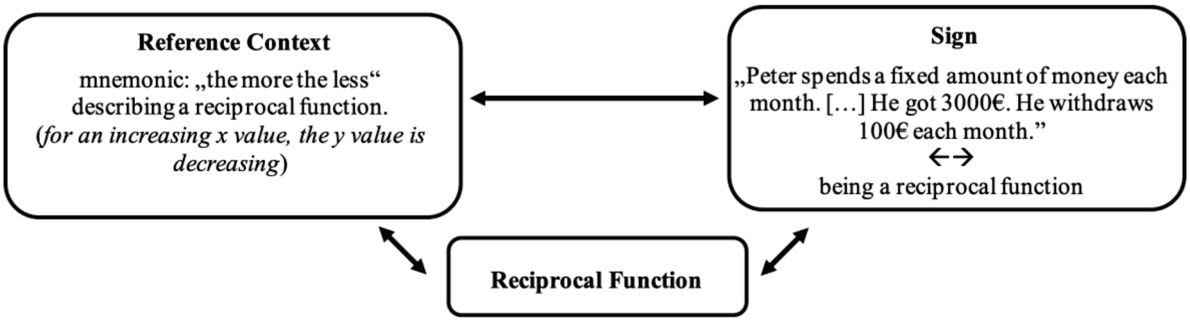

Figure 2: Mechanic significance with a mnemonic

Since no group member protests against the above interpretation, it appears to be a shared one within the group. Now, the use of Padlet comes into play: the posts by other groups inflict a conflict to the formerly shared interpretation – a "productive irritation" (Nührenbörger & Schwarzkopf, 2019).

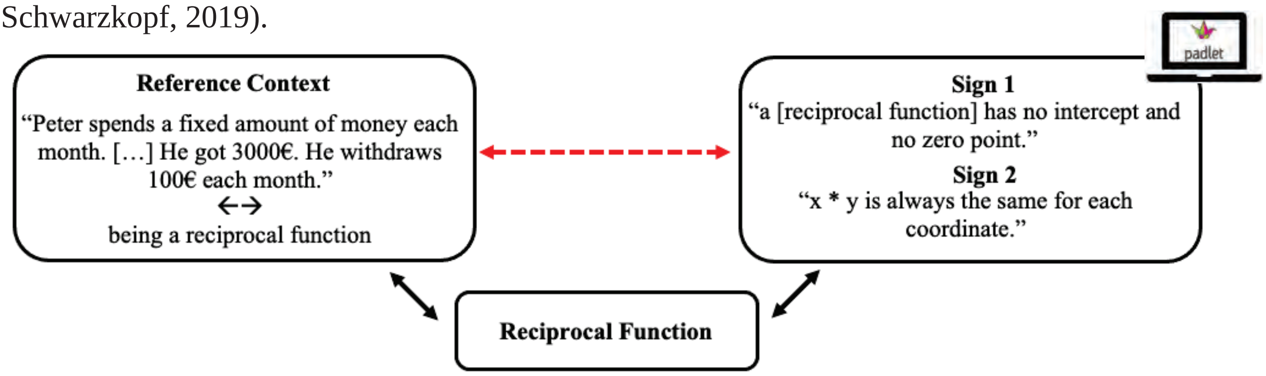

Figure 3: Productive irritation through the confrontation with a content post

The new sign is the statement ("a [reciprocal function] has no intercept and no zero point") in Padlet. The group tries to understand that statement by mediating it to a reference context. In this case, the reference context is the former sign. However, the group is irritated in a twofold mode. On the one hand, the mediation cannot be done successfully (6). On the other hand, they have to interpret the meaning of the second statement ("x * y is always the same for each coordinate") (17) (Fig. 3). When the group fails to interpret the new signs in relation to their reference context, they question their original assumption (16). Therefore, the group looks at another one of the given context-situations: The one with the "*workers*". During their discussion, they realise that both of the statements made in Padlet can be explained using this context (14 & 19). The connection between the "new" context situation and those "new" statements acts as the new signs (Fig. 4). Finally, the students successfully mediate them to fitting reference contexts (Fig 4). The epistemological analysis of these sequences clearly shows a shift from a vague understanding and interpretation of functions to a more sophisticated one – for example by (unknowingly) referring to the anti-proportionality factor (Heiderich & Hußmann, 2013). Padlet seems to induce these mathematical learning processes by confronting students with different ideas and interpretations.

An evaluation of the questionnaires (n = 220) supports those findings: 88% of students said that when using Padlet "they were confronted with new ideas, approaches and representations" and 75% said it led to "more intense discussions".

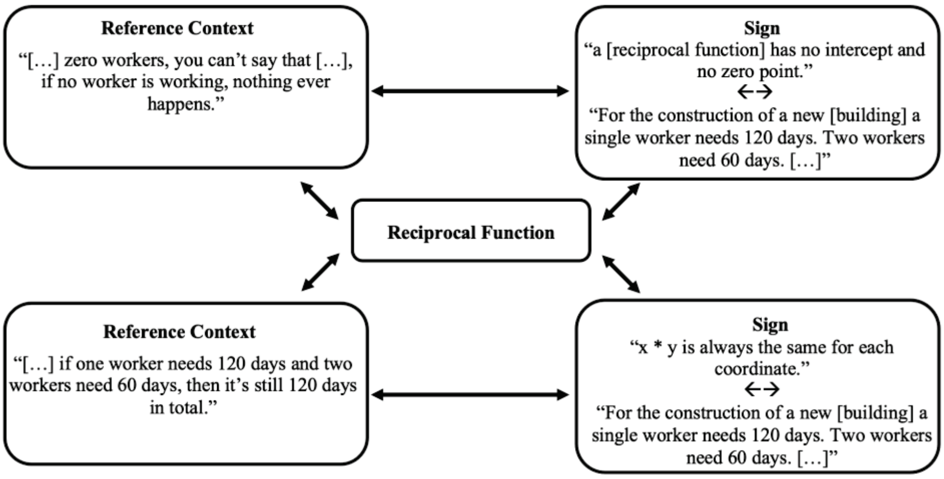

Figure 4: Structural significance on reciprocal functions

Conclusion

This paper highlights specific potentials of the use of technology (by using Padlet) in mathematics education. The first results hint at Padlet being a useful tool to induce and to support mathematical discourse and mathematical thinking. With Padlet, students can be confronted with multiple different ideas and approaches, while at the same time communicating and discussing in a small group. This combines aspects of collaborative learning in smaller groups and class-wide discussions at the same time, resulting in manifold occasions for productive irritations and mathematical discourse. As with every other media or technology, it has to be carefully considered when and how to use them in order to do so most efficiently. Therefore, more research will be done to identify specific design elements and to deepen the understanding of how the use of interactive pinboards, like Padlet, affects mathematical knowledge building.

Acknowledgment

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Evaluating digital student work through model backtracking

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Designing a digital assessment system for mathematics brings along a number of challenges, such as creating an intuitive interface or providing the right feedback. We investigate how these challenges are met in different methods for evaluating digital student work. After done so, we formulate an idea for a new approach: model backtracking. That is, retracing students' calculations through their final or intermediate answers. We explain the mathematical principle behind model backtracking and conclude that the method is promising, despite its limitations. Finally, we hypothesize about possible ways of meeting design challenges using model backtracking and provide recommendations for further research.

Keywords: Digital assessment, Intelligent Tutoring System, Model tracing

Introduction

Digitally assessing mathematics has become increasingly important in the field, since there are many different applications that can lead to increased potential for student learning and summative assessment as well as didactical insight in students' problem-solving behavior. When designing a digital assessment system, a number of challenges needs to be overcome. In this paper we formulate a new approach for evaluating digital mathematical student work, that may help to overcome such challenges: model backtracking (MBT). With this technique, the model the student employs is derived from the final or intermediate answers. The aim of this paper is to explain the idea behind MBT and hypothesize about possible applications, this paper is therefore purely theoretical.

Although MBT could be used in summative settings, in this paper we mainly focus on the applications of MBT in formative settings. We single out intelligent tutoring systems (ITSs) to discuss in detail and outline the general structure of ITSs. We then proceed to investigate feedback and relate specific feedback to the granularity of the student input. We elaborate on various challenges that arise when designing ITSs. The principles behind MBT are explained and we proceed to hypothesize on how MBT may help to overcome design challenges. After done so we discuss the limitations of MBT and give suggestions for further research. Since MBT is still very much in a developmental stage, the results in this paper are of a hypothetical nature, indicating which design difficulties could be overcome using MBT.

Intelligent tutoring systems

ITS can guide the learning process by providing students with feedback. There are essentially two types of feedback: inner loop feedback which focusses on in-task guidance, and outer loop feedback which guides the learning process over several tasks (Santos & Jorge, 2013; VanLehn, 2006). Inner loop feedback can provide appropriate hints if a student fails to make a correct step or is unable to successfully complete the task (Heeren & Jeuring, 2014). Outer loop feedback can include an indication of the degree of mastery of different learning goals, based on a student model covering the domain. From such a model a suggestion on the next task to be completed by the student can be provided adaptively (Heeren et al., 2018).

Generally, ITSs consist of four modules (Nwana, 1990), as Figure 1 illustrates.

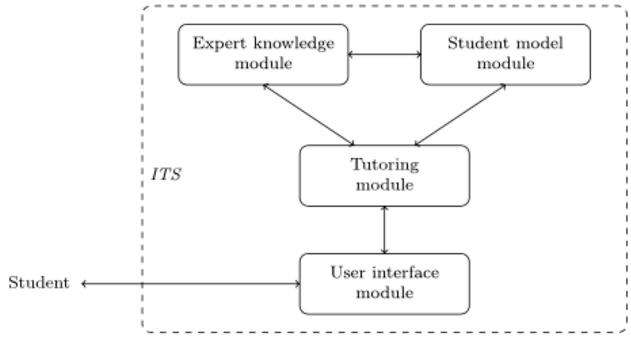

Figure 1: structure of ITSs from Heeren & Jeuring, 2014, p.112, with permission

The *user interface module* is used for communication between the system and the student. It is an important part of the ITS since the way in which information is displayed can affect, for instance, the willingness of the student to work with the ITS. The user interface module communicates directly with the *tutoring module*; this module is also known as the pedagogical module, it can make educational decisions such as suggesting the next tasks best suited for the student or how much and what type of feedback to provide. The tutoring module draws on the *student model* module which contains information about the knowledge of the student. This knowledge is often represented as a subset of the expert knowledge in the expert knowledge module (Brusilovsky & Millán, 2007). The expert knowledge module contains rules governing the way objects in a domain may be manipulated. It also contains so-called buggy rules. These rules model well known student errors within the domain. Currently, there are two approaches to evaluating student input: model tracing (Anderson et al., 1995) and constraint-based modelling (Mitrovic et al., 2007). In a model tracing approach, a step in the user input is compared to an expert solution for the problem. If a step deviates from the expert model, a buggy rule can be applied to check which specific error occurred. In constraint-based modelling the input is evaluated using two conditions: a relevance condition and a satisfaction condition. If a relevance condition of the error applies, then the satisfaction condition also needs to be satisfied to flag the error.

Feedback

Both Shute (2008) and Narciss (2008) studied literature on feedback, in this section we compare both studies with regard to specific errors. In both papers we find evidence that specific student errors need to be known in order to provide meaningful feedback. Both authors give a definition of feedback:

Formative feedback is information communicated to the learner that is intended to modify his or her thinking or behavior for the purpose of improving learning. (Shute)

Feedback in instructional contexts is all the post response information that is provided to a learner to inform the learner on his or her actual state of learning. (Narciss)

In both definitions feedback is information provided with the intent of informing the learner on the state of learning. Where Shute takes it one step further by demanding that the purpose of feedback is to modify behavior to improve learning. How may learning be improved by communicating information on the actual state of learning? Shute argues that a factor inhibiting learning is uncertainty. Since uncertainty is an unpleasant state that needs to be reduced it induces a cognitive load that could distract attention from actual task completion. Feedback could reduce this uncertainty; however, feedback can also serve the function of informing the learner on possible solving strategies or mistakes. To accomplish this, the feedback needs to be specific (Shute, 2008) or equivalently: elaborate (Narciss, 2008), although, the feedback should not contain the actual solution. Both Shute and Narciss support the claim that specific feedback leads to better learning outcome than feedback only on the correctness of answers. Narciss formulates five cognitive functions for feedback four of which require the specific error to be known. According to Narciss, a key issue is: how well a learning medium (i.e. teacher or ITS) is able to transform a discrepancy between the current state of learning and the required state into feedback that contains relevant information to mastering the requirements. The performance of a learning medium in this respect increases if it is more able to identify specific errors.

Granularity (VanLehn, 2006) of the student input plays an important role in digitally assessing students reasoning. The granularity can range from students inputting every aspect of their reasoning to inputting only the final answer. When there is less information available the difficulty of correctly assessing student reasoning increases. Typically, ITSs require that students input every step of their calculation in order to detect specific errors and provide specific feedback in the sense of Shute (2008) and Narciss (2008). Drijvers (2019) stipulates that a task shouldn't be essentially altered to accommodate a digital environment. However, inputting every step, in comparison to just calculating steps using pencil and paper, forms an additional requirement for completing a task. Furthermore, inputting every step could be cumbersome and might discourage students from working with ITSs.

Challenges

When designing ITSs there are certain factors that need to be considered.

Interface

One needs to avoid that proficiency in editing mathematical text digitally is required (Drijvers, 2019). Heeren et al. (2018) deal with this problem by allowing all kinds of input, even textual or mathematically nonsensical. The drawback however is that it is very difficult to evaluate such input to a high degree of accuracy.

Stepwise evaluation

When dealing with tasks concerning equivalent expressions, such as solving equations and simplifying algebraic expressions, often assessment systems require stepwise input. For example, Heeren and Jeuring (2014) describe such a system. Through such a system the required feedback can be provided with a high level of expediency if all the steps are inputted. Although inputting each step separately in the environment can be quite time consuming. Additionally, step by step input isn't feasible on devices with small screens such as smartphones, making this technology less accessible.

Granularity

The granularity of the expected student input concerns the step size of the input by the student. If the student skips a few steps the assessment system was expecting, the efficiency of diagnosing possible errors drops. In the ASSISTment system (Feng et al., 2009) this problem is solved by asking additional questions to the student. However, in a summative setting there is a risk that these questions give away part of the solution.

Non-stepwise evaluation

Some tasks consist of the calculation of several different non-equivalent components such as for instance, linear extrapolation where first the average change is computed before computing a future value. Since stepwise evaluation generally evaluates equivalency of steps it is not applicable here. One way to deal with this problem is to ask the student to input each of the different components of the calculation. The drawback however, is that this gives away part of the structure of the calculation. Tacoma et al. (2019) solve this problem by letting the student select the next step from a dropdown menu before performing the actual calculation. However, this only partly solves the problem since the menu only contains steps necessary to perform the required computation.

The general idea of model backtracking

We propose to retrace the students' computation from the final answer. There will be obvious limitations to this method, however many of the problems mentioned above could be solved for certain classes of digital assessment environments.

Consider the task of computing the derivative of $f(x) = (3x + 2)^3$, if the student arrives at the answer $f'(x) = 3 \cdot (3x + 2)^2$ it is unclear if the student forgot to multiply with the exponent 3, or if the student forgot to apply the chain rule. In this example these two errors are indistinguishable. However, when computing the derivative of $f(x) = (3x + 2)^5$ these errors are in fact distinguishable since they lead to different answers. So, when implementing a task for computing derivatives of functions of the type: $f(x) = (p_1x + p_2)^{p_3}$ one should choose the parameters such that $p_1 \neq p_3$.

The idea behind model backtracking is to design tasks in ITSs with parameters that make it possible to distinguish different errors. Along with a structure in the expert model module as a tree containing all possible paths of errors through the steps of the task given the predetermined buggy rules. When certain conditions are satisfied one can retrace a student error by means of just the final answer. Of course, model backtracking can also be used for evaluating intermediate steps in a computation.

One immediately notices that not all tasks can be designed using MBT (since there are restrictions on the initial values of the task), for instance not all real-world tasks can be modelled using MBT. However, MBT works well on tasks that draw on random parameters, these parameters can simply be restricted to a domain that makes backtracking possible.

The mathematics of MBT: an example

In this section we will describe how model backtracking works, doing so we will formulate certain definitions that can be extended to more general settings. We start out by defining what we mean by a computational task. We proceed to show how, in this setting, buggy rules can be represented by functions and that various errors are comprised of compositions of these functions. We then find sufficient conditions for determining the path, that is, which composition of functions, the student used to arrive at a certain answer. When the path is known, appropriate feedback can be given to the student.

A computational task is the pair (Ω, F) where $\Omega \subseteq \mathbb{R}^n$ is called the parameter space and $F: \Omega \longrightarrow \mathbb{R}^m$ is called the computation. An element $(p_1, p_2, \cdots, p_n) \in \Omega$ represents the given parameters in the task.

Given the table

t	…	t_1	…	t_2	…
N	…	N_1	…	N_2	…

Assume N depends on t exponentially.

Compute the growth factor per unit of t.

Figure 2: computational task on computing a growth factor from a table

This task is a computational task with $\Omega = \{(t_1, t_2, N_1, N_2) \in \mathbb{R}^4 \mid t_1 < t_2, N_1 \neq N_2\}$ and $F(t_1, t_2, N_1, N_2) = \left(\frac{N_2}{N_1}\right)^{\frac{1}{t_2 - t_1}}$

On didactical bases we can distinguish between two steps in this calculation:

Step 1: Computing the growth factor for $[t_1, t_2]$: $f_1(t_1, t_2, N_1, N_2) := \left(t_1, t_2, \frac{N_2}{N_1}\right)$

Step 2: Computing the growth factor per unit of time: $f_2(t_1, t_2, x) := x^{\frac{1}{t_2 - t_1}}$

We see that:

$$F = f_2 \circ f_1$$

In each of the two steps we can now introduce buggy rules for errors in the steps, the compositions of these rules represent different calculations by the student:

$E_1 := \{e_1^1, e_2^1, e_3^1\}$ denotes the set of rules for the first step where: $e_1^1 := f_1$ and e_2^1 and e_3^1 are buggy rules defined by:

Wrong time direction: $e_2^1(t_1, t_2, N_1, N_2) := \left(t_1, t_2, \frac{N_1}{N_2}\right)$

Calculating the difference: $e_3^1(t_1, t_2, N_1, N_2) := (t_1, t_2, N_2 - N_1)$

$E_2 := \{e_1^2, e_2^2, e_3^2\}$ denotes the set of rules for the second step where: $e_1^2 := f_2$ and e_2^2 and e_3^2 are buggy rules defined by:

Dividing by elapsed time: $e_2^2(t_1, t_2, x) := \frac{x}{t_2 - t_1}$

Forgetting to recalculate for time: $e_3^1(t_1, t_2, x) := x$

Now for fixed $p \in \Omega$ we look at the 3×3 matrix $\bar{M}(p)$ with entries: $\bar{M}_{i,j} := e_j^2 \circ e_i^1(p)$. We can compare the input from the user to the entries in this matrix. For instance, if the input matches the entry $\bar{M}_{3,2}(p)$, we know the growth factor was calculated as a slope.

Two examples of $\bar{M}(p)$:

$$\bar{M}(22, 27, 29, 329) = \begin{pmatrix} 1.63 & 2.3 & 11.34 \\ 0.62 & 0.02 & 0.09 \\ 3.12 & 60 & 300 \end{pmatrix}$$

$$\bar{M}(24, 28, 8, 16) = \begin{pmatrix} 1.19 & 0.5 & 2 \\ 0.84 & 0.13 & 0.5 \\ 1.68 & 2 & 8 \end{pmatrix}$$

Above we see all the entries of $\bar{M}$ are different, which means the steps through the various errors are uniquely determined by the outcome; whereas below we see that various entries of $\bar{M}$ have the same value. So, for $p = (24, 28, 8, 16)$ backtracking the student error isn't possible. We wish to avoid this. Having the entries in $\bar{M}$ differ can be characterized by the following equivalent statement:

For fixed $p \in \Omega$ we define the function:

$$M_p: \{1,2,3\} \times \{1,2,3\} \longrightarrow \mathbb{R}$$
$$M_p(i,j) := \bar{M}_{i,j}(p)$$

Then M_p needs to be injective.

For the computational task, we only select parameters from: $RS \coloneqq \{p \in \Omega | M_p \text{ is injective}\}$. This ensures that we will be able to backtrack the students' computation. These definitions can be generalized to tasks with more than two steps. And depending on the structure of the ITS[1] one can find relaxations on the injectivity condition allowing for a wider selection of possible parameters.

The definition of a computational task as the pair (Ω, F) where $\Omega \subseteq \mathbb{R}^n$ and $F: \Omega \longrightarrow \mathbb{R}^m$ seems to exclude tasks involving computations done on functions. However, when the class of functions is known, these functions may often be expressed in terms of their parameters. Therefore, MBT can also be used to develop ITSs involving functions and algebraic expressions. Arguably MBT can even be used for evaluating formulae derived from geometrical representations.

[1] For examples of ITS with MBT see:
Linear interpolation: https://www.geogebra.org/m/e5ewfude
Linear equations: https://www.geogebra.org/m/rqa4w5wb

Possible applications of MBT

In this section we conjecture how MBT could contribute to overcome design challenges. However, currently these conjectures remain to be proven.

Interface

When working in an ITS for the first time, students often start by inputting only their final answer. Therefore, just demanding the final answer is very intuitive. With the use of MBT, feedback can be provided on the basis of just this input. A student can make a calculation using pen and paper and input only the final answer, thereby staying very close to the way mathematics is normally practiced (Drijvers, 2019). As very little input is needed ITSs can be created for small screen devices, which make ITS technology more accessible. Of course, major drawbacks are that sloppy notation isn't detected nor corrected.

Stepwise evaluation and Granularity

By adding the identity function to the set of rules at each step, MBT could also be used to inspect intermediate steps in a calculation. Where, if steps are skipped or haven't been inputted, MBT could still provide feedback. This could constitute an addition to existing systems improving error detection. MBT could be used in combination with constraint-based modelling and model tracing. This way expert knowledge modules that already exist can be employed. When a student for instance inputs step i and then inputs step j $(> i + 1)$ without inputting the intermediate steps, the parameters at step i can be seen as starting parameters p for a computational task. The corresponding function M_p generally won't be injective however a list of possible paths can still be produced.

Non-stepwise evaluation

If the task consists of the calculation of several different non-equivalent components it is possible to let students work in a digital environment without adding any structure: an empty page. After which the computations are scanned for intermediate answers (Heeren et al., 2018). By construction the intermediate (wrong) answers differ. Therefore, it may be possible to recognize intermediate steps by just the values they produce and provide specific feedback.

Limitations and suggestions for further research

A clear drawback is that MBT can only be implemented for certain tasks since the possible parameters will be subject to constraints. Furthermore, the class of computational tasks does not contain many higher order tasks, aside from deriving expressions from geometrical representations. Further research might extend the possibilities of ITS beyond computational tasks. One of the drawbacks of the design principle of MBT is that it is very specific for the computational task. Usually ITSs make use of expert knowledge modules which contain rules for an entire mathematical domain. This way different tasks can be designed using the same set of rules (Heeren & Jeuring, 2014). Perhaps MBT could also be used over entire domains, possibly in combination with existing expert knowledge modules. Currently it is unclear what the performance of error detection trough MBT is, nor is it clear if specific feedback on the final answer enhances learning. Empirical experiments will be necessary to indicate if MBT can contribute to a positive learning outcome.

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Numerical cognition with Touchcounts or Rakin? An enactive and ecological approach to finger gnosis

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Our paper aims to provide a proof of concept about the theoretical framework of enactive and ecological approaches to the field of perceptual learning of mathematics with digital technology. We reporst on the finger gnosis or finger knowledge that school children deploy when engaging with digital technologies such as Touchcounts and Rakin. From our theoretical lens we contrast both applications and conclude that they offer rich possibilities for number learning, however, Touchcounts adheres better with enactive and ecological foundations by prompting finger gnosis in different ways, while the Rakin technology application restricts actions and gestures, leaning toward representationalist cognitivism.

Introduction

In this article we emphasize how digital technology promotes new ways of learning by doing within the framework of new contemporary approaches to enactive and ecological cognition. The evidence on technology and mathematics education from an embodied, intersubjective and instrumental approach is auspicious, see Sinclair and Freitas (2014) and Drijvers (2019). In the framework of these studies, we place special attention to the understanding of cognition and digital learning from a dynamic sensorimotor theory based on the progressive structural coupling between the agent and the sociotechnological environment (Videla et al., 2021). We highlight and share the contribution of Shvarts (2021) on the importance of instrumented action constituted by a body-artefact dynamic functional system that regulates the actions of the agent and the sociomaterial environment. As a complement and from our enactive and ecological theoretical lens, we argue that digital technologies should be presented as ecological niches of skilled sensorimotor expansion, that is to say, they enhance the ecological control of actions for the achievement of learning goals. We highlight the movement of hands and fingers in the framework of perceptual and gestural exploration with tactile technology for the understanding of number. Particularly, we emphasize "finger gnosis", which consists in the knowledge of numbers through finger movements on touch screens. The aim of our study is to contrast the use of Jackiw and Sinclair's TouchCounts (2014) and CEDETI's Rakin (2021) applications for number comprehension, in light of our theoretical approach that highlights the specialization and expansion of actions and gestures that make it possible to bring a world of numbers and motion.

Theoretical framework

Enactivism: cognitive agency

In this section we provide brief clarifications of the role of perception in enactivism in order to illustrate the sub-personal dimension of agency. In face of the persistence of the traditional cognitivist approach based on representationalism, computationalism and internalism, enactivism emphasizes the role of action for perception by pointing out that: "(1) perception consists of perceptually guided action and (2) cognitive structures emerge from recurrent sensorimotor patterns that allow action to be perceptually guided" (Varela et al., 1991). This implies that cognition does not require mental representations to cause skilled bodily activity, since enactivism is framed within the circular dynamics of the perception-action loop in which cognition is action. When the enactive approach is alluded to, it refutes the idea of cognitive processing and emphasizes the notion of a decentralized cognitive system based on autonomy, sense-making, embodiment, emergence and experience (Di Paolo, 2020). Embodiment refers to movements, gestures and multimodal perception intertwined and constitutive of cognition through experience. Also, enactivism embraces the principles of the biology of knowledge, in which living beings are considered adaptive, autonomous agents and creators of their own worlds as a result of their history of coupling with the environment (Maturana and Varela, 1980). A cognitive system means that interchange with the world are inherently meaningful for the knower, given that movements are at the center of mental activity.

Ecological psychology: skillful expansion

Next, the role of perception from ecological psychology is presented in order to understand the interpersonal relationship between agent and sociomaterial environment. Gibson's (1979) ecological psychology proposes that cognition is enacted, shaped and structured by reciprocal interactions between the organism and the environment. Gibson (1979) argues that the environment is hierarchical in ecological information available to support everyday activities, think of; roads, rocks, slopes, forests, buildings, technology, and actions among others (Heft, 2020). Ecological psychology proposes a relevant epistemological debate by promoting an understanding of the world that overcomes dichotomies: perception/action; organism/environment; subjective/objective and mind/body. These dichotomies are at the basis of the theoretical assumptions of behaviorist and cognitivist psychology, which adscribe to the poverty of the stimulus, the passiveness of perception and the processing of information. Hence the importance of ecological information for Gibson, since it depends on the specification of the relationship established by the legal covariation between energy-optical, mechanical and chemical patterns, when actively participating in the environment (Chemero, 2009). This mode of participation, highlights affordances that are conceived as a relationship between an aspect of the sociomaterial environment and an ability available in a life form (Kiverstein and Rietveld 2015).

Brief Overview Tactile Technology and Numerical Cognition

In this article, we ascribe to the radically enactive approaches of numerical cognition developed by Zahidi and Myin (2016) on the importance of the body in the understanding of numbers and counting: correspondence, ordinality, and cardinality.

Rakin

Rakin is an inclusive Chilean application that seeks to promote the learning of mathematics in preschool children through a virtual desktop interface where skills such as seriation, classification, conservation of quantities, numbers, counting, cardinality and ordinality can be stimulated. In addition to being aligned with many of Chile's preschool mathematics learning objectives (CEDETI, 2021).

Touchcounts

TouchCounts (Jackiw & Sinclair, 2014) is an app that allows young learners to simultaneously coordinate various forms of numbers: number names such as 'three', number touches on the screen, number of records on the screen, and number symbols such as 3. It represents a multimodal correspondence between touching with fingers, seeing numbers, and hearing number words (a one-to-one correspondence of touch, sight, and sound). The application has two worlds: enumeration and performance.

Our enactive-ecological proposal: learning number with Touchcounts or Rakin?

Our proposal consists of a dynamic approach to sensorimotor agency for understanding the enactive-ecological approach to numerical cognition, see Videla et al (2021). The unification of these approaches seen as a continuum relieves the human cognitive-technological tool assemblage beyond the subpersonal dimension of enactivism and the interpersonal dimension of ecological psychology (Heras-Escribano, 2018). In the case of number learning with digital technology, we propose that a form of human-tool assembly is only possible from the ecological niche. This meta-relationship is flexible, dynamic and expansive, therefore, the technological niche grows in relation to the cognitive agent's experience. For example, in the case we address here is corresponding to finger gnosis, we assume that changes in the position and movements of the fingers make it possible to bring a world of numbers and not to have or objectively grasp a world of numbers as is characteristic of representationalist cognitivism. Below we present a brief description of the key concepts that nourish our proposal and that we have borrowed from other articles:

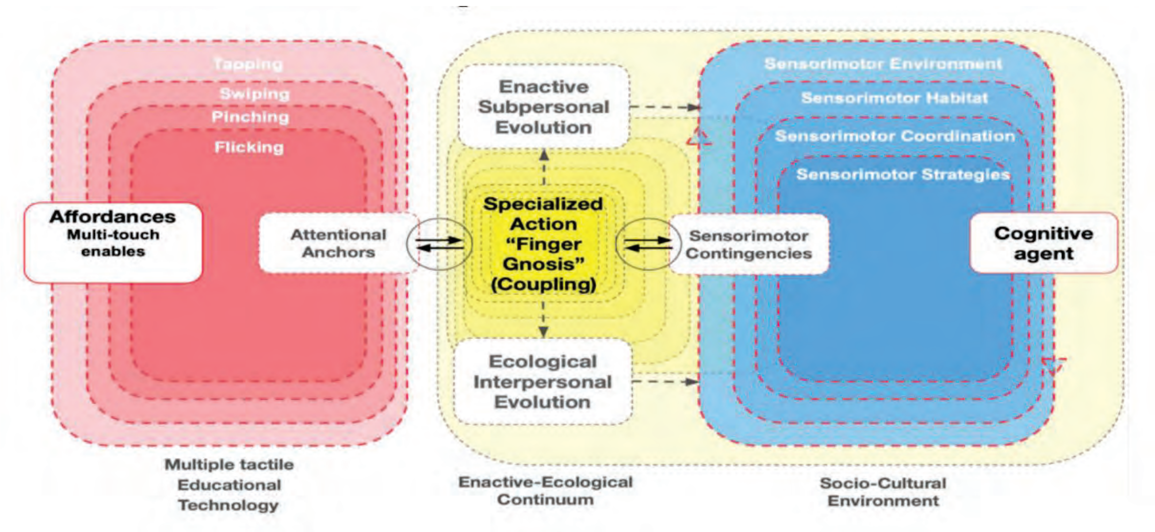
(i) Attentional anchors: during the flow of sensorimotor contingencies of cognitive activity, dynamic equilibrium is instantiated from the attentional anchors that interpolate between the internal dynamics of the agent and the environment in which it participates facilitating emergent understanding (Hutto et al., 2015).

(ii) Sensorimotor contingencies: distinguish four types of (SMC) that contribute to specialization of action: (a) sensorimotor environment: intuitive movements of perceptual exploration, without considering sensory feedback (b) sensorimotor habitat: sensory feedback movements between sensory and motor activity as a function of the agent's internal dynamics (c) sensorimotor coordination: specific action patterns that tend to dynamic control according to task goals (d) sensorimotor strategies: optimal balance of the cognitive agent within a normative framework that solves specialized actions (Buhrmann et al., 2013).

(iii) Finger gnosis: Butterworth (1999) proposed that fingers are important for representing numerosity. Sinclair and Pimm (2015) have evidenced in their studies that number sense in general is dependent on finger knowledge. Finger knowledge involves a performative gestural act such as (tapping, swiping, pinching and flicking) to produce numbered objects on a multi-touch screen.

In what follows we present Figure 1 as a representation of our proposal. Here we show how the evolution of the performative gesture of finger gnosis (tapping, swiping, pinching and flicking) is linked to the sensorimotor contingencies (sensorimotor environment, sensorimotor habitat, sensorimotor coordination and sensorimotor strategies), as can be seen in the red and blue figures. The segmented lines indicate the dynamic coupling between the cognitive agent and the environment reaffirming the co-dependence or mutualism of reciprocal constitution. The bidirectional arrows related to finger gnosis indicate that the specialization of the finger action is co-dependent on the attentional anchors and the sensorimotor contingencies. As the anchors interpolate between the cognitive agent and the touch screen, the sensory variety decreases as a result of the digital affordances that engage the cognitive agent with increasingly specialized actions that transduce number and its operations.

Figure 1. The enactive-ecological continuum model of finger gnosis in numerical cognition with multitouch technology

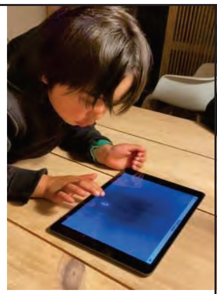
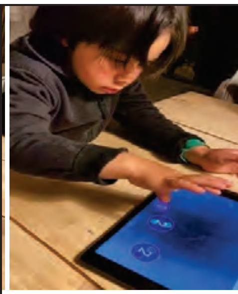
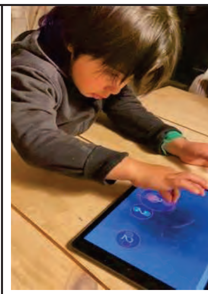
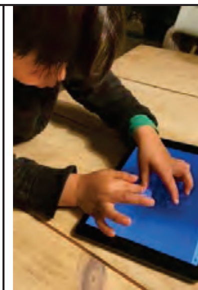
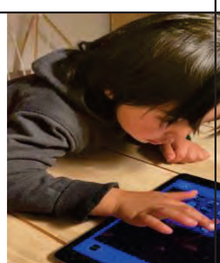
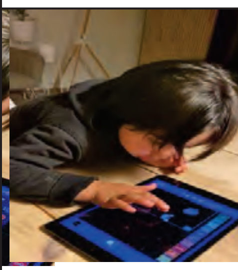


Empirical findings

To assert our theoretical proposition, we present the excerpt in which author Videla participated in a clinical interview with a 4-year-old kindergarten student named Andy, who interacts for the first time with TouchCounts and Rakin. We chose this excerpt from a larger investigation, as it illustrates a variety of gestures co-dependent on their coupling with the material affordances of the touchscreen. We illustrate that hand and finger movements are becoming specialized in Touchcounts, whereas in Rakin the movements to enact numerical comprehension are restricted. The resulting finger movements have not been explicitly taught, but have emerged in the sensorimotor flow of contingencies in the tactile ecological niche.

Table 1. Specialization of the action of the finger's gnosis in the understanding of the number

Technological ecological niche (Digital affordances)	**Finger Gnosis** **Specialization of tactile action (enacting-ecological numerical comprehension)**			
Multiple Tactile	**Tapping (sensoriomotor environment)**	**Swipping** **(sensoriomotor habitat)**	**Pinching** **(sensoriomotor coordination)**	**Flicking** **(sensoriomotor strategies)**

Touchcounts				
Dialogue	**Andy:** I touch and 1 comes out. **Research:** Yes, try again. **Andy**: I play again and it comes out 1, you can also hear 1. **Research:** What else can you do? What appears on the screen? **Andy:** tap and move, numbers come up.	**Andy:** oooh! tap twice and it comes up 2. If I tap three times it comes up 3? **Research:** Yes, and if you tap 4 times how much comes out? **Andy:** 4. Also, you can put the balls together. **Researcher:** Yes, show me. **Andy:** Look, I put two little balls together that say 1 and I get 2.	**Andy:** Now I can form different circles of balls. **Researcher:** How do you do it? **Andy:** I do the little fingers like this (shows the pinching motion by putting thumb and forefinger together on the screen). **Researcher:** When you do those movements, what do you get? Andy: The sum.	**Andy:** You can also separate the balls from the big circle. **Researcher:**Yes, how? Show me. **Andy:** Look, with my two hands I can do it. With one I squeeze and with the other I pull. **Researcher:** What happens to the number in the big circle? Does it increase or decrease? **Andy:** It decreases. **Research:** Why? **Andy:** Because we remove balls.
Rakin				
Dialogue	**Andy:** up come the numbers, I touch 1 and it sounds 1, I touch 2 and it sounds 2. **Research:** Do you know the numbers 1 to 10? **Andy**: Yes. **Research:** What else can you do? What appears on the screen? **Andy**: On the side there are different things that I touch and they sound.	**Andy:** I can touch things and move them here (indicating the central screen divided in two parts). **Research:** Do you see that it changes in the first quadrant? **Andy:** Yes, it changes the amount. **Researcher:** How do you know? What appears on the screen? **Andy:** I move an object and it appears and it sounds 1. I move another one and it sounds 2. **Researcher:** What else can you do with your		

		fingers? Andy: Just move objects to the center.		

Table 1 illustrates the specialization of the action of the knowledge of the fingers for the understanding of the number with the Touchcounts and Rakin applications. This specialization of the action can be seen in the changes presented by the movements of the fingers on the touch screen, of which we have selected (tapping, swiping, pinching and flicking) reported by Sinclair and Pimm (2015) as finger gnosis. In relation to the contrast presented in table 1, we can identify that Andy when participating with Touchcounts begins tapping the multi-touch screen and makes a disc with the number one visually appear and an audio that says one. Later, after several tapping where other discs 1, 2 and 3 appear, he reconfigures this movement for swiping, and where the possibility emerges of making collections of discs that represent new numbers. These two movements correspond to changes in the structural couplings with the touch screen, of which the first are more exploratory and ingenuous where a touch is enough, to move to another sliding movement that makes it possible to join different discs and configure new numbers from the numbers of objects in the popup collection. The reorganization of these actions is due to the emergence of attentional anchors that interpolate in Andy's internal dynamics and the emerging content of the touch screen. From a numerical understanding, these movements promulgate the notions of ordinality, correspondence, and cardinality. In turn, within the framework of sensorimotor contingencies, it can be established that the tapping and swiping movements in Andy are gestated from the sensorimotor reconfiguration of the environment to the sensorimotor habitat.

This occurs within the framework of changes in the movement of the fingers, the first ingenious and without contact with Andy's internal dynamics (environment) to one with sensory feedback that triggers the sense of agency (habitat). At a phenomenological level of qualitative changes in what we call “finger gnosis”, it is possible to observe the alternation of the ring finger with the thumb. Subsequently and in relation to pinching, it can be seen that the movement of the index finger in tapping and the thumb or index finger in swiping, ceases to be one or the other, but transforms into a specialized gesture that encompasses both fingers simultaneously of one hand to respond to emerging objectives. This is relevant, since the effective movement requested by the touch screen is precisely the click to form collections of discs. Andy finds himself moving in some way to bring out new collections in a specialized way. This is what is known from the flow of contingencies in the performative gesture as sensorimotor coordination. Once this movement is coupled to his experience with the pop-up content on the touch screen, he realizes that the collections of discs formed by the pinching movement can also be separated or removed from the collection through a more specialized movement that requires not only sensorimotor coordination of one hand, but it is strategically discovered that you must use both hands to respond effectively to the emerging target. This is known from contingencies as sensorimotor strategies and it is where attentional anchors contribute to sensorimotor ecological control in the understanding of the number: ordinality, correspondence and cardinality.

Regarding the Rakin application, Andy exhibits only the tapping and swiping movements, as evidenced in Table 1. This is not to say that Andy fails to engage in a sustained manner by integrating broader sensorimotor contingencies that contribute to dynamic equilibrium such as sensorimotor coordination and sensorimotor strategies. Rakin offers an environment rich in digital affordances in relation to ordinality, correspondence and cardinality as evidenced by tapping and swiping in which Andy can tap the ordinal sequence of numbers on top and can select from a set of available objects (animals, plants and toys) to establish correspondence and cardinality. Nevertheless, the movements that Rakin requests as digital affordances, do not allow the tactile specialization of the action that does justice to the finger gnosis that leads to enacting numerical comprehension. In this area, we consider that Andy is deployed in a restricted environment of actions that the fingers only lead to tapping and swiping to understand and learn the number. Finally, we consider that Rakin is an important application for the development of the fundamental notions of numerical cognition such as ordinality, correspondence and cardinality, however, it is more framed in a representationalist perspective in which cognition is more linked with representing pre-existing contents that make the contents emerge in the specialization of the action.

Conclusion

In this paper we have considered a proof of concept for enactive and ecological unification in the framework of finger gnosis of number learning with multi-touch technologies such as Touchcounts and Rakin. If our proposal is correct, we contribute to the field of research in embodied perspectives of mathematical cognition that allow to favor instances of learning with technology that resonate with basic forms of actions and gestures.

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How the students chased perpendicular lines in GeoGebra Classroom

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This paper focuses on the issue of what solution strategies do secondary school students provide while performing non-standard geometric constructions. For this purpose, we developed an interactive electronic environment in GeoGebra Classroom and, in that environment, we let students construct perpendicular lines in multiple ways. We formulated the task for students so that it corresponded with their school geometry curriculum but in a non-standard open-ended way. Then we analyzed students' constructions qualitatively. Findings showed that the students were usually able to solve the task but just a few of them were able to generate multiple strategies. However, creative constructional approach appeared in data indicating the significance of the assignment or instruction where a variety of mathematical approaches are possible.

Keywords: Dynamic geometry environments; multiple solution strategies; plane geometry; secondary school students.

Introduction

Recently, school geometry has got a rather dual role: it seeks to reflect ongoing technological progress but at the same time still arises from fundamental concepts described in ancient Euclid's *Elements*. One of the significant aspects of the *Elements* consists in stressing on acknowledging the definitions and proving the theorems, while one of the goals of school geometry is developing students' justification skills or, in general, students' conceptual knowledge in the domain. In the past decades, educational researchers have been naturally concerned with the impact of the use of modern electronics technologies on learners' mathematical reasoning. They have explored how dynamic geometry environments (DGE) such as Cabri Geometry or GeoGebra could assist students to improve their understanding of proof in geometry (e.g. Marrades & Gutiérrez, 2000). In this context, an important characteristic of DGE lies in the ability to modify geometrical objects and to observe their features during the modifications. It has become clear that this feature of DGE provides students with opportunities for deep explorations and heuristics on their way to argue the properties of geometrical objects (Lawson & Chinnappan, 2000; Sträßer, 2001). Through the dynamic process of observing modifications of geometrical objects, DGE does not only facilitate students' justification skills but can also provide visual information that a particular property does not apply in general (Prusak et al., 2012). Altogether, the research has revealed the importance of DGE for stimulation of formulating conjectures and creating proofs (Komatsu & Jones, 2017).

Along with the use of DGE, other activities in school geometry could contribute to formulating geometrical conjectures, namely geometric constructions (Herbst et al., 2017: see p. 106). However, there is a lack of investigations that would deeply focus on the relation between constructional problems and reasoning in geometry, or, in general, on explaining how performing geometric constructions can contribute to improving students' conceptual and procedural knowledge and flexibility in the domain. In this contribution, we present an introductory exploratory study on the topic. We proceed from a construction of perpendicular lines, formulate a task that is not standard compared to common activities in school geometry in our country, and analyze students' solutions of such a task. The task is not standard for three reasons:

(i) we let students construct two perpendicular lines on tablets in an online application GeoGebra Classroom whereas it is standard for them to perform geometric constructions on paper;
(ii) we ask students to solve the problem in multiple different ways;
(iii) we allow students to use only two tools: one that creates a line through two points, and another one that creates a circle from a center and point or a center and radius – despite the fact that the standard approach to school geometry also allows them to use a tool that creates a perpendicular line to a given line.

Our explorations correspond with the theoretical framework of flexibility which is defined as the knowledge of multiple solution methods and the ability to generate and perform them appropriately and effectively (Rittle-Johnson et al., 2012). With respect to notes on learning and teaching geometry mentioned above, it becomes clear that developing students' capability to solve mathematical problems in different ways has a significant potential for increasing their conceptual and procedural knowledge (e.g. Star et al., 2015). Since including the facet of multiple solution procedures required from students, we follow up the studies on so-called multiple solution tasks (MSTs) that might be used as a means of developing students' creativity and flexibility in geometry (Levav-Waynberg & Leikin, 2012; Gridos et al., 2019). From the general perspective of research on incorporating an artefact of ICT into the teaching/learning system, we focus on an online application, learners, and knowledge, i.e., on all three edges of the face "ALK" of the corresponding didactic tetrahedron (Donevska-Todorova & Trgalova, 2017).

Within this context, we introduce an exploratory qualitative empirical study with the following research question: "What solution strategies do secondary school students provide when asked for multiple constructions of perpendicular lines while using only circles and lines within the GeoGebra Classroom environment?"

Theoretical background

Geometric constructions

Geometric constructions are understood as a specific category of mathematical problems that ask solvers to draw a geometrical object in a precise and exact way. According to the given circumstances, it requires choosing an appropriate construction method that would correspond to geometrical characteristics of the desired object (Kuřina, 1996). Therefore, geometric constructions are considered a form of mathematical activity combining the process of manipulating objects with the processes of visualization and reasoning (Duval, 2006). What remains to be discussed, is the way how performing geometric constructions could stimulate doing proofs in geometry and, in particular, how it could provide students with significant ideas that would be helpful during the proving process (Herbst & Brach, 2006).

In the Czech Republic, where our research study took place, geometric constructions as a part of school geometry are commonly performed on paper while using a pencil, straightedge, compass, and a special ruler called *a triangle with a guideline*. This special ruler has a form of a transparent piece of plastic in a shape of an isosceles right-angled triangle with a guideline (an impressed line segment) connecting the center of the hypotenuse with the opposite vertex. The tool is frequently used in school geometry from early elementary school grades, where it provides students with an easy way of drawing a line that is perpendicular to a given line – the solver just has to match the guideline with the given line, and then outline the longest side of the ruler with a pencil. From the perspective of the GeoGebra environment, a triangle with a guideline corresponds to the tool *Perpendicular line*.

As mentioned above, we do not allow the use of the tool *Perpendicular line* in our study. It means an obstacle for the students but, on the other hand, it challenges them to come up with an original solution and opens a rich space for multiple correct ways of constructing the object. This arrangement provides us with an opportunity to explore various students' ideas on a problem that is unfamiliar for them but still fully corresponds with the content of school geometry that is appropriate to their age. This way, our approach shifts the issue of geometric constructions into the frame of open-ended problems in mathematics education (Pehkonen, 1997), and thus mediates a suitable environment for investigating students' knowledge.

GeoGebra Classroom

The use of DGE in our research follows the above-mentioned studies that focus on the benefits of the modern technologies for teaching and learning geometry. We work in an online application GeoGebra Classroom provided by GeoGebra software. This software was originally created by Markus Hohenwarter (2002) and has been under continuous development and addition of new elements in the past two decades. These days, GeoGebra has the form of a rich open-source application covering the topics of geometry, algebra, calculus and statistics, in the range from primary school to university levels (GeoGebra, 2021).

GeoGebra Classroom is one of the latest features of GeoGebra software which was introduced to public in May 2020. It allows teachers (or researchers) to assign various tasks to students, then ask them to join the environment via entering a code and solve the tasks individually. Teachers or researchers can observe updated students' advances in the solving process live, and record the step-by-step course of all the constructions contributed by individual students. Such interactive features provide us with complex data on students' work. In this contribution, we show how these features might be used in research focusing on student knowledge.

Design of the study

Participants

Participants of our research study were 19 students from the same ninth-grade class (age 15 to 16 years) at a suburban school. At the time of the study, they had already discussed whole mathematical curriculum belonging to the lower-secondary school level (grades 6 to 9), including prescribed parts of Euclidean planar geometry and geometric constructions such as copying a line segment, an angle, or a triangle, constructing circles, triangles and quadrilaterals on the basis of certain requirements, or constructing an image of an object in particular symmetry. During some of their previous mathematics lessons, they had also gotten to experience exercises in the GeoGebra environment. Nevertheless, they had not been asked to perform geometric constructions in GeoGebra Classroom before.

Data collection

As indicated above, we prepared GeoGebra Classroom based on three activities with GeoGebra applet where we let the students construct perpendicular lines. The first activity was named “Right angle for the first time” and its assignment was formulated as “Make a right angle! I.e., construct two lines, about which you can safely say that they are perpendicular to each other.” The second activity was named “Right angle differently” and it requested to “Make a right angle again, but now construct two perpendicular lines in a different way than in the previous task.” The third activity was named “Right angle still differently?” and it asked “Can you do it again? Construct a right angle by a different method than in the two previous tasks.”

For the purpose of our research, the GeoGebra toolbar was customized in all three activities. Solvers were able to use only the tools *Move*, *Point*, *Intersection*, *Line segment*, *Ray*, *Line*, *Circle with center through point*, *Compass* and *Delete*. They could also go through the steps of their construction using *Navigation bar*, return to the previous step pressing *Undo*, redone an undone action by pressing *Redo*, or start again with the button *Reset construction*.

Data collection was conducted during a mathematics lesson, while working face-to-face with the participants in the classroom. At the beginning of the lesson, each participant has got its own school tablet, i.e., a device with touchscreen and internet connection. They were used to working with this equipment. We gave them the code of the arranged GeoGebra Classroom to type in the box and join the environment. The participants had 20 minutes to complete all three assignments. They worked individually and independently; we provided them only with technical support.

Task analysis

As a preparation for data analysis, we investigated the assigned problem from the geometrical point of view. We looked for different construction strategies that would lead to creating perpendicular lines using only the given tools. We identified eight basic strategies; others might be derived from them by minor changes (e.g. in the order of steps) or by combining various basic strategies together. In their background, they all have a construction of a geometrical object that contains a right angle as a general property of its basic attributes (e.g. between a line segment and its axis, between diagonals in a rhombus, between a side and its median in an equilateral triangle, between a base and its median in an isosceles triangle) or as a consequence of a general principle used to construct the object (the Thales’s or Pythagorean theorem – Proposition 31 in Book 3 or Proposition 47 in Book 1 of Euclid's *Elements*). We ordered the strategies according to the interrelations between them. In the next paragraph, we illustrate the issue by introducing three of the identified strategies and their interrelations.

The strategy that can be considered as the simplest one is based on the construction of the perpendicular bisector of a line segment consisting of creating two circles with the centers at the end points of the line segment, the circles having the same radii greater than half of the line segment length. This construction had been well-known to the participants as a method of finding the axis (perpendicular bisector) of a line segment. Such an approach leads to obtaining two points (the intersections of the circles) that are at the same distance from the end points of the line segment. The set of all such points is exactly the same as the line that divides the given line segment into two halves forming right angles at the intersection point. This, in its substance, is a geometrical theorem that can be proven on the basis of the congruence of triangles or the properties of the diagonals of a rhombus. When the radii of the circles are equal to the length of the line segment, the circles go through the end points; we labeled this strategy as PL1. The general case when the radii are arbitrary but same and not equal to the length of the line segment, we labeled as PL2. The modification of the general strategy, where the radii are not the same, we labeled as PL3. The PL3 case leads to properties of diagonals of a kite (a quadrilateral with two pairs of adjacent sides of the same length has its diagonals perpendicular).

Data analysis

We analyzed collected data qualitatively, using open coding and constant comparison (Miles, Huberman & Saldaña, 2014). We carefully observed all students' solutions from the perspective of correctness and relevance of (i) individual construction steps, (ii) the figure presented as final in the construction, (iii) the sequence of the GeoGebra tools used during the construction. The latter information was available through the GeoGebra functionality *Construction protocol*. We were also comparing the ongoing findings with our list of basic strategies.

Findings

Strategies provided by participants – an overview

During data analysis, we identified three different basic strategies used by the respondents: PL1 mentioned above, a strategy based on the Thales's theorem (an angle inscribed across a circle's diameter is a right angle) that had also been stated in our list of strategies, and a new strategy that had not been stated in the list. The new strategy consists of the construction of three circles with collinear centers and the same radii, and two rays passing the intersections of the circles and forming the equilateral triangle. Generally speaking, the method is based on the properties of triangular lattice. We added this new strategy to our list, and reordered the list of basic strategies to still follow the sequence of relations between them. After the reordering, PL1 stayed PL1, the new strategy became the fifth one, i.e. PL5, and the strategy based on the Thales's theorem became PL9.

For details on the three strategies see Fig. 1: for each of the strategies, we present the number of respondents that provided the strategy (in square brackets), the list of GeoGebra icons that were available for the respondents during the task, the final figure, the sequence of the construction steps expressed through the GeoGebra tools icons, an explanatory drawing proving the perpendicularity, and a note on the geometrical background of the method.

In all of the assigned activities together (i.e., among the $3 \cdot 19 = 57$ attempts to create perpendicular lines), 16 students completed the strategy PL1 (one of them twice, with two different orders of construction steps), one student completed PL5, and four students completed PL9. All other attempts were unsuccessful: the participants either did not present any perpendicular lines, or presented just a freehand sketch (they drew two lines that looked like they were perpendicular but, in reality, they were not).

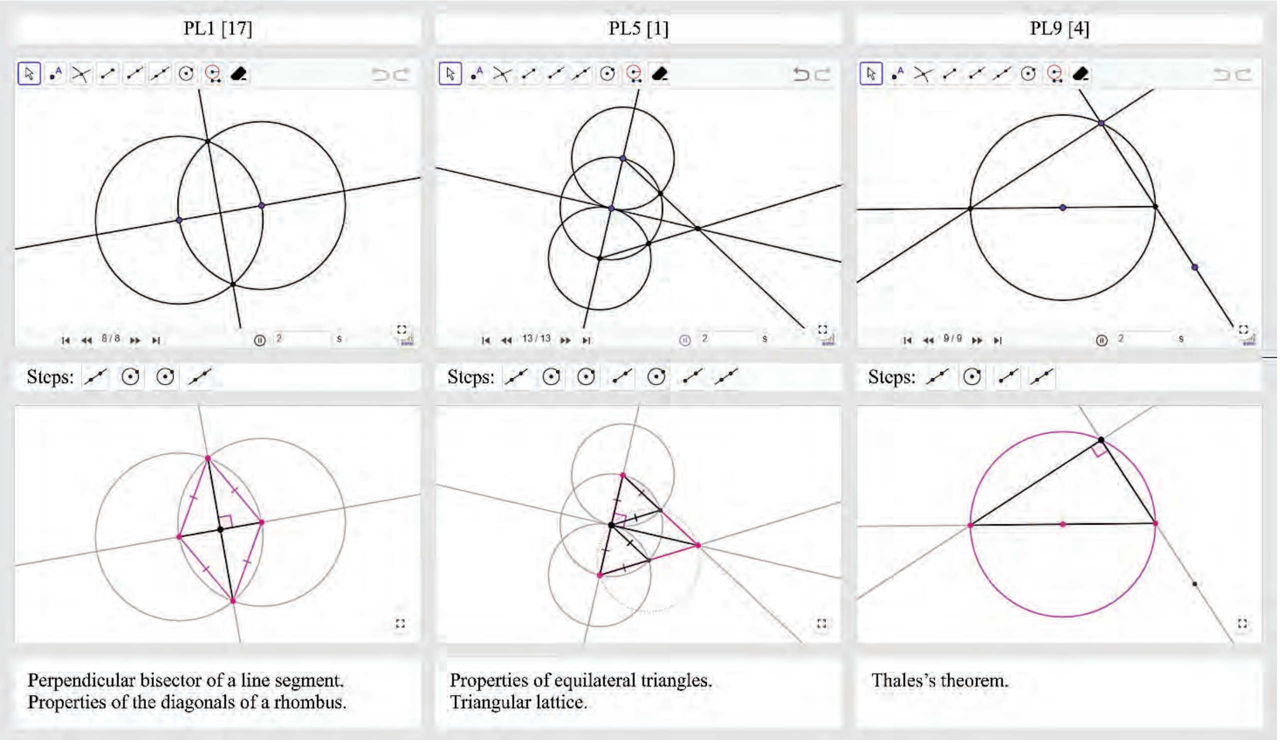

Figure 1: Three constructional strategies that appeared in data with, in square brackets, the number of respondents that provided the strategy

Individual participants

From the perspective of individual participants across the three activities, nobody was able to provide three different strategies. One student provided three different constructions that were based on two different strategies (one of the strategies was presented twice, with two different orders of steps), three students provided two different strategies, 12 students provided one strategy (all of them PL1), and three students did not manage to provide any strategy. The diagram of individual strategies within individual assigned activities that also captures individual participants' shifts in strategies across the three activities is shown in Fig. 2.

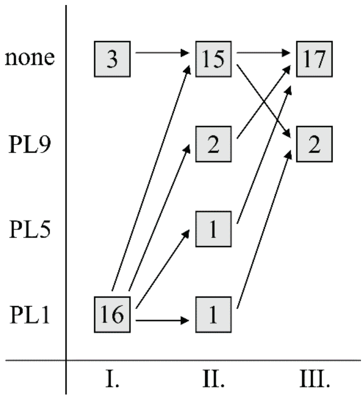

Figure 2: The number of students using individual strategies (PL1, PL5, PL9, none) within individual activities (I., II., III.), the arrows indicate participants' shifts in strategies

Discussion and conclusion

In this study, we focused on the variety of solution strategies that secondary school students provided while performing non-standard geometric constructions. For this purpose, we developed an interactive electronic environment in GeoGebra Classroom, and asked the students to construct perpendicular lines in multiple ways. To accomplish the non-standard nature of the task, we allowed just lines and circles to be used as tools during the construction. Using the GeoGebra Classroom environment, we obtained detailed complex data on students' work which confirmed the potential of DGE in rendering students the opportunities for deep explorations and heuristics (Lawson & Chinnappan, 2000). On the other hand, most of the students provided just one construction strategy, although we asked them to come up with three different ways of constructing perpendicular lines. The variability of the strategies was also low, we detected only three different students' approaches whereas we found at least eight solving methods available and appropriate for the given students. Therefore, as a plan for the future, we see the need to investigate further the relationship between students' performance in non-standard geometric constructions and multiple solution tasks, their conceptual understanding and their justification skills.

Among the construction strategies provided by students, two were based on a fundamental feature of a basic geometric figure or a basic construction – a perpendicular bisector of a line segment (PL1), Thales's theorem (PL9). These methods could be considered mere applications of a known geometrical shape property or construction procedure. However, the approach PL5 that emerged from data represents a shift from the use of the known individual object or construction procedure to a creative synthesis of various geometrical objects and constructions. We could perceive this method as a construct that arises from an enhanced connectedness of student's knowledge in the domain (Levav-Waynberg & Leikin, 2012) or from student's ability to use concepts and procedures in a flexible way (Rittle-Johnson et al., 2012).

The strategy PL5 also highlights the significance of open-ended approach in mathematics teaching and learning (Pehkonen, 1997) that goes hand in hand with the need for stimulation of using various construction strategies as in our assignment. It can be concluded that, without this request, the strategy PL5 would not have been explored, contemplated and developed by the student. For the future, it would be helpful to investigate which other impulses can lead students to such innovative and original solutions of geometric problems.

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Embodied approaches to functional thinking using digital technology: A bibliometrics-guided review

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Digital technology offers many opportunities for embodied approaches to mathematics education. To investigate what is known from literature about such approaches for the case of Functional Thinking, we carried out a systematic literature review, followed by a bibliometric and an expert content analysis. We included 36 peer-reviewed articles from 1986 to 2020 in the study. As a result, we identified five research themes in the field, which are further merged into three categories labelled Embodiment not central, Pseudo embodiment and Embodiment.

Keywords: Bibliometric analysis, digital technology, embodied cognition, functional thinking.

Introduction

In cognitive science, it is emphasised that cognition originates or is grounded in bodily motions and perceptual experience (Barsalou, 1999; Barsalou, 2008; Lakoff & Johnson, 1999; Lakoff & Núñez, 2001). In this context, recently developed digital technology, including motion detectors and augmented reality, seems to offer opportunities for an embodied approach to mathematics education (Bos et al., 2021; Drijvers, 2019; Nemirovsky et al., 2013). A bibliometric approach, which is an objective method that provides an overview of the knowledge structure of the domain (Li et al., 2019), was applied to explore these opportunities for the case of Functional Thinking (FT), a fundamental learning goal in mathematics education (Thompson, 1994; Vollrath,1989). The research question addressed is as follows: *What is known about the use of digital technology for an embodied approach to the teaching and learning of FT?*

Theoretical underpinnings

We draw on theoretical notions from embodied cognition and design research, digital technology research, and research on functional thinking.

Embodied cognition and embodied design

Several theories concern the role of the body in cognition and learning. Based on Conceptual Metaphor Theory in cognitive linguistics (Lakoff & Johnson, 1980), Lakoff and Núñez (2001) analysed the cognitive structure of mathematics and argued that the kinds of everyday conceptual mechanisms, image schemas, aspectual schemas, conceptual metaphor, and conceptual blends are central to mathematics. Some studies carry a similar idea about mathematics cognition concerning embodied design in function learning (e.g., Font et al., 2010; Oehrtman et al., 2019; Paz & Leron, 2009). From a perceptual perspective, Barsalou frames embodiment through grounding experiences, which is also advocated by Schwartz (1999) and Abrahamson et al. (2016) and employed in their own research. In addition, Shvarts et al. (2021) emphasise that knowledge emerges as part of a complex dynamic behavioural system that is constituted through multiple perception-action loops.

For educational materials, Abrahamson (2009) defined embodied design, which was first proposed by Rompay and Hekkert (2001), as a systematic and procedural design method, helpful in guiding the student's construction of meaning. At first, embodied design was classified into two categories: perception-based design and action-based design (Abrahamson, 2009, 2014; Abrahamson & Lindgren, 2014). Action-based designs aim to ground mathematical concepts in students' natural capacity to adaptively solve sensorimotor problems. Perception-based designs aim to ground mathematical concepts in students' natural perceptual ability in their naive views relating to a situation. Similar to the action-based genre, it is followed by a phase of reflection in which these views are developed. Concerning the role of artefact in learning design, Bos et al. (2021) propose a new type of embodied design, incorporation-based design, which is in a sense the opposite of outsourcing a task to an artefact instead of a person.

Digital technology in mathematics education

A major consideration in designing and using technology in mathematics classrooms is how to identify and use the different didactical functionalities. According to task-based interviews, Günster and Weigand (2020) set up a category system. We followed some of the categories for our study: (1) Feedback through the learning arrangement, (2) Use of sliders, (3) Creating objects, and (4) Adjusting existing objects to analyse the digital technology dimension. These four usages are related to doing mathematics and developing conceptual understanding with possible embodied elements. What's more, an embodied instrumentation approach, which can offer a design heuristic for ICT activities, was proposed by Drijvers (2019). This integrated approach, in which digital technology, mathematical cognition and sensorimotor schemes co-emerge, helps us better understand the relationship between embodied approaches, digital technology, and FT.

Functional thinking

Since the beginning of the twentieth century, functional thinking has been a central aspect of mathematical education throughout primary, secondary, and tertiary education (Vollrath, 1986). Although there is no widely adopted definition of FT, we propose that FT encompasses the process of building, describing, and reasoning with and about functions (Pittalis et al., 2020; Stephens et al., 2017). In a broader interpretation, FT connects to the four main aspects of function distinguished in literature (Confrey & Smith,1995; Doorman et al., 2012; Thompson, 1994; Vinner and Dreyfus, 1989; Vollrath, 1989): a) Function as an input-output assignment; b) Function as a dynamic process of covariation; c) Function as a correspondence relation; d) Function as a mathematical object.

Methods

To address the research question, we carried out a systematic literature search, followed by a bibliometric clustering (BC) and expert content analysis (Drijvers, Grauwin, & Trouche, 2020). The first step was part of the FunThink Erasmus+ project, a European research project.

Systematic literature search

The literature search was conducted in four databases: ERIC, PsycINFO, Scopus, and Web of Science. We searched for relevant studies published in peer-reviewed journals and written in English without restricting the publication date. Qualitative studies, quantitative studies and mixed-method studies were included. The query focused on Functional Thinking × (Embodiment OR Digital Technology). Our initial search yielded 278 journal articles. After deduplication, 257 unique publications remained. Next, we carried out two rounds of screening. The first round concerned a scan of title, abstract and keywords, to judge each article's relevance to each of the three aspects: Functional Thinking (FT), Embodiment (EM), and Digital Technology (DT). This led to 93 papers – empirical as well as theoretical papers - being selected with the help of ten coders from FunThink project. In the second round, eleven coders participated in the literature appraisal round, during which each coder read full texts and filled in a spreadsheet with the core ideas of each article. We removed the articles coded 0 to 2 as they are perceived as less helpful to our project. As a result, thirty-six articles were included in the final corpus.

Bibliometric clustering and expert content analysis

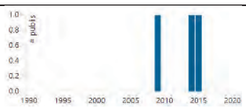
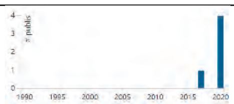
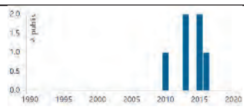
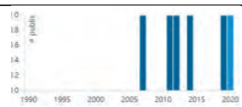
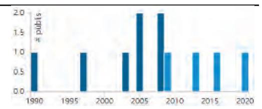
The studies in the final selection were classified with the help of BC techniques (Drijvers, Grauwin, & Trouche, 2020), which provides a sense-making sketch of the 'landscape' of our topic. We did not regard the bibliometric results as strict, exclusive categories; rather, we saw them as analytic tools that help us make sense of the rich diversity in this research field and to locate the main areas of embodied elements. Triangulating the bibliometric findings with expert content analysis helped us to find out a new taxonomy of the studies in relation to the different embodied approaches / embodied elements, which formed a basis for the study's results.

Results

Results from BC techniques include clusters that gather thematically close (based on the references) publications of the studied corpus; overall descriptions of the clusters, including an analysis of publication year, numbers involved, reference, and global meaning (see Table 1); and categories of embodied approaches to the use of digital technology for FT (see Table 2).

The bibliometric clustering leads to five clusters, containing 31 thematically close publications. The quality measure Q=0.429 suggests a meaningful partition. In each cluster, the most frequent references, the most frequent subjects, and the most cited authors are analysed. Table 1 presents some of these features.

Table 1 Description of the five clusters

	Cluster 1 (n=3)	Cluster 2 (n=5)	Cluster 3 (n=6)	Cluster 4 (n=6)	Cluster 5 (n=11)
Time distribution					
	between 2009 and 2015	mainly written after 2020	mainly after 2013	since 2007	since 1990
Top three cited	O' Callaghan (1998) Kennewell(2001) Keong (2005)	Drijvers(2015) Dubinsky (2013) Ellis (2011)	Artigue (2002) Zbiek (2007) Lagrange (2008)	Falcade (2007) Tall (1996) Vollrath (1989)	Nemirovsky (1998) Lakoff (2000) Mariotti (2002)

Our global analysis of tasks	based on computer algebra systems (CAS) but without explicit embodiment	based on animation and dynamic geometry software (DGS) but without explicit embodiment	based on TI-Nspire software/ calculator but without explicit embodiment	involving mouse movements (dragging, sliders, etc.)	based on embodied design

In light of the research question, our main goal is to explore how an embodied approach in learning design can affect developing functional thinking. We merged Cluster 1, Cluster 2 and Cluster 3 into one category labelled as *Embodiment not central*. This category focuses on digital technology-enhanced function learning and teaching without elaborate embodied designs, but has different types of technology. Next, we labelled Cluster 4 *Pseudo embodiment* category, which describes technology-enhanced designs with sort of embodied elements (like dragging). Finally, we labelled Cluster 5 *Embodiment* category, including eleven articles focus on embodied designs for function learning. Figure 1 depicts the results of the BC method, where the node size is proportional to the number of publications contained therein and the line thickness is proportional to the average similarity between the publications of the two linked clusters (in terms of shared references).

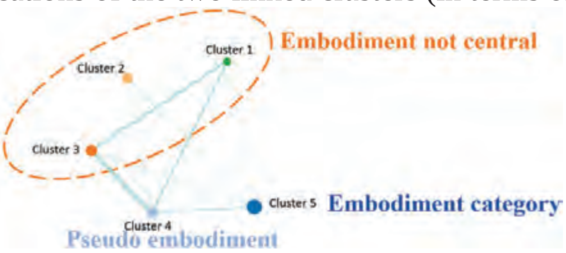
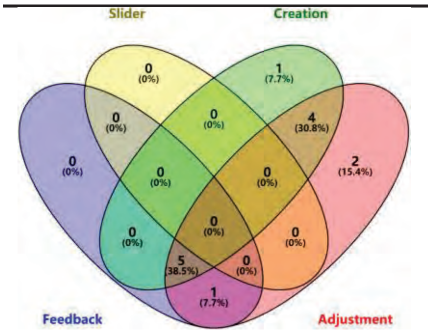
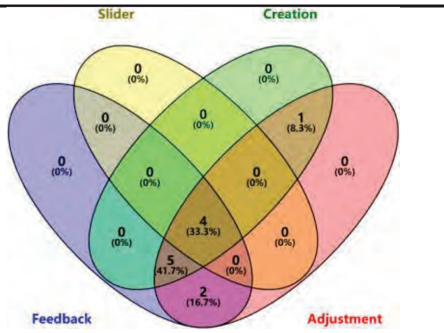
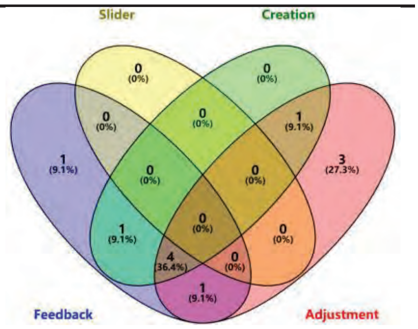


Figure 1 The network of clusters

Following the labelled categories, Table 2 illustrates how the possible embodied elements (the use of slider, create object, feedback through the learning arrangement and adjust object) are involved in the three types of embodiments. First, the use of sliders only appears in the Pseudo embodiment, that is, most designs only allow students to control sliders in the digital environment by mouse. Second, students are given many opportunities to create and adjust functional objects, mainly digital geometric objects, in the designs from Embodiment not central and Pseudo embodiment. But in the Embodiment category, about half of the designs offer the existing, elaborate objects that students only need to adjust. Finally, feedback from the digital environment appears more frequently in the Pseudo embodiment category.

Table 2 Categories of Articles Based on the Use of Digital Technology and Embodiment

Embodiment not central (n=13)	Pseudo embodiment (n=12)	Embodiment (n=11)
		

As a final result, given the theoretical underpinnings and bibliometric results described above, our content analysis led to the identification of three categories of embodied approaches: Embodiment not central, Pseudo embodiment and Embodiment.

Embodiment not central

In the Embodiment not central category, the most common configuration of the designs is creating and adjusting objects with/without feedback. And in these designs, students are allowed to adjust objects by inputting different values or pressing buttons on the calculators. Considering the mathematical object aspect of FT, especially the aspect graphing, GeoGebra, Graphmatica and TI-Nspire software/calculator are used to help students detect the effects of changing parameters in function on its graphical representation through supporting the modelling of different scenarios that allow students to study the effect of changes in the value of one variable on the other (Duijzer et al., 2019; Jon, 2013; Ogbonnaya, 2010). Along with the adjusting functionality, feedback from digital technology was also emphasised in the studies. For example, Asli Özgün-Koca (2016) pointed out that the feedback from the representations on the screen might help students recognise their misconceptions and overcome them through additional interactions with the digital tool. In addition, digital technology has the potential to motivate students and instil a curiosity that enables them to learn more when receiving real-time feedback from the tool (Ogbonnaya, 2010).

Pseudo embodied approach

Compared to the first category, embodied elements in this second category are more visible in the learning design, such as slider using and mouse dragging tasks. Mouse movements play an important role in using DGS (e.g., Cabri and GeoGebra) or in other digital environments (e.g., the Digital Mathematics Environment DME). There are two different settings of sliders: a) continuously slider (free movement on a bar without restriction), b) discrete slider (static selection of particular values). In Liang and Moore's case (2020), students could drag the endpoint (without restriction) to vary the length of the bar, which leads to the dynamic point on the circle moving correspondingly. Students can recognise amounts of change (covariation aspect of FT) when the perceptual material was given, but can not anticipate, represent or regenerate the changes when the perceptual material is absent.

Apart from using sliders, this category includes diverse mouse movements that provide more opportunities for students to create or explore relationships between entities and variables. The Arrow Chain module in DME, for example, is designed to foster conceptual understanding of the notion of function, where the main aspects of function in this design are input-output assignment, dynamic process of covariation, and mathematical object with different representations (Doorman et al., 2012). Students are able to drag and connect machines into function chains. In doing so, the idea of embodying the functional level to compose as well as the input-output assignment is clear. This design could offer educators and researchers some informed directions or ideas for using the technologies to achieve specific learning goals.

In addition to the different mouse movements of embodied elements, the type of feedback from digital technology can also differ among designs. The abovementioned design shows that the movement of connecting embodies the input-output process essential for the function notion, but there is no feedback on the movement itself. Falcade et al. (2007) designed two tasks with real-time feedback on the screen that allows the user to feel functional dependency in the domain of space and time. Students can find the effect of moving one of these points at a time and observe the traces they make through the Trace tool. The traces of points on the screen provide real-time feedback and serve as a cognitive anchor for learning about and understanding abstract concepts (Cox, 1999; Reiner, 2009).

Embodied approach through digital technology

This category of embodied studies in our corpus includes physical motions with the help of digital technology, especially registering movement digitally, processing, and providing feedback. A main similarity between the papers is the presence of (adaptive) motor tasks. In accordance with the three types of embodied task designs, action-based, perception-based, and incorporation-based (Bos et al., 2021), the following analyses provide insight into the possible embodied instrumentation approach (Drijvers, 2019) used in the studies of the third category.

Most studies used function-related tasks from an action-based perspective (8 out of 11). The embodiment of actions can supplement the input received from other modalities (e.g., vision), enabling students to construct richer multimodal representations to support more complex understanding (Drijvers, 2019). Distinctive regarding the understanding of functions, Nemirovsky et al. (2013) designed a mathematical instrument called Drawing in Motion, which is a prototype exhibit that requires physical engagement and collaboration between two people who jointly produce a graph on a displayed Cartesian coordinate plane through a large LCD screen. It did provide a new perspective of understanding function using the embodied instrumentation approach, compared to the conventional ways of thinking about functions (e.g., dynamic/process and static/structural conceptions). The authors claim that, given suitable mathematical instruments and practices, even young learners can engage in the learning of functions with the emphasis on the parameterisation of time.

Studies in the genres of perception-based and incorporation-based designs concerning FT so far are rare. Ferrara & Ferrari (2020) used WiiGraph software to engage pairs of students with functions through graphing motion, and one of their tasks, named *Line option for a+b*, showed the perception features. They even drew a conclusion that aspects of coordination and imagination push the mathematical activity further no matter whether the tool is in use or not. Again, the significance of perceptual experiences in the learning of function has been proved. The graphing motion technology, which allows working with couples of positions over time graphs, provides students with the opportunity to observe in real-time the graph of the sum of two functions on the screen, and then gain perceptual experiences supporting a concrete understanding of function.

Conclusion

With the aim of exploring how an embodied approach in learning design can affect students' FT with the support of digital technology, this study performed an expert content analysis using the BC approach. The literature analysis based on the selected corpus revealed three categories: (1) Embodiment not central; (2) Pseudo embodiment; and (3) Embodiment. All three categories have distinctive features in characterising the embodied elements of technology-enhanced learning designs. In the Embodiment not central category, embodiment remains implicit with keyboard strokes tasks and mouse-clicking tasks occupying the most learning designs. In the Pseudo embodiment category, mouse movements, as a distal movement, play an important role that can be made more proximal through touch screen technology and gestures that more closely correspond to the actual movements intended in using sliders or adjusting geometric objects. In the Embodiment category, digital technology allows for an embodied approach to register movement, process, provide feedback. From a methodological point of view, the bibliometric clustering technique did not offer new insights but did confirm our impression on how embodied approaches are involved in the domain of function.

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Developing a framework for creating heuristic worked example videos to enhance students' modeling competencies

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As a large number of instructional videos can be found online, the question arises to what extent videos can be used in mathematics classroom practice. While many videos address algorithmic domains, this theoretically informed paper focusses on the possibilities of using videos to enhance heuristic skills in the domain of modeling. It introduces a framework for developing heuristic worked example videos based on multimedia learning principles, (heuristic) worked example research and criteria for creating instructional videos. In addition, the framework and directions for future research are discussed.

Keywords: Interactive video, instructional design, cognitive load theory, mathematical modeling, secondary education.

Objective and rationale

Videos are gaining importance in the educational setting. The production of videos has become easier in recent years and many videos are available online (Kay, 2012; Kay & Edwards, 2012). Studies have shown that videos can have a positive effect on learning performance (for an overview, see Kay (2012)). Video formats that contain an instructional method, which is considered particularly beneficial for novices, are so-called worked example videos (Kay & Edwards, 2012). Worked examples present a problem and a step-by-step solution. In mathematics, the efficiency of (text-based) worked examples has been shown in several studies (e.g., Sweller & Cooper, 1985; Renkl et al., 1998). The efficiency is not only observable in algorithmic domains but also in less-structured domains such as proving or modeling using heuristic worked examples (Reiss et al., 2008; Zöttl et al., 2010; Tropper, 2019). First indications of the effectiveness of worked example videos in an algorithmic domain are provided by the study of Kay and Edwards (2012). The question arises, to what extent videos can be used to enhance heuristic skills in mathematics as videos offer new possibilities opposed to the medium "text", such as dynamic visualization. A domain that requires heuristic skills is mathematical modeling because it involves the whole process of mapping a real-world problem to a mathematical model by structuring information, searching for data that is not given, working mathematically and translating the results back to the real world (Niss et al., 2007). The process can be demanding for students because they face many obstacles along the way (Niss et al., 2007). This is why research on teaching through heuristic worked examples to enhance modeling competencies should be continued (Renkl, 2017) and a video provides a new approach. A video can not only explain a step-by-step solution of a modeling problem but it can also establish reality references by presenting scenes from the real world. On the contrary, one main criticism regarding videos is the frequent lack of interactivity and the correspondingly low level of student activation (Brame, 2016). Hence, this paper aims at developing a framework for effective heuristic worked example videos which are interactive and student-activating. To achieve this, principles derived from the cognitive theory of multimedia learning (Mayer, 2020) are considered as well as principles from (heuristic) worked example research. Both are closely related to cognitive load theory (Sweller et al., 1998), which makes this a key component when developing the framework. Moreover, existing frameworks for creating worked example videos (Kay, 2014), educational videos (Brame, 2016) and effective science explanation videos (Kulgemeyer, 2018) are analyzed for the specific demands of developing heuristic worked example videos. The goal is to provide a framework that can be used as an orientation for teachers, educational researchers or anyone who wants to produce interactive videos addressing heuristic domains in mathematics.

Cognitive load theory

When developing a framework for heuristic worked example videos (see Figure 1), criteria for creating educational video/multimedia and (heuristic) worked examples should be taken into account. The effectiveness of both is largely explained by cognitive load theory and design criteria are aiming at minimizing cognitive load. Three different kinds of cognitive load can be distinguished: intrinsic, extraneous and germane cognitive load (Sweller et al., 1998). While intrinsic cognitive load is caused by the intrinsic nature of the material and thus cannot be reduced by the instructor, extraneous cognitive load is imposed by poorly designed material. Germane cognitive load is necessary to construct schemas and is, thereby, an important requirement for storing knowledge in long-term memory (Sweller et al., 1998). Instructors should seek to increase germane cognitive load (e.g., consider suitable cognitive activities), decrease extraneous cognitive load (e.g., avoid confusing instructions, design material carefully) while keeping in mind that each subject imposes an intrinsic cognitive load on learners.

Criteria for developing heuristic worked example videos

In the following, each category for developing heuristic worked example videos (see Figure 1) is described and explained by video/multimedia research and research on (heuristic) worked examples.

Segmentation of the video based on a solution plan

One possibility to enhance germane cognitive load is following the *segmenting principle* by splitting a video into meaningful segments with a break in between two segments (Mayer, 2020, p. 247 ff.). When presenting videos as a continuous unit, learners might have problems processing preceding steps resulting in difficulties connecting them to following steps (Mayer, 2020, p. 252). In a study conducted by Biard et al. (2018), breaks after each main segment of a video and continuing the video by manually pressing the “play-button” led to a reduction of cognitive load. This did not apply when learners could pause a continuous video at self-determined points which might be due to the fact that learners made very little use of the pause-button (Biard et al., 2018). The *segmenting principle* has also been shown as advantageous for (text-based) worked examples (“modular worked examples”) in the domain of probability problems as it resulted in less study time, more correctly solved problems, less cognitive load and a higher feeling of success (Gerjets et al., 2006). If the instructor assigns labels to subgoals and makes the sense of each solution step salient, it can help learners to encode the structure of a problem (Renkl, 2014). This is also included in Kay’s (2014) framework for creating worked example videos as “meaningful steps”. Considering the heuristic worked example literature, it is advised to display a heuristic worked example by following a solution plan (Reiss & Renkl, 2002). Solution plans have been shown as a promising method to scaffold the modeling process of students (Schukajlow et al., 2015; Beckschulte, 2020). Thus, a solution plan provides guidance on how to segment a video into meaningful steps. Furthermore, pauses in between two steps offer the opportunity to prompt for self-explanations (see below).

Implementation of self-explanation prompts

When teaching mathematics, self-explanation prompts can help students build up procedural knowledge, conceptual knowledge and procedural transfer (Rittle-Johnson et al., 2017). Writing an explanation after each section of a video is considered as a *generative activity* and enhanced learning from video (especially for low-knowledge learners) (Mayer et al., 2020). This technique has been superior to rewatching the video or generating a drawing (Fiorella et al., 2020). Other options for prompting during videos are having the learners compare the content observed to prior knowledge or other material (Chi & Wylie, 2014). This corresponds to the *self-explanation and comparison principle* in worked example research (Renkl, 2014). In the case of novices or learners being unable to produce self-explanations, instructors should offer scaffolds such as structuring self-explanation responses (Renkl, 2017; Rittle-Johnson et al., 2017). When implemented into heuristic worked examples in modeling, principle-based prompts helped students elaborate the underlying procedures of an example (Tropper, 2019, p. 235).

Category	Indication from video and multimedia research	Indication from (heuristic) worked example research
Segmentation	segmenting principle (Mayer, 2020) meaningful steps (Kay, 2014)	usage of modular examples (Gerjets et al., 2006) following a solution plan (Reiss & Renkl, 2002)
Self-explanation prompts	generative activity principle (Fiorella et al., 2020; Mayer et al., 2020)	self-explanation and comparison principle (Renkl, 2014, 2017) elaboration of underlying procedures (Tropper, 2019)
Larger concept	follow-up learning tasks (Kulgemeyer, 2018) larger homework assignment (Brame, 2016)	example-set principle (Renkl, 2014) fade worked steps (Renkl, 2017)
Heuristic strategies	clear problem label (Kay, 2014) explain key elements (Kay, 2014)	make used heuristics explicit (Reiss & Renkl, 2002)
Layout decisions	weeding (Ibrahim et al., 2012) / seductive details principle (Mayer et al., 2020) / length shorter than 6 minutes (Guo et al., 2014) dynamic drawing principle (Mayer et al., 2020) / video style: preferably showing instructor's hand (Mayer et al., 2020) signaling (Brame, 2016) / highlighting, write down key information (Kay, 2014) / signaling principle (Mayer, 2020) match modality (Brame, 2016) / modality principle (Mayer, 2020) / use visuals (Kay, 2014)	
Conversational language	conversational language (Brame, 2016) / direct addressing (Kulgemeyer, 2018) / personalization principle (Mayer, 2020) engaging voice (Kay, 2014) / voice principle (Mayer, 2020) pace: 185-254 word per minute (Guo et al., 2014)	

Figure 1: Framework for developing heuristic worked example videos

Integration of the video into a larger concept

Rather than presenting standalone videos, videos should be integrated into a larger learning concept. Especially when learning complex skills, presenting only one (video) example might not be sufficient. Brame (2016) suggests to embed videos into a larger homework assignment. Moreover, Kulgemeyer (2018) recommends supplying learners with follow-up learning tasks, so that they have an opportunity to use the explained information for problem-solving. Considering worked example research, there may be limitations when presenting example-problem pairs, as they do not necessarily lead to a greater learning outcome than presenting worked examples only (van Gog et al., 2011). The *interleaving by fading principle* (Renkl, 2014) suggests to rather design a fading procedure with a complete example presented first before gradually fading solution steps. Atkinson et al. (2003) found that this is especially fruitful when combined with self-explanation prompts as it fostered near- and far-transfer performance in the domain of probability calculation. The integration of the video into a larger concept does not only suggest to fade worked steps and to integrate self-explanation prompts on the level of the video, but also implies different usage scenarios: Students could watch a heuristic worked example video at home in order to initiate in-class (group-)work. Moreover, a heuristic worked example video with its integrated prompts and faded worked steps could be used to support group work in class by stimulating discussion and structuring the solution process.

Explication of heuristic strategies

When solving problems, experts in mathematics usually employ heuristic strategies (Collins et al., 1987, p. 12). In order to present an approach to solve a problem, a video should at first clearly label the problem addressed (Kay, 2014). Subsequently, structuring the video on the basis of a solution plan not only offers guidance for the video segmentation as described above, but it can also be used to depict an experts' approach by outlining applied heuristics explicitly along the way (Reiss & Renkl, 2002). This opens up the possibility of explaining key elements in a way that learners understand the problem structure and the essential elements required to solve the problem (Kay, 2014). Transferred to a video aiming at the enhancement of students' modeling competencies, explaining key elements could involve visually displaying how to search for a comparison value. Depending on the example demonstrated, many other modeling-specific heuristic strategies are imaginable. Presenting those in a video compared to explaining them in a text-based heuristic worked example might be promising because a video supports the modality principle as described below.

Minimizing cognitive load through layout decisions

The underlying principle of the following design recommendations is the reduction of extraneous cognitive load through layout decisions. Since the medium "video" largely differs in the layout possibilities from the medium "text", the criteria are solely based on video/multimedia research. A video containing additional interesting but irrelevant features might increase the precepted learning difficulty and decrease the focus on essential content of the video (Ibrahim et al., 2012). This is called the *seductive details principle* (Mayer et al., 2020) and *weeding* (Ibrahim et al., 2012) describes the process of excluding interesting but irrelevant word or graphics from a video. *Weeding* might also help reducing the length of a video, which has been shown to be an important factor considering student engagement (Guo et al., 2014). Based on the analysis of 6.9 million MOOC videos, Guo et al. (2014) advise that video chunks should not exceed 6 minutes. A video dealing with a complex example probably outranges this recommendation. Nevertheless, it provides orientation when planning the video segmentation as described above, arranging less than six-minute-long segments. Another design feature which should be considered is the display style of the video. Different kinds of videos such as voice-over slide presentations, drawing on a paper and filming from above or drawing on a digital tablet are possible. Mayer et al. (2020) outline that learning outcome increases when an instructor draws graphics on a board rather than adding a voice-over to already drawn graphics (*dynamic drawing principle*), with the instructor's hand visible being an important factor. However, a video containing dynamic drawing without the hand visible still outrated the voice-over to the same already drawn graphics (Fiorella et al., 2019). Altogether, the decision of the video style might be significantly influenced by the content and task presented. When presenting a modeling task through a video including realistic film scenes, the video material may become an essential part to solve the task, if information can only be retrieved from the video (Greefrath & Vos, 2021). In this case, a video including drawings on a digital tablet might be a suitable format. Frames can be used to highlight important information even though the dynamic drawing principle might not unfold its complete potential. Nevertheless, it still offers possibilities for signaling through color codes or arrows. This is seen as an important feature in videos (Brame, 2016; Kay, 2014) explained by the *signaling principle* (Mayer, 2020, p. 166 ff.) because signals provide guidance for learners' attention. Finally, this style allows to address both the audio and visual channel, taking advantage of the *modality principle* (Mayer, 2020, p. 281 ff.), which states that people learn better from pictures and spoken word than from pictures and printed words.

Usage of conversational and engaging language

Other than a text, a video usually relies on visuals and narrated words. This means that the language can have an impact on learning. In videos, a conversational language is preferred over a formal language (Brame, 2016) and using direct addressing is preferred over a passive voice (Kulgemeyer, 2018). The underlying principle is called the *personalization principle* (Mayer, 2020, p. 305 ff.) and its efficiency is explained by learners trying harder to understand the content when the feeling arises that the instructor is talking to them. This is the same reason why speaking in an appealing human voice fosters learning (*voice principle*) (Mayer, 2020, p. 322 ff.) and is considered as an important component of worked example videos (Kay, 2014). It might also be essential to speak relatively quickly (Brame, 2016), with 185-254 words per minute recommended (Guo et al., 2014).

Discussion

The developed framework offers theoretical considerations about how to support student learning from heuristic worked example videos in the domain of modeling by considering the impact of design features on cognitive load.

Even though a comprehensive review of educational video design criteria was conducted and principles of generative learning were taken into account, the criticism that videos frequently lack interactivity cannot be denied completely. The approach of (heuristic) worked examples more or less implies direct instruction. This tension cannot – and should not – be completely resolved in the case of heuristic worked example videos. Providing learners with solution steps is one (if not the) key component of (heuristic) worked examples. It results in a possible limitation of the proposed framework though: Videos based on this framework are probably more suitable for novice learners and may not be a promising approach for advanced learners. The fact that knowledgeable learners might profit more from a minimal instructional approach has been reported as the *expertise reversal effect* in worked example research (for an overview, see Kalyuga (2007)).

Since this theoretical framework has not been tested yet, the next step should be to put it into practice by developing videos based on this approach. Observing students' behavior while working with those videos can allow drawing conclusions about which categories achieve the desired impact. Moreover, the usage of heuristic strategies while working with the videos (especially with the self-explanation prompts and the faded solution steps) should be examined in order to understand to what extent videos can be used to enhance modeling competencies. This may contribute to the proposed further research on worked examples in the domain of modeling (Renkl, 2017) and to the little research done on worked example videos in mathematics (Kay & Edwards, 2012). It is also conceivable to use this framework to produce videos that target other heuristic domains such as problem solving or proving. For this purpose, an adapted solution plan can be used and problem solving or proving strategies can be made explicit analogously to the modeling-specific strategies.

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TWG17: Theoretical Perspectives and Approaches in Mathematics Education Research

Introduction to the Thematic Working Group 17 on Theoretical Perspectives and Approaches in Mathematics Education Research of CERME 12. Horizontal and vertical theorizing.

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Starting from previous lines of thought

Theories are essential parts of each scientific discipline; they can be used to solve problems, to answer research questions, to capture phenomena, to predict what can be expected or prescribe what can be done in practice (Prediger, 2019). However, in the field of mathematics education, there is no consensus of the notion of 'theory' (Assude et al., 2008). What is agreed upon in most communities in our field is that theories are "… individual or social constructions which serve to understand and describe a part of reality in a consistent manner" (see. Maier & Beck, 2001, p. 45, own translation). They provide a language and a lens "…. to understand what are taken to be the things that can be questioned and what counts as an answer to that questioning.'' (Mason & Waywood, 1996, p. 1056). Theories afford the coherence of a research framework and provide the space for consistent argumentation. As Bishop (1992) summarizes, a theory "is the way in which we represent the knowledge and understanding that comes from any particular research study. Theory is the essential product of the research activities, and theorizing, therefore, its essential goal." (p. 711)

Radford (2008, 2012) has conceptualized theory as a way of understanding based on a collection of principles (P), methods connected to those principles (M), and paradigmatic research questions (Q). When researchers draw on a theory for research to produce results (R), those results can contribute to further theory development. Hence theories are dynamic, rather than static, and they continue to evolve via researchers' engagement with them.

Previous TWG17s have addressed the question of how researchers and designers work with theories in the field of mathematics education (Kidron et al., 2018), taking tasks and tools into consideration. Two key, interrelated issues have been (1) how to grasp the complexity of teaching and learning of mathematics and (2) how to deal with the diversity of theories in the field. Working with theories is embedded in the culture of educational systems, and we witness a broad diversity of educational systems worldwide. Accordingly, our working group acknowledges diversity of theories as a kind of richness in mathematics education, through which learning from and with each other provides the potential of advancing the field of mathematics education as a whole.

Coherence of a research framework and consistency of argumentation are guiding principles in the use of theories. This allows researchers to advance knowledge, which may be relevant even beyond mathematics education contexts. In its final discussion at CERME11, the TWG17 has agreed upon the dynamic, evolving nature of theories in research, as Bikner-Ahsbahs et al. (2019) summarize:

> … scholars should neither demand that theories be used with absolute rigor nor allow arbitrarily applications of theory. To form coherent research frameworks, scholars engage in reconsidering, reinterpreting and reusing theories to investigate new phenomena, solve new problems and serve new purposes. Thus, theories develop and evolve through research. Working on coherence and consistency is an ongoing research task, particularly necessary for the Networking of Theories, in which reconsidering the compatibility of the theories or theoretical approaches is an additional epistemological necessity. Achieving generativity, generalizability, and generality affords the potential of the research results to be useful for answering new questions. (p. 3026)

The networking of theories approach (Bikner-Ahsbahs & Prediger, 2014) has been intensively discussed in previous TWG17s (Kidron et al., 2018). It has supported researchers' navigation of a variety of epistemological stances that may underlie the different theories involved in research and their exploration of how various theories can be used to investigate complex situations of teaching and learning. Yet, further elaboration of the networking of theories is needed to expand its research potential. A landscape of networking strategies has proven fruitful for guiding such efforts (Prediger et al., 2008). Specifically, the networking strategies of coordinating and locally integrating theoretical approaches have led to advancing our knowledge (Bikner-Ahsbahs & Prediger, 2014).

The TWG17 of CERME 11 identified the important role of certain sensitivities in theoretical work (e.g., the theory's ecology, its grain size, the nature of the mathematical content it considers, the research objects, etc.), and distinguished between generalizability, generality, and generativity in research methodologies: generalizability in empirical, generality in theoretical and generativity in design research. Chan and Clarke (2019) elaborated how our choices in research are based on mutual affordances between these theoretical sensitivities and the methodologies and methods to be used. Thus, researchers should be aware that theoretical, methodological, and methodical work are intermingled and related to the cultural context in which research is conducted.

Issues addressed in the call: connection to previous TWG17s

Design research has been prevalent in discussions of previous TWG17s. This pointed to various ways that researchers' decision-making could influence their theoretical approaches, specifically with respect to transforming versus depicting in mathematics education research processes. In this TWG17, we addressed this distinction by asking for contributions on theories related to design research, technology use and conditions for a productive dialogue between theorists.

What has been implicit in previous TWG17s were basic commitments underlying theoretical work. For this TWG17 we called for more explicit discussion of ethical commitments in theorizing and theory networking. We also called for attention to ontological, epistemological and axiological presumptions of theories. Adapted from Patterson and Williams (1998), Daene (2018, n.p.) distinguished and described four such commitments. These were “the nature of reality and what really exists (ontology); the relationship between the knower and what is known (epistemology), what we value and how we determine that value (axiology), the strategy and justifications in constructing a specific type of knowledge (methodology), as linked to individual techniques (method/s).” In theories on teaching and learning we–sometimes implicitly–could make ontological assumptions about the piece of reality addressed (e.g., whether teaching-learning is an irreducible entity or consists of two different processes). In epistemology we could build on this assumption to ask how we can know something about the ontological entity and how this knowledge must be so designed. In methodology we could address the question of how knowledge can be produced (and by what means). Ethics, being part of axiology, could enable us “to rethink and re-evaluate some of the taken-for-granted commonplaces of our practices” (Ernest, 2012, as cited in Stinson, 2017, p. 2), “which opens up new possibilities for theorizing and researching mathematics teaching and learning.” (Stinson, 2017, p. 2)

Main contributions achieved

Fifteen papers and four posters were presented in this working group. In sum, 52 authors from 17 countries worldwide were involved. This indicated that doing research happens in communities rather than individually. Fifteen out of 19 contributions followed a networking of theories approach, indicating that networking of theories is being normalized in research although it still needs to be further developed. Design research became more prominent, with many papers involving this approach. In line with previous TWG17s many theory elements were addressed. There were 22 theories and 27 theoretical concepts or ideas addressed across the contributions. The awareness of our own achievements in our home field of mathematics education seemed to be growing. This was not so obvious in the papers but more so in our ways of talking about theories. Scholars became more often named as originators of the theories together with the theories, for example when talking about Duval’s cognitive theory of representation or Schoenfeld’s Resources-Orientations-Goals theory.

A key outcome of our discussion is a new characterization of theorizing, in terms of two dimensions: *horizontal* and *vertical* theorizing, which we illustrate in Figure 1. Horizontal theorizing happens when researchers draw on a theory (or theories) to make sense of problems or phenomena. With horizontal theorizing, researchers focus on the *how* of theory use, for the purpose of illuminating new aspects of complex phenomena. For example, researchers may network different theories to further the investigation of empirical phenomena, such as students’ reasoning in engineering education. In contrast, vertical theorizing happens in the semiosphere, the cultural semiotic space of theory cultures (Radford, 2008; Lotman, 1990). With vertical theorizing, researchers focus on meta-issues of theory use for the purpose of *understanding theories as entities in and of themselves.* For example, researchers may weigh the epistemological ramifications of networking different theories, and argue for the viability of doing so. Researchers’ theorizing may address horizontal or vertical theorizing, or both. As suggested by our examples, the networking of theories entails both. The rule of *keeping our feet on the empirical ground* (Figure 1) takes seriously the purpose of a theory to act as a tool for understanding the empirical world. It means that vertical theorizing can only increase our understanding of theories when it is grounded in horizontal theorizing.

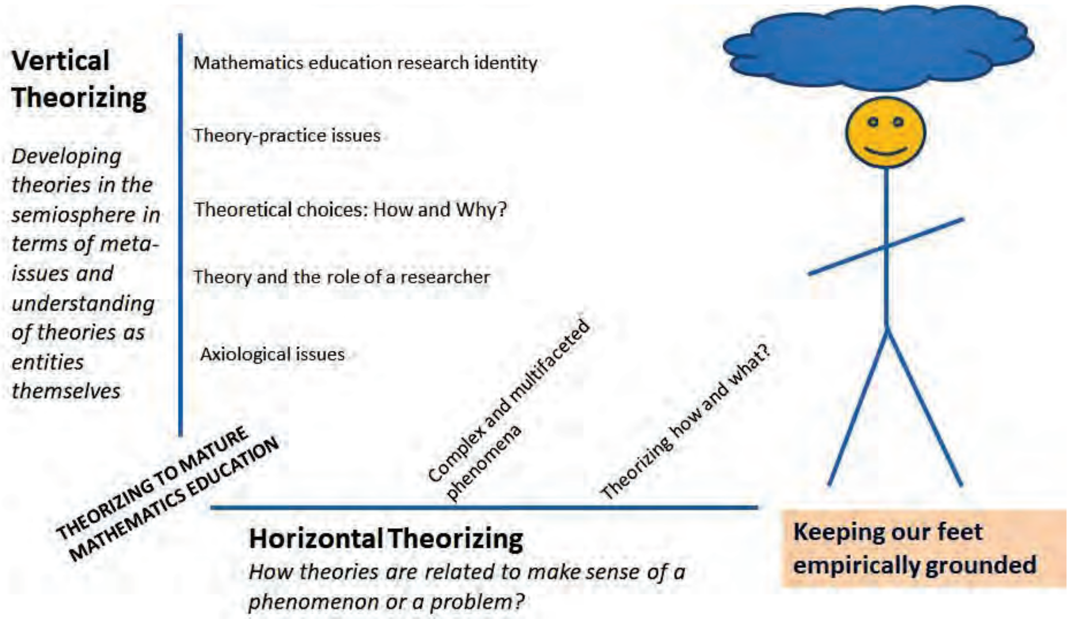

Figure 1: two dimensions of theorizing

Contributions of horizontal theorizing

Some papers addressed complex and multifaceted phenomena as a unity, a balance or tension between theoretical parts. Fosse et al. used Cultural Historical Activity Theory (CHAT) to investigate expansive learning when students crossed institutional cultures in their career. They explored masters students' transitions from university to school to identify if contradictions spark their learning and how and what teacher educators can learn from it for their own teaching. Herbst et al. proposed to use networking of theories to investigate teacher decision making and the role of structure and agency in these processes. Kuzniak and Nechache presented the Mathematical Working Space as a framework to be linked with other theories (e.g., Abstraction in Context) to describe how cognitive actions start from the epistemological level and how it develops with respect to semiotic, discursive and instrumental genesis. They posited that actions may link mathematical work with other theories.

Several contributions theorized the use of artifacts or tools and considered their role for progressing. Shvarts et al. analysed Freudenthal's and Davydov's work disclosing their common understanding of mathematical perception in learning to advance the embodied design framework. Starting with concrete actions, *ascending from the abstract to the concrete* entailed viewing the concrete in a completely new way through the lens of acquired artifacts. Thus, one's perception of the concrete would change as learning proceeds. Salinas-Hernandez et al. linked a cultural-historical approach with a semiotic view. They explored the "Imaging Teaching Scheme" of a physics teacher who used various artifact representations for the production of signs to guide learning. Santi et al. networked the theory of objectification with a differentiation approach for design research. Open Activity Theory Lesson Plan is an artifact they developed to guide designing stations to foster inclusive instruction by processes of objectification and subjectification of all learners. Kanwal networked a CHAT frame with the concept of creative and imitative reasoning to investigate teaching mathematics in engineering education at university. Computer software provided the artifact that shaped the conditions for the students' operating that constrained their reasoning actions.

In some contributions, horizontal theorizing pursued specific purposes. This way, researchers could clarify why networking was needed and helpful. Bach et al. networked Duval's (2017) cognitive theory of representation and the Instrumental Approach in a design research approach for learning functions to foster the representation competency defined in the Danish KOM framework. In their paper, they showed how the networking strategy of coordinating helped them to structure developing design principles. Petersen networked concept image and concept definition also with the cognitive theory of representation to enlighten mathematical thinking related to the KOM framework with a focus on differentiability. Kanwal's theoretical approach allowed for the consideration of students within the activity system as a wider entity. Researchers also could use theories to make new aspects visible. Tuktamyshov offered a "picture of the world." Mali et al. drew on the concept of "perezhivanie" to expose learners' life-changing experiences, and Zagorianakos linked perezhivanie with phenomenology to explore what this might add to previous insights.

Another purpose of horizontal theorizing was to take up and explore new challenges. Bikner-Ahsbahs et al. met the challenge of the pandemic and shifted the summer school YESS10 to an online conference format. This new situation allowed them to conceptualize the rhythmic orchestration of the research pentagon (Bikner-Ahsbahs, 2019). Gardesten coordinated the Knowledge Quartet and the Pedagogical Relational Teachership to build a methodological tool to explore teachers' pedagogical content knowledge and relational abilities when teaching mathematics in inclusive settings. Barquero et al. identified points of contact between the Antropological Theory of the Didactic and the Theory of Didactic Situations to explore constraints of a new paradigm of instruction, the Study and Research Path.

Contributions to vertical theorizing

Vertical theorizing can happen in many ways. Lensing has provided an example of vertical theorizing as he looked at the three essential systems in mathematics education, the social, the individual and the body, and the problem of the impossibility to theorize them jointly. His solution of this complexity problem was to look at regularities the systems share and, thus, build theories on a more formal level. Other ways of vertical theorizing ask for how we understand and practice theorizing, and what kind of ethical, ontological, epistemological, and methodological commitments are present in theorizing. Critical for the ethical dimension are the aspects to which a theory attends while leaving others left aside, which means that some aspects are valued over others. For example, the theory of objectification (TO) addresses the dialectic between objectification and subjectification in teaching and learning mediated by tools (Radford, 2021), but it does not tell Santi et al. how to differentiate tasks for an inclusive setting. Thus, differentiation is invisible in the TO. Santi et al.'s networking theories approach is a way to display differentiation and simultaneously adhering to the TO.

To date it is not so clear what, specifically, theorizing entails when the aim is not to apply a theory to an empirical situation but rather to create new theoretical steps. Valdés-Zorrilla et al. have proposed to consider theorizing as a kind of *metaphorizing*, starting from the source domain of a metaphor to theorize the target domain with the help of the metaphor. In such a process, the role of the researcher as a user, borrower, adaptor, developer, or creator of theories comes into play. This necessitates being explicit about how, why, and for what theoretical steps are made. In this context, two purposes of theory use become relevant: using a theory to depict versus using a theory to transform a teaching-learning situation. The former requires quite different kinds of reasoning than the latter. This has consequences for methodological choices as Chan and Clarke (2019) have pointed out. They have explored the reciprocity of theory and methodology showing the mutual affordance between theoretical and methodical choices. It requires vertical theorizing to describe particularities of how mutual affordance is involved in research. Following this path of vertical theorizing, Johnson et al. have elaborated what mutual affordance might mean for the networking of theories. They have proposed to use the metaphor of a multifocal lens to guide mutual affordances between theory networking and methodical choices.

Vertical theorizing is also present in the researchers' sensitivity to decide how, why, and for what purpose a theoretical step is needed and where to start with theoretical choices. Researchers are not free in their choices, as these are constrained by theory traditions. Theory development is only possible within certain limitations of a theory culture. When a transformative step goes beyond the realm of the theory culture, it can create epistemological obstacles. However, some theories such as CHAT may inform quite different research directions. Fosse et al. offer contradiction as a starting point to explore expansive learning. Kanwal also draws on CHAT to relate students' reasoning to the software the students used and to embed both into a wider activity system. When a new challenge such as the pandemic situation emerges, it may constrain research. However, it also may provide opportunities to see calls for new theorizing steps, such as Bikner-Ahsbahs et al.'s description of rhythm to extend the concept of instrumental orchestration.

Vertical theorizing also may address the relation between theory and practice and contribute to shape a disciplinary identity. In the practice of research, a theory needs an appropriate methodology to be put to work and lead to relevant insights. However, theory and methodology do not transform teaching and learning by itself. The practice of teaching and learning can be transformed by design research, which may in turn also transform research and gain new insights for theories. Design research may even be considered a research genre that is specific for mathematics education, thus contributing to form a mathematics education research identity. This can allow researchers to line up and navigate the power-filled landscape of disciplines and institutions. Theoretical work, including further developing the networking of theories approach and design-based research, may help the field of mathematics education to become aware of its strengths and of what it has to offer to other disciplines. In turn, this can strengthen mathematics education's status as a research discipline.

Lessons learned and future directions

Did we proceed with respect to consider the dimensions of ontology, epistemology, and axiology in processes of theorizing more explicitly? Three contributions point to progress in this area. Gardesten and Santi et al. are aiming at establishing inclusive mathematics education research. The dimension of ethics, and hence, axiology, is very explicit in their contributions. Santi et al. appeal to Radford's (2021) theory of objectification (TO) to position students and teachers as "reflective and ethical subjects" who engage in "joint labour" to develop new knowledge. In the analysis of Shvarts et al., we may consider the relationship between the works from Freudenthal and Davydov as being based on the common epistemological assumption that abstraction leads to a new vision of reality.

Ontologically, this means that reality is structured in a new way when it is perceived through using a mathematical artifact. In future TWG17s, there needs to be continued and more explicit attention to ontological, epistemological and axiological dimensions grounding our research.

The distinction between horizontal and vertical theorizing is an insightful step this TWG17 has made because it accredits us as researchers with the sensitivity of what theorizing means and in what kind of theorizing we are involved. Horizontal theorizing on concrete phenomena or problems is a necessary step in research, and we also need to strengthen our practice of vertical theorizing in order to be able to clarify basic assumptions in our research. Through vertical theorizing we can decide whether and how networking of theories is a consistent approach to reveal a coherent body of results. Whereas the objects of horizontal theorizing are phenomena in mathematics education, the objects of vertical theorizing are the theories or theory elements themselves (i.e., abstract entities in the semiosphere). Vertical theorizing is therefore more difficult to communicate, much more difficult than communicating how theories frame our research. Reflecting on concrete work may strengthen our theoretical understanding of a piece of reality allowing us to advance vertical theorizing.

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Networking practices in design research: refining design principles

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In a design research (DR) study concerning how students exercise their representation competency when using digital tools, we aim to investigate the potential of networking practices to support the purpose of developing theoretically grounded design principles in DR. We provide and discuss an empirical example of how design principles for students' activation of representation competency, when using GeoGebra, are adjusted through retrospective analysis in DR. We further show how such analyses may be structured through the coordinating strategy of networking of theories.

Keywords: Design research, networking of theories, predictive theory elements, design principles.

Introduction

DR aims to develop both theory and practice in an intertwined manner. An important part of DR is to develop principles for designing tasks through careful analyses using theoretical perspectives (Prediger, 2019). However, the development of theory only appears to a limited extent in some DR studies (diSessa & Cobb, 2004). Networking of theories offers strategies for understanding and relating theoretical constructs, while at the same time aiming to connect the many unrelated theoretical perspectives in mathematics education (Prediger et al., 2008). In this paper, we aim to investigate how practices from networking of theories may strengthen theory development in DR. To achieve this, we use an illustrative case involving task design aiming to exercise students' mathematical representation competency when using digital tools, such as dynamic geometry environments (DGE)). The Danish competency framework (KOM) defines eight competencies, among them is the *representation competency*, which involves the capability to make use of, reflectively choose, interpret, translate between mathematical representations and understanding the scope, limitations, and strengths of representations being used (Niss & Højgaard, 2019). For instance, students exercise representation competency when choosing the most appropriate representation for given phenomena or understanding relations between representations (e.g., graphs and equations). KOM does not address the use of digital tools at large, why coordination of two fine-grained theoretical perspectives is relevant. For the use of digital tools, we apply the instrumental approach (e.g., Trouche, 2005), and for further insight on the mathematical representations, we utilise Duval's (2017) perspectives on semiotic registers. Our research question is *Applying practices from networking of theories, how may design principles about students' exercise of representation competency when using digital tools be refined*? We present our DR process focusing on humble design elements. Then, we present perspectives on networking of theories and our theoretical perspectives, followed by our illustrative case of which we carry out parallel analyses conducted to define our design principles for exercising representation competency while using digital tools based on the humble design elements. Finally, we refine the design principles using a networked analysis in a discussion and networking practices in DR.

Design research and importance of design principles for task design

DR includes different phases: 1) *preparation of design*, involving the development of design principles; 2) *testing design* by gathering data in classrooms; and 3) *retrospective analyses*, which serve to analyse the design and refine the design principles (diSessa & Cobb, 2004). Theorising exits in all steps of DR. Different theory elements in DR may be characterised as categorical, normative, explanatory, descriptive and predictive theory elements. In particular, *predictive theory elements* are key for this paper, as it involves the development of *design principles* (Prediger, 2019). Design principles may have the following structure:

> If you want to design intervention X [for the purpose/function Y in context Z], then you are best advised to give that intervention the characteristics A, B, and C […], and to do that via procedures K, L, and M […], because of arguments P, Q, and R (van den Akker, 1999, p. 9)

Based on a previous literature review (Pedersen et al., 2021), involving the representation competency of KOM (Niss & Højgaard, 2019) and the perspective of semiotic registers (Duval, 2017), we present the *humble predictive heuristics*, meaning the design principles that have not yet been tested. The review revealed that students could engage in processes of dealing with different representations and their reciprocal relations in a multi-representational digital tool. Yet, there is a risk that the digital tools outsource essential aspects of mathematical activities by automatically generating translations between representations. As part of the literature review, we proposed five recommendations for task design aiming to have students exercise their representation competency in interplay with digital tools (Pedersen et al., 2021). Three out of the five recommendations are relevant for this study. According to van den Akker's (1999) definition, we regard them as *humble design characteristics* (HDC):

- HDC 1. Including the linguistic register by formulating the task in natural language and asking for answers in natural language.
- HDC 2. Breaking objects and windows of the tool into smaller units by introducing them gradually.
- HDC 3. Using sliders, dragging and tracing for explorations of different representation and representation forms' reciprocal relations.

Networking of theories and presentation of selected theoretical perspectives

Networking of theories provides strategies for connecting theories at different levels: From 'understanding others'/'making understandable' to 'integrating locally'/'synthesising'. We focus on the strategies 'combining' and 'coordinating', both focus on understanding an empirical phenomenon. Often, more than one theory is necessary to understand an empirical problem. When 'coordinating', the involved theories must be *compatible*, meaning that their *background theories outside mathematics education* (Bach et al., 2021) are not in conflict (Prediger et al., 2008).

We aim to network theoretical perspectives to understand the interplay of students' representation competency and their use of digital tools, which are compatible with KOM. KOM plays the role of what Bach et al. (2021) term as a 'framing theory', meaning a background theory inside mathematics education research. Although, it may rather be a framing framework. KOM is of a cognitive nature focussing on an individual's expression of cognition in particular mathematical situations (Niss &

Højgaard, 2019). Semiotic registers and the Instrumental Approach are in line with KOM as both address cognition. As the theoretical perspectives look at different objects and phenomena (mathematical representations and use of digital tools, respectively), they may complement each other. Hence, their backgrounds appear not to conflict indicating that 'coordination' may be possible.

Transformations between semiotic representations are essential in mathematical activity. In their nature, mathematical objects and processes are abstract and only accessible and transformable through their representations. Consequently, the process of mathematical activity involves transforming representations within or between registers. Four registers are distinguished depending on if a register is multi- or monofunctional and discursive or non-discursive (Duval, 2017). Discursive registers (the *symbolic* and the *linguistic* registers, named as such in Pedersen et al., 2021) involve written and spoken languages expressing meaning units of thoughts. Non-discursive registers (the *graphic* and the *figurative* register named as such in Pedersen et al., 2021), only display one (or more) static, visual object(s). Mono-functional registers (the *sy*mbolic and the *graphic* registers) involve algorithmic processes. Multifunctional registers (the *linguistic* and the *figurative* registers) then involve non-algorithmic processes, such as communication and imagination. Each register may contain more representation systems with their own rules. A transformation between representations within the same register is called *treatment.* A transformation between representations from one register to another is called *conversion* (Duval, 2017).

The instrumental approach to mathematics education concerns the cognitive process of turning an artefact (e.g., a ruler, calculator, or dragging in GeoGebra) into an instrument for the activity of solving any kind of task. When distinguishing between artefact and instrument, it is inherent that the artefact in itself is just a material object, and only becomes an instrument when an individual can appropriate parts of the artefact for a certain task. This process is called *instrumental genesis* and is developed over time and shaped through the two-way processes of instrumentation and instrumentalisation. *Instrumentation* is directed from the artefact towards a user, as its constraints and possibilities shape how an individual uses the artefact. Instrumentation can go from a phase of exploiting different possibilities of using the artefact to solve the task relying on prior knowledge to a more stable phase, where the same usage is applied for similar tasks (Trouche, 2005). *Instrumentalisation* is directed from the user towards the artefact, as the user uses and manipulates the artefact (based on the user's knowledge), which can be different from the original intent of the designer of the artefact and can both lead to enrichment of an artefact, or its impoverishment. Within this two-way process of instrumental genesis, an instrument involves an artefact (a material thing) and schemes concerning the mathematical content and its relation to the use of the artefact (Trouche, 2005).

Presentation of the case: Ivy and Aya

The data excerpt and task presented in this paper were chosen because the task represents humble design characteristics. The task (inspired by Johnson & McClintock, 2018) was originally developed to activate the communication competency, which often also involves one or more mathematical representations (Niss & Højgaard, 2019). The task relates different representations and different mathematical contents (geometry and functions) in a GeoGebra template. Further, in this paper, we relate the task to the representation competency. In the graphic view (i.e., the graphic environment in GeoGebra), a rectangle is constructed to the left of a coordinate system (first quadrant) with a point *P*. Point *A* of the rectangle can only be dragged vertically, such that the height changes and point *B* and C are fixed. Point P's *x*-coordinate corresponds to the rectangle's height and the *y*-coordinate to the area of the rectangle. When dragging point *A*, point *P* moves on the line $y=3x$. Unintentionally in the task design, it is also possible to drag point D on the rectangle horizontally. When dragging point *D*, point *P* moves on the vertical line, $x=|AB|$. The algebra view is also accessible in the template. The task and the transcript are translated from Danish. The task has four phases: 1) Investigate the rectangle and features related to it without dragging; 2) investigate the relationship between the rectangle and point *P* by dragging point *A*; 3) fill a table for related values; and 4) represent the functional relationship (to be $y=3x$), by its formula and a graph. The following excerpt presents two students' dialogue about the relationship between point *P* and the rectangle in phase 2, see Figure 1.

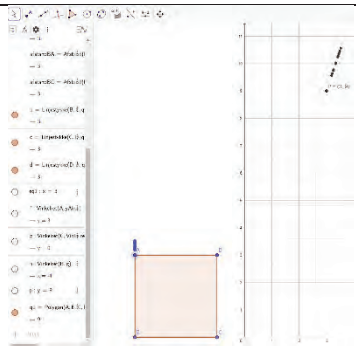
Investigate the construction on your computers by dragging point A in the figure. Describe the relationship between point P and the shown figure. For point P, describe what characterises the x-coordinate and the y-coordinate. Explain why the relationship exists.	

Figure 1: Task and illustration of the template with traces (Johnson & McClintock, 2018)

Before discussing the task, the students investigated the template individually. Prior to this excerpt, the students explored the dragging possibilities, e.g., dragging points *A* and *D*. Dragging point *D* was neither intended nor part of the task description. The students began to discuss, dragging point *A*:

1	Ivy	The area moves *P*. Okay, the area moves *P* up.
2	Aya	No, it is not the area. You mean the width.
3	Ivy	Yes, but the width changes the area
4	Aya	So does the height…
5	Ivy	Yes, yes.
6	Aya	But both factors move the area
7	Ivy	No, but, no… Try to see… try to see. If you move this one here, it changes both the area and the height, then it [point *P*.] moves askew. [Ivy drags point *D* back and forth, slowly. The figure is no longer a rectangle.].
8	Aya	Yes, yes, but Ivy… I know that both elements depend on the area. The area is the height times the width.
		[...]
9	Aya	What have you written?
10	Ivy	I have written that if you change the rectangle's height to 0, the intersection *P* ends at (0,0). I think that the intersection is determined by the rectangle's height and area. The area moves *P* up, up by the *y*-axis, and the height moves it by the *x*-axis. When you change the height, you also change the area and *P* therefore moves askew away from (0,0).

Line 10 shows that Ivy concludes on the relationship when dragging point *A* (not *D*), as point *P* moves "askew" and not vertically.

Analyses of the case

To ‘coordinate’, we analyse data in parallel, meaning that we analyse using (1) semiotic registers (Duval, 2017), and (2) the instrumental approach (Trouche, 2005). Then, we ‘coordinate’ the analyses (Prediger et al., 2008). We have regarded the students’ meanings and productions as collective, hence, considering them as a pair rather than individuals.

Analysis through semiotic representations: Ivy and Aya orally discuss the relationship between point *P* and the rectangle. First, they characterise what defines the area, that is the width (line 3) and the height (line 4). Secondly, Ivy drags point *D*, performing treatments on the quadrilateral, while the students do conversions between height and area of the rectangle to point *P* (line 7). Thirdly, they connect the height with *P*’s *x*-coordinate and the area with *P*’s *y*-coordinate, which is specified at the end (line 10). Finally, the students use the word “intersection” in their natural languages in line 10 in connection with point *P*. It is unclear how the students interpret/understand “intersection”, but it appears to stem from the description of point *P* in the algebra view. Possibly, they use it as a description of point *P* as they say “the intersection *P*”, meaning where the height of *AB* and the area intersect.

Analysis through instrumental genesis: Prior to the excerpt and at the beginning of the transcript (lines 1–6), both Ivy and Aya explore the GeoGebra template using their previous knowledge of rectangles. As they drag point *D*, point *P* does not follow the intended line (line 7). The possibility of dragging point *D* is a constraint of the task related to the process of instrumental genesis, particularly for the instrumentation process, when the students explore the tool. This is reflected in their actions (see line 7). As part of their instrumentalisation, they use GeoGebra for what they believe was the intention of the task and realise (see line 10) that they should drag point *A*, which is the intent of the task. The term “intersection” appears in the algebra view, and may consequently be regarded as a constraint of the tool: it leads the students to use a term that stems from the tool but is inappropriate in the given situation. This may even lead to an incorrect understanding of point *P*. To use dragging as an instrument for this particular situation, Ivy and Aya should be able to find the relation between the rectangle and point P while dragging. Hence, the instrument of dragging for Ivy and Aya is developed for answering the task and the instrument relates both dragging (i.e., the artefact) and schemes relate the content of linear functions and geometrical knowledge on rectangles with the specific actions of dragging.

Summary of analyses: ‘Coordinated’ analysis of Ivy and Aya working with the task

When Ivy and Aya explored the template using prior knowledge on rectangles (lines 1-6), it involved understanding and making conversions between and treatments within the involved registers (i.e., the linguistic, the graphical and the figurative) by relating and expressing what defines the area. Dragging point *D* created constant treatments of ‘rectangle’ *DABC* to be a quadrilateral. For the instrumentation, dragging point *D* (line 7) provided the impression of changing the relationship since point P then moved on a vertical line, and for the instrumentalisation process, dragging point *D* was not in line with the intentions of the task. It was necessary for the evolution of the instrumentation and instrumentalisation processes that the students understood the constraint of dragging point *D* indicating another relationship than the intended one. Ivy and Aya did not continue with the attempt to investigate the conversions between point *P* and the area of the rectangle by dragging point *D*. Thus, for dragging to be an instrument, the students develop schemes that relate treatments within the individual representations and conversions between the point *P* and the area of rectangle *DABC* as results of their dragging. For example, they expressed this relationship between point *P* and the rectangle by making conversions to the symbolic or linguistic registers (line 10).

The term “intersection” is a word adopted from its appearance together with point *P* in the algebra view. This is a constraint of using GeoGebra as a recourse for natural language along with the symbolic register. Ivy’s and Aya’s use of intersection is a result of their instrumentation process, which led to inappropriate language in the situation and, potentially, to an incorrect understanding of point *P*.

Refining the design principles based on the ‘coordinated’ analysis

Based on our analyses, we refine the design principles and discuss how the networked theoretical perspectives have influenced the development of refined design principles using networking practices.

For HDC 1, the students use natural language throughout the task. At the beginning of the students’ instrumentation, they express their interpretations of the representations and the properties of the rectangle. Through their communication, their interpretations develop. As they are expected to give a written answer, they express the discovered relationship using the natural language that appears in the algebra view. In this case, this is unfortunate. However, if the tool’s expression is in line with the task, it can support the students’ use of natural language.

For HDC 2, the students only investigate the rectangle at the beginning of the task sequence. In the transcript, Ivy and Aya also investigate the rectangle on their individual computers, when dragging and discussing that both the width and height of the rectangle influence the area (lines 2-6). Hence, the treatments in the rectangle are related to point *P* (lines 1, 7, 10). About the word “intersection”, it is also important to be aware of the vast information provided in the algebra view. Hiding the algebra view could cause less confusion for the students to only focus on the graphical view.

For HDC 3, Ivy and Aya explore the template by dragging both points *A* and *D* at the beginning of their instrumentation process. When they drag, they can test their conjectures, e.g., regarding what happens to the rectangle when dragging either point *A* or point *D*. Through dragging, the students explore and investigate the relationships between the representations.

Applying the analyses to van den Akker’s (1999) structure, the design principle may be defined as: If you want to design teaching with a digital tool for exercising and developing the representation competency in schools, then you are best advised to *gradually relate all four registers (linguistic, symbolic, figurative and graphic) in a task sequence when using digital tools*. Collaboration between students is essential for all three characteristics and procedures. This may be realised by:

1. including the linguistic register (both written and oral) throughout the task (i.e., when formulating the task, exploring the environment and representations, and writing answers), and being aware of how natural language appearing in the environment relates to the intentions of the task, as the linguistic register is often neglected in a digital tool, although it

is an important register expressing the relations between representations, formulating interpretations of different representations and for developing an artefact into an instrument.

2. breaking objects and tool windows into smaller units by introducing different representational registers and components of the artefact gradually (e.g., hiding the algebra view), because gradually introducing representations makes it possible to identify the specific characteristics of a representation and to help the students' processes of instrumental genesis by limiting their explorations to specific parts of the tool.
3. using sliders, dragging and tracing for explorations about different representations and the treatments and conversions caused by the dynamic features since these features make it possible to quickly move and translate between representations, and for developing conjectures about the representations' reciprocal relations.

Discussion of the potentials for networking of theories in DR

Reflecting on our work as networking of theories, the empirical case guided our work. Concerning the eight networking strategies (Prediger et al., 2008), and the compatibility of the theoretical perspectives, the two theoretical perspectives' backgrounds will be discussed. The background for the semiotics register approach may be defined as "Understanding mathematics is a cognitive process, which involves 'coordination of at least two registers'" (Duval, 2017, p. 110). The background for the instrumental approach we define as 'Using digital tools in mathematics is a cognitive process of developing suitable schemes for using the artefact'. Both theoretical perspectives are cognitive in a similar way as KOM. Hence, the perspectives are not too far apart and the networking process is identified as 'coordinating'. The theoretical perspectives have different objects, i.e., semiotic representations (Duval, 2017) and using digital tools for instrumental genesis (Trouche, 2005). Yet, the perspectives are possible to network as the process of instrumental genesis takes place through the representations present in the involved digital tool, and the students' understanding of conversion and treatments of the involved representations. The instrumental approach is informed by the semiotic register approach concerning how students understand and relate mathematical representations, related to mathematical understanding and activity (Duval, 2017). The semiotic register approach is enlightened by the use of digital tools and students' instrumental genesis (Trouche, 2005). The understanding of the students' use of the tool, while working with several mathematical representations, thus becomes deeper by coordinating the theoretical perspectives. The design principles reflect both theoretical perspectives. Yet, the design is only illustrated for one case.

In this paper, we have aimed to investigate how practices from networking of theories could strengthen the development of theory in DR. The implementation of networking practices in DR helps when having a situation involving two perspectives, which for our study is the use of digital tools and mathematical representations. The strategies of 'coordinating' and 'combining' fit the aims of DR with regard to understanding an empirical case and developing theories (Prediger et al., 2008; Prediger, 2019). Both in DR and networking of theories, the processes of analysis are determined by the objects of research. It is of course possible to conduct DR without the implementation of networking practices. Still, for a case such as ours, the relationships between the involved theories are not necessarily taken deeply into account related to each other. However, if aiming at using networking practices, such as 'coordinating' or 'combining', parallel analyses and investigation of the compatibility of the theoretical perspectives involved are important. Implementing networking practices into DR, thus appear to strengthen the development of theories and theoretical perspectives.

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Points of contact between the ATD and the TDS: questions raised by the implementation of study and research paths

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The Anthropological Theory of the Didactic (ATD) aims to capture the complexity of real inquiries by using a type of instructional proposal called study and research path (SRP). Experimental research shows that certain notions of the Theory of Didactic Situations (TDS) appear specially adapted to analyse implementations of SRPs and point at some didactic phenomena related to the conditions needed for their implementation and management. We report this experience by first identifying some contact points between the ATD and the TDS and then describing the borrowings done. The analysis of the commonalities and specificities of both approaches helps to question the framework of the ATD and to point at some needed theoretical and methodological developments.

Keywords: Anthropological Theory of the Didactic, Theory of Didactic Situations, study and research paths, didactic contract, didactic situation, adidacticity, networking.

Introduction: ATD, TDS and networking

In this paper, we are considering two of the main theoretical frameworks of what is known as the French tradition in didactics of mathematics (Artigue et al., 2019), namely the Theory of Didactic Situations (TDS, Brousseau, 1997) and the Anthropological Theory of the Didactic (ATD, Chevallard, 2015). We present a networking activity that emerged when using notions from the TDS within ATD analyses and continues by considering their differences and commonalities about the kind of tools they propose to problematise, model and develop teaching and learning processes.

In 2007, in an invited lecture at the First International Conference of the Anthropological Theory of the Didactic, Guy Brousseau established some connections between key notions of the TDS and what could be considered the analogous ones in the ATD. He analysed the differences in how both theories model human activities in their collective and dynamic dimensions, including doing mathematics and teaching and learning mathematics. His comparison involved the notions of situation, milieu, situated knowledges (*connaissances*) and didactic processes, in contrast with some elements of the praxeologies (types of tasks and techniques) and the moments of the study. Brousseau considered these notions "more as points of contact, points of articulation that allow moving from one [theory] to the other, rather than as borders" (Brousseau, 2007, p. 3, our translation). He concluded:

> The ATD and the TDS complement each other. But in my opinion, it would be absurd to juxtapose them simply. In many issues, they are intertwined; they must be considered together. What

problems do they pose for each other? What answers do they offer each other? What advances do they promise together? (Brousseau, 2007, p. 22, our translation)

The relationships between the TDS and the ATD have been the object of previous research in the project of networking of theories initiated by Angelika Bikner-Ahsbahs in 2005 at CERME4 (Bikner-Ahsbahs & Prediger, 2014). For instance, Artigue et al. (2010) show how a teaching episode can be described from both approaches and relates the TDS analysis about the limitations of an *adidactic milieu* with the ATD one in terms of the functioning of the *media-milieu dialectic*. Ghemansi and Lecorre (2019) propose combining some TDS and ATD methods in design-based research related to the university teaching of calculus. These works and our experience as researchers are in tune with Brousseau's proposal of considering notions as contact points to question one theory from the other's perspective. This paper presents a bottom-up networking strategy that arose during the analysis of some empirical work carried out within the ATD, where the resource to some TDS notions appeared in a rather spontaneous way. This research practice nourished a reflection about the contact points between both theories, together with some specificities and differences.

Contact points between the TDS and the ATD

Since 2005, our ATD-research team has been implementing and analysing a new type of instructional proposal named *study and research paths* (SRP) based on the continued inquiry of problematic questions. In a way, SRPs include aspects of project-based and inquiry-based learning (IBL). However, they also provide new perspectives and methodologies currently not elaborated in the IBL literature (Markulin et al., 2021). Barquero et al. (2020) summarised that the implementation of SRPs faces several challenges related to the coexistence of two pedagogical paradigms in the school institutions in a very explicit way. Historically, the one that prevails is the *paradigm of visiting works* (Chevallard, 2015) and is based on the proposal of a set of bodies of knowledge – the *works* – for students to study under the teacher's guidance. The new paradigm that is struggling to emerge is the *paradigm of questioning the world*, in which the bodies of knowledge are replaced by open questions to address (or inquire). Between both extremes, some compromise situations can exist. In any case, our research shows how the conditions needed to implement a study process framed in the paradigm of questioning the world are diverse and fragile. We then identify many constraints coming from the prevailing paradigm of visiting works. In our analysis of such conditions and constraints, four points of contact between the TDS and the ATD appeared, as we said, very naturally. We first present these four points before examining each one in more detail in the following two sections.

First point of contact: new epistemological demands. One of the conditions for the normalised running of SPRs is the availability of tools to describe and manage the different types of knowledge mobilised during an inquiry. The existence of new study processes based on the paradigm of questioning the world highlights the need to deal with knowledge-related aspects, such as uncertainty, temporariness, and validation, that often remain implicit in visiting-works instructional proposals, where knowledge is conceived as static and crystallised. Brousseau was one of the first to point out the necessity for researchers in didactics to question and rebuild school mathematics organisations to avoid assuming the prevailing school epistemology as the only and appropriate one. The TDS proposal is to model mathematical knowledge in terms of *fundamental situations*, defined as games against a *milieu*, that is, an environment without any didactic intention towards the student and providing feedback to the actions received (Brousseau, 1997). The ATD, which is part of the TDS project of building didactics of mathematics as a science, joins its epistemological background about the need to question mathematical practices. It, however, differs in the models chosen and introduces a clearer institutional perspective by pointing at the existence of different conceptions and knowledge constructions connected through didactic transposition processes (Chevallard & Bosch, 2020).

Second point of contact: dealing with the curriculum constraint. Even if competency-based curriculums are on the agenda, to a large extent, curricular content is still described in terms of labelled pieces of preestablished knowledge. In this conception, inquiry-based proposals are often seen as means or methodologies to acquire the specific labelled content. In contrast to this conception, SRPs do not oppose inquiry and transmission but subordinate the learning of contents to the advance of the inquiry, and the elaboration of a final answer to the question initially addressed. This change in the knowledge conception is particularly challenging in a setting where the two paradigms coexist and meets what we call the *curriculum constraint* (Barquero et al., 2020). The TDS' proposal of modelling knowledge as the answers to problematic *situations* appears as a first move towards the paradigm of questioning the world while remaining in the frontier of the paradigm of visiting works because situations are always supposed to model a given piece of knowledge.

Third point of contact: the evolution of the didactic contract. A third and more evident contact relies on the changes produced by SRPs in the traditional sharing of responsibilities among the teachers and the students, which we interpret as an evolution of the *didactic contract* (Brousseau, 1997) that prevails in the paradigm of visiting works. Implementing an SRP requires students to assume different roles in the inquiry process, such as seeking available answers, validating or rejecting them, raising new questions, deciding which ones to follow or discard, planning the work to do, etc. Teachers also experience essential changes in their tasks: they are no longer the "knowledge holders" nor the sole person bringing new knowledge into the classroom. The coexistence of paradigms in the same school institution – and even in the same course – makes this contract negotiation complicated.

Fourth point of contact: didactic and adidactic situations. The students' assumption of new responsibilities in the inquiry process depends on how they engage in the SRP's generating question and the importance they attribute to developing an answer. In the paradigm of questioning the world, the study of works is essential as far as it helps to progress in elaborating a response to the initial question. The SRP generating question needs to remain at stake during the entire process for this to happen. In the analysis of the implemented SRPs, to identify the episodes where students seemed to be engaged in the inquiry for the sake of the considered generating question, we find it helpful to use the TDS distinction between *didactic* and *adidactic* situations.

The four contact points correspond to assumptions and modelling strategies that are common to both approaches. They need to be complemented with some comments about their differences and specificities. We will now approach them in two blocks by first addressing the question of the modelling of mathematical knowledge (points 1 and 2) and then the role played by the notions of didactic contract and adidactic situations in the ATD analysis (points 3 and 4).

The ATD inherits the epistemological programme initiated by the TDS

The first and second contact points correspond to a solid and fundamental connection between the TDS and the ATD. As explained by Artigue and Trouche (2021), Brousseau's project aimed "to find a genuine experimental epistemology of mathematics based on the construction of situations able to make mathematical knowledge emerge from autonomous students' interactions with their environment in the social context of classrooms" (p.3). Moreover, the TDS experimental epistemology relies on the observation and analysis of classroom implementations. Therefore, the central object here is not the cognitive subject, i.e., the pupil or student, but the situations organising the relationship of such subjects with mathematical knowledge and its raisons d'être. The ATD participates in this *epistemological research programme* in mathematics education (Gascón, 2003), which locates the description, modelling and reconstruction of the knowledge to be taught at the core of didactics research. An important difference between the TDS and the ATD is in the type of reference epistemological models (Bosch & Gascón, 2006) they elaborate to model mathematics, as knowledge and as a human activity. The TDS models are formulated as *fundamental situations*, defined as games against a *milieu*, with different dialectics or phases (Brousseau, 1997). Brousseau clarifies the scope he gives to this concept:

> One of the approaches of didactics of mathematics consists in modelling not only the knowledge to teach or learn, but also the conditions in which it manifests itself. Situations are minimal models that "explain" how such knowledge intervenes in the particular relationships a subject establishes with a milieu to exert a determined influence on it. (Brousseau, 2000, p. 4)

When we look at the types of epistemological models proposed by TDS, we can distinguish between a general model in terms of fundamental situations (or games against a milieu) and specific models in terms of sequences of situations. Without going into detail here, it is worth noting that in most of the work developed in TDS, the division of the modelled knowledge does not follow the classic school cartography of knowledge. It corresponds to rather vast domains of the mathematics to be taught, as shown by the work on numeration, measurement, and decimal numbers, which go far beyond what their name indicates. These models also make it possible to identify elements that do not exist in the dominant school epistemological models, a well-known example being that of "enumeration" (Briand, 1993; Rivière, 2017). The international diffusion of the TDS has been often limited by the confusion between epistemological and didactic models. Fundamental situations indeed play a double role in this respect because they define knowledge not by what it is but by the processes that allow it to be constructed. This fusion between didactic and epistemological proposals – or between didactic and mathematical situations – is at the heart of the TDS (Brousseau, 2007).

As said before, the ATD emerged within the scientific project of the TDS. According to Artigue and Trouche (2021, p.4): "ATD then broadened the perspective by placing at the centre of the analysis the diverse institutions, institutional positions, and institutional relationships with the knowledge at stake, and how they condition and constrain what is taught and, therefore, what students ultimately have the possibility to learn or not. To this end, new concepts were introduced, notably the concept of praxeology and the scale of didactic codeterminacy." The notion of *praxeology* is introduced as the basic unit to analyse human action in general and mathematical knowledge in particular. The dissemination of praxeologies takes place through what we call didactic systems. A didactic system is a triplet $S(X; Y; \mathscr{P})$ formed by a person or a group of persons Y (the teachers) who do something to help another group of persons X (the students) to learn a given body of knowledge or praxeology $\mathscr{P}$.

In the paradigm of questioning the world, didactic systems are not formed around a given praxeology to be studied, but rather around a question Q, to which X, with the help of Y, has to provide an answer $A^\heartsuit$. One tool used to analyse inquiry processes is the Herbartian schema: $[S(X; Y; Q) \rightarrow M] \hookrightarrow A^\heartsuit$. The schema considers that the study of Q generates an *inquiry* process involving a *didactic milieu M* including questions Q_i derived from the initial one, "ready-made" answers $A_j^\lozenge$ one can find in the literature or by consulting works and experts (the *media*), together with empirical data D_k and other material and knowledge works W_l: $[S(X; Y; Q) \rightarrow \{Q_i; A_j^\lozenge; D_k; W_l\}] \hookrightarrow A^\heartsuit$. Obviously, an alteration in the available *media* or aspects related to the *milieu* can lead to completely different constructions of $A^\heartsuit$. The schema is helpful to design and carry out *a priori analyses* about potential paths to follow when approaching Q, and also *in vivo* and *a posteriori* analyses about the path actually taken and the means used to do so. The Herbartian schema identifies some critical elements of the inquiry, namely $\{Q_i; A_j^\lozenge; D_k; W_l\}$, which can be detailed in terms of praxeologies. As we will see below, the inquiry dynamics are described in terms of dialectics, like those of *questions and answers* (Bosch & Winslow, 2014) and *media and milieus* (Kidron et al., 2014).

A first important difference between the TDS and the ATD proposals to model mathematical knowledge and activities is terminological (and, consequently, also conceptual). The use of situations enables one to model knowledge in an implicit way, not by pointing at its elements (for what a specific language is required) but by describing the situations it allows to solve. In the ATD, the notions of praxeology and Herbartian schema do propose descriptions of the knowledge elements and therefore require a specific language, which cannot be neutral. It is sometimes provided by the scholarly knowledge complemented with ad hoc elaborations. A second difference is the role given to the *media* in both approaches, which is often reduced to the teacher in the TDS and accessed through *didactic situations* – that is, situations where the teacher intervenes in the students' knowledge construction.

Developing TDS notions in the ATD analysis

The generating question(s) of an SRP and the notion of situation

In the implemented SRPs (Barquero et al., 2020), the generating question Q always plays a critical role. It should lead the inquiry and be always prominent, appearing as an end in itself – instead of an excuse to visit some preestablished works. And it should also conduct to a final answer $A^\heartsuit$ that is of a particular value for somebody, not only for X and Y. The experiences carried out in university engineering degrees showed a visible change in the students' involvement in an SRP depending on the conditions under which Q is formulated and $A^\heartsuit$ has to be received: an imaginary client, a fictitious client or a team of students taking part of a car race from the same university. We find here one of the main raison d'être of the TDS notion of *situation*. It includes the question raised but also the conditions under which it arises – the initial *milieu* – and the requirements for its reception and validation – the rules of the game and the winning strategy. In a way, when we talk about a question Q in the ATD, we are also implicitly including the institution and corresponding contract that ensures the reception of Q as such a question, together with the type of acceptable answers A_i and $A^\heartsuit$. Also, the importance of Q depends on the destiny of the potential final answer $A^{♥}$. Therefore, in the design of an SRP, it is essential to require not only some general interest for Q, but also a kind of external contract with an instance Z about Q, the type of expected answer $A^{♥}$ and how it will be valued or validated.

The notion of milieu

The concept of *milieu* proposed by the TDS corresponds to the environment with which students interact and provides them with objective feedback. It can comprise material objects, informative texts, digital tools, and other collaborating or competing students (Artigue & Trouche, 2021). In an SRP, the *milieu* is built along the inquiry process and can be analysed through the elements previously described in the Herbartian schema: $\{Q_i; A_j^{\lozenge}; D_k; W_l\}$. Its dynamics is then explained through the development of some dialectics. In particular, a central one is the *media-milieus* dialectic. This dialectic includes access to already available answers (or pieces of works found in the media) and their integration into the inquirers' milieu. During this integration, answers need to be validated to become useful for the inquiry progress and appear as potential works to be used by others.

Didactic contract and "adidacticity"

The management of an inquiry process in a "question-driven way" requires a new sharing of responsibilities between teachers and students and can be interpreted as the evolution of the didactic contract (Brousseau, 1997). For instance, students must assume new responsibilities, like planning the work to do, proposing the questions to address and those to discard, deciding what media to consult, validating the answers they find or propose, etc. In a way, in the didactic contract of the paradigm of visiting works, teachers are expected to boost the media-milieu dialectics students will then follow, as if the "question-driven" inquiry was only driven by the teacher, not by the students.

Analysing the changes in the didactic contract led us to use the distinction between *didactic* and *adidactic situations*, two key notions of the TDS. However, in our use of these notions, we do not include its functioning as *fundamental situations*, that is, as reconstructions and epistemological models of the mathematical knowledge to be taught. We only include in this notion the assumption that any question or problem is never raised in the vacuum but always appears to X (and Y) under specific circumstances or conditions and with some available resources (and other unavailable ones), a certain *milieu*. In this context, we use the term *adidacticity* to refer to the very moments when students make decisions primarily considering the generating question Q and the targeted final answer $A^{♥}$, without prioritising the instructional process that supposedly envelops the inquiry, that is, the didactic situation in which the SRP takes place. Looking for these adidactic moments can help measure to what extent an SRP is overcoming the constraints of the paradigm of visiting works.

Open questions in the dialogue between TDS and ATD

This paper presents several contact points between the TDS and the ATD, showing many ways, in which the latter is an heir of the former. But there are also aspects in which they seem to differ sufficiently to make it unavoidable to try to refine the formulation of the ingredients that mark the differences. A work that is still in progress.

The paradigm of questioning the world is the paradigm par excellence in research. Accordingly, the choice of theories when addressing research questions should always be subordinated to the question's sake. In other words, theories are good, helpful, valid, or interesting as far as they help researchers elaborate good, helpful, valid, or interesting answers to the questions addressed. In this context, the bottom-up strategy of networking theories is not a big deal. What specific relevance can we find in the networking experience here presented? First, it illustrates the do-it-yourself character of didactics research, and the freedom researchers should adopt in problematising, experiencing and analysing teaching and learning processes. Second, it provides a new perspective of each considered approach when viewed from the other side, establishing a *dialogue between theories* as proposed by Bosch et al. (2017). In this sense, our paper responds to Brousseau's analysis of ATD notions from the TDS perspective, by proposing some analysis of TDS notions from an ATD perspective. Finally, pointing at their complementary aspects is a way to question the scope of the ATD methodology by pointing at some needs in the proposed analyses, and contributes to its evolution.

We believe that such a bottom-up networking strategy is possible because of the robust commonalities shared by the TDS and the ATD. The most important one is the questioning of the mathematical activities to be taught and the need to elaborate scientific models to reproduce these activities without assuming the prevailing visions of school and scholarly institutions. And we should also add the inclusion of new instructional practices that intent to approach the teaching of mathematics to a questioning the world perspective. These are the critical points of contact that enable researchers to easily switch from one approach to the other without distorting any of them.

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Rhythmic instrumental orchestration: Joining two theoretical perspectives in designing an online summer school

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Due to the pandemic situation in 2020 the ERME summer school (YESS10) was designed for an online format using a conference system. Our design choices were based on previous experiences with YESS and the use of the research pentagon as a tool to think about research. This paper elaborates theoretically and empirically the specification of the concept of instrumental orchestration of the pentagon use through its rhythmic implementation into the summer school. Research results indicate that this specification had two main effects: The students described their instrumentation of the pentagon as a structuring tool in various ways. The most relevant pentagon use for the students' experience of growing expertise was listening to and observing how the others used the pentagon.

Keywords: Design-based research, instrumental genesis, rhythm, online learning, summer school

Introduction and background

Due to the pandemic in spring 2020 the Young ERME Summer School (YESS) of the European Society for Research in Mathematics Education (ERME) switched to using a synchronic online conference system. Two out of seven thematic working groups (TWG7, TWG5) run in a virtual environment. As neither clear conceptions nor research were available for the design of the school with respect to distance learning of PhD-students via a conference system, the two experts and two brokers (previous PhD-students of YESS) decided to conduct a design-based research study involving the conference system Adobe Connect. Planning jointly these TWGs led the brokers to decide to act as assistances and to provide several Adobe tools to be shared in the TWGs. The leading experts made common design choices based on their experiences in the previous YESS-TWGs. As each student had to submit a short description of their research project, all the students were asked to read all the papers before the school and prepare a presentation of their project to be discussed during the school. Prior to the school, pairs of critical friends were built within both TWGs to make them engage deeper into the friend's paper and present their critical feedback in an explicit presentation. The main design choice concerned the research pentagon (Bikner-Ahsbahs, 2019) as a tool to think and reflect about research in terms of research aims, questions, objects, methods, and situations involving the situation of the project in the field and creating a specific situation for the research in the project (Figure 1, left). The design of how to use the pentagon took into account previous experiences with the pentagon as well as the fact that the students coming from different time zones, and simultaneously being involved in their families during the school, required a clear timetable so that they would be able to manage their pandemic situation and attend YESS10. This resulted in a sequence of five situations of using the pentagon (Figure 1, middle). This sequence was repeated in all the ten sessions in the same order in both TWGs, however with some variations (Figure 1, middle & right).

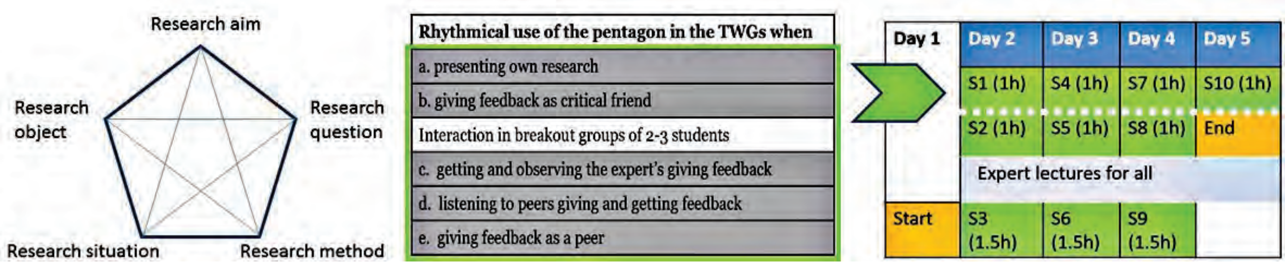

Figure 1: The research pentagon (left), sequence of pentagon use in a session (middle) and the sequence of sessions S1-S10 (right)

Based on these design choices, we ask: How did the students experience the rhythmic orchestration of the pentagon use and the pentagon use itself as contributing to their learning about research?

Theoretical framework

Instrumental approach and instrumental orchestration

Considering the pentagon as a tool that participants need to appropriate in order to use it efficiently, *instrumental approach* (Vérillon & Rabardel, 1995; Rabardel, 2002) is one of two theoretical approaches in our research framework. The key idea is the distinction between an *artefact* – material or symbolic object available to a subject, and an *instrument* – a merging construct that consists of the artefact and the related mental schemes coming from the use of the artefact by the subject in a given context. The process of transformation of an artefact into an instrument, called *instrumental genesis*, consists of two interrelated processes: *instrumentation* leading to the constitution and the evolution of schemes of using the artefact in the subject, and *instrumentalisation* during which the subject adapts and personalizes the artefact according to her knowledge and beliefs. The development of schemes of use manifests itself in the subject's invariant organization of behavior in a given class of situations (Vergnaud, 1990). Trouche (2020) suggests seeing instrumentation both as "an action (by which someone acquires an instrument)" and as "the influence of this action on a subject's activity and knowledge" (p. 307). Thus, using an artefact yields both pragmatic and epistemic outcomes.

In the context of students' learning, students' instrumental geneses need to be accompanied by a teacher. Trouche (2004) introduced the notion of *instrumental orchestration* to describe how a teacher can plan and organize students' interactions with available artefacts in order to enhance individual and collective instrumental geneses. Referring to Trouche (2004), Drijvers et al. (2010, p. 214) define an instrumental orchestration as "the teacher's intentional and systematic organization and use of the various artefacts available in a […] learning environment in a given […] task situation, in order to guide students' instrumental geneses". It consists of three elements:

1. A *didactical configuration*, which is "a configuration of the teaching setting and the artefacts involved in it" (Drijvers et al., 2010, p. 215);

2. An *exploitation mode*, which is “the way the teacher decides to exploit a didactical configuration for the benefit of his or her didactical intentions” and includes in particular the teacher’s decisions “on the possible roles of the artefacts to be played” (p. 215);
3. A *didactical performance*, which involves “the ad hoc decisions taken while teaching on how to actually perform in the chosen didactic configuration and exploitation mode” (p. 215).

Design approach to orchestrate the pentagon use

The pentagon was implemented in the design of the sessions as a tool to structure and reflect on research (Bikner-Ahsbahs, 2019) individually and in interaction with others. Pairs of critical friends provided mutually more in-depth feedback in their presentation in addition to the spontaneous feedback coming from all the other peers. Whereas TWG7 introduced an article about the research pentagon before the school to be used for the preparation of the presentations, the expert of TWG5 introduced this tool in the first session. Besides this distinction, both experts used the tool permanently during the school as a reference object in their feedback and the discussion. We organized each of the 10 sessions in the TWGs by five situations of pentagon use (Figure 1, left) so that, in the course of these sessions, the students’ roles as presenter, critical friend, observer and listener, feedback provider and receiver of peer and expert feedback varied regularly in time and space of the virtual environment. This organization aimed at enhancing participants’ instrumental geneses to improve their research expertise.

The change from a face-to-face summer school to an online format required to theorize the meeting space. According to Lefebvre (1991), any space is established with respect to three dimensions: (1) A space is represented materially (also technologically), (2) it is representational in that it represents some idea (here: switching YESS to an online school), and (3) it is (re-)produced as a social space by the people involved. All the three dimension apply to YESS10. The digital conference system Adobe Connect used as a technological space linked to the individual material home computer spaces, constituted a representation of the virtual space. It provided tools for sharing and distributing information among the students, hence, for creating the social space of acting and interacting of the participants supported by these tools, e.g. the breakout rooms, chat, and note taking possibilities were included into the didactic configuration. This notion of space developed structured in time with a frequency of three sessions per day and breaks of different lengths (session–short break–session–lecture break–session–night break) repeated day after day.

In the summer school, the collective nature of teaching and learning is essential and this depends on the possibility of creating joint attention on the research presented. In face-to-face situations, this is established by building a reference space through deictic expressions, shown e.g., by linguistic means, through a projector, by gestures, gaze, body posture, moving (Balantani & Lázaro, 2021; Stukenbrock, 2015). However, most of these deictic expressions could not be shared in our virtual space. So, we designed the pentagon use in a way that it could facilitate creating a reference space because all the students could refer to this diagram sharing its structure, such as the vertices and their relations, with their peers, and simultaneously talking about their own research.

An implicit key idea in this design was rhythm as a way to structure time. Rhythm emerged as a critical design element when analyzing the data.

Rhythm as a design element

Lefebvre (2004) stressed that space, time, and energy are interrelated by rhythm, the latter being bodily grounded and thus always active. Rhythm as a structure of time is created in the course of sessions by an "ordered variation of changes" (Dewey, 1934, p. 160). Figure 1 (right) shows the *linear* rhythm as a repetition of sessions (three per day) with pentagon use and breaks. Within each session, the order of five situations of pentagon use is *cyclically* repeated involving regular and spontaneous variations by providing all students the opportunity to act in their way in all the roles and addressing various research topics and foci. This rhythmic characteristic of the *didactical configuration* was enriched by a rhythmic *exploitation mode* in which the experts regularly referred to the pentagon explaining while pointing to vertices and connections in varying research projects, and hence, inviting the students to use the pentagon as a reference space, too. Thus, the group could create joint attention on the research presented and discussed. As the experts have used the pentagon in previous summer schools, they related their feedback (*didactical performance*) to students' actual needs, highlighting blind spots, missing aspects, or a change of view, in using the pentagon as an organizing tool for research. So there is constructive interaction of various rhythms, so-called "eurhythmia" (Lefebvre, 2004, p. 16). Lefebvre emphasizes that rhythm allows measuring changes: "No rhythm without repetition in time and in space, without *reprises*, without returns, in short without measure [*measure*]" (p. 6). As the same situations of pentagon use are repeated the students had the opportunity to experience and compare these uses within eurhythmia, thus measure changes of own and others' pentagon uses related to research and hence, measure their own development of expertise on research.

Methodology

We use *rhythmanalysis* (Lefebvre, 2004) to answer the research question focusing on how the students experienced the rhythmic instrumental orchestration, which instrumentation processes they reflect and how the instrumental geneses of the pentagon may have contributed to their learning about research. As we could not observe these processes in time, we shaped this research similar to action research. We drew on our experience of conducting the TWGs, our design choices and note taking. An external expert interviewed 11 volunteers of the 20 participants about the efforts they invested into the school, their experience of the pentagon use and of our design to develop their learning about research. These interviews were video recorded and transcribed verbatim (names were replaced with pseudonyms). Thus, we did not observe the rhythmic processes directly, but rather approached them through the students' individual experience of the (eu)rhythmic orchestration in their interviews. This experience begins with a first contact with the pentagon and ends with reflections on own research. Related to this frame, we conduct our analyses in three steps considering the five different roles (Figure 1, middle) in which each student acted and interacted. First, we identified quotes in the interviews that refer to these roles, and described how each student experienced his/her individual instrumental genesis of the pentagon distinguishing between instances of instrumentation and instrumentalisation within the orchestration (Figure 2, vertical analyses expressed as vertical arrows).

For each role, we secondly compared these descriptions across all the interviews to situate the individual experience into the collective (Figure 2, horizontal analyses expressed by horizontal arrows). In the third step, we identified the students' individual reflections addressing their individual experience on learning about research in the interviews, interpreted them related to their instrumental genesis and through the lens of rhythmanalysis, and compared these across all the students.

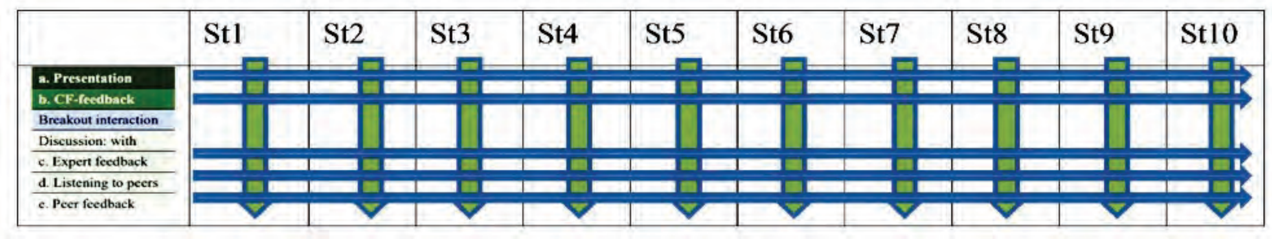

Figure 2: Rhythmanalysis (Stn: Student No. n)

Results

Based on our framework, we extracted quotes from the interview data, which express the students' experience in their reflections immediately after the summer school. Therefore, these results indicate what the students declare to have taken from the summer school rather than the learning processes themselves. The experiences of expertise are regarded as relevant outcomes for the PhD students who after the summer school may then continue their research with a refreshed expertise.

Step 1: The pentagon becoming an individual instrument

From the *case of Emma* we learned that some students' instrumental geneses started before YESS as Emma had read the paper and used the pentagon before the school already. The rhythmic organization of the situations in which she used the pentagon enhanced it. In the interview, she expresses a variety of instrumentation processes developed in different situations. She developed an *instrument for analyzing own research and checking its coherence* ("Also, just to see that you are making progress or making your project more coherent because I think it's very useful to talk about; if there's some parts of your research project that is not coherent.") and for *analyzing other's research projects* **("**I used the pentagon when I read the papers, for instance when I read through my critical friend's paper and whether I found some indications of the research aim and the research object, research questions, the method and…"). She also developed instruments related to feedback, for example for *structuring feedback for a critical friend* ("when I did the presentation for my critical friend **…** I could give feedback on that so what I thought was making a lot of sense") and for *structuring the feedback presentation* ("it can also be quite hard to understand where they [the peers] are exactly in the process and what they would need the feedback on to make them come further because. I think the research pentagon helped in that matter because I could use it to structure my presentation of my feedback"). We identified similar instrumentation processes in other interviews, too.

The *case of Nordy* made us aware that students in TWG5 came to know the pentagon use in the first session when the expert introduced it for the first time. As Nordy was the first presenter, he did not have time ("I need some more time to read the whole text and to incorporate it, to think about this research pentagon") for appropriating the pentagon although he perceived its potential ("I really think that it could be a great thing to organize your work [with the pentagon]"). Compared with the case of Emma, this shows the *lack of and need for instrumentation.* However, his instrumental genesis was about to start. Other students from TWG5 showed similar experiences.

Nordy's *beginning instrumentation process* was based on the expert's didactical performance, when listening to the expert's systematically and repeatedly referring to the pentagon. Nordy says: "as long as our research topic group was going on, then her [the expert's] feedback was more and more referred to it". He relates his individual to the collective instrumental genesis, when stressing "we should do our feedback depending on the research pentagon to our critical friend, I try to do that, I think we handled it quite good".

Through the *case of Kira* we learned that the first parts the students grasp are *vertices* of the pentagon as an instrumentalisation in the sense that students select the artefact features they are going to use. The links between the vertices are more complex to make sense of and need perhaps a specific orchestration (e.g. a 'technical demo' by the expert). For example, Kira used the pentagon as a feedback-structuring instrument identifying missing vertices in a peer's paper ("I could more clearly identify what was missing for me and what was also good"). Then she points to the need to fill the links between two vertices, the research question, and the methods ("filling the link to the research question would mean to ask myself, or for him to ask himself the question 'How can I measure this?' What method can I use? …"). Kira also expresses another instance of instrumentalisation in the idea to take the pentagon home for her peers: "I will take the research pentagon into a small group discussion at the university." Thus, within one week, the instrumental genesis entailed instrumentation as well as instances of instrumentalisation for Kira as well as other students.

Our vertical analyses show that the individual processes of instrumental genesis led to various instruments where the vertices are the primary foci while the links between them are more difficult to understand. Instances of instrumentalisation appeared already as the students decided which vertices they take up. Thus, the pentagon provides instrumentation and instrumentalisation possibilities from the beginning.

Step 2: Rhythm and listening as essential parts for contributing to learning

Kira connected the individual and the common use of the pentagon when she pointed to it as a diagram expressing a common reference space and resource in the didactical configuration of the virtual space ("what appeared were the edges and also a definition of what can be found, what can be identified as a research aim … the research pentagon in the middle on the whiteboard and then it was easy to just point somewhere"). The configuration of the virtual space where the pentagon was used as a diagram in screen sharing allowed sharing different research projects in the group ("This [the vertices] is very very different concerning what kind of research your project is about."). The repeated use of the pentagon emerged as a rhythmic exploitation mode. It helped the students to develop their expertise by listening to and observing others as well as practicing themselves to use the pentagon repeatedly. For example, Peter points to listening to and observing its repeated use by others ("I had the ability to see the pentagon in *action for ten times*.**"**). Dan highlights practicing its repeated use themselves **(**"doing it like *a few times to other colleagues,* each one in a different subject area, and *hearing the other teammates,* how they raise questions, was really really valuable. I mean, I learnt from it a lot."). Therefore, the rhythmic configuration and exploitation mode seem to have the potential to support developing expertise on research. Mirka is a bit more precise as she expressed *how* the repeated use of the pentagon supports growing in expertise. It can happen during one session in the *rhythm of* *different situations* ("I think there were some students who couldn't find… who couldn't use it in a specific way, but during the discussions of their feedback, they realized that they have to be more specific. […]"), as well as through the repetition of the *same situational use* in the course of the school *during the cyclic rhythm* ("once you see how other students used it, I think it's a very nice way to use it [the pentagon] your own way, personally, everyone used it; they didn't use it the same way.")

Step 3: Summarizing students' reflections on learning about research

The students highlighted the relevant role of rhythm, linear as well as cyclic, for the different uses of the pentagon and their learning about research. The most important use, as shown in our *horizontal analysis*, is observing and listening to how others use the pentagon, hence, they benefit from the rhythm in the didactical performance of the experts as well as from observing and listening to the processes of instrumental genesis of peers to learn how to improve their own research. Lefebvre elaborates on rhythm related to measure change with respect to a reference and hence, enables to explain the phenomenon of experience of change in learning. For example, a student may identify that the research object is still a blind spot in the own research project. Through repeatedly listening to others talking about their research objects, he or she can compare and thus measure the change of view on the own object and the experience of learning on own research. For example, Surgeryfish reflects: "what I see from others and fit into my work. … A kind of learning, becoming aware."

Discussion on the theoretical specification of instrumental orchestration

The YESS10-study is deeply rooted in the common experience of an ontological change in the pandemic situation. This change raised the necessity to enable the participants of YESS10 to align their home situations with the summer school in space and time via a conference system. As this study shows, we have achieved this by a rhythmic synchronizing (cf. Akkerman et al., 2021) of the students' instrumental geneses of the research pentagon in our virtual space and time with the goal to advance the students' projects and by that improve their expertise on research. As a result, a *rhythmic instrumental orchestration* emerged as a specification of instrumental orchestration, theorized with the help of rhythmanalysis of the implemented rhythmic structure. This rhythmic instrumental orchestration involves a didactical configuration that provides a rhythmic organization of the various situations and kinds of tool use. Within this configuration, a rhythmic exploitation mode emerged, which involves the expert as well as the students allowing them to use the tool in various ways as well as listening to and observing how their peers use the tool. Within this mode, the expert synchronizes her didactical performance on the tool with the students' needs. In sum, the design choices in the rhythmic orchestration led to eurhythmia, which allowed the students to develop their expertise in a communal way by learning from their peers, particularly through listening.

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Issues with using Activity Theory to understand how master's students view their research skills as contributing to their future teaching

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Using Engeström's third generation of activity theory, we explore three master's students' views of their completed research projects on children's argumentation in number stories as potentially contributing to their forthcoming teaching of Grade 1 students. Activity theory was chosen because it provided opportunities to consider how two different activity systems, research as part of teacher education and mathematics teaching in Grade 1, might overlap around the shared artefact of mathematical argumentation through number stories. The three interviews are analysed using Engeström's description of four levels of contradictions identified in a matrix of principles and questions. The analysis raised some issues with the use of Activity Theory to understand the master's students' learning from the contradictions between the two activity systems they were in-between.

Keywords: Master's students, research skills, number stories, third generation of activity theory.

Tensions from looking back, looking forward

From 2017, all preservice teachers for the compulsory years of school (Grades 1-10) in Norway are required to complete a five-year master's degree, but this has raised some concerns about the value to teachers of learning to do research (Smith, 2021; Aam et al., 2017). As Rørnes (2017) stated, "a research-based education must … be built on R&D work related to real-world issues connected to being a teacher" (p. 8). Yet, there has been little research on how teacher educators could improve their practices by understanding master's students' (MSs) learning of research skills. Learning about research could contribute to expansive learning activity. For Engeström (2001), "expansive learning activity produces culturally new patterns of activity. Expansive learning at work produces new forms of work activity" (p. 139), but this sort of learning may not be able to be predicted beforehand. Non-explicit learning is common, in that "people and organizations are all the time learning something that is not stable, not even defined or understood ahead of time" (Engeström, 2001, p. 137). In this paper, we explore Engeström's (2001) third generation activity theory (3GAT) to determine its suitability for identifying the kinds of learning activity, that could occur from MSs undertaking research. We chose 3GAT because we anticipated that contradictions would appear in the interviews when the MSs discussed the two activity systems of teacher education and school mathematics teaching in relationship to their research projects.

In research, which focused on contradictions in mathematics education, Engeström's (2001) 3GAT has been used to provide insights into when and how learning can occur, when different activity systems are juxtaposed. For example, Engeström's (2014) contradictions has been used to identify the difficulties that students had transitioning from school to university mathematics (Anastasakis et al., 2020). Solomon et al. (2014) identified the contradictions raised by university students and mathematics lecturers in how they viewed the teaching of university mathematics as a way of identifying expansive learning. Whereas in the Solomon et al. (2014) paper, contradictions were between participants in two overlapping activity systems, Anastasakis et al. (2020) focused on survey responses where students looked backwards at both their school education and their university education. Psycharis and Potari (2017) investigated the contradictions that teachers identified as they worked to develop modelling tasks that used workplace contexts. Although the teachers involved were participating in master's courses, their reflections on the role of research in their teaching practices were not in focus.

As part of a wider research project, we wanted to investigate the contradictions identified by three MSs when reflecting on what they had learnt from doing research both before they began as Grade 1 teachers and after a year of doing their teaching. It was important to understand not just the benefits the MSs saw from doing research on students' mathematical argumentation, but also the difficulties that might hinder them when they were the teacher. Part of our aim for the project was to see what we could learn as teacher educators to improve our practices. In this paper, we explore 3GAT (Engeström, 2001) as a theoretical framework for analysing interviews with the MSs before they began teaching and after they submitted their master's theses. We recognised that we were using activity theory differently to earlier mathematics education research, as we were investigating the contradictions raised, when the MSs looked forward and looked backward regarding doing research about teaching. In particular, we wanted to see whether analysing what the MSs told us would help us to identify and articulate contradictions with the teacher education activity system in which we operated. In his discussion of the Finnish health system around frequently ill children, Engeström (2001) described how difficult it can be for professions to articulate contradictions. Therefore, it was important to determine if 3GAT would provide us with relevant insights about expansive learning.

Theoretical Framework

In elaborating on his theory of expansive learning, Engeström (2001) outlined four questions and five principles. The four questions were described as being essential for any learning theory: 1. Who were the subjects of learning? 2. Why do they learn? 3. What do they learn? 4. How do they learn? The five principles were: the activity system as the unit of analysis; multi-voicedness; historicity; contradictions; and transformations. An activity system involves individual and group actions that operate together. The object of the actions is a cultural entity, formed through mediating artefacts. In the teacher education activity system, the object for the MSs was to complete a master's thesis appropriately, using number stories and research skills as mediating artefacts. In the school mathematics teaching activity system, the object was to teach mathematics appropriately, in our case mathematical argumentation using number stories and potentially research skills.

As "the object of activity is a moving target, not reducible to conscious short-term goals" (Engeström, 2001, p. 136), it is always being reformed both within an activity system and when activity systems meet. As a result, "object-oriented actions are always, explicitly or implicitly, characterized by ambiguity, surprise, interpretation, sense making, and potential for change" (Engeström, 2001, p. 134). This contributes to a multitude of views about the actions within each system, which contribute to its multivoicedness. Multivoicedness can illustrate the similarities and differences between activity systems, because there may be a cross-over in participants. Historicity, or the historical development and the circumstances of that development, of the activity systems determines the actions available. Contradictions are part of each activity system and occur when activity systems come in close contact with each other. Acting upon the contradictions could lead to transforming the activity systems or creating a new one.

Engeström (2014) described four levels of contradictions. At the primary level, contradictions occur within one constituent component of an activity system, such as when a new value system is connected to that component, but other components are not affected. The secondary level is where one constituent component is altered, requiring adjustment in other components in the same activity system. This was the focus for research, such as Psycharis and Potari (2017), where a new teaching method, integrating real-world modelling tasks into school classrooms, was introduced. The tertiary level of contradictions occurs when the object of a culturally more advanced activity system is introduced into another activity system, causing resistance within the other activity system. Anastasakis et al.'s (2020) research about the differences between school and university mathematics could be viewed as being about contradictions at this level. The quaternary level of contradictions is when it is not just the object, but also other elements of the juxtaposed activity systems that are disturbed. Engeström (2014) described this level of contradiction as "conflicts and resistances appearing in the course of the 'implementation' of the outcomes of the central activity in the system of the object activity" (p. 72). Solomon et al.'s (2014) research on the views of students and lecturers about university mathematics can be considered as illustrating contradictions across more components of different activity systems, not just the objects.

In examining the interviews with the MSs, we focused on the contradictions that could appear from the juxtapositioning of the two activity systems of teacher education and mathematics school teaching and so were interested in tertiary and quaternary levels. We were interested in determining where and how expansive learning could occur when research skills and understandings, from the teacher education activity system, were introduced as an object into the school mathematics teaching activity system. We wanted to determine if the interview data could be analysed with 3GAT to provide insights that could be used in reflections on our own practices.

Methodology

The MSs were interviewed a few weeks after they had submitted their theses, but before they had received their grades (grading was done by others, not supervisors). As supervisors of two of the MSs, the choice of interviewees was one of convenience. In their master's projects, the MSs had collaboratively designed and implemented a teaching task for Grade 2 students where the students were expected to develop individual number stories, or *regnefortelling* in Norwegian. A regnefortelling usually includes a written problem, a drawing and a solution, sometimes provided through a symbolic algorithm. The MSs collected the regnefortelling and interviewed students to better understand their written and oral mathematical argumentation. Each MS had analysed different aspects of the collected data to produce individual theses. At the time of the interviews, all three MSs had accepted jobs as Grade 1 teachers for the coming academic year.

The interviews were semi-structured, undertaken by the first and third authors, in a mix of Norwegian and English. All responses were translated into English. The focus of the interviews was on how the MSs saw the usefulness of their newly acquired research understandings to their future work as teachers. The analysis was done by first finding the utterances in each individual interview around one topic, that was often connected to a particular thread of questioning. Sometimes this was one utterance, at other times it was two or more consecutive utterances spread across several minutes. For each of these topic discussions, we completed Engeström's (2001) matrix formed from the four questions and five principles. We then identified where contradictions occurred and their level.

We focussed on the contradictions that the MSs described when reflecting on the juxtaposed activity systems because the MSs were in between activity systems. At the time of the interview, they had not started their roles as teachers in the school mathematics teaching activity system but had left the teacher education activity system. Thus, the MSs did not have possibilities to transform either activity system. If expansive learning were to come from reflections on the activity systems, it was for us, as teacher educators, to use knowledge of the contradictions to reconfigure, at least aspects of, the teacher education activity system, while accepting that changing activity system takes time (Engeström, 2001). Although the interviews were individual, sometimes the same contradictions appeared. As discussed in the results section, it was not always straightforward to identify whether the contradictions were at the tertiary and quaternary levels because of the complexity that surrounded the contradictions.

Results and Discussions

In this section, we first describe the discussion topics that produce contradictions at the tertiary level, then at the quaternary level, before discussing the contradictions that were not clearly one or another. In each section, the identification of the contradictions is not straight forward, requiring consideration of what Engeström's (2014) contradictions can contribute to understanding learning.

Tertiary level contradictions

Tertiary level contradictions occur, according to Engeström (2014), when an object and motive from a more culturally advanced activity system is introduced into the central activity of a related activity system. In our data, we took this to mean that tertiary level contradictions would appear when the object of the teacher education system, research skills and understanding, was introduced into the related activity system, that of school mathematics teaching.

One example of a tertiary level contradiction was when two of the MSs reflected on the use of different representations or modes in children's argumentation in their regnefortelling in their individual interviews. From their research, they were able to see how the students were helped in explaining their thinking by using different representations. For example, MS3 described how they decided to not just look at the students' writing and drawings but to interview them, as another way to understand their thinking. This was connected to having the freedom to design their research projects, "I think that is the most interesting part of the project, and also that we could do whatever we wanted to do" (Utterance 12). The valuing of different representations, thus, originated in the teacher education activity system. MS1 highlighted how having students talk about their regnefortelling could be useful in her teaching of Grade 1 students, who may not yet know how to write. Having students talk about their regnefortelling was at this point imagined in relationship to the activity system of school mathematics teaching. MS2 had similar reflections about how drawings, as another mode of representation, could support her as a teacher to understand children's mathematical thinking but also act as a mediating artefact that could help the students to think mathematically, "I as a teacher can see how they have thought, but also that they themselves may be able to see clearly how they think, that they can use drawing for themselves as a thinking tool" (Utterance 11).

The valuing of the use of different representations as tools for students' thinking arose from the MSs' research but gave the MSs insights into different practices that they could use in their future classroom teaching, both for their own benefit but also for their students'. This could be considered an example of a reworking of what Engeström (2014) described as the established structures to do with the usual text production in schools:

> This object is molded by pupils in a curious manner: the outcome of their activity is above all the same text reproduced and modified orally or in written form (summarized, classified, organized, recombined, and applied in a strictly predetermined manner to solve well-structured, "closed" problems). (p. 80)

By valuing students' thinking and supporting them to use a range of modes to engage in mathematical argumentation, the MSs would be providing their imagined, future students with culturally more advanced forms of mathematical argumentation, in which the students had more control of the kind of thinking they were doing. However, it is not clear if the MSs considered this to be a contradiction that arose as result of the object being imposed on the school mathematics activity systems. For the MSs, the contradiction may be at the secondary level in that having students make choices about how to represent their thinking, or even to think, would result in changes within the school mathematics teaching activity system as it redefined who did what in mathematics classrooms. As a result, the possibilities for using research skills to develop their teaching may be de-valued.

For teacher educators, the contradiction between the activity systems from adding research skills and understandings was more obvious in that it made us reflect on aspects of the teacher education activity system, beyond the research skills. MS2 told that in their teacher education, no one had made them aware of the value of drawings to support students to think mathematically and to show their thinking. The contradiction between the two activity systems is complex in that it was not clear if the MSs saw the valuing of other ways for students to show their mathematical thinking as being in contradiction with the existing methods or just something extra to be added on top. Yet, the implied contradiction with what was provided in teacher education can only have an impact on the teacher education activity system if teacher educators take note of what the MSs state they were missing.

Quaternary level of contradictions

Quaternary contradictions occur when there are tensions between two or more neighbouring activity systems to do with a range of constituent components. An example of this could be when the MSs discussed how they could work collaboratively when they became Grade 1 teachers. They were asked about this because all three MSs described how valuable they had found working together on their research projects, "I thought it was very positive, because it was very nice to have several master students together as well" (MS1, Utterance 62). MS3 indicated that the MSs had already thought about how they could do this:

> We are talking about having these Google docs sites of all the (master's) students that are working together now, to just share ideas. And I think it will also be possible to – like take some parts of this project and try it out, absolutely. And we are in so different parts of Bergen as well, so it will be nice to see. (Utterance 40)

In the interviews, there were no reference to multivoicedness or historicity to do with either activity system, except to the other MSs' voices about their experiences of their collaboration. The traditions about teachers working together in schools, which perhaps could have had some connection to what they were suggesting, was not referred to. Without knowledge of how collaborations are expected to occur in the school mathematics teaching activity system, the MSs did not articulate any potential contradiction in trying to collaborate across schools. On the other hand, as teacher educators, we were aware that a contradiction could arise if this form of collaboration that had occurred during research were implemented when the MSs worked as teachers. It did provide us with information that we could take into consideration in designing our future teacher education.

Contradictions that have aspects of both tertiary and quaternary levels

Some discussion topics, raised by the MSs, seemed to be between the tertiary level and the quaternary level of contradictions. One such discussion topic, which appeared in all three interviews, was when the MSs reflected on how asking students about their mathematical thinking provided them with other kinds of information than when teachers usually asked children questions in mathematics classrooms. For example, MS1 described that by showing curiosity about the students' thinking, the students talked to the MSs in a different way, "(we) said that we just want to know what you think, and in a way started talking a little bit about the number story they had, then they started talking a little more freely" (Utterance 11). MS3 discussed how this led her to understand the importance of asking the students about their thinking:

> Just to talk with the students, like … Yeah, just ask the student "oh, what were you thinking here?" That I also think is an important starter to do, and then maybe it will be easier to talk about regnefortelling together afterwards. It will start with saying it's okay to explain how you're thinking, and it's okay to not have the right answer, but it's better to know how to explain it than to have the right answer and not explaining it. (Utterance 79)

This indicated that interviewing skills from her data collection changed her ideas about what was important when talking with students. MS3 seemed to consider that the mediating artefact of doing research interviews could support her to achieve the goal of improving her teaching mathematics in school by moving beyond just being interested in the correct answer. The multivoicedness about this discussion topic can be seen when MS3 used "you" to refer to the students. There were also traces of historicity in that there is a sense that traditionally mathematics teaching has focused on the correct answer and not on the students' thinking.

Although this discussion topic is about the object of research interviews being integrated into teaching, it is difficult to know if the contradiction with traditional classroom discussion practices is likely to stay within the object component and thus be at the tertiary level or affect other components of the school mathematics activity system and be at the quaternary level. The object, research interviews, from teacher education activity system, did disrupt the object of classroom discussions in the school mathematics activity system:

> I have seen that it is very interesting to talk a little more with students about how they think, than to just stand at the front and question them one by one like that, that you get more out if you ask a few questions where you can dig into a little with each individual student. (MS2, Utterance 47)

However, the complexity of implementing these alternative types of classroom discussion may mean that other components of the activity system would not be affected:

> It takes a lot of time, and you have to be so focused on one student, so then you cannot have twenty students sitting there, you have to have control over them as well. So maybe you have to

> collaborate with more teachers, divide into stations, do it a little differently so that you can get a little… Yes, a little help from others. Maybe also sign up for some such research projects, and … Mhm, so you can get help there. (MS2, Utterance 54)

MS1 also suggested ways of getting the same benefits from interviewing students, while managing the rest of the class, “maybe they could have had a conversation with each other and in a way argued with each other, then. But then we must have worked with a lot of argumentation” (Utterance 33). This indicates that even when imagining alternative ways of gaining useful information about the students’ thinking, there were issues in that the students would need to be taught argumentation before they could engage in talking to each other about their regnefortelling.

These quotes indicate that the contradiction raised by valuing the information provided by the interviews was in conflict with the MSs’ understanding of the realities of classrooms, which would not allow for individual interviews and where argumentation needed to be taught before students could use it. At this point in their careers, the historicity to do with mathematics classroom teaching had provided the MSs with a view that their main job was to ensure all students in the class were occupied appropriately, making it difficult to do individual interviews. However, identifying the contradiction between wanting to hear about students’ thinking and keeping all students occupied made the MSs imagine potential solutions. The MSs seemed to say that the contradiction could affect several components of the school mathematics teaching activity system, but their suggestions for implementing alternative approaches seemed uncertain. This could be because the MSs were not yet working as teachers so imagining system level changes was difficult and, as Engeström (2001) stated, transforming activity systems is a collective endeavour, taken over time. Nonetheless, the multiple voices about the school mathematics teaching activity system for these MSs now included the researcher voice, which allowed them to query a focus on the students having the right answer.

The MSs’ suggestions of alternative ways of hearing about students’ thinking suggested that the contradiction was at the quaternary level because more components than just the goal of the activity system were discussed. However, it is not clear if this could lead to school mathematics activity system changing. For the teacher education activity system to change, we as teacher educators need to also reflect upon what the MSs described as difficulties in the implementation of new practices around listening to students’ mathematical thinking.

Conclusion

Although activity theory has been used in different mathematics education projects (Anastasakis et al., 2020; Psycharis & Potari, 2017; Solomon et al., 2014), our analysis had a different purpose. We wanted to determine if it could provide useful insights from MSs’ interviews that could help us to reflect on our work as teacher educators. 3GAT did provide insights on how the MSs considered research understandings and mathematical argumentation using regnefortelling from the teacher education activity system, could be integrated into the other activity system, school mathematics teaching. In Engeström’s (2001) description of a project with Finnish health care system, there was a need for participants to be able to articulate the contradictions across activity systems in order for possible alternative practices to be designed. MSs did not always seem to identify a contradiction between the activity systems. As teacher educators, we could identify the contradictions but the connection to how to change our practices and connect them to the school mathematics activity system was not clear. Some of these issues may be with the data rather than with 3GAT. As the MSs were not yet in the school mathematics teaching activity system, they may not have been able to imagine difficulties often associated with implementing new practices in schools (see for example, Psycharis & Potari, 2017). The value of 3GAT was that the aspirations, challenges and obstacles, that the MSs identify, are reconfigured from being problems of individuals into what they should be perceived as, i.e., tensions and contradictions in and between activity systems. This provided other insights, which had the potential to contribute to more substantial changes in the activity systems.

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Coordinating conceptual frameworks for an in-depth understanding of knowledge when teaching mathematics

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This paper is methodological and theoretical, focusing on how teachers' knowledge in action and interaction when teaching mathematics can be explored. The two conceptual frameworks adopted, The Knowledge Quartet and the Pedagogical Relational Teachership, are coordinated, as they grasp disparate aspects of knowledge in action and interaction when teaching mathematics. An empirical example is given to illustrate how these two conceptual frameworks can be adopted in coordination with each other. The results show that the coordination of these two conceptual frameworks used as a methodological framework can contribute to a deeper understanding of the characteristics of teachers' pedagogical content knowledge and relational abilities when teaching mathematics.

Keywords: Mathematics teaching, networking frameworks, relational abilities, teachers' knowledge.

Introduction

This paper is methodological and theoretical, focusing on how teachers' knowledge in action and interaction when teaching mathematics can be explored. This exploration is based on a classroom study conducted in a Swedish Grade 5 class. It is part of a research project intended to contribute to a more profound understanding of inclusive mathematics education regarding the connections between teachers' knowledge and relational abilities when teaching mathematics. Thus, this paper focuses on the use of a methodological framework based on two conceptual frameworks rather than interpreting the implications for mathematics education in the classroom.

Previous studies have shown that relational leadership promotes inclusive mathematics education (Schmidt, 2015). Furthermore, Roos (2019) concluded that inclusive mathematics education requires the teacher to have mathematical, didactic, and pedagogical skills as well as relational competencies in seeing each student as a person and understanding their needs. Roos' conclusion concurs with that of Ernest (2019). The latter stated that teaching mathematics involves the responsibility to treat students with care and respect and teach mathematics in a manner that benefits students effectively. Additionally, Valero (2005) pointed to the need to view the student as a whole human being, not just as a cognitive subject of mathematical thinking, to fully understand the student's intentions when acting in social mathematics teaching and learning situations.

Several analytical models deal with mathematics teaching, for example, Mathematical Knowledge for Teaching (Ball et al., 2008), The Teaching Triad (Jaworski, 1992; Potari & Jaworski, 2002), and The Knowledge Quartet (KQ) (Rowland, 2013). Ball et al. (2008) aimed to elaborate theoretically on the concepts of subject matter knowledge and pedagogical content knowledge (Rowland, 2013). Potari and Jaworski (2002) acknowledged sensitivity to students as one of the three keystones, making their framework useful as an analytic tool. However, they gave no further descriptions of how teachers act in interactions.

Based on the above, this paper is concerned with coordinating conceptual frameworks that can surface the complexity of teachers' knowledge when teaching mathematics in inclusive classrooms, considering knowledge in action and interaction with students. This complexity is addressed in this paper by illustrating two conceptual frameworks complementing each other using networking strategies (Bikner-Ahsbahs & Prediger, 2010) as guiding heuristics. The two conceptual frameworks are the KQ (Rowland, 2013), which frames how mathematical subject knowledge plays out in teaching, and the Pedagogical Relational Teachership (PeRT) (Ljungblad, 2019), which frames teachers' relational abilities, are coordinated. Thus, this paper aims to illustrate how these two different conceptual frameworks may contribute to exploring teachers' knowledge in action and interaction with students when teaching mathematics in inclusive classrooms.

The two conceptual frameworks

First, the two conceptual frameworks are introduced. Then the common core elements between the conceptual frameworks are presented.

The Knowledge Quartet

The KQ is a conceptual framework of four categories: *foundation*, *transformation*, *connection*, and *contingency*. Each of the four categories consists of several methodological codes to be used when analyzing empirical material (Table 1), from which the four dimensions of the KQ are extracted.

Foundation is related to the teacher's theoretical background in and knowledge of mathematics, making it possible for the teacher to use mathematical terminology deliberately, be aware of its purpose and identify errors (Rowland, 2013). Epistemologically, foundation can be seen as propositional knowledge, involving knowledge about mathematics and the purpose of mathematics teaching (Hundeland et al., 2017). *Transformation* refers to how a teacher's foundational knowledge is transformed into action when teaching, such as demonstrating mathematical content through examples, instructional materials, and mathematical representations (Rowland et al., 2005). Epistemologically, transformation can be seen as knowledge in action, as this knowledge is situated in and visible in teaching moments (Hundeland et al., 2017). *Connection* refers to the connections made by the teacher concerning the coherence of the teaching across a shorter or longer period, for example, connections between procedures, concepts, and sequenced examples (Rowland et al., 2005). Epistemologically, connection can be seen as knowledge in action, as this knowledge is situated in and visible in teaching moments (Hundeland et al., 2017). *Contingency* refers to the teacher responding to ideas from students for which it is impossible to plan and represents deviations from the intended actions in a planned lesson, but which make the teaching meaningful for students (Rowland et al., 2005). Epistemologically, contingency can be seen as knowledge in interaction, as this knowledge is situated in and visible in teaching moments when interacting with students (Hundeland et al., 2017).

The Pedagogical Relational Teachership

The PeRT is a theoretical relational perspective involving a taxonomy categorized under two themes, *tact* and *an inclusive stance* (Ljungblad, 2019), which is then further organized into smaller units of codes to be used when analyzing empirical material (see Table 2) (Ljungblad, 2021).

Tact is situated and improvised and can be observed in the teacher's different ways of relating to the students and their situated needs. Tact can be expressed verbally and non-verbally. The verbal expressions of tact are connected to how teachers use the tone of voice, degree of loudness, and rate of speech. Tact is expressed non-verbally through the teacher's eye contact, movements, gestures, and facial expressions. Epistemologically, tact can be seen as knowledge in action as well as in interaction, as tact is situated in the teaching and visible in the teaching as the teacher relates to the students (Ljungblad, 2016). *An inclusive stance* is connected to the teachers' sensible choices and can be observed in how the teacher takes responsibility for teaching and developing sustainable relationships within the classroom. Epistemologically, an inclusive stance can be seen as knowledge in action as well as in interaction and is both situated in and visible in teaching (Ljungblad, 2016).

Common core elements

Strategies of networking theoretical approaches are research practices that may provide a more comprehensive view of the complexity of teaching and learning mathematics. Various strategies to use more than one theoretical approach have been systematized described as a continuum, from understanding others and making their theories understandable to synthesizing and integrating locally. Somewhere in the middle of the spectrum of the degree of integration is to *coordinate* theoretical approaches. Coordinating multiple theoretical approaches can contribute to a conceptual framework for understanding empirical data (Bikner-Ahsbahs & Prediger, 2010).

In this paper, coordinating two conceptual frameworks implies three consecutive analyses on the same data source, complementing each other in a methodological framework. The two conceptual frameworks are chosen to elaborate on the interactions between a teacher and a student and the teacher's action reflecting the student's responses. The teacher's foundational mathematical knowledge plays out in the teaching, referring to the dimensions of transformation, connection and contingency. Unlike, the teacher's responses to the individual student's actions refer to tact and an inclusive stance dimensions, as well as contingency. Epistemologically, the dimension of contingency intersects with the dimensions of tact and inclusive stance. They are both seen as knowledge in interaction, addressing how the teacher's mathematical knowledge and relational ability unfold in the teaching in unpredictable ways. The common core elements are based on knowledge in interaction between the teacher and the student since they are connected to a context. When teaching mathematics, the teacher's actions are influenced by the context. However, when interacting with the student, the teacher responds to the student's ideas and emotions, which means the teacher influences the context.

Coordinating the two conceptual frameworks

The two conceptual frameworks were used to analyze video-recorded observations of mathematics lessons carried out in an intervention that was ongoing during a school year. The video-recorded classroom observations from this study were transcribed and coded in NVivo. In the analysis, first, the codes from the KQ (Table 1) were used to mark up the data material.

Table 1: Methodological codes from the KQ (Rowland, 2013)

Foundation	awareness of purpose (AP); identifying errors (IE); overt subject knowledge (OSK); the theoretical underpinning of pedagogy (TUP); use of terminology (UT); use of textbook (ATB); and reliance on procedures (COP)
Transformation	teacher demonstration (DT); use of instructional materials (UIM); choice of representation (CUR); and choice of examples (CUE)
Connection	making connections between procedures (MCP); making connections between concepts (MCC); anticipation of complexity (AC); decisions about sequencing (DS); and recognition of conceptual appropriateness (RCA)
Contingency	responding to students' ideas (RSI); deviation from agenda (DA); teacher insight (TI); and (un)availability of resources (RAT)

Then, as a second layer on the same data material, the codes from the PeRT (Table 2) were used. The sequence was first coded by me and then again by three senior researchers and two doctoral students to increase interrater reliability. Discussions were carried out among the experts involved until a consensus was reached regarding the codes used.

Table 2: Methodological codes from the PeRT (Ljungblad, 2021)

Tact	improvising shifts of tact (A11); seeking contact by showing interest in students' different ways of working and reasoning (A21); seeking contact with students by showing interest in the person (A22); meeting the students with respect (A31); and meeting the students in different ways in the same teaching situation (A32)
An Inclusive stance	taking responsibility for the teaching (B11); taking responsibility for the relationships (B12); listening to students when creating space for them to speak in their own way (B21); creating space for students to listen to each other (B22); showing students different possible ways to explore the content (B31); and encouraging students by showing trust in the students' ability and willingness to explore the content mutually (B32)

The third step implied an inductive analysis of students' participation in mathematics education on instances categorized into three groups: i) coded to both KQ and PeRT, ii) coded to only KQ, and iii) coded to only PeRT, exploring differences and similarities of students possibilities to participate in mathematics education. In this paper however, only the first and second deductive steps are focused on and thus not the third inductive step.

An empirical example using the KQ and the PeRT as lenses

In this section, one empirical example is first introduced and coded to illustrate how the coordination of the conceptual frameworks was carried out. The empirical example is from a Swedish Grade 5 classroom with two teachers, a mathematics teacher and a special education teacher in mathematics.

This empirical example was chosen for this paper because the codes from the two conceptual frameworks sometimes overlapped. The codes are written in brackets to the right. Codes from the KQ are in *italics*, and codes from the PeRT are underlined.

First, the mathematics teacher, Felicia, introduces a task about a jogging tour to the whole class. The special education teacher in mathematics (Selma) is present during the lesson. The task describes a jogging tour that took the form of a circle. There is a picture of the circle on the task paper, where the circle's circumference (3,140 m) and radius (500 m) are written. The jogging tour has a shortcut straight across the circle, along its diameter. Subtask b is: Fatima is jogging half the distance and then taking the shortcut home. How long is Fatima's jogging round? Tanja, a student, raises her hand when the transcript below starts. The teachers consider Tanja a student in special educational needs in mathematics (SEM), as she often struggles to participate in mathematics. Selma, who is nearby, stops beside Tanja's right side, bends down, leans her forearms on the desk, and quietly asks Tanja about her thoughts on subtask b while looking at her task paper. Tanja asks about how to figure out the distance.

62	Selma:	How are we about to figure out the half then?	(A11, A21, B32, *RAT*)
63		(Quiet, 7 sec.)	(B21)
64	Selma:	How can we do it?	(B32)

Next, after a passage where Tanja is being silent, Selma asks:

67	Selma:	How long is the whole [circle]?	(*RSI*)

After a wrongly read number, Tanja quickly answers correctly.

71	Selma:	Yes. Suppose we pretend that this is three thousand. How long would the half be then?	(A11, B32, *MCP*)
72	Tanja:	Two and a half. No.	(B21)
74	Selma:	Not two and a half. Instead? What if you and I would split these three?	(A11, B32, *RAT*)
75	Tanja:	Mm.	(B21)
76	Selma:	How much do each of us get?	(A11, *RAT*)
77	Tanja:	Three and a half.	(B21)
78	Selma:	Do we get three and a half if we split three?!	(A11, *IE*, *RAT*)
79	Tanja:	No. (Giggles.)	(B21)
80	Selma:	No. (Laughs.)	(A11)
81	Tanja:	One and a half.	(B21)
82	Selma:	One and a half. And, what does it mean in meters in this task?	(A11, *MCP*)
83		(Quiet, 2 sec.)	(B21)
84	Selma:	Instead of three thousand?	(A11, *MCP*)
85		(Quiet, 2 sec.)	(B21)
86	Selma:	If we split, instead of meters, if there were [Swedish] crowns instead. Suppose we split three	

		thousand [Swedish crowns] you and me.	(A11, *UIM*, *RAT*)
87	Tanja:	Mm.	(B21)
88	Selma:	How many [Swedish crowns] do we get each?	(B32, *UIM*, *RAT*)
89		(Quiet, 4 sec.)	(B21, *RSI*)
90	Tanja:	One and a half.	(B21)

Above is an example where most lines are only coded with codes from the PeRT. In these lines, the special education teacher actively listens to the student or make room for her to speak (code B21, lines 63, 72, 75, 77, 79, 81, 83, 85, 87, 90). In another line, the special education teacher acknowledges that they work together using "we" in her talk, signalling that they will complete the task together (code B32, line 64). She changes her voice tone to a joyful tone (code A11, line 80) and seeks contact, interested in the student's reasoning (code A21, line 91). These lines coded only to the PeRT could be seen as the teacher responding to the student's needs as a person.

One line is coded only to the KQ. Here, the teacher responds to the student's silence by asking an easy question to which the information for the answer is written on the task paper (code *RSI*, line 67). The line could be seen as the teacher wants the student to give an answer related to the mathematical content, as she chooses a question to which the student already knows the answer.

There are also lines with codes from both the KQ and the PeRT. Lines coded to both the KQ and the PeRT display the special education teacher shifts tact while she contingently uses her fingers as manipulatives (codes A11 and *RAT*, lines 62, 74, 76, 78, 86). In some of these lines, the teacher also shows interest in the student's way of reasoning (code A21, line 62) and acknowledges that they together, using "we" in her talk, will be able to complete the task (code B32 line 62, 74). Yet another line shows the teacher suggesting they together, "we", can pretend the number is 3000 instead of 3140 while making a connection between the procedures of dividing 3140 by 2 and dividing 3000 by 2 (codes B32 and *MCP*, line 71). Further, one line displays the teacher pointing to her three fingers when identifying a wrong answer and changing her voice to a joking, surprising tone when asking a question (codes *RAT*, *IE* and A11, line 78). More, there are lines where the teacher connects one and a half to 1500 while changing the direction of her gaze to look at the student and changing her speech rate to slow and gentle (codes *MCP* and A11, lines 82, 84). Another line implies that the teacher again gazes at the student talking about money as imagined manipulatives (three thousand Swedish crowns) while pointing at her three fingers (codes A11, *RAT* and *UIM*, line 86). Again, the teacher uses "we", asking the student about how they can split the money (B32, *RAT* and *UIM*, line 88). The teacher is silent long enough, so the student finally answers (B21, *RSI*, line 89). The lines coded to both the KQ and the PeRT could be seen as the teacher responding to the student's needs as a person as well as adapting her foundational mathematical knowledge to this particular social setting.

Conclusions

This paper aimed to illustrate how the two conceptual frameworks used as a methodological framework can contribute to exploring a teacher's knowledge in action and interaction with a student when teaching mathematics in an inclusive classroom. The empirical example illustrates how the two frameworks make aspects beyond teachers' knowledge respective relational abilities surfaced. That is, from the dimensions of the KQ (Rowland, 2013), the special education teacher in mathematics shows foundational mathematical knowledge connected to procedures (lines 71, 82, 84), choosing (unavailable) materials and representations (lines 62, 74, 76,78, 86, 88), and responding to a student's silence (lines 67, 88, 89). From the dimensions of the PeRT (Ljungblad, 2021), the special education teacher uses verbal and non-verbal expressions to seek contact with the student and shows interest in her ways of reasoning. That is, the special education teacher creates space for the student to speak with her voice (lines 63, 72, 75, 77, 79, 81, 83, 85, 87, 89, 90), acknowledging her answers, although her answers are sometimes incorrect (lines 71, 82). The teacher tactfully makes jokes (line 78), supports the student, and shows trust that they two will solve the task (lines 62, 64, 71, 74, 78, 86, 88). Thus, based on the interaction with the student, the special education teacher adapts her actions to the specific student within the particular situation in time and space, using relational abilities. When using relational abilities, the special education teacher's foundational mathematical knowledge is transformed through adaption to the specific student in the interaction (lines 62, 71, 74, 76, 78, 82, 84, 86, 88, 89). Thus, when coordinating the KQ and the PeRT, the features of a teachers' relational abilities concerning a specific student's diverse interests and prerequisites contribute to our understanding of the complex situations of mathematics teaching and how they evolve.

Earlier studies on mathematics teaching have either focused on teachers' knowledge (for example, Ball et al., 2008; Jaworski, 1992; Potari & Jaworski, 2002; Rowland, 2013) or teachers' relational abilities (Roos, 2019; Schmidt, 2015; Valero, 205) and the approaches of the KQ and the PeRT function as frameworks of their own. They both can be used to analyze mathematics teaching but grasp disparate aspects of teachers' actions or interactions with students. However, coordinating the KQ and the PeRT may contribute to an understanding of mathematics teaching in inclusive classrooms from an extended perspective. The above-illustrated two-step deductive analysis makes it possible to carry out a forthcoming third inductive analysis of the instances where the codes from the KQ and the PeRT overlap and don't overlap. This analysis will make visible if and then how teachers relational and/or mathematical knowledge influence student participation in mathematics.

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Teachers' decision-making as a context for networking theories of mathematics teaching

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This paper illustrates how teacher decision making, as an arena of teaching practice, can be a locus for networking structural and individualist theories of teaching. We use the decision of what problem to give to students as a context for illustrating how those diverse perspectives can be coordinated conceptually in the context of a middle-range theory of teaching practice and the decision of how to respond to students to illustrate how such coordination can combine perspectives in the context of empirical studies of mathematics teaching.

Keywords: Teaching practice, theories of teaching, teacher knowledge, decision making

Introduction: Middle-range theorizing

Variability in theoretical accounts of the work of teaching has existed in education research for decades. This is also the case in mathematics education, where accounts of teachers' actions, beliefs, knowledge, noticing, or of teaching contracts, patterns, or norms, have run somewhat in parallel for decades. In this paper, we discuss teachers' decision-making as a context for networking different theoretical perspectives on the work of mathematics teaching and propose how research using scenario-based assessments could facilitate such networking. Our perspective on the networking of theories builds on Merton's (1967) notion of *middle-range theory* as an alternative between grand theories and local empirical hypotheses: A middle-range theory starts by scoping a terrain of human practice (e.g., mathematics teaching) and builds itself through adapting language to coherently accommodate potentially competing empirical assertions and developing conjectures that could be studied empirically to increase what is known. Middle-range theorizing is thus conceptual and empirical, and it includes activities that Prediger et al. (2008) called *comparing and contrasting* (e.g., by identifying the different observables and predictions each theory would attend to when looking at a complex phenomenon). It also includes what Prediger et al. (2008) call *coordinating and combining* (e.g., by identifying research designs, designing instruments, and specifying statistical models that can help discern the contribution of each theory to the understanding of the phenomenon).

Setting the stage: Different takes on the work of teaching

Scholars who study educational systems have often seen teaching as a production function shaped by institutional structures and policies such as evaluations, incentives, and requirements as well as the characteristics of classrooms (Doyle, 1986; Firestone & Pennell, 1993). In mathematics education, this structural conditioning of the work of teaching, has been apparent in research that describes teaching as the implementation of a curriculum through the enactment of role-relationships among teacher, students, and content established by a hypothetical didactical contract (Brousseau, 1997). Building on the latter, in our own work (e.g., Herbst, 2002, 2006) we have described observations of teaching in terms of effecting symbolic exchanges between student work and knowledge at stake required by the didactical contract—we refer to this as the theory of instructional exchanges. From this structural perspective, the work of teaching includes adapting to these structural demands.

A completely different approach is seen in work by scholars who seek to capture the experience of individual teachers, often drawing on psychological concepts. In these accounts, the sense that teachers bring to the work of teaching their individual characteristics (e.g., identity) or assets (e.g., knowledge) have often been used to explain unique episodes of teaching actions as expressions of self. Schoenfeld's (2010) ROG (Resources-Orientations-Goals) theory, for example, attempts to account for individual teacher actions as a moment-by-moment cognitive calculation of what to do, fed by the individual's Resources, Orientations, and Goals, with resources including teacher knowledge and orientations including teacher beliefs. Likewise, research on teacher noticing (e.g., Sherin et al., 2011) has produced models of the cognitive processes that lead teachers to act on the basis of what they notice. Other approaches, such as those that explain teaching as a projection of a teacher's individual beliefs may vary in the extent to which they hypothesize consciousness but also seek explanations of action inside the individual (Mason, 2002). The number of different perspectives taken to account for teaching as an individual activity evinces how dominant this perspective has been in mathematics education research.

Progress in research on teaching has benefitted from both structural and individual perspectives, yet each has its drawbacks. The first approach risks deprofessionalizing the work of mathematics teaching while the second approach risks burdening individual teachers with an ever-increasing set of resources and orientations to be acquired and used. These observations suggest the need for coordinating and combining what we learn from these perspectives. The development of a middle-range theory of the practice of mathematics teaching can support such coordination. A variety of practice theories have emerged in social research (Nicolini, 2012); among these perspectives, Bourdieu's (1990) logic of practice supports theoretical development that allows researchers to coordinate structural and individual perspectives. Such development can incorporate Stigler and Hiebert's (1998) perspective of teaching as a cultural practice and the notion that improving instruction requires improving teaching rather than teachers (Hiebert & Morris, 2012). We use research on teacher decision-making to sketch how such networking could proceed.

Coordinating theories for conceptualizing teacher decision-making

Decision-making is an important locus for the coordination of theories of mathematics teaching partly because of the marked contrast that can be gleaned between objective (structural) and subjective (individual) perspectives. An objective perspective that sees teaching as a response to structural conditions and constraints might see a teacher's decisions as discrete events: within a lesson, choosing which work to assign to students and how much scaffolding to provide would likely be one of such decisions, and see it as capable of being optimized on the basis of structural characteristics such as learning objectives and class level. A subjective perspective might, in contrast, account for the work of teaching as the continual enactment of individual agency in response to information from the specific context of action, where the actions of one person depend on their noticing of the context and the activation of personal commitments to address perceived needs of the context.

The conceptual work involved in coordinating these perspectives can be seen in the example Lampert (1985) offered to motivate her conceptualization of teaching as managing dilemmas. Lampert had noted the boisterous nature of boys in her class and decided to seat them closer to the front of the class so that she could manage their behavior. After doing that, she realized that her decision left girls farther from the blackboard having fewer opportunities to hear and see the mathematical content of instruction. She found herself second-guessing whether a decision could solve a problem in teaching. Instead, she came to embrace the notion that while decisions need to be made, no choice can be made once and forever. Two important generalizations can be abducted from this example. First, the field of education provides structured structures such as the need to attend to associations between gender differences and achievement and between achievement and opportunity to learn that require a teacher to be a steward of the learning environment. Second, the lived experience of an experienced teacher provides structuring structures such as the disposition to see boisterousness as not beneficial to a productive learning environment and physical proximity to a teacher as conducive for individual students to engage in productive behaviors. These two types of elements played a role in Lampert's analysis, requiring her to accept some structural basis for her decision making and act first to move boys to seats where she could monitor them. This decision changed some structural characteristics of the class so that she might feel less called to attend to discipline. In analyzing this new position, her personal, gendered experience, enabled her to notice a new gender imbalance issued from her decision (e.g., boys receiving more attention) that might need a different decision (viz., to spend more of her time in the back of the room).

Like Lampert, we want to embrace, but complicate, the theme of decision-making when it is applied to teaching mathematics. Yet given how prominent the expectation is among the mathematics education research community that teachers will exercise agency and enact personal commitments as they work in their context, we state our thesis in an alternative way: The study of teachers' decision-making needs to coordinate the usual attention to teachers' personal characteristics and assets with the recognition that there are structures in the field of mathematics teaching that require teachers to make decisions and that activate expectations for what those decisions ought to be. Even when teachers' personal commitments may compel them to deviate from such expectations as well as vary in the decisions they make as they handle specific contextual information, these decisions are best understood in terms of their fit with structures of the environment in which those decisions are made. In the two following sections we take two examples of decisions in mathematics teaching to illustrate two aspects of the coordinating approach we propose, which involves (1) the conceptual articulation that leads to (2) the empirical work that fleshes out of a middle-range theory. The cases are (1) What problem to give to students, and (2) What to say in response to students' contributions.

The decision of what problem to give to students

We examine the decision of what problem to give to students to demonstrate how a middle-range theory of practice can develop conceptual resources to coordinate structural and individual perspectives. The literature on the cognitive demands of tasks (e.g., Stein et al., 1996) has noted that cognitive demands usually lower in the transit from how the tasks are stated in curriculum materials, through how they are presented to students, to when students enact them. From a structural perspective, one could explain this in reference to how such tasks fit in the didactical contract. There are structures in the field of mathematics teaching that situate the practitioner and her students in a place from which they are more or less likely to do things. The course of studies with its institutionalized learning goals and expectations of students' attainment are such structures. Thus, the notion that a mathematics teacher must assign to students work related to the learning goals of the course of studies and the officially acknowledged capabilities of students is a structural characteristic of the field of mathematics teaching. It suggests that tasks that do not fit with courses of studies and learning objectives would likely receive little attention; both teachers and students would adapt to the point of neglecting tasks that fit little with the institutional expectations on them. Similar structural analysis also led Doyle (1988) to explain why novel tasks tend to be transformed into familiar ones.

An individual-centered perspective (e.g., Schoenfeld, 2010), would prompt researchers to see the maintenance of tasks' cognitive demand as a call to understand teachers' use of their personal resources (e.g., relational skills with students, personal goals), shaped by individual commitments or orientations to a variety of other issues emerging from personal readings of the context and mediated by the teacher's own noticing capacities. The subjectivist perspective would affirm teachers' individual agency and identify beliefs that some topics in the curriculum are more important than others, the need for this or that knowledge, and commitment to particular meanings of what learning those topics means as sources for explaining what happens with a task in the classroom.

We argue that the job of mathematics teacher in an educational institution positions the teacher to exercise their individual agency in ways that, while not determined by, can be described as habitual responses to structures such as the didactical contract. Furthermore, as Bourdieu would have it, these habitual responses also function as structuring structures, that is they further constitute ways of shaping future action even if the experience of individual agency continues. The notion of teaching as a cultural activity (Stigler & Hiebert, 1998) accounts for the variability in what can fit as appropriate kinds of things to ask students to do--the cultural forms that students' academic work can take: whether students may be expected to interact with the real world, be occupied in the same task for a long time, or speak to the whole class are examples of structuring structures that will shape the likelihood for a teacher to ask one or another question and for the students to interpret the request the teacher makes.

But there are also such structuring structures within a didactical contract. In our work studying algebra 1 and geometry classrooms in American high schools, we had conceptualized *instructional situations* as local didactical contracts which are specific to courses of studies and organize what tasks teachers assign to students and how they manage the framing of novel problems. Examples of instructional situations include solving equations and doing proofs—these situations include normative task statements (e.g., the use of the word "prove" vs. the word "solve") as well as expectations on the teacher's action (e.g., to provide a diagram that labels all the points to be used in the proof vs. providing all the information needed and no more in a word problem) and expectations on the students' work (e.g., to justify each statement before moving on to the next statement in a proof vs. doing one algebraic step at the time and writing it below the prior statement of equality). These situations initially appear as tasks canonically adapted to curricular structures (and may even be exemplified explicitly for students in study material), and one might use them to recognize the structural difference between novel and familiar tasks (Doyle, 1988). But instructional situations can also be recognized as capable of structuring how teachers see their field of possibilities when deciding how to engage students in more novel work. While teachers may design or adapt a variety of tasks for their students, the instructional situations available to them in a course of studies will structure how they exercise their agency, particularly how they engage in repair strategies with students to support students' identification of what is possible for them to do when they have chosen a task for students to work on. The instructional situations in a course of studies exemplify elements of the habitus of a mathematics teacher not necessarily by dictating what the teacher ought to do when assigning students' work, but by providing a set of expectations that the teacher needs to consider as they negotiate what will get students to engage with a mathematical task. Thus, in the case of engaging students with novel problems such as the question "what can you say about the angle bisectors of a parallelogram?" and the goal of using this question to practice proving, a teacher may feel compelled to suggest that students draw a figure and later label the vertices of the parallelogram and of the quadrilateral formed by the intersection of its angle bisectors. While this compulsion may be experienced as a personal decision to respond to noticed students' difficulties getting started thinking about the problem or getting started with a proof, the norms of the instructional situation of doing proofs (which stipulate that problems are posed in a diagrammatic register) will have likely structured the teacher's attention to expect students' difficulties writing the statements for the proof and to anticipate what they might be expected to do to preempt those difficulties.

The foregoing considerations suggest that research on teacher decision-making can be a productive site for empirical research to support the networking of theories of mathematics teaching. Research can and should investigate this terrain in a variety of ways, including (1) finding patterns in how textbooks and tests present work for students and how that influences what problems teachers choose to assign to their classes and (2) describing from the teachers' perspective how they plan and devolve responsibility for students' work on novel tasks. If this conceptual development seems worthwhile as one that can help us understand the specific ways in which teachers might feel compelled to adapt novel tasks for their students, empirical work can help uncover the structural resources available to teachers in different courses of study: What are the instructional situations available for teachers of a course of studies to frame students' work on novel problems and how do they help account for the ways in which practitioners notice the difficulties students have engaging in those problems and the particular moves they feel compelled to make? Particularly in the context of deciding to use tasks that are cognitively demanding (Stein et al., 1996), identification of the instructional situations that can help a teacher frame those tasks can help refine the instruments for empirical research that builds middle-range theory by combining structural and individual perspectives.

Studying how teachers respond to students' contributions

To show how the development of a middle-range theory of practice benefits from conceptual developments, such as the proposition of instructional situation to describe a type of resource for the teacher's framing of novel task, we sketch out empirical work that could help discern how variability of decisions responds to individual and structural factors. Scenario-based assessments (Herbst & Chazan, 2015) allow the study at scale of questions of teaching practice while also enabling rich representations of the contexts of teachers' work by using cartoon characters to render nonverbal aspects of practice and hence summon their feel for the game in real practice. Scenario-based items can be used to immerse teachers in classroom episodes framed or not framed by an instructional situation. They can be used, in the context of experimental design and regression analysis with covariates to study the complementarity of theoretical concepts suggested by structural and individual perspectives on decision-making. We illustrate it in the case of responding to students' contributions.

The interest in teacher responding to students' contributions (e.g., Jacobs & Empson, 2016) has emerged in mathematics education against what is often described as the pervasiveness of teacher evaluation of student responses, connected to the I-R-F (Initiation-Reply-Feedback) pattern. The research on teacher noticing has contributed to describing an individual-centered perspective on decision making that seeks to understand cognitive mechanisms for a decision: Attending to students' thinking enables the teacher to interpret what the student said, which in turn leads to deciding how to act. While individual centered perspectives might emphasize teachers' pedagogical knowledge and commitments to students' thinking as important to account for how teachers' respond to students, structural perspectives might put the emphasis on class level (e.g., advanced vs remedial) and the status of the student's response (correct, incorrect) as predictors of responses.

Our approach to the practical rationality of mathematics teaching models the context in which teachers are called to make such decisions. Accordingly, *instructional situations*, originally defined as structures behind similar recurrent tasks observed in courses of studies, do more than distinguish familiar from novel tasks: As frames that teachers can use to structure students' mathematical work, they can be used to understand the variability of teachers' responding to students' contributions. To understand a teacher's decision on how to respond to students' contributions we consider that noticing, interpreting, and responding are resources not only available to the individual teacher but also primed by the framing that prior instructional situations can provide to those interactions. The following sketches an experiment that we think could be used to investigate the sources of variability in teacher responding, including individual resources and structural characteristics.

Consider teachers who vary in terms of personal resources (knowledge, beliefs, discursive competence, and experience teaching beginning Algebra), matched in cohorts of 4 individuals who are equivalent on those personal characteristics to make 4 equivalent groups that receive A-control, R-control, A-treatment, and R-treatment items. Each group receives items that depict scenarios described explicitly as happening in an introductory algebra class (A-items presenting cases in an avowedly advanced algebra class and R-items in an avowedly remedial algebra class). In all cases students have already learned how to represent linear functions using tables, graphs, and formulas and participants are told that the goal of the lesson is to develop the idea of inverse of a linear function. In all conditions participants see the teacher present the same mis-en-scene:

Someone comes to the front desk of Parks and Recreations to pay a team's fees for the upcoming softball season. Two other teams came in earlier and paid but you are not sure how fees were calculated. You know there is a league fee per team and a separate fee per player. You see a few handwritten receipts still on the counter. One says 12 players and $554. Another says 17 players and $639. The supervisor has stepped away and cannot be reached. The customer insists on paying now and he was told over the phone how much to pay. The customer gives you a check for $605 and leaves.

After that mis-en-scene, the two control groups see the teacher ask the class "You now need to record the team fee and determine the number of players on the team," while the two treatment groups see the teacher ask “How could you express the relationship between team players and team fees to solve the problem of how many players a team has?”. This last question purportedly frames the problem as one of posing and solving an equation, while the control statement of the problem does not do that.

All groups would then be offered opportunities to respond to students’ ideas ranging from numerical guesses, accounts of incomplete procedures, disorganized calculations, tables, and equations, including equal number of correct and incorrect, partial and complete responses. Teachers are asked to compose how they would respond to each student. A structural conjecture would suggest that teachers would evaluate students’ responses varying in their response depending on the correctness of the answer and the level of the class. An individualist perspective would suggest that individual characteristics of teachers (e.g., beliefs, knowledge of students thinking) would predict their responses to students. A practice-based perspective would suggest that the group which saw the teacher frame the problem as one about equations, would tend to respond to students in ways that privilege equation-based solutions, though recognizing variations depending on correctness and also showing individual variability. The analysis of the various regression models that can be proposed to investigate variables that significantly account for teachers’ responses is then the instrument for coordination between theoretical perspectives in a middle range theory of teachers’ decision making.

Conclusion

The experiment sketched above illustrates one way to combine theories of mathematics teaching: In the context of researching decision making in mathematics teaching, regression analysis of teacher responses to scenario-based assessments can summon the concrete experiences of practitioners in contexts framed by different practical and structural considerations, which can be operationalized using experimental design. While sketchy especially on how to examine teacher’s decisions, the design makes visible the possibility that resources from various perspectives could be integrated to allocating sources of variability to all the factors involved.

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A multifocal lens on qualitative data analysis: an affordance of networking theoretical approaches

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We address the problem of interplay between methods of qualitative data analysis and the networking of theories. When we call a multifocal lens on data analysis an "affordance," we mean a benefit or resource that people may perceive possible when networking theoretical approaches. To situate our argument, we draw parallels between challenges in the fields of mixed methods and mathematics education. Examining the analysis methods in an empirical study of students' conceptions of what graphs represent, we argue that a multifocal lens can help to explain complexities when investigating students' reasoning. Our work contributes to efforts to advance the scope and depth of data analysis techniques employed when researchers network theories.

Keywords: Theories, data analysis, qualitative research, graphs.

Mathematics education researchers have appealed to a plethora of theoretical approaches, and this diversity brings richness to the field (Bikner-Ahsbahs, 2009). By networking, or connecting, different theories, researchers can embrace this diversity (Bikner et al., 2019; Prediger et al., 2008). The networking of theories is more than an intellectual endeavor; it has pragmatic roots, to contend with problems (Bikner et al., 2019). Those problems can encompass the enactment of research in mathematics education. The problem we address is methodological: the interplay between methods of qualitative data analysis and the networking of theories. Our aim is to contribute to efforts to advance the scope and depth of data analysis techniques employed in the field.

We argue that a multifocal lens on qualitative data analysis is an affordance of the networking of theories. By affordance, we mean a benefit or resource that people may perceive to be possible with a particular approach. One reason we use the term "affordance" is because of the reflexivity it implies. Our use is consistent with that of Chan and Clarke (2019), who offered the term "mutual affordance" to describe a back-and-forth relationship between theory and method, one that it is flexible and responsive rather than prescriptive. While Clarke and Chan (2019) address a broader scope of methods, we focus on data analysis. We do this in part because of the crucial role that competent analysis plays in qualitative research (Miles & Huberman, 1994).

The work of theorizing can bring forth many images for researchers. Simon (2009) put forward two images, tools and lenses, as metaphors for ways in which researchers can employ theories. A tool can function for a particular purpose; some tools serve multiple purposes, while others are more specialized. A lens can influence how people perceive a situation; different lenses can result in different explanations of a situation.

We have chosen the term "multifocal lens" to communicate how an analytic lens can make room for multiple perspectives of a single situation. One way to think of each perspective is in regard to a "focal construct" (Chan & Clarke, 2021), on which researchers may focus when analyzing a source of data. For example, researchers may focus on students' thinking and their affect. With a multifocal lens on data analysis, researchers can investigate and coordinate different focal constructs. Multifocal approaches can strengthen qualitative data analysis, by allowing researchers to triangulate within and across different analysis methods (Leech & Onwuegbuzie, 2007).

We contend that researchers' responses to challenges facing the field of mixed methods (Onwuegbuzie & Leech, 2005) can offer insights into the interplay between the networking of theories and methods of qualitative data analysis. We highlight two such challenges, and make connections to the field of mathematics education. The first addresses a relationship between the research methodology (quantitative, qualitative, or mixed) and data analysis methods. This challenge is analogous to Chan and Clarke's (2019) argument that a relationship between theory and method be one of affordance, rather than prescription. The second addresses whether and how researchers may mix quantitative and qualitative methodologies. This challenge is analogous to the networking of theories, as researchers grapple with whether and how theories may be connected (e.g., Bikner-Ahsbahs, 2009; Prediger et al., 2008).

To organize this paper, first we draw parallels between obstacles to mixed methods research and the networking of theories. Second, we explain why we examine interplay between the networking of theories and methods of data analysis. Third, we look at analysis methods reported on Johnson et al. (2020), in which the researchers networked different theories. Our purpose is to show how a multifocal lens on qualitative data analysis can explain complexities in students' reasoning.

Parallels between networking theories and mixed methods

One aim for researchers who network, or connect, theories is to solve problems that demand multiple lenses (Bikner-Ahsbahs et al., 2019). There is a continuum of ways in which researchers may network theories (Bikner-Ahsbahs & Prediger, 2010). On one end of the continuum, researchers can develop understanding of the assumptions underlying different theories. Moving along the continuum, researchers may compare or contrast, combine, synthesize, or locally integrate different theories.

An obstacle to the networking of theories is theoretical competition in response to a quest for coherence in mathematics education. If researchers view different theoretical perspectives to be in competition with each other, the field may appear disjointed (e.g., Prediger et al., 2008). Rather than theoretical competition, researchers who advocate for the networking of theories take a pluralistic approach (Bikner-Ahsbahs, 2009; Bikner-Ahsbahs et al., 2019; Prediger et al., 2008). Meaning, a goal is to interconnect theories, rather than to advocate for the adoption of singular, unifying theories (Bikner-Ahsbahs et al., 2019). This approach can benefit empirical research, by allowing researchers "to gain an increasing explanatory, descriptive, or prescriptive power" (Prediger et al., 2008, p. 169).

An aim for researchers in the field of mixed methods is to solve problems via qualitative and quantitative methods (Onwuegbuzie & Leech, 2005). As with networking theories, there is a continuum of ways in which researchers may mix methods (Leech & Onwuegbuzie, 2009). After employing both qualitative and quantitative methods, researchers may mix these methods within and/or across different phases of the research.

An obstacle to mixed methods research is a perception of dichotomies between qualitative and qualitative methodologies (Onwuegbuzie & Leech, 2005). When researchers identify with only a qualitative or quantitative paradigm, it can create polarization. Onwuegbuzie and Leech (2005) argue for methodological pluralism. Rather than binding methods to a qualitative or quantitative paradigm, they offer a reconceptualization, such that the same type of method may cut across paradigms. For example, researchers may employ exploratory methods from qualitative and quantitative paradigms. Such an approach can serve to dismantle boundaries between qualitative and quantitative research traditions (Onwuegbuzie & Leech, 2005).

The term “pragmatic researcher” describes researchers who embrace methodological pluralism:

> Becoming a pragmatic researcher offers a myriad of advantages for individuals. First and foremost, it enables researchers to be flexible in their investigative techniques, as they attempt to address a range of research questions that arise. (Onwuegbuzie & Leech, 2005, p. 383)

The field of mixed methods is a pragmatic response to the obstacle of a perceived dichotomy between qualitative and quantitative methods. In a similar way, the networking of theories is pragmatic response to the obstacle of theoretical competition in a quest for coherence in mathematics education. One way to conceive of the networking of theories is as “pragmatic theorizing.” A pragmatic approach to theorizing can allow researchers to leverage different theoretical perspectives to contend with researchable problems. Such an approach demands attention to methods, which we discuss next.

Interplay between the networking of theories and methods of data analysis

The networking of theories happens in conjunction with other aspects of research; it is entangled with researchers’ methodological decisions (Bikner-Ahsbahs et al., 2019). Radford (2008) has posited a conceptualization of theories as triplets that include systems of guiding principles (P), collections of methods and methodologies (M), and sets of overarching research questions (Q). From this perspective, theorizing extends beyond assumptions and principles to practical aspects of research (methods and questions). We view the elements of Radford’s triplets, to afford, rather than prescribe each other. While certain methodologies and research questions may be more typical for researchers operating with a certain system of guiding principles, those connections are not lock step. In our view, Radford’s triplet can extend beyond individual theories. In the networking of theories, researchers weigh principles central to different theories (P), examine how methods and methodologies may intertwine with different assumptions (M), and reflect on how theoretical assumptions can impact responses to research questions (Q).

The networking of theories is something more than triangulation via different data analysis methods (Bikner-Ahsbahs & Prediger, 2014). It is a way of employing a multifocal lens on a research setting. Researchers who network theories can employ different theoretical lenses on a single source of data. In turn, different theoretical lenses transform what gets counted as data. Hence, there is an interplay between theory and method. Drijvers et al. (2013) illuminate this interplay in their comparison of methods between two different theoretical lenses that they employed to investigate a student’s work on a computer algebra task.

In qualitative studies, the use of more than one analytic tool strengthens data analysis, because researchers examine a source of data from different viewpoints (Leech & Onwuegbuzie, 2007). We contend that a multifocal lens on qualitative data analysis is one affordance of the networking of theories. This multifocal lens is something more than a collection of different analytic tools, because the foci of data analysis are intertwined with theoretical perspectives. When researchers employ a multifocal lens on analysis, they can relate contributions from multiple analytic methods to illuminate new dimensions of a phenomenon.

A multifocal lens to explain complexities in students' reasoning

In a survey of recent research on students' mathematical thinking, Goos and Kaya (2020) note the increase in theoretical perspectives as the field has grown. They point to the networking of theories as a promising approach to address coherence amidst diversity in theoretical perspectives. To illustrate how a multifocal lens on data analysis can explain complexities in students' reasoning, we look at an empirical study from Johnson et al. (2020). First, we describe the setting of the study and discuss the theories networked. Second, we describe their data analysis methods, and insights gleaned from their approach. Third, we draw connections between theory and method.

Networking theories to investigate students' conceptions of what graphs represent

Johnson et al. (2020) conducted a qualitative study investigating high school students' conceptions of what graphs represent. There were 13 students in the study; each participated in a series of three individual task-based interviews. Students interacted with digital task sequences, and Johnson served as the interviewer. The digital tasks consisted of an animation of a situation, followed by a series of screens in which students could vary one attribute, then another, then both together. Students then could sketch a graph to relate the attributes. For example, one task involved a situation where a toy car moves along a curved track. Near the track was a small shrub. Students were to focus on two attributes: the toy car's distance traveled along the track and the toy car's distance from the shrub. Students could vary each distance, then both together, then sketch a graph relating those distances. Students' work on these tasks served as a primary source of data.

In their study design, Johnson et al. (2020) networked two theories: Thompson's theory of quantitative reasoning (Thompson 1994, 2011; Thompson & Carlson, 2017) and Marton's variation theory (Kullberg et al., 2017; Marton 2015). To argue for the viability of networking the theories, Johnson et al. (2020) identified a key assumption underlying both: researchers and participants bring different perspectives to the research setting, and hence can have different, yet viable goals.

Thompson's theory focuses on students' conceptualizations of attributes as being possible to measure (Thompson, 1994, 2011). Thompson calls this kind of conception a quantity. Per Thompson's theory, a quantity is something more than a unit attached to a number (e.g., 5 "feet"); it is how a student conceives of the attribute itself. For example, a student may encounter a graph that relates two different distances. Employing Thompson's theory as a lens, researchers may investigate what distance means for students, how students might think about measuring distance attributes, or how students might conceive of relationships between different distance attributes.

Marton's theory focuses on students' discernment, or separation, of some feature from an instance of which it is a feature (Kullberg et al., 2017; Marton, 2015). Marton (2015) proposes conditions under which teachers or researchers may engineer opportunities for learners' discernment. First, juxtapose two features, such that each differs with respect to a certain aspect. Second, let one aspect vary, while the other remains invariant. Employing Marton's theory as a lens, researchers may investigate how students discern aspects of graphs, such as a variable represented on an axis.

Employing a multifocal lens to analyze students' conceptions of graphs: Johnson et al. (2020)

Johnson et al. (2020) employed multiple phases of data analysis, following Wolcott's (1994) process of description, analysis, and interpretation. In the first phase, they described what students sketched (or tried to sketch), how students explained their sketches, and students' physical motions related to their sketches. In the second phase, they coded for students' conceptions of what graphs represented. Codes distinguished conceptions of attributes as being possible to measure (e.g., a distance traveled by the toy car) from the physical objects themselves (e.g., the motion of the toy car). In the third pass, they interpreted students' shifts in their goals for graphing, appealing to the different theoretical lenses.

By analyzing for both students' conceptualization (Thompson's theory) and discernment (Marton's theory), Johnson et al. (2020) embraced pluralism in their analysis methods as well as their theorizing. With Thompson's theory (1994, 2011), they identified three goals for students' graphing, which they linked to different conceptions of what graphs represent. With Marton's theory (2015), they distinguished between what researchers intended for students to discern (intended object of learning), what was made possible for students to discern in the task setting (enacted object of learning), and what students discerned as a result (lived object of learning). Looking across both interpretations, a fourth goal for graphing emerged, what graphs should represent. This fourth goal helped explain why some students had persistent conceptions of graphs as representing aspects of physical motion in a situation (e.g., a graph will turn like the toy car).

Drawing connections between theory and method

In Wolcott's (1994) interpretation phase of analysis, researchers strive to make meaning from the data. One way to make meaning is to turn to theory. In Table 1 we show the theoretical lenses, guiding questions, and student goals for graphing (bold) from Johnson et al. (2020). The text in italics addresses how interpretations from different theoretical lenses informed each other in the data analysis. For instance, employing Marton's theoretical lens has illuminated why some goals for graphing are more stable than others.

In the empirical study from Johnson et al. (2020), there is a relationship of "mutual affordance" between theory and method, as put forward by Chan and Clarke (2019). Neither Thompson's theory (1994, 2011) nor Marton's theory (2015) prescribed analytic techniques to follow Wolcott's (1994) process of description, analysis, and interpretation. Yet, the analytic approach allowed for interpretation from multiple theoretical lenses. While Johnson et al. (2020) employed a multifocal lens in the interpretation phase of Wolcott's process (see Table 1), it is not the only possibility. For instance, Johnson et al. (2020) could have conducted parallel passes of description, analysis, and interpretation for each theoretical lens, then made connections across those passes.

Table 1: A multifocal theoretical lens on Wolcott's (1994) interpretation phase of data analysis

Theoretical Lenses	Quantitative Reasoning Theory: Conceptualization (Thompson 1994; 2011)	Variation Theory: Discernment (Kullberg et al., 2017; Marton, 2015)
Guiding Questions	What are students' conceptions of attributes? How do students conceive of what graphs represent?	What is made possible for students to discern? What do students discern?
Three Student Goals for graphing	**(1) Graphs represent observable features of a situation.** **(2) Graphs represent change in a single attribute.** **(3) Graphs represent relationships between attributes.**	*Researchers intended goal 3 for students. All three goals were enacted objects of learning. Only goals 1 and 3 became lived objects of learning.*
A fourth goal	**(4) There are things that graphs "should" do.** *Students' notions of what a graph "should" represent can impact their graphing.*	

Discussion

Our aim was to address interplay between methods of qualitative data analysis and the networking of theories. We illustrated this interplay within Wolcott's interpretation phase of data analysis, putting forward a multifocal lens on data analysis to be an affordance of theory networking. Looking at the empirical study from Johnson et al. (2020), we illustrated how a multifocal lens on qualitative data analysis can explain complexities in students' reasoning.

We drew parallels between obstacles to mixing methods from qualitative and quantitative paradigms and networking theories in mathematics education. Researchers in both fields (Bikner-Ahsbahs, 2009; Bikner-Ahsbahs et al., 2019; Onwuegbuzie & Leech, 2005; Prediger et al., 2008) advocated for pluralism as a response to obstacles which framed challenges in terms of dichotomies (mixed methods) or competitions (mathematics education). With our discussion of Johnson et al. (2020), we intended to illuminate how pluralism can extend to both theory and method.

A look at researchers' methods, in conjunction with theories and paradigms, can be a way to respond to challenges related to a quest for coherence in mathematics education. To attempt to account for some of the theoretical diversity among researchers investigating students' mathematical reasoning, Goos and Kaya (2020) looked at methods employed across studies. Interestingly, they found the methods implemented to be less diverse than the theoretical perspectives employed. Furthermore, looking at methods helped them to draw connections between these studies, and earlier studies, conducted during an era in which there was less diversity in theoretical perspectives employed by mathematics education researchers. This approach dovetailed with Onwuegbuzie and Leech (2005), who recommended a focus on methods, rather than paradigms, to help to overcome perceived dichotomies between quantitative and qualitative methodologies. Although we limited the scope of this paper to qualitative data analysis, we believe it could apply to quantitative and mixed methods as well. Future steps could include principles and/or typologies for methodological approaches in studies in which researchers network theories.

We put forward a multifocal lens on qualitative data analysis as an affordance of networking theoretical approaches. With such a lens, researchers could analyze data sources from different theoretical perspectives, addressing multiple "focal constructs" (Chan & Clarke, 2021) in a single study. With this approach, we aim to advance the scope and depth of data analysis techniques when researchers employ multiple theoretical perspectives in a study.

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Developing an analytical model for mathematical reasoning

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In this paper, I present an analytical model for mathematical reasoning that considers the effect of the learning environment in the formation of mathematical reasoning. The model is developed by incorporating elements from the conceptual framework of mathematical reasoning (Lithner, 2008) into cultural historical activity theory's (Engeström, 2014; Leont'ev, 1974) concept of activity. The principles of both theoretical stances and the networking approach are explained. The resulting model of reasoning processes is presented along with a brief discussion of its affordances and limitations. An example of data analysis is presented for the purpose of illustration.

Keywords: Mathematical reasoning framework, technological environment, cultural historical activity theory.

Introduction

Theory takes an intermediate position between research, problems, and practices (Silver & Herbst, 2007). It facilitates transforming commonsensical problems into research problems, understanding real life practices through research, and enables identifying problems from certain state of affairs. Mathematics education has no grand theory of its own that could guide on all sorts of matters of mathematics teaching and learning and could distinguish mathematical learning from learning in general (Silver & Herbst, 2007). Multiple theoretical perspectives, with mutually exclusive principles, guide research practices in mathematics education. The diversity of theories is considered as a resource as well as a challenge for the development of the field (Artigue et al., 2006; Bikner-Ahsbahs & Prediger, 2014; Ernest, 1998). Networking of theories is a research practice that promotes dialogue between different theories and seeks to find solutions to problems on the intersection of different theories. Lester (2005) suggests that mathematics education researchers should act as *bricoleurs* by adapting ideas from various theoretical sources to deepen understanding of teaching and learning of mathematics as well as to gain practical wisdom about the problems practitioners care about. Silver and Herbst (2007) call for developing mid-range theories that could inform a discrete variety of practices and study the subfields in mathematics education such as individual mathematical thinking, teaching and learning in classrooms, or mathematics teacher education. Lesh and Sriraman (2005) argue for approaching mathematics education research as a design science and call for developing conceptual systems that address the complex learning problems from multiple theoretical perspectives.

In this paper, I provide an example of networking of theories from my PhD research project in which I combine the conceptual framework of mathematical reasoning (Lithner, 2008) with cultural historical activity theory (CHAT) (Engeström, 2014; Leont'ev, 1974) to develop an analytical model of mathematical reasoning. In what follows, I argue for the need of the new model concerning the research aims. After that, I describe the two theoretical resources used and elaborate on the networking of them. I then present an illustrative empirical example for the developed model. At the end, I attend to limitations and affordances of the developed model.

Mathematical reasoning with regard to the learning environment

Mathematical reasoning is regarded as a vital constituent of mathematical learning. Mathematics curricula at different levels advocate for developing mathematical reasoning competence (Alpers et al., 2013; Niss & Højgaard, 2011). Some theoretical positions regard mathematical reasoning as an individual competence (Niss & Højgaard, 2011) while others regard the effect of the learning environment in forming the individual reasoning competencies (Lithner, 2008). However, previous studies consider the role of individual components of the learning environment, such as textbook, teacher, individual tools, and examination (tasks), in forming mathematical reasoning (cf. Granberg & Olsson, 2015; Lithner, 2000; Lithner, 2004; Olsson, 2019). A holistic perspective on learning environment accounting for both tools and social elements in the analysis of mathematical reasoning has rarely been taken up. In this paper, I seek to form an analytical model by benefitting from existing theoretical stances, which could guide the analysis of the role of the learning environment in the formation of mathematical reasoning.

Conceptual framework for mathematical reasoning

In the conceptual framework of mathematical reasoning (Lithner, 2008), reasoning is manifested in a task solving activity, which comprises the following four steps: i) getting a task or a problematic situation, ii) selecting a strategy, iii) implementing the strategy, and iv) reaching to solution. The step of selecting the strategy entails predictive argumentation, which concerns learners' reasons about why the strategy will work. The step of implementing the strategy entails verificative argumentation, which concerns reasoning for why the strategy did work.

The framework characterizes individual learner's mathematical reasoning as imitative or creative based on the anchoring of mathematical arguments. In creative mathematical reasoning, the learner creates a novel reasoning sequence in which the arguments are rooted in properties of mathematical objects. Imitative reasoning is founded in application methods, which are, for instance, given in textbooks or told by the teacher. The predictive and verificative argumentation is based on the authority of sources of information instead of the mathematical properties involved.

The research framework considers the learner's reasoning sequence guided or limited by individual competencies formed in the learning environment. Lithner's framework has also been used to study the effects of individual components of the learning environment such as textbook, teachers, and examinations (cf. Granberg & Olsson, 2015; Lithner, 2000; Lithner, 2004; Olsson, 2019).

Regarding methodology, the main data source are observations of students' solving of mathematical tasks along with written solutions, think-aloud protocols, pre- and post- interviews, and the textbook materials (Lithner, 2008). Multiple sources of data aid in triangulation and ultimately support trustworthiness of the interpretations. The research questions that the framework seeks to answer focus on distinguishing between creative and imitative mathematical reasoning of individual students, required in tasks administered in textbooks and examination tasks (Lithner, 2003; Palm et al., 2011), and facilitated by digital tools (Granberg & Olsson, 2015).

Cultural historical activity theory

In cultural historical activity theory (CHAT), object-oriented activities are arenas for human learning (Engeström, 2014; Leont'ev, 1974). Learning takes place when individuals participate in human activities. An activity is conceptualized as subject-object interaction at three subject-levels: collective subject, individual subject, and non-conscious subject (Engeström, 2014). Each level entails different forms of learning. The collective subject refers to a group of individuals as a whole and learning at this level concerns mastering the whole activity systems bringing changes into the activity systems.

The individual subject carries out actions through which the activity is realized. The individual actions are directed towards goals linked to the object of an activity. To perform goal-directed actions, the individual devises a plan, a tacit representation of the method to reach the goals, termed as a model. The model is regarded as the tool for carrying out the actions. Based on the way the model is selected, the learning is categorized as: a) productive, or b) reproductive. The productive learning happens when the subject finds a new model through careful experimentation. Reproductive learning refers to the subject selecting the model from previously known methods through blind search or through trial-and-error approach (Engeström, 2014).

The non-conscious human functioning refers to performing automatized operations during the activity. The learning at this level refers to formation of automatic operations and relates to tools in use. The tools at this level are the production tools such as writing instruments (Cole, 1996). The object is perceived as a fixed end and the subject attempts to reach to the object by making simple adaptations with regard to the conditions of the tools.

The action-goal and operation-condition layers are interlinked. That is, the actions upon enough practice may become operations and the operations upon alteration of the conditions of execution may rise back to the level of actions. In this sense, the learning at individual and non-conscious levels are intertwined and have implications for each other. That is, the model chosen at the action-goal layer influences the operations and the operations have implications for the action-goal level.

Regarding methodology, Nardi (1996) infers from CHAT the following four methodological aspects. First, the research frame should include time as human activities evolve over long periods of time. Second, the attention should be paid to large patterns and narrow episodic frames should only be used in view of large patterns. Third, multiple data sources should be considered to conceive the activity system from all possible angles. Fourth, the researcher should be committed to understand the subject’s object. The analysis of shorter episodes enables micro-analyses of processes within an activity and is to be linked to macro aspects of the activity at hand.

The common type of research questions concern exploring relationships between elements of an activity system that affect the realization of the activity’s object into the outcome. The individual subject’s functioning can be analysed in view of the conditions of the activity system.

Combining the two theories: A model of reasoning processes through a cultural historical perspective

The analysis of individual cognition lies at the core of the conceptual framework of mathematical reasoning (Lithner, 2008). In activity theory, the individual learner’s acts only make sense in view of other elements of the activity system. In the reasoning framework, the search and implementation of the strategy are the two main steps of the reasoning sequence. These two steps parallel to searching and implementing of the model in CHAT (see Table 1 for other parallel terms).

The reasoning framework regards mathematical reasoning as a product of the learning that already has taken place in a learning environment. Through activity theory, mathematical reasoning can be associated to the ongoing learning in the activity as the model selection links to productive or reproductive learning. Mathematical reasoning can be viewed as woven in the action-operation dynamics of mathematical learning activities.

Table 1: Parallel terminologies in reasoning research framework and activity theory

Mathematical reasoning framework	Activity theory
Mathematical task	Problem
Strategy	Model
Choosing a strategy	Selecting a model in action-goal layer
Implementing the strategy	Implementing the model in operation-condition layer
Creative mathematical reasoning	Productive learning
Imitative reasoning	Reproductive learning

Thus, I argue that the conceptual framework of reasoning and activity theory can be combined to deepen the understanding of the effect of the learning environment on reasoning processes. The mathematical reasoning process model is achieved by putting reasoning steps from the conceptual framework of reasoning into CHAT's concept of activity. The process of achieving the model is depicted in Figure 1.

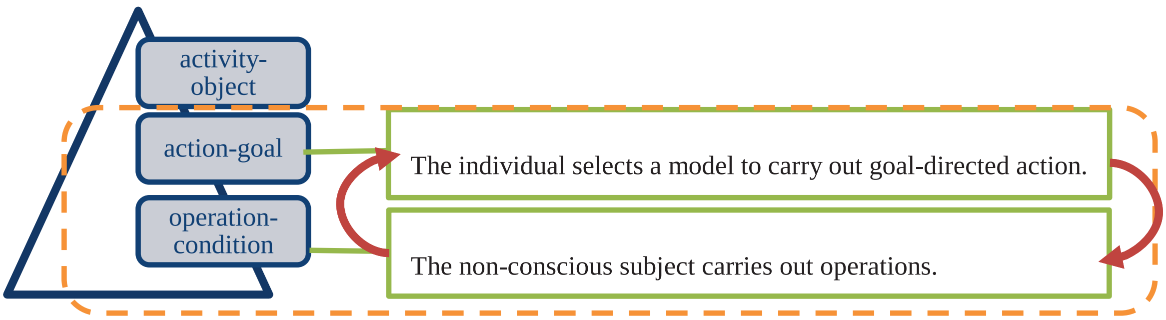

Figure 1: Conceptualizing reasoning processes through CHAT

In Figure 1, the triangle represents the hierarchical layers of the activity (Leont'ev, 1974) while the dotted border shows that the reasoning is entailed in the bottom two layers. That is, the steps of selecting and implementing strategies, the main constituent steps of reasoning when solving a task, are weaved in the action-operation dynamics of the activity (Figure 1). The step of selecting a strategy takes place in the action-goal layer in the form of model selection. The implementation of the strategy takes place in the operation-condition layer (as shown in Figure 1). The fact that the actions and operations in CHAT affect each other implies that the steps of selecting a strategy and implementing the strategy also affect each other (shown with arrows in Figure 1).

The conceptual framework of reasoning provides an outer structure to mathematical reasoning through the two main steps of reasoning. CHAT provides an additional layer to how- and why-aspects of mathematical reasoning and adds an internal dynamic to the reasoning process. That is, CHAT not only allows looking into how reasoning takes place but facilitates understanding why certain actions (e.g., choice of strategies) take place in relation to the condition of the environment (e.g., tools, division of labour). In this way, it allows not only distinguishing between creative and imitative aspects of reasoning to its parallel forms of learning in CHAT, productive and reproductive learning, but also enables analysing the underlying contributing factors from the learning environment. In particular, the action-operation dynamics allow analysing effects of one step onto the other (as shown in Figure 1). Moreover, the role of tools can be analysed clearly and systematically. For instance, the model (Figure 1) enables analysing whether and how the tool operations effect the goals and the selections of models.

In terms of networking strategies (Bikner-Ahsbahs & Prediger, 2014), the practice is regarded as *combining* because the elements from the two theories in concern are fitted together to gain understanding of the empirical phenomenon of mathematical reasoning. In combining theories, the theories do not need to be compatible with regards to their principles as is the case with the theories in consideration — the conceptual framework and CHAT. The resulting conceptual framework is not a theoretical framework but a bricolage for understanding of the phenomenon of reasoning.

An illustrative example of the use of the model

Below, I present an episode from undergraduate engineering students' activity in a digital environment. The data belongs to a larger study, which focuses on student's activities in paper and pencil and digital environment (cf. Kanwal, 2019). This example concerns two students' work on the following task: Solve a definite integral $\int_{-1}^{1} e^{-j\omega t}\, dt$, where j is the complex number with $j^2 = -1$. The definite integral can be solved as follows.

$$\int_{-1}^{1} e^{-j\omega t}\, dt = \frac{-1}{j\omega}\int_{-1}^{1} e^{-j\omega t}\,(-j\omega)dt = \frac{-1}{j\omega} e^{-j\omega t}\Big|_{-1}^{1} = \frac{-1}{j\omega}\left(e^{-j\omega} - e^{j\omega}\right)$$
$$= \frac{j^2}{j\omega}\left(e^{-j\omega} - e^{j\omega}\right) = \frac{j(e^{-j\omega} - e^{j\omega})}{\omega} \quad (1)$$

By using Euler's formula, $e^{j\omega} = cos\omega + j\ sin\omega$ for any real ω, in (1), one gets:

$$\frac{j(e^{-j\omega} - e^{j\omega})}{\omega} = \frac{j[(cos\omega - j\ sin\omega) - (cos\omega + j\ sin\omega)]}{\omega} = \frac{j(-2jsin\omega)}{\omega} = \frac{2sin\omega}{\omega} \quad (2)$$

In the participants' activity, Per and Jan searched on the Internet to make the Maxima code, "j: sqrt (-1); A: e^(-j*w*t); integrate (A, t, -1, 1)". The maxima code served as the model being constructed through the use of Internet with the focus on syntax in Maxima. Involving this model, the task was translated into a Maxima code without the need of taking any integration into consideration.

Later in the implementation phase, Per ran this command, which generated the output, $\frac{\%\mathrm{i}}{\mathrm{e}^{\%\mathrm{iw}}\log(\mathrm{e})\mathrm{w}}$ - $\frac{\%\mathrm{ie}^{\%\mathrm{iw}}}{\log(\mathrm{e})\mathrm{w}}$. The solution is apparently different than the solution given in the textbook ($\frac{j(e^{-j\omega}-e^{j\omega})}{\omega}$ or $\frac{2sin\omega}{\omega}$). The difference is due to the appearance of the two additional terms, %i and $log(e)$, in the Maxima output. In Maxima, $log(e)$ refers to natural logarithm of e. The reason why Maxima did not evaluate $log(e)$ into 1 and produced this term in the output is that Per did not specify 'e' as Euler's number in his input command. The Euler number e is specified by %e in Maxima whereas Per just used 'e'. Also, the term %i represents the imaginary unit, sqrt (-1) in Maxima. Per denoted the imaginary unit with the symbol 'j' in his input command and Maxima replaced it with %i in the output. Replacing $log(e) = 1$ and $\%i = j$ in the output, one gets the textbook solution form.

$$\frac{\%i}{e^{\%iw}log(e)w} - \frac{\%ie^{\%iw}}{\log(e)w} = \frac{j}{e^{jw}w} - \frac{je^{jw}}{w} = \frac{j(e^{-jw}-e^{jw})}{w}$$

Per opened the textbook on the page where the task and its solution were given. Per then remarked on the solution as follows.

131	Per:	That's probably correct, right? It is just that it is written in a way that is crazy hard to…
132	Jan:	Yeah. Probably… Look at the command line for it. What did you write there? [Jan looks at Per's screen whilst preparing to write on his own laptop]
133	Per:	First you have to write… define j.
134	Jan:	Is it so important to write semicolon at the end and that you hold down shift enter?
135	Per:	Yeah. Then you avoid those things… Then it becomes %i and %i is just a symbol that it is…
136	Jan:	It is something.
137	Per:	Yeah, that it is complex…
138	Jan:	Yeah.

Per's suggestive question (131) "that's probably correct, right?" indicates that Per was speculating that the solution was correct. The second part of Per's statement, "it is just that it is written in a way that is crazy hard to…", elucidates that he was aware that the form of the solution was different, and that the participant found the different form difficult to comprehend. Jan also did not seem to be sure, as he replied, "yeah, probably" and started examining the Maxima code by saying "look at the command line for it. What did you write there?". Per responded by reading the command line and pointed out (135) that "%i is just a symbol that it is…", and Jan adds instantly, "it is something". Per completed his sentence by saying that "yes, it is complex" which shows that Per was aware that "%i" represented the imaginary unit in the output. There were no comments regarding the term $log(e)$ and it was probably the main term that was inconceivable. Later, the students ended the discussion by saying that they would ask the lecturer about the correctness of the solution.

This example elucidates only one aspect of the developed reasoning process model, i.e., the manner of model selection affects the model implementation (indicated with the downward arrow in Figure 1). The example shows that the students selected the model in the form of the Maxima code that led them bypass the mathematical operations. The nature of required operations shifted from mathematics to syntax in Maxima. The final solution was correct; however, they were not able to comprehend it due to unfamiliarity with language and symbolism used in Maxima. Based on the implementation phase, the students revisited the initial model to see if it was correct. As the model was a conversion of the integral into Maxima code, the only thing to check was the syntax in Maxima. There were no mistakes in the model (code) and, therefore, the model was not changed. In this particular task, the availability of Maxima shifted the focus away from the integration in itself and the students became more engaged with syntax related issues. The availability of Maxima thus affected the reasoning process in an undesired fashion in this particular example, as the students did not engage with the involved mathematics. Using the developed model (Figure 1) in this example enabled to analyse the reasoning process in the form of individual actions and tool operations rooted in the conditions of the environment. The material conditions of Maxima affected the individual actions and the execution of operations, and hence the reasoning processes, as seen above.

Limitations of the model

The proposed model can only be used in the analysis of reasoning within activities and cannot be used for the analysis of short episodes without understanding the overall activity system. The model also does not guide the analysis of peer interactions while it may enable to consider social actors as division of labour, rules, and the community. Moreover, the reasoning will be interpreted from actions and operations, which may or may not be accompanied by utterances. This requires additional data in terms of stimulated recall interviews in order to make trustworthy interpretations from the data although it will only give access to the activity in an indirect way.

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Understanding mathematical work and mathematical thinking through individuals' actions analyses: a networking approach

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The theory of Mathematical Working Space (ThMWS) provides theoretical and methodological tools to describe and characterize mathematical work. A particular methodology has been developed to analyse actions produced by subjects when they perform mathematical tasks. In this paper, this methodology is introduced by an example of a geometrical task. Then, we show how two theories on knowledge construction based on actions were combined with ThMWS. Finally, we present the contributions drawn from these networking cases which provide insight into mathematical work as well as the thinking of subjects by articulating their knowledge and actions.

Keywords: Mathematical Working Space, mathematical work and mathematical thinking, action and knowledge, methodology.

Introduction

Within the theory of Mathematical Working Space (ThMWS), the understanding of mathematical work is mainly supported by the analysis of the actions, which students and teachers perform in the school context. In this paper, we wonder how to better understand the cognitive dimension of mathematical work and we explore this issue through common analysis of actions with some cognitive approaches.

We first justify why actions can be used as a common key-point for discussion between ThMWS and cognitive theories. After giving some of the key-constructs of the ThMWS used in the paper, we provide an example to introduce the methodology we have implemented to account for the mathematical work of individuals performing mathematical tasks. We then consider how research using both ThMWS and Abstraction in Context (AiC), or APOS theory, has opened up other avenues to describe and characterize the articulations between actions of individuals and their knowledge. It is thus possible to better understand subjects' mathematical work and mathematical thinking.

Articulating mathematical work and mathematical thinking through subjects' actions analysis

The approach of teaching and learning issues using the notion of mathematical work is relatively new in the field of research on mathematics education. Indeed, the great majority of research on teaching and learning refers to the idea of mathematical thinking as CERME conferences indicate through the topics of the working groups on mathematical contents. These topics have long been expressed in the form of specific modes of thinking such as algebraic thinking, geometric thinking, and probabilistic thinking. The focus on "mathematical thinking" urges to attend “thinking” with the real risk of dissolving the mathematical specificity in all the fields, which are concerned with thinking. Indeed, the study of thinking supposes a multiplicity of approaches, which are often far apart from mathematical concerns.

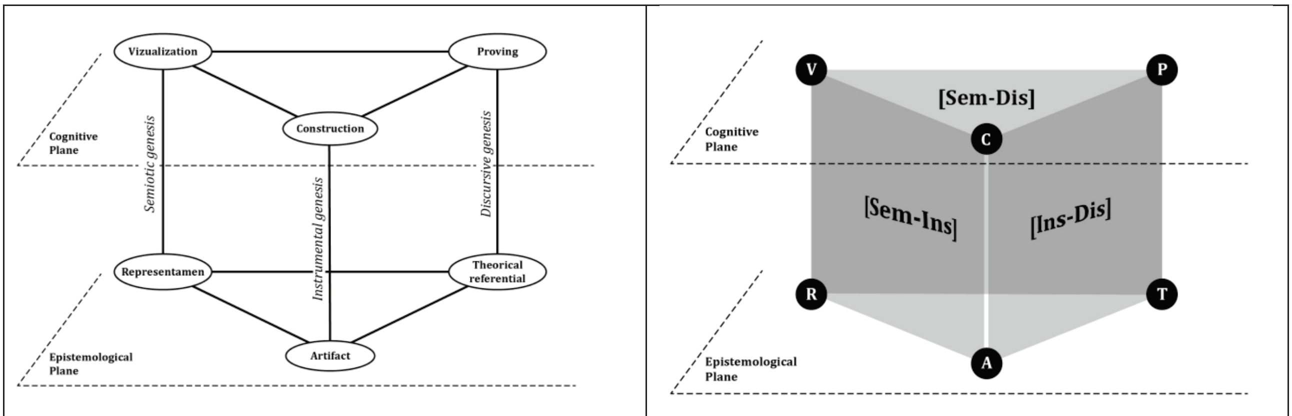
Recently, cognitive sciences have also approached thinking through cognition with the objective of being "the science of thinking". The cognitivist current is itself a composite combination of disciplines such as neuroscience, artificial intelligence, the phenomenology of perception, etc. This current is probably at the origin of the collapse of the primacy of mental representations for the benefit of action in the field of mathematics education. In this view, action is prior to mental representation as the neurobiologist Berthoz (2008) endorses it, and thinking and consciousness relate directly to action and perception. This viewpoint is not new in the field and was already pointed out in his *genetic epistemology* by Piaget who considered that mathematical thinking is not “pre-formed but a creation constantly pursued by actual and new actions which open a new set of possibilities hitherto inconceivable” (Piaget, 1957, p. 34). The support of cognitive sciences gives this approach a new visibility.

This change of perspective on mathematical thinking is valued from the point of view of ThMWS and more generally of studies that favour an entry through mathematical work. In this case, the focus on work naturally places action as a key theoretical and methodological tool in the study of mathematics teaching and learning in school contexts. However, considering mathematical thinking and mathematical work as complementary seems to be necessary to account for the complexity of school mathematics. In particular, some cognitive theories, such as AiC and APOS, focus on the construction of abstract mathematical knowledge and can both complement and benefit from approaches to mathematical work that focus more on the outcomes and processes of work.

Describing and characterizing mathematical work: the ThMWS approach

The theory of MWS: motivations and issues

In the theory of MWS (Kuzniak et al., 2016), mathematical work is considered as intellectual work that involves significant cognitive activity of which the orientation and purpose are defined and supported by mathematics. In this framework, mathematics is envisaged as a specific human work and therefore, mathematical work is seen in both its cognitive and epistemological dimensions.



The Mathematical Working Space (MWS) | The vertical planes of the MWS

Figure 1: The MWS diagrams

To grasp the specific activity of students solving problems in mathematics, the two epistemological and cognitive facets of mathematics are present and articulated in two planes: one of an epistemological nature in close relation to the mathematical content; the other of a cognitive nature, related to the visible action of the individual solving problems (Figure 1, left). Three components in interaction are introduced for the purpose of describing the work in its epistemological dimension: a set of concrete and tangible signs (R), a set of artefacts (A), and a theoretical system of reference (T). The second level is centred in the subject; three cognitive components are introduced as follows: visualisation (V) related to deciphering and interpreting signs; construction (C) depending on the used artefacts and the associated techniques; proving (P) conveyed through discourses producing validations, and based on the theoretical system of reference.

Analysing mathematical work through ThMWS aims at tracking down the process of bridging the epistemological plane and the cognitive plane in accordance with three different geneses: An instrumental genesis (Ins) related to the use of artefacts, a discursive genesis (Dis) related to theoretical part of work, and finally a semiotic genesis (Sem) that expresses the importance of signs for mathematical work. The elaboration of mathematical work is perceived as the harmonious networking and intertwining of these geneses within the vertical planes of the diagram (Figure 1, right).

To achieve the description and characterization of mathematical work, studies performed within ThMWS theory are based on the analysis of the subject's various actions. Thus, the methodology of studying subjects' actions is central in the approach and is developed in the next section.

A cognitive analysis of subject's actions: The example of Alphonse's Parcel

Our starting point is that cognitive activities of a subject at work are not directly observable but they can be deduced from the visible actions of the subjects when they are performing mathematical tasks. The methodology we use must both consider mathematical content and give access to subjects' cognitive processes. We have chosen to introduce our methodology by an example, which is used along the text to show the combination of the different theories involved in the networking study: Alphonse's parcel (Kuzniak & Nechache, 2021). The task statement is given in the form of a text to be read.

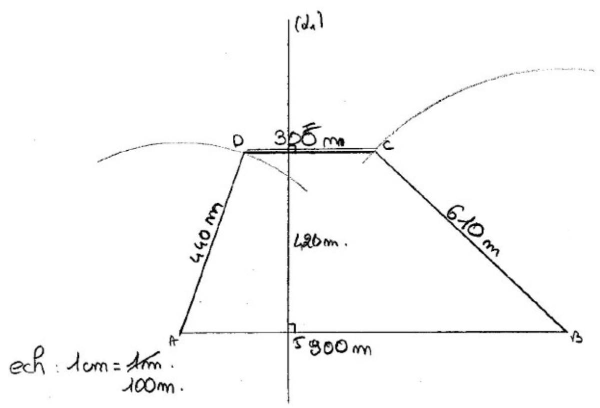
> Alphonse has just returned from a trip in Périgord where he saw a parcel of land in the shape of a quadrilateral that had interested his family. He would like to estimate its area. To do this, during his trip, he successively measured the four sides of the plot and found, approximately, 300 m, 900 m, 610 m, 440 m. He's having a hard time finding the area. Can you help him by showing him the method to be followed?

Data given in the task does not enable to find a unique solution to the problem because quadrilaterals with the same side lengths can have different forms and areas. However, the majority of students used the following very common Theorem in Act (TiA): figures with the same perimeter or the same side lengths have the same area. The main objective of the whole didactic situation explicitly destabilized the students and helped them to banish this false Theorem in Act.

To give a general overview of our methodology, we provide an analysis of Francis's solution of the problem. It is based on a corpus of data collected during the resolution of the mathematical task.

This corpus consists of his written and oral productions. We start dividing his activity into episodes. Each of the episodes is related to a sub-task self-prescribed by the student to achieve the global task: it includes a sequence of mathematical actions that lead to the realization of the sub-task. All the actions are analysed in detail by using ThMWS. The MWS diagrams are also used to visualize the circulation of work during the resolution of the task (Kuzniak & Nechache, 2021).

Table 1. Francis' work division in three episodes each with two actions.

Episode 1: Construction of the figure	Episode 2: Justification of the nature of the figure
	I took the large base 900 m then from the two ends of the 900m with the compass, I made 440 on one side and 610 on each side and then with the ruler I tried to find the 300 with the two arcs of circle. I drew a perpendicular to the large base and this line perpendicular to the large base was also perpendicular to the small base of 300m and as two lines are perpendicular to the same line, they are parallel so it makes a trapezoid.
Action 1: Choice of scale Action 2. Construction of the figure	Action 3. Francis' explanation of his construction Action 4. Francis' description of the fact that it is a trapezium.
Episode 3: Determining the area of the figure. Action 5: Measurement of the height h of the trapezoid Action 6: Calculation of the area from the formula *(B+b). h/2*	

Then a bottom-up analysis gives a summary overview of the different episodes, and helps to deduce the logical organization of Francis' mathematical actions. The analysis seeks to identify how the work-generating processes interact within the components of MWS. From the data, we surmise that mathematical work produced by Francis is triggered by the referential and initiated in the [Sem-Ins] plane to construct a convex quadrilateral with the shape of a trapezoid using technological tools. These tools are used by the student as semiotic tools to collect data and make measurements. Construction of the quadrilateral is then justified with the use of theoretical tools characterizing the trapezoid. The formula for calculating the area of a trapezoid (technological tool) is applied to produce the result (in this case, the area of the land). To conclude the analysis, observations made during the two previous steps are used to characterize the mathematical work. In Francis' case, we may consider that his work circulates through the different geneses and planes of the MWS diagram. He uses drawing and measure tools to support his reasoning and argues about a particular figure, so his work is compliant with a geometrical paradigm based on measurement and drawing

(Geometry I, see Kuzniak & Nechache, 2021). Finally, his mathematical work is not mathematically correct, as the result is false and based on a construction of an inappropriate figure and TiA theorem.

The description above provides the most accurate account possible of the student's choices and actions. However, it is sometimes difficult to be sure, of what really triggered and motivated these actions. Some of Francis' actions are material and easy to interpret, such as constructing the trapezium, but others are more challenging. Why did he choose to use a scale for the construction and then measure in the drawing? When did he focus on the trapezoid? Why didn't he finally see his mistake? To progress with the interpretation of subjects' actions, we explore other methods of action analysis based on theories, which are rather focused on mathematical thinking.

Combining ThMWS with cognitive theories

Several studies have combined ThMWS and cognitive theories. In this paper, we only refer on the studies with AiC and APOS. For each research, we first present its general purpose and introduce the method, which the authors used. Then, we apply this method to Alphonse's task and show the new contribution that can be drawn from the approaches to advance the analysis of Francis' actions and progress in its understanding.

A joint study between the AiC and MWS theories

In their study of the development of functional thinking of students working in four different MWS, Psycharis et al. (2021) combined Abstraction in Context (AiC theory, Dreyfus et al., 2015) and ThMWS. Using a task on the optimal shape of a gutter, they analysed students' progress in their understanding of covariation by having them work successively on four gutter models in different environments (sheet of paper, DGS, etc.). Each of these models is associated with a different MWS. The connections between these MWS are described and explained thanks to the MWS geneses involved. This first analysis through the MWS theory is completed with the AiC theory to account for the processes of constructing abstract mathematical knowledge. In this theory, abstraction is defined as an activity of vertical reorganization of previous mathematical constructs into a new abstract construct. According to the authors, the process of abstraction passes through three stages of knowledge construction shaped by three epistemic actions: recognizing (R), building-with (B) and constructing (C). These actions are nested. They allow tracing the process of knowledge construction and are used as a methodological tool for analysis.

Using AiC for Alphonse's solution of the task, we first focus on *recognizing*. If we look again at episode 1 (figure construction), we surmise that the student *recognized* a construction problem as a previous meta-construct (R0) for which he knew mathematical knowledge involved in the figure construction. While constructing, using the given side lengths but ignoring the word "successive", he recognized (R1) a partially correct construct, a trapezoid, and with it its area formula, another previous construct (R2). Thus, he realized its shape by an approximation and calculated its area via measuring, the latter being a building-with (B) action built on the recognized constructs. The mathematical object "trapezoid" determines the follow-up of Francis' work and as it is only partially correct, he cannot construct the expected knowledge about the various areas of quadrilaterals having the same side lengths –construction of a new construct (C). In the example of the work carried out by Francis, the mathematical object named “trapezoid” is considered under different aspects during the completion of the task and referring to the three components of the epistemological plane: its drawing aspect (representamen), the material artefact aspect (ruler, compass, square) or symbolic artefact (area calculation formula) associated with the trapezoid, and its property and definition. All these elements constitute what we call the Epistemological Entity associated to the trapezoid overlapping with what is called Context in AiC.

The subject’s visualization of the "trapezoid" object as being a quadrilateral with two parallel lines, his construction of a figure and the area with techniques associated with the artefacts to develop the trapezoid, and his justification of these constructions by citing the properties that characterize the trapezoid are all part of what we name the Cognitive Unit associated with the mathematical object “trapezoid”. Identifying the existence and role of the Cognitive Unit and Epistemological Entity associated with the mathematical object “trapezoid” enables to characterize the mathematical work and understand mathematical thinking in terms of epistemic actions developed by AiC.

A joint study between the APOS (Action, Process, Object, Schemas) and MWS theories

In her research, Camacho (2021) uses the APOS (Arnon et al., 2014) and ThMWS together to study the linear algebraic concepts of eigenvalues and eigenvectors in relation to their geometrical representation in the plane. This study helps to further consider *action* by underlining the diversity of meanings of this term. Indeed, in the APOS theory, which assumes the heritage of Piaget, the term Action is one of the basic terms of the theory. The acronym APOS stands for Action, Process, Object and Schema. The action to which it refers is a mental structure that must be distinguished from "material action", which Dubinsky interpreted as actions that are performed by a subject but are external to the subject (Arnon et al., 2014, p. 7). From a set of actions on objects, the subject can derive a property arriving at a more general Action that allows passing from Action to Process through the mental mechanism of *interiorization.* The mechanism of *encapsulation* then transforms the Process into an Object as a cognitive entity. A Schema is then build by acting on the mathematical object. In her research methodology, Camacho applies an approach, specific to APOS theory, which consists of discovering the cognitive functioning of subjects according to their failure or success about an Object in some mathematics tasks. It is then a matter of recognizing, which of the components of the quadruplet (APOS) is operative or not. In Alphonse task, the starting point of the analysis is an initial reflection on the mathematical Object, with its Schemas, which is at stake in the situation. For us, it is the Area of a quadrilateral (Object) and this area is associated to a decomposition by triangulation (Schema) or more generally to the triangulation of any polygon, (Kuzniak & Nechache, 2021). Francis however, activates another scheme acting on another mathematical object: he constructs a trapezoid by approximation and calculates its area with a formula via measuring. The actions as conducted do not lead to processes of building the area of a quadrilateral with the given side lengths because he elaborates the schema of an already existing mathematical object (trapezoid area), which is insufficient for the task.

In APOS, authors seek to elaborate the genetic decomposition of the Object of concern to be more precise in their analysis. Camacho elaborates the genetic decomposition of the Object "collinearity" within a digital environment and in terms of the different MWS geneses. When we analyse Francis' work with the MWS, we may notice that he relies on his old and well-established knowledge of the trapezium of which he knows both the different elements (signs, artefacts, properties) in relation to what we referred to as the Epistemological Entity associated with the trapezium. He also knows the cognitive processes related to the use of the corresponding cognitive unit. He knows well the Object "trapezoid", unfortunately, it is not the right Object, which should be used to perform this task about the area of a generic quadrilateral. The case of Francis shows that students do not really regulate their work once they have embarked on a solution that seems to be in line with their knowledge. An illusion of knowing-how that fosters certainty in action emerges. In the example, it was however possible to see that the order of the sides used in the construction was not the one indicated by the statement (semiotic control). It was also possible to see that different quadrilaterals fulfilled the same condition since Francis searched, by approximation, among these figures the one with parallel sides (semiotic and instrumental control). An internal control of the instruments (instrumental control) should have been based on another construction without approximation. Finally, it is interesting to see that his proof reasoning is biased to reach the desired result (discursive control).

Conclusion and perspectives

All of the studies we reviewed emphasise that ThMWS and its organization around the three geneses was an important structuring element in the analysis of actions and helped to describe and better understand the relationship between cognitive and epistemological aspects. Conversely, the triadic and prismatic geometric structure of the MWS generates theoretical notions that are not always easy to interpret as a tool for studying issues of mathematics teaching and learning. However, these joint studies on students' mathematical actions focussing on cognitive aspects enrich and flesh out a number of emerging notions in ThMWS. In particular, they have provided us with a better understanding of the importance and value of the following ideas:

1. The Epistemological Entities and Cognitive Units account for the importance of knowledge and cognitive functioning of individuals who perform a task using particular mathematical objects. To achieve description and characterization of mathematical work, which captures adequately interactions between cognitive and epistemological aspects, it is important to make visible the mathematical knowledge involved in the solution by identifying precisely the components of the Epistemological Entity and the Cognitive Unit associated with a mathematical object

2. These Cognitive units and Epistemological Entities depend on forms of work implemented, which can be described by different paradigms associated with the teaching contents. Based on AiC epistemic actions and on APOS contributions, it is thus possible to describe how the observed work conforms to particular modes of mathematical reasoning.

3. Finally, we have also made progress in understanding the regulation and control of outcomes through the importance of knowledge and cognitive repository as tools for regulating actions. However, we also met the attracting force of an “illusion of knowing” on acting.

The question of actions regulation and control is a research direction that we plan to develop further. Controls and regulation questions are not new in the field of mathematics education, it has existed since at least the first research on problem solving, seeking to make students more autonomous. Particularly Arzarello and Sabena (2011), who also emphasise the variety of actions implemented by students and the teacher, have recently addressed this. They have analysed the way in which gestures, considered as a semiotic resource, intervene in the learning and teaching process of mathematics. This extension of research to gestures and the body raises the question of how to analyse and identify their real impact on students' actions and beyond on their mathematical work and thinking. We are now engaged in a new project (E-ESMEA) dealing with enacting in astronomy education. We are mainly interested in the mathematical part of the project: relations between proportionality and velocity. The project focuses on the concept of embodied action, which assigns a key role to perceptual and bodily processes in the formation of abstract concepts and knowledge. The sensory-motor engagement of the body carries these processes in situations out that constitute the functional system, on which actions are based. From this study, we expect that it will help us to identify the limits and the right levels of analysis of mathematical action that are really necessary to advance in the mastery of mathematical work and in the learning of mathematical knowledge. This is, in our opinion, a fundamental question to develop a didactic approach in phase with the school reality.

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Methodological Aspects in the Theory of Mathematical Working Spaces

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Keywords: Methodological tool, describing mathematical work, characterizing mathematical work, forming mathematical work

The theory of Mathematical Working Space (Kuzniak & Nechache, 2021), provides theoretical and methodological tools to study three main issues on mathematical work: its description, its characterization, and its formation and/or transformation. Description and characterization of mathematical work help to understand and then to transform it, especially when it is not correct. We intend here to show how focusing on these three issues can help to design didactical situations based on the theory of Mathematical Working Space (MWS).

Describing mathematical work

Describing mathematical work aims at capturing the work undertaken in an educational context. The description is based on the constructs of MWS theory and has an explanatory purpose.

Describing the development of mathematical work within a mathematical domain

In their research on the teaching of geometry, Kuzniak and Nechache (2015) used the theory of MWS to describe the different components and processes of the work as they emerged during a teaching sequence (consisting of 5 sessions) on the notion of the circle in Grade 4 - 6 lessons. In their study, they introduce the use of a "comics" as a set of MWS diagrams to give a global and quick vision of the development of mathematical work throughout a teaching sequence.

Describing the circulation of work between different mathematical domains

Derouet's (2017) research on the teaching of continued probability distributions in high school classes to describe the interactions and circulations between mathematical domains and subdomains. She used of coding (red for what is the students' responsibility, green for the teacher) combined with the MWS diagram. This enabled to highlight the various transitions between different MWS associated with three domains: Probability, Statistics, and Integral Calculus during the completion of task by students (high school).

Describing the subject's personal work

To provide an account of the mathematical work actually produced by student teachers, Kuzniak and Nechache (2021) designed an analysis method based on a division of the students' activity into episodes. Each of these episodes comprises a sequence of mathematical actions, which lead to achieve a subtask. In this view, each episode corresponds to a sub-task self-prescribed by a student and includes the sequence of mathematical actions used to solve the task. All these actions are then analysed in great detail using the theory of MWS.

Characterizing mathematical work

The objective of characterizing mathematical work is to identify certain invariants from the description of work.

Cognitive unit associated with a mathematical entity

To Kuzniak and Nechache (2015), the mathematical work is structured according to a cognitive unit associated with a mathematical entity. In this way, they differentiate the cognitive and epistemological aspects of a mathematical object. The mathematical entity associated with a mathematical object is defined as being a triplet of elements of the epistemological plane, composed of representamen or signs, artefacts, and the mathematical properties and definition associated with the object. This Cognitive Unit is generated from the interaction of the three MWS geneses: instrumental, semiotic, and discursive (Kuzniak & Nechache, 2021), which trigger the cognitive processes.

Forms of mathematical work

Kuzniak and Nechache (2021) sought to characterise forms of mathematical work developed by students preparing for a master's degree. The authors gave three criteria for assessing the process, outcome, and circulation of mathematical work in the MWS: compliance, correctness, and completeness.

Forming mathematical work

Research on the formation or transformation of mathematical work aims to build a new structure for the epistemological plane of work or to provide a new structure for the cognitive plane. The method used by Reyes (2020), to develop and form mathematical work around the notion of 'function' to make it more accessible to students, is based on task design using the theory of MWS. Reyes uses MWS theory to account for the flow of mathematical work produced by students (using the MWS diagram) and to analyse how the work was completed.

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Struggling with the problem of complexity or how to (not) arrive at a general theory of what happens in mathematics classrooms

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For about half a century now, it has been known and widely discussed that the field of study of mathematics education research is extremely complex. In mathematics classrooms, mathematics emerges from a complex interplay between bodily, cognitive, and social processes; and the crucial question for mathematics education research is: How can we develop methodical and theoretical means to 'tackle' these complexities? Given this challenge, which we will call the complexity problem, this paper aims at two things: First, to indicate at what point we think two well-known theories, namely Ernst von Glasersfeld's Radical Constructivism and Anna Sfard's Theory of Commognition fail at the complexity problem; and second, to attempt to point out an alternative route to a general theory that might be able to account for the complex dynamics between the bodily, cognitive, and social dimension of mathematics classrooms.

Keywords: complexity, transdisciplinarity, the social turn, commognition, radical constructivism

Introduction: The complexity problem

For about half a century now, it has been known and discussed that the field of study of mathematics education research is "characterized by an *extreme complexity*" (Steiner, 1985, p. 11). In order to provide some evidence for this thesis, it is, we think, helpful to take a look at the 'prototypical' object of study – the mathematics classroom – and ask: What does actually go on in there?

First of all, in every mathematics classroom there is communication that takes place. Teachers and students talk about mathematical topics, about mathematical problems and their solutions, and of course also about extra-mathematical issues (e.g. about an upcoming class trip). Regardless of how much the *particular forms* of communication may vary from one classroom to another, one will always observe *some* sort of communication processes between teachers and students. So, the first aspect that must be considered when answering the question above is communication – or, more general: *the social dimension* of mathematics classrooms. Now when social processes unfold, at the same time, numerous cognitive activities take place. Both teachers and students are presumably thinking, imagining, expecting, or fearing something when participating in classroom communication. Thus, besides the social dimension, there is also *the cognitive dimension* of mathematics classrooms that must be taken into account. Finally, a third area also comes into play: *the bodily dimension* of mathematics classrooms. Whether teacher or student, everyone participates with his or her body in the classroom – be it sitting, standing, walking or lying – and, for instance, reacts completely different to social events on the neuronal level than on the cognitive level.

So, one is always confronted with the *simultaneity* of at least three different kinds of processes – namely, the simultaneity of bodily, cognitive, and social processes – that are involved in one way or another when mathematics is constituted in the classroom. Now the crucial point about these three kinds of processes is that their simultaneity rules out the possibility of one of them being considered ‘the’ cause of the other two. It is rather that each of the three dimensions obeys its own logic. Each dimension (and that justifies the use of this metaphor) is able to vary independently from the other two dimensions. And yet it is precisely this complex web of bodily, cognitive, and social processes that one inevitably encounters in the study of mathematics classrooms.

This brief description of what happens in mathematics classrooms leads to a more general problem, which lies at the core of research in mathematics education: How can we, as mathematics education researchers, handle the extreme complexity of our field of study? How can we design theoretical and methodical tools which can account for the complex relationships between the bodily, cognitive, and social dimension of mathematical activity in its many facets and forms? It shall be this fundamental challenge to research in mathematics education which we will call *the complexity problem*. Once the complexity problem has been identified, it becomes apparent in what a difficult situation research in mathematics education finds itself in. For to arrive at a theory capable of accounting for the complexities of mathematical activities in educational contexts, the analytical tools of psychology, sociology, or biology are not sufficient. If we ask, for instance, for the conditions of success of a teaching-learning process in a mathematics classroom, we ask about a process that lies perpendicular to the boundaries between these three disciplines: Mathematical teaching-learning processes are processes that require cognitive activities on the part of both students and teachers, and these activities, in turn, depend on neural activities and other bodily processes. If one now continues and asks how teachers can teach and students can learn what they are taught, one encounters the mediating function of communication. Teachers affect the cognitive activities of their students and students affect the cognitive activities of their teachers by participating in communication. To ask about the conditions of success for such complex processes, thus, means to choose a unit of analysis that goes beyond the respective ‘fields of responsibility’ of the three disciplines. Because here it is about the question of the possible *relationships* between the social, cognitive, and bodily dimension of classroom events.

But how do we deal with this situation in our field? Is the complexity problem already solved? Are there theories that are not only able to describe either the bodily, the cognitive, or the social dimension, but all of them together with their interrelationships? And on what grounds can such a general theory be built? These are precisely the questions we would like to explore in this paper.

From the cognitive to the social dimension and back again

We begin with a brief look at the history of mathematics education research and discuss how two theories – Glasersfeld’s *Radical Constructivism* (1) and Sfard’s *Theory of Commognition* (2) – struggle with the complexity problem. By choosing these two much-discussed theories, we aim to show that the complexity problem is indeed of key importance for research in mathematics education. Thereby, we will focus on one aspect of the complexity problem, namely, how to theorize the relationship between the cognitive and the social dimension of teaching and learning in mathematics classrooms.

(1) *Radical Constructivism* is based on the assumption that only the mind of an individual can be considered as the ‘bearer’ of knowing and knowledge. To Glasersfeld, all processes of knowing –

and thus also: all knowledge structures – must be located in the mind of an individual. But how does one get from this focus on cognitive activities of the individual to the social dimension? Against all critics who accuse Piaget of not paying attention to the social dimension, Glasersfeld argues:

> But the child's experiential world also comes to contain other people, and the almost constant interaction with them is an even richer source of perturbation and consequent accommodations. Piaget has stressed many times that the most frequent cause of accommodation is the interaction, especially linguistic interaction, with others. Yet he is often criticized for not having taken into account the social component. (Glasersfeld, 2003, p. 66)

If a child calls a rectangle 'square', then it is corrected in communication; and this communicative correction then leads in many cases to a *perturbation* (i.e. to a disappointment of its expectation that the word 'square' fits these environmental circumstances) and then perhaps to an *accommodation* (i.e. to an alteration of its knowledge base through an adjustment of this expectation). Thus, without a doubt, it is possible to account for the social dimension from the standpoint of Radical Constructivism. But the social is considered only insofar as it appears on the 'screen' of an individual mind. Social events can be investigated for their causal contributions on the individual's cognitive development. They have the character of an *external* 'source of perturbation', which may then lead (or not lead) to an *internal* reorganization of what an individual knows and believes. Hence, Radical Constructivism is and remains a purely psychological perspective on the social. The 'unit of analysis' is the individual knower and his or her processes of knowing.

It was *this* unit of analysis that a number of researchers objected to in the second half of the 1980s. The representatives of this new 'movement', which Stephan Lerman called 'the social turn in mathematics education research' a decade later, no longer had in mind the study of the social from the perspective of an individual mind.

> The *social turn* is intended to signal something different; namely, the emergence into the mathematics education research community of theories that see meaning, thinking, and reasoning as products of social activity. This goes beyond the idea that social interactions provide a spark that generates or stimulates an individual's internal meaning-making activity. (Lerman, 2000, p. 23)

From this standpoint, the social should thus be understood from within itself. To account for the fact that the social obeys its own logic, the unit of analysis was shifted from the cognitive to the social dimension. Although familiar concepts such as knowing, learning, or meaning continued to be used, these concepts were no longer tied to the mind of the individual, but were located in the social. The question was no longer, 'How do social events influence the constructions of an individual mind?'. But rather, 'How is the individual mind participating in the social?'. Researchers of the social turn began to study cognitive activities from the perspective of the social, and the crucial question then became whether the individuality of the individual could still be accounted for in this way:

> A major challenge for theories from the social turn is to account for individual cognition and difference, and to incorporate the substantial body of research on mathematical cognition, as products of social activity. (Lerman, 2000, p. 23)

The challenge for Lerman, then, is to make the insights of psychologically inspired theories accessible again from the sociological perspective without having to return to a primacy of the individual mind. The question that has emerged from the social turn is therefore: If one apparently cannot get from a theory of mind to a substantial theory of the social, does it perhaps work the other way around?

(2) It is, we believe, Anna Sfard who has most decisively pursued this question with her *Theory of Commognition*. Sfard's theoretical program can be read as a systematic exploration of the question of how far one can get with a theory of mind when starting from a theory of the social. In the introduction to her book, Sfard writes:

> In this book […] thinking is defined as the *individualized version of interpersonal communication* – as a communicative interaction in which one person plays the roles of all interlocutors. The term *commognition*, a combination of *communication* and *cognition*, stresses that interpersonal communication and individual thinking are two facets of the same phenomenon. (Sfard, 2010, p. xvii)

The key word for a proper understanding of this theoretical program is: 'defined'. Sfard does not claim that there is no need to distinguish empirically between thinking and communication, between the cognitive and the social dimension. Rather, her thesis is that one can *define* the *concept* of thinking by means of the *concept* of communication. Sfard does not make a statement about the empirical world, but about the theoretical description of it. She claims that one can get to a substantial theory of mind from a theory of the social. For this, Sfard argues, one can start with a concept of communication and then characterize thinking as self-communication–more precisely: as an 'individualized version of communication in which one person plays the role of all interlocutors.' But it is precisely the extraordinarily clear way in which Sfard presents her theoretical program that also reveals a potential weakness: Thinking is *by definition* declared to be a special case of communication. If one can now come up with an example of a cognitive function that is different *in type* from communication in Sfard's sense, then the theoretical program is, at best, incomplete. Sfard herself states: "In my case, no such instance comes to mind" (Sfard, 2010, p. 82). In contrast, we believe that the *cognitive function of perception* can be deployed to generate such a counterexample. Sfard admits that she is "prepared to compromise and leave the more primitive form of perceiving, that which leads to immediate instinctive reactions, out of the realm of thinking" (Sfard, 2010, p. 82). This move, however, which may seem 'generous' at first glance, undermines, we believe, the whole theory of mind. For it is precisely the primitive form of perceiving in which all higher cognitive functions – especially all those functions that presuppose the use of signs, such as all sorts of mathematical thinking (Duval, 2006) – are ultimately founded.[1] Although a comprehensive development of the counterexample is beyond the scope of this paper (see Lensing, 2021, pp. 43–52), we will at least hint at how the argument runs: Sfard defines a communicational action recursively as an action A which is followed by an action B so that A is interpreted by B as being an action *about* an object (cf. Sfard, 2010, pp. 86–89). A communication is thus seen by Sfard as an operation that

[1] It was probably Edmund Husserl (1992) who showed most impressively in *Experience and Judgement* that every theory of logical and mathematical thinking finds its ultimate support in a theory of experience (cf. §1-14, especially: §10).

always processes a distinction. Action A must be ‘seen’ by action B in a dual way, namely as a communication about a certain object. In a second step, it is then argued that elementary perceptual processes are not processing distinctions. For example, one sees a figure in front of a ground and not the distinction between figure and ground. It is then concluded that the cognitive function of perception cannot be characterized as a specific mode of communication in Sfard’s sense.

The social turn had emerged in mathematics education research around the problem that one could not get from a theory of mind to a substantial theory of the social. However, as we have argued in this section, more recent attempts to travel the same route in opposite direction do not seem to be completely successful either. But if there is no clear evidence so far that the path is viable in either direction, what should we do then? Should we consider ourselves defeated by the complexity problem and accept that, while we may well be able to study certain *aspects* of what happens in mathematics classrooms, we will probably never make the *whole picture* accessible to theoretical description?

We believe that there is in fact another way out. It comes into sight if one asks: What do the two aforementioned attempts at bridging the gap between the cognitive and the social dimension have in common? Both, Glasersfeld as well as Sfard, begin their theoretical endeavors by aligning themselves with a particular discipline: psychology in Glaserfeld’s case, sociology in Sfard’s case.[2] While Glasersfeld takes a psychological perspective and thus does not get through to the social dimension, Sfard starts from a sociological perspective and does not get a grip of the cognitive dimension. And it is this beginning, the alignment of one’s theoretical program to one particular discipline, be it psychology or sociology, that we believe to be undermining the possibility of arriving at a general theory. To put it bluntly: *Developing a general theory that can actually ‘grasp’ the complexities of mathematics classrooms must be considered as a transdisciplinary endeavor from the very beginning*. However, such a theory can only be developed from a theoretical standpoint that is not associated with any of the classical disciplines. But how can this be possible? What ways of theorizing might lead to such a transdisciplinary theory?

The formal method: Following the example of modern mathematics

In order to address those questions, we believe it is helpful to take modern mathematics as an example and adopt a theorizing strategy that has revolutionized the field of mathematics over the last two centuries. We will call this theorizing strategy the *formal method*. Now what is the formal method and how can it be employed for theorizing in mathematics education research?

In his or her undergraduate studies, every mathematics student learns concepts from abstract algebra, order theory, topology, measure theory, and so on these days. These theories provide concepts (e.g. the algebraic concepts of group, ring, and field) that allow for relating the seemingly most remote

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[2] That we make this assignment arises from the unit of analysis that the two theorists choose as their starting point: While Glasersfeld chooses the individual knower and his or her processes of knowing as his unit of analysis (= psychological perspective), Sfard starts from the concept of communication and defines it in such a way that communication can only emerge through a certain kind of recursive linking of the actions of at least two different individuals (= sociological perspective).

areas of mathematics (e.g. permutations, geometric transformations, and numbers). Although these *new concepts* have become part of the standard repertoire of a mathematician, surprisingly, it is rarely stated explicitly from which *new way of theorizing* those concepts have actually emerged.

It was most likely Edmund Husserl who, in 1900, in the first volume of his *Logical Investigations*, first provided a satisfactory answer to the question of what was new about these revolutionary developments in mathematics from a methodological standpoint. Husserl argued that the key idea was to *formalize* existing mathematical theories *in their entirety*. Thus, instead of investigating mathematical domains consisting of entities with a determinate content (e.g. numbers and their operations or geometrical transformations and their compositions), mathematicians proceeded to investigate their purely formal counterparts. That is to say, they no longer investigated *particular* mathematical domains, but rather *general* 'domain-*forms*' (Husserl, 2012, §70): In this sense, abstract algebra is no longer concerned with *particular* operations on *particular* kinds of numbers. Rather, it is concerned with the question of what can be said about mathematical domains of a particular *form*, namely about all those domain-forms in which certain operations on objects are defined by their operational laws. When the addition symbol '+' is used in group theory, it does not stand for the operation of addition, which composes two numbers giving their sum, but for *any* operation that satisfies the group axioms. What these operations and objects on which they are performed then actually look like remains completely indeterminate in terms of content. Not the content of the objects and operations is determined, but only certain conditions for their form are demanded. A theory in abstract algebra, such as group theory, is actually a *theory-form*. It is freed from all numerical or geometrical content. By means of formalization, these contents are "converted into indeterminates, modes of the empty 'anything-whatever'" (Husserl, 1969, §29). And it is this particular way of theorizing, this 'conversion' of a theory (or of a whole class of theories) with a determinate content into its (their) corresponding theory-form which we want to call the *formal method*.

With this in mind, it is no longer difficult to understand the networking power of 'theories' such as abstract algebra or topology. Since these 'theories' are theory-forms, they do not really consist of concepts, theorems, and proofs, but of concept-*forms*, theorem-*forms*, and proof-*forms* (cf. Husserl, 1969, §29). And since a theory-form is precisely something that many theories can have in common, it seems quite natural that one can study connections between remote mathematical domains this way. But what, one may now ask, does any of this have to do with research in mathematics education?

Towards a transdisciplinary approach to research in mathematics education

The crucial point is that a proper understanding of the formal method suggests an entirely new approach to theorizing in mathematics education research. Rather than concluding from the striking similarities between cognitive and social processes (e.g., both kinds of processes utilize signs, process meanings, and refer to all sorts of objects through these very meanings) that they are essentially "two facets of the same phenomenon" (Sfard, 2010, p. xvii), our proposal is that they are in fact two quite different phenomena that share a common form. And we hence believe that it may be better to start from an appropriate *theory-form* to explain similarities *and* differences between these processes. In order to avoid a potential misunderstanding right away: Our proposal here is not to apply mathematical theories in the human sciences, but rather to adopt a particular way of theorizing, namely the formal method, and apply it for theorizing in these fields. Fortunately, such a project does not need to be started from scratch. By formalizing and enriching the theory of autopoietic systems, which was first developed by Maturana and Varela as a *biological* theory (Maturana, 2002), Niklas Luhmann was able to arrive at a general theory of autopoietic systems (or more correctly: at a general theory-*form* of autopoietic system-*forms*). And just as the formalized complement of the concept of addition in group theory is no longer about numbers, the formalized complement of the concept of autopoiesis in this general theory is no longer about organic processes. Rather, the formalized version of the concept (= its concept-form) merely captures a certain self-referential form of (re-)production: “We want to call systems *autopoietic*, which produce and reproduce the elements of which they are composed by the elements of which they are composed” (Luhmann, 1985, p. 403, translated by F.L.). The criticism of an illegitimate transfer of biological concepts to social and cognitive phenomena is hence unfounded, because such a transfer simply does not take place. Instead what Luhmann does is formalizing the biological concept of autopoiesis. He first converts all biological terms into empty forms (‘system’, ‘element’, and so on) and then asks for the concrete modes of reproduction that characterize social and cognitive systems. In this way, Luhmann is able to conduct a systematic study of similarities *and* differences between organic, cognitive, and social systems and arrives at a *theory of mind* (Luhmann, 1985) as well as a *theory of the social* (Luhmann, 1995). And it is this work that Lensing (2021) takes as a point of departure to develop the outlines of a general theory which can account for the bodily, cognitive, and social dimension of mathematical activity and model such phenomena as mathematical teaching-learning processes within a single theoretical framework.

Since a detailed exposition of this theoretical program hardly seems possible on the few lines we have left, we would like to conclude this paper with a brief indication of how the formal method may operate for mathematics education research. For this purpose, we have chosen a concept that is used quite frequently in our field, but almost never explicitly discussed with respect to its conceptual content: the concept of *structure*. In Lerman (2020), for example, authors speak of ‘knowledge structures’, ‘mental structures’, ‘cognitive structures’, but also of ‘social structures’, the ‘structure of classroom discussions’, ‘structures of power and control’, and many more. What is striking about this list is the ‘parallel terminology’ that cuts across various disciplines: Psychology deals with cognitive or mental structures, sociology examines all sorts of social structures, and mathematics education research is concerned with both areas and how the formation of structures in one area might have a bearing on the formation of structures in another (i.e., how certain structures of communication in mathematics classrooms affect the formation of students’ cognitive structures). These considerations indicate that it might be of great value to our field to gain possession of a concept of structure that is not bound to any of the disciplines relevant to mathematics education research, a concept that, due to its formality, can be employed in both the cognitive as well as the social sphere.

But how do we actually get to such a formal level of analysis? The answer is: by starting off at the empirical level and then employing the formal method to work our way up. We may start, for instance, with the well-known example of the ‘IRF-Pattern’ (Initiation – Response – Feedback) and the observation that this social structure constraints the possibilities of the occurrence and linking of communications in the classroom. The pattern thus structures communication events as they occur in the classroom in two respects: communications should be of certain types (initiation, response, feedback) and they should be linked in a certain way (first, initiation, then, response, and so on). If we now continue and include other kinds of social structures, such as norms, roles, social positions, etc., we can see that the way in which these structures exceed constraints on social events varies greatly from case to case, but that all instances of social structure at least agree in that there is some sort of constraints: A structure of a social system, quite generally, constraints the possibilities of the occurrence and linking of social events in that system. With this step we still reside in the realm of social theory and the next step of formalization would then be to abandon the qualification of events as social events and thus elevate our 'definition' of structure to the transdisciplinary level of a general theory of autopoietic systems: A structure of an autopoietic system constraints the possibilities of the occurrence and linking of elements in that system.

Clearly, this brief example is only a very rough sketch of how the formal method may be employed for the purposes of theorizing in mathematics education research. Yet we hope that it will at least motivate the thesis that this method allows for theorizing that yields to a theory that is beyond all disciplinary boundaries. And it is the transition to this transdisciplinary level of theory that we believe provides a possible starting point for solving the complexity problem.

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The interplay between person and environment, cognition and emotion: Using the concept of perezhivanie in mathematics education

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Vygotsky's concept of perezhivanie considers the interplay between cognition and emotion of an individual's development within an environment. Empirical research using perezhivanie has mainly focused on early years education and has hardly ever been used in mathematics education. This theoretical paper starts with the problematic issue of the role of the environment in cognition and proposes the use of perezhivanie as a unifying concept to study an individual's development in mathematics education. We provide an exposition of the concept of perezhivanie, seeking to understand its relation to key memorable events and threshold concepts, which are currently being used to analyse classroom activities and events. We conclude by pointing out the importance of perezhivanie as an analytical approach to four areas of study in mathematics education.

Keywords: Perezhivanie, university mathematics education, emotion, cognition.

The Russian word perezhivanie (plural pereshivaniya; from *zhivat* – to live, and *pere* – carrying something over something) has been used in educational research to describe a series of personally-significant, emotionally-charged experiences and how an individual has overcome them to become the person they are today. The concept has rarely been used in mathematics or science education research literature (e.g., Black et al., 2019; Fragkiadaki & Ravanis, 2021), and it has predominantly been used in research on early years education (e.g., Ng & Renshaw, 2019). Perezhivanie nevertheless is a powerful concept that considers the unity of person-environment and cognition-emotion as central to explain an individual's development, something that in general has not been prominent in mathematics education research. For instance, some theoretical perspectives have looked through a mental cognitive lens (e.g., the development of student's algebraic thinking is characterised by the use of alphanumeric symbolism). Some others have studied emotions from the individual's perspective without regard to how socio-cultural factors shape those emotions and hence the cognitive processes of the person. These perspectives neglect or downgrade the inherent relation between cognition and emotion. In response, we agree with Radford (2014) that a study of the development of students' mathematical thinking should be thought of as an unfolding dialectic process of culturally and historically constituted forms of knowing which involves, for example, embodied forms of thinking and emotional and ethical engagement, amongst others.

Hence, we take the position that perezhivanie is a meaningful research tool for a developmental account of an individual but how to conceptualise and investigate it is still “an open research problem for sure” (Radford, 2014, p.275). Therefore, this paper aims to advance the understanding and use of perezhivanie in mathematics education research. We explore the concept of perezhivanie within the research literature of mathematics education and beyond. We start with an introduction to the problem of the environment as it was conceived by Vygotsky (1934/1994) and an exposition of the concept of perezhivanie as understood by him and later expanded by other scholars. We then seek to understand its relation with other concepts used in mathematics education literature, which relate to emotion (i.e., key memorable events) or cognition (i.e., threshold concepts) and that seem to be used to study similar problems to those that can be studied with perezhivanie. We ask: How does perezhivanie provide a deep and detailed understanding of individual development, not captured by key memorable events or threshold concepts? We conclude with a discussion on the added value of perezhivanie in four areas of study in mathematics education.

The problem of the environment

In *The Problem of the Enviroment,* Vygotsky used the word perezhivanie in relation to the study of the role that a child’s social environment has in the formation of their personality, as opposed to the part played by their genetic inheritance. Central to Vygotsky’s thought was the idea that rather than a setting, the environment should be considered as dialectically related to a person’s development:

> “environment should not be regarded as a condition of development which purely objectively determines the development of a child by virtue of the fact that it contains certain qualities or features, but one should always approach environment from the point of view of the relationship which exists between the child and its environment at a given stage of his development.” (Vygotsky, 1934/1994, p. 338)

Vygotsky exemplified the two-way relation between person and environment from his own research. The example was that of three children experiencing the same environmental condition (mother with alcoholism) in different ways. The youngest child developed neurotic symptoms, the second demonstrated contrasting emotional attitudes towards his mother, whereas the third and eldest acted as a senior member of his family to protect his youngest siblings.

Through the interactions with the different aspects of a given environment, an individual’s perezhivanie determines what influence this environment has on their development. In that sense, in order to analyse the impact of a learning environment on developmental processes, one should consider both cognitive and emotional aspects. In the research field of mathematics education there are studies that consider these aspects, but separately. For instance, Key Memorable Events (Marmur & Koichu, 2018) focus on emotion to study events in the classroom with which students identify or strongly remember. By using this concept one can point out various prominent events that take place in the mathematics classroom and that could potentially be important for students’ learning; but research on these events has no direct link to students’ development as (mathematics) learners. Further, Threshold Concepts have been used in mathematics education to study cognitive events. The learning of concepts such as the limit, continuity, etc. in calculus, are fundamental for students' mathematical development. The existing literature nevertheless leaves aside the emotional aspects or the contextual factors that the learning of these concepts entails. Hence, there is a need in the field for the use of a concept that takes into account both cognitive and emotional aspects in the interaction of an individual with their learning environment and that highlights the influences on their development.

What is perezhivanie?

Perezhivaniya are the events, episodes, activities, happenings or experiences in which people are active participants. The perezhivaniya in someone's life help them answer the question "Who am I?", by showing who they were and how they came to be the person they are now – the experiences they had and how these formed them.

In relation to learning, Kim (2021) argued that perezhivanie experiences can facilitate the learning process when the learner recognises meaning in the environment, evaluating or appraising the experience as important. In other words, the experience provides "what is available to the person to do something with" (van Lier, 2004, p. 90), thereby becoming developmentally significant. Hence, perezhivaniya become for the person that which proves to be personally significant for them (Rubenstein, 1957). They are the chapters of one's autobiography, the events that were life-changing and emotionally-charged.

As a research tool, perezhivaniya are the units of consciousness or of the personality as a whole. They encompass intellect, affect, memory, attention, will, etc. They are experiences as a whole, and each has their own plot, its own inception and movement towards its close (Dewey, 1939). Perezhivaniya are *both* subjective and objective, they are both personal and environmental (e.g., the loss of a job for a young person is not the same as for someone whose job is their entire career). A perezhivanie is the prism through which the subject refracts the influence of the environment. Vygotsky (1934/1994, p. 342) remarked that "in perezhivanie we are always dealing with an indivisible unity of personal characteristics and situational characteristics".

These experiences are life-changing in the sense that they are problematic, and one has to overcome, survive or over-live (überleben) them. Perezhivaniya do not have to be only painful, there can be good perezhivaniya (e.g., a risk that paid off and opened a new life phase) (Kotik-Friedgut, 2007).

Success in overcoming a perezhivanie entails changing the social situation, either by transforming the object of activity, transforming the subject or both. Blunden (2016) asserts that development comes from a process of catharsis. In ancient Greece, catharsis was the experience of an audience who, when watching a play at the theatre, externalised their emotions by empathising with the performers who were acting the emotional experience out for them. This had a "purging" effect. Blunden (2016) points out that Freud (1914) was also familiar with perezhivanie, and that he used psychoanalysis to make a patient remember and repeat an emotional experience and working through it, overcoming and "surviving", 'transcending" it, or "sublating" it (in the manner of Hegel). The psychoanalytic process seems to suggest that the aid of another person, who is capable of objectifying and reflecting back the feelings of the person going through a perezhivanie, of guiding them and making use of the resources of the culture to assist them in finding and accommodating their new situation, is usually needed (e.g., teacher, therapist).

The concept of perezhivanie also allows us to understand experiences that are not so dramatic, and what has been said above applies to those relatively minor joys and embarrassments that “stick in our minds” and become part of our development, that evoke an emotional response and are connected with our motivation, without becoming life-changing traumas.

How is the concept of perezhivanie related to other concepts of emotion or cognition that are used in mathematics education literature?

In spite of its potential, perezhivanie is rarely used in mathematics education research. In this section, we seek to understand the concept of perezhivanie by critically analysing the concepts of Key Memorable Events and Threshold Concepts that seek to explain similar phenomena with perezhivanie in mathematics education.

Key memorable events

The concept of Key Memorable Events (KMEs) refers to classroom events that are accompanied by either positive or negative strong emotions and are imprinted as meaningful in students’ memory, because of their contribution to students’ learning (Marmur & Koichu, 2018). In that sense, KMEs can be used as a theoretical lens for analysing affective moments and gaining an insight into the learning process. With their use, one can recollect memories and thoughts of emotionally loaded events that seem to be particularly interesting for the individual. In this way one can consider the role that affect plays in students’ experiences during the learning of mathematics and in this regard we can recognise a similarity with the concept of perezhivanie. However, for us, the difference is in the consequences that this experience has for the development of the individual. With KMEs we have a tool for listing and categorising the kind of events that stay in the individual’s memory and can have an effect on their learning. Perezhivanie provides us with a very detailed process, where different perspectives and aspects are considered with the scope to give us a deeper understanding of the situation and of the impact on the individual’s learning trajectory and development.

The use of KMEs as an analytical tool in mathematics education research requires particular methodologies, such as the stimulated recall methodology. Students are assisted in recollecting their experiences from a particular lesson and provide information regarding their thought-processes. This raises criticism regarding the degree of importance of each event in students’ learning, since the students are reminded of specific classroom events and then comment on them, instead of recalling themselves what was distinctly important for them. We assert that an experience becomes significant for an individual (and not necessarily for others) and changes outcomes for that individual by requiring a challenging situation that has been overcome and resulted in a cathartic outcome.

Furthermore, although with the KMEs one can identify and classify emotionally charged events that take place in the learning of mathematics, there is not an explicit way to understand whether the students act or not upon these events. For instance, in a recent work by Marmur and Koichu (2018) where the concept was used in order to pinpoint the incidents that were memorable for students during undergraduate mathematics tutorials, most of the students referred to events that played a role in their motivations regarding the learning of mathematics (e.g., “It was beautiful. It’s an eye opener.”), rather than their engagement. Based on these recollected events it is not clear if students took action in changing their ways of learning or if these events had an impact on their identities as mathematics learners, something that is crucial in perezhivanie. Overall, KMEs can be a useful instrument for classifying strong affective moments and explain various phenomena that take place during the learning of mathematics in the classroom (or in the lecture hall). They can improve the lesson design by analysing students' emotional reactions to particular teaching practices, but they do not give us a clear picture of the impact that these emotionally-charged situations have on students' further learning and engagement with the subject.

Threshold concepts

Threshold concepts are topics that are considered crucial when a person tries to master a certain subject. Meyer and Land (2006) described threshold concepts as portals which open a new and previously inaccessible way of thinking about a subject. According to them, threshold concepts have five characteristics that can be used in identifying them: (1) they are transformative i.e., they result in a shift of a person's perception of a subject; (2) they are irreversible, meaning that the change in perception is irreversible; (3) they are integrative due to their ability to allow connections of previously unrelated concepts to be made; (4) they are bounded because they might lie at the conceptual border between disciplinary areas; and (5) they are troublesome because they are inherently difficult to grasp.

In the mathematics education literature, Breen and O'Shea (2016) proposed that limits, functions, cosets and quotient groups are mathematical topics that hold the above characteristics and can thus be considered threshold concepts in analysis and group theory. They argued that traditional teaching approaches and curricula in undergraduate mathematics often result in memorising content and procedures which leads to shallow or rote learning. In order to address these issues, Breen and O'Shea proposed to change the curriculum (often built around the structure of mathematics and a linear fashion of teaching content) to one structured around threshold concepts (leading to a recursive design that enables learners to revisit threshold concepts throughout the duration of the course).

Threshold concepts offer ways of identifying topics that act as portals and thus represent important stages of a learner's mathematical development. As a construct, threshold concepts relate to the cognitive aspects of learning mathematics, i.e., what has to be learned in order to master a certain topic. However, their transformative and troublesome nature suggests that a learner's experience in understanding a threshold concept could result in an emotionally charged situation, an aspect that is not acknowledged or taken into account by threshold concepts. In that sense, perezhivanie offers a more holistic account that can capture the interplay between both the cognitive and emotional phenomena involved while a learner tries to master a threshold concept.

In summary, choosing a threshold concept as context may be useful for researching an individual's mathematical development and for introducing perezhivanie as a research tool. In contrast to key memorable events, perezhivanie identifies significant experiences of an individual in a certain environment; importantly, perezhivanie considers the working over of such experiences that characterise the development of an individual as well as who they came to be regarding those experiences.

Final reflections on perezhivanie in mathematics education

The concept of perezhivanie was left unfinished by Vygotsky. Many scholars have since contributed to its development and the concept has recently received increased attention by researchers interested in the topics of emotion, motivation and subjectivity. In empirical research in education, perezhivanie has been used as an analytical approach to the unity of emotion and cognition (e.g., Fleer & Hammer, 2013) and/or the unity of individual and environment (e.g., Chen, 2015). Researchers using perezhivanie have mainly focused on understanding the refracting prism of an individual who interacts with an environment and interprets it in a subjective way (e.g., Adams & March, 2015). Perezhivanie has rarely been used to study the experiences that are significant for the individual and, due to its *cathartic* nature, play a significant role in their development and life-long learning. We contend that this was Vygotsky's main aim at developing the concept of perezhivanie.

We take the position that perezhivanie is a useful tool and concept in mathematics education research, by providing an analytical approach to a phenomenon under study that considers the unity of cognition and emotion and/or individual and environment, while allowing researchers to see how these units become the key pieces in an individual's development. These are powerful features that can offer new, in-depth understandings in mathematics education research and practice. Based on our readings of the literature, we present four areas of mathematics education research as examples of where perezhivanie could be a useful research tool. The majority of empirical research using perezhivanie has focused on early years; we will focus on areas of the study of older students.

The first area is the study of educational transitions (e.g., mathematics transition to university) and the identification of students at risk of dropping out. For many students, the transition to university is a highly emotional event where past experiences merge with the social relations and material aspects of the new environment. Studying the prism through which individuals refract transitional events, researchers can understand the situations that help or hinder students in different ways to make an un/successful transition and document a pool of perezhivaniya experiences from longitudinal data, or from cross-cultural comparisons. The cathartic nature of these experiences is evident since the transition to university is a defining event in students' lives; some see it as a 'rite of passage', or an opportunity to become someone new, hence determining their development as individuals.

The second area we suggest is the study of mathematical resilience. Researchers have related resilience to individuals' cognitive and affective abilities to overcome obstacles and challenging situations in the learning process, turning them into situations that support them (Hutauruk & Priatna, 2017). A view of resilience as relational between the individual and their environment can be potently studied through perezhivanie. Research on the perezhivaniya of resilient learners (i.e., the reflected lived experience of overcoming a challenging situation) might illuminate practices that encourage dispositions that allow individuals to negotiate their contexts in advantageous ways (cf. Hernandez-Martinez & Williams, 2013).

A third area of study is the research of life-changing decisions with respect to mathematics education, particularly the choices of women or other underrepresented groups in mathematically-demanding careers. Perezhivanie can be used to understand those choices where objective circumstances are refracted through the prism of gendered or racialised subjectivities. Using perezhivanie as the lenses through which we understand these choices allows for a holistic perspective of the life trajectories of these individuals.

A final area is the study of how individuals become professional mathematicians. A study of this type, which uses perezhivanie as a research tool, would look for events that steered an individual to become interested in mathematics, the contexts and circumstances that made possible for that individual to sustain, or even nurture, their mathematical identity. Surprisingly, research into narratives of being or becoming a mathematician, or different ways to be a mathematician, is still limited; stereotypical views of mathematicians as “geniuses” or “eccentric” are still widespread among students and society at large, which in turn discourage many from pursuing a career in mathematics.

This paper described the concept of perezhivanie as first conceived by Vygotsky and subsequently extended by many other scholars. We compared this concept with others in the mathematics education literature that share commonalities with perezhivanie, and described how the concept could be used in mathematics education research. We believe that this concept is useful in understanding the interplay between cognition and emotion, and personal and environmental factors in a way that other theoretical frameworks do not.

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Students’ concept images of differentiability from the perspective of the mathematical thinking competency

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Keywords: Mathematical thinking competency, concept definition and concept image, semiotic registers, differentiability

This poster presents an empirical case of two students working with a task sequence on the concept of differentiability, designed with the purpose of investigating students’ mathematical thinking competency (MTC), as defined in the Danish competency framework (KOM). The MTC is one of eight distinct, yet mutually related, mathematical competencies that all together constitute the KOM framework (Niss & Højgaard, 2019). The MTC involves the ability to distinguish “between different types and roles of mathematical statements (including definitions, if-then claims, universal claims, existence claims, statements concerning singular cases, and conjectures), and navigating with regard to the role of logical connectives and quantifiers in such statements” (ibid., p. 15). Hence, students’ work with definitions is part of their MTC.

In relation to the topic of analysis, the theoretical distinction between concept definition and concept image has proved useful (e.g., Tall & Vinner, 1981). The concept image is “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (p. 152). Thus, concept images and the distinction to concept definition are part of the MTC. Due to the MTC’s focus on processes of doing mathematics and to access the students’ concept images, it is suitable, in this case, to study what the students are saying, writing and doing as part of their work with the concept of differentiability. For this, Duval’s (2006) perspective of semiotic registers offers a promising lens. Transformations of representations, namely treatments within one register (e.g., symbol manipulation) and, in particular, conversions between registers (e.g., from a symbolic expression to its corresponding graph) are key to mathematical activity and the mathematical way of thinking. Duval (2006) argues that considering comprehension as being conceptual and mental and semiotic representations as being external is a deceptive division. “In fact, mental representations that are useful or pertinent in mathematics are always interiorized semiotic representations” (ibid., p. 126). Hence, the semiotic representations are part of students’ concept images. Thus, with this poster, I address the question: *How does the ‘concept definition’-‘concept image’ distinction in the case of differentiability assist in characterizing students’ MTC, and how may an attention to conversions between different registers add to such characterization?*

The empirical case reported stemmed from a study carried out in a class of 29 students in Danish upper secondary school (age 16-17) during two 90-minute lessons on the topic of differentiability. The students were working in groups of two or three sharing one computer. Their actions on the computer were recorded using screencasts, also capturing webcam and audio, resulting in 14 recordings. The present case was selected due to the students’ thorough work with the concept definition of differentiability, while still not connecting the concept definition to their work with the graphs of non-differentiable functions. The two students were working on the final set of tasks, where they are to explain with their own words, when a function is differentiable and when it is not; then to construct an example of a function that is not differentiable, and then a function that is continuous but not differentiable. The case illustrated how the students shifted, both written and orally, between the semiotic registers of natural language, symbolic representations, and graphic representations. However, using natural language, the students elaborated on the symbolic representation of differentiability as a limit in terms of graphic representations. Afterwards, when attempting to construct a discontinuous function, the students typed in a piecewise defined function f_1 using a template in TI-nspire, such that f_1 consisted of a linear function for x<0 and a quadratic function for x>0. Thus, f_1 was not defined for x=0. When the students drew the graph of f_1 in TI-nspire, they saw that the graph "jumped" just around x=0, why they concluded that f_1 was discontinuous, hence non-differentiable, without taking into account that the domain of the function did not include 0. Notably, even though the students made shifts between representations of different semiotic registers, and related to the definition of differentiability, their concept image of differentiability was based on their notion that a graphic representation of a function should not "jump."

The three theoretical perspectives serve different roles. KOM's MTC frames the interest and purpose of the study: to investigate students' MTC. For such investigation, the distinction between concept definition and concept image provides a terminology for the students' associations with their work related to the formal concept definition (of differentiability). Furthermore, it focuses on the cognitive processes by which mathematical concepts are conceived (Tall & Vinner, 1981). The perspective of semiotic registers helps to elaborate students' concept images by explaining cognitive processes related to transformations of representations (Duval, 2006). This is to say that they are both of cognitive nature, as is the KOM framework, but focusing on different objects as part of the cognitive processes. Hence, with the application of the two lenses, there is a potential for a 'coordinated' analysis. Such 'coordination', focusing on the students' concept images, elaborated through the lens of semiotic registers, can help characterize students' work related to the formal concept definition. From the perspective of MTC, the two perspectives can elaborate on the processes of relating to the roles and types of mathematical statements, even though the students may not work with consistent concept images coherent with the formal concept definition. Thus, this case implies that Duval's (2006) perspective of semiotic registers offers an analytic tool to focus on the students' activities with different representations and thereby elaborate on students' concept images related to the cognitive processes of MTC.

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Analyzing semiotics in an imaging teaching scheme through a case study in physics virtual classes: towards a conceptual framework

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Keywords: Imaging teaching scheme, semiotics, artifacts.

In this poster, we explore the contributions that a cultural-historical approach and a semiotic point of view can provide to the analysis of teacher's conceptualization processes in the frame of the notion of *scheme* (Vergnaud, 1998). This notion has been introduced as the invariant organization of the activity for addressing a class of professional situations corresponding to the same aim. Pointing out that conceptualization is crucial in the development of mathematical and professional competences; Vergnaud (1998) adds that "the fact that this process of conceptualization is not directly observable, and has to be reconstructed from observed behaviors, makes it a challenge for research" (p. 231). The study was carried out by observing physics classes at distance through Microsoft Teams, during the COVID-19 pandemic. Thinking about the characteristics of the empirical data collected, the conceptual framework is structured considering the networking strategies for connecting theoretical approaches (Prediger et al., 2008); thus, we *combine* the theoretical elements mentioned below.

Theoretical elements: scheme, artifacts, semiotics

Vergnaud (1998) defines *scheme* as: "Invariant organization of behavior for a certain class of situations" (p. 229). The organization consists of four components: (1) Goals, subgoals and expectations of the activity; (2) Rules of action; (3) Operational invariants: mainly *concept-in-action*, and *theorems-in-action* (proposition considered true); and (4) Possibilities of inference. In addition, a concept considers a set of linguistic and symbolic representations (Vergnaud, 2009).

Wartofsky (1979) conceives of *artifacts* as cognitive means for representing the mode of activity in which they are used or produced. He highlights that modes of representation in mathematics and science have their genesis in the modes of representation that emerge simultaneously with social and linguistic practice. Thus, the production of artifacts (e.g., tools, signs, language, gestures, social organizations and interactions) links closely to their sociocultural context and historical development.

Wartofsky (1979) places representations as a central element of cognitive practice. He states that concepts "are the ways in which the scientist has learned to understand complex phenomena, to realize their mutual relations and to represent them in communicable form." (Wartofsky, 1981, p. 21). Their importance lies in the fact that once a concept is articulated, its meanings and relationships can be studied by themselves; so, one can study not only what the concepts refer to, but also the concepts themselves and their influence on the modes of activity. In a particular way, Wartofsky (1979) says that, in mathematics and in the other sciences, we organize symbols of our experience or our thinking with the aim of understanding this experience or explaining it to others. Therefore, the teaching-learning processes constitute semiotic conceptualization processes where teachers and students mobilize artifacts, seeking to give meaning to the mathematical objects that are presented to them.

Method

We observed (from France) physics classes at distance through Microsoft Teams, during the COVID-19 pandemic, of a Mexican teacher with 7 years of teaching experience. The classes took place in a high school in Mexico City, and we observed nine sessions (of 2 hours each). Each session was recorded. The observation took place during the teaching of different contents related to the subject of optics. Here we report findings from two sessions on the topic of "image formation" in the unit of "optical systems and geometric optics" where there is a close relationship between mathematical and physical concepts.

To analyze the data, we define the *Imaging Teaching Scheme* (ITS) as a stable invariant organization of the teaching activity corresponding to the process of the generation of images through mirrors and lenses; that is to say, how to teach that an object, with a defined visual structure, produces a visual representation (image) that manifests its visual appearance. The objectives of ITS correspond to what are the artifacts the teacher uses to teach the process of the generation of images. The rules of actions and the operational invariants correspond to the use of two geometric laws associated with the formation of images: law of reflection and law of refraction (Snell's law). We carried out the data analysis with the objective of identifying the operational invariants in the ITS based on the use of artifacts by the teacher and relating them to semiotic representations (mobilized by the same teacher).

Results and implications

In each session, we analyzed the teacher-student interactions and we distinguish a situation related to ITS: (1) *imaging* in a convergent spherical mirror, and (2) *imaging* in converging thin lenses. We analyzed the semiotics involved in ITS and its operational invariants through the coordinated mobilization of three main artifacts: a Microsoft Word document, ray diagrams (representation of the possible paths light can take to get from one place to another), and gestures. The semiotic analysis allowed us to deepen the analysis of a teacher's conceptualization processes to the extent that the teacher's activity, besides being organized in schemes, was perceived by the researchers in this study as an activity of signification carried out by means of cultural elements: artifacts.

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A dialogue between mathematics education and special education: ethics, inclusion and differentiation for all

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Ethical issues play an important role in moulding the philosophy of mathematics education. The present study spells out ethical features of mathematical learning in terms of inclusion. We present the OPEN-MATH project that aims at accomplishing inclusive mathematics learning environments and a teaching learning model based in such a framework.

Keywords: Ethics, inclusion, differentiation, theory of objectification, networking.

Introduction

In this contribution, we present a networking of theories that goes beyond the realm of mathematics education. We are interested in the possible connections between mathematics education and special education. The enlarging of the networking space beyond mathematics education has been prompted by the need to realize inclusive learning of mathematics. The study has been carried out within the OPEN MATH project funded by the University of Bolzano[1], which aimed at the design of a teaching model that fosters inclusive learning of mathematics. The project acknowledges the growing interest in foundational issues regarding ethics, equity and the political in Mathematics Education (Ernest, 2018; Radford, 2021). Inclusion is located at the point of intersection of cognition and learning in mathematics, equity, ethics and the political, and we believe that research in the issue of inclusion in mathematics is a breeding ground both for the foundational features of mathematics education and inclusive education. Furthermore, an inclusive outlook on mathematics education, besides addressing ethical issues, fosters further reflections and investigations about the principles behind the learning of mathematics. The OPEN-MATH project addresses the need for research in the field of inclusion, outlined by Roos (2018) that intertwines theoretical framing of this feature of Mathematics Education and its translation into effective school practices. We focus on the dialogue between the theory of objectification (Radford, 2021) and differentiation (Tomlinson, 2014) to frame inclusion in mathematics and thereby design a model implemented to embody inclusion in the reality of the mathematics practices. The aim of our study is creating a conceptual framework for inclusive mathematical teaching–learning activities. The outcome is a consistent definition of inclusion and a teaching model that accomplishes inclusion in mathematics.

[1] Grant number BW2086

Didactical Differentiation

We refer to a broad definition of inclusion, not focused solely on students with disabilities: inclusion is understood as a process that aims to identify and overcome any barriers that hinder students from accessing education and achieving optimal outcomes, both in learning and socialization (Ainscow, 2016), therefore, it applies to each student, and not only to a specific category.

To promote participation and learning for all, teaching must simultaneously be sensitive to differences and recognize the similarities of all in being part of a shared community. Teachers and educators are required to face a great challenge. They cannot disregard the use of specific measures for some, which risks an unfair flattening of differences, but neither enhance such differences thereby stigmatizing some students.

Based on these premises, the methodological plan of inclusive teaching requires a design that alternates settings aimed at creating a space in which each student can follow his or her own individual and unique path of learning and others that offer an opportunity for collaborative action, in which each can contribute in his or her own way to a common project. The methodology of teaching differentiation, introduced by Tomlinson (2014), attempts to strike this balance. In the OPEN-MATH, the development of classroom environments encourages both plural and decentralized learning processes, and personal paths in which the student is led to choose according to their needs. This allows for differentiation for all pupils. Furthermore, the centrality of freedom of choice on the part of pupils introduces the possibility of "self-determined differentiation": in a learning landscape that offers multiple opportunities, teachers no longer need to design the most appropriate delivery for each student but support subjective and competent choice on the part of girls and boys.

We can say that the vision of the student according to the perspective of differentiation is that which sees them capable of self-determination and of positioning themselves with respect to the community, they belong to, without, however, losing the link with it. In the OPEN-MATH project, this perspective leads the teacher to pay specific attention to the preparation of each student, and their interests and motivations to learn. The design of activities is adapted to the individual because the assignment is broad or because the single task becomes plural, divided into multiple activities that the student can choose autonomously. An example of this approach can be found in *stations* (Demo, 2016), where different learning activities related to a main topic characterized by specific learning objectives are structured and made available in the classroom. Students can move from station to station and choose which ones to complete and which ones not to complete. The activities that characterize each station allow differentiated forms of thinking and action. A final passport serves as a way for the student to make note of the stations completed, the difficulties encountered, and what enabled them to learn in the most effective or enjoyable way, but it also allows the teacher to keep track of and understand individual differences in math learning.

The Theory of Objectification

In this section, we outline to basic features of the theory of objectification that allow us to realize inclusive mathematics inclusion as differentiation for all, described in the previous section. We will discuss learning as a double-sided construct whose elements are *processes of objectification* and *processes of subjectification*.

The theory of objectification, embedded in sociocultural perspectives, stems from a profound intertwining between culture and the individuals, accomplished in activity. Activity is the ontological category of the theory of objectification as it realizes the consubstantiality between individuals and their culture. In the stance of the TO, mathematical thinking and learning are not processes confined in the mind, but they are intertwined with individuals' activity. Signs and artefacts play an important role in the TO, beyond the role of something that stands for something else or as mediators of activity. They are considered an integral part of human thinking and human activity (Radford, 2021).

Learning as objectification

The issue of learning is rooted in the dialectics between the individual and their culture. Learning is a movement pushed by the intrinsic differential between the individual and cultural knowledge. In fact, in attending to knowledge the student has to cope with something that in the beginning is different from him, an alterity that challenges, resists and opposes him. Learning is the process that erases such a difference in order to make sense of cultural knowledge and transform it into something familiar that allows new forms of action, thinking, imagination and feeling. In order to reduce the distance between the individual and cultural knowledge, students engage in processes of objectification, i.e., the sensuous activity interrelated with signs and artifacts that allows them to meaningfully encounter mathematical knowledge (Radford, 2021).

We remark that according to the TO, signs and artefacts are constitutive of the activity (Radford, 2021) that leads students to notice mathematical knowledge. They are bearers of an embodied intelligence and culturally endowed with specific patterns of activity that individuals use in their meaning-making processes.

In view of accomplishing inclusive mathematical learning, objectification has a fundamental feature in realizing inclusion as differentiation. Objectification is a multimodal process deriving from the variety of sensorial channels (sight, touch, movement, imagination etc.) and the diversity of semiotic means of objectification (objects, tools, gestures, linguistic devices, icons, symbolic language etc.) interwoven with mathematical activity. The multimodality of mathematical thinking and learning allows each student to learn according to their specific and unique potentials, difficulties, cognitive style and sensorial channels that suit him best.

The dialectics between the individual and his cultural-historical environment is not fully accomplished by objectification. The TO pays special attention to the production of subjectivities in the learning processes. The basic idea is that humans are unfinished projects of life, subjects in continuous transformation in the creation of a singular and unique cultural and historical subject. Learning is a double-faced construct. We analysed objectification as one of the two facets. Its entangled counterpart is *subjectification.*

Learning as subjectification

"Processes of subjectification are based on the idea that humans are always unfinished projects of life, subjects perpetually in the making". (Radford, 2021, p. 35). Subjectification is related to the production and transformation of subjectivities and to the continuous co-production of singular and unique individuals against the backdrop of their cultural-historical domain. It allows students to critically position themselves in cultural-historical mathematical practice, realizing new ways of thought and action (Radford, 2021).

The notion of subjectification is coherently linked to the basic tenet of the theory of objectification that establishes the intertwining between individuals, their sociocultural environment and activity. The individual and their culture live in a dialectical relationship in which individuals produce reality as much as reality produces them. On the one hand, the student acts, thinks and feels according to the social and cultural reality he encounters in the classroom. On the other hand, in realizing their unique and specific project of life, the student critically and creatively establishes a reflexive and agentic relationship with the world that allows him to change it.

Subjectification accomplishes the ethical, social, and psychological nature of inclusion fostering self-determination, self-realization, attention, and respect of other human beings contributing to the creation of an educational and social environment where teachers and students co-produce themselves in the dialectical relationship with their cultural-historical context.

Networking strategies: a conceptual framework for inclusive mathematics learning

Mathematical learning develops along two complementary plots: social interaction and self-determination. Social interaction is the fabric of the learning of mathematics envisaged as a joint activity that involves teacher and students towards the production of mathematical knowledge. Self-determination nurtures individual's unique distinctive traits, needs, potentials, desires as they creatively engage in mathematical learning. True inclusion cannot be accomplished if we neglect one of the two plots. If we disregard self-determination, the student might not have the possibility to enter the mathematical practices, thus being excluded, or learning mathematics clashes against his needs, desires and individual features. If we disregard social interaction, a meaningful learning of mathematics cannot occur, and we miss the political and ethical issues of mathematics education that call for the production of ethical subjects open to others and their culture. A feature that our conceptual framework should encompass is the multimodal mode of activity students deploy in the learning of mathematics.

In the previous section, we have described the tenets of the TO and differentiated learning featured as open learning. We now sketch how we used the networking strategies devised by mathematics education to construct a conceptual framework for inclusive learning in mathematics.

Lotman's semiosphere

We will carry out networking within Lotman's (1990) semiosphere. A semiosphere is characterized by the following elements: a system of practices, a meta-language, themes, plots that can be developed within this sociocultural space, and the coexistence of multicultural identities, which in our project refer to mathematics and special education.

The semiosphere is extremely effective to study the connection of theories since it is a space that fosters interaction and dialogue between different theories. Networking can be conceived as a dialogue between theories that takes place in the semiosphere using a metalanguage that allows the different theoretical perspectives to communicate and interact. In a networking perspective, the semiosphere blends two important plots: integration that refers to the intertwining between theories and identity that refers to the internal consistency and distinctive traits of a theory. The dialogue between theories overcomes barriers respecting their identity and consistency.

The semiosphere plays a prominent role in our study, in the light of establishing a dialogue between two theoretical perspectives belonging to different disciplines within the educational domain. In fact, we enlarged the semiosphere to include mathematics education and special education in the meta-theoretical dialogue. Within the enlarged semiosphere, we established a specific meta-language, we acknowledged the common system of practices, we allowed the co-existence of the culture of math education and inclusive education in our research, and we developed the plots of integration and identity.

Connecting strategies

Prediger and Bikner-Ahsbahs (2014) outline a "landscape" of possible connecting strategies, which balance the plots of identity and integration. For the scope of our work, we reckoned *coordinating* as an effective networking strategy. Coordinating leads to a conceptual framework built by "fitting together elements from different theories for making sense of an empirical phenomenon" (Bikner-Ahsbahs & Prediger, 2014, p. 120). Combining is an appropriate strategy when networking theories whose systems of principle are complementary to one another.

For the objectives of OPEN-MATH, the outcome of the networking strategy should encompass the following features connected to inclusion as differentiation: i) Outline the notion of inclusion in mathematics ii) Provide learning activities that meet personal needs, potentials, and talents of each student, allowing them to be included in the sociocultural activity. iii) Nurture both the individual's distinctive traits and social interaction. iv) Outline a teaching–learning model to be implemented in everyday mathematics classroom, which in our study involves grade seven students.

Within the enlarged semiosphere of educational sciences, the Theory of Objectification and the Open Learning Approach are not completely compatible. The former rests on sociocultural underpinnings centred on joint labour and being with others, whereas the latter on socio-constructivist underpinnings that stress the role of autonomy and self-determination. Social interaction and the individual's agency are present in both theories but with a different hierarchical position in their system of principles. They are not conflicting theories, and they can be coordinated to get a multi-faceted insight into inclusive mathematical practices and fit together elements from the two theories in view of a conceptual framework for inclusion in mathematics.

A conceptual framework for inclusion in mathematics education

To frame the project, we introduced a tentative framework of inclusion that links the attention to the individual with their specificities and mathematics learning as a process of objectification. We list here the *defining principles* of our conceptual framework.

Inclusion is conceived from the student perspective (subjectification), as "the dialectical and critical positioning of all students in the cultural-historical practice of mathematics, who act, feel and think according to their individual distinctive traits to pursue their project of life*."* (Demo et al., 2021 p. 8)

Mathematical activity, in its multimodal acceptation, is the meeting point of the social and individual dimension of mathematical learning. Multimodality and sensuous cognition link the individual self-determination and the social interaction. In particular, the semiotic means of objectification, which define the modes of activity of both the single student and the class, can be considered the bridge between the subject and the culture.

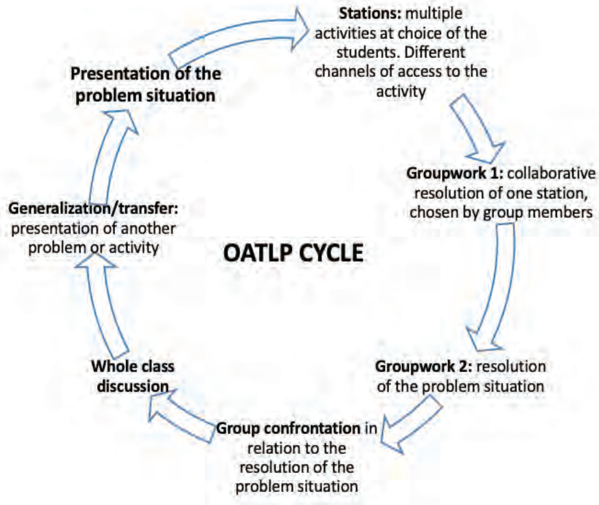

Figure 1: The Open Activity Theory Lesson Plan cycle

Starting from these two principles, we designed a new model of activity that blends self-determination with social interaction. Such a blending is realized by alternating group work with stations. Both group work and stations are informed by the multimodality of objectification and the co-production of subjectivities as they position, according to the notion of subjectification mentioned above, in the sociocultural environment of the classroom. Figure 1 shows the structure of the model that we termed Open Activity Theory Lesson Plan (OATLP) to underline the coordination of activity and open learning that characterize the theory of objectification and differentiation. OATLP is the outcome of the experimentation with a grade 7 class carried out in school year 2020/2021 during the research project.

An example of OATLP

To show how the main phases of the OATLP cycle are built and interact in accordance with the networking of Theory of objectification and Didactical Differentiation we give here an example. The topic of the implemented cycles was chosen in agreement with the mathematics teacher, consistently with the planning of the classroom: in March, we proposed a cycle where the main learning goal could be identified in the resolution of problems related to the estimation of areas. The first individual phase, station work, consisted in several activities that could be chosen by the single student in order to work on the fixed learning goal. In Figure 2, we show as an example two stations: the first, on the left, is more related to a verbal learning approach, while the second, on the right, is more related to a visual learning approach. In the second one, the student is asked to colour the squares in the figure of a tree first considering only the squares that are fully contained within the tree and to count them, and then she is asked to colour the squares that are also partially contained. Then the student is required to estimate the area of the figure as required by the task.

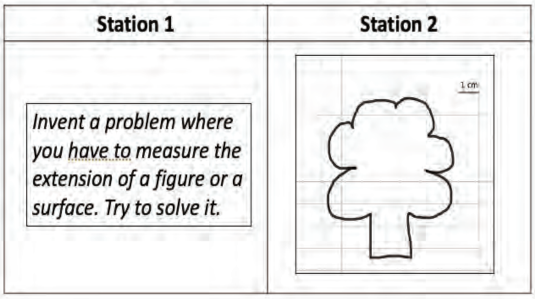

Figure 2: Two stations out of six from the OATLP cycle of March.

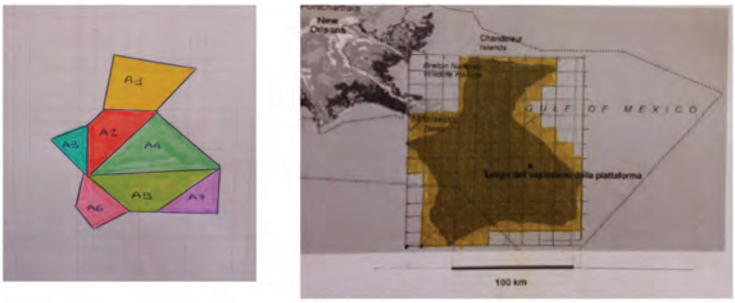

Figure 3: The resolution to the problem situation given by two different groups.

Figure 2 shows two different activities the students were exposed to in two learning stations devoted to the learning of geometry. We can see in this brief example how stations, a didactical methodology that characterizes Open Learning, helps us to exploit the multimodality of mathematics in order to allow each student to choose her path in the learning process, positioning herself with respect to a shared aim. This is also strongly consistent with the principles of Didactical Differentiation: Tomlinson (2014) highlights how differentiation is possible only when the aim of the activity is clear and the essential learning goals are defined. In this example, the aim of the activity is the estimation of areas and the learning goal is defined is a multimodal approach to solving problem.

The individual phase of the OATLP cycle helps the students to access the group work activities according to their individual characteristics and potentialities, and to have a moment to reflect on eventual personal difficulties with the topic, sharing them with a peer or with the teacher before the group work starts. Continuing in the description of the OATLP cycle, after the stations, students were asked to solve a problem, again about area estimation that was more contextualized, and requested the application of a proportion. In Figure 3, we show two solutions of a problem concerning an oil slick in the sea, and the student had to find a method to calculate its area and to justify it.

In addition, within the group work attention was given with respect to the possibility for every student to position herself in the learning goal: in this case, the positioning of the student is made with the other group members in an effort for mutual understanding and joint labour. The phase of group confrontation closes the problem-solving phase allowing each group, and student to justify her work, and to ask for justification to the other: This allows the students to be reciprocal critical friends, and to deepen the understanding of their respective work.

Summarizing, the whole cycle is designed so that there is a balance among the social and the individual dimension of learning, and in order to give every student the possibility to be included, in the sense made explicit above, that is related to the process of becoming a subject that critically position herself with respect to mathematical culture, according to her own characteristics.

Conclusions

Ernest (2018) has singled out ontology and metaphysics, aesthetics, epistemology, learning theory, social and political philosophy, and ethics as the prominent elements of a philosophy of mathematics education. The author deems ethics the *first* philosophy of mathematics education advocating the awareness that we are all the same but different and we are in our present condition (fortunate or unfortunate) only by luck and contingency. He recalls that our subjectivity is formed in and through our subjected-ness to the other, prior to the development of cognition, language, modelling etc. Being-with-others (Radford, 2021) is the true and meaningful motor of learning mathematics, in that the student is engaged in the dialectical interplay between the sociocultural environment of the classroom and their reflexive agency. Resorting to coordinating as a networking strategy, we networked the theory of objectification with differentiation for all, conceived as open learning, to provide a conceptual framework that defines inclusion in mathematics as the intertwining of objectification and subjectification processes. We envisage inclusive mathematics as the production of reflective and ethical subjects who position themselves critically and creatively in mathematical practices, a condition that should be available to all students and respectful of their differences. Inclusion is accomplished via the OATLP teaching model that reflects the coordinating strategy by fostering both social interaction and individual activity in a teaching learning project that involves all students.

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Concrete–abstract–new-concrete: Freudenthal and Davydov in advancing embodied design framework

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Freudenthal and Davydov—two giants of mathematics education research—had a similar sophisticated vision on the position of mathematical models and symbols in abstract–concrete dialectics. In our vertical analysis of those approaches, we integrate them in a spiral *vision of mathematics learning. Starting from concrete enactment, students abstract previously latent aspects of their reality. Further, students fixate abstract ideas in mathematical artifacts, which in turn enable new concrete experiences. We show how local integration of two theoretical approaches can support an empirical analysis of embodied design for proportions and act as backing a design heuristic for embodied technologically supported activities.*

Introduction

The relation between abstraction and concrete experience inevitably lies in the core of mathematics education studies. Should concrete experience become a starting point for teaching abstraction? Can abstract ideas be derived from empirical observation or enactment with concrete objects, or shall abstract ideas be exposed to students directly, through 'symbolically structured environments' (Coles & Sinclair, 2019)? Traditional views on abstract and concrete—established within inductive empirical science—are questioned in contemporary investigations of mathematics learning, particularly by those who take an embodied stand.

The variety of visions on the problem of abstraction in mathematics learning creates a need to base theoretical work on a solid ground. In this paper, we coordinate the approaches of two giants in the field of mathematics education research, Davydov and Freudenthal, with respect to the role of mathematical artifacts and students' concrete experiences in teaching abstract ideas. Revealed similarities in the views of such major yet independent figures makes, in our opinion, the emerging theoretical proposal particularly strong and related design heuristics well backed up. The compatibility of views is based on Davydov's approach belonging to the Marxist tradition of cultural-historical activity theory, which can be aligned with Freudenthal's approach to mathematics as a human activity of mathematizing. Neither Davydov, nor Freudenthal used the term *artifacts*, however symbols, models, and visuals lied in the core of their ideas. We refer to these and any other instances of material culture developed within mathematical activity as *mathematical artifacts*. We interpret *cultural artifacts* as a broader category of entities developed within cultural practices.

The paper consists of two main parts. Firstly, we integrate Davydov's and Freudenthal's thinking into a joint view on concrete and abstract. For this local integration of theories (Bikner-Ahsbahs & Prediger, 2014) we conduct a *vertical analysis* (Shvarts & Bakker, 2021) as we go beyond comparing current states of the theories and dive into the history and philosophical roots of their development. Secondly, we apply the results of this theoretical analysis to guide an analysis of students' interaction with (cultural) artifacts within embodied design (Abrahamson & Sánchez-

García, 2016) and further suggest some design heuristics for technologically supported embodied activities. Two research questions guide our study: (1) What are Freudenthal's and Davydov's positions towards the role of mathematical artifacts in facilitating concrete experiences and abstract ideas? (2) How to introduce mathematical artifacts in embodied designs informed by the integrated Freudenthal-Davydov approach to mathematical abstraction?

Freudenthal and Davydov: concrete, abstraction, and mathematical artifacts

Freudenthal dedicated quite some publications to Davydov's approach (Freudenthal, 1974, 1977, 1979). Moreover, the Davydov's curriculum has been implemented in some Dutch schools, and compared to Wiskobas—a curriculum inspired by Freudenthal (Nelissen, 1987). In our view, Freudenthals' interest in Davydov is rooted in a deep agreement on the research and teaching methods. Within both research programs, intensive investigatory implementations were conducted in schools and referred to as *formative experiments* in Russia and *developmental research* in the Netherlands, presenting historical variants of what we know now as design research (Bakker, 2018). As we explain below, the similarities in the teaching methods convey insights on abstraction, the role of cultural artifacts, and the progressive development of concrete experiences.

(1) An abstraction is not based on recollection of empirical impressions

The fundamental innovation of the Davydov approach lies in an intensive critique of empirical thinking and pedagogy that treats abstraction as deriving from empirical examples (Davydov, 1990). As van Oers (2019) explains, Ernest Cassirer was apparently the first to criticize this type of abstraction because of the impossibility to limit the set of empirical observations from which to derive abstract qualities. Per Davydov (1990), new classes of objects are created within human practical activity, and theoretical thinking later describes those classes not through empirical observation but through transformative actions that reveal otherwise hidden properties. In the course of learning, students are to "develop special object-related actions by which they can disclose in the instructional material and reproduce in models the essential connection in an entity, then study its properties" (p. 174). Analyzing Davydov's approach, Freudenthal highly appreciated this perspective on abstraction and stressed that “abstraction and generality are—in many cases—not reached by abstracting and generalizing from a large number of concrete and special cases” (Freudenthal, 1974, p. 412). Later, Freudenthal tried out Davydov's approach of deriving arithmetic operations from practical actions with magnitudes—such as length and volume—with his grandson and found this approach to be effective (Freudenthal, 1977, 2002b, p. 102).

(2) Children need to reinvent mathematical models and symbols

Another point of the clear coordination between the approaches of Davydov and Freudenthal lies in addressing the role of mathematical models and symbols. Per Freudenthal, the mathematical activity consists of progressive schematization and algorithmization of solving problems that are meaningful for students. Those schematizations and algorithmizations are later preserved in the form of mathematical models and formalized rules (Freudenthal, 2002a). The rules preserve the history of problem-solving for those who came up with them in their own problem-solving. So, the only way for the learners to meaningfully extend their understanding of reality through mathematics, lies in reinventing mathematical rules and symbols. Otherwise, “having been imposed, they [rules and symbols], never had a real chance to develop into common sense of a higher order" (p. 8).

Davydov similarly assigned a primary role in scientific thinking to models, symbols, and signs. "Symbols and signs, as well as mixed forms of them, serve as the material means of idealizing and constructing scientific objectness" (Davydov, 1990, p. 121). Constructing these material means (artifacts) is exactly a process of abstraction, which fixates (reifies) the essential (for a practical activity) aspects of the object under investigation: "The construction of this new object [idealized model] functions as a certain mode of activity—as abstraction" (p. 117). In learning, children pass through a quasi-investigation, in which they uncover an essential (theoretical, mathematical) aspect of an object and reproduce it "in particular object-related, graphic, or symbolic models" (p. 174).

(3) Progressive development of concrete experiences

The origins of Freudenthal's ideas lie in the observation that mathematics education tends to inverse the development of mathematical ideas by presenting students the final products. He referred to his approach as phenomenological, and—although he insisted on the divergence with Husserl, Hegel, and Heidegger (Freudenthal, 2002b, p. 28)—he was apparently essentially influenced by phenomenological thinking. This influence can be traced in his ideas of developing *common sense*: in "the course of life, common sense generates common habits, in particular, where arithmetic is concerned, algorithms and patterns of actions and thoughts, initially supported by paradigms, which in the long run are superseded by abstractions" (Freudenthal, 2002a, p. 7). So, mathematical abstraction, such as arithmetic, derives from the common sense experience of acting and thinking. Further, those mathematical abstractions support later common sense experiences: "These products of common sense acquire in turn the behavioural status of common sense, while their common sense ancestry may have even been forgotten" (p. 7). Per Freudenthal, good mathematical education develops students' ability to *see* reality mathematically; mathematical symbolism is a lens for this *newly developed common sense*.

We find Freudenthal's idea of developing common sense to be close to the dialectical materialist method of *ascending from abstract to concrete* that Davydov exploited. This method does not mean presenting abstraction from the beginning. As Ilyenkov explains, "the ascent from the abstract to the concrete without its opposite, without the ascent from the concrete to the abstract would become a purely scholastic linking up of ready-made meager abstractions borrowed uncritically" (Ilyenkov, 2008, p. 137-138). So, abstraction starts from concrete experience, as well as progresses towards concrete experience: "the ascent from the concrete to the abstract and the ascent from the abstract to the concrete, are two mutually assuming forms of theoretical assimilation of the world, of abstract thinking" (p. 137). However, those two directions are not forward and backward. Abstraction reveals the latent aspects of initially experienced concrete reality, and those aspects further become salient in the theoretically grounded concrete perception of the objects. So, *ascending from the abstract to concrete* does not mean a detachment from the initial concrete experiences but a transformation of perception towards *seeing concrete objects in a new way—through* the lens of abstraction, facilitated by the artifacts as if superimposed on the perceptual field.

Concluding the theoretical analysis

Answering our first research question, we interpret Freudenthal's and Davydov's positions towards concrete experiences and abstract notions preserved by mathematical artifacts as converging in the following vision of the learning process. Students derive an abstract understanding from the concrete experiences within a specially organized practical problem-solving activity. This practical transformative activity elicits latent aspects of the world, which students fixate in cultural artifacts, such as mathematical models and symbols (movement from concrete to abstract). Having constituted an abstraction supported by the artifacts, students can put these artifacts into action and distinguish new initially latent aspects (movement from abstract to new concrete). In this iteration, students develop their *common sense* (in Freudenthal's words) or *ascend from abstract to concrete* (in Davydov's words) in establishing a theoretical vision of an object. Thus, we see students' development as *a spiral*: from concrete approaching the world in practical activities to abstracting latent aspects and fixating them in mathematical artifacts, and further towards establishing *new-concrete* perception mediated by those artifacts. From this approach, cultural artifacts emerge as reifications of the actions, which have elicited abstract features. Students need to actively constitute those artifacts to preserve the history of initial concrete enactment.

Concrete–abstract–new-concrete in implementing embodied design

Embodied action-based design is one of the quickly developing paradigms related to radical-embodied-enactivist-phenomenological reconsiderations within cognitive science (Abrahamson & Sánchez-García, 2016). The learning sequence in this design genre consists of a few major steps (Alberto et al., 2021; Abrahamson et al., 2020). At first, students are invited to solve a motor problem, i.e. discover a new coordination between their hands based on continuous feedback, and uncover the rule of positive feedback to their actions. Later, artifacts are introduced into the problem space, and students are guided towards the quantification of their experiences. Within this paradigm, the researchers have intensively questioned the position of embodied activities and cultural artifacts within abstract–concrete dialectics. In particular, they consider a sensorimotor scheme as "the epistemological core of mathematical learning and knowing" (Rosen et al., 2016, p. 1509), which can be further developed in both directions: towards abstract notions through semiotic signification by cultural artifacts, and towards concrete situations through providing context. Our empirical analysis of embodied activities through the lens of a joint view of Freudenthal and Davydov hints towards another role of the artifacts in abstract–concrete dialectics and further advances the design framework.

Stage 1. Action-based abstraction: Seeing new structures in concrete embodied experience

When solving a motor problem, students discover new abstract qualities,—at first at the embodied level and later in conversations with tutors—such as a proportional relation between the length of two bars, or a coordination of a unit circle circumference and an x-coordinate of a sine graph. Solving a motor problem is coherent with Freudenthal's and Davydov's ideas about abstraction as emerging from a goal-oriented practical actions. Technological environments restrict students' degrees of freedom, thus facilitating quasi-investigation, as Davydov would insist. Although restricted, the students appear to come up with a multiplicity of personal strategies and perceptual orientations, thus meeting Freudenthal's idea of reinventing rather than exposing culture. So, initial embodied enactment with concrete, tangible objects enables the discovery of abstract mathematical relations.

Stage 2. New-concrete: Looking through the lens of emerging cultural artifacts

While the phase of solving a motor problem has been extensively studied, researchers paid relatively less attention to the stage when artifacts are introduced. Therefore, we bring forth a small classroom episode from an experimental tryout of an embodied action-based design for proportions in an ordinary third grade (8-9 years old) classroom in the Netherlands (see a description of the data collection in Alberto, van Helden, Bakker, submitted). Four student pairs were video recorded and the following episode is selected to provide the best insights on the use of mathematical artifacts in establishing new understanding. Two girls (Iris and Frida) collaborated in the tablet-based activity: they manipulated two bars on a screen, which turned green when their lengths were in a particular fixed ratio. The girls were required to keep the bars green while moving and later "to guess a code" that determines the bars' green color, thus describing the proportional relation between the green bars' lengths. In the analysis, we contrast the use of two cultural artifacts: **a dice**, which was spontaneously appropriated by the students, and **a grid**, which was imposed by the educators.

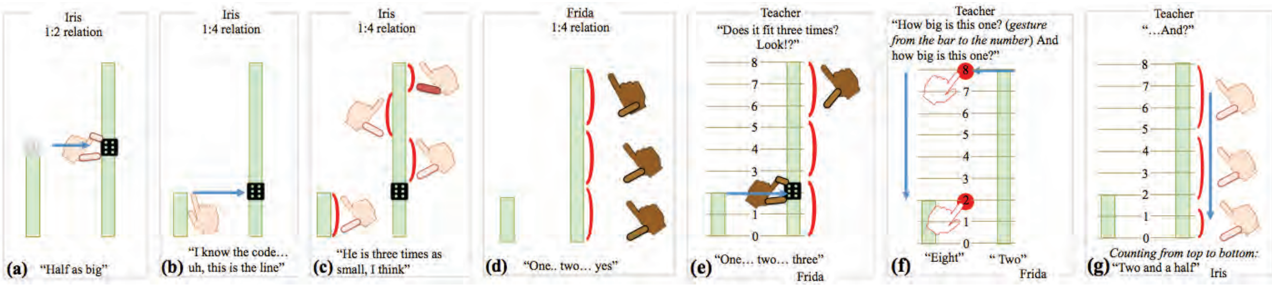

Fig. 1. (a, b, c, d): Seeing proportional relation through a dice. (e, f, g,): Missing proportional relation when applying a grid

By the moment of the episode, the girls have already solved the task with the bars being in ratio 1:2. In order to see that the length of one bar is doubled in the length of another bar, the girls spontaneously used a dice that was occasionally lying on a table: They positioned the dice in the middle of the big bar, marking that small bar fits in it two times (Fig. 1a). In the next task, the bars turned green at a ratio 1:4, but the girls did not know this yet. Adjusting one hand upwards somewhat slower than the other one, Iris found several green positions. She exhibited a general abstract strategy of maintaining two lengths in the same proportional relation, which needs to be concretized in quantifying the exact relation. Iris again took a dice and marked the length of the shorted bar on the longer bar (Fig. 1b), thus marking a unit of measurement that would help assess how many times the small bars would fit into the big one—a concretization of the abstract relation of "fitting into the other one." Then both girls tried to measure with their fingers how many times the length of the short bar would fit into the long bar (Fig. 1 c, d). The relation between two bars is seen through the lens of a cultural artifact, a dice, which here means *marking down a measurement unit*. The dice served as a reification of previous sensory-motor coordination; it allowed for distinguishing a *new-concrete*, new previously invisible aspects of the bars, namely a measurement unit that helps assess how many times one bar fits in another bar. A common sense action, in Freudenthal's terms, developed for Iris towards seeing the big bar as containing some number of small bars. Iris ascended from abstract coordination of two lengths to concrete quantification of their relation mediated by a dice (in a very physical sense). Unfortunately, a dice was barely an appropriate artifact for marking the small bar length: its own size distorted the measurement. As a result, the girls efficiently exploit an abstract idea of proportional relation as "fitting some number times in" but miscalculated the relation as 1:3 (combining two possible mistakes, see Fig 1 c, d).

The activity progressed towards the next stage of introducing cultural artifacts (Abrahamson et al., 2020), in which the girls were asked to confirm their code using an imposed grid (Fig. 1 e, f, g). With the help of a teacher, two bars were positioned at lines 8 and 2 and the bars were green. The teacher asked: "What does it mean?" expecting that 2:8 relation was obvious enough to dissolve students' 1:3 hypothesis. "Three times smaller" was the answer. The teacher invited the students to check: "Does it fit in three times? Look?" Frida did not use the grid, but took the dice (!), and marked the length of a small bar on the large bar by a horizontal gesture (Fig. 1e). Despite ignoring the imposed grid, by this horizontal gesture, Frida *re-invented* the functionality of grid's horizontal lines, as both artifacts serve the same function of marking equal units of measurement. The dice was big, and an approximate measurement led to the answer 1:3 again. Supporting the use of the imposed mathematical artifact, the teacher guided the students' perception towards the grid (Abrahamson & Sánchez-García, 2016): She gestured the horizontal alignment of the large bar and number 8 and then pointed at number 2, Frida read the numbers (Fig. 1f). However, their relation did not guide further enactment. Following the sequence of the teacher's gestures from top to bottom, Iris made a new measuring attempt counting from the top without clear measurement unit (Fig. 1g). She came up with an answer 2,5. The initial abstraction of "fitting in" was lost, and the students could not concretize (quantify) abstract proportional relation using the grid.

The dice was a natural continuation of the students' thinking and bodily enactment (see Shvarts et al., 2021 for conceptualization of this situation as a body-artifact functional system), and it allowed the girls' common sense development. By exploiting the dice, the girls could mark a measurement unit and concretize an abstract embodied idea of proportional relations in the given situation. Contrary, an imposed grid was not reinvented and stayed alien to the emerging abstraction. The teacher could see the bars' proportional relation naturally through the grid, while the girls could use the grid in this way. Their phenomenological realm did not the grid, contrary, a dice that became a mediator for a *new concrete*, i.e. for distinguishing new mathematical aspects of reality.

Towards a new design heuristic

As the theoretical and empirical analyses reveal, a cultural artifact might become a reification of practical actions, which helped to distinguish an abstract relation—a proportional relation between the bars, tangible as "small bar fitting into the big one." Importantly, an imposed mathematical artifact (a grid, see Abrahamson et al., 2020) did not fulfill this function, even with the teacher's guidance. Another artifact (a dice) spontaneously was appropriated by the students to reify an action of distinguishing a unit of measurement and served as an instrument in concretizing the ratio. Yet, this other artifact was barely appropriate for fulfilling this function. A design solution to this dilemma might be in creating an environment where students could spontaneously find suitable materials for creating the target artifacts. Such material might include thin sticks to mark the horizontal position of a small bar, paper stripes for creating a measuring unit and overlaying it on the big bar, or even a ruler, which was spontaneously and efficiently appropriated by some other pairs in the study. This way, a classroom can be enriched by appropriate means for progressive mathematizing/modeling of the situation, which could support establishing the perception and use of new mathematical aspects of concrete situations, thus distinguishing *new concrete*.

Concluding remarks

Freudenthal and Davydov are unique figures by the scale of their influence on the mathematics education and educational psychology communities. However, the program of each of them does not flourish nowadays in the countries where they were working despite intensive and successful experiment-based elaborations. Our analysis brings forth the complexity of their understanding of mathematical abstraction and concrete experience. Those approaches aim to develop in students a new ability to see an object concretely within its mathematical interrelations, i.e., developing *a new common sense*. From a concrete action-based experience, students come to distinguish abstract relations that are later reified in cultural artifacts. Further, the artifacts come to illuminate their *new-concrete* experiences. We hope that contemporary technologies can support students and teachers in fulfilling the aim of learning to see the world mathematically. This type of mathematics learning is in particular valuable for the 21st century with routine calculations being outsourced to the machines and increasing importance of skills such as mathematical modeling and recognizing mathematical patterns in everyday situations (Gravemeijer et al., 2017).

Looking back at the interaction of theoretical ideas and design heuristics, we notice that we used two prominent theoretical approaches as a way of backing a design idea that has been already emerging in our design work and empirical data. We uncovered the essential coherence of two approaches in seeing cultural artifacts as instruments that transform students' concrete experiences. The fact that those approaches are widely recognized as highly valuable strengthens the design heuristic of re-inventing mathematical artifacts and its potential for curriculum design.

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Mathematical picture of the world and understanding

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The aim of the article is to analyze the phenomenon of understanding in mathematics. Two concepts are introduced: mathematical picture of the world (MPW) and individual mathematical picture of the world (IMPW). A distinctive feature of the work is that understanding in mathematics is treated as a process of forming IMPW. The basic element of IMPW and its structure are distinguished. It is accepted that a coherent and complete representation of a basic element of IMPW means understanding of a mathematical concept. The paper offers a scheme enabling the formation of mathematical concepts among students.

Keywords: Understanding, image, meaning, personal meaning

Introduction

Problems related to understanding in mathematics have long been the subject of discussion (e.g., Pirie & Kieren, 1989; Sierpinska, 1990; Skemp, 1976; Tall & Vinner, 1981). Skemp (1976) divides understanding into two categories: relational understanding and instrumental understanding. The difficulty is that remembering a large number of common problems or tasks is already a problem. Tall and Vinner (1981) introduce the *concept image*, which is a total cognitive structure bearing a definition of a mathematical concept. Sierpinska (1990) offers a methodology of "acts of understanding". Pirie and Kieren (1989) conceive of understanding as a dynamic, transcendent, recursive, non-linear process. Even though decades have passed since these studies were published, but the problem of understanding in mathematics continues to be of interest to researchers.

It is clear that a subject cannot identify, classify, name any object unless they have a corresponding *mathematical picture of the world* (MPW), an image of the world (according to Leontiev's (1975, p. 73) terminology). Since mathematical concepts cannot exist in isolation from each other, learning mathematics is the construction of *an individual mathematical picture of the world* (IMPW) in the learner's mind. Mathematical development occurs as the formation of a holistic structure (network) through its relationship with other concepts, namely, through the formation of MPW. The aim of this paper is to investigate understanding as the formation of an MPW of the learner.

Theoretical background

Further, we will proceed from the definition: "A world picture is a general idea of the world, its structure, types of objects and their interrelations. All world pictures are distinguished according to two main features: 1) degree of generality and 2) means of modeling reality" (Lebedev, 2004, p. 48). Stepin (2003) has made a classification of *scientific pictures of the world*.

The objective mathematical picture of the world is a "horizon of systematization of knowledge" (Stepin, 2003, p. 213, own translation) in mathematics. This MPW represents a base of objective mathematical data for the students; it participates in the formation of mathematical concepts and relations. The problem in teaching math deals with how to form an MPW in the consciousness of a learner. The formation of an IMPW means that the learner is able to identify the basic concepts of a studied area of mathematics and to build a network linking the basic selected mathematical concepts.

The ideas of Vygotsky's cultural-historical approach (2005) and Leontiev's theory of activity (1975) became a psychological foundation for constructing the basic element of the IMPW. Vygotsky proposed to consider the social environment as the main source of personality development that determines the formation of higher mental functions of an individual. The main feature of higher mental functions is that they are mediated by certain *psychological tools*, namely, the sign systems that have arisen as a result of the long socio-historical development of mankind. Sign systems include, for example, language, writing, speech, and counting systems. Mental function, such as attention, memory, thinking, and imagination pass the stage of external activity in which the cultural means—*psychological tools*—have an object form and are related to a certain sequence of actions. This sequence is then interiorised (i.e. it is transferred from the external to the internal plane). From Leontiev's point of view (1975), the condition for adequate perception of an individual object is the adequate perception of the object world as a whole and the object's relation to this world. Leontiev stressed: "The predetermination of this indicated, meaningful subject world to a concrete act of perception, the necessity of "inclusion" of this act in an already ready picture of the world" (1983, p. 36, own translation). According to Leontiev's theory of activity, the existence of the IMPW and the MPW is interdependent and necessary, since the formation of the IMPW can be imagined as "drawing out" a subjective image of the world (world picture) from the objective MPW. The IMPW is the imprint of the learner's interaction with the MPW.

The thinking process is carried out with the help of sign system and the accompanying idea of semiotic mediation (Presmeg et al., 2016). As a result of multi-level abstraction, each mathematical object is represented by a special sign, which forms, together with the rule of operations on them, a mathematical language. A prerequisite for students to build their MPW is the knowledge content of the names of the objects being studied and their meanings, as well as having motivation. Any sign denotes a concept and connects a concept and a name into a single whole, therefore finding out the structure of a sign reveal the structure of a concept as well. According to Leontiev, any sign consists of three components: an *image*, which is "an encoded representation of the object of activity; *meaning* as a transformed, coiled in the structure of language ideal form of existence of the objective world, its properties, connections, relations revealed by social practice" (1975, p. 68, own translation); *personal meaning* or "meaning for me, which connects the available experience of the subject of activity, and the *meaning* of the object. Let us use the structural representation of the sign given in Panov (2015, p. 42) and specify that the sign is interesting for us first of all because it denotes some mathematical concept. The sign and its structure can be represented in the form of a pyramid. A subject’s awareness of the pyramid structure and relations between its components means understanding of the concept. The basic element of the subject's MPW is this pyramid, which can be called a pyramid of understanding (Figure 1). The base of the pyramid represents the essence of a mathematical concept (triangle of understanding). Consider the components of the pyramid of understanding.

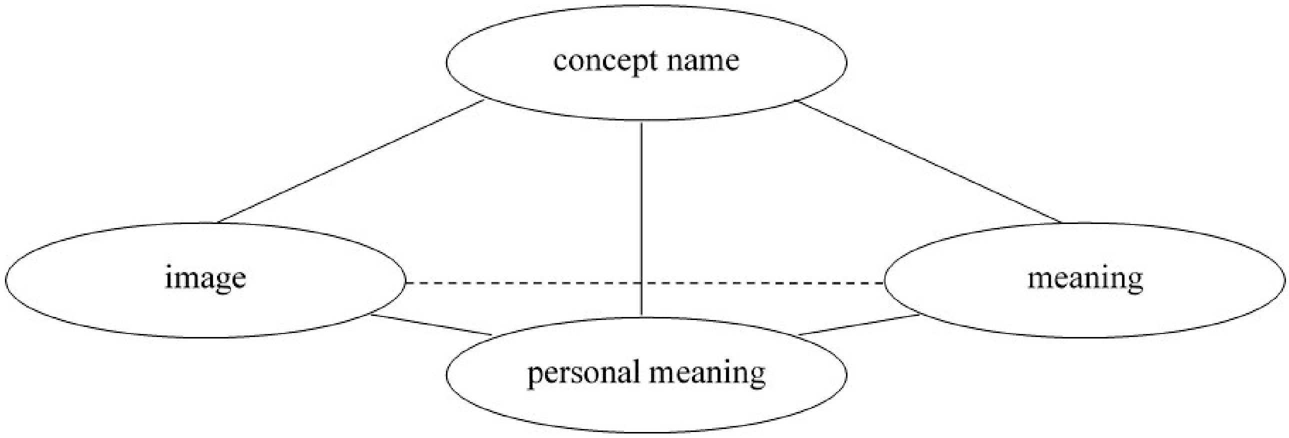

Figure 1: The Pyramid of Understanding

The *image* of a mathematical concept is created not so much by direct perception, but with the help of five processes: imagination, memory, a certain idea in the process of abstraction, generalization, and idealization of objects and phenomena. In contrast to perceptions, which have a sense-object character, the mental image of a concept reflects its content, which expands and enriches as the mathematical experience of the cognizing subject is refined. Mathematical imagery, on the one hand, has the function of supporting mathematical thought and, on the other hand, it is an inseparable part of the conceptual structure. If mathematical images are formed, the learner operates not so much with the given algorithms of working with mathematical objects, as with the mathematical objects themselves. The presence of images allows the mathematical concept to be presented holistically. Herewith only controlled images (Aspinwall et al., 1997) serve the formation of adequate mathematical representations.

Meanings are the result of cognitive activity of people, a system of laws. Meaning is transferable without loss of sense. A set of meanings is a systematically organized knowledge base that exists objectively. The concept of meaning expresses the connectivity of individual consciousness to social consciousness. The concept of meaning captures the fact that human consciousness develops within a cultural whole, that is, within an MPW, in which the experience of mathematical activity, communication and world view is historically crystallized, which the individual must *appropriate*, thereby creating an IMPW.

Personal meaning or "meaning for me" connects objects and properties of reality with the experience and present needs of the subject, and is always emotional and sensitive in nature, giving the picture of the world a special individual nuance. Personal meaning determines the extent to which a given object or process or attitude corresponds to the needs the subject's needs. The concept of personal meaning indicates that understanding cannot be reduced to impersonal knowledge. Rather, it expresses the rootedness of individual consciousness in human being. Understanding presupposes a close connection between the learner's interests and the problem to be solved.

Osipov and colleagues state that "an image without meaning reflects incompleteness of perception as a categorization process, while *meaning* without *personal meaning* reflects the learner's non-involvement in the activity" (2017, p. 75, own translation). Naming connects the three components into a unified structure, so that the sign and its components become elements of the subject's mathematical language system. In this way, the concept is incorporated into the IMPW.

If image, meaning and personal meaning do not form a coherent whole in the learning process, the image is called a *percept*, that is, an image of perception; in this case the meaning is only used for this particular situation and called a *functional meaning* (Panov, 2015, p. 38). The functional meaning of an object allows the learner to use it to carry out specific operations to solve concrete tasks.

Most often in the course of learning elements of a concept (triangle of understanding) are formed at the beginning. Here is the scheme of thought movement from percept to the functional meaning and from here to the choice of *goal* (i.e. to satisfaction of a concrete, temporary need). Because in this case the pyramid of understanding is not closed, this approach does not allow looking at a mathematical concept holistically, so there is a need to give the name of the concept (Osipov et al., 2017). The model constructed allows us to outline a six-item scheme of reasoning (presented next), which is useful for facilitating an understanding of mathematical concept.

1. Based on the similarity or difference with an object known to the subject, the learner's perception is formed independently or with the help of the teacher. The teacher's presentation of the image presupposes the preservation of the essential geometrical properties of the object and should take into account students' practice of using it. A mathematical concept should be presented in different ways, because different students usually have different visualizations.

2. The initial percept is superimposed on the student's past experience of using the given or similar image, which forms the functional meaning of the given concept and leads to the primary pair *percept – functional meaning*. On the basis of the percept formed in item 1 and the existing experience, the student develops an idea of what can be done or what this percept allows to be done. Thus, the formation of a functional meaning takes place. The process of forming this relationship is dynamic (i.e. an initial perception emerges). For this perception the student looks for a suitable image for a perception with an appropriate application, which may not be accurate. Then the initial image is changed or refined depending on the function of the image. At this stage, the role of the teacher is important in helping in establishing the *percept – functional meaning* connection.

3. The functional meaning obtained in item 2 is then compared with the goal (i.e. the suitability of the functional meaning for solving the problem is assessed). If the obtained functional meaning does not enable the problem to be solved, the student revises the initial perception independently or with the help of the teacher. This process (percept ↔ functional meaning) is repeated as many times as necessary until the functional meaning suitable for the solution of this particular problem is determined and the process is usually interrupted by the teacher.

4. The functional meaning obtained in item 3 is compared with the name characterizing this functional meaning taken from the MPW. If the meaning of a concept accepted by mathematicians is close to the functional meaning, a *name of the sign – meaning* relationship emerges. It is linked to the original percept, which from this point can be called an image. If it turns out that the functional meaning and the exact accepted meaning are not close, there is a return to the formation of the percept. This convergence of meanings is carried out through communication with the teacher in the form of a

dialogue. It is desirable to carry out a dialogue in a flexible, probabilistic form. In general, this item implies a repeated transition to item 2, which creates a certain cyclical process (item 4 $\rightarrow$ item 2 $\rightarrow$ item 3 $\rightarrow$ item 4), which is interrupted when the student achieves an adequate image concept. The teacher intuitively knows when interrupt the process.

5. Image formation is a dynamic process because it is, usually, the result of repeatedly going through the sequence from item 1 to item 4, and it influences the student's initial need for solving the problem. In addition, it enriches and generalizes the student's idea of the problem, generating personal meaning from the temporal need. Thus, through the pair *name – functional meaning* there is a connection with the name of the concept, which means that the functional meaning has turned into meaning and the temporal need has turned into personal meaning.

6. Ultimately, understanding means that a learner knows the sign structure of the given concept, which is made up by binding image, personal meaning and meaning through the name of a concept. Independently or with the help of the teacher, the learner interconnects all components of the sign.

In contrast to Panov (2015), the scheme presented here is more flexible, as the discussion of a concept can start not only with the image, but also with other items, and also takes into account the fact that formation of a concept usually takes place with the help of the teacher. Let us consider an example of a dialogue, which shows how the above scheme functions in real practice. The conversation is with the 1st year bachelor students of the Institute of Transport Structures of Kazan State University of Architecture and Engineering. At the lecture before the conversation the definition of a function and, in particular, linear function was given. Below P stands for the professor, S1 to S5 for the students.

1	P:	What can you say about the function $y = 2x + 3$?
2	S1:	It is a straight line on the plane.
3	P:	How is such a function called?
4	S:	I don't know exactly, but e.g. The function $y = x + 3$ is the same.
5	P:	Is the function $y = 3x$ a straight line in the plane?
6	S:	Yes. (Makes a drawing).
7	P:	Is a straight line a function $y = -2x$ or a species $y = ax + b$ in general?
8	S1:	(Thinking). Yes. Such functions always define a straight line in the plane.
9	P:	How are the variables x y related to each other and in the case of a straight line?
10	S1:	When the argument changes by the same amount, the value of the function also changes by the same amount. Increases x, increases also y, for example, for a function $y = 2x + 3$.
11	P:	And how exactly does the value of the function change when the value of the argument changes?
12	S1:	It simply increases.
13	P:	Take the function $y = 2x + 3$ and calculate the value of the function at $x = 1$, and then calculate the value of the function at $x = 2$.
14	S:	For $x = 1$ we have $y = 5$, and for $x = 2$ we have $y = 7$.
15	P:	Now calculate the difference of the function values and find the value of the function at $x = 3$.
16	S1:	This difference is 2 and the value at $x = 3$ will be $y = 9$.
17	P:	Find now the difference of the values at $x = 3$ and $x = 2$.
18	S1:	The difference is equal to 2. The same difference as was before.
19	P:	Yes. When the argument changes by the certain amount, the value of the function also changes by the certain amount. This is true for any function of the form $y = ax + b$. You can see it on your own. (Students do it in general

		form and confirm the conclusion). This function is called a linear function. In which problems can you observe a dependence of the form $y = ax + b$?
20	S2:	In speed problems. When moving at a constant speed. When moving at a constant speed, the distance as a function of time changes according to the formula $s = vt$. Here s is distance, v - speed, t - time.
21	P:	What do you imagine when you see a linear function of the form $y = ax + b$?
22	S3:	I imagine a straight line on a plane.
23	S4:	And I see uniform motion.
24	P:	Uniform rectilinear motion?
25	S4:	Yes. It turns out that a linear function in the plane is always represented by a straight line. And are there other applications of functions of the form $y = ax + b$?
26	P:	There are many such applications in physics, mechanics. For example, Hooke's law $F = k\Delta l$ is known in the theory of elasticity, where F is force, which stretches (compresses) a rod, Δl is absolute elongation (compression) of the rod, k is known coefficient of elasticity. We see that the force is a linear function of Δl.
27	S5:	Ah ... ah, we had Hooke's law. It turns out to be a linear function.
28	P:	So, a function of the form $y = ax + b$ is called linear, here a called the angular coefficient, b is the free member. So, what is a function of the form $y = ax + b$ called?

The analysis of the dialogue shows that in lines 4, 6, and 8 a percept of the concept of linear function is formed (this corresponds to the first item of the scheme of actualization of understanding), and from line 10 the formation of a connection between the percept and the functional meaning begins. This continues in lines 12, 14, 16, and 18 (items 2 and 3 of the scheme apply here). The student at this stage can use the concept of linear function only concretely, without presenting the general properties of a linear function. In lines 20, 22, 23, and 25 the functional meaning is associated with the name, which leads to an awareness of the image of the linear function (corresponding to item 4 of the aforementioned scheme). Line 25 shows the manifestation of personal meaning, as it becomes important for the students to apply the concept of linear function in other areas, which means the implementation of item 5 of the scheme. Line 27 already shows that the name of the function has been associated with the meaning and personal meaning. Line 29 shows the students' understanding of the concept of linear function within our model. Lines 25, 27, and 29 correspond to item 6 of the scheme.

Results

This article offers a model how students learn mathematics by forming an IMPW. The IMPW is a network consisting of basic elements of mathematical picture of the world (network nodes). The basic element of the IMPW is seen as a structure having the following components: image, meaning, and personal meaning. By connecting these three components through naming, a holistic structure, the pyramid of understanding, can be formed. In this paper it is accepted that a holistic and coherent representation of the pyramid of understanding means the student's understanding of the concept.

Discussion and conclusions

Understanding does not happen in isolation, it means *getting into* a network of known relationships and connections (i.e. fitting into a certain picture of the world, so understanding can only be realized within a certain picture of the world). The subject's picture of the world is a sign-mediated picture of the world.

Tall and Vinner (1981) introduced *concept image* and *concept definition* to carefully analyze students' understanding of limits and continuity, as well as other concepts. The concept image can be attributed to the image component of a sign in our model, while the concept definition is consonant with a functional meaning.

Serpienska's model (1990) proposes a methodology of acts of understanding through the categories of identification, discrimination, generalization, synthesis. Skemp (1976) defines understanding in terms of *knowledge* rather the functional meaning of the concept. Pierie and Kieren's model (1989) takes into account both image representation of mathematical objects and abstract, operational representation. The proposed model of understanding in mathematics does not contradict the main known models of understanding, but complements them, considering the social and psychological dimensions of understanding in unity.

This paper proposes a new model for understanding mathematical concepts that link the components of understanding (meaning, image, and personal meaning) into a coherent whole through naming. Particularly, the model allowed us to recognize the importance, value and power of the following ideas:

1. Understanding cannot be viewed only in a social or psychological framework (as most of the mentioned authors in the article suggest), nor can it be reduced to one of these.

2. The introduction of personal meaning into the structure of understanding is a new view. Personal meaning is not only a condition of understanding, but is a necessary component in the structure of understanding. Understanding cannot be reduced to impersonal knowledge.

3. The introduction of the concept of mathematical picture of the world is a new idea. The MPW is intrinsically necessary as a fundamental condition of cognitive activity of students. The IMPW is an imprint of the MPW in the learner's mind.

It seems that further research could go in the following direction: wide testing of this approach for different sections of mathematics.

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Metaphor, Enaction and Adidactical Situations in Mathematics Education

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We aim at exploring the relationships and intertwinings of the theories of metaphor, a-didactic situations and enactivism in mathematics education. We present and discuss some illustrative examples of the didactical implications of these theoretical approaches, involving geometry and probability, drawn from our teaching to a rather broad spectrum of learners, including humanistic first year university students, besides mathematically inclined students.

Keywords: Metaphor, Enaction, Enactivism, Adidactic Situation, Mathematics Education.

Introduction.

Our aim in this paper is to focus on three cognitive theories which are especially relevant to mathematics education, and which we have been exploring for more than a decade, to wit: the theory of metaphor (English 1997; Lakoff & Núñez, 2000; Sfard, 2008, 2009; Soto-Andrade, 2014, 2018), the theory of enaction (Varela, Rosch & Thompson, 1991; Riegler & Vörös, 2017) and the theory of (a)-didactic situations (Brousseau, 1998; Brousseau & Warfield, 2014). We are interested in the relationships and intertwining of these theories as well as in their epistemological stances. Operationally these relations may be complex. For instance, in what follows, we use metaphorising as a meta-theory to describe theories, which we see just as "unfoldings" of *poietic* metaphors (Soto-Andrade, 2014, 2018). This applies also to metaphor itself, in a circular way (cf. Ricoeur, 1975). Also, Brousseau's theory of a-didactic situations, an unfolding of the emergence metaphor for mathematical concepts, may be seen as a forerunner of enactivist approaches to mathematics education. Indeed Brousseau coined himself the term "experimental epistemology of mathematics" to refer to his theoretical approach, in the 60's, at the same time that Maturana and Varela (1992) set up their laboratory with the same name at the University of Chile.

We recall first the main tenets of metaphor theory, enaction (and enactivism) and a-didactic situation theory. We discuss then some concrete examples of their implementation with a rather broad spectrum of learners in Chile, involving math inclined as well as humanistic first year university students.

Metaphorising in cognitive science and mathematics education.

Notice that we focus on *metaphorising*, the *action* of "looking" at something and "seeing" something else, carried out by a cognitive subject, instead of *metaphor*, the *object*, independent of an observer. It is acknowledged nowadays that metaphor serves as the often-unknowing foundation for human thought (Gibbs 2008, Sfard, 2009) since our ordinary conceptual system, in terms of which we both think and act, is fundamentally metaphorical in nature (Johnson and Lakoff, 2003). In our view, though, the classical theory of metaphor is just "the tip of the iceberg", regarding human cognition. Metaphors are not just rhetorical devices but powerful cognitive tools that help us in building or grasping new concepts, as well as in solving problems in an efficient and friendly way (Chiu 2000;

Diaz-Rojas and Soto-Andrade 2015; English 1997; Lakoff and Núñez 2000; Sfard 2008, 2009; Soto-Andrade, 2014; and many others).

Lakoff and Núñez (2000) highlight the intensive (and often unconscious) use we make of *conceptual metaphors* that appear—metaphorically—as inference-preserving mappings from a more concrete or down to earth 'source domain' into a more abstract or opaque 'target domain', enabling us to fathom the latter in terms of the former. In our view, *representations* go the other way around. Indeed, we *re-present* something given beforehand, typically to explain it, and we usually *metaphorise* to try to fathom something unknown or not yet constructed. Our approach to the learning of mathematics emphasises the *poietic* (from the Greek *poiesis* = creation) role of metaphorising, which brings concepts into existence. For instance, we create the concept of probability when, while watching the random walk of a "fair frog" jumping equally likely right or left on a row of stones in a pond, we "hallucinate" (Seth, 2021) seeing it *splitting* into two equal halves that go right and left! This "metaphoric sleight of hand" turns a random process into a deterministic one, and we just need to keep track of the frog's splitting into pieces to answer "impossible questions" like "Where will the frog be after *n* jumps?". The probability of finding the frog at a given stone after *n* jumps is *constructed* as the *portion of the frog* landing there after *n* splittings. (Soto-Andrade, 2018).

In this connection, it is pertinent to recall that in the German school of didactics of mathematics, originally mostly concerned with primary mathematics education and going back to Pestalozzi (vom Hofe 1995), representation and metaphor were ubiquitous: as *Darstellung*—representation, aiming at explaining something to others—and *Vorstellung*—a personal way to figure out or fathom something, operationally equivalent to metaphor (Reyes-Santander and Soto-Andrade 2012). So metaphorising has a long history in German didactics of mathematics, well before its irruption from cognitive psychology and linguistics into mathematics education (Lakoff and Núñez 2000).

The ubiquity of metaphorising in mathematics education should not be underestimated: Besides bringing into existence mathematical concepts or objects and helping learners to fathom them, unconscious metaphorising often dramatically shapes the way teachers teach, for instance. A foremost example is afforded by the "second nature" metaphor 'teaching is transmitting knowledge'. Unperceived here is the 'acquisition metaphor' (Sfard 2009, Soto-Andrade 2014), that sees learning as acquiring an accumulated commodity. This is criticised in Plutarch's metaphor: 'A mind is a fire to be kindled, not a vessel to be filled' (Sfard 2009). Paraphrasing Bachelard (1938), who advocated epistemological vigilance, we suggest nowadays to practise metaphorical vigilance, i.e., the art of noticing our unconscious or implicit metaphors, that shape our way of interacting with the world (even to construct it) and particularly our approach to teaching and learning.

Last but not least, metaphorising plays also a key epistemological role. We have claimed elsewhere (Diaz-Rojas and Soto-Andrade 2015) that, metaphorically, a theory is just the 'unfolding' of a metaphor (a process, that may be laborious and technical, though). A paradigmatic example is the 'tree of life' metaphor in Darwin's theory of evolution. We will use below metaphorising as a meta-theory to describe other theories relevant to us in terms of their generating metaphors, something more helpful to fathoming how they arose than just describing them a posteriori.

Enaction and enactivism

An unfolding metaphor for enaction is Machado's poem (Machado 1988, p. 142): "Caminante, son tus huellas el camino, y nada más; caminante, no hay camino, se hace camino al andar" [Wanderer, your footsteps are the path, nothing else; there is no path, you lay down a path in walking]. Indeed, Varela himself metaphorized enaction in this verse (Varela, 1987, p. 63), when he introduced the enactive approach in cognitive science (Varela, Thompson, & Rosch, 1991) writing: "The world is not something that is given to us but something we engage in by moving, touching, breathing, and eating. This is what I call cognition as enaction since enaction connotes this bringing forth by concrete handling". He also metaphorised it with the well-known *Drawing Hands* lithograph by Escher where each hand draws the other into existence (Varela, 1984).

Key aspects of enaction are: perceptually guided action, embodiment and structural coupling through recurrent sensorimotor patterns (Varela et al. 1991; Reid and Mgombelo 2015). In an aphorism: 'All doing is knowing, and all knowing is doing' (Maturana and Varela 1992, p. 26). Notice *en passant* how the 'laying a path in walking' metaphor is transversal to the traditional one for learning as following a pre- given well-marked path.

Enaction in mathematics education may be traced back to Bruner (1953), who was following the traces of Dewey's (1910) "learning by doing", when he described the *enactive, iconic and symbolic* modes of representation. Bruner's enaction, which means essentially acting out or doing, is however far less radical than Varela's, in that it does not challenge the notion of a reality 'out there', that we represent to our selves more or less successfully. Dewey, however, emphasised the role of sensorimotor coordination in perception, acknowledging that movement is primary and sensation is secondary (Gallagher and Lindgren 2015).

In what follows, to to avoid confusion between Bruner's and Varela's enaction, we use the now prevalent terms *enactivism* and *enactivist* to refer to Varela's anti-representationalist 'enactive program', which sees cognition as embodied action, more precisely, cognition as enaction. We will speak then of an *enactivist approach* to problem solving or to mathematics education. On the other hand, unless otherwise explicitly stated, 'enact', 'enacting' and 'enactive' are to be understood in the sense of everyday language and also in the sense of Dewey (1997) and Bruner (1966), i.e., as synonyms of 'acting out' or 'acted out', in an embodied way. So 'enacting a metaphor' just means 'to act it out', with your body, as in Gallagher and Lindgren (2015), where they refer to 'enactive metaphors' (metaphors in action, that we act out bodily) as opposed to what they call 'sitting metaphors'. We use 'enactive metaphorising' below in this sense.

Learning is neither determined by a didactical environment nor a result of teaching; it arises from the interaction of the learner's structure and environment, which plays at most the role of a 'trigger'. Traditionally, however, problem solving involves problems given beforehand, lying 'out there', waiting to be solved, independently of us as.. In the enactivist perspective, because of our structural coupling with the world (Varela et al. 1991), we bring forth emergent problematic situations instead. This is what Varela calls problem posing, opposing the usual *gas fitter metaphor*, where solvers look into their toolboxes of predefined strategies and choose the appropriate one for solving the problem at hand, instead of mathematical strategies emerging continually in the interaction of solver and problematic situation. In an enactivist didactics of mathematics, the teacher is an enactivist practitioner acting in situation and learning is an emergent, situated and embodied process (Proulx &Simmt 2013). For a survey of enactivism in mathematics education, see Goodchild (2014).

According to Varela, we are always 'enacting' a world, most of the time unconsciously. So we cannot choose to enact or not to enact (in his sense); enaction is just the way we cognise as living beings. We may nevertheless entertain the 'representationalist illusion' (a privilege of humankind!) that we are perceiving and representing an objective reality 'out there'. Also, we can choose to enact (in the everyday sense of the term, of bodily acting out) a given metaphor or situation or not, for instance. Paradoxically, we are definitely able to teach in a way that ignores enaction (in Varela's sense) and does not allow for enacting (as bodily acting out): a non-enactivist stance that paves the way for cognitive bullying to the learners. Our enactivist approach to education, distilled in the 'lying down a path in walking' metaphor for cognition and learning, leads us on the contrary to foster metaphor enacting among the learners.

Adidactic situations and didactic contract

The theory of didactic situations (Brousseau, 1998; Brousseau & Warfield, 2014) may be seen as an unfolding of the "emergence metaphor" for mathematical content. Indeed, mathematical concepts or procedures we intend to teach, instead of being parachuted from Olympus as in traditional and abusive teaching, should *emerge* in a suitable challenging situation the learner is enmeshed in, as the only means to "save her/his life". This type of situation is called a *didactic situation*, because of the avowed didactical intent of the teacher who set it up. It becomes an *a-didactic* situation when the teacher steps back, to let the learners interact (enactively) on their own with the situation, unable to fathom beforehand the teacher's didactical design or the mathematical content she is aiming at.

Brousseau (1998), aiming at accounting for the actions and reactions of the partners involved in a didactic situation, also introduced the metaphor of a "didactic contract" as an interpretative embodiment of their mutual expectations, beliefs and commitments (Brousseau, Sarrazy & Novotna, 2014). This sort of contract is necessarily tacit and unspoken; its effects are nevertheless often perverse and dramatic. Indeed, the prevailing didactic contract in our classrooms usually thwarts idiosyncratic metaphorising and enacting among the learners.

Illustrative examples

We discuss some examples, in geometry and probability, drawn from our teaching at the University of Chile, which illustrate important aspects of the implementation of our different theoretical perspectives and their intertwining. Our learners include prospective secondary school math teachers, undergraduate math students, humanistic first year students and in service primary school teachers. They usually work in (random) small groups of 3 to 4, and they were observed by teacher and assistants, in the spirit of ethnomethodology (Ingram & Elliot, 2020).

Example 1. Angles in polygons and stars.

One radically enactivist way in which we approached this topic (which includes the classical "sum of inner angles and exterior angles" problems) was to just show the students or participants in a workshop, the weird irregular seven-pointed star of Figure 1 and keep silent, waiting for their reactions to arise. If a student asks: "But, which is the question?", our answer is: "That is the question!"

Participants and students were at first puzzled and intrigued. Some even said “Splash!” Especially, in service teachers and mathematics educators tried to make sense of the star by *decomposing* it somehow, into triangles. The reaction of *deforming* the given weird star to a friendlier regular, one did not appear spontaneously. After some prompting: Do you like this star? What would you like to do with it? they recalled the magician’s star, or the ubiquitous regular star in our flag. Most students did not dare to change anything, because of the prevailing didactical contract: the star was not an object to be explored. Primary school teachers noticed the way the star was drawn, like the usual regular five pointed star, approaching the problem in an enactive way in the sense of Bruner (1966), see Fig. 2. They recalled that for the regular five pointed star, they could calculate easily the values of the inner acute angles, but not so with irregular stars. However, by deforming enactively their drawing of the star they realised that stretching one arm of the star decreases the corresponding inner acute angle and increases the others. So the conjecture emerged that maybe the sum of all inner acute angles was *preserved* under deformation, a clever insight indeed! Notice that this way of thinking: looking for invarians under deformation of a system, is unfortunately not much fostered in our math classrooms.

Figure 1: A weird seven- pointed star.

Our approach was radically enactivist in that no specific problem or task was given to our subjects to solve, but only a situational seed or germ for them to explore and make sense of. They were enactive in Bruner’s sense when they drew the stars themselves and deformed it. This enaction allowed them to bodily feel the compensation between increasing and decreasing inner angles, when varying their free hand drawing.

Nevertheless, a more enactive approach to the angle sum problem is possible, inspired by the “laying down a path in walking”. We have indeed described elsewhere (Soto-Andrade, 2018) how the value of the sum of the *exterior* angles of a polygon may be “seen”, with no calculation, just by “lying down a polygon in walking”. When you go around the polygon, inflecting your rectilinear trajectory at each vertex as needed, you are summing up all exterior angles! In our case, students were able to “lay down a star in walking”! And so they gleaned immediately the value of the sum of its inner acute angles. They realised that the most clever walking, among various possibilities, is to follow the path that we lay down when drawing the star, starting as indicated by the arrow in Figure 2. To add up the inner angles at the points of the star, we should be able to “span” or “sweep” them somehow, while walking across the star. If metaphorising is already unleashed among the students, many avatars of the walker may emerge. Among them, a metaphorical hummingbird flying back and forth along our star.

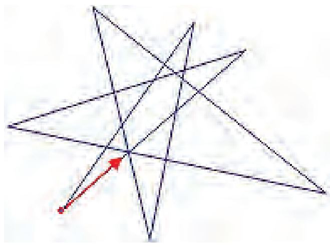

Figure 2: The irregular seven pointed star revisited.

The bird begins its flight as indicated by the red arrow in Fig. 2, but when arriving to the next vertex, it sweeps (i.e. spans) the corresponding inner angle with its *tail* and then flies *backwards* along the next edge. Then it sweeps the inner angle at the next vertex with its beak and flies forwards, and so on. So it will sweep the inner angles: tail-beak-tail-beak-tail-beak-tail, to end up at the initial vertex again, but looking in the opposite direction. Moreover, during its flight, it always turned its beak clockwise at each vertex. So *our flying hummingbird teaches us that the sum of all inner acute angles is just half a turn*. A playful enactment arises, where one student enacts the hummingbird's flight, wearing a fake beak and tail, and another keeps track of the rotation of the beak or tail. In one of our workshops, a couple of primary school teachers discovered this idea on their own, walking across the star holding a duster.

We see that in this case an enactivist approach to a "concrete" geometric problem triggers all the same significant idiosyncratic metaphorising among the learners, which is also enactive (in Bruner's sense).

Example 2. Various avatars of the frog's random walk.

We claim that random walks constitute a royal road to probability, because they are ubiquitous, they constitute models for sundry probabilistic problems, which can be easily enacted and simulated, and when approached metaphorically, they allow learners to construct along the way the "abstract" notion of probability (Diaz-Rojas & Soto-Andrade, 2015; Soto-Andrade, 2013; Soto-Andrade, Diaz-Rojas, Reyes-Santander, 2018).

A typical example is the case of a frog random jumping asymmetrically between just two stones A and B in a pond (with a non-zero probability of remaining stationary), which may be seen as a metaphor (or a model) of an unmerciful market struggle between two yoghurt producers A and B, which month after month seduce each other's consumers according to a fixed pattern. The random walk is a metaphor for the market struggle, or the other way around, according to whether the involved learner is more acquainted with jumping frogs or with market struggles.

In our courses, we told the students the tale of Filomena, a frog which jumps randomly but symmetrically (like heads or tails, to its two next neighbours each time), on a row of 6 stones (tagged 0 to 5), starting at stone 3. Stones 0 and 5, however, are in fact two lurking alligators, called Anibal and Artemio, camouflaged as stones. Filomena will be instantly swallowed if it ever lands on the head of either alligator. Naturally, the students wonder about Filomena's fate...

We also told these students about the classical "ruin problem" : two players *A* and *B*, having a certain amount of euros each, flip repeatedly a fair coin to decide who wins. After each flip, the winner receives one euro from the loser. No credit is available, so the game ends when any player runs out of money (is "ruined"). In our case, *A* would have an initial "fortune" of three euros, and *B* would have only two (metaphorically, A's ruin means that Filomena was eaten by Anibal). Sundry questions arise, like: *How likely is that in the long run A, or B, becomes ruined? How likely is that the game goes on forever?* Students tackling the ruin problem might move metaphorically to the frog's walk, and then in turn, to a splitting process or to a water draining process. Notice here that metaphorising involves a cascade of metaphors, so the skill to move between them becomes significant for the learners.

We were interested in investigating students' reaction when being proposed both the frog's random walk and the ruin problem (with the same data) in a (prepandemic) test. We found that our students, who have being exposed to traditional teaching leaving no room for metaphorising, had trouble in recognising both problems as the "same", but in different guises. Some even solved one right, and the other wrong! A few, less than 5%, realising that they were given the same problem twice, did not dare to say so; another effect of the prevailing didactic contract, we deem.

Discussion

From our viewpoint, the theories associated with enaction, metaphorising and a-didactic situations, besides helping us understand complex phenomena in mathematics education, provide us with an epistemological perspective, which positions us in a particular relationship with what we observe. That is, we see the enactive as inherent in the learning-teaching processes of mathematics, beyond the merely embodied. Likewise, metaphorising would be a ubiquitous emergent, whether or not we are aware of its presence or its role in mathematics education. They would both be, therefore, inevitable phenomena.

The perspective resulting from embracing these theories forces us to rethink the pedagogical chore, the didactics, the methodologies, the methods. In this way, it becomes necessary to open up spaces (construct situations, contexts) where the emergence of (idiosyncratic) metaphorising is welcomed as a legitimate possibility of mathematical thinking, exploring what it allows, and being aware of its limitations. This exploration, accompanied by an enactivist approach, means valuing "learning by doing" in the context of problem-posing instead of problem-solving. In other words, we will need to propose situations where students have the freedom to interact and re-construct, giving raise to the emergence of their own problems. In this context, the approach from a-didactic situations provides not only a theoretical frame of reference in the didactics of mathematics education, but also proposes to the teacher a way of doing, where she will need to step back and allow students to interact with each other and with embodied mathematical knowledge, where enaction and metaphorising are even more visible than in traditional didactics.

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The tree of perception as being phenomenologically grounded, intuition as a mathematical space generator and Vygotsky's perezhivanie as emotional transcendental lived experience

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Keywords: Perception, perezhivanie, lived experience, phenomenology, intuition.

This study is based on revisiting a previously analysed (Zagorianakos & Shvarts, 2015) learning episode. Mary, a prospective teacher of mathematics, engaged in a mathematical task that was initiated in an embodied manner, which utterly permeated her noetic treatment of the task. The study opens new space for discussing *perception* in mathematics education, as being phenomenologically grounded, while linking it with the late Vygotskyan *perezhivanie* (Vygotsky, 1994), as being a critical learning phenomenon, concerning the challenge of a rich environment. What Vygotsky (2012, p. 110) calls "the ascent to concept formation" is investigated here through a phenomenological methodological lens (Zagorianakos, 2019, p. 3081), with a movement from the *natural* to the *phenomenological attitude*.

This study introduces allusions concerning crossroads between phenomenology and Vygotskyan cultural historical activity theory, crossroads that emanate from perezhivanie, a so far untranslated Vygotskian term. For Vygotsky (1994, p. 343) perezhivanie is an "emotional experience", a (psychological) "unit", "a unity of environmental and personal features". He sees perezhivanie as a *refraction* of the environment through the personal psychological state (p. 341) and the current research takes it as pointing towards the analysis of "how I, myself, am experiencing this (i.e., the interplay of) all the personal characteristics and all the environmental characteristics" (p. 342), aiming at reviving discussions on perezhivanie in the existing literature (e.g., Roth & Jornet, 2016).

Mary's bird's-eye-view intuition, her tree of perception and perezhivanie

The task was set for the students by asking them to embody the line that is equidistant from a wall and a fixed point that is set at a distance of 10 paces from the wall (Zagoriankos & Shvarts, 2015). As Mary was attempting to embody the fixed point in her group of three students, she was thinking that the sought, equidistant line must be a parallel line to the wall, halfway between the wall and the fixed point. Hence, Mary was utterly confused during her initial perezhivanie (lived experience) of the task. When she came home, she used grid paper, a ruler and coloured pencils in order to represent the classroom setting, as she drew the wall and the fixed point. And then the *bird's-eye-view* intuition of her diagram surfaced. Her intuition transformed both her diagram and her classroom experience, allowing her to acquire a new sense for both of them, starting with the wall, which was perceived as infinitely long. With her *bird's-eye-view* intuition Mary entered into a *constitutional mode*, drawing from the communicating vessels of *pre-reflective and reflective perception* (see Figure 1: soil-trunk/pre-reflective and branches-crops/reflective parts of the tree). Her task *perception* was filled with crops/mathematical results, serving as an exemplification of how intuitions operate as *mathematical space generators*.

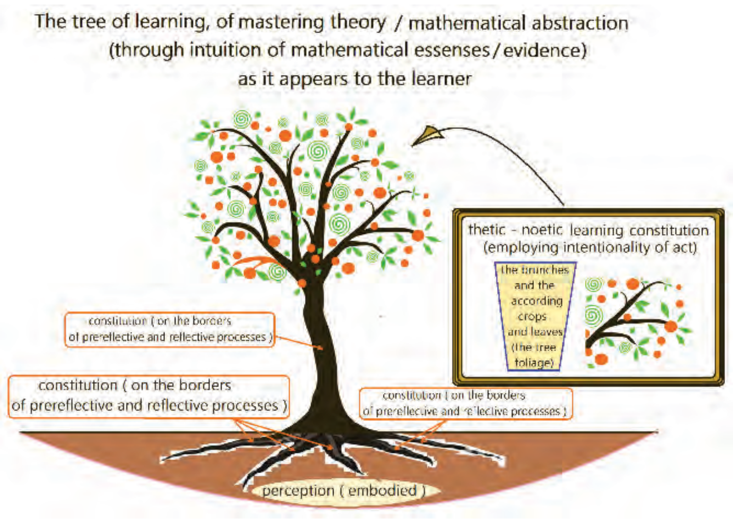

Figure 1: The tree of perception

The tree of perception (Figure 1) exemplifies the personal characteristics that “are represented in an emotional experience [perezhivanie]” (Vygotsky, 1994, p. 342). The environment set by the teacher was allowing ample breadth of scope that each student could push towards. Mary’s bird’s-eye-view intuition transformed her class experience, as the cultural artefacts (ruler, pencils, grid paper, Cartesian plane) opened up abstract geometrical/algebraic affordances. Hence, the growing of her *tree of task perception* also represents the features of the student’s environment, in the sense that for Vygotsky (1994) perezhivanie is a unit, namely a vital and irreducible part of the whole, a significant unit in which personal and environmental components are represented. The suggested crossroad between phenomenology and activity theory is concerned with the inextricable interplay between the complementary and irreducible poles of personal perceptions and of the environment’s affordances, for the constitutions emanating from the former, as they are refracting the latter, rendering perezhivanie as an emotional transcendental lived experience. To better understand “how a child becomes aware of, interprets, [and] emotionally relates to a certain [learning] event” (p. 341).

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TWG18: Mathematics teacher education and professional development

Introduction to the Thematic Working Group 18: International perspectives on mathematics teacher education and professional development: Current and emerging research

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In this paper, we review all of the contributions to TWG18, focusing on the range of research interests, theoretical perspectives and frameworks, and methodological approaches. From this review, the presentations, and discussions, the following future directions have emerged in relation to teacher education (TE): 1) Establishing and exploring research informed design principles; 2) summarising relevant theoretical directions; 3) exploring teacher change from an ethical perspective; and 4) scaling up innovative approaches within TE. In relation to professional development (PD), the following future directions have emerged: 1) Exploring the different roles of participants in PD; 2) exploring what makes change difficult and how professional growth best can be supported; and 3) understanding how we can best build on previous research, and each other, in order to develop the field of mathematics PD research.

Keywords: Educational research, mathematics teacher education, professional development.

Introduction.

Over recent decades, the study of mathematics teacher education (TE) and professional development (PD) has been a central focus of research. During previous ERME conferences, various research activities regarding these topics have been presented and discussed. Hošpesová et al. (2018) thematised the history of Thematic Working Group 18 (TWG18) and linked those topics addressed since CERME1 to questions related to theory and practice, collaborative environments, and reflection. In the CERME12 call for papers, TWG18 addressed a focus on research into prospective mathematics teachers' professional preparation as well as in-service teachers' professional development. The call invited discussions in relation to models and programs of PD as well as related practices (e.g., contents, methods, tools, and impacts). Within TWG18, 27 papers and 14 poster proposals were presented and discussed. Due to this large number of submissions, the TWG was divided into two sub-groups, specifically:

- TWG18a: **Mathematics Teacher Education** (TE) and Professional Development
- TWG18b: Mathematics Teacher Education and **Professional Development** (PD)

By offering a communicative, collegial, and critical forum for the discussion of these and other related issues, TWG18 attracts research from a diverse range of perspectives and theoretical approaches and contributes to the development of our knowledge and understanding as researchers, educators, and practitioners. TWG sessions 18a and 18b comprised both plenary and sub-group working phases. During the plenary phases, two or three papers and/or poster proposals were presented for a maximum of five minutes each, in which the authors highlighted their research interests, the main perspectives and frameworks (explicit, conceptual, theoretical, or practical), as well as the main methodological considerations in their research. These short presentations were followed by one or two discussants reacting to the research presented. Plenaries were followed by parallel sub-group discussions, which were each chaired by one of the discussants. Participants were able to choose and join a sub-group, where discussions lasted for approximately 20 minutes. All sub-group participants then came back together where central topics and issues were shared from each of the discussions. These central topics and issues, which came out of group discussions, often highlighted issues beyond the papers' scope and were recorded in the TWG's Padlet. Examples of issues beyond the papers' scope are ethical considerations in relation to teacher change and exploring how we can learn more about different participants' roles in PD. The content recorded in the Padlet was used when arranging specific topic discussions within TWG 18. The outcome, in form of emerging issues for the future, is described in the final section of this paper.

Building on Eisenhart (1991) and Lester (2005), this review of submissions to TWG18 is organised around three key aspects concerning the research process. The first aspect concerns the phenomenon of interest in the specific study, and how that research interest is justified, positioned, and explained. The second aspect is the choice of framework (theoretical, conceptual, or practical). According to Eisenhart (1991), theoretical perspectives include pre-defined concepts and assumptions that guide the research design. A *theoretical* framework uses a formal theory to establish explanations about a phenomenon. A *conceptual* framework can be viewed as a skeleton that justifies the study in relation to the aim (i.e., a set of assumptions about reality that underlies the research). A *practical* framework is guided by finding approaches that work in practice. Lester (2005) points out that the aim of research is not solely to select and use a conceptual framework, rather, researchers need to adjust and justify the conceptual framework in relation to their specific study. The final aspect concerning the research process is the set of methodological considerations concerning how to reduce the empirical material into meaningful data and how to present the results.

Based on this, this review paper will include sections aiming at answering the following questions:

- What were the **research interests** in the papers and poster proposals within TWG18?
- What were the main **perspectives and frameworks** in the papers and poster proposals?
- What were the main **methodological considerations** in the papers and poster proposals?

In addition to these three aspects, as explained above, we also summarise the emergent issues from the presentations and discussions within the groups, which we present towards the end of the paper.

Research interests.

A diverse range of research interests were presented within TWG18. Within TWG18a, research presented involved all phases of mathematics TE (primary, middle, and secondary phases). Most commonly, research was focused on prospective mathematics teachers (PMTs) and their responses to: tasks and resources; hypothetical or real student(s); hypothetical or real interactions between teacher(s) and student(s); other prospective teachers; teacher educators; learning environments and tools. In TWG18b, many papers and poster proposals explored the design of research-based PD interventions as well as researching the different aspects of these interventions. The main aim of the PDs studied and presented in TWG18b can be summarised as collaboration between teachers or teachers and teacher educators or researchers in order to develop teaching practices, mainly towards more explorative approaches to teaching. In the PD research presented, the roles as facilitators, teachers, students (one or several) were explored.

Research interests: Teacher education.

Concerning the way PMTs respond to tasks and resources, *Sødal* researched PMTs' views in relation to the benefits of different aspects of working with resources in university coursework in preparing them for teaching mathematics. She found that working with resources can provide PMTs with practical and useful experiences as well as a way of combining content knowledge and core practices, to close the perceived gap between theory and practice. With regards to researching how PMTs respond to students' mathematics, *Henriques and Oliveira* researched the development of PMTs' knowledge in relation to students' mathematics reasoning. In this study, the PMTs' interpretations of students' mathematical reasoning processes were analysed and findings suggested that, over time, PMTs knowledge level improved. With a focus on the interactions between teachers and students, *Schnell and Fellenz* researched the role of 'Perspective Taking' in relation to PMTs' noticing students' mathematical thinking. In their study, they analysed the content of PMTs' written responses to a video clip of a mathematical interview. They found the most common perspective taken was one of task solver as opposed to teacher or student. Only two papers presented placed their focus upon the mathematics teacher educators (MTEs). *Longwe-Mandala and Fauskanger* explored ways in which MTEs in Malawi invite PMTs to participate during teacher education programmes, to better understand how these PMTs are enculturated into the practice of inviting learners to participate actively in lessons about number concepts and operations. *Ebbelind and Helliwell* explored the experiences of a group of primary PMTs during their teacher education programme in relation to the language-in-use of one MTE in Sweden.

Research interests also included ways to develop meaningful designs of mathematics teacher education programmes. Across the papers and poster proposals, researchers utilised well-established, as well as innovative pedagogical approaches within mathematics TE which, in some cases, became their focus of research. Examples of innovations include *Frejd et al's* use of a team-teaching approach called Socratic lectures to develop PMTs' communication skills and *Samková's* use of concept cartoons in primary teacher education to help assess prospective primary school teachers' knowledge on topics related to the primary school curriculum.

Research interests: Professional development.

As an example of research that explored the design of research-based PD interventions, *Grimeland et al.* investigated what kind of co-learning and learning gaps could be identified in a PD session on the topic of programming, a topic newly included in the Norwegian curriculum across grade levels.

Their findings indicate that both teachers and teacher educators learn about programming and lesson planning for programming during the PD. In addition, teacher educators learn about teachers' programming knowledge. One of the learning gaps identified is teacher educators' knowledge about use of programming in school.

Aiming at understanding how teachers reason about the role of high-quality mathematical tasks, a second example study analyses three groups of mathematics teachers engaged in collegial discussions as part of a national large-scale PD programme in Sweden. In this study, *Kaufmann* explores how the teachers reflect upon and explain the role of high-quality mathematical tasks when choosing tasks for use in their lessons. *Kaufmann*'s results indicate that the teachers appreciate high-quality tasks for providing student-to-student talk and for supporting students' collaborative efforts to solve problems. However, although these teachers appreciate high-quality tasks, they referred to such tasks as inappropriate for their students, blaming their students' capabilities, their lack of motivation to engage in such tasks, and their lack of experience with such tasks.

Problem solving was a focus of attention in several contributions. As an example, *Keller and Kohen* explore the learning processes occurring in online discussion forums as part of a 2-year PD programme where the teachers first acted as learners through collaborative solution of complex mathematical problems in small groups. Secondly, they led collaborative problem solving in forums as mentors. Based on exploring what is reflected in the teachers' pedagogical activities, *Keller and Kohen* conclude that problem solving forums have a high potential for developing teachers' own self-regulation skills, increasing their effectiveness in collaborative problem solving and empowering their support to students in solving complex mathematical problems.

In relation to exploring the roles of those participating in PD, one example is the study by *Skott and Ding* who focussed on the facilitator's role in lesson study by comparing how facilitators talk with teachers and what they focus their talk on. They use a framework consisting of mentoring strategies and content categories; both developed empirically in a Chinese and European context respectively. Their analysis showed big differences in the facilitators' ways of engaging in talk with teachers, including the dynamic and relational patterns in the Danish case as compared to the lengthy talk in the Chinese context. Based on their analysis, *Skott and Ding* argue that these differences are not only related to the fact that lesson study is new in Denmark, but also to social and cultural differences.

Perspectives and frameworks.

The studies shared within TWG18 were based on a wide variety of frameworks, depending on the research questions being answered and on the researchers' perspectives. In TWG18a different theoretical perspectives were used for the different facets of the teaching profession that the PMTs were prepared for. In TWG18b, different theoretical perspectives were used to guide the design of the PD programmes and also as underlying sets of concepts and ideas guiding the research design and the analysis.

Perspectives and frameworks: Teacher education.

Among the frameworks informing the facets of the teaching profession that the PMTs were prepared for, all three main objectives could be found in the papers and posters: knowledge, beliefs, and practices, both individually and combined. From the perspective of Shulman's (1986) subject-matter knowledge (SMK) and pedagogical content knowledge (PCK), the majority of the papers and posters focused on PCK. In several cases (e.g., *Reitz-Koncebovski et al.*), SMK was connected to PCK via the model of school-related content knowledge (Dreher et al., 2018). The contributions studying PCK focused on one or more components of mathematical knowledge for teaching according to Ball et al. (2008) (e.g., *Schreiber*), on professional vision and noticing according to van Es and Sherin (2021) (e.g., *Karatsioli et al.*), or on diagnostic judgement according to Loibl et al. (2020) (e.g., *Schreiter et al.*). From the perspective of PMTs' beliefs and motivations, there were papers focusing on beliefs about resources for teaching mathematics (*Sødal*) or on motivation and communication skills (*Frejd et al.*). From the perspective of PMTs' practices, *Chikiwa and Graven* proceeded from the six-lens framework for guiding teachers' reflections on video-recorded lessons (Karsenty & Arcavi, 2017). One of the papers (*Karagoz Akar et al.*) focused on knowledge, beliefs, and practices all at once, and studied consistencies among them.

In TWG18a a wide range of different approaches to TE were introduced, including some recent or innovative approaches: for instance, transferring the concept of lesson study from PD to TE (*Ponte and Quaresma*), or transferring clinical simulations from the context of professional preparation of pilots, medical doctors, or nurses to the context of professional preparation of mathematics teachers (*Schreiber*).

Perspectives and frameworks: Professional development.

Among the frameworks informing the structure of the presented PD programmes, collaborative work between teachers is a common factor. The implementation of lesson study in new contexts was the focus of some studies including *Haringová and Medová* who studied the implementation of the lesson study approach in Slovakia. A further example of collaborative work in PD programmes can be found in the study by *Nurick et al.* where teachers participated in the VIDEO-LM project and discussed videotaped mathematics lessons using the "six-lens framework" (Karsenty & Arcavi, 2017).

There was a wide variety of approaches among the frameworks used to analyse research data and explain the phenomena behind them. The meta-didactical transposition framework (Chevallard, 1999) was used by *Pocalana et al.* as an interpretative lens to describe the interactions between teachers participating in PD and researchers acting as facilitators in the PD. In particular, they focused on the evolution of the praxeologies of both communities. In addition, *Pocalana et al.* used the boundary objects (BO) framework (Akkerman & Bakker, 2011) to explain the development of a shared praxeology between teachers and researchers, the internalisation processes of new elements for both communities, and the learning mechanisms activated by the design choices made by researchers for the PD program. Another study using the BO framework is the one by *Casi and Sabena* who interpreted museum collections to be BO connecting communities of students, teachers, and museum staff. As BOs, components of non-scientific museums acted as prompts for epistemological discussions about mathematics.

Collaboration among teachers has also been the focus of research analysis. *Keller and Kohen* studied the interactions of teachers in an online environment, where teachers participated in online forums. They analysed the participation of one teacher in the forum using the framework of collaborative mathematics problem solving and its taxonomy of interactions (Clark et al., 2014), together with the components included in the Self-Regulated Learning framework (Boekaerts et al., 2000).

Methodological considerations.

Methodology can be understood as methods used for creating, gathering, or collecting empirical material and the specific reasons researchers have for using such techniques. As highlighted by Eisenhart (1991), this step also concerns how to reduce the empirical material into meaningful data worth highlighting in the result section. This section focuses therefore on the research design and sample size, methods for generating empirical material, structuring information, and generating data material for the results section in TWG18a and TWG18b.

Methodological considerations: Teacher education.

Research presented in TWG18a covered various research designs: qualitative, quantitative, and mixed methods research. Quantitative research included intervention studies involving a pre– and post– intervention test. For instance, the study by *Volkmer*. Other quantitative studies used written formats as data, such as *Dröse*, who used written diagnostic judgements from a group of PMTs (n = 26). These written judgements were coded with relation to knowledge elements for current or prior learning content using two dimensions of procedures and concepts. Quantitative studies consisted of up to 300 participants.

Half of all papers and posters related to TE were qualitative, covering single-case studies to those consisting of more than 50 participants. For example, *Samková* used indicative questions concerning a concept cartoon. The participants worked on tasks individually, and the data collected was processed using open coding and constant comparison to display subject-matter knowledge. *Ebbelind and Helliwell*, on the other hand, used a methodological tool to structure their empirical material from different contexts. Four contributions involved mixed methods. *Schreiter's* use of eye-tracking as a data collection method was a novel methodology within the TWG18a group.

Because of the various sample sizes and methods, the nature of the data is diverse. Methods for generating empirical material in TE research related to either: written reflections, answers to indicative questions, task solutions, lesson plans, questionnaires, recorded or transcribed interviews, video-recorded lessons, task analyses and movement tracking (mouse and eye). It is sometimes mentioned that theory sets standards for methodology, however, those researchers that use Loibl et al's (2020) diagnostic judgement use the full range of methods displayed within the TWG18a group.

Methodological considerations: Professional development.

Studies on teachers' PD in TWG18b included a systematic review of the literature on PD programmes and their relationship with student achievement (*Peri & Gomez Zaccarelli*) and a survey project on professional journals for mathematics teachers (*Asami-Johansson & Otaki*). The sample sizes in the presented studies ranged from two to 47 and up to several hundred participants: *Österling* conducted a visual and fine-grained analysis of two teachers' lessons in order to develop a framework for representing changes in mathematics teaching over time; *Keller and Kohen* explored the learning processes of 47 teachers when engaging in collaborative problem solving in online discussion forums. *Knaudt et al.* aim at developing adaptive training modules for in-service teachers at 125 primary schools with the aim to foster teachers' diagnostic competences whilst considering the heterogeneity in the participants' individual learning prerequisites.

The majority of the presented studies in TWG18b used qualitative methods for investigating the participating teachers' professional growth: Data sources were videos from lessons (e.g., *Österling*), videos from PD sessions (*Nurick et al.*; *Haringová & Medová*) or interviews (e.g., *Neururer & Shúilleabháin*). In their comparative case study, for instance, *Koellner et al.* conducted semi-structured interviews using think aloud protocols for an in-depth investigation into teachers' practices five years after taking part in a PD programme. Many of the studies analysed written reflections or responses from teachers (e.g., *Keller & Kohen; Knaudt et al.*). *Kaufmann*, for example, analysed data on the reflections from teachers engaged in collegial discussion about the role of high-quality mathematical tasks. *Skott and Ding* investigated teachers' planning of lessons as data sources and explored potential changes. Only two studies (*Pocalana et al.; Knaudt et al.*) also made use of questionnaires: *Pocalana et al.*, for instance, administered a questionnaire to investigate teachers' praxeologies including teachers' beliefs about their students and about the teaching and learning of mathematics to complement qualitative data collected through interviews, written protocols, video recordings and reports about teachers' classroom experimentations. Looking across all studies addressing the professional growth of in-service teachers, various analytical frameworks were applied to analyse and structure the collected data: the commognitive framework (*Österling*); boundary objects theory (*Casi et al.; Pocalana et al.*); directed content analysis (*Keller & Kohen*); co-learning dimensions (*Grimeland et al.*), abductive process analysis (*Kaufmann*) or comparative case study analysis (*Koellner et al.*). Corresponding to the research focus of the respective studies, coding of participants' responses was, for instance, also based on levels of teacher noticing (*Fauskanger & Bjuland*) or stages of teachers' concern (*Neururer & Shúilleabháin*).

Emerging issues.

As highlighted in the introduction of this paper, group discussions often highlighted issues beyond the papers' scope. Therefore TWG18a and TWG18b arranged specific topic discussions during the conference. These issues were highlighted as emerging issues for the future and can serve as inspiration for forthcoming CERME conferences.

Issues from the discussions during TWG18a related to future TE research can be summarised as follows. Firstly, exploring whether it would be possible to establish a set of research based design principles for TE courses that would be applicable across different contexts, and what such principles might look like is a next step for TE research. To accomplish this, there is a need to share concrete task and course designs for TE programmes. Secondly, there is a need to summarise relevant theoretical directions in TE, to get a better sense of the different frameworks used within our community. A better understanding of different perspectives will not only strengthen our own research, but also contribute to more nuanced discussions in the future. Thirdly, teacher change was considered from an ethical perspective, with questions asked such as whether it is right to try to change teachers in certain ways, and who should make those decisions. Such ethical perspectives is in need for future explorations. Lastly, the question of how our research community might scale up innovative approaches should be explored.

Issues from the discussions during TWG18b related to future PD research can be summarised as follows. Firstly, future research should aim at exploring how we can learn more about different participants' roles in PD (i.e., students, teachers, teacher educators and researchers) whilst, at the same time, facilitate collaboration between all participants. Secondly, there is a need for exploring what makes change difficult and how professional growth best can be supported. Lastly, better understanding how we can build on previous research and each other in order to develop the field of mathematics PD research is an implication for future research.

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A survey project on professional journals for mathematics teachers

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Keywords: Levels of didactic co-determinacy, teaching profession, paradidactic stake.

Journals for disseminating professional scholarship of mathematics teachers

This presentation describes a framework and an analytical method we employ in our project, the aim of which is investigating the characteristics of professional journals for mathematics teachers in different countries. Professional journals for schoolteachers in mathematics function as an important venue to disseminate different kinds of professional knowledge necessary for carrying out teaching. The contents of journals can deal with individual teacher's work for designing lessons in certain area, proposals for different methods for assessment, philosophical articles about mathematics education, columns by mathematicians, presentation of results from research, etc. The selecting and editing from such possible contents depends on the needs of the readers—the roles and responsibilities of schoolteachers that are devolved from a given society to which they belong. What similarities and differences are there between different journals around the world? As a first step for answering this question, we report here a preliminary analysis on a Swedish journal *Nämnaren* (The Denominator).

Levels of didactic co-determinacy and paradidactic stakes

A sequence of didactic situations in classroom depends upon a family of *conditions* of different kinds and origins. The hierarchy of such conditions is modeled by the anthropological theory of the didactic as a *scale of levels of didactic co-determinacy* (e.g., Chevallard, 2019). We apply the scale to situate foci—or *paradidactic stakes* (Otaki et al., 2020)—of journals, for identifying the properties of the journals. We have categorised the contents of the articles and related them to the different levels of the co-determinacy as the following: *Society*, related to general issues, e.g., Swedish national traits, culture and world trend. *School*, related to school educational issues, e.g., national curriculum, Swedish school and its policies. *Pedagogy*, related to generic teaching principles e.g., methods for promoting students' interaction. *Discipline*, related to mathematics itself as an academic subject or school subject. *Domain*, related to relatively general areas in mathematics such as algebra, geometry, function, etc. *Sector*, related to the general topics of mathematical objects such as equations, similarity, operations. *Theme*, related to contents of teachers' activity such as how to teach the concept of triangles, etc. *Subject*, related to one simple type of task (in a lesson) and a corresponding solving method. From this point of view, our research question in this survey project is formulated as the following: *what levels of paradidactic stakes do professional journals for mathematics teachers include or exclude?*

Pilot analysis on *Nämnaren*

Nämnaren has been established in 1974 and publishes four issues a year. The journal is currently directed by the National Center for Mathematics Education, the task of which is to support the development of mathematics education from preschool to secondary school levels. Authors of articles are mixture between teachers in service in different levels, teacher educators, researchers in mathematics education and mathematicians. In this pilot study, four numbers from the year 2020 are analysed. The total number of published articles in these four numbers is 58, and the main issues of the submitted articles are following: teachers' activities, students' phycology and special pedagogy, history of mathematics, research and survey reports, and use of manipulatives and ICT. Of the 58 articles, 49 of them aim at primary and (mostly lower) secondary levels. The rest aims at pre-school education, adult education, and education in general. *Nämnaren* has three standard topics in every issue: book reviews on newly published educational books, problem bank and language column. The last two topics often deal with the issues related to lower levels such as theme and subject. Contents of several articles are related to more than one level of the co-determinacy. For example, an article, which describes using double number lines for better understanding for algebra, deals with conceptual description of algebra and equations (sector), but its actual focus was giving suggestions for teaching design (theme) and task examples (subject). Even though two articles present activities on number theory, the focus of one article is on the level of theme, while the other is related strongly to the pedagogy level. In the same way, one article treats a topic of Pascal's triangle for an analysis of a sports context (theme), while the other describes conceptual differences of algorithms using Pascal's triangle in different institutions in several countries (sector). As a summary, two articles are related to the society level, five to the school level, 15 to the pedagogy level, four to the discipline level, three to the domain level, seven to the sector level, 24 to the theme level, and 15 to the subject level.

The tendency that the paradidactic stake of the journal tend to gather in the lower levels *and* the level of pedagogy is not surprising, since the readers of the journal could have a disposition so called *the paradidactic bipolarization* (Otaki et al., 2020). This phenomenon indicates that schoolteachers' interest and their professional knowledge lie exclusively either pinpoint mathematical matters or general-purpose teaching methods. A notable fact is that a professional journal, one of the missions of which is disseminating teachers' professional scholarship, traces the pattern of the bipolarisation of schoolteachers' scholarship. In our view, this illustrates an aspect of the functioning of the *teaching profession*, whereby schoolteachers are reproduced together with the didactic worldview supported by certain *norms* (remember the expression "normal school"). Such institutional structure may hinder forthcoming reforms of mathematics education. Understanding the epistemological economy (and ecology) of the teaching profession—which is often recognised as a *semi-profession*—is crucial, not only for research but also for practice. The information on current states of the teaching profession should be useful for developmental actions for teacher education in the future.

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Criteria used by pre-service mathematics teachers to design teaching plans

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Keywords: Didactic suitability, normative system, teaching.

Teaching planning as a process and as a tool to develop didactic analysis

This work addresses the question: What are the criteria used in planning a class on the perimeter, by preservice teachers? The objective refers to identifying the criteria that preservice mathematics teachers (PMT) use to assess the didactic suitability (DS) in managing a study process, in the class plans for teaching the perimeter. The inquiry process to construct plausible plans is guided by the Onto-semiotic Approach (OSA) of Mathematical Knowledge and Instruction (Godino et al., 2007).

Theoretical framework

The Ontosemiotic Approach (OSA) of Mathematical Knowledge and Instruction (Godino et al., 2007) is assumed; several features related to didactic analysis (DA) are studied, such as the notion of didactic suitability (DS) and normative system (NS) in the context of pre-service mathematics teachers' teaching practices.

Methodological framework

The research was conducted with six preservice mathematics teachers based on a qualitative research approach which allows obtaining complex details related to human thought and emotions. Data was obtained through interviews including previous personal ideas about planning before the interventions and then about actions, evaluation and improvements done to their previously presented lesson plans regarding the topic of perimeter for sixth grade students.

The general fieldwork was organized in three phases: in phase 1, the object of this report, previous and personal ideas about planning and evaluation that students use when designing a class plan are studied; in phase 2, an intervention-training is implanted, to analyze the didactic suitability in the study process; and in phase 3, the actions and evaluation and improvement criteria used in the redesign of the plans to teach the perimeter object projected in phase 1 are studied. In the first phase information was obtained by recording video audios, transcripts and reviewing student responses (I-1), and data on the reflection on the planning of "my class in mathematics (I-2)".

Analysis

The First Phase of the research, focused on the elaboration and a priori analysis of the lesson plans designed by the PMT, has four moments: moment 1) contextualization and investigation of previous ideas about teaching planning; moment 2) application of I-1, and discussion of designed plans;

moment 3) application of the I-2 instrument and discussion; moment 4) analysis in teams on a lesson plan based on I-1 and I-2.

On the a priori criteria emerging

Based on the coding, initially open and then axial (Strauss & Corbin, 2012) of the actions and reasons (A-R) present in the plans, we could to identify eight groups of actions-reasons with different levels of presence- shown in Table 1-. Some of the configured A-R groups presented in the PMT plans are described and explained:

Table 1: Action groups and reasons (A-R) emerging in planning. Source: Authors

Action groups and reasons (A-R) emerging in planning	Emerging A-R in lesson plans and level of prevalence
A-R1. According to training area	Curricular (32%), didactic (7%), disciplinary (44%) and formative-pedagogical (17%)
A-R2. According to the nature of the meanings	Institutional (68%) and personal (32%)
A-R3. According to the teaching model or trajectory	Active-constructivist (11%), problem-based (10%), collaborative (10%), lecture-mechanistic (17%), traditional-expositive (52%)
A-R4. Depending on the time of the class	Start (32%), development (32%) and closure (36%)
A-R5. According to the role of the students	Passive receiver (30%), collaborative (15%), individualist (11%) and active receiver (44%)
A-R6. According to the teaching intentions	Use of technological tools (54%) and motivation with games (46%)
A-R7. On the approach to the mathematical object to be taught	Routine exercises (29%), associated properties (2%), related concepts (20%), definition-concept of perimeter (7%), procedures (14%), representations (13%) and problem situations (14%)
A-R8. On processes associated with the development of mathematical thinking	Problem solving and problem posing (14%), reasoning (14%), communication (8%), formulation and application of procedures (47%)

These results confirm that the criteria and norms used by the PMT, emerging from the plans, are descriptive and evaluative. They are organized based on indicators associated with the different components that make up the didactic suitability criteria. The other A-R groups identified in the plans are related to interactional aspects in the A-R linked to the teaching model. In this sense, according to Table 2, A-R groups identified and organized according to lesson plans, are related to interactional ID criteria. The same occurs with A-Rs referring to the teaching model (A-R3). Concerning epistemic and cognitive ID criteria, they are found in the approach to the mathematical object to be taught (A-R7), as well as in the processes associated with the development of mathematical thinking (A-R8).

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Informal mathematics experiences in museums: what potential for teacher professional development?

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Keywords: In-service teacher professional development, informal mathematics education, co-design.

Informal mathematics education

In informal mathematics education research, learning spaces other than school are explored, and experiences with mathematics that are different from the traditional ones offered by the educational institution are lived. According to Nemirovsky and colleagues (Nemirovsky et al., 2017), informal mathematics education contexts differ from school-based mathematics activities mainly due to three structural features: the voluntariness of learners' participation, the fluidity of disciplinary boundaries, and the absence of traditional forms of assessment. In recent years, we have been involved in projects aiming at preventing school dropout in disadvantaged neighborhoods of some Italian cities, namely the Proud of You project (Carotenuto et al., 2020) in Napoli and the Next-land project in Torino. In both projects informal mathematics learning was chosen for its potential to convey alternative visions of mathematics and to engage all learners in ways that are creative and different from usual school practice, with the underlying hypothesis that engaging students and changing their vision of mathematics may contribute to prevent school dropout. In accordance with the perspective of Culturally Responsive Mathematical Education (Gay, 2010), we decided to situate the informal mathematical learning activities in the history and culture of the students' own territory, allowing them to create, recreate, and shape their meanings. In our experience, two elements emerged as crucial for reaching the desired aims: the collaboration with teachers in co-designing the activities, and the relationship between informal and formal mathematics learning. They need further investigation, and in this contribution we will focus on the idea of exploiting the informal settings provided by museums (and specifically non-scientific museums) in a teacher development perspective. Our starting point is our experience of co-designing informal mathematics learning activities with museum experts. This was accomplished within the Next-land project, which we will briefly introduce.

The co-design experience in the Next-land project

In the Next-land project (https://www.next-level.it/progetti/next-land/), grade 7 students from disadvantaged areas of Torino (Italy) are involved in out-of-school workshops, during the first two weeks of school in their curricular hours. Workshops vary a lot as regarding locations and content. Our research group[1] is responsible for the mathematics workshops, which are located in four museums (Egyptian Museum, Museum of the Risorgimento, Palazzo Madama, Park of Living Art). In collaboration with the staff of each museum, we co-designed four workshops aimed at learning mathematics through the discovery of historical and artistic heritage of the City of Torino, involving students in experiences of observation, exploration and manipulation.

[1] The research group is composed by the authors and by two teacher-researchers: Valentina Leo and Chiara Pizzarelli.

In Autumn 2020, about 200 students, accompanied by their teachers, took part in the workshops, led by the staff of the museums. From our own field observation and the feedback given by the teachers, the students and the museum staff, we gained evidence of some positive results with respect to the project aims. However, we found also that the lack of the teachers' engagement in co-designing the activities limited the workshop experience to just an interesting activity that had no further implication on the students' mathematics experience, and therefore on the possibility of changing their view of mathematics on the long run. But how may teachers be involved in co-designing informal mathematics education activities in collaboration with mathematics educators and museum experts? The need for teacher professional development emerged in a striking way.

What potential for teacher professional development?

Based on our co-design experience with museum experts, we are convinced that informal learning in non-scientific museums may be exploited to engage teachers in rethinking their teaching practice and their relationship with mathematics. The informal character of teaching by workshops in museums and the encounter with different knowledge coming from other domains (like history and art) will favor the teachers' creativity in designing tasks to be carried out, with the perspective to develop in students an *emergent learning* rather than achieving predefined goals. Since informal education is still under-researched in relation to student learning, and to our knowledge has not yet been studied for teacher education, we plan to work by letting teachers first experience the workshops of the Next-land project, and then co-design more workshops with museum experts and mathematics educators. It is our hypothesis that the museum collections can be the boundary objects (Akkerman & Bakker, 2011) connecting the three communities, prompting a discussion about an epistemological analysis of mathematics and bringing to the fore different visions of mathematics itself.

The teacher development process will be the focus of a PhD study aiming to answer the following research questions: What difficulties may teachers experience in designing informal mathematics workshops, and how may they overcome these difficulties? What actions can be implemented to support teachers? What changes can be generated in teachers' practices and beliefs? What impact on the usual classroom activity? To answer these questions, we will collect data through questionnaires, focus groups and audio/video recordings of meetings, which will be analyzed under a qualitative lens.

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How the six-lens framework supports pre-service teachers' reflections

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Keywords: Pre-service teachers/teacher education, six-lens framework, reflective practice.

Introduction.

The ability to reflect on one's practice is an essential skill for all mathematics teachers because of its significance in promoting effective teaching and learning. Pre-service teacher education (PTE) in South Africa is foregrounding this skill as one way to redress the long-standing record of poor performance in mathematics. This poor mathematics performance is often attributed to poor teaching which result from teachers' inadequate knowledge for teaching mathematics. Reflective practice (RP) is potentially useful for developing this knowledge. A mathematics methods lecturer at a South African university engaged her third-year pre-service teachers (PSTs) in three sessions of video-based lesson analysis to develop their RP for mathematics teaching. She employed the six-lens framework (SLF) developed by Karsenty et al. (2015) to guide mathematics teachers as they reflected on selected video-recorded mathematics lessons. Our research explored how the SLF supported the primary PSTs during the process of reflective practice development. We sought to answer the question: How do the various lenses of the six-lens framework influence how pre-service teachers reflect on video lessons?

Reflective practice and the six-lens framework

Although RP is a difficult concept to define or understand, it has many educational benefits. Scholars such as Dewey as early as 1933 suggested "it is impossible to become and remain an effective teacher without commitment to reflective practice" (p.9). The multiple perspectives on RP however have led to a lack of common definition and furthermore it is increasingly acknowledged that it is a skill that is difficult to develop (Russell, 2005). However, the skill is critically important as teachers reflecting on their teaching regularly tend to improve their own professional practices and adapt to the day-to-day ever-changing teaching demands (Dewey, 1933). In support of developing this critical skill Karsenty at al. (2015) developed the SLF to assist teachers to consider and reflect on six important aspects of lessons, namely: mathematical and meta-mathematical ideas (MMI); explicit and implicit goals; tasks and activities (T&A); dilemmas and decision making (DDM); interactions with the students, and teacher beliefs.

Methodology

The empirical field for the research was three lecture sessions on video-based lesson analysis (using the SLF) provided by a lecturer to her 52 third year PSTs at a South African university in 2017. The lecturer explained each lens to the PSTs before requesting them to use these lenses as they analysed the given video-recorded lessons. We employed a qualitative case study with 19 PSTs who volunteered to take part in the research. All written reflections of these participants were analysed across the three 2-hour lecture sessions. Our poster focuses on the written reflections during the third session where PSTs used all six lenses. We analyzed the reflections of the 14 PSTs who indicated each lens in structuring their reflections (5 PSTs did not indicate lenses). We analysed the reflections per lens to understand and ascertain how the different lenses influenced the nature and levels of PST reflections. We used content analysis and our four levels of reflection model (adapted from Lee, 2005 and Muir & Beswick, 2007) to analyse and summarise the data. The levels of reflection from first to fourth were Description; Explanation; Suggestion and Reflectivity. We also classified reflections as general (G) or mathematical (M).

Findings and discussions

The 14 PSTs wrote 628 reflections using the six lenses. Table 1 below shows how the reflections were spread over the six lenses and over general versus mathematical reflections.

Table 1 The results of the analysis of 14 PSTs reflections by lens

Lens	Level 1		Level 2		Level 3		Level 4		Total	G		M	
MMI	116	85%	19	14%	1	1%	0	0%	136	49	36%	87	64%
Goals	64	82%	15	19%	0	0%	0	0%	79	17	22%	62	78%
T&A	128	84%	22	14%	3	2%	0	0%	153	50	33%	103	67%
Interactions	89	82%	18	17%	1	1%	0	0%	108	98	91%	10	9%
DDM	62	83%	10	13%	3	4%	0	0%	75	47	63%	28	37%
Beliefs	63	82%	11		3		0	0%	77	48	62%	28	38%

From the table we noted that some lenses like T&A and MMI attracted more reflections than others and some (e.g. goals, MMI and T&A) tended to lean students towards greater mathematical rather than general focus. Of interest across all lenses PSTs tended to focus on description and explanation rather than suggestion and reflectivity. In the presentation we will talk to some further insights and implications for teacher education.

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Lesson study in prospective mathematics teachers' education

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This communication presents a lesson study conducted with prospective mathematics teachers and seeks to know what the participants consider having learned and what is their perception of the lesson study. The participants are 14 prospective mathematics teachers with a master's degree in teaching. Data collection was made by participant observation with a research journal and individual interviews with six prospective teachers. The results show that the participants needed some time to understand this formative process but recognize that they made significant learning in several aspects concerning mathematics teaching.

Keywords: Lesson study, initial teacher education, prospective mathematics teachers, knowledge of mathematics teaching.

Introduction

Lesson study is a process of professional development of teachers originated from Japan (as *jugyokenkyuu*) in the late 19th century, popularized since the late 20th century from the USA (Stigler & Hiebert, 1999), and currently used worldwide (Huang et al., 2019). The work done in a lesson study focuses on students' learning. This work is informed by the curriculum guidelines and research related to a given theme, assuming a collaborative nature and center on school practice.

A lesson study is developed from the identification of a learning problem identified by the participants, who study curriculum documents, research and professional papers, textbooks, and other relevant material that can help to understand how to deal with this problem and, based on this study, plan a lesson in detail. This lesson, called research lesson, is taught by one of the participants and observed by the others, and is then the subject of a post-lesson reflection. Finally, the participants present their experience, in particular to other teachers. It is a process close to an investigation on professional practice, involving the definition of a research problem, the realization of preparatory work that includes a literature review, the realization of an experiment (the research lesson), the analysis of data related to that experience, and the dissemination of results.

Lesson study is a formative process developed for in-service teachers. Given the agenda of initial teacher education (ITE) in preparing future teachers regarding all aspects of teaching practice, the question arises if lesson study can be used in this field and how that may be done (Larssen et al., 2018; Ponte, 2017). Given the specificities and constraints related to ITE, there are several questions that need to be considered, such as: (i) How to define the research question? (ii) In which class can the research lesson be taught? (iii) Who teaches this lesson, the class teacher, a prospective teacher, or the university teacher? (iv) Who observes the research lesson? In cases where the prospective teachers constitute a large group, it is not easy to establish an organization to carry out the lesson study, and microteaching experiments have been carried out with the prospective teachers teaching each other in small groups (Fernández, 2005). In this communication, we present the main aspects of a lesson study conducted with prospective mathematics teachers during the 2020-2021 school year. Our aim is to know what the participants consider having learned and what is their perception of the lesson study.

Theoretical framework

Many different organizational arrangements have been used to introduce lesson study in prospective teacher education (Ponte, 2017). Two main situations may arise: the lesson study is made with a relatively large group of prospective teachers, in the frame of a university course, or the lesson study is carried out during student teaching with very small groups of prospective teachers. For example, in an experience carried out in the USA, Burroughs and Luebeck (2010) describe a lesson study made in the frame of a middle school mathematics methods course with 24 prospective teachers. The lesson study was carried out by a team of seven experienced mathematics teachers from local schools, with the prospective teachers in the role of observers of this activity, collecting and analyzing data from specific moments of the lesson study sessions. The research lessons were observed by just a few prospective teachers that reported to the whole class.

An example of an experience carried out during student teaching is described by Nakamura (2019), with the case of a prospective teacher in Japan involved in lesson study that focused on problem solving. The work was done in three weeks, with teaching in three mathematics classes, following the steps of preparation, teaching and reflection. The participants were the cooperating teacher and other prospective teachers (5 or 6) but there was no participation of a university teacher. The preparation of the lessons was made by the prospective teacher in interaction with the cooperating teacher but with no interaction with other prospective teachers. The most distinctive aspects of this activity were the intensity of the work carried out and the key role of the cooperating teacher.

One of the main aspects of the preparation of prospective mathematics teachers concerns the development of their knowledge about mathematics and mathematics teaching. Current discussions about such knowledge are framed in a decisive way by the seminal work of Shulman (1987) who called attention for a kind of knowledge that he considered neglected in initial teacher education—pedagogical content knowledge. Following the European tradition of defining didactics as the study of teaching and learning in different curriculum subjects, (Ponte, 2012) highlights the central role of the teacher's knowledge of the teaching process, that includes key concepts necessary to develop a practice aligned with curriculum frameworks, particularly the notions of task and classroom communication. Besides the content of knowledge to develop in future teachers, another important issue concerns the processes of development of such knowledge. In this respect, a key contribution was provided by Ball and Cohen (1999) who consider the work in practice-based situations a fundamental strategy for teacher education.

Lesson study allows to establish a relation among these theoretical perspectives. On one hand, it has a strong connection with practice, developing around the preparation, undertaking, and reflection of a lesson. On the other hand, lesson study requires the mobilization of knowledge of the content, regarding concepts, procedures, solving strategies and representations and of didactics knowledge, in key issues such as lesson planning, selection of tasks, and analysis of students thinking processes and of classroom communication. What prospective teachers learn in a lesson study depends very much of the emphasis of the work (Ponte, 2017). They may learn about mathematics teaching including the planning of a lesson, the features of tasks that best support students' learning, the strategies and difficulties of students, the dynamics of a mathematics class (Burroughs & Luebeck, 2010; Ponte 2017) and develop professional competences such as reflection and collaboration (Gunnarsdóttir & Pálsdóttir, 2011; Lewis, 2019).

Research methodology

This study took place in a semester course (in February to May 2021). The participants were 14 prospective teachers who attended the first year of master's degree in teaching (a 2-year course after a 3-year bachelor's degree). Two of these prospective teachers, come from the bachelor's degree in mathematics and 12 come from degrees in other areas. The course had two co-teaching instructors (the authors of this paper). We chose to organize the 14 participants in 4 groups, doing 4 lesson studies in parallel, with some superimpositions in the preparatory phase. Although the teacher education program does not require the prospective teachers to teach lessons during this semester (but only in the 3rd semester), the research lessons were taught by two prospective teachers in their classes since they were already teaching. The analysis of the school curriculum planning and the university calendar circumscribed the possible topics to address. Clarisse, who was teaching grade 7, choose the theme "Internal and external angles of a triangle" and Joana, who was teaching grade 8, choose the theme "Isometries". Thus, two lessons were held in grade 7 with the topics (i) Justification of the properties of the internal and external angles of a triangle and (ii) Problem solving involving internal and external angles of a triangle. Two other lessons were held in grade 8 with the topics (i) Rotation, and (ii) Central Symmetry. This option for the organization of the class into 4 groups was based on the idea that work, to be productive, must be organized into groups with a small number of elements.

The work on the four lesson studies had 12 sessions (S) and developed in four strands, addressing different aspects of teacher knowledge. The first strand (S1-S2) focused on mathematics, including solving mathematical tasks and reflecting on solving strategies and representations. The second strand (S3-S4) addressed students' strategies and difficulties on the topics. The third strand (S5-S8) concerned the detailed planning of four lessons, considering the curriculum documents. Finally, the fourth strand (S9-S12) included the observation and reflection on the lessons. A very important emphasis of all activity was work from tasks (Swan, 2017-2018). Prospective teachers were required to look for tasks that could be used in the research lessons. The tasks were solved and their features were discussed in detail according to the learning objectives defined, as well as the possible difficulties of the students in solving them. Another important emphasis was the organization of the mathematics class based on a work on tasks that can contribute to students' learning, in a perspective identical to what Japanese researchers call "structured problem solving" (Fujii, 2018).

Data collection was made by participant observation with a research journal of sessions (S*x*) and individual final interviews (FI) with 6 prospective teachers. For these interviews (all transcribed), we selected prospective teachers with different profiles, with some or no professional experience and with high or low participation in the classes. As data analysis strategies, we use content analysis (Bardin, 1979). Data analysis began by identifying significant moments in the sessions and interviews, that is, moments in which future teachers' discourse may be a signal of teachers' learning, taking into account the prospective teachers' perspectives about the work done in the different sessions, namely about the (1) lesson study's organization in the course; (2) the work on tasks; (3) the preparation of the lesson plan; and (4) the preparation of the observation of the research lesson. Regarding their perspectives about mathematics knowledge, we analyzed, (5) their awareness of own difficulties in mathematics, and, regarding their knowledge of mathematics teaching, we considered (6) lesson planning; (7) selecting tasks; (8) anticipating the reasoning and difficulties of students; and (9) conducting classroom communication. The interpretation of data was originally made by one of the authors and cross-checked by the other author. In some cases, the original interpretation was revised.

Results

Understanding and getting involved in the lesson study process

In the first session of the course, the work to be carried out throughout the semester was presented, with a complete script indicating all phases of the lesson study. It was in the second session that the work in the lesson study really began. A document was presented explaining in detail the work that was to be carried out and an article with a detailed description of a lesson study was discussed. However, for the prospective teachers, it was not sufficiently clear what was to be done. In the final summing up, carried out in the last session of the course, several prospective teachers indicated that they were confused regarding the work of the lesson study, namely the exploration of tasks related to the chosen topics:

Joana: Well, I think that at first the [course] was a little confusing. [...] First, we were looking at textbooks for a long-time and... I think that it was too much time for that. When we started being more oriented [by the teachers] and we realized what it was to do, I think it went very well. I think it was quite interesting. (S12)

Carolina: I confess that only from the third session I begun reading about it and I realized what we were doing [laughs]. Really, we were studying textbooks and [I thought] "and now we're going to have to study the textbooks?" Then I understood what we were actually doing... And I really enjoyed it. (S12)

Clearly, the prospective teachers were not used to do the type of work that was proposed – working in detail with mathematics tasks. For the first time, they were asked to select and analyze tasks in depth, anticipate possible strategies and difficulties of students, and plan a lesson. They began by not understanding why doing that. At the very beginning, they showed some difficulty in foreseeing and understanding the whole process. However, this problem was overcome in later sessions, with the continuation of the work, as they saw the structure of the lessons to be taught getting shape.

The planning of four research lessons at distance was very demanding and the prospective teachers suggested the need for more time for the planning, to prepare the observation, and to elaborate and reflect on the observation script:

Joana: So, I think if we had a little more time there to think a little more about these issues and write a better script... Not that our scripts were horrendous, but I think I could have a little more time there on the observation part. Especially in this issue, I think it was very fast. It was just one lesson, basically. (FI)

In a similar direction, Carolina indicated that observation is a very demanding process:

Carolina: The issue of focus... Of the focus spoken by [the teacher] in the last session… That was important to me because I dispersed a bit and tried to focus and even then after the lesson I ended up dispersing myself because I had not pointed [some aspects] and even as I had the script and still did not record everything I wanted... I ended up not being able to point well the times... And other things... (FI)

Learning about mathematics and mathematics teaching

Despite the initial difficulties of understanding the formative model, the prospective teachers considered important the opportunity they had to put into practice the knowledge recommended in the course. They valued that they had the opportunity to plan a lesson in great detail, even if the course does not yet provide for prospective teachers to teach:

Olga: I ended up liking this course very much, because it was a course where I could feel a little bit of what my future will be like as a teacher [...] and then do that with a grade 7 class. I managed to have participative students, enthusiastic about the tasks.

> I do not know if it was good for me or not, but I will continue with the positive feeling that students will like the tasks ... And I always try to enthusiasm them. (FI)

For Olga, the lesson study was a complete experience with the preparation of a lesson that was then taught by another prospective teacher. Olga was excited with the outcome that the lesson that she planned with her group and, more specifically, the tasks, had on the students, and expressed a wish to maintain a similar practice in the future.

The whole group was involved in a deep analysis and reflection on the nature of tasks, their relevance to the learning of the selected topics, as well as the possible difficulties that could arise for students, culminating in the observation of students' work in these tasks. The prospective teachers had already studied in other courses the classification and characteristics of the different types of tasks, so they already knew some of the language associated to this theme. They acknowledged, however, that the deep work of solving and analyzing tasks, considering the objectives defined for the research lessons, provided them with new knowledge: "the work on tasks added something to what we already knew..." (Beatriz, FI). It also led them to deepen the previous knowledge "about the type of tasks we had already seen in the 1st semester" (Carolina, FI). They highlighted that this work became more productive because it was practical: "the most productive work on tasks is as practical as possible" (Carolina, FI).

Clarisse, the prospective teacher who taught the grade 7 research lessons, pointed out that, despite already having some teaching experience, she did not use to solve the tasks before proposing them to her students, "most of the tasks I did not solve and took immediately to lessons" (Clarisse, FI). The solution of the tasks and the analysis of their suitability for the objectives of the lesson and the students made her learn and value the solution of the tasks and the anticipation of the students' strategies and difficulties "Ah! I've learned a lot about teaching. The planning part, the part of bringing responses already foreseen, possible answers from students" (Clarisse, FI).

Joana, the prospective teacher who taught the grade 8 research lessons, also mentioned that she learned several issues about tasks. She became to value exploratory tasks, recognizing that they provide the opportunity for students to do practical work and build their own knowledge:

> Joana: To take a task that is to practice, it is to do something, it is not to stay there just listening to the formula, hear the concept, but start from something and go working that task to learn the concepts, I found it very interesting... (FI)

Although Joana already knew some of the characteristics of exploratory tasks from the work previously carried out in other courses, it was the planning of a lesson based on such tasks that had more novelty for her:

> Joana: In fact, I had never thought of beginning a lesson, although we had heard enough in [a previous course] about investigation and exploration tasks, I had never thought of doing a lesson around the task itself. […] I found it interesting also that certain lessons could unfold just from the task... Really, beginning from a task and preparing the lesson based on it and observing the difficulties of the students and what is expected from them and what can be done from this is very interesting. (FI)

In the lessons led by Joana, on isometries, the moment of whole class discussion did not go as planned. She had difficulty in orchestrating the discussion because the students were not very involved and did not know this way of working. However, Joana valued the moment of whole class discussion: "and also, yes, the moments of discussion, although in my lesson it did not go very well..." (FI).

The prospective teachers had already worked on many of the ideas on mathematics teaching considered in the lesson study. However, when they had to put these ideas into practice, supervised by the teachers of the course, these ideas acquired greater meaning, much closer of what it will be in their profession. This is what Vítor says: “I had heard of exploratory lesson in [previous courses], but it is one thing to speak in a theoretical way and it is another thing to do it” (FI).

For Clarisse, who taught the grade 7 lessons, the experience with a new approach to teaching was positive: “I think it went very well (…) and the students also liked it very much”. The lesson ran according to her expectations and the students were very involved. This led her to value exploratory teaching based on the learning that students may develop, “but I think it’s really essential, I even think that learning is much faster achieved in this type of teaching than in a type of teaching that I have being doing perhaps wrongly” (FI).

The detailed planning work of the lessons also caused initial confusion to the prospective teachers. For example, Clarisse pointed out that “we had some initial difficulty in planning” (FI) while Vítor found some difficulty in determining the desirable level of detail “I don’t know if we didn’t exaggerate in detail” (FI). Despite the difficulties with the preparation of the lesson plan, Carolina stressed that the realization of the plan, in the context of the lesson study, allowed them to understand the relevance of each of the elements of this plan and how it was thought:

Carolina: I had heard of planning and one person ends up copying the plan from some other place... Or from the internet. But the fact that we […] realize that column, that topic of planning, is important and then, in practice we realize that it is effectively important. […] This here in the planning […] it even seemed like it wasn’t important and then we ended up... The students ended up having a different reaction in that part and even should have been more thought of, wasn’t it? [...] (FI)

In addition, Carolina recognized that the fact that they had also observed the lesson in practice makes each of the reflections and elements of the lesson plan more important to understand the work and learning of students. She recognized the importance of elaborating such a detailed lesson plan.

For Beatriz, all this detail in the preparation of the lesson plan made her more aware of the fact that there are different ways to solve a task and also of the importance of the teacher to be attentive to different solutions and value them in the classroom:

Beatriz: Not so much in the way of looking at students, more in the way of looking at the diversity of student solutions. That’s right, because sometimes we only think in one way and when we see, there are several students who have several ways to solve and that is also correct, and I think it is important to give due value. (FI)

All prospective teachers assumed that they made significant learning on mathematics teaching issues, in some cases with new ideas, in other cases bringing a different perspective on things that they already knew. This contrasts with their views regarding mathematics, in which they did not identify any significant learning. However, we note that there were several discussions on mathematical issues in which the prospective teachers seemed somehow confused. This happened, for example, with the issue of how to prove that the sum of the measures of the internal angles of a triangle is 360º, as they were willing to accept that a physical demonstration involving cutting vertices from a triangular sheet of paper could be seen as a mathematical proof. It seems that the fact that they had made many courses in mathematics previously, in their undergraduate study, led them to assume that they had a strong mathematical preparation despite the discussions in the course sessions seemed to suggest that they had several ideas to revise and clarify.

Conclusion

Several prospective teachers indicated an initial confusion regarding the work to be carried out in the lesson study, namely the exploration of tasks related to the chosen topics. More than a difficulty in understand the lesson study process, the teachers had difficulty in understanding why to do a so detailed analysis of tasks related to the chosen topics. Perhaps this needs to be better explained in the future.

In previous courses, the prospective teachers had already studied the classification and characterization of different types of tasks. However, they tended to look at the tasks superficially. They selected tasks to propose to students without solving them and without analyzing the possible strategies and difficulties involved in their solution. The deep work of task analysis and anticipation of students' strategies, as well as the detailed planning of an exploratory lesson, as well as its observation, provided significant learning on the teaching of mathematics. As indicated in Ni Shuilleabhain and Bjuland (2019), the future teachers highlight the value of going to the school in order to plan a lesson that is then put into practice, emphasizing that this experience favors the understanding of the characteristics of the tasks. In the end of the course, the future teachers also emphasized the importance of the practical work of selection, analysis and discussion of tasks, underlying the value of working from practice for the development of teachers' knowledge (Ball & Cohen, 1999; Smith, 2001).

The format proposed for the planning of research lessons also caused some initial confusion. However, after observing the lesson, the prospective teachers tended to value the realization of a detailed planning, recognizing the importance of anticipating the possible strategies and difficulties of students, as pointed by Burroughs and Luebeck (2010). Overall, the prospective teachers highlighted significant learning about the teaching of mathematics but not on mathematics. This may be regarded as natural, given that so far, they had many courses in mathematics, but they still do not have much experience in mathematics teaching.

The pandemic brought some challenges to the lesson study, namely the difficulty in accessing classrooms. The definition of an organizational model with a group of 14 participants proved to be challenging. As in the study by Burroughs and Luebeck (2010), the proposed format contemplated the division of the group into 4 subgroups. However, contrary to this experience, two future teachers taught the research lesson. In previous experiences, the research lesson was led by a cooperating teacher who contributes with his practical experience to the definition of research questions. The option that we assumed in this course, provided an important opportunity for these two future teachers, who are already teaching with other qualification, to reflect on their own practice. Despite the restrictions imposed, it should be noted that, contrary to what happened in the study by Burroughs and Luebeck (2010), this division into 4 subgroups allowed all future teachers to attend the research lesson they planned, completing the entire cycle of the lesson study, with the observation of the research lesson, so valued by them. Although the overall view taken by the prospective teachers regarding the work done is very positive, it is important to adjust the distribution of lesson study sessions to the number of research lessons to plan, considering, in particular, also the need for a deep preparation of the observation of the research lessons.

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Measuring mathematical knowledge for teaching: the case of algebra in pre-service teacher education in the Netherlands

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Keywords: Mathematical Knowledge for Teaching, measurement instrument, algebra, lower secondary education, pre-service teacher training

Introduction

Algebra is central in mathematics education in secondary school: in problem solving, in representing relationships and functions, in describing patterns and structures, as a language. Algebra education aims at integrating procedural skills and conceptual understanding (Drijvers et al., 2011). This requires good mathematics teachers, who have sufficient content knowledge (CK) as well as pedagogical content knowledge (PCK). Learning to teach, however, takes time and is complex (Korthagen, 2010). At the same time, in the Netherlands, teachers are given full responsibility for their own classes from the start of their career. This means that the mathematical knowledge for teaching of pre-service teachers needs to be at a high level. To improve their pedagogical content knowledge (PCK), and, if necessary, the corresponding content knowledge (CK), we need to identify the omissions in their knowledge. PCK, however, is a complex concept that is hard to measure. Ball et al. (2008) have elaborated the knowledge needed to teach mathematics into the Mathematical Knowledge for Teaching (MKT) framework. The goal of the study presented here is to find out whether there exist instruments to measure MKT for algebra teaching in lower secondary education, appropriate for the Dutch context. This is done by conducting a meta-study on existing instruments. The poster will report on the meta-study, its methods, its results and its conclusion.

Theoretical Framework

Next to CK, teachers also need PCK (Shulman, 1986). Within mathematics education, especially at the University of Michigan, research has been conducted into the nature of CK and PCK of mathematics teachers, by collecting data from mathematics lessons in practice, and by analyzing and categorizing these data. This resulted in 2008 in a subdivision of MKT. Parts of the subdivision focus on general CK and PCK, and parts of the subdivision focus on the teaching practice itself: Specialized Content Knowledge (SCK), Knowledge of Content and Students (KCS) and Knowledge of Content and Teachers (KCT). SCK is the mathematical knowledge needed to teach, such as: being able to answer a “why” question. Central to KCS is the knowledge of students' thinking, and of their misconceptions about mathematical subjects. Central to KCT is the design of the instruction, for instance the order of the examples that are used.

Research question

The research question we want to answer is: to what extent are instruments, developed to measure mathematical knowledge for teaching algebra in grade 7-9, appropriate for measuring pre-service teachers’ SCK, KCS and KCT in the Netherlands?

Method

We have conducted a literature review. Keywords used were SCK, KCS and KCT, algebra, (variants of) pre-service education. The databases searched were ERIC, Web of Science, Scopus and PsychInfo. Because the first search yielded a review article (Kim, 2018) in which all measurement instruments of MKT up to 2012 are listed, we restricted the search to the timeframe from 2012 to present. The articles found were analyzed on the measurement of SCK, KCS and KCT for algebra within pre-service teacher education. For example, the target group, tasks of teaching and categories of knowledge were considered to decide to what extent the instrument would be appropriate for the Dutch context. Criteria used in this evaluation, such as the role of conceptual understanding and the extent to which SCK, KCS and KCT are measured separately, will be presented in the poster. The search resulted in fifteen articles.

Results

Findings show thirteen instruments, ranging from the instrument for measuring Knowledge for Algebra Teaching (McCrory et al., 2012) which has been extensively tested and validated, to a questionnaire. Some instruments focus on one particular topic (for example variable). Most instruments consist of items that mainly measure SCK, next to Common Content Knowledge. In general there is no breakdown into the three different domains SCK, KCS and KCT. The instruments and their applicability to the Dutch context will be presented in the poster.

Conclusion

The articles we found in our review show that instruments developed for measuring MKT seem to pay little attention to separately measuring KCS and KCT, which raises the question why this is the case, and whether this is necessary. To measure SCK, KCS and KCT for teaching algebra of pre-service teachers in lower secondary education in the Netherlands, we need to combine items from different instruments and we need to extend this with items to measure KCS and KCT.

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Prospective teachers' diagnostic judgments on students' understanding of conditional probabilities

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In recent years, different studies have emphasized the necessity to improve prospective teachers' professional knowledge and teaching competences, especially prospective teachers' diagnostic competence, which accounts for perception, interpretation, and decision-making concerning students' individual thinking processes and learning obstacles. In order to do so, this paper follows a content-related approach that focuses specifically on diagnostic judgments on conceptual under-standing and procedural skills of students' understanding of conditional probabilities and their underlying prior knowledge elements. Prospective teachers are known to face challenges in adequately focusing on these aspects in their diagnostic judgments. Therefore, written diagnostic judgments of 26 prospective teachers on two transcript vignettes are investigated. The results indicate that prospective teachers show a high focus on prior knowledge and on procedures in their diagnostic judgments.

Keywords: Diagnostic judgments, conditional probabilities, prospective teachers, vignettes.

Introduction

As part of prospective teachers' professional preparation student-centered teaching plays an important role. The latter requires teachers to diagnose skillfully, i.e. to master the mental processes of perceiving, interpreting, and decision-making (Empson & Jacobs, 2008). In order to prepare prospective teachers for adequate diagnostic judgements, it is necessary to gain deeper insights into their diagnostic thinking processes, that can be inferred from their diagnostic judgements. Already existing studies line out that prospective teachers are known to struggle with addressing conceptual understanding and procedural skills in their diagnostic judgments (Bartell et al., 2013). This underlines the importance of focusing on the mathematical content, such as relevant knowledge elements of the current learning content as well as the underlying procedural skills and conceptual understanding in prospective teachers' diagnostic judgements (Prediger, 2020). Therefore, our research interest is to investigate which elements of the mathematical content teachers perceive and interpret (as processes of diagnostic thinking) in students' understanding of conditional probabilities in a one-to-one teacher-student discussion. This topic was chosen as it is a pivotal, but often challenging concept and holds relevance in vocational contexts (Binder et al., 2020).

Theoretical Background

Prospective teachers' content-related diagnostic judgments

Teachers' diagnostic competence has been found to be important for student-centered teaching (Empson & Jacobs, 2008). Synthesizing different approaches on diagnostic thinking and diagnostic competence, Loibl et al. (2020) provide a framework for locating different research approaches (see Figure 1). In addition, the framework displays the impact of the diagnostic thinking processes on diagnostic behaviors.

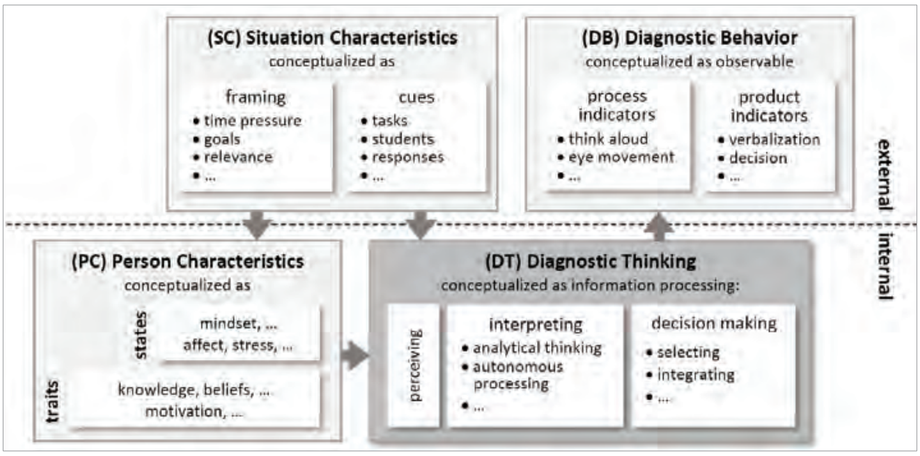


Figure 1: DiaCom framework of diagnostic thinking (Loibl et al., 2020, p. 3)

Therefore, those processes can be derived from diagnostic judgements, influenced by situation and person characteristics. Stahnke et al. (2016), with their systematic review of different studies and research approaches, show that although most of the studies investigating diagnostic competence use a particular mathematical content, only few studies explicitly integrate the content into the analysis of the diagnostic judgments. The study presented in this paper also focuses on content-related aspects in analyzing the diagnostic judgments, following Dröse and Prediger (submitted).

When taking this content-related approach, the principal distinction (as in Kilpatrick et al., 2001) is between conceptual understanding (as meaning of mathematical concepts, operations, and terms) and procedural skills (as procedures for algorithms and solution strategies). This allows for conceptualizing students' understanding as a network of the main mathematical knowledge elements, conceptual and procedural (Hiebert & Carpenter, 1992). The ensuing network contains knowledge elements of the current learning content as well as prior knowledge elements, i.e. conceptual understanding and procedural skills from previous years as a foundation (Prediger, 2020; Dröse & Prediger, submitted).

Prospective teachers in particular have been shown to focus (in the sense of perceiving and interpreting) more on general aspects of learning in their diagnostic judgments than on mathematics- and content-specific aspects (Jansen & Spitzer, 2009). Moreover, when focusing on content-related aspects, prospective teachers are more likely to perceive procedural obstacles – albeit not their cause (Cooper, 2009). What is more, they often misinterpret conceptual obstacles as procedural (Son, 2013). While obstacles are mostly interpreted by prospective teachers as lack of a procedural skill, when it comes to students' resources, prospective teachers often interpret students' statements as indicating conceptual understanding – even if the aspects they refer to are procedural skills (Bartell et al., 2013).

Against this background, it makes sense to first take a closer look at the conceptual understanding and procedural skills concerning conditional probabilities and the related students' obstacles.

Knowledge elements and students' understanding of conditional probabilities

For the *current mathematical content* of conditional probabilities, procedural skills as well as conceptual understanding are relevant learning goals. Conceptual understanding of conditional probabilities implies concepts concerning stochastic (in)dependence for describing the likelihood of an event under certain conditions or independent of conditions (Hoffrage et al., 2015). Students face various obstacles in the area of conditional probabilities, e.g. distinguishing joint and conditional probabilities (Shaughnessy, 1992). For determining conditional probabilities and solving Bayesian problems, different visualizations and solution strategies can be used, e.g. tree diagrams, frequency grids, unit squares, or 2x2 tables, viewed as procedural skills (see overview in Binder et al., 2020).

These knowledge elements for conditional probabilities build on *prior mathematical content* knowledge, especially the part-whole relationship as well as the part-of-part determination as an important mental model for the multiplication of fractions. Both belong to the conceptual understanding of fractions (Post & Prediger, 2020; Prediger & Schink, 2009) and are known to present serious obstacles, e.g. in identifying the appropriate whole (Prediger & Schink, 2009). In this context, procedural skills (e.g. routine calculations with fractions, decimal numbers, and percentages) may also constitute a learning difficulty (e.g. Prediger & Schink, 2009).

In this paper, we will focus on the procedural skills and conceptual understanding of the current learning content, conditional probabilities, and its underlying prior knowledge elements, in connection to possible students' obstacles and individual mental models, found in the diagnostic judgments of prospective teachers. For designing adequate learning opportunities for prospective teachers, it is important to investigate the knowledge elements addressed in their diagnostic judgments.

Considering this design interest as well as the research areas and gaps, our research question reads: *Which concepts or procedures of the current / prior learning content do prospective teachers include in their diagnostic judgments on students' understanding of conditional probabilities*?

Methods

Data collection

The data was collected in a university mathematics education course for German prospective secondary school teachers. The sample consists of n = 26 prospective teachers, 81% studying for secondary and upper secondary school, and 19% for vocational schools. All students have reached the last year of their bachelor program, 69% after three and 31% after four years. They attended the first and second author's university course, which covered content knowledge as well as pedagogical content knowledge on conditional probabilities, e.g. students' errors, and related knowledge elements.

The prospective teachers' written diagnostic judgments were gained by analyzing a vignette as part of the weekly assignments (see Figure 2 for our vignette). Vignettes can be seen as an established instrument for investigating prospective teachers' competences (cf. overview in Buchbinder & Kuntze, 2018) and have been used to investigate content-related diagnostic judgements on procedural and conceptual knowledge elements in the mathematical content of arithmetic (Dröse & Prediger, submitted).

Our vignette consists of a task, two written student solutions and transcripts of subsequent dialogues between student and teacher. For the transcript, a real dialogue (based on transcripts in Post & Prediger, 2020; Post, in preparation) was chosen as basis, which was adapted taking theoretical considerations on conditional probabilities into account (current learning content from Hoffrage et al., 2015; Shaughnessy, 1992; Binder et al., 2020; prior learning content perspectives from Post & Prediger, 2020; Prediger & Schink, 2009).

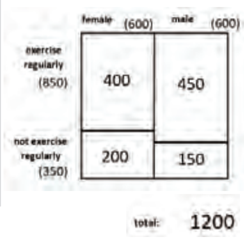
Transcript vignette

For the following task you should put yourself in the teachers' position and react appropriately within the situation. Background information for the following scene: The class in a German upper secondary school has covered conditional probabilities and their calculation. In the following you will read two transcripts displaying excerpts from conversations between two students (Ole and Nazan) and their teacher, regarding the task displayed on the right.

Task: Exercising Teenagers

In a survey, 1200 teenagers were asked if they exercise regularly. 600 out of the 1200 teenagers are female. $\frac{1}{3}$ of the female teenagers do not exercise regularly. $\frac{3}{8}$ of the teenagers are male and exercise on a regular basis. What is the probability that a random male person exercises regularly?

(JIM study 2018)



Part 1: Ole solves the task. He writes down the following solution.

$\frac{3}{8}$ = 0.375= 0.375%, the probability is 0.375%.

The following interaction with the teacher evolves:

1 T: Ole, how did you solve the task?

2 Ole: Actually, there is not much to calculate. The text says that '$\frac{3}{8}$ of all teenager are male and do sports' [reads from the text]. And then I just have to convert this into percentages and that is 0.375%.

3 T: Aha. This share is very small. Hm. Perhaps it might help you, if we had a look at the unit square below the task again? *[Points at the unit square that is printed below the task.]*

4 Ole: So, look at this numbers here, 600 male *[points at the labelling "male (600)"]* and 850 exercising. The 600 are male and the 450 are the ones who exercise, in addition. So, these are the 450 *[points to the area with the number 450]*.

5 T: And what's the size the share that is sought in the task?

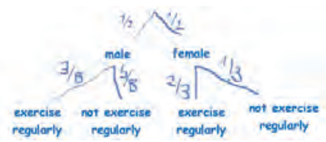
6 Ole: Ehm, the share is 450 of the whole, 1,200? Ehm *[reads the question again]* no, in the question there is just this group here *[points at the area with the labelling "male (600)"]* so this, these are the males, but these here *[points at the two areas on the left]*, those are not considered in the denominator, and the counter would be 450 *[points to the area with 450]*. Or in other words: This is the whole group, these are the males and this is the share of them exercising regularly.

7 T: Good, so you have the fraction $\frac{450}{600}$. But what about the probability or… ehm the share that you calculated before, so the $\frac{3}{8}$?

8 Ole: Perhaps it can be cancelled, and then it is equal.

Part 2: Nazan also solves the task. She writes down the following solution.

$$\frac{3}{8} \cdot \frac{1}{2} = \frac{3}{8} \cdot \frac{4}{8} = \frac{12}{64} = 0{,}1875 = 18.75\ \%$$



The following interaction with the teacher evolves:

1 T: Nazan, how did you solve the task?

2 Nazan: So at first I calculated that the probability for boys is 50%, that is $\frac{1}{2}$. Than I can write this into the tree diagram *[points at her tree diagram*] and then I have to calculate $\frac{1}{2}$ times $\frac{3}{8}$. So you convert them to the same denominator, and $\frac{1}{2}$ is equal to $\frac{2}{4}$ *[points to her written calculation]* and multiplied this is $\frac{12}{64}$. And this is 18.75 percent. So the probability is 18.75%.

3 T: Let's have a look at the unit square below the task to be sure. Which parts do we have to look at? *[Points at the unit square below the task.]*

4 Nazan: So for the numbers we have 600 here *[points at the label "male (600)"]* and 850 exercising *[points at the label "exercising (850)"]*. Yes, and then this is $\frac{600}{1200}$ times $\frac{850}{1200}$, because ehm those are the important issues, male and exercising.

5 T: And how did you transfer that into your tree diagram?

6 Nazan: So, the $\frac{600}{1200}$, that is $\frac{1}{2}$ in my tree diagramm, the probability for boys. $\frac{600}{1200} = \frac{1}{2}$, cancelled out. Are $\frac{3}{8}$ the same as $\frac{850}{1200}$, if you cancel? Somehow this has to beo, because in the text says teenager, who are male and exercise regularly, and that is the same as in the question. Just the "and" here, that has changed in the question *[points at the "and" in the text and then at the question]*.

Task: Analyze the two transcripts:

(1) Describe which prior knowledge and resources (conceptual knowledge, procedural knowledge, representations, etc.) Ole and Nazan draw on.

(2) Describe which obstacles Ole and Nazan display. Explain the possible causes of these obstacles.

Give transcript lines for (1) and (2) that underpin where you locate the aspect within the transcript or the notes.

Figure 2: Transcript vignette with task for prospective teachers

The transcript vignette therefore provides sample insights into students' understanding of conditional probabilities and the underlying prior knowledge elements, and thus sufficient possibilities for diagnostic judgements on conceptual understanding and procedural skills as well as knowledge elements of the current learning content and prior knowledge elements. Cues for the different knowledge elements are presented in equal numbers.

Data analysis

The 26 written diagnostic judgments were coded with respect to the knowledge elements for the current or prior learning content in the two dimensions of procedures and concepts. The coding scheme was deducted from the theoretical analysis of the knowledge elements and inductively enriched by the knowledge elements named by the prospective teachers. Two raters following the coding scheme yielded an interrater reliability of Cohen's $\kappa = 0.86$, which is almost perfect. Table 1 displays the codes used and excerpts of exemplary diagnostic judgements.

Table 1: Excerpt of a written diagnostic judgement and knowledge elements assigned to them

	Conceptual understanding	**Procedural skills**
Current learning content	Knowledge element: understanding conditional and joint probabilities Excerpt: "Does not know the difference between $P(A\cap B)$ and $P(B\|A)$."	Knowledge element: calculating and solution strategies for conditional probabilities Excerpt: "Difficulties in calculating conditional probabilities."
Prior learning content	Knowledge element: understanding of fractions (part-of-part), unit square Excerpt: "Ole has an understanding of part-of-parts"	Knowledge element: calculating with fractions, decimal numbers and percentages Excerpt: "Ole can transform fractions into decimal numbers."

Empirical findings on prospective teachers' diagnostic judgments

In total, 327 codes were set for the statements of the 26 prospective teachers. Figure 3 displays the coded knowledge elements in prospective teachers' diagnostic judgments. Table 1 provides first excerpts of written diagnostic judgements. In the following the relationships between the coded knowledge elements are described and enriched by excerpts of the written diagnostic judgements.

Comparing the knowledge elements, the results indicate that the prospective teachers tend to focus more on the *prior learning content* in their diagnostic judgments than on knowledge elements of the *current learning content* (Figure 3, first line). In Addition, their statements address a higher amount of *procedural knowledge elements* than of *conceptual knowledge elements* (Figure 3, second line). For the procedural elements there seems to be a higher number of obstacles addressed, while for conceptual knowledge elements a higher amount of statements is related to *resources*.

For the *current learning content* (Figure 3, third line), *procedural* and *conceptual* elements seem to be addressed equally. While the conceptual elements are equally described as *resources* and *obstacles* (e.g. conceptual resource: "has a concept of probabilities", conceptual obstacle: "doesn't know the difference between joint and conditional probabilities"), only few statements concerning procedural skills address them as resources (e.g. procedural resource: “knows how to calculate joint probabilities”, procedural obstacle: “has difficulties in calculating conditional probabilities”).

Most of the statements for the *prior learning content* as well as most of the statements overall concern *procedural* skills, e.g. operating with fractions, decimal numbers, and percentages (Figure 3, third line). These elements are equally addressed as *resources* and as *obstacles* (e.g. procedural resource: “can shorten fractions”, procedural obstacles: “cannot convert fractions into decimal numbers”). The addressed elements of *conceptual* understanding are expressed more often as *resources* than as *obstacles* (e.g. conceptual resource: “can interpret parts in the unit square”, conceptual obstacle: “cannot derive part-of-part relations from the unit square”).

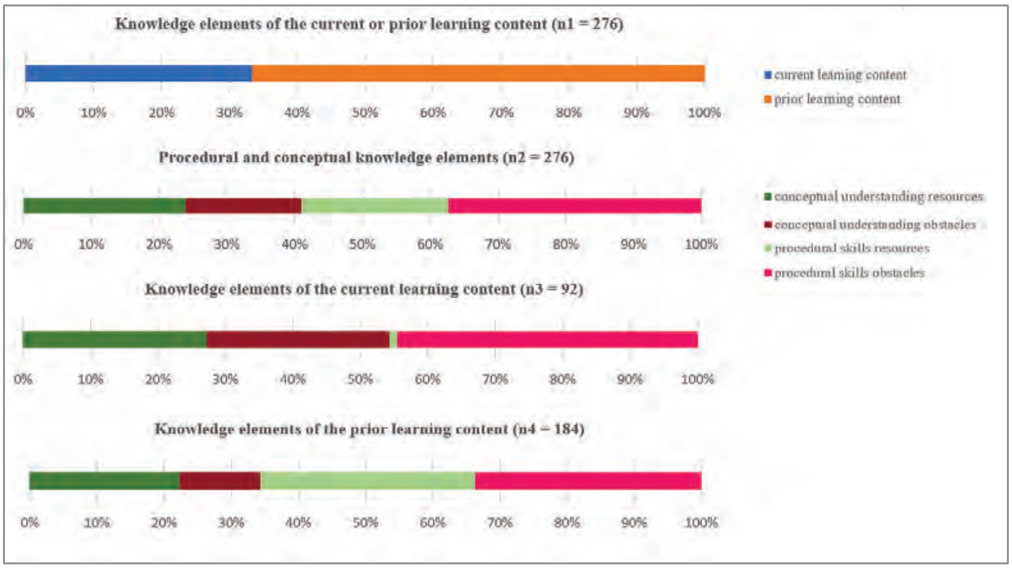


Figure 3: Knowledge elements in diagnostic judgments

Discussion and outlook

Referring to the research question (*Which concepts or procedures of the current / prior learning content do prospective teachers include in their diagnostic judgments on students' understanding of conditional probabilities?*), our content-related approach revealed the following findings:

In general, the prospective teachers in our study focus more extensively on *procedural skills* than on *conceptual understanding*. These findings are in line with previous research (Cooper, 2009; Son, 2013; Bartell et al., 2013). For the current learning content, however, statements on conceptual understanding are dominant. Further qualitative investigations (e.g. interviews) are needed to interpret this result, which is divergent to previous research. By distinguishing resources and obstacles, we saw that procedural skills are far more often addressed as obstacles than as resources overall, although they are categorized equally as obstacles and resources for the prior learning content. As Cooper (2009) indicates, prospective teachers are more likely to perceive procedural obstacles in students' utterances but not their origins which might be conceptual. Future studies could ask prospective teachers to classify their judgements as referring to conceptual or to procedural knowledge elements, as Son (2013) revealed that prospective teachers misinterpret conceptual obstacles as procedural. However, the current study can provide no insights here. Yet, it might present a problem if prospective teachers (in terms of the DiaCom framework) mostly perceive and identify procedural obstacles in students’ utterances, as that might influence their decision-making as active teachers, hindering them to address conceptual learning adequately (see Loibl et al., 2020, for details). On the basis of these findings, starting points for redesigned learning opportunities can be identified, e.g. using authentic tasks for diagnosis and discussing with prospective teachers’ options for student activities that explicitly promote conceptual understanding.

In addition, by distinguishing *prior and current learning content*, we found that prospective teachers are indeed able to describe different content elements building upon each other. This is relevant, as an interconnected network of mathematical knowledge is paramount for the sustainable learning of mathematics. In particular, low-performing students have been shown to lack sufficient prior content knowledge for keeping up with the current learning content (Prediger, 2020). Therefore, our analysis reveals possible potentials of prospective teachers’ diagnostic judgements that have not yet been investigated in depth and can provide first starting-points for offering further learning opportunities and developing teaching-learning arrangements, e.g combining and connecting university courses concerning CK and PCK more deeply.

Due to the aforementioned aspects it would be beneficial to compare the prospective teachers’ diagnostic judgements and the identified knowledge elements to the diagnostic judgments of experienced in-service teachers or teacher educators. By doing so, it might be possible to identify further aspects that could in turn be integrated as learning content in prospective teachers’ courses at university.

Our research is limited due to the following aspects: (a) Our sample comprises only 26 prospective teachers all from the same university. Future research is planned to extend the sample size and the sample itself to in-service teachers. (b) Our research addresses a specific content. As diagnostic judgments might vary between contents, other content should also be investigated. (c) The use of a transcript vignette meant that it could have been read several times. It is possible that the diagnostic judgments would focus on less aspects in a different format, e.g. video vignettes (Buchbinder & Kuntze, 2018). (d) The vignette displays a teacher-student one-to-one interaction and not a whole group classroom discussion. Where the vignette can be extended to in the future. Future research should also clarify and investigate these limitations. In addition, it should be explored how transcript vignettes can be used to foster prospective teachers’ diagnostic judgments.

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What you teach is what you get? Exploring the experiences of prospective mathematics teachers during a teacher education programme

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The role of the teacher education programme in preparing mathematics teachers to teach mathematics is an under-researched area within mathematics education. In this paper, we analyse four components of empirical material, each captured from a teacher education programme based in Sweden. Using a methodological approach informed by enactivism and Systemic Functional Linguistics, we build on previous analysis of the language-in-use of one mathematics teacher educator to situate a further layer of analysis, this time, from the perspective of a prospective mathematics teacher. Our findings suggest the prospective teacher in this study, who had positive experiences of mathematics at school, learns to align linguistically with the mathematics teacher educator's contrasting views of mathematics teaching and learning, and in doing so, creates herself a safe space.

Keywords: Mathematics teacher education programmes, mathematics teacher educator language, prospective mathematics teachers, enactivism, systemic functional linguistics.

Background and context of the study

This paper sets out to illustrate a problem related to mathematics teacher education, as part of a more extensive study, to understand how prospective mathematics teachers negotiate meaning from the language of mathematics teacher educators during teacher education situations (Ebbelind, 2020). We set out to address the relationship between interactions during teacher education situations and the kinds of meanings realised by the prospective mathematics teachers. Previously our focus has been on developing a methodology for studying the language-in-use of the mathematics teacher educator (see, Ebbelind & Helliwell, 2021), whereas in this paper, we foreground the prospective mathematics teacher. Specifically, we turn our attention to how one prospective teacher, Lisa, discursively assembles language-in-use when taking part in her teacher education programme. Our research question is: *How does one prospective primary teacher navigate her initial mathematics teacher education programme?* We are interested in what can be learned, as mathematics teacher educators, from exploring the experiences of prospective teachers during their mathematics teacher education programmes in relation to the language-in-use of mathematics teacher educators.

The empirical material used for this paper consists of four components, each captured during a teacher education situation in Sweden. The first component is a transcript from a mathematics teacher education lecture, the second is a transcript from a seminar where a group of prospective teachers worked on a task set during the lecture, the third is a transcript from an interview with one of those prospective teachers, Lisa, and the fourth is a set of extended field notes (Delamont, 2008) that were taken based on Lisa's reflections on the lecture. Lisa is in her early 20s and began teacher education directly after finishing school. She described herself as an extremely competitive athlete playing soccer (football) at a high national level. Much of her terminology can be interpreted as related to sports and management. During Lisa's teacher education programme, she studied 30 ECTS credits (one full semester) in mathematics education. Lisa perceived herself as a "good" mathematician when re-engaging in her past school mathematical experiences at upper secondary school (aged 17-19 years). After her education she will work as a teacher in an upper primary school (aged 10-12 years). The context in Swedish mathematics teacher education is the reform mathematics movement that "promotes a vision of school mathematics that focuses on students' creative engagement in exploratory and problem-solving activities as they develop their understandings of significant mathematical concepts and procedures" (Skott et al., 2018, p. 164). In Sweden, prospective teachers at primary level (aged 7-12 years) educate to become generalists, as opposed to subject specialists. As a consequence, primary teachers in Sweden will teach a range of different subjects as well as mathematics and their level of education in each of the school subjects is often modest. In Sweden (Ebbelind, 2020), and other western countries (Stoehr, 2015), the professional background of a primary teacher is linked less to the teaching of specific subjects than to the profession as a whole. The subject of mathematics itself becomes subordinated for these prospective teachers, and social development becomes the primary aim of schooling. As a result, prospective teachers create a "safe space" for themselves and their future students (Gellert, 2000).

The role of teacher education programmes for prospective mathematics teachers

The role of teacher education programmes in the educational system is increasingly discussed and problematised. There is an underlying assumption in the research field of mathematics education that teachers matter in relation to students' learning. To become a mathematics teacher at primary level concerns a shift from viewing oneself as a learner of mathematics in school to a perspective of oneself as a mathematics teacher who teaches others to learn mathematics. Being enrolled in a teacher education programme has been shown to change the relationship one has to mathematics teaching and learning (Hošpesová et al., 2018).

Some researchers regard becoming a primary school teacher through teacher education as problematic because of the relationship prospective teachers have regarding their own past school-related experience, even concluding that teacher education has little impact on prospective teachers' beliefs, with a limited chance of affecting their future teaching (Ebbelind, 2020). Prospective teachers past experiences will shape the way they become teachers. Those prospective teachers who struggled with mathematics at school continue to struggle during their teacher education programmes, whilst the prospective teachers who enjoyed mathematics in school also continue to do so (Player-Koro, 2011). The political agenda, as well as some research into teachers' knowledge and beliefs, have both established a deficit model of prospective mathematics teachers (Askew, 2008). Much attention has been given to what prospective teachers do *not* know (based on formulations of what they *need* to know) and how to improve their knowledge by incorporating more mathematical content within teacher education programmes. However, as Hemmi and Ryve (2015) explain, this deficit story needs to be used gently by the research community, otherwise the therapeutic (pastoral) aspect of teacher education becomes too dominant (Hannula, 2002). The deficit story, and the pastoral aspect of teacher education, are considered a part of Swedish teacher education (Hemmi & Ryve, 2015; Player-Koro, 2011). Gaining insights into the experiences of prospective primary school teachers in relation to their development during teacher education programmes is a gap that this study seeks to address.

Methodology: A recursive inquiry

In our research, we draw on enactivism as our theoretical and methodological basis. From an enactivist perspective cognition is viewed as “the enactment of a world and a mind on the basis of a history of the variety of actions that a being in the world performs” (Varela et al., 1991, p. 9). One corollary of this view is that, as individuals, the way we see and act in the world is determined by our own unique history of experiences and interactions. On this basis, we look to explore how prospective mathematics teachers, whose past experiences of mathematics teaching and learning are many and diverse, experience, and thus navigate, their teacher education programmes, specifically in relation to the practices and language-in-use of mathematics teacher educators.

Reid (1996) sets out two features of enactivist research, derived from key principles of enactivism (Varela et al., 1991), that are “the importance of working from and with multiple perspectives, and the creation of models and theories which are good-enough for, not definitively of” (p. 207). From an enactivist perspective, any model or theory used to describe and explain a phenomenon cannot be definitively of some external truth. This does not mean that models and theories are of no use, rather, models and theories are accepted as being good-enough descriptions and explanations for the phenomenon under study rather than mirrors of reality. Thus, utilising multiple perspectives is one way of expanding what is possible to grasp during the research process. In combining our different methodological approaches, we also aim to disrupt the potential interdependency of theory and research findings that are often associated with the use of single perspectives, particularly in relation to sociologically related research within mathematics education (Gellert, 2008). As researchers we bring with us our “theories, beliefs, and biases” (Reid, 1996, p. 206), determined by our own history of experiences and interactions which shape the way we view and analyse the data that we collect. Thus, in addition to utilising multiple perspectives, an important methodological principle for us is to share some of our history as teachers and researchers so that our analysis can be framed within this context.

Since an enactivist perspective endorses a view of learning (and thus researching) as “a recursive process linked to actions in the world giving feedback leading to adapted actions” (Brown, 2015, p. 192), we have adopted a recursive approach to our research design. For an inquiry to be recursive, it involves an iterative process of data collection and analysis, where for example, the initial analysis of data feeds into subsequent analyses in an ongoing process. In this recursive inquiry, we utilise our multiple perspectives through looking at the same empirical data but through different lenses, making multiple revisitations of the data. In this paper, we use our previous analysis of the language-in-use of one mathematics teacher educator (Ebbelind & Helliwell, 2021) to situate a further layer of analysis, this time, from the perspective of a prospective mathematics teacher. We include our own reflections on the process of analysis alongside any observations and findings that emerge as a result of the analysis, which we share with one another in an ongoing dialogue, which then feeds into the next stage of analysis. Specifically, we see enactivism as an overarching methodological framework within which systematic functional linguistics (SFL) provides a complementary analytical tool.

Previously we have developed a detailed description of our research methodology for analysing the language-in-use of a mathematics teacher educator. This research methodology was developed by extending the enactivist informed “methodology for studying talk” described by Coles (2015, p. 235) through utilising SFL as a systematic tool for identifying patterns in the transcript of the mathematics teacher education lecture. In the next section we begin by sharing some of our own history and context (as explained above) before summarising findings from our previous analysis (see Ebbelind & Helliwell, 2021) where our focus was on the interpersonal aspects of the mathematics teacher educator's language-in-use such as voice, tense, polarity, and modality (or certainty) during an early mathematics teacher education lecture for a group of prospective primary teachers in Sweden (the transcript of this lecture providing the first component of empirical material used in this study). Having established our reading of the mathematics teacher education lecture from analysing the language of the teacher educator, we then move on to explore the perspective of one prospective teacher, Lisa, in relation to the same lecture and the mathematics teacher education programme more generally.

Analysis

The authors of this paper were both mathematics teachers before becoming university-based mathematics teacher educators and researchers. Andreas teaches prospective pre-school (aged 1-6 years) and prospective primary (aged 7-12 years) school mathematics teachers in Sweden. His research background links in different ways to social practice theory and symbolic interaction. Tracy teaches prospective mathematics teachers on a one-year postgraduate course for secondary (aged 11-18 years) mathematics teachers in the UK. Her research background links to the perspective used in this study, enactivism, which she has used in researching mathematics teacher and mathematics teacher educator learning and development.

During the lecture, the mathematics teacher educator, who from this point we refer to as Ian, talks about what it means to know something, for example, "one has an understanding of things when one does not have to remember what one must remember to be able to know". He positions the prospective teachers as a unit, and ascribes the prospective teachers as all having had negative experiences of mathematics [note: in the transcripts that follow, ... represents a pause, [...] represents some missing text]:

Ian: I think most students here today… who have gone through the whole school system and high school do not feel that way... was mathematics not really something you had to remember ... do this here and it will be alright [...] Students often do not have the skills needed to be able to present their thinking in for example writing ... It is not simply [...] how many doors do you have at home? What you come up with, we will then try to bring into this lecture. You should start thinking ...

One possibility here is that Ian, perhaps unintentionally, is suggesting that the prospective teachers have had negative past experiences of mathematics as a way of promoting the reform agenda. When Ian focuses on current and future practices, we interpret this as relating to the deficit story of prospective teachers:

Ian: Too many students have not understood anything ... We know that from the national board of education reports. If you understand, you do not have to keep such a lot in mind because you know why it is as it is, and you can pick it up and use it, and we want our students to be able to do that in the future.

The level of certainty in Ian's utterances is low when he speaks in relation to the prospective teachers. For instance, when he talks about the prospective teachers as solving the problem. The level of certainty is also low when Ian addresses the prospective teachers, encouraging the prospective teachers to make sense for themselves of the lecture's content. There are also low levels of certainty relating to mathematics as something for students to master.

In the background, there seems to be a theme of general failure in relation to the past teaching of mathematics. This theme of failure is potentially being used to promote another type of teaching by Ian, "We must… you must in the future be able to write yourself mathematically ... we have to give students these tools to pass the national tests." We interpret that Ian is (perhaps unintentionally) positioning all prospective teachers within a deficit story. Even though he promotes another agenda, the reform agenda, the perplexity, or exclusivity of mathematics is still a part of how the lecture is conveyed. There seems to be a narrative style that can be identified within the transcript of the lecture in that there is a story that unfolds, a story that involves the prospective teachers themselves. Throughout this lecture, Ian promotes the idea that there is another story to tell about teaching and learning mathematics than the expected experience of the prospective teachers.

Following the lecture was a seminar where Ian set tasks for the prospective teachers to complete in their study groups (each study group consisting of a subgroup of the prospective primary teachers). The lecture and seminar were both based around the prospective teachers' past participation in mathematics education, whether that participation was experienced as positive or negative. However, Ian made no reference to any positive experiences during the lecture, and neither did any of the prospective teachers. During the lecture, the majority of the prospective teachers commented on their lack of positive experiences as mathematics students. For prospective teacher Lisa, it became clear that her positive experiences of learning mathematics at school were uncommon within the group. When the different study groups were given a task related to the lecture concerning how their own experiences may have differed from those portrayed during the lecture, Lisa and her peers, Kira, and Dina, expressed how the reform-oriented ideas troubled them. This is reflected in the extended field notes from this session.

> On the way to the group work, Kira [one of Lisa's study-group members] says that this is different from what she expected, something different from what she had done in school. The prospective teachers in this study group had good experiences of mathematics, they are interested in mathematics and seem to understand the mathematics to a higher extent than many of the others.

During the seminar, members of Lisa's study group indicate that some of their own experiences do not align with those emphasised in the seminar questions introduced by Ian, the teacher educator. Kira contrasts her past in relation to the present teaching during the teacher education programme, "[my past mathematics teachers] did not demonstrate mathematics in that way, with all these words on the board, that is said to be so good." Lisa adds, "but doing mathematics was something that I was good at... I liked the textbook." In relation to the reform-oriented teaching promoted by Ian, Dina concluded that "it was probably only later on in the national test you had to [do problem-solving questions] but otherwise, we were not allowed discussions in our class." The three prospective teachers reacted to Ian's utterances which emphasised the failure of schools, the prospective teachers' "old" mathematics teachers, and the school system more generally, all of which contributed, according to Ian, to the prospective teachers' "dislike" of the subject of mathematics. However, no one within this study group "disliked" mathematics, instead, they enjoyed it. They recall enjoying being challenged either by the mathematics textbook or the competitive elements or competitive teaching strategies.

Before calling the study groups back to the lecture hall, Ian has brief discussions with each study group. Lisa's study group is the last group that Ian attends. He seems hurried and begins by discussing their negative feelings against the subject without asking them about their experiences:

Ian:	It seems that you got into the same thing as all the groups, I think, and that is probably it. Typical, that what you bring with you into this course. Is it possible to change this then? It will be a nice task ahead. Yes, you are to become excellent maths teachers.

However, Lisa, Kira and Dina do not challenge Ian during this discussion, instead, they seem to play along and have a tentative conversation with him, aligning with his views. This deviation is brought up in an interview with Lisa one week after the seminar:

Andreas:	I attended your seminar when you had a group discussion ... and from what I perceived, you had a pretty good discussion, but then I thought, when [Ian] came, suddenly, he started talking ...
Lisa:	About many things that we did not discuss.
Andreas:	I was wondering ...
Lisa:	It felt a bit strange because it felt like what we had done was very wrong ... at the same time, it felt like we answered the questions ... and he replied ... it did not feel as if things were connected [...] Yes, it was bizarre [laughing]. But at the same time, you understand what he says, but it is not like you have considered ... often when you think back, you remember what was good ... one does not think in this way when you are young ... then, everything was good, and you quickly calculated things in the mathematics textbook, and it was damn good ... great fun and then when he puts it ... so maybe it was not so good ... it was probably a little awakening for us all...

Discussion

We set out, in this paper, to research the ways in which a prospective primary teacher navigates her initial mathematics teacher education programme and what we can learn, as mathematics teacher educators, by exploring the experiences of prospective teachers in relation to the language-in-use of mathematics teacher educators. In this section we present some tentative findings that prospective mathematics teacher Lisa, who had positive experiences of mathematics at school, learns to align linguistically with Ian's (the mathematics teacher educator) contrasting views of mathematics teaching and learning, and in doing so, creates herself a safe space.

The prospective teachers Lisa, Dina, and Kira all expressed their appreciation of mathematics as related to the way it was taught. They each remember the positive feelings that they had when participating in mathematics as students in school. However, Ian's argumentation is somewhat different. He promotes another way of teaching mathematics, together with rejecting the teaching that, for example, Lisa recalls as being positive. There seem to be conflicting stories about effective teaching and learning of mathematics. When the prospective teachers participate in these teacher education situations, they can experience a form of tension, realising that their view of effective mathematics teaching is not accepted as valid within the teacher education programme. Lisa, Kira, and Dina, who enjoyed mathematics at school, re-negotiated the content taught by Ian. One interpretation here is that the prospective teachers do this as a way of creating a "safe space" (Gellert, 2000) allowing them to continue enjoying mathematics in their own ways (Player-Koro, 2011).

The deficit model of prospective teachers (Askew, 2008) seems to guide the story of the lecture, but at the same time, Ian seems to be convincing the prospective teachers that more (and different) subject knowledge is needed as part of mathematics teacher education. The complex phenomenon arising is that while expressing mathematics as being difficult, Ian is simultaneously enacting the pastoral side of teacher education (Hannula, 2002). In other words, one potential challenge for the mathematics teacher educator is to create an environment where prospective teachers, who may be looked upon as problematic, are convinced that a problem indeed exists (within mathematics teaching and learning), without themselves becoming dejected. The majority of the prospective teachers embrace the ideas presented at the mathematics education course as a revelation, and Lisa, Dina and Kira also embrace the intentions *linguistically*. This means that when they participate in the teacher education setting, they align their language with the language of the teacher educator. However, beyond the formal setting, these prospective teachers express a different view of mathematics teaching and learning, to that being encouraged within the teacher education programme. The prospective teachers engage in different social practices, or in enactivist terms, they bring forth different "worlds of significance" (Reid & Mgombelo, 2015, p.181), triggered by the different environments (the formal and the informal) in which they participate.

From an enactivist perspective, the way we respond to situations is determined by our history of experiences. The situation itself does not (and cannot) determine how an individual might respond, rather, different environments trigger different responses. From this position, a vital aspect of mathematics education programmes is acknowledging and working with the experiences (however varied these experiences may be) of the group of prospective teachers. By foregrounding the experiences of the prospective teachers, we need not abandon the quest to improve mathematics teaching and learning, but the emphasis shifts to supporting the prospective teachers in realising, for themselves, what might be possible in the mathematics classroom, that they have not yet themselves encountered. We are left with a set of new questions, both as mathematics teacher educators and researchers: How do prospective teachers in teacher education situations prioritise the content taught during mathematics teacher education programmes? Do prospective teachers prioritise the content taught during the teacher education programme, or do they prioritise something other than what seems to be expected from the mathematics teacher educator's point of view? What is essential or relevant from the perspective of prospective teachers' when attending the teacher education programme in general and in mathematics education in particular? How can we foreground the diverse experiences of the prospective teachers whilst supporting them to explore potentially new ways of teaching and learning mathematics? These questions we take with us into the future.

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Opportunities to learn professional noticing while co-planning, rehearsing, co-enacting and reflecting on mathematics instruction

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The study explores teachers' opportunities to learn professional noticing while co-planning, rehearsing, co-enacting and reflecting on mathematics instruction in learning cycles of enactment and investigation. Fourteen primary school in-service teachers collaborated with teacher educators in the cycles and the study focuses on exploring what and how these teachers noticed. A framework of noticing was applied in the analyses with the aim of shedding light on the ways in which the participation in learning cycles enables teachers to collectively learn professional noticing. Findings reveal that teachers were provided with opportunities to learn high-level noticing. For instance, they attended to particular students' mathematical thinking and made connections between teaching strategies and students' mathematical thinking.

Keywords: Professional development, learning cycles, professional noticing.

Introduction

Equipping teachers with practices that support students from diverse backgrounds is a critical role of professional development (*PD*). Such teaching practices "aim to deepen students' understanding of mathematical ideas" and support the learning of all students "across ethnic, racial, class and gender categories" (McDonald et al., 2013, p. 385). Teachers' professional *noticing* – a process through which teachers make sense of what occurs during instruction and make plans to respond to students' mathematical thinking (*SMT*) – has become widely accepted as a key teaching practice. What and how teachers notice, matters for student learning (van Es & Sherin, 2021). Novice teachers are often able to talk about SMT (i.e., students' strategies, representations and reasonings), but they find the enactment of practices based on what they noticed challenging (e.g., Thompson et al., 2013). Also experienced teachers are unprepared to notice SMT (Empson & Jakobs, 2008). Learning to enact practices that support all students takes time and because "teachers can be responsive only to what has been noticed" (Jacobs & Spangler, 2017, p. 192) learning such practices is important for PD (e.g., Kavanagh et al., 2019). Such learning is explored in this study.

Noticing is an awareness that enables action and skilled teachers are quicker to identify situations that require intervention (van Es, 2011). Because it can lead to changed teaching practices, noticing is "a key component of teaching expertise and of mathematics teaching expertise in particular" (Sherin et al., 2011, p. 79). Teacher noticing is conceptualised in a variety of ways (van Es & Sherin, 2021). The term is here considered to include a) attending to SMT throughout *learning cycles* of enactment and investigation (Figure 1), b) reasoning about SMT, and c) making informed teaching decisions according to an analysis of observations of SMT.

Developing the ability to notice can be learned through collaboration and scaffolded support (e.g., Star et al., 2011). In the *Mastering Ambitious Mathematics teaching* project (*MAM*), in-service *teachers* collaborated in learning cycles so they could develop their ability to notice SMT and enact on what they noticed. The analysis meets the call from previous research (Stockero, 2021) when drawing on sociocultural perspectives while aiming at shedding light on the ways in which learning cycles might enable teachers to collectively learn to notice SMT. Within MAM, an exploration *what* teachers prepare to notice in co-planning discussions suggests that the teachers focused both on particular SMT and on pedagogy (Fauskanger & Bjuland, 2021b). In addition, Fauskanger and Bjuland (2021a) suggest that developing the ability to notice – both *what* to notice and *how* to notice (van Es, 2011) – can be learned through scaffolded support and collaboration in co-enactments in MAM. While these previous studies within MAM offer the field a glimpse into co-planning and co-enactment in the context of PD, they point out that in order to make clearer conclusions, we need to develop our understanding of how all the different elements in whole learning cycles provide teachers with opportunities to learn professional noticing. Bearing this in mind, the present study examines one representative learning cycle. It draws to a large degree on the analysis of teachers' learning from participation in video-based programs (for a review, see Santagata et al., 2021). However, our work augments the literature by situating practicing teachers in the authentic work of teaching (asked for by Stockero, 2021). By also exploring co-planning, this study meets limitations from previous research, namely the lack of focus on preparation to notice (Choy et al., 2017). Exploring *what* as well as *how* teachers notice in whole learning cycles is one way to meet this call. The following research question is addressed: How can teachers' engagement in learning cycles provide them with opportunities for learning *what* and *how* to notice?

Methods

Sociocultural views on teacher learning inform the presented study which draws on a description of learning, thinking and knowing as "relations among people engaged in activity in, with, and arising from the socially and culturally structured world" (Lave, 1991, p. 67). We see learning cycles as contexts for having reasoned dialogues (i.e, dialogues where everyone engages critically but constructively with each other's ideas and where everyone's ideas are treated as worthy of consideration) providing "affordances for changing participation and practice" and thus opportunities for the participants to learn (Greeno & Gresalfi, 2008, p. 172). *Opportunities to learn* is understood as emerging in activities, and from this perspective, teacher learning includes developing the ability to engage in particular practices. Learning cycles (Figure 1) were designed to engage teachers in learning professional noticing.

Figure 1: Cycle of enactment and investigation for PD (Wæge & Fauskanger, 2021)

In designing the cycles, we gave the teachers repeated opportunities to co-plan, rehearse, co-enact and reflect upon a set of intentionally selected instructional activities (e.g., choral counting, quick images, number strings) with teacher educators as supervisors. The activities supported the teachers in noticing SMT and in making judgments on how to respond in principled, instructive ways (Kavanagh et al., 2019). Throughout the cycles, the teachers were encouraged to 1) ask questions, explain and justify their mathematical and instructional ideas, 2) find multiple strategies and 3) try to understand what other participants said and did. Thus, a setting was developed where teachers could be engaged in the joint enterprise of learning to notice in which questions and disagreements were viewed as a productive part of the enterprise. Fourteen Norwegian primary-school teachers (divided into two groups) met for nine full learning cycles over the course of two years, producing 18 videotaped cycles. In this paper, the analysed data material has been taken from video recordings of one representative cycle (shaded in Table 1) where the instructional activity worked on was a quick image (Figure 2, more about quick images at https://tedd.org/quick-images-2/).

Table 1: Video material analysed – an overview (group 2, session 4)

MAM cycle/ group	Collective analysis (all groups)	Co-plan-ning (gr2)	Re-hearsal (gr2)	Co-enact-ment (gr2)	Collective analysis (gr2)	Discussion (all groups)	Discussion (supervisors/researchers)
4/2	24:51	59:21	21:29	26:46	19:56	29:42	58:18

A framework developed by van Es (2011) was used to analyse the depth and analytic stance of noticing, focusing on the teachers' discussions. This framework includes four levels. The first two levels (*baseline* and *mixed*) are considered as low-level noticing since the noticing is related to the class as a whole and teacher pedagogy (*what* teachers notice), and the teachers provide descriptive and evaluative comments from particular events throughout the learning cycle (*how* teachers notice). van Es (2011) denotes the two next levels (*focused* and *extended*) as high-level noticing since the teachers are then attending to particular SMT, or they are concerned with the connections between teaching strategies and particular SMT (*what* they notice). The extended level of noticing (*how* teachers notice) is described by van Es (2011, p. 139) as highlighting "noteworthy events", providing "interpretive comments", referring "to specific events and interactions as evidence", elaborating on "events and interactions", making "connections between events and principles of teaching and learning", "using interpretations", and proposing "alternative pedagogical solutions." The co-planning and the collective reflection session were divided into small sequences of utterances which were coded baseline, mixed, focused and extended respectively. Each Teacher Time Out (TTO) in the rehearsal as well as in the co-enactment were coded including these four levels of noticing.

Our analytical stance has primarily been to focus on high-level noticing, identifying whether the teachers are attending SMT and highlighting noteworthy events. When presenting findings from the exploration of how the teachers' engagement in this particular learning cycle provides them with opportunities for learning *what* and *how* to notice (whether they interpret, explain and give reasons in their discussions), we will follow the learning cycle (Figure 1) step by step, starting with the co-planning session.

Findings and discussion

The focus of attention in the co-planning discussion is summarised in Table 2.

Table 2: The co-planning session

Time	Focus of the co-planning discussion
00:00-22:00	The teachers present student strategies based on student responses from trying out the quick image in their own classrooms. The supervisor writes the strategies on the smartboard. The strategies are then illustrated on a quick image before the participants discuss how to write the students' strategies by using symbols. Based on this discussion, the supervisor initiates a discussion of the commutative property (i.e., 8 × 3 = 3 × 8).
22:00-51:40	The supervisor initiates a discussion about what goal for students learning they should aim at. The associative property of multiplication, as well as the idea of generalisation (extending the quick image) is discussed. About 45 minutes into the discussion, the supervisor suggests a goal from their previous discussion: "Yes, what mathematical idea should we focus on? Is it the associative property or is it more to develop a general expression for the total amount of chocolates in the boxes? [the dots in the quick image *represent* pieces of chocolate]." They agree to aim at the associative property.
51:00-59:21	Discussion of practical teaching strategies and how to structure and teach the activity for the students.

Whereas practicalities were the focus of attention in the last part of the co-planning discussion, different levels of noticing were visible in the first two parts. When predicting student strategies and representing them in the quick image as well as by using symbols, the teachers attend both to the relationship between particular SMT (*what*, *focused*) and between teaching strategies and SMT (*how*, *extended*). As an example, the suggestion "I see four times three, twice" from one of the teachers is followed by a discussion of how this strategy might be represented in the quick image as well as by using mathematical symbols. The teachers make connections between SMT and teaching strategies (*how*, *extended*). In the mid part of the co-planning session, a discussion of the connection between student strategies and teaching is visible when the aim for the lesson is in focus. It ends by a decision that they will challenge the students to develop their strategies into three factors (e.g., 8 × 3 = 4 × 2 × 3) in order to aim for the goal for the lesson: the associative property of multiplication (e.g., (4 × 2) × 3 = 4 × (2 × 3)). Throughout the co-planning session the participants return to the commutative property of multiplication and some of the participants seem to mix commutativity and associativity.

However – and similar to findings from Fauskanger and Bjuland (2021b) – the participants' preparation to notice (Choy et al., 2017) seems to provide the teachers with affordances for changing practices of *how* and *what* to notice on extended levels (van Es, 2011) and thus opportunities for the participants to learn (Greeno & Gresalfi, 2008) professional noticing.

In the rehearsal as well as in co-enactment in MAM, the participants can pause the instruction by initiating a TTO (Figure 1) so they can think out loud together in the moment, discuss how the teacher might respond to students' contributions and determine the direction of the further instruction. In the rehearsals across all learning cycles in MAM, 175 TTOs were asked for (Wæge & Fauskanger, 2021), 18 in the particular rehearsal analysed here. In four of the TTOs from this particular rehearsal, the participants focused on representing anticipated students' strategies in a quick image, discussing how to write the mathematical ideas in a symbolic language (2 × 12 and 4 × 6). We will show one example from the participants' discussion in one of these TTOs to illustrate their opportunity to reason and to gain insights into SMT. This is an important component of high-level noticing (van Es, 2011). One of the teachers came up with the following strategy: "I saw another pattern: eight in each row," and went to the board and showed three groups of eight dots (Figure 2b). Based on this initiative, the teacher, who was chosen as the instructor in the enactment, wrote 3 × 8 on the board and said: "In which way is it possible to see the eight here [points to the first row]." Another teacher went to the board and showed two groups of four dots in the quick image. Based on this initiative, the instructor wrote 3 × (2 × 4) (Figure 2c). This example shows how predicted student strategies are discussed and represented in the quick image by the participants, gaining insights into SMT (*focused*). The instructor expressed the following utterance: "We have three factors in each and then they [the students] can talk together about what they see." This utterance indicates that the participants are attending to the relationship between SMT and between teaching strategies and SMT (*extended*). Teacher noticing is conceptualised in a variety of ways (van Es & Sherin, 2021). This example in one of these TTOs illustrates signs of the two interrelated and cyclical processes of attending and making sense of particular events (SMT and teacher strategies), often involved in teacher noticing (Sherin et al., 2011). At the end of this rehearsal, the goal set for the enactment, focusing on the associative property of multiplication was briefly mentioned.

In the co-enactments across all learning cycles in MAM, 189 TTOs were asked for (Fauskanger, 2019) and out of these 125 were identified as opportunities to learn professional noticing on various levels (Fauskanger & Bjuland, 2021b). The co-enactment in the learning cycle analysed here included three TTOs and two out of these were instances of high-level noticing (van Es, 2011). In the following, we delve into one of these TTOs to illustrate the teachers' opportunity to reason about SMT and to make informed teaching decisions based on these observations made in the moment of instruction. Just before the TTO, a student presents her strategy as seeing four six times in the quick image. The teacher circles four dots six times in the image (Figure 2d). When invited to the board, this student writes 4 × 6 = 24.

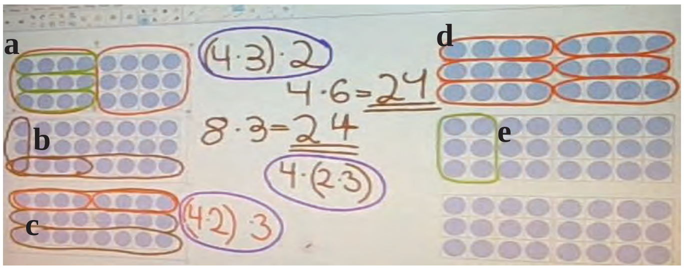

Figure 2: All student strategies from the co-enactment (The multiplication sign used is "·". This is the most common sign to use in Norway. In this paper we decided to use "×" for easier reading.)

The teacher asks the students how to split the 6 in 4 × 6 into "another multiplication task." A student answers: "two times three" and the instructor writes 4 × (2 × 3) on the board and says that she realised that 4 × (2 × 3) could not be represented as the six fours in the image (Figure 2d). From what follows, the teacher obviously is thinking of 2 × 3 as six dots and is confused. She draws six in Figure 2e. The supervisor asks for TTO saying: "I think I see the six in the upper image (Figure 2d)" and asks the students if they "see six" in Figure 2d. They point to the six fours. This TTO invites the participants to attend to the relationship between particular SMT (i.e., that six in this quick image could be represented as six groups of four dots (Figure 2c)), SMT (i.e., six in this quick image represented as six dots (Figure 2e)) and teaching strategies such as representing SMT (*what*) and to work on alternative pedagogical solutions (*how*) as alternative ways of representing students' strategies. Whereas this TTO provides the teachers with opportunities for learning *extended* noticing, in the second TTO the teachers attended to particular students' mathematical thinking (*focused*). In the third TTO, time issues were discussed. Similar to findings from Fauskanger and Bjuland (2021a), our findings indicate that by being provided with affordances for changing practices, the participants have opportunities to learn (Greeno & Gresalfi, 2008) extended levels of noticing (van Es, 2011) in-the-moment of teaching in the co-enactment analysed.

In the collective reflection discussion following the co-enactment, all participants looked at the smartboard from the co-enactment and focused on the different student strategies represented in the quick image (Figure 2). The focus of attention is summarised in Table 3.

Table 2: The co-planning session

Time	Focus of the collective reflection after co-enactment
00:00-05:30	The participants focus particularly on two ideas introduced by two individual students from enactment. One of the students expressed that he/she "saw a box with four across and three upwards, then there were two boxes and I took [multiplied] eight times three", writing 8 × 3 = 24 on the board (see Figure 2a). Another idea was to see four six times in the quick image, writing 4 × 6 = 24 on the board (Figure 2d).
05:30-13:20	They put efforts into practical issues related to the instructional activity of the quick image. One of the participants thinks that only spending 20 minutes on this activity is too demanding and suggests that 45

	minutes would be a more realistic time frame. They are also discussing the use of talk moves and mathematical language related to mathematical terms and concepts.
13:20-19:57	The participants focus particularly on the commutative and associative property of multiplication. The teacher (instructor) posed quite a few questions related to these properties. This is exemplified by the following questions, presented in Fauskanger and Bjuland (2019, p. 126): "Why is it called commutative [property for multiplication] when there are two [factors] and associative [property for multiplication] when there are three [factors]? What's the difference? Why couldn't we just use [the word] commutative, why is another word used there? It's just the same, isn't it? It's all about the order of the factors, or are they [commutative and associative properties of multiplication] two different things?"

As seen in Table 3, the second part of the discussion after the enactment attends to the whole class environment with general descriptions and teacher pedagogy related to time frames, the use of mathematical language and the use of talk moves. This indicates low levels of noticing. Whereas a discussion related to the commutative and the associative property of multiplication was the focus of attention in the last part of this session (no levels of noticing), higher levels of noticing were visible in the first part. From one of the student strategies, seeing eight times three (*what*, *focused*), the participants agreed that it seemed to be difficult for the students to split the eight ($8 \times 3 = 24$), illustrating the mathematical representation in the quick image, $(4 \times 2) \times 3$ (*how*, *focused*). The supervisor elaborated on this particular event (SMT) and challenged the participants to consider how to use this idea with the aim for three factors which are related to their mathematical learning goal for the lesson (*how*, *extended*). In a similar way, they discussed the other student's idea ($4 \times 6 = 24$ on the board), how to "split the six", reiterating this situation from the enactment. The supervisor summarises this discussion by concluding that "the number six could be represented in the quick image in different ways, depending on whether six is seen as the number of groups or as dots within a group" (Fauskanger & Bjuland, 2019, p. 139).

Conclusion and implications

Meeting the call from Stockero (2021) – to explore noticing in-the-moment during the act of teaching – and from Chong et al. (2017) – to explore teachers' noticing while planning instruction – this study explores how teachers' engagement in learning cycles (Figure 1) provides them with opportunities for learning professional noticing. Our analysis indicates that all parts of the learning cycles are contexts where teachers can learn to notice students' ideas and respond to them. When working together in learning cycles, the participants practise how to build on SMT (Empson & Jacobs, 2008), as has been endorsed in many reform documents. In conclusion, developing the ability to notice – both *what* to notice and *how* to notice (van Es, 2011) – can be supported through collaboration (e.g., Star et al., 2011) as in the learning cycles in MAM.

While this study provides the field with a glimpse into one learning cycle in the context of PD, more research is needed. Compared to studies of teacher noticing in video clubs (Santagata et al., 2021), the learning cycles (McDonald et al., 2013) appear to provide teachers with opportunities to learn higher levels of noticing. However, in order to be able to make clearer conclusions we need to provide systematic explorations of more learning cycles, inside as well as outside of the MAM project.

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Developing pre-service teachers' communications skills using Socratic lectures: The audience's perspective

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This study investigates the potential of engaging secondary pre-service teachers in a team-teaching approach called Socratic lectures to develop their communicational skills. An intervention was designed centered around a workshop sandwiched between a pre- and a post-presentation on natural logarithmic and derivatives to an audience of undergraduate students. A quantitative and qualitative analysis of 155 pre- and 92 documented post-experiences of the undergraduate students' shows that the pre-service teachers' presentations and communicating skills developed positively over the course of the intervention. The accompanying statistical analysis of Likert evaluation items shows a significant increase of multiple communicational aspects of the pre-service teachers, and the line-by-line analysis of written evaluations support this claim regarding the pre-service teachers' interactivity. Issues about generalizability and future research is also discussed.

Keywords: Interaction, pre-service teachers, socratic lectures, team teaching, questions.

Introduction

Asking the right questions to support, motivate and engage students in learning is a central part of the teaching profession (Paoletti et al. 2018). Mathematics teaching in Sweden has a long history of *traditional teaching*, a method based on directing students' learning by presenting mathematical content at the blackboard followed by individual work in the textbook. Research reports (Jablonka & Johanssson, 2010) show that the traditional teaching influence students to lose interest in mathematics. In this research project, we investigate the potential of so-called *Socratic lectures* to develop pre-service secondary teachers' strategies to ask different types of questions with different purposes to motivate and engage students to learn mathematics. We report on preliminary results from an intervention, organized around a workshop introducing the Socratic lecture strategy, where 48 pre-service secondary teachers (PSSTs) prepared and presented a focused mathematical content to undergraduate students before (*Pre*) and after (*Post*) a workshop. In this paper, we focus on a set of empirical data gathered on the audience's experiences of their level of engagement in the intervention (the undergraduate students') during the PSSTs' the pre- and post-presentations.

Socratic lectures is a teaching strategy we have developed from research literature on *Self-Determination Theory* (SDT, see e.g. Ryan & Decis, 2000), *rhetoric* and *team-teaching/co-teaching* (see e.g. De Backer et al., 2021; Kamens, 2007). The teaching strategy, which is based on the Socratic method (Thompson, 1998), is constructed in the form of a play, where two lecturers use the Socratic method by asking questions to each other (and to the audience) in the form of a dialogue and thereby engaging the audience to make them interested. The teaching strategy is to some extent applied in computer courses at the Department of Computer and Information Science at Linköping University and has been praised by students as an engaging and productive way of teaching. Students' course evaluations have emphasized communicational aspects, such as developing a friendly and pedagogic atmosphere for discussions, as fruitful for their learning. Motivated by these positive evaluations and that team teaching has shown many valuable effects in research literature, such as *increased dialog about learning and teaching* (De Backer et al., 2021), we aim to explore if applying Socratic lectures by PSSTs also develop their communication skills in the teaching of mathematics. Communication is here broadly understood as "a process by which information is exchanged between individuals through a common system of symbols, signs, or behavior" (Merriam-Webster, n.d.), which include written, verbal and non-verbal interactions. The research question posed to address the aim is:

How, if at all, did the PSSTs' communication skills in the mathematics of teaching, viewed from undergraduate students' expressed experiences, develop after participating in the workshop on Socratic lectures?

Socratic lectures

Self-Determination Theory (SDT, see e.g. Ryan & Decis, 2000) emphasizes *competence*, *autonomy*, and *relatedness* as central psychological features for increasing motivation. Thus, participation in social contexts (relatedness) affects individuals' motivation to learn if the contexts entail opportunities to influence (autonomy) and to draw own conclusions (competence). Adopting a feeling of being interested in something and being involved in one's own learning are keys to success in school, which also applies to the subject of mathematics (Murayama et al., 2013). In trade and sales, it is important to ask the right questions to make customers feel that they 'really' need a certain product. Salespeople often use rhetorical tricks and techniques based on asking questions, which lead the customer to think in a certain direction, so that they desire to buy the product. One such technique used in courtrooms and in political discussions is the Socratic method (Thompson, 1998), which is based on asking leading questions to steer other people's thinking in a certain direction. The central parts of the method are the questions themselves and questioning, because they can facilitate students' knowledge production (Lew et al., 2017). There are several categorizations of the types of questions used in teaching, such as *fact*, *next step*, *proof framework*, *warrant*, *evaluation*, and *convention* (Paoletti et al. 2018). Frequently the different types of questions are connected to factual-, procedural-, conceptual-, or meta cognitive knowledge (Smith & Julie, 2014), which also illustrates the importance of using different types of questions to enhance the development of all types of knowledge.

Socratic lectures as a teaching strategy is related the paradigm of *team teaching* sometimes, also sometimes called *co-teaching*, which refers to "two or more professionals jointly deliver substantive instruction to a diverse, or blended, group of students in a single physical space" (Cook & Friend, 1995, p. 1). In teaching practices, however, the level of collaboration between teachers and how they jointly deliver instructions may vary. Based on a research literature review, Baeten and Simons (2014) identified five models for team teaching: *the observation model*, *the coaching model*, *the assistant teaching model*, *the equal status model* and *the teaming model*. The models capture the roles of the actors in terms of taking passive or active responsibility for the teaching. The models range from passive to active participation in the classroom. One of the teachers can act as a passive observer, to coaching students to assisting the other teacher or even taking more active responsibilities by sharing the working load and splitting the number of students and/or the content to teach as in the equal status model. The teaming model is the ideal model where the teachers collaborate in all phases of planning, delivery, and evaluation, with the characteristics of two teachers “in front of the entire class group and there is a lot of interaction and dialogue between them” (Baeten & Simons, 2014, p. 94). Socratic lectures has these features and should be categorized in accordance to this model. A recent literature review (De Backer et al., 2021) concluded that team teaching has the advantages of increased support, increased dialog about learning and teaching, professional growth, and personal growth, but also potentially the disadvantages of lack of compatibility of peers, comparison between peers, difficulty of providing constructive feedback and increased workload. In this paper, we only investigate on the potential of increased dialog about learning and teaching, which exactly emphasize the frequent opportunities of team teaching for PSSTs to discuss and evaluate teaching approaches between each other. While there is a wide corpus of literature about team teaching in educational research generally, to date this strand of research in mathematics education it is more limited. The existing literature concerns how experienced teachers can teach together with teacher students (Yopp-Edwards et al., 2014), and the use of "team teaching" for developing special education courses on mathematics teaching and learning (Guðjónsdóttir & Kristinsdóttir, 2011), but to our knowledge no research focusing on how PSSTs develop their ability to teach in dialogue. This study will contribute with new knowledge in this area of research as well as address the call by Kamens (2007, p. 156) “to include realistic classroom-based experiences in co-teaching” in teacher education.

The intervention and data collection

The 48 participating PSSTs were enrolled in a first course on mathematics education in a secondary teacher training programme. About two thirds (30 out of 48) of the PSSTs had completed a 2,5-week introductory course in educational science and started taking a mathematics course in algebra prior to participating in the study, the other third had taken a course in geometry as well as started the algebra course, in addition to courses at the university for a year on other subjects, such as English or Biology. The intervention was designed in three parts a *pre-presentation*, a *workshop* and a *post-presentation*. After the pre-presentation the PSSTs had one week of teaching practical practice a at regular school before attending the workshop, thereafter, a few days until the post-presentation.

For the pre-presentation the PSSTs were grouped in 24 pairs and instructed to prepare and present a 15-minute presentation on a given mathematical content for a set of undergraduate students enrolled in a preparatory course on secondary mathematics for achieving qualification for entering the university. The PSSTs were asked to present either the basic rules for differentiating (i) polynomials and power functions; (ii) exponential functions of type $y=a^{kx}$; or (iii) natural exponential function and how work with the logarithm rules. The aim for the pre-presentation was to deepen and consolidate the undergraduate students’ knowledge in these areas, since the undergraduate students previously had received two lectures within their course on these topics. The PSSTs were not instructed how to organize and arrange their presentation more than to observe basic principle such as engage and motivate the students, write clear and concise, use proper mathematical notations, speak so the audience can hear, don’t talk to fast, etc. In other words, the instructions provided the PSSTs with freedom to independently select presentation strategy. The undergraduate students were organized in ten classrooms and experienced two or three pairs of Socratic lectures by the PSSTs.

The 4-hour workshop on Socratic lectures was mandatory for all PSSTs and was taught and organized by four teachers from the Department of Computer and Information Science. The first and third authors were observing and videotaping the activities during the workshop. The workshop included both theory on the teaching strategy and practical training. In short, the workshop was organized in the following blocks: *an introduction (30 min)* describing the background of Socratic lectures and the basic principles of using a dialog conversation mode in pair teaching; *a first practical training session (45 min)* where students were instructed to re-work their pre-presentation using the Socratic teaching strategy, including a presentation of one of the PSSTs pairs whom also received feedback from the four teachers; *more theory (15 min)* a presentation on how to engage the audience using questions; *a second practical training session (45 min)* were organized as the first practical session, but with the exception that the PSSTs were instructed re-work their presentation again to also engage the audience.; *a model example (20 min)* on recursion presented by two of the programming teachers to show how the method is used in one of their courses; and finally *a concluding discussions (35min)* where the teachers summarized the day and opened up the floor for more discussions and comments. The four teachers teaching the workshop applied the strategy of Socratic lectures during their theoretical presentations and assisted and coached the students during the practical training blocks.

In preparing their 15-minute post-presentation the pairs of PSSTs were instructed to use their notes from the workshop and to implement the Socratic lectures strategy to the same undergraduate students as in the pre-presentation. The PSSTs tasks in their post-presentation were to demonstrate and discuss solutions of assessment tasks on differentiation and logarithms, such as: *Find the local maximum and minimum values to* $f(x)= x^2- 2x^4+ (5/8)$ and *Find the maximum area of a perpendicular triangle inscribed with its base on the x axis and its upper corner on the parabola* $y = 36 - 3x^2$.

The data collected document: the video recordings of all PSSTs pre- and post-presentations; the PSSTs' written reflection of their experiences and expressed views after each presentation; and the undergraduate students' evaluations of the pairs of PSSTs' presentations. This paper focuses on the undergraduate students' evaluation forms, which were distributed and filled out individually directly after each presentation. The pre- and post-presentation evaluation forms were identical and included seven statements to be ranked on a 5-point Likert scale, and three open questions. The Likert items concerned: *if the content was relevant*; *if the presentation was instructive*; *if the presentation was at the right level*; *if the presentation was inspiring*; *if the PSSTs engaged in good dialogs with the students*; *if the PSSTs asked enough questions*; and *if they like mathematics*. The three open questions were *What aspects have you experienced as positive in the presentation? What can be further developed in the presentations?* and *Additional comments*. There was also a question about the undergraduate students' last grades in mathematics from secondary education. The question *if they like mathematics* and last grades do not focus on communication and are not analyzed in this paper.

Analysis and results

The undergraduate students' responses, 155 on the pre- and 92 on the post-evaluation forms were, copied into a spreadsheet (SPSS). Answers on all Likert items were analyzed and compared using descriptive statistics and paired sample tests (dependent t-tests). The answers to the two first open questions in the evaluation forms filled out by the undergraduate students were short in length (from just a few words to a few sentences). These answers were explored through a qualitative iterative process of open coding using a line-by-line analysis (cf. Chenail, 2012), with a focus on what communicational aspects in the presentations were emphasized and what aspects were expressed to be potentially improved. The open codes were grouped into six themes, which are discussed below.

First however, we report on the statistical analysis of the undergraduate students' responses on the evaluation forms regarding their expressed experiences about the presentations in terms of the presentations being *relevant*; at the right *level*; *instructive*; *inspiring*; engaging the in *dialogs;* and regarding the use of *adequately many questions* (see Table. 1 below).

Table 1: Undergraduates' expressed experiences of PSSTs' presentations

		Relevant	Level	Instructive	Inspiring	Dialogs	Question.
Pre	***Mean***	4,47	3,77	3,60	3,52	3,18	3,51
	N	155	155	154	155	155	153
	Std. Dev.	0,70	0,95	1,09	1,03	1,10	1,12
Post	***Mean***	4,60	4,22	3,88	3,89	4,08	4,01
	N	92	92	92	92	92	91
	Std. Dev.	0,54	0,75	1,00	0,89	1,06	0,94

With respect to the undergraduates' experience of the PSSTs' engagements in *dialogs*, Table 1 indicates that the workshop made a significant positive impact on the PSSTs' interactions with the undergraduate students (M_{pre} = 3,18, SD_{pre} = 1,10; M_{post} = 4,08, SD_{post} = 1,06; t(91)= -5,75, p = .000). There was also a significant increase found with respect to the undergraduates' experiences of the PSSTs use of *questions* (M_{pre} = 3,51, SD_{pre} = 1,12; M_{post} = 4,01, SD_{post} = 0,94; t(91)= -3,46, p = .001), as well as the workshop having a significant positive impact on the PSSTs communication skill in terms of adopting an accurate *level* of presentation (t(91)= -3,68, p = .000); being *relevant* (t(91)= -2,23, p = .028); being *instructive* (t(90)= -2,49, p = .015); and being *inspiring* (t(91)= -2,05, p = .043).

The line-by-line analysis of the undergraduate students' answers on the pre- and post-evaluation forms to the open questions *What aspects have you experienced positive in the presentation?* and *What can be further developed in the presentations?* generated six themes: *(i) personal attributes* such as if the PSSTs were experienced as friendly, confident, nervous, etc.; *(ii) interactivity* focuses on aspects of discussions, checking, opening th floor for questions, developing a positive atmosphere, etc.; *(iii) planning* captures the experienced prepared communication in terms of suitable level of instructions, how to end, choice of tasks, etc.; *(iv) delivering* focuses on the instrumental aspects of communicating such as black board skills, being loud and clear, pedagogic, etc.; *(v) mathematics* relate to aspects of how the content of mathematics was presented such as step-by-step explanation, mathematical rules, use of mathematical language, arguments for why, etc.; and *(vi) collaboration* concerns the communication dynamics between the presenting PSSTs. To compare the undergraduates' answers between the pre- and post- evaluation forms we report, in table 2 below, on the relative frequency of codes in the themes, as well as the frequency of those undergraduates that explicitly answered the question about what aspects of the presentation that could further be developed with lines such as "everything was good" or "hard to find aspects to improve".

Table 2: Undergraduates' answers to the two open questions in the evaluation form

Themes:	***Attri.***	***Interac.***	***Plan.***	***Deliver.***	***Math***	***Collab.***	***Good***	**N**
Pre +	11%	13%	25%	38%	10%	3%		313
Post +	7%	25%	25%	39%	4%	0%		177
Pre -	10%	15%	14%	38%	16%	2%	6%	150
Post -	5%	8%	16%	36%	14%	2%	20%	64

The thematic categorization of the undergraduate students' answers as displayed in table 2 show that the proportion of codes in the themes of *planning* and *delivering* the presentations was stable between the two occasions both regarding what was positive (+) and what can be further developed (-). The same can be concluded with respect to the question of what could further be developed when presenting the *mathematical content*. Commonly among the third of the undergraduates that provided an answer within this theme in the pre-evaluation suggested the PSSTs to be more explicit: "Please ask more questions and explain **WHY** you do as you do and why I should learn this" or "Explain why it is like that, not just say it is". In the post-evaluation forms these types of suggestions were sparse. The first quote above indicates a suggestion to the PSSTs to try to interact more with the audience by asking question. This was a frequently found argument in the pre-evaluation answers, but a type of argument that decreased in number in the post-evaluation. The undergraduates' suggestions on the post workshop presentations instead commented on the use of too many questions: "The organization of the presentation was messy and they asked too many questions just like being students". Regarding the relative frequency of codes in the *interactivity* theme there is an increase of positive experiences after the workshop and a decrease in suggestions for improvement. Although the workshop focused on the collaboration between the presenters, the undergraduates only provide a few comments within this theme (*collaboration*) and the relative frequency was low and quite stable between the pre- and post-presentations. The results from the Likert items showed that the undergraduates were more satisfied with the post-presentation, which also is supported by the increase of relative frequency within the category addressing that presentation was *good* (an increase from 6% to 20%) and that hardly any further suggestions for improvement were provided in the post-evaluation.

Conclusion and discussion

The statistical analysis of the answers on the Likert questions showed a significant increase between the pre- and post-presentation of many communicational aspects of the PSSTs. We therefore conclude that the PSSTs' presentations and their communicating skills developed during the intervention as experienced by the undergraduate students. This conclusion is also supported by the line-by-line analysis regarding the increase of the relative frequency of *interactivity* and the quality of the presentations as experienced as *good*. The results may be summarized by the following quote of an undergraduate students on the post-evaluation arguing: "*They asked a lot of questions to all of us in the group, good pace, an incredible improvement since last time*".

The results presented in this paper are in accordance with other research studies (e.g. Baeten & Simons, 2014; De Backer et al., 2021) showing that team teaching provides a platform where interaction and dialogue between the presenters and their audience is a central component of the activity. From the line-by-line analysis of the pre-presentation experiences, the undergraduates voiced that the PSSTs showed the procedures to apply when solving tasks, but they were missing explanations of "why" the mathematical procedures were applied. A possible reason for this is the PSSTs' lack of experience of doing presentation. Indeed, for two thirds of the PSSTs this might have been their first chance to act as a teacher and presenting mathematical content. In addition, since all PSSTs just had begun to study mathematics at the university level, they might not have felt confident enough regarding the mathematical topics to get into conceptual issues. The type and number of questions asked by the PSSTs could better be explored using video analysis of the actual situations, which we will follow up with in an upcoming study. The results show, however, that undergraduate students experienced that the interactivity increased after the workshop possibly indicating that the number of questions increased.

What effect the workshop had on the results, require further analysis. On the one hand, all PSSTs attended the workshop and practically tested the new teaching strategy of Socratic lectures focusing on their communicational skills. But, on the other hand, the students also had a week of teaching practice in regular school where they also had opportunities to work on their communicational skills when presenting to a group of students. In addition, the decrease between the undergraduate students participating in the pre- and post-presentations could be another reason for the increase in positive experiences. The students that did not enjoy the first presentation may not have attended the second presentation, which may impact on the results. In addition, the mathematical content presented in the two presentations (logarithmic rules and differentiation), might have invited the PSSTs to focus on standardized solution strategies and procedures for calculating and simplifying, and hence impacted on the type of communication they engaged in. Likely other mathematical content would have generated another result. We are careful not to generalize our results beyond the intervention, the mathematical content presented, the participating PSSTs and undergraduates. Therefore, our first results, that the intervention had a positive effect on the development of the participating PSSTs' communication skills, needs further justifications of new research studies on Socratic lectures.

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Pre-service teachers learn to analyse students’ problem-solving strategies with cartoon vignettes

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The flexible use of different problem-solving strategies is key for processing mathematical problems successfully. The question of how future teachers can be supported to learn about such strategies and their use in the mathematics classroom is consequently of high relevance. In a one-semester university course, we provided a unit on problem solving focusing on the use of primary-school students’ problem-solving strategies. To support the pre-service teachers’ learning, we designed two types of cartoon vignettes: short cartoons, each illustrating a different problem-solving strategy (e.g., work backwards) and more complex cartoons providing the opportunity to analyse how students make use of different strategies and/or struggle while solving a non-routine problem. Our evaluation showed that the participants perceived the cartoons as valuable learning opportunities. Learning to analyse students’ problem-solving strategies could be supported but remained a challenging task.

Keywords: Vignettes, cartoons, problem-solving strategies, teachers’ competence of analysing.

Introduction

Mathematical problem solving is considered a core mathematical activity and it has been incorporated as a key component in mathematics curricula and national standards worldwide (Liljedahl et al., 2016). Related studies have provided evidence that the flexible use of so-called heuristics or problem-solving strategies can enhance problem solving (e.g., Carlson & Bloom, 2005). Various mathematical problem-solving strategies, such as *make a table*, *draw a picture* or *look for a pattern*, can already be introduced for primary school children (Charles et al., 1992) and corresponding instruction achieved a positive impact on students’ use of strategies and problem-solving abilities (Verschaffel et al., 1999). However, there are also studies showing that many students do not spontaneously use problem-solving strategies or struggle when applying them to non-routine problems (e.g., Schoenfeld, 1992). It can therefore be assumed that mathematics teachers do not only need professional knowledge about students’ problem-solving strategies but must also be able to analyse their students’ strategy use in order to provide them with support. With this study, we build on prior research into mathematics teachers’ analysing of classroom situations, described as the competence to link classroom observations with relevant professional knowledge in order to make sense of what has been observed (e.g., Friesen & Kuntze, 2020). In the present study, we were particularly interested in the question of how future primary-school teachers can be supported in their professional learning to analyse students’ problem-solving strategies. The study was conducted in the framework of the ERASMUS+ project coReflect@maths, aiming at designing and evaluating vignette-based learning material in mathematics teacher education courses with a particular focus on cartoon vignettes (cf. Ivars et al., 2020). Correspondingly, we explored in this study the potential of different types of cartoon-based vignettes for (1) developing pre-service teachers’ professional knowledge about various problem-solving strategies and (2) their competence to analyse primary-school students’ use of such strategies in classroom situations. The following sections outline the theoretical background of the study, in particular with respect to mathematical problem-solving strategies and teachers' corresponding competence to analyse classroom situations. We then introduce our vignette-based approach and describe the unit about problem solving in which the pre-service teachers of this study participated.

Mathematical problems and the use of problem-solving strategies

We follow Schoenfeld's (1983) definition of a mathematical problem as an *unfamiliar situation for which an individual does not know how to carry out its solution* since no routine or familiar procedures are (yet) available to him or her. Such problems are often described as non-routine problems in contrast to routine problems, where students can apply familiar or routine solution methods to solve them (Elia et al., 2009). Solving non-routine mathematical problems, in contrast, requires the use of so-called heuristics or problem-solving strategies (Liljedahl et al., 2016). Various problem-solving strategies can already be introduced to primary-school children, such as guess–check–revise, draw a picture, act out the problem, use objects, solve a simpler problem, make a table, look for a pattern, make an organized list, write an equation or work backwards (cf. Charles et al., 1992). A study with primary-school children by Elia et al. (2009) showed that the flexible use of strategies (adapting the strategy to the given task or changing in case the chosen strategy was not successful) led to higher performance in problem solving. Knowing various problem-solving strategies and being able to change or modify a chosen strategy is therefore an important prerequisite for the successful solving of problems. However, the same study also showed that many of the primary-school children struggled with applying different strategies or could not reach a correct answer since they did not have a sufficient understanding of the problem (Elia et al., 2009). These findings highlight in particular the important role of the teacher in supporting the students in understanding the given problem as well as in choosing, applying and reflecting on the use of different problem-solving strategies.

Analysing classroom situations regarding the use of problem-solving strategies

In order to support students in using different problem-solving strategies in a flexible manner, a mathematics teacher must know about different strategies, how students can apply them to problem-solving tasks and where they might struggle. Taken together with the definition of problem solving which is characterised by the fact that the problem-solver cannot simply apply a routine or familiar solution method (e.g., Schoenfeld, 1983), a mathematics teacher's knowledge about *content and students* (KCS, cf. Ball et al., 2008) can be assumed to form an important prerequisite in this context. However, to provide learners with appropriate support, teachers must also be able to *apply their knowledge* in the course of instruction, meaning that they have to link relevant observations in the classroom with their knowledge to make sense of what they have observed in order to make follow-up decisions. Based on the concept of teacher noticing (Sherin et al., 2011), we described such ability as teachers' competence of *analysing classroom situations* (e.g., Friesen & Kuntze, 2020). Numerous studies have shown that teachers' analysing of classroom situations informs their decision-making and is highly relevant for instructional quality and student learning (Sherin et al., 2011). A teacher who is competent in analysing classroom situations regarding the use of problem-solving strategies is consequently able to identify the problem-solving strategies used by his or her students and to evaluate the strategies' potential for solving a particular task. Such competence is an essential prerequisite for supporting students in choosing and applying problem-solving strategies, for helping students where they struggle and for engaging them in reflection on the use of different strategies. In the following section, we describe the potential of using vignettes for developing and evaluating pre-service teachers' professional growth in this context alongside with the cartoon-based approach we take in the framework of the ERASMUS+ project coReflect@maths (Ivars et al., 2020).

Vignette-based learning and the potential of cartoons for teacher education

Using representations of classroom practice, so-called *vignettes*, in the form of short video clips, teacher-student dialogues or cartoons, has proven to be an effective approach in teacher education and corresponding research (e.g., Buchbinder & Kuntze, 2018). Vignettes provide the opportunity to engage in classroom practice without the pressure to act and can help future teachers to analyse classroom situations and to discuss and reflect on alternative approaches. In this way, vignettes can become both, representations of practice *and* theory: As learning opportunities, vignettes are often used to illustrate particular aspects of quality teaching or where norms are challenged and can therefore support future teachers in applying their theoretical knowledge to teaching practice in a safe context. Vignettes have, therefore, also proved to be an effective method to assess development in teacher education courses (Jeffries & Maeder, 2005). Correspondingly, questions regarding possible designs of vignette-based learning and testing environments have become essential and the potential of cartoon vignettes has gained increased attention. Cartoon vignettes combine numerous advantages ascribed to vignettes in the formats video and text: They allow the systematic, theory-based design and variation of classroom situations where individual characteristics that are important to comprehend a classroom situation can be added easily (Friesen, 2017). Compared to the formats of video and text, cartoon-based vignettes can be equally suitable to assess and foster teachers' competence in analysing classroom situations (Friesen, 2017). The design and evaluation of cartoon-based learning material is a main objective of the ERASMUS+ project coReflect@maths and the present study has been conducted in the framework of this project. The following section describes the organisation and contents of the university course in which the study was conducted as well as the design and use of the cartoon-based learning material.

The university course: organisation and cartoon-based learning material

The present study was conducted in a one-semester university course (April-July 2021) for pre-service teachers studying to teach mathematics at primary schools. The course took place at a University of Education in the south of Germany where primary school comprises the first four years of schooling for students aged between six and ten years. Due to the pandemic situation, the course had to be delivered completely online with meetings organised as video calls and study material provided via the university's online learning platform. The course covered different topics related to the teaching and learning of mathematics at the primary-school level. Problem solving was one unit within the course covering four weeks and three group meetings (3 x 90 minutes). The problem-solving unit contained key topics with special regard to the primary-school context (e.g., what are mathematical non-routine problems, problem solving according to Polya, problem-solving strategies, how can learners be supported). Figure 1 (on the left) shows the organisation and contents of the unit. In order to support the pre-service teachers' learning about students' problem-solving strategies, we designed two types of cartoon-based vignettes: (1) short cartoons with the aim to develop the participants' professional knowledge about different problem-solving strategies and (2) more complex cartoons with the aim to develop their competence to analyse primary-school students' use of such strategies in classroom situations. Figure 1 (on the right) shows one of the short cartoons illustrating the problem-solving strategy "work backwards". Based on Haering (2016) who describes various problem-solving strategies used by primary-school students, 16 of such short cartoons were designed, each depicting a problem-solving task and how one or several students solve that task using a particular strategy. The other cartoons of that type show strategies such as: guess–check–revise, draw a picture, act out the problem, use objects, solve a simpler problem, make a table, look for a pattern, make an organized list, write an equation, etc. (cf. Charles et al., 1992).

Week I	Week II
online meeting I: introduction of unit; release of online learning material and study plan	online meeting II: questions, group work
Week III	Week IV
self-study time	online meeting III: presentation of tasks, questions, **course evaluation**
between meetings: self-study time with online material supported by study groups and online forum	

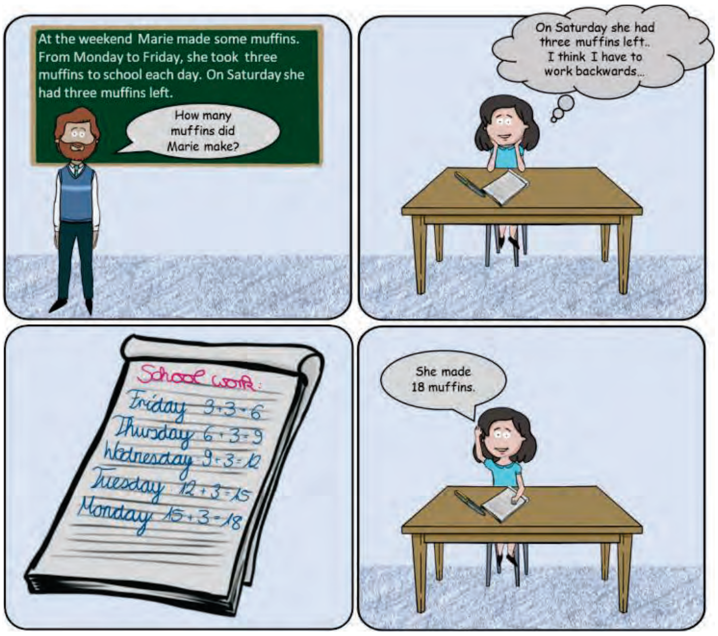

Figure 1: Organisation and contents of the unit on problem solving (left); Short cartoon vignette illustrating the problem-solving strategy "work backwards" (right); (cartoon designed by Alyssa Knox based on Haering (2016); characters drawn by Michael Weninger)

During the unit, the pre-service teachers worked on different tasks provided with the vignettes. They explained, e.g., the strategies used in the cartoons and verified their use with the cartoon characters' solutions or comments. They were also asked what alternative strategies students could have made use of for solving the given task and to compare and reflect on the potential of different strategies.

Figure 2 shows one of the more complex cartoon vignettes. These vignettes were designed based on lesson transcripts by Rasch (2016), representing typical classroom scenarios around primary-school childrens' solving of non-routine word problems. Four vignettes of that type were used in the unit about problem solving, each showing one or several students who use various strategies while solving a non-routine task, who struggle in choosing and/or applying a strategy and who are supported by their teacher. The participants of the course were asked to identify the strategies used by the cartoon students and to evaluate their potential for solving the task. They were also asked to make assumptions about reasons for the students' difficulties in solving the task and to verify their answers with the students' use of strategies as shown in written solutions, use of material or comments. Finally, the participants were given the task to evaluate the teacher's support and to think about alternative approaches. For sharing these approaches, they received a cartoon with empty speech bubbles that could be adapted. In this article, we focus on the students' use of problem-solving strategies.

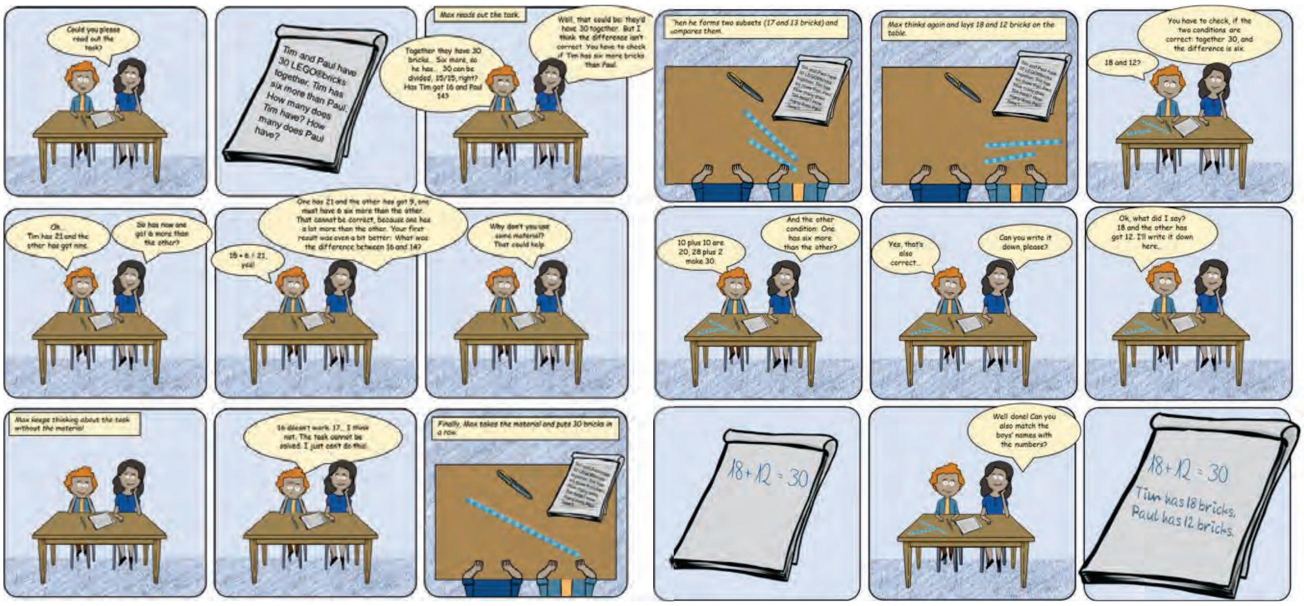

Figure 2: Complex cartoon vignette illustrating different strategies and need for learning support (designed by Alyssa Knox based on Rasch, 2016; cartoon characters drawn by Michael Weninger)

Research interest and research questions

This study aimed at exploring the potential of cartoon-based vignettes for pre-service teachers' professional learning regarding students' use of problem-solving strategies. Accordingly, we addressed the participants' own perceptions regarding their learning with the cartoons and their competence to analyse students' use of problem-solving strategies in classroom situations at the end of the unit. The research questions were the following: (1) How do the participating pre-service teachers perceive the potential of cartoon vignettes in terms of their professional learning? (2) To what extend are the participating pre-service teachers able to identify the strategies used by students when analysing a complex cartoon vignette? (3) Can they suggest alternative strategies?

Sample and methods

The course evaluation took place at the end of the unit on problem-solving after the fourth online meeting (cf. Figure 1). The pre-service teachers received an online questionnaire and were asked to analyse a complex cartoon vignette and to evaluate the cartoon-based learning material in terms of their professional learning. They could use their course material and as much time as they needed. *N*=42 pre-service teachers gave consent to have their data analysed in an anonymised way and participated in the study. The items for the pre-service teachers' evaluation of the cartoon vignettes in terms of their learning (RQ 1) are displayed in Table 3. Since participants' engagement with vignettes is an essential pre-requisite for analysing them (cf. Friesen, 2017), items 1–3 addressed the pre-service teachers' perceived motivation, authenticity and immersion when working with the cartoons. Items 4–5 evaluated to what extent the participants felt supported by the use of cartoons in their learning about problem solving and items 6–7 evaluated the perceived potential of cartoons for teacher education in a more general way. Figure 2 shows the cartoon used for the course evaluation (RQ 2+3) together with the following task: *In the presented situation from a second grade classroom, Max is working out the solution to a problem-solving task while his teacher is supporting him. Please read through the situation and answer the following questions: What problem-solving strategies does Max use to solve the task? What other problem-solving strategies could he have used to find a solution?*

Data analysis and results

Related to the first research question, Figure 3 displays the results of the participants' evaluation of the cartoon vignettes in terms of their learning. Regarding pre-service teachers' engagement with the cartoon vignettes, they rated their perceived authenticity, motivation and immersion on average as positive. The pre-service teachers perceived the cartoons as supportive for their learning regarding students' different strategies and difficulties in solving problems. They evaluated the use of cartoons in teacher education as helpful and as a good connection to practice.

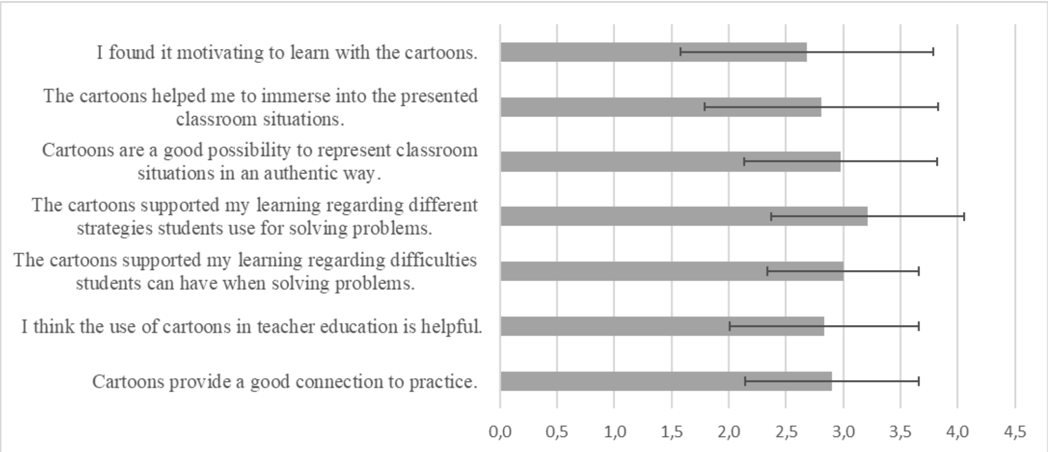

Figure 3: Pre-service teachers' own perceptions of their learning with cartoons (means and standard deviations; *0* = I strongly disagree; *4* = I strongly agree.)

To answer research question 2, an a priori analysis of the cartoon vignette used for the evaluation (see Figure 2) was conducted. In this way, the strategies used by the student in the cartoon vignette (e.g., work backwards, use objects, guess-check-revise) and further potential strategies for solving the given task were identified (e.g., try systematically, make a table). The pre-service teachers' written analyses for the complex cartoon vignette were coded accordingly. The findings show that all of the participants were able to correctly identify at least one strategy used by the student in the complex cartoon; 32 pre-service teachers (76.2%) were able to correctly identify two or three strategies and eight pre-service teachers (19.0%) were able to correctly identify four or five strategies. There were, however, also participants who identified strategies that were not appropriate (e.g., “write an equation” although the student did not use the equation to solve the problem, but only when writing down the answer). In this context, 21 pre-service teachers (50.0%) described one or two such unsuitable strategies and five participants (11.9%) matched three or four strategies in an incorrect way. Although not explicitly asked for, providing explanations for the choice of the strategy was found to be significantly correlated ($r(40) = .41$, $p = .008$) with the number of correctly identified strategies. All of the participants were able to suggest between one and four suitable alternative strategies for solving the task in the vignette. However, 27 participants (64.3%) suggested also one or two unsuitable alternatives; 11 participants (26.2%) suggested three or four.

Table 1: Sample answers showing the relation between correctly identified strategies and explanation for the choice of strategy by referring to relevant events in the vignette

(…)“Try systematically” as he goes through examples in order (first 16, then 17, then 18) and comes to a result this way (PST #2)	(…)“Look for a pattern” as he always has to check for two numbers (difference 6 and result 30) (PST #2)	(…) When he works with the objects later, he also tries systematically, because he notices for 17 and 13 that the difference is still too small and moves one brick from the bar with 13 bricks to the bar with 17 bricks. (PST #22)

Summary and discussion

Vignettes provide safe environments for future teachers’ analysing of classroom situations and can support them in connecting classroom observations to theory – an essential prerequisite for making sense of what has been observed when teaching and for making suitable follow-up decisions (Buchbinder & Kuntze, 2018). This study shows that cartoons hold great potential in this context: they can be purposefully designed to represent key aspects of practice and theory on the learning of mathematics while considering different levels of complexity (cf. Friesen, 2017). Although the study has limitations (e.g., no control group or pre-post design), it encourages further research into the use of cartoons in teacher education: The participants perceived the cartoon vignettes as a valuable learning opportunity, not only in the context of problem solving but also for teacher education in general. The findings show that pre-service teachers’ analysing results were more often correct when they explained them by referring to relevant events in the given classroom situation. When working with vignettes, validating observations with specific classroom events should be part of that work and pre-service teachers need to be supported in a systematic way. Whereas numerous studies have shown the importance of teachers' analysing for instructional quality and student learning (Sherin et al., 2011), this study contributes by showing how pre-service teachers can learn to analyse in a university course on problem solving and how different types of cartoon vignettes can support their learning.

Acknowledgement

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MathCityMap in pre-service teacher education

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Keywords: Mathematics trails, pre-service teacher education, digital learning.

The maths trail seminar

A maths trail is a walk through the neighbourhood where one can find objects to solve mathematical problems, such as calculating the volume of a pond (cf. Shoaf et al., 2004). The MathCityMap project enriches the idea of maths trails with new technologies like smartphones and the internet to create an analogue and digital learning experience. After establishing the MathCityMap project to revive the idea of mathematics trails (cf. Ludwig & Jesberg, 2012), there was a need to educate and train teachers using it in classrooms. Because it is known how hard task design is for in-service teachers (Jones & Pepin, 2016). Furthermore, recent studies have shown positive effects of maths trails on pupils' learning outcomes and motivation (cf. Zender et al., 2021). All this leads to the decision to implement maths trails in pre-service teacher education. So that maths trail seminars were offered for pre-service primary school teachers in the summer semester of 2021 at the University of Koblenz-Landau, at which 45 students took part. The idea was to offer a varied learning experience, analogue and digital, building competencies for the upcoming teachers in outdoor mathematics. Therefore, theory and practice have been equal parts of the seminars: Learning about the theory of outdoor education on the one hand and creating an outdoor learning environment on the other hand. The mathematical topics of the seminar are related to algebraic and geometric content for primary school. The seminar closed with an oral exam: the students produced a video of a task they had created and analysed it from a didactic perspective (cf. Geisen & Zender, 2022).

The conceptualisation of the seminar is based on the interconnected model of professional growth of Clarke and Hollingsworth (2002) for professional development in teacher training, since the seminar for pre-service teachers can be considered similar to training for in-service teachers (see Geisen & Zender, 2022). In this regard, Clarke and Hollingsworth (2002) identified the following four domains as relevant: Personal Domain ("Knowledge, Beliefs and Attitude"), External Domain ("External Source of Information or Stimulus"), Domain of Practice ("Professional Experimentation") and Domain of Cobcequences ("Salient Outcomes") (ibid., p. p. 951). Changes concerning these domains can occur through reflection and enactment processes (ibid., p. 950). The practical implementation of theoretical content imparted in teacher training impacts the learners' learning processes. That can be reflected by the teachers and ultimately lead to a changed practise (ibid.). At university and in particular, in terms of the maths trail seminars, theoretical content is conveyed to pre-service teachers like that in teacher training. Nevertheless, in most cases, they cannot implement this input in school practice. However, they can analyse and reflect on the level of lesson preparation, which can also lead to changes concerning the Domain of consequences and the Personal Domain.

Exploratory evaluation study

The aim of the study is accompanying research to check the effectiveness of the maths trail seminars based on the model for professional growth (cf. Clarke & Hollingsworth, 2002) and the conceptualisation and implementation of the seminar. Therefore the following research questions were fundamental: Is it possible to consider the four domains of the model of professional growth of students through the conceptualisation and implementation of the maths trail seminars? Can this also be confirmed based on the results of the evaluation?

Two digital surveys were used to answer the research questions, the first concerning the seminar itself and the second for the oral exam. The first survey was the general survey of the university to evaluate the lectures and seminars. It was inserted at the end of the semester and contained 64 items, most of them evaluating demographical data or being six-point Likert scales. The second survey contained seven items (4-point Likert scale) and the possibility of leaving a comment for checking the oral exams about its suitability for querying acquired competencies, enabling a multi-stage learning process and a long-term learning effect.

Results and Conclusions

The seminar addressed all four domains of importance for learning identified by Clarke and Hollingsworth (2002): Personal, External, Practice, and Consequences. In the survey, most students agreed that the seminar "established a connection between theory and practice", "makes me interested in the topic", and "makes me want to work on the content". In addition, students reported being highly motivated and voluntarily invested much time into the seminar. Finally, in written feedback, students highlighted the experience and new perspective on mathematics and stated that they felt confident implementing outdoor mathematics in school. Hopefully, the students remember their impressions when being in-service teachers at schools.

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Co-learning in an IBL-inspired PD session on programming

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In this study we investigated what kind of co-learning and learning gaps could be identified in a professional development session on the topic of programming. We found that both teachers and teacher educators learned about programming and lesson planning for programming. Teacher educators also learned about what kinds of knowledge teachers possessed about programming. Learning gaps identified were teacher educators' knowledge about didactical aspects of programming as well as the use of programming in school. We found that teachers express a need to learn more programming.

Keywords: Mathematical literacy, teacher professional development, inquiry-based learning, co-learning, programming.

Background and theory

Programming

In the new national curriculum in Norway launched in 2020, programming is introduced as an integrated part of mathematics and science. This comes in addition to digital skills as one of the basic skills that are to embrace all subjects. At year 5 the mathematics curriculum specifies that “The pupil is expected to be able to create and programme algorithms with the use of variables, conditions and loops” (Utdanningsdirektoratet, 2020a, bullet 10). The science curriculum for years 5–7 specifies that “The pupil is expected to explore, create and programme technological systems that consist of parts that work together” (Utdanningsdirektoratet, 2020b, bullet 7). Programming is a new theme in the national curriculum in science and mathematics for primary school, which implies that there is an authentic need for professional development. Programming is a new theme for teachers, both because they have not been teaching it but also because they themselves have never studied this theme. At the same time programming is new for teacher educators. Even if they may have studied programming at university, they have to a very limited degree been teaching it as teacher educators. Inquiry-based learning (IBL) is a way to investigate and explore new ideas in the classroom and may be a helpful approach in introducing a new curriculum. This leads us to our research question in this paper: What kind of co-learning and which learning gaps can be identified in an IBL-inspired professional development session on programming?

Inquiry-based learning (IBL)

IBL is seen as a way to organize education so as to make students able to function in a society which is changing and where the ability to think, reason and ask new questions are more important and higher valued than the ability to re-answer questions already asked. Bruder and Prescott (2013) point out that IBL is beneficial for motivation and understanding of mathematics and science, as well as beliefs about relevance for society. Our understanding of IBL builds on the model developed in the EU Primas project (e.g., Artigue & Blomhøj, 2013; Maaß et al., 2017). In IBL students inquire, pose questions and engage in exploration in collaborative settings. Teachers foster students' reasoning, connect to student experiences and scaffold learning. The classroom culture is dialogic, and questions are open, experienced as real and relevant with multiple solution strategies. It is thus a goal to foster inquiring minds and build understanding of the nature of science and mathematics. Referring to Jaworski (2006) we consider three forms of inquiry practice in an inquiry community of teachers and educators: inquiry in mathematics (pupils engage in inquiry to learn mathematics), inquiry in teaching mathematics (teachers engage in inquiry to develop the teaching of mathematics), inquiry in research on developing teaching of mathematics (teachers and educators engage in inquiry into the research).

Four co-learning dimensions

Jaworski (2003) launched a four-dimensional framework for investigating the development of mathematics teaching and learning. Development is seen as processes of inquiry, involving critical reflection, in a community of inquiry. In the community experiences are shared, this supports individual inquiry and the development of norms. Hence, there are elements of both individual and community relationships that allow the inquiry and reflection to take place, which is seen as the learning and development. Participants can be insiders in the community or outsiders supporting the research and development. This gives rise to four interrelated reflexive pairs: Knowledge and learning, Inquiry and reflection, Insider and outsider, Individual and community. Investigations along these four dimensions are suitable and will provide insights into the inquiries and co-learning taking place in a learning community consisting of teachers, educators and students. Jaworski (2003) suggests guiding questions for each reflexive pair. To answer our research question, it will be helpful to answer questions related to each of the four dimensions:

Knowledge and learning: Who is learning and what knowledge is being developed?

Inquiry and reflection: Who is inquiring and into what? What kind of reflections are taking place?

Insider and outsider: Who are the insiders and who are the outsiders, and in what situations?

Individual and community: How is the community comprised, and how is the dialectical relationship between individuals and community played out?

By searching within each of the four dimensions, a more complete picture may appear of what kind of co-learning is taking place and of which learning gaps can remain.

Methods

The professional development session studied in this paper was carried out as part of a larger, longitudinal research project with focus on mathematical and scientific literacy and IBL. Participants were three mathematics and two science teacher educators from the university, and five teachers from a local school (teaching grade 4, 5 and 6, and one teacher leader). All teachers were generalist teachers. The TEs all had PhDs in mathematics or science, some with teacher education in addition. The lesson study was carried out according to the following plan: a) Teachers suggested a theme and a lesson plan, b) Teacher educators (TEs), including both mathematics teacher educators (MTEs) and science teacher educators (STEs), discussed the suggested theme and plan, c) Teachers and TEs met in a joint planning meeting, d) Two lesson iterations with reflection and redesign in between. Teaching carried out by one teacher, with other teachers and TEs observing. Final reflection after the second iteration.

The theme suggested by the teachers for this session was programming with micro:bit (https://microbit.org/). Micro:bit is a small computer originally launched by BBC in 2015 with the purpose to be used in schools for pupils to learn programming. To program the micro:bit, pupils use a block-based visual programming language on a computer or tablet, and then transfer the program to the micro:bit using Bluetooth or a USB cable. In the lesson plan suggested by the teachers, pairs of pupils were given the task to make a flashing heart on the micro:bit's LEDs, and then try to make the heart flash with the same rate as their own heart rate. The teachers based this plan on an example found on the Norwegian website Lær Kidsa Koding, which provides resources related to teaching programming to children (Lær Kidsa Koding, n.d.). In the TEs' meeting, the main discussion related to whether the proposed plan opened up for sufficient inquiry or whether it was too focused on following recipes. An alternative task involving using the micro:bit to steer a car was launched. In the joint planning meeting, it was agreed that the lesson plan suggested initially by the teachers would be kept, but some changes were made in the material that the pupils would use.

The following data material was collected: Proposed lesson plan and final lesson plan, field notes from the TEs' meeting, audio recordings of the joint planning meeting and the two reflection and redesign meetings.

The audio recordings from the joint planning meeting and the first reflection and redesign meeting were transcribed. The three authors coded the transcript from the joint planning meeting separately, using the four dimensions from Jaworski's framework as categories (Jaworski, 2003). The three different versions of the coded data were then discussed to come to a joint understanding of each category. To help in guiding the discussion, the lesson plan and field notes were consulted. Subsequently the transcript of the first reflection and redesign meeting was coded separately using the new understanding of the categories. The audio recording of the second reflection and redesign meeting was also analysed using the common understanding of the categories. After this the findings were compiled. The project has been submitted to, and approved by, NSD – Norwegian centre for research data. In this paper, teachers are referred to as T name, where all names are pseudonyms.

Findings

The findings are structured according to the four dimensions suggested by Jaworski (2003).

Knowledge and learning

Our data shows multiple answers to questions related to knowledge and learning. The participants can be divided into three groups according to their prior knowledge of programming, see Table 1.

Table 1. Participants' prior knowledge of programming

Extensive knowledge	Some knowledge	No knowledge

MTE Øystein, STE Arne	MTE Benedikte, MTE Svein	STE Ragnhild, T Fay, T Camilla, T Marie, T Erik, T Jenny

Teachers Fay and Erik had some experience with micro:bit in fourth and sixth grade, respectively. Apart from that, no participants had any prior knowledge of programming in school. Four of the TEs had knowledge of programming from university courses and have used it at some level. Having sufficient programming knowledge at university level is however not sufficient to design tasks that function well at primary school level. The tasks discussed by the TEs in their prior meeting turned out to progress too fast to a too advanced level.

MTE Svein: It can be that, when we talked about it, that it took off and became too advanced.
T Jenny: Is it too advanced?
STE Arne: It is hard for me to say, really, since I have the knowledge.

STE Arne was not able to judge whether the task at hand would be too advanced for the pupils. He had extensive knowledge of programming, so could do the task himself, but did not have experience with programming in primary school, and so needed teachers' inside knowledge of their pupils' abilities and prior knowledge as feedback to judge the appropriability of the task.

Teachers' lack of knowledge about programming is clearly an obstacle for developing good teaching plans that would enhance pupil knowledge.

T Fay: We don't have sufficient knowledge in programming to design good enough tasks.

TEs had knowledge of programming, but not in a school setting. Teachers had knowledge of their pupils, but not of programming. Through the cooperation, both groups had opportunity to enhance their knowledge into new ground. Teachers were given opportunities to develop their knowledge of programming in school:

T Camilla: I ask a lot of questions because I need input on how to continue teaching programming.

We thus see that teachers are learning about teaching programming by cooperating with TEs who have programming knowledge. TEs learn about teachers' knowledge of pupils, and how to integrate knowledge about programming with knowledge of pupils to develop lesson plans. So both teachers and TEs learn how to plan a programming lesson in mathematics or science.

Inquiry and reflection

Several types of inquiry and reflection are evident in the data. During the planning meetings and reflections, ideas were discussed and assessed according to whether they were easy or difficult for the students (mainly with input from teachers) and easy or difficult to program (mainly with input from TEs). Together these constitute inquiry into teaching. Teachers and TEs inquired into pupil knowledge and learning after observing lessons, with reflection on the teaching and how the pupils reacted to the teaching, including how they were able to cooperate and work together in pairs.

As the lessons are to be IBL-inspired, a common quest regards whether the suggested activities offer opportunities for the pupils to inquire and explore. In our case this quest led to a discussion about whether doing programming would provide opportunities for inquiry or whether the programming had to be embedded in something else.

STE Arne: In this particular task at hand, there is little inquiry involved (…) Because, in terms of programming it is not so much exploring.

T Jenny: I believe, we were satisfied when we came up with this about heart rate, since that may be inquiring.

STE Arne wanted to make the programming itself more explorable, while the teachers had put programming into the context of heart rate to make it into something the pupils could explore. Discussions continued regarding how to make it more IBL-like, and inquire in programming:

T Jenny: Do you think that, it is possible to do inquiry with the programming; and that it will be too difficult to combine it with exploring their own heart-rate in addition?

The question relates to what types of inquiry is possible if you haven't done any programming? Teachers expressed thoughts that when something is new to you, then maybe it does exactly give you opportunities to explore. Since the pupils have not worked with programming or micro:bit before, starting playing is exactly to engage in exploration and inquiry. This was expressed at various points during the planning.

T Fay: We were thinking that it will be some kind of inquiry since they haven't worked with it before.

T Fay: Like this, now, like when I am trying it out, it is inquiry for me; will it not be the same for the pupils?

A teacher asks whether there will be multiple solutions and strategies, relating to an important part of IBL:

T Erik: Will there be many different methods for how they can solve it? Will they come up with more examples of how they have done it?

In order to solve the task, the pupils had to find an appropriate pause (in number of milliseconds) between each heart beat shown on the micro:bit, in order to make it match their own heart-rate. Two possible strategies were mentioned:

T Fay: Some will do as I do now, just trying to make it equal, just moving the numbers around. Or some will try to compute the number of heart beats in a minute.

During the planning meeting, only these two solution strategies were suggested. In the reflection meetings, it became clear that this did not correspond well to what actually happened during teaching. Most of the students did, to a certain extent, follow one of these two strategies, but many of them did so in ways that were not expected. Of those following the strategy of trying out different numbers, many pupils changed the numbers in so small increments that there was no noticeable effect of the change. Those following the strategy of finding the number of heart beats in a minute did not continue in the expected way by computing the correct pause to use in their programs, but rather used the number of heart beats as the length of the pause.

At the reflection after the first lesson, it became apparent that the way the task was designed made it difficult to observe what impact changing the input in the program had. For example, if pupils changed input from 60 to 70, they were not able to observe any change since those numbers are in milliseconds. In addition, the program block used to display a heart on the LEDs had a built-in pause of 600 ms, meaning that even if pupils put their own correct numbers into the program, the output would be wrong by 0.6 seconds.

MTE Øystein: So that is the biggest problem. That if someone has understood it, and apparently does everything right, they get the wrong result.

As we have seen, the main type of inquiry was inquiry into the teaching, with critical reflection on the lesson and the task.

Insider and outsider

The data reveal several pairs of insider-outsider configurations. An obvious instance is that teachers are insiders in school while TEs are outsiders. Among the teachers, class-teachers are insiders regarding knowledge of the particular group of students, while other teachers and TEs are outsiders. Moving to the classroom, both teachers and TEs are outsiders inquiring into how pupils work during the lesson.

The insider-outsider perspective also relates to programming. Those who know programming are insiders, while those who don't are outsiders. Without pre-knowledge of programming, you don't know what the possibilities and options are, e.g., what is easy to accomplish and what is more difficult. Those who don't know programming are bound by following given recipes. Those who know programming can supply information/knowledge on how to use the recipe, and change it, for their own means. Even so, pupils have to be guided, like STE Arne is guiding the group during the planning:

STE Arne: They either need a recipe, or you have to tell them, like I do now.
T Fay: Yes, if we have a recipe, then it is quite OK.

Part of being an insider in programming is knowing about the terms and concepts and what possibilities these may constitute. Before you reach that level, having a recipe constitutes a safety net.

Individual and community

A prominent example of how the community of teachers, TEs (and pupils) relate to the wider society is the fact that programming as a theme in mathematics and science in the school curriculum was imposed from the government without much support from teachers, and to a degree resistance, from teacher organisations and teacher educators. The competence aims formulated in the national curriculum provided the basis for the session, and were referred to several times during the discussions, to assure that the task designed for the lesson was appropriate for fulfilling the learning goal formulated in the curriculum.

STE Arne: But then it is not that competence aim, then it is another competence aim.
T Marie: Because it says "design and make a program based on user needs".

An interesting aspect is connected to what resources that are available, e.g., the micro:bit technology that was used in this session. Why and how had the micro:bit technology been chosen? Was the decision made by the teachers or the school or at some higher level? It turns out that the micro:bit technology has been launched at national level:

T Fay:	All schools in our commune have been given class-sets of these.
STE Ragnhild:	All schools in the whole country have been given such class-sets.

Thus the society at large influenced the lesson both concerning the theme (the curriculum) and the resources (micro:bit). The influence of the national curriculum is obvious and natural. The choice of technology is more surprising, and shows that apparently random choices have big influence on lesson design.

Discussion and conclusion

This lesson study provided ample possibilities for co-learning for teachers and TEs as programming as a theme in school was new to all participants. Misfeldt et al. (2019) point out that even if teachers have a positive attitude towards working with programming, they do not feel prepared to take on the task. Miller (2019) states that the school as a system needs to provide support and opportunities for teachers to gain skills in how to implement computational thinking and coding. Since programming in school was new to all participants in our project, all were learning at some level: a) Teachers learned about teaching in/with programming, b) TEs learned about what teachers know about programming, c) Teachers and TEs learned how to plan a lesson in/about programming, d) Teachers and TEs learned about programming in school.

Learning to teach programming concerns the interplay between teachers' knowledge about teaching and pupils, and TEs' knowledge about programming. How to find or develop a common language? Through the cooperation, teachers provided knowledge about what may work in school so that the two types of knowledge, teacher educators' knowledge of programming and teachers' knowledge of the school setting, resulted in co-learning about programming in primary school.

How easy is it to understand that programming is open to inquiry? Programming involves making algorithms, recipes, which make programming appear "closed". On the other hand, making changes to the code is easy, just do it and see what happens, thus programming is very open to inquiry. Maybe there is an analogy with Lego? You can follow the recipe on the box or play with the pieces freely. We found that teachers with little experience in programming tended to view programming as a closed activity that simply consists of following pre-made recipes. The teachers did, however, also express the view that there is inquiry involved in learning to program.

In the professional development session studied here, both teachers and TEs neglected to work completely through the planned task before the lesson. Everyone simply agreed that the task could be solved, and there was only a very brief discussion about possible solution strategies. It seems clear to us that all teachers and TEs should have attempted to solve the task completely during the planning meeting. This would have made the teachers better prepared for teaching the lesson. But perhaps more importantly, it would mean that the teachers and TEs would sit together actually doing programming (as opposed to just talking about programming), which we believe would have given good opportunities for additional co-learning.

Our findings indicate that teacher educators need to acquire more knowledge about didactical aspects of programming and the use of programming in mathematics and science for primary school. Teachers expressed a need to learn more about programming, learn coding, and thereby gain knowledge that will help them choose or design tasks that are relevant, both in the sense of being relevant for pupils in their learning of programming, and relevant in the sense that they are attainable for the age group.

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Math trails supporting the collaboration between mathematics teachers in professional development

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Keywords: Math trail, lesson study, professional development, teachers' knowledge, beliefs.

Professional development and collaboration

Nowadays, collaboration among teachers in activities closely linked to mathematics lessons is considered to be the most appropriate form of professional development of in-service (mathematics) teachers (Prediger, 2020). Current Slovak legislation makes it possible to situate in-service teacher training directly in schools. A well-designed in-service professional development programme influences the knowledge and beliefs of mathematics teachers.

Lesson study, when experienced teachers of mathematics and didactitians work together to design an optimal lesson on a pre-agreed topic, seems to be a suitable method of professional development. It has a cyclic nature consisting of four stages: study, plan, teach and reflect (Lewis et al., 2019) intervening changes in teachers' knowledge, beliefs, routines of professional learning and pedagogies which influence the instruction and therefore students' learning. Beliefs play an important role in assessing curricula, teaching, learning and assessing students' knowledge and are grounded in teachers' knowledge and experience (Carrillo-Yañez et al., 2018). Teacher knowledge and beliefs influence various areas of mathematics lesson planning, implementation, and reflection (Ball et al., 2008; Carrillo-Yañez et al., 2018) including the use of student-oriented teaching methods such as a math trail.

Mathematical trails in teachers' professional development

A math trail is a path during which students can discover and solve mathematical problems related to real objects. MathCityMap trails (MCM trails) are part of outdoor education supported by mobile technologies (Barlovits & Ludwig, 2020). They use a 'bring your own device' approach in school as well as in out-of-school contexts. The purpose of the MCM trails is not only to popularize mathematics for the general public, but also to offer students the opportunity for collaborative problem solving related to their lives. In addition to the mathematical version of geocaching, students engaged in MCM trails solve mathematical problems related to real objects (Čeretková & Bulková, 2020), create the original solutions of the problems and communicate their ideas, reasoning and strategies during collaboration in teams what makes math trails a suitable tool to develop the competences for 21st century.

Methodology

Implementation of the lesson study varies across the countries and needs to be tailored to the national context. Based on the interviews with experienced mathematics teachers, the students in Slovakia lack the opportunities to develop the competences for 21st century. We suggest that it may be caused by teachers' insufficient repertoire and experience with instructional tools and routines. The following research question is addressed: What mathematics teachers' specialised knowledge is addressed during reflection on enacted MCM trails in form of lesson study? The lesson studies will be conducted with mathematics teachers in Slovakia. Didactitians will be participant-observers. First, the teachers solve the MCM trail prepared by didactitians to allow them to experience the MCM trail from the students' view. During the design of the MCM trail for grade 8 students, the teachers gain a different perspective on problem-based learning. Communication with students during the trails and subsequent whole-class discussion after its implementation is different than in transmissive teaching prevalent in Slovakia. All the four steps of lesson study will be audio- or videotaped to enable thorough analysis of obtained data. The mathematics teachers' specialised knowledge will be categorised according to Carrillo-Yañez et al. (2018). The preliminary results from the lesson study conducted in autumn 2021 will be presented.

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The growing knowledge of prospective teachers about mathematical justification in a teacher education course

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The importance of developing students' mathematical reasoning (MR) is emphasized, both in curriculum documents and research, as an essential skill for their learning. Nevertheless, as the teaching practice for promoting MR is often challenging for prospective mathematics teachers (PTs), it is essential to highlight MR as a goal for preservice teacher education. This study aims to analyse the growing knowledge of a group of 13 middle school and secondary mathematics PTs about the justification process, a critical component of MR, throughout a teacher education course. The results show that before working on this theme in the course, PTs revealed difficulties in the interpretation of MR processes and lower levels of knowledge about justification. However, their knowledge level has improved, which may confirm that the type of work carried out in the course was valuable to a deeper and adequate MR knowledge necessary for their future practices.

Keywords: Mathematical reasoning, justification, pre-service teacher education, prospective teachers´ knowledge.

Introduction

The development of students' mathematical reasoning (MR) is currently emphasized in national and international curriculum documents for teaching mathematics (ME, 2018; NCTM, 2000), as it is recognized as an essential skill for students to succeed in understanding mathematics by making sense of concepts and actively constructing mathematical ideas (Lannin et al., 2011; Stylianides et al., 2013). From this perspective, it is important that teachers place greater emphasis on MR in their practices, by providing students the opportunity to engage in and develop MR processes, since their learning depends on the experiences that teachers offer to them in mathematics lessons (Boston & Smith, 2011; Breen & O'Shea 2019).

This recommendation is significant in Portugal, since the curricular documents emphasize the development of students' MR ability, including justifying as an essential MR process to also help them reach a broad nature generalization (ME, 2018). However, school practices are still not consolidated and focused on the promotion of this MR process, since it is challenging for prospective teachers (PTs). Even though they struggle to produce and identify justifications, commonly judge empirical arguments by seeing examples as justifications for a generalization, as they may not have that experience as students and are unfamiliar with the MR meaning and processes (Melhuish et al., 2018; Stylianides et al., 2013). Therefore, it is essential highlighting MR as a goal for pre-service teacher education, to develop the required prospective teachers' knowledge about MR (Stylianides & Stylianides, 2006) to be proficient in promoting this students' skill in their future practices (Buchbinder & McCrone, 2020; Loong et al., 2017; Ponte & Chapman, 2015).

Mathematical tasks involving the analysis of students' reasoning processes may provide opportunities to develop the PTs' knowledge in the context of initial teacher education focused on MR, but how they interpret and understand it and notice students' justification process is still an under researched issue (Melhuish et al., 2018; Park & Magiera, 2020). The current study aims to analyse the growing knowledge of middle school and secondary mathematics PTs about the process of justification, throughout a teacher education course focused on MR. In particular, we address the following question: What improvement do the PTs evidence in their knowledge about the justification process, when they analyse examples of students' work in solving MR tasks, before and after a teacher education course?

Thus, this study may contribute to support the development of PTs' reasoning and the necessary enhancement of the design principles of the teacher education course to better assist prospective teachers in developing the needed knowledge of MR to carry out practices focused on it.

Theoretical framework

Mathematical reasoning

Mathematical reasoning, a process of making justified inferences based on previously known information (Jeannotte & Kieran, 2017; Mata-Pereira & Ponte, 2018), covers diverse kinds of reasoning: inductive and abductive to infer and draw new information, particularly plausible conjectures, and a general conclusion, based on examples and empirical experience; and deductive reasoning using formal rules to validate the inferences made from that information.

In addition, the MR conceptualization and the engagement in this activity involves diverse reasoning processes (Jeannotte & Kieran, 2017): generalizing and justifying processes, which are recognized as central for mathematical reasoning and may be achieved by inductive, abductive, and deductive reasoning; and also exemplifying, which is conceived as a relevant support for generalization and justification (Mata-Pereira & Ponte, 2018). Generalization starts from a conclusion or conjecture about a property or procedure, often based on patterns identification after an exploration of specific examples using inductive reasoning, to assert that it is common or valid to a set of objects. Afterwards, the conclusion or conjecture needs to be validated by testing examples or find counterexamples, checking if it works for diverse objects, or also justifying it through deductive reasoning (Jeannotte & Kieran, 2017; Lannin et al., 2011). Justification involves generating arguments, and explaining why they are true, to provide compelling reasons for established conjectures, but students often evidence difficulties in writing their justifications (Bersch, 2019). The understanding of the definitions and counterexamples' roles in this process allows students to make their reasoning clear and to increase their conceptual understanding, recognizing that a justification validity is a critical component of MR that is expected to be supported by mathematical procedures, properties, and definitions.

Finally, exemplifying appears associated with other reasoning processes, and involvies finding examples to support the identification of similarities and differences, and performing a validation to support the justification (Jeannotte & Kieran, 2017).

Prospective teachers' knowledge of mathematical reasoning

As mentioned in literature, it is essential that PTs learn how to create learning environments for promoting their students' reasoning, attending to the mathematical education demands (Davidson et al., 2019). PTs' knowledge of MR involves different dimensions, particularly the concept of reasoning and the kinds and processes it encompasses. Moreover, that knowledge involves recognizing the importance of developing students' MR as it is essential for their learning, and to be able to identify the reasoning processes they are able to use, namely justification, to support them through mathematical reasoning tasks (Bersch, 2019; Cihan & Akkoç, 2019; Park & Magiera, 2020;). However, several studies (e.g., Lannin et al., 2011; Stylianides & Stylianides, 2009) show that PTs usually express incorrect ideas on MR processes, especially about justification, even after having attending courses targeting this process.

Regarding these difficulties, there is a need to study how PTs may develop that knowledge during their teacher education programs. For PTs access to MR, it is essential that in teacher education they read and discuss documents focus on framework concerning MR (meaning, kinds and processes) and are asked to solve mathematical learning tasks from a school student's perspective (Jeannote & Kieran, 2017; Mata-Pereira & Ponte, 2017). Complementarily, the PTs' engagement on tasks, focused on noticing evidenced distinct ways of student's reasoning, require them to justify and sharing interpretations of MR. This PTs' analysis of students' work, in a first stage of teacher education course, may provide information that helps them to increase the required justification knowledge throughout the training course (Park & Magiera, 2020). To analyse this improvement of PTs' knowledge on MR, their interpretation of justification process should be assessed at initial and final phases of the course, using distinct levels which may characterize the diverse understanding of specific aspects associated to MR processes. For this purpose, we adopt a framework to describe the PTs' knowledge about MR processes, focusing on generalizing and justifying (Rodrigues et al., 2021).

This framework identifies six important levels (subcategories) to classify the knowledge of the reasoning processes and to characterize distinctions among main activities (Table 1), which provide "a tool to assess the learning of the prospective teachers in this domain" (p.10).

Table 1: Subcategories for classifying the knowledge of the reasoning process (Rodrigues et al., 2021, p. 6)

Category	Subcategories
Knowledge of the reasoning process	5. Knowledge of the process fits the definition presented, and includes its relationship with the other reasoning processes 4. Knowledge of the process fits the definition presented, and is explicitly outlined by enunciating the properties of the process 3. Knowledge of the process fits the definition presented, and is explicitly outlined through illustrative example(s) 2. Recognising a reasoning process though considering only 'correct' processes 1. Knowledge of the process takes on the meaning of the term in everyday language 0. The process is confused with other processes

According to these authors, the lower levels (0, 1 and 2) are recognized as insufficient knowledge of the reasoning processes, as do not reveal understanding of their distinct characteristics by attributing meanings to associated terms. The other upper levels (3, 4 and 5) already correspond to better MR knowledge of processes, illustrated mainly by the identification of examples and specific properties that characterize the process as defined in literature, and being able to presuppose relationships of inclusion.

Context and methods

This study has a qualitative nature (Erickson, 1986), and assumes a descriptive and interpretative analysis of PTs' growing knowledge about justification. It developed during a preservice teacher education course in the context of the REASON project (http://reason.ie.ulisboa.pt/en/english/) that aims to develop PTs' knowledge to promote students' MR. The participants are 13 (3 male and 10 female) Portuguese PTs who attended the 1st year of a master's program in the teaching of mathematics for middle and secondary school levels. The majority of them had a degree in mathematics, with about 3 years of advanced mathematics study, or a degree in an area with a strong mathematical component, complement with mathematics subjects in the master's program. All PTs volunteered to participate in the study and are identified by a random capital letter for guaranteeing their anonymity.

The eight sessions (two hours each) of this course on MR were conducted by the first author and observed by second, as researcher, and were part of one semester mathematics education methods course. The sessions were conceived to deepen PTs' knowledge about MR, where they autonomously (individually and in pairs) solved mathematical tasks involving Algebra and Geometry topics that required them to make generalizations and justifications. And after that they analysed episodes revealing some students' reasoning processes in solving the same tasks. In addition, their work on the tasks was collectively discussed with the intention of giving them opportunities to better understand the MR processes. The sessions also considered the reading and whole class discussion of documents focused on a framework concerning MR (Jeannotte & Kieran, 2017) and theoretical principles for task design to promote RM.

Data collection for this study concerns two phases of the PTs' work which includes their written answers to an initial task proposed on the course before working on MR and one task at the end of the course, in order to analyse their knowledge development. Both proposed tasks include some students' resolutions of them, and a question that asks PTs' analysis of the reasoning processes illustrated in students' answers, as following, intending to gather a wide diversity of data regarding their knowledge of justification process:

> ***Initial task question*:** For each of the students' written work on the task, identify how they attempted to justify the stated property. Which students' answers evidence "convincing justifications"? Justify your answer.
>
> ***Final task question*:** Analyse each of the students' written work on the task and characterize the justifications they present. Specify the ones you consider most "convincing" and explain why.

In data analysis, to answer the question of this study, we used the framework reported above (Rodrigues et al., 2021) to analyse PT's knowledge about the reasoning process of justification in both tasks. Based on that, we present in table 2 a quantitative description levels of knowledge exhibited by the PTs in the process of justifying based on their answers of the tasks, and comparing them to illustrate their knowledge development. The PTs' answers were independently coded by the authors, focus on the identification of the categories proposed. Divergent interpretations or doubts concerning a codification were then discussed by both authors to reach a full agreement. After that, we also present some examples of the PTs' answers to the tasks that show the discrepancies of the process meaning attributed by them and their interpretation of students' answers, associated to each subcategory. This examples may provide evidence of PTs' MR knowledge and difficulties they reveal in their work.

Results: PTs' knowledge about justification

Table 2 shows the PTs' different levels of knowledge about justification, based on their answers in each task. Generally, the results show that PTs have improved their knowledge in the reported answers and the interpretation of the students' answers.

Table 2: Levels of PTs' knowledge about justification

	PTs	**Levels of reasoning process**	
		Initial task (*)	**Final task**
justifying	**A**	1	5
	D	0	4
	F	0	3
	K	0	4
	L	0	3
	M	0	4
	N	-	2
	O	3	4
	P	3	4
	R	0	3
	S	4	4
	T	-	4
	U	-	3

* Three PTs did not answer this question in the initial task

In the *initial task*, most PTs (6 + 1) evidenced lower levels (0 and 1) of knowledge about reasoning process of justifying, which is indicative of the common PTs' difficulties in interpretation of MR processes. They mainly confuse the justifying process with others, namely with the generalizing. For example, to contend for the convincing justifications in students' task answers, in their answers they argue that:

> The most convincing justification is the one that reaches generalization … He concluded that all even numbers added to odd numbers give odd … without exemplifying or generalizing. (D)

> The student used the generalization and verified its divisibility by 2. (F)

> They perceive the property and generalize. (M)

One PT also associated this process with the use of a mathematical procedure, a term used in everyday language as illustration of the process, although the procedure in the student's answer is clearly not justified: "The resolution is accessible and assertive. They seek to justify from the use of algebraic addition" (A).

Only few PTs (2 + 1) showed higher levels (3 and 4), that comply with the definition process, although two of them just identified the justification through transcribed examples of the students' answers associated to this process, mentioning for instance that: "Use of equations and acquired knowledge of the sum of even and odd numbers" (P) and "Uses algebraic language and concept of parity" (O).

We also observe that one PT already evidenced deeper knowledge about justification, outlining its properties, recognizing an algebraic expression and mathematical properties as relevant characteristics of this process: "Generalizes algebraically … and uses the property that says the sum of two even numbers is an even number" (S).

At the *final task*, compared to the initial one, the PTs' levels of knowledge of justification have increased (2 to 5). The only PT at level 2 assumes that, for considering the justification as a reasoning process, it needs to be properly complete and correct, namely using mathematical logic: "he is not able to come to a logical justification" (N).

We also find, in this task, that the most common levels are 3 and 4, as evidenced by 11 PTs (4 + 7). The four PTs showing level 3 were able to identify the justification process through transcribed examples of the students' answers, for example:

> it was based on previous knowledge and reached a new conjecture and validated; justified through examples and search for patterns. (F)

> part of a general case to justify this property. (R)

The answers of the PTs' most evidenced level (4) express knowledge of justification and outline their properties, recognizing an algebraic expression and mathematical properties as relevant characteristics of this process. For example:

> The justifications presented use inductions, for generic examples and formal demonstrations; reveal the understanding of the concepts involved and necessary for the demonstration, even though ... general algebraic expressions are used. (T)
>
> He relies on geometric construction…, on knowledge that he already possesses… and gives a deductive generic example. (M)
>
> Resolution D is an example of a justification through deductive reasoning characterizing justifications; justifies through examples, …where its representation allowed reaching the property and justifying. (P)

Finally, we remark that one PT has reached the highest level of knowledge (5), as in her answer she identifies the justification and highlights the relationship between this process and generalization, by assuming that the last is inherent to the first: "The student … ended up finding two regularities that justify his reasoning quite well … discovers the regularity, making a generalization" (A).

Conclusions

This study investigated the improvement of middle school and secondary mathematics PTs' knowledge about the MR process of justification, in the context of a pre-service teacher education course, particularly focused on how they recognize this process in students' solution approaches in MR tasks which are analysed by them at an initial and final moment of the course. The PTs' work on those two moments of the course and its comparative analysis provided an opportunity to evidence their growing levels of knowledge regarding justification process, based on the framework by Rodrigues et al. (2021) and recognised by the authors as essential to establish the progression of MR knowledge. Although almost all PTs had a solid mathematical background, with about 3 years of advanced mathematics study, at the beginning of the course, most of them still revealed difficulties with this MR process that is central to the mathematical activity. They seem not to be familiar with MR knowledge, as they reveal lower levels of knowledge about justification. This is indicative of common and serious PTs' difficulties in the interpretation of MR processes before working on this theme in teacher education, as pointed out also by other studies (Buchbinder & McCrone, 2020; Lannin et al., 2011; Stylianides & Stylianides, 2009).

As expected, after being involved in the teacher education course, it seems that in the final task, the PTs were using knowledge they have learn during the course, as all of them show a progress in their level of justification knowledge, although revealing a wide diversity among them.

Finally, we remark, concerning the research question, that the results of the study evidence that the PTs have improved their knowledge, which confirm the importance of providing PTs opportunity to develop their MR knowledge in initial teacher education. The results of this research could also indicate that the privileged type of work carried out in the course, particularly the discussion of theoretical texts about MR meaning (definition, types, and processes) and the exploration and analysis of tasks proposed to students focused on their types and processes of MR, was valuable to bringing them closer to a deeper MR knowledge necessary for their future practices.

The limitations of this study are related to the fact it only focuses on the PTs' justification process and does not analyse other relevant MR processes worked on the course, but the research will be extended to adress more dimensions of PTs' MR knowledge.

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A concept to handle teachers' heterogeneity within PD trainings

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Keywords: Heterogeneity, primary school, professional development training, teacher.

Introduction and theoretical background

On the basis of the adjustments in the school system and the accompanying requirements of the teachers, in-service training aiming at continuous professional development (PD) is indispensable. In this context, Prediger et al. (2019) differentiate between three different levels: classroom level, teacher PD level and facilitator PD level. On the 'classroom level', many studies have been carried out looking at the heterogeneity of students and teachers dealing with this heterogeneity. However, within the 'teacher PD level', hardly any research was found dealing with the heterogeneity of teachers within PD trainings. In order to design effective PD trainings, Barzel and Selter (2015) identified six design principles: 1) competence orientation, 2) participant orientation, 3) teaching-learning diversity, 4) case reference, 5) stimulating cooperation and 6) promoting reflection. Participant orientation means taking into account the "individual, heterogeneous prerequisites and needs of the participants" and "fostering and demanding their active and autonomous participation" (p. 271, translated by RH).

Based on these six design principles, Korten et al. (2019) developed an evaluated, blended-learning training concept. This is divided into an individual distance learning program and a joint attendance program. The aim of the study was to support teachers in inclusive classrooms and to investigate their diagnostic competence. They found that although they considered participant orientation, there was a great deal of heterogeneity among teachers in terms of their diagnostic competence when attending the training. This implies that their blended-learning concept needs further development in order to successfully address in-service teachers' heterogeneity within all learning phases of a PD training.

Research interest and design of the study

Embedded in the broader project 'School makes strong' (Schule macht stark), we investigate the following questions: (1) How could a PD training be designed (e.g. using design principle 'participant orientation') to address teachers' heterogeneity in a holistic way? (2) By which elements it could be enriched meaningfully for heterogenous in-service teachers? The project 'School makes strong' aims at supporting schools in socially challenging situations. To do so, trainings for in-service teachers at a total of 125 primary schools are being developed. Content wise, we focus on mathematical basic skills, e.g. basic arithmetic operations, number sense and the interlinking of theory and practice.

Our study is conducted using topic-specific didactical design research (Hußmann et. al., 2016). Three cycles – each consisting of a design and a research process – will be passed through. At the moment, we carry out the first cycle, in which we develop a concept for the structure of the in-service trainings and material to design the PD trainings. Within the research process of the first cycle, we will collect various data: (1) questionnaires to evaluate if and how the teachers work with the mini-modules; (2) vignette-based interviews to learn more about the teachers' heterogeneity and their learning progress by the mini-modules and (3) observations of the teachers' lessons to identify how new learning is implemented within mathematics teaching.

Structure of a PD training focusing on teachers' heterogeneity

Building on the theoretical background, the in-service training courses are planned in a holistic way (see Figure 1). Classically, there will be built two training sessions according to the six design principles, focusing on participant orientation. Each training session is structured in the same way: welcome, (reflection phase), theoretical input, planning of practical phase. A differentiated, individual study phase is integrated between the trainings consisting of mini-modules and a practical phase.

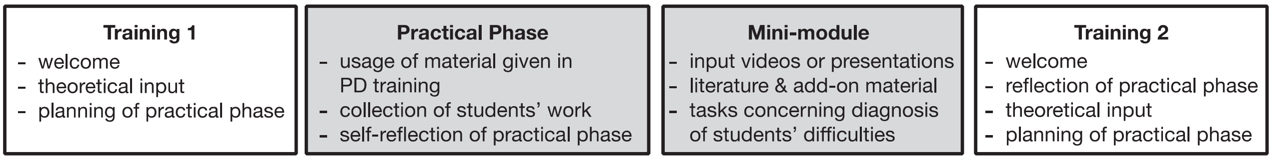

Figure 1: Concept to handle teachers' heterogeneity in PD trainings

Within the *practical phase*, teachers are encouraged to implement their new knowledge in their regular lessons by implementing the new material in their lessons. The teachers are then asked to collect the students' work and reflect their teaching after the lesson. The students' work and the teachers' self-reflection build the basis of the group discussion in the reflection phase of the next training.

Between the trainings, all teachers are offered *mini-modules*. These aim at building and activating knowledge about mathematical or didactical issues, e.g. understanding of mathematical operations.

Future steps

In the future, the concept will be piloted, evaluated and revised within the next design research cycles. A focus will be on the design, planning and reflection of the practical phase and the mini-modules.

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Teacher Perspectives and Mathematical Knowledge for Teaching

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In this study, we report on data from videotapes of two prospective secondary mathematics teachers' practicum teachings. The aim of the study was to investigate the relationship between prospective teachers' mathematical knowledge for teaching and their perspectives on mathematics, mathematics learning and mathematics teaching. We analyzed data using both the Teacher Perspectives and Knowledge Quartet frameworks. Results showed that prospective teachers having a progressive incorporation perspective or perception-based perspective depicted all the codes in Knowledge Quartet; indicating that they held mathematical knowledge for teaching. Yet, results pointed that they had different reasons behind their actions indicating their mathematical knowledge for teaching. We discuss implications of these results on teacher education.

Keywords: Teacher perspectives, mathematical knowledge for teaching, prospective secondary mathematics teachers.

Background and Rationale

Teachers' having mathematical knowledge for teaching is important (e.g., Rowland et al., 2005). In this regard, considerable amount of research conducted with both in-service and prospective teachers pointed that although focusing on what teachers have or lack has importantly informed the field (e.g., Wilson & Cooney, 2002), consistencies among knowledge, beliefs, and practices (prospective) teachers might hold need to be given further attention (Chapman, 2016).

Teacher perspectives as a robust construct is one of the ways to investigate such consistency. Researchers stated that perspectives "…can be thought about as … paradigms with respect to the development of mathematical knowledge...the term paradigm emphasizes the existence of internally coherent systems..." (Simon et al., 2000, p. 599). That is, a teacher's perspective (i.e., meaning making systems) underlies the teaching practices that indicate not only what teachers think about, know, believe and do but also everything that contributes to their teaching (planning, assessing, interacting with students) (Simon et al., 2000). Therefore, teacher perspectives go beyond understanding particular knowledge and beliefs in the context of practice of teachers in transition (Simon et al., 2000) which have affordances and limitations on their teaching (Jin & Tzur, 2011). Particularly, doing research with in-service teachers, researchers reported on teacher perspectives in a continuum between traditional perspective, perception-based perspective (PBP) and conception-based perspective (CBP) on mathematics, mathematics learning and mathematics teaching (Simon et al., 2000; Heinz et al., 2000; Tzur et al., 2001). Later, Jin and Tzur (2011) have placed the progressive incorporation perspective (PIP) between the (PBP) and the (CBP). Tzur et al. (2001) stated that teachers holding (CBP) act accordingly with the views of radical and social-constructivist epistemology such that knowledge is actively built by the learner; so knowledge is re-invented. And, individual and social mathematical learning are reflexively related. Therefore, CBP has two important aspects: First, teachers are aware of his/her current mathematical understandings being qualitatively different from their students' understandings (Jin & Tzur, 2011); and second, teachers focus on what students currently know rather than what they don't know (e.g., Heinz et.al., 2000). On the contrary, (PBP) like the traditional approach, views mathematics as part of the external world independent of the knower, compatible with Platonist view of knowledge (Jin & Tzur, 2011). Thus teachers expect their students to see mathematical situations in the same way as they do. That is, mathematics learning means coming to see a first-hand experience of mathematical reality shared by all through discovery. Mathematics teaching occurs through teachers creating situations that reveal the mathematical ideas. Teachers having this view, in contrast to having a CBP, focus on what students *do not* understand (Heinz, et.al., 2000). Jin and Tzur (2011) postulating an intermediate perspective (PIP) stated that PIP "underlies an integrated stance on knowing and learning---reflecting both 'existence outside the learner' (hence, teacher involvement) and 'dependent on what a learner knows' (hence, student active problem solving)". Mathematics learning is therefore an active mental process. Teachers' main goal is to create learning opportunities for all students to activate their existing knowledge related to the intended mathematics, the old incorporating the new rather than being transformed as in CBP. Thus, we concur that perspectives not only include the foundational knowledge one needs to hold but also how such knowledge is embedded into teaching.

Table 1: Placing PIP within teacher perspectives (Jin & Tzur, 2011, p. 20)

Perspectives	View of knowing	View of learning	View of teaching
Traditional Perspective (TP)	Independent of the knower, out there	Learning is passive reception	Transmission, lecturing instructor
Perception-Based Perspective (PB)	Independent of the knower, out there	Learning is discovery via active perception	Teachers as explainer (points out)
PIP (PIP)	Dialecticallyindependent and dependent on the knower	Learning is active (mentally); focus on the known required as start, new is incorporated in to known	Teacher as guide and engineer of learning conducive conditions
Conception-based Perspective (CBP)	Dynamic; depends on theknower's assimilatory schemes	Active construction of the new as transformation in the known (via reflection)	Engaging students in problem solving; Orienting reflection; facilitator

This takes us to mathematics knowledge for teaching in action (MKT) depicted in the Knowledge Quartet framework (Rowland et al., 2005). Knowledge Quartet (KQ) framework is based on four main categories (See Table 2). The foundation dimension relates to teachers' beliefs on the nature of mathematics and mathematics learning and teaching. It is also about teachers' knowing 'why' behind the mathematics they teach. Transformation regards teachers' presentation of ideas to learners in the form of illustrations, examples, explanations and demonstrations. Connection includes sequencing the material for instruction and awareness of the relative cognitive demands of different topics and tasks. Finally, contingency is the ability to make cogent, reasoned and well-informed responses to unanticipated and unplanned events during the lessons (Thwaites et al., 2010).

Table 2: The Knowledge Quartet: dimensions and contributory codes (Thwaites et al., 2010, p.86)

Dimension	Contibutory Codes
Foundation (1)	(1.a) theoretical underpinning of pedagogy (1.b) awareness of purpose (1.c) identifying pupil errors (1.d) overt display of subject knowledge (1.e) use of mathematical terminology (1.f) adherence to textbook (1.g) concentration on procedures
Transformation (2)	(2.a) teacher demonstration (2.b) use of instructional materials (2.c) choice of representations (2.d) choice of examples
Connection (3)	(3.a) making connections between procedures (3.b) making connections between concepts, (3.c) anticipation of complexity (3.d) decisions about sequencing (3.e) recognition of conceptual appropriateness
Contingency (4)	(4.a) responding to students' ideas (4.b) deviation from lesson agenda (4.c) teacher insight (4.d) responding to the (un)availability of tools and resources

So, we hypothesized that the three dimensions, the nature of mathematics, mathematics learning and mathematics teaching expressed in the perspectives corresponds with the (KQ) framework (Rowland et al., 2005). This is because as well the theoretical knowledge and beliefs related to mathematics and mathematics education are handled in KQ, the theoretical knowledge possessed by teachers is transformed into teaching through connections and the existence of contingency moments revealing students' thoughts, mistakes and difficulties. This implies a link between teacher perspectives and MKT; yet, there is few studies focusing on their relationship (Karagoz Akar, 2016; Bukova Guzel et al., 2019). Also, we hypothesize that (prospective) teachers holding different perspectives might depict different codes in KQ (Karagoz Akar, 2016). By the same token, even if (prospective) teachers depict the same codes referring to their MKT they might do so with having different reasons (Weston, 2013). Therefore, we conjectured that once prospective teachers had different perspectives, their MKT would reveal itself at different levels during their teaching. Scrutinizing the coherency between teacher perspectives and MKT is important: it might help uncover the reasoning behind (prospective) teachers' MKT. Diagnosing the reasons might provide teacher educators with particular steps to follow towards establishing more sophisticated perspectives and a full grasp of MKT on part of not only (prospective) teachers but also in-service teachers during professional development studies. With the results reported in this study we also aim to contribute to the field in the following ways:

First, how the practices of prospective mathematics teachers with PBP and PIP before, during, and after teaching will be depicted. Secondly, how such practices comprehensively revealing the relation between these perspectives' characteristics and the codes of KQ through empirical data, including before-during-after teachings and interviews, will be shown. It is in this respect that, in this study, we investigated the following research questions: What are the indicators that prospective teachers have a particular teacher perspective? How is their mathematical knowledge for teaching revealed in the act of teaching? How are prospective teachers' perspectives reflected in their mathematical knowledge for teaching?

Methods

Participants and Data Collection

In the larger study, participants were six prospective secondary mathematics teachers who were senior students at one of the universities, in which the medium of language is English, in Turkey. We chose to work with these participants since they volunteered to participate in the continuing set of interviews and the practice teaching sessions till the end of the study. Data from six prospective teachers depicted that four of them hold PBP perspective and two of them hold PIP perspective. For the report in this paper, we specifically depict data from two prospective teachers, Alin and Meryem for illustrative purposes as the data from them were providing context to examine the relationship between the two perspectives (e.g., (PBP) and (PIP)) and (MKT). For the larger study, six prospective teachers' two practicum teachings and interviews were videotaped and transcribed. Also, we conducted pre-interviews on their lesson plans, observed the teachings and conducted post-interviews upon completion of the teachings within the same week. For instance, in the pre-interviews, we asked the rationale behind prospective teachers' choice of learning goal(s), the tasks and how they consider the tasks they have chosen would allow students to learn meaningfully. In this paper, the reports will be on Alin's and Meryem's practice teachings. Alin taught an 80-minute lesson to the 10th grade students at a private high school. Alin had created a task for her students to make sense of the graph of the function $f(x) = ax^2 + bx + c$ in terms of the meaning of real coefficients a, b, and c. Meryem taught a 40-minute lesson to the 11th grade students at a public high school. Meryem had modified a task for her students to make sense of piecewise functions.

Data Analysis

We analysed the data using coded analysis (Clement, 2000). First, based on previous research results (e.g., Jin & Tzur, 2011; Simon et al., 2000) we created a chart regarding teacher perspective characteristics. Following, each researcher read the lesson plans, transcripts from the pre-interviews, the practice teachings and the post-interviews line by line, looking for participants' explanations regarding their perspective. Using the characteristics of teacher perspectives, we looked for their existing meanings. Once we spotted a line of explanation regarding their meanings in any of the data sources in terms of mathematics, mathematics learning, and mathematics teaching, we also checked other sources to possibly provide further evidence of such meaning. Then, we came together to have a consensus on the data set and our analyses going back to the whole data set to challenge our conjectures. Following, we created the table showing the frequencies of the teacher perspectives. The reason was that in different data sources, the same characteristic has been represented more than once.

For instance, the code PIP.1a1 was depicted five times within the lesson plan. Secondly, engaging in the same process using the codes from KQ we examined both this same data set and read further each of the data sources line by line to determine their MKT. Then, again we came together to have a consensus on the whole data set to challenge our conjectures. For example, the code KQ.1a was depicted five times in all the data sources for Alin. Finally, we wrote the narratives.

Results

Data showed that, Alin had a PIP and Meryem had a PBP. Also, compared to Meryem's practice teaching, the number of KQ codes from each data source from Alin's practice teaching indicated that Alin had shown more actions regarding the MKT in every aspect of the teaching. Now, based on some data from Alin and Meryem (the pre-interview data), we explicate how different perspectives and MKT are depicted and how some of the same KQ codes might reveal different reasoning patterns pertaining to different teacher perspectives. As mentioned earlier, Alin had created a task (PIP 1a) for her students to make sense of the graph of the quadratics functions, $f(x) = ax^2 + bx + c$ in terms of the meanings of coefficients a, b, and c. During the pre-interview, when asked how the lesson plan would possibly promote the learning goal Alin had in mind for her students, Alin stated the following:

Researcher: Why do you think this [lesson] plan will promote your students' learning?

Alin: I'm starting with the amount of change in $y = x^2$; therefore, students need to recognize the arms of the graph gets open and there is a decrease, I mean, there is a change in slope... Starting always with $y = x^2$, how this change is formed and how this change affects the graph, so thinking this point… I mean, my activity provides quantitative operations by playing with something existing in their mind that they know.

Researcher: You said playing with something they know, what do they know? Like could you explain one more time what is quantitative operations?

Alin: They know what $y = x^2$ is, what its roots are, how the change occurs in $y = x^2$, I mean, how the slope is changing and how it looks in the graph. However, they don't have any idea about what happens to the graph when "a" changes, because they don't observe $ax^2 + bx + c$ for changing "a" values. Therefore, the quantitative operations formed in their mind when they changed "a", I mean the thing they know in their mind, is like how the slope in $y = x^2$ is, how the amount of increase is, and drawing the graph…. we need to observe the change of "a" one by one, and keep "b" and "c" constant so that we can only be aware of the change in "a"…Let me say the amount of change in y in terms of x, rather than amount of increase, because "a" can be negative too. It is necessary for students to observe how the amount of change in y is changing. When "a" changed and x changed as one unit, they can compare the amount of change corresponding to y, so that they can have an idea about the shape of the graph, I mean the arms (referring to the parts of the parabola)… Actually, what I am learning is to compare the amount of change in y with respect to change in x as one unit for different "a" values.

In terms of MKT, data showed that Alin had an awareness of purpose for her teaching (KQ 1b). She also effectively analysed which mental operations students need to engage in to make sense of the effect of the coefficient "a" on the graph, in her own words: "compare the amount of change in y with respect to change in x as one unit for different "a" values". Her analyses revealed that she has a strong subject knowledge and theoretical background about coefficient "a" for quadratic functions (KQ 1a and 1d). Also, she anticipated the complexity of the concept: She planned her lesson in such a way that by keeping "b" and "c" constant, students' examining the change of "a" would be more efficient (KQ 3c). In addition, her consideration of graphical demonstrations such as starting with the graph of $y = x^2$ while students were examining the coefficient "a" together with the tables showing x and y values showed that Alin had knowledge about different representations (KQ 2c) and she could integrate these representations into her lesson with relations to each other (KQ 2d). Regarding the perspectives, data showed that Alin's statement about comparing the amount of change in the dependent variable with respect to the one unit change in the independent variable suggested that for Alin students might create an idea about the graph of the quadratic functions and its structure through their mental operations (covariation). This suggested that Alin considered mathematics as constructed on the learner's mind. Alin's planning her lesson hypothetically depending on her students' mental operations and actions (PIP 1a1) also suggested that she considered mathematics dependent on the knower. In fact, further data in her written lesson plan pointed to more evidence for this claim: She articulated how students might possibly reason on the questions in the task sequence for different values of a such as a=1, 2, 4 and 10. Alin wrote: "By giving "a" different values and obtaining related y values, this time students compare the respective rate of changes in y values for different values of "a". For the same change in x values, the rate of change in y gets bigger as absolute value of "a" increases. Meanwhile, in students' minds legs of the parabola gets steeper and hence the width of the opening of the parabola gets narrower." Data suggested that she was hypothetically envisioning how students would reason given different values of "a". In other words, she would expect students to go through the mental activity, the simultaneous comparison of the change in the variables x and y, so that they would know the reason behind the effects of the changes in the values of "a" on the graph (PIP 1a1). In addition, her choice of example y= x^2 to start the lesson indicated that she has chosen this example as a conceptual anchor to activate what they already know for the intended learning to take place (PIP1b). Alin stated "It is because $y = x^2$, I mean it will be easier for them to understand, to start with, they can start from what they can compare, like for that reason I did not include bx+c first, so that they don't get confused. This way, because $y = x^2$ stays on the symmetry axis, like this is what they already know, they do not have to deal with finding the roots, they can focus on the changes on "a" more easily". On the contrary, regarding the same interview question, Meryem stated:

Researcher: Why do you think this [lesson] plan will promote your students' learning?

Meryem: I will be asking "could you explain for each of the graph?" I mean, after they have written for the first graph, the second graph, the third graph, I think they will realize the sets of domain and range will differ, like for different intervals we will be writing them. I think they will realize this. I mean I will ask again after they work on the examples, "if anything gets your attention", "if you see anything similar in those examples?" If they can see, they can say for different intervals of domain corresponds to different intervals of range, then I will ask them how they can name it. Like I will ask for their guess, like they can say this or that. If they like they cannot say anything then I will ask them to write down domain and range sets for each example…If like noting comes from them, then I will tell them if we can call these functions as piecewise, "do you think this makes sense?" Then I will ask if they can support my explanation by different examples. I think they can say that. I mean I do not expect them to write it but they can see it in the graphs like they can explain that the graphs start and end at some points and then start again at the same point. I think they can give examples. Then I will summarize the lesson.

In fact, Meryem had planned to provide three problem situations to the students based on which she expected them to draw the graphs. Regarding MKT, Meryem had a sense of awareness of purpose such that she had modified the task based on her learning goal for the students (KQ1b). Similarly, once asked how she determined the leaning goal, Meryem stated that she had read about a research article on students' understanding of piecewise functions. The article was about how students made sense of domain and range of different functions. Such analyses also revealed that she had a strong subject knowledge and theoretical background about piecewise functions (KQ1a and 1d). She further stated "If I have enough time, like I will ask them to draw graphs of linear, quadratic and constant piecewise functions". Her choice of examples (KQ 2d) in her lesson plan together with her further planning of including different piecewise functions also indicated that she had sequenced examples supporting students' deductions (KQ 3d). In terms of the perspectives, though, Meryem expected her students to see a similarity among the graphs they would construct based on the problem scenarios. That is, after their examination of the graphs visually students would realize that the domain and range of those functions would differ. This suggested that Meryem had expected her students to see the mathematics obvious to her in those representations (PBP 1b). Her use of problem scenarios having similar characteristic suggested that she wanted to create a learning trajectory for students so that those problems would make the mathematical relationships as apparent to them as possible (PBP 1a). In contrast to Alin's planning of her lesson on her students' background knowledge, Meryem's focus was on mathematics of other students. That is, albeit important, Meryem's rationale for how she has chosen the learning goal for her students was based on some other students' understanding of the domain and range of different functions. This also suggested that she might have believed that mathematics as part of objective reality existed in those representations ready for students' perception through engaging in tasks that would allow them to 'see' and connect the intended ideas (PBP 1c).

Conclusion

This study investigated prospective teachers' mathematical knowledge for teaching and their perspectives. In particular, data from two prospective teachers having a PIP and a PBP revealed that prospective teachers having both PBP and PIP depicted all the codes in Knowledge Quartet; indicating that they held mathematical knowledge for teaching (MKT). Data from the pre-interviews both from Alin and Meryem had shown that they did not adhere to the textbooks, rather they focused on students' prior knowledge while they were deciding and arranging the lesson activities, they had a sense of choice of examples and sequence of ideas within the lesson. Similarly, they had strong subject matter knowledge and theoretical background about the topics they have taught. Yet, their reasons behind such knowledge were different: Alin's focus on students' mind activities and their abstracting the relationships between the covarying quantities (the dependent and independent variables) suggested that she thought of mathematics as dependent on the knower. Similarly, this suggested that Alin viewed mathematics learning occurring through students' own activities. On the other hand, Meryem had different reasoning behind such mathematical knowledge that she viewed mathematics as depicted in the graphs and examples she expected students to perceive. These results are consistent with Weston (2013) results, who found that although different prospective teachers demonstrated the same codes in KQ, the nature of such demonstration differed from one prospective teacher to the other in terms of how much of such knowledge they had. Results suggest that teachers with different teacher perspectives might depict a strong MKT albeit different reasons behind such knowledge. This therefore suggests the need to pay attention to teachers' mathematical knowledge together with their rationale behind it. Yet we acknowledge that the data came from only two prospective teachers. Therefore, we suggest further examination of MKT of prospective and in-service teachers having different perspectives. Also, since the results suggest that prospective teachers having PIP might have a strong rationale behind their MKT we suggest to promote at least the development of a PIP on the part of prospective and in-service teachers during professional development studies.

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The proposition of an analytical tool for the evaluation of mathematics teachers' diagnostic competencies in the noticing framework

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This paper proposes an analytical tool for the evaluation of prospective mathematics teachers' (PTs) diagnostic competencies through noticing critical incidents in hypothetical classroom situations (mathtasks). Data were collected from nineteen PTs attending an undergraduate course for one semester. Data analysis highlighted variations within the four characteristics of teachers' diagnostic competencies that are described by four internal levels inspired by how teachers notice. This analysis resulted to the analytical rubric we present and exemplify in this paper. We see the potential of the proposed rubric in research and mathematics teacher professional development.

Keywords: Noticing, mathtask, prospective teachers (PTs), professional development.

Introduction

Research has highlighted the significant role of critical incidents, which, according to the Goodell (2006), are classroom incidents that have the potential to trigger teacher reflections on students' mathematical learning. Such reflections have been connected to the development of diagnostic competencies, namely their competencies to interpret students' mathematical actions by identifying the rationale behind these actions (Prediger, 2010). Very often, teachers encounter unanticipated situations in their lessons to which they are expected to respond on the spot. Especially prospective teachers, with limited teaching experience, often face difficulties to give immediate interpretations and multifaceted responses when they needed. For that reason, teachers' education on ways of analyzing students' thinking and different teaching practices is pertinent (Grossman, 2011). Such education can be supported by the analysis of critical incidents (Psycharis & Potari, 2017). Also, studies report improvement in teachers' engagement with mathematical and pedagogical terminology when they interpret students' reasoning (Grisham et al., 2002).

Recently, researchers have been using videos, pictures or texts from classroom incidents in order to support teachers' observation of students' reactions (Prediger & Zindel, 2017; Sherin & Van Es, 2003; Van Es, 2011). The ability of teachers to observe students' mathematical thinking is attributed by Van Es (2011) to their ability of *noticing*. She argues that teachers need to learn how to observe and interpret classroom interactions that affect learning. Teacher noticing is analyzed in three dimensions: the monitoring of noteworthy events; their justification; and, their interpretation in order to make an appropriate teaching plan. The key element for teachers' noticing ability is whether the substantiation of their arguments is based on the principles of teaching and learning. Van Es proposed a two-component analytical framework for teacher noticing: *What teachers notice* and *How teachers notice*. The How teachers notice component can be categorized into four levels: *Level 1*, when the analysis of a fact they notice includes general impressions, providing descriptive and evaluative comments, with little or no evidence to support it; *Level 2*, when in addition to Level

1 the analysis includes some interpretive comments referring to noteworthy events or interactions as evidence; *Level 3*, when in addition to Level 2 the analysis includes interpretive comments and the elaboration of specific noteworthy events and interactions; and, *Level 4*, when in addition to Level 3 the analysis include connections between the events and principles of teaching and learning and proposals for alternative pedagogical solutions (Van Es, 2011).

Noticing has also been connected to the diagnostic competences of PTs through their interpretation of aspects they have noticed in familiar or non-familiar instructional episodes (Prediger & Zindel, 2017). Characterisation of mathematics teachers' diagnostic competencies is related to the work of Biza et al. (2018). In their work, Biza and colleagues design hypothetical classroom situations (mathtasks[1], see Figure 1) that are inspired by mathematical and pedagogical issues likely to occur in the mathematics teaching practice, and they invite mathematics PTs and in-service teachers to reflect on these situations. The classroom situations, although hypothetical, are designed with potential critical incidents in mind (in the sense of Goodell, 2006) that PTs are invited to notice, interpret, and propose intended actions. Mathtasks were used in the study we present here towards the familiarisation of PTs with critical incidents that may arise in their classroom as we explain in the next section. In earlier work of Biza et al. (2018), the analysis of teacher responses to mathtasks proposed a typology of four characteristics of teachers' diagnostic competencies in recognising the issues in the incident described in the classroom situation and in responding to students' needs: *consistency*, how *consistent* a response is in the way it conveys the link between the respondent's stated pedagogical priorities and their intended actions; *specificity*, how *contextualized and specific* a response is to the incident under consideration; *reification of pedagogical discourse (RPD)*, how *reified*[2] the pedagogical discourse of the response is in order to describe and interpret the pedagogical and mathematical issues of the incident and to propose appropriate actions; and, *reification of mathematical discourse (RMD)*, how *reified* the *mathematical discourse* of the response is in order to describe and interpret the underpinning mathematical content of the incident and to propose appropriate actions (Biza et al., 2018). The four characteristics are attentive to teachers' both mathematical and pedagogical discourses. However, the operationalization of the typology in analysis requires more transparency on the level of sophistication within each one of the four characteristics (e.g., what does justify a high level of RPD?). Such lack of transparency is addressed by this study that examines the research question: "what levels of variation can be identified within each one of the four characteristics of PTs' diagnostic competencies as evident in their responses to hypothetical classroom situations (mathtasks)". To address this issue, we propose an analytical tool that draws on Van Es's (2011) levels of *how teachers notice* in order to describe variations within each one of the four characteristics as we exemplify through the analysis of empirical data in the next section.

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[1]Mathtasks are designed in the context of the MathTASK research and development program on mathematics teachers' pedagogical and mathematical discourses (https://www.uea.ac.uk/groups-and-centres/a-z/mathtask).

[2] Reified means that the mathematical discourse (the mathematical content and practices PTs have become familiar during their studies) and the pedagogical discourse (the theories and findings from mathematics education research PTs have become familiar during their studies) have been integrated productively into PTs' responses.

Methodology

The research took place in the context of a 14-week mathematics education undergraduate course at a Greek university with nineteen PTs. The 19 PTs who participated in the research were also those who participated in the semester course.The condition for attending the course was the completion by PTs of mathematical courses and at least four courses related to psychology, philosophy and mathematics education. During the course, PTs engaged with school based activities – such as lesson observations, lesson planning, delivering sessions, noticing of critical events from the classroom and the interpretation of these events – and university based activities – such as introduction to theories and findings from research into the teaching and learning of mathematics, engagement with mathtasks and discussion on PTs' school school based activities. One of the aims of the course was the development of PTs' pedagogical discourse also through their engagement with research in mathematics education literature.

Reasoning

In a class of maths, students are asked to solve the following problem:
"Can you make the two columns of numbers below add up to the same total by swapping just two numbers between the columns? Explain why or why not."

1	7
3	2
8	4
5	9

The following conversation between students A and B takes place:

Student A: If I add up the number in the columns, I get totals of, umm…17 and 22. So we need to make these the same.

Student B: How about we just try swapping some numbers and see what happens!

Student A: Okay, let's try the top two numbers first…If we swap 1 and 7, we get new totals of 23 and 16. That's worse than before!

Student B: Let's try some others…what about swapping 5 and 7?

Student A: No, that gives 19 and 20.

Student B: We're getting closer, though!

Student A: What about if we swap two numbers that are close together, like 2 and 3?

Student B: Ummm…that gives 16 and 23, that can't be right.

Student A: We could be here doing this forever!

Student C joins the conversation.

Student C: Maybe it can't be done and we have to show why not.

Student A: How would we do that then? We can't try every single possible swap…that would take too long!

You have just heard this exchange between students A, B and C.

Questions: **Lesson for: (specify the class)**

a. Solve this mathematical problem. What is the main goal of this problem?

b. For what reasons do you believe that this episode is important? (from mathematical and pedagogical view)

c. How you would interpret this dialogue? (refer to the literature)

d. How would you respond to Students A, B and C and to the whole class?

In this paper we draw on PTs responses to the mathtask of Figure 1 (translated from Greek), the last of the three mathtasks used in the course as instances of potential critical incidents, that was given to PTs towards the end of the semester. PTs' responses were collected electonically one week after the assignment. The hypothetical scenario of the mathtask is based on an open mathematical problem, different from what is considered as usual in the Greek mathematics school curriculum. We expected PTs to analyze the goals of the activity, to identify the flexibility of using it in different classes (and specify the class), to notice the issues in students' dialogue, to interpret these issues and to respond to them accordingly. Thus, we aimed to trigger PTs' noticing in the given incident and through this to analyze and evaluate their diagnostic competencies. All the PTs who attended the course consented to the use of their work for research purposes. The research was implemented within the framework of the qualitative research methodology.

The data were coded with the analytical tool of a rubric (Andrande, 2000) with two-dimentions (see Table 1): the typology of the four characteristics (criteria) proposed by Biza et al. (2018) in rows; and within each of the characteristics, in columns, the four levels inspired by Van Es's (2011) levels of *how teachers notice* and concern quality differences within each characteristic.

Initially, the first author analysed PTs' responses in relation to the four characteristics by looking for quality differences within each characteristic. Then, the quality differences between these levels were described with an adaptation of Van Es's (2011) levels. The rubric in Table 1 is the outcome of this phase of the analysis. Finally, PTs' responses were analysed again with the use of the rubric. The other two authors validated the analysis in each one of the three phases described above. In this paper we present the rubric together with examples from the final analysis that concerns the RPD characteristic with some reference to the other characteristics.

Results

In Table 1 we present the rubric with the four characteristics in each row and the levels in each column. Then, we exemplify the RPD levels from the responses of PTs to the mathtask of Figure 1.

Table 1: The rubric of four characteristics

	Level 1: Irrelevant	Level 2: Superficial	Level 3: Evolving	Level 4: Multidimensional
Consistency	There is no consistency in the interpretation of the incident and the proposed actions.	There is consistency in the interpretation of the incident and the proposed actions. There are general references to the incident with superficial interpretation of what is happening in it.	There is consistency in the interpretation of the incident and the proposed actions. There are specific references to the incident with interpretations and identification of connections of what is happening in it.	There is consistency in the interpretation of the incident and the proposed actions. There are targeted suggestions based on evidence from what is happening in the incident, specific and/or alternative approaches, links to principles of teaching and learning related to them.
Specificity	There is no accuracy in the descriptions of the incident.	There is accuracy in the descriptions of the incident. There is a general reference to what is happening in the incident.	There is accuracy in the description of the incident. There are specific references to the incident with interpretations and identification of connections of what is happening in it.	There is accuracy in the description of the incident. There are detailed interpretations based on evidence of what is happening in the incident, which are connected to teaching and learning principles related to them.

RPD	Wrong, irrelevant or no use of pedagogical terminology.	Limited use of pedagogical terminology. There is general reference to pedagogical issues from the incident.	Good use of pedagogical terminology. There is interpretation of interactions happening in the incident with evidence, using pedagogical terms related to principles of teaching and learning.	Very good use of pedagogical terminology. There are interpretations and identification of connections in what is happening in the incident with reference to relevant literature of principles of teaching and learning and alternative pedagogical suggestions.
RMD	Wrong or no use of mathematical terminology.	Limited use of mathematical terminology. There is general reference to pedagogical issues from the incident.	Good use of mathematical terminology. There is interpretation of interactions happening in the incident with evidence, using mathematical terms and description of suggestions based on the mathematical content.	Very good use of mathematical terminology. There are interpretations and identification of connections in what is happening in the incident with reference to the relevant literature of the basic principles of teaching and learning and alternative suggestions focused on the mathematical content.

Exemplification of RPD levels

The Levels in Table 1 are inspired by Van Es's (2011) levels and the headings in the table were named in order to attribute the quality differences from one level to another. To exemplify the levels of the RPD characteristic we present characteristic examples from four different PTs' interpretations at various levels in terms of the RPD. The levels within each one of the characteristics are interconnected in the responses of each one of the PTs. Due to space limitation, we only exemplify the rubric from PTs' responses to questions b and c of the mathtask in Figure 1.

Specifically, at the *Irrelevant level* (L1), Mania responds to question b:

> From the teaching point of view, we can see how the students manage a problem without solution and how the need of the concept of the proof emerges, as student C quote**.**

Mania seems to recognize two teaching objectives: managing a problem without solution and the emergence of a need for proving. Then, in question c, she writes:

> The concept of proof is not a standalone concept, it comes together with the sense of "legitimizing a proposition" and "theory" [her quotation marks]. … It requires a substantial transition of the student to an epistemological state: the transition from a practical state (ruled by a kind of practical logic) to a theoretical state (ruled by the physical particularity of a theory).

Her response above includes text retrieved from the internet. This is not necessarily a problem when the right reference is used, which she has not done. Also, it seems that she is using terminology without connection to elements arising from the dialogue in Figure 1. Yet it is not clear, when she refers to "theory" and "epistemological state" if she means proof and, while she refers to "a practical state", if she means the tests that students make when they try to give a solution.

Mania’s responses are coded as L1 for the other three characteristics as well, which means that she does not interpret the elements from the dialogue, neither she proposes how she could manage this open-ended task in a classroom. Especially for L1 in RMD, she presents a few tests from MATLAB as solution and insisted on experimentation, which alone is not enough to help students to think of an algebraic explanation.

At the *Superficial level* (L2), we present part of Anna’s response to question b in which she highlights the pedagogical interest of the episode because:

> The dialogue, the exchange of views and the collaboration between students are encouraged in this episode. Moreover, the intervention of the teacher cultivates the mathematical thinking and ability of abstraction. It is an open-ended problem that allows students to take initiatives.

Then, in question c, she writes:

> [she describes a mathematical solution] This problem was given to the class with the aim [to make] students to distinguish the meaning of even and odd [numbers].

She notes that through an open-ended activity, dialogue and different ways of resolving are favored. Nevertheless, the comments in her interpretation are general without connection to points of the episode. Moreover, she doesn’t cite any reference from relevant literature. Overall, Anna focuses on the mathematical content both to her response the problem and to her interpretation of students’ answers, without making any reference or connection to the principles of teaching and learning. We notice that comments such as “exchange of views and cooperation are encouraged” or “teacher cultivates mathematical thinking and abstract abilities” and “open-ended problems allow students take initiatives” are general in terms of the pedagogical content and brief that they can be applied to other cases as well. In addition, she does not support her points with evidence from the incident under consideration and with references to student interactions. The difference between Mania and Anna is the relevance of Anna’s pedagogical comments to the critical incidents she identified in the dialogue. Moreover, in relation to the characteristics of Consistency and Specificity, Anna’s response is at L2. However, in relation to the RMD characteristic, her response is at L3: Anna’s interpretations are based on the mathematical content of the problem and she proposes specific interventions as a response to the difficulties of the students she identified in the dialogue.

At the *Evolving level* (L3), Vaso’s response in question b included the following:

> The teacher lets the students discuss the exercise with each other without intervening. This practice encourages the exploration and the exchange of views. The problem is a good example of a task that sharpens students’ mathematical thinking and curiosity. The students are expected to engage actively and try to think of a shorter way to solve it.

Here she points out that the teacher promotes the dialogue without guiding the students. Furthermore, she commends students’ involvement as important in the demanding activity of the incident. However, she neither makes any comment on students' approaches nor she makes connections to the dialogue. Continuing with her interpretation, she comments in question c:

> …students face difficulties in relating the proof with their attempts to find a solution. The literature confirms that a large percentage of students find it difficult to acknowledge the importance and usefulness of proof. This often happens because students do not understand why the proof validates the original claim or they do not yet fully understand the meaning of proof. In this example, proof occurs as a result of a conjecture-and-test process to solve a problem which turns into an algorithmic exercise. The students seem to find it difficult to move from conjecture to proof.

Finally, we notice that Vaso refers to basic principles of teaching and learning such as tasks of high or low demand, key points of the proof (tests, conjecture, and proof), students' difficulties in understanding the meaning of a task, student habits with algorithmic solutions. This indicates that Vaso has studied the relevant literature suggested by the course although she does not refer directly to it. This approach is more detailed and focused in comparison to Anna's. Vaso's interpretation is more justified as she uses evidence from the situation to describe students' interactions through pedagogical terms. In relation to the other characteristics, Vaso's response is at L4.

At the *Multidimensional level* (L4), Anastasia's response involves pedagogical terms connected with the incident and references to relevant literature. In question b, Anastasia noticed some significant points of the dialogue like the cooperation between the students through the discussion, the investigation through tests to lead to the conjecture and the proof. Moreover, she focuses on the type of activity and the flexibility of strategies it allows to the students. She also refers to time effectiveness, number of tests and the conjecture-proof relationship. Later in question c she writes:

> According to F. Furinghetti, open problems include activities with a short formulation that does not imply the solution method, but instead stimulate the production of conjectures and encourage discovery. As M. Mariotti points out, the teacher plays a key role in helping students dealing with such problems. […] Student A observes that the solution to the problem seems time consuming. This student concern could clearly be a sign of despair or lack of willingness on his part to be further involved in the solution process. On the other hand, it may be a way by which the student expresses the need to change the way they work, or try to come up with a shorter solution.[…] It is possible, however, that student A's observation that the problem is time-consuming may have caused student C to "think cunningly" and assume that something else might have happened. [...] In any case, he seems [student C] to have realized part - if not all - of the functions of the mathematical proof (according to Bell, Hanna and de Villiers), that is, to verify the truth of a proposition, to explain it, to contribute to the discovery and exploration of new situations, concepts and properties and to "communicate" the new knowledge. The evolution of the event is expected to be critical too.

Here Anastasia describes in detail what happens in the incident by commenting on students' reactions. Her answers reveal familiarity with the curriculum and the relevant literature on teaching and learning. She refers to the open problems that motivate the production of conjectures and she comments on the role of the teacher in managing such problems in the classroom, with references to relevant literature. The pedagogical terminology she uses is constantly in connection with points of the incident. The importance she gives to the interaction is crucial in order to provoke further dialogue about the mathematical proof. Overall, she approached the incident comprehensively by interpreting the students' reactions and their interactions and offered alternative pedagogical solutions. All the above with the fact that she connects the pedagogical terms with the literature, classify her response as L4. Finally, Anastasia’s response is at L4 for all the other characteristics.

Conclusion

In this paper, we address the lack of transparency on the level of sophistication of PTs diagnostic competencies and we proposed an analytical rubric that describes how teachers notice within each one of the four characteristics of PTs’ diagnostic competencies (Biza et al., 2018). At the broader research, in which this rubric was used, we explore the levels of the characteristics of PTs in three consecutive mathtasks, the third one (Figure 1) is discussed in this paper, in order to study the development of PTs diagnostic competencies across the course. The analysis evidence that the typology of the four characteristics and the proposed quality differences within them provide a detailed picture of PTs’ diagnostic competencies and their development as PTs moved from the first mathtask to the third one across the course. We note that such development is attributed to the course design that prioritized appropriate connections to the teaching and learning of mathematics literature and supported PTs’ reflective activities. Therefore, we would say that the results agree with studies that report the benefit of teachers’ interpretation of their students’ mathematical thinking (e.g., Psycharis & Potari 2017; Van Es & Sherin 2010). Also these results reinforce research findings that emphasize the critical role of the kind of intervention (in our case based on the use of critical incidents in hypothetical classroom situations) on the improvement of PTs’ diagnostic competence (Prediger & Zindel, 2017). The proposed rubric has affordances to map out PTs’ development across the course and to identify areas for further enhancement. Following Grisham er al. (2002) observations about teachers’ engagement with terminology, the co-existence of mathematical and pedagogical aspects in the MathTASK activities and the proposed rubric makes them valuable tools for PTs’ education and teacher professional development as well.

In this paper, we chose to present the levels of RPD in a classroom situation related to a mathematical problem that was not familiar to PTs due to the critical role of the teacher in such situations and the variety of responses we elicited in our data. The choice of the classroom situation and its impact on PTs’ response is of interest for future research. Moreover, we observed interrelations between the four characteristics. For instance, PTs who improved their RMD level across the course, they demonstrated similar improvement in the Consistency levels of their responses. In the future, it would be interesting to see in more detail the factors that influence the level change of one characteristic in comparison to the level change of another.

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These tasks are very good but inappropriate for my students

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This paper investigates how teachers reflect on and explain the role of high-quality mathematical tasks when selecting tasks for use in lessons. By analysing data from three groups of mathematics teachers engaged in collegial discussions, this study aims to elucidate how teachers rationalize the role of high-quality mathematical tasks. The results indicate that teachers appreciate high-quality tasks in providing discussions among students and supporting the collaborative efforts to solve problems. Conversely, despite this appreciation, teachers refer to such tasks as inappropriate for their students. For this reason, they pointed to the capabilities, lack of motivation to engage, and the lack experience of their students.

Keywords: Professional development, collegial discussion, high-quality tasks.

Introduction

The reform movement in mathematics education, which emphasizes the learning of additional mathematical competencies apart from procedural fluency, typically encourages the development and reorganization of syllabi, curriculum materials, and classroom practices. A central policy initiative that facilitates change frequently involves professional development (PD), which aims to support teachers in establishing productive classroom practices. A typically important component of such PD programs is the collegial discussions of teachers (Cobb & Jackson, 2011; Cobb et al., 2018). Within PD programs, collegial discussions may be viewed as a means for facilitating teacher development, which, in turn, is conceptualized as a means for improving and changing classroom practices (Desimone, 2009). According to Munter (2014), collaborations between teachers are less effective unless they share vision of high-quality instruction that gives meaning and purpose. High-quality instruction can be defined in three related dimensions of classroom instruction (Munter, 2014). The first is the role of the teacher, where the teacher supports students in learning mathematics by facilitating understanding. The second is developing a classroom community. Teachers are responsible for orchestrating discussions, such that students can share multiple problem-solving strategies, analyse relationships among strategies, and explore contradictions in ideas to provide more insight into mathematical thinking. The third dimension is the role of mathematical tasks. High-quality tasks should support students in developing problem-solving strategies (Hiebert et al., 1997) and should hold the potential to "engage students in solving challenging, ambiguously defined problems without the suggestion of a particular procedure or path to a solution" (Munter, 2014, p. 607). A central aspect of a concept labeled "ambitions instruction" (Kazemi et al., 2009) is using cognitively demanding tasks to challenge students. Engaging students in cognitively demanding and challenging tasks is characteristic of a reform-oriented approach to mathematics instruction. The study focuses on the third dimension, that is, the role of mathematical tasks and, specifically, how teachers rationalize the role of mathematical tasks in collegial discussions as they engage with a PD program in mathematics education. Investigating the explanation of teachers regarding the role of mathematical tasks when selecting tasks to use in lessons is important for understanding their reflection on the role of mathematical tasks. In this context, the following research question guides this paper: "How do three groups of teachers, who are participating in a PD program in Sweden, rationalize the role of mathematical tasks in collective planning with colleagues?" I seek to answer this question by analysing three groups of mathematics teachers engaged in collegial discussions as part of a national large-scale PD program in Sweden. In the context of the current study, I understand collegial discussion in a pragmatic manner. In other words, when teachers work in teams with the support of the large-scale program, the boost in mathematics is designed to support such groups of teachers to engage in collegial discussions regarding resources as well as planning lessons and collective reflections on classroom instruction.

Relevant research

The notion of high-quality or challenging tasks is relatively common (Ingram et al., 2020). Through tasks in the mathematics classroom, compared with other methods, opportunities to learn are made available to students (Munter, 2014). Therefore, considering the mathematical ideas behind a task, the potential to engage students in solving such challenging problems, and possible solutions, strategies, and misconceptions that students may provide when attempting to solve a task is important prior to teaching (Munter, 2014). Based on the importance of high-quality tasks in mathematics teaching and learning, scholars have investigated the justification and characterization of teachers regarding the tasks they opt to use in terms of the potential of the task for students' work. For example, Heyd-Metzuyanim et al. (2019) interviewed two teachers as they participated in a PD program. The authors found that the main justification of teachers for selecting tasks was their location in a certain place in the curriculum, instead of mathematical goals. Another justification for selecting a task was that the task would lead to a discussion. However, the two teachers did not explicate the nature of this discussion during the interview. Through analyses of teacher interviews, Sun (2019) examined the beliefs of four teachers about mathematical tasks. The author found that their beliefs are frequently related to the concept that certain forms of mathematical activity are not viable for certain groups of students due to their different innate abilities. Thus, students with low achievement tend to be excluded from engaging in high-quality tasks.

Researchers also distinguish between high- and low-quality tasks. Cobb et al. (2018) investigated the aspect required to support the development of ambitious instructional practice among teachers. One of Cobb et al.'s. (2018) perspectives in this large-scale PD program was the nature of the task. One distinction is whether a task is of low or high cognitive demand. For tasks with low cognitive demand, students apply known procedures. Thus, little ambiguity exists in solving such tasks. High-cognitive tasks are frequently open-ended and can be solved using various strategies. In other words, students tend to struggle with such tasks for a certain period without intervention from the teacher. One of the findings by Cobb et al. (2018) was that maintaining the cognitive demand of a task is challenging for teachers. As a result, they frequently reduce the challenge of the task over the course of the lesson. Their views on high-quality tasks are that these tasks do not align with their structure of the lesson. Moreover, these tasks are considered inappropriate for the students. Munter (2014) developed a framework for characterizing the perceptions of teachers toward high- and low-quality tasks on the basis of more than 900 interviews. In this manner, he modeled the trajectories of the perceptions of high-quality instruction along the findings in the literature. At the lowest level, teachers fail to view tasks as being of high or low quality. At the next level, the responses of teachers suggest that tasks can vary in quality. However, those performed by the students should first enable procedural practice before problem-solving and application. At the third level, teachers refer to more sophisticated descriptions of high-quality tasks, such as tasks that require multiple solutions or support the conceptual understanding of students. At the highest level, teachers refer to the rationale that high-quality tasks support students in learning and doing mathematics, such as making and testing conjectures, opening up for examining, and comparing several strategies. To better understand how teachers discuss and reflect on the role of mathematical tasks, I have opted to focus on the collective planning of teachers with colleagues during a PD project.

Method

The Swedish National Agency for Education launched a curriculum-based PD project. Called "Boost for Mathematics[1]" (Skolverket, 2018), this project intends to improve the teaching of mathematics. Its major components are 24 modules, where eight are disseminated per grade level (1–3, 4–6, and 7–9). Each module focuses on certain mathematical contents, the manner in which students learn these contents, and how teachers can support learning. A central part of these modules is high-quality tasks.[1] Each module presents several high-quality tasks and encourages teachers to discuss these tasks, such as selecting which ones to use in lessons and adjusting the task to be suitable for their class. The curriculum, which is distributed digitally on a website, includes articles, instructions, high-quality tasks, and videos. Each module is designed to support groups of teachers in engaging in eight iterations, comprising individual preparation, conducting collective planning with colleagues, teaching individual classrooms, and facilitating collective reflections in classroom instruction. This study focuses on the collective planning of teachers with colleagues at three selected schools. The selection process was based on two factors. Selecting one group of teachers from each of these three grade levels was convenient, because the data material was intended teachers from each grade level (i.e., 1–3, 4–6, and 7–9). The other process of selecting groups was including groups that opted to study the module in terms of problem-solving, which included high-quality tasks. Data were collected by videotaping two meetings with each group for a total of six sessions. The first meeting was based on collective planning with colleagues during the first semester, whereas the second was based on collective planning with colleagues during the second semester.

Analysis

To understand what teachers engage in when focusing on high-quality tasks in collegial discussions, as previously described, I have chosen to deeply examine the collective planning of the three groups of teachers for their classroom instruction. As part of data reduction, I identified and transcribed all discussion episodes that involved the teachers in the discussion of tasks. For analysis, I defined an episode of pedagogical reasoning as a coding unit:

> Units of teacher-to-teacher talk allow teachers to exhibit their understanding of an issue in their practice. Specifically, episodes of pedagogical reasoning are moments within teachers' interactions in which they describe issues in, or raise questions about, teaching practices that are accompanied by some elaboration of reasons, explanations, or justifications. (Horn, 2007, p. 46)

Episodes of pedagogical reasoning, in which the teachers explicitly discussed the tasks presented in the PD, were analysed. The abductive process was used to develop the analytical framework for the research (Bryman, 2016). Inspired by the framework of Munter (2014), I made modifications to their

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1 The modules and the high-quality tasks: https://larportalen.skolverket.se/#/moduler/1-matematik/alla/alla

categories to create a total of three categories. Developing the analytical framework has been a continuous process, which required moving back and forth between the data and the analytical framework. Gradually, I made clear distinctions between different categories with a focus on entire episodes of the discussions among teachers regarding high-quality tasks. Munter (2014) categorized the different views of teachers about high-quality tasks. At the first level, they are aware that tasks can vary in quality; however, students require procedural practice before working with high-quality tasks. In the data material, several utterances occurred about the difficulties, limitations, and inabilities of students in working with high-quality tasks. These utterances were categorized as appropriateness for the students. At the next level, the teachers described the nature of tasks as being oriented toward reform. However, they fail to describe a function or describe it in terms of increasing interest levels and student engagement (Munter, 2014). Moreover, teachers were concerned about leisure gained from tasks and their potential to lead to discussions. Such views were categorized as the function of tasks. At the highest level, Munter (2014) described the view of teachers about high-quality tasks to support student learning and performing mathematics and to provide content for the entire class discussion. In the data material, the teachers were concerned about the structure of the lesson and the presentation and discussion of tasks across the phases of the lesson. These concerns were, therefore, categorized as the structure of the lesson. The results section provides further elaboration on these categories. Table 1 demonstrates the overview of schools and number of episodes identified for each category.

Table 1: Analysis of video materials — an overview

School (pseudonyms) and modules	Teachers (pseudonyms)	Video-Recorded Meetings.	Function of tasks	Structure of lessons	Appropriateness for students
Rafford 1–3. Problem-solving and number sense	Amy, Maria, Helen, Grace, Julie, and Hannah	First session: 50 min. Second session: 49 min.	3	5	12
Hadlow 4–6. Problem-solving and number sense	Mary, Mona, Celia, Fred, Josie, and Nicole	First session: 83 min. Second session: 55 min.	3	5	6
Padstow 7–9. Problem-solving and teaching mathematics using IT.	Emely, Michael, Lily, Tyler, and Stella	First session: 63 min. Second session: 57 min.	5	3	9

Results

This section presents an elaborate characterization of the rationalization of the three groups of teachers with regard to high-quality tasks during their participation in the PD program for mathematics education. The results from the three teacher groups were combined and discussed in the following sections in terms of appropriateness for students, the function of tasks, and structure of lessons.

Appropriateness for students

This category was the most dominant when the teachers discussed high-quality tasks. Although the teachers agreed that high-quality tasks are beneficial, because they support the conceptual understanding of students and enable multiple solutions, they were relatively concerned regarding whether the tasks were appropriate for students. Nearly all arguments involved these tasks as being extremely difficult for students. I categorized these arguments of the appropriateness of high-quality tasks, which resulted in five aspects, namely, 1) students are only concerned with one correct answer and are satisfied when they arrive at a solution; 2) students do not challenge themselves but want an immediate answer; 3) students lack the patience to work on a task over time; 4) students are *locked* and unable to *think outside the box*, an aspect frequently required by such tasks; lastly, 5) students are preoccupied with the mathematics textbook and believe that all work apart from those that involve the textbook is not mathematics. For these reasons, the teachers explained that the students are unwilling to work with high-quality tasks. The following excerpt illustrate teachers refer to students being extremely *locked* and unable to *think outside the box*:

Mary: What about these tasks? How do you think the students are able to place a number of given fractions on a number line? What would it look like in your groups? Are they able to put them on a number line?

Mona: I think this is very difficult, at least for my students. Many of these tasks, it feels like it's too high a level. So, it's good for us to think, maybe, but it's not, not for my students anyway, so it feels too difficult.

Celia: No, it's too difficult.

Mary: Is it too difficult?

Fred: Yes. And my students are too locked, they'll get stuck, they're not able to think outside the box.

Mary raises an issue about her students' ability to mark fractions on a number line. This scenario is viewed as extremely difficult for Mona's students with agreement from Fred and Celia. The difficulty is partially related to the students' abilities and partially to the high level of task difficulty. As Fred mentioned, these students are too locked and unable to think outside the box, which Fred believes is a demand of such high-level tasks.

Function of tasks

The teachers presented three main arguments regarding the function of tasks. Two of these, which are the most common, are that the tasks should be fun for students and that the tasks should lead to and open discussions. In these categories, the teachers did not emphasize how these tasks can support the learning of mathematics. In one case out of all arguments, however, they discussed how tasks can support student learning, which is illustrated by the following excerpt:

Stella: We want to capture students' knowledge, whether they know this or that. Then we should choose the tasks based on that, I think, what we should work on or think that we benefit from getting to know about them. What gives us the most. How I as a teacher have intended to continue working, so that I choose a task that suits how I think I can continue working with it later.

In this case, Stella is looking for assignments that reveal the knowledge of students in mathematics and how the assignment can be used as a starting point for a further understanding of mathematics. This excerpt is an exception to the arguments that the teachers made regarding the function of tasks. The most common argument was that the tasks should be fun and motivating for the students, as shown in the next excerpt:

Josie:	This one might not be so exciting for them, so they should … this with decimals, they should just …. I think they would think this was cool.

This case is an example of the fairly common argument that teachers use tasks that they consider fun for students. Thus, no argument was raised about the type of mathematics that students should work with or how the tasks can lead to learning.

Structure of lessons

A three-phase classroom activity structure, namely, the phases of introduction, students working on mathematical tasks, and finalizing the lesson, was recurrent in the collegial discussions. The teachers frequently referred to this structure as introduction, pair or alone, and all. According to Jackson et al. (2013), a common lesson structure in a reform-oriented mathematics curriculum is the three-phase lesson (which these teachers refer to), where a complex task is introduced, students work on solving it, and the teacher orchestrates a conclusive discussion with the entire class. This structure is typical of lessons in PD. These descriptions of the classroom structure espouse the reform-oriented view on the structure of lessons, although such characterizations may fail to describe the introduction of the task and the content of the interaction among students. According to the content of the interaction, the teachers are more concerned about holding a discussion or the tasks leading to a discussion instead of the quality and content of the discussion based on high-quality tasks. Furthermore, in the discussions, they frequently emphasized lowering the cognitive demand for high-quality tasks during their introduction:

Helen:	Actually (...) what should I do, should we do a problem first together or should we just (..).
Amy:	Don't you think yours (students) can do one?
Helen:	Yeah, some of them.
Hannah:	Mmm (...) mine are rather weak.
Helen:	But eh (Amy (C): mm), yes but we'll do one, we'll formulate a problem based on an image.
Maria:	Yes, you do a similar problem then (Hannah: mm).
Julie:	With your group.
Maria:	So they (Helen: mm) have a similar (...) structure to follow (Hannah: mm).

Helen expresses concern about whether the students should tackle tasks/solve problems immediately or if the lesson should start with an introduction. The teachers plan to introduce a problem based on arguments about the capacity of the students and the necessity for a teacher to provide structure for them. Other studies (e.g., Boston & Smith, 2009) have demonstrated that teachers experience difficulty in maintaining the cognitive demand of tasks during teaching. As seen in this example, the teachers are concerned about the difficulty of high-quality tasks. Therefore, they decide to lower the cognitive demand of the task to ensure that each student has the opportunity to work with it.

Discussion and conclusion

This article contributes to research on the collegial discussions of teachers engaged in a PD program on teachers' development as a process of change toward a reform-oriented educational practice (Cobb et al., 2018; Jackson et al., 2013). Specifically, this study adds to the literature by highlighting the views of teachers about high-quality tasks. Analysis indicates that the teachers are relatively ambivalent about using high-quality tasks in their lessons. On the one hand, they share certain elements of high-quality instruction that are aligned with the reform-oriented teacher practice. In other words, they emphasize and appreciate high-quality tasks as they correspond to their structure of the lesson according to the three-phase lesson structure called introduction, working together in pairs or alone, or both and summaries the lesson in whole-group discussions (Jackson et al., 2013). Within this lesson structure, the teachers are aware of the value and importance of high-quality tasks in promoting discussions among students and supporting their collaborative efforts to solve problems without relying on the teacher for explanations or to offer solution strategies. They emphasize high-quality tasks, because such tasks will support students' discussions better than low-quality tasks. Such discourse communities are unlikely to develop unless students gain opportunities to engage in rich mathematical work, which is typically initiated by a high-quality task (Munter, 2014). This notion forms part of a reform-oriented classroom practice (Cobb et al., 2018). An important aspect of reform-oriented teacher practice is the use of cognitively demanding tasks (Kazemi et al., 2009). On the other hand, although these teachers appreciate high-quality tasks, they stated, nearly in unison, that they referred to such tasks as inappropriate for students. For this reason, they blamed the capabilities, lack of motivation to engage, and lack of experience of their students with such tasks. The teachers' discussion about the role of high-quality tasks may be helpful in understanding their potential for learning in collaborative meetings to improve and change classroom practices (Desimone, 2009). The current findings reveal an ambivalent vision regarding high-quality tasks in relation to the reform-oriented teacher practice (Jackson et al., 2013). Such ambivalent views may influence the potential offered by implementing the reform-oriented classroom practice, given that teachers hold an unproductive framing of the capabilities of their students (Jackson et al., 2017; Sun, 2019).

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Teachers’ development of self-regulation and problem solving skills in online forums as learners and mentors

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The current study explores the learning processes occurring in online discussion forums, from a perspective of self-regulation learning which facilitates in planning, monitoring and reflecting on learning. Participants were 47 teachers, who took part in a 2-year mathematics teachers professional development program. The study was conducted in two stages. In the first stage the teachers acted as learners through collaborative solution of complex mathematical problems in small groups, while in the second stage they led collaborative problem solving in forums as mentors. The study showed that problem solving forums have a high potential for developing teachers' own self-regulation skills and increasing their effectiveness in collaborative problem solving. This experience is reflected in their pedagogical activities, aimed at developing self-regulation skills of their students. It also empowers teachers to support students in solving complex mathematical problems.

Keywords: Problem solving, online forums in social networks, self-regulated learning, professional development of teachers.

Introduction

Currently, the mathematical community has largely reached a consensus on the importance of mathematics problem-solving (PS) as a central goal for mathematics teachers. Mathematics problem-solving involves active role of students, which are involved in a discussion with each other for solving the mathematical problem. Technology allows the application of such collaborative mathematical problem-solving through the use of Problem-Solving Forums (PSF). Studies of the effectiveness of PSF for teaching mathematics has shown that PSF has great teaching potential due to a number of key features such as a shared discussion space, opportunity for asynchrony communication, or preservation of the content of the discussion for a long time. The virtual property of PSF significantly expands the boundaries of classroom both in the space and in time (Kramarski, 2012). Studies show that over time, students participating in PSF develop norms of collaboration, the most important of which is the habit of sharing the ideas in the process of solving problems, rather than publishing the final result or the complete solution (Koichu & Keller, 2019). However, research has shown that realizing the potential of discussion forums is challenging (Koichu & Keller, 2019). One of the main reasons for these difficulties, according to the literature, is the lack of skills in non-traditional forms of education for both teachers and students (Kilpatrick, 2016; Lester & Cai, 2016; Schoenfeld, 2013).

Participation in PSF requires students to be motivated, able to plan, track ideas and results, and reflect on the work done. All of these skills are key components of Self-Regulatory Learning (SRL) (Pintrich, 2000; Zimmerman, 2000). Thus, the effectiveness of the learning process, which includes PSF as an integral part, requires the teacher to use teaching methods that focus on the development of SRL skills and use them in collaborative problem solving (Hogan, et al., 2015). A number of studies have shown that one of the priority ways for teachers to acquire the skills needed to successfully implement new methods is to first test those methods on themselves as learners (e.g. Kramarski & Kohen, 2017).

Specifically, it was found that projects aimed at developing SRL skills in teachers also develop their pedagogical skills focused on developing SRL skills among their students. This approach to education has been called self-regulated teaching (SRT) (Kramarski & Kohen, 2017; Kramarski & Mikhalsky, 2009). The purpose of this study is to study the processes of forming SRL and SRT skills among teachers who participate in PSF, first in the role of learners, and t-hen in the role of mentors.

Theoretical framework

Collaborative mathematics problem solving

In recent years, collaborative problem solving in small groups as form of learning based on problem solving has been widely studied. Proponents of this method argue that it is based on mathematical discourse that fosters mathematical thinking (Hogar & Harney, 2014; Shindler & Bakker, 2020). The effectiveness of group work largely depends on the type of interactions between its participants (Carlson, & Bloom, 2005). Clark and colleagues have suggested a taxonomy of interactions in the process of collaborative problem solving in two dimensions. The first of these are questions directed to another member or the whole group, such as questions to clarify or define the group's status in moving towards an intended goal. The second-dimension concerns group synergy, which the authors define as long-term collaboration in which participants continue and develop each other's ideas. According to the researchers, the group progresses in learning when group synergy is the most significant interaction in its activities (Clark, et al., 2014). Modern technologies allow using online forums in social networks as one of the possible platforms for collaborative solving of math problems in small groups. This is possible due to the characteristics of online forums as having a common space for discussion, asynchrony, preservation of discussion content and virtuality.

Self-Regulated Learning (SRL)

In this study, SRL is defined according to Pitrich and Zimmerman, as a cyclic recursive process combining motivational, cognitive and meta-cognitive components (Pitrich, 2000; Zimmerman, 2000). In this definition, cognition refers to direct mathematical actions aimed at solving a problem; meta-cognition refers to the ability to understand and control cognitive processes, and motivation refers to the student's attitude to his/her abilities to solve a problem. According to Schoenfeld (2013), the meta-cognitive aspect – including the ability to plan, monitor and reflect on the work done – is especially important for the successful solution of mathematical problems. To successfully tackle complex challenges in collaborative learning, in addition to each student's SRL, meta-cognitive collaboration is required, where team members or groups act as a unit, defining common goals and choosing common strategies (e.g. Hogar & Harney, 2014). Research shows that SRL skills are not acquired spontaneously. Their development requires an environment, targeted learning, and guidance referred to as Self-Regulated Teaching (SRT) (Kramarski & Michalsky, 2009).

Many studies point out the need of developing advanced training programs for mathematics teachers, that will help them develop the SRL skills of their students (Hoover et al., 2016; Kramarski & Michalsky, 2009). In recent years, a theory of the teacher's dual role in the acquisition of SRL skills has been developed. The theory has proven the effectivity of developing SRL skills in teachers, to increase their ability to develop these skills in students (e.g., Kramarski & Kohen, 2017). Research on teacher professional development (PD) in online environment, which has become widespread in the past few years, has shown that the online environment has a positive effect on the development of teacher skills in SRL (e.g., Broadbent et al., 2021).

Research questions (RQs) are: a) How do teachers' SRL and collaborative PS skills evolve during their continued participation in PSF as learners? and b) How are the teachers' SRT and PS skills reflected in leading a PSF as mentors?

Methodology

The study took place in two stages. At the first stage, within the framework of the course "Fundamentals of Geometry: Plane Transformations", each teacher participated in six PSF meetings as a learner in a group of four. Each meeting was devoted to joint discussion and solution of a complex geometric problem. At the second stage, within the framework of the course "Methods of teaching mathematics", each of the participants acted as a mentor (teacher) in two forums, where students, under their guidance, also solved complex geometric problems. Throughout the study, teachers received targeted SRL support by the facilitator of the PD program, through collaborative discussions on forum processes, and guidelines during the forums themselves.

Participants and research background

The study was conducted as part of a 2-year Mathematics Teacher PD Program at the Faculty of Education in Science and Technology at the Technion, Israel. The program aims at expanding the mathematical and pedagogical knowledge and skills of teachers, for the goal of reaching a teaching license that will enable them to teach all levels of mathematics. The study involved 47 teachers with 5 to 15 years of experience, teaching in high schools in the north of the country. This paper focuses on the activity of one of the teachers, named John. John (35 years old) is a math teacher at a school in the north of Israel with 12 years of experience. John was selected as a case study for two reasons. On the one hand, his activity in the forums in which he acted as a learner, which was evaluated by the number of statements he posed, was similar to the average number of statements posed by the entire group of teachers. On the other hand, John's example shows clear patterns of change relating to his PS and SRL skills, as was reflected in the forum in which he acted in the role of a mentor.

Data and data analysis

In the course of the study, 96 protocols of the forums' work were obtained and analyzed. Of these, 72 protocols of the work of the forums, in which teachers acted as learners (12 groups with a permanent composition) and 24 forums, in which some of the teachers acted as mentors. This paper focuses on the analysis of the PSF protocols of three forums of John's group, where John and the other participants act as learners in three forums (with the facilitator of the PD program as a mentor) and one forum in which John acts as a mentor, while the other forum participants are in the role of learners. For responding the first RQ, we present PSF protocols analysis that reflects the development of all participating teachers' SRL and PS. For responding the second RQ, we focus the analysis of the PSF protocols on the activity of John.

We have applied directed content analysis (Hsieh & Shannon, 2005) to code forums in light of SRL and collaborative PS theories. For the research unit, the message (post) sent by the participant was selected. Most of them were text messages, but sometimes participants posted a photo of a drawing or part of a solution. This method of analysis allowed us to analyze the data qualitatively as well as quantify it as described below. First, we distributed the forum to posts, where each post was assigned two codes – one for collaborative PS and one for SRL. For collaborative PS, data analysis was based on the taxonomy proposed by Clark et al. (2014). This study examines the success of the PSF in terms of the level of group synergy achieved in the forum. There was evidence of group synergy in cases where a group of related posts from several participants discussed ideas for solving a problem. We calculated the ratio between the number of posts included in the synergy chains and the total number of messages on this forum. On the SRL aspect, forum posts were codified in accordance with the definitions of SRL models by Pintrich (2000) and Zimmerman (2000). We determined the level of SRL in each of the forums, both as a whole and per individual components. The SRL attributes were defined as follows: a post was classified as motivational if it indicated the participant's attitude to his knowledge, capabilities, and teamwork in the forum. A post was classified as cognitive if it contains specific mathematical actions aimed at solving a problem, such as: in my opinion point B will move to B'. With regard to metacognitive attributes, it was determined that a post refers to planning if it concerns the general analysis of the problem, the choice of the solution strategy, the definition of the general and intermediate goals, such as: Shall we start rotation around the center of a circle? Monitoring attribution was assigned to a post if it evaluates progress towards a solution in terms of correctness and effectiveness in achieving a goal, such as: Moment, checking. So, after the rotation K moved to M, then… Finally, a post was classified as reflection if it includes a retrospective look at the process of solving a solution problem, such as: The problem is interesting in that its first part solves a standard with a section of means. After assigning a classification for each message, we calculated the ratio between the number of messages for each component and the total number of messages in a given forum. These values are presented in the findings as percentages.

Findings

We will present three forums out of six in which John participated as learner: at the beginning (PSF1), in the middle (PSF3) and at the end (PSF6) of the part of the study in which the teachers were in the role of learners. All problems proposed for discussion in the forums were carefully selected in accordance with the criteria specified in the theoretical base and were equal in complexity. As an example, we will give a problem that was proposed to the participants for discussion in the first forum: "Given a square ABCD. Point K is on segment BC, and point M is on segment DC. Line AM bisects the angle KAD. Prove that the length of the segment AK is equal to the sum of the lengths of the segments BK and DM". The following table summarizes the performance of the group of teachers that John was a member of throughout the study.

Responding to the first RQ request, Table 1 and Table 2 present the SRL attributes and group synergy scores (respectively) that were assigned to all posts made by the entire group of teachers in these forums. Data is presented in percentage of the total number of posts in the presented forum.

The data indicate that with long-term participation in the PSF, the weight of group synergy in the discussion increases significantly, in parallel with increase in the number of messages related to the collaborative PS. In the final forum, almost all posts are related to SRL: 48% of them are meta-cognitive, while in the initial forums only 28% of posts were meta-cognitive.

Table 1: Indicators of SRL attributes at the initial, middle and final stages of teacher participation in forums as learners (P = planning; M = monitoring; R = reflection; Other = non-SRL messages)

SRL components		PSF 1	PSF 3	PSF 6
Metacognition	P	5	13	17
	M	14	16	17
	R	9	3	14
gCognition		23	22	24
Motivation		19	22	24
Other		30	24	4

Table 2: Indicators of group synergy at the initial, middle and final stages of teacher participation in forums as learners

PS interaction	PSF 1	PSF 3	PSF 6
Group synergy	9	32	61
Others	91	68	39

Moving forward to John's case, we might delve into these changes, as well as responding to the second RQ that aims to explore the SRT and collaborative PS skills of John's who were reflected in the forum in which he took part in the role of a mentor. Figure 1 demonstrates the dynamics of changes in the SRL components and the participation in the group synergy of John as an individual teacher. This figure contains characteristics that describe the forums in which John was in the role of learner, as well as the one forum he was in the role of mentor. The data in the figure is presented as the percentage of John's posts that we attributed to a certain type when analyzing the data, out of the total number of John's posts in the presented forum. For example, 66% of John's posts in the first forum had been classified as "Cognition". An additional column shows John's participation in the group collaboration (the percentage of his posts that make up the group synergy of all his posts).

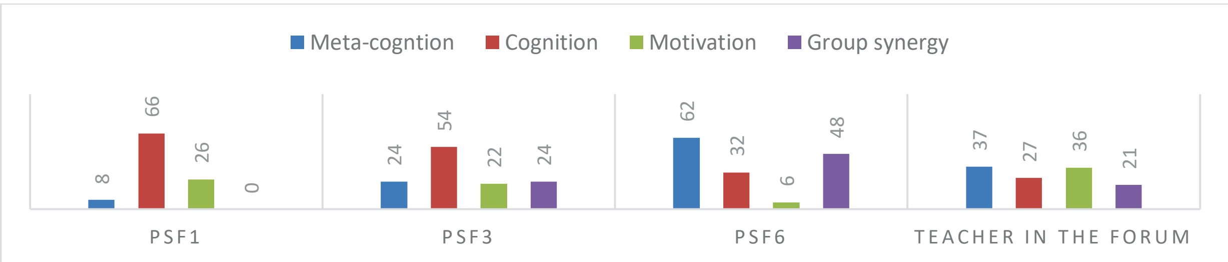

Figure 1: Jonn's activities in PSF as a learner and as a teacher

Figure 1 shows that, as in the group as a whole, an increase relating to the meta-cognitive component was seen in John’s posts, in the forums where he participates as a learner. While low indication for metacognitive activity was revealed in the first forum, in the third forum over 60% of his posts were characterized as metacognitive. Another increase had been seen in the participation of John in group synergy. Particularly, in the third forum, group synergy was accounted for over 60% of group interactions and half of John's interactions. Below we describe the events that took place in each of the forums, focusing on John's activity.

Forum 1 is John's first attempt at PSF as a student. On this forum after a while (about 20 minutes) he published a complete solution to the problem. Soon, he removed the solution, saying it was wrong. John did not participate in the discussion. His motivational messages were about appeals to think, indications that the task is simple and multiple references to his successful solution. Forum 3 was John's third attempt to collaborate on the forum as a student. This time, John also did not participate in the discussion for a relatively significant period of time (about 15 minutes). He solved the problem again on his own. However, he did not publish the complete solution, but took part in the discussion, sharing his ideas. His reasoning was meta-cognitive in nature. For example, he shared with the group: "To prove that a polygon is a square, I started looking for segments that could be equal, and angles that should be 90. This gave me the idea to prove the equality of triangles in advance ...". In the last forum where John participated in the student role (Forum 6), he unexpectedly took the lead in the discussion. He reasoned and involved others in the discussion. The share of group synergy in this forum was 61%. And the average length of the synergetic chain (a number of interconnected and continuing each other messages related to the solution of the problem) was equal to 10.5 messages. This demonstrates the collective nature of the search for a solution. John was very much involved in the discussion. Often, a new round of discussion began with his proposal to think about other tasks of this kind or a proposal to return and analyze the data.

John's work as a teacher deserves special attention. Figure 1 shows that at this stage he has achieved a balanced proportion of SRL characteristics that are equally expressed. John participated in the discussion to stimulate group synergy in the students' work. He motivated them to jointly search for ways to solve the problem with phrases such as "Listen to each other. It will give you good thoughts." Most of John's advice was meta-cognitive. For example, "Let's talk about the essence of the problem and outline an action plan." The cognitive notes hinted at a way to solve the problem, not a direct solution. Many of John's posts as a teacher had prototypes in his student forums. These were either the lines of the group mentor or colleagues.

Discussion

The results of the study support the conclusions made in previous studies about the beneficial effect of self- experience as learners on the pedagogical activity of teachers, especially when it comes to the use of non-standard teaching methods (Hoover et al., 2016). Using the example of John and other teachers, we traced how their ability to listen and help the development of other people's ideas developed in the process of participation in the PSF, which is especially important in pedagogy aimed at developing independent thinking in students. We recorded a significant increase in the role of the meta-cognitive approach in solving problems by teachers, which was subsequently reflected in the role of John as a mentor of his colleagues. The participating teachers did not require the provision of a complete error-free solution by each learner separately, but facilitated a discussion aimed at understanding the essence of the mathematical ideas inherent in the problems. At the same time, in the development of the idea of the double role of the teacher in light of the theory of SRL-SRT (e.g., Kramarski, 2012), it was found that the process of forming SRT in PSF environment occurs not only sequentially, but also in parallel with the development of SRL. It can be assumed that the explanation of the discovered phenomenon lies in the fact that the development of SRL skills in the process of participation in the work of forums contributes to the desire for a meta-cognitive assessment not only of the cognitive processes pertaining to the solution of the problem, but also a “top-down” view of the entire communicative and cognitive process, occurring in the forum. This conclusion is confirmed by numerous episodes of teachers' reflection on the processes taking place at the forum, both from the point of view of the ways of solving the problem, and from the point of view of the analysis of the interactions that took place. The discovered phenomenon makes it possible to use PSF, in which teachers solve mathematical problems, in a new context, as a model for exploring the mechanisms of development of communicative and mathematical aspects of teaching.

The present study has demonstrated the dynamics in the development of group synergy as participants gain experience of participating in PSF. A friendly, safe atmosphere has emerged in the forums, including mutual interest, willingness to listen and accept someone else's opinion, and not to be afraid to make mistakes in expressing one's opinion, which, according to Schindler & Bakker (2020), is necessary for the successful work of the group. We found that with an increase in the proportion of episodes of group synergy among interactions in the forum, groups progressed not only in terms of efficiency in solving problems, as Clarke and colleagues' research (2014) shows, but also in terms of self-regulation of their participants, in particular, their meta-cognitive approach to problem solving. It might be that the orientation of the participants towards self-regulation allowed them to notice and evaluate the episodes of successful joint work and purposefully strive to expand them, thus contributing to the emergence and strengthening of group synergy in the forums as a factor contributing to success.

In conclusion, this study contributes to the theory of the mutual influence of the collaborative solution of mathematical problems, in particular in the PSF environment, and the development of self-regulation skills. The fact that the participants of the forums are teachers allowed us to move to the next level of understanding of this issue and to trace the reflection of these processes on the pedagogical activities of teachers, in particular as mentors of PSF.

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Enduring impacts of professional development: Three cases that illustrate evolving learning five years after participating in different PD projects

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Mathematics professional development (PD) has been prevalent, and scholars have been investigating interventions that target teacher knowledge, while also maintaining a focus on instructional practice and student learning. Findings have been mixed and we hypothesize that teacher learning takes time and that teachers learn more than studies report. Case study analysis allow for an in-depth investigation into what teachers take up and enact in their classroom context. The study that is the focus of this paper uses comparative case study analysis to examine three different and distinct professional development programs that are geographically situated across the US, focused on different mathematical content, and utilize different PD structures. This paper highlights three cases of enduring and evolving teacher practices resulting from their three respective PDs.

Keywords: Professional development, teacher knowledge, case studies, mathematics.

Introduction

There continues to be a widespread effort within the international research community to carefully consider how to support teachers' on-going professional development (c.f. Bautista & Ortega-Ruiz, 2015). Mathematics professional development (PD) has been prevalent, and scholars have been investigating interventions that target teacher knowledge, while also maintaining a focus on instructional practice and student learning (Jacobs et al., 2020). Funding opportunities typically target large quantitative proposals that study impact using pre post measures usually across one academic year and two at most. Findings have been mixed. Some studies have found incremental gains related to content and instructional practices (Koellner & Jacobs, 2015) and others have found no change at all (Santagata & Yeh, 2014) We hypothesize that teacher learning takes time and that teachers learn more than these studies have reported. This study investigates the impact of three different National Science Foundation (NSF) funded PD projects and what teachers use in their mathematics classrooms today. More specifically the research question that this study addressed is: What do teachers take up and use from participation in mathematics PD four to five years after engagement in the workshops?

Theoretical Frameworks

Situative theorists define learning as changes in participation in socially organized activity (Greeno et al., 1996). They consider the acquisition and use of knowledge as aspects of an individual's participation in social practices. With respect to professional learning, situative theorists focus on the importance of creating opportunities for teachers to work together on improving their practice and locating these learning opportunities in the everyday practice of teaching. All three PDs were designed around this premise. For instance, all three PD studied in this paper used a collaborative approach and focused on relevant content that was currently being used in teachers' classrooms. In addition, artifacts of practice such as student work or video tapes of teachers' classrooms anchored conversations and connected the classroom to the PD. Lastly, resources were used that were aligned with curriculum and standards. A situative perspective suggests that groups of teachers who take part in different PD workshops using different materials, with different facilitators, and are situated within different educational contexts (e.g., different geographical locations within the United States) might have very different learning opportunities and experiences impacted by the role of context.

Overview of TaDD Project

The Taking a Deep Dive (TaDD) project is collecting qualitative data from three large U.S. National Science Foundation PD projects to use case studies and cross case analysis to further inform what teachers take up and use from participating in different PDs in different contexts and examine why some teachers appear to take up and use more than others. The TaDD study is unique and does not follow the typical impact methodology in that it utilizes a qualitative theoretically based case design. Additionally, as opposed to quantitative studies that typically examine impact one year after the intervention, this study examines impact of different PD models in classrooms four to five years after the intervention. This approach allows for a deep dive into the variations among models of PD and across teachers. As noted, the field has conducted numerous scale-up studies focused on the impact of PD on teacher and student learning with positive but not necessarily dramatic results – the TaDD project seeks to uncover what specifically teachers take up from professional learning opportunities and the factors associated with impact. TaDD utilizes a multi-case method (Stake, 2013) which centers on a common focus of what content, pedagogy and materials teachers take up from PD experiences.

Case study analysis allow for an in-depth investigation into what teachers take up and enact in their classroom context five years after participating in PD. The case study allows us to understand the different ways teachers interpret and take up aspects of content and pedagogy from a PD. It allows us to unpack teachers' thinking and their practice through interviews and videotapes respectively. Additionally, case study analysis allows us to consider the context in which teachers work, their philosophy of teaching and learning, and the ways they work with diverse learners in their classroom. The construct of variance between teachers and types of PD has the potential to shed light on the factors that are associated with uptake, and the similarities, differences and subtle nuances of teacher uptake and use. These factors have implications for PD design, PD selection and associated policy and would be an important contribution to the field of research on PD.

Using a specified sampling procedure, TaDD selected teachers from each of three PD projects to serve as case study teachers (n=19). This paper highlights three cases of enduring and evolving teacher practices resulting from their three respective PDs. In the next section, we briefly describe the three different PD projects.

Learning and Teaching Geometry (LTG)

The LTG project is an efficacy study of the learning and teaching geometry professional development materials: Examining impact and context-based adaptations, sought to improve teacher's own knowledge and instructional strategies in transformations-based geometry. This PD consists of 54 hours of highly specified video-based PD that is grounded in modules of dynamic transformations-based geometry aligned with the Common Core State Standards in mathematics (CCSSM). Through video analysis and analysis of student work, teachers work together to solve problems and further their knowledge in mathematics teaching in the domain of geometry. The PD allows teachers to better support students in their attempt to gain a deeper understanding of transformations-based geometry through activities like rate of change on a graph, scaling activities, and similarity tools. The material strongly connects to other critical domains including similarity, proportional reasoning, slope, and linear functions.

Problem-Solving Cycle (PSC)

The PSC project developed two interconnected PD models--the Problem-Solving Cycle (PSC) and the Mathematics Leadership Preparation (MLP) models for preparing PD facilitators (Borko et al., 2015). The PSC model consists of a series of interconnected workshops organized around a problem that can be solved using multiple representations and solutions and can be adapted for multiple grade levels. Each cycle focuses on a different math problem. During the first cycle, teachers collaboratively solve the focal math problem and develop plans for teaching it to their students. Teachers then teach the lesson in their classes and videotape the lessons. Subsequent workshops focus on participants' classroom experiences teaching the problem using videotapes and artifacts of practice to anchor their conversations.

Visual Access to Mathematics (VAM)

The VAM project aimed to build skills in mathematical problem solving and communication using visual representations. This PD consisted of face-to-face PD as well as online workshops where teachers implemented problems from the PD and shared their student work to discuss access for English Learner's (EL's) and all students. The project investigated the instructional strategies and supports that teachers of EL's need to provide access to mathematical learning while advancing academic language development. The approach was grounded in the use of visual representations, such as diagrams and geometric drawings, for mathematical problem-solving with integrated language support strategies. The intended goals of VAM were to help teachers properly select appropriate visual representations for the use of different rational number task types and communication tools to show and explain mathematical thinking.

Methodology

Sampling, Data Collection and Analysis

During Spring 2021 pairs of project researchers conducted two one-hour semi-structured interviews using "think aloud" protocols with the 19 teachers individually (LTG 10; VAM 7; PSC 2). The participants selected two to three video clips that showed their routine daily teaching practices yet potentially could show instructional strategies and/or content from their respective PDs. The participants included time stamps and a justification for why they selected the clip prior to each interview. Pairs of researchers watched the entire video as well as the clips before interviewing participants and analyzed the lesson and the selected clips for instances where the participants could show evidence of instructional impact from the PD that they participated in- including content, resources, or instructional strategies. Then participants watched the clip with the pair of researchers on a video conference that was recorded and the participant discussed why they selected the clip and what it represented in terms of their learning related to their PD. With these data, field notes from the interviews and prior survey data, we created initial profiles for each teacher collaboratively in pairs. These profile documents initiated the development of our case studies – three of which are presented here.

Results: Three Case Studies

Tyra (LTG): Case of a Teacher's Evolving Learning

Tyra has taught math for eight years at two different high schools. She was warm, reflective and engaging throughout our semi structured interviews and enthusiastic about how much she learned from PD over the last five years. Interestingly, she team teaches with Diantha, another teacher with six years of teaching experience. Tyra and Diantha regularly talk about their own learning and reflect together especially when they attended the same PDs. Tyra has taught 9-12 graders throughout her career and typically teaches Algebra, Algebra II, Geometry. Tyra and Diantha teach in an urban high school in the northeastern part of the United States where more than half of the students receive free/reduced breakfast and lunch. Trya spoke highly about her administration and mentioned that her assistant principal serves as a mentor by providing feedback and suggesting PD opportunities that she believes will assist Tyra in her ongoing professional learning. There appears to be mutual respect between them.

Tyra participated in the LTG PD in the 2016-2017 school year and shared with us that she was "very committed" to the PD sessions – and that she always completed homework (which was voluntary). She recounted the transformations-based concepts that she learned and how she uses the content and resources from the PD in her classroom. This was also evidenced in the videotapes that she shared with us both when she was teaching with Diantha and without. Tyra explained that attending the LTG PD that was focused on geometry, and another PD offered through a university in her area that was focused on algebra, dramatically changed the way she teaches in the classroom.

She explained that these two PDs that she had taken at the same time (five years ago) moved her from using "chalk and talk" (direct instruction) to using mathematical practices and collaborative learning techniques. Tyra indicated that "before the PD she would give the rule and then give students problems to apply the rule." She also shared that she did not originally think that these methods would "work for her kids" but once she started using them her students responded and seemed more engaged and learned more. In addition, she explained that she wants her students to know how to derive formulas through tools such as tracing paper, but had not thought about using tracing paper when learning geometry concepts. She explained that when students used tracing paper (when solving dilation problems) she said, "they are remembering a process rather than just a rule." She reflected that talking to her peers during the LTG PD influenced much content and pedagogy that she learned, as a result she started incorporating more time for students in her own class to talk to one another for more engagement and learning to take place.

When we spoke about the school climate, she stated that “her environment was great because she had professional freedom in her classroom and a collaborative math department.” Diantha and Trya shared what they learned from LTG with the math department and the following year they made some significant changes to how they teach geometry. In addition, they incorporated a transformations-based geometry unit to begin their Algebra II course. They saw the connections between the two and realized that understanding proportionality and similarity are keys to understanding the foundations of algebra.

Megan (VAM): Case of a veteran teacher’s evolving learning

Megan is a veteran high school math teacher that lives and teaches in a northeastern town in the US. She has taught for 28 years and courses ranging from 6th grade math through Calculus. Megan teaches in a school where most of her students are English Learners (EL’s) and more than half of the students receive free/reduced breakfast and lunch. Megan appeared to be a very committed teacher and she relayed her excitement about participating in the VAM PD to learn more about how she could reach her EL’s. She told us some stories of her students and the difficulty she had teaching students that spoke a variety of languages in one class. She felt that she needed to learn strategies to support the different languages. Megan participated in the VAM PD in 2016-2017. She shared that she really liked the PD and that the philosophy aligned with her own thinking about teaching mathematics, “Some of the strategies I had already been using but the others were new and built on the directions of the district.”

A highlight of the PD for Megan was “sharing instructional strategies and resources with teachers from other districts.” She remembers learning how to “create visual representations to solve problems and apply language strategies to make sense of word problems.” She selected a video clip to share with us that showed aspects of her learning from the VAM PD that she continues to use in her classroom today. In the video clip, she teaches remotely with a white board application using a double number line. The double number line represented two quantities from a word problem that the kids had to solve. She explained to us that she had never used or heard of a double number line before VAM. After we watched the clip, she emphasized how cool she thought the double number line was because it allowed her to show students the varying quantities on the number line and how the quantities translated to a coordinate plane visually. She attributed this to have a big impact on students’ conceptual understanding. She also commented that she learned how to use color in representations to help students “see” the math and she believed the use of color had “visual impact.”

Megan continues to use representations for access that she learned in the VAM PD as well as other strategies that she has researched or developed to provide more access to her students. Megan shared a story about how she decided to use easels to support five students with Individualized Education Plans in her class that struggled with the amount of time they had to sit during remote instruction. In her school, some students came to class in person while others joined remotely through a video-based conferencing application. These boys were in the classroom with her during instruction. Megan decided to bring in easels and the students worked in pairs to work on the math problems. They only had one pen and they had to take turns using the pen, again strategies that address access and collaborative working relationships. At the end of the class, the students wanted to do another problem. She said that this ended up being “the best teaching day all year.” She shared the “vertical workspace” strategy with the rest of the staff at her school and they have implemented this technique in their school - using easels, chart paper and white boards, which Megan believed helped the students collaborate and talk with one another. This is an example of how instructional strategies from PD can be modified and translated into best practices in schools. Noticing and listening, aspects important in the VAM PD, potentially provided Megan new perspectives on addressing student needs.

Linda PSC: Case of a teacher leader’s continual learning with other leaders

Linda is a math and science teacher as well as a PD leader in her school and district located on the west coast of the US. She is a veteran teacher that has taught in the district for many years. Linda served as a teacher leader, a person that was trained to lead the PSC PD with teachers in the math department at her school. Linda explained that her involvement in the PSC PD “has been a defining and productive exploration of herself as a math teacher.” She said she appreciated being part of a professional community and she noticed growth in her math teaching. Linda found the PSC PD experience “productive and valuable”, and she valued being around people passionate about teaching math and teaching in general. Linda believes that the involvement in the facilitator group helped her and the other participants to become better mathematics teachers and bring back instructional strategies to their math departments. She also said that this group supported the facilitators in becoming leaders and helping each other as colleagues.

Linda reported that participating in our TaDD project helped her reflect on what she learned six years ago when she participated in the PSC project. She compared it to other PDs focused on learning strategies and found it challenging to incorporate something new without practice and reinforcement. Linda found great value in participation because the PSC PD was ongoing, and she was involved in it for 2-years. She explained that extended involvement was important for her as well as developing as a teacher. She appreciated the opportunity to be part of a community and that revisiting her learning by “doing math, practicing math” was a more valuable experience than other PDs because she learned “good math teaching strategies” and gained more confidence as a math teacher.

Linda told us that one of her current goals is trying to use academic vocabulary with her students. She also mentioned that she is in support of “productive struggle” in math and trying to engage students and have them “do the work” for three reasons. She believed productive struggle helps to amplify student voice, respond with academic language, and allows students to answer informally.

Laura felt positively about the facilitators in the PSC PD—she appreciated how they set up the PD so that the participants could explore ideas together. She said it is hard to separate what she learned from how they facilitated the PD. When asked about her role as a facilitator of the PSC at her school, she admitted that it didn’t go well, in fact “it was a disaster”. She loved her own participation with the community of facilitators but struggled with how to productively facilitate the PSC with the teachers at her own school stating that they did not want to participate.

Discussion and Implications

Our current study aims to understand and capture long-term learning from a more nuanced perspective. Teachers have shared their own perceptions of learning that is more detailed and provides clarity about the evolution of their practice including new ideas from other PD opportunities that may or may not have aligned with the various PDs in our study (VAM, LTG, PSC). These other professional opportunities ranged from additional formal professional development workshops to professional conferences, to collaborations and conversations with colleagues, as well as advice from key constituents, such as a principal. In the US and in other countries as well, professional learning is required in most contracts, but teachers also participate because they are motivated to learn formally and enjoy debriefing with colleagues either formally or informally. We believe these additional learning opportunities do not shade the results of our study but rather illustrate the actual ways teachers learn. Even though as stated above, our goal is to understand how teachers use the content, pedagogy, and resources from PDs in our study, we believe it is necessary to consider these additional learning situations and how they may have also contributed to the learning of a teacher not only because it is the reality of their job but also in order to be transparent in our research.

The three case studies illustrate continued learning at least five years post PD experience. The full case studies illustrate much more complex and nuanced variance among cases but the shortened versions in this paper show the ways in which three different teachers' learning has evolved over time. They all have shifted practices and are trying new teaching strategies, noticing and listening to students in new ways, and recognizing the importance of PD and reflecting on their everyday teaching practice.

All three case study teachers are motivated to learn and had a purpose for attending and learning from their respective PDs. Tyra was motivated by the geometry content and resources for teaching transformations-based similarity. Megan was motivated by a need to learn more strategies for supporting her English learner students. Linda was motivated by an interest in becoming a teacher leader and participating in a learning community of facilitators from others outside of her school. Yet they were each different as well. Tyra was interested in learning content and better ways to reach more students. Megan needed new resources and strategies to serve her students while Linda wanted a new challenge as a teacher leader. Each of our 19 teachers are on their own journey and we are continuing to investigate the trajectories of change which vary across participants yet have the potential to provide and contribute understandings of variance across different teachers and different types of PD. We anticipate a cross case analysis will help us disentangle similarities and differences within and across the three PDs and contribute a more in depth understanding into what teachers take up and enact in their classroom context.

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Structured lesson preparation improves teacher behaviour

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Keywords: Higher order skills, problem solving, primary teacher education.

People need mathematical higher order skills to participate successfully in society (Hoogland, 2016). Working together on and discussing non-routine math problems promotes the development of these skills. In Dutch primary education however non-routine math problems are most of the time presented to high-performing students only (Van Zanten & Van den Heuvel-Panhuizen, 2018) and whole-class discussions on solving problems to develop higher order skills are rare. In what way can whole-class discussions be organised to foster the development of higher order skills of all students?

Theoretical framework

A productive whole-class activity on problem solving should start with a non-routine mathematical 'low-floor-high-ceiling task' (Boaler, 2016). Teaching such problems involves: (1) explaining the problem, (2) putting students to work, (3) observing their problem-solving approaches, (4) guiding whole-class discussion and (5) summarising learning outcomes (Stein et al., 2008). In the whole-class discussion the teacher encourages his students to present, compare, evaluate and defend their problem-solving approaches. This places high demands on teachers. They must deal with a broad spectrum of students' problem-solving approaches and simultaneously challenge all students to think and reason about those approaches. To achieve this, mathematical and pedagogical content knowledge is needed (Ball et al., 2008), together with specific teaching skills as described in the Mathematical Quality of Instruction (MQI) instrument, like mathematical sense-making, using multiple problem solving-approaches and using students' ideas (Hill et al., 2008). Stein et al. (2008) advise teachers to prepare their problem-solving lessons thoroughly, by (1) solving the problem in multiple ways, (2) predicting mistakes of students, (3) devising hints and support, (4) selecting and order problem-solving approaches to be used in the discussion and (5) predicting learning outcomes.

Research question

To support preservice teachers in preparing and performing whole-class activities with non-routine problems we developed a preparation form based on the aspects mentioned by Stein et al. (2008). During the activity, while the students work on their problem, the preservice teacher observes their problem-solving approaches. Having his own preparation in mind he then makes a selection to be used for the whole-class discussion. During this discussion the preservice teacher invites students to present their problem-solving approach. The preservice teacher then challenges the other students to discuss and evaluate these. The preservice teacher guides the discussion and ensures participation and understanding. His thorough preparation will support him in responding to and developing the thinking and reasoning process of the students. The discussion is more productive when the preservice teacher has explored beforehand how students might solve the problem (Stein et al., 2008). In light of the above, the following research questions emerge:

How and to what extent are preservice teachers able to prepare a whole-class activity on a non-routine mathematical problem using a structured preparation form?

How and to what extent does structured preparation of a whole-class activity on a non-routine mathematical problem improves the quality of teacher behaviour during this activity?

Method

Approximately 80 Dutch preservice primary teachers will participate. For their first lesson they will use a preparation form completed by a teacher educator (the so-called “expert”). For the next two problems, chosen from a selection of 30 problems, they will complete the preparation form themselves. After teaching with the three problems, they will complete a questionnaire evaluating their own teacher behaviour while performing the third lesson. They will score themselves on certain aspects of the MQI instrument using a four-point Likert scale (Hill et al., 2008). They will answer questions about the perceived impact of their preparation on various aspects of their teacher behaviour and about experienced differences in using their own preparations and that of an expert.

For the first research question, the preparations of the preservice teachers will be analysed and compared with the experts’ preparations. For the second question, the scores of the preservice teachers will be analysed and how they attribute these to their preparations and those of the experts.

To be continued

Rich preparation of problem-solving activities will support teacher behaviour (Stein et al., 2008). We plan to publish the problems and experts’ preparations in the near future. Should it turn out that teachers perform better while using their own preparations we will publish the problems with empty forms. In any case we will encourage the use and preparation of problems in whole-class activities.

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What can be causing this error? - Preservice primary school teachers' interpretations of students' errors

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Keywords: Preservice teacher education, diagnostic competence, analysis of students' errors, mathematics teachers' competencies.

Introduction and theoretical framework

Teachers' diagnostic competence is crucial for understanding and evaluating students' thinking and differentiating teaching strategies. What teachers pay attention to in the classroom and their interpretation are decisive for supporting students (Kaiser et al., 2015). In mathematics classrooms, students' errors are a rich source of information about their thinking. They can reveal students' understanding of mathematical concepts and procedures. The *diagnostic competence in error situations* is the competence needed by teachers to identify, interpret and then manage error situations in a way that promotes students' mathematical understanding (Heinrichs & Kaiser, 2018).

The development of preservice teachers' (PST) diagnostic competence in error situations can begin in initial teacher education. Teacher educators can support PST in learning to identify relevant details, ask relevant questions and apply specialized content knowledge to understand students' thinking. However, such learning opportunities are complex, which raises multiple questions about the characteristics they should have to be effective. Moreover, it is necessary to extend our understanding of the characteristics of the competence in the early developmental stages and the interactions between the competence and PST's knowledge, beliefs and practical experience.

Thus, the goal of this study is to contribute to the characterization of preservice primary school teachers' diagnostic competence and its development. We focus on the interpretation dimension of the competence, i.e. on PSTs' competence to formulate hypotheses about causes of students' errors. This dimension of the competence is decisive for the subsequent pedagogical decision-making process and the quality of the teaching strategies that can be implemented.

Method

The goal of this study is to characterize PSTs' competence to formulate hypotheses about causes of students' errors and its development after they participated in a university seminar sequence. In addition, based on the model of competence as a continuum (Blömeke et al., 2015), we examine if beliefs, knowledge and practical experience are related to PSTs' competence. The seminar sequence was aimed at developing PSTs' competence to identify, interpret and deal with students' mathematical errors. It consisted of four 90-minute sessions, in which PST were engaged in individual and group analyses of students' errors and discussions about students' mathematical thinking. Short videos of error situations and samples of students' written work were used as prompts.

Participants were 131 undergraduates from 11 Chilean universities in their initial teacher education for primary schools. Before the seminar sequence, they answered an online survey, that collected demographic data, beliefs questionnaires (from the TEDS-M study) and two error analysis tasks. In addition, they answered a mathematical knowledge for teaching (MKT) assessment. After the seminar sequence, participants completed another two error analysis tasks.

In each error analysis task, they were asked to suggest three possible causes for the student error shown in a video vignette. Their responses to these open items were coded using qualitative text analysis. Five thematic categories were used to classify the types of hypotheses to which PST attributed the error: lack of conceptual understanding, lack of procedural understanding, instructional wrong decisions, ambiguous hypotheses and not valid hypotheses. In addition, the relationship of PSTs' competence to their professional knowledge, beliefs and practical experience was examined.

Results

Results indicated that PST had difficulties formulating hypotheses about causes of students' errors. They often gave incorrect, incomplete or contradictory suggestions. The valid hypotheses mostly attributed the error to a lack of conceptual understanding of some mathematical issue. Further analyses showed that beliefs, knowledge and practical experience were related to PSTs' competence. Their beliefs about learning mathematics as an active process showed the strongest correlation, together with practical experience.

Analyses revealed a significant improvement in the number of valid hypotheses PST could formulate between pre- and post-test. After the seminar sequence, PST formulated more hypotheses attributing the errors to a lack of conceptual and procedural understanding and to a lesser extent, hypotheses focusing on inappropriate instructional decisions. This showed that they were more able to focus on students' thinking. Study progress (number of finished university semesters) was a significant predictor for greater changes in the competence to hypothesize about causes of students' errors.

Our findings highlight the role of beliefs in the competence and its development. They also provide insights into the role of teaching experience and knowledge for the development of the competence. The poster format will provide us with the opportunity to share and further discuss characteristics of the seminar sequence and of PSTs that contribute to the development of the competence.

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What remains? A longitudinal study of the effects of curated field experiences for preservice mathematics teachers

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This paper examines the long-term effects of a teacher education program that prepares preservice elementary mathematics teachers to work specifically in Detroit schools. The study was designed to better understand the degree of alignment between the program's curated field experiences and the work graduates currently do as teachers; the aspects of program design that contributed or detracted from graduates' understanding of culturally relevant practice in mathematics classes; and the reasons for graduates' retention in high-poverty schools over time.

Keywords: Teachers, mathematics teachers, preservice teachers, elementary school teachers.

Objectives

This paper examines the long-term effects of a teacher education program designed to prepare teachers to work specifically in Detroit. Although teacher education programs are frequently studied for their near- and medium-term effects, this study sought to uncover the longer-term traces of the program for its graduates. The TeachDETROIT program was established in 2016, and its first cohort entered the workforce as teachers of record in 2017. The program was founded to contribute to the revitalization of the city with special emphasis on the importance of schools in their efforts to build a just educational system for minoritized people. The TeachDETROIT program welcomed its sixth cohort in fall 2021. As part of program evaluation, the program leadership along with external evaluators sought to understand the longitudinal traces of this context-specific (Matsko & Hammerness, 2014) urban teacher residency program, regarding graduates' perceptions of preparedness to teach mathematics in under-resourced schools; implementation of culturally relevant teaching practices in the context of teaching mathematics; and program graduates' current commitment to remain in under-resourced schools. The study was designed to better understand the graduates' pathway into, through and beyond the program; the degree of alignment between the program's highly curated field experiences and the work graduates currently do as teachers; and the aspects of program design that contributed or detracted from graduates' understanding of culturally relevant practice in mathematics classes.

Theoretical Framework

As one of the most important factors for improving student achievement (Darling-Hammond et al., 2009), teachers need to be prepared to offer their students the high-quality mathematics instruction they deserve and require (National Council of Teachers of Mathematics, 2000; Conference Board of Mathematical Sciences, 2012). Unfortunately, there is evidence that students in the United States are not experiencing high-quality instruction. The 2019 National Assessment of Educational Progress (NAEP) showed that only 41% of fourth-grade and 34% of eighth-grade students were "proficient" at mathematics. For Black students in large, urban school districts, those numbers fall to 18% and

13%, respectively. In Detroit, Michigan the situation is even more dire: only 5% of fourth-grade, and 5% of eighth-grade students were "proficient" in mathematics.

Researchers have long posited a connection between the quality of instruction teachers can offer and the quality of the preservice preparation they experience (Schmidt et al., 2011; Senk et al., 2012). Although the familiar mix of content, methods, and social science foundational courses, together with supervised practice teaching, has endured (Donoghue, 2006; Fraser, 2007), so has the finding that this type of preparation is disconnected from the work of teaching (Feiman-Nemser, 1983, 2001).

After decades of interest in learning to teach at some remove from classroom experience (Feiman-Nemser, 1985; Floden & Buchmann, 1993), the pendulum swung back in the early 2000s to emphasize a focus on learning "in and from practice" (Lampert, 2010). A flurry of work in teacher education called for preservice teacher education to focus on "practice" (Ball & Forzani, 2009; Grossman et al., 2009; Grossman & McDonald, 2008; Lampert, 2010; Zeichner, 2010). The emphasis in this line of work is that preservice teachers should learn "in and from practice" (Ball & Cohen, 1999), not just educational theory learned at a distance from classroom life. Today clinical residency programs abound, where preservice teacher education is predominantly a full immersion in a field placement classroom. Yet the quality and specifications of these field experiences can vary widely: the insides of the classroom environments, the quality of cooperating teachers and school principals, and the experiences that preservice teachers have can often go largely unspecified (Ronfeldt, 2012). Furthermore, there are few studies that investigate how well preservice teachers learn to do the interactive work of teaching mathematics in such programs. In this study, we describe an early field experience designed for preservice teachers to learn to carry out mathematics instruction in elementary classrooms, and we report on its relevance to participants five years later.

The curated field experience in TeachDETROIT is composed of four primary elements: 1) Highly scaffolded instructional activities in mathematics; 2) Shared classroom spaces where mathematics teaching and review of teaching is public; 3) Ongoing study and conversation about teaching mathematics to children of color living in poverty; 4) Group explorations of the city of Detroit.

In contrast to reports that the quality of field experiences is often poor (Greenberg et al., 2011; Ronfeldt, 2012), the field experiences described in this paper are carefully designed and sequenced. We have named this intentionally designed field experience the *curated field experience*, one in which the cooperating teachers and the instructional activities in field experience classrooms are highly specified and aligned, bringing together a team of teacher educators, cooperating teachers, school administrators and support staff. We used the word *curated* to refer to the careful selection and sequencing of field experiences, including the preparation of cooperating teachers for this program, the matching of cooperating teacher to intern, and the progression of field experiences where the complexity and demands grow according to the needs of each intern. Researchers designed this study to better understand how the early curated field experience was viewed by graduates of the program some five years later. In this section we provide some detail about the four elements of TeachDETROIT's early curated field experience.

Highly scaffolded instructional activities. One important feature of the curated early field experience is that interactive instructional work is woven into the mathematics methods class from preservice teachers' first day in TeachDETROIT. Each methods class session is held in an elementary school, and interns enter one 3rd grade classroom for 30-40 minutes during the methods class of 150 minutes. Preservice teachers work with small groups of children in the same classroom. Preservice teachers learn to conduct four instructional activities in mathematics to rising 3rd grade students in a summer school setting: teaching a mathematical game using compositions of ten; interval counting (Franke et al., 2018); modeling number facts with concrete materials; and using a variety of strategies to represent multidigit addition and subtraction. In this setting, preservice teachers learn to enact a short instructional activity by observing an experienced teacher conduct the activity, rehearsing the activity with a peer using a script, rehearsing the activity in front of the larger group of preservice teachers for feedback, and finally enacting the short activity with a small group of children. In the first weeks of the early field experience, the instructional activity is highly scripted. Although scripted teaching has been the subject of lively critique in teacher education, in the early field experience the script serves to quiet the complexity of classroom work by reducing some of the cognitive load for novices until strong mathematics teaching practices become more familiar and internalized. As the term progresses, novices rely less and less on scripted lessons and take on increasingly complex facets of planning and enactment. Preparing for teaching includes analysis of video from the previous session's lesson, reviewing and adapting lesson plans, and rehearsing for teaching enactment. The rehearsals underscore the unspoken notion that teaching is something to be studied, practiced, and constantly improved. This feature of TeachDETROIT draws on the scholarship of Lampert et al. (2013).

Shared classroom spaces where teaching is made public. Preservice teachers, host teachers, and the methods instructor are all present in one classroom for every class session, so that all teaching is always conducted in the company of others. From the very first day of TeachDETROIT, preservice teachers experience teaching as an activity that is witnessed by others, and the subject of collegial conversations towards improvement. The preservice teachers videorecord their teaching enactments and review them each day; following a teaching enactment, preservice teachers share excerpts of their recordings and write about them in journals and video platforms designed for such purposes. In this way, teaching is positioned as work that is done in the presence of other colleagues, with frequent collaboration, and guided by review of prior lessons with ongoing attention to evidence of student learning. Each lesson plan is shaped by the review of the previous day's teaching and learning as documented in video records, student work, and in consultation with peers, the instructor, and the host teachers. This feature of TeachDETROIT, making practice visible and shared, follows the work of Lewis (2007).

Teaching children of color living in poverty. In a course titled *Detroit Families, Communities, and Schools,* TeachDETROIT participants learn about the history of schooling in Detroit, and study the effects of poverty and systemic racism on families. The course is part of an ongoing open conversation in the program about working with children of color living in poverty and their families and caregivers, and promotes a strengths-and-assets view of students and their families. Curated field experiences are especially important in preparing preservice teachers to do ambitious teaching in high-poverty settings with children of color, since some preservice teachers may not have had such experiences themselves, either with children of color or with ambitious teaching practice—and the challenges of the two combined can be especially daunting. The empirical experience provided by curated fieldwork in conducting ambitious practice with children of color living in poverty serves as a kind of existence proof to preservice teachers: that such practice is possible, that teachers and their students are capable of doing such work, and that this kind of work is what is done in "regular" schools, real-time, day in and day out. Ambitious mathematics teaching may not be the norm in many schools that serve Black children living in poverty, or preservice teachers may not imagine that such practice can be done with minoritized populations. For this reason, the curated field experiences described in this article are intentionally situated in schools that serve children of color living in poverty. This feature draws on the work of Matsko and Hammerness (2014).

Exploring the city of Detroit. The TeachDETROIT experience includes multiple excursions around the city, especially to venues and events where families are present. Experiencing Detroit as a place of vibrant cultural and social ferment emphasizes the possibilities of a city that has been portrayed in the media as a blighted, scary place. Program participants come to see Detroit as a compelling place to work; being part of the city by visiting libraries, parks, artist studios, outdoor murals, urban farms, museums, coffee shops and lectures together is part of the education that TeachDETROIT provides beyond the preparation for teaching academic subjects. The required course *Detroit Families, Communities, and Schools* features such excursions, and optional social gatherings in the city are an integral part of the TeachDETROIT experience. This feature of TeachDETROIT was introduced following the literature on place-based education (cf. Gruenewald & Smith, 2014).

"Slow food, small batches." Although not a primary feature of the curated field experience, it bears mentioning here that an important characteristic of the TeachDETROIT program is the maintenance of small cohort size so that the preservice teachers receive individualized attention and wraparound services as needed. Each preservice teacher progresses through the program on pace with their own growing proficiency; individual strengths, resources, and interests are addressed as much as possible. For example, by knowing the preservice teachers well, the program is able to place them with mentor teachers who are matched to their needs. The program endeavors to be the educational analog to the Slow Food movement, which prizes the production of high-quality, environmentally responsible food over industrialized food production. Thus, TeachDETROIT cohorts are purposely small, and the program resists the trend to abridge the time and requirements for teacher certification 10.1111/0161-4681.00141

increasingly allowed by the Michigan Department of Education.

Methods

All members of the inaugural TeachDETROIT cohort were invited to participate in the study by external evaluators for the program. TeachDETROIT preservice teachers are eligible to join the program if they have a bachelor's degree with a 2.75 grade point average, some experience working with children (even minimal experience), and an expressed interest in working with children in Detroit. Candidates are often returning to school after other careers or raising families, so their prior educational experience may be distant and not necessarily a predictor of success in TeachDETROIT; for this reason, weak academic history is not necessarily a barrier to entry in the program.

Program graduates received three email invitations and three phone calls to take part in 45-60-minute interviews. Study participants were offered a gift card of $75 for joining the study, and were approached by the external evaluation team, not program personnel, and interviewed by the external evaluators to encourage study participants to speak candidly about their experiences. Stimulated recall (Calderhead, 1981; Kennedy, 1991) using photographs of the interns themselves during their first semester of program participation was used to prompt graduate recollections about their curated field experiences. These photographs were taken for marketing purposes at the time, and show each intern engaged in the instructional activities with children as detailed above, including the conduct of small group work, writing on the whiteboard, and peering over students' shoulders as they work, for example. The photographs were not staged; they were taken during regular instruction. A structured interview was developed making use of the photographs and the memories they sparked for study participants. As mentioned above, interviewers were external to the program and had conducted external evaluation activities in years prior to the current study. The study was conducted with IRB permission.

Data sources

67% of the cohort (N=4) participated in the study. Interviews were audio recorded and transcribed; the transcripts were then coded using an approach that closely follows methods explicated by Miles et al. (2019). This approach emphasizes well-defined study variables to ensure the comparability of data and reduction of data using data displays and matrices so that the common themes can be identified.

Findings

Findings of the study are summarized here.

1. Most graduates found out about the program through their academic advisors at the University.

2. Study participants reported that memories of their preparation experiences in the field were triggered when shown a photo of themselves in the classroom. Participants recalled particular instructional routines and named them.

3. All study participants said that their mathematics lessons did not go as planned in their first months in the program, and that they had to learn to adjust to what students knew in the moment. The following exchange illustrates this finding:

Interviewer: Do you remember any of the content of the lessons that you taught?

Graduate: Oh, my gosh. Yes. Shared reading, guided reading and read alouds for reading. And then for math, we did number talks.

Interviewer: Okay. And do you remember feeling that the lessons went as planned?

Graduate: Sometimes. I mean you never know what children are going to say and they're going to throw you off, but most of it, I would say that the majority of the time, yes, with the exception of you just don't know how many children are going to respond. And, for example, when we would do a number talk and talk to children about math, your expectation is that at some point, they're going to get it while sometimes they didn't. So in that regard, no.

4. All study participants valued the interactions with peers, found those relationships to be educative and sustaining, and reported that they continue to stay in contact five years later.

Interviewer: Do you remember discussing your teaching with your peers? And if you do, what do you remember kind of taking away from those discussions?

Graduate: Oh, absolutely. It was very beneficial because we would talk about what our strengths were during that lesson, what we could improve upon and how we would change it if we had to do it again. And the thing is we were talking to our peers, so some of the feedback came from teachers, mentor teachers, but then other times, it came back from our peers, which helps a lot.

5. Reflecting on their early mathematics teaching experiences, all graduates noted that those early preparation experiences were aligned with the teaching work they did after graduation**.**

Interviewer: So looking back on these experiences, were they kind of aligned or misaligned with the teaching work that you would do once you graduated?

Graduate: Oh, they were very aligned.

6. Graduates named classroom management as an area for which they felt underprepared upon entering the profession. Graduates felt that the program did not address this topic adequately, and noted that the demands of managing a classroom alone were greater than they had anticipated.

7. All graduates felt that the program prepared them to teach mathematics in a culturally responsive way. They reported learning about culturally responsive instruction through their coursework, specifically in their methods course and the course titled *Detroit Families, Schools, and Communities*; during discussions with teachers and university instructors and coaches; and by participating in a series on trauma-informed teaching.

8. Nearly all program graduates from all cohorts have remained in teaching and are working in high-needs schools. Typically, graduates stay for two years in their first job and then move to another school, to follow a principal or to work closer to home.

Scholarly significance of the study

Teachers, especially minority teachers, leave the profession at astonishing rates and soon after graduation (Ingersoll & May, 2011). This problem is especially acute in schools for children of color living in poverty. TeachDETROIT graduates defy this national trend: nearly every graduate from all six cohorts of TeachDETROIT has remained in teaching and serves in high-poverty schools. We hypothesize that the curated early field experience is a significant factor contributing to increased teacher retention in hard-to-staff schools. This study provides some preliminary evidence that teacher retention, particularly in schools for children of color living in poverty, can be strengthened by preparing teachers to teach ambitious mathematics, along with attention to teaching for equity. We hypothesize that the curated field experience provides teachers with the skills and knowledge to teach ambitious mathematics and work in schools much like those they will teach in, and this in turn leads to feelings of self-efficacy and job satisfaction once graduates are teachers of record. Their shared explorations of the city of Detroit further cement program graduates' sense of commitment to the families in the city and the sense that as teachers, they are part of something bigger than themselves. Although limited by the small number of participants, this study sheds some light on the longer term effects of a program that is designed for preservice teachers to learn "in and from practice" (Lampert, 2010). Further investigation is needed to understand the intersection of culturally relevant teaching practices and the instructional routines in mathematics that graduates learned while in the program. Additionally, external validation of graduates' perceptions would greatly strengthen the claims made here, and to obtain the quality of their instruction Finally, although retention in hard-to-staff schools is strong for TeachDETROIT graduates, we have yet to study the quality of teaching and the extent to which graduates continue to carry out ambitious mathematics instruction.

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An exploration of how mathematics teacher educators invite pre-service teachers to participate in lessons about the teaching of number concepts and operations in early years

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This study explores the nature of pre-service teachers' participation in their lessons on how to teach number concepts and operations to learners in early years of primary school. The paper is part of a qualitative case study aimed at exploring how pre-service teacher education prepares pre-service teachers to teach number concepts and operations in early years (Grades 1-4) in Malawi. This paper reports on findings from two mathematics teacher educators. The lessons were analyzed using the Mathematics Discourse in Instruction framework. The analysis involved segmenting the lessons into episodes, each of which was recognized by change in content focus. Findings indicate that the pre-service teachers were mostly invited to participate through answering yes/no questions or supplying one-word responses to the teacher educators' unfinished sentences. Implications of these findings are discussed.

Keywords: Pre-service teacher education, early years mathematics, learner participation, malawi.

Introduction

In any mathematics lesson, meaningful learner participation is fundamental for understanding the mathematics made available for learners to learn (Carpenter et al., 2003; Trocki et al., 2014). Mathematical understanding develops best when learners participate actively and are encouraged to discuss mathematical concepts and to generate and argue mathematical solutions with one another (Carpenter et al., 2003). However, Trocki et al. (2014) argue that it is usually challenging for teachers to facilitate such type of discourse in their mathematics lessons. This suggests that teachers need to be supported to learn the practice of inviting learners to active participation. Inviting learners to participate in the lessons is one of the practices that teachers perform regularly in their work of teaching. As such, teachers need to learn how to do it (Adler & Pournara, 2019). This learning needs to begin from pre-service teacher education because pre-service teacher education is expected to expose pre-service teachers (PSTs) to learning environments that help them to experience active and meaningful learning, while at the same time helping them to learn to create the same learning opportunities for their learners (Taylan, 2017).

In Malawi, little is known about how teacher educators (TEs) invite PSTs to participate in their mathematics lessons and how these PSTs are enculturated into the practice of inviting learners to participate actively in the lessons about number concepts and operations. Number concepts and operations is the predominant focus of early years mathematics in Malawi; it also takes a big part of mathematics teacher education content (Ministry of Education Science and Technology [MoEST], 2017). Mastery of number concepts and operations plays an important role in the development of learners' future mathematical abilities. Yet, studies continue to show that early years learners in Malawi perform below the expected achievement level in mathematics, including in the core-element of number concepts and operations (Brombacher, 2011, 2019). A study by Saka (2019) indicated some challenges in the way teachers teach number concepts and operations in early years. Saka argued that the way teaching is done in early years does not fully support the development of number concepts and operations for learners. This finding may indicate some gaps in the way PSTs are helped to learn to teach number concepts and operations. Research indicates that what teachers learn during their pre-service training greatly influences how they teach (Ball & Forzani, 2009). Thus, if PSTs are engaged in active and meaningful participation during their teacher education, they are likely to engage their learners in active and meaningful participation during their work of teaching. This makes it necessary to explore how the teaching practice of inviting PSTs to participate in the lesson is enacted in pre-service teacher education. The main argument being put forward here is that teachers' ability to encourage meaningful participation among learners does not come naturally; it is a function of how they were enculturated into the practice during their pre-service teacher education. Thus, this study focused on answering the question 'How do mathematics TEs invite PSTs to participate in the lessons about how to teach number concepts and operations in early years?' In this paper, 'teacher educator' refers to the one teaching pre-service teachers how to teach, 'pre-service teacher' is the one learning the work of teaching, while 'learner' is the one whom the PSTs are expected to teach at the end of their pre-service teacher education.

Theoretical Framework

This study was guided by the Mathematics Discourse in Instruction (MDI) framework, developed by Adler and Ronda (2015). This framework was chosen for use in this study because it helps in describing the mathematics that is made available during teaching (Adler & Alshwaikh, 2019), and it specifically targets mathematics teaching practices that teachers meet regularly in their teaching. The framework considers four key elements to the teaching of mathematics: object of learning, exemplification, explanatory talk and learner participation as shown in Figure 1.

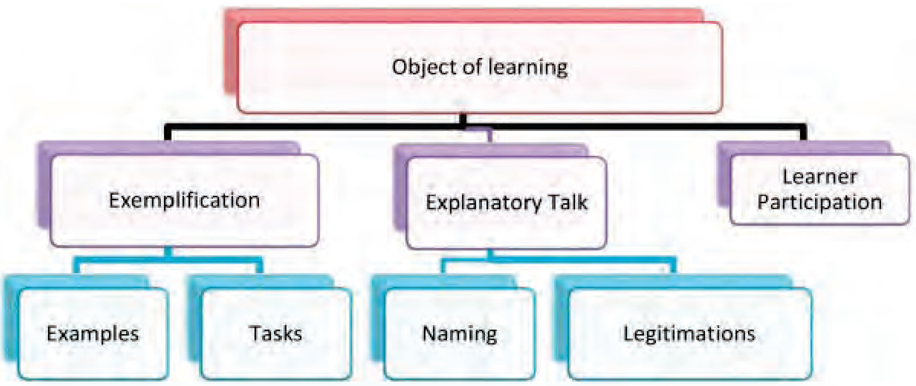

Figure 1: Constitutive elements of the MDI framework (Adler & Ronda, 2015, p. 239)

The object of learning is what learners are expected to know and be able to do. In a mathematics lesson, the object of learning is brought into focus through three mediational means: exemplification, explanatory talk and learner participation. Exemplification is concerned with examples and tasks used in the lesson, and how these provide opportunities for learners to learn mathematics. Explanatory talk is talk which names and legitimates important aspects of the object of learning, while learner participation is about how learners are invited to participate in the lesson (Adler & Ronda, 2015).

In the larger study, which explored how pre-service teacher education prepares PSTs to teach number concepts and operations in early years (Longwe, 2021), all these elements of the MDI framework were used to analyze how the TEs enacted the teaching practices of exemplification, explanation and PST participation in helping PSTs to learn how to teach number concepts and operations. However, the focus in this paper is on the element of learner participation, which is concerned with interactions that happen in a mathematics lesson. With this element, attention is focused on how learners are invited to talk mathematically and verbally show their mathematical reasoning (Adler & Ronda, 2015). Adler and Ronda (2015) characterize learner participation in three categories depending on the level of participation and opportunities they provide for learners to learn mathematics. Level one is where learners are invited to either answer yes/no or supply words to teachers' unfinished sentences. Level two is where learners are invited to answer what or how questions in phrases or sentences, while level three is where learners are invited to answer why questions, present ideas in discussion and the teacher re-voices, confirms or asks questions (Adler & Ronda, 2015). This framework was used in the present study to analyze how mathematics TEs invited PSTs to participate in their lessons about the teaching of number concepts and operations.

Methodology

This was a qualitative case study (Creswell, 2014). Four mathematics TEs were purposively selected for participation in the larger study where all the lessons were conducted through face-to-face mode of learning. Data being reported here is from two TEs. These were selected for this paper because data from one of the two TEs showed some differences in the way PSTs were invited to participate while the other one was selected as a representative of the three TEs who presented some similarities in the way they invited PSTs to participate. Data were collected through lesson observations. From the first TE (TE1), six lessons were observed. Two lessons focused on the teaching of place value of whole numbers and four lessons focused on the teaching of addition. This class had a total of 37 PSTs. From the second TE (TE2) four lessons on addition of whole numbers were observed. In TE2's class, there were 39 PSTs. It is important to note that the core-element of number concepts and operations is a wide area which encompasses many topics, and the topics of place value and addition of whole numbers also fall under this core-element (MoEST, 2017). Each of the lessons was videotaped and subsequently transcribed.

Data analysis was done by dividing the transcribed data into episodes. Coding for PST participation was done by indicating beside the utterances whether the PSTs participated by answering yes/no; answering what/how questions; or answering why questions, involved in discussions, or asking questions. At the end of each episode, a descriptive summary of all forms of participation was provided. The summary also included the number of occurrences of each form of participation. This quantification guided the analysis in determining the extent to which each form of participation was enacted and later make claims of how PSTs were invited to participate based on how each form of participation was enacted.

Findings

As stated earlier, this study explored how mathematics TEs invited PSTs to participate in the lessons on number concepts and operations. Following the MDI framework, analysis focused on whether PSTs were invited to either answer yes/no questions or one-word responses; answer what/how questions; answer why questions, present ideas in discussion, or ask questions (Adler & Ronda, 2015). Attention was also paid to how these forms of participation provided opportunities for PSTs to further their knowledge of number concepts and operations and learn how to teach these concepts to their learners. The findings are presented by first providing a summary of instances of how PSTs were invited to participate in all the lessons as shown in Figure 2. In presenting the findings in Figure 2, the forms of participation are presented as 'Y/N' where PSTs were invited to participate through answering yes/no or provide one-word responses, 'what/how' where they were invited to answer what or how questions, 'discussion' where they were invited to present ideas in discussion, 'why questions' where they were invited to answer why questions, and 'ask questions' where they were invited to ask questions. As explained in the methodology section, six lessons were observed in TE1's class (presented as L1-L6 in the graph) and four lessons were observed in TE2's class (L1-L4).

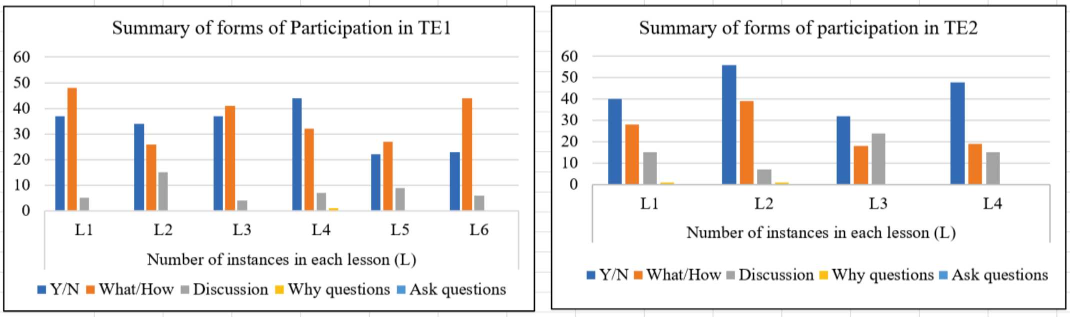

Figure 2: Summary of forms of PSTs participation in TE1 and TE2 lessons

As Figure 2 shows, TEs invited PSTs to participate through different forms of participation, as categorized in the MDI framework. PSTs were invited to participate through answering yes/no or one-word response, answering what/how questions, presenting ideas in discussion and answering why questions. However, these forms of participation were enacted to varying degrees.

Participation through answering yes/no or supplying one-word responses

While all forms of participation were observed in the lessons, findings indicate that TE2 mostly invited PST to participate through answering yes/no or supplying one-word responses. In all the 4 lessons the most common form of participation was where he invited PSTs to participate through this form of participation. For TE1, however, findings indicate that it was in 2 out of 6 lessons (L2 and L4), where he mostly invited PSTs to participate through answering yes/no or supplying one-word responses (see Figure 2). The following excerpt offers a representative example of how TEs invited PSTs' to participate through answering yes/no or supplying one-word responses (TE2, lesson 2 episode 1.2).

35 TE:	Yes. If I say ten plus nine, will this be a basic addition fact?
36 PSTs:	No
37 TE:	Why say no?
38 PST:	Because there's two-digit number

39 TE: Because we have used two-digit number, are we together?
40 PSTs: Yes
41 TE: Otherwise, we would like to concentrate on one-digit number added to one-digit number…In addition, what are addends? When you put two addends together, you get a sum, so what are addends. Yes!
42 PST: Are numbers which can be added
43 TE: The numbers that can be added. Are we together? So, in this case, if ten is added to nine, therefore, ten is a what?
44 PSTs: Addend
45 TE: Nine is a what?
46 PSTs: Addend
47 TE: And nineteen is what?
48 PSTs: Sum
49 TE: The sum, the result, are we together?
50 PSTs: Yes

In this dialogue, the TE focused on the basic facts of addition and how to come up with addition sentences from the basic facts of addition table. Throughout this dialogue, PSTs participated by answering yes/no or supplying one-word responses, except in utterances 38 and 42 where they went beyond giving yes/no or supplying one-word responses. This implies that most of the mathematical talk was being done by the TE while the PSTs were only supplying one-word responses.

Participation through answering what/how questions in phrases or sentences

Findings from data analysis indicate that PSTs were also invited to participate through answering what/how questions in phrases or sentences. For TE1, findings indicate that this was the most prevailing form of participation through which he invited PSTs to participate (see Figure 2). For TE2, findings have indicated that the most common form of participation was through answering yes/no or one-word responses, as indicated in the section above, but participation through answering what/how questions was also present as shown in Figure 2. Instances where TEs asked PSTs to answer what/how questions, such as to define concepts about number concepts and operations were observed. Below is an example of how PSTs were invited to participate through answering what/how questions.

TE: Our today's lesson is about place value, (writes place value on the board). Have you ever heard of that word, place value? Or what comes into your mind when you hear about these two words, place value? (TE1, lesson 1, episode 1.1)

In this excerpt, the TE invited the PSTs to explain what the concept 'place value' means. This form of participation invited PSTs to go beyond supplying single words to TE's questions, to reasoning about the concept of place value.

Participation through presenting ideas in discussion, answering why questions, and asking questions

In all the lessons, findings indicate some instances where PSTs were invited to participate through presenting their ideas in discussion. Some of the activities in which PSTs were invited to participate through discussion include discussing how to model addition of whole numbers using different resources such as place value box, spike abacus, and on a number line. Findings also show few instances where PSTs participated through answering why questions—one instance for TE1 and two instances for TE2 (see Figure 2). In these forms of participation, PSTs were seen to be exposed to more open discussions where they shared their mathematical thinking. Below is a representative example of instances where PSTs were invited to participate by answering why questions.

TE: Now, can you explain why addends in the basic addition facts do not go beyond nine? (TE1, lesson 3, episode 2.1)

This question provided opportunities for PSTs to go beyond just giving a definition, to applying their mathematical thinking considering the properties of basic facts of addition. Findings also indicate that participation through asking questions was not observed in any of the lessons (see Figure 2), implying that the PSTs were not exposed to the practice of getting to ask questions from their TEs.

Discussion and conclusion

In this study, an exploration of how TEs invited PSTs to participate in their lessons about how to teach number concepts and operations in early years, the observed TEs invited PSTs to participate through all the three forms of participation as characterized by the MDI framework, namely, participation through answering yes/no or supplying one-word responses, participation through answering what/how questions in phrases and sentences, and participation through discussions and answering why questions (Adler & Ronda, 2015). However, it was revealed that PSTs were mostly invited to participate through answering yes/no or supplying one-word responses and through answering what/how questions. In few instances, they were invited to participate through discussion and answering why questions. Also, findings have revealed that in these observed forms of participation, there were some variation in the way the two TEs enacted them. It was observed that TE1 mostly invited PSTs to participate through answering what/how questions, while TE2 mostly invited PSTs to participate through answering yes/no or supplying one-word responses. Although there were such differences, both forms of participation—participation through answering yes/no and participation through answering what/how questions—as enacted by the TEs, did not appear to provide enough opportunities for PSTs to engage in more active and meaningful participation. This finding appears to be in contrast with what Taylan (2017) argues about engaging PSTs in meaningful participation. Taylan (2017) contends that PSTs are expected to create active and meaningful learning situations for their learners. But for these PSTs to be able to do this with their learners, there is need for them to encounter similar learning opportunities during their pre-service teacher education. This implies that if PSTs are given opportunities to participate meaningfully in their mathematics lessons, they are likely to provide the same learning opportunities to their learners during their work of teaching. Thus, this finding from the current study suggests that the PSTs had limited opportunities to learn to do this to their learners.

During mathematics teaching, What learners are invited to say determines their opportunities to talk mathematically and demonstrate their mathematical reasoning (Adler & Ronda, 2015). The same can be said about PSTs, implying that exposing PSTs more to participations that required them to answer yes/no or one-word responses limited their opportunities to talk mathematically, develop and demonstrate knowledge about number concepts and operations. Also, when PSTs are only invited to supply one-word responses, it is difficult for TEs to spot misconceptions that PSTs might have about the mathematics they are learning (Taylan, 2017).

On the other hand, inviting PSTs to participate through answering what/how questions in phrases or sentences appeared to create opportunities for them to show what they know and also opportunities to learn about number concepts and operations, but it may not have provided enough opportunities for them to argue mathematically and expose their mathematical thinking. The few instances where PSTs were invited to participate through discussions and making presentations appeared to have provided opportunities for them to engage in greater and more meaningful participation where they shared their ideas and were able to make mathematical arguments. The MDI framework advocates inviting students to participations where they present their ideas in discussion, answer why questions and also ask questions (Adler & Pournara, 2019). However, the finding that the practice of inviting PSTs to answer why questions was rare and that there was no instance where they asked questions may indicate that the PSTs had limited opportunities to demonstrate their mathematical reasoning and to learn to provide such kind of participation to their learners. It might also indicate that the practice of facilitating meaningful mathematical discussions is challenging for TEs. Encouraging PSTs to ask questions is an important practice for teachers to learn to do as it promotes active learning and also helps to unveil misconceptions that PSTs might have (Taylan, 2017).

The findings of this study suggest that PSTs were mostly exposed to participations that provided limited opportunities for them to talk mathematically and verbally show their mathematical reasoning. The challenges in the way teachers teach number concepts and operations, and the resultant poor learner performance (Brombacher, 2019; Saka, 2019), as indicated in the introduction section, might be related to the way these teachers were prepared to teach during their pre-service teacher education— how they were invited to participate in their mathematics lessons as PSTs, as visualized in this study. In order for early years learners in Malawi to improve from performing under the expected achievement level in the core-element of number concepts and operations (Brombacher, 2011, 2019), an implication from this study might be to extend the ways PSTs are invited to participate in Malawi pre-service teacher education. Learners learn more from their mathematics lessons when they are given the opportunity to discuss mathematical concepts and argue mathematical solutions with others (Carpenter et al., 2003; Taylan, 2017). Therefore, exposing PSTs to such a learning environment would be useful in helping them understand primary school mathematics better, and at the same time help them to develop knowledge of how to facilitate such type of participations with their learners.

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Pre-service teachers using programming to design learning objects: A case study

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In this paper we are interested in understanding how pre-service teachers use programming to design digital learning situations (learning objects). We discuss a case study of a pre-service teacher creating a learning object to teach a mathematics concept. Using a development-process model and the instrumental approach, with its concept of scheme, we analyze the pre-service teacher's engagement with the activity of creating the learning object and identify two schemes that she developed and mobilized for articulating the learning trajectory and articulating it in programming language. The analysis of schemes highlights the need for understanding operational knowledge in the context of pre-service teachers' experiences of using programming to design learning objects.

Keywords: Instrumental approach, programming, pre-service teachers, learning objects.

Introduction

Integrating computer programming into education is increasingly becoming a necessity in all levels and fields, from preschool to life-long learning (Schina et al., 2021). In the decades since Seymour Papert published his seminal work *Mindstorms* in 1980, the increasing research has emphasized the importance of programming in supporting students' understanding of mathematical concepts (Wilensky, 1995). Accordingly, teacher education and instructional programs are creating new learning paths and integrating programming. In particular, the Department of Mathematics and Statistics at Brock University has integrated programming into mathematics education for mathematics majors and future mathematics teachers through a sequence of Mathematics Integrated with Computers and Applications (MICA) courses (MICA I, II, III –for math and science majors / III* –for pre-service teachers). The MICA program is the context of the study reported in this paper. The learning objectives of MICA courses are to develop mathematics concepts in conjunction with programming skills and to encourage mathematical creativity. In the progression of the sequence of these courses, students engage in 14 programming-based mathematics investigation projects (4 in MICA I, 5 in MICA II, and 5 in MICA III or III*). Unlike other projects where instructors specify the topics, in each MICA course final project, students choose a topic of their interests and the type of project. Pre-service teachers may choose to create a "learning object (LO)" (Muller et al., 2009), i.e., a step-by-step guided learning interactive object of a school mathematics concept, and to work individually or in pairs in its design, which may be relevant to their future profession. Research highlights the importance of developing pre-service teachers' understanding of computational thinking in the context of the subject matter, such as mathematics (Yadav et al., 2014). However, there is limited research on pre-service teacher' learning experiences (Aslan & Zhu, 2016). This paper focuses on MICA pre-service teachers' learning experiences of creating LOs. Specifically, the paper discusses a case study aimed at exploring a pre-service teacher's engagement, and her development and mobilization of schemes through the activity of creating a learning object.

Theoretical Framework

We frame the pre-service teachers' engagement with the activity of designing an LO using a development-process (dp) model (Figure 1), proposed by Buteau and Muller (2010), that represents the student engagement in the activity, involving multiples steps that arise in a dynamic and non-linear way.

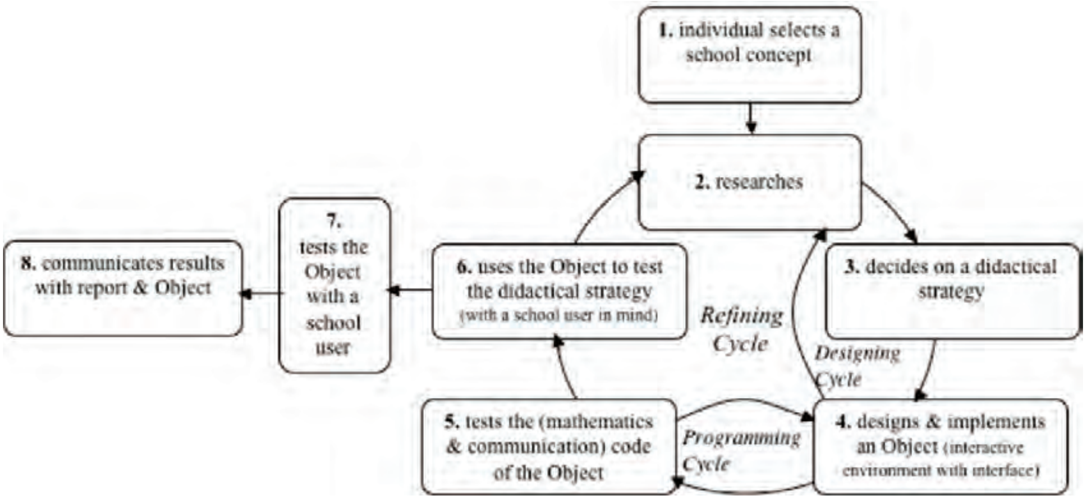

Figure 1: Development process model of an LO of a mathematical concept (Buteau & Muller, 2010)

Our understanding of pre-service teachers' development and mobilization of schemes is framed by an instrument-mediated activity approach –the instrumental approach– which was developed in the field of ergonomics to account for the active role that a user of an artefact plays, and the development of competence during his/her activity (Rabardel & Beguin, 2005). The instrumental approach has been further articulated and used in mathematics education research to conceptualize teaching and learning situations involving artefacts (Guin et al., 2005). In contrast with the dyadic subject-object interaction, the approach highlights the triad interactions among the subject, the instrument and the object towards which instrumented action is directed.

Critical to our study is the theory of instrumental genesis, which articulates a distinction between an artefact as a material or semiotic construct and an instrument as a psychological construct that emerges from the subject's activity with the artefact for a given goal. Put differently, "…during the activity and in situation… the user constitutes the artifact (whether physical or symbolic) as an instrument" (Rabardel & Beguin, 2005, p. 4) through an instrumental genesis process. The instrument is composed by a part of the artefact and a scheme of its use. A scheme (instrumented action scheme) is a stable organization of the subject's activity for a given goal, which is developed and mobilized by the user in action. It constitutes a whole or a set of mutually dependent components: i) one or several *goals* of the activity; ii) *rules-of-action* (RoA), to generate action, information seeking and control according to the features of the situation; iii) *operational invariants*: concepts-in-action (CiA), which are concepts considered as relevant and theorems-in-action (TiA), which are propositions considered as true and governing the RoAs; and iv) possibilities of *inferences* (Vergnaud, 2009).

Methodology

This paper is part of a 5-year ongoing research study, that seeks to examine how post-secondary mathematics students learn to use programming as a computational thinking instrument for mathematics inquiry. The naturalistic (non-interventional) study utilizes a mixed methodology and an iterative design approach. Data collected include each participant's programming-project assignments (e.g., the LO; and its associated report), and semi structured interviews with each of the participants after completing each of the project assignments. The design of the interview guiding questions was informed by the students' dp-model (Figure 1). In addition, data collected included post-laboratory session reflections and a questionnaire. After each of the 10 weekly 2-hour MICA lab sessions, participants recorded their reflections on their learning during the lab session (guiding questions were provided). All participants filled an online questionnaire before and after their MICA course. In this paper, we discuss the case study of a MICA II student, Kassie (pseudonym). Kassie was among eight MICA II participants recruited in year 2 of our larger study. Kassie was selected because she was particularly reflective and elaborative in her interview and lab reflections. Data for the case study include Kassie's final project –an LO and report, a semi-structured interview about her final project, and post-laboratory reflections related to her final project. The interviews were recorded as audio files and then transcribed into word documents. To describe Kassie's engagement with the activity, we analysed Kassie's final project interview, her report and the LO by trying to observe her activity in the steps of the dp-model. To analyse Kasie's schemes, first Kassie's interview and lab reflection data were coded individually by two coders, followed by a thematic analysis (Cresswell, 2014) done jointly by two coders. 16 subthemes were grouped in five main themes, two of which corresponded to strategies and perceptions. In addition to those themes, other themes specific to LOs were identified. Using codes under themes for the strategies (associated to rules-of-action) and the perceptions (associated to operational invariants) and informed by the steps in the dp-model, we then analysed the scheme according to its components (*RoAs* and operation invariants).

Findings

We present the results of the case study. First, we describe Kassie's engagement with the activity of creating the LO. Second, we present two examples of Kassie's schemes that were identified through analysis.

Kassie's engagement with the activity

As indicated below in Figure 1, the development process of an LO begins with the student selecting a school concept to teach (step 1). Kassie and her assignment partner started by looking for potential high school mathematics topics on the internet. They decided on the derivatives topic after discussing three possible topics between each other and their professor. They stated on their LO report that the focus was on first derivatives, specifically focusing on velocity. After deciding on the topic, they referred mostly to their personal notes from high school, the internet and library resources to find out when and how the derivatives are taught in the Ontario school curriculum and what the prerequisite mathematical knowledge for the topic would be (step 2). They chose grade 11 and 12 for their LO and assumed that the students would have sufficient background knowledge on derivatives and velocity because they would have just learned or would be in the process of learning about derivatives.

In deciding on didactical strategy (step 3), Kassie and her partner decided to focus on the derivatives as a concept and planned to show the relationship between the derivatives and the graphs. They aimed at enhancing the students' understanding of derivatives, give prompts and fun refreshers to help them understand one of the applications of derivatives–velocity.

After deciding on their didactical strategy, they began to design and implement (through code in the vb.net programming language) an interactive LO with a self- contained interface (step 4). Kassie stated that they gradually built on the LO as they were developing it. For this activity, Kassie and her partner created a random equations generator, derivatives and graphs and led the user to match the graphs to derivatives. They designed their LO in a way as to lead the user to find the derivative, then find the graph, then find the velocity. Kassie and her partner tested the interface in terms of its functioning, communication and/or navigation in several different cases, to ensure the accuracy of their coding (step 5). Kassie indicated that she and her partner had to debug the programming when they randomly generated the questions. She also stated that, after they tested the LO, they improved the look of the picture and textboxes for the user.

Kassie and her partner revisited their didactical strategy by having three university students who are not in a math program to use the LO and complete a survey to control if any change or improvisation was needed in the activities (step 7). Kassie stated that the feedback they received from the university students affirmed that their LO was adequate for high school grade 11 and 12 students, and that it portrayed the information in an entertaining and interactive way. Subsequently, they submitted the LO and their corresponding LO report, which included the didactical purpose and strategies, the target audience and the mathematical background of the target audience, a summary of the school pupil's experience, and a discussion (step 8).

Kassie's examples of schemes

Table 1: Scheme of Articulating the learning trajectory

Goal	Rules of Action	Operational Invariants (TiA or CiA)
Articulating the learning trajectory / development of the topic	I identify the way students learn the best I identify a concept development I identify an interesting context to capture interest (Friends) I identify the possible applications between the selected topic with other topics (not only in mathematics)	When learners can relate the topic that they are learning with something from their lives, they are more interested. (Theorem in action) High school students need interesting situations/contexts to learn math (Theorem in action). A good context or situation to learn is one that relates mathematics to other area (e.g., physics) (Theorem in action) Relation between math concepts and other contexts (Concept in action) Math concepts can be applied to other areas (e.g., physics)

Table 1 summarizes components of Kassie's scheme of articulating the learning trajectory (step 3 of the dp-model). During the interview, Kassie was asked to articulate on what kind of a didactical strategy she used for the LO user to learn, and she noted:

> Well, we wanted them to learn the most basic way possible, but in a way that they could relate to, not something that's like just straight math. So, we put like a storyline and tried to enhance that so they would be interested in actually doing it. (LO, 15)

We interpreted this description by Kassie as indicating a rule of action such as "I identify the way students learn the best" and a theorem in action such as "When learners can relate the topic that they are learning with something from their lives, they are more interested". We interpreted Kassie's reflection on how the students learn the best as she is aiming to establish connections between real life situations. Kassie assumed that a story that LO users can relate to, is likely to draw the targeted age group's interest to the mathematics activity. Kassie was asked if her focus was more on the algebra and calculations, or more on the concept of what a derivative is. She stated:

> I think I focused more on the concept of what a derivative is, because we were looking at velocity and how fast something moves. So, we did a Friends (tv series) theme, and we gave them three options for Ross (a character in Friends) to get to work, and what was the fastest way for him? So, we said that they had a background in derivatives, we just wanted them to understand the concept of velocity and what that means and how you see if something has a faster velocity than something else. (LO, 14)

We interpreted Kassie's answer as indicating many RoAs such as "I identify a concept development", "I identify an interesting context to capture interest", and "I identify the possible applications between the selected topic with other topics that are not necessarily in mathematics". Kassie's emphasis on the derivatives as a concept, indicated that she intended to design the LO to develop the user's understanding of the concept with its relationship to other concepts such as velocity. In her explanation, Kassie identified a learning objective (comprehension of the derivative concept); and its relations to other areas (velocity in physics); and how the mathematics problem could be presented to the learner in a relatable and entertaining context to capture interest (the Friends theme and going to work). On her LO report, she expressed her thoughts on how learners are more interested when they can relate the topic with something from their lives, stating:

> By having our program F.R.I.E.N.D.S. themed, it not only engages the students in what is being taught to them, but it also gives them ideas and leeway into being creative with other concepts of math in order to better their own understanding. The students will gain a better understanding of velocity by comparing the velocities of three different modes of transportation, and actually conducting their derivatives in order to find the fastest velocity. This not only enhances their understanding of derivatives, but also allows them to explore a real-life situation in regard to velocity. (LO Report)

We interpret her explanation as she regards a good context or situation to learn is to be one that relates mathematics to another area such as physics.

Table 2 summarizes the components of scheme of articulating the learning trajectory in programming language. The scheme relates to the planning stage of step 4 of the dp-model.

Table 2: Scheme of articulating the learning trajectory in programming language

Goal	Rules of Action	Operational Invariants (TiAs and CiAs)
Articulating the parts of the learning trajectory in programming language	I identify the parts of the learning trajectory to be coded I code the parts separately step by step (e.g., equations, graphs) I use the story/context of the to code the parts I use my previous programming knowledge from previous assignments to code the parts	Learning trajectory shows how the user will develop understanding of the concepts to be learned The parts of learning trajectory must align with the story/context of the learning Programmed math must help the user to visualize the mathematics

When Kassie was asked how she incorporated the graphs with the concept of velocity and derivatives in her design, she expressed that she envisioned the user to first understand what a graph is and how it looks, then understand what a derivative graph is and how it looks and finally understand the relationship of velocity with the graphs and the concept of derivatives through finding the actual velocity. Kassie noted: "We wanted them to know what the graph looked like, know what the derivative graph looked like, and know like, what the actual velocity was." (LO, 15)

We interpret the above as Kassie developing a RoA "I identify the parts of the learning trajectory to be coded" supported by the TiA, "Learning trajectory shows how the user will develop understanding of the concepts to be learned." To design and execute these steps in the LO, Kassie and her partner started to design and work with the original graphs, then with the derivative graphs and finally find the velocity to imitate the user's steps. Kassie noted: "So, I think we started with the original graphs, and then we started with the derivative graphs and then we went to find the velocity." (LO, 15)

We interpreted this explanation as Kassie developed a RoA, "I code the parts of the learning trajectory separately step by step (e.g., equations, graphs)". Kassie further elaborated on how she and her partner embedded a storyline –in this case, a character from a TV series trying to go to work– as they coded step by step the parts of the LO:

> We built on the learning object, because we explained the story where "he needs to get to work, let's try and find him the fastest way". So, then they had the graphs, the original graphs, and then they had to match them to what they thought it was, and then they took the derivatives, or they practiced derivatives, just like random equations, then they took the derivative graphs and matched it to the derivatives that they took and then they found the velocity. So, it was kind of "here's your first part, then find the derivative, then find the graph, then find the velocity. " (LO, 16).

We interpret this explanation as Kassie developing a RoA, "I use the story/context of the learning task to code the parts" and the TiA, "The parts of learning trajectory must align with the story/context of the learning object."

When Kassie was asked why they wanted to include the graphs and the function with its derivative in their LO, she elaborated:

> I feel like it's really important because they make sure you know it in high school but also for our own understanding, I thought that "you want to know what that looks like", you want to know that the derivative of a parabola is like a straight line. I feel that is really important to know, so that they can visualize it while they're actually learning, the concept of it. (LO, 26)

We interpret the above explanation as Kassie developing a TiA such as “programmed math must help the user to visualize the mathematics”. Kassie was asked if she referred to her previous programming knowledge that had been covered previously in the course and she stated:

> Yes, definitely. Because there was certain things that I didn't remember how to do that, but I know we did it in previous assignments, so it was easy to go back and be like "this is how you do it, okay let's do it like that". (LO, 18)

We interpreted Kassie’s answer as indicating a RoA she developed during her design process of the LO as “I use my programming knowledge from previous assignments to code the parts”.

Discussion and Concluding Remarks

In this paper we presented a case study of a pre-service teacher experience of using programming to design a learning object. Using the dp-model and concept of scheme we described a case of a pre-service teacher engagement, development and mobilization of two schemes for the activity. The concept of scheme is crucial in understanding Vergnaud’s (2009) distinction between operational form of knowledge (action in the physical and social world) and the predicative form of knowledge (linguistic and symbolic expressions of the knowledge). While both kinds of knowledge are important in understanding pre-service teachers’ activity, mathematics education research tends to focus more on predicative knowledge (Vergnaud, 2009). Our analysis of engagement and two schemes developed by the pre-service teacher highlights the need to understand operational knowledge in the context of pre-service teachers’ learning experiences of using programming to design learning objects. As integrating computer programming into mathematics education is increasingly becoming a necessity, more research is needed to understand this kind of knowledge and its implication to mathematics teacher education. In our previous work on exploratory objects for pure or applied mathematics investigations, we have argued that the schemes that students develop and mobilize are associated to steps in the related dp-model (e.g., Gueudet et al., 2020). Likewise, our case study on learning objects indicates that the schemes that pre-service teachers develop and mobilize are associated with the dp-model for LOs. The two pre-service teachers’ schemes identified in the case study are associated with Step 3 and 4 of the dp-model (Figure 1).

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Investigating the concerns of post-primary mathematics teachers towards problem-solving following curriculum reform

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Keywords: Teacher concerns, problem-solving, curriculum reform.

Background to the Research

In line with an international trend of curriculum reform towards a greater focus on problem-solving and mathematical literacy, significant reforms of mathematics curricula have taken place in Ireland in recent years. A new secondary curriculum was introduced in 2010 representing a comprehensive overhaul of content, assessment, and teaching and learning practices (Johnson et al., 2020). Incorporated in this reform was an attempt to move towards a more problem-solving approach to teaching mathematics. In 2015 the mathematics curriculum was revised once again for the junior cycle (ages 12/13 – 14/15). Classroom-based assessments (CBAs) were introduced representing a dramatic change in teachers' roles assessing their students' learning, as formal assessment in Ireland has traditionally been anonymous and centralized without any input from a student's teacher.

Teachers play a central role in the adoption of curriculum reforms and the successful enactment of reform depends on the teachers who will interpret and implement it (Spillane, 1999). However, educational reforms can aggravate teachers' concerns (Charalambous & Philippou, 2010), thereby influencing the implementation of reforms. Identifying and attending to the concerns of teachers can contribute to the successful implementation of educational reforms (Christou et al., 2004).

This research intends to gain an insight into the nature of Irish secondary mathematics teachers' concerns about problem-solving and asks the question: What is the nature of Irish secondary mathematics teachers' concerns with regards to problem-solving and the associated classroom-based assessment following recent curriculum reforms?

Methodology and Theoretical Framework

Hall et al. (1977) proposed seven "Stages of Concern" (SoC) which teachers experience as they implement a reform: awareness, informational, personal, management, consequence, collaboration, and refocusing. More recent studies have identified these stages of concern in teachers' interpretation and implementation of reform (Charalambous & Philippou, 2010) and found a pattern where teachers move through these stages, though not necessarily linearly, as a reform is introduced, implemented, and becomes established (Johnson et al., 2020). It has been suggested that the success of a reform depends on this development of concerns (McKinney et al., 1999).

This research uses a qualitative approach to generate data on teachers' concerns around problem-solving and the CBAs. Semi-structured interviews were conducted with 12 mathematics teachers, representing a range of teaching experiences (e.g. gender, mathematical background, years of experience etc.) and school contexts (e.g. urban/rural; single-sex/co-ed; small/large pupil population etc.). The SoC framework informed the design of the interview and data analysis is being undertaken using this same framework. The anonymized transcripts are being analysed through qualitative content analysis with the seven stages of the SoC framework serving as “a priori” codes.

Preliminary Findings

Following the initial phase of data analysis, the research provides a snapshot of participating teachers' concerns in each of the seven stages of the SoC framework. All participants demonstrated management concerns, most of which were related to time pressures with regards to classroom time within an already over-loaded curriculum.

> Mary: It's just being able to facilitate [problem-solving] in the classroom under the time constraints seems to be a serious challenge to me.

Collaboration was a dominant theme. It was highlighted as a source of support but there was also an explicit desire for more collaboration with colleagues.

> Cillian: I think having more opportunities there to engage with other schools around and share practice there would be the biggest thing.

Findings from this research will feed into the design of a specific teacher development initiative, where curriculum materials will be developed in order to provide targeted support for mathematics teachers enacting these reforms.

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Revisiting the concept of reflection: An empirical investigation of mathematics teachers' articulations on the teaching practice

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The scientific literature points to the importance of reflection and its contribution to teachers' work. However, the reflective process involves some challenges. Definitions of reflection are abstract, and models of reflection are complicated and not always accessible to practitioners. The goal of this study was to unpack the concept of reflection into categories – actions or phases which are part of a comprehensive reflective process. For this aim, we analyzed 11 mathematics teachers' reflective expressions, collected in three different settings of professional development. Six categories of reflection were identified in the study, which suggest different actions that teacher educators can consider when designing guiding tools to support teachers through the reflective process.

Keywords: Reflection, professional development, journal writing, stimulated-recall interviews.

Introduction

Beginning with John Dewey, reflection is a central concept in many fields, among them the field of mathematics education. Reflection is a special kind of thinking, a process of looking back at experiences and learning through and from them (Finlay, 2008). Involvement in a reflective process is considered to be important and beneficial for teachers' professional development: Reflection enables teachers to be more aware of their actions and of the beliefs and assumptions that stand at the basis of these actions, so decision-making processes become more deliberate (Finlay, 2008; Karsenty & Arcavi, 2017). Reflection is linked to the mechanism of knowledge development (Clarke, 2000; Karsenty et al., 2015; Schön, 1983), and was also found to be a key aspect in processes of change regarding teaching practices (Clarke, 2000; Karsenty & Arcavi, 2017; Schwarts & Karsenty, 2020).

However, there are still some considerable challenges in understanding this concept. Specifically, there is a need to understand what it is that mathematics teachers do when they reflect on the teaching practice. In this study, we analyzed mathematics teachers' reflective expressions, with the goal shedding light on the complex concept of reflection, and to unpack the definition of reflection into categories – actions or phases which are part of a comprehensive reflective process.

Theoretical Background

Based on John Dewey's ideas, Schön (1983) related to reflective practices, where practitioners examine their actions. Schön distinguished between two types of reflection: (1) *reflection in action* – the practitioner's thinking about his/her actions while doing them; and (2) *reflection on action* – a process where the practitioner is consciously looking back at a situation and critically examining, analyzing, and evaluating it, in order to gain new insights which will improve future practice.

Dewey's and Schön's ideas led to development of various definitions for the concept of reflection, which relate to diverse kinds of categories of reflection, to aspects included in the reflective process, to the purposes of the reflective process, or to its results (Finlay, 2008; Lyons, 2010). In the field of mathematics education, Clarke (2000) distinguished between three types of reflection: the first two are similar to Schön's – *reflection in practice* and *reflection on practice*. The third type, *reflection for practice*, refers to the development of teachers' awareness to decision-making processes. Geiger et al. (2016) offer a two-dimensional framework for looking at mathematics teachers' reflections: the level of reflection (technical, deliberate, or critical reflection) and the object on which the reflection is focused (self, practice, or students). As the basis of this study, we chose the following definition, which is concise, yet relates to diverse aspects of reflection which the literature emphasizes:

> [Reflection is a] detailed, analytical, and careful observation of ''what was done'' in order to attempt to understand intentions, plans, actions and utterances and to consider alternative decisions and their possible implementation (Karsenty & Arcavi, 2017, p. 435).

Different models were developed in order to support teachers, and other practitioners, in the process of reflection. Two central models are Gibbs' (1988) Reflective Cycle, which offers six phases for the process, and Korthagen's (2014) ALACT model, comprised of five phases, alongside the 'core reflection' model, which highlights the aspects teachers should relate to in the reflective process. Finlay (2008) stresses that different models for reflection are discerned from each other in the phases and actions they include, the aspects that the reflection relates to and the tools and means that the model offers (guiding questions, peer discussions, writing journals, etc.).

However, although reflection is a keyword in the field of education for many decades now, the literature shows that it is not easy to be involved in a reflective process (Finlay, 2008; Korthagen, 2014; Lyons, 2010). The definitions for reflection are somewhat abstract, and models for reflection include many components. It is difficult for teachers, as well as for teacher educators, to understand what the process of reflection should look like in actuality (Brown & Coles, 2012; Finlay, 2008; Korthagen, 2014; Lyons, 2010). Since the guiding lines for reflection are not clear, teachers tend to reflect on their teaching in technical, superficial, or inefficient ways, which do not necessarily support their teaching (e.g., Korthagen, 2014).

Rationale and Research Question

As we aim to support mathematics teachers as they are involved in reflective processes, the overarching goal of our study is to bridge the gap between the theoretical knowledge that exists on reflection on the one hand, and the way it is implemented and practiced by teachers, on the other hand. In our study, we analyzed mathematics teachers' articulations, in which they reflected on the teaching practice in three different settings of professional development (details follow). When we tried to analyze these articulations with the definition of Karsenty & Arcavi (2017), we found this definition does not capture all the actions that teachers perform when they reflect on their practice. Other existing definitions were not operational either. Therefore, the first step of the study, reported in this paper, was aimed at unpacking the definition of reflection into operational categories which relate to actions that mathematics teachers perform when they reflect on their teaching practices. Accordingly, the research question was: *What categories of reflection can be identified in mathematics teachers' reflective articulations, within different settings, and how do these categories relate to, correspond, or add to the existing literature?*

Methodology

Study participants

70 secondary mathematics teachers took part in one of seven VIDEO-LM PD courses held in Israel in 2015-2016. Out of the 70 teachers, 11 teachers were chosen to participate in this study. The 11 teachers were selected to represent various characteristics of the 70 teachers, such as teaching record, gender, academic education, and teaching experience (high/middle school; low/advanced track, etc.).

Data collection

For each teacher, data was collected from the following sources, which refer to three different settings of professional development that allow teachers to reflect on their mathematics teaching practices:

(1) **VIDEO-LM PD meetings**: The participating teachers all took part in one of seven VIDEO-LM PD courses' meetings held in Israel in the academic year 2015-2016. The VIDEO-LM project (**V**iewing, **I**nvestigating and **D**iscussing **E**nvironments of **L**earning **M**athematics) is a project aimed at enhancing mathematics teachers' reflection skills and mathematical knowledge for teaching. A typical VIDEO-LM course is composed of 7-10 meetings, 30 hours in total. In each meeting, teachers watch a videotaped mathematics lesson and discuss it, while using the "six-lens framework" – an analytic framework focused on aspects of mathematics teaching (see Karsenty & Arcavi, 2017). All meetings in the seven PD courses were videotaped. From these meetings, all excerpts in which the participating teachers spoke were transcribed. The number and length of excerpts vary between the teachers.

(2) **Weekly journals**: The 11 teachers wrote personal journals on a weekly basis, for five months. In these journals, the teachers were asked to write about the most significant event that happened to them during the past week, either while preparing for class or during a mathematics lesson. They were also asked to explain why it was significant for them. There were no additional guiding questions, and the teachers wrote the journals independently and without feedback. Each teacher wrote between 10 to 19 journals (15 in average). The average journal's length was 165 words.

(3) **Stimulated-recall interviews (SRI), based on a videotaped lesson**: One lesson of each of the participating teachers was videotaped. The teachers chose which lesson will be videotaped, with no limitations regarding the class, level of group or the subject/setting of the lesson. A while afterwards, an individual interview was held with each of the teachers, where the first author watched their videotaped lesson together with them. The teachers were asked to stop the video whenever they saw a "issue of interest" which they wanted to talk about. The conversation was held in an open manner, with no specific instructions, with requests for clarifications as needed. The interviews lasted some 60-90 minutes. The teachers stopped their videos about 10 times in average, and in most cases the conversation continued after the video watching has ended. All the interviews were videotaped and transcribed.

The three settings differ in many features, like the manner of expression (written or oral); the object in the basis of reflection (the teacher's own experiences, or another teacher's lesson); and the level of guidance in the settings (high level of guidance in the VIDEO-LM meetings compared to minimal guidance in the weekly journals and in the SRIs). These differences influence the manner of reflective process, and so can highlight different aspects in teachers' reflections.

Data analysis

The data analysis process consisted of moving back and forth between two processes: (1) The teachers' reflections in the different settings were analyzed in an inductive approach, in order to identify actions that teachers perform in actuality when they reflect on their teaching (i.e., analyzing the situation, referring to intentions\plans or considering alternative decisions). (2) Examination of definitions and models of reflection offered by the literature, both inside and outside mathematics education. While the result of the inductive process was in a list of coded categories of reflection, the comparison to the literature enabled us to refine these categories.

Results

The analysis resulted in the identification of six main categories of reflection (some have sub-categories): (1) analysis of the situation; (2) consideration of alternatives, doubts, or dilemmas; (3) re-orientation; (4) consideration of beliefs; (5) addressing the emotions that a situation evokes; and (6) addressing challenges of teaching. Below we describe and demonstrate each of these categories.

Category (1): Analysis of the situation

A central part of a reflective process is to look back at a situation and analyze it. In different models of reflection, such as the 'Reflective Circle' (Gibbs,1988), this is the first phase of the process. However, the literature is not clear on what is included in such an analysis. Therefore, we decompose this category into four subcategories: (a) Observation of 'what was going on' and thinking about reasons for what has happened; (b) Consideration of goals that stand at the basis of the teacher's decisions and actions; (c) Attendance to broad assumptions and contexts, such as social, political, affective, or institutional assumptions; and (d) Evaluation of the situation and of teachers' actions (e.g., practices, assignments, or interactions). In order to exemplify these subcategories, we will describe a section from Diana's SRI, where she watched her own videotaped calculus lesson:

After watching a part of the lesson where she phrased some conclusions to her students, Diana stopped the video and talked. She began with an evaluation of her action: "Horrible! I remember that while I phrased the conclusion, I felt I wasn't phrasing it in a good way. And now when I watch it, I wonder if it is even worthwhile to write down this conclusion". Diana then related to the reason for her action: "I did it because I always think about the weak student that learns in the advance track, and there are many here". Then, Diana explained that "We used to have only 12 students, in average, in the advanced level class […] and now we have 40 students. It is clear to us, the mathematics teachers, that not all of them are in the same level, and they need our help". This explanation referred to the political context of the situation: due to the Israeli Ministry of Education' reform, which aims at increasing the number of students in the advanced track classes, Diana' class became larger and more heterogenous. Diana continued and deliberated her reasons, while considering her goals:

Diana: When I phrase rules and conclusions, I always think about these students, that must have this reminder when they do their homework. It is a good goal, right? But now I think that these things must be well prepared, in advance. If a weak student gets home and reading his notebook – does he understand what is written there?

The first two subcategories were prominent in all the three settings, while the subcategory *evaluation of the situation and of teachers' actions* was identified mainly in the weekly journals and in the SRIs. The subcategory *attendance to broad assumptions and contexts* appeared sometimes in an implicit manner in the teachers' articulations. We will also mention that the subcategories are connected to each other and the distinction between them can be delicate, as can be seen in the example above.

Category (2): Consideration of alternatives, doubts, or dilemmas

This category includes expressions where the teachers discussed alternative actions, practices or perspectives, with correspondence to the second part of the definition of Karsenty & Arcavi (2017). This category also includes expressions where the teachers referred to dilemmas or deliberated some issues, which quite often were accompanied with consideration of alternative actions. The following excerpt exemplifies this category, when in her second VIDEO-LM PD meeting, Mila related to a situation she identified in a videotaped probability lesson in an 11th grade class:

Mila: It seems like the [videotaped] teacher believes she should present many solutions, and so she went along with the student's idea […]. It is a dilemma, to choose whose idea to hear, because some students can make things complicated. For instance, I sometimes choose to hear someone's idea personally, so he will not confuse the other students. I have this dilemma, but I didn't feel this teacher has it. On the contrary, it seems like she encourages the most problematic students, goes along with them: 'show me your way'.

According to Dewey, a situation of dilemma or doubt is the basis for reflective process. Consideration of alternative actions can be found also in models for reflection. For instance, Korthagen's (2014) ALACT model, creating alternative methods of action is the fourth stage, before a new trail. In the data, this category was more prominent in the VIDEO-LM PD meetings than in the other two settings.

Category (3): re-orientation

This category refers to gaining new insights or adjustments, and includes two subcategories:

(a) "looking forward": Expressions where teachers refer to possible future actions, as a result of their analysis. There is a subtle difference between this subcategory and the category of considering alternatives. Expressions were identified as "looking forward" when teachers referred to new insights they gained from their analysis which *could* be used in future actions. However, expressions were identified as considering alternatives when teachers explicitly referred to "what would I have done differently". The following excerpt, taken from Ivan's 8th weekly Journal, exemplifies this category:

Ivan: A concluding lesson in the subject of 'scale', in a low-track 8th grade class. I asked the students to draw a sketch of the class's tables. Then, after a short discussion, the students drew objects they chose. The lesson was more free than usual. During the lesson I felt uncomfortable, like there's no real learning. But in the end of the lesson, I was pleasantly surprised [by the students' cooperation and outcomes]. I plan to combine more activities of this kind in my teaching. I believe I should "let go" more often, to give the students opportunities to learn and to make mistakes.

(b) A change in perspective: Expressions in which a change could be identified in teachers' beliefs, attitudes, or perceptions regarding the teaching practice or the students. The change was either articulated explicitly, for example when teachers used expressions like: "I learned", "I realized" or "I noticed", or it was implied, for instance by a change in the teacher's tone (e.g., decisiveness vs.

hesitancy), or when a teacher expressed openness to new ideas. The following excerpt, taken from Nora's 4th weekly Journal, exemplifies this category:

Nora: This student's case helped me realize that students can go through a real change, and I should not stop believing this can happen (although, unfortunately, sometimes it happens to me).

Re-orientation is emphasized in the literature (Clarke, 2000; Finaly, 2008; Schön, 1983), since gaining new insights and tools for future practice is, in a way, the purpose of a reflective process. In the data, however, this category was identified only in some of the teachers' expressions, and it seems like teachers need more guidance in order to implement it.

Category (4): consideration of beliefs

Expressions where teachers explicitly referred to their beliefs, orientations and attitudes towards mathematics and mathematics teaching and learning – considered, wondered about, or doubted these beliefs. The following excerpt exemplifies this category when Sam, in his 4th VIDEO-LM PD meeting, watched an analytic geometry videotaped lesson and related to the teacher' choice:

Sam: How mathematics should be learned – is a question. In my opinion, this subject gets pretty heavy later, so [the videotaped teacher] tried to soften it in the beginning, when he connected it to worlds which were relevant to the students. I liked it. I teach both stronger classes, where the students are more active, and harder classes, where the students are less confident, less engaged, they don't even try to understand. But we should think about how to connect them [to the mathematics].

Beliefs influence teacher's actions and decisions (Karsenty & Arcavi, 2017), and awareness to teachers' beliefs is mentioned as one of the reflective process' goals (Finlay, 2008; Korthagen, 2014). This category was prominent in the data, in all the three settings and within all teachers' expressions.

Category (5): Addressing the emotions a situation evokes

Expressions where the teachers addressed emotions that certain situations evoked in them. This was realized explicitly ("I was glad to see", "it is hard for me"), but was also realized implicitly: For example, in a change of tone, in gestures, or in use of exclamation marks. The following excerpt, taken from Michelle's 2nd weekly Journal, exemplifies this category:

Michelle: A student does not respond to any of my questions, and not for the first time. For 10 minutes (during the lesson) I tried to get her to say something – but not even one word!! After the lesson – again, nothing! I was very angry about this situation. In similar cases before I was told that this student is "weird". This time I was very disappointed and could not tolerate such behavior. I turned to all the concerned parties and asked for their help […] I'm racking my head, how to deal with her.

This category was identified in some of the weekly journals and SRIs, but its frequency was relatively low. However, we decided to include it since the literature emphasizes the importance of connecting between a situation and the emotions it raises (Gibbs, 1988; Korthagen, 2014; Schön, 1983).

Category (6): addressing challenges of teaching

Expressions where the teachers addressed challenges and difficulties they experienced during their work, and analyzed them: the source of the challenge, its implications on their decision, etc. The teachers addressed various challenges: mathematical-pedagogical challenges (e.g., how to present mathematical subjects); challenges in classroom management; addressing students' pedagogical, personal, or emotional difficulties; challenges the teachers experience as part of their belonging to the institutional system. The following excerpt, taken from Leo's SRI, exemplifies this category:

Leo: Often, when I give assignments, many students have questions. And I still don't know how to, let's say, 'split myself' between them. Many times, I give a direction [to one student] 'do this in the meantime' and then go to another student. And then they tell me 'Stay, don't go!'. They don't understand that they are not alone, that while they do something I can help other students as well. This is something I still didn't figure out how to handle.

This category appeared sometimes with proximity to consideration of doubts or dilemmas. However, we chose to include this category separately since it was prominent in the teachers' expressions, especially in the weekly journals and in the SRIs settings.

Discussion

In this study, we analyzed the concept of reflection in a process that went back and forth between reading the scientific literature and gaining insights from the data. This process resulted in the unpacking of the concept of reflection into six main categories. The identified categories offer actions that extend the categorizations found in the literature. Furthermore, they relate to different aspects of a mathematics teacher's practice and require the teacher to address various factors, some of which are not emphasized in the domain of mathematics education. Observing different teachers through the three different settings simultaneously, resulted in a broad perspective on the concept of reflection. Such perspective might not have been possible from an analysis of only one setting. Connecting between the empirical analysis to the literature suggests validation of different aspects of reflection, in the sense that the categories are congruent with definitions and models offered by the literature.

The study suggests practical implications teacher educators can consider when designing guiding tools to support teachers through the reflective process. Following Brown & Cole's (2012) question "how do we do reflection?", the categories offer actions from which guiding questions for reflection can be derived, such as: What are the goals, assumptions or beliefs that influence your actions and goals? What alternative actions\practices could have been taken, and what are the advantages or disadvantages of each one of them? What emotions does the situation induce? What challenges can be identified in the situation? What can be done differently in a similar situation in the future? This study also offers theoretical contributions: The categories can be used as a theoretical framework for analyzing reflective expressions of teachers. Said framework was implemented in a continuance study, where we looked at opportunities for reflection teachers have in different PD settings.

The literature relates to issues and challenges of involvement in the reflective processes, such as a struggle to deeply analyze the teaching practice, to consider emotions situations evoke or to connect situations to broader contexts (Finlay, 2008). As the categories of reflection were identified in the data, the study indicates that teachers are **able** to perform such actions. However, the variety of categories stresses that reflection is indeed a complex concept, and teachers should have an adequate guidance and tools in order to be involved in a beneficial reflective process. Furthermore, teachers should experience reflective processes in different PD settings, as the results indicate that teachers' actions are influenced upon different settings and contexts in which the reflection takes place.

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Operationalising de-ritualisation for the analysis of teaching-as-usual

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To trace and represent changes in mathematics teaching over time is a challenge for research. This methodological paper proposes a framework for visually representing teacher moves in mathematics lessons. Based on commognitive theory, a framework of de-ritualising moves towards explorative teaching is developed. Two lessons of mathematics student teachers 'teaching-as-usual' in their final practicum are used for the development of a coding-scheme. The analysis enables to represent several de-ritualising moves, and the visual and fine-grained analysis is a promising result for longitudinal research on de-ritualisations in teaching-as-usual.

Keywords: Commognition, teacher education, secondary mathematics, analytic framework.

Introduction

This methodological paper proposes a framework for visually representing teacher moves in mathematics lessons. The challenge is to capture important characteristics in mathematics teaching, and at the same time avoid normative or idealised views of teaching. Research in mathematics teacher education was found to often take a fidelity approach, where teaching is researched relative to particular reform idea or intervention (Österling & Christiansen, 2018). In a classic article, reform mathematics is claimed to depend on cognitively demanding instructional tasks with multiple possible responses (Stein et al., 2008). However, the majority of mathematics lessons are not based on such tasks, and Nachlieli and Tabach (2019) remind us that the resilience among teachers to change in line with reforms may indicate that there are gains in engaging in ritual, so called 'traditional' teaching. In Sweden, contrasting norms are balanced in mathematics teaching. On the one hand, collective reasoning where both learners and teachers contribute with arguments is found (Eriksson & Sumpter, 2021). On the other hand, textbooks do *not* privilege problem solving tasks (Brehmer et al., 2016), and learners value mainly facts and procedures as important in mathematics learning (Andersson & Österling, 2019). In this paper, I use *teaching-as-usual* to describe teaching which is *not* part of a research intervention or reform initiative.

Categorisations, as traditional versus reform-based teaching, imposes a dichotomising view on mathematics teaching. Under dichotomising views, learning to teach means to choose between extremes, rather than becoming aware of complexities. Despite training in so-called reform-based or student-centred teaching, mathematics student teachers find it challenging to pursue such teaching in their classrooms (see f. ex. Bahr et al., 2014). When norms of the school context contradict norms from campus-based education, student teachers abandon the norms of campus-based education (Nolan, 2012). In addition, idealised views of mathematics teaching make norms and practices in the field unintentionally devalued (Horn & Campbell, 2015). However, when norms and practices are implicit in teacher education, they are less available to learn (Christiansen et al., 2019; Helgevold et al., 2015), and the dilemma between normative views on teaching and visible teaching practices raises a need for a descriptive way of talking about mathematics teaching, where dichotomies as good or bad teaching are avoided.

For that purpose, this paper aligns with recent developments in commognitive theory, and operationalises commognitive routines for the analysis of teaching. An assumption in commognitive theory is that mathematics learning, like all human activity, entails a modelling of actions based on what was done in the past in a similar situation, a *routine* (Lavie et al., 2019). Routines range between rituals and explorations, where *explorations* are the desired end-condition for learning, but *rituals* the essential starting point. Ritual activities are performed to please someone else, hence, ritual learning is to recognise and stepwise imitate different processes. Explorations on the other hand, are oriented towards truths about mathematical objects, where the performer has control over both the enactment and the end-conditions for the activity. Keeping in mind to not re-introduce dichotomies, Lavie et al. (2019) claim that no mathematics activity is fully-fledged rituals or explorations. Instead, teaching and learning mathematics means participating in *de-ritualising moves*, moving gradually from ritual routines, towards explorations. Six such moves are suggested by Lavie et al. (2019): flexibility, bondedness, applicability, performer's agentivity, objectification and substantiability, detailed below.

The strength in using de-ritualising moves for describing teaching is that it connects disparate aspects in mathematics teaching to a move towards explorations. The aim of this paper is to develop a coding-scheme for de-ritualising moves in teaching, and provide a visible representation of teaching-as-usual, to be able to compare lessons. The question posed is: What adaptations are necessary to operationalise de-ritualising moves for the analysis of teaching-as-usual?

Methodology

This section describes the data and context for the adaptations, and thereafter the steps taken for developing the analytic framework and visible representations.

Data

Two lessons were used for the development of the coding-scheme. The lessons were led by Eva and Kristin (pseudonyms), two secondary mathematics student teachers. Both Eva and Kristin followed a programme for upper secondary teachers in mathematics. The included lessons were part of their final practicum. Both had substantial mathematical and mathematics educational knowledge, however, commognitive theory, which is the basis of analysis in this paper, had not been part of their education. The students both agreed to participate in the TRACE-project (see acknowledgements), a longitudinal study where new teachers are followed from their final practicum through the first three years of teaching. For 17 student teachers, now teachers, over 90 lessons were video-recorded, with researchers present in the classroom, and with informed consent from participants.

The lessons were selected as examples of teaching-as-usual. They were similar in their mathematical level, and both were the last lesson before a test. The lessons reinforce what is important content, and learners can be expected to be familiar with it. Kristin taught a mathematics intensive class in the first year of upper secondary. The learners had worked on a set of three examples as homework, so-called "real-world situations" described by mathematical models: a linear, a quadratic and an exponential. Eva's lesson was from the third and final year in upper secondary mathematics. Eva had prepared three examples on antiderivatives to present in front of the whole class, thereafter, five different sets of examples were provided for learners to work on in five groups.

Method for developing the analytic framework

The developing framework of de-ritualising moves is based on recent developments in commognitive theory. An important contribution is the work by Nachlieli and Tabach (2019), who researched opportunities to learn in ritual-enabling and exploration-requiring teaching. Their approach informs both the connection between commognition and teaching, but also how the start and end of a task are important aspects in a lesson for determining who has the agency to determine initiations or end-conditions. Rather than using rituals and explorations, the present framework is based on six de-ritualising aspects, as Lavie et al. (2019) suggest are important for moving learning towards explorations. These de-ritualising moves were adapted by Österling (2021) for privileged teaching in mentor conversations, and the present paper presents a further adaptation for the analysis of teaching. The coding-scheme targets descriptions of teacher moves which encourage or legitimise de-ritualisations. A unit of analysis was obtained through two steps: First, when a new example is introduced, and second, when there is a transition into a new way of organising the lesson, i.e. from teacher presentation to learners group work, or from learners group work to presenting solutions.

The operationalisation of objectification, flexibility, bondedness, substantiability, applicability and performers' agentivity are based on first, a theoretical translation from learning processes to teaching moves, and thereafter iteratively coding and empirical fine-tuning of the coding scheme, exemplified below by transcripts from the two lessons. These units were the base for the visible representation.

Results

This result section first presents the development of the analytic operationalisation of de-ritualising moves, and the resulting coding-scheme. Next, the visual representations of the lessons are provided.

Developing a coding-scheme for de-ritualising moves

Objectification in Lavie et al.'s (2019) description is when the story about routines becomes more abstract, a narrative about mathematical objects rather than concrete objects or procedures. In upper secondary calculus, the task situation rarely involves concrete objects, as manipulatives, rather, calculus often involves examples with different mathematical realisations of the object, as graphs, tables, symbols or real-world-situations. Thus, learning objectification is the abstraction from questions and examples rather than from concrete objects. It is often recognised in language by the use of nouns. Objectification in teaching is hence when the teacher encourages discussions through the use of objects (as nouns), or about what characterises objects. Kristin posed several questions to encourage objectification, for instance:

Kristin How can we tell that this is a power function?

Kristin encouraged learners to focus the characteristics of objects, in this case, power functions. For Eva, objectification is traced in her explanations, as in the antiderivative of $f(x) = sin2x$:

Ari: But Eva, the $\sin 2\,x$, why does it get a negative sign?
Eva: It is always like this, now when it is like this [points to $f(x) = sin2x$] this must be sine, which means that it has been cosine. But the derivative of cosine is negative sine. There we have no negative sine [points at $sin2\,x$], therefore, this need to be negative [points at $(-cos2x)/2$].

Eva treated the derivatives of sine and cosine as familiar and abstract objects, as in "the derivative of cosine is negative sine", Even though her questions to learners never encouraged them to engage in objectification, her own explanation legitimised objectification when she referred to familiar content, such as derivatives, as abstract objects.

Flexibility, according to Lavie et al. (2019), is that learners realise there is more than one way of solving a task, or that previously unrelated procedures results in the same outcome. For teachers to encourage flexibility is to ask for different solutions, or the comparison of solutions. In the two lessons, the one instance akin to encouraging flexibility is when Kristin started the lesson by asking learners to discuss their solutions with a peer.

Bondedness is when the routine is changed from a mixture of steps into a compound, where the output of a given step is used as input in latter steps, and hence only necessary steps are performed (Lavie et al., 2019). In teaching, moves towards bondedness is to encourage turning previously disparate steps into a new compound procedure through focusing the connections between steps. Several examples were found in both Eva's and Kristin's teaching, and the transcript below is from the beginning of Eva's lesson on antiderivatives:

Eva:	This is a derivative, yes. [Writes $y' = 2x$]. Now, imagine going backwards, one step up with the derivative and get the function.
Ari:	It is x and remove two…
Eva:	So we are looking for this [writes y=], and we want… what must this have been before to obtain… x^2. [Writes $y = x^2$].

Here, Eva connected a familiar procedure, the derivatives of quadratic functions, to the antiderivative. The antiderivative is not yet treated as an abstract object in teaching, but the encouragement of bondedness between procedures for derivatives and antiderivatives is prevalent throughout the lesson. A similar pattern is found in Kristin's lesson. In both lessons, it was the teacher who made moves towards bondedness, whereas learners provided brief factual or procedural answers.

Substantiability is when learners provide criteria for the performed routine, rather than paying attention to the processes or rely on the judgment of those they regard as authority (Lavie et al., 2019). In teaching, this means an encouragement of substantiations beyond the steps of a procedure. Kristin used "why"-questions to encourage substantiations:

Kristin:	The maximum is 93, why is the maximum 93?
Frida:	Because it is 50, so take $18 - 1{,}5 \cdot 50$
Kristin:	[Writes the solution on the board] Exactly, because this is the amount of time where we leave the Sauna on, and hence the time for the temperature to rise. To 93 degrees.

Learners responded to the why-question by giving the steps of a procedure; however, Kristin reinforced the move of substantiability, and drew on the information given in the example and about the situation of a heating sauna in her substantiation. In the included two lessons, substantiations based on mathematical properties were rare.

Applicability is when the range of task situations for a given routine is expanded beyond the precedent situation (Lavie et al., 2019), and for teaching, this means encouraging the extension of task situations where previously known procedures can be used. One move of applicability was when Kristin applied the range of a function to the real-life situation of a heating sauna (above). Another is when Eva reminded her learners of familiar procedures:

Eva:	Ok, so we must find the antiderivative for this, h(x)=… [Points to $.h(x) = 6\sqrt{x}$]. Do you remember how to write square roots differently?
David:	0,5, the power of 0,5.
Eva:	So we have $6x^{1/2}$, I prefer to write it like this, as one over two. [Writes $h(x) = 6\sqrt{x} = 6x^{1/2}$].

In this example, the procedure of manipulating expressions of powers was applied in the new situation of finding antiderivatives. Thus, applicability moves include both the extension of task situations into real-world situations, but also into new mathematics content.

Performers agentivity contribute to de-ritualisation when performers are free to make a growing number of decisions without the help of others, as in the initiation, on how to proceed, and in the evaluation of a task (Lavie et al., 2019). Teachers' de-ritualising moves are when they encourage or legitimise learners' decisions, as in the transcript below:

Kristin:	What do we need to do to find the temperature after 20 minutes? What are your thoughts?

Here, Kristin asked for learners' thoughts, rather than the right answer or the way to proceed. Thus, she encouraged learners' freedom to make decisions on the way to proceed. Here, the distinction between de-ritualisation as learning, and de-ritualising moves in teaching come to a head. Even though Kristin encouraged learner agentivity, learners provided a correct answer or the steps of a solution, thus, learners interpreted the situation as if they were expected to engage in ritual participation. Nevertheless, for the purpose of this analytic framework, teachers' encouragement of learner participation is enough for the coding of a move for learner participation. Table 1 presents codings:

Table 1: Coding-scheme for de-ritualising moves in teaching

De-ritualising moves	Descriptions
Bondedness	Encourage turning a sequence or previously disparate steps into a new compound procedure through focusing the connections between steps.
Flexibility	Encourage learners to find more than one way of performing a task.
Substantiability	Encourage substantiation of results beyond the steps performed as procedure.
Applicability	Encourage the extension of task situations for previously known procedures.
Learner agentivity	Encourage learner agentive participation, e. g. to decide for themselves what the task is, what to do, and if a procedure worked or not.
Objectification	Encourage discussions of what characterises objects (rather than how to use it), or legitimise objectification in explanations.

A visual representation of de-ritualisation

After sectioning each lesson into time-slots based on the start of a new example, and/or a transition in the organisation of the lesson, the time-slots were marked as black where de-ritualising moves were engaged by the teacher. The shaded columns are sections where learners work in groups or individually, where only the initiation by the teacher was coded. Kristin’s lesson was divided into 12 sections, represented in columns in table 2 below.

Table 2: De-ritualising moves in Kristin's teaching

Total: 55 min.	0:00	1:55	4:05	6:20	7:27	10:21	13:30	17:21	22:00	26:40	30:45	33:10
Example	1	1a	1b	1c	1c	2ab	2ab	3a	3bc	3c	3c	individual
Bondedness												
Flexibility												
Substantiability												
Applicability												
Agentivity												
Objectification												

It is visible how Kristin engaged several de-ritualising moves, where bondedness and substantiability was used throughout the lesson, agentivity was prevalent at the start, and objectification and applicability relate to particular tasks. Eva’s lesson was divided into nine sections, as in table 3 below:

Table 3: The de-ritualising moves in Eva's teaching

Total: 50 min.	1:27	2:25	4:01	9:20	30:00	32:00	32:20	35:40	37:00
Example	1	2	3	1:1-5:3	2:2	3:3	3:3; 4:3	5	individual
Bondedness									
Flexibility									
Substantiability									
Applicability									
Agentivity									
Objectification									

Eva encouraged bondedness in the beginning of the lesson, where she repeated procedures for finding antiderivatives. In the follow-up discussion after group-work, Eva also engaged elements of applicability, objectification and learner agentivity.

The visual representation facilitates the comparison of the two lessons. In Kristin's lesson, the grey and black sections show how she uses several transitions between tasks, and between whole-class or group-work, and that these transitions initiate new de-ritualising moves. Although she encourages learner agentivity in the beginning, she abandons this ambition towards the end. Eva's lesson has visibly fewer black sections, and learner agentivity is only engaged after group-work section. Hence, although bondedness is a repeated move, more de-ritualising moves are only engaged after the group-work section.

Discussion

The visual representation provides an overview of the de-ritualising moves in mathematics teaching-as-usual. Thereby it facilitates comparisons of what moves are present and when they are used, and the model has the potential to capture developments in teaching over time.

The analysis of the two lessons adds aspects to be further investigated. First, objectification was found to be legitimised mainly in relation to familiar content, a pattern which is important to engage further. Second, bondedness is dominant, and it is interesting to enquire whether this is a characteristic for secondary mathematics. A third pattern in Kristin's teaching when she attributes learners agentivity, learners return to ritual responses, as asking for correct answers. Thus, the relation between de-ritualising moves in teaching and learning is a central investigation to continue. Analytically, the boundaries between the de-ritualising moves are not always sharp. There may be a need to add more distinctions or sub-moves in future analysis.

In the two included lessons, teachers engaged in teaching-as-usual, where a dichotomising analysis can be insufficient to capture the complexity of what is going on. Instead, the analysis of de-ritualisations has the potential of making the complexity visible, and thus possible to describe. To continue the logic of Nachlieli and Tabach (2019), teachers engage where they see potential for substantial gains. Therefore, I propose the analysis of de-ritualising moves in teaching has the potential to reveal not only what is going on, but also what teachers find important in teaching.

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Systematic review: Teacher professional development programs in mathematics education and student achievement

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Keywords: Professional development, mathematics education, student achievement.

Introduction

The present poster proposal will show a Systematic Review of the Literature (SRL) on Teacher Professional Development (PD) programs in mathematics education and their relationship with Student Achievements (SA). A relationship will be established between the theoretical framework of effective PD programs and research outcomes that address and consider student achievements as a source of data on the programs' effectiveness.

Theoretical framework

Over the past few decades, the number of PD programs in mathematics education has increased significantly, as has research in the field (Sztajn et al., 2017). This has impacted guidelines, and characteristics suggested when designing an effective PD (Desimone, 2009; Sztajn et al., 2017). In particular, the most used characteristics are those proposed by Desimone (2009): 1) focus on content, 2) active learning, 3) coherence with the context and reality of teachers, 4) duration, and 5) collective participation. These characteristics have established a debate about their relevance, which has had researchers in favor (e.g., Sztajn et al., 2017) as researchers in opposition (e.g., Lindvall, 2017). This broad debate has led to progress in establishing those key elements that should be present in the design of PD programs even if there is not 100% agreement in the community about the minimum. In addition, it has made the scientific community debate about the most crucial objective of PD programs. Many academics agree that the main aim of PD is to improve the classroom practices of teachers and the SA (Jacob et al., 2017; Lindvall, 2017).

Based on the above, the problems addressed by this SRL seek to answer the question: What is the evidence in research on the approach to the relationship between effective PD programs and student achievements in mathematics?

Methodology

In this research, the methodology of a SRL was used (Boote & Beile, 2016). The search for articles was carried out in the two central education databases: Web of Science (WoS) and Scopus, with the criteria summarized in Table 1. Then the articles were reviewed on two more occasions to establish their relevance to the research, which finally obtained 35 studies.

Table 1: Search criteria in WoS and Scopus

Criteria	Data

Keywords included	1. Professional development; 2. Mathema*; 3. Student* effect*
Excluded keywords	1. STEM; 2. Science*; 3. Higher educ*
Type of documents	1. Articles
Languages	1. Spanish; 2. English
Years	2008-2020
Results found WoS	111
Results found Scopus	174

Results

From the analysis process, we identified a sparse presence of articles focused on student achievements between 2008 and 2018 (12 out of 24), while this trend changes in 2019 and 2020 (7 out of 11) with a greater focus on them. In particular, the 35 reports offer student information, but most present them as second priority data or a source used to support that the PD studied is effective in teachers and students. Indeed, only 19 projects explicitly state their objectives to measure, evaluate or relate the PD to student achievements. In addition, the results of this review show that 100% of qualitative studies (8 out of 8), 66.6% (2 out of 3) of mixed studies, and 79.17% (19 out of 24) of quantitative studies achieve favorable changes in student achievements in the analyzed programs. Finally, another relevant finding of this review is that the six studies that show no change in student achievements used some test as a measuring instrument.

Discussion

Based on the findings made in this SRL, we identify a tendency for studies that use PD effective characteristics (Desimone, 2009) to incorporate SA as one of their primary objectives. We also note the dominance of tests as an instrument to measure SA, where there is a clear trend to use standardized tests, which may not comprehensively determine the effects of PDs on the students. Finally, we recommend that to assess SA, instruments should be extended to standardized tests constructed by third parties and should increase the utilization of more than one of them.

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Researchers and teachers involved in professional development: is convergence possible?

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The aim of this study is to discuss how specific educational researchers' choices, regarding the design of a professional development (PD) program for in-service mathematics teachers, influence the praxeologies of both the teachers and the researchers. In particular, we aim to highlight what learning mechanisms are activated by these choices. We use the Meta-Didactical Transposition (MDT) and the Boundary Object (BO) frameworks to describe the interactions between the two communities involved in the PD program, the researchers' and the teachers' one. The choices regarding the design of the program, made by the researchers, trigger internalization processes for both communities, as a result of the learning mechanisms activated during the joint work on a BO. These internalization processes led to an evolution of the initial praxeologies of the two communities and to the creation of a new, shared praxeology.

Keywords: Teacher professional development, meta-didactical transposition, boundary object, inquiry.

Introduction and theoretical framework

The study of mathematics teacher professional development has been an important focus of research over recent decades. Within CERME11 TWG18, open questions and issues emerged regarding, among the others, the crucial role of teacher educators (Zehetmeier et al., 2019). The present study is focused on the analysis of a teacher professional development (PD) program for lower secondary school (6^{th}-8^{th} grade), in-service mathematics teachers, held by a group of educational researchers (the authors), with particular regard to the researchers' choices in the design of the program itself. We aim to highlight the evolutionary process undergone by both communities, the teachers' and the researchers' one, and to describe how convergence on some theoretical and practical aspects was favored by specific researchers' choices.

Meta-Didactical Transposition

We use the MDT framework as an interpretative lens, which allows us to describe the interactions between two communities, one of teachers and one of researchers, in a dynamic way, showing the evolution over time of their respective praxeologies. The term "praxeology" was introduced by Chevallard (1999): it refers to the combination of a practical component, or *praxis*, and a theoretical component, or *logos*: in the case of didactical praxeologies, this combination is referred to teachers' activity in the class. The practical component includes tasks to be solved and techniques to solve them, while the theoretical component includes the justification and the explanation of the techniques. In the case of MDT, the praxeologies involved are meta-didactical ones, because they are referred to teachers' activities in the context of a PD program, not in their everyday classes. The evolution of the praxeologies of the two involved communities is described as the result of internalization processes of elements that were previously external to the communities.

Boundary Object

Robutti et al. (2020), in order to explain the evolution of praxeological elements from external to internal, suggest conceptualizing the meetings between mathematics teachers and educational researchers as “boundary encounters” between communities with different perspectives and objectives, which work together on a BO. Following Star (2010), we can say that a BO allows different communities to work together without preliminary consensus, due to its interpretive flexibility. Cusi et al. (in press) focus their attention on the internalization process during a professional development program. They show an example in which the MDT framework is integrated with the BO construct, to explain how the evolution of the praxeologies has been triggered.

Akkerman and Bakker (2011) identify four potential learning mechanisms activated by the joint work of two communities on a BO: *identification* of the practices of the different involved communities; *coordination* through cooperative exchanges between the two communities; *reflection* on the practices of both communities and *transformation* through the co-development of (new) practices and perspectives. *Transformation* has several steps, including: *confrontation* with some lack or problem, *recognition* of a shared problem space and *hybridization*, in which a hybrid cultural form emerges from the encounter and dialogue of different practices.

Aim and research questions

In this study, we use the MDT and BO frameworks as lenses to discuss how the researchers’ design choices for the PD program influence the praxeologies of both communities, in terms of activation of learning mechanisms. The learning mechanisms, identified by Akkerman e Bakker (2011), can explain how the development of a (completely or partially) shared meta-didactical praxeology between the community of the teachers and the community of the researchers took place. We will describe, in the following sections, the evolution of both the *praxis* and the *logos* components of the praxeologies of the two communities, the consequent internalization processes and the learning mechanisms triggered by specific design choices for the PD program.

In the light of what has been said, our research question is: *What researchers’ design choices for the PD program trigger learning mechanisms towards a shared praxeology, in the context of the joint work on the BO?* In order to better understand the effects of the design choices made by the researchers for the PD program, we will also answer two sub-questions: *Which internalization processes can be identified in the two communities, following the activated learning mechanisms?* and *Which evolution of the praxeologies of the two communities can be highlighted and which shared meta-didactical praxeology, if any, is developed?*

Context and features of the PD program

The PD program we are studying is included in the University project SSPM: Scuole Secondarie Potenziate in Matematica, which is part of the national project Liceo Matematico (Commodari et al., 2020). The program is directed to lower secondary school mathematics teachers, who participate because their schools signed an institutional agreement with the University. They are all experienced teachers, only two of them have less than 10 years of service, and they have already participated in the previous three years of the program. The PD program consists, in fact, of two phases: the first lasted for the first three years of the program, while the second began with the fourth year and will continue for two more years. The object of this study is, in particular, the first year of the second phase of the program.

During the first phase (10 meetings of two hours per year), the two communities worked on inquiry-based learning (IBL) (Maaß & Artigue 2013) in mathematics, with a specific focus on problem solving and laboratory teaching. The researchers designed mathematics IBL tasks for students, already in a complete form, plus tasks for teachers. The teachers, during the meetings, had to solve the students' tasks in small groups, and the teachers' tasks, based on educational requests. Later, they had to experiment with the resources in their classrooms and to provide reports.

During the second phase of the PD program (10 meetings of two hours per year, as for the first phase), the teachers have been engaged in the design of resources for their students in a collective work, with the cooperation of the researchers, who provided to teachers these tasks in a not completed formulation. After the group work, in every meeting there was a collective discussion on the potentials and the limits of the tasks designed by the different groups and on the most suitable learning environment to be experimented in the classroom. The last two meetings were devoted to teachers' reports about their classrooms experimentations, with the resources designed during the PD program, and to collective discussions about possible improvements for the re-design of the resources.

Methodology

The authors, who were also the researchers/educators in the PD program, collected data by written protocols and videos related to the 18 teachers who participated in the first year of the second phase of the PD program, which took place online in a synchronous mode during the pandemic period. In particular, data come from:

1. A written preliminary questionnaire, administered during the first meeting, with open questions and a Likert Scale. The questionnaire had the aim to investigate the teachers' praxeologies, with a particular focus on their *logos* component, including teachers' beliefs about their students and about the teaching and learning of mathematics.
2. Video-recorded semi-structured interviews, conducted in order to better understand the answers to the questionnaire about which we had doubts or we wanted a deeper insight.
3. Teachers' written protocols related to the design of resources, carried out during the group work, and written reflections about it.
4. Video-recordings of the collective discussions that took place during the meetings.
5. Teachers' written and oral reports about their classroom experimentations.

In all the collected material, we looked for elements of the teachers' praxeologies referred to the design of resources, both regarding the *praxis* and the *logos* component and we traced their evolution over the ten months of the fourth year of the PD program.

As a hypothesis for our study, we considered the IBL approach as the BO on which the researchers' community and the teachers' community have been working for many years. A BO should be conceptualized, according to Star (2010), as a dynamic object consisting of several components, so that, when teachers and researchers work together, they often focus only on some components and not on the whole object. Specifically, during the second phase of the PD program, the BO on which the two communities interacted consisted in the design of resources for students, according to the IBL approach.

In order to answer our research questions, we looked for clues of the activation of learning mechanisms, due to the joint work on the BO. According to Akkerman & Bakker (2011), *identification* occurs when a new insight is gained into what the diverse practices concern. *Coordination* entails a "communicative connection" between diverse practices or perspectives, allowing cooperation between two communities. *Reflection* entails the realization of differences between practices and thus the learning of something new about their own and others' practices. Finally, *transformation* is identifiable when profound changes in practices occurred, potentially leading to the creation of a new, in-between practice.

We also looked for evidence of internalization processes, occurring in both communities as a consequence of the learning mechanisms, that led to a shared meta-didactical praxeology. We particularly focus on the elements of the praxeologies of both communities, which refer to the design of resources for students: they include the mathematics tasks and all the elements of the learning environment. Each of the three authors worked, at first, autonomously, proposing also a possible correlation between the choices made by the researchers and the mechanisms observed. After numerous meetings, the three authors agreed on a shared proposal.

Results

The first and most general design choice for the second phase of the PD program was to involve teachers directly in the design of resources for their students, and it was inspired by what emerged in the ICMI Study 22 (Watson & Ohtani, 2015). This researchers' choice contributed to shed light on issues that had not emerged in previous years and constituted the basis of the activation of all the learning mechanisms that we will describe, in chronological order, in the following.

Identification

With the analysis of the preliminary questionnaire and of the collective discussions of the first meetings, we could identify some elements of the teachers' initial praxeologies, which did not emerge clearly during the previous phase. At the same time, during the collective discussions, the teachers could come to understand some elements of the initial researchers' praxeologies, that probably previously were not totally explicit. This objective has been pursued through the conscious choices made by the researchers in the design of the PD program and it led to the first learning mechanisms, i.e. the *identification* of the practices of the different involved communities, as well as of the justifications (*logos*) for these practices. The *logos* component of the initial researchers' praxeologies can be identified in the educational literature on IBL, recalled above. In the light of the literature and on the basis of their experience with classroom experimentations, the researchers deem it appropriate to propose inquiry-based tasks to all the students, without excluding anyone from this learning opportunity. For this reason, at the beginning of the second phase of the PD program, they asked teachers to design resources inspired by the IBL approach, in which all their students could have the opportunity to think, to explore, to conjecture and to share their ideas. We can, therefore, identify the *praxis* component of the initial researchers' praxeology, related to the design of resources for the students, with the proposal of inquiry-based tasks to the whole class, without reducing for anyone the learning opportunity to the resolution of too guided tasks. In this respect, there was an initial praxeological 'distance' between teachers and researchers. For example, in the written questionnaire, many teachers expressed doubts and reservations regarding the possibility to engage all the students in inquiry-based tasks and open problems:

Teacher 1:	At lower secondary school, we have to build problem solving ability, the horizon is limited. We have to work slowly, with guided problems. Only sometimes, someone in the class is able to treat a situation with something a little different from the acquired method.
Teacher 2:	Some [students] manage to identify the solution to a problem only among previously proposed strategies […]

During the first meetings, this kind of reservation emerged from different teachers in different moments, leading the researchers to understand that the teachers were used to propose inquiry-based tasks only to the high-achieving students. This can be considered part of the *praxis* component of the initial teachers' praxeologies, regarding the design of resources for their students. During the discussion related to the resources designed by the teachers during the second meeting, one of the teachers clearly stated her perplexities about the proposal of IBL tasks to the whole class:

Teacher 3:	We are always a lower secondary school, we have very heterogeneous classes, there is nothing to do, there is the one who, with all due respect, will learn a job, will live peacefully, knowing how to calculate the rest […], get up to there, is needless to hide it. […] There are one or two in each class that you need to stimulate, but there are one or two in the whole class.

We assume that these considerations, related to the idea that only few students should be exposed to inquiry-based tasks and the others should do the "basic things", have to be included in the *logos* component of the initial teachers' praxeologies, because they are justification for practices deriving from the teachers' experience in the classroom and from their didactical praxeologies.

Coordination and reflection

In order to foster a productive collaboration with the teachers, the researchers decided to prompt, during the third meeting, a collective reflection on some answers to the preliminary questionnaire and on some resources designed by the teachers, in order to highlight possible inconsistences and to promote awareness about them. For example, in the questionnaire the teachers were asked to express with a score, from 1 to 6, how much they felt that certain goals are central to the role of the mathematics teacher. The options that received the highest scores were: "To create situations in which students have to make decisions and choices", "To promote freedom of thought and creativity", and "To promote students' awareness and critical sense". The researchers showed these answers during the third meeting and asked the teachers to think if the resources they had designed were coherent with the goals chosen in the questionnaire. This discussion led to a *coordination* and *reflection* on the practices related to the design of resources, as exemplified by the words of this teacher:

Teacher 5: We actually gave some suggestions, so maybe we didn't think about a real exploration activity, it seems to me. At this point, if I reflect, the exploratory activity on the part of the students is missing.

Coordination and *reflection* laid the foundations for the subsequent learning mechanisms of *transformation.*

Transformation

The mechanisms outlined above led to the *confrontation* with a lack of coherence and to the *recognition of a shared problem space* (Akkerman & Bakker, 2011). The researchers understood that there was the need to build a hybrid, shared praxeology, fruit of the evolution of the meta-didactical praxeologies of both communities and of internalization processes on both sides. To do that, they began to look for an expansion of the theoretical framework of reference for the PD program and they found what they needed in the work of Cusi et al. (2017), performed in the context of the European Project FAsMEd. The researchers decided to propose to the teachers “theory pills” extrapolated from this work, regarding methods for formative assessment, which can be adapted, in this case, to inquiry-based resources for the whole class. In particular, the researchers proposed to integrate to the already established practices, the subdivision of students in level groups and the creation of “helping worksheets” (Cusi et al., 2012, p. 758) with specific suggestions or guiding questions, besides “problem worksheets”, to be given to the different groups, if they request or need it. This decision triggered the *hybridization* step of the *transformation* learning mechanism and the teachers began to elaborate on these new ideas, evolving their praxeologies. During the following meetings, the teachers introduced new elements, namely the “help cards”, that would constitute components of the hybrid, shared praxeology between the researchers and the teachers:

Teacher 6: […] we thought of preparing helping sheets, but without already addressing them to strong groups or weak groups, we have prepared them by points […] So, we give help cards to those who ask for them. We didn’t think of a single worksheet but, we give different cards with specific suggestions, depending on how the work in the groups goes. Because we create homogeneous groups but we cannot a priori know where they get stuck, what their difficulty is […].

An internalization process of praxeological elements occurred for both communities (as summarized in Table1), because of the introduction of “help cards” as part of the shared praxeology related to the design of resources for the students. The subdivision of students in level groups was a practice already internal to the teachers’ community, but, at the beginning, only the groups of high achieving students received inquiry-based tasks to work on. At the end of the fourth year, instead, the praxeology element related to the proposal of inquiry-based tasks to all the students became internal also to the teachers’ community, thanks to the introduction of “help cards” for the groups who needed them.

Table 1: Internalization processes for the two communities involved in the PD program

Internalization processes	Task design
Researchers’ community	• Level groups

	• Help cards with hints for the groups that need them
Teachers' community	• Inquiry tasks for the whole class • Help cards with hints for the groups that need them

The researchers asked the teachers to experiment with the new resources in their classes and to report the results during the last two meetings of the PD program. During the collective discussion about the experimentations, an ex-post reflection on what had been designed took place attested the creation of a new praxeology for the design of resources for students:

Teacher 7: The idea was to have a series of cards and, as we faced their difficulties, […] we gave them that help card just to help them take a step, without giving them too many elements, too much information.

In Table 2, we summarized the researchers' choices at the basis of the described learning mechanisms.

Table 2: Researchers' choices triggering learning mechanisms

Researchers' choices	**Learning mechanisms**
• Preliminary questionnaire • Teachers' involvement in the design of the resources to be used with their students • Collective discussions, during the first meetings, about teachers' practices and the justifications of their practices.	Identification
• Teachers' involvement in the design of the resources to be used with their students • Collective reflections on the questionnaire and on the task design proposed by the teachers	Coordination and reflection
• Teachers' involvement in the design of the resources to be used with their students • Integration of new elements in the theoretical framework of the PD program • Proposal of the integration of new practices • Request to experiment with the new practices	Transformation

Discussion

In this study, we discussed how specific design choices, made by a group of educational researchers for a PD program devoted to in-service mathematics teachers, influenced the evolution of the praxeologies of both the involved communities. We studied, in particular, the learning mechanisms activated thanks to these design choices and the subsequent development of a shared meta-didactical praxeology between the researchers' community and the teachers' community. We described the evolution of the *praxis* and the *logos* components of the praxeologies of both communities and the occurred internalization processes, relating them to the researchers' choices.

We highlighted how the choice to involve the teachers in the design of resources based on the IBL approach, including the tasks for their students and the learning environment, was at the basis of all the learning mechanisms activated during the PD program. Besides that, the choice to administer a preliminary questionnaire to the teachers, with questions about their practices and justifications for these practices, as well as the choice to make collective discussions about teachers' answers to the questionnaire and about their task design, triggered the learning mechanisms of *identification*, *coordination* and *reflection*. The subsequent choice to integrate new theoretical elements and new praxeologies, inspired by the work of Cusi et al. (2017), related to the FAsMEd project, and the request to incorporate these integrations in the design of resources and in the classroom experimentations, triggered the mechanism of *transformation*. In particular, the *hybridization* step led to the construction of a new, shared praxeology between the two communities, and to internalization processes of new elements for both communities.

In this way, we showed the process which led from an initial praxeological 'distance' between the researchers and the teachers, towards a shared 'in-between' praxeology, as a result of the contribution of both communities. The shared praxeology, at the end of the PD program, included the proposal of inquiry-based resources for all the students, the subdivision of students in level groups and the creation of "help cards" to be given to the groups that needed some guidance. Further research is needed to study if the shared praxeology between the two communities can be considered stable or other evolutions could be triggered during the future years. The analysis we conducted can be also extended to other PD programs, held in different contexts, in which other types of BO are involved.

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Relevance and adaptations of a lesson study process within the context of initial teacher education in the canton of Vaud, Switzerland

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Keywords: Lesson study, initial teacher education, didactical situation.

Research context and aim

Lesson study (LS) is a model of teachers' professional development that originated in Japan at the end of the 19th century and gained popularity outside Asia almost a hundred years later. Since then, it has been at the centre of a growing interest by researchers in mathematics education, both in the case of in-service and pre-service teaching.

In the latter context, a large amount of experimentation has been made over the past twenty years. Research on this topic shows a variety of promising results for prospective teachers, such as increasing knowledge in and for teaching (Ni Shuilleabhain & Bjuland, 2019). Nevertheless, some issues have been highlighted, such as the risk of simplifying the process, and the subsequent changes in key features of LS itself (Ponte, 2017).

The PhD project outlined in this poster takes place in the context of the training program held at the Lausanne University of Teacher Education, in the French-speaking part of Switzerland, for pre-service mathematics teachers at the secondary level. The project aims to describe and analyse an LS-based class to pursue the following double-folded objective. On the one hand, it seeks to investigate the relevance of introducing LS in this particular context and the potential learning outcomes to prospective teachers. On the other hand, it intends to explore the adaptations and changes needed for LS to fit in with this new setting. For this poster proposal, I focused mainly on the first objective.

Theoretical background

Investigating the relevance of introducing LS into a new context can be done from multiple points of view, such as analysing pre-service teachers' knowledge or perspective-taking. In this doctoral research, these aspects are taken into account together with the study of the characteristics of LS as a didactical situation, according to Clivaz's research (2018) and the Theory of Didactical Situations (TDS, Brousseau, 2002).

In particular, Clivaz considers that in LS teachers are engaged in some *adidactical* situations, in which they can acquire new knowledge under the pretext of designing a lesson, which, in TDS terms, constitutes the *milieu* of these situations. To analyse a situation's potential, Hersant (2010) identified three properties of the *milieu*, namely the capability to provide feedback, bring out the desired knowledge and ensure that the knowledge needed to enter the situation is available to the learners.

Moreover, Hersant extended TDS investigation on situations where the *milieu* is less robust, and the didactical contract is more binding. This seems to be the case of LS when applied to initial teacher education. In fact, pre-service teachers' limited teaching experience makes the situation's *milieu* weaker. At the same time, the new setting implies constraints including a strict schedule, partially established content and evaluation, which entail a stronger didactical contract. In this case, Hersant analysed the didactical contract according to four dimensions: the mathematical domain, the didactical status of the knowledge at stake, the distribution of responsibility between the learners and the teacher and the characteristics of the didactical situation.

Methodology

The experimental design of the research is that of didactical engineering, in accordance with the TDS framework. The study of the LS class has been hence organised into three parts: *a priori* analysis of the *milieu*, data collection, *a posteriori* analysis of the *milieu* and the didactical contract.

The LS-based class included 12 sessions with five prospective lower-secondary school mathematics teachers. The class was organised around studying a teaching problem and the subsequent preparation, teaching and discussion of a research lesson related to it. The mathematical topic, chosen in advance, was that of integers. A university trainer with experience in LS served as the facilitator, while experienced schoolteachers supported the group's work during certain LS phases.

Data collected consists of the video recordings of each session and preparation meeting, the lesson plan, the prospective teachers' notes, and the documents shared within the group. These are used to reconstruct the situation's *milieu* during *a posteriori* analysis and examine the four dimensions of the didactical contract. To gain better insight into the participants' learning, *a posteriori* analysis will be supported by the course final assessment, and it will be completed by semi-structured interviews.

Since the data collection was postponed due to Covid-19, data analysis is at a preliminary stage. Some expected results include a dynamic description of the situation's *milieu* and an evaluation of LS potential as a didactical situation, despite some limitations due to the initial teacher training setting.

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Is there a number in-between, and if so, how many? Analysis of pre-service primary teachers' knowledge of rational numbers

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Research shows that a considerable number of pre-service primary mathematics teachers enter university with deficient mathematical knowledge, especially on fractions. As part of a design research project that aimed at improving the pre-service education of primary teachers (Grades 1–6), a normative question and a descriptive question arose: What should pre-service mathematics teachers know about rational numbers? And what do they know after completing an arithmetic course at university? The normative question was answered based on mathematical knowledge for teaching models. To answer the descriptive question, a test instrument was designed and administered to 123 pre-service teachers. The results show pre-service teachers' well-developed didactic knowledge, but difficulties in taking a meta-perspective on their knowledge. Some failed to catch up on school knowledge gaps. The results are discussed regarding their theoretical and practical implications.

Keywords: Pre-service teacher education, mathematical knowledge for teaching, school-related content knowledge, teaching fractions, natural number bias.

Motivation: A fractional math problem and pre-service teachers' solutions

Consider the numbers 1/9 and 1/8.

a) Is it possible to find a rational number that is between these two numbers? If so, please provide this number.

b) Could there be more than one number? If so, include any of the numbers that fall between these two numbers.

This problem might be posed to Grade 6 students. Pre-service teachers, as a matter of course, should be able to solve it correctly. — This is not as self-evident as it first appears. Equivalent tasks were given to all pre-service teachers that were enrolled in the 2020 (N = 111) and 2021 (N = 84) arithmetic courses in their first year of bachelor degree studies. They had to complete a test on the fractional computing ability (Stampfer et al., 2019) before starting the topic "rational numbers." The first task was as follows: a) "Enter a number between 12/4 and 13/4. If you think there is no such number, click impossible." In total, 32.4% of the pre-service teachers of the 2021 course stated "impossible," another 8.1% stated an incorrect intermediate number. The results of a 2020 survey were similar: 33.3% stated "impossible," and even 28.6% an incorrect intermediate. The latter result can be explained by the fact that this time there were fractions given with the same numerator and different denominators (6/18 and 6/19), whereby it was apparently more difficult for the pre-service teachers to calculate an intermediate number. The second task was as follows: b) "How many numbers are there between 10/4 and 10/8?". It is distressing that 69.4% of the pre-service teachers of the 2021 course gave a wrong solution. Particularly often (21.6%) the solutions 4, 3 or 2 were provided. This can be interpreted as an indicator of a natural number bias (Stampfer et al., 2019; van Hoof et al., 2015). Presumably, the participants simply counted how many fractions there were with the same numerator 10, and various denominators between 4 and 8. Again, the 2020 survey results were similar with 58.3% of wrong answers. These incorrect answers, and the underlying misconceptions, remained persistent even after six weeks of intensive study of rational numbers and the didactics of fractions. Concretely, in the 2021 post-test with similar tasks, still 21.6% of the pre-service teachers (compared to 32.4% in the pre-test) answered “impossible” for task a), and 24.3% for task b) (compared to 69.4% in the pre-test) gave incorrect answers. These results were used as a challenging starting point for a design research project which aimed to improve, among others, the arithmetic course for pre-service teachers (Grades 1–6), who will be teaching fractions in their future classes. Many of them, as our data showed, enter university with rather poor mathematical school knowledge, and more than a fifth of the total number of pre-service teachers do not manage to catch up on their knowledge gaps concerning fractions during their studies, as we found out with several further tests.

In this paper, we will first investigate what knowledge pre-service teachers should acquire, especially concerning fractions. This analysis formed the basis for the design of new university courses, in particular the arithmetic course mentioned above. The second part of the paper deals with the effectiveness of the newly designed university courses with respect to the first goal.

Theoretical background: What should teachers know?

Broadly speaking, mathematics teachers should know the subject they teach and they should know how to help learners to acquire that same knowledge in the setting of school lessons. When describing the content knowledge teachers need to carry out their work, Shulman (1986) discerned between *subject matter knowledge* (SMK) and *pedagogical content knowledge* (PCK). In the course of several decades of research on teachers’ knowledge, Shulman’s foundational work was differentiated in various directions. Ball et al. (2008) developed a framework of *mathematical knowledge for teaching* (MKT), distinguishing, inter alia, two subdomains within PCK, *knowledge of content and students* and *knowledge of content and teaching*. Furthermore, they recognized a type of SMK which was exclusive to teachers, namely *specialized content knowledge* (SCK). Another approach to define the content knowledge specific for the mathematics teacher profession is Dreher’s et al. (2018) model of *school-related content knowledge* (SRCK). This model is based on the fundamental distinction between academic and school mathematics[1], and describes the knowledge teachers need about the interrelations between the two in top-down and in bottom-up directions, as well as knowledge of the curricular structure and its legitimation. The latter includes knowledge about fundamental ideas of mathematics, and the significance of certain topics regarding these ideas (e.g., Why are fractions taught at school?) (Dreher et al., 2018). When we deal with content knowledge in this paper, then with the focus on SRCK. In the area of PCK or, to use the term common in most parts of Europe, *subject didactic knowledge,* we are particularly interested in pedagogical content knowledge that derives from mathematics itself (see Carrillo-Yañez et al., 2018). Thereby, we will follow the distinction of a learning and a teaching perspective (Ball et al., 2008; Carrillo-Yañez et al., 2018).

[1] … rather than that between common and specialized content knowledge as defined by Ball et al. (2008).

We exemplify the different relevant types of knowledge using the initial example. Starting with the *teaching* perspective of PCK, pre-service teachers need to acquire the knowledge of how to explain mathematical content to students, for instance a strategy to find or calculate an intermediate number. And not only one strategy, but several strategies, as their students might suggest different approaches, and they should be able to support each of them. Concerning the *learning* perspective of PCK, they need knowledge about students' typical errors, misconceptions, and difficulties in understanding in order to assist them in their learning. As to our example, a typical misconception might be that there is no (natural) number between 8 and 9, and subsequently, there is also no number between 1/9 and 1/8. An algorithmic hurdle could be that students come up with the strategy of expanding to the main denominator, but still cannot find another fraction between 8/72 and 9/72. Or students might find a few fractions between 1/8 and 1/9, but have no idea of how to provide all the numbers.

In order to explain the task correctly, when considering different levels of students' knowledge, pre-service teachers need to understand the mathematics involved deeper than their students. What mathematical content knowledge exactly, above mere school knowledge, should they possess? With respect to our example, we would argue that they need to know specific properties of rational numbers as distinct from those of integers, in particular the properties of order and density (Freudenthal, 2002; Padberg et al., 1995). For instance, a rational number, unlike an integer, does not have a clearly defined successor and one can always specify an intermediate between any two rational numbers (i.e., arithmetic mean). Pre-service teachers also need to know how rational numbers are mathematically constructed and defined as equivalence classes of pairs of integers (Padberg et al., 1995). This knowledge would allow them to flexibly shift between representatives within equivalence classes. This is academic mathematical knowledge, and pre-service teachers will not teach fractions in this way to their students. Sometimes, as we experience, pre-service teachers use this as an argument why they do not need university mathematics. We, however, consider important for them to possess this knowledge and we regard it to be close to or even coincide with what was described as SRCK above.

The idea of using equivalence relations to construct new number systems can be regarded as a fundamental idea or "big idea" (Kuntze et al., 2011) of mathematics, and the knowledge about fundamental ideas is part of SRCK (Dreher et al., 2018). Pre-service teachers should have answers to the above mentioned question "Why are fractions taught at school?" (Dreher et al., 2018) or more specifically, "Why is it interesting to know whether there is a number between two fractions and, if so, how many?", as the reasoning on the legitimation of curricular structure and content belongs to the SRCK.

Constructing new number systems as extensions of natural numbers by preserving some number properties and altering others is a typical mathematical procedure. We consider the knowledge about typical procedures, strategies for solving problems and paths of generating mathematical knowledge as an important (meta-)knowledge for teachers and would also assign it to SRCK. In a similar way, Carrillo-Yañez et al. (2018) distinguished the *knowledge of practices in mathematics* as an essential subdomain of teachers' mathematical knowledge in their *mathematics teacher's specialized knowledge* (MTSK) model. In order to convey an adequate image of mathematics, pre-service teachers should *do* mathematics themselves, and experience mathematics as a science. Such expectations are aligned with the demands paced on modern school mathematics: In the German educational standards for mathematics education (Ständige Konferenz der Kultusminister der Länder in der Bundesrepublik Deutschland, 2005), great importance is given to process-related competencies, such as communicating, arguing, problem-solving and modeling. In this sense, not only mathematics lessons, but also mathematics teacher education needs to meet these general educational requirements. Doing mathematics is an integral part of human culture, and aspiring mathematics teachers should embody that culture.

Different conceptualizations of MKT (Ball et al., 2008; Carrillo-Yañez et al., 2018; Dreher et al., 2018) together with the above-mentioned considerations, led to a completely new design of courses for pre-service teacher education at the primary level which connect subject content and subject didactics at the curricular level (Mayer et al., 2018). The intended facets of MKT were mapped into specific design principles for these courses (Reitz-Koncebovski et al., 2020), namely *follow fundamental ideas of mathematics*, *connect fundamental ideas to main principles of mathematics didactics*, and *experience the process nature of mathematics.* Moreover, the principles of mathematics didactics are not only being taught, but are also implemented while teaching following the idea of a "*pedagogical biplane*" (Wahl, 2013), and by *making learning processes of students experienceable*. Lastly, a *cross-sectional principle* demands the design principles as well as the processes of teaching be made explicit and reflected with pre-service teachers *on a meta-level* (Reitz-Koncebovski et al., 2020). At the end of each course, we research to what extent this has been achieved. Especially for the arithmetic course, we were interested in the following research questions: To what extent have the pre-service teachers acquired the desired knowledge of rational numbers? What understanding (and what misconceptions) can we identify at the end of the two-semester arithmetic course?

Test instrument and methods of analysis: What do pre-service teachers know?

To evaluate the effectiveness of the arithmetic course and to answer the descriptive research questions just mentioned, we developed a knowledge test. This included, among others, an item on pre-service teachers' knowledge of the density of rational numbers in the real numbers which was an attempt to operationalize the aspired content knowledge of fractions with its different facets as was described earlier in the paper. The item comprised four subitems with respect to our introductory math problem (see above):

(1) Solve the problem parts a) and b) in the way you expect students in Grade 6 might solve it.

(2) Now give an explanation for a student how the problem can be solved correctly.

(3) What difficulties may arise when a student is working on the problem?

(4) Name two essential mathematical ideas that you as a teacher need to understand in order to use the problem in class.

Subitem (1) concerns mathematical school knowledge, the other subitems relate to different facets of MKT: (2) PCK in the teaching perspective, (3) PCK in the learning perspective, and (4) SRCK in the form of knowledge about the interrelations between school mathematics and academic mathematics in bottom-up direction: Starting point is a school mathematics problem (1), and the last subitem focuses on essential or "fundamental" mathematical ideas, in particular on the academic mathematics behind it. Both mathematical content and didactic knowledge are necessary for answering the test item. These were discussed in detail in the course, so the task appeared to be reasonable.

The knowledge test was carried out at the end of the two-semester arithmetic course as part of the final examination. The written answers to the exemplary item of all 123 pre-service primary teachers enrolled in the course were analyzed using the *qualitative content analysis* method (Kuckartz, 2019; Mayring, 2014). For it, a deductive-inductive category system was employed. Main categories were developed deductively (i.e., concept-driven) from a sample solution to the task which was derived from the desired teacher knowledge presented in the theoretical part. The inductive categories emerged from the data (i.e., pre-service teachers' answers) building a basis for subcategories. For an easier understanding, more details about the categories are reported below, together with the results. We created a coding guide which was supplemented by definitions and anchor examples, and used it for the coding of the entire written material. This approach made it possible to analyze the pre-service teachers' answers in great detail and to make visible different, often unpredictable, expressions of knowledge and understanding. In a subsequent step, a quantitative analysis was carried out using descriptive statistics to determine the frequency of occurrence of certain categories or subcategories (Kuckartz, 2019) as well as of correct and incorrect, complete or missing answers per participant.

Results

Table 1 shows that between 15% and 27% of the pre-service teachers did not give an answer to one or more subitems. Only very few (4.7%) answered all subitems correctly. The first subitem (1a), which requested school knowledge alone, was answered incorrectly, inadequately, or not at all by nearly 40% of the participants. The results to subitem (2) relating to teaching as a facet of PCK as well as subitem (4) relating to SRCK were particularly weak: nearly half of the explanations for Grade 6 students (2) were incorrect or missing, and less than 20% of the test participants were able to name adequate essential mathematical ideas.

Table 1: Rough quantitative evaluation of the test item on rational numbers (total N = 123)

Item with subitems	Completely correct answer	Incomplete or partially correct answer	Incorrect answer	No answer
(1a) „Is there a number between …“	61.8%	1.6%	21.1%	15.5%
(1b) „are there more … how many“	43.9%	26.0%	6.5%	23.6%
(2) Explain to a student	15.4%	36.6%	21.1%	26.9%
(3) Difficulties for a student	47.2%	27.6%	6.5%	18.7%
(4) Essential mathematical ideas	18.7%	56.9%	4.9%	19.5%

Qualitative content analysis allowed a more in-depth investigation of pre-service teachers' answers. In subitem (1a), the pre-service teachers chose different strategies to specify a number between 1/9 and 1/8, namely determining the arithmetic mean, expanding the fractions to the same denominator or the same numerator (possibly several times), the "wrong addition" (divide the sum of the numerators by the sum of the denominators) or converting into decimal numbers. Almost a quarter of the test participants chose the latter way. This preference of converting to decimal numbers was also seen in pre-service teachers' explanations for subitem (2), whereby most of them did not explain the procedure in detail. In general, a lack of accuracy and concreteness was the greatest shortcoming of many of the explanations for subitem (2). In addition to referring to decimal numbers, there was often a reference to a visual model (i.e., circle, rectangle, number line), but without an explanation of how exactly this was supposed to help. The didactically impeccable explanations (15.4%) used the strategies of arithmetic mean or expansion, and described in detail how this can be done several times in order to find more and more numbers in-between.

Subitem (3) that focused on possible students' difficulties tended to be answered much better than subitem (2). Many test participants considered problem part a) and mentioned difficulties at the procedural level (e.g., expanding and shortening, calculating the arithmetic mean, converting to a decimal number), but also typical misconceptions about the order of fractions (e.g., no number between 1/9 and 1/8, as there is no natural number between 8 and 9, or 1/8 is smaller than 1/9) were mentioned by 51.2% of the participants. Only 7.3% of them mentioned misconceptions concerning the density of fractions or lack of understanding of the concept of infinity required in part b).

Less than 20% of the test participants succeeded in naming essential mathematical ideas in subitem (4). Test participants could get partial credits for this subitem by naming suitable keywords. Most of those keywords were quite unspecific like "knowledge on basic ideas about fractions" (66.7%), without specifying which basic ideas, "general understanding of fractions" (37.4%) or "knowledge of number systems" (13.8%). Here again many test participants referred to procedural knowledge (60.2%). The intended answers regarding knowledge on the equivalence of fractions (7.3%), understanding of the order (10.6%), density (4.0%) or infinity of rational numbers (5.7%) were mentioned to a limited extent. Moreover, some test participants (4.9%) described knowledge requirements of the students, and not those of the teachers.

Discussion and Conclusion

In our study, we investigated what kind of knowledge of rational numbers the pre-service teachers had acquired at the end of the two-semester arithmetic course. Thereby, we focused on mathematical content knowledge at school and at university level, in particular SRCK, as well as subject didactic knowledge (i.e., PCK). Two major difficulties faced by the pre-service teachers stood out:

The first major difficulty concerns mathematical school knowledge. Obviously, a sizable group of pre-service teachers have the same or similar problems with the task as school students and thus lack basic subject matter knowledge (Dreher et al., 2018; Shulman, 1986). This can be derived from the incorrect and missing answers to subitem (1), incorrect explanations for students in subitem (2), and also from the results of the arithmetic test reported in the beginning. Hence, the pre-service teachers obviously need to catch up on school knowledge and skills in fractions. The arithmetic course alone seems not to suffice to compensate for their deficits. We might also ask: What are the causes of these deficits? Is it pre-service teachers' lack of motivation for their mathematics studies in general? Have there been obstacles as a result of mathematics anxiety since school? For pre-service teachers in the Inclusive Pedagogy Teacher Education Program, mathematics is a compulsory subject, and some of them have reported about own mathematics anxiety. Or do they perceive the mathematics university course having low relevance for their future professional work, which resulted in poor performance?

The second major difficulty concerns perspective adoption. As to different perspectives of the PCK considered in theory (Ball et al., 2008; Carrillo-Yañez et al., 2018), it seems to be easier for pre-service teachers to adopt the learning perspective addressed in subitem (3) than the teaching perspective addressed in subitem (2). This may be due to the fact that they are only in the second semester of their bachelor's degree. The results showed that it was particularly difficult for the pre-service teachers to adopt a meta-perspective, as was required in subitem (4). This finding is consistent with those by Kuntze et al. (2011) who reported that the pre-service teachers in their study often were unable to link mathematical content to the big ideas of mathematics. In view of this difficulty, the design principles for our course (Reitz-Koncebovski et al., 2020) require the lecturers to occasionally adopt the meta-level, and to make explicit what "school-related content knowledge" means in certain contexts, as outlined in the theory chapter above. Consequently, the lecturers explicitly motivated their audience to take a meta-perspective themselves. The participants' poor results regarding the latter aspect raise a number of questions: Why was subitem (4) so difficult? Is it generally difficult to adopt a meta-perspective? Do pre-service teachers lack a deeper knowledge about rational numbers (Freudenthal, 2002; Padberg et al., 1995) which we consider an essential part of SRCK? Or do they actually possess this knowledge, but rather have difficulties in articulating it in a written form?

A critical view of the test results implies further development of the course, in particular regarding the implementation of the *cross-sectional design principle* (Reitz-Koncebovski et al., 2020), namely adopting the meta-level and making explicit what SRCK (Dreher et al., 2018) means in certain contexts, and why this knowledge is important and useful for pre-service teachers. Our assertion is that it is not only necessary to allow pre-service teachers to question the usefulness of the content of the mathematics courses at university, but also to explicitly discuss their usefulness in seminars, and to a greater extent than has been done in the past. Some arguments for this kind of discourse with pre-service teachers can be drawn from the lines of argument in the theoretical considerations above. Another, perhaps even more difficult endeavor, is to address the level of attitudes and emotions. We assume that some pre-service teachers' poor performance on fraction tasks was related to specific aspects of the affective domain since the MTSK model of mathematics teacher knowledge (Carrillo-Yañez et al., 2018) places beliefs at the center of the various facets of knowledge.

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Using Concept Cartoons in primary school teacher training: the case of a mathematics content course

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The contribution focuses on an educational tool called Concept Cartoons and the possibilities to use the tool in teacher education. It perceives Concept Cartoons as educational vignettes and show how they can be incorporated into mathematics content courses to help assessing future primary school teachers' knowledge on topics related to the primary school curriculum. The paper introduces one of the Concept Cartoons created on the topic of divisibility and a qualitative empirical study conducted with 67 future primary school teachers within a mathematics content course. The aim of the presented study is to observe and investigate the nature of knowledge displayed in written data collected via the Concept Cartoon. The results of the study confirm the potential that educational vignettes such as Concept Cartoons have in future primary school teacher education.

Keywords: Concept cartoons, educational vignettes, elementary school teachers, mathematics education, preservice teacher education.

Introduction

An integral part of primary school teacher education consists of mathematics content courses where mathematical content related to the primary school curriculum is reviewed and applied in contexts outside the primary school level. For instance, the primary school curriculum in the Czech Republic covers the four operations with natural numbers (addition, subtraction, multiplication, division) and their properties, and the secondary school curriculum in this area starts with prime and composite numbers, common multiples and divisors, and criteria of divisibility. So that the content course for future primary school teachers contains all these secondary school topics, to bring the future teachers a broader perspective and to engage them in intensive argumentation related to the mathematical content they are supposed to teach in their future school practice. Such an arrangement helps verifying that the future primary school teachers understand and comprehend primary school concepts properly.

This contribution focuses particularly on the topic of divisibility and on subject-matter knowledge (Shulman, 1986) of future primary school teachers. From the more detailed perspective of mathematical content, it focuses on conditional reasoning within the topic of divisibility – an area that appears to be difficult for future teachers as they often tend to handling the topic of divisibility procedurally rather than conceptually (Zazkis et al., 2013) and having deeply rooted misconceptions about argumentation that affect their conditional reasoning (Simon & Blume, 1996).

The following text describes a qualitative study of an explorative character that uses an educational vignette (Skilling & Stylianides, 2020) in the form of a Concept Cartoon (Samková, 2020) as a tool for collecting data. The study addresses the research question *"What kind of subject-matter knowledge can be revealed in future primary school teachers when using Concept Cartoons as a written assessment tool within a mathematics content course?"* The paper follows up on the contributions from previous ERME conferences where various educational vignettes were used in teacher professional preparation: e.g. for investigating professional vocabulary of future teachers (Friesen et al., 2019), assessing how future teachers respond to hypothetical student ideas within primary school topics (Buforn et al., 2017; Samková & Hošpesová, 2015), how they respond to hypothetical learning support situations (Kuntze & Friesen, 2017) or what awareness they have about potential student ideas (Samková, 2019b). The presented study was conducted within the framework of the ERASMUS+ project *coReflect@maths* that aims at designing and evaluating vignette-based learning environments for various mathematics teacher education courses, with a particular focus on cartoon vignettes (Krummenauer et al., 2020).

Vignettes and Concept Cartoons

In this paper, educational vignettes are understood as stories representing school practice (Buchbinder & Kuntze, 2018), i.e. as representations of classroom situations or situations that relate to content taught and learnt in the classroom. In that sense, vignettes are rather short, descriptive episodes that may take the form of texts, single or multiple images, videos, or their combinations (Skilling & Stylianides, 2020). The protagonists of vignettes might be various combinations of teachers and students, e.g. a teacher with one or more students, one or more students without a teacher, one or more teachers without students. With future teachers as respondents of research or intervention, the purposes for implementing vignettes are wide (Herbst & Chazan, 2011); they usually lay in aiming for development or analysis of professional knowledge and skills such as noticing (Schack et al., 2017), professional vocabulary (Schleppegrell, 2007), etc.

Among vignettes, we may also include Concept Cartoons – individual pictures showing a content-related situation and a group of several children discussing the situation via a bubble-dialog. The opinions in the bubbles may be correct, incorrect, unclear or incomplete (Keogh & Naylor, 1993). Originally, Concept Cartoons were developed as a means of supporting the quality of discussion in primary school classrooms (Naylor et al., 2007), with the key aspects for the discussion being the absence of the teacher in the picture (i.e. the presence of just the peers) and the diversity of opinions given in the bubbles. However, Concept Cartoons may be created for different target groups and different purposes, including the target group of future primary school teachers and the purpose of analysing their pedagogical content knowledge (Samková & Hošpesová, 2015) or subject-matter knowledge (in this paper). Within this context, the protagonists in the picture may not be just children but also adults (future teachers, i.e. peers of the respondents). For the purpose of collecting data on teacher knowledge, Concept Cartoons are usually accompanied by some set of indicative questions, and this combination appears to be able to provide data that are ample and relevant (Samková, 2019a).

When creating a new Concept Cartoon, one has to choose the focus of the mathematical task in the background (calculation, proposition, application), its openness (e.g. single vs multiple correct solution procedures), determine the nature of correctness of individual bubbles (ambiguous, unambiguous, conditioned), and choose the form of texts in bubbles (results, procedures, statements); for more details on the typology of Concept Cartoons see Samková (2020). This study is based on a Concept Cartoon that has a group of future teachers as protagonists, a propositional task with multiple correct solution procedures in the background, four bubbles with unambiguous correctness, and texts in bubbles in the form of statements (see Figure 1).

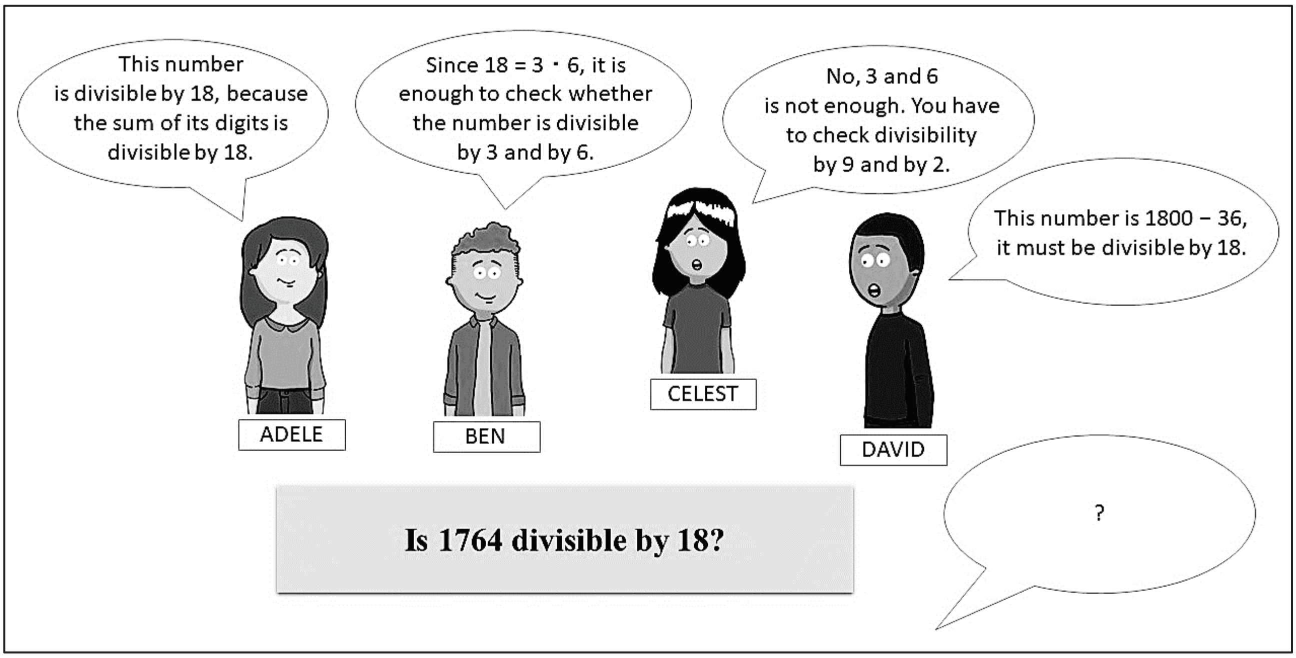

Figure 1: The Concept Cartoon on divisibility

The Concept Cartoon in Figure 1 presents two statements that are correct (Celest, David), and two that are incorrect (Adele, Ben). One of the statements (David) refers to a manipulation with numbers (based on finding a nearby multiple of 18 that is easily identified), while each of the other three statements (Adele, Ben, Celest) informally refers to an application of a general rule. The rules can be formally rewritten as follows:

Adele:	If the sum of digits of a given number is divisible by 18, then the number is divisible by 18.
Ben:	If a given number is divisible by 3 and by 6, then it is divisible by 18.
Celest:	If a given number is divisible by 9 and by 2, then it is divisible by 18.

For the rule behind the Celest bubble, the condition in the statement is necessary as well as sufficient, i.e., the rule is valid and can be also rewritten in the form of equivalence. For the rule behind the Ben bubble, the condition in the statement is necessary but not sufficient, since 3 and 6 are not coprime numbers; numbers 6, 12, 24, 30 are some of the counter-examples for the rule. For the rule behind the Adele bubble, the condition in the statement is not necessary (even the number 18 itself does not have the sum of digits divisible by 18) nor sufficient (swapping the order of digits does not change the sum of digits but may easily create a number that is not even and thus not divisible by the even number 18; e.g. 1467, 7641).

Such an arrangement creates an environment that challenges skills in conditional reasoning, by requiring proper differentiation between necessary, unnecessary, sufficient and insufficient conditions in an informally worded statement (Buchbinder & McCrone, 2019). The statement in the Adele bubble is also closely related to *overgeneralizing* – a frequent misconception consisting in improper use of analogical reasoning (Hemmi et al., 2017); here the overgeneralizing stems from criteria for divisibility by 3 and by 9 that are both based on the sum of digits.

Design of the study

Participants of the research study were 67 future primary school teachers – full time students of the first year of the 5-year teacher training program at the University of South Bohemia in České Budějovice. In the time of the study, they were attending the content course on arithmetic. They have not worked with Concept Cartoons before the study. The participants were randomly labelled by code names V1 to V67.

In the data collection stage, the participants were assigned the Concept Cartoon from Figure 1 and a set of indicative questions to respond. Having the new environment where the protagonists of the Concept Cartoon were not children but future teachers, also the set of indicative questions had to be newly created. Taking inspiration from various sets of indicative questions verified in previous research and proceeding from the fact that it has proved useful to have the indicative questions purposefully fragmented in their focus (Samková, 2019a), the following three indicative questions were distributed to the participants in order to find out about how they draw on their subject-matter knowledge: (1) What thoughts could be behind the student teachers' thinking? (2) How could you help the other student teachers to correct their answers or to improve their argumentation? (3) Write YOUR solution into the empty speech bubble. The participants worked on the task individually, in the form of a compulsory written homework.

Collected data were processed qualitatively, using open coding and constant comparison (Miles et al., 2014). The process of open coding focused on various displays of subject-matter knowledge or lack of it, and their interrelations. Data were compared repeatedly across participants, across bubbles, and across indicative questions.

Findings

The four following code categories appeared as relevant at the end of the analytic process: Coprime condition (codes *coprime forgotten*; *missing coprime reported*, *prime factorization misused*), Language (codes *inaccurate terminology*, *shifted meaning*, *shifted interpretation*), Argumentation modes (codes *counter-example for sufficient*, *counter-example for necessary*, *objection towards coincidence, overgeneralizing*, *rule followed instead of verified*, *use of assumptions not mentioned in the bubble*), and Alternative ideas (codes *favour on the use of rules*, *favour on the non-use of rules*). Below, we describe the code categories in detail and provide illustrative data excerpts related to them.

Coprime condition

The first of the code categories refer directly to weak or good knowledge of divisibility concepts. The most occurring concept in focus appeared to be the concept of verifying divisibility by decomposing the divisor into a product of two coprime numbers (e.g. $18 = 9 \cdot 2$) and verifying the divisibility by these two numbers. Almost half of the respondents (33 out of 67) forgot about the coprime condition and agreed with Ben who decomposed 18 into a product of two numbers that are not coprime. Usually, they then (incorrectly) included a prime factorization as a proposed enhancement of Ben's reasoning. As a direct consequence, these 33 respondents labelled Celest as incorrect:

V15 Ben: $18 = 3 \cdot 6 \rightarrow 6 = 3 \cdot 2$ divisibility criteria for 3 and 2 must be met
$3 \rightarrow$ sum of digits is divisible by 3 $2 \rightarrow$ must be even
$\rightarrow$ holds good (meets both criteria)

		I would be more specific and decompose as 18 = 3 · 3 · 2, it is enough to check whether the number is even and its sum of digits is divisible by 3.
	Celest:	She just made another decomposition. She is not right, both decompositions are good.
V31	Celest:	I think she is not right. It is enough. She must decompose the 6.
V65	Ben:	I think that the divisibility by six is a little extra. It is enough to verify divisibility by two and by three.

In responses to the third question, 22 of these 33 respondents offered as their own solution the decomposition into 9 · 2 or 2 · 9, and 5 respondents offered the decomposition 3 · 3 · 2 or 2 · 3 · 3. On the other hand, there were respondents who remembered the coprime condition and pointed it out:

V64	Ben:	We cannot decompose this way. The numbers you decompose into must be coprime (cannot be divisible by the same number) → 3 and 6 are divisible by 3, we do not want it.

Language

Some of the respondents displayed shortcomings in the language of mathematics that transpired in the form of an inaccurate terminology (V30), a shift of a meaning of a mathematical concept (V36), a shift in an interpretation of the text in a bubble (V47), or a combination of them (V34 – terminology & interpretation):

V30	David:	1800 is the closest whole number.
V36	Ben:	The idea is good, but it is not sufficient to have the number divisible by 3 and by 6, it must also, after dividing by one of the numbers, be divisible by the other.
V47	Celest:	According to Celest, we have to check divisibility by 3, 6, 9, and 2. It is sufficient to check just divisibility by 9 and 2.
V34	Celest:	She thinks that we have to check all variants of multiples, but it is not true.

Argumentation modes

The third code category refers to modes of argumentation and logical aspects in general. Among the proper argumentation modes, it included counter-examples that some of the respondents provided as a reaction to Adele. These counter-examples referred either to a condition that is not sufficient (V14) or a condition that is not necessary (V29):

V14	Adele:	She tried to sum the digits of 1764 → it came out 18, and 18 : 18 = 1, so that she thinks this is a rule for divisibility by 18. However, when we take e.g. the number 4455, the sum of its digits is also 18 but the number is not divisible by 18.
V29	Adele:	Her opinion surprised me. The sum of the digits is 18, so it is divisible by 18, but I did not find this kind of criterion anywhere. … I chose the number 126 (a multiple of 18) to check it → 126 : 18 = 7 → 1 + 2 + 6 = 9 → 9 : 18 = 0,5. Other example: 1710 : 18 = 95 → (1 + 7 + 1 + 0 = 9) 118764 : 18 = 6598 (1 + 1 + 8 + 7 + 6 + 4 = 27 → 27 : 18 = 1,5). In my opinion, it implies that we cannot use the sum of digits this way as decisive. It was just a coincidence that it worked out for her.

Surprisingly, none of the respondents offered a counter-example as a reaction to Ben. However, several of them provided to Ben an objection towards coincidence similar as the one by V29 to Adele:

V46	Ben:	That Ben's claim comes out in this particular case is, in my opinion, just a coincidence.

The improper argumentation modes included using an assumption that was not mentioned in the bubble (V18), overgeneralizing (V46/David) or following a rule in the bubble instead of verifying it (V46/Adele):

V18	Adele:	She is right, because if a number is even and a sum is divisible by 18 → it's true.
V46	David:	Number 1800 is divisible by 18, number 36 as well, in this case she is right. Check: $18 \cdot 4 = 72 = 100 - 28$ 100 is not divisible by 18 28 is not divisible by 18 In my case, it did not work out, which means that it was just a coincidence.
	Adele:	1 + 7 + 6 + 4 = 18 → 18 : 18 = 1 → she is right, it will work.

Alternative ideas

The last code category summarizes how respondents reflected the fact that there were alternative opinions shown in bubbles. Aside from the discourse between Ben and Celest that got assigned its own code category (*Coprime condition*, see above), there were also two different correct statements presented by Celest and David. Here, some of the respondents favoured the Celest‘s way based on a well-known rule (V13), others appreciated that David had managed without the rule (V29, V5); one of the respondents favoured both the statements (V44):

V13	David:	This procedure is logically correct, but might be time consuming. It is better to use divisibility criteria instead.
V29	David:	His opinion is interesting and might also be considered correct … He came to the conclusion logically even without knowledge of the divisibility criteria.
V5	David:	Nice, quick reasoning!
V44	(3)	I myself would support both opinions (C and D). C is a classical method. For D, we have to think a bit but, for one, it is faster.

Discussion and conclusion

As illustrated in the previous section, using educational vignettes, namely Concept Cartoons, as an assessment tool within a mathematics content course for future teachers might bring a broad insight into various facets of knowledge that is more or less related to the mathematical content that the future teachers would teach in their future teaching practice. The environment consisting in a Concept Cartoon presenting various correct and incorrect opinions on a chosen topic (divisibility by 18) and a set of three differently aimed indicative questions has proved to be able to indirectly provoke reasoning of future teachers and obtain rather talkative responses from them (even if only in writing). These responses reflected in detail how future teachers reasoned about the topic, how they understood key concepts, what mathematical language they used, what kind of arguments they were able to provide, and how they reacted to various alternative ideas.

The results of the study highlighted the advantage that Concept Cartoons have over standard written tests: 33 of the 67 respondents labelled as correct a solution that was not correct (Ben, missing coprime condition), however, 22 of them offered as their own solution a solution that was correct. It is reasonable to assume that if only a standard test were used as a method of assessment (e.g. with a task *"Is 1764 divisible by 18?"*), these 22 respondents would succeed in the test and there would be no doubt about their subject-matter knowledge. Moreover, using the format of Concept Cartoons for assessing subject-matter knowledge also allowed to learn about future teachers' mathematical language and argumentation. The findings of the study confirmed weaknesses in conditional reasoning (cf. Simon & Blume, 1996; Buchbinder & McCrone, 2019) as well as a tendency to overgeneralizing (cf. Hemmi et al., 2017), a tendency to handling the topic of divisibility rather procedurally than conceptually (cf. Zazkis et al., 2013), insecurities in mathematical language (cf. Schleppegrell, 2007). Such findings show that vignettes might be implemented meaningfully into teacher training not only to advanced courses focusing on pedagogical content knowledge and teaching practice (Buchbinder & Kuntze, 2018) but also to initial content courses.

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‘I personally would have done it the other way’ – Pre-service teachers’ Perspective Taking when noticing mathematical thinking

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Learning to notice is perceived as an important aspect of the professional development of pre-service mathematics teachers in Germany. It includes attending to specific elements of a complex situation and using one’s knowledge to make sense of them. This article makes the argument that taking over a different perspective might provide a valuable starting point to facilitate noticing of mathematical thinking processes. The qualitative content analysis of 280 pre-service teachers’ written statements shows that some participants take over different perspectives, which plays an important role in their describing, evaluating, and interpreting of the situation.

Keywords: Pre-service teachers, noticing, perspective taking, qualitative content analysis.

Introduction

Over the last decades, there has been a lot of research into teachers’ practices with a focus on professional vision and noticing (van Es & Sherin, 2021). Especially noticing students’ mathematical thinking is perceived as a prerequisite to providing rich learning environments and a productive mathematical discourse (Jacobs et al., 2010). Nevertheless, not only *what* teachers notice is important, but also *how* they analyze what they have noticed (van Es & Sherin, 2008; van Es, 2011). Several studies have shown how regular video club meetings can improve the way participants reason (van Es & Sherin, 2008; van Es & Sherin, 2021). Mathematical content knowledge (Friesen et al., 2015) and pedagogical content knowledge (Prediger & Zindel, 2017) provide categories along which certain aspects of a situation (including student thinking) can be noticed. However, more research is needed into to what extent pre-service teachers (PSTs) are able to notice specific aspects if they are provided with video material that allows them to focus more closely on a mathematical situation rather than a whole-classroom activity. In this paper, we will make the argument that *Perspective Taking* – i.e. imaging oneself in the teaching-learning situation presented in the video clip – facilitates noticing and especially interpreting rather than merely describing events.

Noticing as a goal in teacher education

Important aspects of teaching are to provide students with learning opportunities and to react adequately to their needs and potentials. The construct ‘teacher noticing’ refers to the process of making sense of complex classroom environments (Jacobs et al., 2010, p. 170). Generally, noticing is divided into three dimensions: (1) identifying what is noteworthy in the situation (*attending* or *selective attention*), (2) using one’s knowledge and experience to make sense of what is observed (*interpreting* or *knowledge-based reasoning*), and (3) constructing interactions to gain further information during the noticing process (*shaping*) (van Es & Sherin, 2021). According to van Es (2011) and van Es and Sherin (2021), there are three general stances to interpret what is noticed: descriptive, evaluative, and interpretive. Interpreting is perceived to be the most sophisticated action and is driven by a stance of inquiry, i.e. the intrinsic motivation to figure out what is going on: "interpreting is not only about trying to make sense of a phenomena [sic] but also involves seeing observed phenomena as something worth trying to figure out" (van Es & Sherin, 2021, p. 22; see also van Es, 2011).

As research has shown, the ability to notice can be facilitated by (regularly) reflecting on video clips from situations related to teaching and/or learning (Friesen et al., 2015; Prediger & Zindel, 2017; van Es & Sherin, 2021). Providing regular meetings of video clubs with in-service teachers, van Es (2011) and Sherin and Han (2004) found that the participants had a tendency to take a descriptive and evaluative stance in the first session, but adopted more often an interpretative stance in the last. In-service and pre-service teachers alike tend to focus on the teachers in the beginning, but over time shift their attention to students' actions and thinking (van Es & Sherin, 2008; Star & Strickland, 2008). Regarding the evaluative stance, Karsenty et al. (2019) illustrate how in-service teachers move from predominantly judgmental comments to justifying different perspectives over the course of a video club. While they argue that this allows for more productive discussions, they also find an overall decline in evaluating actions in favor of openly discussing specific aspects of practice and pedagogy.

Looking specifically at PSTs, it is not surprising that they tend to be less proficient in noticing as they normally lack the experience of teaching as well as the theoretical categories from content knowledge and pedagogical content knowledge. They tend to chronologically report on events rather than interpreting them (van Es & Sherin, 2002) and provide limited or no evidence for interpreting children's understandings (van Es & Sherin, 2002; Jacobs et al., 2010). However, the depth of the analyses by PSTs may depend on the type of video clip: Using a full-length video of a whole-classroom period, Star and Strickland (2008) found that PSTs had trouble focusing on important aspects and "generally do not enter teaching methods courses with well-developed observation skills" (p. 107) – but were able to develop them over time. In contrast to that, Prediger and Zindel (2017) used a shorter sequence from a one-on-one interview with a student in a study with 159 PSTs. Their findings illustrate how – even before an intervention - a majority (roughly 65 %) is able to focus on student thinking rather than making only surface judgments.

While the goal is to deal with the full complexity of a whole classroom, these findings indicate that it might be fruitful to start with a reduced setting for PSTs. Overall, learning to notice is complex, requires time and experience. Especially for developing an interpretative stance, PSTs should learn to be aware of themselves and their own expectations as a prerequisite for noticing (Mason, 2009).

Perspective Taking as a personal interpretative frame

Mason (2009) describes the importance of self-awareness to being able to notice and facilitate student thinking:

> Ongoing enquiry into the actions available to learners and into ways of both triggering them and making them available for inspection involves ongoing enquiry into the origins of the actions, which underpin different topics. Ongoing enquiry into the origins and variations of those actions in one's own experience is necessary in order to be sensitive to learners. (p. 207)

Thus, the own experiences and approach to a certain task can serve as an important starting point for understanding and interpreting students' actions. Freudenthal (2002) uses the term *reflection* for "mirroring oneself in someone else in order to look through his skin, to explore him, to take him in" (p. 104). This type of reflection leads to a gain of knowledge not only about the other person but also about their own behavior and knowledge. The own experience serves as an anchor and/or priming for perceiving and analyzing the student's behavior. It also plays into the idea of recognizing choices and alternatives for a specific situation (Mason, 2009).

Regarding the process of diagnosing, Nickerson (1999) proposes a descriptive framework, in which experts use their own knowledge as a model to compare to another (a child's) model of knowledge. As Philipp (2018) points out, using the own (expert) perspective as a starting point for assessing a learner might yield potential biases, such as over-or underestimating the challenge for a novice. However, carefully comparing and contrasting the own and the child's actions might lead to a higher awareness of both differences and commonalities, and support the focus on student thinking.

The idea of *Perspective Taking* as a means for noticing is included in the construct of interpretative frames proposed by Sherin and Russ (2014). These frames function as lenses, through which a situation is perceived: "In each interpretative frame, the way [a teacher] makes sense of the video ([i.e.] knowledge-based reasoning) is both constrained by and contributes to what [he/she] notices in the video ([i.e.] selective attention)" (p. 10). In an interview study with 15 in-service teachers, the authors identify 13 interpretative frames, one of which is *Perspective Taking*. This frame was found in regard to teachers imagining themselves in the role of the teacher in the video clip, but only four out of the 15 participants showed this frame. The authors offer as a brief potential explanation that the teachers only use *Perspective Taking* when the depicted instruction is in line with their own pedagogical approach. However, we see two problems with this explanation: First, by only taking the perspective of the shown teacher (rather than a student), *Perspective Taking* might not lead to a better understanding of student thinking. Second, if *Perspective Taking* only occurs in situations close to the own approach, it cannot help with getting to know alternative interpretations and actions.

Overall, one can assume that *Perspective Taking* has the potential to facilitate PSTs' learning to notice student thinking processes. In this paper, we address to what extent it might be found in PSTs' written analyses even at the beginning of their university studies in order to identify potential starting points for subsequent teacher education material. Thus, our presented study aims at the following research questions:

1. To what extent analyze PSTs the video clip by taking over a different perspective?
2. Which role does the *Perspective Taking* play in PSTs' analyses?

Study design & methodology

Data Gathering: To pursue the research questions, the study was carried out in the context of a lecture on mathematics education aimed at prospective primary school teachers (PSTs) in their first year of their three-year-long university studies in Germany. We collected written analyses of a video clip by 280 PSTs (35 male (12.5 %), 244 female (87.1 %), 1 diverse), of which 256 were in their first or second semester.

The video clip is roughly four minutes long and shows an excerpt from a one-on-one mathematical interview with the second-grader Mats, which is situated in the context of arithmetic learning. In the video, Mats first is working on a task where he has to find pairs of red and blue cards (unstructured and structured representations) with the same number of dots. Later, he describes his strategies. The video was uploaded to the lecture's accompanying online learning platform and was used as a preparation for the following lesson on counting and number sense. The PSTs were asked to answer the following questions in a text of up to one page: "Describe how Mats solves the task. What do you notice and which parts are especially interesting to you?"

Data Analysis: In order to see if PSTs were using a *Perspective Taking* frame and which role it played in their analyses, we used a qualitative content analysis approach (Kuckartz, 2019). The content analysis performed included the following steps:

1. Preparing the data
2. Segmenting written products of the 280 participants into idea units for analysis and coding each idea unit with one or more of the concept-driven categories based on the interpretative frames by Sherin and Russ (2014)
3. Focusing on all instances of the interpretative frame *Perspective Taking*: inductively identifying sub-categories from the data by comparing and contrasting the codings in regard to the questions whose perspective was taken (cf. resulting set of subcategories in Table 2). In addition to Sherin and Russ's (2014) definition of *Perspective Taking*, we included not only instances where someone in the situation was mentally replaced, but also where the participants imagined themselves (possibly additionally) in the situation.
4. Focusing on all instances of *Perspective of > PSTs as Task Solver*: coding each idea unit with one Stance identified by van Es and Sherin (2021) and van Es (2011) (cf. Table 1)

All steps were carried out by both authors and results were consensually validated.

Table 1: Deductively derived categories relating to the Stance, slightly modified to fit the situation

Stance (van Es, 2011; van Es & Sherin, 2021)		
Descriptive		Simply stating identified differences or commonalities
Evaluative		Evaluating the quality of the child's approach against the own perspective or proposing a (better) alternative
Interpre-tative	Reasoning	Identifying a cause/explanation for observed differences or commonalities
	Generalizing	Drawing a generalized conclusion in terms of either attribution of competence to the child or communicating a personal 'lesson learned'

Findings

Out of the 280 PSTs, 32 persons were identified to use a *Perspective Taking* frame with 38 different instances in total. Table 2 gives an overview of the findings. Overall, four different sub-categories were identified in terms of *whose* perspective was taken.

In nearly all instances (32), the PSTs imagined themselves in the situation *as the ones who had to solve the task* (but with their current mindset and skills). We identified three instances of taking the *child's perspective*, in which actions and/or statements of Mats were either explained or questioned from a student's point of view (as in the example in Table 2). There were two instances in which we used the category of *"Teacher"*, however, both are far less explicit than what Sherin and Russ (2014) found. Rather, in the example shown below, the assessment that the second part of the video is "where the actually important process of the task takes place" was interpreted as taking over a diagnostic perspective by identifying Mats' explanations as especially meaningful for understanding the student's thinking. The second utterance in this category also shows a diagnostic perspective referring to "immediately being aware of Mats' problems" (f1215). Lastly, one PST took the perspective of the *course instructors* who had administered the video analysis task (cf. Table 2).

Table 2: Inductively derived sub-categories of *Perspective Taking* > *Perspective of*

Sub-category (number of instances)	**Description for the sub-category**	**Exemplary quote from the PSTs' written statements (identification-code in parentheses)**
PSTs as Task Solver (32)	*Taking the perspective of being the one who has to solve the task presented in the video but with the current mind- and skill-set*	"What I find noteworthy here is that Mats split the blue card with seven points into six and one, while I myself would rather split between the lines into three and four." (f1059)
Child (3)	*Taking the perspective of Mats or another child with their mind- and skill-set*	"I would have thought - especially as a child - that he thinks of a die, which he knows from playing with it, and not of two rows of four with a dot in the middle" (f1148)
Teacher (2)	*Taking the perspective of the teacher/interviewer or another teacher with their mind- and skill-set*	"Afterwards, the student is asked to explain his approach. This is where the actually important process of the task takes place." (f1207)
Course Instructors (1)	*Taking the perspective of being the course instructor who administered the video analysis task to the PSTs*	"I think this assignment is trying to make us students aware of how important good arrangement and structured representations are for elementary school children to be able to understand abstract mathematics based on them." (f1017)

Due to the dominance of the *PSTs as Task Solver* category, further analyses were concentrated on these 32 instances. Of these, 26 instances focused on differences between their own approach and the one shown by the child in the clip, five referred to commonalities, and one did neither.

As Table 3 shows, all stances identified by Sherin and van Es (2021) and van Es (2011) were found when looking at *Perspective Taking > PSTs as Task solver*. In eleven cases, there was a mere *description* of differences (none of these instances related to commonalities) between the own and the child's approach. Looking at the *evaluative* stance, we included similar to Karsenty et al. (2019) assessments of the situation in which the different perspectives served as a sort of measure or suggestions of alternatives (as shown in Table 3). Here, in six cases, the PST's approach served as a better way to solve the task than what Mats did. In the only instance where a commonality was described, the PST gave a positive evaluation, stating how they were surprised that a child was able to tackle the task similarly to themselves.

Table 3: Identified sub-categories of *Perspective Taking > PSTs as Task Solver > Stance*

Sub-category	Exemplary quote from the PSTs written statements
Descriptive Stance (11)	"What I find noteworthy here is that Mats divided the blue card with seven points into six and one, while I myself would rather divide between the lines, so into three and four." (f1059)
Evaluative Stance (7)	"Here I notice in particular that I personally would probably have done it the other way around, [... because] I think it would be quicker to find the matching card." (f1083)
Interpretative – Reasoning (8)	"For me it was very obvious to calculate 3+4=7 but for him there were six dots and one more. Probably he compared the structure of the dots with a structure known to him, namely the two times three dots on the card with six dots." (f1062b)
Interpretative – Generalizing (6)	"What I personally also find very interesting is how each person uses a different method for counting points, as I myself would have approached some numbers very differently." (f1068)

Belonging to an *interpretative* stance, we identified two categories: *reasoning* and *generalizing*. There were eight instances in which PSTs described differences (5) or commonalities (3) and formulated possible reasons for these observations. Here, the participants used background knowledge for example on developmental levels or everyday experiences of young children (cf. Table 3). Lastly, there were six instances in which the *Perspective Taking* resulted in stating generalizations. Here, in two instances, the PSTs used the comparison with their own perspective to attribute certain competencies to the child. In the other cases, they communicated how they themselves as PSTs learned something from the differences perceived.

Discussion

Regarding the first research question, 32 out of the 280 PSTs did use an interpretative Frame by taking a different perspective in their analysis of the video clip. Comparing our results (Table 2) to Sherin and Russ (2014), it might be surprising how often our participants imagined themselves in the situation and how rarely they took the perspective of the teacher or the child. One explanation for this might be the type of video clip, in which the interviewer is staying mostly in the background so that there is not much of a teacher's perspective in the situation to take over - especially as academically young PSTs lack experience with teaching. Furthermore, relating to the thinking of a second-grader might be challenging due to the age difference, and therefore neither the perspective of a teacher nor a child is close to the PSTs. However, most of our participants recently graduated from school and might well remember how they themselves solved tasks. As Philipp (2018) pointed out, there is a potential risk of a biased judgment when using oneself as the sole point of reference. However, according to Mason (2009), drawing on differences between the own approach and the child might lead to widening the own knowledge of possibilities and make PSTs more receptive to student thinking.

As our analysis regarding the second research question shows, only in a third of the cases was the own perspective used for a mere description of the video clip. Some PSTs managed to use their own perspective in an evaluative or even interpretative stance. Keeping in mind that these PSTs were participating in their first course on mathematics education, not all did use adequate categories of content knowledge or pedagogical content knowledge in their analysis. However, they seemed to be inclined to not only describe the student's behavior in contrast to their own but also look for possible explanations. This might be an indicator of an inquiry stance, which was argued by van Es and Sherin (2021), van Es (2011), and Mason (2009) as the driving force in noticing. While one could argue that the generalization from one case (or video clip) might lead to an over-generalization and thus to an assessment bias (Philipp, 2018), we believe that especially those PSTs who communicated a 'lesson learned' widened their knowledge horizon on possibilities how young students make sense of a task, which is an important prerequisite for noticing (Mason, 2009).

Furthermore, the number of evaluating instances is lower than what was expected with respect to the study by Karsenty et al. (2019). A possible explanation lies in the sample of looking only at *Perspective Taking* instances – the ongoing analysis of other frames indicates a much higher number of evaluative stances overall. A carefully formulated hypothesis is that *Perspective Taking* might be less likely to lead to evaluating, which would be in line with the benefits described by Mason (2009).

Implications for teacher education and further research

Overall, the presented study showed that even though there was no specific prompt for it, roughly 10 % of the PSTs used *Perspective Taking* in their written analyses of a video clip from a mathematical interview. Imagining oneself as a PST being presented with the same task as the child in the video can lead to different interpretative stances. Similar to Sherin and Han (2004) and van Es (2011), describing and evaluating dominated overall in our results. However, roughly half of the idea units were identified as interpretative, which not only showed how PSTs without prior experience in mathematics education reason about student thinking, but also provided insights into how some of the PSTs became aware of their own personal learning from *Perspective Taking*. This is in line with Freudenthal's (1991) suggestion that reflection on another person's behavior by taking their perspective does not only provide insights into their thinking but also about *oneself*.

We perceive the results from this ongoing study as possible starting points for designing a teacher education course, which aims at facilitating PSTs' noticing from the beginning of their university studies. The next steps will be to finish the analysis on the other interpretative frames in order to acquire a more comprehensive picture of PSTs resources for noticing mathematical thinking.

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Clinical simulations for mathematics teachers' training: the impact on preservice teachers' sense of self-efficacy regarding their knowledge

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Forty-four preservice teachers participated in clinical simulations as part of their training. The simulations were designed to improve teachers' knowledge. The study investigated the advantages and disadvantages of this method concerning the teachers' self-efficacy regarding their knowledge. Results demonstrate that the simulative experience did not enhance teachers' confidence and exposed gaps in teachers' knowledge.

Keywords: Mathematics, preservice-teachers, knowledge, self-efficacy, clinical simulations.

Introduction and theoretical background.

Clinical simulation is a learning process in which the learner copes with a simulated action or occurrence from the real world of employment. It is unique in that it allows the learner to experience an authentic scenario that may occur in real professional life (Sauvé et al., 2007). The main advantage of simulative experiences is the formation of a direct connection between abstract concepts and the reality of which one learns, allowing the learner to implement theoretical ideas and terms on a realistic event (Davidovitch et al., 2008). Such experience is considered more effective than mere theoretical training on the professional knowledge demonstrated by the learners, their self-confidence in their ability to cope with different situations, and their inner motivation (Kolb et al., 2014). It is extensively used in the context of vocational training for pilots, medical doctors, nurses, social workers, and more.

In recent years, clinical simulation has become common in teacher training institutions, as part of a general tendency to incorporate more real-life scenarios. Traditionally, mathematics teachers' training courses include content aimed at improving the knowledge needed for teaching and enhancing the sense of self-efficacy regarding the ability to teach; this could be achieved in diverse methods, such as workshops, lectures, and practical work in school. Nowadays, the use of analyzing real-life situations, rehearsing lessons, and discussing realistic scenarios is commonly used (Horn, 2010), sometimes using virtual reality that simulates a classroom (Barmaki, & Hughes, 2015). The present study focused and explored a method of clinical simulations, not by rehearsing scenarios in front of fellow students or colleagues, nor engaging in an imaginary scenario, but a simulation with actors. As part of the simulations, the preservice teacher (PST) takes on the role of the teacher, and the actors the role of the students. Each simulation can be conducted around a scenario that portrays either an event in the classroom (a teaching-learning event, or a disciplinary event) or an individual meeting conversation (with a student or a parent). The clinical simulation allows the PST to experience (or watch others experience) the event in a safe environment, so they can practice coping with different and varied confrontations, learn from them and receive professional feedback about them. Studies dealing with teacher training through clinical simulation with actors have mainly dealt with conflict situations (Cil & Dotger, 2015; De Coninck et al., 2021), and focused on discussions with parents or colleagues. These studies have found that this method of learning has improved teachers' awareness and empathy, and their sense of confidence for dealing with such situations in the future. The present study used simulations not for conflict situations, but for everyday teaching-learning experiences, to promote teachers' knowledge and to enhance teachers' self-efficacy regarding their knowledge.

What is the knowledge required for teaching? In the context of mathematics teaching, four components of knowledge were defined (Ball et al., 2008): Common Content Knowledge, CCK (e.g., knowing how to solve or calculate); Specialized Content Knowledge, SCK (e.g., knowing how to solve a problem in more than one way); Knowledge of Content and Teaching, KCT (e.g., knowing how to choose the suitable examples for presenting a topic); and Knowledge of Content and Students, KCS (e.g., familiarity with common student errors). Teachers' knowledge is a very significant factor in the teaching-learning processes: the broader the teachers' knowledge is, the better they cope with students' difficulties, the better intervention programs they prepare, and the better attainments their students have (Tchoshanov, 2011; Van Inger et al., 2016).

Another factor that may affect the teaching-learning processes is the teacher's sense of self-efficacy (Bandura, 1977): one's belief in his ability to successfully organize and perform a series of actions necessary to achieve the desired result. Performing a task properly requires both suitable skills and confidence in the ability to employ them. Thus, self-efficacy affects people's functioning: the higher self-efficacy is, the greater efforts and time they invest. Self-efficacy may affect teaching-learning processes: the higher teachers' self-efficacy, the better relations they have with school staff, the higher job satisfaction they have, and the more involved they are in promoting students (Dellinger et al., 2008; Sarıçam & Sakız, 2014). Studies exploring teachers' self-efficacy regarding their knowledge found a difference between the different components of knowledge. The highest level of self-efficacy is related to common content knowledge, and the lowest level of self-efficacy is related to knowledge of content and students (Schreiber & Fillo, 2019).

As mentioned above, simulative experiences might be used to promote teachers' knowledge and their sense of self-efficacy regarding their knowledge. Thus, it is important to explore the influence of simulative experience on the teachers' knowledge and the associated sense of self-efficacy. As far as I know, no simulative studies have been conducted that focused on the effect of simulations on mathematics teachers' knowledge and their sense of self-efficacy regarding their knowledge.

The aim of the study: To examine the effect of simulations on the teachers' confidence in their knowledge (both their knowledge in general and specific knowledge components). The research question: (1) Are PSTs able to express their knowledge during simulations? (2) What is the impact of clinical simulation on PSTs' level of self-efficacy regarding their knowledge?

Methodology

Participants: Forty-four PSTs, who participated in a practice-based course for mathematics teachers training, as part of their studies for a teaching certificate in mathematics. During the course, they participated in several simulation sessions with professional actors who played school students. In these sessions, 23 PSTs actively participated, playing the teacher in the simulation, while 21 participants watched them. Among the 23 active participants, 14 PSTs participated as a teacher with

3 players (class simulation), and 9 PSTs participated as a teacher with one player (individual meeting simulation).

The participants have a bachelor's degree in mathematics education/science/economy, and they have very little experience in teaching (only a few hours a week). During the course, they acquired theoretical knowledge regarding students' conceptions and misconceptions, and knowledge regarding teaching specific mathematical concepts and topics. They also watched and analyzed lessons videos.

Research tools:

1. Scenarios that were written as scripts and simulated a teaching-learning event. Two types of scenarios were constructed: a scenario that simulates a classroom event, in which the teacher teaches three actors, and a scenario that simulates an individual conversation, in which the teacher teaches one actor. In each script, there is a math problem, given to the teacher in advance. The actors (professional actors who do not know mathematics) get in advance the solution that they are supposed to present during the simulation (correct/wrong/correct but unusual solution) and are instructed how to react to every possible response of the teacher. The teacher must choose how to explain the problem to them, whether to answer their questions and how to respond to their suggestions for solutions or to continue in the path he has chosen. The solutions presented by the actors were aimed at probing the knowledge component needed for mathematics teaching (Ball et al., 2008): an unusual solution is aimed at SCK, the wrong solution is aimed at KCS.

An example for a scenario involving an arithmetic sequence problem: "the third number in an arithmetic sequence is 10, and the seventh number is 20. Find the fifth number". Each of the actors presented one of the following solutions: (1) A correct solution: using the equation $a_n=a_1+(n-1)d$ for the third and seventh numbers, and then solving a system of two linear equations; (2) A correct answer: finding the fifth number by performing arithmetic mean of the two given numbers; (3) An incorrect solution: the student wanted to find the constant difference between the consecutive numbers, so he wrote: 10, a4, a5, a6, 20, subtracted the values of the two given numbers (20-10) and divided the result by 5, instead of by 4, because he counted 5 numbers and not the 4 differences between them.

2. Simulation videos: Each simulation is filmed. The videos were transcribed and then coded - mapping events to the relevant knowledge component: CCK - participants knew how to solve the problem correctly, SCK - participants chose to show the students alternative or unconventional solution, KCT - participants prepared an explanation or a teaching method to the problem, KCS - participants addressed student difficulty or error. For the data analysis, the number of events in which participants displayed each knowledge component was counted, as well as the numbers of misses - events in which the participant was expected to demonstrate knowledge and did not (for example, an occasion in which a participant refused to acknowledge an easier solution suggested by a student).

3. Self-efficacy questionnaires: The participants were asked to address thirteen different statements that dealt with different components of knowledge and indicate whether their level of confidence increased, decreased, or did not change after participation in the simulations' workshops. For example, they were asked to refer their level of confidence in their ability to examine an unconventional student solution to a problem and determine whether it is correct (SCK). Or their

level of confidence in their ability to anticipate students' common errors (KCS). In addition, participants were asked to answer a concluding question: whether they generally felt that their level of confidence in their knowledge increased, decreased, or did not change after participation in the simulations' workshops.

4. Interviews: To elaborate on the self-efficacy questionnaire findings, interviews were conducted with 10 PSTs who participated as teachers in the clinical simulation. Participants explained their answers regarding their general level of confidence in their knowledge and elaborated on the reasons for the change in their confidence level.

Procedure: The actors were instructed how to respond and what to say to the PST. The active participant, who played the role of teacher, was given in advance the mathematical problem the simulation deals with. He could think about it and choose the solution and the way he wanted to explain it. But he did not know what solutions the actors would present, what their errors would be, and what they were going to ask. This was intended to imitate a real experience in the classroom, in which a teacher does not know in advance what the students will ask and what solutions they will offer. Each simulation lasted from 6 minutes to 15 minutes and immediately after it a feedback discussion was held with all present. During this feedback session, the teacher explained his motives, what guided him in his responses, the difficulties he had in managing the scenario his conclusions for a similar situation in the future; the actors explained how they felt as students, what the teacher’s strengths were and what they would offer him to do differently; viewers highlighted the teachers' strengths, his actions they approved of, and suggested additional approaches and additional possible responses to the simulative situations.

The participants could (if they wanted) participate again in another simulation.

Results

Teachers' knowledge: As previously written, in each simulation, the PSTs coped with a problem they received in advance. The solution they chose to present, their reaction to an error or an unconventional solution, revealed their knowledge and their ability to use it in real-time. Watching the videos, I analyzed the responses and pedagogical decisions the teachers had made and examined what components of teachers' knowledge were expressed during the simulation.

The PSTs showed common content knowledge, but they had partial and even lacking knowledge in specialized content knowledge and knowledge of content and students. Regarding CCK, the teachers had no gaps: all the PSTs but one solved the problem correctly. The one teacher who brought a wrong solution noticed it right at the beginning of the simulation and corrected her solution right away. But, regarding SCK, some PSTs were not familiar with solutions presented by the actors, either choosing to ignore it or forbidding the actor to present it. Most teachers did not know whether and how to present a second solution after one was presented, so they chose to adhere only to one solution –in all the scenarios but two, no solution was discussed other than the one that the teacher had prepared in advance. Furthermore, no connection was made between different representations, and many PSTs found it difficult to explain the mathematical principle underlying the solution they chose. All these expressions of SCK were missing or partial among the PSTs. Another gap in teachers' knowledge was regarding KCS: it was found that there was great difficulty among most PSTs in dealing with a student's error or wrong solution: most of them could not identify the difficulty, and none understood what the source of the error was. While most of them did pay attention to the actor who played the struggling student, their communication was merely personal, expressing empathy and patience, rather than educational; it lacked the necessary instruction tailored to a specific difficulty or specific error. Since the teachers did not understand exactly what the difficulty was, they chose to repeat the same solution instead of explaining to the student what the mistake was and how it could be avoided in the future.

Teachers' self-efficacy: The findings of the concluding question in the self-efficacy questionnaire (Table 1) vary depending on how the PSTs participated in the simulation.

Table 1: Changes in the level of confidence in the knowledge

	Increase in confidence		No change in the confidence level		Decreased confidence	
Active participants:						
All active participants N = 23	4	17%	10	44%	9	39%
Active with 3 actors N = 14	1	7%	7	50%	6	43%
Active with one actor N = 9	3	33%	3	33%	3	33%
Inactive participants:						
Only audience N = 21	8	38%	12	57%	1	5%
Total N=44	12	27%	22	50%	10	23%

Table 2 demonstrates that in general, only a quarter of those present in the simulation reported an increase in the level of confidence in their knowledge. Among those who were observers only, the simulation seemed to raise the confidence level or at least did not cause a decrease. Among those who were active with one actor, there was no difference in reporting between increase and decrease of confidence: a third of them reported an increase, a third of them reported a decrease, and a third of them reported no change in their confidence. In contrast, participants who were active as teachers with three actors were more likely than other participants to report a decrease in their level of confidence in their knowledge, even though in the discussions that took place immediately after the simulations, the strengths of the teacher were emphasized.

For a better understanding of these findings, interviews were conducted with some of the participants to explore the reasons for the effect of the simulation on their level of confidence. The interviews, as will be explained in detail below, highlighted that the change in confidence level mainly depends on the knowledge component the teachers inadvertently focused on.

Four PSTs reported that coping with the scenarios strengthened their confidence in their knowledge and in their ability to deal with similar situations in the future. In their interviews, they all referred to their knowledge of content and students. One of them, David, had a successful simulation experience with three actors. When one of them posed a question, David felt he answered it well. In the self-efficacy questionnaire, he indicated that his level of confidence in the knowledge of content and students increased, and so did his overall level of confidence. In the interview, he said: "When I was at school and my students had difficulties, I realized that I had already experienced something similar during the simulation. That was the moment I felt more confident. The scenarios we were given are very similar to what is happening in a real class with real students. The simulations improved my sense of confidence in my ability to deal with different situations in the classroom". Another one, Eva, also reported an increase in her general level of confidence and referred in her explanation mainly to the knowledge of content and students: "A situation like the one I experienced in the simulation happened to me in a classroom, and then I realized I was given tools to deal with it. I knew what to focus my explanation on. It strengthened my confidence that I could deal with situations in a real classroom". In these statements, the PST emphasized that coping with a scenario helps to cope with a similar experience in the future and strength the sense of confidence in the ability to handle similar situations.

However, it seems that first-time encounters with real-life situations might have the opposite effect. Unlike the above statements, many PSTs reported that the simulations decreased their confidence. They also explained their feelings in the interview by referring to what they know or don't know. For example, Ron said: "I feel that I don't always know how to explain a solution that I am not familiar with, or how to explain errors to the student who made them". Ron's confidence was influenced by two components of knowledge: specialized content knowledge, e.g., dealing with an unconventional way of solving, and knowledge of content and students, e.g., dealing with a student's error. Lina, another PST, also focused on these two types of knowledge, because during the simulation she had to deal both with an unusual solution, and with a wrong solution: "During the simulation, I felt the need to bridge between my desire to show a nice mathematical solution and the actors' desire to get an answer... there were situations where I felt that I directed my explanations at a particular actor, while the others were not silent… On the one hand, it feels like a real classroom, but on the other hand, here it was a little harder for me, and it undermined my confidence. All I wanted at that moment was for the actors to listen to me and to understand their error, but I felt my explanation was lacking".

Summary and conclusions

In mathematics teacher training courses, PSTs are aimed to gain the knowledge required for mathematics teachers: learn teaching methods, different representations for mathematical objects, and common errors of students. One of the tools used in the training process is simulation workshops, in which the PST can experience a reality-like scenario that allows him to use the theoretical knowledge he has acquired. The rationale for conducting these is that teacher knowledge is best developed through the experience of teaching a classroom; for this reason, it is believed that PSTs should be trained to develop a wide range of skills experientially and practically (Ball, & Forzani, 2010). The advantage of this experience is that the scenario's focus is on specific types of knowledge and enables the participants to watch themselves and discuss with others their decisions and their responses.

The questions of the present study were: (1) Is the knowledge imparted to PSTs reflected during the simulation workshops? (2) Does attending workshops increase PSTs' sense of self-efficacy regarding their knowledge?

The study described in this paper does not seem to give clear-cut answers to these questions. Regarding the knowledge of the PSTs, the findings show that differences were found between the levels of proficiency and control the PSTs demonstrated in each component of knowledge. In common content knowledge, the teachers were more knowledgeable; in pedagogical-content knowledge, they were less knowledgeable, which may adversely affect their teaching and the learning process of their students; and in specialized content knowledge and knowledge of content and students, they were least knowledgeable. These findings support the findings of past studies, in which it was found that teachers in general, and particularly novice teachers, lack these components of knowledge (Schreiber & Fillo, 2019). If the teaching quality of teachers is an outgrowth of their knowledge, the findings suggest that PSTs should strengthen mainly the two components of knowledge that were found missing (SCK and KCS), and therefore training should include workshops with scenarios that require an online active use of these components; for example, unusual (SCK) and wrong (KCS) solutions.

Regarding the PSTs' sense of self-efficacy, the results indicated that the simulation might not be the best experience for the participants, since only a few participants reported improved confidence**.** It was found that after participating in the simulation workshops, most of the participants reported that their general confidence was undermined, as well as their confidence in their knowledge. A possible explanation for the results is that the teachers lack certain knowledge components. In the interviews, the explanations that teachers gave for their decrease in their general sense of confidence, usually focused on their difficulty to handle unconventional solutions and students' difficulties, which is linked to SCK and KCS. Most of the participants' explanations focused on a particular knowledge component that lowered their confidence. It can be deduced from the PSTs' answers that for each of them there was one component of knowledge in which they were less knowledgeable, and thus has more influence on their sense of confidence. Another possible factor for the decrease in the general confidence that was expressed in the interviews is the difficulty of addressing three actors at once while being subjected to criticism from many spectators, as well as from the actors themselves.

How can the benefits of simulative experience be maximized so that they improve participants' sense of competence and increase their level of confidence? How to avoid a situation of feeling 'failure' in dealing with the scenario? The results suggest that some measures should be taken to increase the positive effect of clinical simulations on preservice mathematics teachers. First, more work needs to be done to support the PSTs' development of SCK and KCS in positive experiences, to increase their self-efficacy before engaging in this type of simulation. Second, the self-efficacy aspect should be strengthened in the training process in general, and in the feedback for a simulation in particular; more emphasis should be put on the strengths of the participant and his empowerment. Third, simulations with one actor are less intimidating and should be the only kind used; the course and the simulations should focus on one type of knowledge. The present study and its findings imply that while providing guidance and consultation to novice teachers, it is important to strengthen their confidence while promoting their knowledge, especially concerning specialized content knowledge and knowledge of content and students. This recommendation has also been mentioned in previous studies (Schreiber & Fillo, 2019; Van Inger et al., 2016).

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Improving pre-service teachers' diagnostic judgments regarding task difficulties in the domain of fractions and angles – A process-oriented investigation with Eye-Tracking.

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In this ongoing study, the focus is put on the potential influence of specific knowledge on pre-service teachers' judgment processes when diagnosing the difficulty of mathematics tasks. It is assumed that pre-service teachers use their knowledge in order to identify difficulty-generating task characteristics and to evaluate them in terms of their difficulty for students. To examine this assumption, an experimental group, in which participating pre-service teachers acquire knowledge about typical student difficulties and specific difficulty-generating task characteristics is compared with a control group. Using eye-tracking technology and stimulated recall interviews, indicators for diagnostic judgment processes are collected and compared between conditions. First results suggest that specific knowledge leads to a more efficient judgment process and enables pre-service teachers to identify and correctly evaluate difficulty-generating task characteristics more frequently.

Keywords: Diagnostic judgment processes, knowledge, eye-tracking, stimulated recall interview.

Introduction

For effective mathematics instruction, the use of tasks that are adapted to the students' ability level in terms of difficulty is considered central (Leuders & Prediger, 2016). A prerequisite for this is that teachers are able to identify task difficulties and evaluate them adequately. The difficulty of mathematics tasks is influenced by mathematical task characteristics (e.g., in the domain of fractions: like vs. unlike fractions; Padberg & Wartha, 2017) and instructional characteristics (according to cognitive load theory (CLT): e.g., split-attention vs. integrated task design; Sweller et al., 2011), among other characteristics. However, research findings show that teachers often make inadequate judgments about task difficulties (Karing & Artelt, 2013) and do not sufficiently consider the task's instructional design (Schreiter et al., 2021). Specific knowledge of difficulty-generating task characteristics was found to play a significant role for the accuracy of teachers' diagnostic judgments regarding task difficulty (Ostermann et al., 2017). However, most of the research on teachers' diagnostic judgments has so far focused on judgment accuracy. Therefore, little is known about how teachers get to their result and what role knowledge plays in the judgment process (Loibl et al., 2020). To gain insight into internal judgment processes, eye-tracking in combination with eye-tracking stimulated recall interviews has proven to be an effective method (Schindler & Lilienthal, 2019).

Theoretical background: Diagnostic judgment processes

It is assumed that in the genesis of diagnostic judgments, teachers use the information available in a diagnostic situation and process it on the basis of their knowledge to get to their result (Herppich et al., 2018). When judging the difficulty of a task for students, the relevant information constitutes the task characteristics that hold information about the tasks' difficulty ("difficulty-generating task characteristics"; Leuders & Prediger, 2016). In the judgment process (Loibl et al., 2020), these difficulty-generating task characteristics should be identified by teachers and adequately evaluated in terms of difficulty for students. Diagnostic judgment processes, such as identifying and evaluating task characteristics, constitute internal cognitive processes that cannot directly be observed. Research showed that eye-tracking is an effective method to investigate prospective teachers' diagnostic judgment processes (Brunner et al., 2021). However, as the interpretation of eye-tracking data can be challenging and ambiguous (Moreno-Esteva et al., 2017), a triangulation with qualitative data, such as eye-tracking stimulated recall interviews (ET SRI), is recommended (Schindler & Lilienthal, 2019). ET SRI is a research method to investigate cognitive processes by asking probands to retrospectively describe their own thoughts using a video sequence of their eye movements. In their systematic review on teacher's professional vision, Grub et al. (2020) report that numerous eye-tracking studies determined significant differences in gaze behavior between experts and novices. For example, experts were found to have shorter fixation durations, which were interpreted as an indicator of faster information processing. However, in most of these studies, experts were distinguished from novices only by the number of years of job experience, and other knowledge components were not considered. It therefore remains unclear how and whether teachers' knowledge – regardless of job experience – influences their gaze behavior during diagnostic activities.

This Study

This ongoing study aims to examine pre-service teachers' diagnostic judgment process when diagnosing the difficulty of mathematics tasks in the domain of fractions and angles. In this regard, it is intended to examine which task characteristics (mathematical vs. instructional) pre-service teachers identify and evaluate when judging task difficulties. In addition, a particular research interest is to investigate the potential influence of specific knowledge about difficulty-generating task characteristics on identification and evaluation processes. Please note that this contribution constitutes a pre-report of an ongoing study and only covers a sample of the data and results of a larger data set. Based on the above-mentioned findings of Schreiter et al. (2021), we expect for both content areas (fractions and angles) that a) mathematical task characteristics are identified and correctly evaluated more frequently compared to instructional characteristics (H1a) and b) instructional task characteristics are to a large extent not identified and correctly evaluated (H1b). Furthermore, building on the results of existing research on the influence of specific knowledge on judgment accuracy (e.g., Ostermann et al., 2017), we assume that specific knowledge enables pre-service teachers to identify and correctly evaluate more difficulty-generating task characteristics (mathematical and instructional) (H2). As eye-tracking studies focusing on teachers' professional vision revealed significant differences in gaze behavior between experts and novices (here: teachers with and without job experience; Grub et al., 2020), we further aim to exploratively investigate whether specific knowledge influences the gaze behavior of pre-service teachers (who have no job experience) during their diagnostic judgment of task difficulties.

Methods

Participants. N = 25 pre-service mathematics teachers were assigned to two conditions: an experimental group (n = 11), that received a 90-minute intervention and a control group (n = 14) without treatment. The intervention addressed specific mathematical task characteristics in the domain of fractions and angles that are known to cause difficulties for learners. Furthermore, theoretical foundations of CLT and related instructional task characteristics were addressed.

Material. Eight mathematics tasks were created (four fraction tasks and four angles tasks). Between these tasks, difficulty-generating mathematical and instructional task characteristics were systematically varied. The mathematical difficulty of the fraction tasks was varied by modifying the denominators (like vs. unlike), by mixing natural numbers and fractions, and by using mixed fractions (Padberg & Wartha, 2017). The mathematical difficulty of the angle tasks was varied by the number of argumentation steps needed to solve the task and by the angle values used (tens vs. units) (e.g., Reiss, 2002). The instructional difficulty of both content areas (fraction and angle tasks) was varied according to specific CLT design principles (Sweller et al., 2011): based on the *split-attention effect*, the tasks' relevant information is presented either close or distant from each other. Furthermore, building on the *redundancy effect*, the tasks were created in such a way that a) one and the same information is presented by different information sources or b) additional information irrelevant for the solution is included or c) no redundant information is included. The instructional design of the sample fraction task (figure 1) causes potential difficulties, as different information sources (the graphic and the time information) are presented distant from each other (split-attention effect) and redundant information (the route from Neckarsteinach to Ziegelhausen) is included (redundancy effect). The instructional design of the sample angle task (figure 1) causes potential difficulties, as the same information (values of the given and missing angles, tip) is presented by the text as well as in the graphic which causes unnecessary processing procedures (redundancy effect).

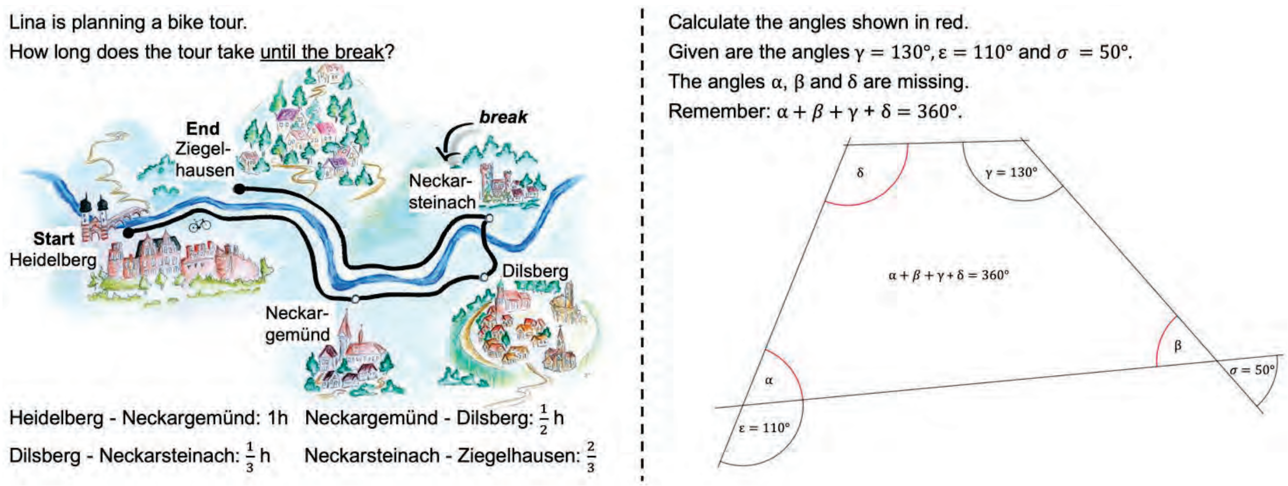

Figure 1: Sample of a fraction task and an angle task with specific difficulty-generating mathematical and instructional task characteristics

Procedure. The participating pre-service teachers were asked to assess eight mathematics tasks regarding the question "What makes the task easy/difficult for students?" The tasks were presented individually and in randomized order on a 24'' LCD screen. Eye-tracking data was collected using a monitor-based eye-tracker (Tobii Pro Fusion) that captures binocular eye movements at a sampling rate of 120 Hz. For adjusting the eye-tracker, a 9-point calibration was performed before each task. The time interval between the diagnostic task on the eye-tracker and the subsequent ET SRI was kept as short as possible to avoid loss of memory (approx. 1-3 min.). During the interview, subjects described what they did and thought during the diagnostic task, based on their shown gaze behavior. For the recording of the ET SRI, the software OBS was used, which records screen contents including sound, so that the videos of the eye movements with the corresponding comments of the subjects were available for the later analysis.

Data analysis. Tobii Pro Lab software was used to analyze the eye-tracking data. In each task, specific Areas of Interest (AOIs) were defined around the varied mathematical and instructional task characteristics. The number of fixations and fixation durations were determined using the Tobii I-VT Fixation Filter. To determine the number of transitions between two AOIs, the videos of eye movements were visually inspected. A mixed-methods approach was used to analyze the ET SRI data: The ET SRI were first transcribed and coded deductively using qualitative content analysis. The following category system was used and binary coded (in parentheses): difficulty-generating task characteristics can be *identified* (1), or *not identified* (0) when diagnosing a task. It turned out that some task characteristics are only identified when reflecting on one's own gaze behavior during the SRI. This resulted in another category *retrospectively identified*, which was evaluated as a subcategory of *not identified*. Identified task characteristics can be *correctly evaluated* in terms of difficulty for students (1) or *incorrectly / not further evaluated* (0). Task characteristics that are only evaluated during the SRI are assigned to the category *retrospectively evaluated* that counted as a subcategory of *not further evaluated*. Transcripts were coded by two raters with high interrater reliability (Cohen's Kappa = .88). The assigned codes were then integrated into a quantitative data set to examine differences across experimental conditions using variance analysis.

Preliminary Results

Identification and evaluation of difficulty-generating task characteristics

Two repeated-measures ANOVAs were calculated with the within subject factors *task characteristics* (mathematical/instructional) and the between subject factor *condition* (experimental/control group), separately for the fraction and angle tasks. Figure 2 gives an overview of the average percentage of difficulty-generating task characteristics that were identified and correctly evaluated in terms of difficulty for students.

Regarding the fraction tasks, the results showed that there is a significant difference with high effect size between the identification and evaluation of mathematical vs. instructional task characteristics ($F(1,23) = 15.01$, $p < .001$, $\eta 2 = .40$). This effect is, however, dependent on the experimental condition. Bonferroni-adjusted post-hoc analysis showed that differences between mathematical and instructional task characteristics can only be determined for participants of the control group (cf. figure 2). Here, significantly more mathematical task characteristics were identified and correctly evaluated compared to instructional characteristics (H1a). Instructional task characteristics were to a large extent not identified and adequately evaluated (H1b). Without specific knowledge, less than half of the instructional task characteristics were identified and correctly evaluated on average (M = .42, SD = .12). A significant difference with high effect size could be determined between experimental conditions ($F(1,23) = 31.29$, $p < .001$, $\eta 2 = .58$). Bonferroni-adjusted post-hoc analysis revealed that participants of the experimental group identified and correctly evaluated a significantly higher number of both mathematical and instructional task characteristics (cf. figure 2). Specific knowledge about difficulty-generating task characteristics has thus enabled pre-service teachers to identify and adequately evaluate more difficulty-generating task characteristics when judging task difficulties for students (H2).

Regarding the angle tasks, however, no significant difference could be determined between the identification and evaluation of mathematical vs. instructional task characteristics ($F(1,23) = 0.43$, $p = .521$, $\eta 2 = .02$). In the case of the angle tasks, both the mathematical ($M = .34$, $SD = .24$) and the instructional task characteristics ($M = .29$, $SD = .30$) were to a large extent not identified and correctly evaluated by the participants of the control group. Thus, only H1b can be confirmed for the angle tasks. Comparing the two study groups, a significant difference with high effect size could be determined ($F(1,23) = 26.29$, $p < .001$, $\eta 2 = .53$). As can be seen in figure 2, participants of the experimental group identified and correctly evaluated a significantly higher number of both mathematical and instructional task characteristics. H2 can thus also be confirmed for the angle tasks.

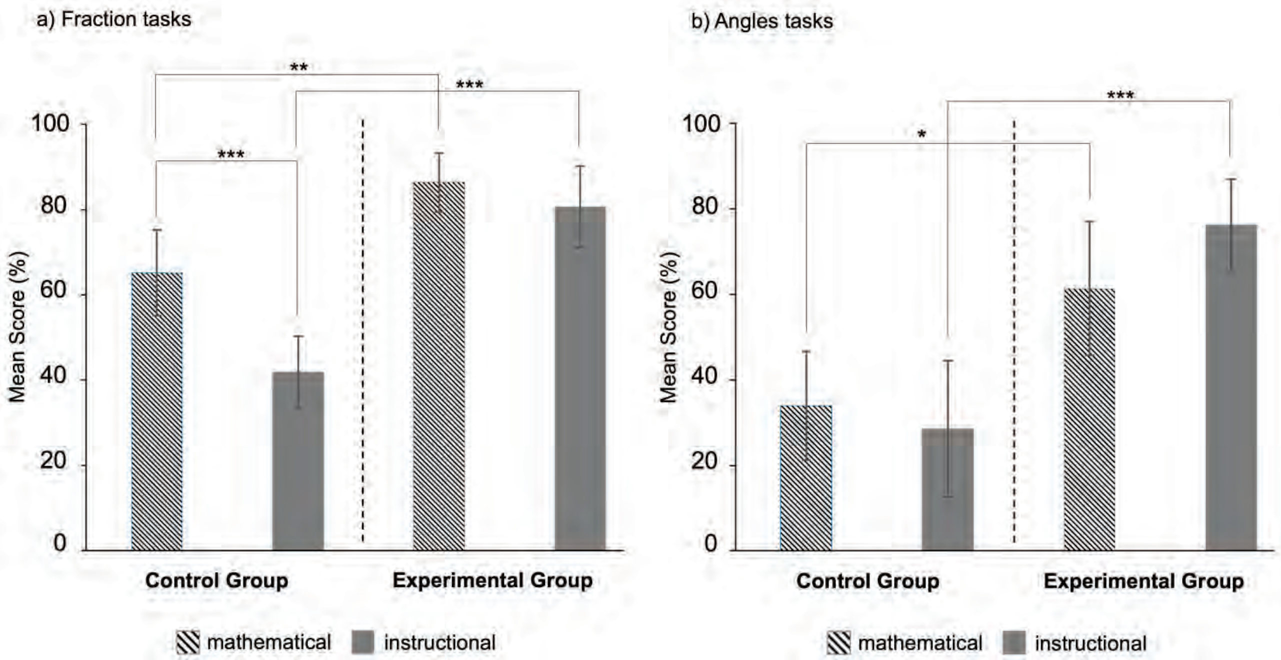

Figure 2: Means and Standard Error for identified and correctly evaluated difficulty-generating task characteristics (mathematical and instructional). *p<.05, **p<.01, ***p<.001.

Analysis of gaze behavior

The collected eye-tracking measures were used as indicators of visual attention in diagnosing task difficulty. Regarding the fraction tasks, a lower number of fixations and lower fixation durations within predefined mathematical AOIs as well as shorter total recording durations could be determined for the participants of the experimental group compared to the control group. These group differences are statistically significant with high effect sizes (cf. table 1). Against the background that participants of the experimental group identified and correctly evaluated significantly more difficulty-generating task characteristics, these eye-tracking measures may indicate a more efficient approach to diagnosing with specific knowledge. For the instructional AOIs, however, no significant differences in gaze behavior were found between the control and experimental group. Regarding the angle tasks, the eye-tracking data analysis is not yet complete but will be presented at the conference.

Table 1: Eye-Tracking measures (only fraction tasks)

AOI	ET measure	Control Group M (SD)	Experimental Group M (SD)	t	p	Cohens d
Mathematical	Fixation count	28.71 (12.71)	19.14 (7.84)	2.19	.039	0.91
	Fixation duration	8.31 (4.34)	4.89 (2.22)	2.55	.019	1.00
Instructional	Fixation count	34.35 (6.95)	34.21 (11.03)	0.04	.972	0.02
	Fixation duration	6.92 (1.37)	7.68 (2.93)	0.79	.442	0.33
	Transition count	9.04 (3.90)	8.68 (4.24)	0.22	.830	0.09
Overall	Total record duration	118.62 (12.74)	102.07 (12.03)	3.30	.003	1.34

Discussion

The aim of this ongoing study is to investigate which task characteristics (mathematical vs. instructional) pre-service teachers identify and evaluate when judging the difficulty of mathematics tasks in the domain of fractions and angles. Furthermore, a particular research interest is to explore the potential influence of specific knowledge about difficulty-generating task characteristics on identification and evaluation processes during the judgment.

In line with expectations, instructional task characteristics were to a large extent not identified and adequately evaluated, both regarding the fraction as well as the angle tasks. These findings support existing research on diagnostic teacher judgments (Schreiter et al., 2021), which showed that instructional task characteristics are insufficiently considered by teachers. In comparison to the instructional task characteristics, difficulty-generating mathematical task characteristics were identified and adequately evaluated significantly more frequently in the fraction tasks. In the case of the angle tasks, however, both mathematical and instructional task characteristics were to a large extent not identified and adequately evaluated. One explanation for this finding could be differences in prior knowledge. The results of the prior knowledge test of this study (results will be presented at the conference) showed that the participants of both study groups have more prior knowledge regarding difficulty-generating mathematical task characteristics in the domain of fractions compared to the domain of angles. Overall, the results of the study point to deficits with regard to the identification and evaluation of difficulty-generating instructional task characteristics, and, depending on the content area, also regarding mathematical task characteristics. However, for an effective mathematics instruction, teachers should choose, modify, or create tasks in such a way that task difficulties occur in appropriate doses and students are neither over- nor under-challenged (Leuders & Prediger, 2016). To do so, teachers need to be able to identify difficulty-generating task characteristics and to adequately evaluate them in terms of difficulty for students.

In line with expectations, the results showed that specific knowledge enables pre-service teachers to identify and correctly evaluate more difficulty-generating task characteristics. This effect could be found for both mathematical as well as instructional task characteristics. These results are consistent with existing research that highlights the importance of specific knowledge for the accuracy of teachers' diagnostic judgments (e.g., Ostermann et al., 2017). In our study, the collection of direct process indicators allowed to gain insights into the positive influence of specific knowledge on identification and evaluation processes that underlie teachers' judgment. Regarding the fraction tasks, the collected eye-tracking data further suggested a more efficient approach to diagnosing with specific knowledge: pre-service teachers with specific knowledge showed fewer fixations with shorter durations in task areas with difficulty-generating mathematical task characteristics. At the same time, they identified and correctly evaluated more difficulty-generating task characteristics. These results indicate that prospective teachers with specific knowledge are able to process relevant information faster, which is often seen in experts compared to novices (Grub et al., 2020). For instructional task characteristics, however, no significant group differences were found in terms of gaze behavior. It might be possible that there are no observable differences between persons who focus on task characteristics and process them fast and those who only pay little attention to the same characteristics. One possible explanation for our study finding could be that pre-service teachers without specific knowledge may have paid little attention to instructional task characteristics overall.

Overall, impulses for teacher training can be derived: An intervention on specific difficulty-generating task characteristics enables pre-service teachers to identify and adequately evaluate more difficulty-generating task characteristics and allows a more efficient approach to diagnosing. These results point to a need for learning opportunities to build specific knowledge during teacher training. Such learning opportunities should cover a wide range of difficulty-generating task characteristics (Leuders & Prediger, 2016) and typical student difficulties.

In addition to the practical relevance, the research strategic approach of this study should be emphasized: In the present study, theoretical predictions were made about how person characteristics (here: specific knowledge of prospective teachers) and situation characteristics (here: difficulty-generating task characteristics, content area of the task) influence the assumed information processing processes (here: identification and evaluation of task characteristics). These hypothesized relationships were then experimentally tested by systematically varying both situation and person characteristics and collecting process indicators (here: Eye-tracking data, and ET SRI) to examine internal judgment processes. This research strategy allows to generate knowledge about the information processing involved in the formation of diagnostic judgments, as has been widely called for (e.g., Herppich et al. 2018).

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A comparative study of the role of the external facilitator in lesson studies in Denmark and China

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Collaborative forms of mathematics teachers' professional development, such as lesson study, are integrated parts of the educational systems in many East Asian countries, while these forms are often new in Western countries. In this paper, we focus on a central role in lesson study, that of the facilitator. We compare how they talk with teachers and what they focus on in their talk in a lesson study context in Denmark and in China. We use a framework consisting of mentoring strategies and content categories; both developed empirically in a Chinese respectively a European context. Our analysis shows big differences in the facilitators' ways of engaging in talk with teachers. One big difference is the dynamic and relational patterns in the Danish case as compared to the lengthy talk of the Chinese facilitator. We analyze these patterns deeply and argue that their differences are not only related to the fact that lesson study is new in Denmark, but also to social and cultural differences.

Keywords: Comparative study, lesson study, the facilitator, mentoring strategy, content categories.

Introduction

In this paper, we compare the role of the external facilitator (the expert teacher, the knowledgeable other, etc.) in the varied forms of lesson study (LS) in Denmark and Shanghai, China. While LS is new in Denmark, it has long been a school-based teacher professional development activity in China which vary in forms for teachers with different teaching experiences (Huang et al., 2017). Given the challenge of importing a routine developed in one culture and one educational system into countries with different cultures and systems (Stigler & Hiebert, 2016), it is important to develop insights into the roles of central actors in LS, in particular the role of the facilitator. At the level of interaction between facilitators and mathematics teachers, we investigate the research question: To what extent are there similarities and differences between LS in Denmark and Shanghai in relation to *how* the facilitators talk to the teachers and *what* content aspects they talk about?

The role of external facilitators in lesson study

Research into the role of the external facilitator in LS is scare, though studies emphasize the role as crucial (Takahashi, 2014). In countries, where LS is new, there is a lack of external facilitators able to support and qualify LS, and the role become often to scaffold the LS processes, not to enhance their quality (Hart et al., 2011). Studies from East Asia warn against oversimplifying the role, emphasizing the complexity of conducting LS (Takahashi, 2014; Ding et al., 2019). We will highlight aspects of the role that have been pointed out in different countries.

In a study in the US, Lewis (2016) examines how teacher educators new to LS learn to lead this work. The author followed two teacher educators for 18 months, who, among other things, were apprenticed to experienced LS leaders before leading LS themselves. For our purpose, especially one challenge experienced by the teacher educators seems important: to define a form of leadership that is credible and valued and at the same time respectful of teachers' choices in directing the LS processes.

In a Japanese context, Takahashi (2014) investigates the nature of the final comment of the facilitator (i.e., the knowledgeable other). The three selected and popular facilitators (in the Tokyo area) focused on presenting new knowledge from research and the curriculum, showing the connection between theory and practice, and helping others learn how to reflect on teaching and learning.

Huang et al. (2017) contribute to a better understanding of Chinese LS as regards its social, cultural, and institutional aspects. They emphasize that the culture of respect to seniors makes it legitimate for teachers to learn from experienced facilitators and from watching exemplary lessons. They suggest that in a culture, where less respect is given to seniority and authority, modeling good lessons, and getting feedback from facilitators, which are crucial elements of CLS, may not work as effectively.

In another study in China, Gu and Gu (2016) examine the role of the facilitator (i.e., the Teacher Research Specialist) in post-lesson debriefings based on more than 100 h. of videos of 50 facilitators. They develop a two-dimensional framework for analyzing the mentoring activity: the first dimension encompasses the dynamic between the facilitator and the teachers, *the mentoring strategies* (see below), and the second dimension is the knowledge that mentors pay attention to (i.e., mathematical, pedagogical and practical knowledge). Regarding the mentoring strategies, Gu and Gu found that "the conversations between [the facilitator] and teachers were…monologues rather than dialogic in nature", with the facilitators paying most attention to "what they know and what they anticipated, rather than…what teachers were concerned about in their teaching" (p. 451). Regarding the knowledge, the facilitators focused on practical knowledge, helping teachers to analyze concrete cases that embraced mathematical and pedagogical ideas.

Theoretical approach

Given our research questions, we need a theoretical approach that allows us to capture both how the facilitators talk with teachers, and what content aspects they focus on. Regarding the 'how' question, we are inspired by Gu and Gu's (2016) four types of mentoring strategies, especially because they are developed in a Chinese LS-context. The strategies are: 1) *General comments*: what teachers in general should know and do, regardless of the specific LS. 2) *Comments on anticipated problems*: focused on problems, that teachers were expected to encounter, and advice on how to deal with them. 3) *Responses to teachers' questions*: related to issues occurring in the observed lesson. 4) *Dialogues with teachers*: the facilitator and teachers discuss and share their views on these issues. Gu and Gu (2016) characterize the conversation between facilitator and teachers as *authoritative*, if the first two types dominate, and as *dialogic*, if the last two types dominate.

Regarding the 'what' question, we are inspired by the *Knowledge Quartet* (KQ) (Turner & Rowland, 2011). The KQ consists of four categories: *foundation, transformation, connection*, and *contingency*. *Foundation* refers to the teacher's theoretical background and beliefs in terms of what they learned at school and teacher education etc. It includes knowledge of mathematics and of research on mathematics education, and beliefs about mathematics, its teaching and learning. The three other categories are different as they refer to ways in which "content knowledge comes into play in the classroom" (p. 199) as *knowledge-in-action*. *Transformation* relates to the choices, teachers make, when transforming their own content knowledge into pedagogical forms targeted at students, while *Connection* refers to teachers' choices and decisions about establishing coherence in students' learning across lessons and class levels. The last *Contingency* category concerns the teacher's response to unexpected classroom events. The KQ is suited for our purpose since it focuses on content that comes into play in classrooms (or in conversations as in this paper), and not explicitly on the knowledge possessed by the participants or the knowledge they ought to possess. This perspective on knowledge is different from the one in Gu and Gu (2016).

Methodological approach

We conduct our comparative study as a multiple case study based on two cases from existing research projects, one in Denmark (Skott & Møller, 2020) and the other in Shanghai (Ding et al., 2019). We introduce briefly the two projects, their contexts in each country, and our analytical strategy.

The Danish LS case

Introduced around 2010, LS is new in a Danish context. A LS project is typically initiated by persons outside a school as short-termed initiatives that are self-contained and aim to learn teachers to do LS on their own. The 1½ year long project, that was initiated by the first author's research group in 2014 at a school in the Copenhagen area, fits this description (for details see Skott & Møller, 2020).

At the time of the project, there were radical educational changes at the political level in Denmark. Particularity, one change was important: the steering documents now encouraged teachers to plan in a certain way and to formulate measurable learning goals. This challenged teachers in general.

The project used a Japanese approach to LS (Murata, 2011), but with repeated teaching of the revised lesson plans. The selected case was from the project's second year, and the group consisted of three experienced mathematics teachers and two facilitators (teacher educators) from the research group (Ea and Pia) of which only Ea had experience with LS. Ea and Pia facilitated all the LS processes (three two-hour planning sessions, three repeated teaching – one by each teacher – and three one-hour post-lesson debriefings) and participated in them on equal terms with the teachers. The teachers aimed to design a new teaching approach to the solution of linear equations in which their 6th grade students would be supported in developing a structural understanding of the equal sign.

For this paper, we selected the second planning session since it was representative of the facilitators' ways of engaging in conversations with the teachers. During this session, the teachers presented their pre-prepared learning goals and tasks, which the participants further developed together. Transcripts of the video-recordings of this session comprise our primary data, but we also include lesson plans.

The Shanghai Lesson Design Study case (SH LDS)

The LDS is a variation of the forms of LS (called "Keli" in Chinese) in the school context in Shanghai. LS is one form of school-based development in which each teacher participates as part of their work (Huang et al., 2017). The LDS model has three LS cycles (L1, L2 and L3) (for more details see Ding et al., 2019). It was conducted at an international school in a suburb of Shanghai from 2012 to 2015.

The selected case was one of seven Keli topics focused on in the LDS. The LDS group consisted of seven Keli elementary teachers (Grades 1-5), two expert teachers invited by the school (one was the mathematics Teacher Research Specialist, Zhang, in this paper), one researcher (the second author), and five other mathematics teachers (Grades 1-8) from the school's teacher research group (TRG). The Keli teacher was a junior with two years of teaching experience. The case topic was to investigate the relationship between perimeter and area of rectangles in the Shanghai Grade 3 textbook.

In this paper, we focus on Zhang's talk in the first post-lesson debriefing of the selected Keli case. There were roughly two parts of his talk. The first part lasted 40 min. and focused on the problems in L1 and how to redesign the lesson in L2. The second part lasted 35 min. and focused on how to reimplement the redesigned lesson. The meeting lasted 75 min. Transcripts of video-recorded TRG-meetings are our primary data materials, but we also draw on lesson plans and a teacher interview.

Strategy of analysis

To examine both the 'how' and the 'what' questions, we analyzed the facilitators' utterances line-by-line and coded them in a two-folded way: mentoring strategies and KQ-categories. However, in the Danish case, Gu and Gu's four types proved to be insufficient to capture all the strategies used by the facilitators. We thus added three more types based on our preliminary data analysis: 1) *Encouraging comments*, such as emotional recognition of teachers' ideas and suggestions (e.g., "I think you did the right thing by choosing goals"). 2) *Challenging comments*, such as disagreeing with teachers' proposals and understandings (e.g.," I think [your learning goals] are too comprehensive to be reached in a single lesson"). 3) *Building on or reformulating teachers' ideas*, that are expressed in the conversation (e.g.," but as you said [this] could have been taught in the previous lesson, so in this lesson we could start from …").

Regarding the KQ-categories we were inspired by the codes provided by Turner and Rowland (2011). We did not code short utterances that only had a clarifying purpose and was unrelated to the KQ.

Results of the analysis

The Danish case

Few of the facilitators' utterances in the second planning session fell within the first two of Gu and Gu's (2016) mentoring strategies, while slightly less than half fell within their last two types, and slightly more than half fell within in the three added types. According to Gu and Gu, the facilitators, thus, seemed to talk with the teachers in a dialogic way. However, we nuance this characteristic later.

The main part of the facilitators' utterances fell within the KQ categories. Of these, the majority fell within the connection and transformation categories, while the rest was of a foundational nature. There was almost none in the contingency category, which is not surprising, as the focus was on planning. The contributions outside the KQ, was primarily encouraging comments, such as "What you suggest sounds reasonable" (Ea). This indicates that the facilitators primarily focused their contributions on knowledge-in-action, which we will elaborate below.

We will give three examples of different combinations of mentoring strategies and categories. The first is an example of: response to teachers' question (third type) and foundation. The example is interesting, since this is the only combination where the facilitators brought foundation in the form of mathematics education research into play. Initially, a teacher asked, “what does research say, should it [the context of the task they were designing] be something that the students can relate to, or could it be purely mathematical?”. Ea answered that “research does not say anything about that is has to be a context from students’ everyday life, but Realistic Mathematics Education emphasizes that it should be a context that the students can imagine and experience as meaningful”. Here, the facilitator transformed her foundational knowledge of the specific research result into forms that made sense for the teacher in relation to the specific issue and that helped all the teachers to broaden their perspective on the issue. Note that it was a teacher asking for this kind of knowledge.

The second is an example of the combination: building on teachers’ ideas (added type) and foundation. The participants discussed learning goals in relation to the task they were designing, when a teacher claimed that an equation “can be interpreted in many ways as” something to do with concrete materials and “something about x”. Pia replied, “That is exactly why it makes sense to break down goals … you need to focus only on parts of them in specific lessons”. Breaking down goals is a term introduced by the new steering documents, which the teachers have difficulties assigning meaning to in practical situations. As such, the term can be said to be part of the foundation as it needs to be learned formally for instance by studying the documents. The facilitator built on the teacher’s idea of many layers of goals to provide practical meaning to the term by meta-communicating its purpose and how it can affect their planning. We argue again that the facilitator transformed her foundational knowledge into forms that were meaningful for the teachers in the specific situations.

The third is an example of the most frequent combination: dialogues with teachers (the fourth type) and connection. The participants discussed how to introduce the task to students. Pia suggested to use scenarios that they had formulated because “it is a difficult process for students to make up a scenario that can be solved without including weird numbers. Then you can also formulate different scenarios to meet the needs of different students”. This example is one of many where the facilitators intended to make the teachers themselves understand and realize what would be the most appropriate decision in a particular situation, instead of telling them what to do. The facilitators formulated their advice as suggestions, which they provided reasons for. In this case, the reason was the anticipated epistemological difficulty for students (i.e., the connection category).

In summary, the facilitators contributed to establish a conversation with the teachers, that was much more dialogic than authoritative in nature. It was characterized by being dynamics in terms of short contributions (less than two min.) from all participants, open and negotiable. The negotiations were on the terms of the teachers (as the aim was to produce a lesson plan that suited their needs) and based on their contributions (i.e., building on their ideas, questions, and concerns). The facilitators did not make decisions (not even when asked) but attempted to support the teachers in making these by encouraging and supporting pedagogical reflections. The content of the facilitators’ contributions was mostly related to knowledge-in-action (transformation and connection), but also to the foundation category. However, then the facilitators contributed something of a foundational nature, they tended to transform it into forms that were meaningful for the teachers in specific situations and not to present it as knowledge per se. Hence, contributions of this kind tended also to be knowledge-in-action.

Instead of describing this type of conversation as dialogic, we suggest to characterizing it as *relational*, as it was crucial for the facilitators to establish a good relationship with the teachers (cf. the nature of the added types). This seemed to be a prerequisite for producing a joint lesson plan.

The SH LDS case

During the 75 min. long post-lesson debriefing, Zhang talked most of the time and the teachers listened. The conversation can, thus, according to Gu and Gu (2016), be characterized as authoritative. However, after a close analysis of the content of Zhang's talk, we consider the conversation as dialogic, which we will illustrate by two examples. First, Zhang initiated the debriefing by saying to the teacher, "In LS, you need to pay more attention to other's critical and creative ideas about your lesson…merely praising your lesson will not help you to improve your teaching". This comment can be perceived as contributing to an authoritative conversation. However, given the limited time of the school-based LDS and the business of teachers, the general cultural atmosphere in China is to be humble and to first learn from the facilitator's input to the LDS by listening carefully. Second, in the end of the first part Zhang said to the teacher, "So now you understand what problems you had in L1, and why we must modify the lesson plan. The key learning goal of this lesson is to enable pupils to participate into and thus gain learning experience of the whole process of plausible reasoning in mathematics". Hereafter all the participating teachers smiled to Zhang. They thus used a professional sign in China to show their respect and high appreciation of his input and practical wisdom. This indicates that Zhang's talk was not driven by his power or position above teachers, and that the questions he posed helped them to reflect on alternative ways to deal with their problems of using the reformed textbooks. Hence, when we consider Zhang's talk as a whole, we will characterize it as contributing to a predominantly dialogic conversation.

Regarding the 'what' question, in the first part of Zhang's long talk we could identify the foundation category, but with strong links to the transformation and connection categories. For instance, "In the west the focus is on the ground theories of the cognitive/psychological processes. We focus on craft art on the application of the theories. That is, how we deliberately use these learning theories to improve our lesson plan and classroom teaching and learning". The same picture emerged in the second part but combined with an emphasis on contingency of how to design and implement tasks to support students' different needs in class as regards their anticipated reasoning through inquiry-based activities. Thus, it was difficult to categorize Zhang's utterances into the individual KQ categories.

We will give an example of the most frequent combination of mentoring strategies and KQ categories. The example combines the two types: responses to teachers' (not formulated) questions and (implicit) dialogues with teachers, with the foundation category that links to transformation and connection. In the first part of his talk, Zhang posed a sequence of questions, "Why chose this topic in the textbook? … From the van Hiele theory of levels of geometrical thinking …". Zhang further posed questions for teachers to reflect on the updated educational assessment of deep learning, such as "whether the construction of the lesson matches our fundamentally shared educational value by the majority in the field". Though, Zhang's talk may be considered as "monologic rather than dialogic in nature" (Gu & Gu, 2016, p. 451), we interpret his sequence of questions "why … (because)" and "whether … or" as a mean to support teachers to reflect on the use of updated theories and educational assessment in LS.

In the utterances, Zhang expressed foundational knowledge such as awareness of educational purpose, the van Hiele theory, and theories of deep learning of mathematics. However, this foundation knowledge is specifically targeted the transformation and connection categories that are related to teacher's struggle in understanding the concepts of mathematics inquiry and inquiry-based teaching addressed in the reformed curriculum and textbooks. That is, Zhang links foundational knowledge to the transformation and connection categories by relating to the teacher's actions in L1 and her struggle with understanding the key problems embedded in the reformed textbook. Note that the teacher's problems were only evident in an interview with her after L1. It was not uttered during the debriefing.

In summary, Zhang's talk was considerably long and dominated the post-lesson debriefing. However, our analysis illustrates that he addressed the teachers' shared problems with the reformed textbooks and supported their learning of craft skills according to updated theories and educational assessment.

Concluding discussion

We identified major differences by comparing how the facilitators talked with the teachers and what content aspects they talked about in the Danish LS and the SH LDS. In the Danish case, the facilitators contributed to a type of conversation that we called *a relational dialogue*. Contributing encouraging comments, challenging comments, and building on teachers' ideas (the three added types of mentoring strategies), it was crucial for the facilitators to establish good relationships with the teachers to work together in the new LS context. Moreover, the facilitators' way of talking with teachers (incl. also the strategies related to a dialogic conversation) aimed to encourage and support teachers in their reflections on appropriate decisions regarding specific issues related to their classroom teaching. Our analysis shows that the facilitators supported the teachers' reflections, among other things, by providing reasoned suggestions in close response to the teachers' expressed needs. This is contrary to how the facilitators in Gu and Gu's study (2016) talked with teachers.

In contrast, the Chinese facilitator did not need to build up relationships with the teachers due to the school-based LS systems (Huang et al., 2017). Thus, Zhang could directly use the limited school-based meeting time to guide the teachers to deliberately learn craft skills. The use of Gu and Gu's (2016) mentoring strategies shows that Zhang contributed to an authoritative conversation. However, our analysis shows that it is more dialogic in nature. We see roughly two dialogical layers. One layer is self-evident as different teachers' utterances were recorded and thus noticeable in the analysis. The second layer concerns the teachers' gesture (e.g., smiling) and appreciation of Zhang's contribution. We interpret this as an expertise of the facilitator, being able to notice (implicitly) the teacher's struggle and provide a way to reflect pedagogically through a sequence of why and whether questions.

Regarding the 'what', our analysis shows that the facilitators focused their talk on the content, but with emphasis on different aspects. In the Danish case, the foundation category played a lesser role than connection and transformation, but the analysis shows, that regardless of which category the facilitators related to, their contribution took the form of knowledge-in-action. That is, they tended to transform their knowledge into forms that were meaningful for the teachers in relation to specific classroom situations and not to present it as knowledge per se. This seems to some extent to be different in the Chinese case, where the foundation category was prominent in Zhang's talk. However, Zhang was expected to provide foundational knowledge and gained respect by doing it, while the

Danish facilitators only contributed such knowledge, when requested to do so. This difference might be explained in terms of the differences between the two countries. In Denmark, teachers generally have a high degree of self-determination, a so-called *methodological autonomy* (Skott & Møller, 2020). This means that the single teacher tends to acquire the legitimacy to decide on teaching matters themselves, not only on those related to methods. For a facilitator, it is thus often demanding to balance between orienting teachers' work in certain directions and at the same time being valued and credited, as in Lewis' study (2016). While in China, there is a cultural belief: 'there must be a teacher from whom one can learn when one works together with other persons', which conveys a collective belief that learning is a process that involves acting, speaking, listening, observing and thinking in a group where some may be more knowledgeable than yourself. Combined with the cultural values of seniority and authority (Huang et al., 2017), the facilitator is thus from the outset positioned as a competent actor to be listened to. Although, the participating teachers differed in terms of experience – the Chinese teacher was a junior – this did not seem to influence our results significantly.

One important contribution of this initial comparative study is that although LS is new in Denmark and requires a new actor role, this novelty alone is insufficient to explain the differences in 'how' and 'what' the facilitators talk with teachers. The study indicates that cultural, social, and power related issues at both the interactional and a broader level influence the role of the facilitator. We shall examine such issues and suitable analytical frameworks further in our future comparative work.

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A resource approach to mathematics teacher education

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The relation between pre-service teacher education and the day-to-day practice of mathematics teachers in schools is a recurring topic in research on teacher education. Furthermore, the Norwegian teacher education programme has been criticised for polarity between theory and praxis. The following pilot study investigates what aspects of working with resources in university coursework pre-service mathematics teachers perceive as valuable preparation for teaching mathematics. I found that university coursework can alleviate the gap between theory and praxis through resources for teaching mathematics. Working with resources can provide learning opportunities that pre-service mathematics teachers experience as practical and useful.

Keywords: Documentational approach to didactics, resources, mathematics teacher education, preparedness, theory-praxis gap.

Introduction

Novice teachers leaving the profession is a challenge in many countries. In Norway, nearly 30% leave their teaching position within the first four years (Boyesen, 2021). The relation between pre-service teacher education and the day-to-day praxis of mathematics teachers in schools is a recurring topic in research on teacher education, and there is a call for better alignment of the knowledge development that takes place within universities and schools (Winsløw et al., 2009). Norwegian teacher education has been criticised for polarity between theory and practice, and developing a teacher education programme which better prepares pre-service teachers for their professional lives is pivotal (Mosvold et al., 2018).

In Norway, pedagogical content knowledge and, in this case, mathematical knowledge for teaching has been prominent in teacher education. However, the past decade witnessed a shift of focus from mathematical knowledge for teaching to core practices in the work of teaching mathematics (e.g., leading group discussions), both in the Norwegian teacher education programmes and in the international research literature on teacher education (Mosvold et al., 2018). Nevertheless, several researchers have expressed concern about this shift in ignoring content knowledge while highlighting the teachers' practice (Hovtun et al., 2021). I suggest that both approaches hold promise with respect to educating mathematics teachers who are well prepared for their profession.

Paving the way for a more extensive Ph. D study, the research question for this pilot study is: What aspects of working with resources in university coursework do pre-service mathematics teachers (PSMTs) accentuate as beneficial in preparing them for teaching mathematics in elementary school? In line with the shift towards core practices, the participants in this study showed an inclination towards teaching and experiences derived from praxis and school placement. However, as will be elaborated upon in due course, they also expressed appreciation of university courses that facilitate connections between knowledge acquired in university and in school placement.

The overarching goal of my inquiry is to contribute to mitigating the transition from pre-service teacher to novice teacher, aspiring to retain novice teachers in teaching positions. I contend that working with resources "hands-on" in mathematics teacher education can be beneficial in addressing the gap between theory and praxis, albeit not confined to core practices but rather as an activity that might amalgamate core practices and mathematical knowledge for teaching.

Theory

I propose taking a resource perspective on the issue of connecting theory and praxis. Hence, I chose to use the documentational approach to didactics (DAD) as my framework. Moreover, for the pilot, I rely on the concept of schemes as retained in the instrumental approach by Vergnaud (1998), which draws on activity theory and sociocultural theory of teaching and learning (Gueudet, 2017; Gueudet & Trouche, 2009). Further, I consider the process of instrumentation (how an artifact influences the subjects' activity) and instrumentalization (how the subject shapes the artifact) as operationalising the PSMTs' interplay with resources.

Pepin (2018) argue that research on curriculum materials (in mathematics education) to date has largely focused on how these support student learning with less attention to teacher learning. She further claims that simply exploring how teachers interact with curriculum resources “as is” is not sufficient (e.g., investigating schemes of use), but that extending the research area by also addressing ways of how teachers can usefully work with these “tools” is equally important (ibid., p. 366). For this purpose, Pepin distinguishes between curriculum materials and educative curriculum materials. The latter are materials designed to support teacher learning as well as student learning. Retaining this perspective on teacher learning, I contend that this is also pivotal to pre-service mathematics teachers learning and teacher education. However, as described by Pepin (ibid.), this perspective focuses largely on materials designed for teachers (e.g., by researchers/ teacher educators) to further help teachers design learning sequences. In comparison, the tentative position derived in this pilot study is that (student) teachers learn from choosing, transforming resources, implementing them, revising them, etc. This is in line with the perspective proposed by Gueudet and Trouche (2012) in terms of taking a broad view on resources as anything likely to re-source the teachers’ practice. Hence, I argue that involving PSMTs in all aspects of working with and developing resources, not limited to curriculum resources, is pertinent. Therefore, I consider the DAD framework apposite for this study as it regards teachers’ work with resources as pivotal in their professional activity and professional growth (ibid.).

Thus, following the DAD, the aspects accentuated by the PSMTs in preparing them for teaching mathematics are operationalised through the concept of schemes, and studying the first three elements of a scheme. According to Gueudet (2017, p. 200), a scheme has four parts:

- An aim (can correspond with the goal in activity theory);
- Rules of action: regular ways of acting for the same aim;
- Operational invariants of two kinds: theorems-in-actions (propositions considered as true by the subject) and concepts-in-action (concepts considered as relevant for the subject);
- Possibilities of inferences: the subject can adapt his/her activity to the special features of a given situation corresponding to the same aim.

Delineated by Vergnaud (1998), a scheme developed by a subject is associated with a class (“set”) of situations which in turn correspond to the same aim of the activity. In this case, the aim I have chosen to consider (inferred from the data material) is *knowing how to teach mathematics* corresponding to the situation of being part of a teacher education programme attending classes on mathematics education. Hence, the activity considered in this case is at a rather general level and associated with a “big scheme” (Gueudet, 2017). One might be critical of this level of generality. However, as I have chosen to focus on consistent schemes in all the participants' responses, I suggest that a “big scheme” is applicable for rendering the different aspects in this context.

As the participants in this pilot study have not been followed and observed over time, rules of action represent what the PSMTs perceive to be regular ways of receiving lectures and teaching in their education programme. This is operationalised through their answers in this inquiry, consisting of a questionnaire and written reflections. Moreover, rules of action also represent omissions in relation to the same education programme as evidence of what the PSMTs would have preferred their education to contain. The operational invariants are inferred from these rules of action as the reasons/beliefs behind said omissions and experiences. As long-term follow-up was not done, the last element of schemes, possibilities of inferences, is beyond the scope of this interim study.

Finally, the process of instrumentation/instrumentalization details the PSMTs' interaction with resources. Resources in this case, denotes an activity that, in their own view, prepares the PSMTs for teaching mathematics supporting their learning process for the aim presented above.

Methodology

Since 2017, the Norwegian teacher education has been a five-year education programme (previously four years), and the graduates receive a master’s degree. In their fourth year, the student teachers choose their specialisation. The participants in this pilot are in their fourth year. They specialise in mathematics education, which also entails having at least 60 ECTS credits in mathematics from previous courses, which are didactical courses focusing on the learning and teaching of mathematics in primary schools. The data collection for the pilot was based on the participants’ efforts in coursework in the fourth year of their educational trajectory.

As I try to solicit the opinions and experiences of the participants, the pilot study is conducted from an interpretivist position (Scheiner, 2019). Through a qualitative case study involving seven student mathematics teachers, I investigated what aspects of working with resources this group of PSMTs accentuate as beneficial in preparing them for teaching mathematics in school. The participants attended a course on problem-based mathematics education at a university in Norway. In this course, the PSMTs are introduced to a variety of resources on the subject of problem-solving and exploration as work methods in mathematics. At the end of the semester, the PSMTs were required to write down some overall reflections relating to their experience in this course, at which point they had only had one week of school placement. Seven responses were produced and gathered from the participants. Further, the participants took part in a qualitative questionnaire with open-ended questions at the end of the academic year. The questions focus on resources and the feeling of preparedness for teaching mathematics in elementary school. When completing the questionnaire, the PSMTs had been in school placement for five weeks, giving them opportunities to test and use their resources.

The overarching aim of the pilot was to gain an account of what the PSMTs regard as important in their mathematics teacher education. Simultaneously, I also ventured to discern what encourages or stymies their vocation for teaching mathematics. The concept of schemes as described within the DAD is pertinent for both objectives. Moreover, the data have been analysed using a qualitative content analysis approach where the theoretical categories (Kuckartz, 2019) are derived from the concept of schemes, namely aims, rules of action, and operational invariants, and further, the resources associated with the aim. The data analysed are the participants' answers to the given questionnaire and their written reflections from the coursework. The questionnaire was in Norwegian while they wrote the "overall reflections" in English.

As the data material in this pilot is limited, a discussion on trustworthiness is apposite. It is important to observe that this paper should be viewed in the context of a report on a pilot study. With respect to transferability, the sampling has indeed been purposeful as these participants are the predecessors in line of study to the participants selected for the main study. Hence, despite the lack of "thick" descriptions, I argue that the collected data give a reasonable foundation for interim inferences. Further, some degree of triangulation is applied by juxtaposing the overall reflections and the questionnaire.

Results

Because the scope of this pilot is not to study the resources used in a specific teaching unit nor the PSMTs documentation work per se, I do not present the data using specific tools from the DAD, such as a document's table. Rather, in the following, I present excerpts from both the PSMTs' reflection notes and their answers in the questionnaire, emphasising shared features or attributes in the PSMTs' schemes that might inform the overall goal of better alignment of the knowledge development happening within schools and universities. For the situation of studying to be a mathematics teacher, the aim at the crux of all the PSMTs responses was knowing how to teach mathematics. In this paper, I have chosen to focus on this aim as it is salient both for theoretical and practical knowledge development.

Aim: Knowing how to teach mathematics

In the questionnaire, the PSMTs were asked what they had found useful in their teacher education so far and what made them (better) prepared to teach mathematics:

PSMT 1: Lectures in didactics of mathematics, so how to teach mathematics.

PSMT 2: Another thing that I think has been very useful is learning about different strategies for teaching which we can use to plan good lessons for teaching mathematics.

PSMT 4: More school placement and to learn more about ways to teach [mathematics][1].

These excerpts serve as examples of an attribute present in all the participants' responses, without exception, namely the expressed desire to know more about how to teach mathematics to children in elementary school. Typically, they emphasised learning about methods for teaching mathematics as particularly useful so far in their teacher education and simultaneously as something they wanted to

[1] Words enclosed by brackets are added for the sake of context when not evident from the excerpts themselves.

learn more about to be more prepared for teaching. Several of the PSMTs put this in contrast to what they discern has been the norm in their teacher education:

PSMT 3: They often focus more on teaching us mathematics and not on how to teach mathematics.

PSMT 7: The introductory course was mainly doing mathematics. A lot of the curriculum was repetition from upper secondary school.

As demonstrated in the quotes above, some of the participants criticised previous mathematics courses in their teacher education programme, claiming that they focused too much on learning mathematics (which they, e.g., PSMT 7, described as a repetition from upper secondary school) and not on teaching mathematics. Triangulating these results with the PSMTs' "overall reflections", the PSMTs have a want for understanding how the knowledge acquired in their teacher education programme can have practical applications:

PSMT 3: This is the first course [problem-based mathematics education] that has allowed us to use what we already know and have learned before to make something that is going to be useful for us as teachers in the classroom.

The PSMTs express a lack of relevance and a call for practicable knowledge in their teacher education. Investigating this further, I now turn my attention to the rules of action and operational invariants interconnected with the aim described in this section and proposed resources thereof.

Rules of action and operational invariants

A possible operational invariant inferred from what has been explored thus far can be that the PSMTs consider themselves proficient in mathematics and that teacher education should focus on learning how to teach mathematics in elementary school practically. The rule of action in the PSMTs experience has, to a large extent, been learning mathematics. However, drawing on the operational invariant expressed above, what they would have liked to be the rule in their education is rather learning practical ways to teach mathematics. Examples they list, among others, are teaching methods, how to introduce new mathematical topics to young students, learning about how to differentiate their teaching to include all the students in a class, and learning about students with learning difficulties in mathematics.

In the question of what could help them to be better prepared for teaching mathematics, five out of seven participants answered more school placement:

PSMT 5: We could use more school placement, and to work with connecting what we are taught [in university] to practical situations and how to use it.

When comparing with the "overall reflections", all the participants have contemplated the connection between what they learn in university and in praxis. Drawing on these responses, another operational invariant is deduced. The connection between what is taught at university and the field of praxis is, in many cases, not apparent to the PSMTs. Hence, learning in and from praxis is seen as more relevant and useful. Albeit this is by no means a new discovery (Hammerness, 2013; Solomon et al., 2017), it is still salient in context of this study as an overarching goal is to find a viable way to attend to the gap between theory and praxis. The PSMTs then perceive theoretical learning as the rule of action while they prefer learning in and from school placement. However, when they receive help in connecting the two, they are more positive as elaborated upon in the following section.

Resources: What helps the PSMTs in learning how to teach mathematics

With respect to preparing them for teaching mathematics, the PSMTs endorse activities that help them connect theoretical and practical considerations – here represented by two of the participants reflections:

PSMT 6: Some of the [problem-solving] task [sic] has been hard to find the "right" learning goal [from national curriculum], and then we had some discussions around different goals. This has been so learning full [instructive] for me[2].

PSMT 7: It has been exciting to read about what authors think students think and feel about problem-solving, and afterwards go out and actually see that a lot is true. I use a lot of what I read, and have used it both when planning a lesson plan, but also during planning in practice [school placement] and as a substitute teacher.

All the participants accentuated how working practically with tasks in coursework at the university helped them connect research on mathematics education to practical situations. Further, as PSMT 6 exemplifies, connecting this activity to the national curriculum was experienced as positive and instructive. Through a process of instrumentation, problem-solving help the PSMTs reflect on practical and theoretical considerations for teaching. Moreover, as PSMT 7 shows, it leads them to enact what they have learned in coursework in their own teaching and thus, appropriating the knowledge from their coursework (instrumentalization). This holds implications for mathematics teacher educators as it exemplifies what activity PSMTs might experience as meaningful and relevant in their further work as pre-service teachers.

Moreover, the PSMTs appraise problem-solving tasks as an asset both for their understanding of students (experience how the students might think/feel) and as a method for teaching which supports them in several ways:

PSMT 1: Having the opportunity to work practically with tasks we can bring to the students ourselves has given me a greater experience of what it is like to work with such tasks with the students.

PSMT 7: Now I am so lucky that I have many problem-solving tasks in my folder, as well as which learning goal they are under. This helps me a lot later. I have already planned to have a folder with problem-solving tasks that are sorted by year level. This way they are easy to retrieve when needed.

Thus, in working with resources for problem-solving, some of the PSMTs' requests for their education appear to be met (instrumentation) e.g., learning methods for teaching (problem-based teaching) and ways to differentiate their teaching through using mathematically rich tasks. Moreover, as PSMT 7 shows, it can also affect their overall practice as mathematics teachers.

Discussion and conclusion

The RQ of this study was the following: *What aspects of working with resources in university coursework do pre-service mathematics teachers accentuate as beneficial in preparing them for teaching mathematics in elementary school?* I venture that explicitly working with resources for teaching mathematics might be one feasible way to align better what pre-service mathematics teachers learn in university coursework with what PSMTs themselves accentuate as beneficial in

[2] I have not corrected their English writing to prevent conveying misconstrued reproductions.

preparing them to teach mathematics in elementary school (as expressed by PSMT 7 above). The PSMTs in this study express discontent with the current allocation of theoretical and practical learning in their mathematics teacher education, claiming that praxis and school placement provides more relevant and applicable knowledge. The PSMTs express that when entering mathematics teacher education, they expect coursework and teaching to give instruction on how to teach mathematics to students in elementary school. In contrast, their experience is that more attention is given to teaching them (the PSMTs) mathematics rather than how to teach the subject themselves (e.g., PSMT 3). However, when they work with resources for problem-solving in university coursework, they perceive this activity as practical, instructive, and relevant to their future professional work as mathematics teachers. Following this, I claim that working with resources in university coursework can be auxiliary in amending the strategy of mathematics teacher education.

Further, two important aspects of working with resources is evident from the participants' responses. First, it is the mathematical content of the problem-solving tasks themselves and the opportunities these provide in the teaching of mathematics as expressed by both PSMT 6 and 7. They appreciate both solving the tasks themselves in preparing them for what meets the students, but also how these tasks facilitate learning with respect to the national curriculum. Hence, they regard content knowledge as relevant in preparing them for teaching mathematics. Second, they accentuate the methods for teaching that problem-solving tasks provide, such as problem-based teaching, differentiation, and the more general activity of gathering problem-solving tasks they can use later in their own teaching. As PSMT 3 exemplifies, such activity is "useful for us as teachers in the classroom". This suggests that working with resources may also promote core practices in university coursework.

A recent study in Norway shows that student teachers often see theory and praxis as separate entities and that theory does not have a place in discussing their experience when in school placement (Brekke & Leikvoll Eide, 2021). The results presented in this paper are in line with these findings and show that PSMTs need help establishing meaningful connections between knowledge acquired in university and in school placement. Evidently, these results are the expressed experiences of a small group of PSMTs. However, from what has been conveyed in this pilot I consider it reasonable to suggest that there is an imbalance between teaching related to praxis and strictly teaching on mathematical content in their educational trajectory. Moreover, Hammerness (2013, p. 411) found that when asked about learning opportunities grounded in practice, teacher educators, in line with their student teachers, saw the school sites as the places that provided those opportunities suggesting a discrepancy in the structure of the teacher education and how student teachers prefer to be educated. As seen from the results of this pilot study, the PSMTs accentuate working practically with resources for teaching mathematics in university coursework as meaningful and relevant to them, hence, supporting their vocation for teaching. Thus, resources for teaching mathematics can provide learning opportunities grounded in practice while at university, showing that coursework can also contribute to alleviating the gap between theory and praxis. How such work can facilitate professional growth (as described by Gueudet and Trouche (2012)) for PSMTs in mathematics teacher education requires further study, and is the scope of my ongoing PhD study.

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Comparing and contrasting solutions to foster diagnostic competence

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A main focus in the field of research on competences of teachers is on diagnostic competence. Although it is stated that prospective teachers' diagnostic competences are trainable, they are still described as too low. Therefore, the present study aims to foster diagnostic competence of prospective primary teachers. In the study diagnostic competence is conceptualised with so-called epistemic activities. The treatment bases on empirically known factors about fostering diagnostic competence and follows the teaching and learning method of comparing and contrasting. The success of the method is proven for various fields but has not yet been researched in the field of professional development of prospective teachers. The article introduces the theoretical framework, the method and results of the data analysis.

Keywords: Diagnostic competence, developing diagnostic competence, measuring diagnostic competence, comparing and contrasting, inquiry-based learning.

Introduction

Within the field of teachers' competences an important focus lies on diagnostic competence that we understand as a teacher's ability to identify weaknesses and strengths in students' mathematical work. It is considered as an essential part of teachers' professional knowledge which impacts the quality of teaching and students' learning (Baumert & Kunter, 2006). Research showed that teachers' competences are trainable (Herppich et al., 2018) but it is also stated that teachers' diagnostic competences are too low (van Ophuysen & Behrmann, 2015). Therefore, the systematic development of teachers' diagnostic competence has become a relevant issue of educational research (Chernikova et al., 2020; Larrain, 2019; Hoth, 2017).

For this reason, the aim of this PhD-project is to design interventions that improve prospective primary teachers' diagnostic competence and to analyse the effect of these interventions. The PhD-project is called "KoVe-Dif" (Comparing and contrasting solutions to inquiry-based leaning tasks as a basis for improving diagnostic competences of prospective teachers). The design of our interventions is particularly emphasising different processes of comparing and contrasting products of students' mathematical work (Alfieri et al., 2013). Comparing and contrasting is also an integral part of the diagnostic process of teachers (Philipp, 2018). The teachers' ability of identifying students' competences based on students' written mathematical work is the main focus referring to teachers' diagnostic competence. The project addresses the following main question:

(RQ): What is the effect of different interventions based on the teaching and learning method of contrasting and comparing on diagnostic competence?

Diagnostic competence

Competences are defined as "[…] context-specific cognitive dispositions that are acquired and needed to successfully cope with certain situations or tasks in specific domains" (Koeppen et al., 2008).

Context specificity can be interpreted in different ways. On the one hand, it can refer to different school classes and, on the other hand, to different subject areas. In the project described below we are focussing on a specific set of tasks from the arithmetic which will be described in the description of the treatment. We assume that the diagnostic skills developed in the context of the specific topic are transferable to different school classes.

In the past, diagnostic competence was conceptualised in three different ways (see also Philipp, 2018). First, there is research about cognitive dispositions such as knowledge or motivation. Focusing on the cognitive disposition such as knowledge, diagnostic competence is part of professional knowledge of teachers. For example, Brunner et al. (2011) conceptualised diagnostic competence as a part of pedagogical content knowledge and of pedagogical psychological knowledge. Second, diagnostic competence is equated with the so-called judgement accuracy (Südkamp et al., 2012). This research line focused on the performance of teachers regarding the comparison between a teacher's estimation of students' performance and the actual test scores of their students. This approach got criticized for being distant to a teacher's daily work (Praetorius et al., 2012). As an answer to this critique, the diagnostic competence started to get framed as a process. According to this conceptualization, research focuses on the diagnostic process itself (e.g. Klug, 2013) and on the cognitive process during the teacher's diagnosis (e.g. Philipp, 2018).

In the framework of Loibl and Leuders (2020) diagnostic competence is understood as a process of diagnostic thinking influenced by situation characteristics and person characteristics. While a situation is defined through framing and cues, the person characteristics include cognitive dispositions. The external diagnostic behaviour is either observable by process indicators (such as "think aloud") or by product indicators (such as a "decision"). Diagnostic thinking is the core of the framework, and Leuders et al. (2018) state that it is possible to conceptualise this thinking process similar to the "clinical reasoning". In our study we use this approach and follow Chernikova et al. (2020) as well as Fischer et al. (2014) who conceptualise processes of diagnostic thinking based on clinical reasoning by epistemic activities. These activities are characterised by "(a) identifying a problem, (b) questioning, (c) generating hypotheses, (d) constructing artefacts, (e) generalizing evidence, (f) evaluating evidence, (g) drawing conclusions, and (h) communicating process and results" (Chernikova et al. 2020, p. 161; see also Fischer et al. 2014). These epistemic activities are an adequate basis for modelling the process of a teacher's diagnosis of students' written solutions. For example, a teacher may identify a problem in a students' solution by perceiving an error. The teacher he/she asks for the reason of the error and develops a hypothesis. If possible, the teacher sustains his/her hypothesis by other errors and generalise the hypothesis by evaluating evidence. In this PhD-project, we analyse the diagnostic processes conceptualised with the epistemic activities of prospective teachers while looking at school students' solutions to tasks. In our research, we refer to students' solutions of inquiry-based tasks that are comprehensive and offer many possibilities for teachers to use their diagnostic competence to draw conclusions about the students' solutions.

Fostering diagnostic competence

Recently, Chernikova et al. (2020) published a meta-study investigating different approaches to foster diagnostic competence. The meta-study contains studies that investigate teacher or medical education. In these studies, a special context, a specific problem or the method of scaffolding was used to foster diagnostic competence. While all aspects showed positive effects, orientation towards specific problems (problem orientation) stands out most positive. So-called inquiry-based tasks are specific problems and, particularly, diagnosing cases of extensive students' solutions of these inquiry-based task comprise the orientation towards specific problems of diagnosing. Therefore, the present study is based on recommendations on how to design a course aiming to foster diagnostic competences concerning problem orientation.

The epistemic activities that are the basis of diagnostic thinking rely more or less on contrasting and comparing. Chernikova et al. (2020) state: „More generally, diagnosing first focuses on comparing the current state of learners' knowledge and skills to predefined learning objectives“. Referring to Nickerson (1999), Philipp (2018) describes the ongoing comparison of a solution with relevant prior knowledge or further information as a central part of the diagnostic process. Beyond the specific subject of developing diagnostic competence, Alfieri et al. (2013) pointed out contrasting and comparing to be an effective teaching and learning strategy. However, it has not been investigated regarding the professional development of prospective teachers yet. With our research we target to fill the gap by addressing contrasting and comparing to foster teachers' diagnostic competence.

Method

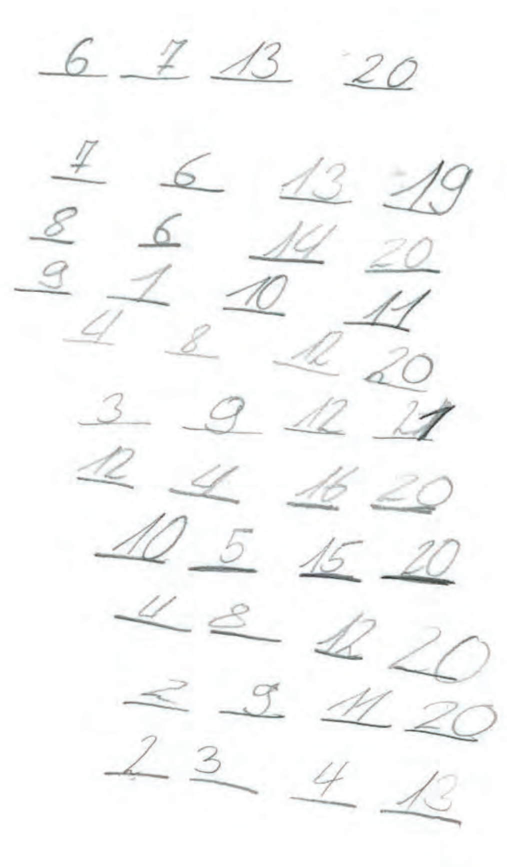
The design of our study is shown in Table 1. Both, the three treatments and the control group were conducted in winter 2020 and in summer 2021. All three treatments (see Table 1) follow well-known characteristics to effectively support the development of teachers' diagnostic competence (Chernikova et al., 2020).

Table 1: Treatment-Design

Treatment 1 (n = 37)	Treatment 2 (n = 40)	Treatment 3 (n = 35)	Control group (n = 25)
Pre-test	Pre-test	Pre-test	Pre-test
Students solve inquiry-based tasks	Analysing primary school students' solutions to inquiry-based tasks	Students solve inquiry-based and analyse primary students' solutions to these tasks	Neither analysis of students' solutions nor analysing of arithmetic tasks
Post-test	Post-test	Post-test	Post-test

The main element of the three treatments were inquiry-based tasks. In each treatment group we provided the same inquiry-based tasks; one example is shown in Table 2. Those tasks are challenging for prospective teachers and primary school students. As our example in Table 2 shows also primary school students are able to provide substantial solutions. In addition, all solutions to be analysed are genuine, to make them as authentic as possible.

Table 2: Example

Task-Example	Solution example of a student
How can you get to the number 20? Choose two numbers for the first and second position in the row. Add these two numbers up to get the third number. Finally, add the second and third number to get the number at the fourth position. Your goal is to reach the number 20 with this step (position 4). For example, you choose 3 and 4 the result will be: 3, 4, 7, 11. This means you cannot get to 20 if you start with the numbers 3 and 4. Task is to find all possibilities to get to 20. (Birnstengel-Höft and Feldhaus 2006)	

In the first treatment group, prospective teachers got a brief introduction into inquiry-based tasks. Subsequently, they were encouraged to solve these tasks and to compare their solutions in pairs (first and second week). In the third week, the prospective teachers got the prompt to compare their approaches and solutions with the whole group. The prospective teachers repeated this proceeding for three inquiry-based tasks. Referring to Chernikova et al. (2020) the first treatment group followed the perspective of learners that solve mathematical tasks and got related prompts.

The second treatment group did not get the prompt to solve the inquiry-based tasks. Instead, there was a brief introduction in the process of diagnosing students' solution based on epistemic activities. Subsequently, the prospective teachers got the prompt to individually analyse the solution of school students and to afterwards compare their analysis of students' solutions in pairs (first and second week). In the third week, the analyses of all pairs were compared in the whole group. In this treatment group the same inquiry-based tasks were used as in treatment group 1. Similar to group 1 the procedure was repeated three times. Referring to Chernikova et al. (2020) the second treatment group followed the perspective of teachers that analyse primary school students' solutions to mathematical tasks and got related prompts (problem orientation).

The third group firstly got a brief introduction to inquiry-based tasks as well as in diagnosing students' solutions according to epistemic activities. Further, this group solved one inquiry-based task and compared their own solutions in the same way as treatment group 1 (weeks 2 and 3). Afterwards the group analysed the students' solutions and compared their analyses in the same way as treatment group 2 (weeks 4 and 5). Finally, the prospective teachers in the third treatment group solved a second inquiry-based task, analysed related students' solutions and compared their own solutions with the solutions of primary students (week 6 and 7).

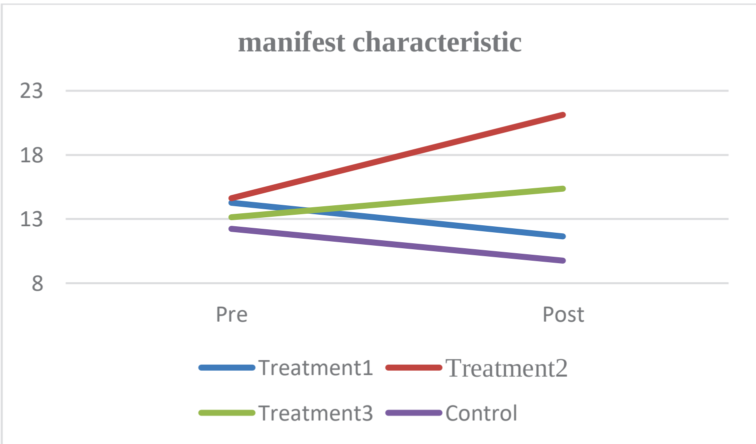
We conducted the pre-test before the semester started and the post-test after eight weeks. We developed these tests to measure diagnostic competence. Each test comprises three items with solutions of primary school students. The three items refer to three different inquiry-based tasks (in Table 2 an example of the pre-test is displayed). To prevent learning effects, we changed two items by slightly modifying the external form of primary school students' solutions, and we exchanged one item completely. Primary school students' solutions of the inquiry-based tasks provided in the pre- and post-test are analysed by the prospective teachers who participated in the treatment and control groups. The prospective teachers' analyses are the basis to investigate their diagnostic competence. Therefore, these analyses are categorized with content analysis regarding two dimensions. In the first dimension we distinguish between statements about manifest characteristics and hypotheses. Manifest characteristics are for example a student's wrong solution, a specific way of a student to write something down, or a specific way of a student to develop different examples regarding a problem. Statements that we classify as hypotheses are for example, prospective teachers' interpretations of (school) students' abilities shown in their solutions that were sometimes backed by manifest characteristics. With this dimension we want to measure the diagnostic thinking conceptualised by the epistemic activities described earlier. In the second dimension we assigned single statements to spheres of competence inspired by Rathgeb-Schnierer and Schütte (2011). The coded data was further analysed by descriptive and inferential statistics.

Results

In the following, the data is analysed regarding the two epistemic activities "identify manifest characteristics" and "generate hypotheses" while comparing the three treatment groups and the control group.

Table 3: identify manifest characteristics

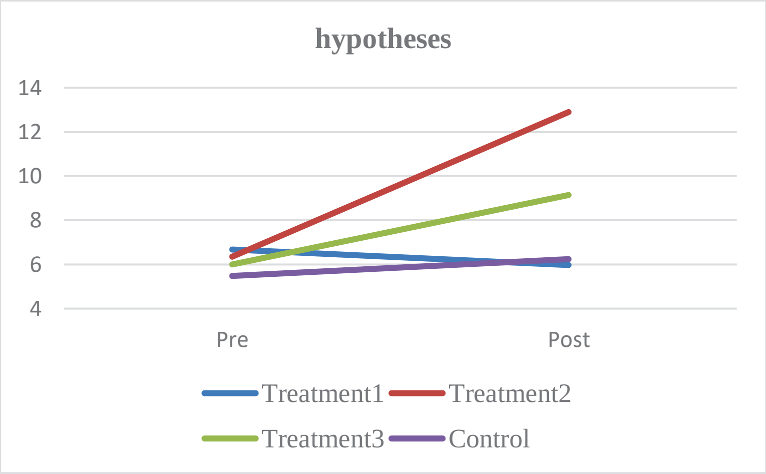
Treatment	N	Pre-M	Post-M
1	37	14,27	11,65
2	40	14,62	21,12
3	35	13,14	15,37
C	25	12,24	9,76



Regarding the identification of manifest characteristics (Table 3) in primary school students' solutions our results exhibit that all groups start at nearly the same level but differ at the post-test. Mixed-anova shows a significant interaction effect between time and group ($F(3,133) = 11,253$; $p < 0,00001$, $\eta^2 = 0,202$). Pairwise t-tests reveal that the difference between the groups is not significant in the pre-test ($p = 1$ with Bonferroni correction), but in the post-test significant differences were revealed. For example, treatment group 2 and treatment group 1 differ significantly (pairwise t-test $p < 0,0000001$ with Bonferroni correction). Only the pairwise t-tests between group 1 and the control group ($p = 1$ with Bonferroni correction) and between group 1 and group 3 ($p = 0,0548$ with Bonferroni correction) reveal no significant differences.

Table 4: generate hypotheses

Treatment	N	Pre-M	Post-M
1	37	6,68	5,97
2	40	6,35	12,9
3	35	6	9,14
C	25	5,48	6,24



Similar results are shown concerning the generation of hypotheses (Table 4). Mixed-anova shows a significant interaction effect between time and group ($F(3,133) = 12,209$; $p < 0,000001$, $\eta^2 = 0,216$). Again, all groups start on a close level in the pre-test (pairwise t-test $p = 1$ with Bonferroni correction) but differ in the post-test. For example, the difference between group 2 and group 1 is significant ($p < 0,0000001$ with Bonferroni correction). Again, the difference between treatment group 1 and the control group is not significant ($p = 1$ with Bonferroni correction).

In the presented study, diagnostic competence is conceptualised and measured with epistemic activities (Fischer et al., 2014). Presented results indicate that the analysis and the contrasting and comparing of primary school student solutions (treatment 2) have a significant impact on the development of diagnostic competence. In contrast, creating and then contrasting and comparing the own solutions with peers (treatment 1) does not seem to have influence on the diagnostic competence. Finally, creating and then contrasting and comparing the own solution with peers followed by analysing primary school students' solutions (treatment 3) also leads to an increase regarding the epistemic activities. Although this increase is not as high as focussing only on the analysis of primary school students' solutions (treatment 1). The results presented allow the interpretation that this applies independently of the specific epistemic activity. Accordingly, contrasting and comparing student solutions seems to be another aspect of effectively promoting the diagnostic competence of prospective teachers (Chernikova et al., 2020). We are left with the open question: Why is solving tasks and contrast and compare the solutions to peers not increasing or even decreasing the amount of epistemic activities? At this stage, only hypotheses are possible. It could be due to motivational reasons. Or the fact that it is about peers with whom is compared, hinders the development. In the future, we want to use different research approaches to gain deeper insight into how our treatments change diagnostic competence.

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Pre-service teachers' use of information when diagnosing first graders' number sense in text-image vignettes

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Keywords: Number sense, use of information, diagnostic competence, restricted focus viewer.

Introduction

A central goal of mathematics education in primary school is to develop flexible calculating and solid arithmetic competencies. These competencies require a comprehensive understanding of numbers and operations (number sense), which should be established during the early school years (Rathgeb-Schnierer & Rechtsteiner, 2018). In this context, teachers face every day the challenging task of comprehending their students' ways of thinking in order to provide them with appropriate learning opportunities. An essential condition for succeeding in this task is the teachers' diagnostic competence: When mathematics teachers want to find out what number sense a child has, they use information about the student's behaviour when solving arithmetic tasks to draw conclusions about his or her thinking and knowledge in this area. The quality of corresponding diagnostic judgements appears to depend on whether information valid for the diagnosis can be identified and used for this purpose. There is evidence that novices have limited ability to select valid information when diagnosing complex situations and also use invalid information, often resulting in lower quality of their judgements (Kellman & Massey, 2013). However, it is still unclear how and what information is used by pre-service teachers when diagnosing first graders' number sense, and whether the quality of their diagnostic judgments can be attributed to their use of information.

Theoretical background

In order to investigate how teachers form diagnostic judgments, a line of research has developed that focuses on the cognitive processes of information use that underlie diagnostic teacher judgments. Studies of this type examine cognitive processes, which are situational skills that mediate between a teacher's dispositions and actions (Loibl et al., 2020): Teachers, in order to accomplish a professional, complex task such as diagnosing learning processes, must (1) perceive information, (2) interpret it and (3) make decisions. Additionally, the influence of information use on judgment quality is illustrated in Brunswik's lens model (Brunswik, 1956): The better a person is able to select and use information with high validity, the higher the judgment quality will be. Although findings suggest that pre-service teachers are less likely to focus on children's learning and thinking processes when assessing their mathematical concepts, there is still little evidence on how pre-service teachers use information of different validity when diagnosing first graders' number sense and whether judgment quality depends on the information use. Accordingly, our research questions are: (1) How do pre-service teachers use information of different validity when diagnosing first graders' number sense?

(2) Does the quality of the pre-service teachers' diagnostic judgment increase when valid information is made explicit?

Methodology

The presented study uses an experimental design to investigate pre-service teachers' information use when diagnosing first graders' number sense. For this purpose, nine authentic text-image vignettes were designed based on literature, each showing a typical classroom situation with a first grader's learning of number sense. Each of the vignettes comprise of a task, a student's solution, notes on the observed solution process and a short teacher-student dialogue. After the validation with experts, four vignettes were provided in three varying information environments which contain: a) additional valid and invalid information about student behaviour, b) additional valid and invalid information about student behaviour, where validity is made explicit and c) only additional valid information about student behaviour. A valid information in the context of number sense is for example the use of manipulatives, an invalid information is for example the social behaviour of the student while solving an arithmetic task. The sample consists of N=173 pre-service teachers at the end of a one-semester course covering key topics on the development of number sense. The participants were randomly allocated to the three experimental conditions. Using the Restricted Focus Viewer (RFV) (Jansen et al. 2003), frequency, order, and duration in which the pre-service teachers accessed the different units of information were collected during the experiment. The analysis and comparison of the data between the experimental groups will provide information about the use of the presented information. Additionally, the participants will be asked to write down their diagnostic judgement in order to choose a suitable follow-up task. The analysis and comparison of the answers with the expert norm will show the quality of the diagnostic judgment. The poster presents the theoretical framework and methodology of the study, highlighting in particular the design of the text-image-vignettes and the use of the RFV for investigating the pre-service teachers' use of information.

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TWG19: Mathematics Teaching and Teacher Practice(s)

Introduction to the work of TWG19: Mathematics Teaching and Teacher Practice(s)

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This introduction for TWG19 offers a brief history of the group and describes past challenges the group has experienced when discussing papers — seeing papers as related and as contributing to a common effort. These challenges led us as TWG19 team leaders to develop three initiatives to support communication among researchers who work in different contexts with different purposes. The initiatives are presented and used to discuss the papers. We conclude with implications for the future.

Introduction

Thematic Working Group 19 (TWG19) is concerned with the study of mathematics teaching and teacher practice(s). From its inception, it has struggled to clarify its focus. TWG19 was created after CERME8 from a division of a TWG titled *From a Study of Teaching Practices to Issues in Teacher Education*. Having grown too large, this group split into three: one targeting teacher education and professional development (TWG18); another targeting teacher knowledge, beliefs, and identity (TWG20); and a third with an uncertain target (TWG19). Participants at CERMEs 9, 10, and 11 spoke of challenges in thinking and talking across papers. Studies had different purposes, were conducted in different contexts, and used different conceptualizations, theories, and methods. How should the group understand the "theme" and how should it make sense of submitted papers? The title was changed from *Mathematics Teacher and Classroom Practices* to *Mathematics Teaching and Teacher Practice(s)* after CERME10 to clarify interest in studying *teaching* rather than *teachers* and *teacher practices* rather than broader *classroom practices*. Still, the distinction between *teaching* and *teacher practice(s)* remains unclear and communicating across papers continues to be challenging.

Perhaps challenges should not be surprising. Despite its long history, research on mathematics teaching remains relatively underdeveloped. Well-developed learning theories abound, but teaching theories are scant. Important frameworks capture key aspects of teaching but fall short of comprehensive, elaborated theories. Two compelling reasons come to mind. First, teaching is extraordinarily complex. It is an involved human activity that navigates other complex human activities, such as learning, communication, child development, and social life in and beyond schools. Second, teaching has a familiarity that breeds a presumption that teaching is little more than common sense. As Lortie (1975) argues, the "apprenticeship of observation," carried out by everyone from the perspective of being a student for countless hours, renders teaching either invisible or distorted in ways that have to be unlearned. Nor are researchers immune. It is little wonder then that participants struggle to think and talk across the fragmented glimpses of teaching that papers provide.

To meet these challenges, as organizers of TWG19 who are both equal participants and responsible for leading, we have invited participants to experiment with three initiatives:

1. Think, write, and talk more explicitly about the meaning of "teaching"
2. Use five proposed analytic domains to stimulate communication and reflection
3. Contribute to and use shared datasets to provide common referents.

In addition, we have set goals in the spirit of ERME's commitment to communication, cooperation, and collaboration. First, we aim to give all participants opportunities to develop as scholars. We invest in thoughtful reviews and design sessions for participants to present their work, have colleagues react, and respond to others in turn. Second, we aim to support communication and collaboration among participants by having them explore the meaning of teaching across studies, use the domains to organise and reflect on studies, and study shared data. We offer these as ways to build bridges among diverse studies. Third, we aim to characterise TWG19's collective effort to study teaching and teacher practice(s), including reflecting on what we are learning and priorities for future work. This is no small feat, and our efforts are nascent at best, but this is an important goal.

Exploring the meaning of teaching

Papers studying teaching often take the concept for granted. The call for proposals for CERME12 asked that papers explicitly say what is meant by "teaching" in the paper. Nevertheless, only five papers did so (all involving an organiser as an author). Mosvold and Wæge offer a practice-based conception grounded in instructional interactions as collaboratively constructed between teacher and students around particular content, and situated in broader environments, where teaching is the work involved in managing these interactions. Grundén offers a linguistics-based conception grounded in sociolinguistics of power relations and power struggles, where teaching is a social, cultural, and political practice of situated and habitual actions and interactions. And Nowinska offers a discipline-based conception grounded in socio-mathematical norms and a developing theory of metacognitive-discursive activities that shape them, where teaching is expert participation in and scaffolding of the doing of mathematics. More implicit definitions differ as well. For example, Holmedal draws on discourse analysis to explore the idea of teaching as identity performance, and Arnesen and Dahl draw on commognition to explore the idea of teaching as expert participation in classroom discourse.

At CERME12, participants considered the meaning of teaching implicated each paper in group discussions and posted their in-the-moment thoughts on a *Padlet*[1] page for the paper. In additional sessions, participants discussed the meaning of teaching across papers and posted thoughts on another page. Discussions appeared meaningful and lively. Posts reflected thinking more deeply and together about what is meant by teaching. One participant reflected on teaching as a complex set of tasks and professional judgements made in real-time, and another wrote about the importance of finding a balance among six dimensions of a teaching-learning process: epistemic, cognitive, mediational, ecological, emotional, and interactional. Others highlighted teaching as creating opportunities to learn or as managing dilemmas. Some raised questions about how values and theories shape views of

[1] *Padlet* is an online digital canvas where users can post text, videos, and images.

teaching. One post asked for strengths and limitations of thinking about teaching as “what teachers do” versus “work to be done,” an idea discussed at CERME11 (Sakonidis et al., 2019). Another suggested that the two might be used alternately by looking for what teachers do in the classroom, analysing and discussing these observations, and then considering the specialised work to be done.

Some participants raised questions about the meaning of teaching in relation to teacher education and teacher development: What do we teach teachers when we teach teaching? Another asked where teachers learn what we as researchers expect them to do. These questions emphasised that how we as researchers and teacher educators think of teaching has consequences for what is possible for prospective and practicing teachers to learn. Participants also pondered whether a choice of how to conceptualise teaching might differ for mathematics teaching versus teaching other subjects; whether norms, values, beliefs, and emotions (as related to mathematics) influence what might be meant by teaching; and how intuition and habits might play a role.

Several insights emerged from attending to the meaning of teaching. First, meaning is not as obvious as often presumed. Furthermore, what is meant by teaching matters for understanding all aspects of a study (questions, methods, and claims). In addition, attending to the meaning of teaching made it easier to think across papers. No one argued for a single, shared meaning, but many saw a need for being clearer about what is meant and how this would help us make sense of research.

Domains of research on teaching

The second initiative is to experiment with using five domains as an analytic frame to locate and discuss studies being conducted. The domains are:

(i) Consideration of *mathematics* and the central endeavour of extending the subject to students.
(ii) Organizing and *enacting* the design, interactions, and discourse of teaching and learning.
(iii) Becoming acquainted with, relating to, and responding to *students* as people and learners.
(iv) Attending to broader social, cultural, and political *issues* that matter for teaching and learning, including imperatives of social justice.
(v) Addressing all domains of teaching in a *comprehensive* way, with that as an explicit aim.

The domains are based on the didactical/instructional triangle evident in studies of teaching across intellectual traditions (cf., Brousseau, 1970-1990/2002; Cohen et al., 2003; Jaworski, 1994; Steinbring, 2011; see Goodchild & Sriraman, 2012, for a discussion of this breadth). More recently, attention has been given to how the interactions of teaching and learning are situated in broad socio-political, historical, and cultural environments (cf., Ball, 2018; Jaworski & Potari, 2009). Any study of teaching necessarily involves all domains, but a study may foreground one while maintaining regard for others. Hoover et al. (2022) posit capacity for maintaining mutual regard for all aspects of teaching while focusing on a single aspect as key to professional growth of mathematics teacher educators but propose that this capacity is likely crucial to both doing and understanding teaching. The goal is to support thinking, not to prescribe studies. One can imagine a study investigating the interplay between two domains, with others in the background. Yet, we identify only these five, where a dual-regard paper could be addressed in either. The fifth domain, *comprehensive*, is meant for studies that deliberately attempt to attend to all aspects of teaching together.

The domains were first presented at the virtual meeting of TWG19 in February 2021 and, with some modifications, were used to organise papers and to stimulate discussion at CERME12. The intention is to use the domains to support elaboration and understanding of the contribution of each paper in relation to others. It is also hoped that the domains offer an analytic lens that might characterise our collective research and help make sense of research on teaching and teacher practice. All papers grouped together belonged, at least loosely, to the same domain. After individual papers were discussed, the papers and then the domain itself were discussed. Participants reflected on the types of research questions being asked and what other important questions might be asked for the domain.

At CERME12, papers were presented in the domains of *Mathematics*, *Enactment,* and *Issues,* with no papers submitted in the domains of *Students* or *Comprehensive*. The absence of papers in the *Students* domain was noted by participants but not much discussed.

Mathematics

Studies in this domain foreground mathematics and how it can be extended to students, but the meaning of mathematics and what it means to foreground it vary across studies. Hummes et al. and Lovemore et al. consider teacher engagement in collaborative processes, namely lesson study and a community of practice respectively, with a focus on irrational numbers in the Pythagorean theorem (Hummes et al.) and using music to support the teaching of fractions (Lovemore et al.). Within the context of teaching geometrical patterns, Gray and Kleve examine the demand on mathematical knowledge for teaching when supporting students' agency, identity and access to mathematics. Papadaki and Biza present a method of analysis suitable for investigating teachers' use of opportunities to go beyond the "mathematics of the moment" in the context of teaching geometry. Finally, Adler and Mosvold use shared data to illustrate how the Mathematical Discourse in Instruction framework identifies four tasks of teaching central to making mathematics available to students, regardless of the nature of the pedagogy employed (e.g., traditional or reform).

Studies in this domain are different, yet participants appeared to find it provocative and rewarding to reflect on these papers as located in a common territory that emphasizes mathematical considerations. All are concerned with teaching, but specifically with questions of creating mathematical opportunities for students and recognizing and taking them up as they arise. They focus on discourse, as well as identity, agency, and autonomy, as distinctly mathematical, central to the doing and learning of mathematics. Comments on the *Padlet* for this domain wondered how mathematics teaching is distinct from the teaching of other subjects and how conceptions of mathematics might shape conceptions of mathematics teaching. Considering this domain led to reflections on both the essential role it needs to play in studying teaching and how incomplete it is as a lens.

Enactment

Twelve papers and one poster foreground the enactment of mathematics teaching, with diversity in the focus and intent of individual studies, the underpinning theory, and the nature of the methods employed. Papers involve empirical investigations of teacher actions. In some cases, specific hypotheses are investigated, for example, whether teachers' use of why-questions supports student reasoning (Arnesen & Dahl). In other papers, efforts are made to investigate or describe aspects of teaching. Drawing on data from different classrooms, van Bommel et al. analyse teachers' communication of learning goals and Sigurjónsson investigates features of teaching in classrooms where cognitive activation was high. In a more fine-grained analysis, Gray et al. use data from a single lesson to report on the role of progressing and focusing actions in students' appropriation of mathematical ideas, while Gobede and Mosvold use data from a single classroom to identify dilemmas of teaching arithmetical notation to young learners. Some papers report descriptions and investigations of teacher or student actions after, or in the context of, interventions to support particular forms of teaching and learning. Kovács-Kószó et al. investigate a teacher's responses to significant moments of student thinking after participation in a professional development programme; Rösseland et al. report on students' participation in mathematics tasks where roles and positions were used to foster explorative talk.

Several papers explicitly aim to build or test theory or methodological approaches. Nowińska illustrates a set of negative discursive activities that may lead to negative socio-mathematical norms, while Svensson and Wester propose a method for identifying socio-mathematical norms through analysis of student and teacher responses in mathematical activity. Drawing on discourse and positioning theory, Drageset and Eidissen develop and test a framework using shared data gathered for TWG19. Their framework describes teacher positions based on an analysis of their interactions. Concepts from theatre practitioners, in addition to organisational and educational theory, inform the framework proposed by McIvor for identifying and analysing the practice of improvisation in the secondary mathematics classroom. Most papers in this domain analyse recordings of mathematics lessons. In contrast, Mosvold and Wæge examine collective planning activities within a professional development initiative to investigate and propose entailments of questioning practices.

Comments about this domain on the *Padlet* reflected on how much of the work in this domain focuses on what is easily observable, both with a sense that this domain is central to teaching yet also with questions about what we might be missing, the less observable. One participant commented that the students seemed to be missing in this domain, and others commented that studies in this domain do not attend to context, or to teaching as influenced by larger structural systems. Another wondered whether a focus on frameworks in different studies might hide similarities across studies.

Issues

Boundaries, influences, and contradictions are highlighted in the set of four papers in this domain. In the context of a collaboration between mathematics teaching and visual art teaching communities, Choutou and Potari use grounded theory to analyse data from group meetings to identify the boundaries that emerged, as well as the ways in which these were handled. In a paper where mathematics teaching is seen as a social, cultural, and political practice, Grundén investigates textbooks as actors in the transformation of the intended curriculum by conducting thematic analysis on focus group discussions among primary school teachers. Holmedal draws on the notion of big D Discourse and figured worlds to unpack the role of teacher identity in navigating contradictions. Finally, Mwale and Jakobsen use the Mathematical Discourse in Instruction framework to investigate teachers' practices when teaching mass.

Padlet posts that reflected on this domain surfaced concerns and new insights. The "name" of this domain, especially shortening it to "issues," is seen as inadequate. It lacks clarity, but participants have different ways of thinking about this domain and widely agreeable terms are hard to find. Participants discussed and commented on the advantages of “ecology” and “environment.” Part of the concern is about language; part is about meaning. Some participants wondered how broader social, cultural, and political issues matter for doing research as well as teaching. Several commented on the need to acknowledge differences among countries, both for teaching and for research. Another theme in the *Padlet* posts was about how much the teacher and teaching matter for how these issues influence what happens in classrooms. This insight prompted comments about needing to consider methodological tools for addressing such complexity. Prominent across comments was a newfound appreciation for needing to keep this domain in mind. One post described a view of teaching as being about classroom interactions and how the papers in this domain seemed initially unrelated to teaching. The participant went on to write:

> Thinking about them in light of this domain, however, at least helped me see how and why these can be considered as studies of mathematics teaching. Whereas many other studies focus on dialogue, moves or interactions, these papers identify and analyze different kinds of structures that might influence teaching and learning. This is a useful way of looking at teaching.

Views of this domain varied, but having it as a domain seemed to help the group see how these studies were contributing to research on teaching and to see the work as having challenging yet important implications for all studies in TWG19.

Sharing data to study teaching

The third initiative is to contribute to and use data that others in the group have shared — for analysis in papers, as illustrations in presentations, or more generally as a common reference point in discussions. For CERME12, five contributions were available. Most included video, along with transcripts, a document that provides information about the context, and a document containing information from the owners of the data about restrictions. Seven papers made use of the shared data for CERME11, and a session was devoted to discussion of these papers. The initiative appeared to create enthusiasm and productive discussions. For CERME12, only two papers used the shared data.

Gathering and preparing data for sharing and reuse can be rewarding but comes with challenges. One is navigating the General Data Protection Regulation (GDPR), which provides directives for protecting personal data. These regulations can discourage, but there are important reasons to work to make data available. Besides reproducibility and replicability of research, as Dewey (1904) points out, to teach is to be a student of teaching. Teaching is a public act, and its study is a foundational professional activity. In part, we need to educate ourselves and others in the role of responsibly generating and sharing records of practice. Responsible data sharing requires careful planning, crafting proper consent, and establishing secure infrastructure for storage and access. Among the benefits of sharing and reusing data, scholars have highlighted the promise of better utilization of data, access to rich and unique data, saved costs for data production, reduced burden for participants, validation and extension of previous work, and increased quality in analysis. In TWG19, we have experienced potential for new and productive research collaboration and having a common ground for improving communication as significant benefits. Going forward, we hope to explore new ways of organising this work.

Implications for the future of TWG19

The focus on the meaning of teaching, the use of domains to organise and examine studies, and the establishment of shared data have provided greater cohesion for TWG19. They are also helping the group realise CERME's commitment to communication, cooperation, and collaboration. Attending to what is meant by "teaching" is pushing participants' thinking regarding individual studies and is occasioning new conversations across studies. Seeing other's thinking about teaching and what it affords for their research adds perspective to our own thinking. Likewise, the use of domains is helping us think together about each study as well as across studies. So far, the domains appear to be functioning without privileging some work over other work. Instead, they are helping us see diverse studies as part of a larger, concerted effort to investigate teaching. Last, while establishing and using shared data is challenging, it remains a promising tool for supporting the work of TWG19.

In addition to these three innovations, the organisers have also been exploring ways of working together. The pandemic forced us into virtual meetings. Designing online sessions was not easy, but several new approaches may be worth keeping. Participants recorded 5-minute presentations beforehand, which seemed to encourage preparation, lead to better and efficient presentations, and helped everyone get familiar with the work. Viewing presentations does not replace reading all papers, but it helps everyone recall what they read and provides a quick way to revisit and remember.

A second novelty was designing sessions for authors to hear reactions from a small group of colleagues. These author sessions were highly structured to engage authors in listening and participants in honing professional skills. After viewing the 5-minute recorded presentation, one volunteer had 3 minutes to describe what they heard, then two other volunteers had 2 minutes each to say what stood out to them as significant about the work. This was followed by time for brief reaction from the author and then discussion of the meaning of teaching evident in the work and its place in the identified domains. Perhaps the skills identified in these sessions should be more scaffolded and the sessions facilitated. Perhaps participants should know ahead of time which papers they will be reacting to, so they might prepare. Notwithstanding potential improvements, participants appeared to appreciate deliberate attention to building professional skills, and authors appeared to appreciate the opportunity to listen to how their work was being understood and taken up.

In this design, individual papers were not presented to the whole group. However, the author sessions were seen as valuable to both authors and participants, and authors received several other types of feedback. Having an organiser comment on a set of papers after everyone had a chance to engage with one paper in a small group seemed to help participants go deeper with the one paper and extend thinking across the papers. In addition, having individual papers discussed in parallel allowed for more whole-group discussions across the papers, where authors again had a chance to hear their paper discussed, albeit in relation to other papers, a specific domain, or meanings of teaching. Last, we created a *Padlet* page for each paper. This allowed authors to ask for pointed feedback and provided all participants with an expedient way of offering comments and raising questions. We now have better ideas about how to support participants' productive use of these online tools and plan to continue exploring their use when we meet in person.

In planning for the conference, we strove to provide opportunities for in-depth discussions of teaching and teachers' practice(s). Our sense is that the three initiatives together with thoughtfully designed sessions contributed to such in-depth discussions. However, we think there is more to do. As we plan for CERME13, we will continue to consider how to design sessions that support each participant while also supporting our collective efforts to advance research on teaching.

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Mathematics Discourse in Instruction: How it helps us think about research on mathematics teaching

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Mathematics Discourse in Instruction (MDI) is a framework developed to describe, interpret, and support development of mathematics teaching. Since its inception, it has been successfully used as a tool for research and professional development in South Africa, where it was developed, but it has been less used in other contexts. In this paper, we use shared data as a starting point to explore how the MDI framework can contribute to thinking about research on mathematics teaching. We elaborate on the conception of teaching underlying the framework, describe its elements and their rationale, and show how these can illuminate four core tasks of teaching mathematics.

Keywords: Exemplification, explanatory communication, learner participation, discourse, mathematics.

Introduction

In the past decades, multiple frameworks have been developed, each offering different perspectives on mathematics teaching. The underlying conceptualization of teaching is often not made explicit in these frameworks, and it can be challenging to communicate across theories and frameworks. To stimulate communication and move the group forward, sets of shared data were provided for participants of thematic working group 19 (TWG19) to use at CERME11. Papers applied different theoretical and analytical lenses on these data. For instance, Nic Mhuirí's (2019) application of the Teaching for Robust Understanding (TRU) framework enabled her to compare the quality of teaching across datasets according to the pedagogical norms of the TRU framework, but it also raised questions about underlying values of frameworks and usability across cultural contexts. Bass and Mosvold (2019) applied the instructional triangle as a conceptual framework, adding theoretical perspectives on agency, authority and identity. This allowed them to explore how various teacher moves may influence student agency and position. These examples illustrate how frameworks can offer different perspectives and allow researchers to notice different aspects of the data. Having shared data available in the group thus provided a productive space for discussion. In the present paper, we continue this effort by applying the Mathematics Discourse in Instruction (MDI) framework on one of the shared datasets to investigate what the framework might help us see, what underlying conception of teaching it has, and how a framework like this can contribute to research on mathematics teaching.

The framework[1]

MDI was developed in the Wits Maths Connect Secondary (WMCS) research and professional development project working with teachers in schools serving low-income communities in one province in South Africa. The goal was to improve teaching and learning. Mathematics teaching in

[1] References to 'we' and 'our' in this section are to the WMCS research team, and not to the authors of this paper.

secondary school classrooms is predominantly direct instruction, and typically described as ‘traditional’. Our initial observation was that within this similarity were both important teaching differences and incoherence in the mathematical messages across lessons (Adler, 2017). In this context, we sought a framework that describes what mathematically is made available (or not) to learn, irrespective of pedagogical norms, and provide for developmental trajectories that we could use with teachers to improve the coherence of teaching and opportunities for learning mathematics. In its initial form MDI was used analytically to describe and compare lessons (see Adler & Ronda, 2015). Between 2015 and 2019 the project focused on using it to develop practice. It is thus a living framework and has functioned as a boundary object, shifting flexibly across teaching and research practices in the project (Adler, 2017). This paper contributes to current work of refining MDI.

MDI is underpinned by key tenets of sociocultural theory. Briefly, these include an orientation to mathematics as coherent and connected scientific knowledge (Vygotsky, 1978); and to mathematics teaching as goal directed with mediation towards learners’ appropriation of increasingly sophisticated and increasingly general ways of thinking to progress in the discipline. Critically, teaching is always about something – an object of learning (Marton, 2015) – and the coherent mediation of that ‘something’ is the teacher’s work. To focus the project’s work, we foregrounded what we considered were high leverage practices in this work, and specifically in preparing for and teaching a lesson.

If teaching is always about ‘something’, a first core task of teaching is to identify the object of learning – that which students are to come to know and be able to do in a lesson. Our analysis below will show that while this task is obvious, its enactment is not trivial. Key next core tasks are selecting and sequencing examples, their related tasks, and representational forms (*exemplification*), attending to explicit mathematical word use and justifications/substantiations (*explanatory communication*) and to what learners are invited to do, say and write (*learner participation*). Our conceptualization of each of these tasks has been informed by two literature strands: on exemplification and variation in mathematics and mathematics education (e.g. Al-Murani et al., 2019), and on language as a resource in mathematics teaching and learning – including attention to lexicalisation (Planas, 2021) – and explicit criteria for mathematical explanations (Prediger, 2019). Our sociocultural perspective sees tasks of teaching as mediational, drawing in cultural tools that shape and are shaped by contexts (see Figure 1). The further salience of the four elements (tasks of teaching) in the framework was their resonance with practice and possibilities for connection and developmental work with teachers.

Task 1: Identifying the object of learning requires both mathematical and curriculum analysis. The ‘object’ in the lesson we study here is placing a fractional number on the number line (for more information about the context and task, see Ball, 2017). This task requires analysing (1) where this ‘content’ is located in the curriculum (and so on the trajectory of student learning), (2) its mathematical meaning and what other mathematical concepts, procedures and practices are connected and entailed. In analysis of shared data, we only had access to the enacted object (what is actually taught) and can only infer what was intended. Below, we show that the intended object of learning is provided in the description of the lesson, enabling mathematical and curriculum analysis.

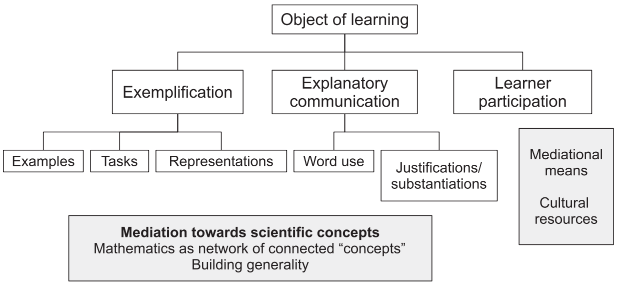

Figure 1. The MDI conceptual framework, adapted and refined from Adler & Ronda (2015).

Task 2: There is considerable literature foregrounding exemplification in mathematics education – as specialised knowledge for teaching, and as teaching and learning through discerning variation (for a review, see Adler & Pournara, 2020). Watson and Mason (e.g., 2006) brought these two strands together in mathematics. Al-Murani et al. (2019) build on this earlier work and use principles of variation to analyse example sets in mathematics lessons. They argue that discerning similarities and differences in an example set – what is changing over a stable background, or what is invariant as features change – is particularly useful in illustrating opportunities created for building generality and/or recognizing structure in mathematics learning. Resonant with this approach, two key principles of variation were recruited into MDI: similarity and contrast. We analyse variation amidst invariance (what is the same and what is different) and contrast (what is and what is not) across example sets to interpret opportunities made available for mathematics learning.

Task 3: Research in mathematics education has attended to important language practices – referred to as language responsive mathematics teaching (e.g., Prediger, 2019). That such teaching needs to extend beyond the communicative function of language to its epistemic function has long been recognised (e.g., Pimm, 1987). The epistemic function includes both vocabulary work, explaining words and phrases, thus requiring teaching to create opportunities for students to participate in (hear, see and use) specialised discourse (e.g., Planas, 2021). This is well illustrated in Planas' (2021) study of teaching equation solving, where she argues for the specific language practice of lexical elaboration. She shows how deliberate attention to the equivalence relation, and the appropriate words to support this meaning create opportunities for learning specialised discourse. The two elements of *explanatory communicatio*n (word use and justifications) in MDI link with this work.

Task 4: Learners obviously need to engage with mathematics to learn. MDI focuses on how students are invited to hear, see, write and speak mathematics, the latter beyond chorusing and one-word answers to teacher questions, and so to use specialised discourse themselves.

Exploration of framework with shared data

The MDI framework was developed for analysis of a lesson. Adler and Ronda (2015) operationalise key constructs of each task, with codes developed to distinguish, for example, similar and contrasting examples, differing task demands, colloquial and mathematical word use, criteria for explanations and learner participation. They describe how lessons are chunked into episodes and the analysis of episodes accumulate to describe the example set and other elements in terms of levels of a trajectory towards more coherent teaching. The exploration of the shared data below cannot do this kind of analysis as we only have one episode. Given the coherence of the overall lesson, the tasks would likely all be coded the highest level 3. As a coherent lesson, it thus is more useful to elaborate our exploration descriptively, and this is how we proceed.

The shared data we use – the Hoover data set – is a short episode (video recording and transcript) within a lesson together with a context document describing the class (rising fifth graders who attend a summer school in the United States), the motive for the lesson, its mathematical focus and student responses to a warm-up problem at the start of the lesson, and a similar exit task at the end. We use all of this for our analysis and discussion of what the MDI framework allows us to see.

The first step in MDI analysis is to identify the "ostensible" object of learning: analysis of the example set and related tasks, and the explanatory communication are all in relation to this 'object'. The warm-up problem asks: "What number does the orange arrow point to? Explain how you figured it out". The arrow points to the position of $\frac{1}{3}$ on a number line that extends from just before 0 to just past $2\frac{1}{3}$, with thirds marked on the line. In the context document accompanying this shared data set, the lesson is described as *marking a turning point in mathematical work* from *"naming fractions as parts of areas to identifying fractions as points on the number line"* (p. 1), with a critical aspect of the object of learning being the important *"shift" in understanding* that "on the number line the whole is defined as the distance from 0 to 1. With area models, the whole can be greater than 1" (p. 2). This is something students are to know. What they need to be able to do – the focus of the lesson – is to carry out "steps for naming a fraction correctly". This key procedure is evident in the chart provided. It is presented as the interpretation for teaching of *an extract from curriculum documents* (p. 7). The intensive mathematical and curriculum analysis highlighted in the italicized phrases that was done in preparation for this lesson are made visible. These foci for the lesson are not trivial and require specialized teacher knowledge of fractions as part of an area, and fraction as a number, where and how the progression from one to the other is located in the curriculum, and consequently what prior learning brings to the shift learners need to make. The next analytic step is to analyze the mediation of these object(s).

Exemplification – examples, tasks and representations

The focal example in the episode is the fraction $\frac{1}{3}$ and the task for the students is high demand. They are required to name the fraction being pointed to on the given number line which extends from 0 and past 2, and to explain their reasoning. There are no other given examples in the episode. The example in the concluding task is the fraction $\frac{2}{3}$, and the task is the same. The fraction $\frac{1}{7}$ is also discussed in the episode, bringing in a second fraction example. We see other fractions ($\frac{1}{4}$, $\frac{1}{5}$) written on the board and possibly offered prior to the $\frac{1}{7}$, and still other responses to the warm-up task such as $\frac{2}{4}$. We do not have access to other examples of fractions and their location on the number line that were likely to have been discussed later in the lesson. Specifically, we do not know whether different 'thirds' were discussed – where the 'whole' in thirds (the denominator) remained invariant, and the number of 'parts' (numerator) varied (e. g. $\frac{2}{3}$, $\frac{3}{3}$, $\frac{5}{3}$, $\frac{7}{3}$) – or whether there was discussion of examples with a different denominator, with a similar range of numerators (e.g. $\frac{1}{5}$, $\frac{2}{5}$ ...). There are thus limits to what can be said about the example set across the lesson and variation with respect to similarity, and so what of the written fractions were varying and invariant and how these might have been publicly discussed. The intended generality is the procedure for naming a fraction on a number line. What is observable is the chart that outlines the steps for the procedure in general terms, indicating that there were probably more examples of fractions with varying numerators and denominators, with the generalized fraction name written as $\frac{\mathrm{n}}{\mathrm{d}}$. The extent of how these varied, and what was invariant are what is salient to what is made available to learn.

What is observable in the episode is variation through contrast through the selection of $\frac{1}{7}$ for public discussion. This brings 'what is not' the name of the fraction, into focus i.e. specifically naming the fraction by counting the hash marks, or the 'parts' without reference to the unit whole. The selection of $\frac{1}{7}$ (as opposed to $\frac{1}{4}$, *or* $\frac{1}{5}$), is salient, provoked by the number line extending past 2. Unfortunately, the episode does not include the discussion of $\frac{1}{7}$ only some clarifications of the thinking that produced this, and so counting all the parts, making available a discussion following on the unit whole and not the length given (as in the case of the area model). Other student questions about starting at 'zero' and what was counted draw attention to the hash marks (as opposed to spaces). In MDI terms, and the application of principles of variation of similarity and contrast, we would conclude from what is available in the video, transcript and the context document, that the lesson provided opportunities for learning to discern the correct place of a fraction between 0 and 2 on the number line, specifically for identifying the whole, and focusing on counting spaces not hash marks.

Looking at the students' successful responses at the end of the lesson leads one to speculate that the whole lesson included discussion of a varied example set, with purposefully selected similar and contrasting examples, making available opportunity for generalizing the mathematical steps for marking a fraction on the number line (and through contrast not over-generalizing).

What is made possible to learn is, of course, not only a function of the selection and sequencing of examples, tasks and representations[2], but how the example set is discursively mediated.

Explanatory communication – word use and justifications

As the episode is restricted to the presentation and clarification only of the answer $\frac{1}{7}$, and questions posed by other students (the teacher's task here is focused on students hearing the explanation offered), there is limited possible analysis of word use towards the object of learning in the transcript. However, the description of the teachers' work through the rest of the lesson, and the extracts of students' explanations of their responses to the exit task in the context document enable analysis of what was or what was likely to have been made available in the classroom discussion.

From the selected written student explanations in the document, words they used initially to explain their answer included "count", "equal parts", "count from zero" indicating that they associated fractions with equal parts that needed to be counted, but not what or where these equal parts were nor the whole they were referring to. In contrast, in the exit task, we see the following: "You have to

[2] Of course, in this lesson and episodes the shift in representation from area to number line is built into the lesson and not focused on here.

count the space"; "all the spaces have to be equal"; "… because I saw what the howl (whole) was. Then I made sure it had equal parts. Last I counted the spaces."; "I counted the spaces"

The students' use of specialized discourse relevant to fraction as number is evident. Contributions to how this was made available to learn in the lesson is observable in the chart at the end of the context document that emphasizes the words: "Whole", "equal parts", "unit fraction", "count parts"; as well as in the extract below that elaborates the discussion (though we are not privy to who said what, when). From the context document (p. 4):

> … the discussion emphasizes the importance of partitioning the *unit interval* in *equal part*s and being sure to *count spaces (i.e. the intervals, not hash marks)* to *determine the distance from* 0 for a given point on the line. The students practice naming points on the line and also explaining carefully with *reference to the whole* and to *the equal parts* and to *counting spaces to determine the number*.

We have italicized the specialized discourse and interpret that these words and aspects of justifications for the fraction name were used in the lesson by both teacher and students. Moreover, as indicated in the underlined sections, we observe that students practiced this specialized word use and justifications, probably repeated on a range of different fraction examples.

Using MDI, we would analyze all episodes in a lesson for specialised discourse coding word use by distinguishing colloquial from mathematical, and coding justifications by distinguishing those that are non-or pseudo-mathematical (assertions – e.g. because the textbook says, or visual cues – e.g. the hash marks), from those that are local, partial, and then full explanations. Our data here is different and coding the given episode would put word use as colloquial and mathematical in name only. However, specialized mathematical word use coherent with the object of learning (whole, equal parts, counting spaces) was made available, as was a full explanation of the steps for correctly placing a number on the number line. If we had the full lesson transcript, MDI analysis by episode would show what and how word use and justifications build up through the lesson, and so through the temporal flow of the lesson, as well as who says what, and so what the teacher inserts, repeats, revoices and reinforces (as this is her work), and what students get to hear as well as say for themselves. This last point is apposite, for specialized discourse (word use and criteria for valid mathematical argument) is not spontaneously available, and thus a crucial aspect of knowledge for teaching and task of teaching (Planas, 2021; Prediger, 2019).

Learner participation

Given the traditional teaching context in which MDI was developed, the focus of research in the WMCS project was on the extent to which students themselves had opportunities to hear, speak and write mathematically. These opportunities are extensive even in this one episode. We discuss this further below.

Concluding discussion

Different frameworks enable scholars to notice different things about mathematics teaching. For instance, the TRU framework allowed Nic Mhuiri (2019) to evaluate and compare the quality of teaching across contexts in analysis of shared data. Theoretical constructs like agency and authority enabled Bass and Mosvold (2019) to explore different aspects of the same data sets. Their framework allowed them to notice how the pedagogical moves that teachers make can influence the agency and position of students in mathematics classrooms, and they argue that this is particularly important to notice when responding to apparent student errors. While these perspectives are indeed salient, the MDI framework was developed with the aim of describing the *mathematics* that is made available to learn. The emphasis is thus primarily on the mathematics as it is made available through examples, tasks and representations, and through word use and justifications. As we have tried to show in our discussion of shared data, selection and sequencing of examples is a key task in mathematics teaching, which is not straightforward, yet often overlooked. One thing the MDI framework allows for, is to highlight exemplification and how it may or may not contribute to the set learning goal.

A potential weakness of the framework is that the underlying conception of teaching is left implicit, linked only with being goal directed and involving mediation. We highlight two aspects of the conception of teaching underlying the MDI framework. First, like Ball (2017), it useful to think about mathematics teaching as a special mathematical work. Others have suggested a distinction between considering teaching as something teachers do, as opposed to a work to be done. This distinction was not considered in developing the MDI framework. However, the framework operationalizes four core mathematical tasks of teaching, and so the kind of work to be done: 1) deciding a mathematical object of learning, 2) selecting and sequencing examples, tasks, and representational forms, 3) attending to mathematical word use and justifications, and 4) considering what learners are invited to do, say, and write. Second is an underlying issue of teaching methods. Some frameworks of mathematics teaching are bound to specific teaching methods or pedagogical values. Nic Mhuiri (2019) highlights this in her discussion of the TRU framework, which appears to value an approach to mathematics teaching that corresponds with ongoing reform efforts in the United States. In contrast, the MDI framework aims at being useful across methods or pedagogies of teaching. It was developed in a context that is dominated by traditional teaching, but our effort to apply it to Deborah Ball's teaching in the shared data set indicate that it can be useful also for analysis of nontraditional mathematics teaching.

In the call for papers to TWG19, five domains were identified to facilitate communication and collaboration. When considering the potential contribution of MDI in relation to these domains, we first notice that MDI has a primary emphasis on extending mathematics to learners. Whereas the organization of interactions is less in focus, the framework does focus on the mathematical discourse. The MDI framework was not developed with an emphasis on responding to students, but equity was an underlying issue of concern. Although social, cultural, and political issues are not directly visible in use of the framework, access to mathematics *is* an issue of equity in many countries. Since learners in areas with significant poverty often do not get the opportunity to learn mathematics, improving their access to mathematics in the classroom is thus an equity issue – and a key motivation for the development and use of the MDI framework.

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“Why have you written six times five?” Teachers’ use of *why* in Norwegian mathematics classrooms

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From classroom data consisting of transcripts from 20 videotaped lessons, we studied five Norwegian mathematics teachers’ use of the word why. *It is our initial assumption that asking why in mathematics is a means to support students’ mathematical reasoning. We found evidence for this assumption in our data. However, the analysis of the 114 occurrences of* why *indicated that the most common purpose for asking why was to make students re-state thinking they had already done, thus not prompting new mathematical reasoning. Moreover, when teachers used* why *in questions not about mathematics, they usually did so in a rhetorical and reproachful manner. We discuss these ambiguities and implications for teaching.*

Keywords: Mathematical reasoning, questioning, mathematical discourse, interactions.

Introduction

In the mathematics classroom, teachers’ questioning is acknowledged as crucial in supporting students’ learning (see e.g., Franke et al., 2009). Martino and Maher (1999) claim that there is “a strong relationship between (1) careful monitoring of students' constructions leading to a problem solution, and (2) the posing of a timely question which can challenge learners to advance their understanding” (p. 53). Recent studies have pointed out that *asking why* is a type of question that particularly holds the potential to encourage students to explain or reason (Ingram et al., 2016; Drageset, 2021). In those studies, the researchers first identified students’ explanations in classroom data, and then considered what the teacher did to prompt the explanations. In contrast, we are taking as a starting point all teacher-initiations given in the form of a why-question, and we aim to categorize the possible implication of the question in the given situation, not unlike Enright et al.’s (2016) typology of questions according to the function they serve in instruction. We are particularly interested in the potential why-questions hold to challenge students to advance their reasoning, reflecting Martino and Maher’s (1999) “timely questions” mentioned above.

In our study, we take a commognitive view of mathematics teaching and learning (Sfard, 2008). Within this tradition, learning is seen as increased participation in a discourse, and discursive behavior is characterized in terms of routines (Lavie et al., 2018). Teachers have the role of expert participants in the classroom discourse, and the theory of commognition states that imitation of expert participants is a central aspect of the learning process (Sfard, 2008). Martino and Maher (1999) point out that students do not naturally seek to build a justification or proof of the validity of a solution. Thus, routines related to reasoning and proving in mathematics must be made visible by the teacher. Therefore, teachers’ use of why-questions influences students’ routinization (i.e., individualization of routines in the community), regarding when it is a timely question to ask, and what kind of actions such a question requires.

Thus, to get the whole picture of how teachers' why-questions relate to students' mathematical work, we need to extend our knowledge beyond the why-questions identified by Drageset (2021) and Ingram (2016) that actually led to students' conceptual explanations. Other uses of why-questions will also affect how students routinize asking why in mathematical discourse. To this end, we employ the following research question: *How is the word "why" used by five middle-grade teachers in mathematical classroom discourse?* The data for the study consists of transcripts of 20 lessons given by the five teachers. As all occurrences of the word "why" in the data material are part of a question (this includes a few instances of implicit questions), we will use "why" and "why-questions" synonymously in this paper.

The routines of asking why

Within the commognitive framework, *routines* are important features of discourse in a community (Sfard, 2008). Routines are discursive patterns which are evoked whenever a participant finds herself in a situation similar to one she has experienced before, and acts thereafter (Nachlieli & Tabach, 2018). Routines consist of an initiation, a procedure, and a closure (Nachlieli & Tabach, 2018). The initiation is the clue that is recognized and thus prompts the participant to perform the expected actions, and the closure is "the conditions under which a procedure is considered complete" (Nachlieli & Tabach 2018, p. 3), together they form the "when" of a routine. The procedure is the action taken in between the initiation and the closure, also referred to as the "how" of the routine. Lavie et.al (2018) consider two types of discursive routines in mathematical discourse: Explorations and rituals. Explorations are outcome-oriented routines, where the aim is to produce or endorse new narratives (i.e., statements about mathematical objects and their relations). They are motivated by the question "What do I want to achieve?" and are often performed by the more experienced participants of the discourse. In contrast, rituals are process-oriented routines, guided by the question "How do I proceed?". Rituals are often performed by novices in the discourse, typically by imitating the teacher's actions, and the motivation is social acceptance. Routines can be nested – a new routine can be initiated, carried out and closed as a sub-routine of another routine (Nachlieli & Tabach, 2018).

As previously mentioned, we are particularly interested in routines that are connected to students' mathematical reasoning (MR). Therefore, we need to elaborate on what we mean by MR. We employ Jeanotte and Kieran's (2017) conceptual model of MR for school mathematics, which is constructed in coherence with the commognitive framework. The model consists of two interrelated aspects: The *structural aspect* addresses the underlying construction of an argument (e.g., deductive, inductive, abductive), while the *process aspect* capture activities related to searching for similarities and differences, validating, and exemplifying. A search for similarities and differences is aimed at producing new narratives, a validation is aimed at establishing the correctness (or wrongness) of narratives, while an exemplification works as a support for the first two types of MR processes. We note that all the MR processes – when considered as overall routines performed by students – are exploratory routines, because they seek to produce or endorse narratives. We thus use the term *MR exploration* for an exploratory routine involving one or more MR processes.

Both Drageset (2021) and Ingram et al. (2016) found that why-questions were connected to student explanations of reasoning. However, in their studies, it is not possible to separate students' explorations and rituals, although it is likely that many of the explanations of reasoning were connected to an exploratory routine. Nor is it possible to see whether the teacher's question prompted the student to share his reasoning, or whether it invited the student to start to reason. This distinction between starting and explaining routines is also not evident in the typology of questions suggested by Enright et al. (2016). Our study aims to give more insights into these matters.

Methods

The study presented in this paper is part of an ongoing, large-scale study with a focus on reasoning and proving in primary education (ProPrimEd). As a first part of the ProPrimEd study, we collected classroom data (videotaped lessons, and interviews with some students and all teachers who participated in the lessons) to describe the "state of the art" of work with reasoning and proving in ordinary mathematics education. The data for the study reported here consist of the transcripts of the videotaped lessons. In total, we observed five teachers (T1, fifth grade; T2, sixth grade; T4, fourth grade; T5, seventh grade; T6, sixth grade) for two weeks each, a total number of 20 lessons of various lengths. All the five teachers were male. T1 and T2 worked at the same school, so did T4, T5 and T6. All were educated as primary school teachers with mathematics as one of their subjects. Some had additional professional development courses in mathematics. Their teaching experience varied between 2 and 17 years, with an average of 9 years. The researchers took the role of non-participating observants during the data collection and played no part in lesson planning. Topics of lessons were whole-number multiplication, addition and subtraction with rational numbers, problem solving involving geometrical shapes and multiplication, prime numbers, and operator precedence. Most classes consisted of a variety of activities. Although the data is limited to two schools and five teachers, we consider it to represent ordinary mathematics discourse in Norwegian classrooms.

The transcript documents consist of a total number of 143 448 words (including meta-data). From this material, we identified (using the search functionality in Microsoft Word) every occurrence of the word *why* used by a teacher in whole-class discussion and in discussion with (groups of) students. We found 114 occurrences.[1] For each teacher, we made a new document of cut-outs from the original transcript, consisting of a separate text block for each occurrence of *why* – including enough text before and after the word to make the context clear, so that we would be able to interpret the possible implication of the *why*. Sometimes, during the coding process, we would go back to the complete transcripts to review more of the context, and we also had the opportunity to consult the video recordings. We did not seek to identify differences between the teachers, and therefore treated all why-questions as a pool.

We were inspired by other studies on the use of certain words in mathematical discourse, such as Monaghan (1999) and Wagner and Herbel-Eisenmann (2008), who studied the words *diagonal* and *just*, respectively. These studies pointed to the influence of single words in mathematical discourse: Monaghan showed how an "informal" meaning of *diagonal* could possibly weaken the mathematical content at stake, while Wagner and Herbel-Eisenmann showed how a seemingly innocent word like

[1] The lessons, and thus the transcripts, were in Norwegian. We searched for the Norwegian word "hvorfor", which holds the same meaning as the English word "why". The data excerpts used in this paper are translated to English by the authors.

just could open or close discourse. In our study, we were guided by the commognitive concept of routines when analyzing and discussing data. The two authors read some of the material together and agreed on a temporary set of codes. These codes were:

- *initiating*, later renamed *routine-initiating*: *Why*'s that prompt new reasoning (based on the underlying assumption of the study)
- *recalling*, later renamed *routine-re-initiating*: *Why*'s that prompt students to share (aspects of) mathematical thinking they have already done
- *non-mathematical*: *Why*'s used in contexts not about mathematics.

Each author then coded the material separately. When discussing the coding afterwards, we found that we agreed for most occurrences and came to agreement on the few we had coded differently. We added a category *Other* for why's occurring within a mathematical context, but not fitting into the first two categories. We started the process of analysis with an open coding that was as close as possible to the data. As we interpreted our interim findings, we employed Sfard's and Jeannotte and Kieran's theories to further nuance the content of each category, and the categories were appropriately renamed. In the next section, we elaborate on some typical uses of *why* from each category.

Findings

Table 1 shows the number of occurrences of the word *why* in each of the categories routine-initiating, routine-re-initiating and non-mathematical (and other).

Table 1: Number of occurrences of the word *why*

routine-initiating	routine-re-initiating	non-mathematical	other
34	65	7	8

We note that the most common use of *why* is to recall mathematical thinking (routine-re-initiating). In the following, we provide examples from the four categories. The examples are chosen to indicate both "typical" usage of *why* and to demonstrate the variation within a category.

Teachers' use of routine-initiating why-questions

Routine-initiating why-questions invite the students to choose and take action. Almost all instances of routine-initiating why-questions in our data material called for an MR exploration. The data material provides examples that why-questions were used both to prompt students to search for similarities and differences, thereby producing a narrative, and to validate a conjecture. An example of the former is given below. Two students are working on a jigsaw puzzle of the multiplication table (the pieces in the puzzle consist of more than one number).

Teacher: What I'm wondering is why does it [a piece] fit so well here?
Student: Because here it says 36 and here it says 32
Teacher: Why do 36 and 32 fit together? And together with 28 and 24? What's going on with these numbers?
Student: Because (…) because. It's 4 between them.

In this example, it seems that the students first placed the piece in the puzzle because of its shape. The teachers' questions made them reason about the mathematical relationships, and eventually identify the pattern in the 4-times table. Next is an example of a why-question that is used to initiate a validation process. The excerpt is from a situation where the students are asked to calculate the surface area of the (glassed-in) walls of different buildings, all having the shape of rectangular prisms, and all having the same circumference of the ground floor and the same height.

Oline:	Like. Around the walls. It's the same for all around the glass? Isn't it the same for all?
Teacher:	Yes, why is it so?
Oline:	It's the same outline, it, it [Norw. outline is omriss]
Teacher:	The same circumference? [Norw. circumference is omkrets]
Oline:	Yes.
Teacher:	Yes. Why is the glass the same when it's the same circumference?
Oline:	I don't know, but it feels that way.

Here, the student claims that there is a relationship between the circumference of the ground floor and the area of the walls but has no ready-made reasoning to support the claim. Hence the two why-questions serve as possible initiators for a validation process of the student's hypothesis.

We conclude this section with two examples of routine-initiating *why*'s taken from a lesson on prime numbers. Common for these *why*'s is that they are not self-contained as routine initiators; they are a part of a more extensive initiation. In the first example, the students are investigating whether all whole numbers between 4 and 20 can be written as a sum of two prime numbers. During the students' group work the teacher reminds them of the task:

Teacher:	You'll get 3 more minutes. Remember that, although I claim, I claim that it's possible for all the numbers from 4 to 20, but it's not certain it's the case though. So it could be that you find some numbers where it doesn't work. But then maybe you have to say something about why it doesn't work. How can we be completely sure that it doesn't work? So we work for 3-4 more minutes (…)

Here, the *why* is a support for the process of validating a hypothesis, which is already going on in many students' work (thus, the *why* initiates a sub-routine nested within the routine initiated earlier, when the task was given). The students are working on a task, and this *why* clearly does not refer to an answer provided by a student. Later, the teacher discusses an answer with some students:

Teacher:	Yes, there you only have to explain why some numbers aren't prime numbers. Such that he said that 8 isn't a prime number, since you can get that by using the factors 2 and 4. You can just write down that computation.

The use of *why* here points to a validation, and the teacher add some information about the possible nature of this explanation. Thus, this *why* is connected to the structural aspects of a reasoning process. In our data material, we found very few examples of such *why*'s.

Teachers' use of routine-re-initiating why-questions

Routine-re-initiating why-questions invite students to recall, share and elaborate on the routine they have already performed. In most cases, this happens during a whole-class discussion after the students have worked on a task. As with the previous category, we find different uses of *why* within this category. In many cases, the why-questions were (mainly) addressing the procedure aspect, or the "how" of the routine performed. This is the case in the following example, taken from a lesson on negative numbers. The teacher is requesting details of a calculation:

Teacher:	Seven minus negative three. What will that be, Anja?
Anja:	Ten
Teacher:	That is ten. Why?
Anja:	Because it is the same as seven plus three.
Teacher:	It is the same as seven plus three. Yes.

In the next example, the teacher's why-question addresses the procedure of regrouping in addition with decimal numbers. Again, this excerpt addresses the "how" of the student's performed routine, but the "when" is also addressed as the teacher probably seeks to highlight conditions for taking this action (when to regroup: if we get ten tenths).

Teacher:	Yes, we get in a way zero tenths at the end. But why do we get zero tenths? What have you done then?
Anniken:	Because we transfer to the ones' place.
Teacher:	Yes. That is right. As soon as we get ten tenths, we have to regroup. That is the way the decimal system works.

In other cases, re-initiating why-questions were used mainly to highlight features related to the "when" of a routine. The next example, taken from a lesson on early multiplication, shows how a why-question seeks to emphasize the connection between the given context and the numerical expression, i.e., in what situations a multiplication procedure is called for.

Teacher:	Six times five. Ok. Adele, why have you written six times five?
Adele:	Because it [the worm] climbs six centimeter each hour.
Teacher:	Mm, it climbs six centimeters each hour. So it does.
Adele:	And it climbs for five hours.

There were also some examples where the teacher asked *why* to make students elaborate on a routine, but where the teacher (most likely) did not know what the student had done (e.g., when the student had provided a wrong answer). Why-questions used in this way have the potential to provide the student with an opportunity to revise his/her thinking, which would be a routine-initiating why. However, in our data we only find examples where the teacher guides the student to the correct answer by asking closed questions (like Drageset's (2014) *closed progress detail*). Thus, the student's possible contributions are limited to yes/no-replies and short answers, with few or no opportunities to engage in mathematical reasoning.

Non-mathematical and other uses of why

The few *why*'s in the *Other* category are either not part of a question to students, or meta-questions which we do not discuss further in this paper. Similarly, the non-mathematical why's are rare in the material, and occur in data from only two of the teachers. Yet, they are interesting because all of them are used in a rhetorical and even reproachful manner, such as:

Teacher: Six. Why don't you raise your hand when you know it?

This example, as well as the remaining six instances of this code, are rhetorical and reproachful because the teacher does not expect an answer and the aim of the questions is to address something unwanted in the student's behavior.

Discussion

We started this paper with an assumption that why-questions can serve as initiators of exploratory routines involving MR processes, in line with Martino and Maher's (1999) "timely question[s] which can challenge learners to advance their understanding" (p. 53). In our data, we found evidence that this indeed happens. Yet, we found that by far most examples of why-questions are routine-re-initiating. They are usually asked after the students have finished their work on a task. Thus, in commognitive terms, we could consider those why-questions initiators of subroutines that are nested within the closure of the outer routine, the task itself ("work on the task is done after the teacher has got all the details"). Still, they have an indisputable potential for developing students' mathematical reasoning: The student gets an opportunity to express her thinking; the (teacher and) other students get access to this reasoning; and the teacher is given the opportunity to emphasize for the whole class what he considers to be important ideas. Moreover, some of the routines expressed by the students may initially have been exploratory, and insights into explorations could model other students' later attempts at explorations.

So, is there a problem with this bias towards routine-re-initiating why-questions? Although earlier studies recognize the potential of why-questions as routine-initiating (Drageset, 2021; Ingram et al., 2016), we must remember that if students should interpret a why-question as a "signal" to start an MR exploration, then they must be exposed to why-questions used in this way. Lavie et al. (2018) write that "[t]he teacher's own mathematical discourse is the model for the learners to follow, and the question is whether its explorative nature gets through to the students" (p. 20). If why-questions are mostly restricted to the recalling of performed work (which may even not have been explorations), or to aspects that are non-mathematical, the connection between asking *why* and the need for MR explorations is not likely to be routinized by the students. Accordingly, we note from further reading of the transcripts that in many of the cases where routine-initiating *why*'s are used, explorations did not consequently occur. We do not elaborate on the reasons for that here – that would be outside the scope of the paper – but we remark that a possible reason could be that students simply does not interpret the question as a "signal" to start an MR exploration. One implication for teaching could be for teachers to be careful of how they use the word *why*. In situations where the aim is not to invite students to reason, but rather to invite them to share their thinking (and in particular when it concerns rituals and concepts rather than explorations), could there be other phrases to choose from? "What does that mean", "how does it work" and so on. But most important is probably that teachers should stress to use *why* in connection with MR explorations, also when it is the teacher himself who takes the lead in the process.

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Boundaries between mathematics and visual art teaching communities

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Focusing on mathematics and visual art teacher collaboration and drawing on Communities of Practice and boundary crossing, this study aims to examine the boundaries that emerge between mathematics teaching and visual art teaching communities in regard to mathematical practices and tools, as well as the way they are handled among collaborating members. Using a grounded theory approach, data from 10 group meetings of art and mathematics secondary school teachers who try to develop ways of linking mathematics and art teaching were analyzed. Results indicate the emerging boundaries by means of epistemological differences regarding "formal vs informal", "theoretical vs empirical", "analytical vs visual", "terms vs meaning" and "terms vs representations". In handling these, brokering, identification, coordination, reflection, transparency, and negotiability emerged.

Keywords: Mathematics teaching practice, visual arts teaching practice, interdisciplinary mathematics teacher collaboration, boundaries, boundary crossing.

Introduction

Recent research examines *arts integration* into other subjects and the benefits it provides in learning, such as supporting different learning styles (An & Tillman, 2014), enhancing motivation-emotions and constructing meaningful knowledge (Baird, 2015). Students of mathematics may also benefit from such an integration, by engaging in processes such as problem solving and modeling (Jacobs, 2000), supporting exploratory learning approaches through the process of creating art (von Renesse & Ecke, 2016), developing mathematical thinking and connecting classroom mathematics with students' personal experiences (Presmeg, 2009). After all, the connection between mathematics and art is not new and can be used in common curriculum content as well as in integrated curriculum, where teachers in both subjects are required to work together to reshape them (Bickley-Green, 1995). This interdisciplinary collaboration seems crucial, as teachers alone naturally do not have the knowledge needed to integrate art in their classroom. However, according to Akkerman and Bakker (2011), people collaborating across disciplines may face *boundaries* between different perspectives or practices. In response to these challenges, interest exists in how continuity in action or interaction is established despite of sociocultural differences. In describing potential forms of continuity across sites, *boundary crossing* and *boundary objects* seem central. According to Kisiel (2010), a successful collaboration "would rely on clarifying the potential boundaries between communities" and "on the introduction of appropriate boundary objects as well as the utilization of brokering to make fruitful connections" (p. 99). This inherent learning potential makes boundaries important to education fields, mathematics education included (Potari et al., 2019).

In mathematics education, interdisciplinary collaborations are often placed in a professional development context, with interactions between communities being studied in terms of boundaries and boundary crossing, often framed by Communities of Practice (CoP) (Wenger, 1998). For example, Psycharis and Potari (2017), by adopting an activity theory framework, use boundary crossing to study teachers' learning in relation to integrating workplace contexts into mathematics teaching; and Bouwma-Gearhart et al. (2014) focus on interdisciplinary collaboration between postsecondary STEM and education faculty, highlighting the critical roles of brokers. In our study, we focus on collaboration between mathematics teaching (MT) and art teaching (AT) communities of practice, via collaboration of teachers from these communities, as well as on boundaries and boundary crossing between the two practices. The research questions are: 1) What are the emerging boundaries between MT and AT communities of practice in regard to mathematical practices and tools? and 2) How are these boundaries handled by members in collaborative group discussions?

Theoretical perspectives

Arts integration, communities of practice, boundaries and boundary crossing

Burnaford et al. (2007) recognize arts integration as learning through/with the arts, a curricular connections process and collaborative engagement (pp. 11–12), citing aspects such as building links between learning in arts and in another subject area and professional development as a key element that defines it as an integration of people (p. 18). In addition, Silverstein and Layne (2010) define it as an approach to teaching where students construct understanding through an art form, engage in a creative process that connects the two fields and evolve in both. They consider that it can be used as incentive to enter the field being taught, as a tool for verifying existing knowledge or as equivalent to the other subject. This definition is embraced by our own conceptualization of mathematics and visual art integration. However, it implies a shift in daily teaching practice, something that is not easily achieved. It would seem, though, that it is the kind of engagement that the students will be drawn in that would be most crucial (e.g., problem solving, instead of coloring tiles in a worksheet).

Furthermore, the "integration of people" mentioned above implies the concept of community. On to this, Wenger (1998) describes CoP as groups of people who share a practice reflecting their mutual learning, being mutually engaged in a joint enterprise generating a shared repertoire, where negotiation of meaning is generated through participation and reification. Focusing on interactions between different communities, he is not the only one to talk about boundaries and boundary crossing. Boundaries are defined as "sociocultural differences leading to discontinuity in action or interaction", simultaneously suggesting "a sameness and discontinuity in the sense that within discontinuity two or more sites are relevant to one another in a particular way" (Akkerman & Bakker, 2011, p. 133). In relation to this, boundary crossing refers to a person's interactions across different sites (Suchman, 1994), facing the challenge of "negotiating and combining ingredients from different contexts to achieve hybrid situations" (Engeström et al., 1995, p. 319) in order to establish continuity across them (in Akkerman & Bakker 2011, p. 133). In this, people and objects that transcend boundaries are central: brokers introduce elements of one practice into another via creating connections, moving knowledge, exploring new territories (e.g., Wenger, 1998), and boundary objects enable multiple practices to negotiate relationships and connect perspectives, including artifacts (tools or documents), discourses (common language) and processes (shared routines or procedures) (Wenger, 1998).

Thus, although boundaries can be sources of misunderstanding, as moving across them means moving into unfamiliar contexts, so being unqualified (Suchman, 1994, as cited in Akkerman & Bakker 2011, p. 133), they can also be resources for learning, as moving across them also means engaging in dialogue and negotiating perspectives, through which new meaning and new possibilities may arise. In fact, Wenger (1998) suggests that communities must exceed their own boundaries so to preserve their dynamism and refers to three dimensions for boundary processes to create bridges that connect practices in deep ways: a) coordination (enabling coordinated actions, accommodation and standardization of practices so that everyone can engage in them); b) transparency (access to meanings and understandings of practices); and c) negotiability (negotiation of perspectives and multiple voices). Moreover, Akkerman and Bakker (2011) name four boundary-learning mechanisms: a) identification (recognizing what each practice is about and the boundaries that exist between them: how they do and do not relate to each other, how either differs from the other); b) coordination (ways that make transitions between practices smoother: a communicative connection, efforts of translation [with boundary objects serving as mediating artifacts], enhancement of boundary permeability and processes of standardization [organizing activity as smoothly as possible] and routinization [creating routines to rely on]); c) reflection (perspective making, i.e. defining the different perspectives that each practice brings or amplifying one's understanding and knowledge of a particular issue; and perspective taking, i.e. looking at one's own practice through the eyes of the other); and d) transformation (changes in practices or creation of an in-between practice [e.g., Wenger's "boundary practice", p. 114]: a confrontation with a situation, a recognition of a shared problem space, a hybridization and a crystallization of the new practices [Wenger's reification], maintenance of the characteristics of each practice and continuous work at the boundary).

Methodology

Situated in two Greek art-based schools (grades 7–12), the study initially adopted an ethnographic approach (Allan, 2017). R1 (first author) visited each school twice a week for a whole day for a period of 8 months and kept field notes from MT and AT classroom observations, audio-visual records of lessons and events and informal discussions with the teachers. In the next school year, 2 collaborative groups were formed, 1 at each school (2 mathematics teachers, 5 visual art teachers and R1 in each one (R2 [second author] was present in each of the first meetings). The meetings (sum of 17) were held in teachers' office once every two weeks and embedded in the daily school setting. In the first 10 the meetings, members engaged in a familiarization phase between the two communities. Special resources (visual-artistic artifacts, students' actions, common ground in curriculums, examples from teaching issues) were chosen by R1 as potential boundary objects and given for the group to negotiate upon. In the last 7 meetings, teachers were asked to co-design and co-implement integrated tasks in their classrooms and to reflect on what happened. Data included field notes, audio-visual records of the meetings as well as teachers' lesson plans and written reflection reports. In Choutou (2019), a more detailed view on the design of the study and R1's role as facilitator is provided.

In the analysis, grounded theory approach is used (Charmaz, 2014), identifying themes and categories in Atlas.ti. We are interested in the emerging boundaries (based on Akkerman's & Bakker's definition) in connecting MT and AT. Here, we draw on the data from the first ten meetings of one of the groups. We report the boundaries identified as epistemological differences or tensions that lead to discontinuity in action or interaction in regard to mathematical practices and tools, and the ways these are handled by members, tracing any action or interaction that relate to establishing continuity and characterizing them using terms related to boundary crossing.

Results

Boundaries between MT and AT communities

Several boundaries between MT and AT, in the form of epistemological differences between the two contexts, emerged, in relation to *mathematical practices* (including constructions, processes, procedures, ways of thinking and inquiring) and to *tools* (concerning language and artifacts [hands-on materials]). In Figure 1, we present the systemic network developed from the data analysis in relation to these emerging boundaries. Following, we discuss the categories and the subcategories offering examples from the data to illustrate them. MT1, MT2 stands for the two mathematics teachers and AT1, …, AT5 for the art teachers accordingly.

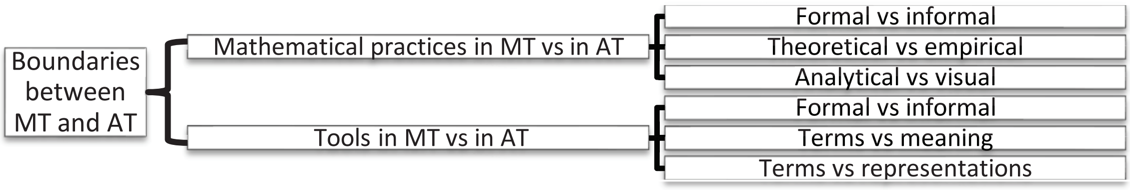

Figure 1: The systemic network describing the emerging boundaries between AT and MT

In regard to differences in *mathematical practices* a first difference concerns *formal (in MT) vs informal (in AT)* mathematical practices. For example, the following extract illustrates an artist's informal construction of a circle starting from a square:

R1: In the beginning he makes a square. And then he focuses on the middle of that distance and marks a little further inside. He does it in each of these points, he joins them as circularly as he can and so he gets a circle.
AT3: "A little further in" is not very mathematical… it can be visual-artistic (laughs) but it does not seem very mathematical to me (laughs).
R1: (laughs) It's a little over the mean… if you do it algebraically, a square root emerges. That's why it does not come out exactly in the middle.

A second difference concerns *theoretical (in MT) vs empirical (in AT)* mathematical practices. For example, towards the construction of the development of a solid in art, AT4 says: "it's not enough to construct a theoretical development. The development you'll make (e.g., out of cardboard) will probably have to have edges (left/glued around the sides of the development) so you can glue it all-closed-together." An interesting highlight here, is the non-existence of mathematical proof in AT as we know it in MT. More specifically, AT1 says that her interest lies in how she can support her students, so that they can learn mathematics more easily and without hating it, "because, theoretically, mathematics is the basis of everything. I cannot prove it, this is what science is for, but I really feel this", thus pointing to an empirical sense of feeling (vs proof) coming from her experience as an artist.

A third difference concerns *analytical (in MT) vs visual (in AT)* mathematical practices. For example, when inquiring about why the pentagon does not tessellate the plane, MT1 came up with the algebraic formula, whereas AT1 engaged in a visual inquiry in dividing the regular polygons into triangles in order to see what is happening with the angles. Ways of thinking in AT seem more visual in general:

AT1: Are there indeed two equal lines?
MT1: Mmm… it says… 10, 10. Yes, 10, 10.

AT1: So out of the 4, two are equal. And this helps (in making the tile tessellate the plane), I imagine… look. In the drawing (tile), here, this is equal to this, this is equal to this… and I imagine that this side is two times another side.

MT1: It's true! (enthusiastically).

AT1: You can see it with your eye.

MT1: YOU can see it with your eye...

AT1: Because, this is equal to this, so… this is true because, this side is equal to this, and this side here, I mean the one that lies next… is two times the first ones.

MT1: So they can complement the big one… hm...

AT1: Yes, because, the way this side is here, that's how these two should be here also. And then, the pattern can develop further. If it wasn't for this, the development of the pattern would not be feasible to develop...

In regard to differences in tools, a first difference concerns *formal (in MT) vs informal (in AT) tools*. These concern differences in *language*, as for example, mirror instead of axial symmetry and cookie instead of circle. Metaphorical descriptions are included, for example, directions and meetings of the lines (informal) instead of angles (formal), turning like a windmill, but at equal distances (informal) instead of rotation of equal degrees (formal). They also concern differences in *artifacts*, for example, rope and string or circular objects (informal) instead of compasses (formal) for making circles and "eye scaling" and measuring needle (informal) as means of comparison instead of ruler (formal), or even working only with their hands and drawing instinct. It must also be noted that hands-on materials "guide" the mathematical practices the artist will use (informal or empirical), as in the case of the construction of the development of a solid mentioned above.

Moreover, some of the concepts that appear both in AT and MT, such as positive-negative, rhythm, symmetry, small-big, shrinkage-enlargement, pattern, share the *same terms* but have *different meanings*. For example, the concept of pattern in AT, it seems to have the meaning of a single shape or a decoration instead of a sequence of shapes. In the next extract, R2 focuses on the pattern as a whole, whereas AT5 focuses on the shape that is repeated:

AT5: I am thinking of Minoan decorative rhythmic shapes that could be very useful in 10th grade. They can be made into friezes. And then move on to mandalas and what MTL was saying (fibonacci spiral) and end up in the costume (students' artwork).

AT3: Yes, because it's a repetition, it still has the (notion of) rhythm.

R2: Ok, that, in the sense of what we might say "pattern" in math.

AT5: A pattern and a blank space then a pattern and a blank space and so on.

In addition, the opposite relation emerged too, pointing to artifacts that share the *same meaning* in the two contexts, but appear with a *different term* in each. For example, during a discussion on the connections that teachers find between a grid resource for artwork and mathematics, MT1 says: "I see the Cartesian (coordinate) system, with squares. Like a millimeter sheet." Also, the concept of vanishing point (AT) (the point where two parallel lines seem to meet, in linear perspective), shares the same meaning in both contexts, but appears with a different term in MT, i.e., a point at infinity.

In addition, other concepts appearing with the *same term* have *different representations*. For example, AT2 reports for the stretching of a line in graphic design, that they have different thicknesses and sparsity, so when the artist chooses to do a line (in the program), he must choose the thickness he wants, thus, "… say I have scanned a line. The distortion starts and the line thins… because the pixels are removed. But the line doesn't have a second dimension, only one, so in theory this isn't possible!"

Handling of the emerging boundaries among group members

Results in regard to how boundaries were handled indicate several actions and interactions among members, including: identification processes (identifying other's way of thinking, recognizing own' disciplines features), perspective making (defining other's perspective in light of owns perspective), efforts of translation and explanations for coordination and transparency [translating and linking perspectives, translating terminology/perspectives in ones' own context, (AT) describing artifacts using formal language, translating informal/empirical/visual into formal/theoretical/analytical accordingly, explaining relative/according AT or MT content, proposing coordination of teaching, reducing teaching expectations/requirements, designing properly an activity], brokering (bringing mathematical knowledge), transparency (reasoning one's own context's practices) and negotiability (sharing an expert's perspectives on fixing students' negative feelings, counter-proposing own perspective). The following example from the data illustrates how a specific boundary was handled.

The group discussed on two students' construction of a ballerinas' tutu (a resource found in the ethnographic phase), where students made a circular ring out of cardboard and after cutting it (like a radius), crossed one edge over the other, forcing it to tilt (informal). According to AT1, "It was a very good job, but the mathematics teacher didn't interfere there at all. The mathematics used there is empirical. Clearly." However, MT2 disagrees, saying that what students did is mathematics indeed: "well, they've done mathematics so far. The fact that they want it to curve and the lines they made, it's still mathematics. It's fine, it's not bad." Here, AT1 stresses that the students had to construct a cone which would become the tutu, adding that "there is the form of truncated cone" (formal). At this point, AT2 comes to sooth things down by pointing out that using formal mathematics for this construction would be using stereometry (formal) for calculating the development of the shape and the dimensions of the tutu, thus proposing it as the solution: "We used to do stereometry at school…we took cardboard and they were teaching us how to cut it to make a cone, a pyramid. These are needed," adding that "the equivalent, if we collaborated, when I do this (tutu), you teach stereometry, even for 1 hour, so that we can have coordination, so students can see these important stuff," stressing the importance of having "an epistemological stance to coordinate our speeds." AT1 enthusiastically agrees and adds: "Well done, that's it! We (AT) go in (classroom) and show them: how to make a cube, how a cone is constructed. Basically, it's not exactly our stuff. We rely on it. But we introduce it. So, all I am saying is, I did not see this here, that's why I come a little like that. This is empirical." In addition to this, MT1 also identifies the matter of transfer from the empirical implementation of mathematics in AT (the tutu construction here) to the theoretical knowledge in MT, saying: "to start from what AT1 said about students liking to use geometric instruments, and see how far they can go from this to the most theoretical. That there are formulas, for how we calculate (and) construct, areas, triangles, taking advantage of the aim of this situation, this construction."

The tension described above seems to be generated by the difference between the formal (stereometry) and the informal practices of students. A trigger for it seems to be AT1's *brokering* ability to *bring and compare the formal mathematical practices* used in this construction with the existing formal "form of truncated cone". In negotiating with AT1, MT2 suggests the acceptance of the informal practices as "mathematics indeed" (negotiability), and the boundary is, finally, handled through AT2's suggestion of *coordination* of the respective teaching sides, for *linking the two for* *boosting students' learning*. AT2, here, acts as a *broker* between the two communities, bringing and naming formal mathematical knowledge of stereometry from MT community to the AT community and relating it to the according informal practices the students used. Side with AT2 lies MT1, who propose linking the empirical implementation of mathematics and the theoretical mathematical knowledge for achieving better results in respect to students' learning. It is evident that, here, the *coordination* mechanism/dimension between the two sides plays a central role, as being proposed to being the solution itself for bridging the formal-informal and theoretical-empirical gap.

Discussion

In this paper, we focus on the emerging boundaries between AT and MT communities and the ways these are handled among members in respect to boundary crossing. First, taking under consideration Akkerman's and Bakker's (2011) definition of boundaries, we focused on emerging epistemological differences between the two contexts in regard to mathematical practices and tools. Mathematical practices emerge as formal/theoretical/analytical in MT and informal/empirical/visual in AT accordingly. Tools in relation to language and artifacts bring to the fore different terms, representations and meanings of concepts and practices. For example, formal in MT and corresponding informal language in AT, as well as formal mathematical tools compared to informal in AT and tools with same meaning but different terms, describe ways in which MT and AT differ from one another. All these differences create forks between MT and AT, leaving "blanks" in member's actions and interactions and nevertheless, juxtaposition the two, suggesting that "within discontinuity (the) two…sites are relevant to one another in a particular way" (Akkerman & Bakker 2011, p. 133), implying a set of relationships between learning in art and learning in mathematics (Burnaford et al., 2007), thus giving meaning in an effort of crossing them. Secondly, for locating how boundaries are handled, we used terms related to boundary crossing. Here, brokering (bringing M knowledge) and coordination (e.g., linking theoretical and empirical ways of thinking), seem to be crucial, along with transparency and identification, as well as reflection and negotiability (Wenger, 1998; Akkerman & Bakker, 2011). Overall, our findings seem to indicate aspects of mathematics and their teaching, highlighting this teaching as something that is being or can be done, not only in the mathematics classroom but also in non-mathematical settings, and describing the nature of mathematical practices taught and tools used in either of them. Through their inevitable comparison, the co-existence of the two poles of the boundaries links these aspects and a more complete "image" of mathematics and teaching mathematics is provided. Their interplay is commented by members to be beneficial for mathematics teaching and students learning. Thus, the feasibility of building bridges to connect MT and AT communities and create learning potentials for both teachers and students is suggested. Future research will focus on the negotiation among members towards the integration of the MT and AT and the potential creation of a hybrid, art and mathematics integrated practice.

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How do teachers share and develop student ideas?

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This article aims to try out a framework describing teacher positions based on an analysis of their interactions, and through this, gather ideas for further development of the framework. This article uses shared data from TWG 19, with data from five classrooms. These were analyzed by categorizing interaction from transcripts of the data. The analysis focuses mainly on the interactions related to one position: A teacher that shares and develops student ideas. The findings illustrate how this framework can characterize and contrast these five classrooms.

Keywords: Positioning theory, interactions.

Introduction

In all social settings, including classroom discourse, the participants position themselves and each other based on personal preferences and social interactions. In the classroom context, a teacher typically will position students continually, and this positioning can be seen by the type of questions asked, the types of tasks given, and who the teacher groups together (Harré & Langenhove 1999). This study aims to use the shared data in TWG19 to try out a framework describing five teacher positions: A teacher that knows and tells, a teacher that supports, a teacher that shares and develops student ideas, a teacher that participates, and a teacher that facilitates (Drageset, 2021). The results of this will be used as a base for further developing the framework. We will particularly study the third position with this research question: How can a study of how teachers enacted the third position, *a teacher that shares and develops student ideas*, be used to characterize and contrast different mathematics classrooms?

Theory

It is well established that in a discourse, a turn is thoroughly dependent on previous turns (Linell, 1998). Consequently, it is possible to argue that student interactions are affected by the teacher, and this is particularly true when the teacher speaks every second time as in the Initiation-Response-Evaluation (IRE) pattern. However, there is more to communication than responding to the prior turn. Other factors are in play, such as when a teacher asks a question, students might know what is accepted as an answer and not. This is not only related to the content of the prior turn (the teacher question) but might also be related to established classroom norms. Such norms are described by Yackel and Cobb (1996) as socio-mathematical norms. These are developed over time in all classrooms, whether worked on deliberately or not. Another factor that affects communication is related to positions in the classroom. When joining a social setting, one might have dispositions that guide one's preferences for positions, such as trying to keep silent because of insecurity or a feeling of safety and curiosity in mathematics, making you an active participant. On the social level, such positioning is a reciprocal, sometimes competitive, process where you do not choose a position freely but might be positioned by others' positions or actions, and your positioning affects others (Harré & Langenhove 1999). In the classroom context, a teacher typically will position students continually based on the teacher's beliefs about the competency and personality of the student (Harré &

Langenhove 1999). Also, this positioning of others, being by the teachers or fellow students, might be intentional or not. However, it is not just the teacher that positions students. It is more plausible that positioning is a form of negotiation, such as in socio-mathematic norms. One basic example could be when a teacher presents a rule to be used, and the student wants to know why. This can be seen as the student trying to position the teacher as something other than a teller of methods. Moreover, Drageset (2021), in a study of how explanations are initiated and responded to, has found that teacher interactions are far less dependent on the prior turn than student interactions. This illustrates how teacher typically has other agendas than responding to prior turns, and positioning might be a tool to describe the difference between how teachers and students contribute to classroom talk.

According to both Harré and Langenhove (1999) and Wagner and Herbel-Eisenmann (2010), positioning can be observed through a study of interactions. This means that one way to explore and describe different positions in a classroom is to study what each participant says. A framework that is being developed from a review of literature on classroom discourse in mathematics suggests five broad teacher positions (Drageset, 2021). The first position is a teacher that knows and tells. A teacher that knows and tells takes a position as somebody who knows something and has the authority to decide and evaluate. This position can be further detailed into three ways of telling: telling as initiation (such as teacher as initiator by Lobato et al., 2005), telling what or how to do (such as teacher explanation by Henning et al., 2012), and telling about connections (such as connections by Rowland et al., 2005). The second position is a teacher that supports. A teacher that supports helps students in their work to reach answers and develop their mathematical understanding. Also, this position can be further detailed into three ways of supporting: supporting by reducing the complexity (such as simplification by Drageset, 2014), supporting by assessing (such as confirmation by Henning et al., 2012), and supporting by progressing student thinking (such as probing guidance by Warshauer, 2014). The third position is a teacher that shares and develops student ideas. Such a teacher position focuses on student thinking as the source for discussions and learning. This position is based on a large number of concepts from the literature and can be further detailed into three parts. First, the position is based on the teacher accessing and sharing student thinking (such as eliciting student thinking by Fraivillig et al., 1999). When ideas are accessed and shared, the teacher might point out what is important in different ways (such as clarifying statements by Conner, 2014) or using student thinking as the core of the discourse (such as by encouraging reflection by Cengiz et al., 2011). Such use of student thinking has been described as uptake by Correnti et al. (2015). The fourth position is a teacher that participates. When a teacher participates, the teacher works together with students to find solutions or understand new concepts. This might be done in two ways, either by being a real collaborator in an inquiry where the teacher does not know the answer (such as in a landscape of investigation by Skovsmose, 2001) or by playing a role (such as in teacher in role, by Drageset et al. 2021). The fifth position is a teacher that facilitates. When taking this position, the teacher facilitates the discourse and mathematical work without engaging in the content. This can be further detailed into three ways of teacher facilitating: facilitating by orchestrating the discourse (such as guiding participation and norms by Drageset, 2019), facilitating the development of ideas (such as turn-and-talk by Kazemi et al., 2014), facilitating a focus on peer thinking (such as requesting evaluation by Conner et al., 2014).

Table 1: Five positions and further detailing (first row) and characterization of each (second row)

A teacher that knows and tells…	A teacher that supports…	A teacher that shares and develops student ideas…	A teacher that participates…	A teacher that facilitates…
… as initiation … what to do … about connections	… by reducing complexity … by assessing … by progressing student thinking	… by accessing and sharing … by pointing out … by using student thinking (uptake)	… as a real collaboration … as a teacher in role	… by orchestrating the discourse … by developing ideas … focus on peer thinking

Some teachers may maintain one position most of the time, while other teachers might change position frequently, deliberately or not. It is also probable that the students sometimes are willing to take positions aligned to the teacher positions, and at other times not. If the teacher takes the position as *a teacher that knows and tells*, this only works if the students align themselves by taking a position of *listeners*. Accordingly, a teacher can only facilitate the discourse without engaging in the content if the students are willing to take positions as active participants by asking questions, explain, evaluate, and argue. This negotiation can also be seen as part of developing classroom norms, where different teacher positions are accepted and aligned student positions are accepted.

Method

This article reports from a study of shared data within TWG19 for the CERME12 conference. Different participants of the thematic working group shared five videos from different classrooms. This is done so that the participants can achieve a deeper understanding of each other's analysis and frameworks when not only the results but also the data is shared. The article is also connected to the SUM project that aims to develop teachers' capacity to teach through inquiry-based learning.

We use a framework describing five teacher positions (Drageset, 2021) to explore characteristics and contrasts between the five classrooms, aiming at trying out and further develop the framework. The analysis is based on a turn-by-turn analysis, categorizing all teacher interactions related to the five positions and their further detailing (see table 1). Then we chose to explore the most frequently used position further (a teacher that shares and develops student ideas). We then used the characterizations from table 1 as categories (access and share, pointing out, using student thinking) and categorized all teacher interactions in this position. We discovered a need for a new category through the data analysis, which we named *requesting mere answers*.

The data from each classroom is too limited to say much about the classrooms. However, this type of data from different classrooms is valuable because one can try out how a framework might be useful to characterize different practices and use this for further development of the framework. This result also illustrates different ways teacher positions themselves during mathematical classroom discourse, which may be a foundation for conducting larger studies.

Findings

Table 2: Overview of the frequency of positions taken in each classroom (empty means zero)

	Classroom 1 Drageset 1 (King's birthday)	Classroom 2 Drageset 2 (whiteboards)	Classroom 3 Hoover	Classroom 4 Sakonidis	Classroom 5 Santos
A teacher that knows and tells	7 (16%)	27 (31%)		11 (34%)	21 (22%)
A teacher that supports	10 (22%)	9 (10%)			
A teacher that shares and develops student ideas	17 (38%)	42 (47%)	2 (15%)	18 (56%)	62 (67%)
A teacher that participates					
A teacher that facilitates	11 (24%)	11 (13%)	11 (85%)	3 (9%)	10 (11%)

When looking at table 2, we can see a considerable difference between the classrooms. However, the data from the classrooms vary in length and type of interaction to such a degree that it is not possible to use table 2 as a basis for comparison. But as Table 2 illustrates, the most frequent teacher position in these five classrooms is to *share and develop student ideas*. Since this position is most frequent in total and in four of the five classrooms, we will look more into how the interactions of this position look like and what these can tell us about classroom characteristics.

According to Drageset (2021), the teacher position of *share and develop student ideas* can be characterized by three types of teacher interactions: access and share, pointing out, and using student thinking (uptake). During the analysis, we discovered that these could not characterize all teacher interactions, and therefore, we added a new type: requesting mere answers. With these four types, we were able to categorize all teacher interactions related to this position.

Table 3: Frequency of each type of interaction within the teacher position of *sharing and developing student ideas* (empty means zero)

	Classroom 1 Drageset 1 (King's birthday)	Classroom 2 Drageset 2 (whiteboards)	Classroom 3 Hoover	Classroom 4 Sakonidis	Classroom 5 Santos
Access and share	7 (16%)	28 (31%)	2 (15%)	3 (9%)	12 (13%)
Pointing out	7 (16%)	8 (9%)			1 (11%)

Requesting mere answers		5 (6%)		14 (44%)	40 (43%)
Using student thinking (uptake)	3 (6%)	1 (1%)		1 (3%)	

As seen in Table 3, the first type of interaction, *access and share*, is the only interaction found in all the classrooms. Drageset's second classroom (whiteboards) is the classroom that uses this interaction most frequently, while the data from Hoover's classroom only have two interactions related to this position. Otherwise, we see a common pattern in these classrooms. To access student ideas, they all ask how, what, and why. Examples include "*How did you think?*", *"What do you mean?"* or simply "*why?"*. Typically, when asking what and how the teachers request a chronological explanation of how an answer was reached or what to do to reach an answer. When asking why, they request a reason or argument for why one should use this method or why an answer should be seen as correct. These three types of questions (how, what, why) are about what Fraivillig (1999) labels eliciting student thinking. When doing this in plenary talk, it is also about sharing student ideas with their peers. This can be seen as the foundation of the position of sharing and developing student ideas, as accessing is about sharing, and sharing is needed to develop the student ideas further, particularly to include other students in the development of the ideas.

The second type, *pointing out*, was found in three of the five classrooms. There was little difference in the frequency of the use between the classrooms, but we found two quite different ways of pointing out. Most frequently, the teachers used this position to revoice parts or all of what the students had said. There were two different applications of revoicing, either simply revoicing the student's point or revoicing while adding additional information. One example of simply revoicing is, "*Okay. So you used the fact that you know twenty present is one-fifth.".* The other way of pointing out was when the teachers summarized students' explanations. We saw this most frequently in Drageset's first classroom (King's birthday), where the teacher frequently gave a thorough summary of how a student had solved a given task, based on the student's explanation. One of these summaries is this one:

Teacher: First, she wanted to go to the year 2000 first, and then go from 2000 to the year the King was born. Right? So you started with 80 minus 17, and got 63. She removed 17 from 80. Then you got 63. Then she wanted to find what 2000, from here, and jumping 63 years backward from the year 2000. That's how she thought. So you tried to do 2000 minus 63, but this is a really difficult equation, to do it like this. Because... You haven't really learned this, with such large numbers. Did you manage to finish it, too?

This way of pointing out still focuses on the student's ideas, but the teacher takes an active role in repeating and emphasizing the student's solution, possibly also modeling how to explain.

The third type, *requesting mere answers*, is the most used interaction in this position, but we only found it in three of the five classrooms. There are mainly three ways the teachers requested mere answers. One way whereby asking yes and no questions, such as *"Did you agree that this was the best way?".* Often we recognized these types by how the students answered the given question. As we mentioned earlier, the student's response to the teacher's questions seems to be based on what their experience tells them is an acceptable answer for that type of question, in what looks to be an example of socio-mathematical norms (Yackel & Cobb, 1996). An interesting variant of these yes and no questions were questions that, in reality, were yes-questions (the correct answer was obviously yes) and no-questions (the correct answer was obviously no). A second way of requesting mere answers was by requesting answers to tasks, where the answers typically were a number, such as *"and here, how many do I need for a whole pizza?"*. A third way of requesting mere answers was to ask for the meaning, such as *"What did percentage mean then Ole?"*, which we only saw in Drageset's second classroom (whiteboard). Since we only saw this at the beginning of the lesson, it seemed that the teacher wanted to get the students thinking about the subject they were going to focus on in that lesson.

The fourth type, *using student thinking (uptake)*, was not observed in these classrooms. However, in Hoover's classroom, the teacher did use student thinking in a comparable way but without interfering in the mathematical content. Consequently, these interactions were seen as part of the position called *a teacher that facilitates* and the code *focus on peer thinking*.

Discussion and conclusion

This part will discuss what the findings related to the position of *sharing and developing student ideas* mean for each classroom and use this to characterize and contrast the classrooms. According to Harré and Langenhove (1999), teachers position students continually, so we will also comment on how the teachers' choice of positions might position the students.

Classroom 1: Drageset 1 (King's birthday)

In this classroom, we see that 38% of the teacher's interactions belong to the position of *sharing and developing student ideas*, primarily by *accessing and sharing*, and *pointing out*. This teacher was the one with the most thorough summaries of student's solutions when pointing out. This shows a teacher focused on getting access to and sharing the students' ideas and then gathering any loose ends into a coherent whole. This can position the students as owners of ideas while the teacher maintains the authority to define and model the solutions.

Classroom 2: Drageset 2 (Whiteboard)

In this classroom, 47% of the teacher's interactions belong to the position of *sharing and developing student ideas*, primarily be *accessing and sharing*, where the teacher's goal seems to be to share several different student ideas and methods. This teacher rarely summarizes the student's ideas but instead lets the students' answers stand on their own. This means that the students also here are positioned as owners of ideas, but without the teacher using the authority to point out a more precise or correct solution.

Classroom 3: Hoover

In this classroom, the teacher mainly takes the position of *facilitating*, but only twice the position of *sharing and developing* (see table 2), which is not enough to analyze related to the latter position.

Classroom 4: Sakonidis

56% of this teacher's interactions belong to the position of *sharing and developing student ideas*, primarily by *requesting mere answers* and occasionally *accessing and sharing*. This means that the teacher focuses on sharing answers and quite rarely on sharing solutions. This teacher positions students as task solvers and the teacher continually uses the authority to confirm or reject the answers.

Classroom 5: Santos

In this classroom, 67% of the teacher's interactions belong to the position of *sharing and developing student ideas*, primarily by *requesting mere answers*. However, this teacher also uses *access and share* as well as *pointing out* to a certain extent. This means that the teacher positions the students as task solvers and occasionally as the owner of ideas. At the same time, the teacher continually uses the authority to confirm and reject answers.

Conclusion

As illustrated above, the teacher position of *sharing and developing student ideas* is the most frequent one in this data set. However, while four of the five teaches use this position frequently, there are clear differences. One is most focused on sharing student solution methods and let them be the end product. Another is also focused on sharing solution methods but also refine them through long summaries. Furthermore, two others seem most focused on sharing answers and using their authority to confirm or reject them. This work illustrated how a study of the position of a teac*her that shares and develops student ideas* can be used to characterize and contrast different classroom practices, which indicates that this framework might be used for studies of larger datasets. This work has also revealed that the three original types of teacher interactions suggested by Drageset (2021) were not sufficient to characterize all teacher interactions related to the position of a teac*her that shares and develops student ideas*. By adding one new type, requesting mere answers, we were able to categorize all teacher interactions related to this position.

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Dilemmas of teaching arithmetical notation to young learners

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This study takes the analysis of a Malawian Grade 1 teacher's mediation of mathematics as a starting point for discussing dilemmas that might be entailed in the teaching of arithmetical notation to young learners. Two exemplar episodes are selected from six video-recorded lessons that were analysed using the Mediating Primary Mathematics Framework. The teacher introduced the writing of numbers and mathematical symbols with their corresponding hand movements and used these movements as the criteria for enabling learners to assess the correctness of their written inscriptions. Two inherent dilemmas of this complex work of teaching are identified and discussed.

Keywords: Early years, mediation, gesture, inscriptions, interactions.

Introduction and theoretical background

It has recently been suggested to distinguish between considering teaching as something teachers do and as work to be done. The latter corresponds with Ball's (1993) investigation of dilemmas that arise in mathematics teaching and the more recent conceptualization of the special work of teaching mathematics (Ball, 2017). Our study further explores this distinction. As a case, we use the introduction of elementary mathematical notation in a Malawian Grade 1 classroom.

When learners are introduced to new mathematical concepts and notation during the early years of primary school, they often make errors. The teacher's reaction to a learner's error made during whole-class activities has implications for the individual learner and the whole class (Bass & Mosvold, 2019). For instance, when a learner has made a writing error, the teacher may just compare the wrong inscription or notation made by the learner with the correct one presented on a chart or workbook. However, as observed by Venkat and Askew (2018), young learners may not have yet developed the mental faculties for distinguishing features of seemingly related representations and may require appropriate teacher's mediating talk and gesture to make these features apparent.

Handling learner errors is complex (Sapire et al., 2016). For persistent errors and misconceptions, learners may need to be equipped with strategies for checking the correctness of their work even in the absence of the teacher. One possible strategy is to embody some mathematical concepts and processes that learners often find difficult to remember. The embodiment of mathematical concepts enables learners to view the subject as an activity involving physical actions and gestures (Edwards et al., 2014). This makes the association of mathematical concepts and processes with their corresponding physical actions essential, especially to young and inexperienced learners, who are just being inducted into school mathematics. Eventually, the teacher is supposed to help the learners progress from the embodied physical representations to their corresponding abstract mental structures (Venkat & Askew, 2018). If the teacher sticks to the physical representations, the learners may no longer see the need to look for mental conceptual structures to make some necessary connections and generalisations (Askew, 2019; Wilson, 2002). This implies that teaching can either enhance or constrain what is made available to learn in a lesson—thus constituting numerous dilemmas. In this paper, we share the same understanding of a dilemma as Ball (1993) to mean paradoxical situations where the teacher has several alternative choices to make, each having varied consequences, and oftentimes the decision is required instantly.

Drawing on these insights from research on early mediation of mathematics, and on the perspective of considering the teaching of mathematics as work to be done, we ask:

What dilemmas can be entailed in the work of teaching mathematical notation to young learners?

To answer this question, we first apply the Mediating Primary Mathematics (MPM) framework by Venkat and Askew (2018) in the analysis of data to enable careful description of the mediating work that is performed by the teacher, before we discuss dilemmas entailed in this work. Like Bass and Mosvold (2019), we focus only on a small slice of the work of teaching mathematics here, namely what may be involved in attending to learners' errors.

Analytic framework

The MPM framework is guided by Vygotsky's sociocultural view of teaching as a set of mediated transactions, with the teacher as the main mediating agent in the classroom—who works with a set of sociocultural tools of mediation (Kozulin, 2003). We adopted the MPM framework to understand the sociocultural tools for mediating mathematics in the early years of primary school. The MPM framework identifies four strands or means of mediating primary mathematics, namely: Tasks and examples, artefacts, inscriptions, talk and gesture. The framework further subdivides talk and gesture into three sub-strands: Talk and gesture for generating solutions to problems, talk and gesture for making mathematical connections, as well as talk and gesture for building learning connections. Even though the teacher works with all the four means of mediation when teaching, we were particularly interested in the teacher's mediating talk and gesture for building learning connections (see Table 1). As shown in Table 1, teachers' mediating talk for building learning connections is manifested when handling errors from learners' offers. Venkat and Askew (2018) observed that learners' errors provide a context for richer mediation as teachers are prompted to make "responsive moves" to address the errors. In some cases, the learners' errors provide an ideal moment (or "teachable moment") where they could be more receptive to the teacher's remedial actions than if the error was just ignored or the teacher's response was deferred to a later time (Muir, 2008).

Research design and methodology

A qualitative case study design was adopted, which enabled an in-depth inquiry into the complex task of teaching mathematics to young children. Our case is a Grade 1 teacher with an overall teaching experience of seven and half years after graduating from a two-year teacher training programme. The teacher had been teaching mathematics to different cohorts of Grade 1 learners for four consecutive years. The teacher was selected as a paradigmatic case (Flyvbjerg, 2006), exemplifying outstanding learner achievement in resource-limited settings. The school consistently outperformed other primary schools in the same geographical area both during the standardised end of primary school examinations as well as during quiz competitions with nearby schools. The school was based in a remote village where learners mostly relied on the teacher as the sole source of mathematical instruction, as learners had limited access to extra tuition through books, parents and relatives, or educational television programs. The rural setting increased the possibility of attributing the school's exemplary learner achievement to the classroom practices of its teachers.

Data collection was scheduled for the week when the teacher was introducing the addition of whole numbers to the Grade 1 learners. This was done across six lessons that were observed and video recorded. Unstructured interviews were conducted at the end of each lesson to seek clarification on some observations made in the classroom. An in-depth video-stimulated recall interview was conducted with the teacher after a preliminary analysis of the lesson transcripts. Interview data were analysed thematically. Themes were centred around the reasons for the teacher's choices made in the use of the means of mediation observed during the lesson. Analysis of the affordances that were made possible through the teacher's presentation of the chalkboard inscriptions was done through the application of variation theory (Kullberg et al., 2017), which is one of the theoretical foundations of the MPM framework.

The recorded videos were segmented into instructional episodes that are considered as a unit of analysis in the MPM framework. Even though this paper reports on the observations that were made while the teacher was working with written inscriptions, the analytical focus was on the teacher's mediating talk and gesture for building learning connections that accompanied the inscriptions in each episode. We referred to the indicators for mediating talk and gesture for building learning connections provided by the MPM framework, as shown in Table 1 that follows:

Table 1: Mediating talk and gesture for building learning connections. Adapted from Venkat and Askew (2018, p. 90)

Indicators for mediating talk and gesture for building learning connections	Level
Pull back to naïve methods OR No evaluation of offers (correct or incorrect).	0
Accepts/evaluates offers Accepts learner strategies or offers a strategy OR Notes or questions incorrect offer.	1
Advances or verifies offers. Builds on, acknowledges or offers a more sophisticated strategy OR Addresses errors/misconceptions through some elaboration e.g., "can it be….?" Would this be correct, or this? Non-example offers.	2
Advances and explains offers. Explains strategic choices for efficiency moves OR provides rationales in response to learner offers related to common misconceptions OR Provides rationale in anticipation of a common misconception.	3

As indicated in Table 1, analysis of the teacher's mediating talk for building learning connections examines the extent to which a teacher handles the evaluation of learners' offers in a lesson episode. A learner's offer could be a correct or an incorrect response to the teacher's question or a strategy for solving a problem. In some cases, the teacher may verify an offer or build on it in response to "teachable moments" (Muir, 2008). Ultimately, the teacher may make a "responsive move" (Venkat

& Askew, 2018) or an “asset oriented response” (Bass & Mosvold, 2019) in the form of explaining strategic choices made or providing the rationale for each option while taking into account the common misconceptions.

Findings and discussion

During the study, the teacher was introducing the addition of two whole numbers with a sum not exceeding 5. By this time, the learners had been in Grade 1 for ten weeks, and the teacher had taught them how to write the numbers 0 to 5. In the Malawi context, about 60 per cent of learners do not attend pre-school education before starting Grade 1 (Robertson et al., 2017), hence the preceding ten weeks were their first school experience in life. During classwork, the teacher asked learners to write the worked-out solutions on pre-written papers and the chalkboard, thereby opening up more opportunities for making “responsive moves” (Venkat & Askew, 2018) to the “teachable moments” (Muir, 2008) made possible through the writing errors made by the learners.

Verbalisation of hand movements when working with the plus sign

The teacher introduced the writing of the plus (+) sign during the first lesson of the study by demonstrating how to write the sign in the air while verbalising the hand movements as “Dot! Down! Cut-in-the-middle!” She then asked the learners to do the same:

Teacher: Aa-aah! We have not yet started writing! Just raise your hand and get ready to write *[inaudible]*, alright? Everybody use your right hand! Begin!
Class and Teacher: *[Verbalise the movement of the hand while tracing the + sign in the air]* Dot! Down! Cut-in-the-middle!
Teacher: Again!
Class: *[Verbalise the movement of the hand while tracing the + sign in the air]* Dot! Down! Cut-in-the-middle!
Teacher: Again!
Class: *[Verbalise the movement of the hand while tracing the + sign in the air]* Dot! Down! Cut-in-the-middle!

After teaching the hand-movement for the plus sign, the teacher asked learners to suggest the hand movement for the equal sign. The hand movements verbalised by the teacher acted as the basis for the justification of the correctness or incorrectness of the notations made by the learners during the subsequent lessons. For instance, when reviewing how to write the plus sign in the introduction of the fourth lesson in the study, one learner wrote a plus sign that was not accepted by the class (see part (a) of Figure 1). The learner required a convincing explanation on why the written sign was rejected by the class. The teacher reasoned with the learner the original hand movement that was verbalised when the plus sign was introduced for the first time; that is, “Dot! Down! Cut-in-the-middle!” To remediate the error, the teacher asked a second learner to write the plus sign, but this time making sure that the downward stroke of the line making the + sign is cut in the middle by the horizontal stroke. The explanation of the hand movement clarified the contrasting feature of the sign offered by the first learner and the expected sign. After a second learner had written the correct sign, the teacher asked a third learner to re-write the correct notation for the plus sign (Figure 1, part b).

 (a) Rejected + sign.	 (b) Two accepted + signs.

Figure 1: Correction of an incorrectly written plus sign

The teacher used the verbalised hand movements as the rationale for either rejecting or justifying the offers provided by the three learners. We coded this as the highest level of the MPM framework's mediating talk and gesture for building learning connections (see Table 1) – where a teacher "provides rationales in response to learner offers related to common misconceptions."

The teacher's approach to verbalize hand movements when introducing new inscriptions was not only done for the plus sign. In the interview excerpt that follows, the teacher explained that even when teaching her learners to write numerals before the observed lessons, the same strategy of verbalising hand movements was used:

214	Teacher:	How to write? We have several ways. Aah, first, we start to write in the air.
215	Researcher:	Okay?
216	Teacher:	If you had come when I was teaching numbers you could see that. Because when we say: "Let's write four!" We say: "Dot! Then down! Then right! Then…." Those things. We first start in the air, then after in the air, it's when we go on the ground, before they write in the exercise book.

In the interview excerpt above, the teacher referred to an example of how she introduced the writing of the number 4 based on learners' aptitude. We notice towards the end of Utterance 216 that the teacher must decide when it is appropriate to present the notation as verbal, gestural, written, or any combination of these forms, without losing the mathematical meaning of the notation―thus constituting a dilemma to the teacher.

Using verbalised hand movements when remediating errors related to the writing of 4

The teacher demonstrated the third level of the MPM framework's mediating talk and gesture for building learning connections (Table 1) when responding to learners' errors related to the writing of 4 during the fifth observed lesson. The teacher gave strategic explanations targeting the main source of the observed errors. In that lesson, learners were given pre-written problems on pieces of paper and were asked to find the sum in their groups. The worked-out solutions were then pasted on the chalkboard. One group wrote the answer on their paper as shown in Figure 2.

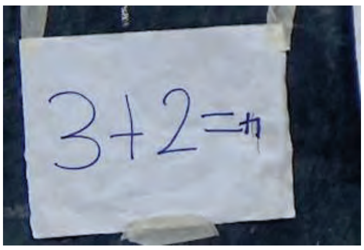

Figure 2: A wrong answer that was written as flipped 4

The representative of the group that was assigned the task shown in Figure 2 read the statement as "three plus two answer four". Rather than quickly dismissing the answer by looking at the expected sum of the given addends 3 and 2, the teacher approached the remediation in phases. Firstly, the teacher asked the class if 4 was supposed to be written as shown in Figure 2, and the class was divided. This posed another dilemma to the teacher—whether to quickly dismiss the incorrectly written numeral and work out the correct sum with the class or consider this as the right moment for remediating the inscription error first, even though the lesson's focus was on the addition of numbers.

The teacher started with remediating the writing error in Figure 2 by asking another learner to write 4 on the chalkboard (see part (a) of Figure 3) and asked the class if the 4 was written correctly. Instead of applying common logic or sentimentality, the teacher reminded the class of the verbalised hand movements for writing 4 ("Dot! Down! Turn-right! Cut-in-the-middle!") while moving a pointing stick. Thus, the teacher used the verbalised hand movement as the basis for justifying the correctness of the 4 offered by the group. The teacher repeated the verbalisation of the hand movement while writing another 4 above the one written by the learner as shown in part (b) of Figure 3.

		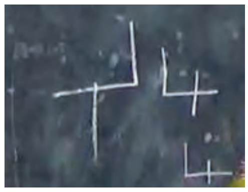
(a) A correct 4 written by a learner.	(b) Another correct 4 written by the teacher above the one written in part (a) by a learner.	(c) An incorrect 4 written by the teacher alongside the two correct 4s in (b).

Figure 3: Remediating errors related to the writing of 4 using similarity and contrast

The teacher employed similarity to emphasize the correct way of writing 4 (Figure 3, b), and she continued the discussion by providing a plausible contrast of an incorrectly written 4 alongside the two correctly written 4s (Figure 3, c). The teacher wrote the wrong "4" while simultaneously verbalising its hand movement as: "Dot! Down! Turn-left! Cut-in-the-middle". Rather than providing the contrast by rewriting the disoriented 4 shown in Figure 2, the teacher probably noted from the learners' inscriptions that the common error was the horizontal direction of the hand. After remediating the inscription error of 4 in Figure 2, the teacher then prompted the class to check if 4 was the correct answer to 3 + 2 as initially proposed by the assigned group. After working out the expected sum for 3 + 2, the next group of learners had been assigned to find 1 + 3 and wrote their answer (Figure 4, a).

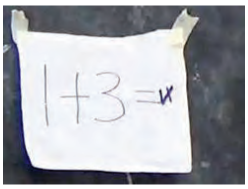	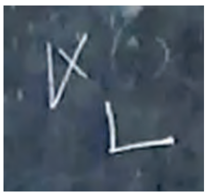
(a) Distorted 4 written by learners.	(b) Teacher's emulation of the error.

Figure 4: A distorted 4 written by one group and emulated by the teacher

The teacher used her technique of verbalising hand movements to check if the 4 shown in Figure 4 was correctly written. The teacher expressed the hand movement "Dot! Down! Go-up! Cut-in-the-middle!" while simultaneously writing the movements on the chalkboard (Figure 4, b). Next, the teacher isolated the feature that made the just written 4 incorrect, that is, the expected angular turn in the acceptable hand movement. This was verbalised by the teacher with an emphasis on the turn as "Dot! Down! Turn-right!" while simultaneously writing the hand movements on the chalkboard.

Concluding discussion

The findings indicate how the teacher worked with the MPM framework's mediating talk and gesture for building learning connections (shown in Table 1) related to the writing of arithmetic notations. After receiving an offer from the learners, the teacher first checked with the whole class whether the offer was correct or not. When the error was not apparent, the class was not sure if the offered notation was correct. By providing the rationales (verbalised hand movements) for justifying the learners' offers, the teacher achieved the highest level of the MPM framework's mediating talk and gesture for building learning connections (Venkat & Askew, 2018). Whereas the findings highlight remediation of notations written on the chalkboard, the teacher explained in an interview that the remediation starts with writing in the air, followed by writing on the ground, before using notebooks.

Analysis of data from this Malawian classroom illustrates two common dilemmas entailed in the work of teaching early mathematics. The first dilemma relates to deciding on how to establish correct mathematical notation in a way that is suitable for the learners' age and development (Ball, 1993), while at the same time maintaining mathematical integrity. In the episode analysed, the teacher used similarity and contrast (cf. Kullberg et al., 2017) to help learners identify key characteristics of correct mathematical notation, and she also used verbalised hand movement. Still, there is a risk of confusing learners about the underlying mathematical idea in the process. Instead of just telling learners if the offered inscriptions were correct, the teacher in this study attempted to justify the acceptance or rejections. This may provide the learners with an opportunity to learn about the importance of justification in mathematics. However, young learners may lack the necessary understanding on which the teacher can base justifications for actions taken during lessons (Venkat & Askew, 2018), and this provides a risk that the teacher must attend to. In this episode, the teacher used gestures to justify the correctness of the written arithmetic notations. This use of bodily based resources such as hands and fingers can make the learners feel competent to work out mathematical tasks anywhere anytime (Wilson, 2002).

A second dilemma relates to identifying and interpreting student errors on the fly and deciding on what errors to attend to first when several errors are present (Muir, 2008). The reasoning that is required for probing learners' errors on the fly tends to be one of the highest and complex forms of teacher knowledge (Sapire et al., 2016). In the second episode of the lesson, a learner presented an incorrectly written 4 as the sum of 3 + 2. The teacher then had to decide on whether to attend to the error, or to use other pedagogic moves that do not attend directly to the errors—like assigning competence to learners and positioning them as contributors (Bass & Mosvold, 2019). Deciding on whether a situation constitutes a teachable moment (Muir, 2008), and deciding on how to act in ways that provide opportunities for learning, constitutes a common dilemma for mathematics teachers.

The MPM framework provides a useful lens to describe observations in mathematics classrooms, and to evaluate the level of teacher mediation, and it provides a useful language to describe what teachers do. This can be useful to some point, and it provides a simplification of teaching that can be useful to teachers as well as researchers. However, shifting attention toward the work of teaching opens the way to understanding what is actually involved in carrying out the complex, dynamic, and situated work of teaching (Ball, 1993; Ball, 2017). This does not simplify the picture, and it does not provide immediate solutions for how to act, but it approves of the real nature of the complex work of teaching mathematics. The dilemmas of this special work of teaching cannot be easily solved, and their management requires professional knowledge and judgment.

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Mathematical knowledge for teaching: challenges and potential in the case of geometrical patterns

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We report on an observation of a lesson where students were working on geometrical pattern tasks in pairs. The teacher's intentions for the paired setting and his reflections upon seeing a video of the lesson are presented, and through the lens of a TRU analysis and the Knowledge Quartet, we identify potential for exploiting the task and the environment to afford students' mathematical engagement. We focus on the interplay between the Contingency and Foundation dimensions in the Knowledge Quartet, and the potential of developing retrospective awareness into in-the-moment awareness.

Keywords: Paired learning, geometrical patterns, knowledge quartet, awareness, teacher insight.

Introduction and theoretical background

The new mathematics curriculum in Norway (LK20) has a specified learning goal that 9th grade students should be able to explore in the sense of describing, explaining, and presenting the structure and development of geometrical patterns and number patterns. Although geometrical patterns were not outside the remit of the previous curriculum, they were not mentioned specifically and the requirement to teach them is new. Within mathematics education literature, geometrical patterns have been promoted as a rich entry point to algebraic and functional thinking (Bednarz et al., 1996; Carraher et al., 2008). However, working with geometrical patterns in this way presents both possibilities and challenges for students and teachers (Orton & Orton, 1999). Moss and McNab (2011) argue that the challenges are due to pedagogical choices and not due to an inherent difficulty in subject material. Thus, mathematical knowledge for and in teaching plays a central role. In this article, we will study one teacher, here called Erik, while working with geometrical patterns as a means for developing students' ability to express relationships algebraically in a 9th grade class. Specifically, we will address the research questions:

> What demands were placed on the teacher's mathematical knowledge for teaching in exploiting the task and the environment to afford students' engagement with mathematics?

We explore how the teacher's intentions in carrying out exploratory activities were enacted in a setting where working in pairs was supposed to encourage collaboration. In that respect, we pay attention to how the potential in paired learning and the mathematical potential of the task were exploited and what demands it put on the teacher's mathematical knowledge for teaching, MKT. Several frameworks for investigating MKT have been developed, and much research on the theme has been reported (Skott et al., 2018). The Knowledge Quartet, KQ, (Rowland et al., 2005) is based on the work of Shulman and his categories of knowledge. KQ was developed through a grounded approach to video data focusing on how teachers' mathematical knowledge surfaced in mathematics lessons. 18 codes from the categorisation were grouped into 4 broad dimensions; Foundation (teacher's mathematical knowledge possessed), Transformation (teacher's capacity to transform his/her foundational knowledge to accessible knowledge for others), Connection (connections between concepts which bind different parts of the mathematics together), and Contingency.

Contingency is informed by the three other dimensions, and concerns situations in the classroom which are not planned for. In later studies, this dimension has been further explored. A new code, “Teacher’s Insight” which is demonstrated when a teacher is aware that students construct mathematical ideas and something which sounds “half baked”, was included in the Contingency dimension (Rowland, 2012). Rowland and Zazkis (2013) related the contingency dimension to Mason’s (1998) description of “knowing to act in the moment” (p.139). This in-the-moment-pedagogy is “the teacher’s capacity to engage flexibly and productively with their students” (Mason & Davis, 2013, p. 184). Knowing to act-in-the-moment, puts a demand on the teacher’s mathematical knowledge as well as on how he or she is aware of this knowledge and how it is used and exemplified. “A teacher who is aware [...] is in a position to direct student attention to what really matters” (ibid, p. 189). Contrary to teacher’s insight or awareness is what is known as funnelling of which the effects are described by Wood (1998): “students’ thinking is focused on trying to figure out the response the teacher wants instead of thinking mathematically himself” (p.172). A challenge for teachers will often be to balance between funnelling and scaffolding. A pitfall may be “fostering dependency on teachers and cap opportunities for more independent learning” (Mazenod et al., 2019, p. 2).

Methodology

We observed five mathematics lessons with one teacher (Erik) in two different mixed ability ninth grade classes. Here we report from one of these lessons. The students were seated in pairs and worked on growth pattern tasks. Prior to our first classroom observation, we had a semi-structured pre-interview with the teacher. We asked him to describe a typical mathematics lesson, his intentions in teaching mathematics and how he dealt with mixed abilities, adapted education, and grouping. Additionally, we interviewed some of the students in the classes. We also conducted a semi-structured post-interview with Erik one month after having observed in his classes. The interview included a session in which we showed him several episodes from the five observed lessons, so he could share his reflections with us. The lessons were videotaped, and the teacher wore a microphone. The camera followed the teacher as he moved from one pair to another. After the observations, we studied the videos and selected episodes which were transcribed and analysed.

During the classroom observation, we used the Teaching for Robust Understanding (TRU) Framework (Schoenfeld & Floden, 2014) as an observation scheme. Schoenfeld (2016) emphasizes that there is “no one ‘right way’ to teach. The key idea is that TRU specifies the attributes of learning environments in which students flourish” (p.2). In advance the researchers had prepared for the application of the framework to ensure reliability. The Framework offers ways to reflect along five dimensions: 1-The richness of the mathematical content (How accurate, coherent, and well justified is the mathematical content?). 2-Cognitive demand (To what extent are students supported in grappling with and making sense of mathematical concepts?). 3-Access to mathematics (To what extent are all students supported in meaningful participation in (group) discussions?). 4-Agency, Ownership, and Identity (To what extent did teacher support and/or group dynamics provide access to "voice" for students?) and 5-Uses of Assessment (To what extent does the teacher monitor and help students refine their thinking within small groups?). The Framework has different rubrics: Whole class activities; Small Group Work, Student Presentation, and Individual Work. Since students worked in pairs, we used the rubric for Small Group Work. TRU assigns scores of 1, 2, and 3 for each dimension. Although we assigned approximate scores when observing, we find it more appropriate to report the analysis with aspects of the content within the levels rather than scores, and we relate to the dimensions of KQ with a focus on teacher's insight and awareness in the Contingency dimension.

Findings and discussion

In the pre-interview, Erik explained his thinking in choosing the tasks that students were working on. We see that his intentions are inclusion based on differentiation:

> To include everyone, even the ones who aren't so good, I try to find self-differentiating tasks, that have a low entry threshold and that one can do a lot with them [e.g.] a figure that is growing.

In the post-interview, we asked Erik if he thought it made sense for the students to sit in pairs in mathematics lessons. He responded that he intended for them to problem-solve together:

> After all, maths is a language... When they are going to solve a problem, they should talk together and come up with ideas on how to solve it. And then the other partner comes up with "maybe that way" and that they …[in] a social way ... find some solution together.

Commenting in the post-interview he talked about the need for student agency:

> [There are] many ways of seeing the figure. Then they make a formula that looks different at first, then they simplify it and then they get the same formula in the end. [You] can solve the problem with different approaches. What do they want to be left with? Well … their agency that should be their own. [It] shouldn't be that the subject has its agency and on the other side the students have their identity and their agency and [it] should be … that they experience that they have a sense of determination over the subject…The subject should not define everything for them. They should also define what it means to have flexible solution strategies then, I think. For they should have their own strategies at the end of the day.

Erik used learning pairs actively in all his lessons. At the start of the lesson Erik encouraged the students to "help each other". Based on what the students said in interviews, this seemed to be an established way of working: "if I cannot complete an exercise, I ask my partner for help". This tells us that Erik's intention that students should collaborate in pairs was exploited by the students.

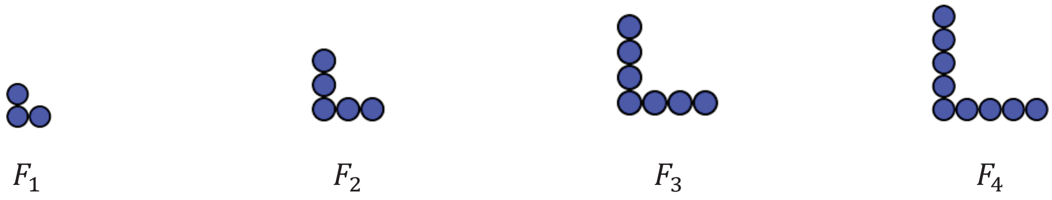

Figure 1: Task: "Find a formula for the n-th figure"

In the following we explore to what extent Erik realised his aims. The episode below was typical of the interactions that we observed between Erik and different pairs. Here Finn, who was working at a desk with another boy, had his hand up. Erik had just finished helping another pair and went over to Finn. The task they were working on is shown in figure 1.

Finn: Is the exercise there … is it … F_n plus … F_n plus two? Or?

Erik: erm ... n[o] .. [He takes the pen from the student] Add the bit that comes up here [He draws a dot to represent the figure F_0 on the task sheet.] This one will always be there. Then it's the arms that grow.

Finn: Yeah

Erik: It's always there. Plus one is always there. Right. That one, if you go back. F_0, I mean, then you will get rid of those two, then you will just be left with … with one, right. [He holds up one finger to Finn.]

Finn: Yeah

Erik: So there is plus one in each one there.

Finn: Hmm

Erik: Here is F_1. What do you times one with to get two? What do you times two with to get four? Three to get six? Four to get eight? What do you times with there?

Finn: Four to get eight[?]

Erik: Yeah

Finn: Two

Erik: Two yeah. [Holds up two fingers.] Then it's always plus one and then it's n times by…[Knocks the table several times with his finger and then holds up two fingers.]

Finn: Two

Erik: $2n + 1$

Finn: Yeah … yeah … $2n + 1$ [?]

Erik: Yeah. Because there you have … this one, if you look at this here, you will always have it … that there, right … that will always be there

Finn: Hmm

Erik: It becomes plus one.

Finn: Yeah

Erik: When you see… Here is F_1, then it is times by two. Here is F_2. Then it is times by … here there are four. Here there are six. Here there are…?

Finn: Four

Erik: Eight

Finn: Ah [Sounds resigned and disappointed]

Erik: It's $2n$. Two times by one is two, plus one is three. Two times by two is four, plus one is five.

Finn: Hmm

Erik: Here there are eight arms [points to the next task on the sheet]. You may see it better there because it is a bit better version. [He stands up and walks away.]

Finn: Yeah

Initially there was discrepancy between the student's question and the teacher's response. In the task, the number of dots per figure were increasing by two. The student therefore asked: "is it $F_n + 2$?". The student was trying to express the solution as a recurrence relation and was grappling with "half baked" mathematical ideas. Here was a potential for the teacher together with the students to explore the relationship and difference between recurrence relations and closed formulas for geometrical patterns. As reported in the research literature, this is a challenge both for students and teachers (Moss & McNab, 2011) which leads to mathematical and pedagogical demands. The Foundation dimension of the KQ, as well as teacher's insight and awareness are of crucial importance in order to exploit this potential. Finn's idea was not followed up by the teacher, who guided him to a full solution to the task based on the closed formula. During this episode Erik addressed only the student who asked for help while the other student paid attention without contributing nor being invited to contribute to the conversation. The potential in paired seating was not exploited. This differs from Erik's intention about the students working in pairs to find a solution together.

In terms of the mathematical content offered in this classroom environment - the first dimension of the TRU framework (Schoenfeld & Floden, 2014) - the teacher's focus was on the answer and how to get there. The question "What do you times one with to get two?", is an example of funnelling (Wood, 1998). The task in hand has been reduced to a simple question about multiplication with answer "two". In the TRU framework, this would indicate a low score on cognitive demand as teacher intervention constrained students to activities such as applying straightforward or memorized procedures, and was explaining how he would solve the task (Schoenfeld & Floden, 2014). It appears that Finn did not see the relevance of the answer "two", since when Erik tried to get him to relate it to the closed formula, he needed to hold up two fingers as a prompt. Later Finn answered wrongly ("four") on a related line of questioning, another indication that he was being funnelled into giving correct responses. As in Wood's (1998) description of the effects of funnelling, Finn appeared to be focused on answering the teacher and abandoned his own recurrence-relation based ideas.

In the conversation Erik asked closed-ended questions, and Finn's contributions after the initial question were just single word responses. Thus, the two boys were not given the opportunity to discuss, explain or reflect on their mathematical ideas, processes that would indicate a fostering of the students' sense of agency. Much of the episode had the feel of a teacher monologue because Finn interjected the "yeah" responses rather than them being invited by pauses from Erik. The monologue with closed questions indicated the teacher's ownership of the mathematics.

In the post interview the teacher realised that he had been too eager to give the solution to the task, remarking on his impatience both before and after seeing himself in the video. In response to the video of the episode, Erik said:

> I remember this one here, yes, the problem here is that he really struggled to understand - to crack the code here [What we] see immediately here [in the video] is that here I probably explain a bit more than…am probably, in a way, a little too quick to give him an answer or solution to the problem. I think. That he himself should have had a little longer to ponder the problem. I see that now, yeah. I should have done that. Eh, but it is clear that, on the other hand, if I help him with one task, then maybe he understands; can maybe move on to the next task and understand it …

We see that Erik's focus is on helping the student. However, he expressed an insight about the student's mathematical capacity - that he should have given him some more time to ponder himself. We consider this expressed insight and awareness as a potential for the teacher to develop his in-the-moment pedagogy so he can act flexibly and productively with his students so he instead of funnelling can direct students' attention to what really matters. However, in the actual moment of this lesson, Erik did not seem to try to analyse and understand Finn's thinking, what he has said and why he said it. In the post interview, we suggested to Erik that when Finn said "F_n plus two" he was trying to express how the number of dots increases, and we asked Erik how he might get from there to the closed formula. He said:

> He was thinking of this [the recurrence relation] I would assume. Because we spent a lot of time last year finding patterns in how it grows. We have been working on that. Then clearly to go from there and find that it is $2n + 1$. It is clear that we have not [done that]… We have worked a lot

> with the graphical representation. Because we should have had a … a type of square number variant then.

Here he went on talking about certain shapes (square numbers, rectangular numbers, and triangles numbers) that he had given the closed formula for on a hand-out sheet. In Norwegian, use of “clearly” in the sentence “Then clearly to go from there and find that it is $2n + 1$” is expressing that it is clearly difficult to do. We see that Erik struggled to suggest a way to support this transition. This quote indicates the heart of the matter: Erik struggled to understand the students’ utterances and he had to rely on his own solution and understanding due to not being able to transform the connection between the different parts of mathematics to something accessible for students.

The fourth dimension of the TRU framework considers the “extent to which students have opportunities to conjecture, explain, make mathematical arguments, and build on one another’s ideas, in ways that contribute to students’ development of agency, authority, and identities as doers of mathematics” (Shoenfeld & Floden, 2014, p.1). In the episode above the environment offered the chance for Finn to make a conjecture “F_n plus two”. Finn’s subsequent contributions were constrained by the teacher’s questions. The episode above was picked out because it was typical of the interaction between Erik and the students. Thus, we argue that there was little opportunity for developing a sense of agency, authority, and an identity as a doer of mathematics in this classroom environment.

There was one interaction between Erik and another pair of students (Anna and Berit) which was atypical due to the students explicitly saying that they wanted to retain authority over their work. As Erik approached the desk Berit said “No, we don’t want any help from you” and waved her hand as if to indicate that Erik should go. Erik persisted to ask questions about Anna and Berit’s progress, but, before they would show anything, Anna demanded that Erik promise not to “give any hints at all”. Anna underlined this point by saying it three times. Apart from confirming how the classroom environment constrained the students’ ownership of the mathematics, the interaction between Anna, Berit and Erik indicates potential within this environment. We noted earlier that the mathematical authority was retained by the teacher in his interaction with Finn. In the interaction with Anna and Berit, we see a willingness from Erik to give that authority to the students, and a relaxed classroom environment that allows the students to challenge established patterns of social interaction.

The ensuing conversation with Anna and Berit returns to a pattern that resembles the other interactions with the students. Erik asked Anna about her solutions to the questions on the worksheet, and sometimes told her the solution before she had time to answer. Erik commented on one of the expressions that Anna had found by saying “No, this one I would multiply with this, to make it look prettier”. Anna objected but eventually said: “Ah, but OK” in a resigned way. Schoenfeld’s introduction to TRU refers to Engle when defining agency:

> Learners have intellectual agency when they … share what they actually think about the problem in focus rather than feeling the need to come up with a response that they may or may not believe in, but that matches what some other authority like a teacher or textbook would say is correct. (Engle, 2011, in Schoenfeld (2016), p.9).

In the exchange between Anna and Erik, the response from Erik constrained Anna to writing expressions that conformed to his idea of pretty.

Discussion and Conclusions

We would like to highlight that there are many positive aspects that we observed in this classroom: the teacher's intentions were for the students to collaborate and explore mathematics; he was using low threshold high ceiling tasks to promote inclusion and collaborative learning; there was a safe working environment as demonstrated by the girls' frank comments to the teacher; and the teacher was aware of the ideas of agency and ownership in the context of mathematics. The analysis of the mathematics revealed that students were directed to the answers and their form, not included as a pair, and the mathematics was reduced to a step-by-step procedure. Thus, the mathematics being offered by the interactions with the teacher is only limited in nature and scope. From the pre interview, this seemed not to be his intention. However, when reflecting on this episode in the post interview, rather than having shown in-the-moment awareness, which is an important aspect of the contingency dimension of the KQ (Mason & Davis, 2013; Rowland & Zazkis, 2013) he displayed a retrospective awareness. It is unclear if the teacher was aware that the way he spoke mathematics with the students reduced their access to it. Rather than the potential of students' half-baked ideas being exploited, learning was capped by over-nurturing, fostering teacher dependency. In particular, there was little awareness of and planning to address the difficulties of the transition from recursive relations to closed formula. For teachers to understand students' starting points, and how these can be built on to develop genuine understanding and agency, this study has demonstrated the importance of developing awareness of the interplay between the Foundation and Contingency dimensions of the KQ, especially the aspects of teacher insight and awareness in the Contingency dimension. There is potential here in the teacher's retrospective awareness, and further research could look at how retrospective awareness can develop to an in the moment awareness.

Furthermore, we saw that it is not sufficient to place students in pairs to promote collaboration. There is a possible mismatch between the teacher's intention (problem solving together) and the teacher's instruction to the students (to help each other) - the latter does not necessarily imply working on a joint product. The teacher's intention also contrasts with the manner of the help (a conversation with an individual not a pair) and the nature (demonstration through funnelling). Exploiting the environment with paired learning thus places demand on teachers' skills to initiate a mathematically rich discussion. We have identified the need for Teacher Insight to be able to build on the students' utterances. Thus, the interplay between the Contingency and Foundation dimensions of the teacher's MKT plays a crucial role here. Initiating a mathematically rich discussion in paired learning activities may place other demands on a teacher's MKT than orchestrating a whole class discussion does. We see this as an avenue for further research. In our experience, paired learning has become widespread in Norwegian schools in the last decade and, thus, the demands of paired learning for teacher knowledge have general implications beyond the teacher in the present study.

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Appropriation: The role of progressing and focusing actions

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In a lesson in a challenging 5th grade classroom environment, we observed interactions between a teacher and a student that displayed characteristics of an appropriation process. We analysed the conversations focusing on the teacher's contributions that build on the student's ideas using a framework to identify redirecting, progressing, and focusing actions. A pattern emerged from the analysis: focusing actions dominate initially with a later shift to progressing actions. The implications of this pattern, and also how the teacher introduced the progressing actions, are discussed with regard to the fostering of the student's appropriation process.

Keywords: Appropriation, classroom observation, conversation analysis.

Introduction

The way teachers and students interact with each other in the social context of the mathematics classroom is of central importance to student learning (Franke et al., 2007). In every classroom, no matter how challenging such a demand might be, there needs to be room for students to engage in sense making and productive struggle (Schoenfeld, 2019). Maintaining this room creates a dilemma for the teacher: How to help without "robbing the student of initiative" (Schoenfeld, 2019, p. 367)? Drageset (2014) provides a framework for analysing teacher responses but points out that "in order to understand classroom communication, it is necessary to study both a single [teacher] question…and the larger picture" (p. 288). To this end, this paper studies all the interactions between a 5th grade mathematics teacher (Joe) and one of his students (Mira) during a lesson where they engage in sense making. With the intention of setting the scene, we start by providing a short narration from the beginning of the lesson:

While most students have found their seats, some keep walking around. It is not quiet: Some throw rubbers in the air – there are constant movements of arms and legs. In this tumult, Joe starts the lesson by asking his students to write down different calculations that give 36 as an answer. Then he begins a whole class conversation, trying to tune them in on the connection between multiplication and division. During this 10-minute whole class conversation, Mira leaves her desk five times. One of her many 'errands' is changing a bin liner, causing her to leave her desk for three minutes.

Despite this start, the lesson ended with Mira presenting a detailed solution to a fairly complicated division on the blackboard. By viewing Joe and Mira's interactions through the lens of Drageset (2014), we search to understand a teacher's role in how a student can *appropriate mathematics*.

Theoretical framework

When studying how a student appropriates aspects of division and its close connection to multiplication, we draw on the work of Moschkovich (2004). As most research on different forms of appropriation, Moschkovich (2004) takes the work of both Newman et al. (1989), Rogoff (1990) and Radford (2001) as her point of departure, and uses it as a foundation when elaborating on the notion of appropriation. Appropriation is a central Neo-Vygotskian concept that has been used to describe how learning in students is mediated by interaction with others and how students learn through the teaching and guidance of a teacher (Newman et al., 1989). It involves joint productive activity, a shared focus of attention, and shared meaning (Rogoff, 1990). Additionally, Newman et al. (1989) focused on the situation of an expert helping a novice, where the expert provided alternative interpretations of the novice's actions. Grounded in these rudimentary insights, Moschkovich (2004) set forward two aspects of appropriation: what the learners appropriate, and how learners actively transform what they appropriate (pp. 49–50). This involves "taking what someone else produces during joint activity for one's own use in subsequent productive activity while using new meanings for words, new perspective, and new goals and action" (Moschkovich, 2004, p. 51).

By drawing particularly on Newman et al. (1989) who used appropriation to describe in detail how interaction with an adult can affect cognitive change in children, we focus on the teacher's role in an appropriation process where both aspects set forward by Moschkovich (2004) are under investigation. That is, we seek to see the teacher's role in what Mira appropriates, and in how she actively transforms what she appropriates. Drageset (2014) reminds us that "an appropriation process often includes actions that in isolation can be labelled as funnelling, teacher-dominated communication, or IRE [where the teacher does the main work], but as part of the appropriation process these actions might be both beneficial and necessary" (p. 288). While acknowledging the many possible theoretical lenses that can shed light on classroom communication (such as wait time (Ingram & Elliott, 2016); funnelling (Wood, 1998); and revoicing (O'Connor & Michaels, 1993)), we turn to Drageset (2014) who states that, in so doing, there is a need to consider both single questions and the larger picture, such as an appropriation process. In this quest, it is necessary to take utterances, dialogue and sequences of dialogues into consideration.

Viewing a dialogue as a joint construction "made possible by the reciprocally and mutually coordinated actions and interactions by different actors" (Linell, 1998, p. 86), Drageset (2014) focuses on the value that is hidden in the details in teachers' comments and questions. He proposes a framework consisting of 13 categories of teacher comments that are grouped into redirecting, progressing, and focusing actions (the detailed categories are given in Table 1 below). These categories summarise how communication can contribute to students progressing towards a conclusion, "or to redirect the students into alternative approaches focusing on the mathematical content" (Drageset, 2014, p. 281).

Drageset (2014) proposes several ways of using his framework, one of which is to study how combinations of his categories occur and if there are patterns that can have "explanatory power beyond the study of single comments, for example by studying how a teacher uses different actions or categories as part of an appropriation process when the students have to learn something new" (p. 303). Solomon et al. (2021) used Drageset's framework to analyse an appropriation processes in the case of whole class discussions and found that, in addition to focusing actions, "the teacher is forced to intervene with a series of … progressing actions in order to progress the lesson" (p. 186), and "although teachers keep the intellectual authority in such actions, their strong focus on the students as originators of the appropriated contributions appears to provide the means by which they …leave students with the responsibility of solving the problem" (p. 187). This paper builds on Solomon et al.'s (2021) approach, but in the case of a one-to-one interaction.

Table 1: Redirecting, progressing, and focusing actions (Drageset, 2014)

1. Redirecting actions a Put aside b Advising a new strategy c Correcting questions	2. Progressing Actions a Demonstration b Simplification c Closed progress details d Open progress initiatives	3. Focusing actions a Requests for student input i Enlighten details ii Justification iii Apply to similar problems iv Request assessment from other students b Pointing out i Recap ii Notice

As both researchers and teacher educators, we are drawn to the transformation of Mira's practice during this lesson and wonder what it was about her interactions with Joe that fostered Mira's process in this case. Hence, we use Drageset's (2014) framework to analyse the nature of different sequences of conversations. Assuming that there is an appropriation process that has happened, we ask the following research question:

> How do redirecting, progressing, and focusing actions facilitate the appropriation process in a one-to-one interaction between a teacher and a student?

Methodology

As part of a larger research project on inclusive mathematics teaching in Norway, a series of observations of 5th grade mathematics classes (ages 10–11) in an inner-city school were conducted. There were three parallel classes of between 23 and 27 students. The school had a diverse student population with many students who had Norwegian as a second language. The classrooms were organized with rows of paired desks, facilitating students working in pairs with their assigned learning partners. All classes were video recorded using a fixed camera at the back of the classroom. A wireless microphone recorded the teacher's voice and voices of nearby students.

The video allowed us to identify a possible appropriation process in the interactions between Joe and Mira, and additionally the way in which Joe acts in the classroom. These interactions appeared to change over the course of the lesson, and Mira seemed to be gradually more engaged in the mathematics. We chose therefore to analyse the dialogue in the interactions to see what they revealed about the teacher's role in the emerging mathematical processes. This was performed in a three-step process. We first transcribed all of their conversation in the original language (Norwegian) and watched the video paying careful attention to their movements and interactions. Then, we coded the transcriptions using the 13 categories from Drageset (2014) framework, which we operationalised in close connection with the understanding put forward by Drageset (2014) (see Table 1). Each of the authors coded the transcripts, the codes were discussed until agreement was reached, and associated discussions were noted as these gave us a deeper understanding of the material. The dialogues were translated to plausible English, making sure that the intended meaning was kept. We used the transcription conventions given in Table 2 to indicate the rhythm and intonation of the original. Finally, we conducted a holistic reading of the data in order to capture their changing actions and the development of the conversations and interactions.

In presenting our analysis, we have referred to the numbering of the categories given in Table 1. The three dialogues between Mira and the teacher appear in the analysis in chronological order.

Table 2: Summary of transcription conventions adapted from Jefferson (2004)

(1)	Numbers in brackets represent elapsed time measured in seconds
(.)	Brief pause of less than a second
=	No pause.
↑word	Noticeable rise in pitch
[word]	Overlapping talk.
wo:rd	Colons indicate a stretched sound
((description))	Indicates the transcriber's description

Findings

We continue Joe and Mira's story where we left off in the introduction. At the end of the initial whole class conversation, the students were assigned a new task – to discuss in pairs the connection between multiplication and division. Joe went straight to Mira's pair:

Joe:	How are timesing and dividing related? ((Interruption from another student. Joe turns back to Mira)) [How]
Mira:	[Times]ing and dividing (.) Timesing and dividing are just about the same, it's just that when you times, then you sort of add (.) but instead of adding (.) you are to (9)
Joe:	What do you think? ((addressed to Mira's learning partner who looks down and then up again but does not speak)) (19) If you look at these ((points at the blackboard)) calculations, how do you think they connect?=It says 36 divided by 9 (.) is 4, (.) because 4 times 9 (.) equals 36.
Mira:	It is because it's backwards=
Joe:	=Yes, it's backwards. (.) Is it possible that it's the opposite, is that another way to look at it?=
Mira:	=Yes (.)
Joe:	So the opposite of timesing is dividing and the other way around? (1) So can we use (.) ((looks at the blackboard)) timesing to (.) calculate dividing ↑perhaps?
Mira:	↑↑Maybe= ((sounds satisfied, not doubtful))
Joe:	=↑Maybe? (.) If you know that 4 times 9 is 36, (.) can you turn it around? (.) Then 36 (.) ((He looks at the blackboard, and then back at Mira)) divided by 9 must be 4?
Mira:	Yes ((she sounds very satisfied and nods eagerly))
Joe:	↑Yes. (.) If you know how to times, then you actually know how to divide, maybe? hm: Good ((Joe leaves))

When Joe is asking for input from Mira we identify his actions as enlighten details (3.a.i) or justification (3.a.ii), while Joe makes details explicit to Mira with recap (3.b.i) and notice (3.b.ii) actions. A possible interpretation of Joe's two turns where he introduces the term "opposite" is as a progressing action either in category (2c) or (2d). Drageset (2014) describes these progressing actions as a way of "moving the process forward" (p. 294) either with closed or open questions. We argue that the purpose is rather that of highlighting and clarifying the connection so that it may be used later. Indeed, Drageset (2014) expands on to the description of the category *notice* (3.b.ii) by adding that "[t]he teacher often slightly changes the statement or adds new information to make the point clearer... to support the students by pointing out... important aspects to notice which they should understand or use in the future" (p. 297).

Two minutes after Joe left Mira a new whole class discussion took place. During this discussion Joe asked if Mira could repeat what she said during their conversation:

Mira: 36 divided by 9 is 4. So if you do it (.) ba:ckwards, first you take 4 divided by (.) 4 times 9 is 36, so, it is just the opposite!

We note that Mira has adopted Joe's academic language using the word "opposite". Joe highlights this to the whole class by writing "The opposite of multiplication is division" on the blackboard at the end of the discussion. Then a new task was given: 264 divided by 4.

After four minutes working on the task, Mira left her chair and interrupted Joe (who was speaking with other students) asking "Does 200 divided by 4 equal 50?" He confirmed and continued his ongoing conversation. Mira listened for a while before she started doing dance moves. A few seconds later, Joe followed Mira back to her desk, where she immediately started to explain her thinking:

Mira: 100 divided by 2 is 50=
Joe: =Yes=
Mira: =100 divided by 4 is 25=no 100 divided by 4 is not 25, is it? (1)
Joe: Yes, (.) because 25 times 4 is 100.=
Mira: =And then, 200 divided by 2 is 100 and then I thought, then it has to be like, since 100 divided by 4 is 25, 25 plus 25 is 50. So if 200 divided by 4 (.) then 200 divided by 4 needs to be 50.
Joe: =Mm. I agree. And then we have spent 200, and we are left with 64. (.) Ok. Then we know that they all get 50 each (.) and we are left with 64. ((Interruption from one student and then another. Joe encourages them and says he is coming.)) Yes, and next it is 64 divided by 4. (1) Can we make this number any easier?
Mira: Yes, maybe ((Mira sounds positive. Joe leaves to get students back in place.))

During this dialogue, Joe begins with a notice action (3.b.ii) and then makes an open progress initiative (2.d): "Can we make this number any easier?"

Mira worked for a couple of minutes before she started wandering around looking thoughtful counting on her fingers. Suddenly, she jumped, turned around, and ran to Joe while shouting "Joe, I have the answer!". She continued to shout it six times, and "104. 104. It is 104". He took her back to her desk:

Mira: Because I divided it by 2, and it is 36= ((referring to her calculation 64 : 2 =32))
Joe: =Yes
Mira: And, (.) it is sixty (.) 64=And six (.) 50, 50 pl[us 64]
Joe: [but, look,] you took 62 divided by 2 is 32, and then half of that is?
Mira: Half of tha[t]
Joe: [Then] 64 divided by 4 must be half of this one? ((points at something in Mira's notebook, presumably 32))
Mira: Yes [Don't look]
Joe: [What is half] of 32? (.) That was very smart. ((Presumably 64 : 2 = 32 in Mira's notebook.)) (.) What is half of 32? (1) What is half of 30?
Mira: Half of 30 is 15.
Joe: Then half of 32 needs to be one more. (.)
Mira: 16

Joe: Yes (.) So (.) So then 64 divided by 4 is 16.

Mira: Is it 50 + 16? ((Mira sounds unconvinced, but another student grabs Joe’s attention and he leaves to help them.))

Both Mira and Joe mis-spoke in this conversation. Mira said 36 when she meant 32 and Joe said 62 when he meant 64. It is noteworthy that this did not affect the meaning and we will discuss this later.

It seems that Mira had incorrectly calculated 50 + 64 = 104 as the solution when she invited Joe to her desk (see also the right hand column in Mira’s notebook in Figure 1). Hence Joe made a redirecting action which could be interpreted as a correcting question (1.a). However, this turn also has elements of a notice action (3.b.ii), “you took [64] divided by 2 is 32”, and a progressing action, “half of that is?” Even though this progressing action is a closed question it could be interpreted as an open progress initiative (2.d). Drageset (2014) points out that “comments in this category...are also aimed at moving the process forward, but without pointing out the direction” (p.294) and, by asking “half of that is”, Joe was following up a strategy determined by Mira, not Joe. In any case, and crucially, Joe’s response allows Mira to retain at least part of the intellectual responsibility. Joe continues by pointing out both orally and physically, and repeating the question “What is half of 32?” After a short pause, Joe makes a simplification action (2.b) by splitting 32 into 30 + 2 and taking Mira through step by step.

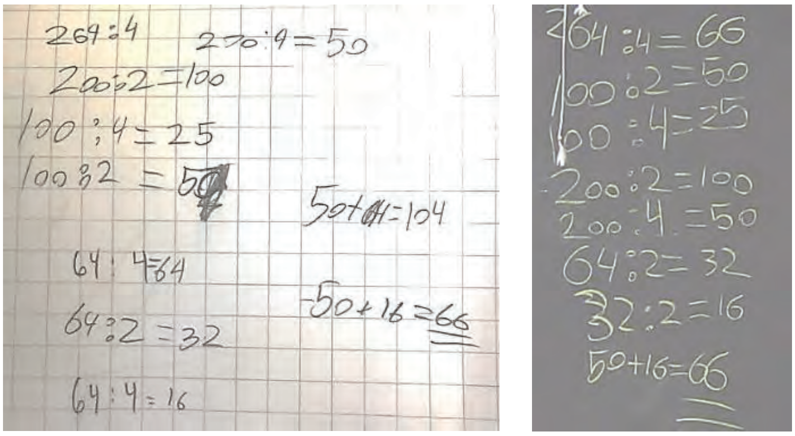

Figure 1: Mira’s notebook (left) and her presentation on the blackboard (right)

Less than one minute after Joe had left her desk, Mira ran to find him again, repeating “Joe, I have the answer!” She was very eager, shaking her book, jumping, saying, “66, can I show my answer on the blackboard?” Joe agreed. She climbed on a shelf placed under the blackboard and began to write. While she wrote nothing was commented upon by her or Joe (see Figure 1 for her presented solution).

We note that Mira’s presentation on the blackboard diverged from her notebook. In particular, she had reorganised the calculations so that they follow a logical progression. Crucially, Mira’s presentation of 64 divided by 4 used her strategy of repeated halving, and she did not split 32 into 30+2. This was now her solution.

Discussion and concluding remarks

Appropriation involves joint productive activity, a shared focus of attention, and shared meanings (Rogoff, 1990). As noted earlier, both Joe and Mira misspoke in the last conversation without it affecting the communication or the dialogical flow. We take this as a sign that they are locked-in to the mathematical process as a shared focus of attention. In addition, appropriation involves taking what someone else produces during joint activity for one's own in subsequent productive activity (Moschkovich, 2004). After the first conversation, Mira adopts Joe's alternative interpretation (Newman, 1989) by using the word "opposite" in her whole class discussion contribution, and, in the end of the previous section, we saw that Mira had made the solution her own when she presented it at the blackboard. We thus argue that the interactions between Joe and Mira displayed characteristics of an appropriation process where the student is working with mathematical practices. But, what has the analysis uncovered?

An overarching view of the analysis with respect to Drageset's framework (2014) reveals a shift in Joe's responses during the lesson. In the first conversation, Joe used focusing actions exclusively, and these were also present in the other conversations. In the second conversation, there was an open progress initiative. Progressing actions were also present in the third conversation in addition to simplification and a correcting question. This pattern may play a part in the appropriation process. The initial focusing actions may have indicated to Mira that her thinking was valued. Once this was established, progress was encouraged with actions that allow the student to retain intellectual authority (Drageset, 2014), wholly or partially. Finally, there was a simplification sequence. This final sequence taken out of the context of the lesson could be interpreted as a case of funnelling (Wood, 1998). However, Joe's previous careful handling of Mira's intellectual offerings allowed for a shared focus to be retained and a shared meaning to be developed (Rogoff, 1990) as evidenced in Mira's presentation to the class. The pattern is similar to the strategies employed by the teachers in Solomon et al.'s (2021) study of whole class discussions. In the whole class setting, the teachers explicitly emphasised student authorship (Solomon et al., 2021). Joe makes one such move ("that was very smart") in the conversations analysed above, but as we have argued, his choice of actions emphasises student authorship and "help maintain the shift of authority away from the teacher towards at least shared responsibility" (Solomon et al., 2021, p. 187).

Our analysis highlights what we believe to be an important feature of this interaction: The way in which Joe starts with focusing actions, and later introduces progressing actions sparingly. However, it is clear that this is not the whole story. Depending on what we focus on, there will always be nuances that come to the fore and we see potential for fruitful further research. For instance, the wait of nine seconds and then 19 seconds after Mira's first turn is unusually long (Ingram & Elliott, 2016) and especially so in this busy classroom environment. It is possible that these indicate to Mira that her thinking is valued. Similarly, we saw several instances where vocal intonations featured. In the first conversation when the idea of "opposite" is introduced, it is accompanied by a high pitched "maybe" that is then repeated by both parties in the following turns. This may function as a way of softening Joe's imposition of intellectual authority on the conversation. These diverse features have a commonality that is also revealed in the analysis: Through the interactions Joe and Mira co-produce a way of thinking that Mira can eventually inherit.

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Textbooks as actors in the transformation of the intended curriculum

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Teachers are central in the process of making learning situations out of intentions expressed in governing documents, such as the national curriculum. However, there is no straight line from intentions to learning situations – the teacher actively designs the planned curriculum and the enacted curriculum. In the process of planning, the teacher interacts with the material, and her decisions are also influenced by, for example, textbooks. This study aims to explore in what ways the textbook participates in the process of planning, i.e., the transformation from intended to planned curriculum. Based on focus group discussions with primary teachers, results show a variety in teachers' relations to the textbook, which have consequences for how the textbook participates in decisions. Results also show that students' positive feelings about the textbook influence the planning and that textbooks sometimes function as "emergency exits" in the process of planning.

Keywords: Curriculum, planning, teachers, textbooks, issues.

Introduction

Often curriculum materials are seen as ways to implement reform and influence teaching. However, teachers do not just transfer the content in them; they rather interact with the curriculum material (Remillard, 2005). Hence, teachers are central in the process of transforming ideas in tasks and pedagogical recommendations into events in the classroom (Lloyd et al., 2009), which means that the teacher – rather than being a transmitter or an implementer – is an active designer of curriculum. Consequently, there is a need to distinguish between the intended and the enacted curriculum (Remillard, 2005). The intended curriculum can, for example, be described as "the overt curriculum that is acknowledged in policy statements as that which schools or other educational institutions or arrangements set out to accomplish" (Kridel, 2010, p. 489). In a Swedish context, this would mean that the national curriculum and other policy documents such as national tests in school years 3, 6, and 9, and the mandatory tests in pre-school class and school year 1 can be seen as the intended curriculum. According to Remillard (2005, p. 213), the enacted curriculum is described as "what actually takes place in the classrooms (Gehrke et al.[, 1992])."

Taking teachers' active role as curriculum designers seriously and using the terms intended and enacted curriculum means that what is stated in the intended curriculum is processed by the teachers and transformed into the enacted curriculum. However, there may be reasons for dividing the process further. Teachers seem to be active designers and make decisions of importance for their teaching both in the process of planning and in the process of transforming the plan to classroom events, which means that there in addition to the intended and the enacted curriculum there is also a planned curriculum (Grundén, 2022). When teachers construct the planned curriculum, they are influenced by actors and structures on different levels – one such actor is the textbook which in a focus group conversation with primary mathematics teachers about planning, emerged as a prominent actor related to the teacher and her decisions (Grundén, 2022). The textbook as a prominent actor means that although the teacher is central in the construction of the planned curriculum – she is the one making the decisions – the textbook influences the decisions to a fairly large extent. A common notion about textbooks is that the use of textbooks increases the older the students get, which is supported by results from the latest national review of mathematics teaching in Sweden (The Swedish School Inspectorate, 2009).

Research on textbooks is a common theme in mathematics education research. According to Rezat et al. (2018), the research field has moved from focusing on the textbook itself to focusing on the design and use of textbooks, and textbooks are seen as one resource among many. Several studies focus on understanding processes involved in teachers' textbook use (Rezat et al., 2018). However, textbooks have consequences for teaching based not only on how they are used and how teachers interact with them but as previously mentioned, also on how they participate in the process of planning – the curriculum transformation. Hence, textbooks' role in the transformation from intended to planned curriculum might be one of the keys to understanding more about why intentions in the intended curriculum do not always reach all the way into the mathematics classroom. This paper aims to shed light on in what ways textbooks as actors participate in transforming the intended curriculum to the planned curriculum in primary school and to discuss possible consequences and implications for mathematics teaching.

Background

Mathematics teaching

There is a diversity in what is meant by mathematics teaching, and depending on how teaching is conceptualized, researchers can contribute in various ways to the expanded understanding of mathematics teaching and learning. In this paper, mathematics teaching is seen as a social, cultural, and political practice, which according to Fairclough (2015), means that there are situated and habitual actions and interactions going on. In a practice, there are people and relations involved, and the people involved act among other things by using language. Included in a practice is also the material world (Fairclough, 2015). In a practice, such as mathematics teaching, actors participate in the actions and interactions, and structures are influencing them. However, the structures are also influenced by the actors in the practice (Fairclough (2015).

When the intended curriculum is transformed into a planned curriculum it is done within the practice of mathematics teaching. Hence, the process is influenced by structures as well as by actors. An actor is, according to Oxford University Press (2021), "a participant in an action or in a process" and according to Enserik et al. (2010), an actor is "able to act on or exert influence on a decision" (Enserik et al., 2010, p. 79). Leaning on a definition of practices as including the material world (Fairclough, 2015) opens for actors as physical objects. In this paper, this means that planning involves several actors. Some actors are human, such as colleagues and school leaders. Some are organizational, such as the National Agency of Education. Others are material, such as textbooks or templates for planning. When a group of primary teachers talked about planning, textbooks were one of the most prominent actors that influenced decisions by virtue of how often they showed up in the discussion (Grundén, 2022).

Textbooks in Sweden

In Sweden, there is no national control of curriculum materials. The national curricula state that each principal is responsible for students having access to and conditions to use teaching materials of good quality (The National Agency of Education, 2019). Although no recent national large-scale studies focus on textbook use, results from prior studies might give indications. TIMSS 2007 and 2011 show that teachers in Sweden use textbooks as a base for mathematics teaching to a high degree compared to other countries. However, there seem to be differences depending on school years. In a national review of mathematics teaching students in school years 1–3 work with tasks in the textbooks 11% of the time in the observed lessons, in school year 4–6 31%, and in school year 7–9 47% of the time (The Swedish School Inspectorate, 2009). Although these numbers indicate that younger students work less with textbooks than older students, textbooks seem to influence teachers' planning (Grundén, 2022).

Traditionally, textbooks in Sweden do not consist of detailed lesson plans or instructions (Van Steenbrugge, & Ryve, 2018). However, as Van Steenbrugge and Ryve point out, a prior study by Boesen showed that teachers follow the content as it is sequenced in textbooks.

Method

This paper builds on four focus group discussions with teachers who teach mathematics and other subjects in school year 1–3. The teachers that participated in the four groups worked in three different schools. In total, the groups consisted of 17 teachers.

In focus group discussions, participants interact with each other in the conversation, which often leads to greater insights into experiences, and hence, richer data than individual interviews would have given (Carey & Asbury, 2012). In the discussions – where teachers were asked to freely talk – the theme was planning for mathematics teaching. At the beginning of the discussion, pieces of paper with words written on them (aspects identified in an earlier study) were placed in the middle of the table and used as stimuli. The words were *students*, *school management*, *national tests*, *template/forms*, *parents*, and *textbook*. The teachers could remove aspects or add things they thought were missing. My role during the discussion was to – by small words and gestures – confirm that I was hearing. I also asked follow-up-questions on themes already introduced by the teachers and invited all participants into the conversation, for example, by asking: "What do you think when you hear her say …"?

Analysis

Before the analysis, passages in the transcript of the four discussions where textbooks influenced considerations and decisions in the process of planning were extracted. This phase can be seen as the first step – getting familiar with data (Braun & Clark, 2006) – in the thematic analysis that followed. Each extract was coded with respect to what it was about. The next step in a thematic analysis is to collect extracts together within each code (Braun & Clark, 2006), and the different codes were sorted into sub-themes. Relations between codes and sub-themes were considered, resulting in four main themes: Teachers' relations to textbooks, Students' relations to textbooks, The teaching and textbooks, and Governing documents and the textbook.

Results

In this section, the four themes that represent the core content of how textbooks influence decisions in the process of planning are presented. Not all teachers or all groups of teachers talked about everything that is in the results. However, some issues came up in all discussions and when those are presented in the result, I emphasize that they came up in all discussions. When only one teacher, or a few teachers, say something, that is marked in the text as well.

Teachers' relations to textbooks

When textbooks are actors, i.e., participate in decisions in the process of planning, teachers' relationship to the textbook has an impact on how the textbook's participation looks like. Many teachers in the study seem to see the use of the textbook as something negative that they would rather avoid. Although mentioned in all the groups, especially teachers in one of the groups talk a lot about "dare to let go of the textbook." In this group, the teachers agreed on that they need something that supports them with the structure, but they are not satisfied with the way the textbook does that. One teacher expresses: "I think we will have to do such thing by ourselves," and another teacher continues, "We simply write a textbook," and the other teachers in the group agree.

All groups agreed that textbooks give teachers confidence – following the book is, according to some of the teachers, a way to ensure that nothing is left out and that students learn everything in the right order. The more experienced teachers get, the more they can let go of the book, and according to one of the teachers, the critical evaluation of textbooks comes with experience.

According to one of the teachers, planning is first and foremost about coming up with fun activities. When the teacher does not have enough energy to do so, she turns to the textbook and lets the students work in it. The teacher ends her post by saying: "But fortunately, the periods that you are so tired are not that long."

Students' relations to textbooks

Students' relationship to textbooks also plays a role in how textbooks participate as actors in teachers' process of planning. In the planning, students' individual work in the textbook seems to be considered an alternative that often brings out positive feelings in students, which is an argument for planning to use it in teaching. The teachers talk, for example, about how the students love their textbook and how they enjoy working with tasks in the book. One teacher says: "They – most of them – can tackle them [the tasks in the book] with life and desire because they think it is so fun when they come to the stop". However, some teachers talk about the importance of thinking about when and how they use the textbook with students in the early school years. For example, how to plan so that students who cannot read the tasks themselves can work independently. Some teachers beforehand choose tasks for individual students while others in their plan think that students shall work with some pages and adjust the plan for those who cannot keep up. When teachers use the textbook for planning, it seems common that they do it with their students in mind. They evaluate the tasks based on their knowledge about the students and decide what students shall do. However, there are also examples where teachers express that the goal is that students do all tasks, for example, when a teacher says: "You can skip these tasks and go back and do them later if you have time."

One of the reasons for students' positive feelings that teachers consider when they plan is that when students work in their textbook they can see a result – what they have done – while when they do other things there is no visible evidence of what they have done. One of the teachers points to her head and says: "The only thing left is in here." This means, for this teacher, that when she plans for activities outside the textbook, she also needs to plan how to make visible for students what they have learned.

The teaching and the textbook

This section presents results concerning the textbook and its relation to the teaching that comes out of the planning. Teachers in the study agree that what textbook is used influence the teaching. Most common seems to be that teachers are the ones who decide what textbook to buy, although there are examples where the school district has decided that all teachers must work with the same book series from year 1 to 9. A reason for keeping the same book for several years is that when teachers have worked with the same textbook for several years, planning gets easier. The book used must not be too advanced; the students must be able to work on their own. However, one teacher talks about how she has her students working in pairs to communicate and figure out together how to solve the tasks.

Some of the teachers talk about textbooks as an obstacle to teach the way they want. According to many of the teachers, using the textbook as a base for planning implies a focus on procedural skills. In one of the groups, teachers agreed that the best would be to make a structure and a plan together. A teacher expressed one of the reasons they emphasized: "I want to decide by myself when and how much procedural training I do [with the students]". However, all groups emphasized the textbook as a source for structure and tasks. Textbooks facilitate the planning work, and by following the book's structure, the teaching is about the same things in parallel classes at the school.

There are also examples where a teacher talks about how she turns to a specific textbook for specific mathematical content, for example, when teaching algorithms with regrouping. One textbook covers this content and has better tasks and activities than the others. Some of the teachers express that when they use the textbook in their planning, they choose what to do in the textbook based on their experience about what students need. For example, one teacher says: "Now we will work with multiplication. There are endless ways to work outside the textbook. You still see all the parts [in the book], but we pick this part out and work experimentally with it. And I think you can actually do that without having to feel that 'oh now we have not done these pages' ". Another teacher who advocates that she as a teacher can be flexible when she plans, i.e., the textbook does not decide what she does and when. She talks about how the students she has now need more challenges than other groups she had. She decided to introduce algorithms in year 2, although it was not in the textbook until later.

Teachers in the study seem to agree that procedural skill training is an important part of mathematics learning, and when they plan for that, they use the textbook. When teachers need to keep students busy, for example, when the teachers are absent or when they know some students will work faster than others, they also plan for students to work individually with tasks in the textbook. As one teacher expressed it: "It [the textbooks] is self-propelled. The students know what to do". Several teachers express that the textbook functions as an "emergency exit" they can use when the energy is running low or when the teacher is to be absent. Then the planning becomes “the students shall work individually in their books.”

Governing documents and the textbook

When textbooks participate as actors when teachers plan, the relation between governing documents and the textbook plays a role. In the discussions, teachers refer to the governing documents: national curriculum, national tests for school year 3, and mandatory assessment material for school year 1, and how they influence decisions in the process of planning. However, references to the national curriculum are rare and sometimes implicit. There are differences in how the teachers in the study think of the textbook in relation to the national curriculum. For some of them, it is obvious that textbooks do not cover everything in the national curriculum, while others state that textbooks are approved by the National Agency of Education and aligned with the national curriculum. Many of the teachers agree that the textbook needs to be evaluated against the national tests and mandatory assessment material when planning. What is not in the book, the teachers need to supplement with. According to several teachers in the study, there are differences in what students need to know to manage the mandatory assessment material and the national test and what is in the textbooks for that age group. Hence, following the textbook’s structure might make students less likely to pass all the tasks on the tests.

Some teachers state that the national curriculum is where the goals for teaching are presented, and the textbook is just a resource among others that the teachers can choose to use in their teaching. According to teachers in one group, textbooks participate in planning by offering the “what” – the content – while the “how” – which according to one teacher is the abilities [stated in the national curriculum] – is what teachers need to come up with by themselves.

Discussion

The purpose of this paper is based on results showing that for teachers in one of the focus groups, the textbook was an actor in the process of planning (Grundén, 2022). The results of this study confirm that the textbook participates in the planning for the other groups of primary teachers as well. In this section, the most prominent results about in what ways textbooks are actors in primary teachers’ process of planning will be discussed. The discussion will also highlight some possible consequences and implications for mathematics teaching.

Firstly, the structure and the content of the textbooks are often used to do rough planning. However, teachers in the study see the influence of textbooks as negative, as an obstacle to teaching the way they want. At the same time using the textbook when planning gives a feeling of confidence. This can be interpreted as teachers wanting support when they plan their teaching. The national curriculum with its overarching goals and content that should be covered during three years (National Agency of Education, 2019) is not enough. Or at least, it is not enough when teachers plan on their own. Several teachers in the study expressed that during the in-service teacher development program *Matematiklyftet*, textbooks did not participate in the planning as much as they usually do. Instead, the teachers cooperated with colleagues, which raises questions about whether collegial work with transformation using textbooks as resources is a way to develop the support.

Secondly, there is a great deal of variation in how textbooks participate in planning – from using them as a “smorgasbord” to seeing the textbook as an extension of the national curriculum, from choosing what tasks to work with to working from page to page. Sometimes, the textbook function as an “emergency exit” in the process of planning. These results are not surprising given previous research on textbooks (e.g., Van Steenbrugge & Ryve, 2018). However, looking at them in the light of textbooks as actors in the transformation from intended to planned curriculum gives new insights. The variation raises questions about what would be the best support in the transformation process? Is it to go the way some countries do – detailed textbooks and teacher guides teachers can follow? According to Remillard (2005), the answer is no, which also the results in this study indicate. The use of textbooks in the process of planning is not just a practical operation for teachers but is deeply associated with assumptions about mathematics and the learning of mathematics. It seems reasonable to believe that these assumptions remain although the textbook is changed. Hence, instead of “teacher-proof” textbooks, results indicate that teachers would benefit from support in another way. In addition to various textbooks, perhaps discussions with colleagues where assumptions are made visible and challenged is one alternative support.

Thirdly, students’ positive feelings about working individually with their textbooks is an argument for planning for such work. However, the positive feelings teachers refer to do not build on an idea that students learn more when working in textbooks, but rather that students gain good self-confidence by working on tasks they can manage to do on their own. The teachers also emphasize the evidence of “what has been done” that the textbook offers. One may wonder if this has to do with a tension between *performing mathematics* and *learning mathematics,* and one of the teachers might sense this tension when she emphasizes the importance of making visible to students what they have learned when working with other activities than textbooks – what they are able to do and talk about that they could not do before. When students’ positive feelings about *performing mathematics* are an argument for the participation of the textbook when planning, this might build on students’ ideas about what counts as mathematics and mathematics learning that might not benefit their learning. Hence, rather than letting the textbook be the planning to meet the students’ preferences, teachers can involve them in discussions about learning and signs of learning.

This study indicates reasons for learning more about the transformation from the intended to the planned curriculum. In this paper, the focus is on the textbook as an actor. However, there are also other actors – and perhaps other assumptions - that somehow participate in the transformation and might be obstacles to intentions formulated to benefit students’ learning.

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Navigating the contradiction between attainment grouping and inclusion in mathematics: the role of teacher identity

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This paper focuses on teaching mathematics in attainment groups as a means of fostering "equal opportunities for learning" (Norwegian: tilpasset opplæring, TPO) and thus meeting the Norwegian education system's historical aim of inclusive teaching. I report on interviews and classroom observations of one teacher working in a school which has introduced attainment grouping, focusing on how she explains her practice in the context of TPO. Applying Gee's (2014) theory of language in use and situated meaning, I focus on her enactment of practice in relation to teacher identity and the notion of big D Discourse. The analysis highlights the contradiction between the practice of attainment grouping and the policy of TPO and its implications for the role of teacher identity.

Keywords: Attainment grouping, inclusion, big d discourse, teacher identity, issues.

Introduction

The Norwegian school system has deep roots in inclusive mainstream schooling where teaching is mainly organized in whole class mixed groups. Indeed, the Education Act § 8-2 (Opplæringslova) states that "students shall not *normally* be organised according to level of ability, gender or ethnic affiliation" (my emphasis) (Opplæringslova, 1998). In addition, the national curriculum emphasises the pedagogic principle of "equal opportunities for learning" (*tilpasset opplæring* in Norwegian, or TPO) which emphasizes that education should develop each student's full potential; it is the teacher's responsibility to facilitate this. Despite this background, some schools have introduced attainment grouping as a means of organizing TPO, particularly in mathematics. This appears contradictory in the context of an inclusive approach to mathematics teaching and based on what we already know from research on attainment grouping. Drawing on observation and interview data, this paper explores how one teacher navigates this situation in both her teaching and her account of attainment grouping as a means of organizing for TPO. It argues that exploring teacher identity is crucial if we are to understand her stance.

Background literature

There is little research in Norway on the impact of teaching mathematics in attainment groups, particularly on classroom level practices. However, international research reports on differences in teaching practices between groups: teaching in lower attainment groups tends to be more traditional and is dominated by teacher-led teaching and the use of restricted and repetitive tasks. Teachers' questions are often closed with little opening for critical reflections on mathematical thinking (Kaur & Ghani, 2011). High attainment groups on the other hand are often characterized by more reform-oriented teaching, emphasizing critical thinking and deep learning through problem solving and open-ended tasks. However, work can also be fast-paced, emphasising fluency in procedural algorithms (Beswick, 2017; Francis et al., 2019; Solomon, 2007). Research also suggests that teaching in mixed groups may be less restricted and more investigative: teaching in mixed groups tends to be more differentiated, while attainment grouping treats students as a homogenous group on the same level (Francome & Hewitt, 2020; Taylor et al., 2017). Attainment grouping may therefore lead to a more restricted access to mathematics in terms of both pedagogy and content, leading to student labelling and a fixed ability view of both low and high attainers (Francis et al., 2017; Taylor et al., 2017). Teachers' perceptions of students are important. Beswick (2017) asked teachers to describe "poor" and "rich" students, finding that poor students were described as lacking proficiency, understanding and ability to explain mathematics. "Rich" students were described as being proficient in describing skills and knowledge. On this basis the "poor" students were offered restricted tasks and "good" students more open-ended tasks (Beswick, 2017). Similarly, Mazenod et al. (2019) found that teachers of lower attainers took a nurturing approach, believing that students should not be over-challenged; this led to an "over supportive" pedagogy which limited development.

As noted above, the literature indicates that attainment grouping leads to limited access to mathematics and differences in teaching approach which suggest that the practice of attainment grouping is not consistent with the view of inclusive mathematics teaching encapsulated in the Norwegian emphasis on inclusion through TPO. This tension is observed by education researchers in Norway, but the increasing practice of attainment grouping is largely unquestioned, driven as it is by pressure from international tests which show Norway underperforming in comparison to other countries (OECD, 2016) and arguments that grouping benefits higher attainers (National Centre for Science Education, 2015). Hence, to explore how teachers meet TPO policy in the context of attainment grouping, this paper addresses the Research Question: *How do teachers navigate the relationship between TPO and attainment grouping*?

Theoretical framework: "big D Discourse"

In this paper I draw on Gee's (2014) theory of language in use and situated meaning to enable a focus on teachers' enactment of their classroom practice within the context of policy requirements and school organisation. Gee's theory emphasizes the role of "big D" Discourses which are distinct from "small d" discourse and its focus on language. "Big D" Discourse captures actions as well as words, and in this sense, it also captures identity performance, which involves

> ways of enacting socially situated identities and associated practices in society through language and ways of acting, interacting, valuing, knowing, believing and using things, tools and technologies at appropriate times and places. (Gee, 2014, p. 127)

Enacting and being recognized in a Discourse requires more than language. When people are engaged in Discourse, they use language to engage in a practice to do things, but also to *be* things as they take on socially situated identities. Gee emphasizes that saying-doing-being gains its meaning from the practice it is a part of and enacts (Gee, 2014, p. 11). He foregrounds identity, arguing that saying things "never goes without also doing things and being things" (Gee, 2014, p. 3), and thus concerns recognition as a certain kind of person engaged in a certain kind of practice. To "pull off" a Discourse therefore requires the individual to both "talk the talk" and "walk the walk" (Gee, 2014, p. 24).

Gee also draws on the idea of figured worlds (Holland et al., 1998) to understand how situated meanings are constructed:

> Figured worlds is a theory, story, model or image of a simplified world that captures what is taken to be typical or normal about people, practices, things or interactions. (Gee, 2014, p. 226)

A figured world is thus a local simplification which mediates between local social interactions and Discourses, enabling enactment of a Discourse. This simplification aspect of figured worlds means that they often relate to particular values about how things are, or should be. The concepts of big D Discourse and figured worlds provide not only a theoretical perspective on the nature of situated meanings, but also a method of inquiry as outlined below.

Methodology

The work presented here is part of a larger study focusing on attainment grouping in mathematics teaching and TPO, involving four 9th grade mathematics teachers in one lower secondary school (Berg School) in Norway. Berg School had organized mathematics teaching so that the four 9th grade class groups were taught for two of their three weekly lessons according to attainment level, and in their remaining lesson as a whole class mixed group. Each of the four teachers were responsible for one attainment group and one whole class mixed group. This paper focuses on a case study of Lena, who teaches group 4, the highest of the attainment groups. The data includes two semi-structured interviews and classroom observations of three of Lena's lessons (one week of teaching). The first interview took place before observation and focused on her view of teaching in attainment grouping and TPO; the second interview took place after the classroom observations and included reflections on the lessons observed. All names of people and places are pseudonyms.

Interviews were transcribed in full and analysed by searching for references to "big D Discourse" (what kind of teacher Lena described herself as or how she wanted to be) and "figured worlds" (Lena's theories of teaching and learning, in particular her references to values about how mathematics teaching is or should be). The observation episodes were also transcribed and annotated to record Lena's and students' movement about the classroom, student hands up and so on. My analysis focused on Lena's use of questions and discussion, wait time, her use of tasks, and her use of explanations. I was interested in her choice of whole class teaching or individual work for particular activities. I also noticed how she distributed time among students and her use of positioning. Seeing big D Discourse as enactment of identity and associated practices, these references enabled an analysis of Lena's teaching practice as socially situated identity performance – that is, as performed within the context of TPO and the school's emphasis on attainment grouping.

Analysis

In this section, I draw on both the interview and observation data to analyse Lena's enactment of teaching in attainment groups and TPO, bearing in mind the big D Discourse emphasis on "saying, doing and being". Hence the interview and observation data are presented together in the analysis, since they mutually support each other in the application of big D Discourse.

A figured world of fixed ability

In the interview, Lena is clear that she sees attainment grouping as the best way for organising mathematics teaching for TPO. Her arguments suggest that she draws on a figured world in which mathematical ability is fixed. This becomes evident in her descriptions of teaching in different attainment groups and students learning according to different levels: “everyone gets something on their own level (…) and everyone is about the same level then (…) it will not be too easy or too difficult. That it is right where someone is”. Lena talks about the students as “different kinds of students” and describes them in the context of homogeneous groups based on different levels, where the students in one group are alike and have the same ability, both in terms of understanding mathematics as well as in their way of thinking: “...the others [in the group] have the same opinion, ... and everyone really thinks the same way”.

This view of students being alike can also be seen in how she describes what characterizes the mathematics and teaching approaches in the different groups. Referring to teaching for the low attainers she talks about “the “didactics” of the weak”, in which teaching is “a bit easier and a bit more practical”. In contrast she describes high attainers as taking a more formal approach to mathematics. This approach was also evident in the observation data, where her group 4 teaching valued a procedural approach to practicing Pythagoras’ theorem and how to write it down correctly:

> Lena: And then when we write down and solve those tasks, what do we always start with then? …
> Kari: Writes that formula.
> Lena: Writes that formula $k^2 + k^2 = h^2$ (Lena writes the formula on the board)

She also contrasts the low and high attainers in terms of what she sees as the usefulness of drawings for the low attainers, while the high attainers are not in need of images and examples to the same extent as the low attainers “…because that’s the way their brains are made”.

Lena’s figured world of fixed ability is also apparent when she refers to how she limits the content of talk in mixed whole class discussions because of all the different levels represented in the class.

> When I have a whole [mixed] class, I often set the level on use of concepts and… like I do not go into depth in the class talks…. Have them explain to me what they think… Instead of (…) problem solving tasks where there is a little ... high level then.

This assumption of a figured world in which there is an average level of whole class mixed group was also evident in her teaching in this group. Lena started the lesson by giving a short repetition of how to make diagrams in Excel before the students were asked to try this out as a repetition activity.

Being a caring teacher

Lena is concerned about having a good relationship to the students and she refers to this as “most of her job”, assuming a figured world in which good relations between teacher and students is an important basis for learning. This perspective indicates a Discourse of being a kind teacher who cares about the students. This is also evident in how she justifies parts of her teaching based on what the students “like to do” and what they think is “fun”: “…they like to do tasks. (…) at least in the high group, they learn a lot from working for themselves. (…) And at the same time, it is the task that they want for themselves”

The Discourse of a caring teacher is also visible in her theory of the importance of providing a comfortable learning environment for all the students in an attainment group. Comparing teaching in group 4 with teaching in whole class mixed groups, she says that it is “better” with group 4, because:

"… the students may feel a bit more comfortable, that it's kind of okay to be quite good. Because here all are good".

This caring aspect is also evident in her teaching. Often, Lena's discussion with the students does not focus on mathematics but is more about everyday life. Her language can be characterized as youthful and friendly in tone, almost like chatting. For example, Lena joins in when some of the girls start to talk about the price of the food in the school canteen, and when other students talk about the next tests and assignments in other subjects that week. Lena also enacts the caring teacher when the students work individually on tasks. As she circulates around the classroom, her comments are mainly "how is it going" rather than on the mathematics in the tasks – she does not probe what lies behind the students' frequent answers of "fine". There is thus a lot of social work going on in the lesson. Lena's position in the classroom in this part of the lesson is more as a "mate" to the students than as the teacher in a position of authority.

Lena's argument for teaching in the different attainment groups also draws on a theory of the need to take a nurturing approach to teaching. She is concerned that the mathematics teaching and content should be manageable for the students, and especially the low attainers, and that teaching should not expose the students to "too much or too difficult" stuff. As a caring teacher, Lena argues that they should be exposed to a limited mathematics content, just enough to get by: "…they should at least be able to ..., enough to do well enough on the exam anyway".

This theory of teaching is also evident in a group 4 lesson where one of the students, Tom, stops Lena in her teaching of the procedural solution of Pythagoras' theorem. Although this is the higher attainment group, Tom is unhappy with the pace, and asks (implicitly) for things to slow down. Lena's immediate response and her subsequent action suggests that she positions him as weak in the group and in need of individual attention:

Tom: It's going too fast
Lena: Am I going too fast? ... We'll look at that a bit afterwards.
Tom: Yes

Lena does not treat Tom's interruption as a request for the whole group to engage in further explanation of the mathematics. Instead, she finishes her teaching with the whole group, then comes back to talk to Tom individually. She repeats the procedure for him, going through it step by step, but now at a slower pace. Importantly, she limits her explanation to how to write the solution down, telling him to use the example as a model for the next questions. It seems as though she tries to reduce the demands on Tom, making the question merely manageable for him.

"I need control in teaching"

Lena is also a teacher who needs to be in control in her teaching. She explains that she chooses teaching approaches which are comfortable for her. One of these is talking to the students individually instead of in whole class discussion, so that she can maintain control, as in the example with Tom above. She also explains that she prefers to teach the high attainers in group 4, and that she is not comfortable with teaching the low attainers: "And, you maybe need to go down to the practical level, which I am not fond of. And then I get uncomfortable too, it just gets messy all together".

For Lena, attainment grouping makes it possible to avoid “uncomfortable” teaching where she is not in control. She argues that Jon, one of the other teachers, is the best person to teach the low attainers, simultaneously ensuring that she should not have to teach this group.

> … the one who has group 1, he has actually always had group 1, is very good at the [pedagogy]/didactics of the weak. (...) and is very good with that kind of student ... So, he wanted to have that group.

Although Lena argues for attainment grouping as the best way to teach TPO she also argues for it as better for teachers, because they are more in control: “And then we wanted to try it out to make it a little better for us teachers, to have a little more, control of the lesson then”. This emphasis on control is enacted in her practice in her emphasis on ensuring that the students write solutions in “the right way”. For example, she tells the students to start each new question by writing up Pythagoras' formula:

> And what is it that is important to watch out for when we are going to WRITE pieces like this? (…) The equal symbol below each other (Lena points to her correct notation on the board) … because then it looks much tidier.

Identifying with the high attainers

Lena's wish to teach in group 4 is not just related to her need for control in teaching. When she describes her own mathematical thinking, she identifies herself with the group 4 students:

> I like, I like it best in group 4. Because they, eh, I'm a bit bound by rules myself. Because I'm kind of the same type. (…) Eh, so that's a bit like that, there I can see how it, why they think what they do too. Because that's also the direction I'm going.

She describes herself as the same kind of mathematics person as the students, as a mathematically strong teacher. In her teaching, this view appears in how she explains the mathematics to the students, positioning herself as the authority in the classroom. She appears to emphasize the mathematics in group 4 as the most valued, enacting the Discourse of a mathematically strong teacher which assumes a figured world of fixed ability both about the students and herself as the teacher. She appears to see herself as ideally suited to teaching the high attainers.

Discussion

In this paper I have addressed the research question, “how do teachers navigate the relationship between TPO and attainment grouping?” I have focused on the story of one teacher, Lena, and her enactment of teaching in attainment groups. The analysis reveals that Lena brings TPO and the way her school organises mathematics teaching together by identifying as a caring teacher, and by drawing on a figured world of fixed ability which enables her to enact the big D Discourse of the mathematically able teacher meeting the needs of mathematically able students.

Inclusive mathematics teaching means that all students are included regardless of assumptions we might make about their potential for learning. Lena argues for attainment grouping as the best way to address TPO and inclusive teaching, but her enactment of teaching does not necessarily lead to an inclusive mathematics teaching for all students. In her fast-paced work on applying the Pythagorean algorithm she excludes Tom from taking part in the teaching with the rest of the class. Rather than opening up a whole class discussion, Lena isolates Tom and enacts the approach of the caring teacher, giving him a barely modified instruction that repeats her original teaching more slowly, with an explicit instruction that he should just follow the procedure with other questions. Lena's teaching can also be seen as non-inclusive in that she sees procedural knowledge as valuable for the high attainers, compared to limited content and restricted tasks for low attainers. It appears that her view of TPO and inclusive mathematics teaching concerns adapting teaching approaches and mathematical content in accordance with a figured world in which ability is fixed. Furthermore, her procedural approach excludes group 4 students from an explorative approach to mathematics and discussion for deep learning. This too is closely connected to a figured world in which ability is fixed. Coincidentally, this appears to serve a need for control, which may itself be an element of the same figured world in which teachers are authority figures.

Lena's enactment of teaching in attainment groups and her figured world of fixed ability is also related to the Discourse of being a caring teacher, in line with the nurturing approach to low attainers identified by Mazenod et al. (2019). Lena justifies limiting content for the low attainers on the grounds that they need only to pass the exams, and for her, seemingly as a good way of organising for TPO and inclusion. Although the Discourse of being a caring, nurturing teacher may be a way of enacting inclusion, Lena appears to prioritise good relationships and care for her students as a basis for their learning, but the result is their exclusion from engagement with mathematics learning.

This big D Discourse analysis of Lena's identity as a mathematics teacher brings together observation of her enactment of teaching in attainment groups and her account of her practice within TPO. It reveals complexity and tensions in her practice, values and enactment which make sense when we take the context she operates in into account. As noted above, in Norway the move to attainment grouping is not contested despite research evidence that it is not beneficial. Locally, Berg School has compounded this situation by deciding that TPO in mathematics teaching will be addressed through attainment grouping. Although Lena has been party to this decision, it is not hers alone; additionally, there are pressures outside of the school which prioritise examination performance. In this general context, Lena's socially situated identity as a mathematics teacher draws on particular figured worlds in which doing mathematics is seen as procedural and fixed in order to support her enactment of the mathematically able and competent teacher who supports all her students.

Lena seems unaware that her approach to teaching in attainment groups can lead to exclusion from mathematics. She seems also unaware about the tension between her figured world of fixed ability and the idea of inclusive teaching, and the potential impact of a nurturing approach on inclusion. An implication of this study is that it is important for teacher educators to work with teachers to explore teacher identity and their "big D Discourse" in order to support a more reflective enactment of their teaching practise for TPO and inclusion.

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Irrational numbers in the teaching of the Pythagorean theorem: practical argumentation in a Lesson Study and Didactic Suitability training course

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This work investigates the role of the Didactic Suitability Criteria in the argumentation oriented towards action when introducing irrational numbers in the teaching of the Pythagorean Theorem in a teacher training course that combines Lesson Study and Didactic Suitability. We analysed the dialogues of the lesson planning stage of the Lesson Study cycle by means of an argumentative trajectory and the Didactic Suitability Criteria, also considering practical argumentation. We evidenced the conflicts between the cognitive and the epistemic suitability criteria in which teachers gave more importance to one criterion than another depending on the context. Epistemic suitability gives rise to considering a representative sample of problems for a partial meaning of the Pythagorean Theorem, and cognitive suitability promotes argumentation about which intended meanings can be achieved.

Keywords: Irrational numbers, pythagorean theorem, practical argumentation, lesson study, didactic suitability.

Introduction

Research in Mathematics Education highlights the need to consider students' previous knowledge in teaching and learning processes. This entails the teacher should know and comprehend students' previous knowledge and then decide what to do in the instructional process. Among some trends that regard the importance of previous knowledge, we focus on Lesson Study (Huang et al., 2019), which originated in Japan and later spread to other countries, and the Didactic Suitability Criteria (Breda et al., 2018), proposed by the Onto-Semiotic Approach (Godino et al., 2019).

In Lesson Study, implicit agreements between participants on the aspects that are positively valued emerge. These aspects can be reinterpreted in terms of the components and indicators of the Didactic Suitability Criteria. In Lesson Study, some of these components and indicators may emerge as consensuses from the reflection of the group of teachers. This justifies the inclusion of the Didactic Suitability Criteria in a training course involving Lesson Study. This phenomenon is the origin of proposals for teacher training which focus on the development of reflection on teaching practice by combining Lesson Study and Didactic Suitability Criteria (Hummes et al., 2019). In addition, analysis of teachers' reflections indicates that teachers engage in argumentation about the actions that they agree to undertake, which some authors call practical argumentation (Gómez, 2017). This argumentation occurs when the participant teachers try to find a consensus and explain their reasons as equals. The structure of this type of argument is of interest in our research.

This work is part of broader research whose objectives are designing and implementing training experiences aimed at developing reflection and teachers' argumentative competence in order to study: i) How they argue in their lessons; ii) What is their practical reasoning about the actions they decide to carry out in lessons; and iii) How Didactic Suitability Criteria relate to (teachers') arguments about their practice during the teaching and learning processes. In particular, this study focuses on an episode from a training course for mathematics teachers that combines the Lesson Study and the Didactic Suitability Criteria. It aims to analyse the argumentation oriented towards action when introducing irrational numbers in the teaching of the Pythagorean Theorem and the role of the Didactic Suitability Criteria in this argumentation.

Theoretical Framework

Lesson Study: This is the collaborative and detailed design of a lesson, its implementation and direct observation in the classroom, and its joint analysis after its implementation (Huang et al., 2019). A group of teachers and experts with a common concern about their students' learning gather, plan a lesson, and finally analyse and discuss what they observed in the implementation. Multiple iterations of this process bring the teachers many opportunities to discuss the students' learning and how the teaching influences this learning. A Lesson Study cycle should follow these stages (Lim-Ratnam, 2013): 1) study of the curriculum and goals, when participants choose content to teach and set the learning goals; 2) lesson planning, when participants set the objectives of the lesson and describe its development; 3) implementation and observation of the lesson, when they record the impact of the planning on the students' learning and collect data generated from the observation; 4) joint reflection on the collected data, when participants use the data from the observation to reflect on the implemented lesson, the students' learning and the previous planning.

Didactic Suitability: In the Onto-Semiotic Approach, the didactic suitability of an instructional process is the degree to which it meets certain characteristics that allow it to be described as suitable (optimal or adequate) to achieve an adaptation between the personal meanings developed by students (learning) and the institutional meanings intended or implemented (teaching), taking into account the circumstances and available resources (environment) (Godino et al., 2019). This is a multidimensional construct that consists of six suitability criteria: 1) epistemic criterion, to assess whether the mathematics that is taught is 'good mathematics'; 2) cognitive criterion, to assess, before starting the instructional process, whether what is intended to be taught is at a reasonable distance from what students know; 3) interactional criterion, to assess whether the interaction solves students' doubts and difficulties; 4) mediational criterion, to assess the adequacy of time and material resources used in the instructional process; 5) emotional criterion, to assess the students' involvement in the classroom; and 6) ecological criterion, to assess the adequacy of the instructional process to the educational project of the school, the curricular guidelines, and the conditions of the social and professional environment, among others factors (Godino et al., 2019).

Argumentation oriented towards action: In the Theory of Communicative Action, argumentation is defined as a "type of speech in which participants state the validity claims that have turned dubious and try to accept or decline them using arguments" (Habermas, 1987, p. 37). We focus on argumentation oriented towards action (or practical argumentation) and on the consensus that emerges from the reflection of a group of teachers on their own practice. Gómez (2017) defines practical argumentation as a “reasoning in social contexts directed to choose an action to solve a practical problem” (p. 215). The argumentative speech may be considered as a process, a procedure, and a product (Habermas, 1987). As a process, one may exclude any coaction and focus on the search for truth as an action directed to understanding. As a procedure, the discursive process of understanding is regulated as a cooperative work division between proponents (P) and opponents (O), where a validity claim (VC) that has become a problem is stated, examining with arguments whether this claim is recognized or not. In the case addressed here, some VC are discussed and the teachers can be distinguished, based on their reasons -supporting and providing argumentative strength (AS) to the VC, or trying to invalidate it (InvVC)- to recognise or not its validity. Finally, as a product, the argumentative speech is producing appropriate arguments that are convincing by their intrinsic properties and can be used to accept or refuse the VC. Arguments are the ways to obtain a common acknowledgment of the VC that the proponent states hypothetically, and through which an opinion can become a knowledge or an action. The argument trajectory (Ramos, 2006) is a tool that allows the elements of the argumentative speech mentioned above to be identified. In this study, we used an argumentative trajectory to analyse the participants’ practical argumentation.

Methodology

This is a qualitative-interpretive study. Eight mathematics teachers working in schools in the south of Brazil (students aged 11-18) participated in this research. They were graduated in mathematics and had three-to-fifteen-years of teaching experience. In addition, three of them had a master’s degree in Mathematics Education. The training program was planned as a face-to-face implementation, but it had to be restructured as a virtual implementation due to the COVID-19 pandemic. The sessions were conducted using Skype and were recorded with the participants’ permission. The training course followed these phases: 1) implementation of two complete Lesson Study cycles (two groups of four teachers, each group developed a cycle); 2) introduction of the Didactic Suitability Criteria as a tool to guide the teacher reflections; 3) new analysis of the implemented lesson and its redesign using these criteria. The first author led the training course and acted as a participant observer.

We identified different episodes of practical argumentation between the participants along the phases of the course. We also identified the role of the Didactic Suitability Criteria within the arguments given in each session, with different relevance. The analysis presented in this work can be reproduced for each of these episodes. However, due to a lack of space, we present the analysis of only one episode from the lesson-planning stage of the Lesson Study conducted by one of the groups. The participants chose the Pythagorean Theorem as the topic to be taught as it allowed them to implement the original lesson and its redesign for two different student groups (aged 14-15) at two different moments of the semester.

With the argumentative trajectory, we considered the three aspects of the argumentation in the episode: ‘process’, aiming to achieve ideal conditions for communication between participants; ‘procedure’, considering the teachers’ argumentations in the form of cooperative work division between proponents and opponents; and ‘product’, obtaining appropriate arguments to accept or not the VC about including irrational numbers in the teaching of the Pythagorean Theorem. In addition, the argumentative trajectory allowed the argumentation to be related to the components and indicators of the Didactic Suitability Criteria, in order to identify the role of these criteria in the argumentation oriented towards action about introducing irrational numbers in the teaching of the Pythagorean Theorem. The analysis followed some phases similar to those used by Ramos (2006): i) We reviewed the sessions to identify the episodes of practical argumentation between the teachers, meaning discussions between them about the decisions made to plan the lesson. ii) Once we identified the episodes of argumentation in the videos, we recognised the teachers' roles, the achieved consensus, the invalidated claims, and the reasons to invalidate them, using the constructs described in the theoretical section (P, O, VC, InvVC, AS) plus RMC (rationally motivated consensus), and CO (consensus by omission). iii) We made an argumentative trajectory to visualise the relation between VC, InvVC, and AS in the episode, and thus conclude with the RMC. We distinguished the participants as P or O, and identify who participated in the CO. Regarding the latter, some elements of the argumentation are implicitly expressed in the sessions. These elements were inferred in the argumentative trajectory from the general context where the episodes of argumentation occur. To do this, we reviewed the videos recorded during the first and second stages of the Lesson Study cycle several times. iv) We analysed the argumentative trajectory using the components and descriptors of the Didactic Suitability Criteria to identify the role of these criteria in the participants' argumentation. v) We used triangulation by experts to validate the obtained results.

Analysis

In this section, we present the analysis of an argumentation episode that occured in the planning phase of the Lesson Study cycle implemented by four participant teachers (P1, P2, P3, P4). In the first phase of the Lesson Study cycle, where learning goals were established, P1 suggested a problem about finding the length of the diagonal of a square of side one using the Pythagorean Theorem, aiming to include examples of triangles with irrational lengths, in addition to examples with natural lengths. However, at that moment, the teachers did not discuss this proposal. Nevertheless, in the phase of planning the lesson, the teachers considered presenting examples of right-angled triangles with irrational lengths. They suggested using the Pythagorean Theorem to identify the length of the diagonal of a square of side one. In the following lines, we present the argumentation episode developed by the teachers:

P1:	I think that this will be light for them [he refers to the Pythagorean Theorem as a relation between natural lengths in a right-angled triangle]. They will have problems when we talk about the square root of two, since […] students have a great difficulty understanding the irrational numbers and, when we make the square of side one, I think that it will generate certain difficulty. Don't you think so? (VC1).
P2:	But the idea was only showing that it also works for the square root, was it not? (InvVC1).
P1:	This is the idea. In this case, we should remind that the square of the square root of two is two. (AS1 to VC1).
P4:	They didn't learn that. (AS2 to VC1).
P1:	Then, as they have not seen that, there should be an easy way to verify… Did they not learn the notion of the square root of two in the number line last year? (AS3 to VC1)

P4:	We worked on the irrational numbers quite well. But what you said is true, they have difficulties accepting that the square root of two is a number. I doubt that they remember it. (AS4 to VC1)
P1:	[…] we could propose how a square of area two is. For instance, you have a square of side one, then the area is one. In order to obtain the area two, what must happen? If the square side is two, then the area will be four. Thus, they will have a notion that, in order to obtain an area two, the length of the square side should be a number between one and two […]. (AS5 to VC1)
P4:	I think that this is a wonderful idea when the lesson is face-to-face. (InvAS5 to VC1)
P3:	Because many times they see a root and get blocked. They do not even wait to see what you want to do. I think that we should do a previous class to delve into the content of roots. (VC2)
P4:	I think that, after learning the Pythagorean Theorem and really showing that it works, we could make a triangle with two sides one and try to find the third side. Then, I can mention that the square root of two is an irrational number and that length that we have just found is an irrational number. (VC3)
P1:	What about using the ruler and the compass to make, for example, a square of side one and using the compass to show the length of the diagonal on the number line? What do you think? (VC4)
P4:	I like it. (AS1 to VC4)

We identified proponents (P1, P3, P4) and opponents (P2, P4). The theses are: i) introducing irrational numbers with the Pythagorean Theorem applied to the right-angled triangle of cathetuses one; ii) the sample of problems cannot be extended due to a lack of previous knowledge and the classroom conditions.

Argumentative trajectory

i) P1 posed the problem of working on the Pythagorean Theorem with irrational lengths. (**VC1**) ii) P2 tried to invalidate VC1 returning to the initial proposal of just showing some examples of right-angled triangles with irrational lengths. (**InvVC1**) iii) P1 considered that it is important to bear in mind the properties of irrational numbers, specifically, the operations with roots. (**AS1 to VC1**) iv) P4 indicated that the students did not learn operations with irrational numbers. (**AS2 to VC1**) v) P1 tried to discover what the students know about irrational numbers. He said that they should find an easy way to verify the Pythagorean Theorem with irrational numbers. (**AS3 to VC1**) vi) P4 explained that she has worked on that, but she was not sure whether the students remember it. She confirms that students also had difficulties understanding that the square root of two is a number. (**AS4 to VC1**) vii) P1 proposed to remind the students that the square root of two is an irrational number, doing an approximation of the decimal expansion. (**AS5 to VC1**) viii) P4 argued that this would be a good idea if this were a face-to-face implementation. (**InvAS5 to VC1**) ix) P3 highlighted the need of doing a previous lesson about operations with roots to address the students' difficulties. (**VC2**) x) P4 only mentioned that the number found is an irrational number. (**VC3**) xi) P1 proposed that they could use the ruler and the compass to make a square of side one and show the irrational number that corresponds to the length of the diagonal, representing it on the number line. (**VC4**) xii) P4 said that she liked that idea. (**AS1 to VC4**) xiii) Participants achieved a rationally motivated consensus (**RMC**). P2 and P3 did it by omission (**CO**).

Participants continued discussing about the resources for the lesson. The following questions arose: How could they use the ruler and the compass in a virtual lesson? How will they record the lesson? Which should be the position of P4 during the recording with the resources?

Analysis of the argumentative trajectory from the perspective of the Didactic Suitability

i) P1 highlighted the students' difficulties during the learning of irrational numbers. He questioned whether "the intended meanings can be achieved", an indicator of the component "prior knowledge" of the cognitive criterion. ii) P2 proposed to present a situation with irrational numbers as the lengths of the sides of a right-angled triangle, without delving into the concept of irrational numbers and operations with them. P2 tried to maintain the idea of having a representative sample of problems within the same partial meaning of the Pythagorean Theorem. This is related to the indicator "for one or more partial meanings, a representative sample of problems is provided" of the component "representativeness of the complexity of the mathematical object" of the epistemic criterion. iii) P1 considered whether "the students have the previous knowledge necessary to learn the topic", particularly the knowledge on operations with irrational numbers. It is related to the component "prior knowledge" of the cognitive criterion. iv) P4, who implemented the lesson and knew the students, confirmed the students' lack of knowledge on operations with irrational numbers. The component "prior knowledge" of the cognitive criterion emerged again. v) When P1 asked if the students placed irrational numbers on the number line, he looked into the students' previous knowledge. Thus, the component "prior knowledge" of the cognitive criterion was present again. vi) P4 confirmed the students' difficulty understanding the square root of two as a number. This assertion also corresponds to the component "prior knowledge" of the cognitive criterion. vii) P1 proposed an activity to work on the idea of irrational number as a relation between the area of a square and its sides (particularly, he proposed the square of area two), searching for an approximation of the decimal expansion, as a way to review irrational numbers for the students. He tried to propose a task that included a relevant mathematical process. This is an indicator of the component "richness of processes" of the epistemic criterion. viii) P4 explained that the idea of P1 would be wonderful for a face-to-face lesson. Although she did not mention it, P4 implicitly refered to the "classroom conditions" and "teacher-student interaction". These are components of the mediational and the interactional criteria respectively. ix) P3 mentioned a possible aversion to the lesson, when the students see the roots of non-perfect square numbers. This idea is related to the component "emotions" of the emotional criterion. In addition, she proposed to do a pre-lesson activity to address the prior knowledge necessary to comprehend the mathematical object that would be taught. This proposal is related to the indicator "adaptation of the intended meanings" of the cognitive criterion. x) P4 proposed only to show that using the Pythagorean Theorem they could obtain an irrational number (in this case, the square root of two), without doing a mathematically rich process (a component of the epistemic criterion) or considering the students' lack of previous knowledge on irrational numbers (a component of the cognitive criterion). xi) P1 proposed the use of material resources (ruler and compass), a component of the mediational criterion. At the same time, he proposed an activity rich in mathematical processes (using the ruler to translate the length of the diagonal of the square on the number line). It is related to the component "richness of processes" of the epistemic criterion. This episode ends with the discussion about the means and resources for the lesson. Thus, both the mediational and interactional criteria were highlighted.

In this episode, the practical argument took place when the teachers considered an indicator of the epistemic criterion: a representative sample of problems must be provided. The group tried to find a certain balance between the epistemic and the cognitive criteria and, in this search, they resorted to other Didactic Suitability Criteria (mediational and interactional criteria). The group agreed to briefly introduce the square root of two as an irrational number after applying the Pythagorean Theorem to a right triangle with length one. This was, in a certain way, a reformulation of the initial proposal that triggered the episode. The consensus obtained depended on the relevance of the different didactic suitability criteria in the argumentative trajectory.

Conclusions

The episode of argumentation that we analysed occured in the lesson-planning phase. When the teachers were planning to work on the Pythagorean Theorem as a relation between the lengths of the sides of a right-angled triangle, they proposed to extend the sample of problems with tasks that included right-angled triangles with irrational lengths, in addition to tasks with natural numbers (Pythagorean triples). This generated an episode of practical argumentation about which actions to do, that evidences how the teachers gave importance to the students' knowledge of irrational numbers to teach the Pythagorean Theorem. Throughout the episode of argumentation, the teachers became aware of the students' knowledge through the validity claims that they made.

In order to answer our study aims, first, we could identify moments of practical argumentation in the course of Lesson Study and Didactic Suitability Criteria. As explained in the last section, the analysis of the argumentative trajectory from the perspective of the didactic suitability made evident the important role of the Didactic Suitability Criteria to argue either for or against a certain action. Didactic Suitability Criteria were essential to provide argumentative strength to the proponent. In terms of Toulmin's (Molina et al., 2019) model, the Didactic Suitability Criteria took the role of the warrant. An area of interest is why Didactic Suitability Criteria provide argumentative strength and are present even when they were not taught. Our interpretation is that this happens because the Didactic Suitability Criteria emerge from a wide consensus among the educational community.

Moreover, it is worth noting that some criteria that are recommendable a priori, such as working on the Pythagorean Theorem in a mathematically rich way or considering the students' previous knowledge, can be in conflict when they are applied to a certain context. Regarding the epistemic suitability, the sample of problems for a partial meaning of the Pythagorean Theorem (the relation between the lengths of the sides of a right-angled triangle) should be representative. The cognitive suitability fosters arguments about the intended partial meanings that are achievable and the students' previous knowledge. In this case, in the practical argumentation about introducing irrational numbers in the teaching of the Pythagorean Theorem, conflict arose between the cognitive suitability criterion and the epistemic suitability criterion. Then, the teachers focused on one criterion or the other, considering the specific context. We observed that the teachers who are in favor of extending the sample of problems, focus on epistemic suitability. While the teachers in favor of limiting the extension of the sample focus on cognitive aspects.

In the broader project, where this study is located, the main objective is to develop teacher reflection. In that sense, it is already known that teaching the Didactic Suitability Criteria to the participants improves reflection. In addition, with the design and implementation of the course that combines the Lesson Study and Didactic Suitability Criteria, it is expected to obtain data to analyse the practical argumentation in different stages of the course and, in this way, verify the influence of the combination of Lesson Study and Didactic Suitability Criteria in the promotion of teacher reflection as a professional competence. From the analysis carried out, our conclusion is that we could identify several moments of practical argumentation in which the Didactic Suitability Criteria had a relevant role. In our opinion, this occured because the training course implemented facilitates collective argumentation, among other reasons.

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Planning and implementation: the impact of a professional development program on teachers' and learners' oral manifestations

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This paper investigates a professional development program's (PD) effect on classroom discourses. The PD was based on problem-based curriculum processing and intensive use of classroom discourse. The authors analyze seven pilot lessons of one of the teacher training's participants, providing an example of how a teacher with 20 years of experience in mathematics teaching, but using traditional teacher-centered methods, can apply the new approach in the classroom. Our analysis is based on a combined theoretical approach that starts from students' mathematical thoughts and examines the teacher's responses. We found numerous possibilities for starting classroom discussions building on students' thinking, and the teacher responded to most of them. Furthermore, the teacher's responses also included several invitations for students to classroom discourse, although these elements were almost absent before the PD.

Keywords: Professional development, problem-oriented instruction, student-centered teaching, class discussion.

Introduction

Based on the Hungarian tradition and inspired by the ideas of Tamás Varga in mathematics teaching (Varga, 1988), the second and third authors have developed a research-based professional development (PD) program. This program was piloted in 2018-2019, with four voluntary teachers as participants. One teacher's lessons are the basis for this paper, since the authors analyze the PD's effect on classroom discourses during the PD's lessons.

The PD program's two main basis points were problem-oriented curriculum processing and the systematic use of classroom discourses. Kónya and Kovács (2021) characterized the problem-oriented approach to learning mathematics by three attributes: (1) students analyze mathematical problem situations; (2) students critically adapt their own and their classmates' thinking; (3) students learn to explain and justify their thinking. This approach is very close to the idea of Roh's definition of problem-based learning: it describes a learning environment in mathematics, where problems drive learning (Roh, 2003).

Cirillo et al. (2014) believe that mathematics classrooms' discursive nature impacts students' identities as mathematical knowers and doers. Leatham et al. (2015) hypothesize that high cognitive demand tasks support the emergence of mathematically significant moments as possible starting points of mathematical discourses in the classroom. However, teachers' and students' proper reactions are crucial. Whether the active knowledge construction takes place in a discourse is decided by the teacher's reaction. Sfard (2003) points out "a productive mathematical discussion (…) turns out to be an extremely demanding and intricate task. The role of discussion coordinator is particularly difficult" (p. 375). Teachers tend to teach and make decisions by routine (Shavelson & Stern, 1981), and it is challenging to change this routine. One of the PD's aims was to break the teacher-centered pattern among practiced teachers and create more complex and precious whole-class discussions.

Therefore, this paper concentrates on two aspects: firstly, the possible starting points of classroom discussion generated by students' thoughts identified by Leatham et al.'s (2015) framework. The second focus is the teacher's reactions to these, analyzed by Sohmer et al.'s (2008) framework. With these tools, our research question is: *How does a problem-oriented professional development with additional focus on classroom discourses impact an experienced teacher's lessons with teacher-centered instructional habits?*

Theoretical frameworks applied for the analysis

A student's action is characterized as a *Mathematically significant pedagogical Opportunity to build on Students Thinking* (MOST) moment when it fulfills six criteria built on each other – each represented by a question (C1…C6) (Leatham et al., 2015), see Figure 1. "In their analytic process, the unit of analysis is an instance (…) Typically an idea unit is one conversational turn or physical expression (such as writing a solution on the board)" (Leatham et al., 2015, p. 92).

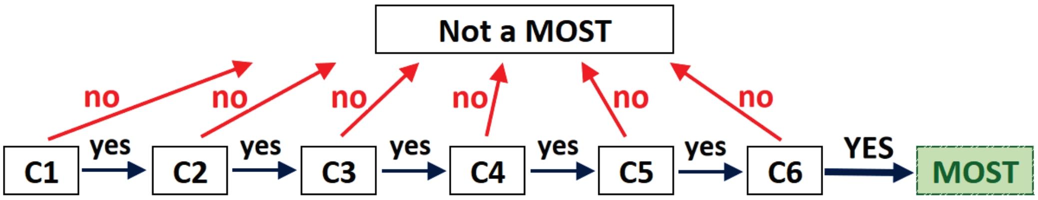

Figure 1: The MOST Analytic Framework

This framework was used to identify MOST moments in the videotaped and transcribed lessons. To find these moments, one should examine each student's utterances with six questions. These questions are built on each other, starting from the very basic point: whether the student's utterance contains understandable mathematical thoughts or not (C1, C2), then examining whether it is accessible for the students (C3) or whether it is the central goal of the lesson (C4). The fifth criteria (C5) can be described by the following question: Does the expression of the student mathematics seem to create an intellectual need (called opening) that, if met, will contribute to understanding the mathematical point of the instance? These expressions can classify into one of the following five groups: (a) a correct answer with novel reasoning, (b) an incorrect answer that involves a common or mathematically rich misconception, (c) a mathematical contradiction, (d) incomplete or incorrect reasoning, (e) why or generalizing questions. The sixth criterium (C6) is about the timing, whether it is worth taking advantage of the student's opening. Later, these criteria will be presented through an example.

Identifying MOST moments provide a quantitative description of a lesson, which can inform about the active and meaningful participation of the students. However, this framework does not provide information about the quality of the teacher's reactions. Therefore, the authors added Sohmer et al.'s

(2008) framework to analyze the teacher's reactions to the MOST moments. This framework seems to provide supportive information to assist teachers in making in-the-moment decisions about whether or how to react to those MOST moments. The authors found the combination of these two frameworks highly fruitful.

Researchers try to identify how some types of teacher interaction influence the following student's utterances to support teachers in activating students more effectively (Dahl et al., 2019). Sohmer et al. (2008) identified talk move as

> a turn at talk that (1) responds to what has gone before; (2) adds to the ongoing discourse; and (3) anticipates or ‘sets up’ what will come next. A talk move is inextricably tied to the context. It reaches beyond a single turn (p. 107).

They studied teachers who have been effective in using talk to promote learning. Six moves were identified that can be useful to model and to elicit academically productive talk: (1) revoicing students' utterances, (2) asking to restate someone else's reasoning, (3) asking students to apply their reasoning to someone else's reasoning, (4) prompting students for further participation, (5) asking students to explicate their reasoning and provide evidence, and (6) challenging or providing a counterexample.

The professional development program

The analysis of this research concentrates on a teacher with 20 years of experience in teaching mathematics. She joined the PD program voluntarily, out of an inner urge to renew her practice. The researchers visited her before the PD, observed her class, and discussed the teacher's professional view as a starting point. She used to prefer a teacher-centered way of teaching: explaining the new material, driving the students with direct questions, and rarely initiating open classroom discussions.

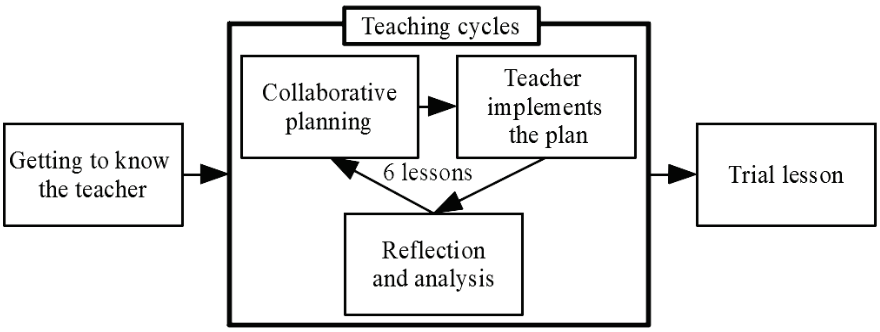

Figure 2: Scheme of the PD-program

In an opening workshop, the researchers explained the design of the program and the principles based on work by Varga (1988). The most important of them were the followings:

1. Problem-solving both alone and in pairs or small groups.
2. Improving students' oral and written communication skills encourage independent opinions.
3. Let students learn through experience and using heuristic strategies.

4. The teacher's role should include encouraging group discussions, planning classroom discussions, and implementing students' proposals into the flow of the lesson.
5. Differentiation and individualized treatment for each student.

After selecting the pilot lessons from the teachers' agenda, the researchers developed detailed lesson plans and provided all the teaching materials. These lesson plans explicitly contained reminders and advice on how to realize the principles of the PD. For example, there were time slots devoted to classroom discussions after solving a problem in small groups.

The teachers gave their opinions and suggestions and finalized the lesson plans. The teacher needed to feel that the lesson plan suited her at the end of the collaborative planning. At the end of the lessons, the teacher should reflect on it both alone and accompanied by the researchers.

The researchers organized six teaching cycles (three per semester during one year of the experiment) and concluded the year-long program with one trial lesson (Figure 2). During the planning for the trial lesson, the teacher had to come up with her own lesson plan.

The whole research process and focus are summarized in Figure 3.

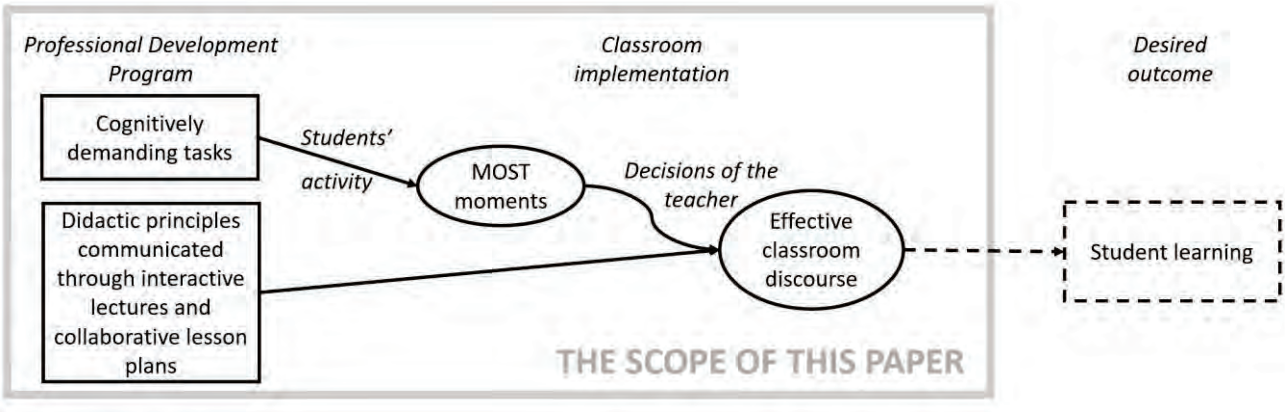

Figure 3: Scheme of the research

The process of analysis

Each 45-minutes lesson was videotaped and transcribed. Two researchers analyzed the transcripts independently and looked for moments where all the six MOST criteria appeared. In a disagreement, the three authors' consensus fixed the MOST moments. The transcripts of each lesson were investigated in the same way:

1. The authors separated those parts of the lesson in which the whole class discussion occurred and identified each observable student utterance according to Leatham et al. (2015).
2. All of these instances were coded according to the six MOST criteria.
3. Further distinctions between the MOST units were made according to the situation that caused it (see criterion C5) and assigned one of the above codes a, ..., e to each unit.
4. After gathering the MOST moments, the authors examined the teacher's response to these MOST moments in a new analysis process according to Sohmer et al.'s (2009) framework. Three categories emerge A) The teacher evaluates the student's action and tells the correct

answer if it was incorrect. B) The teacher starts a classroom discussion. C) The teacher does not notice the MOST moment.

5. Case B was further refined, determining the occurred talk move. In line with the work of Sohmer et al. (2009), the authors used the following codes: the teacher B1) revoicing the student's utterance, or the teacher asks a student for B2) restating someone else's reasoning, B3) commenting on someone else's reasoning, B4) further participating, B5) providing evidence or B6) providing counterexample. The traditional Initiation–Response–Evaluation pattern (IRE) completes the code system as B7.

We illustrate our coding system with an example.

Example - Class 9, fourth pilot lesson

Topic: Divisibility

Episode: Whole class discussion after finding all divisors of 54 in pair work. (Time: 35:49-36:16)

Student:	Can I write them as products? (He writes on the blackboard 1·54,2·27)
Teacher:	[Please use] semicolon…
Student:	(He corrects and writes 1·54;2·27;3·18; 6·9)
Teacher:	How did you know you had to finish here?

This activity is considered as mathematical problem instead of routine task because the student has to define the procedure itself. The teacher does not present the solution as usual, but it appears as the students' activity, as he uses the structure of products to identify all divisors of 54.

The two student manifestations were considered a single action because the teacher's interruption is mathematically insignificant. However, this interruption demonstrates the teacher's accustomed state of controlling everything. The authors classified this student action as a MOST moment because it meets the C1-C6 criteria:

C1. The student is concentrating on mathematical ideas and not offtopic themes.
C2. The mathematical point of the instance is to determine all divisors of a number and decompose it into two-factor products in all possible ways.
C3. The mathematical point is accessible to all students, but at that point not all students realized it as a helpful tool for the task.
C4. A deep understanding of the above procedure is one of the lesson's goals.
C5. A deeper analysis of the fifth criterion shows that this is the case (d), i.e., incomplete reasoning, as the student wrote the products in a logical order but did not verbalize the reason behind it.
C6. Finally, the timing is considered appropriate as all students were paying attention, and there were still nearly 10 minutes left in the lesson.

The teacher recognized the MOST moment and asked the student to provide evidence, i.e., to explain why the presented procedure is appropriate for finding all the divisors. Therefore, the authors coded the teacher action responding to the MOST moment as B5.

Findings and analysis

The emergence of MOST moments

The analysis found numerous MOST scenes that have emerged applying the problem-oriented lesson plans, which were considered satisfactory from the researchers' point of view (Table 1).

Table 1: The emergence of MOST moments

MOST category	a) Novel reasoning	b) Incorrect answer	c) Mathematical contradiction	d) Incorrect reasoning	e) "Why" question	Total
Occurrence during the 7 lessons	13	22	0	16	2	53

Most of the MOST situations in this research emerged from students' incorrect answers (42%), and "MOST" situations based on the student's novel approach or the student's "why" questions were less frequent while mathematical contradictions did not appear.

Teacher's reactions on MOST moments

The teacher identified the MOST scenes effectively (Table 2). The authors attribute this success to the teacher's 20 years of practice, in addition to the influence of the PD. Leatham et al. (2015) also support this view implicitly, as they stated that novice teachers could miss realizing when a MOST scene has developed more often.

Furthermore, 64% of the identified MOST moments do not end with a simple teacher evaluation but lead to a "talk move." The authors consider this to be the result of the PD's approach, as the teacher was aware of the importance of classroom discourse, which she almost neglected in her previous teaching practice.

Therefore, the PD's result is considered to be the realization of the importance of the MOST moments and the use of the appropriate talk moves in the teacher's reactions. In conclusion, the teacher started using the learner's initiative to guide the lesson.

Table 2: Teacher's reactions on MOST moments

MOST category	A (evaluation)	B (talk move)	C (unnoticed)	Total
Occurrence	18	32	3	53

The subtle structure of "talk move" reactions

The dominant "talk move" reaction was that the teacher involved others in the conversation (B4, 20 out of 32 talk moves, 62.5%), see Table 3. Thus, B4 has become an almost permanent behavior, especially in the case of novel reasoning by the students. However, the teacher has also used it when a more detailed explanation of the student's own thinking, i.e., elaboration (B5), would have also been adequate. Moreover, since the problem-based approach requires the learner to think critically about his/her thinking, encouraging this elaborative behavior would also have been part of the problem-based learning approach.

Table 3: The subtle structure of "talk move" reactions

MOST category	B1 revoicing	B4 participate	B5 elaboration	B6 counterexample	B7 IRE	Total
a) Novel reasoning	0	7	0	0	0	7
b) Incorrect answer	1	5	3	2	2	13
d) Incorrect reasoning	0	7	3	1	0	11
e) "Why" question	0	1	0	0	0	1
Total	1	20	6	3	2	32

Conclusions and pedagogical implications

During the PD's lessons, when the teacher got support from training, lesson plans, and collaborative lesson planning, numerous MOST moments mainly emerged from students' incorrect answers or incorrect reasoning. Furthermore, the teacher identified MOST scenes effectively. Most of them do not end with a simple teacher evaluation but lead to a talk move, supporting students' activity. Based on one previously observed lesson and the discussion on her professional view of teaching before the PD, the authors believe that this is the result of the PD's approach. Although further research is needed to prove this finding.

Based on the above result, the authors conclude that the PD is probably suitable to improve the amount and the quality of classroom discourse in an experienced teacher's lessons. It would be worth examining lessons after the PD to explore more about the PD's long-term effect.

It is worth to highlight that identifying mathematically valuable moments is insufficient. Teachers must also be aware of the importance of MOST moments and consciously apply potential talk moves. However, we still know little about how the implementation of talk moves unfold and what is needed to enable teachers to apply them effectively in practice. In connection with it, this paper also argues that the two joined frameworks were beneficial to examine how the demanding mathematics gets leveraged into productive discourse.

Acknowledgment

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Initiating a community of practice amongst primary school mathematics teachers – trials and triumphs

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This paper derives from the early phases of the first author's doctoral study. Eight primary school teachers at a school in South Africa's Eastern Cape Province were invited to be part of a community of practice in which strategies for using music to support the teaching and learning of fractions would be explored. Wenger's theory of community of practice guided the first author as facilitator in this collaborative space. Having immersed herself as a researcher in the school context, she began by inviting the participating teachers to interrogate and trial a series of integrated mathematics and music lessons designed by the authors. In this paper the authors analyse some of the early trials and triumphs of working within a community of practice.

Keywords: Community of practice, curriculum integration, primary school mathematics teachers.

Introduction

The first author has, as part of her doctoral research, initiated a Community of Practice (CoP) (Wenger, 1998) with a group of primary school mathematics teachers, all of whom teach at the same school in South Africa's Eastern Cape Province. The intended purpose for this CoP is to explore strategies for integrating music into the teaching of fractions to Grade 4, 5 and 6 students (children between the ages of 9 and 12). Initiating the CoP has, however, been compromised by a number of factors and the first author has had to make a number of adjustments to her initial research design choices. In this paper, the first author, together with her co-authors (the joint supervisors of her doctoral study), share their analysis of some of the initial data from this early phase of the study.

Researcher-teacher relationships are not new to education research, and much focus has been placed on the fostering of positive interactions. Initiating and working within a CoP is not without challenges. In the present instance, some of the challenges encountered are unique to the particular setting; some derive from the unprecedented current global Covid-19 pandemic situation; some are common to CoP platforms more generally (see for example, Pyrko et al., 2017). Many CoPs fail to "reach maturity" (Bouchamma et al., 2018, p. 91). Reasons for challenges noted in the literature include a lack of the appropriate means and strategies to initiate and develop a CoP, as well as difficulties encountered in ensuring that CoP members' needs are met in ways that sustain mutual engagement (Bouchamma et al., 2018; Pyrko et al., 2017). Our initial findings around some of the challenges, and some of the successes, we encountered in getting our particular CoP up and running, may, we believe, be relevant and helpful to other researchers working in similar CoP contexts.

Literature around national, regional and international bench-marking assessments point to low levels of mathematics achievement in South African schools. In writing about these concerning achievement levels, Venkat and Graven (2017) argued that inadequate support is given to South Africa's mathematics teachers in achieving curriculum aims and that this has contributed to a lowering of teacher morale. Spaull (2019) similarly noted that many South African teachers were found to lack confidence in their mathematics content and pedagogical knowledge which negatively impacted their morale. Such observations highlight a need to better support teachers, providing them with opportunities to enhance their competencies, and ensuring they have access to appropriate teaching and learning resources and associated strategies all of which might also then contribute to boosting teacher morale. The first author's setting up of the CoP thus represents one such initiative. It provides a forum through which to support her own ongoing professional development together with that of the participating teachers while simultaneously being able to interrogate task and learning support materials design decisions. For brief discussion of some of the first author's earlier work on the use of music to help strengthen her own Grade 5 students' fractional understanding, see Lovemore, Robertson and Graven (2021).

Literature review and theoretical framework

The South African mathematics curriculum (South Africa. Department of Basic Education [DBE], 2011) aims to develop deep conceptual understanding of mathematical concepts, as well as recognition of mathematics as an elegant and creative human activity. Furthermore, integration across subjects is encouraged. The authors recognised potential benefits of integrating music into mathematics lessons, which, as literature suggests, can contribute to increasing student motivation and participation, and decrease anxiety (see for example, Edelson & Johnson, 2003).

Wenger's CoP model (1998) was used to explore such integration strategies. A CoP is a space where professionals can share strategies, solve problems and learn from one another (Bouchamma et al., 2018; Pyrko et al., 2017). In particular, Wenger's CoP model interrelates the concepts 'community', 'identity', 'practice' and 'meaning' so providing participating teachers with rich opportunities to reflect critically on their teaching practices and resources. As Wenger (1998) explains, *community* refers to developing "relations of mutual accountability" (p. 81) where information and resources are shared responsibly in supporting members; *identity* reflects members' ways of talking about their role in the context of their community (for example, their professional confidence as mathematics teachers); *practice* implies action within a social context which, in turn, gives *meaning* through community members' shared experiences.

The first author, in her role both as researcher and as CoP facilitator (for the sake of simplicity, throughout the remainder of this paper, the phrase 'first author' will be consistently used), was mindful of the deficit perspective noted in literature where, all too often, teachers are viewed as objects rather than professional partners in research undertakings. Such a perspective makes teachers feel 'used' rather than genuine "research collaborators" (Makar, 2021, p. 440). Setati's (2005) distinction between research 'with' teachers and research 'on' teachers is useful, highlighting as it does the merit in creating reciprocal power dynamics between researcher-teachers whereby all benefit from the research interactions. Consistent with Horne and Makar's recommendations about designing a "win-win project" (2013, p. 769), teacher choice and empowerment needs to remain key throughout. Mutually positive relationships help ensure that all members of the group find the project significant to their own practice. This, as Horne and Makar (2013) note, requires planning a mutually agreed upon vision and values, maintaining mutual respect, valuing all members' input and practicing patience and perseverance. Challenges in sustaining a CoP (Bouchamma et al., 2018) notwithstanding, the first author hoped that once the CoP had got off the ground, participating teachers would want to continue in this collaborative space to explore further opportunities for curriculum integration even after her own withdrawal from the site. In our next section we outline key aspects of the methodological decision-making process.

Methodology

Guided, as noted, by Wenger's CoP principles (1998), the first author issued an invitation to eight practising mathematics teachers to explore with her music/mathematics integration strategies. The CoP first met in February 2021. The school at which all of these teachers are employed is a private one and well-resourced relative to many other schools in the country. The school's teaching and learning philosophy actively encourages its teachers to explore innovative teaching strategies, including more integrated approaches towards curriculum delivery. The school's principal welcomed the ideas informing the first author's proposed CoP, adjudging them as fully consistent with the school's commitment to the ongoing professional development of its teachers and he granted the requisite gatekeeper permission to approach the teachers accordingly. All eight teachers voluntarily gave their informed consent. The first author emphasized to them that she was researching *with* them as co-researchers (Setati, 2005). She emphasised too that the research would focus on teaching strategies and resources, not on individual student or teacher achievement.

The design of the broader qualitative, participatory design research study consists of three phases: (1) preliminary introduction phase, (2) collaborative phase, (3) evaluation phase. It is the first and second phases of initiating and maintaining the CoP that constitute the substance for the present paper.

Ahead of the preliminary introduction phase, the first author immersed herself in the school context to gain the kinds of autoethnographic insight (Du Plooy-Cilliers et al., 2014) that would help inform her researcher interactions with the CoP teachers. Although some have questioned the value of auto-ethnographic studies because of their interpretive and subjective nature, auto-ethnography provides a powerful mechanism for researchers to "self-interrogate" (Denshire, 2014, p. 834) their role within a research setting. The first author strove to consistently and critically evaluate her ongoing researcher/facilitator relationship and interactions with the CoP teachers. As a full member of the social group being studied (Anderson, 2006) she continues with her auto-ethnographic journey, spending full days at the school and observing classes, engaging in reflexive practice by writing entries in her reflective research journal during and after every CoP meeting. Reflective journals and field notes represent, as noted by Farrell et al. (2015), an important part of data collection in autoethnography

The preliminary introduction phase is now complete. It consisted of introducing the music-mathematics connection and sharing with the CoP teachers integrated mathematics and music activities. The second (collaborative) phase is still underway. Teachers have been invited to interrogate and adjust the suggested strategies and resources, to experiment with them, and to then reflect on what worked well or otherwise in their integration of music into their teaching of fractions. Via mainly focus group discussions and interviews, the CoP teachers have been providing the first author with valuable feedback data from their early trialling of the various strategies and resources.

Transcriptions of the audio-recorded CoP meetings, focus group interviews and electronic communication between the researcher and the teachers have been made. Data continue also to be collected via the first author's ongoing reflective research journal entries detailing reflections on critical moments, as well as on her verbal exchanges, both formal and informal, with the teachers. Together with the external data from the other participants, this self-reflection provides a means of triangulating the data.

Farrell et al. (2015) explain that thematic analysis in ethnography requires the researcher to "step back and think about the story" (p. 979), which may then result in the development of themes which describe the social interactions. The first author's transcribed field notes, reflective research journal and supplementary external data such as the CoP meetings and interviews, allowed for thick description (Maxwell, 2013) of some of the trials and triumphs experienced both by the CoP teachers and by the first author. The focus here is on identifying patterns and themes emerging from these transcriptions during these two phases of the study. Themes were identified deductively, based on Wenger's (1998) CoP model, by coding the data according to the four elements, *community* (C), *identity* (I), *practice* (P) and *meaning* (M). Patterns emerged from the data which allowed the authors to identify lessons around the trials and triumphs of initiating a CoP to trial novel integration strategies. These are discussed in our following section.

Discussion on preliminary findings – trials and triumphs

In this section, we share preliminary findings on, and insights deriving from some early trials and triumphs encountered by the first author in her working with the CoP. Before doing so, however, we note also that many aspects of the study have been, and continue to be, affected by the global Covid-19 pandemic. Disruptions included the school having to close for periods of time; some teachers being forced to isolate in quarantine; the need to introduce a blended learning approach. Such circumstances placed additional pressures on the participating teachers, and many planned CoP meetings had to be postponed, resulting in a break in momentum and affecting the first author's communications with the teachers. It also created additional time constraints.

Clear communication to maintain momentum

In facilitating the CoP, the first author found a lack of continuous communication delayed the CoP teachers' trialling of some of the strategies and resources. Weeks would pass between meetings and communication, and lines of communication became blurred. She noted in her reflective research journal, 'There is no clarity as to what communication channels to follow' [06/05/21]. The Head Teacher had, for example, suggested the first author communicate directly with the CoP teachers. The teachers, on the other hand, requested that arrangements should rather go through the Head Teacher. This challenge in communication threatened to compromise the 'community' aspect of the CoP (Wenger, 1998). Graven (2004) emphasizes the importance of access to other members in the community and a wide range of "ongoing activity" (p. 182) in order for a CoP to be successful. Finding time, however, in which all the teachers could meet as a group proved a challenge. Normal expectations on teachers, with the additional demands of blended teaching and learning due to Covid-19, meant that their time to meet were limited, leading to the first author deciding to rather be more of a physical presence at the school, something that had initially been suggested by the school principal. She made her visits to the school more frequent and regular. This provided opportunities for more informal ongoing discussion with CoP members. She found that the teachers responded more favourably to these informal, small group discussions, and welcomed opportunities to share their experiences around their trialling of the various integration strategies and resources. The less formal visits appeared to strengthen the first author's relationships with the teachers, and communication increased. Rather than waiting for the next scheduled, face-to-face meeting, teachers began to show a greater willingness to communicate with her via platforms such as WhatsApp and email when they had questions requiring immediate responses, or simply wanted to share a comment about something that had happened in a lesson. This development led to the first author making the research and ethical decision to continue these informal (individual, or small group) conversations alongside the CoPs more formal discussion sessions. They gave her an additional means of supporting the teachers. She observed in her reflective research journal: "I came to the conclusion that supporting Teacher P individually is ethical, as I do not want the study to cause additional pressure or anxiety for her" [28/05/21]. Key points arising from these informal exchanges were noted for subsequent follow-up at the scheduled whole group CoP meetings.

A further challenge the CoP teachers experienced was trying to fit the integrated lessons into their term plans. These were already under strain due to the school days missed because of Covid-19. The Head Teacher alerted the first author to this, during a Zoom meeting, "… something that the teachers are wrestling with … is to find time to fit in these lessons, over and above their weekly curriculum work" [02/06/21]. The first author realised that CoP members had not yet come to recognise the potential value of the integrated lessons as a time-saving mechanism for their teaching of fractions. Instead, they saw the integration ideas as an 'add on' to their existing workload. Data showed this to be an example of the teachers questioning the 'meaning' (Wenger, 1998) of the CoP goals, but the first author was reassured when some of the teachers acknowledged that, once they had become more confident in delivering their integrated mathematics-music lessons, this approach may indeed have represented a more efficient use of teaching time.

The fostering of equitable power relations

Makar's (2021) description of teacher-researcher collaboration resonated with the first author relative to roles within the CoP. She was aware of possible difficulties to do with differential power relations. In the introductory meeting she had emphasised to the CoP teachers that she intended for them to work together as a team of equals, interrogating and adjusting various ideas around integration strategies and resources. Early on, however, she noticed that teachers were experiencing concerns and challenges in trying to teach the initial suggested lessons to try to 'get it right'. She therefore re-emphasised Makar's point (2021) that there is no 'right' or 'wrong' way to carry out lessons, and that, as co-researchers, teachers were at liberty to adjust initial ideas around the design and teaching of a lesson. Recognising that her status as a researcher may inadvertently have made teachers feel somewhat disempowered, the first author reiterated in follow-up meetings the point about their status as co-researchers meaning that, far from following top-down directions from her, they were free to make their own choices as to how to tackle the lessons. Once they had accepted this point, and due, also, to the challenges limiting the frequency of scheduled CoP meetings, the teachers began discussing amongst themselves elements of their lessons and of the challenges encountered. They nominated Teacher P to communicate the substance of these discussions to the first author. Consistent with Wenger's CoP model, they appeared now to be developing a genuine community and embracing their identity as participants/ co-researchers. This data was interpreted as evidence that the CoP was succeeding in encouraging members to exercise greater autonomy over their decision-making. Reflecting on this in her research journal, she wrote:

> Perhaps this is a sign that the participants in the CoP are starting to work together and share common goals. I don't want to be prescriptive or top down in the CoP but rather facilitate the teachers to act. [06/05/21]

The creation of a safe space to trial and interrogate new ideas

During the introductory meeting, there was every sign of initial buy-in from the teachers. They appeared interested in finding out more about the novel idea of integrating music and mathematics. During the preliminary introduction phase, the teachers pro-actively made suggestions as to how they might adjust the resources to make the application of the lesson easier in their classes. Teachers discussing adaptations early on in the study, the first author noted as a positive element of the CoP, taking this as evidence of shared 'practice' and 'meaning' (Wenger, 1998). Keeping in mind a warning from Krajcik et al. (1998, p. 41), she remained aware, however, that challenges teachers face when trialling new strategies and resources may lead to "premature rejection" of such innovations. Cautious to avoid this happening, she spent time reflecting with the teachers on their challenges in teaching their mathematics-music integrated lessons. Some teachers, for example, expressed uncertainty about the second lesson which was based on Western Staff notation of music note values.

> Teacher K: I am starting to wrap my head around it. [21/05/21]
> Teacher D: I didn't feel like in my mind, I knew what I was doing. So that was my mistake. I think I did it fine, but in my mind I still– I wasn't 100% confident. [10/09/2021]

It was these sorts of aspects that the CoP teachers identified as challenging that led to the co-authors having to further grapple with the wisdom of using Western Staff notation. They adjusted the musical representations, stepping back from Western Staff notation (for example, two eighth notes: ♫) and instead, adopting a percussion representation of music beats per bar (using Xs to indicate claps), which, further along, could then be adjusted to reflect a number line. The following comments indicate that the teachers responded favourably to this adjusted musical representation, expressing greater confidence about being able to integrate the music into their fraction lessons:

> Teacher P: I'm really feeling more confident with that [the Xs representing claps] than with the music notes.
> Teacher D: For me, taking the notes out of it and that you say adding a set of claps makes a total difference. I feel more confident. [10/09/2021]

Seven months into the study, after many disruptions, a CoP meeting was held where Teacher K described to the other teachers how her own teaching of mathematics had evolved in response to her participation in the study. The first author recognised this as positively representing the collaborative intention that ideally informs a CoP's research focus and 'meaning'.

> Teacher K: I've changed the way I've taught fractions. Today we had to do a table on common fractions to decimal fractions to percentages. And for the first time ever, I've said, "Okay, but now this common fraction is five fourths –", because I've always just given them three fourths or two thirds…And then they were fascinated, because

> they said, "But then you can't get a percentage!" And I said, "You actually can. It's 125%." And then we had a very interesting conversation because of how you would use it; you can't get 125% for a test, but you can have 125% mark-up… Yeah, and I would never have thought to do that without that jumping/clapping exercise and talking about a fraction as a measure, rather than part of a whole…I think it's taught me how to teach fractions. [10/09/2021]

Conclusion

Analysis of the critical feedback from CoP members, together with first author's preliminary auto-ethnographic data suggests some key lessons which can support researcher-facilitators when trialling new strategies for teaching mathematics. Three key lessons were identified for this paper. A first key lesson is that clear communication channels are of utmost importance for maintaining momentum within a CoP. Immersing oneself in the CoP's school context provides opportunities for gaining deeper insight into the experiences of the teachers which not only helps build more meaningful relationships with the CoP teachers but also provides more time to support them via informal, small group meetings. This became particularly important given the additional time-constraints caused by Covid-19 circumstances. A second key lesson is that the potential for power inequities need to be countered to foster teacher-researcher teamwork and action essential for this kind of collaborative research endeavour. And finally, a third key lesson is that a CoP context has rich potential to act as an invaluable, supportive and empowering space in which not only researchers, but also teachers are able to reflect upon, interrogate, and adjust innovative strategies for enhancing teaching and learning, such as, in this instance, the integration of mathematics and music.

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Identifying improvisation in the secondary mathematics classroom

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This paper argues that improvisation is a common feature of expert mathematics teaching, but that the processes involved remain under-explored by the research community. Drawing on ideas from organisational theorists and improvisational theatre practitioners as well as educational writers, it proposes a framework for identifying and analysing the practice of improvisation in the secondary mathematics classroom. It then goes on to place a more clearly defined concept of classroom improvisation within a wider understanding of teacher expertise and suggests future directions of study.

Keywords: Secondary mathematics teaching, creative teaching, decision making, interactions.

Why improvisation matters.

'Good teachers think on their feet,' was the opening statement of Robert Yinger's 1986 paper presented at the AERA Conference that year (Yinger, 1986, p.263). He clearly regarded this as a self-evident truth, making no attempt to justify the claim beyond the assertion that 'few educators or researchers of teaching would deny this' (Yinger, 1986, p.263) and going on to use the term 'improvisation' on four separate occasions as he discusses the key skills that underpin successful, interactive teaching.

Yinger's discussion is entirely theoretical and does not focus on any particular subject, but his ideas were developed by Borko and Livingston (1989). They found that the 'expert teachers' of mathematics (identified as such by both school and county leaders) were very skilled at keeping the lesson on track while including the comments and questions of their students in the discussion whereas novices struggled to accommodate student input in their lessons. Borko and Livingston concluded that a key marker of maths teacher expertise is a capacity to improvise productively.

Berliner (1994) draws on a range of different sources including Borko and Livingston in an influential survey exploring the nature of teacher expertise more generally. He proposes eight characteristics that distinguish the 'exemplary performance' of expert practitioners including the practice of being 'more opportunistic and flexible in their teaching than are novices' (Berliner, 1994, p.161), and, like Yinger, repeatedly uses the term 'improvise' to describe this behaviour. Rowland, et al. (2003) choose the term 'contingency' in preference to 'improvisation' as one of the four pillars of 'The Knowledge Quartet', which they propose as a framework for understanding mathematics teacher knowledge, but are clearly describing a very similar phenomenon, echoing Yinger when they define it as 'the ability to "think on one's feet"', (Rowland, et al. 2003, p.98). More recently, Pinto (2017), explicitly linked expert mathematics teaching to jazz improvisation in title of his paper delivered at the CERME conference that year.

Improvisation appears to be a recurring theme in discussions of teacher expertise, including expert mathematics teaching, and the brief account offered above draws on a much wider body of work. Despite these frequent references, however, there are very few detailed descriptions of what improvisation might actually look like in the classroom, and still fewer suggestions about how teachers might develop their skills in this area. Borko and Livingston, for example, offer only a few, quite general indications of teacher practice, such as the post-lesson reflection by expert teacher 'Scott' who explains: "I sort of do a little and then they do a little. And then I do a little and then they do a little" (Borko & Livingston, 1989, p.484). This, seemingly casual approach to defining such a key concept is because Borko and Livingston are not primarily concerned with the practice of improvisation, but with the nature of mathematics teacher expertise. Specifically, they are trying to articulate the distinctive ways in which novices and experts conceptualise their mathematical knowledge and conclude that the experts have 'an extensive network of interconnected, easily accessible schemata' (p.485) which enable them to respond more quickly and productively than novices with less well-connected schemata. For Borko and Livingston, therefore, the capacity to improvise well is a pointer towards the structure of this expert teacher knowledge. The implication is that such schemata are a necessary condition for improvisational teaching; the question that remains unasked is whether it is also a sufficient one.

Borko and Livingston (1989) link the notion of improvisation to the metaphor of the teacher as a performer. Developing this line of inquiry, Barker and Borko (2011) explicitly connect classroom practice to a number of seminal texts on stage improvisation such as Spolin (1963) and Johnstone (1981). This is an appealing prospect, because improvisation for the stage is recognised as a teachable skill by drama schools and theatres across the world. If similarities between successful classroom and theatrical improvisation can be identified, well-established methods for developing improvisatory skills on the stage may offer a route for mathematics teachers to move towards expertise more rapidly. The remainder of this paper is devoted to a consideration of how this might be accomplished, offering a conceptual framework for re-examining existing literature and undertaking further investigation.

Identifying improvisation in existing Maths Education literature.

The near silence of Borko and Livingston (1989) with regard to the processes involved in improvisation has already been noted. A decade later, Remillard (1999) described improvisation as 'on-the-spot curriculum development' (p.331), claiming that the practice is central to the way in which textbooks and other curriculum materials are used in the classroom. In contrast to Borko and Livingston, Remillard's primary concern is on the effect this practice has on the way teachers use textbooks and other materials, rather than the way their knowledge is structured, but like them, her interest in the processes involved is secondary; she does, nonetheless, identify one broad category of improvisatory practice – that of 'task adaptation' (Remillard, 1999, p.328) – as an important feature of expert teaching and gives some thought to the way in which this accomplished.

Remillard situates improvisation within a framework of curriculum development comprising three 'arenas' in which teachers participate: the over-arching arena of 'curriculum mapping' which defines the organisation and content of the entire school mathematics curriculum and two distinct, subsidiary arenas of 'design' and 'construction' where the day-to-day decisions made by individual teachers take place (Remillard, 1999, p.322). In this conception, a central feature of the construction arena is 'improvising in response to students' (p.322), and although it is not her primary focus, Remillard offers some analysis of what she terms the process of 'task enactment', breaking it down into two distinct activities, first ‘*reading* of students' performances, that is, observing and listening to students in order to assess their understandings’ then ‘*improvising* in response’ (Remillard, 1999, p.329, author's italics).

Brown (2009), develops Remillard’s discussion of the way teachers improvise in their use of curriculum materials, explicitly associating the process with the way in which jazz musicians interact with a musical score. Drawing on of Yinger’s proposal that teaching can be seen as a ‘design profession’ (Yinger, 1986, p. 275), he describes this design process as one which treats ‘curriculum artifacts’ – textbooks, slides, worksheets and so on – as tools with which the teacher interacts. In Brown’s model, these interactions can be placed on a 3-point scale that characterises the level of teacher agency involved according to the way the artifact is used. The lowest level of agency is labelled ‘*offloading*’ (Brown, 2009, p. 24, author’s italics), which would be exemplified by a teacher simply issuing a worksheet without offering any guidance to pupils beyond that provided by the publisher. The next level is termed ‘*adapting*’ (ibid., p. 24) and involves the use of some existing artifact but involves the teacher making some adjustment to its original use in response to the needs of the class. It is worth pointing out that this kind or adaptation could potentially be planned prior to the lesson, distinguishing from highest point on Brown’s teacher-agency scale, which he identifies with the term ‘*improvising’* (ibid., p. 24). To be classified as falling into this final category, Brown envisages the teacher moving beyond the scope of the original artifact and devising their own spontaneous strategy.

Like Remillard, Brown’s focus is on the use of curriculum resources, but he also identifies three types of ‘teacher resource’ that have a significant impact on the way those curriculum resources are used, ‘a) subject matter knowledge, (b) pedagogical content knowledge (Shulman, 1986), and (c) goals and beliefs’ (Brown, 2009, p. 27). In terms of relating these concepts to practice, however, the discussion is largely theoretical and focuses on ‘The Design Capacity for Enactment Framework’ which the author proposes as ‘a starting point for identifying and situating the factors that can influence how a teacher adapts, offloads, or improvises’ (ibid. p. 27). While the notion of teacher-as-designer is an intriguing one, and Brown’s framework offers an indication of the factors that may be in play when a teacher is engaged in the kind of ‘on-the-spot curriculum development’ described by Remillard (1999), it offers little insight into the processes involved in classroom improvisation.

Any attempt to explore existing literature on classroom improvisation soon encounters the problem that the term is often deployed in studies that appear to be addressing quite different issues. Even the Borko and Livingston study which uses the term in its title is addressing the broader issue of expertise, while Remillard and Brown are concerned with the use of curriculum materials. To identify what is known about improvisation in the mathematics classroom, the first step is therefore to identify the terms in which it has been discussed in the past. In their systematic review of adaptive teaching in mathematics, Gallagher et al. (2022), use the term ‘Teacher Improvisation’ as one of several for their initial database trawl (although for some reason they ignore ‘contingency’). The mere fact that they have placed improvisation within the scope of their search shows that they are adopting a far broader understanding of ‘adaptation’ than the one proposed by Brown, and indeed some of the studies cited in that review seem to regard the terms ‘improvisation’ and ‘adaptation’ as virtually interchangeable. In fact, the Gallagher et al. review offers a useful collection of related terms for conducting a survey of improvisation-related research in Mathematics Education, including adaptive teaching, responsive teaching, unexpectedness, noticing and orchestrating.

Having established an approach to identifying the work that has addressed improvisation in the past, an even more fundamental question arises, namely: what do all these different concepts have to do with 'expert improvisation' as it is understood in the theatre? To answer that question a clear definition of improvisation is required alongside an account of what constitutes expertise in both the classroom and the theatre. The next section proposes a conceptual framework for understanding both ideas in these two very different settings, and perhaps surprisingly, starts in the office.

Improvisation and expertise in the office, on the stage, in the maths classroom.

Organisational theorists Crossnan and Sorrenti define improvisation as '*intuition guiding action in a spontaneous way*' (2002, p. 27, author's italics). They are concerned with behaviour in commercial settings which are very different from a secondary mathematics classroom, nonetheless, aspects of their theoretical framework shown in Figure 1 offer useful insights into how it might be possible to differentiate between spontaneous classroom actions that might be considered fully improvisatory and those which are to some extent prepared.

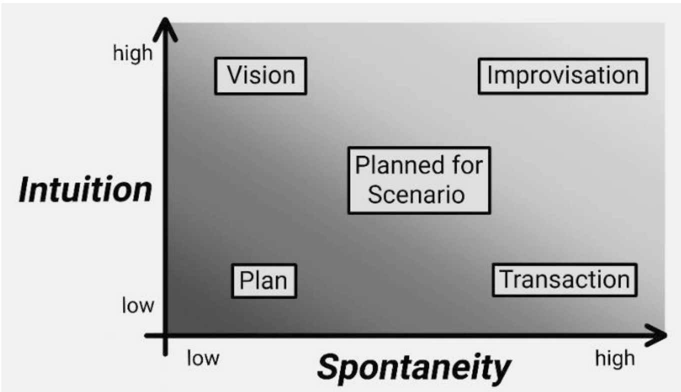

Figure 1: Adapted from Crossnan and Sorrenti (2002)

A key insight of this model is way it places planning and improvising at opposite ends of a spectrum, with the intermediate notion of 'planned for scenario' in between. The proximity of 'planned for scenario' to 'improvisation' in the diagram hint at the possibility that improvisation may be more accessible for teachers who have considered possible scenarios more thoroughly, as they are able to move easily into the semi-improvised region of working within anticipated contingencies, bringing improvisation within easy reach, but it is premature to read too much into what is, after all, a conceptual structure with no obvious scale. The other aspect of the structure which is of interest at this stage is the positioning of the 'transaction' category to describe 'spontaneous but not intuitive' actions. The Crossnan and Sorrenti model therefore allows for the possibility of actions which are spontaneous – in the sense of being immediate but not part of the teacher's formal plan – and yet not improvised. This distinction narrows the concept of improvisation being explored here from the wider category of 'adaptation' described by Gallagher, Parsons and Vaughn as 'any diversion from the lesson plan stimulated by some classroom event' (2020, p.1).

A drawback of Crossnan and Sorrenti's definition of improvisation is that it rests on two other concepts: spontaneity and intuition. Their understanding of spontaneity as acting 'in-the-moment' without time for serious forethought is clear enough, but their view of intuition as 'an unconscious process based on distilled experiences' (2002, p.28) is more elusive. The view taken here is captured in the aphorism: 'intuition and judgment – at least good judgment – are simply analyses frozen into habit' (Simon, 1987, p.63). Intuition is therefore seen as a rapid decision-making process rooted in prior learning which may be inadvertent or deliberate.

While organisational theory has provided a succinct definition, the practical appeal of improvisation is its supposed 'teachability', at least in the sense in which it is understood by stage performers. This raises the obvious question of how the improvisational performances that might win applause in a theatre relate to those that have the potential to support learning in a mathematics classroom. To address these difficulties, the first step is to move the 'theatrical' metaphor from the public setting of the stage to the relative privacy of the rehearsal room. This simplifies the analogy by removing the audience but leaves the teacher in the role of director and continues to situate the pupils as performers. The next step, therefore, is to move yet further from the metaphor of the stage, and rather than envisaging the classroom as a rehearsal space, view it as an 'improvisation workshop.'

One of the most influential figures in the development of improvisational theatre in the mid-twentieth century was the American, Viola Spolin. She talks at length of the 'workshop' (Spolin, 1963, p.18), as the space where performers can develop the skills they need before embarking on formal rehearsals. For Spolin, the key task of the workshop 'teacher' – and it is interesting to note how often she uses the term teacher – is '*giving problems to solve problems*' (Spolin, 1963, p.20, author's italics) through 'problem-solving games and exercises' (Spolin, 1963, p.9). The final step in mapping an improvisational performance to an improvisational mathematics lesson identifies the students as workshop participants the with the teacher adopting the dual role of 'workshop leader' (or 'game-chooser') and player.

To explain the connection between Spolin's notion of a 'problem-solving game' and the mathematics classroom, the key improvisational principle of 'accepting offers' is required. In improvisational theatre, an 'offer' is defined as 'anything that an actor does' (Johnstone, 1981 p. 97); to 'accept' an offer, another performer – or player of the game – must acknowledge that action and build on it, a strategy that is sometimes codified as the 'yes, and…' principle. The brief extract below illustrates this principle in action during a Year 7 class (11– 12 years old) in the autumn of 2020. To make sense of the exchange, the reader needs to know that the pupil has mistakenly interpreted the marker for '–6 degrees' on a temperature scale as 'negative four'.

1	Teacher:	How did you know that one's a negative four?
2	Pupil:	Because it's like, one line behind negative five.
3	Teacher	Yeah, we're at negative five and we've gone one down…
4	Pupil	…negative six!
5	Teacher	Negative six degrees, okay. Remember, when we're going in the negative direction, we're counting down the number line

In line 1, the teacher starts the game by making an 'offer' which involves questioning a pupil who has given an incorrect answer. In line 2 the pupil 'accepts the offer' by answering the teacher using the ambiguous term 'behind' and in line 3, the teacher uses a 'yes, and…' structure to accept the pupil’s offer while simultaneously clarifying the term ‘behind’ through what they say, and by moving their pen one unit to the left along the number line drawn on the board. In line 4 the pupil accepts the teacher’s offer and corrects their earlier answer, then in line 5, the teacher accepts the pupil’s new offer and builds on it by offering a more general description of what is meant by the word negative, again reinforcing their spoken words with actions, this time, walking backwards across their pre-drawn number line in a ‘negative direction’. The way in which the teacher responded, incorporating details of what the pupil had said in their responses provides strong evidence that their remarks were spontaneous rather than planned, and the ease with which the dialogue flowed, with no pause for deliberation, indicates that any decision-making process was intuitive. It is therefore argued that this exchange demonstrates genuine improvisation according to the definition being used here.

According to Johnstone, ‘good improvisers seem telepathic; everything looks prearranged. This is because they accept all offers made – which is something no 'normal' person would do’ (Johnstone, 1981, p. 99). Lines 2 to 5 demonstrate both participants immediately accepting each other’s offers, but the interaction seems rather brief to serve as an exemplar of expert practice. In fact, the offer-acceptance structure continues and is shown below

6	Teacher:	This is just a number line; it’s just a number line disguised a thermometer.
7	Teacher:	Negative six degrees.

At the end of line 5, the teacher had reintroduced the number line (which was discussed earlier in the lesson) and in line 6, they ‘accept their own offer’ taking the idea of the number line and relating it back to the thermometer on which the original question was based. Finally, in line 7, the teacher rejects their own ‘offer’ of the thermometer, and simply restates the correct answer offered by the pupil earlier, ending the improvised episode. Johnstone describes this kind of rejection with the rather pejorative term ‘blocking’ and regards it as something to be avoided, but in this instance, the teacher is using the tactic deliberately to end a diversion from an existing plan, judging that enough time has been spent on this particular question.

The discussion above highlighted several similarities between theatrical and classroom improvisation, but the exchange on which it is based also illustrates several key differences, one being the very different levels of knowledge possessed by the participants. In a workshop situation, it is reasonable to expect all those involved to be aware of the ‘yes and…’ principle, but this does not apply to a mathematics classroom. In a well-run class, the teacher can reasonably expect a pupil to ‘accept’ a direct question and ‘offer’ an answer, which is exactly what happens in line 2. The teacher is then able to incorporate the answer into their next statement. However, the next utterance by the pupil was not so much an offer as an inadvertent calling out of the correct answer, and it was only the skill of the teacher that allowed them to transform line 4 into an offer by responding instantly, accepting, and incorporating it into their next statement. Realising that they could not rely on the pupil to productively maintain the dialogue any longer, the teacher then elected to continue with the offer and acceptance structure in the form of a monologue for as long as they felt necessary. A second important difference between theatrical and classroom improvisation is its purpose: in classroom, it serves to draw in the student by including them in a learning dialogue that serves the teacher’s wider goal. This contrasts sharply with a workshop where the priority is to explore a situation until is mutually agreed that the scene has run its course. Given the different priorities of the classroom, ‘blocking’ by the teacher is legitimate strategy. It is therefore argued that the episode above shows expertise in theatrical improvisation re-interpreted for the classroom.

It is further argued that this brief exchange includes behaviours which are consistent with the account of expert teaching outlined by Winch (2017). In this model, successful learning is identified as ‘epistemic ascent’ (EA), which is explained using the following metaphor: consider a subject expert (who may or may not be an expert teacher) as someone with an overhead view of a room, that room represents the subject, and the view they have of every item within it and all the relationships between those items represents their knowledge. The novice is ‘gradually opening the door to that room, initially gaining partial glimpses of apparently unrelated items’ (Winch, 2017, pp. 80-81), and the process of EA is moving from the view of the novice to the view of the expert. To facilitate this journey among their students, the expert teacher must therefore understand ‘the kind of difficulty that they may be experiencing in learning’ (Winch, 2017, p. 137, author’s italics). To take the learner on this journey, the teacher obviously needs a thorough grasp of the relevant subject knowledge – metaphorically, an ‘expert’ grasp of where all the items within the room lie and the relationships between them – but Winch goes further, suggesting that the expert teacher needs the ability to switch between the perspective of the omniscient expert and that of the novice at will (2017, p. 81). The teacher response in line 3 could be explained by imagining the teacher making such a perspective-shifting move: ‘seeing’ how their pupil has (mis)read the number line, then moving rapidly to teacher mode and homing in on the point where they moved in the wrong direction from the correctly interpreted ‘–5’ marker. The speed of the pupil’s response in line 4 adds weight to this hypothesis.

Conclusion and next steps

This paper adopts Crossnan and Sorrenti’s (2002) definition of improvisation as ‘*intuition guiding action in a spontaneous way*’, going on to argue that the language of theatrical improvisation – in particular, the so-called ‘yes and…’ principle – offers an approach for identifying the practice in mathematics classrooms. It embeds these ideas within a wider understanding of teacher expertise and proposes a mechanism by the which the interconnected framework of knowledge referred to by Borko and Livingston and further explored by Winch might facilitate the practice.

The published evidence cited here is inevitably limited. Borko and Livingston's observations from 1989 are consistent with the view of expert teachers as skilled in perspective shifting, but a more detailed review of existing literature, starting with the terms noted earlier in this paper, needs to be undertaken to ascertain whether there is widespread support for this view. The empirical research described here is still sparser, and the few lines of classroom dialogue merely hint at the ways in which expert teachers might exercise their improvisational skills. More thorough investigation is clearly required. Nonetheless, the prospect of finding simpler ways to articulate, and ultimately teach others to productively engage in spontaneous classroom interactions continue to inspire this author to keep searching.

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Entailments of questions and questioning practices in ambitious mathematics teaching

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The work of teaching mathematics is complex and involves numerous problems and challenges for teachers to handle. Questioning is a slice of mathematics teaching that is both common and challenging. In this paper, we use the context of collective planning in a Norwegian professional development initiative as a space for investigating questions and questioning practices in ambitious mathematics teaching. Based on analysis of nine co-planning sessions, which were carried out over the course of two years, we suggest that questioning practices involve considerations of: 1) what types of questions to ask, 2) purpose of questions and questioning, and 3) timing of questioning. Based on our conceptual framework, we propose a fourth logical entailment relating to the positionality of students, and we suggest that further research is needed to explore this entailment.

Keywords: Questioning, planning, mathematics teaching, interactions.

Introduction

More than a century ago, in her seminal study of questions as a measure of effective teaching, Stevens (1912) investigated if the number of questions asked by teachers could serve as a proxy for efficiency in instruction. Interestingly, she found that it was also necessary to consider the quality of questions. Based on observations across different subjects, Stevens identified three elements of questioning that indicate quality: questions have to stimulate reflection, they must be adapted to students' experience, and questions must aim at moving students' thinking forward. Despite an increasing number of studies of teacher questions and questioning practices, Gall (1970, p. 707) introduced her review of research on the use of questions in teaching by stating that even though everyone agrees that questions are important in teaching, "researchers still do not know much about them. What educational objectives can questions help students to achieve? What are the criteria of an effective question and how can effective questions be identified?" As evident from the review of Gall, as well as from other reviews of research on questions and questioning in (mathematics) education, studies tend to highlight either how questions are used in the classroom (e.g., Gall, 1970), or what constitutes effective questions and questioning practices (e.g., Shahrill, 2013; Wilen & Clegg, 1986). In other words, studies often have a similar focus to the seminal study of Stevens (1912).

Following recent discussions in Thematic Working Group 19 (TWG19), this study attempts to shift focus from *what teachers do* to consider teaching as a *work to be done*. Instead of focusing on activities performed by teachers – and their effects on student learning — we follow Ball (2017) in our attempt to investigate what constitutes the special work of teaching mathematics, and we seek to explore and understand the entailments of this work. In particular, we focus on questioning practices, which constitutes a slice of the work of teaching. Whereas Stevens (1912) focused on questioning in recitation, we focus on questioning in discussions within the context of ambitious mathematics teaching when we approach the following research question:

What can be entailed by questions and questioning practices in ambitious mathematics teaching?

Our use of the word ‘entail’ indicates that something is logically involved or necessitated, and it points toward a view of teaching as work, which implies that it is something to be done, or task(s) to be undertaken (Work, n.d., para. 3). When we use the term ‘questions and questioning practices’, we follow Enright et al. (2016), who use the term with reference to a collection of practices that are carried out within the teaching profession.

Conceptual framework

This study draws on a conceptualization of teaching as instructional interactions that are collaboratively constructed between teacher and students around a particular content, and situated in broader environments (Cohen et al., 2003). Mathematics teaching can never be considered as transmission of content in isolation, but it is “co-constructed in classrooms through a dynamic interplay of relationships, situated in broad socio-political, historical, economic, cultural, community, and family environments” (Ball, 2017, p. 15). In addition, we draw on the notion of teaching as work, as described by Ball (2017) and Lampert (2010). Following their interpretation of the work of teaching, we consider teaching as a complex work that entails many demands and dilemmas that teachers must manage. A study of the work of teaching mathematics, in this sense, thus involves efforts to identify and understand these demands and dilemmas, rather than describing what teachers do and attempting to make sense of these actions.

Our study also draws on a conceptualization of ambitious mathematics teaching where students’ emerging mathematical thinking and sensemaking is at the forefront. Ambitious mathematics teaching aims at positioning students as sense-makers and provide equitable access to learning. To achieve these aims, ambitious mathematics teaching typically includes facilitation of mathematical discussions that elicit and build on students’ thinking (Kazemi & Hintz, 2014). The study draws on data from a larger project, called Mastering Ambitious Mathematics teaching (MAM). This project is organized around cycles of enactment and investigation, and it is inspired by similar projects in the United States (e.g., Lampert et al., 2013).

Methods

Thirty Norwegian elementary teachers participated in the project. In the cycles, teachers were divided into four groups. We analyze data from one group of eight teachers in the present study. The groups participated in 12 sessions over a period of two years, and nine sessions were organized around learning cycles that include the following phases: 1) *Preparation* that involved reading and watching videos; 2) *Collective analysis* guided by a teacher educator around principles and practices central to the instructional activity; 3) *Co-planning* of the instructional activity; 4) *Rehearsal* where one or two teachers tried out the activity with colleagues acting as students; 5) *Enactment* of the instructional activity by the same teacher with a group of students (age 11–12); 6) *Collective analysis* that included reflections on how the principles and practices had worked out. We anticipated that experienced demands and dilemmas would become most visible in the planning phase, so our focus here is therefore on the third phase of the learning cycles.

We analyzed nine co-planning sessions using Sportscode video analysis software, which allowed for direct coding on the videos. Initial deductive coding followed an adapted version of the discussion planning framework of Boerst et al. (2011). We then conducted inductive coding of the segments that were coded as *questioning* in the initial coding phase, with an emphasis on trying to identify tasks that are entailed in the work. Instead of asking what teachers did, and what they discussed in the planning sessions, we asked what kinds of problems, dilemmas, or decisions they were faced with. Three categories emerged from the inductive coding, and these are presented and discussed below.

Entailments of questions and questioning practices

In the following, we present excerpts from our data material to illustrate and discuss the proposed categories of entailments.

Considering type of question to ask

In the planning sessions, we observed several instances where the group engaged in discussion of what questions they could ask students. This is not surprising, as questions abound in mathematics lessons; when teachers want to guide students in a certain direction, they often try to use questions instead of teaching by telling.

In the second session, when the teachers planned an activity using quick images, they discussed the challenge of stimulating the students to identify a certain pattern. One teacher suggested that they could ask: "How can you use the 4 times table to get the first number? In other words, how can you get 5 as an answer when you have to use the 4 times table." Another teacher commented that this is hard for students, both to visualize and discuss. One of the teacher educators in the group interrupted,

TE1: Yes, but let's try to come up with a really good question that helps the students understand where we want to go, shall we? Because this is really where we are at, right. If we want them to discuss this, which we might not want to eventually, but if we want them to discuss this…

This was one of several instances where the teachers discussed how to use questioning to guide the students in a particular direction, which is a task that is entailed in the work. In this case, the teachers wanted to help the students discover the pattern of $4n + 1$. From the teacher educator's reference to "a really good question", and from the overall discussion, it appears that the aim here was to identify a particular question that could work as a prompt to stimulate exploration through discussion, rather than to check students' understanding, as in recitation.

Although the primary focus in this discussion seemed to be on finding the best question to ask, we notice that there is also an embedded focus on purpose. This leads to the next, and tightly related, kind of entailment that teachers are faced with in questioning practices.

Considering purpose of questions and questioning

Another entailment is thus to consider the purpose of questions and questioning practices. For instance, teachers can consider how to formulate a question to stimulate students' thinking. We noticed that this is also at play in the above discussion of the type of question to ask. In another session, the teachers discussed how they wanted students to discover connections between a string of tasks, but they hesitated to ask the students straight away if they could see any connections. One teacher reflected, “So, we want to tell them: I want you to consider these numbers now. Do you see any connection between them?” Then she added, “Is it wrong to be so direct about it?” The teacher educator affirmed that they could ask students directly if they could see any connections, but this challenge of how direct they should be, and what kind of questions they should ask, persisted.

As the teachers continued to discuss this, the issue of whether to use an open question came up.

T1:	When we say, in the context of 4000 divided by 160, we could ask: Do you see a connection with the previous task? 400 divided by 16.
T2:	Should we say that?
T1:	Well, if we say that, they have to consider it for a while, and then they will notice that both are multiplied by 10. And, what do you think happens there?
T3:	That is kind of more open. If you consider these expressions, what do you think about?

Although this appears to be another example of considering what type of question to ask, we notice an underlying challenge relating to purpose here. The task of deciding on a type of question to ask is thus entailed by considering the purpose of stimulating students’ thinking, which is important in ambitious mathematics teaching. Sometimes teachers consider using an open question to allow for more creativity and exploration, but, at other times, they might consider using a more closed and directive question to check on students’ understanding. This choice depends on the purpose.

Considering order and timing of questioning

A third entailment of questions and questioning practices relates to timing. In direct response to the discussion above, T2 asked: “But do you think we should say that first? Should we begin by asking if they see a connection, or should we let them think for a little while first, before we hook them on?” This points at two considerations that need to be made in questioning practices: one concerning the *order* of questions, and another concerning the *timing* in the sense of finding the exact right moment to ask a question. If the overall aim of an activity is to stimulate exploration, which is prominent in ambitious mathematics teaching, teachers might want to start by asking a more open question, before asking for connections more directly. T2 seemed to be conscious about this challenge. When one of the other teachers moved on without responding, he continued to press for reflections about this: “But should we ask them: Do you see a connection with the previous task? Or should we allow them to think a little bit on their own first, to see if someone actually discovers it … before we hook them on?” On the one hand, we notice that this comment is indicative of a challenge concerning the order of questions. On the other hand, there appears to be an entailed issue concerning timing, both with respect to timing of the questions in the overall trajectory of the class, but there is also an entailed challenge of how long to wait before providing the students with a more directed question.

The discussion of order and timing above relates to a situation that the teacher controls. Deciding what question to ask first or how long students may think before posing a more directive question, are both examples of issues that teachers can consider beforehand, as they plan the lesson. However, in a classroom with real students, unexpected situations often occur. For instance, a student might immediately come up with a solution that ends the discussion prematurely. This kind of situation was discussed in the third session, where the teachers planned an activity on quick images (see Figure 1).

TE1: What do we ask if this [pattern] comes up quickly?

T1: This relates to our previous discussion about how 9 can be represented in different ways. So it might be a good idea to bring up that one in addition, showing that they are almost similar. Perhaps put this [pattern] below the other one.

TE1: But what if someone says 3 times 9? Yeah, that was nice. Then the discussion is kind of over. What kind of questions do we ask then, in order to move on?

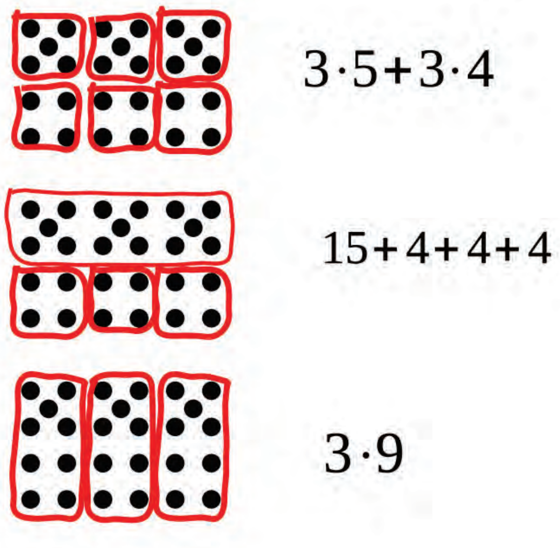

Figure 1: Reconstruction of some of the teachers' annotations on the quick images discussed

This question was followed by a pause in the conversation, indicating that it was considered a dilemma by the teachers. One of the teachers followed up, suggesting that they could prompt the students to explain their thinking, and show the others how they see the 9 in this representation. This suggestion corresponds with the literature on cognitively guided instruction (e.g., Carpenter et al., 2015), and points at eliciting students' thinking. The initial issue of timing in questioning is thus related to the more foundational question about purpose of questions and questioning practices.

Discussion

In her review of research on questioning, Gall (1970) emphasized the *type* of questions being asked, and Stevens (1912) focused on type of questions as well as the number of questions that were asked. Early studies of questioning in teaching had less emphasis, however, on the *purpose* and functions of questions, and they did not emphasize *timing* of teacher questioning. Yet, we can recognize some of these as underlying perspectives in many studies of teacher questioning. Another important difference between our study and other studies of teacher questioning is that most other studies focus on what teachers *do* – what types of questions they ask, what purpose their questioning has, and how their timing of questioning influences students' learning – and not on the entailments of the work of questioning. In the following, we discuss results from our study in relation to previous research, and in relation to our conceptual framework. Analyses of what teachers do often end up in evaluation of their performance, attempts to explain why they performed in a certain way, or in efforts to measure the effects of their performance on students' learning. As opposed to this, analysis of teaching as *work* tends to focus on understanding the complexity and entailments of teaching and its demands. Such analysis thus contributes to unpacking and conceptualizing the work of teaching, and it may also contribute to developing a much-needed professional language to describe this work.

Every review of questioning in teaching that we have seen involves a focus on types of questions that teachers ask. In her review of the use of questions in teaching, Gall (1970) notes that studies have used different taxonomies for classifying types of questions. She highlights Bloom's taxonomy as the one that "best represents the commonalities that exist among the systems" (p. 710). Later reviews, both in general education and in mathematics education, also highlight the importance of considering cognitive levels of questions (e.g., Shahrill, 2013; Wilen & Clegg, 1986). With respect to the instructional triangle, considering the type of question to ask relates to what is described in the triangle as interactions between teachers and students. In ambitious mathematics teaching, deciding on the type of question to ask is particularly important. Whereas use of questions might be appropriate to check on students' understanding, it can be challenging to decide on a good question to prompt discussion.

Considering the purpose and function is another entailment of questions and questioning practices, and Enright et al. (2016) unpacked this entailment in their attempt to develop a typology of questions by instructional function — also by applying the instructional triangle as conceptual framework. Whereas teachers in the context of planning would typically consider the purpose of questions and questioning, the entailments that teachers are faced with inside of the instructional interactions would often be more related to considering the functions of questions. Consideration of purpose and functions of questions would often be located along the axis of interactions between teachers and content, for instance when considering purpose in relation to the learning goal for the lesson. However, purpose and function of questions might also involve considerations around what types of interactions between students and content that teachers wish to facilitate. For instance, teachers might decide to use open questions or why-questions when they aim at facilitating a discussion, whereas more closed questions or what-questions might lead to recitation. Ambitious mathematics teaching aims at facilitating discussions that elicit and build on students' mathematical thinking (Kazemi & Hintz, 2014), and our analysis illustrates some of the challenges teachers are faced with when trying to build on students' thinking without being too directive. This balancing of drawing on students' thinking while at the same time leading a class toward a mathematical goal may constitute a dilemma.

Entailments of considering order and timing of questions might involve similar considerations as those teachers have to make in the context of purpose and function. One aspect of timing that has frequently been described in research on questions and questioning is wait time (e.g., Shahrill, 2013; Wilen & Clegg, 1986), and this was also visible in the planning discussions we analyzed. Considering what is the appropriate time to pose a question has been less focused on in previous research, but it is also something teachers need to consider when interacting with students. Our analysis also indicates that considerations of order and timing of questions and questioning might constitute a task of teaching that is particularly pressing in the context of ambitious mathematics teaching. As one of the teachers noted, a student might come up with a response that are perceived by the class to end the discussion, and the teacher is then faced with a considerable challenge of coming up with a follow-up question or prompt to move the discussion forward.

All three entailments of questions and questioning practices that we have identified thus appear to fit well within the instructional triangle, and we have discussed above how these entailments are particularly pressing in the context of ambitious mathematics teaching. What we did not observe in our analysis of planning sessions, and what we have not seen much of in other studies, are considerations around interactions with environments with respect to questions and questioning practices. Recent studies of the work of teaching mathematics that build on the conceptual frames of the instructional triangle have explored interactions with environment in terms of considering identity and positioning of students. For instance, in her discussion of what constitutes the mathematical work of teaching, Ball (2017, p. 17) states:

> Central to bear in mind is an inherent fact of teaching, namely, that teachers are always communicating, relating, and making sense across differences, including differences in age, gender identities, race and ethnicity, culture and religion, language, and experience. This important dimension of difference in identity and positionality means that a fundamental part of the work of teaching is being aware of and oriented to learning about and coordinating others' perspectives.

Thus, a logical entailment of questions and questioning practices would be to also consider identity and positioning. As can be seen from the quote above, such considerations tap into interactions with environments and is a fundamental — but often overlooked — part of the work of teaching mathematics. There are a couple of possible reasons why positionality did not emerge as a category from our analysis. On the one hand, it might be a feature of the project. The teachers in the MAM project engaged in cycles of planning and enactment in a context where they did not know the students. When planning the activities, they had to make assumptions about students, since they did not know what students they would encounter before the enactment phase. In a context when teachers do not know their students, they might not be expected to consider positionality, which requires knowledge of students and their identities. On the other hand, the lack of emphasis on positionality in planning of these instructional activities might also indicate a general lack of emphasis on positionality among (these) Norwegian mathematics teachers. If that is the case, it is even more important to emphasize this — both as a fundamental part of the work of teaching mathematics, and as a fundamental part of questions and questioning practices in mathematics teaching. We believe that considering *positionality* might constitute a fourth entailment of questions and questioning practices in mathematics teaching, and we call for future studies to investigate it. This is particularly important in the context of ambitious mathematics teaching, which aims at positioning students as sense-makers and provide equitable access to learning mathematics (Kazemi & Hintz, 2014). There is thus a need to explore what this might look like in questions and questioning practices.

Conclusion

Questions and questioning practices in teaching have been studied for more than a century. Whereas much research has focused on how teachers use questions and the effects of these questioning practices on student learning, we propose to instead focus on trying to understand what might be entailed in questions and questioning practices — considered as a slice of the professional work of teaching mathematics. From our analysis of teacher planning sessions, we identified three entailments that relate to considering what *types* of questions to ask, the *purpose* of asking questions, and the *timing* of questioning, which includes considering both the order of questions and deciding on the right moment to ask questions. In addition, we suggest that a fourth entailment is also involved in questions and questioning practices: considering *positionality* of students through questioning practices. This involves attending carefully to how teachers' questions and questioning practices can contribute to providing equitable access for all students and thus serve as part of an overall effort toward justice and inclusion — in school, but also in the society at large.

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An investigation of teacher's practices when teaching mass measurement in grade 4 in Malawi

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This paper reports on a study on teachers' practices when teaching mass measurement in grade 4 in Malawi. Data was collected from video recordings of three grade 4 teachers (two lessons each) who were purposively sampled to ensure they were teaching measurement of mass. The teachers' practices were analysed using the Mathematics Discourse in Instruction (MDI) framework. Findings of the study showed that the common teachers' practices included: having lesson plans with clear lesson goals, using similar examples and tasks throughout the lessons, involving learners in hands-on activities without providing the conceptual understanding of the tasks, and asking low level questions. We argue that teachers' practices when teaching mass measurement should focus more on developing learners' conceptual understandings of mass measurement.

Keywords: Teacher's practices, mathematics discourse in instruction, measurement, mass.

Introduction

Measurement of length, area and mass is a central part of primary school curriculum in many countries (e.g., Ministry of Education, Science and Technology [MoEST], 2006; National Council of Teachers of Mathematics [NCTM], 2020). Research on the teaching of measurement shows it is poorly taught in many countries and is focused on procedures rather than conceptual understanding (Clements, 2003, Irwin et al., 2004, Zacharos, 2006). Stephan and Clements (2003) have attributed this to curricular content, teachers' knowledge and instructional practices. Cheeseman et al. (2011) challenged the traditional-curriculum approach of using informal units for an extended period of time before introducing standard units of measure, and Zacharos (2006) showed that both students and teachers have difficulties in understanding the concepts of measurement.

While several studies have investigated the teaching of length and area measures, less is known about the teaching of mass measurement in primary school. For example, in our review of research reported in CERME11 (see Jankvist et al., 2019), we found 12 studies reporting on the measurements of length and area, but none reporting on the teaching of mass measurement. In addition, research about the teaching of mass measurement in the Malawi context is specifically lacking. This motivated us to investigate Malawi teachers' practices in the teaching of mass measurement. Measurement is one of the core elements of the mathematics curriculum in Malawi, from grade 1 through secondary school. For grades 1 and 2, the curriculum includes mass measurement using non-standard units, while in grades 3 and 4, learners are introduced to the standard units of mass and are taught how to measure mass in kilograms and grams (MoEST, 2006). The learning of measurement involves the use and understanding of procedures and the development of conceptual understandings. In the literature, these are commonly discussed in relation to length measurement (e.g., Battista, 2006; Lehrer et al., 2003) but can be transferred to other measurement concepts like mass.

According to Cheeseman et al. (2011) children need rich experiences involving the measurement of mass, especially in the early grades. Rich experiences are those in which learners are offered opportunities to engage in activities that lead to conceptual understanding in mathematics and challenge them to think and foster the communication of mathematical reasoning.

We describe teacher practices as what teachers do and say in a lesson. Teacher practices matter in mathematics lessons and determine what learners learn and the skills they acquire (Adler, 2017). This is particularly critical in teaching measurement and in early grades where learners depend on teachers to learn. The need to understand practice in the Malawian context motivated our research question:

What are teachers' practices when teaching measurement of mass in grade 4 in Malawi?

Literature review

McDonough et al. (2012) found that although measurement may look simple, insights gained from research into young children's concepts of mass measurement show that the learning of measurement can be complex. Some studies on mass have focused on learning in early grades. McDonough (2010) showed that children in the early years of school have informal knowledge of mass measurement. They develop this knowledge during outdoor play activities prior to formal schooling. Some acquire the knowledge of mass measurement from handling or weighing things at home (Spinillo & Batista, 2009). Cheeseman and McDonough (2013) reported that children's learning about measurement continues from experiences prior to school through formal schooling, where they are taught about attributes of measure including length, mass, time, area, angle, and volume. MacDonald (2010) found that children four to six years old have an awareness of the attribute of mass, as revealed in drawings of measurement situations. These and other findings reveal the importance of underlying knowledge and skills that early grade learners bring to the learning of mass measurement. These include informal knowledge of mass, handling or weighing objects and attributes of mass. Knowing learners' prior knowledge ensures that the teacher works towards building on what is already known and correcting misconceptions.

The teaching of mass in early grades involves measuring in both standard units and non-standard units. McDonough and Cheeseman (2015) found that in learning to measure, children develop other skills, such as how to use a balance scale, and develop understanding of foundational ideas, including awareness of the attribute, comparison and unit iteration. Other skills that learners develop in measuring mass are precision and origin (Lehrer et al., 2003, Sarama & Clements, 2009). Therefore, in teaching measuring mass in standard units, more skills and knowledge are developed in learners. These skills are used or applied in other mathematics topics and subjects like science.

Other researchers have pointed out the need for learners' conceptual understanding and reasoning when they are measuring in standard units. Wilson and Osborne (1992) found out that while the basic idea of direct measurement is simple, there are complex mental accomplishments within measuring that are often downplayed in typical lessons. Opportunities for children to reason, with the purpose of coming to understand foundational or key ideas of measurement, can be enhanced by task design and teacher actions when carrying out those tasks. This makes the teaching practice important as what teachers say and how they say it matters for children's learning (Adler, 2017).

Theoretical framework

This study used the Mathematics Discourse in Instruction (MDI) as its theoretical and analytical framework. The framework was developed from extensive research among poorly resourced schools in South Africa (Adler & Ronda, 2015). The MDI framework describes the lesson “bit by bit” meaning step by step, thereby analysing teaching shifts that take place in a mathematics lesson. This way of analysing the lesson was useful in this study as it enabled thorough understanding of the shifts in teaching practices and how each shift made mathematics available to learners.

In describing the framework, Adler and Ronda (2015) represents it diagrammatically as below:

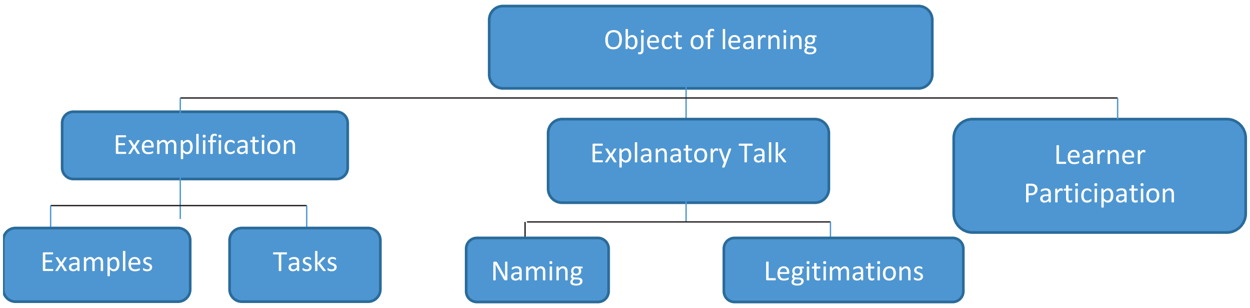

Figure 1: Constitutive elements of MDI (Adler & Ronda, 2015, p. 3)

The four constitutive elements of MDI are object of learning, exemplification, explanatory talk and learner participation. Object of learning is regarded as the lesson goal (that which students are to know and be able to do). In the diagrammatic representation above, Adler and Ronda (2015) separate the object of learning from the other components of MDI. The three components of *exemplification*, *explanatory* talk and *learner participation* are viewed as the key mediational means or cultural tools in a typical mathematics classroom instruction. These tools are used to achieve the object of learning. Exemplification which is further divided into *examples* and *tasks* is a common practice in mathematics teaching where lessons start with examples followed by similar tasks for learners' practice. Examples are categorized into three levels, from Level 1 to Level 3, depending on whether the selection of examples are similar, contrasting, or a combination of the two.

Explanatory talk involves communication by the teacher that takes place during the lesson. It is divided into *naming* (words used to name the mathematics being discussed) and *legitimation* (explanations of what is to be known and done in the lesson). Naming is also categorized into three levels: Level 1 meaning colloquial language is used, including ambiguous referents such as this, that thing, to refer to objects; Level 2 if some math language is used to name the object or component, or the string of symbols is simply read when explaining; and Level 3 if appropriate names of math objects and procedures are used. If non-math legitimation is used (such as visual cues, or metaphors relating to features), it is classified as Level 1 NM (nonmathematical); Level 2M (math) if a specific/single case, real-life application or purely mathematical explanation are used; Level 3M (math) if equivalent representations, definitions, or previously established generalizations are used but explanations are unclear or incomplete; and finally, Level 4M if it is a general full explanation.

Learner participation on the other hand allows learners to participate in the teacher's communication even if it may be in form of mostly listening to the teacher (Adler, 2017). It also involves their participation in answering questions. It is also categorized in three levels: Level 1 if learners simply answer yes/no questions or offer single words to teachers in the form of unfinished sentences; Level 2 if learners answer (what/how) questions in phrases/sentences; and Level 3 if learners answer why questions or present ideas in discussion, or the teacher revoices/confirms learners' questions.

Methodology

This study collected qualitative data from three teachers in three classrooms in two schools in Malawi. The teachers were purposively sampled to ensure that they were teaching mass during the time of data collection. Two lessons from each teacher were observed and video recorded. We used the MDI framework to analyse what the teachers were doing and saying in class to make the idea of mass available to learners. We sought consent from the District Education Office, Head teachers and teachers themselves to record the lessons. Teachers were free to withdrawal from the study anytime within the data collection exercise.

Findings of the study

Due to limited space, in this section we present one sample lesson in detail and its analysis using the MDI analytic framework. The title of this illustrated lesson was "Kulemera kwa Zinthu" meaning mass of objects. The teacher guide gave the following success criteria of the lesson: i) measure mass in kilograms (kg) and ii) measure mass in grams (g).

Following the MDI framework, the lesson was divided into five events, with a new event distinguished by a shift in activity. Below is a detailed description of the events with dialogue between the teacher and the learners.

Event 1: Measuring using non-standard units

The teacher carried a stone and a duster in her hands and asked learners to identify which one of the two was heavier than the other. The learners were able to identify the stone as being heavier than the duster. The activity was repeated using two stones of different sizes. The learners were able to identify the heavier stone, judging from the sizes of the stones.

Teacher	Which stone is heavier between Stone A and B?
Learners	Stone A.

Event 2: Measuring using unmarked simple balance

Learners were shown a simple balance and the teacher explained how it is used to determine which object has more mass than the other or if any two objects have the same mass.

Teacher	The balance tilts on the side where there is a heavier mass. That shows that one object is heavier than the other. When the two objects have the same mass, the balance does not tilt.

The teacher demonstrated how to measure two stones by putting them in bags and hanging the bags on the simple balance. The learners were able to identify the stone with a bigger mass. This activity was repeated using different objects such as duster, stones, books and pencils. Learners carried out the rest of the activities in pairs.

Event 3: Comparing masses of objects with a 1 kg packet of sugar

In this activity learners were comparing mass of an object with a 1 kg packet of sugar on a simple balance. The teacher put a stone in one bag and a packet of sugar in another bag and hung the bags on a simple balance. Learners were able to identify which one of the two had more mass than the other by looking at how the simple balance tilted. The learners repeated this activity using stones of different sizes and a chalkboard duster.

Teacher	Which is heavier between a 1 kg packet of sugar and a stone?
Learners	A packet of sugar.

Event 4: Introducing gram (g)

In this event, the class continued to compare the mass of 1 kg of sugar with various objects using a simple balance. The teacher was careful to choose objects that were less than 1 kg this time around. The learners were then informed that objects that were less than 1 kg are measured in grams. The learners identified items such as 500 g salt and 100 g baking soda that were present in class. The class was comparing masses of items in grams with the 1 kg packet of sugar using the simple balance. Similarly, the teacher put the 2 masses under comparison in bags and hung them on a simple balance.

Teacher	Which is heavier between a 1 kg packet of sugar and a 500 g packet of salt?
Learners	1 kg packet of sugar.

Event 5: The relationship between kg and g

The last event was for learners to establish the relationship between kg and g. The teacher showed that 500 g + 500 g = 1 kg by comparing two packets of 500 g salt and 1 kg packet of sugar on a simple balance. Learners identified the two sides of the simple balance as the same and concluded that 1000 g = 1 kg. The learners were later given the following three exercises to write individually in their exercise books: i) 250 g + 250 g + 500 g = ____ kg, ii) 3 kg = ___ g, and iii) 200 g + 500 g + 300 g = _____ kg.

Discussion

There were two lesson goals given by the teacher: i) measure mass in kilograms, and ii) measure mass in grams. These identified objects of learning were well captured in the lesson plan. Most of what happened in this class was direct comparison of masses of objects using standard and non-standard units. The common question by the teacher in the first four events was: Which is heavier between a stone and a packet of sugar? At this point, learners' judgement of "heavier than" was based on how the simple balance tilted. In event 5, learners were comparing known masses of items such as 1 kg of sugar and 500 g of salt.

The lesson intended to teach learners how to measure masses in kg and g. However, what happened in this class was mostly comparing and ordering masses of objects. According to Cheeseman et al. (2011), comparing, ordering and matching masses of various objects are important skills in mass measurement. They form part of the preliminary skills that children should acquire in measurement. However, more skills and knowledge of mass measurement need to be acquired in the early years of primary schooling.

The second element of MDI is exemplification and it consists of examples and tasks. Mathematical goals are supposed to be achieved through elaborated examples and given tasks. Since the lesson under discussion was mainly hands-on, we analysed the hands-on examples and tasks that the learners were given and the materials with which the children were engaged. However, when analysing the five events of the lesson, it was observed that the tasks were similar in terms of level of difficulty and demands for cognitive ability. The instructions for the tasks were the same and repetitive. Further, the tasks were set procedures in comparing the given masses. The lesson focused on teacher's instructions on how to compare different masses of objects. Conceptual understanding of the underlying idea of mass and measuring mass in standard units of kg and g was missing in the lesson.

The third element of MDI is explanatory talk. The MDI framework divides teacher explanatory talk into naming and legitimations. In this case, naming are mathematical terms used in the lesson while legitimations are explanations of mathematical ideas and procedures. The two are divided into three levels (low to high) depending on the teacher's use of mathematical language in the lesson. In terms of naming, we observed that the teacher mostly operated at Level 2 and 3 where she was able to use appropriate mathematical language to name objects, simple balance, 1 kg, 500 g masses, and measuring mass in kg and g among others.

In terms of legitimations, we observed that teacher talk was generally of Level 1 and 2. The teacher was giving explanations that were simple, single and isolated cases with real life examples. For example, she explained with the help of a simple balance that the mass of two 500 g salt packets is equal to 1 kg of sugar. The teacher's explanations were characterised by the use of unspecialised mathematics to name mathematical terms and compare masses of objects. In line with Adler (2017) reporting that what teachers say and how they say it matters in mathematics lessons, especially in early grades where learners depend more on teachers, we observed that the selection of tools of measuring mass affected the teaching and learning of the lesson.

The fourth element of MDI is learner participation. Learners participated in carrying out activities in measuring masses of various objects and in answering questions from the teacher. The MDI framework describes learner participation in terms of levels of answers provided by learners. It was observed that learners' answers were of both Level 1 (the yes/ no answers) and Level 2 (what, which and how answers). For example, the common question in this lesson was: Which is heavier between stone A and stone B?

Conclusion

This study collected and analysed qualitative data to find out teachers' practices in grade 4 when teaching mass measurement. Using the MDI framework, we divided the lesson into five events, in which each event was characterised by a shift in the activity. We analysed the three cultural tools of a typical mathematics lesson according to MDI; exemplification, explanatory talk and learner participation based on their level of complexity. The object of learning was also analysed to find out if the intended goal of the lesson was achieved.

While we only presented detailed data from one lesson, the study established that all the teachers had lesson plans with clearly stated lesson goals (success criteria); to measure mass of objects in kg and g. However, it was not clear whether learners achieved this intended lesson goal. In terms of examples and tasks, teachers used similar examples in different events of the lesson such that the examples looked like a repetition of what had already been done. The examples were teacher demonstrations of measuring in grams and kilograms. Most of what was called measuring in g and kg activities were direct comparisons of objects of known masses. Teachers' explanations followed the examples and tasks that they did in class with learners. These were mostly instructions on how to compare objects using simple balances.

Possible implications from using these identified practices in the teaching of measuring mass in grade 4 in Malawi is that some learners may not be able to measure mass of objects in kg and g by the time they finish grade 4 because the teaching of mass did not prepare them adequately with the measuring knowledge and skills. We also noticed that teaching is compromised by the lack of resources such as scale balances to use during the teaching of measuring mass. A final observation is that teachers' practices are determined by suggestions from the curriculum material. Therefore, improving curriculum materials would provide better guidance to teachers so they can better support learners to develop conceptual understanding of mass measurement.

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„That is how you do it, when you justify something in math“: Learning opportunities with an unproductive sociomathematical norm

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Promoting students' mathematical reasoning skills is a challenging teaching goal. This paper exemplifies that the construct "negative discursive activities" provides a useful theoretical perspective to capture characteristics of class discussions that might impair students' understanding of what acceptable justifications in mathematics are. By applying this construct to three episodes from mathematics lessons in Grade 7, an unintentional evolvement of a sociomathematical norm with potentially negative consequences for students' reasoning skills is reconstructed. The core purpose of this paper is to broaden researchers' understanding of interactions involved in the process of establishing sociomathematical norms.

Keywords: Justification, reasoning skills, discourse, sociomathematical norms, interactions.

Introduction

In this paper, *teaching* is considered as a continuous process, in which teachers establish and maintain a specific classroom culture of *doing mathematics*, where "doing mathematics" refers to students' activities when they learn and use mathematics in the classroom. According to this understanding of teaching, one goal of teachers' practices in the classroom is to support students' enculturation into the ways of speaking, working and thinking that are specific for mathematics. Teachers do it, among other things, through the negotiation of social and sociomathematical norms (Yackel & Cobb, 1996). Both kinds of norms pertain to normative or students' taken-as-shared interpretations of their mathematical activities in the classroom. They describe what learners are expected to do in certain situations, e.g. when presenting their solutions to others (Yackel & Cobb, 1996). Social norms refer to the expected activities in class discussions, whereas sociomathematical norms to their quality. Examples of a social norm can be *that* learners are expected to explain and justify their own answers. Sociomathematical norms, on the other hand, regulate *what* is an acceptable explanation or justification in their class. For example, they may specify that explanations must be comprehensible for others and that justifications must be based on actions performed on mathematical objects known in the class or on properties of these objects. Both kinds of norms are not simply imposed by teachers and accepted by students. Instead, they are negotiated through student-teacher interactions. In this process teachers play a crucial role as representatives of the mathematical community in a class. They legitimize certain aspects of learners' mathematical activities and sanction others by pointing them out as inadequate or incorrect.

Sociomathematical norms described in research literature are commonly associated with means that have a positive potential to support students' mathematical enculturation and their intellectual autonomy in doing mathematics, i.e. their ability to judge and decide in action on what is an appropriate cognitive activity in a given situation (Yackel & Cobb, 1996). Teachers can intentionally initiate the negotiation of these norms (McClain & Cobb, 2001). However, also negative, i.e. unproductive, norms can emerge unintentionally. Thereby norms are meant that might hinder the mathematical enculturation of students, their intellectual autonomy or their understanding of what they learn. To avoid such norms, the process of their development must be understood. In this paper, the construct "negative discursive activities" (Cohors-Fresenborg & Kaune, 2007) is used to shed light on this process. The paper exemplifies that this construct provides a useful theoretical perspective to identify activities and interactions in class discussions that might lead to an unproductive sociomathematical norm. This is the pragmatic purpose of this paper. The theoretical purpose is to broaden teachers' and researchers' understanding of the role of negative discursive activities in interactions involved in negotiating sociomathematical norms. Given the teachers' interest in improving teaching quality, this paper is also of practical relevance.

The insights presented in this paper result from our[1] reflection on a communicational pattern that we unexpectedly noticed when analysing instructional data from the perspective of a *metacognitive-discursive teaching quality* (Nowińska, 2019) in a broader empirical study (Nowińska & Praetorius, 2017). The following section explains the theoretical perspective that gave rise to these insights.

Metacognitive-discursive teaching quality

The construct *metacognitive-discursive quality* (Nowińska, 2019) has been developed to analyse how class discussions promote students' metacognitive skills, i.e. their ability to spontaneously and adequately regulate (plan, control and reflect on) their own cognitive activities and comprehension in learning mathematics (Cohors-Fresenborg & Kaune, 2007; Nowińska, 2019). This construct has its roots in theoretical considerations (Flavell, 1979) and empirical findings (Veenman et al., 2004) indicating that metacognitive skills improve students' learning in mathematics. Furthermore, it also considers the social context of learning in the classroom, in particular its role in constructing students' shared understanding of what they learn in the classroom (Iiskala et al., 2011). Metacognitive-discursive quality encompasses characteristics of class discussion that are considered favourable to improve the process and the effects of learning mathematics in the class by promoting students' metacognitive skills. One of these characteristics is that teachers provide students with opportunities to engage in *metacognitive activities*, e.g. to *plan* how to approach a given question, task or problem, *control* the validity of answers and solutions generated in class discussions, and *reflect* on their own understanding of what they learn. Another characteristic is that teachers promote students' *discursive activities*, in particular they support students in making their own cognitive and metacognitive thoughts comprehensible to others. That is necessary to construct a shared understanding of the mathematics discussed in the class. The term *discourse* refers thereby to a class discussion, where the participants make an effort to provide precise, extended explanations and justifications and to build on or elaborate previous contributions from themselves or others. They link their contributions to previous questions or comments, provide a justified agreement or disagreement on other contributions, paraphrase what has been said in order to avoid a misunderstanding. These are examples of *discursive activities*. *Negative discursive activities*, on the other hand, refer to activities that might impair or even hinder a discourse, and consequently also students' shared understanding

[1] References to *our* and *we* are to a mathematics research team in the study presented in (Nowińska & Praetorius, 2017).

of the mathematics discussed in the class. Examples of these are: (NDa[2]) providing unclear or fragmentary statements/argumentations or (NDb) statements in which it is not clear what is meant, (NDc) providing inappropriate (with an uncommented substantial change to what was said) summaries or repetitions of what was said, (NDd) uncommented change of the reference point or of the meaning of the issue discussed, (NDe) using false logical structure in an argumentation, and (NDf) asking a sequence of different questions or (NDg) asking a leading question. A metacognitive-discursive teaching quality is characterised by teachers' and students' efforts to prevent negative discursive activities from impairing their communication in the class.

Analysis of negative discursive activities and their effect on the learning process

The instructional data analysed in this paper consists of three consecutive episodes selected from one of four lessons on locus lines videotaped in Grade 7 in Germany. The following task is discussed in this lesson: "The perpendicular bisector of chord ($\overline{BC}$) of a circle with the centre A passes through point A. Justify why." The episodes are chosen for this paper as representative examples of communicational patterns observed in all four lessons, where the students learned examples of locus lines (e.g. a perpendicular bisector and an angle bisector) and used their properties to justify relations between geometrical objects. One goal of doing mathematics in this lesson series is to foster students' reasoning skills and promote their understanding on how mathematical statements can be justified by drawing conclusions from statements accepted as "true" or "given", or justified before (deductive reasoning). Based on tasks that require this kind of reasoning, the teacher created opportunities for students to learn what is an acceptable justification in mathematics. At the end of this lesson series, however, the students were not able to decide in action what to focus on in order to provide a correct justification without asking the teacher if their answers are what the teacher wants them to say. Analysing class discussions in these lessons from the perspective of metacognitive-discursive quality allowed us to identify a communicational pattern based on negative discursive activities. This pattern might have led to the development of one *unproductive sociomathematical norm* of what are acceptable mathematical justifications in this class. It seems plausible that this norm contributed to the problem observed in students' understanding of correct justifications.

In the following, the three lesson episodes are analysed from the perspective of the negative discursive activities. The analytical work was done by two researchers in mathematics education who were trained to code metacognitive, discursive and negative discursive activities using the category system published in Nowińska (2019). Afterwards, it will be explained how these activities might have led to an unintentional evolvement of the unproductive norm.

Episodes from the class discussion

According to the teacher in the observed class, the learners know the definition of a perpendicular bisector (The perpendicular bisector of a line segment $\overline{AB}$ is the locus of all points that are equidistant from points A and B.) and two properties of it (P1: The perpendicular bisector to a line segment is a line which meets the segment at its midpoint perpendicularly. P2: All points that lie on the

[2] The codes given here for negative discursive activities will be used to code these activities in lesson transcripts.

perpendicular bisector to a line segment $\overline{AB}$ are equidistant from A and B.). Since in this context the term “locus line” is defined as a set of all points, whose location is determined by one specified condition, one more property of a perpendicular bisector can be concluded (P3: Every point that is equidistant from A and B belongs to the perpendicular bisector of the segment line AB.).

Although a justification required in the given task can be reduced to three steps (1., 4. and 5. in the following argumentation), its logical structure is quite complex.

1. The perpendicular bisector of $\overline{\mathrm{BC}}$ consists of *all* points that are equidistant from B and C.
   [Reference to the definition of a perpendicular bisector as a locus line (or to the property P3)
2. Points B and C of the chord $\overline{\mathrm{BC}}$ belong to the circle line with the centre A.
   [Reference to the definition of a chord of a circle.]
3. Points B and C are equidistant from point A.
   [Conclusion from 2. with a reference to the definition of a circle line as the locus of all points with the same distance to the centre of the circle.]
4. The centre A of the circle is equidistant from B and C.
   [Conclusion from 3. with a reference to a property of the distance between two points.]
5. The centre A of the circle belongs to the perpendicular bisector of $\overline{\mathrm{BC}}$.
   In other words: The perpendicular bisector of $\overline{\mathrm{BC}}$ passes through the centre A of the circle.
   [Conclusion form 1. and 4.]

Episode 1: The teacher starts the discussion by constructing a circle with the centre A, a chord $\overline{BC}$ of this circle and the perpendicular bisector of $\overline{BC}$ using GeoGebra and asking the following question:

Teacher:	The perpendicular bisector passes through the centre of the circle. (…) Is that always the case? What do you think?
Karen:	I think so because I mean it’s called perpendicular bisector[2]. (NDa)
Teacher:	Mhm. Lea
Lea:	I would also say that it always passes through the centre of the circle, because it is a circle as well, and is equidistant to every uhm to every point, the centre of the circle. (NDa)
Teacher:	Mhm. Tom
Tom:	And the line that passes through the centre of the circle always lies in the middle of point B and C, I mean the one that passes through to the right … so then that’s automatically the centre, because uhm… (NDb)

In Episode 1, negative discursive activities, which turned to be typical for class discussions in all lessons videotaped in this class, can be identified. Karen’s answer refers to the name “perpendicular bisector”, which in German is associated with a middle (Mittelsenkrechte (Ger.) = perpendicular to the middle (Eng.)). She uses the word “because”, but she does not explain how the argument “it is called a perpendicular bisector” is to be used to justify the observed relationship. Her answer can be regarded as an *unclear, fragmentary statement/argumentation* (NDa). Lea, on the other hand, refers to an appropriate argument – to a property of a circle. However, likewise Karen, she does not explain how this argument can be used to draw the conclusion that the perpendicular bisector passes through the point A. Her answer has the form of an *unclear, fragmentary statement/argumentation* (NDa) too. Tom’s contribution consists of broken sentences. Based on them it is difficult to reconstruct his reasoning. Therefore, for the teacher and for the classmates is may by *not clear what exactly he wants to say* (NDb). The negative discursive activities observed in this episode may indicate that the students lack an adequate understanding of what are precise, acceptable justifications. As a consequence of these activities it might be difficult for the classmates to grasp logical relations between the given arguments and the relationship to be justified.

Coding the learners' contributions as negative discursive activities does not mean that their reasoning is fundamentally wrong or irrelevant for the intended argumentation, nor does it mean that perfectly articulated justifications are expected there. These codes are used to mark parts of the discussion that clearly indicate opportunities for the teacher to explicitly negotiate with the students what acceptable justifications are. Instead, the teacher implicitly accepts the contributions ("Mhm.") without challenging the students to explain their reasoning.

Episode 2: Following Tom's answer from Episode 1, the teacher asks:

Teacher:	Have you noticed what those, who have answered the question, actually have done? What have they paid attention to? How have they justified their opinion, their view? What kinds of arguments have they used? Felix. (NDf)
Felix:	Because the circle is round, and B and C are always the same distance away from A. And then uhm that's where that comes from. (NDd)
Teacher:	Mhm. Exactly. Right. One argument is the one about the circle. There has been another argument, or another justification, that has been brought up. (4 sec) Ines.
Ines:	I also think it's the case, that it always passes through the centre, because… The perpendicular bisector divides the lines into right angles, so to speak, and that is why that is, I think, that it's always like that. (NDb)
Teacher:	Mhm, what do you say to Ines? (15 sec) Where in the circle do you have right angles?
Ines:	Well, I mean, if you … if you divide a circle in four equal sized halves, you get four right angles, that's what I meant. (NDb)

The teacher tries to make students' answers from Episode 1 explicit objects of the class discussion. In doing so, she *asks a sequence of different questions one after the other* (NDf). This may impede a clear course of the subsequent class discussion and is therefore to be coded as a negative discursive activity. Each of these questions does definitely have the potential to initiate students' reflection on the logical structure of the arguments mentioned in Episode 1 and to explicitly negotiate the meaning of an acceptable justification – of *what* a justification is based on and *how* it is used. However, the subsequent reactions indicate that this potential is not productively used.

Felix seems to refer to Lea's answer, but he *changes the meaning of her argument* (NDd). Lea named a property of the point A, whereas Felix describes a property of the points B and C. He does not make a mistake here, but he shifts the focus of attention from the centre of the circle to points B and C on the circle line. This activity has the consequence that even more argumentation steps would be needed to justify the relationship in question (see steps 3.-5. in the argumentation). Since Felix responds to one of the teacher's questions about the arguments mentioned by others, his utterance is not a new unclear and fragmentary utterance/argumentation. The teacher explicitly expresses her acceptance of the given answer. She stresses the reference point of this argument ("the one about the circle") and suggests that another relevant argument can be found in Lea's and Tom's answers. Her statement has the potential to initiate students' reflection on their own understanding of their classmates' arguments. Unfortunately, Ines does not directly respond to the teacher's contribution. Instead, she expresses her own reasoning. In doing so, she *changes the reference point of the conversation without any comment* *on this change* (NDd). This activity might impair the clarity of the discussion. Furthermore, in her first answer Ines refers to the results of actions on objects that cannot be interpreted unambiguously ("the lines [divided] into right angles"). Therefore, there is a risk that for the teacher and for her classmates *it is not clear what exactly she means* (NDb) there. Particularly, she does not make it clear which "lines" she refers to. Her answer is the only one, where the teacher asks for clarification. Unfortunately, also in her second answer Ines refers to the results of actions on objects that cannot be unambiguously interpreted ("the circle [divided] into four equal sized halves"). Consequently, it might still be *unclear for others what she means* (NDb). Thus, in Episode 2, the teacher explicitly accepts justifications based on imprecise arguments "about the circle". Implicitly she does not accept justifications based on ambiguously described mathematical objects or actions. As far it is left to the learners themselves to interpret how the accepted argument can serve as a justification for the relationship described in the task.

Episode 3:

Teacher:	(…) Ines wanted to link the circle and the perpendicular bisector, their properties. And you guys also mentioned the properties of the circle before. Lea did it first and Felix just picked it up again. So, what you did is that you looked at the properties of the circle. The points B and C of the circle are always equidistant to A. And you also used another argument. You also linked the circle and the perpendicular bisector, just like Ines tried to do again just now. What did you say and what did you use about the perpendicular bisector? (4 sec) You also used these in your statements, maybe you weren't even aware of it, again the properties of perpendicular bisectors used. Which ones? (5 sec) Eva. (NDc, NDg)
Eva:	Every point on the perpendicular bisector has uhm is equidistant uhm to B and C.
Teacher:	Mhm, exactly. That is what you used. Sometimes you didn't even say that directly because we have always talked and known about that in class. And so that is what you used: B and C are equidistant to the perpendicular bisector and the centre of the circle is equidistant to both points as well. And that is what you linked to justify, that the centre of the circle A has to be on the perpendicular bisector. That's actually a useful strategy or that is how you do it, when you justify something in math, that you take different properties of different objects and try to link them and to look for connections. (NDe, NDc, NDa)

The teacher's comments in Episode 3 indicate a typical feature observed in all four lessons videotaped in this class. She interprets Ines' contributions in Episode 2 as an attempt to "link" the properties of the circle and those of the perpendicular bisector. Ines mentioned both concepts, but clear attempts to "link" them cannot be observed in her statements. Therefore, a part of the teacher's comment can be regarded as *a summary with a substantial change to what was said* (NDc) and to what can be interpreted unambiguously with regard to the learners' reasoning. The "links", i.e. the logical relations between the properties of the circle and those of the angle bisector, are not made explicit at any point in the class discussion. This would be, however, necessary for the students to understand the logical structure of the intended justification, and to construct a shared understanding of how arguments are to be used to provide an acceptable justification.

Since no student answers the teacher's question "What did you say and use about the perpendicular bisector?", the teacher changes it to a *leading question* (NDg) about "Properties of the perpendicular bisector". In her second comment, the teacher points out that the pupils did not explicitly mention this property. Despite this fact, she does not specify the expected chain of arguments and the conclusion derived from this property, nor does she clarify the logical structure underlying the intended justification. In her second contribution, she suggests that since this property is a part of the knowledge shared in the class, it did not need to be explicitly mentioned. This comment might cause the learners to feel not obliged to express their reasoning precisely and in a way that can be understood by others.

The argumentation given by the teacher in her second comment in Episode 3 indicates a *false logical structure* (ND3e). The teacher should have used the property P3 instead of P2. P3 describes conditions that must be fulfilled for a point to belong to the perpendicular bisector. P3, on the other hand, describes properties of points that are known to belong to the perpendicular bisector. This comment is not to be understood as a negative discursive activity just because it includes a mistake. The reason for coding it as a negative discursive activity is that this mistake makes it more difficult for others to understand how the teacher generates her justification based on the mentioned properties of mathematical objects. Moreover, the teacher's comments *differ from what the students actually said* (NDc). No student said that B and C are equidistant to the perpendicular bisector. Since this argument is not relevant for the intended argumentation, this comment must be regarded as an *unclear and fragmentary statement/argument* (NDa). For the learners, it might be unclear how the arguments mentioned by the teacher should be "linked" together to form an adequate justification. Consequently, the learners must find their own interpretation of the statement "that is how you do it, when you justify something in math, that you take different properties of different objects and try to link them together and to look for connections". One possible interpretation, supported by the way how the teacher provides the final justification, is that of linking together as collecting properties of mathematical objects given in a geometrical representation; for an acceptable justification, the arguments on which a justification is based must refer to unambiguously described mathematical objects or actions.

Conclusion

The negative discursive activities identified in the three episodes occur in all four lessons videotaped in this class and create a pattern with the following regularities: (1) The students make an effort to provide justifications based on their knowledge about mathematical objects given in a particular geometrical representation, but their justifications lack precision with regard to the language and logical structure. (2) The teacher usually does not comment on students' imprecise answers nor does she provide comments that could help the students improve their reasoning. Instead, she implicitly accepts arguments and justification despite their negative discursive character. Only if students' contributions refer to ambiguously described mathematical objects, the teacher asks for a clarification. In doing so, she implicitly negotiates the boundaries of what is an acceptable justification. (3) The teacher inaccurately summarizes the students' contributions. There she gives an answer to the question under discussion without referring very precisely to the learners' previous answers. Her explanations support the interpretation of justifications as collections of arguments related to mathematical objects.

Since the regularities occur in all four lessons, the students seem to have it as a rule that the teacher "summarizes" their contributions and gives the expected answer or solution. This is evidenced by the fact that they often try to guess what the teacher expects them do say (e.g. "Do you mean this?") and do not question the teacher's "summaries". Their behaviour can be interpreted as acting according an unproductive sociomathematical norm. According to this norm, acceptable justifications are collection of properties of clearly named mathematical objects. This norm promotes activities and interactions that are unproductive in the sense that they impair students' understanding of mathematical justifications and their reasoning ability.

The paper has demonstrated that negative discursive activities can be used to capture interactions in class discussions that might lead to unproductive sociomathematical norms. In class communication negative discursive activities cannot be completely avoided and this should not be an idealistic teaching goal. Moreover, not each negative discursive activity necessarily implies negative consequences for students' understanding. The consequences depend on whether teachers accept these activities or rather sanction them and use them as learning opportunities to negotiate ways of thinking and talking that are specific for mathematics. In the first case, there is a risk that unproductive sociomathematical norms emerge. The negative norm identified in this paper indicates a challenge for teachers' educators to make teachers aware of consequences that negative discursive activities might have for students' learning and to enable teachers to productively deal with these activities.

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Teaching beyond the mathematics of the moment

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In this paper, we showcase a method of identifying opportunities during the lesson where the communicational patterns go beyond the mathematics of the moment, namely beyond the boundaries of the curriculum. Drawing on the literature around Horizon Content Knowledge and building on the Theory of Commognition, we look at a lesson from one teacher and study instances where her communication with the students goes beyond the mathematics of the moment. Our analysis identifies three types of opportunities (manifested, potentials of the discussion and potentials of the task). Further analysis of manifested opportunities indicates that intersubjectivity is a characteristic of the discourse of the teacher at the mathematical horizon.

Keywords: Discourse, mathematical horizon, teaching practices, mathematics.

Introduction

This paper reports on preliminary results from the first author's doctoral study that aims to explore in-service mathematics teachers' discourse that goes beyond the boundaries of the curriculum, what is described by Ball and Bass (2009) as "the larger significance of what may be only partially revealed in the *mathematics of the moment*" (para. 17, our emphasis). The importance of this idea has been described in the literature in connection to teacher actions in response to students' contributions and towards addressing students' learning needs (Ball & Bass, 2009). However, the boundaries of the curriculum, even more going beyond them, are not always identifiable. Initially, this study, in its effort to describe such boundaries, was inspired by Ball and colleagues' idea of the Horizon Content Knowledge (HCK), one of the mathematical domains of the Mathematical Knowledge for Teaching (MKT) model for teacher professional knowledge:

> an awareness – more as an experienced and appreciative tourist than as a tour guide – of the large mathematical landscape in which the present experience and instruction is situated. (Ball & Bass, 2009, para. 17)

HCK is theorised to influence, among others, teaching practices related to noticing and evaluating mathematical significance in what the students are saying, foreseeing and making connection across educational levels and disciplines and evaluating opportunities (Ball & Bass, 2009). So far, several studies drew on the idea of the HCK with variations in the use and the narratives about the 'horizon' metaphor (see also Papadaki, 2019). Evidence of the role of the HCK in the quality of teaching relys on these different uses and narratives of the 'horizon' metaphor. In our work, we argue that bridging the gap between an all-encompassing mathematical knowledge for teaching and teaching practices aiming to enrich the learners' experience of mathematics as a subject, could lessen the reliance on the metaphor. To this aim we propose a discursive approach to the "horizon" embedded in the communicational patterns of the classroom. Specifically we draw on the commognitive theory (Sfard, 2008) to address the theoretical and methodological question: *How can we identify opportunities during the lesson where the communicational patterns go beyond the mathematics of the moment?*

Theoretical background

To study the communicational patterns in the mathematics classroom, we use the *Theory of Commognition* (Sfard, 2008). According to this theory, a discourse is signified by four elements: *word use*, *endorsed narratives*, *visual mediators* and performed *routines*. Thus, mathematical discourse is the type of communication signified by specific word use (e.g., 'circles'), visual mediators (e.g., diagrams, symbols), endorsed narratives (e.g., theorems, definitions) and routines (e.g., calculating angles). Learning of mathematics is conceptualised as change in learners' discourse (Sfard, 2008). Tabach and Nachlieli (2016) define teaching as "the communicational activity the motive of which is to bring the learners' discourse closer to a canonic discourse" (p. 303). In this study, the canonic discourse is the Mathematical Discourse[1] of the class which is confined by the curriculum. The role of the teacher in mediating changes is vital. The *Mathematical Discourse for Teaching* (*MDT*) model (Cooper, 2016; Mosvold, 2015), an adaptation of the MKT, provides a basis to study teachers' discourses shifting the attention to word use, narratives, visual mediators, routines and "the kinds of mathematical activities that are valued" (Cooper, 2016, p. 21).

MDT consist of two main Discourses that could be viewed as a way to partially distinguish between mathematical narratives situated in the social context of the classroom (Mathematical Discourse) and pedagogical narratives situated in the context of teaching mathematics (Pedagogical Content Discourse). The two Discourses are distinguished into six sub-Discourses. One of them is the *Discourse at the Mathematical Horizon* (*DMH*), which is described as "patterns of mathematical communication that are appropriate in a higher grade level" (Cooper & Karsenty, 2018, p. 242). In our work DMH spans across the Mathematical and Pedagogical Discourses and regards the recognition of mathematical as well as pedagogical significance in students' work beyond the mathematics of the moment by addressing, for example, issues of content (e.g., 'what should be learnt') and access (e.g., 'who should learn').

So far, the MDT has been used in the context of teachers' professional development (e.g., Cooper, 2016; Mosvold, 2015). However, in our work we aim to identify and analyse aspects of DMH in lesson observations, where Mathematical and Pedagogical Discourses are intertwined in teaching actions. To this purpose, we encompass recent developments of the Theory of Commognition. We are interested in identifying teaching practices, seen as routines performed by the teacher in the mathematics classroom, that go beyond the expected Mathematical Discourse of that classroom. Attending to the mathematical elements of the communication between teacher and students could provide evidence about mathematical aspects of DMH. To account for pedagogical aspects, we adopt Nachlieli and Elbaum-Cohen's (2021) view that teaching practices could be considered as objects of Pedagogical Discourse. Furthermore, to investigate situations in which the communication between the teacher and her students makes or does not make sense at both ends, we use the notion of intersubjectivity, namely "an action that makes sense from the perspective of two discourses – the learner's and the expert's–which may be incommensurable" (Cooper & Lavie, 2021, pp. 8–9).

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[1] Here, the term 'Discourse' denotes community established patterns of communication while 'discourse' is used to refer to the individualised version of that communication between the interlocutors.

Data and methods

The study is conducted in England and participants are secondary school mathematics teachers, their students and teacher educators. In England, schools develop a program of study following the guidelines of the curriculum for Key Stages 3 and 4 (KS3, ages 11 to 14 and KS4, ages 15 to 16) and the specifications of the qualification provider the school collaborates with. Thus, what students are expected to learn each year varies across schools. Mathematics teachers prepare their lessons using resources available on the internet, textbooks or their school's repository. Data of lesson observations consist of an audio recording of the lesson and notes produced by the first author during the observation and revised shortly after. The notes include recreations of the writings on the board, the slides used by teachers and accounts of their actions during the lesson. The lesson was transcribed, we also produced a factual account summarising the lesson. The data collection process complies with the code of ethics of the University of East Anglia.

The analysis of the data consists of four phases. First, we created a template to study teacher's contributions by dividing each lesson into sections according to the actions of the teacher (e.g., talking to groups of students, introducing a task, initiating a whole class discussion etc.). Second, all sections were coded in relation to the mathematical objects (e.g., angles) and/or practices (e.g., measurement of angles) accounting for the discursive elements (word use, visual mediators, narratives, and routines) identified in each section. By comparing the codes with the topic of the lesson, as identified by the teacher and the specifications of the curriculum, we eliminated the sections that were directly related to what the students are expected to learn during the academic year. We regarded the remaining sections as opportunities to engage in communication beyond the mathematics of the moment and proceeded analysing them further. Thirdly, we categorised the identified opportunities in three groups: *potential of the task* (the mathematical object or practice is not made explicit during the lesson, it is only attributed by the analysis), *potential of the discussion* (the mathematical object or practice is made explicit in the lesson but is not addressed by the teacher), and *manifested* (the mathematical object or practice is made explicit and addressed by the teacher during the lesson). Finally, we revisited the opportunities proposed in the third phase to identify deviations or alignments in teacher-students communications from the intersubjectivity perspective.

Exemplification of the analysis: The lesson of the 'nine-point circle'

Here, we draw on one lesson of a newly qualified mathematics teacher, Liz (alias), and her 11-year-old students. The students are in Year 7, the first year of lower secondary education. Liz designed the lesson by combining resources available on the internet and the school's repository. For her lessons, Liz shares slides on the interactive whiteboard. At the beginning of the lesson we discuss here, Liz invited the students to calculate the value of an angle in a composite shape. Figure 1, depicts the main task of the lesson that consists of four parts (1, 2, 3 and 4). The common characteristic of the parts is a circle with nine equally spaced points on its circumference. Liz referred to this shape throughout the lesson as "the nine-point circle". In part 1, the students were asked to identify all the different triangles with one vertex on the centre of the circle and the others on two of the nine points and calculate their angles. In part 2, they were asked to find a way to calculate the angles in a triangle with all the vertices on the circle. In part 3, the students were asked to find their own triangles by joining three points on the circle and work on their angles. Finally in part 4, Liz asked the students to calculate the angles ACB and ADB and tell her if they notice anything. By the end, Liz pointed out that the observation made in part 4 was related to one of the 'circle theorems' included in the highest band of their qualification requirements (GCSE, acquired at age 16).

In this lesson, we identified four opportunities to engage in communication beyond the mathematics of the moment. Two of them were categorised as *manifested* ('auxiliary lines' and the 'Star-Trek lemma'), one *potential of the discussion* and one *potential of the task.* In the following sections, we present the opportunities alongside the curriculum expectations.

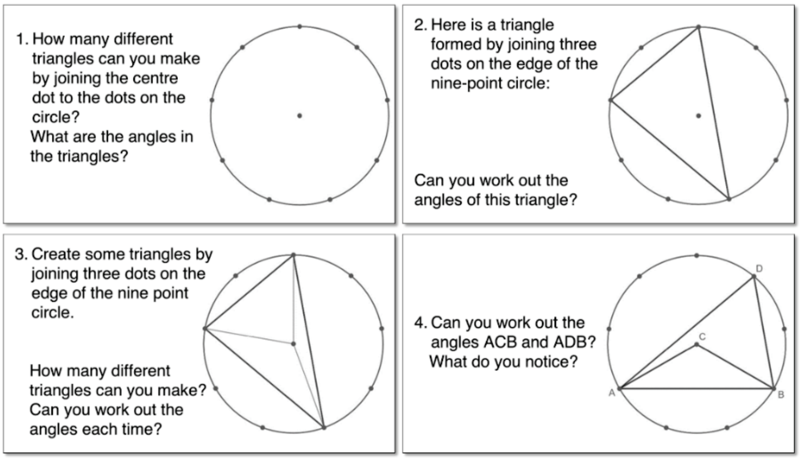


Figure 1: A recreation of the main task of the lesson

Manifested opportunities to go beyond the mathematics of the moment

Auxiliary lines: The completion of the main task, requires drawing auxiliary lines (i.e., additional lines) in order to divide the inscribed triangle in part 2 appropriately and calculate its angles. The process of drawing auxiliary lines was used by Liz as an approach to tackle an open problem.

Liz: Yeah. Right. [slide of part 2 on the board] So, I've got a slightly trickier question for you now. So here is a triangle formed by joining three dots on the edge of a nine-point circle, nine-point circle [points towards the diagram]. However, this time it doesn't go through the centre. Can you work out the angles of this triangle?

The process of drawing additional lines is not new to students at this age. As part of their primary and secondary school education, the students are expected to use auxiliary lines for the purposes of calculating the area of composite shapes or identifying a line of symmetry. Highly attaining students are also expected to learn to use auxiliary lines when proving certain theorems, in known situations, at the end of KS4, but not at Year 7. Also, this task requires the students to explore unfamiliar to them situations where auxiliary lines are necessary. Similar uses of auxiliary lines are observed in more of Liz's lessons suggesting that engagement with this process was not an one-off event.

Initially, Liz prompts the students to explore possible ways of tackling the problem. The students try to measure or estimate the angles of the given triangle. Five minutes later, she addresses the class:

Liz: […] it's taken him [student A] quite a long time to ask me a really, really good question. But, I think I should share it with everybody. He said, can I split it up into other triangles.

Student B: Oooh.

From a commognitive perspective, drawing auxiliary lines to modify the triangle in an appropriate way is a routine. For student A, the initiation of this routine is prompted when other routines failed. However, he requested Liz's approval before moving on. For other students the initiation was prompted by the teacher:

Liz:	Yeah, I'm not going to tell you the answer. I'm going to show you the clue. So, this is the clue. [draws three line segments from the centre to the edges of the triangle]
Student C:	Oh.
Liz:	[…] if you can split it into other triangles and using the triangles where we can where we can start. We know some facts, don't we? An isosceles triangle is nice. I'm not trying to find three unknown angles now, am I? I know that two are the same. […] plus we did all that work beforehand.

Here the procedure is presented to the students as a "clue". The use of the word could signify a step to a direction that was previously hidden from the students who are now invited to follow this clue to complete the task. The use of the phrase "split up into other triangles" and the absence of words like 'draw', 'line segment', that are usually present in formal geometric proofs, indicates that the discussion is mediated by the specific drawings of the task (Figure 1). We interpret the use of these words as evidence of intersubjectivity. For the students, this task is a new situation where they need to draw additional lines to solve a problem - similar to what they are doing when they are asked to calculate the area of a composite shape. While for Liz, this is a routine towards the modification of a shape in order to use properties established in earlier stages of a task. This routine is not part of the canonic discourse (Tabach & Nachlieli, 2016). Liz presents the students with a broad spectrum of situations that promote the routine of auxiliary lines beyond the requirements of the curriculum of Year 11. The instance presented here illustrates her teaching practice of providing gradual support to her students, of expanding the applicability of the routine and of establishing this process as conventional in geometric reasoning. Evidence of this practice were observed across Liz's lessons.

<u>The Star-Trek lemma:</u> Liz uses the 'nine-point circle' to introduce her students to a theorem about angles in circles also known as the Star-Trek lemma:

The angle subtended by an arc at the centre is twice the angle subtended at the circumference.

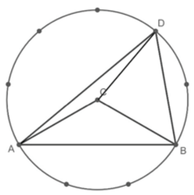
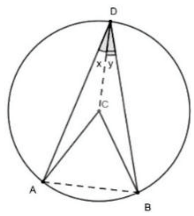
Liz presents the last part of the main task (part 4, Figure 1) as a challenge to the students:

Liz:	So, I'm going to give you one last challenge, which is to take a look at this. [Liz puts up the slide with part 4 of the task]. And you're going to need to start calculating the angles and see if you notice anything. So, I'd like you, actually we can start from, we've got a starting point, haven't we? We know this angle here.

According to the curriculum, the Star Trek Lemma, named after the resemblance of the logo of the popular series, is one of the theorems that only highly attaining students are expected to learn to prove and apply during KS4. According to the school's program of study, Liz's students might come across circle theorems in Year 11 (age 16), subject to their grades until then, but not at Year 7.

The analysis shows that with the help of the nine-point circle and building upon the previous parts of the task, Liz and the students have an opportunity to go through the main ideas of the proof of the Star-Trek lemma despite not having yet engaged with algebraic routines and relevant terminology (e.g., arc, subtended, etc). Throughout the lesson, Liz and the students communicate using the words 'angle', 'triangle', 'circle' and 'point' or 'dot'. The new narratives about angles can be negotiated through routines and other narratives endorsed in previous lessons or earlier parts of the task. In the lesson students engage with simple algebraic routines based on the visual mediator of ‘the nine-point circle’ in comparison to the proof of the lemma seen later in KS4. Table 1 illustrates our reconstruction of the narratives observed during the lesson (left column) alongside a canonic proof of the lemma seen in KS4 (right column). To produce the reconstructed narrative, we collated parts of the recording and the notes in a way that corresponds with the steps of the proof.

Table 1: Reconstruction of narratives observed in lesson (left) and a canonic proof seen in KS4 (right)

The nine-point Circle task	A proof of the Star-Trek Lemma
 [Spliting the shape into the triangles ACD, ACB and CDB.] **Liz:** Two sides when I'm going from the centre point out to the edge of my circle, that line there is exactly the same distance as that line there. DCA=160º [task 1/angles around a point] ADC=10º because the triangle is isosceles. ACB=120º BCD=80º [task 1] CDB=50º because the triangle is isosceles. ADB=60º **Student**: Mmm, Double!	 Draw AB and the radius CD. On the diagram angle ADC=x and CDB=y. Therefore, ADB=x+y Angle CAD=x because the triangle ACD is isosceles, CA=CD radius of the circle. And DCA=180º-2x (i) Also, angle DBC=y because the triangle DCB is isosceles, CB=CD radius of the circle. And BCD=180º-2y (ii) ACB+DCA+DCB=360º because angles around a point sum up to 360º. From (i) and (ii): ACB+180º-2x+180º -2y=360º⇔ ⇔ACB=2x+2y ⇔ ⇔ACB=2(x+y) ⇔ ⇔ACB=2ADB

The students are not yet familiar with advanced algebraic routines, such as rearranging algebraic equivalences with more than one variable. However, with the mediation of ‘the nine-point circle’ Liz and the students negotiate the Star-Trek lemma and its proof, using an example where the angles can be calculated. Students are familiar with routines such as using angle facts to substantiate their actions or naming unknown angles towards their calculating. Additionally, drawing auxiliary lines was introduced in part 2. The action above led Liz to conclude:

Liz: […] this angle here [centre] will always be double the size of that one there [circumference].

Phrases like “this angle here” and “that angle there” act as placeholders for missing words that the students might come across in the future. Liz is aware of the words but does not name them to her students. Yet, the constructed narrative make sense both from the perspective of Liz, as an application of the Star-Trek lemma, and to the students, as an observation that can be confirmed following the steps of the task. Also, she uses the word “always” to signpost the generality of the argument. The task and Liz’s actions gear students towards constructing a narrative about the two angles by the end of the lesson. Actions beyond the mathematics of the moment are seen here as she engages students with mathematical content that students might see in the future in ways that are accessible to them.

Potentials of the discussion and the task to go beyond the mathematics of the moment

We also identified potential opportunities where the discussion could go beyond the mathematics of the moment but were not addressed by Liz. In part 3, the students are asked to create as many different triangles as they can by joining three points on the ‘nine-point circle’, during a specified time. An opportunity that emerged regards the total number of the different triangles:

Student: Miss, do you have us drawing thousands of triangles?
Liz: Are there thousands of triangles to find?
Student: Yeah… [Liz moves on]

Later, Liz makes the following comment during the whole class discussion:

Liz: There are, I believe there are about thirty-six different triangles that you can make out of this. There are lots. But we're running out of time, so I can't show you and I can't load up the the document I wanted to show you.

Time limitations and technical difficulties seem to be the reason why this opportunity did not materialise despite Liz’s intentions. Determining the exact number of different triangles relies on observations about congruent shapes which is part of the students’ compulsory education, as well as elements of problem solving and combinatorics that go beyond the mathematics of the moment.

Finally, one opportunity we observed emerging from the task is to use the same visual mediator, the ‘nine-point circle’, to introduce more circle theorems e.g., exploring the angles of cyclic polygons. As mentioned earlier, circle theorems, including cyclic quadrilaterals, could be mentioned in KS4.

Discussion and Conclusion

The analysis yielded opportunities to engage in mathematical conversation beyond the mathematics of the moment. Some of these opportunities were taken (manifested) and others were missed (potentials of the discussion and the task). Using intersubjectivity, we discuss opportunities that materialised during the lesson, to identify the elements of the teacher communicational patterns at the Mathematical Horizon. The teaching actions exemplified here, regard Liz inviting her students to endorse intersubjective narratives (the Star-Trek lemma with word use and routines which are accessible by the students) or to perform intersubjective routines (auxiliary lines prompted by teacher’s ‘clues’). We observed Liz and her students using words, endorsing narratives and perform routines that make sense from both teacher and student perspectives through the mediation of the ‘nine-point circle’. Liz’s expectation is that students, with her support would utilise known routines to justify their arguments mathematically. Our observations are in line with nested routines and ritual-enabling opportunities to learn as stepping stones for the students to enter a new discourse (Nachlieli

& Tabach, 2019). The missed opportunities identified here indicate that the communication might be constrained by factors such availability of time and resources or teacher's priorities.

The data reported here, illustrate a theoretical frame and a method of analysis to account for teaching situations where the communication goes beyond the boundaries of the curriculum. We consider the opportunities discussed here as evidence of DMH and, thus, contributes to the work on how DMH is conceptualised and investigated in research. The identification of opportunities (manifested and potential) can be utilised to determine what facilitates or hinders the opportunities to go beyond the mathematics of the moment. What, for example, does stop teachers to act upon students' contributions that have potential for a discussion that goes beyond the mathematics of the moments? How could these discussions be beneficial to students? These are the next steps of our work on the analysis of data collected from teachers (interviews and class observations) and teacher educators (interviews). In our work, our attention is on teachers: we study their narratives and how they react to student contributions. We note, however, that this study does not explore connections between DMH and students' learning. Finally, what we discuss here is connected to our own horizons as mathematicians and mathematics education researchers. Therefore, further research and collective work will give a nuanced view of DMH and its implications in teaching and teacher professional development.

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Using roles and positions to foster explorative talk in mathematics

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This article is part of the Theatre in Mathematics project (TIM), where we use drama and roles to involve students' actively in their learning of mathematics. In this article, we report on the use of roles in group work in mathematics. The data comes from one group of six students that were given roles to use during joint task solving, and the analysis is based on transcriptions from the lesson. In this particular group, two of the students enacted the role of the curious very actively. We find that these two ask almost all the questions. Looking further into it, we find four types of questions that are most frequent: requesting answers and claims, requesting explanation, requesting evaluation and clarification, and requesting argumentation. The last three types are essential parts of explorative talk, and we conclude that our study has illustrated how an active role of curios can move the discussion forward in ways similar to explorative talk. This study is an example how the teacher can shift the focus from teacher questioning to student questioning in the mathematics classroom.

Keywords: Positioning, interactions, roles, explorative talk.

Introduction

Our experience from the classroom indicates that many students rarely participate in discussions in mathematics. There are several reasons for this, ranging from some students taking a dominant position that makes other students passive to students lacking sufficient self-confidence in mathematics to dare to express themselves in the classroom. The background for this project was to explore whether we could change the classroom discourse by giving the students different roles and positions, and if this could lead to a more exploratory talk in mathematics and if more students will be actively involved in the mathematical discussions. This included a shift in focus from searching for the correct answer in mathematics to discussion, argumentation, and in-depth explanations.

Both Mortimer and Scott (2003) and Mercer and Wegerif (2002) describe different types of classroom discourse. Mortimer and Scott (2003) suggest four communicative approaches, where one is preferable (the interactive-dialogic approach) as it gives room for several points of view and allows several persons to participate. In the same way do Mercer and Wegerif (2002) present three types of talk where one is preferred. In explorative talk, all partners actively participate, opinions are sought, and decisions are jointly made. This means that the interactive-dialogic approach and explorative talk both emphasize participation and openness to different ideas. Through positioning theory, we might explain and understand why not all classrooms look like this. We can even use positioning theory as a mean to change the classroom towards the ideals of Mortimer and Scott (2003) and Mercer and

Wegerif (2002). In this article, we report from a European project called Theatre in Mathematics (TIM), where we use positions, roles, and drama to create classroom discourses characterized by active participation, openness for ideas, and a clear focus on questions, challenges, explanations, and arguments. Our research question for this article is: How can assigned roles and positions, particularly the curious role, foster a more interactive and explorative talk in mathematics? To answer this question, we needed a framework capable of describing student interactions on a turn-by-turn basis. As we could not find one that suited, we developed one based on literature.

Theory

Mortimer and Scott (2003) suggest a model that describes teacher's communicative approach along two dimensions (table 1). The first is the authoritative-dialogic dimension, which refers to whether only one point of view (authoritative) or more than one point of view (dialogic) is paid attention to. The second dimension is the interactive-non-interactive dimension that separates between approaches that include or exclude people from participating.

Table 1: Communicative approach (Mortimer & Scott, 2003, p. 35)

	INTERACTIVE	NON-INTERACTIVE
DIALOGIC	A Interactive/Dialogic	B Non-interactive/Dialogic
AUTHORITATIVE	C Interactive / Authoritative	D Non-interactive / Authoritative

The result is four different communicative approaches, where the interactive-dialogic approach, which opens for several points of view and includes participants, is preferred. Another one is the interactive-authoritative, where the teacher allows students to participate, but there is only one point of view. This has apparent similarities with the IRE pattern (Initiation-Response-Evaluation) (Cazden, 1988; Mehan,1979), as the students typically are allowed to answer questions and tasks but rarely allowed to introduce other points of view by initiating new ideas or evaluating.

While Mortimer and Scott (2003) present a model for a teacher's communicative approach, Mercer and Wegerif (2002) look at the dialogue per se and suggest three general types. The first is *the cumulative talk* in which each interaction builds on the prior one, in a positive and supportive way, but also uncritically. Repetitions, confirmations, and elaborations characterize cumulative talk, and only one idea is heard. The second is *the disputational talk* which is characterized by disagreement and individual decision making. Even though multiple ideas are heard, there is no genuine attempt to understand each other. Instead, it is characterized by assertions and challenges, and the participants are trying to win the discussion. The third is *the explorative talk*, where the participants engage critically but constructively, and multiple ideas are accepted and even wanted. It is also typical that suggestions are offered, justified, and challenged. It is characterized by making knowledge publicly accountable and making reasoning visible as part of the talk. While explorative talk is preferred by Mercer and Wegerif (2002), the characteristics of Mercer's categories are further explained and nuanced by Sjåstad (2018). A central argument that Sjåstad (2018) argues that all three types of talk have some positive sides. For example, the cumulative talk might be consensus-based or explanatory, and particularly the latter has potential for learning even though only one idea is discussed because the idea through discussion is modified. Also, disputational talk may be argumentative or true disputational, and the first has potential for learning even though they do not try to agree because their ideas are substantiated (Mork, 2006; Sjåstad, 2018).

However, how do we identify the different types of communicative approaches or talk? One way to do so is to study the discourse on a turn-by-turn basis. Scholars have developed a wide range of concepts describing different types of interactions, and some have also developed frameworks. One such is the inquiry co-operation model by Alrö and Skovsmose (2004), suggesting eight types of interactions that both teachers and students use: *getting in contact, locating, identifying, advocating, thinking aloud, reformulating, challenging, and evaluating*. Another framework that separates teachers' and students' interactions is suggested by Drageset (2014, 2015), which describes four main types of student comments: *(mere) answers to mathematical questions, explanations, initiatives, and evaluations.* Such frameworks, and their concepts, are helpful when trying to characterize different types of communication in the classroom, based on a turn-by-turn analysis.

While it is well established that a turn is dependent on the prior turn (Linell, 1998), communication is more than responding to prior turns. For example, some students never talk even when invited in by the teacher, while others tend to dominate any discussion. This might be explained by using positioning theory. According to Harré and Van Langenhove (1999), people have preferences that guide their position in social settings and discourse. Also, taking a position could affect other positions, so positioning may not be taken freely but instead a negotiation. Such positions, and positioning of others, could be intentional sometimes and unintentional at other times. For example, if one student position herself as a helper for those who do not understand during group work, this is also a way to position someone else as needing help. Also, if one or two positions themselves as a solver of a task, this might exclude or passivate others.

Roles relate to positions. Roles are a central part of any drama, and even though roles are used in many ways (Drageset et al., 2021), two key factors separate roles in drama from positions. One is that roles include fiction, while positions do not. The other is that you are always aware of playing a role, while you are not always conscious of your positioning and how this affects your surroundings. At the same time, there are apparent similarities as it is possible to choose both a role and a position deliberately, and it is possible to change to another role or position deliberately. In the TIM project, we use limited roles (which we call role categories) to make students aware of possible positions they can take in the classroom and give them experience in taking them and changing between them. One such role category is the curious, a role where you ask questions until you understand, sometimes rather insistent. It is well known that asking questions may be a scary thing to do as one might be seen as dumb, but when you are given the role of curious, you are asking because it is your task. Another role is the skeptic that tries to find other solutions or challenge ideas. We also use the role of authority, where this is a democratic authority that requests arguments and explanations and several points of view before deciding. Furthermore, we use a mediator that tries to find common ground for a joint decision. Then we try to establish these roles as positions in the classroom by giving students roles and encourage them to use the given position in the discussion around the math problems.

To study the dialogue, we have built an analytical framework based on the above theory. The starting point was a limited search for concepts that could describe student interactions in a group-work setting (without teacher participation). Then we grouped the concepts in different ways and arrived at seven quite distinct types of student interactions (see table 2). The framework was also adjusted during the analysis.

Table 2: Analytical framework describing seven main types of student interactions

Code	Description	Developed from
Answers & claims	These are answers to questions and might be correct, partial, or wrong. No explanation or argument is given. Often part of a flow of questions and answers, which is typical for cumulative talk.	(mere) answers to mathematical questions (Drageset et al., 2021) Cumulative talk (Mercer & Wegerif, 2002)
Argumentation	Argumentation is focused on why something is correct or beneficial, or logical.	Advocating (Alrø & Skovsmose, 2004)
Challenges	Challenges break with the flow, present a new idea, or opposes a presented idea. This is an essential part of explorative talk if it leads to arguments or explanations but might also create a disputational talk if challenges are met with challenges and no arguments or explanations.	Challenging (Alrø & Skovsmose, 2004) Explorative talk and Disputational talk (Mercer & Wegerif, 2002)
Evaluation & clarification	Evaluating is an assessment of any of the other codes, typically related to correctness or logic. It might also be about clarifying, typically seen in reformulating.	Evaluation (Alrø & Skovsmose, 2004) Explorative talk (Mercer & Wegerif, 2002) Reformulating (Alrø & Skovsmose, 2004)
Explanation	Explanations are focused on what is done, or has to be done, to reach an answer, typically chronologically.	Explanations (Drageset et al., 2021)
Questions	Questions about what, how, and why. It is typical for an explorative talk that students take initiatives and ask questions.	Initiatives (Drageset et al., 2021) Explorative talk (Mercer & Wegerif, 2002)
Suggestions	Suggestions are an initiative to a way of solving a task, often related to thinking aloud. These typically will be followed by arguments or explanations.	Thinking aloud (Alrø & Skovsmose, 2004) Initiatives (Drageset et al., 2021)

Method

This article builds on data gathered as part of the Theatre in Mathematics (TIM) project financed by Erasmus+ and partners from Italy, Norway, Greece, and Portugal. The aim is to develop a mathematical teaching methodology that involves students actively in their mathematics lessons by using drama techniques. The methodology builds on two approaches. One approach uses process drama, where the participants take on specific characters or roles in a story. This is a form of drama where there are no fixed lines, but the participants instead interpret how the role or character would act in the different situations of the story. The other approach is called Mathemart, where theatre workshop techniques that include mathematical games and performative activities are used to explore a particular mathematical topic. These activities aim at creating a trusting atmosphere where mistakes are not stigmatized but instead considered elements of a creative process.

The data reported in this paper comes from a lesson in a tenth-grade class with 25 students and their teacher. The students have used role categories since they were in grade 8. In this lesson, they were given a set with nine problem-solving tasks to complete in groups (five to six students) in 60 minutes. The tasks covered statistics, relations between metrics, physics and variables, geometry, equations, and functions. An example is how to make and analyze diagrams that show how many apprentices there are in different educational programs. Another is to find the relations between density and volume in a practical situation. The students had worked on the tasks on their own before the group session. Each group of students was given three different role categories to be used during the work with the tasks: authority, curious, mediatorial. One can see the curios as a role that cultivates what Mercer and Wegerif (2002) call explorative talk, while the mediator cultivates the cumulative talk. Arguably, an authoritarian role could be seen as cultivating disputational talk, but we seek to create a more democratic authority that listens to all before deciding, hence supporting both cumulative and explorative talk. The working process of each group was video-recorded, and a desk microphone captured their speech. For this paper, one group of six girls was chosen as a case study because two students were playing curios in a very active way.

The group discussions were transcribed, and the analysis was done using NVivo, where we first coded all student turns into the seven categories from table 2. Then we identified one group where two of the students were actively playing the role of curios and looked further into this. In the second step we coded the two girls' questions based on what they asked for (of the six other categories), to characterize how they used their role to include others in different ways. The analysis of turns was supported by observation of non-verbal communications seen in the video.

Findings

The group consisted of six girls, one being an authority, three being curios, and two mediators. First, we categorized all turns related to table 2 (See table 3). The first analyzes showed that two of the three students who had a role as curious asked 90% of all the questions in the group. The girls in the role of curious were aware of their roles as questioners. One of them even asks at the beginning of the lesson: Should I ask questions, even though I know the answer? We decided to analyze in more detail the type of questions they asked.

Table 3: Amount of each type of turn for each girl, named by their role category

Categories	Sum	Girl 1 (curious)	Girl 2 (curious)	Girl 3 (curious)	Girl 4 (mediator)	Girl 5 (mediator)	Girl 6 (authoritarian)
Argumentation	3	1	0	0	0	2	0
Evaluation/clarification	76	28	9	3	10	19	7

Explanation	48	12	5	0	3	21	7
Suggestion	35	9	8	0	4	13	1
Question	88	55	18	3	1	4	7
Answer/claim	52	13	11	4	5	12	7
Challenge/initiative	32	15	2	0	0	6	9

In the analysis below, we looked for the qualities of the questions in terms of the type of response they asked for. This means that we coded the questions of these two active curios based on what they asked for (using the categories of table 2). Figure 1 highlights the type of questions the two girls asked. The category challenge/initiative are excluded as we only found one example of that kind.

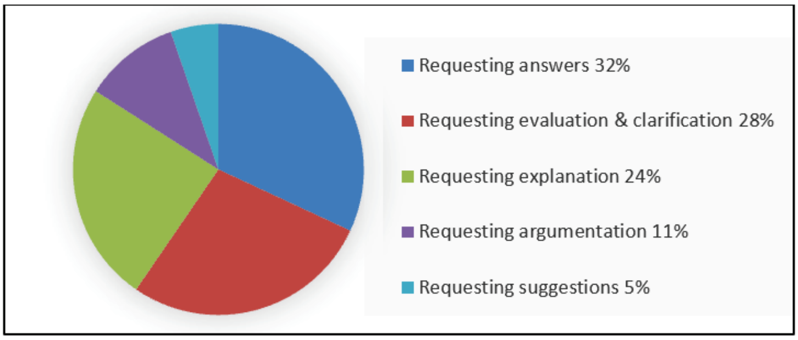

Figure 1: Type of questions in the percentage of all questions

There were four types of questions most frequently asked by these two curious girls. The first, *requesting answers and claims*, could be questions such as “how long was that (line between A and B)?”, “are they all of equal length?” and “what is the formula?” Such questions are essential to create progress and get something on the table to discuss or work on. At the same time, they just request what Drageset et al. (2021) call (mere) answers without any explanation. When such questions and (mere) answers go on with confirmations and no elaboration, it is a typical cumulative pattern described by Mercer and Wegerif (2002).

The second, *requesting explanation*, could be questions such as “hmm, how do we find the average then?” and “how do we construct that (an angle of 120 degrees)?” These questions focus on revealing what was or could be done to reach an answer, often step by step. Such questions and explanations are typically about elaboration, and if these explanations are mostly accepted without further questions, they might be explanatory as part of a cumulative talk. On the other hand, when questions are further worked on or challenged, they might form a basis for Mercer and Wegerif’s (2002) explorative talk.

The third, *requesting evaluation and clarification*, could be questions such as “What if there were no number in the middle, what if so? (talking about the median)”. While these questions request an evaluation, they also typically request clarification of detail, or what Alrø and Skovsmose (2004) call reformulation, which is vital to move the process or understanding forward. Such questions form an essential part of an explorative talk (Mercer & Wegerif, 2002) since clarifications and evaluations form the basis for further developing each other’s ideas.

The fourth, *requesting argumentation*, could be questions such as “Why did you do that?” and “Why is it so that we can add and divide?” These questions could be separated from explanations as they request a logical explanation or advocating (Alrø & Skovsmose, 2004) for a reason and not just a chronological explanation of the steps that lead to the answer. Such questions are an essential part of an exploratory talk, but when they are rhetorical with no genuine interest in the answers, they belong more to Mercer and Wegerif's (2002) disputational talk.

Overall, girls 1 and 2 moves the mathematical talk forward with their different types of questions. When one question is asked, suggestions and explanations are often followed up by additional questions, requiring elaboration and involvement from several students. Since both multiple students and multiple points of view were accepted, the conversation in the group is characterized by having an interactive-dialogic approach (Mortimer & Scott, 2003). One example of how girls 1 and 2 move the talk forward is this, where they support and elaborate on each other's questions:

Girl1:	Which education programs have more apprentices than the average for all education programs? What must be done then?
Girl2:	Then we must find the average
Girl1:	Of all of them?
Girl2:	Yes. How do you do that then? I have forgotten
Girl1:	Hmm, how do you find the average here then?
Girl6:	You add everyone, and then you divide by the number.
Girl1:	Lovely, and then once we have done that, then we find out which ones are over?
Girl2:	Yes, which one is more than average?
Girl1:	Why is it so that we can add and divide? Why is it like that?

The students' discussion can be seen as negotiations about understanding where meaning is constructed together. It is not necessarily the correct answer that gets the most attention, but rather the process that leads to the answer. The students build on each other's input, which is a characteristic of cumulative talk. At the same time, the conversation shows elements from an exploratory talk where statements are being challenged and required for further explanations. The situation above can also be characterized as a dialogical interaction (Mortimer & Scott, 2003). The students are open to each other's ideas, and they build on the suggestions that emerge. They do not respond by pointing out errors but ask for further argumentation and explanation when they do not understand.

Although not all the girls were equally active orally, for example, the third girl who had the role of curious, the video shows that even the students who did not contribute with many statements were involved. They responded with *yes*, *no*, and other statements of support such as *hmm*, and visual expressions such as nodding their heads. Questions, explanations, and clarifications given by some of the students in the group seemed to contribute so that all students became actively involved in the mathematical discussions, some with an active role as listening more than talking.

Discussion and conclusion

This article reports from a study of how assigned roles and positions, particularly the curious role, can foster a more interactive and explorative talk in mathematics. Our findings show that the discussion in the group we studied is characterized by both cumulative talk and explorative talk (as defined by Mercer & Wegerif, 2002). Cumulative talk is most clearly seen when the curios request answers and claims and when they request explanations without using the explanation further. At the same time, almost two thirds of the questions request explanations, evaluations, and arguments. Such questions help to shift the focus from the search for mere answers to inviting peer students to focus on reason. These are signs of explorative talks, and these questions are so frequent that this group discussion is more explorative than cumulative. Further, the questions from girl 1 and 2 invites the other students to share their ideas, which means that the discussion is also characterized by dialogic and interactive communication, as defined by Mortimer and Scott (2003).

These findings illustrate how roles in mathematics, where we especially highlight the role as curious, can influence the mathematical talk towards a more interactive and explorative talk. This is done by requesting explanations, evaluations, and arguments, and in this way inviting other students into the discourse while simultaneously shifting the focus from finding mere answers towards reasoning. More generally, this article illustrates how the teacher can change the emphasis from teacher questioning to student questioning in the mathematics classroom.

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Cognitively activating lessons: A Nordic comparative study

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Keywords: Instructional quality, cognitive activation, classroom observation.

Cognitive activation

Although frameworks differ somewhat in exact conceptualization, there is a general agreement that instructional quality is a multidimensional construct (Croninger et al., 2012; Grossman et al., 2014). Cognitive activation has been established as one of "three basic dimensions" of instructional quality, along with individual learning support and efficient classroom management (Kunter et al., 2013). Furthermore, cognitive activation is among the dimensions of teaching most often represented in both mathematics-specific and content-generic frameworks for analyzing instructional quality (Praetorius & Charalambous, 2018). In an analysis of 12 frameworks, Praetorius and Charalambous (2018) concluded that cognitive activation consists of three aspects of teaching practice: (1) the teacher's selection of challenging tasks and use of mathematically rich practices, (2) facilitation of students' cognitive activity, and (3) supporting students' meta-cognitive learning from cognitively activating tasks.

The aim of this study is to compare the differences and similarities in cognitive activation in mathematics instruction between Iceland and the other Nordic countries.

Method

The study is a part of a larger research initiative, *Quality in Nordic Teaching* (QUINT), whose vision is to investigate instructional quality in the Nordic countries using video-based classroom observations. The sample consisted of ten mathematics classrooms from each country: Iceland, Denmark, Sweden, and Norway. In each classroom, three to four consecutive lessons were video-recorded and scored on a four-point scale by trained observers in the mathematics-adapted Protocol for Language Arts Teaching Observations (PLATO; see Grossman, 2019). For this study, specific lessons were identified where the teacher offered students rich opportunities for cognitive activation by considering the PLATO scores for *intellectual challenge* (IC) and *classroom discourse* (CD). In lessons where IC is at a high level, the teacher provides students with activities where analytical thinking is required by explaining, justifying solutions or reasoning. In lessons where CD is at a high level, the teacher provides students with opportunities for mathematics-related talk and the participants in the discussion expand on each other's ideas. The "top two" lessons in these elements from separate classrooms from each country were selected, for a total of eight cognitively activating lessons from eight separate classrooms.

Results

Analysis of data is ongoing as of this writing. In figure 1, a scatterplot visualizes the average scores across classrooms in IC and CD where chosen lessons have been highlighted.

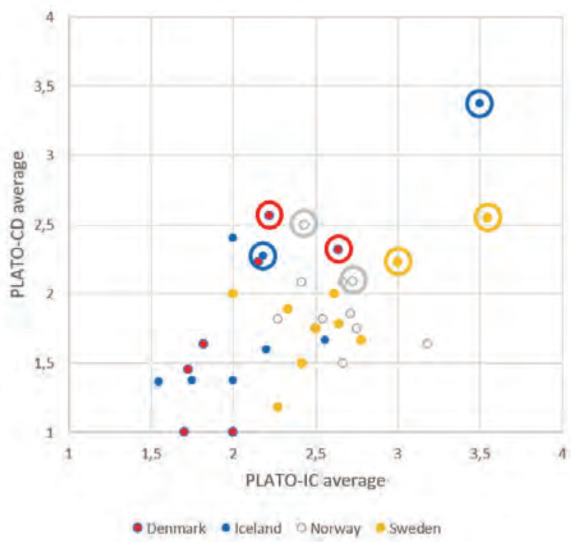

Figure 1: Scatterplot of average scores in IC and CD across classrooms in the four countries with classrooms highlighted from which lessons were chosen for further analysis

It is worth noting that average classroom-level scores only tell a small part of the story. Ongoing analysis of the chosen lessons will shed further light on which aspects of cognitive activation were manifested in classroom interaction and resulted in the high scores, such as choice or implementation of tasks, activity structures or other teaching profiles. The results will address to what extent the lessons may fit the notion of a "Nordic model" of instruction.

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Socio-mathematical norms regulate whole-class discussion

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This study identifies existing socio-mathematical norms through student and teacher responses in mathematical activity in grade eight in Sweden. The data consist of one video observed research lesson, and the analyses contribute to methodological identification of socio-mathematical norms through student responses and connected teacher activity during dialogues. Through these dialogues, the study identifies how students and the teacher expressed aspects of mathematical knowledge in their responses based on socio-mathematical norms in this classroom.

Keywords: Mathematical reasoning, socio-mathematical norms, student responses.

Introduction

Socio-mathematical norms (SMN) are jointly agreed upon between teachers and students in the mathematical classroom, constituted through participation (Cobb, 2002). SMN are common cognitive and social structures that frame what is communally valued by the classroom community (Cobb, 2002). These are shaped and reflected by cultural traditions that emerge specifically in mathematics classrooms through the configuration of activities. Focus is directed towards what knowledge is rewarded during the learning process (Yackel & Cobb, 1996). What distinguishes a mathematics classroom from another is the structure of classroom norms, not their existence or absence (Yackel et al., 2000). According to Yackel and Cobb (1996), the teacher has a significant role in suggesting SMN, which the students can collectively accept. For example, the teacher can do so based on the type of solution and arguments from students (McClain & Cobb, 2001), which influences the type of mathematical knowledge to be valued in the activity.

Some previous studies have identified limitations where social classroom norms limit the establishment of SMN when the focus is more on how teachers or students should behave democratically in an activity (Fredriksdotter et al., 2021; Kazemi & Stipek, 2001), instead of promoting discussions or arguments for which a mathematical model is best suited for the task (Cobb & Yackel, 1996b). SMN frame both the knowledge and structure for interaction (Cobb & Yackel, 1996a). However, sometimes the teacher will intend to gather short student responses with a character of correct answers. Responses like that contrast with occasions where the student response is more comprehensive, containing conceptual understanding and elegant reasoning (Kilhamn & Skodras, 2018).

Methodologically it is challenging to detect classroom norms. These are often indirect and implicit but somehow obvious to participants through their participation in practice (Cobb, 2002). Different perceptions of existing norms may coexist within the classroom (Cobb & Yackel, 1996b). One way to methodologically detect classroom norms is when teachers and students explicitly discuss expectations (Cobb, 2002; Wester, 2015). Another opportunity to identify SMN occurs through participants' reactions when existing norms suddenly get challenged (Cobb, 2002). These reactions will be indicators when existing norms are broken. The current study will contribute to identifying SMN through video observation of interactions between students and teachers in a mathematical teaching activity. Instead of methodologically using teachers' intentions to promote SMN (Wester, 2015), this current study elucidates student response through dialogical patterns in the classroom community of practice aiming to identify SMN. We address the following research question: How can student response and connected teacher activity be used to identify socio-mathematical norms (SMN)?

Method

We analysed a whole-class discussion (50 min) to answer this study's research question. To capture students' responses and connected teacher activity, the students were video-observed during a whole-class discussion in a grade 8 classroom in Sweden (one teacher and 25 students). The video observation was transcribed verbatim and analysed by the two authors. We have used an abductive approach during the analysis of empirical data and use of the framework (see Table 1). First, we divided the whole-class discussion into parts based on dialogue patterns. Each pattern consists of the students standing in front of the class discussing their own solutions to a given task (see Figure 1). Students' responses in the discussion, together with the related teacher activity, form the unit of analysis. By studying several dialogue patterns in the whole-class discussion during the same lesson, the opportunity for detecting how SMN directs and influences the mathematical teaching activity regarding the structure of the whole-class discussion increases. According to Kilhamn and Skodras' (2018), framework dialogue patterns are illustrated in the operationalising approach (see Table 1).

In operationalisation, different characters of reasoning are elucidated, and these have been inspired by Hjelte et al. (2020). Reasoning can be characterised as either general or domain-specific for mathematics (Hjelte et al., 2020). General reasoning is like a logical chain attempting to reach a solution, using necessary mathematics regardless of domain. Domain-specific reasoning belongs to a specific mathematical domain. Hjelte et al. (2020) suggest the following mathematical domains in research-based studies: *Spatial reasoning*, *Informal Inferential Reasoning*, *Additive, multiplicative and distributive reasoning*, *Algebraic reasoning*, *Proportional and covariational reasoning*, *Quantitative reasoning*, and *Transformational reasoning*. Different types of reasoning may be connected to each other and form a network of reasoning.

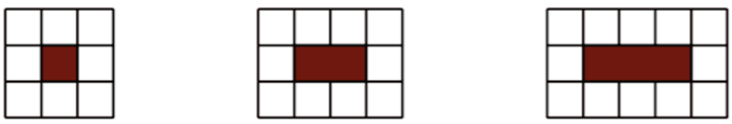
The operationalisation of the framework also focuses on different characters of mathematical knowledge as procedural or conceptual, based on Yackel and Cobb (1996), concerning SMN. Procedural knowledge focuses on solving the already constructed mathematical calculation structures, while conceptual knowledge means that the student knows mathematical concepts and their relations between each other (Yackel & Cobb, 1996). Knowledge is valued as SMN in category LC through the operationalisation procedural, while conceptual understanding is valued as SMN in category HC.

Table 1: Operationalisation of Kilhamn and Skodras' (2018) framework according to student response and associated SMN

Level of response	Student's response associated to SMN
F Factual	An accepted answer related to the task or to teacher's questions. No attempt at descriptive or explanatory response. Ex: Name, identify, repeat, recall known facts or procedures, "correct" answers wanted. (often short answers, including yes/no)
LC Low-level Conceptual	The descriptive response values procedural knowledge (Yackel & Cobb, 1996) containing expressions of step-to-step solutions and may contain general reasoning (Hjelte et al., 2020)
HC High-level Conceptual	The explanatory response values conceptual knowledge (Yackel & Cobb, 1996) and is based on domain-specific reasoning (Hjelte et al., 2020).
E Evaluative	Apply and evaluate domain-specific reasoning (Hjelte et al., 2020) in relation to mathematical content containing in specific task. Different domain-specific reasoning forms a network for explanation.

Ethical procedures of informing participants of their rights to refuse to participate and obtaining verbal and written consent for their voluntary participation were followed using Swedish guidelines for research (Vetenskapsrådet, 2017). Furthermore, the collected data were anonymised and coded to protect participants' confidentiality, and the researchers had no teaching or grading role for the students involved. In addition, the names of the participants and other sensitive data were removed from the stored data, and these data were secured in a safe locker to avoid unauthorised access.

Eva and Ali put white tiles around flowerbeds as the pictures shows.



Figur 1 Figur 2 Figur 3

3. How many white plates are used in figure n?

4. How long will the flower bed (dark tiles) be if they have 158 white tiles to lay?

Figure 1: Task

Result

Analysis of this whole-class discussion reveals the following lesson sequence of dialogue patterns based on the operationalised framework: F-F-**LC-LC-HC**-LC-F. Three dialogue patterns are emphasised (bold) to answer the research question: How can student response and connected teacher activity be used to identify socio-mathematical norms (SMN)?

Dialogue 1 (student's response level LC)

66	Teacher:	We will stick to problem 3 (How many white plates are needed in figure n?) Namely the formula (point to the formula that the students wrote on the board: 2 * n + 6) and then I take problem 4 (How long will the discount be if they have 158 white tiles to put?) which was also to find out how many or which figure how many blacks what discount is it that has 158 tiles around it and then we have student A and student B, welcome.
67	Student A	We have taken 2 times n plus 6 (read out what is written in the formula 2 * n + 6 on the board for the teacher).
68	Teacher.	How have you used this since, in task number 4 where what have you done?
69	Student B	2 was the difference, after all, any number plus 6 (rename the formula for the teacher).
70	Teacher.	How have you used this since how have you done to calculate, ie which figure you are looking for?
71	Student A	We took 2 times 54 plus 6.
72	Teacher	Hm
73	Student A.	It will be 158 minus 6 it was the edges and then 152 divided by 2 and it will be 76.

The teacher (line 66) begins by inviting students to the board and asks them to focus on parts 3 and 4 in the task (Figure 1). Through the introduction (line 66), the students explicitly request no specific student response. Still, it is up to the students themselves to interpret what is expected of them when they stand by their solution at the board. Student A responds by reading out the expression for the formula without mentioning the answer to part 3 in the task (line 67), which constitutes a limited student response. The situation that, as a student, sometimes occurs at the board is not new for these grade eight students. Students have support for their interpretation of what is expected of them at the board based on previous similar experiences, which follows prevailing classroom norms for the situation and similar contexts. Pretending that the students' responses do not contain the direct answer to the task may mean that the students have interpreted that the teacher is looking for something other than just the answer to part 3 in the task, of which the indication of the formula becomes their attempt at what can constitute an acceptable response (level F).

The teacher follows up students' short responses by asking a more investigative question (line 68), aimed at part 4 in the task, which the students also did not answer with their short response. This how-to question has the potential for student responses at higher levels than F. Student A's response continues as before by repeating the formula in different ways (line 69 and line 71), although the formula alone still does not answer the teacher's consecutive how-questions (lines 68 and 70). The student's response remains at level F.

Only when the teacher (line 72) does not provide the expected feedback on the given student response (based on the formula), do the students develop their responses to a description at the LC level. Student A describes how they arithmetically worked their way to answer part 4 in the task. LC level because the student in their response expresses different steps in their calculation against the answer. In their arithmetic description, the student also names a geometric representation (the edges of the figure) to support the calculation. Therefore, we cannot answer whether these students are able to make an explanatory response at the HC level or if the students' interpretations of the norm system prevent them from providing an explanatory response.

In writing, the students show the way to the answer on the board and orally express what they value is most interesting and of the highest mathematical quality (the formula). These responses show how these two students interpret prevailing SMN. Since the teacher is repeatedly not satisfied with the students' responses, the students finally succeed in changing the content in their responses (line 73). Based on its arithmetic calculation, the steps to the answer are described (level LC), and no further teacher questions are then asked.

Dialogue 2 (student response level LC)

80	Teacher:	Now there are two solutions left that do not do exactly the same […] can you come up and tell?

Already in the introduction (line 80) of dialogue 2, the teacher expresses that one should compare two different solutions. This teaching document challenges the accounting students to develop their response since they have different solutions that must be compared in an investigative way.

83	Student C:	May I draw?
84	Teacher:	Yes
85	Student C:	Ok, then we'll see (draw a rectangle and another rectangle inside) if you have, so to speak, the whole figure here, i.e. 158, (draw around the inner rectangle) then you take minus 2 and you get (draw again) you get the cubes. And then you get 156 which is this and that (shows in the drawing) then you divide by 2 so you get 78 and 78 is then this one down as well (shows in the drawing) and since we knew there was one on each side (shows in the drawing) instead of taking it minus 2 you know yes it is the solution (aimed at the teacher).
90	Teacher:	(fills in more the four boxes in the middle) these are the discount in the middle and this is what you want to find out so C removes this and that piece (point to the boxes next to the right and left of the discount) to remove them simply and then he has left the top row and the bottom row (points in the rectangle) and then he divides them by two and then he only gets one row (points to the bottom row) but that row is two pieces longer still than the one in the middle so then he takes minus two again. Are you in? Turns to the class and then you did a formula of this (turns to student C).

Contained in the introduction (line 80) of dialogue 2, the teacher expresses that one should compare two different solutions. This teaching document challenges the accounting students to develop their response since they have different solutions that must be investigated. Student C asks if it is ok to draw (line 83). After the teacher's approval to draw, the student enters a dialogue with the teacher using some verbal caution and then tries to describe the chosen calculations based on a drawn picture. Through general reasoning, the student uses, without verbal expression, concepts in the mathematical domains geometry (perimeter) and algebra (Figure n). Based on the student's response, there are possible conditions to end up at the HC or E level. However, expressions in students' responses are not explanatory and contain domain-specific reasoning. Instead, this student's response is a step-by-step description, strengthened through the drawn picture, of the path to the task solution (level LC).

In line 93, the teacher once again verbally expresses the student's description of the solution based on the picture. For the context, the teacher possesses more functional language than the student, and the teacher's revoicing of C's description becomes more accessible to the listening students.

However, in terms of content, the teacher's narrative is on the same level as the student's (level LC) as it intends to explain the path to the solution. As the content of the teacher's responses remain at the LC level, the prevailing SMN in the internship community are confirmed for the context.

Dialogue 3 (student response level HC)

D: accounting student, Ex: students in the class.

97	Teacher:	[...] can you tell us how that formula came to be?
100	Teacher:	Look at what you have done (point to the formula $Y = (1n + 2) * 3$ and student D looks) 1n plus 2.
101	Student D:	In order to …
102	Teacher:	What is 1 and why 2?
103	Student D:	Hm.
104	Teacher:	Can you tell others what he did? (turns to the class)?
105	Student E1:	2 is probably the difference between the different ones.
106	Student E2:	n is well that in the middle.
107	Student E3:	n is well the figure plus ...
108	Student E1:	not always.
109	Student E2:	Now we'll find out in the middle.
110	Student D:	Wait n is it (points to the flowerbed in the middle, the four coloured squares in the rectangle) plus 2 is equal to it (points to the bottom row) and
111	Student E2:	Why multiplied by 3?
112	Student E3:	Y is the whole set.
113	Teacher:	Yes exactly, he has taken the middle plus the edge pieces (points) and so he has called it y. So he puts someone else's letter. One row is (points) and taken it three times, there are three rows, but after doing that you have to remove what is in the middle to get what is around.

In line 97, student D is asked to unpack the formula they have formulated. Since the student cannot answer the question, it is reformulated by the teacher (lines 100 and 102). When the student is still unable to formulate a response (line 103), the teacher (line 104) turns to the whole class and asks them to investigate what is identified in the formula. This teacher's action breaches norms that open the norm system for investigative conversations. This invites interactions from students who have previously only listened and not been involved in the dialogue. Student D and some listening students then formulate a joint understanding of the formula (lines 105–112). As a collective unit, they unpack the formula and express the meaning of the parts in words. Each of the individual students' responses contains short responses and statements. Through joint interaction in the investigative conversation, the short contributions together explain domain-specific reasoning in algebra reasoning and spatial reasoning. In the common dialogue (lines 105–112), there is a knowledge of mathematical content based on spatial reasoning linked to numbers (numerical), which are allowed to meet algebraic reasoning about the variables and constants of the formula to create a common relational understanding of what the formula expresses. Despite students' mathematical knowledge, their verbal response does not contain expressions of reasoning and mathematical ideas. Instead, the students express different statements about how different parts of the formula can be interpreted in their responses. It does not become an open dialogue about which mathematical ideas and mathematical reasoning are behind the various statements. Instead, statements that are not met during dialogue with alternative statements are silently accepted. Therefore, the common student response remains on the HC level. Both the students' knowledge and the teacher's activity – to encourage students into exploratory conversations – contain the potential for HC and level E. However, there is a lack of support in the SMN for students' responses to be able to reach level E in this situation.

Also, in this episode, the teacher sums up the discussion by re-telling and verbalising the jointly expressed opinions by the students. The teacher thus responds to students' responses at both the LC and HC levels.

Discussion

Through an operationalised framework (Table 1), it becomes possible to categorise students' responses in the dialogue patterns. The level of student response and connected teacher activity is used to identify the prevailing SMN. Based on an analysis of each dialogue pattern, a lesson sequence for the whole lesson is possible. In the current lesson sequence (F-F-LC-LC-HC-LC-F), F and LC levels became the most common student response. This means that prevailing SMN supports dialogue patterns at the F and LC levels through this classroom discussion but lacks support in SMN for both the HC and E levels. However, one dialogue pattern still ends at the HC level during this lesson. This exception happened when the current teacher opened up the prevailing norm system for that moment. After this particular dialogue pattern, the following dialogue patterns once again returned to the normative for F and LC levels. This then saw the identification of the F and LC levels around the mathematic knowledge valued in the activity (McClain & Cobb, 2001).

A complementary methodological way to identify SMN can be done through potential norm breaks. According to Cobb (2002), another way to identify SMN are manifested through participants' reactions when existing norms are challenged. In dialogue 1, the teacher's action (line 72) challenges students' perceptions of their existing valued response. Instead of the students becoming frustrated at expressing the algebraic formula once again (level F), the teacher's activity leads to the students developing a response containing a description of the calculation steps (level LC). In dialogue 2, the student explicitly asks the teacher for permission to break existing SMN about valid representations (line 83). This student's response aims to precede that there will be no reactions to the drawing. This student response leads to be descriptive or explanatory and contains reasoning. In this episode, the student response still ended up at the LC level by being descriptive and containing general reasoning (Hjelte et al., 2020). In dialogue 3, the teacher insinuates norm break (line 104) by challenging the prevailing SMN to change temporarily. This teacher activity made it normative for other students to become temporarily involved in a joint explanation activity. Through the mentioned potential norm breaks, we will receive similar SMN as we would when analysing through the framework (Table 1).

The operationalised framework contains four "levels" of student responses, and these levels in our approach are somehow considered misleading. This is because they are more about four different structures, with different teaching intentions linked to different SMN due to classroom community of practice (Cobb & Yackel, 1996a). From this point of view, we will consider the analysis through the framework as non-normative for teaching, rather a contribution to developing and challenging teacher practice. Based on the findings in this research study, our implications for practice indicate that SMN impact and even regulate the opportunities for developments in the mathematical community of practice. Therefore, implications for practice will be to contribute through this framework teachers' professional learning development about their own teacher practice according to deeper insights in different SMN. We suggest further research in this area, with the aim to contribute to professional teaching-learning development. Such research could also contribute to further elaboration of the operationalisation of the framework and analysis through the framework according to SMN.

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What to do or what to learn – on communicating learning goals

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This paper reports the results from a large-scale video-observation study where 127 mathematics lessons were coded with regard to the way teachers communicated learning goals. Four consecutive lessons were collected in 35 classrooms (grade 7, students age 14) in Sweden. The PLATO-manual was used to code the lessons with a four-grade scale. The results show that teachers mostly communicate vague or inferred learning goals. Tasks are in line with these (inferred) learning goals and the teachers give clear instructions to students regarding what to do. Throughout lessons, teachers seldom refer back to the learning goals. However, half of the teachers occasionally express more detailed learning goals and relate the content of the lesson to a learning outcome.

Keywords: Learning goals, video-observation study, lower secondary school, interactions.

Introduction

> So, the coming lesson we will work from page 64 and work through task 83 to 94. For task 93 and 94 you might need a calculator which you can find in the cupboard here in front of the class. Don't forget to explain your calculations! Now let's get started.

When a teacher gives instructions such as the one above, students are given clear instructions on what to do and can immediately start working on the tasks. What is to be done has been clarified, but what is to be learned is vaguer. Further, no discussion is initiated on how or why students are expected to learn this. Research on instructional techniques in all core content areas has found that explicitly linking classroom activities to learning goals helps students understand the purpose of the instruction and make them feel motivated to engage with the ideas (Baker et al., 2002; Banilower et al., 2010; Spinath & Steinmayr, 2012). Further, clarity and explicitness of learning goals help students to create a context in their learning (Spinath & Steinmayr, 2012).

The situation presented above describes a rather common lesson start as observed in a large-scale video-observation study in grade 7 (14-year-olds) in Sweden (Tengberg et al., 2021). This LISA-study (Linking Instruction and Student Achievement) aims to capture different aspects of teacher instruction, in which the way teachers present their learning goals is one such aspect. In this paper data from the LISA-study will be presented with a specific focus on learning goals. It aims to answer the following question: To what extent and in what detail are learning goals communicated across lessons and classrooms in Swedish 7th grade mathematics?

Previous studies

Detailed and clearly communicated learning goals influence students' learning achievements (Hattie, 2009; Boden et al., 2019; Locke & Latham, 2002, Reed, 2012). Further, clear learning goals can motivate students (Spinath & Steinmayr, 2012) and clarify what is expected as an outcome of the lesson (English & Steffy, 2001; Hattie, 2009). Communicating learning goals enables both students and teachers to see connections between previous lessons and the current one but also to see connections between activities and instruction within a lesson (Vaughn & Bos, 2010).

Within a Swedish context, Hemmi and colleagues (2019) found that Swedish teachers, when working with Finnish curriculum materials, expressed their goals vague and implicit. This is in line with previous studies in Sweden (c.f. Boesen et al., 2014). In an interview-study, Fauskanger and colleagues (2018) found that Norwegian teachers preferred goals focusing on the content and supporting student learning. Yet, observing teachers' instructions on a large scale is relatively uncommon in the Nordic countries. In Sweden, a limited number of observation studies have been conducted either led by The Swedish Schools Inspectorate (2009), or as an evaluation of a nationwide mathematics professional development program (i.e., The Boost for Mathematics, Matematiklyftet) dedicated to the improvement of the teaching of mathematics (Ramböll, 2016: Österholm et al., 2016; 2021). One of the outcomes revealed that while lessons are often structured in terms of planned activities, they seldom start with a presentation of goals and purposes of the lesson, and teachers seldom leave time for reflection or evaluation of what was learned at the end of the lesson (Österholm et al., 2016; 2021).

Since the importance of communicating clear learning goals to students has been discussed and has been the subject of professional development initiatives in Sweden, this element deserves special attention. Also, the Swedish goal-oriented curriculum (Swedish National Agency for Education, 2011) pleads for an attention to communication of learning goals in class.

Background – the LISA-study

The LISA-study aims to capture the quality of teaching of different subjects in the Nordic countries. This article focuses on mathematics taught in Sweden. LISA uses an observation protocol (see below) to explore the quality of teaching. In Sweden, we observed 35 classrooms, taught by 31 teachers at 15 schools. Each classroom was video-recorded for three or four consecutive lessons in the middle of the school year, which resulted in 127 video-recorded mathematics lessons.

To obtain a representative sample and to match the national distribution, the schools were stratified according to different variables such as the schools' locality (urban/rural), the percentage of immigrant students; achievement level; the organization of the school (public/charter). Also, age, gender and qualifications of the teachers included in the sample varied which provided a fair and diverse representation of mathematics teachers in Sweden (for a more detailed description, see Tengberg et al., 2021).

Method

Video observations

In the present study, two cameras and two microphones were used to capture the teaching. One camera in the back of the class, capturing the teacher, and one in front of the class, capturing the students. The teachers wore a microphone so all their talk could be recorded. Another microphone was placed in the middle of the room to capture students' speaking. Students who did not want to be video-recorded were seated on one side of the classroom, and the cameras were adjusted in order not to capture that part.

Four consecutive lessons were recorded in order to enable reliable information and to capture enough variation of teaching practice, this choice was led by findings from previous studies (cf. Kane & Staiger, 2012). However, the question of how many lessons that are needed to capture a teaching phenomenon (like clarifying and communicating learning goals) has been raised by several researchers (Cohen & Goldhaber, 2016; Ho & Kane, 2013) and consensus has, so far, not been obtained.

Each lesson was divided into segments, 15 minutes each, constituting the unit of analysis. If the last minutes of a lesson did not make up a whole segment, these minutes were either included in the previous segment (if the segment was shorter then 7,5 minutes) or added as a new segment (if the segment was at least 7,5 minutes long). When analyzing a lesson, a division into smaller units, following the different stages of a lesson is advisable (Clarke et al., 2006). These stages correspond to the structural level a teacher adopts to a lesson and often follows 15-minute segments.

Observation protocol

The protocol for language arts teaching observation (PLATO) was used to code the video recorded lessons. It was first designed for observing lessons in language arts (Grossman, 2015) but has been used in other subject areas as well. For mathematics the observation protocol has been revised, which has resulted in qualitative criteria similar to those used for language arts, but valid for mathematics. In specific, the PLATO-element *purpose* attempts to focus on whether or not the learning goals are clarified, and if coherence is established between the tasks, activities and the learning goals. By doing so, the quality of instruction is coded on a four-grade scale. In the example at the beginning of this paper, no specific learning goal was communicated. However, an implicit goal (connected to the mathematical content of the exercises) can be inferred. This would score a '2' according to the PLATO-manual. General instructions like "Today we will learn about linear functions" would also score a 2 (Figure 1). If a teacher would communicate a more specific learning goal such as "Today we will learn more about the slope of a linear function in relation to the values a and b in the formula", this could result in a score of '3'. In order to score a '4', there should also be evidence that the students are aware of the purpose or that the teacher or the students refer back to the purpose during the segment. Each segment is coded separately and only for what is present during the segment. However, if a teacher or students refer back to a purpose that was presented more specifically during a previous segment, the coding will take that level of specificity into account in the later segment in order to score a '4'.

1 **Provides almost no evidence**	**2** **Provides limited evidence**	**3** **Provides evidence with some weaknesses**	**4** **Provides consistent strong evidence**
There is no clear learning goal in the class or the learning goal is not related to the development of mathematical knowledge or skills.	There is a learning goal communicated or inferred, that is connected to the development of mathematical knowledge or skills. The goal takes the form of a general topic or activity (e.g., "Today we will learn about linear functions", or "Now we will work on our division"). The lesson's activities may not align to the learning goal.	There is a clearly communicated, specific, learning goal that is connected to the development of mathematical knowledge and skills. The lesson's activities align to and target the specific learning goal. The teacher makes clear how the lesson will support students' development as users of mathematics.	There is a clearly communicated, specific, learning goal that is connected to the development of mathematical knowledge and skills. The lesson's activities align to and target the specific learning goal. There is evidence that students are aware of the purpose. The teacher or students refer back to the purpose during the segment. Teacher makes clear how lesson will support students' development as users of mathematics

Figure 1: Scoring criteria of *Purpose* in mathematics (adapted from Grossman, 2015)

Reliability

Video observations enable repeated analysis to capture important details and patterns to greater extend then observations in classrooms (Borko et al., 2017). Further, they enable joint coding. In the present study, around 40% of the data were scored jointly by two raters, in order to obtain and monitor reliability. The observers were all certified PLATO-raters. Certification was obtained after a four-day course and a test in which .80 reliability per item had to be obtained.

To obtain a reliability of .80 on PLATO, at least five lesson segments should be observed (Cor, 2011). In our study we have an average of 12,7 observed segments per teacher, with a minimum of 7 segments.

Ethics

The principals of all schools were contacted first after which our request for participation was forwarded to the teachers. Teachers and students were informed about the aim of the research and how the data would be used as well as their rights as participants. Questions could be asked prior to data collection. All participating teachers, students, and guardians of students signed an informed consent. As stated, students who did not want to participate were seated on one side of the classroom so they would not be captured on video. The research was conducted in line with Swedish guidelines on research ethics (Swedish Research Council, 2017).

Results

Learning goals per segment

Of all 403 segments, 358 segments scored a ‘2’ (89%) on the four-grade scale. Often this was due to the fact that teachers only described the topic to work with in a broad way (e.g. “algebra”, “functions”, “fractions”), but on some occasion teachers would also refer back to previous lessons and just state that “today we will continue from where we ended yesterday”. Thus, they would not make it clear how the lesson would support the students’ development of mathematical competencies. Just as in the introductory example, teachers frequently instructed students on what to do “work from page 64 and work through task 83 to 94. Don’t forget to explain your calculations”, but did not explain why or what they were supposed to learn during the lesson.

One might assume that communicating goals would mostly occur during the beginning of a lesson, like in the following example (scoring a 3 or a 4): “We start with today’s schedule. The goal is to be able to calculate part of whole and to be able to simplify and reduce fractions”. Indeed, of the 41 instances where a score at the high end (3 or 4) was observed, 23 were observed in the first segment (Table 1). This means that, even when only the scores of the first segments of each lesson are included, still as many as 103 out of 127 segments (81%) scores a ‘2’. One could also argue that the goal of a lesson can be communicated at other stages of the lesson: for instance, at the end of a lesson. Looking at final segments, high scores of *purpose* occurred ten times out of 127 in the last segment of the lesson, which means that in just below 8% of the last segment of a lesson, teachers would come back to previously stated (implicit or explicit, vague or detailed) learning goals: “To sum up, todays lesson was about (…)”. There were in total four segments where no learning goal at all was communicated. One instance occurred in a lesson where a fifth segment was recorded (thus a lesson of consisting of more than 127,5 minutes), the other three segments occurred during one single lesson, indicating that no learning goals were communicated that lesson.

Table 1: Scores on Purpose divided over start, middle and closure of the lessons

Score	First segment	Middle segment(s)	Last segment	Total
1	1	1	2	4
2	103	140	115	358
3	21	6	8	35
4	2	2	2	6
Total	127	149	127	403

Learning goals per teacher

If we look at the different teachers and see in what way the scores were divided, we can see that the majority of teachers scores a mean value at or close to 2,00 (Mean 2,11; SD 0,38). Of all 31 teachers, 16 scored only on the low end (score 1 or 2), the other 15 teachers scored both at the high and the low end (scores ranging from 1 to 4). These teachers, who scored distinctly higher than 2,00 on average, communicated their goals more explicitly and in more detail. On rare occasions, this was done through utterances like: “at the end of the lesson you are expected to be able to (…)” or “yesterday we dealt with functions with one unknown, today we will continue and you will learn how to handle functions with two unknows or variables”. The scores on the high end require that the learning goals are in some way connected to a learning outcome, which could for instance be through exit tickets where students were to write what they had learned during the lesson.

Summary

We set out to answer the research question “To what extent and in what detail are learning goals communicated across lessons and classrooms in Swedish 7th grade mathematics?” and found that learning goals are typically implicitly stated and addressed in a vague way. Half of the teachers, express their learning goals more explicitly and, on rare occasions, learning goals are revisited by the teacher at the end of the lesson.

Discussion

The present study aimed to contribute to previous literature about learning goals through analysing how the purpose of the lesson is communicated in class, in specific in 35 mathematics classes in Sweden. The results showed that learning goals in lower secondary mathematics instruction are often implicitly stated, which is in line with the findings of Hemmi and colleagues (2019). Also, in a study including LISA data from all Nordic countries, similar results were found with a large number of segments scoring a ‘2’ (Selling & Klette, 2021). Their study also examined if the goals were focusing on content or competencies and found that most often content was addressed. If competency goals were included, these addressed procedures rather than conceptual knowledge.

A lack of clarity and detail can lead to a difficulty for students to perceive coherence within a lesson (Vaughn & Bos, 2010). In the present study, we found that even when instructions indicate some kind of coherence (e.g., when teachers say “Today we will continue from where we ended yesterday”) the connection between previous lessons and the current one might nevertheless be unclear. Also, the outcome of a lesson might be unclear for students, as no guidance is given regarding to what they are supposed to learn during the lesson. As there were few deliberately planned closures of lessons, learning goals and learning outcomes were very seldom reflected upon.

In light of the goal-oriented curriculum in Sweden, where it is considered to be important for students to understand what they are expected to accomplish for a specific grade (Swedish National Agency for Education, 2011), implicit goals in class do not offer students such an awareness (English & Steffy, 2001; Hattie, 2009). A critical note to our choice of method (video observations) is that a phenomenon like communicating learning goals might be observable elsewhere, and not only through the teachers’ communication captured on the video-recordings. For instance, detailed plans that explain what students are expected to do are often available for students on digital platforms, and in such plans, learning goals might be more explicitly stated. Furthermore, when giving feedback to students, teachers might indicate more explicitly what an expected learning outcome might be, related to detailed learning goals (Hattie, 2009). PLATO includes feedback as an element, and the analysis of that element might reveal some more insights on the communication of learning goals. The implicitness and vagueness of the orally communicated learning goals do not have to imply that students are not aware of the more detailed ones. During the observations we saw that students were constantly working on their tasks (measured in the PLATO-element ‘time management and behaviour management, for more details, see Tengberg et al., 2021). Students ask relevant questions and teachers replied accordingly, seemingly in line with specific learning goals.

In sum, we argue that since communication of learning goals influences the students’ learning achievement (Hattie, 2009; Boden et al., 2019; Locke & Latham, 2002, Reed, 2012), the results of this study could suggest that the students in half of the classrooms (with teachers scoring at the high end) might be affected in a positive way, whereas the other half of the students (with teachers only scoring on the low end) might obtain lower results. The next step will thus be to link teachers’ instruction to student achievement as our data enables us to do so.

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TWG20: Mathematics teacher knowledge, beliefs, and identity

Introduction to TWG20: mathematics teacher knowledge, beliefs, and identity

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Rationale

The TWG 20 had 28 papers and 8 poster presentations. Due to the high number of participants, we had to split the group into two subgroups, A and B. We had 5 parallel paper sessions for each subgroup and one all group poster session. Like all other CERME TWGs, we used Padlet to share materials and reflections that were visible to both subgroups. The final session has been an all group discussion on the main crucial points discussed along the work developed. The discussions occurred around four main themes: (i) teacher knowledge and research – including the specificities of prospective and in-service teachers; (ii) teacher beliefs and identity; (iii) teacher knowledge and reflections and (iv) collaborative knowledge development. The discussions revealed the need to perceive and study teachers' beliefs, identity, teaching practice, emotions and teacher knowledge has a complex system, and the need to design (and to deepen in how to do it) and research collaborative professional development dynamics for teachers taking into account cultural context.

Here, instead of presenting what has been discussed in each one of the presented papers we opted for a more holistic approach: we present some of the main ideas and questions raised in collective discussions and at the end of TWG20 work, hoping that pursuing finding answers to such questions would lead us to advance to new paths and bring possible insights and approaches also to be discussed in the next CERMEs. For doing so we address the four main themes and conclude with some open questions and provocations for (possible) future work.

Teacher knowledgeand research

Research on teachers' knowledge seems to have assumed naturally the existence of different teachers' knowledge models with all its diversity and focus on particular specificities of the knowledge considered. In both CERME9 and CERME11, there was an effervescence of model development. The focus was on their use for a deeper understanding of teachers' knowledge in various contexts and on various topics. The discussions developed in TWG 20 during CERME 12 revealed a research shift of attention from the knowledge teachers (don't) have to the knowledge teachers use to teach mathematics which lead to the "superman/woman teacher" idea and on how to deal with it.

The ideas, theoretical discussions and methodological viewpoints emerged can be summarized as:

- the role of teacher knowledge frameworks – discussing the importance of the different teachers' knowledge models and knowledge conceptualizations and their use in and for research;
- research tools that "serve" both educators and researchers – focusing on the intertwined way of perceiving how the research tools can (should) be employed in both contexts (for teacher education and for doing research in and for teacher education);
- the need to pay attention to the specificity of teacher knowledge This problematic emerged associated to the large amount of research supposedly being done focusing in teachers' mathematical knowledge that can be carried out similarly with students from the school level those (prospective) teachers are (will be) teaching.

Some of the emerging questions are:

(a) What are the differences amongst prospective teachers and practice teachers in terms of content knowledge? Are we "taking care" of prospective teachers as teachers or as "older" students? And do we consider them in the work we do?
(b) The focus of attention in teachers' knowledge research has been mainly (in TWG 20) in prospective teachers, do we need to shift this focus and what are the affordances of and for doing so?
(c) What are the differences amongst these groups in terms of mathematical and pedagogical knowledge and what are the implications of the research being done and of the conclusions obtained?
(d) When doing research in prospective teachers, are we looking at the transition from student to teachers? What can we learn from that transition?

Teacher beliefs and identity

The themes of beliefs and identify have been addressed mainly in an intertwined manner to other issues. Teachers' beliefs, identity, teaching practice, emotions and teacher knowledge seem to represent a complex system, in which each dimension informs the others in a direct or indirect manner. The discussions carried out on these themes during TWG20 can be synthetized around three main ideas:

(a) the importance for prospective teachers and in-service teachers of collectively discussing and reflecting on core teaching practices for the development of their identity and their knowledge;
(b) different definitions and conceptualizations of identity allow for the study of different aspects of teacher (learner) identity and change;
(c) the relationship between teacher beliefs, culture and traditions: culture and traditions represent factors which inform beliefs.

Teachers’ knowledge and reflections

Reflection is transversal to teachers’ practices and thus, related with all the teachers’ cognitions (and not only). A set of discussion went around reflecting on the relationships involving enactment of teacher knowledge on:

(a) student thinking – associated to the research being done using student productions as a reflective tool and the analysis of students’ assessment items for investigating teachers’ knowledge;

(b) online environment – choosing and implementing tasks for online learning environment, the (dis)advantages of each one of the tasks and its types and the knowledge involved and required;

(c) developing measurement items – metacognitive aspect of developing measurement items to measuring teachers’ knowledge and the different dimensions of knowledge being measured.

In this space of discussions, also influenced by experiences during the pandemic period, the online practices emerged in an interconnected explicit manner and some questions related worth mention:

(i) What kind of practices allows teachers to improve themselves for online learning environment and what knowledge is involved?
(ii) What does it mean to have a strong mathematical knowledge for teaching online? In what terms it differs from other contexts of teaching?
(iii) How can different teacher knowledge models – considering or not explicitly technology – can be used for practical and methodological research purposes?

Collaborative knowledge development

The cultural dimension emerged strongly during the TWG meetings last CERME and TWG20 participants addressed it again during this CERME. There is the need to continue considering the international audience we have in the TWG and its cultural diversity in order to make explicit the particularities of the context research occurs. It will allow also the findings to be important to such a broader audience, and allowing, thus, contributing for the broader picture of improving the quality of research and its impact.

Some questions and reflections that emerged were:

(a) How to design and research collaborative professional development for teachers taking into account cultural context?
   (i) the need to a productive middle way between ‘anything anywhere’ and ‘nothing nowhere’ – the two ends of trying to implement anything anywhere or not implementing due to cultural factors are not beneficial;
   (ii) the need for comparison and exchange between professional development programs in different local contexts for further understanding of both local contexts and important aspects in implementation and the research being done;
(b) How to describe and/or analyze a cultural context that is useful in research relating to professional development?

(i) the need to consider that the awareness of culture/context might not be enough leading to the need to be explicit in how and what ways the specificities of the cultural context and considered and impact on the research being done;
(ii) the need for precise terminology for describing what was actually taken into account in the research being done so it can be also meaningful to a broader audience and impact on the more global level.

These questions were framed with the background of "*direction with potential*", thinking on the research being done and its intertwining with teacher education and professional development programs – both with prospective and practice teachers. In that sense in a transversal manner to the discussions there were to topics of cooperation, comparison and extension of these professional contexts and the tasks involved both in micro, local and global levels of practices.

Transversal ideas and new emerging questions

The structure of the work carried out during TWG 20 was designed to move from the discussion of the specific papers presented (which can be perceived as case studies) to the elaboration of a set of possible new trends for research on teachers' knowledge, beliefs and identity in initial and continuing education contexts.

In doing so we have been able to frame new possible paths and inquiry questions that have, indeed guided some of the research being done afterwards and which reflect the problematic themes from this community. Some of these transversal ideas to all the discussions refers to:

(a) a more concrete and explicit focus of attention in the connections and relationships between research and teacher education, emerging from: (i) the diversity of roles each one of us assume as researchers, teacher educators, and teachers; (ii) the research tools conceptualized and employed – e.g. tasks for data collection and its intertwined required relationship with teacher education;
(b) a specific aim of considering the impact of what we do (in research) in what it's expected to be done in the future – in terms of public policy about teacher education.

Some of the emerging open questions are related to:

(i) How researchers will manage zooming in on knowledge for teaching specific topics within a cultural context while zooming out of teacher knowledge models?

(ii) How can we balance the role of cultural context in conducting research and implementing teacher education with developing common understanding from research results?

(iii) What does it mean to have a "sound mathematical knowledge" allowing students to understand (online or not)? How can this be improved?

(iv) How can we (researchers, educators) address, utilize and examine different definitions of identity?

(v) How should we look into the system of teacher knowledge, identity, beliefs with lens of situated learning?

Teacher Practicum in Turkey at the Time of Pandemic

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Mentoring in pre-service teacher education is defined as a process in which mentor teachers assist the professional development of pre-service teachers. In this study, teacher practicum at the time of pandemic is examined through supervisor and mentor practices. This study aims to unveil how the practices of the two stakeholders in Turkey, and their interaction with each other in the context of teaching practicum evolved during Covid-19 pandemic. The sample of the study was 2 supervisors and 4 mentor teachers from Istanbul who worked with secondary school preservice mathematics teachers. Data was collected by using interviews with supervisors and mentor teachers. The findings of the phenomenology analysis showed that the adaptation of the interaction of stakeholders emerged to be in three categories; i) discontinuity of practicum due to practicum school online teaching policy; ii) online micro-teaching at university as practicum; iii) online teaching at high school courses.

Keywords: Mentoring, teaching practicum, secondary school preservice teacher education.

Introduction

In line with the fast changing world, education has an important role to shape nations' generations who are capable of overcoming the challenges. Therefore, the quality of education is important for countries in order to raise qualified citizens who make their counties much better. The most essential component of education is teachers and raising effective teachers is the first step of education for students. Because, "the quality of teachers determines the quality of education" (Jan, 2017, p.50) and the quality of teachers are determined by the quality of the continuum of teacher education. Teacher education is a life-long learning and preservice teacher education is the first step of it. The most essential part of preservice teacher education is teaching practice because teaching practice gives opportunities for preservice teachers to transform their theoretical knowledge into practice (Azkiyah & Mukminin, 2017). During the practicum process, mentees need guidance and support. The people who are responsible for providing such support and guidance are supervisors-instructors of mentees in the teacher education program and mentor teachers-collaborating teachers from practice schools.

Understanding the practicum process is difficult because it should be evaluated in terms of opinions of three stakeholders, university supervisors, mentor teachers and mentees. When the research studies about teacher practicum are examined, it can be seen that there is a limited number of studies to examine the practicum process in terms of the relationship and collaboration between three stakeholders. Because of the Covid-19, understanding these relationships has gained more importance to provide effective teacher practicum for mentees. Thus, the purpose of this study is to contribute to the literature on how the practices of the triad stakeholders in Turkey, and their interaction with each other in the context of teaching practicum evolved during Covid-19 pandemic.

Even if the importance of teacher has started to increase, many people think that anyone who has such content knowledge can be a teacher (Darling-Hammond, 2000). According to Shulman (1987), being a teacher, and skills that teacher must have are underestimated. Being a teacher contains within itself many roles, responsibilities, knowledge, and skills. Even if content knowledge is an essential to be a good teacher, it is not the only component of teaching because it just forms the “what” part of teaching. Teachers should also have pedagogical knowledge to be qualified to “how” s/he will teach the content.

Knowing how these skills and knowledge can be got as much as important knowing which skills and knowledge are necessary to be a teacher (Darling-Hammond, 2006). According to Shulman (1987) and Darling- Hammond (2000), preparation is an important step for getting these kinds of skills and knowledge because teaching can be learned. Therefore, teacher education has a significant role to raise qualified teachers.Teacher education is not short-term learning, it is a life-long learning. Therefore, it is not true to say that each teacher who graduates from undergraduate teacher education program completes their teacher education. Undergraduate teacher education program is just the first step of the teacher education process, and it has a great role to shape preservice teachers’ lives as a teacher.

Preservice teacher education and practicum

Due to the fact that preservice teacher education is the first step of teacher education, planning and applying the preservice teacher education is important to raise qualified teachers for countries. Even if each county may have its own preservice teacher education program, preservice teacher education has generally two parts to prepare students teachers as a teacher. One part constitutes the theoretical side of preservice teacher education and the second part constitutes the practical side of preservice teacher education.Even if theoretical knowledge is necessary, it is not sufficient to be a teacher. According to Munby et al. (2001), knowledge of teaching can be acquired and improved by the preservice teachers’ own teaching experience. Therefore, preservice teachers should have opportunities to apply their theoretical knowledge into practice (Azkiyah & Mukminin, 2017; Zeichner, 2009; Nguyen, 2020) to build a bridge between theory and practice.

In Turkey, recently, the Ministry of National Education and the Council of Higher Education made regulations on pre-service teachers' practices, the partnership between universities and schools and revised the roles of pre-service teachers, mentors and supervisors (MEB, 2018). Based on these regulations, mentees are expected to take subject-matter courses and teaching methods courses and to experience teaching in practicum schools. Universities are responsible for providing teacher candidates with the courses of subjects and teaching methods. In the field experience, both universities and partner schools have responsibilities for teaching practices of teacher candidates. Especially, mentoring provided by mentor teachers who are the supporters of teacher candidates at practicum schools plays a significant role in the development of teacher candidates' knowledge and skills on teaching.

There is a limited number of studies that focused on triad stakeholders in teacher practicum in Turkey. One of the studies of Yılmaz and Bıkmaz (2020) aims to examine the mentors' needs within the context of Classroom Teacher Education from the perspectives of supervisors, mentors and mentees.

According to findings, each stakeholder stated that mentors had a need for their roles and responsibilities to be clearly stated, which confirms Curran's and Goldrick's (2002) argument that mentoring should be based on clearly stated objectives. Moreover, Yılmaz and Bıkmaz (2020) emphasized that even though the mentor training program was built on a collaborative approach, in practice, mentoring diverges from being a collaborative effort. Even though the literature suggests investigating mentor teacher practices for area specific studies, the pandemic created an unexpected crisis on teacher education. In Turkey, schools were closed for a long period of time which required preservice teachers to finish their senior year practicum with emergency solutions. This study aims to investigate how mentors and supervisors of preservice secondary school mathematics teachers adapted teacher practicum while conducting emergency remote teaching in Turkey.

Methodology

In order to investigate the experiences of two stakeholders, mentors and supervisors, at the time of pandemic in 2020 for teacher practicum, phenomenology design is used. In the context of teacher education, in order to examine preservice teachers' learning processes it was necessary to examine the interaction between these stakeholders. In order to do so, data was collected from the two stakeholders by the end of teacher practicum semester, Spring 2020. There are only two secondary school mathematics teaching programs in Istanbul. Researchers contacted both of these programs and reached mentor teachers (n=4) of their practicum schools and supervisors (n=2) in order to conduct individual interviews. In the semi-structured individual interviews both groups of participants were first asked about pre-pandemic practices of teacher practicum. Then they were asked to talk about their experiences of practicum after all high schools and universities were closed and continued education as emergency remote teaching. Both stakeholders were also asked about their relationship with others during the adaptation of practicum at the time of pandemic. These interviews transcripted and coded for themes.

Findings and Discussion

Findings from two supervisors (one from two universities) and four mentor teachers showed that the pre-pandemic interactions among stakeholders were consistent with each other in terms of how they interacted. The interaction was evident, carefully planned and regulated by both universities and the Ministry of Education. Both parties mentioned these regulations and how the Ministry of Education requirements really regulates mentee preservice teachers' practices but not providing much direction for supervisor-mentor interactions.

One expected finding was the limited direct interaction between these two stakeholders. Supervisors and mentors interacted through the mentee preservice teachers. Their answers for elaboration of the interaction revealed the nature of interaction in two categories; i) cooperation or ii) collaboration. When mentors' practices were limited with Ministry of Education requirements (mentees to do six-hour weekly observations, four teaching practices) the interaction became at cooperation level. For some supervisor-mentor pairs, there was clear differentiation from others in terms of how they interacted beyond the regulations. These supervisor-mentor interactions were extended from mentee evaluation into deliberate communication and visits for topics other than specific mentees. For such cases, stakeholders collaborated with extended share of interest and resources.

As the Turkish government decided emergency curfew, all schools were closed by March 23rd, 2020. After one week of emergency spring break, schools started emergency remote teaching. In addition to the online synchronous teaching, the Ministry also provided TV broadcasting. Meanwhile universities started emergency remote teaching. But it brought a great challenge for teacher education because mentees could not complete their internship. Most of them were able to attend only one third of the internship. Due to curfew, the Ministry waived all the requirements, asked supervisor and mentors to evaluate mentees based on the work during face-to-face education, before curfew.

We found that, when the Ministry of Education waived all practicum requirements as an emergency reaction, the adaptation process was directed by supervisors' priorities for teacher education rather than mentors' practices. This may imply how stakeholders positioned themselves for teacher education during the pandemic. Findings also revealed three types of adaptation: i) discontinuity of practicum due to practicum school online teaching policy; ii) online micro-teaching at university as practicum; iii) online teaching at high school courses as practicum.

In this presentation, authors will discuss these adaptation types which provide insights to how stakeholders considered new normal in teacher education by relating them to pre-pandemic practicum interactions and also how it may transform teacher practicum for the following years.

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How teachers' knowledge and didactic contract evolve when transitioning to student-centered pedagogy – the case of project-based learning

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Despite project-based learning (PBL) benefits, its use when teaching mathematics is not as common as expected, partly because of the required changes in teachers' role and knowledge. In this case study, we map and characterize the changes undergone by one teacher while transitioning from teacher-centered instruction to PBL. This transition was accompanied by a teacher community where the teachers experienced authentic real-life problems from the tech industry as learners, then led project-based learning in their 9th-grade classrooms. The findings indicate a development in the content knowledge, the pedagogical content knowledge, and the general pedagogical knowledge of the teacher and demonstrate the formation and implementation of an up-to-date didactic contract between her and her students. Recommendations include integrating the topics of teacher's knowledge components and didactic contract in teacher training for PBL.

Keywords: Teachers' knowledge, teacher community, didactic contract, pedagogical content knowledge, project-based learning.

Introduction and background

Project-based learning (PBL) is a teaching method known throughout the world for several decades, and there is extensive knowledge and literature on its characteristics and advantages (e.g., Condliffe et al., 2017; Thomas, 2000). It is rooted in Dewey's (1938/1963) educational ideas stating that learning is a social activity and emphasizing the effect of real-life situations, students' choice of context, and ways of learning. The projects allow students to work relatively autonomously, over extended periods of time, with a tangible product created by the students at the end (Thomas, 2000). When students perform PBL in mathematics, they implement strategies such as reducing the complexity of the problem, searching a simple model, and moving from the specific case to generalization (Halverscheid, 2005). In PBL of mathematics, students engage in problem solving, decision making, and investigative activities (Palatnik & Koichu, 2017).

Despite these advantages, PBL is not a common practice in math classes as one might have expected, partly because the planning and the implementation in class are challenging (Thomas, 2000). Being student-oriented and constructivist, PBL challenges teaching practices and learning methods and thus requires a significant change in the work of teachers and students. PBL requires a modification in teachers' role from director to a facilitator of learning and requires them to tolerate ambiguity and higher levels of noise and movement in the classroom (Condliffe et al., 2017). These challenges suggest the need for new models of professional development and teachers' training geared to the specific characteristics of this teaching method to provide teachers with the tools and skills they need to successfully implement PBL in class (Barron et al., 1998).

The purpose of this study is to begin building a conceptual, cultural, and organizational framework that will enable PBL based on real-world industrial problems to be widely implemented in schools. The motivation is based on the aforementioned characteristics of PBL contributing to the development of students' independence, strengthening their sense of ownership of knowledge, and encouraging learning out of curiosity (Thomas, 2000), and on a perception that tackling real-world context problems leads to higher-order thinking processes, integration of critical thinking, and improvement in communication and reflection (OECD, 2018).

Given the importance of the teacher's role and the need to rethink and refine this role while implementing student-centered pedagogy (Condliffe et al., 2017), we chose to focus our research on teachers and the changes that occur when transitioning from teacher-centered instruction to PBL in mathematics. Based on the second author's experience in implementing PBL (Palatnik & Koichu, 2017; Palatnik, 2022) as well as on Brousseau, 1997; Condliffe, 2017 and Thomas, 2000, we hypothesize that this transition creates changes in the teachers' knowledge, as well as changes in the nature of the interaction between the teacher and her students. To map and characterize these changes, we conducted a multiple case study of teachers who, for the first time, used real-life context PBL in their classrooms. This report describes the case of one of these teachers, Sarah.

Theoretical Framework and Research Questions

As mentioned, our study aims to identify and characterize the development of the teacher's knowledge when transitioning to teaching mathematics using PBL and to map and describe the changes in the nature of the interaction and the set of mutual expectations between the teacher and her students as a result of this transition. Thus, we acknowledge Shulman's categorization of teachers' knowledge (1986) and recent developments in the field (e.g., the Mathematics Teacher's Specialized Knowledge (MTSK) model, Carrillo-Yanez et al., 2018). However, as our study focuses on the professional development of mathematics teachers within a learning community and the enactment of new knowledge in class, we sought a more dynamic theoretical framework that will emphasize the knowledge flow between various layers of teachers' knowledge. Hence our theoretical framework draws on the Refined Consensus Model of PCK (Carlson & Daehler, 2019) for science teachers' practice. Their model describes the *flow* of knowledge from the outer circle of *collective* PCK (cPCK) of peers, teacher educators, researchers, and mathematicians to the *personal* PCK of the teacher in question (pPCK), and then on to *enacted* PCK (ePCK), which is "the unique subset of knowledge that a teacher draws on … during the planning of, teaching of, and reflecting on a lesson" (Carlson & Daehler, 2019, p. 82)

When the teacher shifts to student-centered pedagogy, students become more responsible for learning. Hence, we incorporate into the theoretical framework of this study the notion of didactic contract—the set of reciprocal obligations and mutual expectations between the teacher and the students (the theory of didactic situations, Artigue et al., 2014; Brousseau, 1997).

In terms of the theoretical framework, our research questions are:

(1) Which components of teachers' knowledge develop when transitioning from teacher-centered pedagogy to PBL in a real-life context?

(2) What changes does the didactic contract between the teacher and her students undergo due to this transition?

The context of the study – Practimatics program

The study is a pilot study accompanying "Practimatics", a novel program that brings industry system engineers, mathematics education researchers, and mathematics teachers around one table. The program relies on three elements: new mathematical content – tech-related real-world problems, new pedagogy – PBL, and a dedicated teacher community. During the program, we exposed the teachers to several real-life industry-based mathematical problems that were processed by the research team. The mathematical content needed for the inquiry was consistent with the 9th-grade Israeli curriculum (Pythagorean theorem, linear and quadratic equations and functions, the ratio, etc.) In addition, the tasks allowed the students to combine algebraic and geometric content, which is not usual for regular curriculum tasks. The milk container problem (see below) was chosen based on these considerations. The teachers who participated in the program and their 9th-grade students had no prior experience in PBL. As mentioned, a key element in the program was establishing a teacher community led by the research team. The community incorporated means to facilitate PBL implementation, as mentioned in the literature. Namely, it provided initial training, allowing teachers to experience project-based learning as learners (Condliffe et al., 2017), supported and accompanied the teachers through networking and training (Kokotsaki et al., 2016), was a safe meeting place for them to draw emotional support and conduct peer dialogue on the various projects (Tsybulsky et al., 2019), and provided teachers with explicit models of possible student learning trajectories before they enter the classroom (Barron et al., 1998). The first author of this paper is one of the community leaders and is a high school math and physics teacher. His background as an electronics engineer who worked extensively with systems engineers for over a decade before becoming a teacher helped create a common language between the program participants – teachers, engineers and researchers.

The original problem in the described case study

The students' projects originated from "TetraPak", a revolutionary pyramid-shaped milk container presented in the 1950s by a group of engineers and entrepreneurs (Figure 1a). The students were asked to create a model of such a container using as much as possible from an A4 paper sheet and to present it along with calculations and justifications.

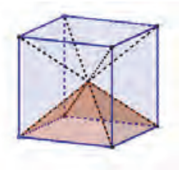
(a)  (b) 

Figure 1: TetraPak pyramid-shaped milk container and a diagram from one of the student projects

Methodology

Our method is an instrumental case study, exploring the issue over time with detailed, in-depth data collection involving multiple sources of information. In this paper we present the case of the teacher Sarah, whom we perceive as a typical case of an experienced teacher with no acquaintance and no prior experience with PBL or any other form of inquiry-based pedagogy. Sarah was very cooperative, verbal, and informative, and she was eager to share with us the nature of the interaction in her classroom, which made her a valuable informant. Data included transcriptions of seven 1:30 hr. community meeting recordings. We used inductive coding when analyzing Sara's account of her experience implementing the new teaching method. Both writers read the transcripts together, associating each teachers' expression with the categories based on our theoretical framework and our hypothesis: subject matter knowledge, pedagogical content knowledge, pedagogical knowledge, didactic contract. We created and examined a new category in cases where an expression did not fit any of these categories. We then cross-referenced data from additional sources: observation of Sarah's class, questionnaires, and documents collected in the field (Sarah's notebook, WhatsApp, and email correspondence), and compiled the account of Sarah's case.

Findings

Sarah teaches at a six-year religious high school for girls in central Israel. At biweekly community meetings, Sarah spoke in detail about PBL implementation in her class. Her students worked on a project that originated from the milk containers problem for about four and a half months. The mathematical content of the projects was mainly geometry (see example Figure 1b) and, in particular, the properties of the pyramids. Starting with reference to the didactic contract, we describe the findings in chronological order as possible to account for the gradual change. All descriptions and entries were made in Hebrew and translated by the authors of this article.

Formation of an up-to-date didactic contract between the teacher and her students

Already at Sarah's first meeting with her students, at which she planned to present only the structure of the program without delving into mathematics, she decided to introduce them to the problem of the milk container. At a learning community meeting, she explains why:

Sarah: Because…. well…, the audience demanded…it was as if they were curious and they said: "no, no, give us the problem now." So, of course, I said OK. Luckily, I prepared in advance.

Four months later, Sarah looks back at the beginning of the process and recalls how the students had to adapt to the new pedagogy and to the clauses of the updated contract with the teacher:

Sarah: …they never worked that way. These students did not even know what it means project-based learning…They are so used to learning in a specific way, to have all knowledge and information 'thrown' on them. It is really like a new language they had to learn.

During the following lesson, in which the students tried to build pyramids using rolls of paper, they asked why one should study this subject, which is not in the curriculum. Sarah replies:

Sarah: There is a reason for that, but wait, you will see soon. That's why I want you to try to look for the subjects yourself... In this program, you will learn on your own, and you will be able to teach me.

She builds on the students' questions to reveal another aspect of the new didactic contract as she tells them: "learning in this program will be different; responsibility is on you now." At the end of the lesson, when the students' attempts to create pyramids were unsuccessful, they asked her to reveal the answer. Sarah insisted they keep trying at home. According to Sarah, the students were surprised. Up to this point, in such situations, students were used to receiving solutions to the problem from her. Sara's actions and students' surprise indicate the start of redesigning the didactic contract—students' responsibility for learning and ability to deal with initial failure as part of mathematical inquiry.

Note, Sara's persistence in not revealing the answer is not arbitrary. She learns to be a learning facilitator to her students (PCK related to PBL). One of the elements of facilitation discussed lengthily in the community was that the teacher needs to avoid revealing the answer to motivate students to complete the inquiry. Thus, through community sessions and WhatsApp group, an element of the *collective* knowledge regarding the implementation of PBL in mathematics becomes part of Sarah's *personal* knowledge. When she acts in class applying this knowledge in the specific situation with specific students, this knowledge becomes *enacted* PCK (we elaborate on this later on).

A few weeks after the start of the project, a student told Sarah that she did not understand what she wanted from her in this project. Sarah was stunned because this was their fourth meeting already. As the project progressed, Sarah found herself constantly explaining to students what she expected of them. Indeed, the didactic contract should be re-clarified repeatedly for students and the teacher.

The final part of the project was an event held at the Hebrew University, in which the students presented their work and answered questions of prospective teacher program students. One of the questions concerned the way of learning. In the words of Ella, one of the students:

Ella: Learning this way was much more interesting because in regular math classes, you often study subjects you are less connected to, that interest you less. But when you choose the subject on your own, out of curiosity, you always have more motivation to learn, discover, and solve.

The student demonstrates the contrast between her motivation and interest during regular math classes and those during the project, highlighting another facet of a didactic contract for PBL: student choice of a problem.

Development of Sarah's pedagogical knowledge

Throughout the project, we witnessed the development of Sarah's pedagogical knowledge in aspects related to PBL, including ways to motivate and recruit students and ways to deal with procedural and social challenges in the working groups in her classroom. For example, at the fourth community meeting, she shared her opinion about the low pace compared to conventional pedagogy, and the teacher's need to be prepared widely for more than just the intended content:

Sarah: It takes a lot of time. It took me at least double the time I planned...I learned that one should be very flexible and be prepared not only for the specific lesson, right? You need to be ready for more.

At the seventh community meeting, she referred to the ways she has to prepare for a PBL session:

Sarah: So what you (members of the learning community) are saying here is that even if I give them this question in class, I should not try to solve it on my own beforehand.

These two excerpts illustrate bilateral knowledge exchange between *enacted*, *personal* and *collective* pedagogical knowledge of teachers. Sara's understanding of PBL planning and implementation includes now notions of teacher's flexibility regarding students’ projects and acting as a research partner of her students.

In yet another session, she shared her thoughts about encouraging low-motivated students who are stuck. Her pedagogical knowledge is challenged here:

Sarah: On the one hand, I can turn to them now and ask, but on the other hand, I wonder maybe it's supposed to come from them? Because if I'm the one asking, then ...well... I have to think, what's better.

However, two months later, she already has a solid opinion regarding ways to engage students:

Sarah: Explain to them the goal, show them the whole process so they can understand what's planned, so they have a sense of belonging. If you want students to feel a sense of belonging, you must share.

After enacting new pedagogy of PBL for a relatively long time, Sarah contributes her insights to collective pedagogical knowledge—sharing the goal and the process creates engagement.

Development of Sarah's content knowledge and pedagogical content knowledge

During the project, Sarah had to acquire new content knowledge. She learned some of it on her own, while in some cases, she relied on the community, as in the following example: Sarah tells the students tried to understand the pyramid volume formula by building a physical model. They managed to put six pyramids into a cube, as seen in Figure 1b, but not three pyramids. Sarah says:

Sarah: Maybe it came out by chance, but to really insert three pyramids into a cube, you cannot... I mean in volume, yes, but not in shape, not in dimensions.

In response, a discussion in the community begins as to whether it is possible to physically insert three pyramids into a cube. The program mentor demonstrated with the help of GeoGebra how it could be done, and Sarah now refers to her own lack of expertise in using GeoGebra:

Sarah: You see, say I would like to show them such a thing, (but) just looking it up... it would take me forever.

This example demonstrates how collective PCK becomes personal. Now Sarah knows it is possible to insert three pyramids into a cube, and she saw a new tool that allows visual illustration. The new knowledge she acquired here will most likely serve her in the future when teaching similar topics, thus becoming enacted PCK.

In a different case, Sarah seeks help from the community to bridge another content knowledge gap. She raises a question in the community's WhatsApp group regarding the characteristics of right pyramids and their heights. One of the community leaders replied without revealing the answer, in a way that encouraged Sarah to investigate and expand her knowledge on her own, while at the same time demonstrating how to communicate with students in PBL.

Discussion and conclusions

This study examines the case of an experienced teacher with no acquaintance and no prior experience with PBL and analyzes the development in her knowledge while transitioning to student-centered pedagogy. As mentioned in the literature, transitioning to student-centered pedagogy presents challenges and difficulties to teachers, as they require to adapt to new roles. This study brings empirical evidence of these challenges in the context of tech-related PBL and of the process of overcoming these challenges and developing relevant knowledge with the help of a teacher community. For instance, Sarah and the other teachers in the community realized that planning needs to be modular and flexible, much beyond a specific lesson, and must rely on broad mathematical knowledge, so they would be able to cope with surprises that arise from student inquiries. Teachers learned how to motivate and engage students in inquiry-based learning. They witnessed the power of learning through curiosity and experience and the notion that "breaking down the isolation of the classroom and designing performance opportunities in which students present to outside audiences can be a powerful way to support learning" (Barron et al., 1998, p. 285). Moreover, our findings demonstrate that PBL summons interpersonal situations and challenges that influence learning progress and requires teachers to develop and apply mediation skills. These insights and nuances regarding the implementation of PBL were new to Sarah and therefore represented an expansion of her knowledge.

In line with Brousseau's theory of didactic situations (1997), transitioning to student-centered pedagogy requires a new didactic contract between teachers and their students. Our study brings empirical evidence to the implementation of such a contract. Throughout the entire project, Sarah finds ways to convey her new set of expectations and re-clarify them when students lack understanding of these expectations. This movement back and forth in students' understanding of the expectations from them and the teacher's need to re-clarify the clauses of the contract is in line with Artigue's notion of an “appropriate didactical contract and the difficulties attached to the progressive negotiation of such a contract” (Artigue & Blomhøj, 2013, p. 804). From the teacher's knowledge perspective and in line with the RCM-PCK model (Carlson & Daehler, 2019), the whole notion of didactic contract, its formation, and the ways to convey and clarify it to the students represent a knowledge flow from the outer circle of community's collective knowledge to Sarah's personal and enacted knowledge.

Applying the RCM-PCK model in science to mathematics education, we demonstrated the flow of knowledge between the community leaders, peers and the teacher's personal knowledge and the development in her subject matter knowledge, pedagogical knowledge, and PCK. The case study exemplified that the notion of didactic contract and the need to redesign it when implementing PBL is a development of pPCK, thus suggesting a theoretical connection between this element of the TDS with a model of teachers' knowledge. Further research is needed to characterize these connections and how the formation of multiple didactic contracts by teachers in the community will enhance cPCK. Sarah's awareness of these developments and processes and her willingness to share and expand her knowledge helped her to implement PBL in her classroom successfully. We therefore recommend incorporating the subjects of didactic contract and the RCM-PCK model of teacher professional knowledge in mathematics teacher PD programs.

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Preservice teachers' knowledge for teaching uncertainty: cases from Slovakia and Spain

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In this paper we inquire about the knowledge of the meanings of probability of preservice primary and secondary school teachers from Slovakia and Spain. At the end of their training on this content, we wonder about their conceptions of randomness and how they quantify its uncertainty. From the results obtained through a questionnaire filled in by 89 preservice mathematics teachers of Primary and Secondary in both countries and the interview to some of these informants, we highlight that they associate randomness mainly to future events and drawing experiments; they show knowledge of the classical meaning of uncertainty and little acquaintance with the subjective meaning. Besides, they seem to separate uncertainty from probability assignment.

Keywords: Probability, uncertainty, preservice teachers, Primary and Secondary School.

Introduction

Compared to other topics in mathematics, probability is especially difficult to understand. Unlike logical reasoning, which deals with statements which are either true or false, probabilistic reasoning deals with events for which there is no complete certitude. The intuitions underlying concepts of probability, such as dependence or fluctuations, are usually obscured by mathematical counting methods. These reasons, among others, make this a particularly difficult content item for both students and teachers, as Batanero *et al*. (2004) and the references therein, point out. These authors note that one aspect of teacher training concerning probability on which special emphasis should be placed is epistemological reflection on the concepts to be taught. We concur, and stress that understanding the inherent nature of probability should be one of the main goals in teaching the topic.

Probabilistic contexts can be understood from different perspectives. One major approach is the classical approach, which imposes the need for objective mathematical rules to explain random processes; in addition, the epistemic approach sees probability as the degree of personal belief about the ocurrence of an event, which is dependent on the information available, and constrained by theoretical decision rules. These two major trends can be included among what many authors have called the different perspectives on, or meanings of, probability, which also include the classical, frequentist and subjective approaches, among others (Batanero et al., 2016).

In our approach, following the Mathematics Teachers' Specialised Knowledge model (MTSK, Carrillo et al., 2018), teachers' knowledge of the different meanings of probability is located in their Knowledge of Topics (KoT). Numerous researchers have highlighted that, in order to offer students a richer learning experience, it is important for teachers to develop a broad understanding of the range of meanings of mathematical objects (e.g. Martín-Fernández *et al*., 2019; Thompson, 2016). In the case of the probability theory, realizing that uncertain situations can be described in terms of probabilistic models is especially significant. Ultimately, from the teachers point of view, we think that probability should be identify as the main mathematical tool for dealing rationally with uncertainty in daily life contexts.

This work is a descriptive exploratory study which focuses on the knowledge of preservice primary and secondary school teachers from Slovakia and Spain at the end of their training on probability. The research questions are: What are their conceptions of randomness, and how do they quantify uncertainty? In other words, our goal is to explore to what extent preservice teachers consider probability as a measure of uncertainty or of partial information.

Theoretical framework

We are especially interested in the subjective interpretation of randomness and its use in decision-making, as a way of teaching inferential reasoning. This approach is based on the probability theory initially presented by Ramsey (1926) and De Finetti (1937) and developed by authors such as Jeffrey (1965) and Savage (1967). Under this theory, randomness is associated to any observation or experimentation for which the observer has no total information or certainty. That is, randomness is a property not related to a situation, but related to the judgement of the subject that observes. That judgement translates in conditions governing a set of personal preferences so as to obtain coherent behavior in uncertain situations. Informally, such preferences and coherence refer to actions and consequences depending on the uncertain events considered. Technically, those concepts are defined inside the axiomatic for decision theory and based on concepts such as betting and Dutch book.

From the subjectivist point of view, probability is seen as a model for partial information, or uncertainty, of the decision-maker. This means that, e.g., two physicians could assign different coherent sets of probabilities (in fact, for possibly different diagnoses) and consider treatments and consequences in different ways according to their personal judgement.

As any other mathematical problem, random contexts can be modeled by different approaches inside the probability theory, the main two being the classical one, based on equiprobability and counting methods, and the decision theoretical one, based on personal information and coherence axiomatic. Their implications in statistical education are discussed in (Batanero et al., 2016).

These approaches are related, though. The axioms for coherence have been shown to be equivalent to Kolmogorov’s, which means that the usual rules of the calculus of probability are coherent, and, inversely, a coherent assignment of probability satisfies the usual properties, for instance, that the probability of the union of disjoint events is the sum of their probabilities. Additionally, the classical interpretation of probability, which assigns equal probability to the elementary events, is a particular case of subjective assignment, when we consider conditions such as the symmetry of the results of an experiment or lack of information. Similarly, the acceptance of conditions that guarantee the limit frequency as an adequate assignment of probability for a given event is a subjective choice, relative to the subject that is mathematically modeling the problem. Finally, the subjectivist interpretation implies that our own assignment of probabilities can change when updating our knowledge. Both prior and posterior assignments must follow coherent consequential rules, equivalently to the relation between conditional probabilities and the Bayes rule.

Several authors mention that preservice teachers are unfamiliar with different meanings of randomness and probability and that they need to be aware of these approaches, because they influence their students' reasoning (e.g., Chernoff & Zazkis, 2011; Batanero, 2016). That is why, in this paper, we focus on the meanings that preservice teachers attribute to randomness and its measure as mentioned above. More specifically, we explore the preservice teachers' conceptions of uncertainty in relation to certain daily situations, and how they quantify it. We locate this knowledge as distributed among knowledge of notions of uncertainty (KoT – definitions, properties, and their foundations), situations that this knowledge models (KoT – phenomenology) and calculation procedures for quantifying this knowledge (KoT – procedures). This is consistent with Di Bernardo *et al*. (2019) and the references therein, where subjective knowledge about probability and knowledge of connections between different meanings was divided into: knowledge of definitions, properties and their bases (intra-conceptual connections between different meanings); knowledge of procedures (calculation of probabilities); knowledge of representations; and knowledge of phenomenology (modeling a situation from the appropriate meaning).

Methodology

Our informants were 89 preservice mathematics teachers (PSTs), in Spain and in Slovakia. The Spanish group consisted of 43 preservice teachers at the University of Huelva, who had completed a course on statistics and probability as part of their degree. Both the classic approach to probability and a preliminary introduction to the subjective perspective had been covered during the course. The Slovak group of informants consisted of 46 preservice secondary school teachers (grades 5-13) from Pavol Jozef Šafárik University in Košice. All the Slovak PSTs had passed the compulsory classes in statistics and probability. During those courses, they were exposed to the classical definition of probability, and worked with the Kolmogorov axioms. The subjectivist view was not directly dealt with on that course. The training of both groups of students in probability has not been equivalent, neither in depth (greater in the case of the preservice Slovak secondary teachers) nor in that the preservice Spanish primary teachers have received some notions of subjective probability (idea of uncertainty and its relation to decision-making). Both groups have in common that they have received their last course on probability. Our goal is to explore whether differences in training and context are reflected in differences in their understanding of probability as uncertainty.

The information was collected through a questionnaire with four sets of questions. We analyze in this paper the first two (Table 1). The first question focused on the concept of randomness in a variety of situations including the result of a lottery taking place both in the past and the future, a social or economic index in the past, the location of a historical event, and a weather forecast. For each situation, we then asked for the PSTs' personal estimation of the probability of specific events.

In the case of Spain, the questionnaires were completed in the training classroom, while in Slovakia the questionnaires were completed online due to COVID-19 restrictions. One group of Slovak PSTs (13 participants) was explicitly asked to explain precisely the thinking behind their responses at the end of each question. A few days after they finished the questionnaires, they had a group discussion in which some of the PSTs explained the reasoning for the answers given in the questionnaire.

Once the data for each country had been collected, they were analyzed by the corresponding country team. Later, they were jointly analyzed by all the authors of this work. The main aspects of this analysis are: the students' conception of randomness, the assignment of probabilities and the coherence of this assignment. In Table 2 we point out how we analyzed each of these aspects, and explain them with more detail in the Section Results.

Table 1: Questions for information gathering

Task 1	Task 2
Indicate if you think that the following situations are random (A) or non-random (N), according to your current information. A: ( ) Next week's result of the 1st prize in the National Lottery. B: ( ) Last week's result of the 1st prize in the National Lottery. C: ( ) The birthplace of Alexander the Great. D: ( ) The temperature in Bratislava/Málaga tomorrow at noon. E: ( ) The Euribor index at close of business yesterday. F: ( ) The proportion of primary school pupils diagnosed with attention deficit disorder in 2019 in Slovakia/Spain.	Indicate your estimation of the probability that each of the following statements is true. Assign the numerical values that you consider appropriate according to your current information. (a) Next week the 1st prize in the National Lottery will be the number 89342. (b) Last week the 1st prize in the National Lottery was the number 89342. (c) Alexander the Great was born in Greece. (d) The temperature in Bratislava/Málaga tomorrow at noon will be between 18°C and 20°C. (e) The Euribor index closed down yesterday. (f) The proportion of primary school pupils diagnosed with attention deficit disorder in 2019, in Slovakia/Spain, is greater than 5%.

Table 2: Process of analysis

Aspect	Research question
Conception of randomness	Is randomness associated with uncertainty or partial information, as opposed to situations where the exact probability can be calculated? (KoT, notion of randomness –*definitions, properties and its foundation*) Task 1: Comparison of events A,B vs C,D,E,F
	Is randomness assigned to past / future events? (KoT, situations that randomness models – *phenomenology*) Task 1: Comparison of events B,C,E,F vs. A,D
Assignment of probabilities	To which events is a subjective probability assigned, and to which a value obtained by Laplace's rule? (KoT, calculation procedures –*procedures*) Task 2: Analysis of the students notes
Coherence (in decision making)	Is probability assigned only to the events associated with random contexts? Task 1 and Task 2: Comparison of corresponding events – focusing on non-random contexts with probability within the interval (0,1)

Results

Conception of randomness

Within this aspect, our goal was to identify the contexts that are believed to be random or nonrandom in the students conception. Two kinds of contexts were stated in the assertions of the Task 1: drawing experiments (A,B) vs. daily life events (C,D,E,F), and events occurring in the past (B,C,E,F) vs events in the future (A,D). Both groups of students agree on what they consider random, as can be seen in Figure 1, where each bar represents the percentage of respondents who regard the situation as random.

The situation described in item A is a random draw, the prototypical context for exemplifying probability in introductory courses at school – and was consequently universally considered random, as expected. The situation described in B, identical to that of item A but formulated in the past, shows a decrease in the assignment of randomness to about 65%. Events C, D, E and F were identified as nonrandom by most PSTs. The contexts of these situations do not offer the possibility of using combinatorics or frequencies, except perhaps for the context based on the weather, which has a slightly higher proportion. Moreover, neither the Spanish nor the Slovak PSTs had received the appropriate training for solving such tasks, nor had Slovak PSTs been introduced to subjective probability. In the Table 3, we associate the results of Task 1 for the past contexts (items B, C, E and F).

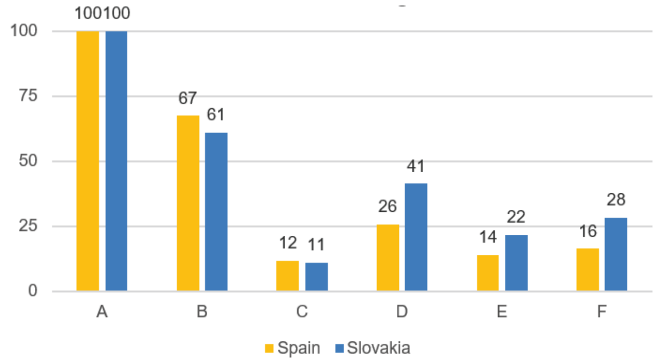

Figure 1: Proportion of answers (in percentage) which consider situations A-F as random

From the Table 3, we can see that 10 out of 43 Spanish PSTs (shaded orange) and 14 out of 46 Slovak PSTs (shaded blue) considered all past contexts as nonrandom. However, if situation B is excluded (a draw – usually connected with randomness) we get 28 out of 43 (18 + 10 shaded orange) Spanish PSTs and 23 out of 46 (14 + 9 shaded blue) Slovak PSTs. These numbers make up at least 50% of PSTs in each of the countries. Five PSTs from the group of 13 PSTs in the Slovak sample that commented on their answers argued that the situation is nonrandom because it happened in the past. The other argument was that a past situation is not random at all, since it is either true or false.

Table 3: Consideration of randomness for the past contexts.

		E – R				E – nR				Total
		F – R		F – nR		F – R		F – nR		
		ESP	SVK	ESP	SVK	ESP	SVK	ESP	SVK	
B – R	C – R	0	0	0	1	3	2	1	1	8
	C – nR	0	0	5	6	2	9	18	9	49
B – nR	C – R	0	0	0	0	0	1	1	0	2
	C – nR	0	1	1	2	2	0	10	14	30
Total		0	1	6	9	7	12	30	24	89

Assignment of probabilities

Two main methods of assignment of probabilities used by the PSTs emerge: the classical calculation from equiprobability and personal reasoning based on their previous knowledge.

For events A and B, a significant number of the PSTs in both countries assigned a probability based on the rules of combinatorics and classical calculus of probability: 33 out of 46 Slovak PSTs for event A, and 16, for B; 20 out of 43 Spanish PSTs for event A, and 12 out of 29 who assigned a probability less than 1, for event B. For the 13 PSTs in the Slovak sample who commented on their answers, a few of them tried to use Laplace's rule for the events C and D (4 PSTs in C, and 2 in D), but the corresponding values were chosen according to subjective considerations. For instance, the *"number of countries at that time"* was used for counting probability in event C, or *"according to the weather forecast, I expect that tomorrow's temperature will be between 15 and 20, so the probability will be 3/5"*, for event D. No Slovak PST used Laplace's rule in connection with events E and F. However, in situation F, two PSTs explained their answers in a way which combine estimation with personal experience, e. g.: *"5% means that every 20th child has been diagnosed with attention deficit in 2019, which is too many in my opinion. In my class we had nobody with such a condition", "If we have 800 thousand pupils 5% means 40 thousand pupils at approximately 2000 schools, that is 20 such pupils at one school – it is too many"*. During the discussion with the 13 Slovak PSTs after the questionnaire filling, 2 PSTs stated that the Task 2 was too challenging: *"Task 2 makes no sense to me. I don't know what to do"; "It was a big problem for me"*. In some of PSTs' explanations, we could explicitly identify assignment of subjective probability: *"First, I tried to find some relevant number in a question about Alexander the Great. Then, I decided that he was either born in Greece or not, so I assigned a probability of ½. I used the same logic in the subsequent questions"*. Spanish PSTs made similar remarks for events they do not have any information about.

Coherence

Finally, the third aspect is related to the coherence of the answers. For each case, the following four possible options can occur: the situation can be considered random or not, and the probability assigned to the specific event can be 0 or 1, indicating certainty, or a number within the interval (0,1), indicating uncertainty. Table 4 shows the coherence state for each pair.

Table 4: Probability assignment and random situations.

	Random	Nonrandom
Probabilities within the interval (0,1)	coherent	incoherent
Probability equals 0 or 1	may be coherent	coherent

From the perspective of our research question, it is most interesting to look at incoherence and therefore to display how many PSTs assigned probability within the interval (0,1) to events in the situations described formerly as nonrandom. We can see that this incoherence is quite common among our informants (see Figure 2).

Conclusions

From the results of our study, we conclude that: 1) regarding the situations associated to randomness (KoT -*phenomenology*), past events seem to be considered mostly as non-random, with the exception of when they are linked to a kind of draw, in which case, especially among the Spanish PSTs, it is mostly considered as random; 2) regarding the assignment of probability ((KoT- *procedures*), the PSTs have little experience in using a subjective or any other approach, even though it can be very useful in daily life. It seems that PSTs do not consider that kind of reasoning to be within the scope of mathematical modeling; and 3) regarding coherence, the results may reflect a narrow use of the concept of randomness (KoT- *definitions, properties and its foundation*), which does not include situations of uncertainty. Although the probabilistic language can be used to make statements about uncertainty, randomness as a mathematical model is not used in daily contexts.

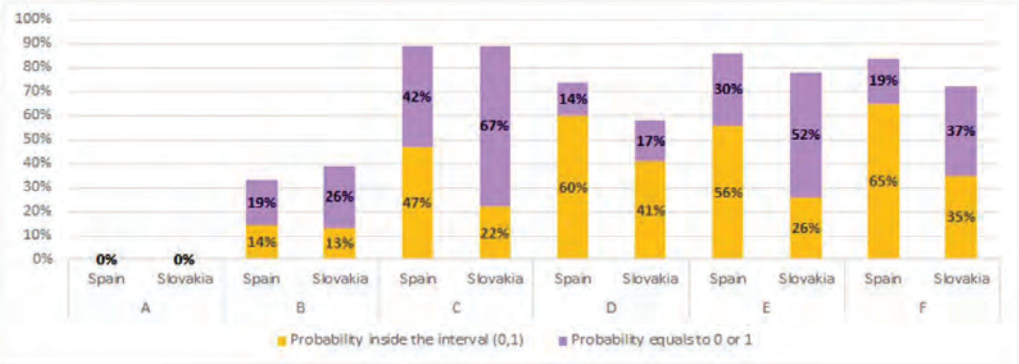

Figure 2: Probability assignation to events in contexts considered non-random

Therefore, PSTs' conception of randomness is not as deep and complex as might be desired. It enables them to use probability as a tool to inquire about contexts where Laplace can be utilized. This may be influenced by the fact that "Laplacian definition is echoed in today's textbooks" (Chernoff & Zazkis, 2011, p. 16). However, out of this box, most of the students do not develop a coherent probabilistic reasoning. Their knowledge of situations that can be modelled by uncertainty is restricted. They seem to be familiar with the classical meaning and do not consider the absence of information as uncertainty, although they use subjective arguments to justify their probability assignments. Contrastingly, they seem to separate uncertainty from probability assignment.

These results indicate that an axiomatic formation in probability, exclusively linked to the classical meaning (Slovak PSTs) leads to a limited knowledge of the topic, both in terms of the notion itself, as well as procedures and situations related to it (KoT – definitions; procedures; and phenomenology). On the other hand, a brief introduction to subjective probability (Spanish PSTs) does not seem to affect such knowledge. It seems that the knowledge of the PSTs is barely expanded in relation to that of secondary school students and does not acquire a specialized profile.

Multiple authors suggest activities for students which comprise subjective probability (e.g. Borovcnik & Kapadia, 2017; Martignon & Krauss, 2009) . The reason consist on large presentation of subjective probability in daily-life. On the other hand, to this time, subjective approach is rarely represented in national curricula. Moreover, our results show that PSTs who received their last course on probability have not developed KoT concerning probability properly, and they are not well-prepared to teach subjective approach to probability. If we want to include subjective probability in the curriculum, then it is necessary to develop their KoT in this area. One possibility is to include activities and tasks as it is suggested e.g. in Di Bernardo *et al.* (2019).

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Preservice teachers' knowledge on area measurement

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This study aims to characterise elements of specialised knowledge of a group of pre-service teachers using procedures related to the comparison and rearrangement of surfaces when solving area tasks. For this purpose, emphasis is placed on the subdomains of knowledge of topics and knowledge of the structure of mathematics. The written justifications and procedures that pre-service teachers used to solve two area tasks are analysed. The results indicate that when tasks condition the strict use of procedures related to the comparison and rearrangement of surfaces, written justifications supporting the procedures become more rigorous. These rigorous justifications appear, exclusively, when pre-service teachers mobilise specific geometric properties and principles, either implicitly or explicitly.

Keywords: *Area measurement, knowledge of the topics, knowledge of the structure of mathematics.*

Introduction

Area measurement has been identified as a problematic topic to both students and pre-service teachers (PST). The literature shows that students' difficulties related to the limited variety of procedures for solving area tasks are also evident in PST (Baturo & Nason, 1996; Caviedes, De Gamboa & Badillo, 2019; Chamberlin & Candelaria, 2018; Hong & Runnalls, 2020; Murphy, 2012). Hong and Runnalls (2020) show that PSTs have difficulties for accepting the conservation of area in non-prototypical figures, as they do not have numerical values to compare the areas of the triangles with the same base and height but different kinds of shapes, so they prioritise visual estimation without reasoned justification. The same authors emphasised that understanding the ideas behind area conservation would enable a better understanding of formulas for PSTs, helping them develop procedural fluency; based on the acquisition of the initial concepts. Similarly, Caviedes, De Gamboa, and Badillo (2019) point out that PSTs have a limited repertoire of strategies to solve area tasks, prioritising formulas rather than geometric procedures (procedures related to the comparison and rearrangement of surfaces) that may simplify a solving process.

This study assumes the specialised nature of mathematics teachers' knowledge in the sense of Carrillo et al. (2018) since it allows a first approach to characterise aspects of knowledge about specific mathematical concepts. In this context, we pose the following research question: *What specialised knowledge about area measurement do PSTs mobilise when solving tasks that require using geometric procedures?* Thus, we attempt to characterise elements of knowledge of topics (KoT) and knowledge of the structure of mathematics (KSM) in a group of PSTs when solving two tasks that require the use of geometrics procedures.

Theoretical framework

Amongst different possible ways of perceiving teachers` knowledge we consider the Mathematics Teachers` Specialised Knowledge -MTSK- (Carrillo et al., 2018). The MTSK model studies, mainly, the knowledge at stake in teachers' practice, but recent studies consider that it is possible to assume the MTSK model as a reference of the desirable components of a teacher's specialised knowledge, and as a consequence, as a first approximation of what PSTs should know for their future practice (Policastro, Ribeiro & Fiorentini, 2019). The MTSK conceptualisation is conceived as a theoretical and analytical tool to better understand teachers' knowledge specificities and it has been shown that MTSK model could be useful in conceptualising tasks for accessing and developing knowledge of procedures, representations and connections in certain concepts (Policastro, Mellone, Ribeiro & Fiorentini, 2019). Specifically, we are interested in two subdomains of knowledge - Knowledge of Topics (KoT) and Knowledge of the Structure of Mathematics (KSM). KoT includes teachers' knowledge of definitions (i.e., what is area mathematically speaking?); properties and their principles (i.e., the role of each element involved in solving an area task); the phenomenology or contexts of use (i.e., comparing and reproducing shapes; measuring, or sharing fairly); procedures (i.e., knowing how, when and why using certain procedures), and representations (i.e., the geometric, numerical and algebraic representations involved in solving area tasks).

The KSM refers to teachers' knowledge of connections, considering four categories: simplification connections and complexities connections, auxiliary connections, and transversal connections. For instance, knowing that the area of an unknown figure can be calculated by decomposing the area into known figures, such as triangles, rectangles, and-or squares is a simplification connection because figure decomposition can be a precursor to formulae; knowing that the area of a scalene right triangle can be calculated using Heron's formula is a complexity connection, because it involves more advanced mathematical knowledge, also considering the perimeter; knowing that the procedure of iterating units of measurement, lined up in rows and columns, can evoke the measurement procedure involving multiplying number of rows by the number of columns, is an auxiliary connection because it uses one procedure to introduce a different one; and knowing that an area model can be used as a basis for working on fractions and algebraic operations is a transversal connection since the concept of the area can relate different mathematical contents. Due to the scope of our work, we focus only on knowledge of auxiliary connections.

Method

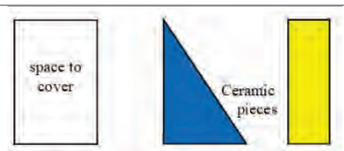
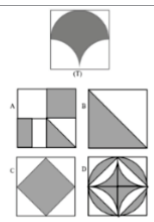
The study is situated in an interpretative paradigm with a qualitative approach (Bassey, 1990) and is part of broader research that seeks to characterise the specialised knowledge about area measurement in a group of PSTs. Data collection was carried out in the first term of the 2020-2021 school year. The participants were non-randomly selected, and they were 70 PSTs studying the third year of the Primary Education Degree at the Autonomous University of Barcelona. The PSTs had previous instruction on different procedures for measuring areas as part of their study program. Content analysis is carried out (Krippendorff, 2004) using two of the subdomains of the MTSK model: The KoT and the KSM. The MTSK defines, for each of these subdomains, specific categories. For the KoT, we consider representations, procedures and justifications, properties and principles, and intra-conceptual connections. For the KSM, the auxiliary connections. In each category there are indicators which have been constructed by the authors based on a previous study (Caviedes, De Gamboa & Badillo, 2020). The MAXQDA software is used to facilitate the process of assigning indicators to the PSTs responses.

Instrument and procedure

A semi-structured open-ended questionnaire (Bailey, 2007) was designed to be completed individually. The PSTs were asked to justify each procedure in writing. To solve the tasks the PSTs could use manipulative material (cut-outs as an annex to the questionnaire), as well as measuring instruments (ruler, square, protractor). The questionnaire consisted of 8 tasks and was structured as follows: three tasks responding to contexts of equal sharing, and comparison and reproduction of shapes (Tasks 1, 2, and 3); two measurement tasks (Tasks 4 and 5); one task to classify statements and one task to define the concept of area (Tasks 6 and 7); finally, one task to analyse students' responses (Task 8). In Tasks 1, 2, and 3 the use of calculations and measuring instruments was prohibited. The questionnaire was administered by the subject teacher in online format due to the COVID-19 health contingency. The official language to administrate and to answer the questionnaire was spanish and the translation was executed by a professional translator. The validation of the instrument considered external research experts, in-service and pre-service primary school teachers.

The PSTs had one week to answer it and send it in pdf or word format. In order to answer our research question, the analysis of the resolutions to Tasks 2 and 3 is presented (Table 1).

Table 1: Tasks proposed to the PST group

Task statement	Graphical representation of the tasks
TASK 2: Camila is making a ceramic mosaic with different colours. To finish her mosaic, she needs to cover a rectangular space, but she doesn't have any ceramic pieces with that shape. **What can Camila do to cover the missing space in her mosaic? Justify your answer.**	 (Adapted from Puig and Guillén, 1983)
TASK 3: Using the shaded surface of Figure (T) as a reference: **Which of the following figures has a shaded surface equivalent to Figure (T)? Why? Justify your answer.**	 (Elaborated by the authors)

Methods of analysis

Since we have not found any studies detailing the KoT and KSM indicators of area measurement, these have been constructed based on the results of a previous study that allowed for the construction of an epistemic configuration of area concept (Caviedes, De Gamboa & Badillo, 2021). From this epistemic configuration we define the KoT indicators to focus on the analysis of the PSTs' responses to the tasks. Each indicator was adapted to the categories that the MTSK model proposes for KoT (representations, procedures and justifications, properties and principles, and intra-conceptual connections) and allowed a deductive coding of the PST responses, with the support of MAXQDA software. The indicator corresponding to the KSM (auxiliary connections) emerges from the analysis of PSTs' responses to the questionnaire (Table 2).

Table 2: Categories of specialised knowledge

Categories of KoT and KSM	Indicators
Representations (R)	(R1) W*ritten:* use of adjectives such as "equal", "thinner" "wider", "double", "half" "a quarter" related to surfaces. (R2) *Manipulative:* use of physical objects or dynamic geometry software. (R3) *Geometric:* use of convenient decompositions to compare and-or estimate surfaces quantities.. (R4) *Symbolic:* use of the R^+ set to compare two or more surfaces, for counting units or adding up areas.
Procedures (P) and justifications (J)	(P1) Compare two or more surfaces directly by total and-or partial overlapping. (P2) Compare two or more surfaces indirectly by cutting and pasting. (P3) Decompose in a convenient way, graphically or mentally, two or more surfaces. (P4) Carry out movements of rotation, translation and superimposition of figures. (P5) Measure areas as an additive process by counting units and-or sub-units that cover the surface. (J1) The act of comparing two or more surfaces by placing one shape over another is useful for establishing equivalence and/or to include relationships

	(J2) The mental act of cutting the two-dimensional space into parts of equal area as a basis for comparing areas. (J3) The act of changing the shape of a surface does not change the area of the surface, as the figures can be decomposed and reorganised while keeping the same "parts".
Properties (Pp) and principles (Pr)	(Pp1) Use of conservation (Pp2) Use of accumulation and additivity (Pp3) Use of transitivity (Pr1) Use of the fact that a parallelogram that has the same base as a triangle, both placed between the same parallels, has twice the area of the triangle. (Pr2) Use of the fact that the triangles placed on equal bases and between same parallel are equal. (Pr3) Use of the fact that two polygons are congruent if their sides and angles are respectively equal or congruent. (Pr4) Use of the fact that every polygon can be broken down into triangles. (Pr5) Use of the fact that every triangle is equidecomposable from a parallelogram.
Auxiliary connections (Cau)	(Cau1) Use of a procedure or concept to introduce a new procedure or concept.

Results

Figures 1, 2 and 3 show examples of resolutions of PST 2 and PST 39, who mobilise specialised knowledge from the KoT and KSM subdomains. These resolutions are considered representative of PSTs' set that evidence mobilization of these subdomains, as these are the type of resolutions that allows for the emergence of different indicators from PSTs' resolutions. Figure 1 shows indicators of KoT in Task 2.

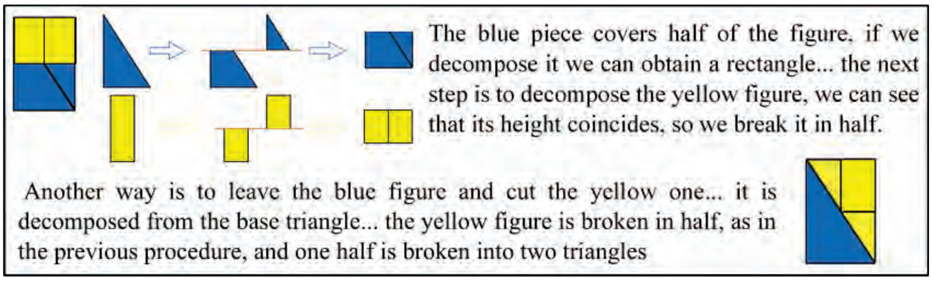


Figure 1: PST 2 resolution for Task 2

PST 2 uses written representations (R1) in her justifications, evidencing use of (J1) and (J2), as she superimposes the surfaces, as well as breaking them, in order to compare them. When breaking surfaces and rearranging their parts, she makes use of (J3). PST 2 uses manipulative representations (R2) to make visible decompositions and reorganizations of figures; and geometric representations (R3) as it compares surfaces indirectly (P2) and decomposes the surface in a convenient way (P3). In addition, she performs rotation and translation movements of the parts (P4), in order to check that both the triangle and the rectangle correspond to half of the piece that needs to be covered. PST 2 also shows an implicit use of the transitivity property (Pp3), while she compares between the surface to be covered and those represented by the pieces. Furthermore, PST2 shows an implicit use of the properties of accumulation and additivity (Pp2) and conservation (Pp1), as PST2 recognizes that figures can be decomposed and recomposed into other figures, while retaining the same "parts". Finally, it is possible to infer that PST 2 recognizes that a triangle is equidecomposable to a parallelogram (Pr5); that is, that a triangle can be decomposed into a finite number of polygons and form a parallelogram (and vice versa), conserving the area. Similarly, she implicitly recognises that a parallelogram with the same base and height as a triangle, both placed between the same parallels, is twice as large as the triangle (Pr1).

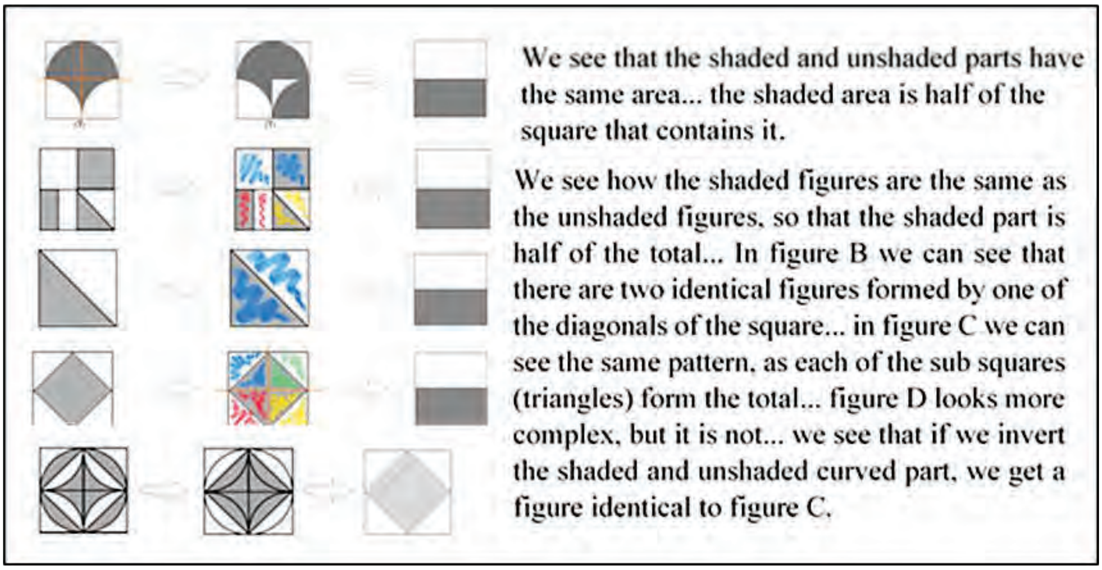


Figure 2: PST 2 resolution for Task 3

Figure 2 shows indicators of KoT knowledge in Task 3. It can be seen that PST 2 uses written (R1), manipulative (R2) and geometric (R3) representations, as she decomposes each of the figures in a convenient way (P3) and performs rotation and translation movements (P4) to verify and illustrate, manipulatively, that each of the figures is equivalent to the model figure. Thus, PST 2 manifests an implicit use of the properties of accumulation and additivity (Pp2) and conservation (Pp1), as she rearranges the shaded surfaces into rectangles that represent half of the square containing them. The property of transitivity (Pp3) is made explicit when PST 2 says that "if Figure D is equal to Figure C, and Figure C is equal to the model, then Figure D is equivalent to the model." PST 2 recognizes that triangles placed on equal bases and between the same parallels are equal (Pr2) and that every polygon can be decomposed into triangles (Pr4).

It is inferred that the PST 2 also makes use of (J2) which indicates that the mental act of cutting two-dimensional space into parts of equal area serves as a basis for comparing areas; and of (J3), since the PST 2 recognizes that changing the shape of a surface does not change the area of the surface.

Figure 3 shows indicators of knowledge of the KoT and KSM of PST 39 in Task 3. The PST 39 uses symbolic representations (R4) in a fractional register; that is, she uses fractions to compare the shaded area of each of the figures, and sets the shaded fraction relative to the total area. PST 39 also shows knowledge of addition of fractions with different denominators and simplification of fractions, showing an auxiliary connection (Cau1) to this concept. By rearranging the parts, PST 39 implements the procedure of surface decomposition (P3) and rotation and translation movements (P4). In this sense, it is inferred that PST 39 recognizes the properties of conservation (Pp1) and accumulation and additivity (Pp2). In turn, the comparisons made between the figures are associated with the property of transitivity using (Pp3). The procedures allow us to infer an implicit use of (J2), since PST 39 recognizes the usefulness of cutting the two-dimensional space.

T= $\frac{1}{2}$ of the figure. We can rearrange the bottom parts and fit them into the upper corners.

A= $\frac{1}{4}+\frac{1}{8}+\frac{1}{8}=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$

B= $\frac{1}{2}$ of the figure

C= $\frac{1}{2}$ of the figure

D= $\frac{1}{2}$ of the figure. D = C

Figure 3: PST 39 resolution for Task 3

Due to the geometric nature of the tasks, the results showed that PTSs solve Tasks 2 and 3 mostly using geometric type representations (R3) that are materialised through convenient surface decomposition procedures (P3), or the use of rotation, translation, and superimposition movements of figures (P4). These procedures, in turn, are supported by some of the properties as the mental act of cutting the two-dimensional space into parts of equal area serves as a basis to compare areas (J2); or that changing the shape of a surface does not produce changes in its area, since the figures can be decomposed and reorganized while retaining the same "parts" (J3).

The resolutions presented in the analysis show the way in which the PSTs mobilise knowledge about the different categories of the KoT and KSM, but do not allow us to identify the tendency of PSTs at the time of solving the task. For this reason, we consider it appropriate to show the frequency of each of the categories mobilised by the PSTs. Table 3 details this frequency and it can be seen that only 11 PSTs establish auxiliary connections (KSM), which are linked to KoT and associated with knowledge about fractions. Thus, knowledge about fractions is made explicit through the use of convenient surface decomposition procedures (P3), or the use of rotation, translation and superposition of figures (P4). Moreover, this type of knowledge is associated with a context that requires establishing equivalence or inclusion relations between different surfaces, linked to the properties of conservation (Pp1), transitivity (Pp3) and accumulation and additivity (Pp2). In this way, the relationship between KoT and the establishment of auxiliary connections becomes clear.

Table 3: Categories of specialised knowledge mobilized by PST (N=70)

Code	Frequency	Code	Frequency
P3	57	R4	13
Pp3	57	P2	11
Pp2	57	Cau 1	11
Pp1	57	R2	6
P4	56	Pr1	3
R3	56	J1	3
J3	56	No response	3
Pr5	56	Pr4	3
J2	53	Pr3	2
Pr2	51	P1	2

Discussion and final remarks

This study focused on the characterisation of PST's knowledge related to the topics and the structures of mathematics, according to the MTSK conceptualization. Results show a relation between descriptive answers that do not justify what is done and why it is done with the lack of use of geometric principles. As it can be seen in the examples presented in the above section (Figures 1, 2 and 3), PST 2 and PST 39 were able to justify and support the procedures they were using based in some of the geometric principles, such as: two polygons are congruent if their sides and angles are respectively equal or congruent (Pr3); every polygon can be broken down into triangles (Pr4); and a parallelogram that has the same base as a triangle, both placed between the same parallels, has twice the area of the triangle (Pr1). However, these principles are not mentioned in most of the PST's resolutions and written justifications, revealing a gap in their KoT and a link between the lack of this kind of knowledge and descriptive answers. Regarding KSM, the auxiliary connections that emerge from the PSTs responses showed a close relationship between knowledge about fractions and the use of procedures, properties and representations of a geometric nature. Although some research highlights the difficulties of PSTs in accepting area conservation (Hong & Runnalls, 2020), in the present study the use of this property is implicit in the justifications and-or procedures used and PSTs do not present major difficulties. This may be due to the time PSTs had to solve the questionnaire, or to the geometric nature of the tasks themselves, since by restricting the use of calculations and measuring instruments, PSTs are forced to use procedures of a geometrical nature.

The indicators proposed for the KoT subdomain serve as a reference of what PSTs should know for their future practice (Policastro, Mellone, Ribeiro & Fiorentini, 2019), as they allow detailing different representations, procedures and justifications, properties and principles underpinning area measurement. This could provide hints for PSTs trainers on how to propose tasks to promote the mobilisation of specialised knowledge, gradually increasing the indicators of knowledge to be developed. The need for further research seems evident, specifically in both at enriching the conceptualisation theoretically and in conceptualising tasks for developing a deeper knowledge on area measurement processes. It can also be seen that there is a need to explore the potential of MTSK model as a tool for promoting and developing an understanding of area measurement in relation to different kind of connections and in relation to other sub-domains of knowledge that have not been considered in this study.

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Teachers' professional competence to pose school problems: the case of transformation of existing problems

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The goal of this study is to categorise transformations of existing problems performed by preservice primary teachers (PPTs). For this purpose, we propose to PPTs from Autonomous University of Barcelona (Spain) to transform two multiplicative problems from two primary school textbooks. Following a content analysis, we carry out a deductive-inductive coding process of the problems posed, establishing as main categories the four main elements of a problem proposed by Malaspina (2003). We describe the categorisation of the context transformations in detail and illustrate its four subcategories. The results highlight the abilities of PPTs to perform context transformations that encourage the engagement of potential solvers in the problem situation.

Keywords: Teacher education, professional competences, problem posing, problem solving.

Introduction

There is a broad consensus about the potential of problem solving as an approach to teaching mathematics, which promotes a deep understanding of mathematical knowledge and the development of mathematical abilities. Therefore, students should approach mathematical problems, being the teacher's responsibility to select the mathematical tasks to be faced by his or her students. Although the ability to formulate problems has been studied in mathematics education research (e.g. Kilpatrick, 1987), the focus has been mainly on students' formulation processes, requiring greater attention on teachers' ability and capacity to propose problems (Crespo, 2003).

We understand that proposing school problems is a professional competence of mathematics teachers in the sense of Weinert's competence (2001), meaning the development of cognitive abilities and motivational and volitional aspects to allow teachers to deal with problems of different nature. In particular, the professional competence of formulation of school problems (Carrillo, et al., in press) is seen as the teacher's ability to propose problems to their students, with the intention of encouraging the construction of mathematical learning. This competence is expressed by three skills: creating, selecting, and transforming problems. We focus on the latter skill, concretely the research question we address here is: how do PPTs transform existing mathematical problems? The aim is to categorise the transformations of multiplicative problems from school textbooks performed by PPTs. We select problems from textbooks as they are a didactic resource widely used by teachers for teaching and learning mathematics in the classroom (Fan, et al., 2013; Hadar, 2017).

Next, we present our theoretical view on problem formulation and transformation, as well as the four main elements of a problem (Malaspina, 2013). Then, the design and qualitative methods of data analysis are introduced. Finally, we describe the categorisation of the context transformations and conclude with reflections on the results and their implications on the PPTs' training.

Theoretical framework

Problem posing has a plural nature. It can be conceived as a professional activity inherent to mathematicians when they create problems for the development of the discipline. It can also be seen as a learning dynamic at different levels of schooling, aimed at promoting a deep and relational understanding of mathematics (Kilpatrick, 1987). Likewise, as a professional activity of mathematics teachers, focused on proposing problems to their students as learning tasks (Milinkovic, 2015). Formulating mathematical problems demands creativity (Silver, 1994), since it requires taking a proactive attitude towards mathematics, coordinating different mathematical elements in the construction of word problems. Therefore, it is a complex and cognitively demanding activity (Silver, 1994). Depending on the nature of the mathematical content, posing a problem could require a greater proficiency on specific abilities, such as visualisation in the case of geometry.

In the case of teachers' training, problem formulation is considered a powerful tool for understanding mathematical and didactical knowledge (e.g. Tichá & Hospesová, 2013). Moreover, problem formulation has been used as a tool to develop professional competences, such as sensitivity to students' mathematical thinking (Xu, et al., 2020). However, teachers' ability to propose problems to their students has received less attention (Crespo, 2003). This ability has multiple ways of being expressed in professional contexts (Carrillo, et al., in press), for instance, the transformation of existing problems (Milinkovic, 2015; Lavy & Hourigan, 2019). From a professional perspective, transforming a problem involves modifying one or more elements of the word problem with a didactic intentionality that may have an impact on the problem solving process (Lavy & Bershadsky, 2003).

The professional competence in the formulation of school problems is transverse in nature to the teaching activity itself. During the preparation of a lesson, a teacher can create, select, or transform problems when designing tasks for the students. For example, Lavy and Bershadsky (2003) propose transforming geometry problems using the "What if not...?" strategy. On the other hand, during a lesson, in response to a student's intervention, a teacher may decide to propose a problem to focus reflection on certain mathematical knowledge. Finally, at the end of a lesson, a teacher may reflect on some possible transformations of the problems proposed to the students, either for future courses or to propose similar problems in following lessons.

Malaspina (2013) suggests that a word problem can be broken down into four main elements: information, i.e., the elements provided explicitly or implicitly in the word problem; problem context, which can be either extra-mathematical or intra-mathematical; problem requirement, usually expressed as a question, which determines the goal to be achieved; and mathematical environment, understood as the mathematical elements useful for solving the problem. In this study we assume that transforming a problem implies changing at least one of the main elements. We consider that the information of a problem is transformed when some mathematical elements (numerical, algebraic, graphical, ...) of the problem are eliminated, added, or changed for others. In the case of the context, a transformation occurs when the situation related to the written word problem is totally or partially changed, extended or eliminated. Likewise, transforming the requirement involves changing, extending, or eliminating the explicit or implicit demand of the problem. Finally, transforming the mathematical environment implies changing the mathematical area useful for approaching the resolution of the problem. For example, when multiplicative problems are changed to others that are intended to be solved through elements of additive structures, or when problems focused on exact division are modified to include a possible remainder. The transformations of different elements entail the coordination of the information eliminated and added with the existence of a solution, as well as the consistency between the different expressions of a mathematical element and the element itself (Wessman-Enzinger and Tobias, 2020, Montes, et al., in press).

Methodology

We design a written protocol addressed to PPTs with four tasks involving the transformation of multiplicative problems from two primary school textbooks widely used in Spain. We restrict the mathematical content to multiplication and division because this crucial content at primary school has already been introduced to the participating PPTs during a previous course. We collect 67 written protocols solved individually by PPTs in the third year of the Primary Education Degree of the Autonomous University of Barcelona (Spain) in 2020-2021. For each task of the protocol, participants proposed one or more modifications to the given problems. We identify one or more transformations in each modification. For example, a PPT can extend the context (context transformation) and eliminate a numerical element (transformation of information) in the same modification. In total, the data consists of 186 modifications, 93 for the first problem and 93 for the second. Here, we present methods and results related to the analysis of the modifications proposed in the first task, which suggests modifying two word problems to promote the mathematical learning of the contents identified (Figure 1). Both problems correspond to the structure of isomorphisms of measures, where the proportion between two measurement spaces is established (Vergnaud, 1983). The first problem belongs to equal groups situation where the total number of elements is unknown. The second is a quotative division problem where the number of sets is unknown. Both problems have an extra-mathematical context with visual models of multiplication and division.

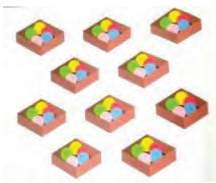
<table>
<tr><td colspan="2">Modify the following two word problems to promote the understanding of the mathematical content you identify. For each modification, indicate in the table: the type of transformation; the reason why you perform it; and the contribution to the students' mathematical learning.</td></tr>
<tr><td>Problem 1: Bruno has 10 boxes with marbles. Each box contains 5 marbles. In total, how many marbles has Bruno? </td><td>Problem 2: Lola has 30 bracelet beads. She uses 5 beads to make each bracelet.  Lola makes … bracelets.</td></tr>
</table>

Figure 1: Word problem of the first task of the protocol

Within a qualitative methodology, we conducted a content analysis in two phases: preparing and organising (Elo & Kyngäs, 2008). The aim is to describe the transformations proposed by PPTs as the result of a deductive-inductive coding process of each modification.

Preparing stage. In this stage, we familiarised with the whole data and established each modification as the unit of analysis. We also determined the central categories of coding: information, context, requirement, and mathematical environment (Malaspina, 2013). Then, we drew inspiration from the data and question codes proposed by Lavy and Bershadsky (2003) to create, a priori, some codes that were refined in the following analytical stage.

Organising stage. We begin the coding process by creating new emergent codes that are refined by comparing, contrasting, and abstracting codes across larger and larger samples of the data. The process is carried out in three rounds with the following structure: first each researcher codes the same sample, then a list of codes is agreed upon by the three authors and, afterwards, each researcher analyses an extension of the initial sample using the new codes. We record the agreements adopted in the joint discussions, which focus on the delimitation of the meanings of each code, the conditions of occurrence and the grouping of codes in each of the main categories. The final codes express the element transformed and the specific action to transform it, considering three actions: changing an element for another of the same class, extending an element, and eliminating one of the problem elements. This leads to descriptive codes such as changing to another extra-mathematical context or eliminating the context. Once a stable code set is established, we count the frequency of each code in the whole data. For each main category, we calculate the percentage of modifications incorporating at least one transformation from that category. As a result, we find that 77,9% of all the modifications incorporate transformations of information, 73,2% of them contain context transformations, 51% requirement transformations, and 15% mathematical environment transformations.

In this paper, we focus on the four emerging subcategories of context transformations. To highlight the nature of each subcategory, we illustrate each variant with some PPTs' modifications along with excerpts from their justifications.

Context transformations

We find that 73.2% of the PPTs' modifications incorporate some context transformation of the problem. In Table 1 we present the four emerging subcategories that constitute the context transformations, along with their frequency in the data.

Table 1: Context transformations subcategories and their frequencies

Context Transformations			
Extending the context	Changing the written expression	Changing to another context	Eliminating context
73	48	13	2

Context extension is the most frequently occurring transformation of context in the data. It refers to changes in which the word problem is expanded by adding characters, actions, descriptions, or relationships to the original context. These transformations seek to improve the comprehension and access of primary students to the problematic situation presented to them. Hence, the new problems are more realistic and closer to primary students' experiences or contains more detailed descriptions of certain information given in the problem. For instance, in the modification of the PPT UAB1.37 (Figure 2), the story that frames the problem is set in a scenario known to primary students- the school -. Also, the characters and actions introduced facilitate the identification of the student with the main character –making boxes with marbles to give to friends–. UAB1.37 states that simulating a situation close to a primary student's daily life can motivate him or her to solve the problem.

UAB 1.37 Modification:

Bruno has bought 10 boxes to give to each classmate. In each box he wanted to put some marbles, the same amount for all of them! Bruno has already made Joan's box and it looks like this:
Can you tell how many marbles he will need to make all the boxes?

Justification:

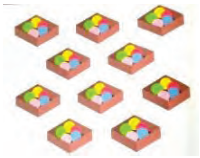
It is a situation that the student can easily find in his daily life, therefore, it gives meaning to the problem. For the student's learning, it motivates him/her to carry out the problem.

Figure 2: Modification and justification of UAB1.37

In contrast, elements of the visual model –quantity and colour of the marbles in each box– are described in greater detail in the UAB1.4 modification (Figure 3). Therefore, the word problem remarks the multiplicative unit, clarifying the elements that make it up, and suggesting a more detailed mental image in the primary student.

UAB1.4 modification:

Bruno has a green ball, a red ball, a green ball, a pink ball and a blue ball in a box. If he has 10 identical boxes, how many balls does Bruno have?



Justification:

Change in the word problem. Longer and more explicit word problem so that students have a better comprehension in reading and understanding it.

Figure 3: Modification and justification of UAB1.4

Changes in the writing of the context are micro-transformations of text segments of the word problem, such as changing the subject of a sentence that entails a change of the main character of the story in which the context is framed (Figure 4). These transformations are based on a social vision of language with the potential to transform sociocultural and affective aspects. For example, UAB1.25 changes the gender of the main character in response to a change in socially established gender roles. On the other hand, UAB1.2 changes the subject to the second person singular which places the reader as the main character of the story. As UAB1.2 explains in her justification, the context is more relevant to a potential solver and encourages his or her immersion in the problem

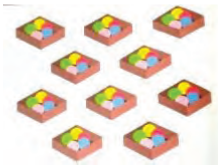
UAB1.2 modification: You have 10 boxes with marbles. Each box contains 5 marbles. How many marbles do you have in total? 	UAB 1.25 modification: Marcos has 30 bracelet beads. To make a bracelet he spends 5 beads. How many bracelets will he be able to make with the beads he has?
Justification: Change in the writing of the word problem. From 3rd person singular to 1st. With this change we make the problem more relevant to the student and his context.	Justification: Change the main character of the exercise to male to reinforce an image of equality.

Figure 4: Modifications and justifications of UAB1.2 and UAB1.25

Less frequently, we found *changes to another* extra-mathematical *context* (in no case was there a change to an intra-mathematical context) that implied a change in the scenario, objects, subjects or actions of the original context, i.e., a change in the phenomenon mathematised in the problem. Mostly, the new context poses a scenario and actions familiar to elementary school students, such as eating at school in the problem proposed by UAB1.7 (Figure 4). On other occasions, we find changes that incorporate references to other mathematical contexts such as measurement in the problem posed by UAB1.63 (Figure 5).

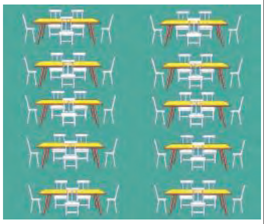
UAB1.7 modification: In the school's dining room, there is only space for 10 tables. At each table there can be a maximum of 5 students. How many students can eat at the same time in the dining room? Can the whole class eat together? 	UAB1.63 modification: Each bracelet beads measures 2 centimetres. If my bracelet has 5 beads, how many centimetres does it measure? 
Justification: The word problem is a bit more contextualised; the exercise allows students to place the use of mathematics in their lives. A second question is added to show such contextualization.	Justification: This change allows to work on measurement units, which is an important concept that is usually forgotten in the first years of primary school.

Figure 5: Modifications and justifications of UAB1.7 and UAB1.63

Finally, we found only two modifications with the *context eliminated*, i.e., the written word problem is deleted. This transformation appears together with a change of requirement. Either the execution of a mathematical operation is demanded, or the word problem is demanded from the graphical or symbolic-algebraic elements given. This is the case of UAB 1.3, who proposes: "show them the operation they have to perform, 30:5, and explain to them that they have to invent a word problem".

Final reflections

This study is part of a broader research project whose final goal is to characterise the professional competence in the formulation of school problems. Specifically, we are interested in the transformation of existing problems. First, we focus on describing and categorising the transformations that PPTs perform. Here, we present the analysis and preliminary results of this first step. We have illustrated how PPTs, when transforming existing problems, mostly focus on extending, changing, or eliminating elements related to the information and context of the problems rather than transformations focused on the requirement or the mathematical environment.

Regarding context transformations, we have identified four subcategories that describe different types of context transformations that result from extending or changing the context, introducing subtle changes in the writing of the word problem, or eliminating the context. With the first three types, the PPTs mostly encourage the immersion of the primary student in the problem and, to a lesser degree, the transmission of education in human values (such as gender equality) or the combination with other mathematical contexts (such as measurement). It should be noted that we only found two transformations in which the context is eliminated and the word problem itself is demanded. From our perspective, this fact supposes a call of attention on the need to train PPTs in problem posing dynamics as a type of classroom instruction (Silver, 1994; Malaspina, 2013).

It must be assumed that if students are to solve problems, their teachers must be active agents in the selection and generation of tasks focused on problem solving. Therefore, this line of research should result in the design of formative tasks for PPTs to develop and systematise their abilities to transform existing problems as the initial step on the route to create entirely new problems (Leavy & Hourigan, 2019). It seems promising to base these tasks on the adaptation and transformation of textbook problems, given their very wide use in Primary Education (Hadar, 2017). This will require transcending the cognitive perspective we assumed here about the professional competence to pose school problems, taking into account volitive and motivational aspects, in order to ensure a whole development of the competence.

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Towards a dialogic analysis of mathematical problem-solving knowledge for teaching in a lesson study group

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This paper presents the construction of the analytical framework of an ongoing research about the development of teacher mathematical knowledge for the teaching of problem solving in a lesson study process. After presenting the context of the research, the article overviews previous research about lesson study and mathematical knowledge for teaching, about mathematical knowledge for teaching problem-solving and about teachers' dialogue in order to introduce the two dimensions of our analytical grid: the knowledge dimension and the dialogue dimension.

Introduction

Mathematical knowledge necessary for teaching has attracted a lot of attention in mathematics education. The same tendency is exhibited in problem-solving, seen from the students' point of view and, more recently, the collaboration in teacher professional development. In our research, the teaching of problem-solving is studied from the point of view of the development of teachers' professional knowledge in a collaborative process. In this paper, we present the context of our study and the analytical framework that we are currently developing, as well as our research questions.

Context and method

Our research is part of the ongoing research of the Lausanne Laboratory Lesson Study (3LS, n.d.) which studies the work of several lesson study (LS) groups in the French-speaking part of Switzerland. We are currently analysing the work of a LS group composed of eight grade 3 and grade 4 teachers from the Lausanne region and two facilitators. The two facilitators consist of a mathematics educator (the first author of this contribution) and a teacher from the institution who participated as a member of a previous LS group in mathematics. From 2018 to 2019, this group has completed three LS cycles with the question: "how to teach grade 3-4 students on how to solve mathematical problems". The fact that the teachers will be in a (professional) problem-solving situation about problem-solving teaching is coherent with Ball and Cohen's (1999) suggestion that professional development programs should situate teacher learning in the types of practice they wish to encourage. The data from these three cycles are analysed using Transana (Woods, 2002-2021) to encode video recordings which are integrated with the transcripts. The first cycle discussed in this article, includes eight meetings that lasted for about 90 minutes each and two research lessons. Interviews with the two facilitators were also analysed as a way of data triangulation. Since other research of our team used the same type of data (e.g. Batteau, 2017; Clivaz & Ni Shuilleabhain, 2019), the obtained results show that this type of data is thorough, systematic, reliable and authentic regarding the perspectives and practices of participants.

The research interests of our research team are two folded. The first series of questions is about the mathematical knowledge related to problem-solving which teachers use during this LS, the second series aims to describe the types of interactions related to the construction of this mathematical knowledge and, ultimately, to describe how the mathematical knowledge is constructed in a LS group. In order to do so, we will present some findings of the previous studies and the analytical frameworks which is currently being developed by our research team.

Lesson study and mathematical knowledge for teaching

Jugyou Kenkyuu, literally lesson study (LS), was born in Japan in the 1890s. It was popularised in the 2000s following international TIMSS comparisons (TIMSS Video, n.d.) and the comparison between mathematics education in Japan, Germany and the USA that Stigler and Hiebert (1999) presented in *The Teaching Gap*. Thanks to the efforts made to promote LS, and in particular to the work of Lewis, who contributed to formalising and popularising LS in the USA (Lewis, 2002, 2015; Lewis & Hurd, 2011), LS was initially introduced in the USA as a professional development approach to improve US mathematical classroom teaching and learning (Yoshida, 2012). As a mode of professional development, LS has developed all over the world and has attracted the interest of many researchers in educational sciences, particularly in mathematics education (see, e. g. among the most recent international edited books, Huang et al., 2019; Quaresma et al., 2018).

LS starts from an area of difficulty in teaching and learning identified by a group of teachers. Teachers analyse the targeted learning, study the mathematical concept, consult the various teaching methods, study articles from professional journals and other resources. This study allows them to plan a lesson together. This lesson is implemented in the classroom of one of the group members. Other teachers observe the lesson in real-time and analyse its impact on students' learning. The group may decide to plan an improved version of the lesson to be conducted in another teacher's classroom and the loop begins again. The results of the work are disseminated, both in the form of a detailed lesson plan for future use by other teachers and also in the form of articles in professional journals.

LS groups are usually led by an experienced teacher or trainer, called a facilitator, who "keeps the conversation moving and fair. Involves all participants. Follows an agreed-upon agenda" (Lewis & Hurd, 2011, p. 124). While in Japan LS is sometimes facilitated by the group members, it almost always involve one knowledgeable other who provides feedback after the research lesson and sometimes another knowledgeable other who can draw attention to key elements during the planning phase (Watanabe & Wang-Iverson, 2005). In countries where LS is developed (particularly in Japan), the role of supporting professionals participating in the group as facilitators and that of occasional external experts is well defined. In contrast, these two roles are often assumed by the same person or are confused in places where LS is starting to take root (Clivaz & Takahashi, 2018).

Mathematical Knowledge for teaching problem-solving

Most of the research about problem-solving have considered the student’s point of view (for a survey on the state-of-the-art, see Liljedahl et al., 2016). A few authors have considered the teachers’ point of view and Mathematical Knowledge for Teaching (MKT, Ball et al., 2008) framework to characterise the knowledge teachers use to teach problem-solving. Wake and his colleagues (Foster et al., 2014; Wake et al., 2014) have attempted to broaden MKT to include mathematical process knowledge and pedagogical process knowledge, by rewriting with ‘concepts and processes’ instead of ‘content’ of the MKT categories (Foster et al., 2014, p. 3.98). The focus on the process and not only on the knowledge is undoubtedly noteworthy. Nevertheless, during our data analysis, we realised that the specificity of the knowledge for teaching problem-solving is not only of it being a process. Similar insights were found in the work of Chapman (2005, 2012, 2015). In her conceptualisation, Chapman (2015) distinguishes six categories of Mathematics Problem-Solving Knowledge for Teaching, divided into Problem Solving Content Knowledge and Pedagogical Problem Solving Knowledge (Figure 1). All these categories are influenced by the teacher's Problem Solving Proficiency and by his/her Affective Factors and Beliefs. In the line of Chapman's findings, for the purpose of bridging the MPSKT to the MKT categories, we propose the following graphical representation of this categorisation (Figure 1).

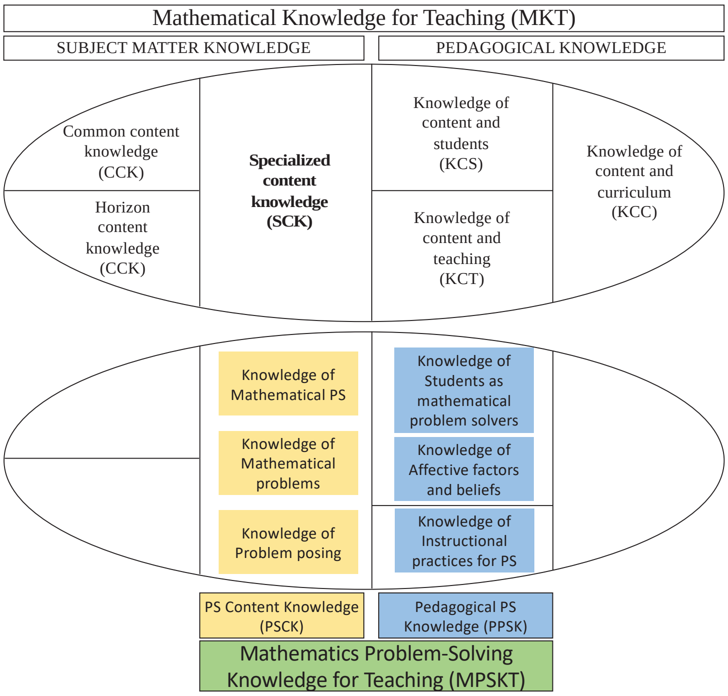

Figure 1: MKT and MPSKT. MKT upper uncoloured part of the figure is from Ball et al. (2008), MPSKT coloured categories are from Chapman (2015). Graphical representation of the coloured categories is by the authors of this paper

Lesson study and teachers' dialogue

It is with the objective of accurately describing how teachers' knowledge is constructed or evolves and, more generally, to better understand what happens between actors within an LS process, that we have been led to focus on discourse analysis in a sociocultural perspective. This perspective is rooted in the work of Vygotsky (1962, 1978), for whom the acquisition and use of language transforms children's thinking. One of our first inspirations was driven from the work of Vermunt and his colleagues (Vermunt et al., 2019; Vrikki et al., 2017; Warwick et al., 2016) who categorised the dialogic processes in LS groups in order to find statistic correlations between certain dialogic features and teachers' meaning-oriented learning in LS. With these categories being too broad for a comprehensive analysis, we were led to study the work of a sister group within the Cambridge Educational Dialogue Research group (CEDiR, n.d.), the Scheme for Educational Dialogue Analysis (SEDA, Hennessy et al., 2016) group. Rooted in the work of Alexander (2008) about dialogic teaching and of Littleton and Mercer (2013) about interthinking, this SEDA group produced a comprehensive grid to analyse classroom dialogue in problem-solving situations. The grid and the method come from an “inductive-deductive cycle that allowed to distil out the essence of dialogic interactions and operationalise them in the form of a new systematic indicators for these productive forms of educational dialogue” (Hennessy et al., 2016, p. 17). The grid and the method seemed to be a good choice to serve as a basis for the construction of a grid of systematic indicators able to capture the forms of professional dialogue within a professional development process. Nevertheless, adaptations have to be made to SEDA scheme to take into account our research context as well as previous research on teacher learning in LS.

Comprehensive research on LS groups and the fact that they appear to have an impact on teachers' professional knowledge often focuses on the essential role of facilitators (e.g. Bjuland & Helgevold, 2018; Lewis & Hurd, 2011; Lewis, 2016) and possible knowledgeable others (e.g. Seino & Foster, 2020; Takahashi, 2014). While many studies mention the importance of these roles and give examples of facilitator interventions or mention statements by teachers saying how important this role is to them, qualitative studies describing precisely how this role allows teachers to build professional knowledge are rare to date.

For our part, in our previous research, we examined the evolution of the trainer's role in terms of knowledge sharing in a series of LS (Clivaz & Clerc-Georgy, 2020) and showed which MKT teachers use during the LS process (Clivaz & Ni Shuilleabhain, 2019; Ni Shuilleabhain & Clivaz, 2017). Nevertheless, interactions within the group, in particular between the facilitators and the teachers, are yet to be explored.

The Construction of the LS Interaction Analysis Grid

In this section, we will describe our grid for analysing interactions as the result of a process that is both deductive and inductive. This grid currently focuses mainly on enunciative modalities but will be linked in the rest of our research with MKT and issues related to the topic of problem-solving.

Composed of 33 codes grouped into 8 entries, the SEDA grid (Hennessy et al., 2016; SEDA, n.d.) had to be adapted from students’ mathematics problem-solving classroom situation to teachers’ professional problem-solving discussion. We started from the SEDA grid, using the same codes every time it was possible and adapting them when necessary. After a one-year coding work, and team discussion of the coding, we were able to set up, in an inductive way, our grid for analysing the interactions within a LS. This required a fairly radical adaptation of the original grid, as we had to take into account our particular context as well as the actors and their intentions.

The process of the modification of the SEDA scheme lies beyond the scope of this paper, but we will highlight here the two main modifications related to the type of exchange in a LS professional dialogue which differs from a students’ dialogue in a classroom situation. The first modification is related to the two SEDA categories “B-built on ideas” and “R–make reasoning explicit” which were close. For example, in our data, it was almost impossible to distinguish “B1-Build on /clarify others' contributions” and “R1–Explain or justify another's contribution”. We therefore merged these two categories into “R – Answer, develop”. Since the Question-Response type of exchange among teachers was often present in our data, the codes for the R category were symmetrised with those of category Q - Prompting development or reasoning (categories Q and R in Table 1), both entries being specified into clarification-justification-hypothesising categories. The second modification is the adaptation of the “connect” category, due to the observation that group participants often make reference to incidents or episodes of the LS cycle or to other teaching experiences (categorie C in Table 1). This observation has already been illustrated in the analysis of the work of a different group (Ni Shuilleabhain & Clivaz, 2017) and we can relate it to the cumulative principle of Alexander (2018) “which underpins enquiry and knowledge growth in academic communities as well as classrooms, ensures that discussion is genuinely dialectical yet builds on what has gone before, advances understanding and is not merely circular” (p. 566).

Lesson study dialogue analysis

The first unit of coding of our analysis is the conversational turn, which allows us to first, code the identity of the speaker. Our LS Dialogue Analysis (LSDA) has 30 codes (33 for SEDA) which are clustered into 6 categories (8 for SEDA) that allow us to characterise interactions within the LS. Each code is characterised by indicators, assuring a good validity of the consensus coding (see Table 1):

- Categories E, Q, R, P and G allow us to highlight a dynamic of talk. Each turn is coded;
- Category C allows us to show what is being used as a reference in the conversational turn. This enables us to be aware of the connections that are made during the exchanges. In this case several turns are coded as a group.

The categories for LSDA are presented as follows in Table 1.

Table 1: Categories of codes for LS Dialogue Analysis

Category	Features
E – Express or invite new ideas	This category marks the entry of a new subject into the discussion, a new idea, an observation. Distinction between invitations to formulate new ideas and expression of a new idea is made.
Q – Arouse development or reasoning	This category is used with the next category R to code a series of exchanges around a subject. The Q-coded turn involves reference to a previous input. The three possible purposes of the Q-coded turn are, to better understand a factual statement or to understand the reasons for a previous statement or to consider other possibilities or hypotheses.
R – Answer, develop	This category has three possible purposes: to provide clarification and explanation, to give a justification, to develop other possibilities or hypotheses.
P – Position or coordinate	This category is used to indicate a turn intended to mark one’s stance or to coordinate ideas in relation to previous exchanges. It may involve synthesising ideas, evaluating different perspectives, challenging an idea or taking a position, approving.
G – Guide	This category is used to indicate a turn intended to guide the course of interaction by encouraging dialogue, by verbalising the rules of communication in order to promote discourse, by proposing an immediate action, by proposing an action in the future, by taking an expert position, by providing feedback, by focusing.
C – Connect	This category is used to show what a series of exchanges refers to. It might refer to: • the content of a past discussion episode • the research lesson (past or future) • one’s teaching experience • one’s personal experience • a theory or to mathematical principles • the LS process (at a meta level) • believes about teaching and learning • the learning trajectory of participants • the mathematical task

Towards an analytical framework

The ongoing coding of our transcribed LS meetings allows us to have three levels of coding:

1) Identity of the speaker
2) Dynamic of talk (categories E, Q, R, P, G) and the topic to which the series of utterances are connected (category C)
3) The category of MKT/MPSKT and the problem-solving proficiency and the affective factors and beliefs expressed by the participants.

We illustrate these three levels of coding, only with one intervention because of space constrain.

T2: Because when you discover [the problem], actually, it's not... it's complicated, you mix everything up, but if you separate it into four questions, you have the answer, so you have to find [the four questions].	1) Teacher 2 2) R: Give a justification, go further, develop 3) Knowledge of mathematical prob.
F2: Yeah, you have to split the problem up.	1) Facilitator 2 2) P: Take a stand 3) Knowledge of mathematical prob.
T4: That's it, and for them to be able to split it up like we do,... I find it interesting	1) Teacher 4 2) R: Give a justification, go further, develop 3) Knowledge of Students as mathematical prob. solvers Knowledge of mathematical prob.
T7: And I think that there's also the cover story of the problem… that's not easy	1) Teacher 7 2) E: Express a new idea, make an observation 3) Knowledge of prob. posing

These three levels and the relationships among them allow us to operationalise our question about the construction of MKT and MPSKT during LS. Level (3) will illustrate the categories of MKT or MPSKT observed in the dialogue. It will also allow us to provide some answers to the questions about the links between one teacher's problem-solving proficiency and his/her MPSKT, and about the links between one teacher's affective factors and beliefs and his/her MPSKT. The type of intervention (2) linked to the identity of the speaker (1) will lead us to determine the specific dialogic role of each facilitator and of each teacher during the phases of the LS process, to answer the questions: Are the facilitators mainly in charge of bringing up the mathematical knowledge? Is it the facilitators' role to develop mathematical knowledge or is it a shared responsibility? Are these roles evolving along the process?

Conclusion

The development of teacher mathematical knowledge about the teaching of problem-solving in a collaborative setting is a complex process. Developing a framework to analyse this process has proven highly challenging for our team, and we consider this framework as a first result. Our data analysis is still ongoing, and we hope that this analysis will bring some fine-grained description of how teachers construct knowledge collaboratively during a lesson study process.

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Integration of two theoretical lenses to analyse the potentialities of a practice-based task in fostering pre-service mathematics teacher specialized knowledge

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This paper presents an a priori analysis of a practice-based task designed to be implemented in the context of primary school pre-service mathematics teacher education. We integrate two different theoretical lenses aimed at describing teachers' practices and knowledge. By means of our analysis, we highlight the potentialities of the task to promote the goals of the educational program and to point out pre-service mathematics teacher specialised knowledge.

Keywords: Pre-service teacher, MTSK model, meta-didatical praxeologies, practice-based tasks, mathematics teacher specialised knowledge.

Background

This paper focuses on a specific task that was designed to be implemented in the context of pre-service mathematics teacher education. As Liljedahl et al. (2009) stressed, pre-service mathematics teachers (in the following, PMTs) live a unique experience within their educational paths, since

> they are both student and teacher, and through the constant shifting between student and teacher they are given the opportunity [...] to recast their initial (pre-conceived) beliefs about what it means to be a teacher, what it means to teach, what it means to learn, and even what it means for something to be mathematics (Liljedahl et al., 2009, p.29).

Researchers in the field of mathematics teacher education have stressed on the importance of focusing on practice-based approaches to mathematics teachers' professional development (Ball & Cohen, 1999), which aim to connect the ongoing professional development of teachers with the actual work of teaching. We think that these approaches could be particularly effective in the case of PMT education, since they could foster PMTs' reflections on teaching practice, filling the gap due to their lack of real classroom experience (Cusi & Morselli, 2018). In fact, by focusing on tasks aimed at fostering "activities that are situated in and organized around components and artifacts of instructional practice that replicate or resemble the work of teaching" (Silver, 2009, p.245) – the so called practice-based tasks. These approaches foster the development of a "useful and usable knowledge that builds mathematics teachers' capacity for the kinds of complex, nuanced judgments required in mathematics teaching" (Silver, 2009, p. 246). Therefore, teachers need to acquire and develop both subject matter knowledge and general pedagogical knowledge for teaching, but, as already highlighted in the 1980s by Shulman (1986), teachers' knowledge is characterised by the combination and amalgam of content knowledge and knowledge about teaching, students and curricula. This characteristic knowledge is defined by Shulman as "pedagogical content knowledge (PCK): "the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (Shulman, 1986, p.9). Starting from Shulman's studies, different models to describe mathematics teachers' knowledge have been developed in the last decades: for example, the Mathematical Knowledge for Teaching - MKT model (Ball, et al. 2008), the Knowledge Quartet (Rowland, et al., 2005) and, more recently, the Mathematics Teacher's Specialised Knowledge-MTSK model (Carrillo-Yañez et al., 2018). In our study we referred to the MTSK model to describe some aspects of the specialized knowledge of PMTs. Teachers used/needed specialized knowledge to do their work and this specialized nature of their knowledge is linked to teaching:

> Our starting point is the assumption that in order to carry out their role (including lesson planning, liaising with colleagues, giving lessons and taking time to reflect on them afterwards) the teacher needs specific knowledge. We associate this specificity with mathematics teaching (Carrillo-Yañez et al., 2018, p. 239).

In this paper we present a specific type of practice-based task and we analyse the PMTs' specialised content knowledge that can emerge. The task combines the design of fictional classroom discussions representing virtual dialogues between a teacher and his/her students with the explicit request of justifying this design by means of specific theoretical lenses introduced during the teacher education courses. The activities fostered through this kind of task represent a fundamental component of a methodology for PMT education aimed at fostering PMTs' reflective practices (Jaworski, 2004), by actively involving them in the analysis of practice through the theoretical lenses provided by research (Cusi & Malara, 2016; Cusi & Morselli, 2018). To develop the *a priori* analysis of this type of task (in the following, FCD task, acronym for "fictional classroom discussions task"), we will integrate different theoretical lenses that are presented in the next section.

Analytical framework

The analytical framework is constituted by two main components. The first component is the Meta-didactical transposition (MDT) model (Arzarello et al., 2014). Based on Chevallard's Anthropological Theory of Didactics (Chevallard, 1985), this model was born to describe and analyze the evolution of mathematics teachers' and didacticians' practices within institutional contexts, when they are jointly engaged in professional development programmes or collaborative research projects (Arzarello et al., 2014). We use the term didacticians as "people from the university with knowledge of research and theory in the didactics of mathematics, interested to work with teachers to promote better opportunities for mathematics learning in classrooms." (Jaworski, 2012, p. 623).

In tune with Chevallard's framework, the MDT model focuses on the notion of praxeology, a tool to model the human activities developed within institutional contexts. A praxeology is structured in two main levels (García et al., 2006): the *praxis* or *know-how level*, which includes the task, or a family of tasks, and the techniques used to face the task; the *logos* or *knowledge level*, which includes the "discourses" developed to justify or frame the techniques for the task. The MDT model distinguished between: *didactical praxeologies*, which refer to tasks related to the knowledge to be taught and the technique being recognized and justified within a specific institution; and *meta-didactical praxeologies*, which focus on teachers' and didacticians' meta-level reflections on contents to be taught and corresponding didactical praxeologies (Arzarello et al., 2014). We chose to refer to the MDT model since: (a) it focuses on the role played by the meta-level reflective practices developed by communities of teachers and didacticians involved in professional development programs or collaborative research projects; and (b) it acknowledges teachers and didacticians' reciprocal influences when they work together in such contexts.

The second component of our analytical framework is aimed at describing PMTs' knowledge. We have chosen to use the interpretive lenses drawn from the MTSK model (Carrillo-Yañez et al., 2018) in order to describe the mathematics teacher specialized knowledge. This model is suitable for the analysis we want to develop since it focuses on the knowledge that teachers may use/need for the analysis and design of educational activities. We chose the interpretative tools provided by this model to understand and interpret teachers' praxeologies focusing on knowledge level. The starting assumption of the MTSK model is that teachers need specialized knowledge to fulfil their role. Therefore, knowledge developed and implemented for teaching is considered specialized. Inspired by Shulman's (1986) studies, the MTSK model distinguishes between mathematical knowledge (MK) and pedagogical content knowledge (PCK), both of which are considered as sub-domains of the teacher's specialized knowledge. The MTSK model describes three sub-domains of mathematical knowledge: Knowledge of Topic – KoT (e.g. knowledge of definitions, properties, procedures, representations, and applications of mathematics); Knowledge of the Structure of Mathematics – KSM (e.g. knowing how to connect activities in different domains of mathematics); and Knowledge of Practices in Mathematics – KPM (e.g. knowing how to prove, justify, define, make inferences and inductions, give examples and counterexamples). Pedagogical Content Knowledge is divided into three sub-domains: Knowledge of Mathematics Teaching – KMT (e.g. knowledge of theories of mathematics teaching or knowledge of teaching resources, materials and technologies, but also knowledge of strategies for introducing and representing contents and concepts, etc.); Knowledge of Features of Learning Mathematics – KFLM (e.g. knowledge of theories of mathematics learning or knowledge of the way in which pupils interact with mathematics); and Knowledge of Mathematics Learning Standards – KMLS (e.g. knowledge of expected learning outcomes and teaching goals in different school segments). The MTSK model in addition to detailing these subdomains of Mathematical Knowledge and PCK explicitly highlights the centrality of teachers' beliefs about mathematics and mathematics teaching-learning.

In the study presented in this paper, we integrate the theoretical lenses belonging to the MDT and MTSK models to develop an *a priori* analysis of a specific FCD task. The *a priori* analysis is performed in order to highlight the potentials of the FCD task in: (a) fostering the educational goals of the professional development programme within which the FCD task has been implemented; and (b) bringing out, consolidating and developing different aspects of PMTs' specialized content knowledge. As regards (a), we refer to the MDT model to frame the educational context in which this task has been implemented, characterizing, on one side, the praxeologies that guided the didacticians' design of the task and, on the other side, the praxeologies that PMTs have to activate in order to face the task. As regards (b), we use the MTSK model to characterize the different aspects of PMTs' specialized content knowledge that can arise when PMTs face the FCD task.

An example of FCD task

The example we present in this paper refers to the context of primary school PMT education. The FCD task on which we focus has been implemented within a 48 hours course for PMTs enrolled at the first year of the master-degree course “Primary education sciences” at Sapienza University of Rome. The course, named “From arithmetic to algebra. From algebra to arithmetic” was aimed, in tune with the studies presented in the background section, at making PMTs develop reflections, in an integrated way, on specific mathematical contents, mathematical processes and on specific pedagogical aspects of mathematics teaching-learning. The main mathematical contents, on which the course was focused, are: the use of algebraic language as a thinking tool; the different meanings of the equal sign; the construction and interpretation of mathematical representations and tools (such as tables, diagrams, graphs…); the study of sequences, relations and functions. As regards the mathematical processes, during the course PMTs had to opportunity to experience and reflect on generalization, argumentation, problem solving and posing. Finally, the main pedagogical aspects of mathematics teaching-learning that were discussed are: the possible approaches to early algebra; the use formative assessment in mathematics; the design of laboratorial activities; the role of the teacher in guiding classroom discussions. All the reflections, in tune with the MDT model, were always developed by referring to institutional aspects, such as the National Guidelines. PMTs faced different FCD tasks during the course together with other kinds of practice-based tasks that involved PMTs in the role of future teachers (classroom tasks, analysis of students’ written answers, and videos from real teaching experiments) and other laboratorial activities that involved PMTs in the main role of learners (activities focused on numerical explorations, conjecture and proof and on problem solving). The FDC tasks we are going to analyse was carried out at the end of this course. PMTs, at that time, had followed two other courses focused both on mathematics and mathematics education, for a total number of 100 hours.

The example of FCD task for PMTs analysed in this paper (Figure 3) requires to: (1) design an excerpt of a fictional classroom discussion, focused on a specific task for students (Figure 1), starting from a collection of six real students’ written answers (the translation of two of these answers is presented in Figure 2); (2) organize the discussion by selecting the students’ answers to be discussed, grouping them according to their characteristics and identifying the order in which to discuss them; (3) justify the choice made when designing the fictional classroom discussion, making explicit reference to the theoretical constructs introduced during the course.

Giovanni and Francesco have prepared a game for us, by constructing this three-floors pyramid. Complete the pyramid, explaining how you reasoned to identify the numbers to be put in the empty bricks.

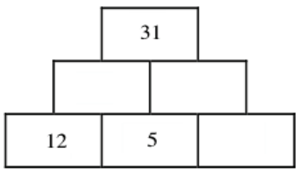

Figure 1: The task for students on which the FCD task is based

The task for students (Figure 1) is part of a sequence of tasks. During the work on the previous tasks, the students have already discovered the relationships between the numbers on the bricks that constitute a mini-pyramid (a pyramid of three bricks), that is “the number on the brick at the top of each mini-pyramid is the sum of the two numbers on the bricks at the base of the mini-pyramid”.

	We first made this pyramid: 17 / 12 5 Then we made the second pyramid. Since there were not two bricks [*they mean that they do not know one of the numbers at the base of this mini-pyramid*], we searched for the number to find out 31. We unified 14 with 17 and the result was 31. Then we made 14-5, which is 9; so, we finished also this pyramid: 14 / 5 9
	In the first line we found out the number 31. Then we formed the same number in the other two [*lines*].

Figure 2: Translation of two of the real students' written answers on which the FCD task is based

The translation of the text of the FCD task (except the problem in Figure 1 and the real students' written answers in Figure 2) is presented in Figure 3.

> Some students of a 2nd grade class have faced the following **problem** during a laboratorial activity.
>
> (*Here the text of problem in Figure 1 is inserted*)
>
> These are the **real answers** written by six of these students.
>
> (*Here the six students' written answers are inserted – two of them are in Figure 2*)
>
> **Design, by referring to the theoretical lenses introduced during the course, an excerpt of a fictional classroom discussion starting from these six students' answers.** Within the excerpt of the fictional classroom discussion, insert both the teacher's and his/her student' hypothetical interventions.
>
> The **aims of the discussion** are:
> (a) to compare the different argumentations proposed by the students;
> (b) to collectively reflect on strategies and mistakes;
> (c) to collectively construct representations aimed at making these strategies explicit.
>
> With these aims in mind, choose how to organize the phases of sharing and reflections on these written answers. For example, you can choose to discuss only some of them, to change the order in which to present the written answers, to group the answers according to the foci of the reflections you want to develop during the discussion.

Figure 3: Translation of the text of the FCD task

During the course different kind of data have been collected: the PMTs' written answers to the FCD task (that is the fictional classroom discussions they designed and the justifications of this design), videos of collective discussions between the didactician and the PMTs, PMTs' final reflections on their experience within the course (with a focus on each activity in which they were involved).

A priori analysis of the FCD task

The data collected during the course could be analysed at different levels: (1) the level of the *a priori* analysis of the FCD task as a tool for PMT education; (2) the level of the analysis of the excerpts of the fictional classroom discussions designed by PMTs and of the ways in which this design was justified referring to the theoretical tools introduced during the course; (3) the level of PMTs' reflections on the role played by their work on the FCD task in their professional development. In this paper we focus on the first level of analysis.

In this paper show an *a priori* analysis of the FCD task presented in the previous paragraph. We frame the activity on FCD tasks by using the theoretical lenses presented in the analytical framework. We refer to the MDT model to characterize the different practices, developed by the didacticians and the PMTs, in relation to this FCD task. First of all, we focus on the *praxeology of the didactician* who have conceived and designed the FCD task within an educational program for PMTs of primary school. This praxeology is related to the *task* of making PMTs experiment the activity of design of classroom discussions starting from students' answers. The adopted *technique* consists in the FCD task design itself. The *logos component* of this praxeology is constituted by the theoretical references that frame the educational program and support the justifying discourses behind the choice of focusing on methodologies for teacher education aimed at fostering PMTs' reflective practices. In relation to this, a fundamental element of the logos component is represented by the research studies focused on the role played by theory as a tool to support practice. As concerns the *praxeologies developed by the PMTs* in their role as authors of fictional classroom discussions, the *task* to which these praxeologies refer is that of designing a classroom discussion focused on a specific task for students and on specific students' written answers. The *technique* to face this task is characterized by different processes that have to be realised: the analysis of the task for students, the analysis of the students' written answers and the identification of the possible interventions that the teacher and her students could make during the classroom discussion. The *logos component* is constituted by the different theoretical lenses shared with PMTs during the whole course. It must be added that the didacticians' choice of the task for students on which the FCD task is focused (Figure 1) and the aims of the classroom discussion to be designed make PMTs direct their attention also on specific aspects related to different mathematical and pedagogical contents faced during the course (for instance, the role of argumentation in mathematics, early algebra, formative assessment…). Other elements could also be part of the logos components of PMTs' praxeologies, such as the PMTs' (mathematical and not mathematical) previous knowledge and their beliefs about teaching and about the mathematical content on which the task for students is focused. In order to analyse which aspect of specialized content knowledge might emerge when pre-service teachers face this task, we use the MTSK model. In the analysis of the task for students, the students' answers and the possible discussion about them, PMTs can use their KoT about: the properties of natural numbers and operations; the additive relation; the procedural/relational meaning of the equality symbol. With regard to KSM, their knowledge about relations among number sets and about pre-algebra should emerge. The role of examples and counterexamples in the production of hypotheses in arithmetic problems and the possibility of using different possible arguments are part of KPM. The task could make the PMTs deeply reflect on different and effective representations that can be used to work with students to explore numerical relations or artefacts and meaningful activities concerning problem solving (KMT). The different procedures that students might carry out or students' possible errors as well as the difficulties in producing arguments can be framed in the KFLM domain. The examples of students' answers (Figure 2) are useful for the development of shared reflections on students' argumentative processes and on the arithmetic relationships they can identify. It is important to be aware that the pyramid task aims not only at implementing computational schemes but, above all, at fostering problem solving and argumentation processes. These processes are key issues in the goals for the development of competences written in Italian National Guidelines (Standards) that PMTs have to know (KMLS). This detailed analysis of PMTs' mathematical knowledge and pedagogical content knowledge allows us to enhance the MDT model with regard to the description of the *logos component* of PMTs meta-didactical praxeologies.

Conclusion

In this paper we have presented a specific practice-based task – the FCD task – designed to be implemented within courses for PMTs. By integrating two different theoretical lenses, we developed an *a priori* analysis of the FCD task aimed at highlighting the potentialities of this kind of task in terms of promotion of both the educational goals of the course within which it was implemented, and development of PMTs' specialised content knowledge. In particular, the MDT model enabled us to reflect on the meta-didactical praxeologies activated by the protagonists of the educational process under scrutiny: on one side, the praxeologies that guided the design of the FCD task by the didactician; on the other side, the praxeologies activated by PMTs during their work on the FCD task. The MTSK model constituted the lenses through which we deepened the characterization of the logos component of the PMTs' praxeologies, by focusing on the specialised content knowledge that could emerge and be consolidated by means of the examined task. We detailed, for each of the subdomains of the MTSK model, what knowledge can emerge when PMTs face this task.

The results of our analysis could have both practical and theoretical implications. At the practical level, by highlighting the potentialities of FCD tasks, our analysis confirmed the effectiveness of the criteria that guided the design of this kind of tasks: focusing on meaningful mathematical problems (to foster the activation of mathematics teachers' specialized knowledge); asking to design fictional classroom discussions starting from students' real written answers (to promote PMTs' reflections on aspects related to different mathematical and pedagogical contents faced during the course); asking to justify the fictional classroom discussions' design by referring to specific theoretical lenses (to better trigger PMTs' reflective practices). At the theoretical level, our analysis shows that the integration of the MDT and the MTSK model was effective in highlighting both the educational aims connected to the design of FCD tasks and the possible results of the implementation of such tasks in terms of potential emergence of PMTs' specialised content knowledge.

In this paper we focused only on the *a priori* analysis of the FCD task. As a further step of our study, we will focus on the different data collected during and after the implementation of the FCD task to perform other levels of analysis in which we will continue to interweave theoretical lenses from the MDT and MTSK models.

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Mathematics teacher educators' thinking about mutuality in teaching

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Stakeholders agree that the mathematical education of teachers needs to focus on mathematical knowledge for teaching, but the practice-based nature of this knowledge poses challenges for mathematics teacher educators — for understanding it, developing tasks that maintain its integrity in practice, and teaching it to teachers in ways that meaningfully support their learning to teach. We know little, however, about how mathematics teacher educators conceptualize the teaching that knowledge is to support. Our analysis reveals that thinking develops from a view of teaching as straightforward, where aspects can be treated in isolation, to a view of it as requiring focused attention while maintaining mutual regard for the whole. This difference has implications for how mathematics teacher educators understand specialized mathematical knowledge and for how to support their understanding and teaching of it.

Keywords: Mathematical knowledge for teaching, teaching, teacher educators.

Introduction

At present there is little doubt about the importance of specialized knowledge for teaching mathematics. Scholars have invested in supporting mathematics teacher educators (including, mathematicians, education specialists, instructional coaches, teacher leaders, and others) to develop their understanding (Even & Ball, 2009). As mathematics teacher educators are a diverse group with disparate backgrounds and uneven expertise and experiences, such work is often challenging and showcases how these differences might matter (Lloyd & Chapman, 2020). Specifically, the practice-based nature of mathematical knowledge for teaching together with uneven understanding of teaching creates challenges in the work that impede progress. A more detailed and systematic unpacking of mathematics teacher educators' thinking would help professional development efforts, however, currently we know very little about how mathematics teacher educators think about teaching or how this thinking connects to their understandings of specialized mathematical knowledge.

Theoretical Background and Research Design

For several years, the research group in which we work (based in the United States) has conducted workshops that seek to develop mathematics teacher educators' understanding of the mathematical demands of teaching. The current study is part of a larger project designed to support mathematics teacher educators' engagement in collegial development of instructional tasks for teachers that address mathematical knowledge for teaching (Ball et al., 2008). We conducted a dozen four-day workshops of approximately 30 participants each. Participants applied as teams that were composed of individuals from different professional settings engaged in the mathematical education of teachers in their local area. Workshops began with a basic introduction to mathematical knowledge for teaching, task writing, and the mathematical demands of attending to justice, where teaching is understood as management of the interactions among teacher, students, and content, occurring in immediate and broader social environments (Ball, 2018; Brousseau, 1970-1990/2002; Jaworski, 1994), and justice is understood as an essential feature of professionally responsible teaching (Ball, 2018). Moving between whole-group and small-group work, we introduced participants to tools for developing tasks with most of the time spent on writing and reviewing tasks.

For this study, we asked, how do mathematics teacher educators think about teaching? Throughout, our focus was on differences that might matter for their thinking about mathematical knowledge for teaching. We selected 12 participants to interview, with varied demographics, professional affiliation, and experience with the ideas. Approximately one third were affiliated with university mathematics departments, one third with schools of education, and one third were in public primary and secondary schools. Roughly half of the participants had over ten years of experience teaching, and the majority were white women. (All names are pseudonyms.) Although this is a small sample, sampling was purposeful across roles and experiences, and differences in thinking (which is our focus) likely reflect differences in the wider community. We collected extensive pre-workshop applications and post-workshop surveys, including responses to specialized content knowledge tasks with detailed explanations. Our guided interview protocol was designed for multiple purposes: to gather feedback on the workshop, to gain insight into participants' understandings, and to inform future development efforts. Interviews were conducted via video conferencing, recorded, and transcribed. Open-ended questions probed four topics: (i) reasons for applying; (ii) what they learned and found useful; (iii) perspectives on a video clip of a workshop discussion; and (iv) thoughts about the mathematical-knowledge-for-teaching needs of mathematics teacher educators. The interview was not designed solely to elicit views of teaching, but teaching was a focus of the workshops and each section of the protocol touched on teaching directly and indirectly. Analysis focused on the interviews but also used applications, surveys, and observations to inform and test interpretations.

To investigate participants' thinking, we used a conceptual analytic approach (Erickson 1986). Our approach was empirically grounded, coordinated different perspectives, and was driven by practical concerns and logical analysis. Analysis involved cycles of attuning initial conceptualizations, relevant literature, and available data with a focus on local meaning and an assumption that what people say is sensible from their perspective. We identified units of text in tandem with coding — selecting text that focused on the evidence for the code and was enough context to stand alone as evidence. We wrote memos, developed a codebook, and coded for salient distinctions in thinking from direct viewing recordings of interviews, and when uncertainty or inconsistency occasionally arose, we re-watched, explored different interpretations, imagined from the participant's perspective, and set aside to revisit later. We developed codes from sections of a subset of interviews until they could be used consistently and then applied and documented them for the full set. Our focus was on interpretive power, with attention to subjective coder reliability when reconciling our independent coding.

Analysis and results

Our analysis reveals a fundamental distinction in mathematics teacher educator's thinking about teaching — that it develops from views of teaching as *straightforward, where one gives attention to an aspect of teaching but with little regard for other aspects*, to views of it as *involved*, *where, while giving attention to an aspect of teaching, one maintains regard for other aspects and the whole, with* *a sense of mutuality*. By *aspect* we mean a focused slice of teaching. By a sense of *mutuality*, we mean regard for the comprehensive interactions of teaching and learning, including the influence of broader social environments and the need to contend with complexity by specifying practice at a finer grain size (Ball & Forzani, 2009; Grossman & McDonald, 2008), while also considering its character as dilemma management (Lampert, 1985).

We coded units of text conveying: (i) regard for only a single aspect of teaching with no regard for other aspects; (ii) mutual regard for mathematics content and student thinking, but where these alone are privileged; and (iii) mutual regard other than this limited purview, for instance, additionally considering classroom culture, students' developing identities, or the pace and remaining time for a lesson. Viewing an aspect of teaching without regard for other aspects is common in the general public but is likely rare among mathematics teacher educators (as in our data). Instead, the second code is prominent, where teaching is considered in ways that foreground mutual regard for mathematics and student thinking to the exclusion of other aspects. Frameworks drawn from mainstream mathematics education shape, or at least reinforce, this thinking. Fuller mutual regard combines experience and more theoretically grounded consideration of the complex nature of teaching as situated human interaction.

As an example of thinking about teaching as straightforward, consider Claire, a white graduate student in mathematics education, with three years of secondary mathematics teaching experience. She describes interpreting and classifying student thinking as an independent task teachers need to do. She talks about not needing to know students' backgrounds or identities and how this can be a distraction. She acknowledges that teaching can seem complex, but in her view, teachers manage complexity by focusing on "narrow pieces" of teaching in isolation.

> *Instead of thinking about like okay there are forty individual approaches in this classroom, there are reasonably like three or four ways my students might be thinking about that, and … based on these [students'] comments, we can try to put students into this model that we already have of how a student is thinking.*

Her comment suggests that understanding and making good use of idealized conceptual models of how students are thinking is the crux of the work in this situation, in almost complete isolation from consideration of other aspects of teaching. From this perspective, interpreting students is less an attempt to grapple with students' ideas on their own terms, and more about considering them in relation to pre-existing conceptions of content as conceived by the teacher. This view conveys that teaching is about being aware of and understanding these conceptions of content and using them to assess students and manage the presentation of mathematics going forward.

Such an understanding is limited. It reduces the work of interpreting student thinking to one where human beings and their interactions are not the primary focus, but abstract conceptions are. Content is given primacy here above all other aspects of teaching, which obscures and distorts essential mathematical work required for the multiple and interrelated concerns in teaching. In this view, we lose sight of how a student might feel when they are the subject of this kind of classification (what if the assigned classification is incorrect?) and the impact such classification might have on future interactions and the subsequent mathematical trajectory of the class. We also miss how certain conceptions (even if accurately assigned and skillfully used in service of content goals) might be viewed as less mathematically advanced by peers and potentially reinforce stereotypes that undermine a productive learning environment. In this way, the central purpose of such an activity shifts away from understanding students or making sense of complex human interaction and toward unpacking abstract conceptualizations of content. Additionally, as teachers often interact with students across cultural and racial differences, their perceptions, unless actively interrogated and disrupted, are likely to impose norms of dominant groups that harm marginalized students.

Despite all of this, Claire trusts that care and good intentions will sufficiently address other aspects of teaching. She has an abiding regard for mathematics and teaches prospective teachers how, for example, to set clear objectives, yet she treats this and other pedagogical tasks as separate matters that do not come into play when classifying student thinking. For Claire, the pedagogical tasks that make up teaching can each be learned relatively independently and used straightforwardly.

A second form of constrained thinking about teaching prioritizes two concerns, mathematical goals and student thinking, with these treated as interdependent but as so primary that exclusive attention to them is given. For example, when debriefing the workshop discussion from the institute, Teresa (a Latina instructional coach with over 10 years of teaching experience, whose focus is the professional development of elementary teachers) explains:

> *The hardest part that we see teachers work with is making connections, helping students connect these different ideas ... What I appreciate about this video is that it makes it real time … it's almost asking teachers to write exactly what questions or what they would say to actually bring together Aniyah and Katherine's work and build on the mathematical thinking that is there to reach this goal.*

This comment highlights mutual consideration of student thinking and mathematical goals, and how their navigation requires thinking through and attending to the details of what a teacher needs to say and do. She describes Stein et al.'s (2008) five practices for leading a productive discussion as the essence of teaching — a dance between mathematical content and student thinking. When she focuses on mathematical issues, student thinking is near at hand, and when she scrutinizes student thinking, she keeps mathematical goals in mind. Relevant mathematical knowledge for teaching in this view is fundamentally shaped by the mutual consideration of student thinking in conjunction with content goals. In the quote above, Teresa expresses the need to help teachers formulate what they might actually say because she sees that specificity is needed to size up whether and how to probe student thinking in a way that maintains mathematical goals. In her interview these two aspects of teaching are often privileged — to the exclusion of others.

Another example of our second code is the following from Andy, a white high school teacher with two years of teaching experience and a graduate degree in mathematics. He talks about how challenging it is to hear the parts of student contributions that are mathematically correct and see how to leverage these to advance mathematics. For Andy, this is the heart of skillful, experienced teaching. As did Teresa, he emphasizes the challenge of evaluating student responses in real time.

> *Getting over that fact that this is so crazy, this student has one seventh, and this student is looking at fourths, and to get to the point of where their fractions are coming from … you know, you have*

five minutes basically in real time … while you're trying to manage all the other behavior issues and the other things going on in the classroom.

From his overall response, we hear Andy describe the need for mutual, interdependent regard in the work of bridging student thinking and mathematical goals, but his reference to "other things going on" posits these as independent tasks that need to be carried out simultaneously, but do not require substantial coordination. Andy does not, for example, note how the real-time nature of this moment shapes the kinds of responses that might be available to a teacher, or say how particular responses to behavior might exclude certain children from the mathematical work, both relevant concerns that significantly impact the mathematical knowledge demands of this situation.

The quotes from Teresa and Andy suggest a view of teaching as coordinated attention to mathematical goals and student thinking, but without mutual regard for other aspects, such as for students' identities, classroom culture, materials available, and practical time constraints. Such a conception of teaching does bring genuine regard for a dual attention to mathematics and student thinking but does not attend much beyond that. Indeed, in this view, teaching is still principally about content, whether student-generated or prescribed by curricular and disciplinary goals. While this conception might consider interactions between students or student ideas as mathematically relevant, they are seemingly only so if they are in service of immediate content goals — to the exclusion of other possible goals of teaching (e.g., development of a longer-term mathematical trajectory, encouraging individual and collective participation, human improvement, empathy, disrupting status hierarchies, or social change). Such a view of teaching continues to idealize content away from context and again misses important mathematical work in the interactions among the environments surrounding instruction and the multiple interacting components of the instructional triad. For example, work concerning student identity development is intertwined with and inseparable from mathematically relevant questions such as who should get to speak next, what content might be useful to surface, which examples can reasonably be done in the remaining class time, or what examples are likely to elicit unconventional responses. Mathematics teacher educators who see mathematical objectives and student identity development as isolated and disconnected concerns miss how one can shape the other. It is worth noting that this limited view of teaching is evident in much mathematics education research and many mathematics-education programs, where learning theory and mathematics are often central and integrated but other issues are treated as separate, e.g., classroom management, time management, moral and civic education, and social (in)justice.

Our third code identifies mutuality that goes beyond regard for mathematical goals and student thinking. As an example, Naima (a black curriculum specialist and professional development facilitator with six years of teaching experience and a graduate degree in public policy) focuses on interpreting students' contributions and using them together to advance instruction, but she also stitches regard for other aspects into her comments. Discussing the mathematical content and the collective trajectory of the class, she comments about carefully choosing who should speak, with reference to each student's strengths and growth, how they are positioned, and the overall classroom culture. She attends to all parts of the instructional triad. Naima routinely focuses on a specific concern yet maintains a sensibility for teaching as complex interactional work, inserting brief asides to other aspects of teaching and offering examples that situate her specific point in an overall picture.

The participants in our study were recognized professionals. They were connected enough to hear about and attend a national workshop. The majority of our codes were of this third type (Figure 1).

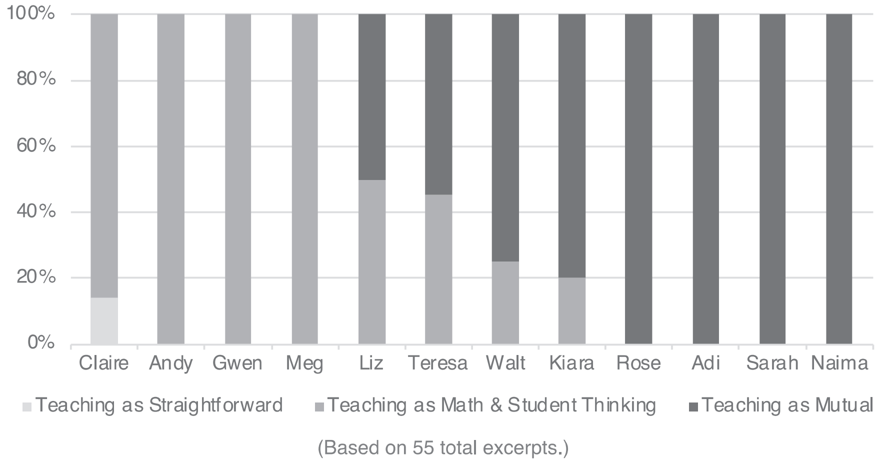

Figure 1: Percents of coded units for each way of thinking about teaching for each participant

Some participants consistently spoke of teaching as requiring mutual regard, while others did not. These two groups are visible at the two ends. The middle four were mixed. For instance, Teresa described how attention to justice issues in the workshop was making her realize she was leaving significant parts of the work implicit in her work with teachers, not only related to justice but to other mutual considerations. She reflected that, in her focus on orchestrating a productive discussion, "the characters, for lack of a better word, the little people, the kids have been absent."

> *I didn't realize how powerful it is to actually paint that context and paint the picture of who is that student with that voice. Like, as a classroom teacher I think I did attend to that, like I understood who kinda- like when you're working with an equalizer board in music, there are times where I need a tone, somebody's voice, and I was- I felt like I was in tune with that. But I didn't realize how to make that visible and explicit to teachers as they think about how they do that with their students. And now I found it seems like a simple solution, tell the whole story. Paint the picture, tell who this student is so that it is part of how they are making their decisions when they have this much time to think about it … I think is my biggest aha! Like make it visible- it can't be something you hope they think to consider in the moment.*

For Teresa, attention to justice issues became an inroad that extended her constrained thinking about teaching as navigation of dynamics only between mathematical objectives and student thinking to navigation involving a much fuller set of dynamics at play in the human interactions of teaching and learning. Similar to Teresa, the three other participants in the middle of Figure 1 all have frames that constrain their thinking about teaching (limited views of mathematics, of what is interesting or important, of students, or of teaching), but also have experiences that allow them at times to take up pedagogical concerns with fuller mutual regard.

Conclusion

The distinction we have unpacked here matters. It is likely a source of miscommunication and misunderstanding among mathematics teacher educators; it reflects something fundamental about the development of mathematics teacher educators' thinking about teaching; and it also has implications for their understanding of specialized mathematical knowledge. Understanding of teaching as straightforward constrains the work that is visible and considered mathematically relevant. Consequently, such a view inhibits understanding of specialized mathematical knowledge. The tendency to idealize or abstract out of context in service of content goals distorts the purposes of teaching and severs its connections to the realities of practice. It reframes a situated activity as one that is almost entirely cognitive. Our analysis suggests that in the absence of a well-developed understanding of teaching, existing orientations and sensibilities get imported to fill the void. For mathematics teacher educators, viewing teaching through a cognitive or content lens is probably a natural step given the backgrounds and training these individuals likely bring to the work. It is perhaps no surprise then that the practice-based nature of specialized mathematical knowledge often creates difficulties for mathematics teacher educators. Writing tasks for teachers that are authentic to practice is difficult when one's understanding of practice is limited.

In addition, understanding how mathematics teacher educators think about teaching can be used in the service of professional development. In a larger study that makes use of the mutuality distinction, we found that it aligns with thinking about specialized mathematical knowledge as practice-based or resource-based, and with thinking about justice as fundamental or optional (Hoover et al., 2022).

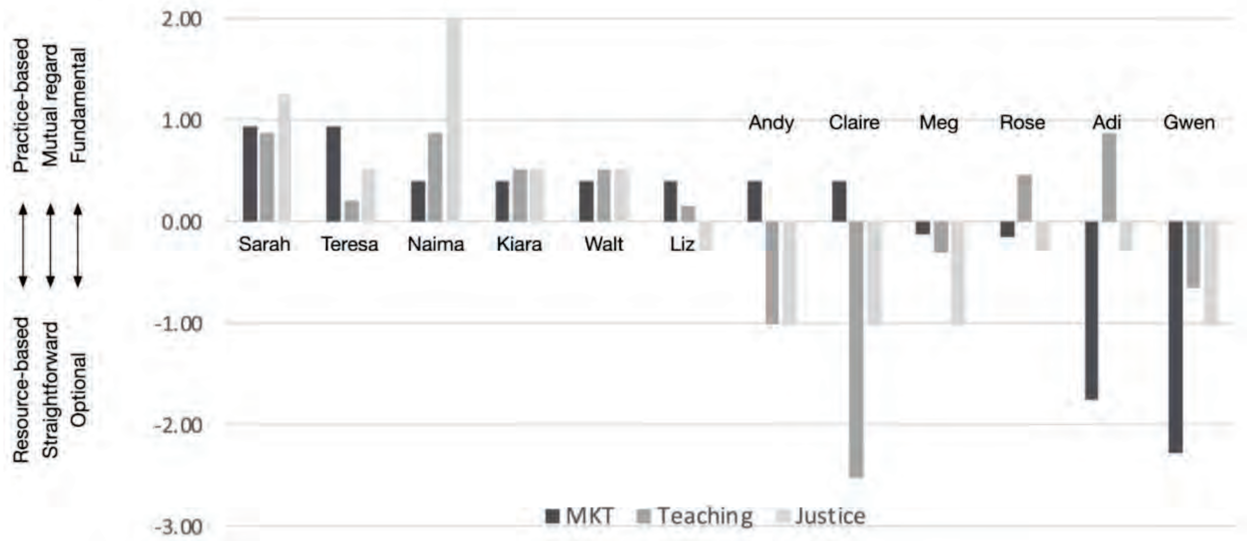

Figure 2: Profiles of the extent to which participants think of mathematical knowledge for teaching as practice-based, teaching as mutually involved, and justice as fundamental and consequential

Deeper examination suggests several clusters of participants with distinctive characterizations that might benefit from more focused professional development that takes alignment (or misalignment) into account. For example, understanding that a mathematics teacher educator thinks of teaching as straightforward allows professional development to be tailored to push on the boundaries of that view and might help them to see the practice-based nature of specialized mathematical knowledge.

While other scholars have described dynamics akin to mutuality, using different language, in different contexts, our analysis is significant in several ways. Conceptual distinctions are significant for specific purposes, and this distinction matters for how mathematics teacher educators work, including how they think about specialized mathematical knowledge, the tasks they write, and the mathematical opportunities they provide to teachers. We recognize that our analysis is more suggestive than definitive, that our language for and elaboration of the concept of mutuality is limited, and that our data draws primarily from mathematics teacher educators working in a U.S. context, but it offers a starting point. Important questions to address are the composition of skillful regard for mutuality and whether attention to mutuality matters for teachers as well as for mathematics teacher educators.

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Using the Knowledge Quartet to quantify the appropriateness of present and absent Mathematical Knowledge in Teaching

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Keywords: Classroom observation techniques, statistics education, statistical inference.

Introduction

The purpose for teachers acquiring mathematical knowledge for teaching (MKT) is to use this in their classrooms. Research on teachers' MKT, therefore, also needs to measure their mathematical knowledge in teaching (MKiT). The Knowledge Quartet (KQ) framework is designed for such situations (Rowland, Huckstep, & Thwaites, 2005). The KQ comprises four dimensions: foundation, transformation, connection, and contingency, each consisting of four to seven aspects (see knowledgequartet.org). Using the KQ, Weston (2013) suggests to quantify the presence and appropriateness of MKiT. However, she does not quantify the appropriateness of the *absence* of MKiT. Building on Weston's method, we also analyze whether the absence of particular aspects of teacher knowledge is appropriate. The research question is: What does the quantified use of the KQ framework reveal about the appropriateness of present and absent MKiT?

Methods

Using a multiple case study, we applied the KQ in the context of a teacher college intervention that aimed to improve the PSTs' MKT of informal statistical inference (ISI; De Vetten, Schoonenboom, Keijzer, & Van Oers, 2018), which can be defined a generalization, based on data, expressed with uncertainty. Part of the intervention was to teach an ISI lesson. We observed three of the 21 participants, Alfred, Celine and Demi (pseudonyms) teaching their 3rd to 5th classrooms. The PSTs taught a lesson, called "What is the most frequently used word in a stack of children novels?" The lesson was modelled at teacher college and was laid out in a lesson plan. The analyses are based on transcripts of video recordings of the classroom interactions. The unit of analysis was a teaching situation, constituted by a fragment of whole class discussion that concerned one substantive topic. Within each fragment the PSTs' actions were analyzed from the idea that these actions "could be construed to be informed by a trainee's mathematics content knowledge or […] mathematical pedagogical knowledge" (Rowland et al., 2005, p. 258). The coding process followed the approach of Weston (2013). First, for each fragment the presence of each of the 20 KQ codes was coded (present versus non-present). Second, the appropriateness of the (non-)presence was evaluated (appropriate versus inappropriate). Present teaching actions were coded as appropriate when these teaching actions helped to attain the learning objectives (p-a), and as inappropriate when they hindered the attainment of the learning objectives (p-i). Absent teaching actions were coded as appropriate when absence did not hinder the attainment of the learning objectives (np-a), and as inappropriate when they were essential to move the lesson towards attainment of the learning objectives (np-i). Coding was discussed with a second coder until consensus was reached. Six out of 20 KQ codes were present in at most one fragment and were excluded from the analyses. These concerned, for example, procedural codes, which are not relevant for the conceptual nature of ISI, and codes related to lesson design issues, which are not relevant for lessons not designed by the PSTs. The analyses involved tabulating the presence and appropriateness codes and searching for patterns.

Results and discussion

Table 1 shows the results of the analysis. It shows, for example, that Alfred and Demi were not aware of the purpose of the lesson in 27 to 28% of the fragments. Another result is that inappropriate (absence of) teaching actions often concern the failure to correctly interpret or respond to students' conceptual input, and to provide correct conceptual explanations of the content. Lastly, the use of mathematical language was largely absent from the lessons, apart from Alfred and Demi defining a sample. This is not surprising given that ISI is informal by definition and stimulates the use of context language. The above examples thus show how quantifying the presence and appropriateness of teaching actions helps to overall impression of PSTs' MKiT. Qualitative analyses could provide more depth to these results.

Table 1: Proportion of appropriate (absence of) teaching actions[1]

	Alfred				Celine				Demi			
KQ code	**p-a**	**p-i**	**np-a**	**np-i**	**p-a**	**p-i**	**np-a**	**np-i**	**p-a**	**p-i**	**np-a**	**np-i**
Adherence to textbook	.09	.09	.82			.08	.92		.06	.22	.72	
Awareness of purpose	.73			.27	.92			.08	.72			.28
Identifying pupils' errors	.45		.27	.27	.50		.42	.08	.72		.28	
Overt display of subject knowledge			.73	.27	.08		.83	.08			.72	.28
Use of mathematical terminology	.09		.91				1.00		.22		.78	
Choice of representations	.27		.64	.09	.17		.83		.11	.06	.83	
Use of instructional materials	.64		.36		.83		.17		.22	.06	.72	
Teacher demonstration	.36		.36	.27	.25		.58	.17	.06	.11	.78	.06
Anticipation of complexity	.18		.45	.36	.42		.58		.33		.39	.28
Decisions about sequencing	.09		.91				1.00		.06	.11	.83	
Making connections between concepts	.18		.45	.36	.08		.75	.17	.17		.56	.28
Recognition of conceptual appropriateness	.45		.27	.27	.67		.25	.08	.50		.33	.17
Deviation from agenda	.09		.91			.08	.92				.83	.17
Responding to students' ideas	.64	.36			.75	.17	.08		.78	.22		

[1]Alfred 11, Celine 12; Demi 18 fragments. [2]Thin border: proportion inappropriate between 0.1 and .25 ; thick border: proportion inappropriate > .25.

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Subjective probability: Mathematical Teachers' Specialised Knowledge in a betting game

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In this paper we present a study regarding a professional development course for primary and lower secondary teachers designed in the frame of a model of Mathematical Teachers' Specialised Knowledge (MTSK) conceptualization. In particular, taking inspiration from the subjectivist approach to probability, the teachers are involved in a context of betting games facing the problem of quantifying the degree of confidence of an event. We will focus on some steps of a collective discussion emerged in one of the meetings analysing them by the MTSK frame. Starting from this analysis we discuss the implications regarding the use of this kind of settings for developing teachers' knowledge in the field of probability.

Keywords: Probability, subjectivist approach, Mathematic Teachers' Specialised Knowledge.

Introduction

In the last years the topic of Probability has become an increasingly central topic in the national and international curriculum around the world (Batanero et al., 2016). Also, the research in mathematics education is giving much more attention to this topic, focusing both on the pupils and the teachers (e.g., Ireland & Watson, 2009; Chernoff & Sriraman, 2014). Some studies (Franklin & Mewborn, 2006; Chick & Pierce, 2008) point out that teachers' training in the field of probability is not enough to enable them to deal with challenges posed by the work with pupils. Several teachers present difficulties similar to their students' ones in managing basic concepts (Prodromou, 2012). Moreover, their pedagogical content knowledge in probability appears to be not appropriate (Batanero et al. 2004) and they seem to have scarce experience in conducting experimental activities with students (Stohl, 2005). In this view, Batanero et al. (2016) have highlighted the necessity of leading research on the components of teachers' knowledge and of designing materials for teachers' professional development concerning probability.

In this paper we propose an analysis regarding a professional development course (PD) for primary and lower secondary mathematics teachers. This PD was designed referring to the subjectivist approach to probability (de Finetti, 1931) and to the model of Mathematical Teachers' Specialised Knowledge – MTSK (Carrillo et al., 2018). In particular, the PD has the aim of challenging the teachers' mathematical knowledge and allowing them to invest in designing meaningful activities and contexts for teaching probability. The simulation of betting games has been the background for this specific PD and one of the games explored was the one related to betting on the sum of the two faces outcoming when rolling a six-sided dice.

In this study we discuss how the autonomous creation and modification of quotas, together with the debate about the associated decisions and the stabilization of quotas, have activated in teachers processes of qualitative and quantitative refinement of the degree of confidence, associated to each possible event. This discussion aims at answering the following question: Which element of specialised Knowledge on the Topic (KoT) of probability is possible to trace in teachers facing the problem of quantifying “quotas” in a context of betting game? These elements could be revisited in a perspective that allows the construction of a probability measure to be rooted on a subjective choice (de Finetti, 1931).

Theoretical framework

Teachers’ knowledge plays a crucial role in students’ learning process (e. g. Ball, Hill & Bass, 2005; Boyd et al., 2009; Nye, Konstantopoulos & Hedges, 2004). For this reason, the research interested in teachers’ knowledge has been an emerging focus of attention in the last years and, in this scenario, different approaches were designed and explored in order to help understanding the specificities of such knowledge. In particular, the MTSK’s model was proposed to catch better the complexity and the specialised nature of this teachers’ particular knowledge. It comprises two main domains: Mathematical Knowledge (MK) and the Pedagogical Content Knowledge (PCK) in which three subdomains are considered.

KoT’s sub-domain (Knowledge of Topics) concerns mathematics content itself and refers to the teachers’ knowledge related to definitions and notions about each specific mathematical topic, its own set of properties, the intra-conceptual connections between the single items’ content of the topic and the understanding of its foundation and history. KoT refers to a mathematical specialised knowledge as it comprises the teacher mathematical knowledge situated in the context of practice and thus it is a knowledge only required for the task of teaching mathematics. It is composed by a set of categories: procedures; definitions; properties and foundations; phenomenology and applications; intra-conceptual connection. This subdomain includes teachers’ knowledge on different representational systems about the specific mathematical topic and the different meanings connected with the procedures related to that topic. According to the MTSK model a sound KoT allows teachers to manage mathematical procedures in an aware way and to master the meanings connected to the different procedures. Mathematical epistemological aspects, and so all the forms of explorable examples in a real context or in connection with other disciplines, characterise the category of phenomenology and application, useful to contextualise a problem and a situation.

In this vision, in the field of probability, it’s important to understand which the kind of specialised knowledge is that a teacher needs to have and which the sub-domains are where these knowledge can be located. Specialised knowledge of probability is identified in this paper with the words "KoT - category name" to simplify the reading.

It is desirable that a teacher has a strong KoT that involves, among other things, the fact that probability needs to be seen not only in the classic probability’s vision, but also with the frequentist and subjectivist approach. These three kinds of approaches are linked to each other (KoT - intra-conceptual connection). It is also relevant that the teacher should have a knowledge of the definition and the meaning of probability, which allows to know the potential of each approach.

Let’s take the example of the "dice game", that it is also the context of the discussion that we will analyse later, and consider the problem of the quantification of the degree of confidence of events “sum of faces of two six-sided dice” (KoT - phenomenology). This issue can be faced using all the three approaches of probability (KoT - definition, properties and foundation).

The classical approach to probability sees the probability’s measure like a fraction between the number of favourable cases of a given event E and the number of all possible cases of a given event space (KoT - definition). Thus, according to the classical approach, in Figure 1 are reported, with the black colour, the probability’s measures associated with each sum (listed in green and unpacked in orange and blue) of the problem (KoT - procedures) presented in this paper.

					6+1					
				5+1	5+2	6+2				
			4+1	4+2	4+3	5+3	6+3			
		3+1	3+2	3+3	3+4	4+4	5+4	6+4		
	2+1	2+2	2+3	2+4	2+5	3+5	4+5	5+5	6+5	
1+1	1+2	1+3	1+4	1+5	1+6	2+6	3+6	4+6	5+6	6+6
2	3	4	5	6	7	8	9	10	11	12
$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Figure 1: probability’s measures associated with sum from 2 to 12

In the frequentist approach, if an experiment is repeated in the same conditions for a considerable number of times (KoT - phenomenology and application), the relative frequency of an event is considering a first reasonable valuation of the degree of reliability of the occurrence of the event itself (KoT - definition, properties and foundation). With “relative frequency” is meant the ratio between the number of occurrences of an event (meaning the number of times that an event happens, that can be "how many times the sum 6 came out of the dice") and the number of accomplished proves (that can be “the number of times we roll the dice”). By the increasing of the number of experiments or accomplished proves (potentially for an infinite number of times), the value of the frequency tends to the theoretical value of probability, that is the value given to the classical probability (law of large numbers- KoT - definition, properties and foundation).

The construction of the probability theory from a subjectivist point of view has emerged from the intention to give to the probability’s meaning a psychological basis. In particular, the probability of an event is a quantitative measure of the degree of confidence based on the judgment that events occur (KoT - definition, properties and foundation). In this perspective, what really matters isn’t the concept *"what I foresee, will happen, because I foresaw it*” but, instead we should focus on the question “*why do I foresee that this event will happen?"* (de Finetti, 1931). The answers to the query “why different degrees of confidence can be attributed to different events” are various: reasoning based on sensations, on statistical analysis or on assessments that rely on the combinatorial calculation. Based on the subjectivist approach, the definition of probability P of an event E can be given supposing that a bookmaker is obliged to accept bets on a certain number of events, including the event E (KoT - phenomenology and application).

The bookmaker (B) has the authority to decide the price p that a bettor (G) has to pay to bet on some events at stake and to collect the amount 1 if the event E verifies. G has the authority of deciding to pay the price related to the chosen event in order to collect 1 or to pay S times that price ($p \times S$) to collect, in case of victory, S times that amount 1 ($S \times 1$).

If G decides to bet on the event E, B collects the price $p \times S$ from G who, if the event E is verifies, collects S. Let's indicate this situation as *pro E bet*. In de Finetti's frame another possibility is provided: if G decides to bet against the verifying of E, B will be obliged to pay $p \times S$ to G if the event E doesn't verify (G-wins), or to collect $(1 - \text{p})\text{S}$ if the event E verifies (G loses). In this way, G is forcing B to play money on an event using the prices set by B. Let's indicate this situation as *cons E bet*.

Regardless of the opportunity to build the definition of subjective probability, however, it seems useful to take into account that, according to this approach, the probability is *"the mathematical theory that teaches to be coherent"* (de Finetti, 1931). In a betting game contest, probability is the measure of the "degree of confidence values" to be attributed to events in order to "not enable competitors to win with certainty" (de Finetti, 1931).

Context and Method

The research presented in this paper is part of a PhD project that the first author is developing with the focus on mathematical teachers' specialised knowledge in the field of probability. For this project of research, a PD was designed and implemented and it involves primary school teachers and lower secondary school teachers together with their pupils. The analysed PD, which took place last year for a total of 12 meetings; it involved 8 teachers, 4 of primary school and 4 of lower secondary school. The teachers of the PD have a very different educational background and so their background in stochastics is very heterogeneous. The second part of the PD was interrupted due to the COVID-19 pandemic. The purpose of the PD was to make explicit the specialised mathematical knowledge around the deepening of the probability construct, starting from the de Finetti (1931) vision about subjectivist approach.

An initial part of the meetings was used to develop and expand the teachers' KoT. For this purpose, from the beginning teachers were put in a situation of game immersion. The game consisted in betting on the sum of the two faces outcoming in the six-sided dice's roll (both white) and in establishing the quotas to be allocated to each possible sortie (sums from 2 to 12, including). The quota is commonly understood as the multiplication factor that is applied to the player's bet to determine the amount that the same player will be entitled to collect if the bet event E occurs. The challenge with the teachers was to establish quotas so as to be coherent to de Finetti's vision (1931) in order to *"not enable competitors to win with certainty"* (de Finetti, 1931).

The setting of the classroom has been arranged in order to have two gaming stations using the game board depicted in Figure 2 that was designed and built for this PD. Some teachers were given the role of bookmaker (those who set quotas), while the other teachers were given the role of bettor. Taking turns, each teacher played both roles in the game. During the betting sessions, the teachers had the opportunity to identify elements in the dynamics of the game and so to rethink the criteria with which they chose the first quotas and, possibly, to modify them.

BET MONEY											
SUM OF THE VALUE OF TWO SIX-SIDES DICE'S FACE	2	3	4	5	6	7	8	9	10	11	12
QUOTES											
POTENTIAL WINNINGS											

Figure 2: Game board to bet

The purpose of KoT phase's meetings, was to study the dynamics, created in the group of teachers, to analyse the idea of probability that an adult has and how this idea of probability is managed and used. All the discussions were recorded and transcribed. For this study, we propose the analyses of some excerpts of the discussion that took place during the fourth meeting. In order to comment and discuss the choices of bookmakers and bettors, in this specific meeting teachers were involved in a simulated game session, that provided the possibility, for the first time, to do *pro E bet* but also *cons E bet*.

For privacy reasons we will use pseudonymous for the teachers' names.

Analysis and discussion

After remembering the *pro E bet* modes and explaining the dynamics of the *cons E bet*, each teacher took some time to set the quotas for each event and to share them with the group.

In this paper are reported, in Figure 3, the productions of Alba (in red), of Giorgio (in pink) and of Mirco (in violet) – in the order in which the quotas are shown in the figure, respecting chronologically the sharing's order of the teachers with the working group.

SUM OF THE VALUE OF TWO SIX-SIDED DICE'S FACE	2	3	4	5	6	7	8	9	10	11	12
ALBA'S QUOTES	10	9	8	6	5	5	6	5	7	8	9
GIORGIO'S QUOTES	4	5	6	8	8	8	8	8	6	5	4
MIRKO'S QUOTES	36	18	12	9	7,2	6	7,2	9	12	18	36

Figure 3: Representation of teachers' quotes

Mirco fixed the quotas symmetrically, using the inverse of the classical probability measure. The quotas set by Alba and Giorgio are very different: Alba chose to assign higher quotas to less probable events (this choice entails an higher profit in a less probable event) and lower quotas to more probable events (KoT - phenomenology and application). Giorgio wrote the quotas using a contrary criterion. But during the simulation of the game and the related discussion, Giorgio realised that his quotas earned less money to the bettor who risked more and vice versa (KoT - phenomenology and application), so at the very first part of the discussion he said:

Giorgio: My reasoning upstream was like Alba's one, but I was wrong writing the numbers.

Alba: If a person thinks about the law of large numbers and calculates all the ratio between favorable cases and possible cases, it should be a Gaussian that peaks on the “sum 7”. Because “sum 7” has more combinations, so quotas need to be symmetrical around this point (referring to “sum 7”). If I wanted to give “symmetrical quotas”, I would really choose quota five to “sum 7”, quota six to “sum 6”, quota seven to “sum 5” ...

Alba, more qualitatively than quantitatively, employed a criterion which is consistent with her classical probability’s knowledge (KoT - definition, phenomenology and application).

The meeting led to the need to create an “ideal” and “unrealistic” simulation game, in which it is possible to roll the dice a very big number of times, and so, to rely on the law of large numbers. Ciro is the educator who mediated the discussion:

Alba: Let’s say I have to play 36 times; I play 36 times.
Ciro: A million times.
Alba: Let’s imagine rolling the dice 36 million times. Let’s consider 36 rolls: I’m extrapolating the 36 rolls from the case of the million. So, out of 36 rolls: the “sum 2” comes out one time, the “sum 12” comes out one time, the “sum 7” comes out six times. Now, what should I do? In 36 rolls I decide that I bet all 36 bets only on one sum. Let’s do the math!

Alba referred to the frequentist approach of probability (KoT - phenomenology and application) to create the "ideal" and "unrealistic" situation described above. Moreover, Alba quantified the “big number of times” with the number "36 million" and normalised this number to “36 rolls” using, in a non-explicit way, the combinatorial calculation with which it is possible to unpack 36 total possible combinations by summing the faces of two six-sided dice (like in Figure 1). The knowledge about the frequentist definition of probability allowed Alba to use the classical definition of probability to predict what would happen with frequencies (KoT - intra-conceptual connection), since supposing an infinite number of throws the relative frequency tends to the classical value of probability.

Paola: If the bookmaker, in this situation, wants to ensure a certain gain, he must choose quotas lower than the probability value. In this sense, if “sum 2” has 1probability out of 36, the quota must not be 36, but must be lower.
Alba: Quota 35 would be enough to make him earn.
Paola: Yes, the bettor would lose a coin.

In this example, in the situation previously normalised by Alba, for 36 rolls the bettor plays *pro “sum 2”* using each time 1 coin and paying 36 coins in total. Ideally, the "sum 2" event occurs once out of 36 launches and the bettor wins 35 coins. He pays 36 coins, winning only 35. The bookmaker earns 1 coin.

Giorgio: But are we talking about the previous game where you couldn’t bet *cons E*?
Ciro: Why are you saying that?
Giorgio: Because in a situation like this, the first thing I would do is pointing *cons “sum 2”*.

Giorgio remembered that there was the possibility to play *cons E* (KoT - phenomenology and application). With this move, the bettor would reverse the results of the "ideal" bet. If the bettor bets 36 times *cons "sum 2"*, at the end of the 36 rolls, he wins.

Ciro: Using what Giorgio said: so, is the bookmaker at risk? How does the bookmaker respond to this move?
Paola: So, I have to put higher quotas.
But the bettor can always bet in both ways.

At first Paola, not explicitly, seemed that she had changed idea radically: previously she proposed to use “quotas lower than the probability value”, then she was proposing to use quotas higher than the probability value. At the same time, she verbalised this "idea’s change", Paola realised that, even doing so, the bettor could overturn the situation, once again, betting *pro E* for 36 times.

Mirco:	And so, this situation is impossible to be solved.
Ciro:	If you use lower quotas?
Alba:	You lose.
Ciro:	If you use higher quotas?
Mirco:	You have to use equal quotas.
Ciro:	Equal to what?
Paola:	To 36 (referring to the “sum 2”).
Ciro:	Yes, quotes equal to the inverse of the probability value.

It is significant that Mirco said that it wasn’t possible to solve that situation: he was the one who - we remember it - from the beginning used the inverse of the probability value to choose his quotas (Figure 3). In this discussion we could appreciate the collective knowledge development in constructing a criterion for establishing quotas (KoT - phenomenology and application) in order to ensure that none of the competitors would win with certainty (de Finetti, 1931).

Conclusion

The goal of this research is twofold: on one hand we tried to develop an innovative educational approach to the topic of probability; on the other one we attempted to identify useful elements for experiences of teachers’ professional development. In particular, taking inspiration from the subjectivist approach to probability (de Finetti, 1931), we designed a particular PD for primary and lower secondary teachers involving them in a betting game context in which they had to quantify the degree of confidence associated to each possible event of the game. The analysis of the excerpt of the discussion showed how the refinement process of the degree of confidence associated to each possible event by the teachers involved both qualitative and quantitative aspects: qualitative, because teachers gave meaning to their knowledge; quantitative, because they used, especially at the end of the process, their knowledge of the classical and frequentist probability definition as a tool to make conscious choices. The actual mathematical systematisation of the subjectivist approach to probability, and in particular the results proving its consistency with the other two approaches to probability, is a fascinating piece of Mathematics and it entails many mathematical formal and complicated structures and techniques. With this study we are not proposing that the subjectivist definition of probability should be explicitly taught during PDs for teachers in service and students at school, but we are exploring its potentiality as educational innovative paths for teachers and students – to avoid proposing not trivial probability activities. In this episode emerges that a betting game like that described above (which follows an idea by de Finetti’s) actually generates a dynamic that leads to the coherent construction of a probability measure based on the choices of the players. This provides a strategy for teachers to design educational paths where a (classical or frequentist) definition of probability is not given a priori, but it is established starting from the degree of confidence expressed by learners. In this case, the probability measure can be given as the reciprocal of the quote. In this perspective, in the second part of the PD we fostered and challenged teachers to design didactical paths of probability for their students, taking inspiration from the experience lived during the PD and the knowledge that it developed. In further studies we will address also these parts of the research.

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Analysis of standardized tests and pre-service teacher education: reflections on developed teachers' specialized knowledge

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In this paper we describe a pre-service teacher professional development program based on Meta-Didactical Transposition model. We focus on the encounter between the praxeologies of researchers/teacher educators and pre-service primary teachers. We show how specific tasks for teachers and information provided by a standardized assessment database (GESTINV) can be used to establish a link between Italian National Evaluation System and pre-service teacher professional development. We present some aspects of mathematics specialized knowledge that pre-service teachers declare to be improved during the educational program. Pre-service teachers' reflections on their specialized knowledge are framed into MTSK model.

Keywords: Teacher education, pre- service primary teachers, teacher knowledge, MTSK model, standardized tests.

Introduction

The use of standardized tests in mathematics for both international surveys (such as PISA and TIMMS) and national evaluations of school systems is widespread, and the results of these tests are also often disseminated in the press and commented on by policymakers and researchers (Doig, 2006). In order to not confine these data to the ranking of students, schools and nations it is essential to develop the dialogue between standardized assessment and educational research (De Lange, 2007). The Standardized Assessment data can also become effective and usable tools for improving teaching and learning processes. Our study fits in this stream of thought; in particular, we will show educational potentials of Italian Standardized Assessment by using theoretical tools to interpret the quantitative data they provide and the macro phenomena that emerge from the complexity of educational systems (Bolondi et al., 2019;, Ferretti, & Bolondi, 2019). Quantitative data should be interpreted and intertwined with a qualitative analysis of tasks. This is possible by means of an encounter between Standardized Assessment, mathematics education research and teacher education. In this paper we present an example of pre-service teachers' professional development that profits from data collected by the Italian National Evaluation Institute for the School System (INVALSI). We will show how to create a link between the Italian National Evaluation System and pre-service teacher professional development programs in order to improve mathematics teaching school practices. In our study we argue how the analysis of standardized assessment results should impact the improvement of the teaching and learning of mathematics. We have been working for many years in this perspective and we have carried out educational projects using data from INVALSI tests as an object of reflection for teachers (Martignone, 2017; Ferretti et al., 2018; Ferretti, et al., 2020, Santi et al., in press). It must be stressed, however, that this use of INVALSI tests is favored by the fact that in Italy the framework of INVALSI tests is closely linked to the National Guidelines and it takes into account the research in mathematics education. In this paper we describe some teacher education activities carried out during a pre-service primary teacher professional development course. During a university course teacher educators/researchers shared with pre-service teachers some theoretical tools in order to conduct an a priori qualitative analysis of tasks from INVALSI tests. As final reflections, after the introduction of the Mathematics Teacher Specialized Knowledge model - MTSK Model (Carrillo-Yañez et al., 2018), the educators/researchers asked pre-service teachers about which aspects of mathematics specialized knowledge the course activities brought out and developed

Theoretical framework

Mathematical knowledge lives in the institutional dimension where mathematical objects emerge from socio-cultural activities shared by individuals belonging to one or more institutions. The development of mathematics, teaching and learning of mathematics and mathematics teachers' education are characterized by the dialectics between the personal and institutional relation to knowledge (Chevallard, 1985). The Anthropological Theory of Didactics (Chevallard, 1985) conceives human activity as a praxeology, which is made up of a set of tasks that drive the practice (praxis), the techniques that allow individuals to solve the problems, and the knowledge and discourses (logos) that ground the techniques. To describe and interpret the practices of mathematics educators/researchers and those of teachers who are engaged in teachers' education activities, in our study we use theoretical lenses from Meta-didactical Transposition Model (Aldon et al., 2013; Martignone, 2015). The teacher professional development program described in this paper can be seen as an instantiation of the Meta-Didactical Transposition model. In regard to the institutional dimension, the teacher education process involves the Italian Ministry of Education via the INVALSI institute, Italian Schools and Universities. Researchers/teacher educators and teachers share didactical praxeologies and reflect on them. Researchers bring to the fore tasks and techniques related to the epistemology of mathematics and mathematics education studies. As we will show in the next paragraph, the shared praxeology is linked to the analysis of mathematical and pedagogical issues in INVALSI tasks about rational numbers, but not just this. Theoretical lenses for reflecting on mathematics teacher specialised knowledge are shared and discussed. In specific, the MTSK model (Carrillo-Yañez et al., 2018) was introduced. MTSK model distinguishes between mathematical knowledge (MK) and pedagogical content knowledge (PCK), both of which are considered as sub-domains of the teacher's specialized knowledge. As concerns the MK, are part of the Knowledge of Topic (KoT) the knowledge of definitions, properties, procedures, representations, etc. The knowledge about how to connect activities in different domains of mathematics is part of the Knowledge of the Structure of Mathematics (KSM). As knowledge of Practices in Mathematics (KPM), we can identify, for example, the knowledge of how to prove, justify, define, make inferences and inductions, give examples and counterexamples. Also Pedagogical Content Knowledge is divided into three sub-domains. The Knowledge of Mathematics Teaching (KMT): knowledge of theories of mathematics teaching or knowledge of teaching resources, materials and technologies, but also knowledge of strategies for introducing and representing contents and concepts, etc. The Knowledge of Features of Learning Mathematics (KFLM): knowledge of theories of mathematics learning or knowledge of the way in which pupils interact with mathematics. The Knowledge of Mathematics

Learning Standards (KMLS): knowledge of expected learning outcomes and teaching goals in different school segments. The MTSK model in addition to detailing these subdomains of Mathematical Knowledge and PCK explicitly highlights the centrality of teachers' beliefs about mathematics and mathematics teaching-learning. For this reason, in our opinion, MTSK model is suitable to be used by teachers to explore, reflect, discuss on what they think/believe about their specialized knowledge developed during the education program. Therefore, in our study we show that MTSK model can be used by pre- service teachers to reflect on their knowledge during a teacher education program. The interpretive lenses provided by MTSK model have become part of the logos of shared meta-didactical praxeologies. Moreover, the MTSK model becomes a theoretical object shared between researchers and teachers to reflect on tasks and practices. Following MDT model, in our teacher educational program the researcher/teacher educator play the role of broker (Rasmussen et al., 2009) who belongs both to the community of mathematics experts and to the community of the teacher education program. The broker facilitates the sharing of knowledge and practices from one community to the other by drawing on boundary objects (Bowker and Star, 1999). Boundary objects are meaningful tools, ideal or material, in both the communities and they put them in touch, although with different nuances and uses that characterize their respective praxeologies. Boundary objects can be material artefacts, digital technologies, mathematical procedures etc. In our teaching education programs the database of INVALSI tests, GESTINV, acts as a boundary object between communities of researchers and future teachers. GESTINV (www.gestinv.it) is a structured database containing all the data of the INVALSI standardized assessments from the 2008 surveys of all surveyed domains. In detail, it contains 2121 items from the INVALSI mathematics tests. There are more than 25,000 registered users in GESTINV and an average of 200 accesses per day. These data, together with its structured information in line with the theoretical framework of INVALSI tests, promote Gestinv as a tool to be implemented in the design of teacher education models (Ferretti et al., 2020). There are many ways in which the database can be used; different forms of research can be carried out within it (National Guidelines, scholastic year, school grade, content, percentage of correct/wrong/invalid answer). In our research, GESTINV plays an important role in providing teachers and researchers with an interactive tool to access a wide range of information and feedback concerning the processes of learning and teaching mathematics. In particular, the results of the INVALSI tests highlight and quantify relevant macro-phenomena, which can be interpreted according to the methods and results of research in mathematics education. By means of specific tasks for teachers the information provided by GESTINV can be used to establish the link between the data of the standardized assessments and pre-service teacher professional development (PTPD), assuming the role of boundary object. In this paper we show some teachers' a-posteriori reflections on specialized knowledge emerged and developed in the analysis of INVALSI tasks selected by means of GESTINV. The pre-service teachers use theoretical lenses from MTSK model to state which aspects of specialized knowledge are improved at the end of the PTPD program.

Methodology

In this paper we show some activities carried out during a pre-service primary teacher professional development university course (conducted in presence in 2019). This course lasted 40 hours and each activity consists of the following phases:

Introduction. The teacher educators address the mathematical contents selected for the educational activity from an epistemological and didactic point of view and they also present GESTINV. The teacher educators share with future teachers' possible interpretative tools and research results that can be useful in the analysis of INVALSI tasks and data. As we already stressed INVALSI tests' theoretical framework is linked to the National Guidelines and also for this reason the macro-phenomena emerging from the INVALSI tests are meaningful for the analysis carried out with future teachers.

Analysis of an example. The teacher educators analyze an example of INVALSI tasks selected by using GESTINV.

Group activity. The teacher educators ask teachers to analyze INVALSI tasks on specific topics. Teachers work in subgroups of a maximum of 4/5 people. They choose mathematical contents with reference to the Italian National Guidelines of primary school. They should identify one or more learning difficulties. The small group activity takes place in line with the characteristics of a Community of Inquiry in the sense of Jaworski (2006). The group activity aims at the construction of a multimedia product, an artefact, the design of an activity for the students, etc. that highlights the reflections, beliefs and convictions of the future teachers involved.

General discussion. The subgroups present their productions to the whole group. Each presentation is discussed in order to highlight beliefs and convictions, address doubts, difficulties and unclear contents. Then teacher educators presented and discuss with teachers the MTSK model.

At the end of the course, we administered an open-ended questionnaire to the pre-service teachers in which they were asked if and how, in their opinion, the activities carried out during the course had increased their knowledge by referring to each sub-domain of the MTSK. Specifically, the teachers answered this question: "Using the MTSK model, explain which topics covered during the course, which tasks and activities contributed, in your opinion, to the development of your specialist knowledge". The teachers knew that answers would not have been assessed. Data relating to 52 questionnaires administered to the pre-service teachers a have been collected and analyzed. The gathered data were analyzed by an inductive content analysis (Patton, 2002); by using the MTSK theoretical lens, a top-down were performed.

INVALSI tasks analysed by pre-service teachers

In this section, we describe an example which is part of a broad prospective primary teacher academic course, in which different contents were involved and macro-phenomena and teaching practices were analyzed and framed with different constructs of mathematics education. The tasks selected, according to cross-research allowed by GESTINV according to categories described above, belong to INVALSI standardized tests administered to all Italian grade 5 students. Figure 1 shows the texts of the INVALSI tasks and the national percentages of correct, wrong and invalid answers.

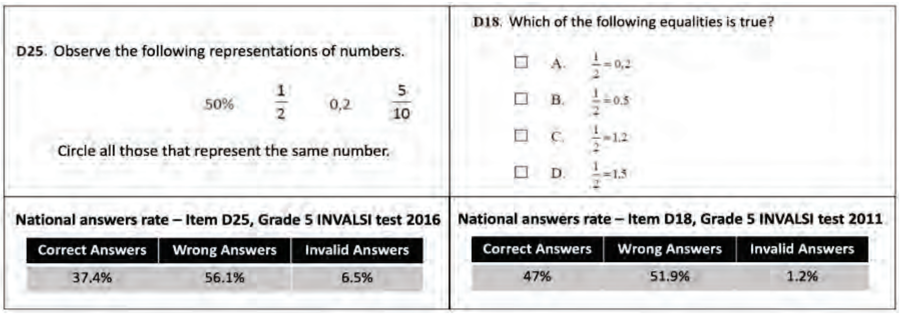

Figure 1: INVALSI items and national rates (www.gestinv.it)

As we can see in the Figure 1, in both tasks less than half of the students nationwide gave the correct answer. INVALSI provides percentages for each option chosen by the students and these data are available in the GESTINV database. These tasks were identified by teachers through the GESTINV database, using the "Guided Search" function by searching for tasks from the grade 5 INVALSI tests concerning rational numbers and with correct response rates below 50%. Among the items returned by the database, the items D25 and D18 were selected because they clearly show some possible errors related to the representations of rational numbers. By means of the qualitative analysis of the tasks and the quantitative results, researchers/TE and pre-service teachers can share ideas and reflections. Starting from the analysis of these data researchers/teacher educators and pre-service teachers discuss about students' difficulties in conceptualizing and making sense of decimal and fractional notations (some of the main findings of research in the field were deepened, e.g. Iuculano & Butterworth, 2011, Ni & Zhou, 2005). As a matter of fact, they notice that quantitative results show that many Italian students gave the wrong answers, in particular in D18 we can notice that 33.5% of the students chose option C. The teacher educators/researchers address the mathematical content selected for the task from an epistemological and educational point of view. Furthermore, they discuss with pre-service teachers the main research findings of the research that can help frame the macro-phenomena that emerged (Ball, 1993; Empson & Levi, 2011). As highlighted in literature, often, many difficulties are related to the management of the different representations of rational numbers, both in terms of the comparison and ordering of fractions and their comparison with decimals. They agreed that, in order to answer both tasks correctly, it is necessary to properly perform treatments between different semiotic representations (Duval, 2006).

Pre-service teachers' reflections

In this paragraph we show some excerpts from per-service teachers' answers to the questionnaire administered at the end of the teacher educational program. Pre-service teachers state that the analysis of the INVALSI tasks was useful in strengthening their knowledge about the process of teaching and learning rational numbers. They stress the importance of discussions about students' learning processes, possible students' difficulties and effective/ineffective mathematics teaching practices. By using the interpretative lenses given by MTSK model, they declare an increase in their Mathematics Teacher Specialized Knowledge, in particular in their Pedagogical Content Knowledge (PCK) declined into three subdomains: Knowledge of Mathematics Teaching (KMT), Knowledge of Features of Learning Mathematics (KFLM) and Knowledge of Mathematics Learning Standards (KMLS).

For reasons of brevity, we will only provide a few examples of statements for each PCK sub-domain.

As far as KMT is concerned, some pre-service teachers declared:

PT_17: As far as fractions are concerned, […] I have become aware that they need to be taught from the beginning to represent the same concept in different semiotic representations.

PT_18: Knowing the different representations of a mathematical concept means recognizing its connections and underlying structure. In this case, we can see how each number is represented differently by a percentage, a fraction or a decimal number. Through the analysis of this item we can see the strong connection between percentages, fractions and decimal numbers, which every teacher needs to know very well in order to foster the most effective teaching-learning process possible from a mathematical point of view.

PT_21: I have therefore understood that the teacher's task will be to offer children different representations of the same mathematical object and will have to make them aware of this knowledge. […]

Below we provide excerpts referring to KFLM:

PT_17: Thanks to this activity, I became aware that for students one representation is not the same as another.

PT_39: I have seen that students have difficulties in both processing and converting, leading to misconceptions about fractions.

Finally, here are some pre-service excerpts referring to their KMLS:

PT_21: As stated in the National Guidelines, children at the end of primary school will need to be able to handle all the different representations and so it is up to the teacher to structure pathways and activities to help students achieve this understanding.

PT_23: With this topic, I got to know the standards of knowledge and skills possessed by the pupils, as well as what learning objectives are required at the end of primary and secondary school. This activity has also underlined that it is necessary to respect the design of the objectives as much as possible, defined in such a way that does to create inconsistencies in later grades.

As we can see in these excerpts, pre-service teachers reflect on their PCK, in terms of KMT, KFLM and KMLS, and recognize aspects of their knowledge that have emerged and developed.

Conclusions and further directions

In this paper we presented an example of a teachers' professional development program for primary pre-service teachers in which meta-didactic praxeologies, linked to the analysis of selected items from the national standardized assessment INVALSI test, were shared. We have shown how the use of GESTINV database, as a boundary object in professional development paths, allows an improvement of pre-service mathematics teacher specialized knowledge. As a concluding activity of the program the pre-service teachers were asked, using the MTSK model, to identify what specialized knowledge they think/believe they had developed during the professional development program. The MTSK model was used by pre-service teachers to reflect on teaching and learning processes. MTSK model become part of logos component of meta-didactical praxeology and it could also be considered a theoretical boundary object between the two communities of researchers/educators and pre-service teachers. This aspect as well as the analysis of the pre-service teachers' declarations concerning the MK domain will be objects of our next studies. As literature highlights (i.e. Doig, 2006), standardised assessments, designed with the aim of assessing mathematical learning at the system level, are increasingly having implications from an educational, didactic, cultural-historical and political perspective at both global and local levels. Using data from standardized assessments in pre-service teachers program fits into this line of research and it increases the potential of the educational implications of linking standardized assessments to mathematics education.

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Exploring Indonesian prospective teachers’ teaching belief and teaching practice

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This study aims to report preliminary findings on prospective teachers’ teaching beliefs and teaching styles when implementing their proposed problems. Three second-year Indonesian prospective teachers who were accustomed to the teacher-centered teaching tradition were involved in a brief collaborative workshop. In the workshop, they get an advantageous opportunity to broaden their perspective about mathematics teaching through sequential activities, such as formulating problems collaboratively, teaching practice, reflecting on their teaching practice, and providing suggestions to each other. After the workshop, they expressed their belief in the ideal problem-solving teaching style and implemented their proposed problem.

Keywords: Teaching belief, teaching practice, prospective teacher.

Introduction

In the early 1990s, the usual mathematics teaching technique in Indonesia was to directly explain the solution of mathematical problems (Kuipers, 2011). Subsequently, the most recent curriculum has been introduced as an attempt to upgrade educational traditions with the cornerstone: place problem-solving as a challenging skill to master and position students at the center of learning to collaborate and share ideas. Nevertheless, some challenges remain in practice, such as the persistence of conventional teaching traditions along with belief that an effective teaching style is to manage the class completely. Thus, when students from this environment become prospective teachers, it is not surprising that this belief naturally exists in their minds.

To support the new curriculum and address the existing belief for better teaching quality, a collaborative workshop was held with two principal activities: problem-posing which cannot be considered part of the conventional teaching tradition and implementing the posed problems in the classroom, which is directed towards a problem-oriented approach. Involving prospective teachers in problem-posing activities and asking them to implement their problems, accompanied by discussions, can be fruitful endeavors to enrich their perspective to mathematics teaching. Implementing their own problems appertains to the classroom activity in which prospective teachers will face in the future. Providing opportunities for prospective teachers to practice teaching from the early years of the training while most teacher preparation programs put it off towards the final year of the program constitutes a complementary point of this study.

This study focuses on the second principal activity and reports preliminary findings through the research questions: (1) What is the prospective teachers’ belief about the ideal problem-solving teaching after the workshop? (2) What is the teaching style of prospective teachers when implementing their proposed problem before and after the workshop? As supported by Ellerton

(2013), disclosing how prospective teachers implement their problems after experiencing problem-posing activities will contribute to the literature.

Theoretical background

When it comes to teaching problem-solving, the diversity of beliefs emerges as an influential constituent of teaching styles. Some believe that teaching should be straightforward, focusing on clear explanations to pupils (Richardson, 1996). This belief seems to have developed due to familiarity with traditional teaching practices that favor teacher-centered learning. On the other hand, some believe that teaching should actively involve students to spur their thinking skills in which can be considered as student-centered learning (Kofa, 2018).

Nowadays, there has been increasing awareness in applying student-centered learning due to its prosperous benefits. One approach to student-centered learning is problem-oriented instruction. According to Kovács and Kónya (2019), problem-oriented instruction has three characteristics: (1) students analyze the situation of a mathematical problem, (2) students critically adapt to their own and their classmates' thinking, and (3) students learn to explain and justify their thinking. Considering one feature of problem-oriented instruction that the classroom activity evolves into a critical discussion of the problem being addressed, the implementation of problem-oriented instruction can be recognized by studying talk formats. Critical discussions are triggered by specific forms of talk that support a deep understanding of concepts and robust reasoning. Sohmer et al. (2009) characterized the talk format into four as shown in Table 1.

Table 1: Talk format

Talk format	Description
Recitation	The teacher completely controls the content and direction of the conversation by presuming special right to ask the questions and evaluate students' answers. Students are positioned as seekers for the correct answers that the teacher is looking for.
Stop-and-talk (Partner talk)	The teacher gives a pointed question to the students and asks them to discuss it with one or more partners. Students are positioned as active reflectors and contributors. During small group discussions, the teacher selects key voices among students to be heard by the entire community.
Student presentation and group critique	The teacher asks the student to present his/her work in front of the class accompanied by follow-up questions proposed by the other students or teacher. The presenter student is positioned as the expert of their work.
Whole-group 'position-driven' discussion	The teacher leads a discussion on a single problem or question which has more than one answer, so that reasonable arguments appear from the students. This kind of discussion promotes active participation of the students by proposing an idea and listening to each other even before being fully competent in the discussed domain.

Supportive talk formats for problem-oriented instruction, such as partner talk, student presentation and group critique, and whole-group position-driven discussion, require appropriate teachers' behavior. Rott (2019) classified teachers' behavior when implementing problem-solving into three. In more detail, he stipulated the behavior at each problem-solving step by Polya (1945), as shown in Table 2. The classification refers to the differentiation between teachers as controllers or facilitators. The indicators show that the closely managed style represents a teacher-centered learning in which the teacher acts as a controller, while the other two styles are closer to the teachers' role as a facilitator which aims to generate mathematically rich and meaningful discussions in the classroom.

Table 2: Teachers' behavior in each problem-solving step

	Closely managed	Neutral	Emphasizing strategies
Understanding the problem	The teacher explains the problem formulation.	The teacher does not comment on the problems and does not answer students' questions.	The teacher gives hints but does not explain the problem.
Devising a plan	The teacher tells the students which approach is correct to use.	The teacher does not give any guidelines and strategic support.	The teacher hints at (ideally) different approaches and encourages the students to follow their ideas.
Carrying out the plan	The teacher gives the students concrete content-related support (often early in the process).	The teacher gives (almost) no (strategic) help and does not answer students' questions related to the problem.	The teacher gives staggered aids (motivational/feedback/general strategic/task-specific strategic/content-related).
Looking back	The teacher fixates on results; perhaps, one (arithmetic) approach is presented.	Different approaches are presented; however, strategic ideas or the differences between approaches are not highlighted explicitly.	The teacher highlights approaches and strategies; results might also be presented, but it is of secondary importance.

By studying talk formats and teaching styles, we will find the connection between those two. For instance, recitation goes hand in hand with a closely managed style, while the other talk formats assist neutral or emphasizing strategies styles. In other words, the teaching style could be detected from the tendency to use a particular talk format, whether it leads to a productive talk or not.

Both talk formats and teaching styles during the lesson might indicate tendencies in beliefs about teaching. As Rott (2019) emphasized, teachers' belief is a component of professional teacher competence that influences their teaching behavior. Moreover, analyzing talk format and teaching style will cue whether the lesson appertains as problem-oriented instruction or not. Figure 1 illustrates the relation between talk formats, teaching styles, and problem-oriented instruction.

Another effort to improve mathematics teaching is to examine teaching practices as has been done in Japan. Through lesson study, Japanese teachers discuss lessons they have planned and observed together which then direct them to look for ways to improve it and broaden their knowledge of the teaching profession (Fernandez & Yoshida, 2004). Thus, it is precious to adapt.

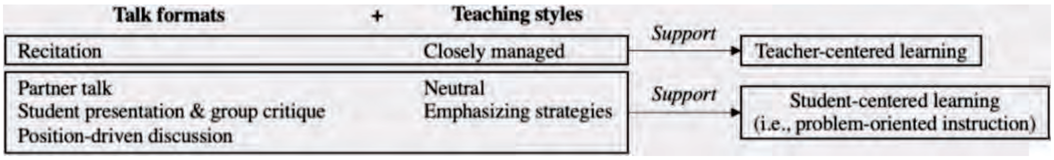

Figure 1: The relation between talk formats, teaching styles, and problem-oriented instruction

Method

The participants of this study were three second-year Indonesian prospective teachers (Vey, Ann, Kay – pseudonym) who enrolled in a 4-year mathematics teacher training program for grades 7-12 in a private university. To select the participants, 22 prospective teachers were offered to participate in the workshop and implement their proposed problem voluntarily. Prospective teachers who had a chance to implement their proposed problems with real pupils, such as those who handled a private course, a non-formal additional class outside of school which aims to strengthen the lesson provided at school, occupied the priority positions to be selected. The number of participants was limited based on several considerations: to get them more actively involved in the workshop and to make the activity during the workshop better observed since the workshop was held online due to the ongoing COVID-19 pandemic. The cornerstone of the workshop is adapting Japanese lesson study by involving prospective teachers. Figure 2 shows the problem-posing backgrounds and the implemented problems by Ann as one of the participants.

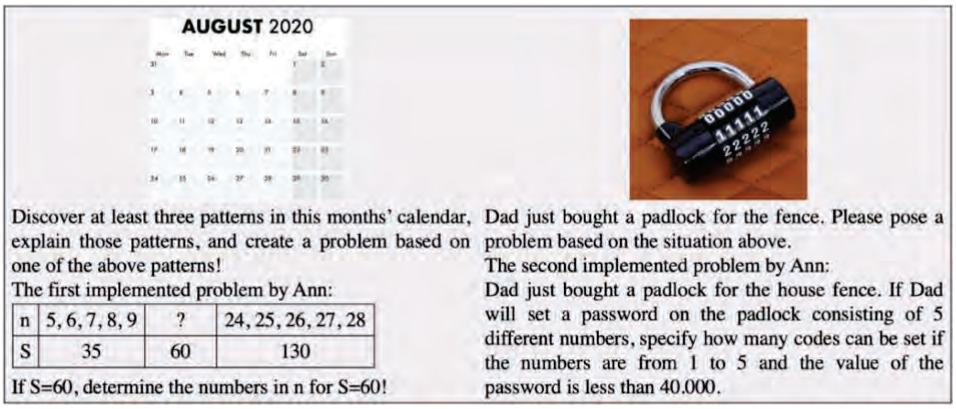

Figure 2: The problem-posing backgrounds and the example of implemented problems

Table 3: Research activities

Pre-test	Posing problem autonomously related to patterns on calendar and implementing the proposed problem to pupils.
1st meeting (50 minutes)	Teaching reflection, giving feedback to each other, and discussing how the situation in mathematics classes should be.
2nd meeting (100 minutes)	Discussing several paradigms of problem-posing: free, semi-structured, and structured problem-posing.

3rd meeting (100 minutes)	Analyzing a less feasible problem, fixing it collaboratively, and posing problem collaboratively by modifying the initial situation.
4th meeting (100 minutes)	Posing problem collaboratively by using a what-if-not strategy.
5th meeting (100 minutes)	Posing problem autonomously based on the given situation and fixing the proposed problems collaboratively.
Post-test	Posing problem autonomously related to the padlock and implementing the proposed problem to pupils.
After post-test	Filling out a questionnaire and teaching reflection.

The questionnaire was directed to capture the prospective teachers' viewpoint on ideal problem-solving teaching implementation. Among three styles, they were asked to choose which one is their ideal teaching style. Since this study is part of a larger research project, only two data are presented: data from the questionnaire after the workshop and data from the videos of teaching implementation at the beginning and at the end of the workshop. The teaching implementation data consists of 24 sections, 8 sections for each prospective teacher for the first and the second implementations. It was analyzed in terms of behavior by Rott (2019) and talk format by Sohmer et al. (2009). All teaching practice data were coded independently by two raters which resulted in 23 of the 24 cases (96%) being agreed. While the divergent code was recoded consensually thereafter.

Research findings and discussion

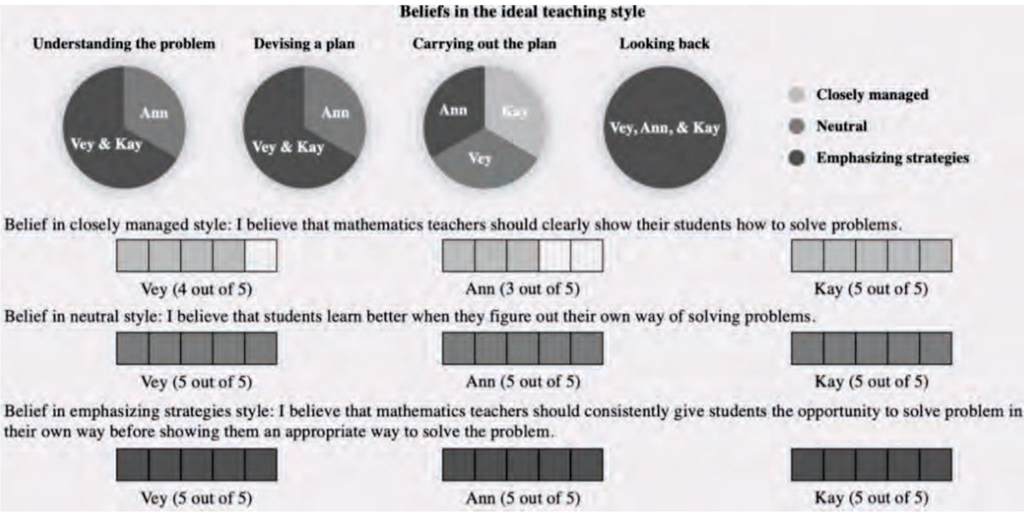

Figure 3: The results of the questionnaire

Based on pie charts in Figure 3 that show prospective teachers' beliefs about the ideal problem-solving teaching, a closely managed style was not the primary choice. It only appeared in Kay who believed in the necessity to control the class when carrying out the plan. Those findings were reinforced by the result of the second part of the questionnaire. All prospective teachers gave the maximum point to neutral and emphasizing strategies styles that signify student-centered learning.

The closely managed style became a secondary preference for them except for Kay, who did not indicate her teaching belief tendency (see Figure 3).

Figure 4 presents the prospective teachers' teaching styles and beliefs about problem-solving teaching. Considering the research of Ellerton (2013), which revealed the impact of problem-posing activities on prospective teachers while they were taking courses in teacher preparation programs and the impossibility to follow those prospective teachers into their classrooms once they became teachers, data in this study regarding the prospective teacher's teaching practice in implementing the problem they posed might be an alternative response to the condition.

In the first teaching implementation, Vey skipped the looking back step that appeared to be deemed unnecessary. As she recognized this step was consistently carried out in the lesson throughout the workshop, she stated, "Usually, if we arrive at the solution, that's all. We don't think if it's reasonable or not. Now that I think about is there should be follow-up activity like we did in this lesson". Realizing the importance of looking back step, she undertook the step in her second teaching implementation. Her teaching styles in the first and the second implementations were neutral and closely managed over the classwork activities. Particularly, in the second teaching implementation she thought that the fundamental counting rule, which is the mathematical background of the implemented problem, had been taught in school but evidently it was an unfamiliar topic for her pupils. There was a school lesson delay due to the covid-19 pandemic. This emerged as her consideration for performing the closely managed style. The only noticeable difference between the first and second implementations was the use of the talk format. In the first teaching implementation, recitation was applied in all problem-solving steps, while in the second teaching implementation, she applied position-driven discussion in the looking back step among recitations in the other steps.

Ann had been involving her pupils since the first teaching implementation. She received positive reinforcement for her behavior from her peers during a reflection and discussion meeting. They said, "I like how she communicates with her students" and "The class is active". In the second implementation, she maintained her behavior and even improved it. She obviously encouraged her pupils to express their idea by asking them to tell the strategy they used and explain their reasons. The closely managed style had been sidelined in favor of being neutral and emphasizing strategies. The combination of position-driven discussion and recitation remained her preferred talk format. She also organized student activities in which students worked on the task individually, rather than just implementing classroom work activities.

In the first teaching implementation, Kay controlled the class completely. She explained how to solve the problem without allowing her pupils to speak and only asked if they understood or not which led them to say they understood but probably not. The closely managed style with full recitation dominated her demeanor aside from trying to be neutral in the step of understanding the problem. At the discussion and reflection meeting, she received feedback from her peers. They said, "It looks like your implementation must be in accordance with your plan. You seem unfree and constrained while teaching" and "You must be confident. We are learning how to teach together. Keep your spirit up". The feedback seemed to spur her on to make improvements. Although the closely managed style continued to dominate in the second teaching implementation, she tried to be neutral in understanding the problem and emphasized strategies in the looking back step. By involving pupils, she attempted to highlight several strategies to solve the problem that did not appear in the pre-test. Both the strategy and the result proposed by her pupils were considered. She let them choose their own preferred approaches. The entire lesson was classwork activity in both implementations, but she combined position-driven discussion and recitation in the second implementation.

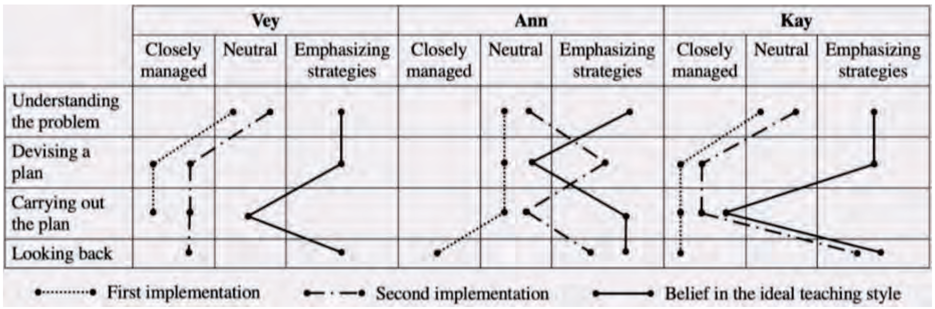

Figure 4: Teaching style and belief

The finding contributes to the literature on whether there is a link between belief and practice or not. This contradictory insight appears in Safrudiannur and Rott (2019). In this study, prospective teachers' beliefs and practices are not always in line. Among those three prospective teachers, only Ann holds a closer attachment between her belief and practice, either neutral or emphasizing strategies that belong to the student-centered paradigm. Meanwhile, belief in student-centered learning accompanied by teacher-centered implementation was found in other prospective teachers. The discrepancy between belief and practice brings out consideration of another aspect, i.e., their pedagogical knowledge since it directs them to a firm understanding on classroom management, lesson planning, and student assessment (Koehler et al., 2009).

The finding also reveals how adapted Japanese lesson study goes among prospective teachers. Discussions, sharing ideas, and giving suggestions to each other during the workshop provide them empirical experience about student-centered paradigm, as they were positioned not as receptive but active members in the class. They can reflect on the activity, make it as a shoot for their teaching belief, which then direct them to increasingly involve their pupils in the second implementation. As stated by Ambrose (2004), beliefs may lead to behavior in ways that could be depicted as habits. Moreover, their beliefs and the second teaching implementation which are more towards student-centered paradigm might be the impact of reflection and feedback activities, since reflection leads to finding new solutions and paths in teaching to improve learning (Šarić & Šteh, 2017) and helps teachers to become more successful (Lee, 2005). Thus, the whole activity during the workshop appears to be a fruitful endeavor.

Limitation, further research, and acknowledgement

This study only involves three prospective teachers in a short action. The contribution is part of ongoing research involving prospective teachers on the role of problem-posing and problem-oriented teaching in active mathematics learning. The workshop is conducted as an attempt to broaden their perspective about mathematics teaching. The author is a member of MTA-ELKH-ELTE Research Group in Mathematics Education, and this study is funded by the Hungarian Academy of Sciences through its Scientific Foundations of Education Research Program.

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Developing Preservice Teachers’ TPACK Through a Virtual Number Talks Field Experience: A Case Study

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Field experiences are an important part of preservice teacher education. However, interruptions caused by the COVID-19 pandemic and the sudden shift to virtual and hybrid learning presented challenges for implementing classroom-based field experience opportunities. In response to this challenge, the authors developed and implemented a virtual field experience based on the Number Talks routine in our elementary mathematics methods courses. We utilize case studies to explore the ***t****echnological,* ***p****edagogical, and* ***c****ontent* ***k****nowledge (TPACK) development of two preservice teachers as they engaged in the Virtual Number Talks field experience.*

Keywords: TPACK, field experience, virtual education, preservice teacher education, elementary mathematics education.

Introduction and Focus

In Fall 2020, schools across the United States were in a time of great instability due to the COVID-19 pandemic, with schools shifting between remote, hybrid, and in-person instruction. Mathematics teacher education was also at a time of great instability as in-person, school-based field experiences were no longer available to preservice teachers. In response to this challenge, the authors collaborated with other teacher educators to develop a Virtual Number Talks Teacher Learning Cycle (Joswick et al., 2020). The authors implemented this virtual mathematics field experience in their preservice mathematics methods courses in Fall 2020. For this study, we explore the following question: How does a virtual field experience based on the Number Talks routine develop preservice elementary teachers’ technological, pedagogical, and content knowledge (TPACK)? The authors utilize case studies to illustrate examples of TPACK development as two preservice teachers engaged in the Virtual Number Talks field experience.

Background and Relevant Literature

Field experiences provide preservice teachers with opportunities to “observe and work with real students, teachers, and curriculum in natural settings (i.e., PK-12 schools)” (Huling, 1998). Due to the COVID-19 pandemic, teacher educators had to develop flexible field experiences that could be implemented in virtual or hybrid classrooms. For example, Zolfaghari et al. (2020) employed the use of multi-perspective 360 videos to provide a virtual preservice teacher field experience. Evagorou et al. (2020) shifted a conventional science, technology, engineering, and mathematics fair to a virtual delivery using technology to provide preservice teachers with this field experience while schools were shuttered. Number Talks were selected as the focus of our virtual field experience due to the structure, content, and length of the routine. A Number Talk is a short activity—usually 5 to 15 minutes long—during which a teacher facilitates a discussion with a whole class or a small group of students about a carefully designed sequence of problems (Parrish, 2010). Students solve problems mentally and the teacher facilitates dialogue about students' strategies with open ended questions. The teacher records student thinking on the board using equations and visual models to help students consider others' ideas and make connections between strategies and representations (Parrish, 2010; Sun et al., 2018).

Number Talks are implemented in classrooms to achieve several important goals: to help students develop number sense and understanding of numerical relationships (Sun et al., 2018) and to shift focus towards making sense of strategies and away from a singular focus on solutions (Parrish, 2010). By centering students' thinking, Number Talks contribute to the development of a community of mathematics thinkers, doers, and learners in which "mathematical authority" is no longer held solely by the teacher and is instead shared among students (Lambert et al., 2017).

The US National Council of Teachers of Mathematics (2014) has identified a framework of essential teaching practices that stimulate meaningful mathematics learning: "establish mathematics goals to focus learning, implement tasks that promote reasoning and problem solving, use and connect mathematical representations, facilitate meaningful mathematical discourse, pose purposeful questions, support productive struggle in learning mathematics, build procedural fluency from conceptual understanding and elicit and use evidence of student thinking" (p. 10). Number Talks were chosen as the focus of our virtual field experience due to the potential for preservice teachers to engage in these essential mathematics teaching practices while facilitating virtual instruction.

Theoretical Framework: Technological, Pedagogical, and Content Knowledge

Built on Shulman's (1987) conceptualization of pedagogical content knowledge, Mishra and Koehler (2006, 2009) introduced the TPACK framework to account for the knowledge that must be developed for effective teaching practice given considerable technological change. The framework identifies the complex relationships that exist from the interactions between content, pedagogical, and technological knowledge for effective teaching practice (see Figure 1). Utilization of the framework, in a theoretical sense, provides a method of communicating the presence or absence of connections between content, pedagogy, and technological knowledge (Mishra & Koehler, 2006).

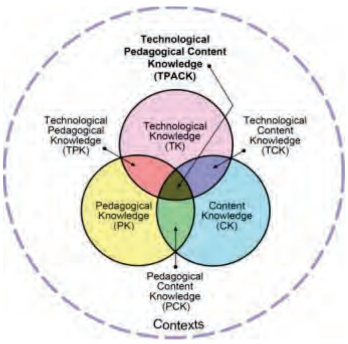

Figure 1: TPACK framework, reproduced by permission of the publisher, © 2012 by tpack.org

Pedagogical and content knowledge components of TPACK used in Number Talks can be categorized into the subdomains of mathematical knowledge for teaching (MKT, Ball et al., 2008). Common content knowledge (Hill & Ball, 2009) of school mathematics serves as the knowledge base needed to facilitate a Number Talk. Teachers need specialized mathematical knowledge (Hill & Ball, 2009) to record number relationships and operations using various visual models during Number Talks.

Knowledge of content and students is needed to anticipate and interpret students' strategies, to be aware of common developing (mis)conceptions or points of confusion, and to make sense of nonstandard strategies (Ball et al., 2008). Knowledge of content and teaching (Ball et al., 2008) is needed in Number Talks to select appropriate tasks and to choose in the moment which representations to use, which strategies to explore deeply or save for later, and which probing questions to ask to best support students' sense making. When facilitating Number Talks virtually, teachers must use their pedagogical and content knowledge and the subdomains of MKT to purposefully select and use technological tools to meet the mathematical goals of the Number Talk.

Methodology

Scientific and Cultural Context

Number Talks have become increasingly common in classrooms in the US as teachers have begun to incorporate the US Standards of Mathematical Practice, such as "make sense of problems and persevere in solving them, reason abstractly and quantitatively, and construct viable arguments and critique the reasoning of others" (NGACBP CCSSO 2010), into daily instruction. However, due to the COVID-19 pandemic and the switch to emergency remote instruction, much of the progress made in shifting mathematics teaching towards conceptual understanding and sense making came to a halt. Schools were forced to put together emergency remote instructional materials in a matter of weeks or even days, and many did so with packets of rote exercise worksheets and videos of direct instruction. As mathematics teacher educators, we sought to offer a field experience that translated best practices in mathematics teaching and learning into the digital environment.

Development of a Teacher Learning Cycle for the Virtual Number Talks Field Experience

The Association of Mathematics Teacher Educators' (2017) Standards for Preparing Teachers of Mathematics calls for teacher preparation programs to incorporate practice-based experiences that "develop core pedagogical practices and pedagogical content knowledge for teaching mathematics" (Indicator P.3.4.). To provide such a field experience while many schools were operating remotely, we created a Teacher Learning Cycle (TLC) in which preservice teachers learned about and engaged in Virtual Number Talks (Joswick et al., 2020). The TLC was developed using Teacher Education by Design's (University of Washington, 2014) learning cycle and includes four phases: learn, plan, implement, and reflect (see Figure 2).

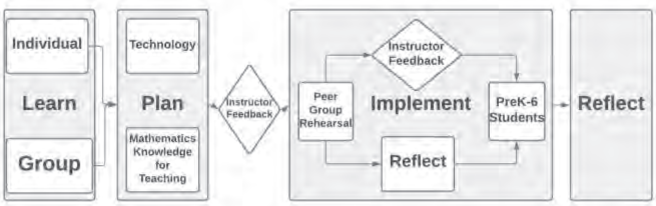

Figure 2: Teacher learning cycle for the Virtual Number Talks Field Experience

Setting and Participant Selection

The Virtual Number Talks Teacher Learning Cycle field experience was implemented in two sections of an elementary mathematics course taught at a university in the northeastern United States in Fall

2020. The course was a requirement for a master's degree in Elementary Education. 29 students, most of whom were preservice teachers, were enrolled between the two sections of the course. Of the 29 students enrolled, 21 students gave permission for their course assignments to be analyzed for research purposes. Most students took the course in the semester prior to their student teaching semester. The course was taught as a fully online course. Many students were working as school-based interns while enrolled in the course. Internship modalities ranged from fully online to hybrid and shifted during the semester due to changes in local COVID situations. Purposeful sampling was utilized to select "information-rich cases" (Patton, 1990, p. 169) to learn about the potential for TPACK development through the Virtual Number Talks field experience. In this paper, we chose to focus on two students, Tatum and Skyler, as illustrative cases (Yin, 2009) because they represented a range of prior teaching experience and prior technological knowledge and because they each demonstrated change over time in their knowledge of various elements of the TPACK framework.

Demographic Data: Tatum and Skyler

Tatum was a preservice teacher who worked for three semesters as an intern in a suburban public school for grades K-4 at the time of the project. Tatum completed the mathematics methods course and Virtual Number Talks Teacher Learning Cycle in their second year of an elementary education Master of Arts program during the fall semester before student teaching. Tatum had no prior teaching experience outside of their internship placement. Tatum's school was implementing a hybrid model at the time of the project; they completed their Virtual Number Talks Teacher Learning Cycle with three third grade students outside of their internship placement due to videotaping not being permitted at the school. Tatum demonstrated basic general technology knowledge at the start of the course.

Skyler was a preservice teacher who was working as an intern in a suburban public school for grades 3-5 at the time of this project. Skyler was in their fifth year of a five-year integrated Bachelor of Arts/Master of Arts program in elementary education, and they completed their mathematics methods course and the Virtual Number Talks Teacher Learning Cycle in the fall semester prior to student teaching. She previously worked as a paraprofessional in a summer school program for grades 9-12. Skyler completed their Virtual Number Talks Teacher Learning Cycle with three third grade students from their internship. Their school was implementing a hybrid cohort model at the time of the project. Skyler was adept at using newer technological tools and demonstrated more advanced technology knowledge at the start of the course.

Analysis

In a deductive approach and prior to analysis, *a priori* codes were developed from the three foundational areas of the TPACK framework. A code book containing these *a priori* codes along with their respective definitions from the existent literature was created to assist the analysis, create a consensus among coding members of the research team, and "maximize coherence among codes" (Creswell & Creswell, 2018, p. 199). Data sources for analysis existed in the form of nine assignments related to Number Talk learning, planning, implementation, and reflection completed by student participants throughout the semester. Each data source was then read and coded by members of the research team to the corresponding *a priori* codes. As a peer-debrief (Creswell & Creswell, 2018), members of the research team discussed themes emerging within the data for consistency, verification, and discrepancies. Validity in qualitative research is identified as one of its methodological strengths and was achieved through the triangulation of themes among the multiple data sources (Creswell & Creswell, 2018), discussion of discrepant codes, and peer-debriefing.

Results

Tatum: Developing Technological Knowledge and TPACK

Tatum demonstrated growth in their technological knowledge as they moved through the components of the teacher learning cycle from planning to rehearsing to implementing their Virtual Number Talk. Though Tatum practiced their use of technology tools during the rehearsal of their Virtual Number Talk with peers, he experienced technical difficulties when working with children. In their reflection on their first Number Talk, Tatum discussed their technical difficulties and the need to practice the use of technology tools simultaneously.

> In the beginning, I think that it was working well, but there was a time where I wasn't sure how to move from slide to slide from my screen in Zoom. I think that I should troubleshoot my number talk with all of my technology instead of both separately. It worked well during my rehearsal, but not as smoothly with my actual number talk. (Assignment 6, Reflection on first Virtual Number Talk)

In their final reflection after conducting two Virtual Number Talks, Tatum reflected on the growth of their technological knowledge through the peer rehearsal assignment and the benefit of teaching with technological tools.

> After our rehearsal and hearing back from my peers, I was able to use controls in the Zoom toolbar to use as ways the students can show me what they're thinking. I also was able to fine tune how to use PowerPoint as a constructive tool to make the virtual lesson feel more like an in-class lesson. (Assignment 9, Final Reflection after conducting two Virtual Number Talks)

Beyond simply learning how to use technological tools, Tatum's knowledge of using technological tools for specific pedagogical purposes in mathematics teaching showed significant development. For both Number Talks, Tatum showed images of dots, asking students to share how many dots they saw and to describe their strategies (see Figure 3). In their first Virtual Number Talk, Tatum showed dot images by screen sharing a PowerPoint but otherwise focused the activity on a verbal discussion.

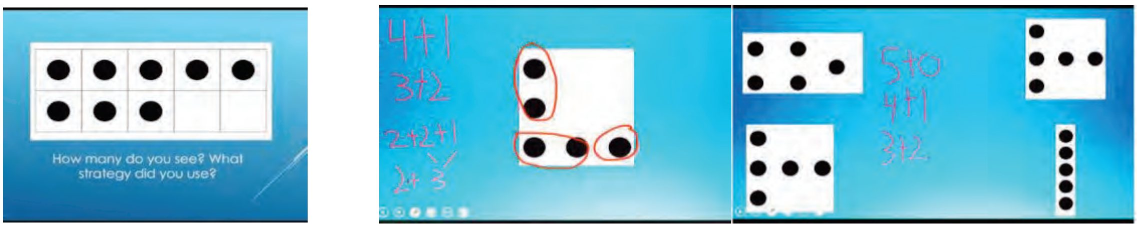

Figure 3: Images from Tatum's first (left) and second (middle and right) Virtual Number Talks

In their second Virtual Number Talk, Tatum once again showed dot images by screen sharing a PowerPoint, but this time used the pen tool to record on the screen throughout the conversation. By circling dots and writing equations to record student thinking, they made children's mathematical ideas visible to one another (see Figure 3). They closed the conversation by showing a slide with all of the dot images from the Virtual Number Talk, using the pen tool as they facilitated a conversation about connections across the dot configurations showing different ways to decompose the number 5, demonstrating the knowledge of content and teaching subdomain of MKT (Ball et al., 2008).

Skyler: Developing Pedagogical Content Knowledge and TPACK

As Skyler engaged in the Virtual Number Talk Teacher Learning Cycle, they demonstrated growth in their both pedagogical content knowledge and TPACK. Skyler focused their first Number Talk on dot patterns, facilitating an exploration of composition and decomposition of small numbers with their students. For their second Number Talk, they focused again on number composition and decomposition, but this time through a series of double-digit addition problems, a significantly more challenging task to facilitate. Skyler's recording of student thinking at the start of their second Number Talk was disconnected from place value. She honored students' thinking by typing what children said verbatim, but they did not ask any follow up questions about the value of the digits being added in the tens place. In a discussion board post early in the teacher learning cycle, Skyler stated the following, "I know in my experiences as a child and adult I have always had a harder time with auditory/nonvisual problems. Once the teacher wrote the problem and strategy on the board it was easier to follow along" (Assignment 2, learning about Number Talks and discussing with peers). Though Skyler recognized the importance of making mathematical ideas visible for children, their use of technological tools did not allow them to achieve the mathematical goal or support student sense making.

Midway through their second Number Talk, Skyler stopped typing students' explanations on virtual sticky notes and started recording using the pen tool. In doing so, Skyler began to "mathematize" the children's thinking by connecting their verbal explanations to mathematical equations, demonstrating specialized mathematics knowledge, knowledge of content and students, and knowledge of content and teaching subdomains of MKT (Ball et al., 2008). Through their recorded equations, Skyler highlighted the place value understanding inherent in the student's partial sums strategy. Skyler's in-the-moment shift in their use of technology tools, shown in Figure 4, made one student's mathematical thinking accessible to the other children and helped them to focus the discussion on important place value concepts, demonstrating a significant development in their TPACK skills.

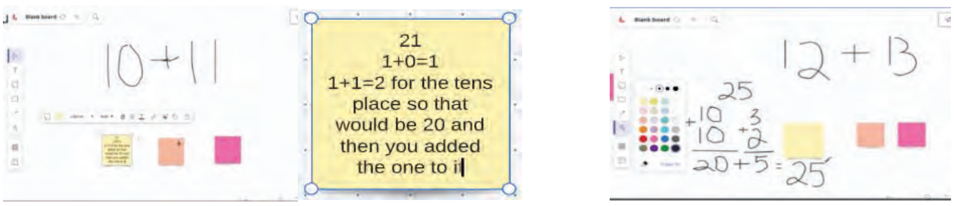

Figure 4: Images from the start of Skyler's second Virtual Number Talk (left and middle) versus the end of the Number Talk (right)

Benefits and Challenges of Virtual Number Talks as a Virtual Field Experience

There were several benefits to using Number Talks as a virtual field experience for preservice teachers. Virtual Number Talks allowed for maximum flexibility of implementation in terms of group size, content, and student location, creating access to teaching experiences for preservice teachers who were not able to enter schools due to COVID-19. While many classrooms shifted to videos of direct instruction and worksheets of practice problems at the start of the pandemic, Virtual Number Talks provided preservice teachers the opportunity to facilitate a mathematical conversation with students, creating windows into student thinking that many of our students had not experienced previously. Planning for and facilitating Virtual Number Talks also aided preservice teachers in developing and practicing their technological pedagogical content knowledge (Koehler, & Mishra, 2009) as they learned to utilize technological tools to create and display visual models, record equations and strategies, and facilitate students' thinking, discussions, and learning in the virtual classroom.

There were also challenges in implementing the Virtual Number Talks field experience for preservice teachers. New privacy concerns have arisen from K-12 virtual instruction, which prevented some preservice teachers from being permitted to facilitate or video record instruction within school-based placements. Virtual classroom management, including using conferencing software and tech tools for displaying and recording models and equations while engaging students online, was challenging for many preservice teachers. Writing on a digital whiteboard space also proved to be difficult for preservice teachers who did not have access to a stylus or touch screen device and had to write with a track pad or mouse, impacting the amount and quality of recording of students' strategies done by preservice teachers during their Virtual Number Talks.

Conclusion

Number Talks contribute to the creation of a mathematics community in which all students' contributions are valued. Number Talks also allow for the development of teachers' MKT (Ball et al., 2008) through selecting tasks, anticipating strategies, facilitating discussions, and recording strategies with equations and visual models. Bringing Number Talks to online classrooms helps teachers to maximize limited synchronous class time with a short but effective instructional routine. Through Virtual Number Talks, teachers can leverage technology to provide high-quality learning opportunities that empower students, center students' thinking, and position students as creators of mathematics. To do so effectively, teachers must start with their mathematical goal in mind and use pedagogical and content knowledge, MKT subdomains, *and* technological knowledge to purposefully choose technological tools best suited for the mathematical goal. As a virtual field experience, Virtual Number Talks can allow preservice teachers to develop their technological, pedagogical, and content knowledge (TPACK) as they learn to engage in essential teaching practices, use technology to achieve mathematical and pedagogical goals, and foster a virtual mathematics classroom community.

Acknowledgment

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Prospective teachers' specialized knowledge on periodic number: mathematics knowledge and beliefs

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In this work we analyze the specialized knowledge about periodic numbers of a group of prospective mathematics teachers by means of a task for teacher development. The results revealed that a KoT and the KPM knowledges were displayed in the transformation of a periodic number to a fraction. In the discussion there appeared a resistance to accept the equality $0,\overline{9}=1$. *The prospective teachers (PTs) based their initial discussion on their beliefs rather than on mathematical arguments.*

Keywords: Periodic and semi-periodic number, mathematical knowledge, beliefs.

Introduction.

One of the causes of difficulties in rational number is the mechanistic style with which fractions (and decimals) are approached, focusing on the application of nonsense rules and due to the underestimation of the difficulty of this subject (Streefland, 1991). Thus, it is necessary to implement strategies from the initial teacher development that include a comprehensive point of view of this issue. In this work we focus on determining this knowledge from a development task focused on periodic (and semiperiodic) decimal numbers from the MTSK perspective (Carrillo et al., 2018).

Theoretical framework.

One issue that can cause confusion within periodic decimal numbers is the two different representations of the number $0,\overline{9}$ and 1, which corresponds to the equality $0,\overline{9}=1$. Rittaud and Vivier (2013) reported the difficulties of university students to establish that $0,\overline{9}=1$, since most of them answered that these numbers were different. We used the Mathematics Specialized Teachers' Knowledge (MTSK) model (Carrillo et al., 2018) particularly in *Mathematics Knowledge* (MK), *Knowledge of Practices in Mathematics* (KPM) and *Beliefs* of and about mathematics and its teaching.

Analysis and Results.

PTs solved a task on periodic number. From question 1 (Figure 1-image 1) they reveal that they know the procedure to transform a periodic number into a fraction, which corresponds to the knowledge of the topics (KoT) and of the procedure category (KoT-p1: Know the method to transform a periodic number into a fraction by applying the usual algorithm). From question 2 (Figure 1-image 1) it is shown that PTs know the method to transform a semi-periodic number to a fraction (KoT-p2: Know the method to transform a semi-periodic number to a fraction applying the usual algorithm).

From question: How would you explain the procedure to solve $0,\overline{16}$? From the resolution (Figure 1-image 2), we observe that the fraction transformation procedure was applied using operations on periodic numbers in an equation - method 1 (KoT-p3: Know a procedure to transform a periodic number into a fraction by performing operations in an equation).

The Figure 1-image 3 shows the use of an example and a mathematical work in which he uses method 1 to mathematically justify the case of a semiperiodic number. Both issues imply that knowledge associated with the KPM was mobilized. From the analysis emerges the label mathematical justification (“j”) that is not yet a category since it requires broader discussions (KPM- “j1”: Know how to mathematically justify the transformation of a semiperiodic number to a fraction from one more case simple). After, despite their knowledge, PTs did not identify than $8,\overline{9} < 9$. They stated that they believe that $0,\overline{9}$<1, which shows a *resistance* to accept that mathematical equality (Bel-1: Belief that different representation records cannot be associated with the same quantity ($0,\overline{9} < 1$)).

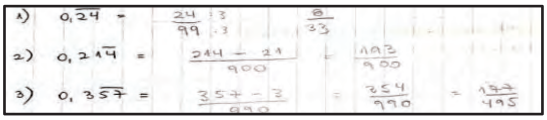


Image 1

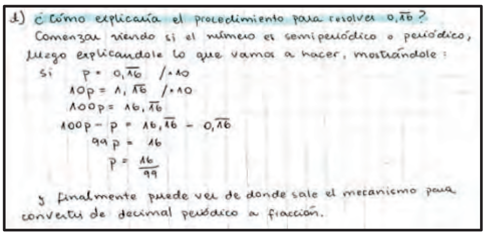
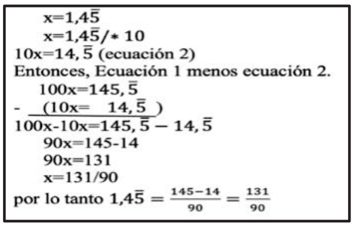


Image 2



x=1,4$\overline{5}$
x=1,4$\overline{5}$/* 10
10x=14,$\overline{5}$ (ecuación 2)
Entonces, Ecuación 1 menos ecuación 2.
100x=145,$\overline{5}$
- (10x= 14,$\overline{5}$)
100x-10x=145,$\overline{5}$ − 14,$\overline{5}$
90x=145-14
90x=131
x=131/90
por lo tanto 1,4$\overline{5}$ = $\frac{145-14}{90} = \frac{131}{90}$

Image 3

Figure 1: Answer on (image 1) periodic numbers; (image 2) semiperiodic numbers; (image 3) transform a periodic number to a fraction.

Conclusion.

Here, mathematical knowledge associated with KoT and KPM was found. In addition, there was a resistance from PTs to accept the equality $0,\overline{9}$=1 and can be explained because there are two numerical representations of the numbers $0,\overline{9}$ and 1 for the same value (Rittaud & Vivier, 2013).

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Teachers’ knowledge in the context of division of fractions

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Keywords: Mathematics Teachers Specialized Knowledge, fraction, teacher education.

Introduction

The amount of research on division of fractions has increased largely since 1970s, however it is still a challenging topic both for teachers and students (e.g., Newton, 2008). Among the operations involving fractions, the division has been considered the most mechanical and less understanding by teachers (Lo & Luo, 2012), which reveal to know how to find the solution, but lack understanding on the how’s and whys grounding the division involving fractions. For students to develop such an understanding, we, has teachers are required to be in possession of a specialized knowledge which we consider in the perspective of the M*athematics Teachers' Specialized Knowledge* – MTSK (Carrillo et.al, 2018). Also, due to the central role of tasks in teachers’ practices, its essential do develop research focusing on the role of such tasks in teacher education. Aiming at enriching our understanding of such a knowledge here we focus on the following research question: What knowledge is revealed by teachers who teach Mathematics in the context of fraction division when solving a task in their continuing education?

Some theoretical dimensions

Students reveal difficulties in understanding fractions and the operation involving them, in particular, division. The same can be said for teachers and such difficulties might be grounded in the lack of understanding multiplicative conceptual field in mathematics curriculum and in previous school experiences teachers have encountered (Lamon, 2007). When considering teachers’ knowledge, we assume its specificities bit in terms of mathematical and pedagogical knowledge and we assume the MTSK conceptualization and aligned with our research focus here we address the Knowledge of Topics subdomain and, in particular: procedures, definition and representation.

Context and method

Here we report on data gathered in a six hours workshop which occurred in two days in a row (3 hours each day) with the participation of six mathematics primary teachers (teaching students aged 7 to 14). Data concern questionnaires, teachers written productions to a task for teacher education and video recordings of the sessions – through google meet. Here we focus on one of the questions of the task: Why when dividing by a fraction do, we invert the divisor and multiply to get the result in the operation 12/15: 3/5= 12/15 x 5/3 = 4/3.? The analysis has been made using the MTSK categories related to KoT.

Data analyses

Preliminary results enhance teachers' knowledge grounding the inability of justifying the IM method –the teachers knew how to use the procedure, but they do not understand and cannot explain why it works. Only one teacher provided a production trying to give meaning to the arithmetic expression. The teacher's representation does not justify the IM and shows that the teacher confuses the concept of partitive with measure.

T1: We do representations to represent fractions, but not their operations.

T5: My students ask me why it is needed to multiply to divide, and I was always sad that it was the rule, but I didn't know how to explain and why. When I learnt it, I learnt it through the rule.

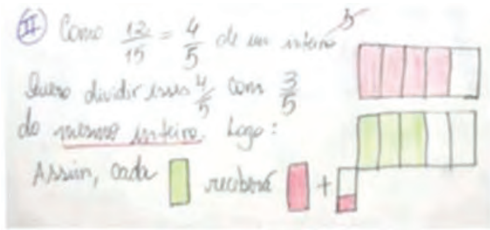

Figure 1: teacher production

The analysis revealed teachers' knowledge related to sense of division (measure/ partitive); different kinds of representation that cope with the sense of each mathematics constructions (beyond the unit fractions) and the relationship between division and other topics, for example, measure (Fazi & Siegler, 2011).

T6: Now, I understand why the algorithm works, because I have always used the IM and I have never thought about its meaning.

T3: I didn't know the division as a measure and now it makes sense to think of rations and in the resolution of problems, in a whole.

Conclusion

With the task, it was possible to know and develop the knowledge of the mathematical content necessary for the teacher in teaching fraction division. The investigation suggests an improvement in training courses *to* develop the teacher's mathematical knowledge in these topics for an improvement in teaching.

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Teachers' Knowledge of Topic revealed via lessons plans for linear function introduction

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In this paper, we present preliminary results of the study where teachers' lesson plans were analysed to inquire about their content knowledge. Our theoretical background stands on the model Mathematics Teacher Specialised Knowledge (MTSK), particularly we focus on the element Knowledge of Topic. In the paper, an analysis of two lesson plans is provided to answer the research question: "What Knowledge of Topic (as described in the MTSK model) and what gaps in it are evident in the teacher lesson plans concerning the topic of a linear function?" The analyses revealed that indicators of all four components of Knowledge of Topic can be observed in the teachers' lesson plans whereas gaps in the particular components could be diagnosed as well.

Keywords: Teacher content knowledge, MTSK, lesson plan, linear function.

Introduction

A list of scholars starting with Shulman (1987) has tried to provide a model describing knowledge specific for (mathematics) teachers. Many others tried to "measure" or "describe" teacher knowledge using different methodologies. The broader objective of the research group is to use the collected knowledge in their practice as mathematics teachers' educators. Particularly, to develop a tool to track the progress of pre-service and in-service mathematics teachers achieved in their practice or during a specialised course. Hopefully, such a tool can support the reflection of mathematics teachers' educators and subsequently enhance the quality of provided courses or informal professional development meetings. This paper refers to the first step of the process.

Our approach was motivated by the work of Zakaryan and Ribeiro (2019) who used the model Mathematics Teacher Specialised Knowledge (MTSK) to classify knowledge about the content of the topic of rational numbers. The other resource of inspiration is the paper by Corey et al. (2021) who used written instructional products of teachers to "understand what knowledge (of student mathematical thinking) is evident in them and how that knowledge is used to justify instructional decisions" (p. 635). This idea encouraged us to use the lesson plans of teachers as a resource to learn about their knowledge. As pinpointed by Carlson and Daehler (2019) teacher knowledge can be enacted in planning, teaching, and reflecting. Therefore, to inquire about planning is as valid as inquiring about teaching practice or teachers' reflections. Naturally, it is not enough to inquire only about lesson plans, when it comes to the knowledge of an individual teacher. However, in terms of having a tool, which should be both – practical and as authentic as possible – we consider a lesson plan (possibly supported by an interview or a group discussion) as a good choice. For the beginning of the tool development, in the paper, we try to answer the research question: "What Knowledge of Topic (as described in the MTSK model) enactments are evident in the teacher lesson plans concerning the topic of a linear function?" Further, we will use LP instead of the full "lesson plan".

Theoretical framework

As already proposed above, we will frame mathematics teachers' knowledge with a focus on the Knowledge of Topic. From many possible ways, we opted for the model Mathematics Teacher Specialised Knowledge (MTSK) introduced by Carrillo et al. (2018). The section Knowledge of Topic, which is part of the model mainly interesting for this paper, consists of four components: (1) The first component (KoTp) is the knowledge of definitions, mathematical properties, and their foundations. (2) The second element (KoTph) is knowledge of phenomenology and applications for related problems in the topic. (3) The third sub-part (KoTmp) is knowledge of the procedures applicable to a specific topic. It includes situations such as "How to do something? When to do something? Why is something done this way? Characteristics of the resulting object." (4) The last sub-area (KoTrp) covers the knowledge of the different registers of representation and representational change. In general, KoT "combines the knowledge that the students are expected to learn with a deeper, and maybe more formal and rigorous understanding" (Carrillo et al, 2018). *The knowledge that the students are expected to learn* is dependent on the country's curriculum. This cultural aspect is not of a small importance as underscored at previous CERME (Ribeiro et al., 2019). Therefore, to describe KoT, it is possible to look at the national textbooks. The following table describes KoT specific for teaching the topic of linear function as introduced in the appropriate Slovak and Czech mathematics textbooks (Šedivý et al. 2004; Šedivý et al., 2006; Šedivý et al., 2011; Molnár et al., 2001; Binterová et al., 2010; Kolbaská, 2014; Berová and Bero, 2015, 2013; Kohanová et al., 2016; Hecht et al., 2001; Odvárko et al., 1985; Odvárko, 1993). The process of analysis and its deeper levels, which are beyond the scope of this paper, is part of the thesis which is being elaborated by the first author.

Table 1: KoT in the topic Linear function

KoTp:	
def	the linear function definition (any of the correct approaches, e.g. function which graph is the straight line, the same change in the "x" causes a constant increment in the "y", or the function which equation is $y=kx+q$)
nF	ability to formulate the examples of the relationship which is not functional (e.g. $x=2$)
q	knowledge of the q (y-intercept) interpretation, both cases acknowledged $q=0$ and $q\neq 0$
k	knowledge of the k (slope) interpretation, acknowledgment of differences between $k<0$, $k=0$ and $k>0$
Df	knowledge of the domain of the linear functions and its restriction due to the real context of the task
KoTmp:	
belong	to determine whether the point is (not) the point of the function
x	to determine the intersection with the x-axis (graph, notation)
y	to determine the intersection with the y-axis (graph, calculation from the equation)

coor	to determine the missing coordinate of the point (not in the context of filling the table)
inters	to determine the intersection of two linear functions
Rf	to determine the range of the function in correspondence with the context
mon	to determine whether the function is increasing/decreasing/constant (not using the interpretation of *k*)
KoTph:	
dts	the context of the task is focused on distance, time, and velocity
eur	the context of the task is money
t	the context of the task is time (but not velocity)
V	the context of the task is a volume of some substance
n	the context of the task is the number of something
m	the context of the task is the weight of something
KoTrp:	
verbal	change of representation from equation / table / graph into verbal description
equation	change of representation from verbal description / table / graph into equation
tab	change of representation from equation / verbal description / graph into table
graph	change of representation from equation / table / verbal description into graph

One might object that such a list of KoT represents also students' knowledge and thus, it is not specialised for teachers. We argue, that firstly (and sadly), it is not granted that teachers have valid “students' knowledge”. Secondly, the context in which we will inquire about it is directly connected to teaching. Thirdly, the planning of teaching in Slovakia is somehow specific, as our curriculum is not well developed and teachers often combine many resources (not always with externally guaranteed mathematical quality) in order to prepare a lesson plan, or they build on their ideas.

Methods and context

In our study, we analysed ten lesson plans (LP) created by lower-secondary mathematics teachers. Only two of them will be presented in the paper. The teachers were from different schools in the East Slovakia region. The LPs consisted of the commented materials (e.g. goals formulations, solved tasks, instructions for the teacher, comments on expected problems, …) and the student worksheet. They were collected during the professional development course provided as part of the National Project IT Academy. Before the course module about functions and functional thinking started, some teachers were asked to submit the LP about linear function using the inquiry method. All LPs were focused on the topic linear function. Some of the teachers elaborated the introduction of a linear function (n=5), the others worked on the examination of its properties (n=5). In this paper, we focus on the first type of LPs.

The main tool of the analysis was the codebook which is presented in the theoretical framework (Table 1). In the case, any other KoT indicator would be presented in the LP, we would inductively add it into the codebook. The LPs were analysed by both authors and the results of the analyses (for the sake of this paper) were summarized in a so-called “KoT map”, where each of the codes is presented and coloured in green/red/yellow (see e.g. Figure 2). The analyses were conducted to find:

(1) the evidence (e.g. correctly solved task, correct mathematical statement, assigned task with the specific context, …) that the KoT indicator is actively used by a teacher. In the KoT map, these codes are highlighted in green.

(2) the indication, that the piece of KoT is not valid (e.g. incorrect solution of the task, tasks which are not in the line with the mathematical goal of the lesson, …). In the KoT map, these codes have a red colour.

Certainly, one LP from the whole topic is not enough to be analysed to see the teacher’s KoT. If our research was aimed at assessing teacher’s level of KoT, it would be true. However, we could use only part of the topic because we aimed to inquire about LP as a resource to “measure” teachers' KoT.

Results

In this paper, we will provide an analysis of two LPs, which were chosen to pinpoint different strengths and difficulties in KoT. The first one, submitted by Jana (names changed), is stated to show the following situation: within many different contexts, only a few correct indicators of KoT are presented. Moreover, some invalidities were found as well. In the other one, proposed by Kristína we want to pinpoint the diversity of correct indicators of KoT revealed within one task for students.

Jana

Jana proposed 5 tasks in her LP. All of them were similar to the one presented in Figure 1.

The brick weighs 6 kg. What is the weight of 2, 3, 4, 5, 6 bricks? Build a table of dependencies between the number of bricks and their weight. Write the equation between the number of bricks and their weight. Plot a graph of the number of bricks with their weight.

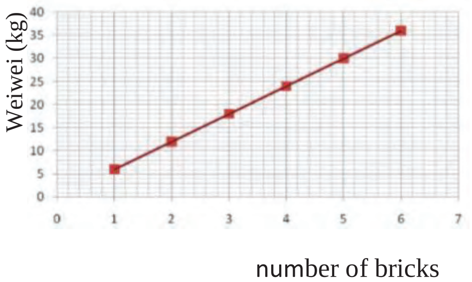

Figure 1: Task to introduce Linear function (Jana)

KoT map (Figure 2) pinpoints, that Jana used many different contexts for her tasks and she was focused on the representational change. Nevertheless, she did not reveal deep KoT in the other two components and what is more, some incorrectness was observed as well.

KoTp	def	nF	q	k	Df		
KoTmp	belong	x	y	coor	inters	Rf	mon
KoTph	dts	eur	t	V	n	m	
KoTrp	verbal	equation	tab	graph			

Figure 2: KoT map (LP by Jana)

KoTp:

- q (red) – The teacher used exclusively tasks where the *y*-intercept, *q*=0, in other words, to introduce the topic of the linear function, she used only direct proportionality, which is in the contrast to the goals she set in her LP and which directly mentioned *linear function*. She specified direct proportionality as the prior knowledge.

KoTph:

- dts, eur, t, n, m (green) – Jana used all marked contexts in her LP.

KoTrp:

- equation (green) – Students were required to write the equation which describes the relationship between the variables – to change the verbal description/table into an equation, and she had the correct solution of this task presented in the LP.
- tab (green) – Students were asked to fill the table which describes the relationship between the variables – to change the verbal description into the table.
- graph (red) – The teacher asked students to create a graph, however, the graph depicted in Figure 4 (which is proposed by her) is not correct in the context of the given task. The graph should start at (0,0) and its continuity is at least disputable. Moreover, it should not end at (6,36).

Kristína

Kristína planned one task to be solved in small groups of students. Each group worked on the same situation; however, they were asked to answer different questions as shown below (Figure 3):

> The mobile provider offers two different tariffs. When choosing tariff A, we will pay € 0.10 for a minute. When choosing tariff B, we will pay 4 € at the beginning of the month and € 0.05 for each minute. 50 text messages and 1.5 GB of data are offered on both tariffs for no extra payment.
>
> Group 1: Which tariff do you think is more advantageous?
>
> Group 2: Which tariff do you think is more advantageous if you estimate that you will call about 0.5 hours a month?
>
> Group 3: Which tariff do you think is more advantageous if you estimate that you will call in about an hour in a month?
>
> Group 4: Which tariff do you think is more advantageous if you estimate that you will call in about 1 hour and 20 minutes per month?

Group 5: Which tariff do you think is more advantageous if you estimate that you will call in about 2 hours a month?

Figure 3: Task to introduce Linear function (Kristína)

Kristína's KoT map (Figure 4) is visualised, and the codes are explained below. We can see that within the narrow context of only one task she revealed many different indicators of correct KoT and did not show any incorrectness.

KoTp	def	nF	q	k	Df		
KoTmp	belong	x	y	coor	inters	Rf	mon
KoTph	dts	eur	t	V	n	m	
KoTrp	verbal	equation	tab	graph			

Figure 4: KoT map (LP by Kristína)

KoTp:

- q (green) – We can see the interpretation of the *y*-intercept $q = 4$ in her task – Tariff B initial payment.
- k (green) – She offered the interpretation of the slope *k* by visualization $k_1 = 0{,}1$ and $k_2 = 0{,}05$, which are different minute prices for the tariffs A and B.
- Df (green) – She worked with the domain of the function concerning the real context of the problem. In her solution, she restricted the graph to the positive *x*-axis.

KoTmp:

- coor (green) – In the subtasks for groups 2-5, she wanted students to determine which tariff is better for the particular number of minutes. She had a correct solution for each of the groups in her LP and therefore she displayed the knowledge of determining the missing coordinate of the point.
- inters (green) – From the LP, it was visible, she prompted pupils to calculate the time in which are both tariffs the same cost and she had solved this task as well.

KoTph:

- eur, t (green) – The variables in the task are money and time.

KoTrp

- tab (green) – Pupils are asked to fill the table to solve the task, in other words, to change the verbal description into the table. The teacher displayed the correct knowledge as well.
- graph (green) – Pupils are prompted to draw a graph to solve the task, in other words, to change the table into the graph. She had the correct solution for this task as well.

Discussion

Coming back to the research question: "What Knowledge of Topic (as described in the MTSK model) enactments are evident in the teacher lesson plans concerning the topic of a linear function?" we can say the following: Firstly, the LPs are a good resource to find the indicators on teachers' knowledge on the application of the topic (KoTph). It was clear, which context of application teachers was familiar with (Kristína – used time and money contexts, Jana – distance-time-speed, money, time, number of something and weight). Comparably, LPs reveal teachers' knowledge on representations and representational change (KoTrp). In this case, we were able to observe not only the indicators of correct KoT but also signs of the invalid KoT, for example (in the case of Jana) in the graph representation, where it was necessary to restrict the line into discrete points (or possibly a ray).

Moreover, concerning teachers' Knowledge of definitions, mathematical properties, and their foundations (KoTp) and the component Knowledge of mathematical procedures (KoTmp), we could observe differences between the LPs, especially in the terms of "being" or "not being" presented. These two components were much more elaborated in the LP of Kristína (we observed determining the intersection and finding the missing coordinate of the point), however, barely presented in the LP of Jana (where we did not observe any indicators and her tasks were focused only on the direct proportionality, therefore she did not cover the topic of linear function).

Based on these two LPs we could exactly name the differences between the two teachers. Therefore, we can see that LPs could serve us as a good tool to discern all four components of a teacher's KoT, of course, if analysed in whole (not only one LP for a topic). In the sense of the broader goal (development of a tool to track the progress), we also see the potential of such analysis. If two presented KoT maps did not describe two different teachers' LPs, but LPs of one teacher at the beginning and the end of the course (or at the beginning of the career and after five years), we could be able to reveal how his/her knowledge was developed. This approach is to be applied in future research.

Furthermore, the other issue, which we would like to discuss, are parts of the LPs, which were not coded, because none of our codes was applicable. For instance, one of the teachers proposed the task focused on reading the statistical information from the bar chart (e.g. to count arithmetic mean) as the activity to revise important knowledge before the linear function is introduced. Possibly, this could be understood as some gap in the Knowledge of the Structure of Mathematics – another element from MTSK. This observation prompts us to the next analysis of the LPs to find out, what – except for KoT – we can learn about the teachers from their written instructional plans.

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Collaborative knowledge creation of teacher subject knowledge using concept maps

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This paper sets out the theoretical foundations and design of a future study planned for early 2022. The study focuses on the use of concept maps as the object of collaborative knowledge creation. The aim of the research is to explore the potential role concept maps have in supporting in-service primary teachers in developing their subject knowledge for teaching through collaboratively constructing a concept map mediated by an online tool developed by Cambridge Mathematics.

Keywords: Teacher knowledge, concept mapping, collaboration.

Introduction

In the pursuit of supporting teachers to develop richer more connected conceptualisations of mathematics this study will explore an example of a technology-mediated knowledge creation activity. The activity will involve multiple steps. Initially, I will invite teachers to explicitly describe their own image of an area of mathematics, detailing the structure and content of their subject knowledge. During the second stage participants will develop a representation of this by sketching and refining a concept map in an online environment. Third, an online community will be created in which teachers will have the opportunity to reflect on their own and others' concept maps through sharing, analysing and questioning each other. Finally, the participants will collaborate to build a single concept map using the online tool. It is hypothesised that the outcome of this collaboration will not only be an accumulation of knowledge, which for participants will include the accommodation of new or alternative perspectives and connections, but also enable new knowledge to be created. This paper discusses the theoretical background to the study and details the phases in the current design.

What do we mean by teachers' subject knowledge?

What teachers need to know to be successful has received a significant amount of attention (Remillard & Kim, 2017). International comparative studies persistently search for reasons for and methods to avoid poor outcomes for learners of mathematics. Policy makers and mathematics educators have placed their focus on the identification, development and improvement of teacher mathematical content knowledge. Understanding what is to be learned enables teaching to commence and in this role a teacher "must understand the structures of subject matter, the principles of conceptual organization" (Shulman, 1987, p. 9).

In considering teacher knowledge, Shulman (1986) identifies (amongst other types) *content knowledge* that extends beyond recognition of assumptions, facts and concepts and necessitates familiarity with the structure of the subject, including how truth is established, why knowledge is worth knowing and the relationships within and beyond the subject. *Pedagogical content knowledge* is described as the range and power of representations, analogies, illustrations, examples, explanations and demonstrations. Also included is understanding pupils' preconceptions, many of which may be misconceptions, and their effect on future learning. Within *curricular knowledge* Shulman highlights the importance of teachers' *lateral* curriculum knowledge, in other words how the content being taught in mathematics relates to that being taught in other subjects. V*ertical* curriculum knowledge, being familiar with the mathematics being taught in preceding and future years, is identified as essential. Throughout Shulman's work, the organisation of mathematical content is identified as an important feature of teaching.

In the study of teachers in China and the United States Ma (1999) describes how successful teachers exhibited profound understanding of fundamental mathematics; connected, curricula-structured, longitudinal knowledge of core mathematical ideas. Such teachers had a breadth and depth of connected knowledge that enabled flexibility in their approaches and an awareness of the implications of what and how things are taught on later learning.

The mental organisation of a teacher's knowledge plays an important role in developing a positive relationship between content knowledge and classroom instruction (Fenneman & Franke, 1992), implying that the deeper, more connected knowledge a teacher holds the more successful they will be in developing learning with understanding. What is deemed to be a valuable measure of a teacher's potential, in these studies and others (Loewenberg Ball et al., 2008; Mason & Spence, 1999), is not just their declarative knowledge but what they understand this knowledge to mean, its place in school mathematics, the representations used, and misconceptions held.

A flexible connected schema of mathematical content knowledge offers a structure into which pedagogical content knowledge can be embedded, explored, connected and also developed. This may be knowledge of content and students (KCS): typical misconceptions, areas of confusion and challenges, learners' interests and motivations, knowledge of content and teaching (KCT): how to teach, worthy examples, representations, or knowledge of content and curriculum (Lowenberg Ball et al., 2008). In turn this knowledge enriches the original schema and supports teachers' understanding of why specific tasks, manipulatives, representations, misconceptions, and learning sequences arise and relate. What representation can be used productively to enable the development, of this schema? What professional activities enable teachers to construct and work flexibly with these schemas, allowing them to represent and develop their subject knowledge for teaching?

Why are concept maps a suitable vehicle to represent mathematical ideas?

Each individual holds their own visualisation or personal mapping of their mathematical knowledge – the schema by which they organise their mathematical ideas (Asiala et al., 1997, Skemp, 1978). This structure is not stable, but dynamic, reflecting the developing knowledge and experiences of each of us. An individual's schema is not possible to study, compare or discuss directly. An instance of it must be elicited into some explicit form that can then be interpreted. A variety of artefacts could support this through being the focus of a discussion, including schemes of work, resources, worked solutions or questions. Alternatively, teachers could be interviewed about or observed in their classroom practice. Such verbal discussions, written solutions and texts tend to represent ideas in a linear manner, offering little insight into the conceptual structure someone holds (Grevholm, 2008).

Graphic organisers show relationships between concepts and processes, employing spatial position, connecting lines and overlapping structures (Nesbit & Adesope, 2006), and therefore offer a tool through which a non-linear structure can be represented. Concept maps are a type of graphic organiser, "a diagram representing the conceptual structure of a subject discipline as a graph in which modes represent concepts and connections represent cognitive links between them" (McGowen & Tall, 1999, p. 2). These links that may be directed or undirected, labelled or unlabelled (Nesbit & Adesope, 2006). Crosslinks identify relationships between concepts in different segments or domains of the map, showing relations between domains (Novak & Cañas, 2008).

An individual's concept map is a visual representation of their concepts (Novak & Gowin, 1984) and a tool for analysing expression of current concept structure (Grevholm, 2008). Any map will be constructed from a personal mental representation of knowledge although, how accurate the representation is, is impossible to know (Williams, 1998). A concept map's structure depends on the context in which knowledge is being applied or considered (Novak & Cañas, 2008). Not only will this adapt and change as the context varies; it will also be altered, refined and elaborated as the creator's own maths develops. Concept maps are not "inert and finished, but interactive and expandable" (Eppler, 2004, p. 200).

The software chosen for this study to represent concept maps is based on an off-the-shelf network graphing tool; Neo4J (n.d.). This has a bespoke visual interface designed by Cambridge Mathematics (n.d.) a curriculum research and design project based at the University of Cambridge. The interface consists of nodes and edges. Nodes can include a title, free text description, images and be tagged to implement colour coding. Edges can be directed or undirected, include free text and be tagged to implement colour coding. As a member of the Cambridge Mathematics writing team, the author was instrumental in the design of this software.

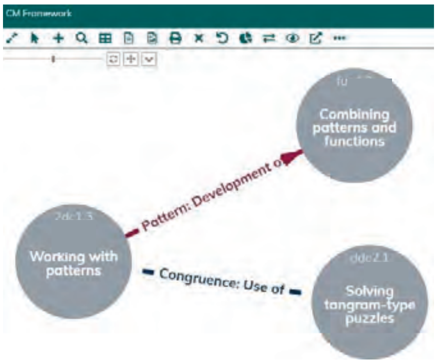

Figure 1: Exemplar concept map using the Cambridge Mathematics interface

What is collaborative knowledge creation?

In their discussion of socio-cultural perspectives on collaborative learning, Hakkarainen et al. (2013) describe three significant metaphors of learning: knowledge-acquisition, participation and knowledge-creation.

In the knowledge acquisition metaphor for learning knowledge is viewed as a personal characteristic that is developed by individuals. Interactions that occur may provoke cognitive conflict enabling learning but not as a product of any collaboration but because of an internal dialogue (monological). From the perspective of the participation metaphor knowledge is viewed as the objects and practices (including rules and beliefs) of communities. It has been influenced by the history and culture of the community. Learning is seen as gradually becoming aware of this knowledge, joining the community through learning how to function according to its rules. Through participating in collaborative activities learners become aware of their own and others knowledge (dialogical) and develop shared meaning.

In describing their third metaphor, collaborative knowledge creation, Hakkarainen et al. (2013) reference ideas of knowledge building (Bereiter, 2002), expansive learning (Engeström, 1987) and organisational knowledge creation (Nonaka & Takeuchi, 1995). Rather than acquainting themselves with a community, as in the participation metaphor, collaborative knowledge creation focuses on the deliberate pursuit of advancing shared objects for future development and use (Hakkarainen et al., 2013). These knowledge artefacts are to be created and mediate activity (Paavola et al., 2012). Often with the help of technology, learners take collective responsibility to design, construct, modify and manage versions of knowledge artefacts that partly pre-existed. and have the "capacity to unfold indefinitely" (Cetina, 2001, p. 181 as cited in Paavola et al., 2012, p. 2).

A concept map: A flexible representation that can be created collaboratively

The knowledge-creation metaphor offers the study a theoretical background which accepts and capitalises on teachers' varied experiences, expertise, beliefs and theories whilst they collaborate to develop a deeper understanding of their subject, thus impacting on their teaching. Each participant holds their own image of any area in mathematics, including the different types of teacher knowledge described above. With the correct mediating object this can be shared, modified and extended through collaboration. A concept map provides an example of an epistemic object, its boundaries non-existent, as multiple connections link ideas across domains and into related subjects. In fact, a map could expand across multiple subject domains. The concept map is being defined as it is being designed, decisions are being made about how far to go in any direction and set its limits. It has no boundaries, whereas a curriculum, learning resource or learning trajectory tends to be a finite linear progression a concept map can expand indefinitely. It is proposed that both the process of collaboratively creating a shared concept map and the map itself emphasise eliciting, enriching and creating new teacher knowledge.

Theoretical framework

The foundations of the learning theory ascribed to in this study are that; mathematical knowledge is created and agreed to by a community because of a need to explain, interpret, communicate or explore (Hersh, 1979); learning is continuous, evident in every aspect of our lives, there is no one final 'knowledge' in any domain (Vollrath, 1994); and participating in activity, including a social activity or personal reflection, impacts on our knowledge, understanding and interpretation of the world, hence results in learning (Engeström, 1999; Vygotsky, 1978).

Research Questions

The aim of this study is to explore if and how collaborative knowledge creation, concerning teacher subject knowledge, can be mediated using an online concept mapping tool (alongside conferencing software). The goal is to develop a practical model that elicits deliberate enrichment and creation of subject knowledge as well as transform the way in which those teachers perceive and reflect on the subject both abstractly and when carrying out the associated activities of teaching.

The research questions are:

1. What types of teacher knowledge do teachers use to create, populate and justify the content and design of a concept map?
2. Does an online concept mapping tool enable teachers to create, share and elaborate upon a representation of mathematical ideas? How?
3. Does an online concept mapping tool enable knowledge creation? How?

Study Design

The study design is detailed in table1. It reflects the aim of collaboratively creating a shared structural representation of mathematics, viewing knowledge as socially created yet considering individual perspectives, experiences, and reflections. An analytical framework has also been designed, details of which can be found in an extended version of this paper, available from the author.

Participants will be purposefully selected, the criteria for participation being that the teachers are in their first five years of teaching, have experience of holding online discussions about teaching mathematics and do not have existing professional relationships (such as being in the same school, co-teaching, training together). The choice of early career teachers was made so that each teacher has a similar number of years of classroom experience on which to draw. Participants with no existing relationships are being chosen in order to encourage them to be clear in their explanations, as no assumptions can be made about each other's experiences, including approaches during their training, specific teaching resources or activities or pupils. In order to promote some heterogeneity, the aim is for those selected to have experience of working with different age groups, to have entered the teaching professional from a variety of routes and have experiences of a variety of institutions (promoting cross-fertilisation). Another aim is for all participants to have had experience of online professional development and therefore, have familiarity with online meeting software and in participating in collaborative discussions with their peers.

Table 1: Study design

Phase	Part	Activity, participants, and format	Aim(s)
1	a	Online demographic questionnaire	To collect demographic information about each participant
	b	Administration and answering questions Online discussion (not recorded) between individual participants and researcher	To familiarise the participants with the research intentions and processes To answer any outstanding questions from the participants To explore and clarify answers given to the demographic questionnaire
		Eliciting initial concept image	To elicit a representation of the participants' concept images for the chosen area of mathematics and

		Individual participants observed by researcher online completing a think aloud activity	To give participants the opportunity to consider the area of mathematics in advance of using a concept map
	c	Introducing concept maps and software Online presentation by researcher and discussion with individual participants	To ensure participants have as much of a shared understanding of concept maps as possible To introduce the Cambridge Mathematics Framework software and develop familiarity in using it
		Eliciting initial individual concept map Individual participants observed by researcher online completing a think aloud activity	To elicit an initial concept map for the chosen area of mathematics To give participants the opportunity to consider the area of mathematics in advance of group discussions using a concept map.
2	a	Paired then group discussions concerning individual concept maps Pairs then group of participants observed by researcher online	To give participants the opportunity to describe, explore and critique their own and other's concept maps
	b	Group collaborative construction of a concept map Group of participants observed by researcher online	To give participants the opportunity to negotiate and collaborate in constructing a shared concept map.
3		Reflection Semi-structured stimulated recall interview between individual participants and researcher	To give participants the opportunity to reflect on the impact taking part in the study has had on their conceptualisation of the chosen area of mathematics To give participants the opportunity to reflect on the impact taking part in the study has had on their own mathematical beliefs; and To give the researcher a chance to explore any observed identified changes in the way that participants discuss mathematics.

Next steps

The data from a pilot study (to the main study detailed above) is currently under analysis. The pilot's aims are to:

- ascertain reasonable expectations for the number and length of interactions;
- assess the clarity and usefulness of the questionnaire, prompts, and other documentation;
- allow the researcher to develop their interview skills;
- verify that the data collection procedures – recording, scanning, transcribing and saving – are appropriate and useful; and
- examine the planned analysis design and process, on a more compact data set.

The pilot provides the opportunity to revisit elements such as specific questions, prompts and planned use of technology, and use these to make adjustments. This iterative process may yet also result in changes to the design of the phases of the full study itself. Whereas there will be four participants in the main study, the pilot study involved two. Initial findings from the pilot are promising with significant commentary from participants on ideas that they had not considered before, ways in which the activity made them re-evaluate existing learning trajectories and their own subject knowledge as well as a request to use the mapping tool outside the study.

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It is my ethical obligation to report that I am employed by Cambridge Mathematics whose software design is being used for this study and who are funding my PhD studies.

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A methodological approach to the development of prospective teachers' interpretative knowledge

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In this paper, we propose a methodological approach—the SCS cycle— suitable to develop prospective teachers Interpretative Knowledge (IK). In particular, this study focus on a group of 19 prospective secondary teachers attending a mathematics education course in Italy who were given an interpretative task involving measurement of the surface area of a rectangle. Their work on the task followed the SCS cycle and was video recorded and later transcribed. The analysis showed that the SCS cycle supported the development of the prospective teachers' IK, but further work is needed to evaluate the effectiveness of the cycle as a way for developing prospective teachers' IK.

Keywords: Interpretative knowledge, SCS cycle, measurement, surface area.

Introduction

Measurement (e.g., of length, area, volume, mass) is an integral part of the school mathematics curriculum (e.g., NCTM, 2000). However, how measurement should be taught for relational understanding (e.g., Skemp, 1976) is often neglected by teachers, as this topic is typically approached as a problem of "finding the correct number" by using a mathematical procedure, indicating that the main goal is finding the final result instead of developing relational understanding.

In order to enrich the understanding of a mathematical topic and use such understanding in the development of fruitful mathematical discussions with students, teachers need to possess a mathematical knowledge which is considered specialized to the practice of teaching mathematics. One of the teaching tasks entailed in the work of teaching is grounded in proposing and discussing tasks with students (Ball et al., 2008). We assume that developing students' relational understanding should be one of the goals of all mathematics teaching, and thus, it is critical to consider the starting point for a mathematical discussion on what the students know and how they know it. This requires that teachers can "listen to the students' thinking" and possess what we have termed as Interpretative Knowledge (IK), which is a specialized kind of knowledge that is not necessarily developed in teaching practice, and is thus an essential focus of teacher education (Mellone et al., 2020).

In our previous work, we discussed the nature and content of (prospective) teachers' IK on the topic of area measurement (e.g., Ribeiro et al., 2018) posed in a context aimed at giving meaning to the area formula for the rectangle. For that purpose, we developed an ad hoc interpretative task. In such a task, prospective teachers (PTs) are situated in a practice-based context where they have to mediate and give meaning to different students' reasonings and justifications for the formula for calculating the surface area of a rectangle. In particular, by proposing this task, we challenged PTs to give a mathematically meaningful justification for the area formula of the rectangle as a direct reading of multiplying the measurement of the length by the measurement of the width. In pursuing the goal of developing PTs' IK, an ad hoc way of implementing the task was developed. We call this method the Small group−Collective−Small group (SCS) cycle, as it is a modification of the Individual−Collective−Individual cycle proposed by Pacelli et al. (2020). In this paper, we focus on the following research questions: What kind of knowledge development can be recognized in PTs when experiencing an SCS cycle involving an interpretative task? In particular, can we recognize any development of IK in these PTs?

Literature review and theory

Understanding a mathematical topic goes beyond knowing the mathematical procedures associated with the topic. This also applies to measurement, considering that it is one of the core mathematical topics in the school curriculum since kindergarten (e.g., NCTM, 2000). For this reason, it is of fundamental importance to develop PTs' and students' understanding of the measurement process in general, and in particular in the context of surface area. Such an understanding can be described by the six principles defined by Clements and Stephan (2004) for the length measurement, which can also be adapted for other magnitudes.

In traditional school practice, the teaching of surface area measurement tends to focus on formulas without meaning, sometimes preceded or followed by teaching of the measurement process where only standardized measurement units are used (Policastro et al., 2017). Moreover, no opportunities are provided for exploring with students the differences and similarities concerning the measurement processes based on different magnitudes. For example, looking at similarities among the measurement processes, it is possible to recognize a common planned action of choosing a convenient unit of measurement to be compared to the quantity being measured, ensuring that both have the same magnitude, and counting how many times the unit of measure fits in the quantity to be measured (e.g., Clements & Stephan, 2004).

The Dynamic Measurement approach is an alternative method of surface area measurement (Parnorkou, 2020). The approach focuses on how space is measured by the lower-dimensional objects that generate it. An inductive approach to visualizing this generation of area (and volume) attributes involves moving objects in space (Parnorkou, 2020). By imagining that a line segment 'a' is swept in a perpendicular direction across a distance 'b', we generate a rectangle with area 'ab'. These two different approaches demonstrate the richness and complexity of a mathematical topic, such as measuring the surface area of an object that is usually considered a trivial task. From this perspective, we argue that teachers should possess a sound and broad mathematical knowledge that contributes to the development of students' mathematical knowledge. Such knowledge will allow them to take the students' own mathematical work, and the differences in the provided representations and argumentations as a starting point—including mathematical work that contains mathematical ambiguities, errors, and non-standard reasoning—assuming that they can be used in practice as learning opportunities (Borasi, 1996). In this sense, the notion of IK (Jakobsen et al., 2014) refers exactly to this deep and wide mathematical knowledge that enables teachers to support students in building their mathematical knowledge by starting from their own reasoning and productions (Di Martino et al., 2019). It includes the ability to expand one's own space of solutions by looking at situations from a wide range of points of view and the capacity for developing specific feedback based on the meaning ascribed to individual students' reasoning (Jakobsen et al., 2014).

Our aim is to develop a methodological approach to support mathematics teachers in the development of IK. Extant research indicates that having in/pre-service teachers working on interpretative tasks can be an effective tool for this purpose (Mellone et al., 2020). By interpretative task we mean a task that in which we ask teachers to solve a mathematical problem and then to interpret students' answers to the same problem (Jakobsen et al., 2014). However, even if PTs' individual work on the interpretative task can contribute to the development of new insights and awareness, it is insufficient for developing a sound IK. As a consequence, referring to the design study methodology (Cobb et al., 2009), we designed an ad hoc methodology for implementing an interpretative task based on the SCS cycle. As implied by the SCS, teachers first work on the interpretative task in small groups of two or three members, after which the task is discussed by all participants in a collective discussion mediated by the teacher educator. Finally, the same small groups of teachers are asked to work on the same interpretative task after one month. This methodology aims at disrupting the vicious cycle of teachers being passive listeners by prompting them to assume an active role in their learning—a strategy we expect they can transpose to their practice.

Moreover, it is widely established that, when teachers work and learn through collaboration, this can have a crucial positive effect on their practices (e.g., Jaworski et al., 2017). Thus, we used the collective mathematical discussion as a collaborative and knowledge-generating activity, in which students' productions are placed at the center of interpretation and feedback construction (e.g., Cobb et al., 2009). Perceiving knowledge as a social elaboration (e.g., Bartolini Bussi, 1996) and recognizing the crucial role of collective discussions in developing awareness about errors and nonstandard strategies (e.g., Levin, 1995) convinced us that we need to allow PTs to be active participants in the learning process and not passive listner, as "when they do talk they ask clarifying questions or acknowledge that they agree or understand" (Spillane, 2005, p. 394). By focusing on the collective discussions about students' productions related to a mathematical problem, our intention is to develop PTs' (teachers') IK from their mathematical social interactions with peers. The task for teacher education (Ribeiro et al., 2021)—an interpretative task in this case—is used both to measure the PTs' IK level and to stimulate subsequent peer discussions. Owing to its nature and structure, the interpretative task aims at prompting PTs to develop novel insights into the mathematical reasoning involved in students' productions. Consequently, IK development is transformed from an individual to a collective activity—a transformation characterized by the evolution of community's norms. This evolution is facilitated by the social setting, where the educator's knowledge is a crucial element for the development of PTs' IK. The collective discussions of PTs' diverse interpretations, reasoning, and reflections upon students' productions is the resource for the educator to orchestrate collective discussions, aimed at identifying mathematical and pedagogical insights and developing the IK. The ultimate goal of this strategy is an evolution from a group of PTs into a professional teaching community (e.g., Cobb et al., 2009), which requires a set of four types of norms pertaining respectively to: (a) general participation; (b) pedagogical reasoning; (c) mathematical reasoning; and (d) institutional reasoning. It is worth noting that the evolution of one type of norms creates conditions within the group for the evolution of norms of another type (Cobb et al., 2009).

Context and method

The context of this study is a Mathematics Education course held in the Autumn of 2020 as a part of a Master's Degree in Mathematics at an Italian University. The requirement for becoming a secondary mathematics teacher in Italy is to have Master's Degree in Mathematics (or Physics) and to successfully pass a public competition organized by Italian government for secondary school teacher recruitment. The students had already completed a Bachelor's Degree in Mathematics or in Physics and are considered to have a strong mathematical knowledge. The Mathematics Education course is a non-compulsory course, but it is typically chosen by students who intend to become secondary teachers, due to which all participants are considered to be PTs. The study participants are the 19 PTs who attended the course which was held online in a synchronous way—through Microsoft Teams—due to the restrictions imposed on mobility and gatherings to prevent the spread of COVID-19. The online teaching was recorded.

Task and activities

The interpretative task we discuss here was proposed to the PTs during the final part of the course. It consisted of three parts. First, the PTs were asked to answer a generic question on how to define the area of a figure in a plane. Then, there they were asked to find the area of a rectangle with sides measuring 3 cm and 4 cm, to provide an argument for their answer, and relate their answer to the first question. Finally, they were asked to interpret four 5th graders' productions to the area of the rectangle problem (Figure 1), focusing on "listening to the students' thinking and reasoning" to make sense of their solutions, and provide a constructive feedback to each one of those reasonings to support students' mathematical understanding (Ribeiro et al., 2018). One of the aims of the task was to discuss the meaning of the area formula for the rectangle and to refine the PTs' understanding and meaning attribution to the product of two lengths.

Consider the following students' productions to the question: Determine, and justify, the area of a rectangle with sides measuring 3 *cm* and 4 *cm*.

Caio: Multiplying the length by the width, we get $4\ cm \times 3\ cm = 12\ cm^2$.

Douglas: The area is a surface measurement and thus it has two dimensions (length and width) so we need to put the 2 in the exponent and we get $3 \times 4 = 12\ cm^2$.

Camila: We just need to count the number of square centimeters needed to cover the square, and thus we get $3\ cm^2 \times 4\ cm^2 = 12\ cm^2$ or, similarly, $4\ cm^2 \times 3\ cm^2 = 12\ cm^2$.

Fernanda: I think the area is $12\ cm^2$ as we have to do $4 \times 3\ cm^2 = 12\ cm^2$ or $3 \times 4\ cm^2 = 12\ cm^2$.

Figure 1: Students' productions included in the interpretative task

Although all students' numerical answers are correct (12 cm^2), their reasoning and argumentation differ and are associated with different interpretations of area, area units, and the meaning associated with the formula (A = length × width).

This interpretative task was implemented in three phases using the SCS cycle methodology. It was implemented in the middle of the master's course and the interpretative activity was proposed to the

PTs during an online class using the “Activity” function of the Microsoft Teams platform. Using the “Breakout rooms” function, the PTs were divided into small groups (of two or three members) for working on the task for one hour. At the end of this period, they had to deliver to the educator a shared written interpretation using the “Activity” function. Upon completion of this phase, the educator orchestrated a 90-minute collective discussion of the task for all the PTs using Microsoft Teams (phase two). In the final phase, the PTs were given Parnorkou’s (2020) paper as a reflection on the Dynamic Measurement approach for measurement of surface area. The PTs had one month to read the paper, reflect upon the task and co-write, using the same small working group as in phase one, a reflection on the lived experience and to hand it in to the educator. This phase was also part of our goal as educators to constitute a professional teaching community (Cobb et al., 2009).

When conducting the analysis of the PTs’ productions in the three phases of the SCS cycle, we focused on identifying the knowledge mobilized. In what follows, we present our analysis of one group’s work during the first and third phase, while attempting to trace the knowledge and awareness developed during the collective discussion (phase two).

Interpretative knowledge revealed and developed – some discussions

Danilo, Pietro, and Caterina, as a group, wrote in the first phase of task implementation (before the collective discussion) the following:

> Caio gave the standard definition and therefore it is not clear to us if he actually understood the meaning of the operation or if he simply applied a definition he had memorized. We would recommend a graphic approach in which the sides are divided into respectively 3 and 4 equal parts and from there we can see that each of these forms a square with a unitary area.

In some sense, we can see that these students are mature in giving feedback. In particular, in their suggestion to divide each side in equal segments—hence linking up to the idea of making a regular grid of squares with unit area—we can recognize their effort to help Caio to link his calculations with the meaning of covering the surface of the rectangle with unit squares.

The idea of making regular grid also emerged in the collective discussion:

Marco:	In my opinion, if we talk about units of measurement and therefore the symbolic expression, only Caio wrote well. The others have all made a mistake, either because they didn't write the unit of measure or because they added too many of them.
Rino:	In Fernanda’s case, I don't see formal errors.
Marco:	In my opinion, Fernanda is wrong because you can't write like this for units of measurement.
Rino:	The area is expressed in square centimeters, and she took four of them.
Pietro:	It is as if she had taken four strips three high, or three strips four long and covered the rectangle. It could be interpreted like this, obviously I don't know if that's what she thought.

We can see that there is a collective effort to give meaning to Fernando’s answer, and to create a link between Caio’s and Fernanda’s answers by making a regular grid covering the rectangle. After this, the educator prompted the PTs to choose between Caio and Fernanda:

Educator: If you have to write on the blackboard which formula would you write?

Caterina: If I had to explain the area, I would always use the covering technique. I would try to highlight this concept every time because it seems to me the central one in the measurement theory. But obviously I also like a unit of measurement written each time and, therefore, a dimensional analysis like Caio's works.

Danilo: But in some way also Caio is a covering if you mean a covering of thin strips, as many strips that would be the height for how long the base is.

Caterina: But in fact he did not make any mistake, he wrote well, if one really has to go to the bottom and ask what the area is, then I would like a child to understand that it is a covering with small cells at will.

Danilo: But you must also go and explain what the elementary area with which he covers is.

Educator: I believe that what Danilo tried to do by reading Caio's formula is to define this elementary square with a side of 1 cm and see it as generated by a segment of one cm which is repeated continuously a certain number of times. This is strange to say sometimes because it is a continuous movement, then cover it with very small strips in this case 1 high, which is precisely the fundamental theorem of integral calculus.

Danilo: Yes, it was how I imagined it when I was as a child.

Educator: So you already had a vision of the fundamental theorem of integral calculus and you didn't know it.

The educator's provocation was overcome by Caterina proposing to look at the area first as a covering process (referring to Fernanda's answer), but also expressing appreciation for the dimensional analysis present in Caio's answer. Caterina's comment created the opportunity to Danilo to present his crucial observation that gave new insight into Caio's answer. He proposed interpreting Caio's answer also as a covering process, but performed by using a "thin" linear segment to repeat for "how long the base is." We can recognize a link between this interpretation provided by Danilo and the Dynamic Measurement approach mentioned earlier that relies on visualizing the generation of a rectangular area by mentally sweeping a segment corresponding to one side of the rectangle along its perpendicular direction, in other words along the other side, for a distance corresponding of the length of the side (Parnorkou, 2020). It is important to underline that this interpretation of Caio's answer was not presented by Danilo in the previous phase in which the group comprising of Danilo, Pietro, and Caterina just expressed the grid covering perspective, and this represents an evolution of the mathematical reasoning norms (Cobb et al., 2009). It is also noteworthy that the evolution of one type of norms created conditions within the group for the evolution of norms of another type. In particular, the evolution in this mathematical reasoning norm also created an evolution in the pedagogical reasoning norms in the sense that the PTs were also expanding their space of solutions by looking at the situation from a wide range of different points of view, consistent with the IK approach.

Caterina's emphasis of the importance of considering the covering process as the underlying meaning of the area measurement shows that she has in fact failed to grasp Danilo's point of view. Still, the fact that Caterina does not understand gives Danilo a new opportunity to advocate for his point of view, challenging Caterina's suggestion for how to define the area of the square, used as unit of measure. The educator tried to respond to Danilo's challenge by proposing that PTs visualize generation of the area of the square (used as a unit of measure) using Danilo's dynamic way of looking at it, and by making an explicit reference to the Fundamental Theorem of integral calculus behind this vision.

One month after Parnorkou's (2020) paper was provided to all the PTs, Danilo, Pietro, and Caterina wrote a new interpretation, part of which is replicated below:

> Beyond some errors concerning the units of measurement, two different approaches emerge from the words of the students, already known in literature: that of covering and that of dynamic measurement. […] In any case, it is important to underline how the teacher must be able to recognize these two different approaches, consider them equally valid, but choose a starting one to present to the class, and then show equivalence with the alternative approach (perhaps following a discussion in the classroom from which this dichotomy may emerge).

In this excerpt, we appreciate the careful and effective summary made by this group of PTs of their new knowledge, which emerged from their participation in the SCS cycle and by working on this interpretative task. We stress that this final writing represents an important part of the cycle. By completing this written reflection, the PTs were able to organize their new IK. In particular, in the text provided by Danilo, Caterina, and Pietro, we see how their initial IK has been developed and enriched by the two approaches to surface measurement, and they are now able to present it in organized manner. The awareness of the possibility of approaching the measurement of the surface in two ways represents an evolution of the PTs' mathematical reasoning norms. This also prompted an evolution of their pedagogical reasoning norms (Cobb et al., 2009), as evident from this quote from one of the PTs: "the teacher must be able to recognize these two different approaches, consider them equally valid."

Some final comments

In this study, we have found that the adoption of the SCS cycle methodology has supported the IK development among the PTs. This is an initial study, and we propose that further research into the effect of the SCS cycle on developing the IK among PTs be conducted.

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Foundation and justification of mathematics teaching decisions

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Keywords: Decision-making, teacher knowledge, mathematics education, research design.

Poster summary

Already in the seventies Alan Bishop considered decision-making at the "heart of the teaching process" (Bishop, 1976, p. 42) and together with other scholars of that time, such as Lee Shulman and Richard Shavelson, thought that if we better understand these decisions, we can better understand teaching (Borko et al., 2008). In more recent years, decision-making has gained a renewed interest as a situation-specific skill, especially within research pertaining mathematics teacher noticing (Dindyal et al., 202, Stahnke et al., 2016). In a review by Stahnke et al. (2016) regarding mathematics teaching situation-specific skills; perception, interpretation and decision-making, the latter was the construct least commonly addressed in the included studies. They also found that especially decision-making was difficult for pre-service teachers.

Within research on pedagogical content knowledge (Depaepe et al., 2013) and situation-specific skills (Stahnke et al., 2016) in mathematics education two perspectives are identified: a cognitive and a situated. The cognitive perspective focuses more on knowledge as a disposition. The situated perspective on the other hand pays more attention to the context where the knowledge is used (Depaepe et al., 2013). Common designs for the first mentioned includes larger samples with less contextual focus and the latter often utilizes smaller case studies in classroom settings.

Both perspectives have their merits and pitfalls (see Depaepe et al., 2013, p.23) that will have consequences for methodological design and study results. Hence, Stahnke et al. (2016) calls for designs that integrates perspectives and further notes that approaches that links mathematics teaching competence, skills and performance are rare. This is also in line with Mason (2016) who underlines the need for both studies with sufficient samples sizes for multivariate analyses and studies that acknowledge teacher actions and their justifications that could be beneficial for teaching development. As Depaepe et al., (2013) points out, teachers' justification of decisions in teaching is a core component of their knowledge base, these decisions and their justifications could provide grounds to further understand mathematics teachers' knowledge base.

In summary, this calls for further attention to mathematics teachers decision-making, its relation to other aspects of teaching competence, and how theory and design could address both disposition and skills, with attention to context and eliciting justification as means for understanding mathematics teaching. This is of importance both to the mathematics education research community but also to teacher education.

The aim of this poster is to contribute to the ongoing discussion regarding mathematics teaching competence by focusing on decision-making in mathematics teaching, its foundation and justifications. This is done by presenting an ongoing project that combines aspects from different perspectives with the purpose of gaining further understanding of mathematics teaching. The project draws from Schoenfeld's (2010) theory of goal-oriented decision-making together with the view of competence as a continuum from Blömeke et al. (2015), where decision-making is seen as a mediator between disposition and observable behavior. The project utilizes a mixed methodology to be able to explore the foundation and justification of teaching decisions and also connecting this to student learning and affective outcomes, which has been rare in previous research.

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Engaging community colleges faculty in developing items for assessing mathematical knowledge for teaching

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In this paper we investigate how experienced community college mathematics faculty, taking part in an item-writing camp designed to generate items to assess mathematical knowledge for teaching college algebra, thought about student thinking and the work of teaching in this context. We conceived of the participants' knowledge as practice-based professional learning. By analyzing video records of item development sessions, drafts of items produced, feedback sessions on the items, and self-reports of their learning during the camp, we identified four major themes related to participants' work highlighting the possible advantages of engaging faculty in item-writing: understanding mathematical knowledge for teaching, understand the item writing process, drawing on their teaching experience in the item-writing process, and noticing the complexity of writing MKT items. These themes make salient the advantages of engaging classroom-based faculty in item-writing.

Keywords: Mathematical knowledge for teaching, tasks of teaching, choosing problems, understanding student's work, student thinking.

Introduction

Over the last two decades, researchers have attempted to identify and describe the relationship between teacher mathematical knowledge for teaching and student achievement (Hill, Rowan, & Ball, 2005). Several research groups have developed instruments for measuring this special type of knowledge targeting specific grade bands (e.g., Hill, Schilling, & Ball, 2004). To develop items that reliably assess this type of teacher knowledge researchers have collaborated with mathematicians and practitioners (teachers, teacher educators, and professional developers), so that both the mathematics at stake and the ways in which it manifests in the classroom are authentically represented in the items. Relevant grade-level teachers have taken part in cognitive interviews, instrument piloting, or assessment item-writing camps (Phelps et al., 2014). Selling and colleagues (2016) have argued that

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item developers need support to make sense of the nuances of the knowledge construct and the best ways to assess it, yet there is little documentation about how classroom-based faculty work in collaboration with mathematical knowledge for teaching (MKT, Hill, Schilling, & Ball, 2004) researchers to create items for MKT instruments, or about their thinking process as they generate ideas and contribute to the item-writing process. This paper presents an exploration of how a group of 13 practitioners (community college mathematics faculty who teach college algebra), chosen from different regions of the United States, thought about their students' thinking, in the context of a two-week item-writing camp in which they were asked to produce 10 different multiple choice or testlet MKT items each. Involving teaching faculty in writing MKT items is not typically done by other MKT-assessment instrument developers—those groups limit practitioner input to feedback through cognitive interviews and pilot studies. We set out to answer this research question: In what ways does leveraging hypothetical examples of student work facilitate the development of MKT items designed to measure instructors' knowledge of students' mathematical reasoning?

The participants were invited as part of a large project that seeks to develop an assessment for mathematical knowledge for teaching college algebra at U.S. community colleges. College algebra is a credit-bearing course taught in postsecondary institutions in the United States, and typically covers content related to linear, rational, and exponential functions, as well as other relevant algebraic concepts in preparation for calculus. The number of students taking this course at U.S community colleges has steadily increased from about 190,000 in 2000 to about 300,000 in 2015 (Conference Board of the Mathematical Sciences, 2015). This course is considered a gateway to degrees in science, technology, engineering, and mathematics (STEM), being a prerequisite for a STEM calculus sequence. The immense number of students who take this course at U.S. community colleges, and consequently the large number of teachers needed to fill this need, creates an opportunity for developing an instrument that assesses the MKT of community college faculty teaching college algebra, (MKT-CCA), in these institutions; such instrument can serve as a tool for understanding the quality of instruction of this key course and its impact on student learning.

Sustained prior work to corroborate that MKT is composed of the various dimensions identified in the literature (e.g., subject matter knowledge, pedagogical content knowledge, Shulman, 1986) has proven very difficult. We chose follow Ko and Herbst (2020) who successfully distinguished knowledge needed when geometry teachers perform two different tasks of teaching, understanding student work and choosing problems. Tasks of teaching are the "recurrent practices that make up the work of teaching [school] subjects" (Phelps, et al., 2014, p. 3). Since these tasks of teaching are indeed foundational for teachers' work and given that there is evidence that it is possible to identify two distinct types of knowledge, we chose this conceptualization to develop our instrument. We used a tasks of teaching framework and adopted a two-dimensional hypothesis for the construct: (1) choosing problems and (2) understanding student work. We hypothesize that the knowledge that community college instructors use to choose problems for teaching college algebra can be distinguished from the knowledge that they use to understand students' work in college algebra. We (four university researchers, four community college faculty-researchers, three graduate students) developed a pool of items after training in instrument development. After developing a set of principles for writing items we invited practitioners to develop items along these two dimensions.

Methods

We recruited 13 community college faculty to participate in a two-week item-writing camp. These practitioners came from different geographical regions, taught at different types of colleges (urban, suburban, large, small), had varied demographic characteristics (e.g., race, gender), and had some experience writing cognitive tests and working collaboratively in small groups. These criteria generated a diverse group and shortened the time needed for training and team building. During the virtual item-writing camp we provided training for the participants that included: (1) an introduction to MKT literature, (2) definitions of the two tasks of teaching, (3) examples of items from our item pool, and (4) ways of checking that items fit the definition of the task of teaching they were targeting. We also provided a set of algebra topics we selected as representative of core mathematical ideas in college algebra, and that relate to linear, rational, and exponential functions. The faculty worked in self-selected teams of 2 to 3 participants; they were asked to draft items with feedback being provided by other participants and members of the research team in several cycles of feedback and revision.

Data and analysis

We collected two types of data (1) audio and video recorded work sessions in which the faculty worked collaboratively in teams to draft items, reviewed draft items from other teams, and used other teams' review comments to change and improve their drafts, and (2) reflection memos from the 13 instructors written at the end of each week, and that sought personal insights on the work of writing items. We used both ongoing and retrospective analyses of the data. Ongoing analysis occurred during the item-writing camp and involved reviewing the reflection memos after Week 1 to identify common themes using an open coding and a constant comparative method (Corbin & Strauss 2008).

Theme: Complexity of Writing MKT Items	
Objective Choices for Multiple Choice Items	It is challenging to create tasks with objectively correct options and objectively incorrect options (Mark, Memo #1, June 2, 2021)
Evidence for Student Understanding	A big reminder from the past week or so has been the idea of "what does it mean to understand __?" For example, to demonstrate understanding if a student computes the annual growth factor based upon 10-year percentage growth, that doesn't guarantee that they understand. (Steph, Memo #1, June 4 2021)
Metacognition	It feels very meta to write not just what student thinking might be but to take it to the level of what a teacher might be thinking about student thinking and how student thinking can be elicited. (Maggie, Memo #1, June 4, 2021)

Figure 1: Sample theme, subthemes, and sample quotes

These themes informed prompts for the reflection memo delivered after Week 2. Figure 1 shows one of the themes, its subthemes, and sample quotes that emerged. During retrospective analysis, we examined all available data (recordings and memo responses) using the coding system used to analyze the reflection memos from Week 1 while adding new themes and sub themes as needed. Four researchers independently coded these records; then met to compare, discuss, and agree on a final list of themes identified across the data set. This paper presents preliminary findings focusing on the data found in participants' memos.

Preliminary Findings

Four themes emerged from our data related to the participants': (a) attempts to understand MKT, (b) attempts to understand the MKT-item writing process, (c) attempts to draw on their teaching experience in the item-writing process, and (d) realizations of the complexity of writing MKT items. The first two themes were not entirely surprising, as they represent issues that anyone doing the work of MKT instrument development has to grapple with. Due to space limitations, we present findings about the last two themes to highlight insights into experiences of MKT writers who were encountering MKT research for the first time. These themes represent ways in which participants wrestled with writing items to assess community college instructors' thinking about student thinking.

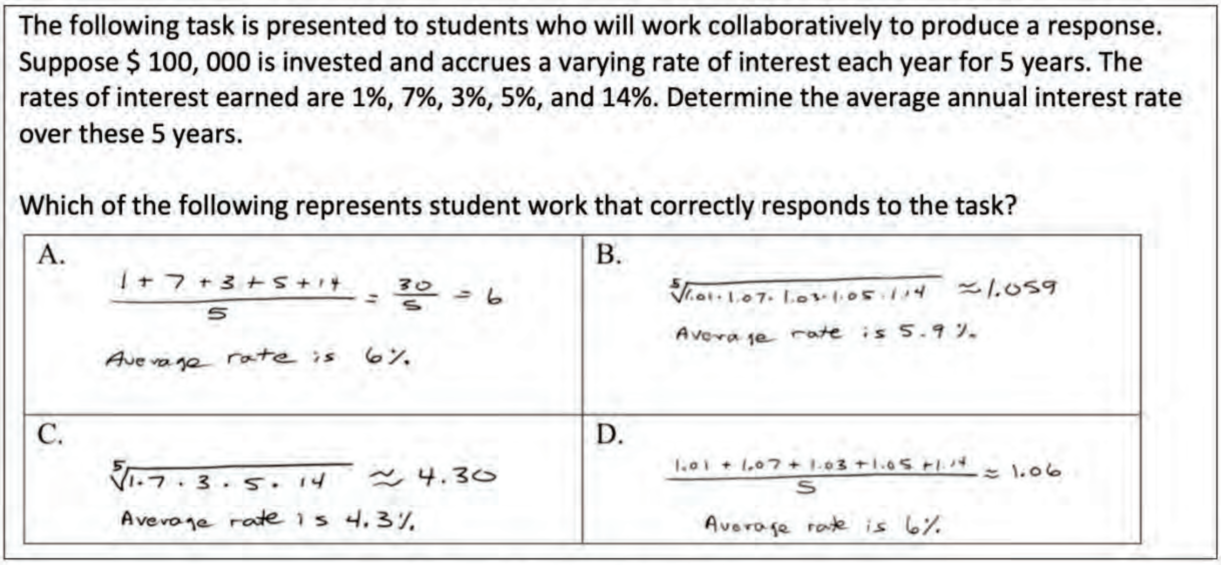
Drawing on Teaching Experience

Participants leveraged their teaching experience in the item-writing process in two different ways: (a) They used common student errors that they have observed in their own teaching of college algebra and (b) negotiating the goals of the college algebra course impacted their decisions in item writing.

Using Common Student Errors. Of the 13 instructors, ten reflected on how they sought to use common student errors they have encountered in their experience teaching college algebra to write items that would assess community college instructors' thinking about student thinking. They wrote that they began with common "mistakes," "errors," or "misconceptions" that they see in their own teaching. For example, Steph (pseudonym) wrote:

> I thought about problematic areas… common (from my experience) struggles [students] face or challenges they have in making sense of the ideas… like percent, geometric mean, ... then I created a couple of items to assess an instructor's ability to connect their math content knowledge with these hypothetical student responses. (Steph, Memo# 2, June 10, 2021).

Steph gave Figure 2 as an example of an item he created by reflecting on his knowledge of students' thinking. His ability to reflect on a common student struggle informed his decision to draft a realistic problem for college algebra at community colleges. Participant's knowledge of students' struggles with mathematical ideas was valuable for the item development process.



The following task is presented to students who will work collaboratively to produce a response. Suppose $ 100, 000 is invested and accrues a varying rate of interest each year for 5 years. The rates of interest earned are 1%, 7%, 3%, 5%, and 14%. Determine the average annual interest rate over these 5 years.

Which of the following represents student work that correctly responds to the task?

A. $\frac{1+7+3+5+14}{5} = \frac{30}{5} = 6$ Average rate is 6%.	B. $\sqrt[5]{1.01 \cdot 1.07 \cdot 1.03 \cdot 1.05 \cdot 1.14} \approx 1.059$ Average rate is 5.9%.
C. $\sqrt[5]{1 \cdot 7 \cdot 3 \cdot 5 \cdot 14} \approx 4.30$ Average rate is 4.3%.	D. $\frac{1.01+1.07+1.03+1.05+1.14}{5} \approx 1.06$ Average rate is 6%.

Figure 2: Draft MKT-CCA Item Created by Steph

Negotiating Goals for College Algebra. Our analysis revealed that participants leveraged their teaching experience as they wrote items to assess instructors' ability to interpret students' work relative to the instructional goals for the course. For example, Taylor wrote about the importance of writing items that focus on community college instructors' ability to understand their students' conceptual and procedural understanding:

> I tried to think about common errors or misconceptions I observe students making consistently... I also try to go back and forth between procedural and conceptual knowledge. Having procedural knowledge doesn't guarantee having conceptual understanding but it was difficult sometimes to pull and separate the two… I think that some of my items were focused on procedural knowledge more so than conceptual knowledge although having conceptual knowledge is key. (Taylor, Memo #2, June 10, 2021).

Taylor's understanding about the goals for college algebra enabled her to be mindful of the different types of understanding that the course seeks to develop, and to then think about the types of items needed to assess instructors' thinking about these types of student understandings. Trent also seemed to be thinking about the goals for this course when he wrote:

> There is still a large divide between what different [other participants] think is most important in college algebra. My team is very much focused on conceptual understanding and big ideas and others appear to still hang on to what I personally think are outdated aspects of the course. (Trent, Memo #2, June 10, 2021).

Trent's comment highlights a challenge related to intended outcomes for college algebra as there is variation in course outcomes for college algebra across institutions and states. Trent's statement seemed to go a step further than Taylor's by thinking about how other participants in the item camp were thinking about teachers' thinking about student thinking. Professional teacher organizations in the U.S. encourage mathematics teachers at all levels to strike a balance between developing their students' conceptual understanding and procedural fluency (e.g., American Mathematical

Association of Two-Year Colleges 2018). The challenge of making sense of a students' depth of understanding using a snapshot of a student's work presented on an MKT-CCA item in the absence of a dialogue with the student complicated the ability of item camp participants to write items that would assess an instructors' ability to think about and make judgements about student thinking. The participants' insights about the varying goals of college algebra contributed to the item-writing process in terms of the development of MKT-CCA items that assess test takers' ability to think about students' understanding in both procedural and conceptual terms.

Realizations of the Complexity of Writing MKT Items

Our data showed participants negotiating the complexity of writing MKT assessment items in three ways: (a) creating multiple-choice options that could be evaluated objectively, (b) evaluating evidence of students' understanding, and (c) the challenge of metacognition.

Creating multiple-choice options. Not only did the participants have to contend with crafting an item stem about an identified common student error but they also had to create either multiple-choice options or binary options for testlets that would be realistic for the stem and for college algebra. Eight participants wrote about their struggles with writing multiple-choice options. They gave varying reasons for why they struggled including: writing options that could be evaluated without subjectivity, writing realistic incorrect student responses, and using snapshots of students' incorrect work produces some unpredictability since students' work cannot be easily explained or packaged into a multiple-choice item. For example, Steph said:

> Multiple choice questions are VERY hard to write. Do the MC choices have to be right or wrong? Or can they all be viable and the response will tell us something about the respondent? In my first case, correct student work is shown and the respondent is asked which question they might ask to follow up on student thinking. It might be argued that all 4 questions (MC options) are possible (that is... not really "wrong" or "right"...) (Steph, Memo #1, June 4, 2021)

Like Steph, most participants struggled to decide on how to orient multiple-choice options to best assess a test taker's understanding of students' work or the test taker's perceptions of the best way a teacher would respond to help a student struggling with a math concept. The latter was further exacerbated by test takers' perceptions of the goals of college algebra and how it should be taught—two instructors could select diverging paths of intervention (e.g., two different follow up tasks) they believe would be appropriate intervention depending on their believes about teaching mathematics.

Evidence of student thinking. Ten participants wrote about the challenge of using students' written work and how that made writing items assessing community college instructors' ability to understand students' work difficult. Of the ten, five wrote about the difficulty of moving from a common student error to an MKT-CCA item, as illustrated in the following excerpt from Star's reflection memo:

> Nearly all of my items were drawn from actual work that my students have done in the past. I found that more often than not, the mistakes that students make cannot easily be explained/do not follow a predictable pattern. While it made it very difficult to write options for those stems, I think those will be the most valuable for testing MKT! Writing the "Understanding" items was an interesting experience because it highlighted the assumptions we make about student work. What

evidence do we base those assumptions on? Or is it all experience? (Star, Memo #2, June 10, 2021).

Star's pondering about the challenge of writing items about students' work, that different test takers who bring different teaching experiences, can interpret in a way that leads them to choose the same correct choice without the option of asking the student follow-up questions was shared by other participants. The jump from identifying common student errors to crafting an item that would meaningfully assess a community college instructor's thinking about the student's thinking posed challenges that Star and other participants had never wrestled with.

The Challenge of Metacognition. The reflection questions prompted participants to make explicit not only their thinking about their knowledge for teaching college algebra but to also think about the test takers' thinking about students' thinking and how that thinking would elicit knowledge needed in the particular task of teaching. The difficulty item writers had in thinking about student thinking suggests that the activity is not standard practice among community college instructors, that is, thinking about other faculty's knowledge is not in the scope of community college instructors' role as teachers, and we hypothesize that it is therefore rare. Star said:

> The line between knowledge of college algebra and knowledge for teaching college algebra is much finer than I thought. My first pass at a lot of the items I wrote ended up testing the participant's [mathematical] knowledge, rather than their knowledge for teaching. I think a lot of my knowledge for teaching college algebra is subconscious, and I still haven't quite figured out what the difference is, in explicit terms. (Star, Memo #2, June 10, 2021).

Star's reflection suggests that the work of writing MKT items is different from the work of writing assessment items for students that focus on college algebra content. MKT items, especially those focused on understanding students' work are inherently different in the sense that they target to assess a test taker's mathematical thinking about students' thinking. Maggie touched on this in her memo:

> The writing of the items is also new for me. It feels very meta to write not just what student thinking might be but to take it to the level of what a teacher might be thinking about student thinking and how student thinking can be elicited. Putting that into parallel foils is hard. Heck, writing the stem is hard! (Maggie, Memo #1, June 4, 2021)

Discussion

Involving practitioners in item writing meant adding a layer of training—as this was a group not intimately engaged in researching MKT—but also brought valuable perspectives to our item-writing process. Practitioners' experiences teaching college algebra at community colleges and their understanding of linear, rational, and exponential functions helped to enrich our database of items. Further, their ability to draw on their experiences to highlight common student errors for students in the targeted college algebra topics and their reflections on the goals for the course impacted the types of MKT-CCA items they drafted, which we believe will add to the diversity of items in the instrument. Finally, we found that the process of writing MKT-CCA items to assess instructors' thinking about student thinking provided the instructors who attended the item camp valuable insights into their own practice. Analysis of the data offered clues about how participants wrestled with issues such as what it means for a student to understand an idea and the kind of evidence needed to show understanding. The discussions held among partners in the item writing or reviewing process were rich and allowed the writers to think about issues around instruction that they do not ordinarily think about outside the context of thinking about a colleagues' thinking about student thinking.

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Functional thinking development in Slovakia

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Keywords: Function, aspects of functional thinking, mathematics teachers.

Introduction

This research deals with the teaching of functions and students' functional thinking development in Slovakia. According to Doorman et al. (2012) and Pittalis et al. (2020), there are four main aspects of the functional concept and functional thinking:

1. The function as an input-output assignment: This view on function as an input-output machine stresses the operational and computational character of the function concept. It also comes into play when patterns and structures are investigated (recursive patterning).
2. The function as a dynamic process of co-variation: This aspect concerns the notion that "two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other." (Thompson & Carlson, 2017, p. 444)
3. The function as a correspondence relation: A correspondence relation includes identifying a correlation between variables, using the function rule to predict far-function values, and finding the value of one variable given the value of the other. (Confrey & Smith, 1995)
4. The function as a mathematical object: "A function is a mathematical object which can be represented in different ways, such as arrow chains, tables, graphs, formulas, and phrases, each providing a different view on the same object." (Doorman et al., 2012, p. 1246)

This set of four views has some taxonomy characteristics, in the sense that it shows an increasing level of sophistication and may also suggest an order in which to acquire functional thinking. Therefore, the main aim of this research is to describe how is the functional thinking perceived by Slovak teachers and how the Slovak textbooks cover this topic.

Method

This study is part of the European project Erasmus+ FunThink that tries to improve functional thinking in a transnational perspective drawing on the partners' specific and complementary expertise. To describe the situation in Slovakia, we conducted six semi-structured interviews with Slovak teachers from primary to tertiary schools and analysis of the Slovak mathematics textbooks. Subsequently, we analyzed the data collected in the terms of the mentioned aspects.

The interviews were focused on the person's understanding of functional thinking, ways to address functional thinking in the class, and the implementation of some design principles. Due to the current situation, the interviews took place online, all of them were recorded. The Slovak mathematics textbooks, by Kubáček (2010-2012), cover all four years of secondary education in Slovakia and are the only mathematics textbooks approved by the Ministry of Education. In the tasks, we monitored which aspect occurs in them and which aspect is used most often in the tasks.

Results and conclusions

All six interviewees mentioned the search for connections and relationships between values or variables when describing functional thinking. They also agreed on what is the main problem in developing functional thinking in Slovakia. That is, as one teacher said: “Mathematics tools - variables, algebraic expressions, equations were reduced in the secondary school curriculum. Therefore, high school students have a big problem with the perception of variables. That in turn hinders the development and inquiry of a mathematical model of a function. The student intuitively perceives the dependence between variables, but the problem pops out when it is necessary to work with a mathematical apparatus.” Teachers emphasized that as many tasks as possible with a real context should be included in the teaching, thus linking knowledge to practice.

Given the four aspects of teachers' way of teaching, we have come to the following conclusion: When teaching some unit e.g., a linear function, they begin with aspect 1. Then their teaching methods differ. They either solve problems with the real-world context or everyday situations (aspect 2) by which they gradually generalize new knowledge (aspect 3) until they are completed with the definition of a linear function (aspect 4), or first they define the concept of a linear function (aspect 4) and then address tasks involving aspects 3 and 2. We can also add that in Slovak textbooks by Kubáček, the first three aspects occur in approximately the same number, tasks for the fourth aspect are rare, in some cases absent. In conclusion, we can add that there is a certain discrepancy between interviews and textbooks, which is influenced by the tradition of teaching functions.

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Didactic suitability criteria related to the reflection on the implementation of mathematical modelling in a virtual context

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We report the results of a research focused on studying what aspects of the instructional process prospective mathematics teachers relate to their reflections on the implementation of mathematical modelling in their didactic proposals. To do this, we used the construct Didactic Suitability Criteria proposed by the Onto-Semiotic Approach, which is the tool used by prospective teachers in their Master's Degree Final Projects to reflect on their own practice. In methodological terms, we carried out a content analysis of 122 Projects written during the 2019-2020 academic year, and whose implementation was performed in a virtual context due to the COVID-19 pandemic. Among the results, we stress that the prospective teachers related their comments on mathematical modelling mainly to the epistemic, affective, and ecological criteria of the implemented instructional process.

Keywords: Didactic suitability criteria, mathematical modelling, teacher reflection.

Introduction

There is a worldwide consensus on the development of competencies that involve the use of mathematics to solve real-world problems, among which mathematical modelling stands out (Kaiser, 2020). This competence is considered as: a) a central aspect of PISA for problem solving (Organisation for Economic Co-operation and Development, 2019); b) a beneficial process for the learning of mathematics (Blum, 2011); c) essential to train individuals capable of linking their mathematical knowledge to contemporary needs (Doerr & Lesh, 2011). Some studies have been reported on the role of modelling in teacher training (e.g., Tekin, 2019; among others), which are in line with the Maaß's (2007) idea that it is not enough just to train teachers in modelling, but they must also experience it. Unlike such studies, the one here reported focuses on the reflection of prospective teachers on the implementation of modelling in their Master's Degree Final Projects (MFP).

In the Spanish context, prospective teachers must obtain a master's degree in order to teach mathematics at secondary and baccalaureate education, for which they must prepare an MFP. This is an original, autonomous, and individual work, which allows the student to show the training content received and the general competencies acquired during the master's program in an integrated way. Furthermore, it must contribute to reflect on and deepen the analysis of their own practice, making it possible to propose elements for its improvement. Thus, due to the importance of both modelling and teacher reflection within teacher training, this study raises the question: What aspects of the instructional process do prospective mathematics teachers relate to mathematical modelling when they reflect on its implementation? In order to answer it, we analysed the reflection that prospective teachers made in their MFPs on the design and implementation of their didactic proposals, in which they included the work with modelling. We analysed this reflection using the Didactic Suitability Criteria (DSC) proposed by the Onto-Semiotic Approach (OSA) (Godino et al., 2007), which was the same tool used by the prospective teachers to reflect on their own practice.

Theoretical framework

Mathematical modelling

In general terms, the modelling process is understood as a transition between the «real world» and «mathematics» for solving a problem-situation taken from reality. At a theoretical level, different cycles have been designed to explain this process (Borromeo, 2006), as well as different perspectives on its implementation in classroom have emerged (Abassian et al., 2020). Since in this study we did not adopt any particular modelling cycle or perspective, we considered some consensual attributes that characterise both the work with this process in classroom and a modelling problem.

The work with modelling in classroom is usually carried out in small groups of students, to whom a real-world problem-situation is posed that they must mathematise (Doerr & English, 2003). Modelling activities involve a cyclical process, with different ways to obtain a plausible and coherent solution with the context of the situation posed (English, 2003). This problem must be *open* (not limited to specific answers or procedures), *complex* (useful information must be distinguished from the rest of the wording of the task), *realistic* (adding elements taken from reality), and *authentic* (a situation consistent with a fact from reality) (Borromeo, 2018).

Didactic suitability criteria

In the OSA (Breda, 2020), the *didactic suitability* of a teaching-learning process is understood as the degree to which it (or a part of it) meets certain characteristics that allow it to be qualified as suitable (optimal or adequate) in order to achieve an adaptation between the *personal meanings* achieved by the students (learning) and the *institutional meanings* intended or implemented (teaching), taking into account the circumstances and available resources (environment). This multidimensional construct consists of six *suitability criteria*: *epistemic criterion*, to assess whether the mathematics that is taught is 'good mathematics'; *cognitive criterion*, to assess, before starting the instructional process, whether what is intended to be taught is at a reasonable distance from what the students know; *interactional criterion*, to assess whether the interaction solves students' doubts and difficulties; *mediational criterion*, to assess the adequacy of resources and time used in the instructional process; *affective* (or *emotional*) *criterion*, to assess the students' involvement (interest, motivation) in the instructional process; *ecological criterion*, to assess the adaptation of the instructional process to the educational project of the school, the curricular guidelines, the conditions of the social and professional environment, etc. Each of these criteria has its respective components, and its utility requires defining a set of observable indicators, which allow assessing the degree of suitability of each of the facets of the instructional process. The list below shows the components of each DSC with the codes used in this research to label them, based on the Breda and collaborators' (2017) guideline.

Epistemic: Errors (ES1); Ambiguities (ES2); Richness of processes (ES3); Representativeness of the complexity of the mathematical object (ES4).
Cognitive: Prior knowledge (CS1); Curricular adaptation (CS2); Learning (CS3); High cognitive demand (CS4).
Interactional: Teacher-student interaction (IS1); Students' interaction (IS2); Autonomy (IS3); Formative assessment (IS4).
Mediational: Material resources (MS1); Number of students, class schedule, and conditions (MS2); Time (MS3).
Affective: Interests and needs (AS1); Attitudes (AS2); Emotions (AS3).

Ecological: Curriculum adaptation (EcS1); Intra and interdisciplinary connections (EcS2); Social and labour usefulness (EcS3); Didactic innovation (EcS4).

In the OSA, mathematical modelling is considered as a *hyper* or *mega process* (Godino et al., 2007), since it involves other more elementary *processes* (representation, argumentation, idealisation, etc.). Furthermore, within this framework, enhancing modelling is an aspect that improves the suitability of the instructional process (Ledezma et al., 2021).

Methodology

In this study we followed a qualitative research methodology from an interpretative paradigm (Cohen et al., 2018), which mainly consists of a content analysis (Schreier, 2012).

Research context

This research was carried out in the context of the Master's Program in Teacher Training for Secondary and Baccalaureate Education (mathematics speciality), taught by the public universities of Catalonia (Spain), during the 2019-2020 academic year. The master's program prescribes (within the Internships module) to carry out educational internships in collaboration with the institutions established through agreements with the universities. In these internships, the prospective teachers must design a didactic unit that they must implement, which is determined by the educational institution, the students' level, and the time of the school year in which they carry out their intervention. Because of this situation, the margin that a prospective teacher has to work exclusively on modelling in his/her didactic unit is subject to certain restrictions, but not in the redesign proposed in his/her MFP. Due to the lockdown because of the COVID-19 pandemic, the prospective teachers of this course had to implement their didactic units, either partially or totally, in a virtual context.

For the elaboration of a MFP, the DSC are presented to the prospective teachers, along with the Breda and collaborators' (2017) guideline that allows their application. With these tools, the prospective teachers are suggested to assess in their MFP the didactic unit that they implemented, so that they propose changes that can help improve the suitability of the instructional process. The structure of an MFP consists of five chapters: *Introduction* (institutional and curricular context of implementation), *Implementation analysis* (assessment of the didactic suitability of the didactic unit, using the DSC), *Redesign proposal* (reformulation of the didactic unit), *Competence self-assessment* (according to the knowledge and competencies acquired in the master's program), and *Annexes*.

Content analysis

For this study we considered 122 MFPs, corresponding to the 2019-2020 academic year. For their qualitative analysis, we followed steps similar to those used by Sánchez and collaborators (2021). In a *first step*, according to the literature review and our knowledge on the topic, we drawn up a list of keywords related to mathematical modelling (context*, model*, problem*, real*), in order to identify the references about this process in the evaluative comments of the MFPs. In a *second step*, we recorded the data of each document (author, title, educational level, mathematical content), in order to have an ordered database to consult the MFPs and, in this way, keep a first record of which MFP mentioned the keywords from the *first step*. In a *third step*, we classified the MFPs according to four levels of reference to modelling that we could identify in their proposals, as detailed below:

Level 0 (L_0): No modelling problems are proposed/implemented; the inclusion of modelling is not considered in the redesign proposal.

Level 1 (L_1): No modelling problems are proposed/implemented; the inclusion of modelling is considered in the redesign proposal.

Level 2 (L_2): Modelling problems are proposed and there is a reflection on their implementation; no improvements are proposed to enhance modelling in the redesign.

Level 3 (L_3): Modelling problems are proposed and there is a reflection on their implementation; improvements are proposed to enhance modelling in the redesign.

In a *fourth step*, we categorised the comments with references to modelling using the DSC. Various studies have addressed the teacher reflection in mathematics teaching training processes (see Breda, 2020, for didactic analysis; Hidalgo-Moncada et al., 2021, for self-regulation practices; Sánchez et al., 2019, for the development of creativity, among others), using a content analysis methodology to make evident the use of the DSC components. In this research, these components are considered as a priori categories (Schreier, 2012), in order to identify the aspects of the instructional process that the prospective teachers related to the work with modelling. More specifically, we considered the evaluative comments from the *Implementation analysis* and *Redesign proposal* chapters, since they include all the prospective teachers' reflections on their own practice, from the MFPs classified in the levels of reference L_2 and L_3. As an example of the content analysis described above, we use the MFP #24. We found several references to the keywords: «model», «modelling», and «problem» in this MFP, which consists of a proposal for the teaching of functions in the third grade of secondary education (students aged 14-15). Among many others, we found this comment in the assessment of the CS4 component: "We tried to have activities and sessions that were more competent to work on cognitive and mathematical processes such as […] modelling and problem solving" (p. 12). However, this MFP did not include improvements to modelling in its redesign proposal, so we classified it in the level L_2 (a more detailed analysis can be found in Ledezma et al., 2021).

Results

Classification of the MFPs according to the levels of reference to modelling

From the search for keywords in the MFPs (*first step* of content analysis), we identified that 86 of the 122 documents referred to the terms related to modelling. After recording each MFP (*second step* of content analysis), we classified them according to the levels of reference to modelling (*third step* of content analysis), thus we obtained the following results: 33 MFPs in L_0, 41 MFPs in L_1, 21 MFPs in L_2, and 24 MFPs in L_3. The *first* and *third step* of content analysis also allowed us to refine the number of MFPs to analyse: a) during the *first step* we identified three MFPs presented under the research project format, so we did not consider them for this study; b) during the *third step* we identified three MFPs which did not explicitly refer to modelling in the *Implementation analysis* chapter, but they referred to implemented modelling activities in the *Annexes* chapter, including comments related to this process but not being attached to any specific DSC. Although we classified these last three MFPs in L_3, we did not consider them in the *fourth step* of content analysis. Taking into account these considerations, the following results show the analysis of 45 MFPs classified in levels L_2 and L_3.

Classification of the comments according to the DSC components

From the previous classification (*third step* of content analysis), we categorised the evaluative comments related to modelling according to the DSC components. To do this, we focused on the

Implementation analysis chapter of the 45 MFPs classified in levels L_2 and L_3, and then we identified in which DSC component the prospective teachers –explicitly or implicitly– made their reflections on the role of modelling in their didactic units (*fourth step* of content analysis). In this way, the results presented in Table 1 show the number of comments that we identified in the assessment of each DSC component, using the codes mentioned in the theoretical section.

Table 1: Number of comments referring to modelling in each DSC component

Components	No. of comments	Components	No. of comments	Components	No. of comments	Components	No. of comments	Components	No. of comments	Components	No. of comments
ES1	1	CS1	5	IS1	3	MS1	4	AS1	32	EcS1	2
ES2	2	CS2	1	IS2	4	MS2	0	AS2	3	EcS2	14
ES3	40	CS3	1	IS3	1	MS3	4	AS3	5	EcS3	20
ES4	11	CS4	12	IS4	0					EcS4	4
Total	54	Total	19	Total	8	Total	8	Total	40	Total	40

Discussion and conclusions

The *first step* of content analysis made us evident that around two thirds of the 122 prospective teachers referred in some way to modelling in their MFPs, through the keywords related to this process. The *second step* of content analysis allowed us to identify the mathematical contents of the MFPs in which the presence of such terms was stressed (in descending order): Functions, Trigonometry, and (linear and quadratic) Equations. This tendency to use modelling to teach functions, on the part of the prospective teachers, is in line with Michelsen's (2006) position, who highlights the role of this mathematical object as a tool to develop the modelling process in classroom. The *third step* of content analysis refined the results of the *first step*, as it allowed us to narrow the number of MFPs to analyse (from 122 to 119), and also showed us that around a quarter of the 122 prospective teachers implemented modelling in their didactic units. A relevant aspect is that, although the master's program includes a submodule on modelling (within the Specific Training module) in which the cycle proposed by Blum and Leiß (2007) is presented, none of the MFPs referred to this cycle (or any other) and neither used it to describe the proposed problems or to analyse their implementation. This suggests, on one hand, that there is insufficient clarity on how to use a modelling cycle to analyse this kind of problems and, on the other hand, that carrying out this kind of analyses in the reflection on their own practice is not considered as a relevant aspect. However, since this is an autonomous work, the prospective teachers are not asked to specifically reflect on modelling from the theoretical perspective in their MFPs, but rather it is an open decision, taking into account the pages number and time constrictions for its elaboration.

The *fourth step* of content analysis made us evident that most of the comments on the implementation of modelling focused on the assessment of the *epistemic*, *affective*, and *ecological criteria*, although also –to a lesser extent– on the *cognitive criterion*. Regarding the *epistemic criterion*, the assessment of the ES3 component (on the realisation of relevant processes in mathematical activity) was the one that gathered the largest number of comments on modelling, because here the processes worked in the didactic unit were defined. A relevant aspect is the tenuous differentiation established in some MFPs to define the processes: «problem solving», «modelling», and «contextualisation», in which, for instance, if a statement was posed within the mathematical world, then they did «problem solving», but if it was posed in the real world, then they did «modelling»; or also, if a problem developed «contextualisation», then it was a «modelling» problem. This situation could indicate that there is insufficient clarity about the theorising around mathematical modelling and the characteristics of this kind of problems (as described in Borromeo, 2018). Derived from above, the comments in the ES4 component (on the meanings and representation modes for the treatment of mathematical objects) mainly refer that modelling problems allowed working the mathematical objects using different semiotic representations. Regarding the *affective criterion*, the assessment of the AS1 component (on the interest and utility of the tasks) was the second with the largest number of comments on modelling. These comments pointed out that this kind of problems, being 'contextualised' and 'realistic', caught (or intended to catch) the students' attention, due to many of them took advantage of the context of COVID-19 and lockdown as a topic in their wordings. Regarding the *ecological criterion*, the assessments of the EcS2 (on the relation between mathematical contents with other disciplines) and EcS3 (on the utility of contents for social and labour insertion) components included comments referring to modelling as a tool to relate mathematics, both to other curricular contents and to the students' context. We make a special mention regarding the *cognitive criterion*, since the assessment of the CS4 component (on the activation of relevant cognitive processes in mathematical activity) stressed that modelling problems made it possible to work on other relevant processes of mathematical activity (in line with Godino et al., 2007). Regarding the *interactional* and *mediational criteria*, it is evident that their assessments included very few comments on modelling. Due to the virtual teaching context, the prospective teachers commented that they had many difficulties in developing a collaborative work between their students (affecting the *interactional criterion*), as suggested for the work with modelling (Doerr & English, 2003). For this reason, in both *criteria* they made comments on the components that would potentially be addressed by the solving of this kind of problems in their redesign proposals.

Resuming the research question, the main conclusion is that the prospective teachers mainly related the *epistemic*, *affective*, and *ecological* (and to a lesser extent, the *cognitive*) aspects when they reflected on the work with modelling in their didactic units implemented in a virtual context (similar result to that found in Breda, 2020). The study reported in this paper is part of a broader research project, which aims to contribute to research on teacher reflection in teacher training, specifically, on the implementation of modelling. Since this part was carried out during the 2019-2020 academic year, the next step is to compare these results with the prospective teachers' reflections in the MFPs from the 2020-2021 academic year, implemented in a face-to-face context. As a final purpose, we intend to design a guideline of specific DSCs for the work with mathematical modelling in the instructional processes implemented both in a face-to-face and in a virtual context.

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The emotions of a mathematics teacher educator in an online context

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The global health emergency caused by COVID-19 has had profound repercussions in education around the globe. In response, many educational institutions, including universities, have shifted teaching to online formats. This sudden and abrupt change has had a significant impact on teachers' emotions. In this study, we use an instrumental case study to explore the effect this unexpected context has had on the emotions of a primary teacher educator. Drawing on the theory of the cognitive structure of emotions, we analyse the educator's emotions, as evidenced and as reported, over a series of teaching sessions. The results show the occurrence of both positive and negative emotions deriving from the situation, which should give pause for thought in the university context.

Keywords: Emotions, online teaching, affective domain, mathematics teacher educator.

Introduction

There is an increasing amount of research focussed on investigating the affective domain and its importance in the teaching and learning of mathematics. Interest in the topic has developed over the last few decades, and this has been reflected in different theoretical and methodological approaches (Hannula, 2011). One of the constructs constituting this domain which has received increasing interest is that of the emotions. It has been demonstrated within the fields of cognition and neuroscience that emotions are fundamental to people's learning and development. The importance of affect and the emotions is now also beginning to be recognised within the area of educational research (Pekrun & Linnenbrink-Garcia, 2012). Furthermore, teacher educators have centered the attention of the field as key professionals in the management of (prospective) teachers professional development (Goos & Beswick, 2021).

As the emotions are profoundly intertwined with thought (Immordino-Yang & Damasio, 2007), it is natural that they should play a role in the relationships and situations that the teacher manages in the classroom. This is because the emotions reflect not only an individual's external relationships with other people, society and the situation, but also their internal relationships with their own reflections and memories (Mayer et al., 2011). Bearing this in mind, and consistent with Zembylas (2007), in order to teach effectively, educators should be able to marshal their emotional knowledge to establish or strengthen links with the topic and with the students, to develop the syllabus, and to make decisions on how to act. Further, it makes sense to assume that an appropriate environment for learning is fostered when the experience is pleasant for educator and learner alike. In this respect, various studies have set out to describe and analyse students' emotions and their influence on learning (e.g., Fernández-Berrocal et al., 2017; Pekrun et al., 2002). Within a supportive environment, the student will be willing to make an effort to learn and to reconfigure their existing knowledge, which is the main purpose of learning.

On the other hand, the pandemic (COVID-19) has triggered significant changes at all levels of education, obliging institutions and teachers to rely on technology to conduct lectures and other teaching-learning activities (Engelbrecht et al., 2020). With so much uncertainty, it is unsurprising that teachers have experienced a range of emotions more intense and keenly felt than usual.

In response to this issue, the research question for this paper is the following: What impact has the imposed online context had on the emotions felt by a primary level teacher educator while teaching the course "Mathematics and its Didactics I"? To answer this question, we carried out an instrumental case study. This involved video recording an educator's classes, and conducting follow-up interviews with her to clarify and explore the emotions she had felt.

Theoretical framework

In our study of emotions, we draw on the Ortony, Clore, and Collins (OCC) Theory (Ortony et al., 1996), based on the cognitive structure of emotions. We assume, following Ortony et al. (1996), that prior to any emotion there is a system of analysis and information processing, that is to say, the subject's cognitive construal of a specific external situation. We use the definition provided by these authors, for whom "emotions are valenced reactions to events, agents, or objects, with their particular nature being determined by the way in which the eliciting situation is construed" (Ortony et al., 1996, p.13). Emotions arise from specific situations and the perception the subject has of them. The subject cognitively construes the situation in a manner which can be more or less conscious, and this construal produces a valenced reaction on the basis of the subject's own internal system of goals, rules and attitudes. The valency refers to the positive or negative charge of the emotion, according to whether the event taking place fulfils or not the subjects goals, rules and attitudes.

When a teacher is giving a class, consciously or unconsciously, they are constantly construing the different interactions and situations which arise, eliciting different emotional responses: some of greater intensity, some longer lasting, and some with a larger cognitive load. One of the functions of an emotion is prime the subject for action (Ortony et al., 1996). In this way, emotions can provide us with information about both the eliciting situation and the reaction this produces, as we might not always react in exactly the same way when faced with similar situations. The explicit analysis of these processes can reveal information that the teacher might not have been consciously aware of, and can thus help them to understand why they acted as they did, and to decide whether such responses would be effective in similar situations.

According to the OCC model, emotions can be classified into three large groups, corresponding to the three major aspects of the world which can trigger them: events, agents and objects (Ortony et al., 1996). Prior to the emotion, the subject appraises the situation and focuses on one of these three aspects, that is to say, either the consequences of the event, or the responsibility of the person (or other agency) involved in bringing about the event, or on the simple pleasure or displeasure that the event has caused. For this reason it is important to describe the situation which has triggered the emotion, and to study emotions in the context in which the event takes place. According to Ortony et al. (1996), each individual has a structure of goals, interests and beliefs underlying their behaviour; in other words, the construals preceding an emotion vary from person to person on the basis of this structure. Naturally, this structure is directly related to the three major aspects of the world mentioned above. Hence, consistent with the OCC theory, the appraisals determining whether or not an event is desirable are underpinned by an internal structure of goals; likewise, whether the action of an agent is praiseworthy aligns with the subject's system of norms; and whether or not an object is considered attractive is based on their system of attitudes. The OCC theory proposes a classification of emotions according to the context that elicits them (Table 1). Hence, it identifies emotions as reactions to events, agents, and objects, or a simultaneous combination of event and agent.

Table 1: Classification based on information from the OCC theory (Ortony et al., 1996)

Reactions to events	
Well-being	*Joy*: pleased about a desirable event
	Distress: displeased about an undesirable event
Fortunes-of-others	*Happy-for*: pleased about an event desirable for someone else
	Sorry-for: displeased about an event undesirable for someone else
	Resentment: displeased about an event desirable for someone else
	Gloating: pleased about an event undesirable for someone else
Prospect-based	*Hope*: pleased about the prospect of a desirable event
	Fear: displeased about the prospect of an undesirable event
	Satisfaction: pleased about the confirmation of the prospect of a desirable event
	Fears-confirmed: displeased about the confirmation of the prospect of an undesirable event
	Relief: pleased about the disconfirmation of the prospect of an undesirable event
	Disappointment: displeased about the disconfirmation of the prospect of a desirable event
Reactions to agents	
Pride: approving of one's own praiseworthy action	
Shame: disapproving of one's own blameworthy action	
Admiration: approving of someone else's praiseworthy action	
Reproach: disapproving of someone else's blameworthy action	
Reactions to objects	
Love: liking an appealing object	
Hate: disliking an unappealing object	
Combined reactions to event and agent	
Gratitude: pleased about a desirable event and someone else's praiseworthy action	
Anger: displeased about an undesirable event and someone else's blameworthy action	
Gratification: pleased about a desirable event and one's own praiseworthy action	
Remorse: displeased about an undesirable event and one's own blameworthy action	

In addition to identifying emotions and interpreting the corresponding situations which elicited them, we will also analyse whether the unexpected online context had an influence on the educator's emotions, or indeed actually acted as a trigger for them. It should be borne in mind that the majority of Universities in Spain opted to switch to online classes during the pandemic. This situation saw teachers having to modify not only their material but also their manner of running classes just to keep students' attention. Although the health crisis has been seen by some as an opportunity for change, there has also been a degree of uncertainty in the sudden shift to online teaching (Engelbrecht et al., 2020). These anxieties are likely to be reflected in the educator's emotions on finding themself in an unfamiliar environment, without the kind of control they are accustomed to having in the physical lecture hall.

At the same time, teachers are aware that if they are to be effective in the new environment, they need to adapt their teaching style (Hollebrands & Lee, 2020), and equally importantly, they need to do so within the confines of the limited resources and facilities their institutions are able to provide. In the best case scenario, they can count on the support of colleagues who generously share their material, time and ideas, or even agree to work as a team. Even so, every teacher has found themselves investing considerable time in learning how to use new applications which might be of use, and in creating or adapting their teaching material so as to fulfil the requirements of the online medium. These new challenges facing teachers can make them feel isolated and uncomfortable (Collazos et al., 2021; Hollebrands & Lee, 2020).

It should be remembered that for most teachers and students the circumstances we find ourselves in are completely new. Due to the lack of feedback, one of the main challenges for teachers in online teaching is how to get the information they need from learners to check that the desired knowledge has been acquired (Collazos et al., 2021). This will be the guiding principle when educators plan their classes and elicit responses during the sessions, as in the new context, the traditional channels – monitoring students for reactions or still less circulating around the lecture hall or classroom – are no longer available.

Methodology

This study follows an instrumental case study design, taking an interpretative perspective with an exploratory intention. The informant for the research, who will be identified as Nora, is a primary teacher educator in Mathematics. At the time of the study, she had been a university teacher for 27 years, the last 7 of which completely focused in teacher education at primary level and in Mathematics teaching. She was also an active participant in research projects into Mathematics education. We chose Nora because of her great communication skills and her willingness to discuss her own emotions, and because of her experience teaching teachers to teach mathematics (more than 10 years).

The teaching observation took place during the delivery of Mathematics and its Didactics I, a course in the second year of the Degree in Primary Education, which focused on teaching arithmetic. Nora had taught the course previously, but this was her first time via online/ mixed modality classes, with students following the class remotely by computer. Data collection was carried out by means of video-recording ten sessions of approximately one and half hours each. After five of these sessions, one of the researchers – who had observed the sessions in a non-participatory capacity– asked Nora to answer a set of questions (formalised as a report) about situations that had arisen; this was followed up by four semi-structured interviews aimed at amplifying and confirming Nora’s responses.

According to Ortony et al. (1996), there are four kinds of evidence which can help us to understand emotions: language, personal reports, the type of behaviour elicited, and physiological cues. In our case, we were principally interested in the first two of these – language and personal reports – as they provide information about the subject’s prior cognitive appraisal leading to the elicitation of the emotion. For the purposes of this study, a content analysis was carried out (Krippendorff, 2018), taking the emotions proposed by the OCC theory (Table 1) as the categories of analysis. This model allows us to work with the cognitive part of the emotion. The analysis will show the triggering situation of the emotion and information about Nora's appraisal and how it makes her feel.

Analysis of the results

This section shows the first emotions identified and classified in the research process, and the influence on them of the online context. Interlaced with the analysis, and marked by italic, we include some of Nora's commentaries providing further evidence of the emotion.

Reactions to events

This category, we remind ourselves, concerns emotions bound up with Nora's system of goals, and focuses on events, adjudged desirable or undesirable, depending on whether or not they enable her to achieve her goals.

At various points, Nora mentions how important it is for her that the students participate and talk to each other; they should be at the centre of things, and it is they who should set the pace of the class. On occasions she finds it difficult to get any kind of response, while the students claim they are having problems activating the camera and even the microphone – when they do respond, they tend to write in the chat window. Typical among Nora's comments is this: *It does cause me a bit of frustration, I would like there to be more interaction through the microphone or with the camera because that makes the session more dynamic*.

Related to this is her sense that sometimes the classes are not going as she would like, according to her own conceptions, in direct contrast to the in-person format: *I think that the activity with the multiplication tables, if they could come up to the blackboard to do it, interact and so on, it would be better. I mean, I feel bad for doing it in such a traditional way*. In addition, at certain times, she would like to be able to deal with her students differently, to be more accessible and elicit the kind of answers and interaction mentioned above: *Those interactions when you get closer: "Come on, what have you seen here?" That leaves me with a sense of disappointment ... in that I don't have a more direct relationship with them*.

This is a new challenge for Nora, and when she achieves the dialogue she wants with the students, as might be expected, she shows her pleasure: *When I began giving the classes online I thought the level of interaction would be higher. When I saw that wasn't the case, each time it happens, I feel pleased*.

When Nora puts herself in her students' place, she evokes the category 'fortunes-of-others' (Table 1) with her empathetic response, appraising events as desirable or undesirable for her students. It seems clear that she perceives an enforced distance, which it is difficult to avoid in the online context. On the one hand, not being able to attend to and manage the emotions of her students is disappointing for her: *I can't be with them, the supportive role of saying "Come on everyone, you can do it"*. On the other hand, she is sympathetic of her students' situation. She recognises that the online context is also proving a challenge for them, particularly with regard to working in groups and to communicating with her when they are not in the classroom: *They may be right that this year, until they get up to speed with deadlines and so on, they need a little more time*.

The final group of reactions to events are prospect-based, that is to say, based on events which the educator hopes or suspects might happen. Here we see how the online context causes her unease: *I've already lost 10 minutes before the class even starts, what with the computer won't boot up, the camera, and whatever ... I don't have any leeway, I'm much more dependent on the time*.

Nora modified the activities so as to be more manageable in the online context, but she worried that in the end something might not work, or the students might get distracted and it might be difficult to gain their attention again: *Fear of starting something and taking a long time to get going, or that the platform crashes; and then it's that idea/emotion that stays with the students about the class (those problems and not the content)*.

To this can be added the background worry that although the class seems to be progressing, nobody is asking any questions: *In class they are more willing to ask questions, even if they are not following you. Online, though, to interrupt, to put their hand up, that is harder for them to do*. When the students are in the classroom, if Nora spots a look of doubt on their faces, she encourages them to ask for clarification. Online, there is no way of seeing whether they understand, there is no visual feedback loop.

In contrast to the above, however, there are also moments of satisfaction, when Nora manages to create a dynamic in which the students do intervene: *Each time they get more involved, even though it is through the chat window, it gives me a certain degree of satisfaction*. The same sense of achievement is felt after she manages to readjust the classes and still meet the stated objectives: *I've covered what I wanted and I felt good. And I think they liked the slides*.

Reactions to agents

This category of emotions concerns the attribution of responsibility for the triggering situation to a person (in our case usually the students or the teacher educator) on the basis of Nora's norms, codes of conduct, rules and so on.

When talking about the process of evaluating her students, Nora expresses a degree of discomfort (a process she feels is very demanding of her integrity), putting the focus on herself: *I'm worried about not being fair in the evaluation ... discriminating who is well trained at that moment and who needs more time*. It is an emotion which could appear in any context, but the fact that the exam is likely to be online, and therefore devoid of the usual measures to prevent cheating, increases her unease and mistrust.

At other times, the focus is on the students, such as when in a group none of them is able intervene to discuss the activities because no-one has a microphone: *I'm not happy about it, I get the impression they're not being honest with me*. Such as things are, the students can, if they wish, blame the tech and disappear, a ruse to avoid participating which is unavailable to them in the face to face classroom.

Combined reactions to events and agents

This category of emotions concerns appraisals involving both goals in relation to events and norms in relation to agents. An example is Nora's sense of gratification for having successfully completed the large amount of extra work involved in adapting the course to an online context, and her feeling that the results have been good: *I've done it, I've worked hard and have reaped the rewards ... Their faces have an expression of 'Oh, that's interesting!'* This particular comment arose when some students, following a mixed online/in-person methodology, had returned to the classroom with Nora and hence she could see their faces. Previously, when all the students were online, she had missed this kind of feedback.

Finally, there is specific moment when Nora feels indignant at the something a student says. The student switches on their microphone to complain about the latency affecting the online students, something she had been at pains to warn them would happen: *I had gone over it a lot to make sure it was clear: I made sure the slides were large, and didn't use the blackboard too much ... It seemed extremely unfair to me and I <u>felt bad</u>.* In this instance, in addition to being, in the terminology used in OCC theory, an undesirable situation for Nora, she also found the action of the student blameworthy, as he had not listened to her or appreciated her efforts.

Final reflections

The analysis foregrounds how the educator's emotions respond to the changes she perceives in her interactions with her students and the teaching-learning environment (Mayer et al., 2011), and how influential the online context is in triggering these emotions. According to Engelbrecht et al. (2020), teachers have to face new problems and may feel uncomfortable and isolated in this online context. From the beginning, we verified how for Nora it was a concern and trigger for many of the emotions we have analysed. It also makes sense of the fact that eleven of the emotions analysed have a negative valence and only four have a positive valence.

The majority of the emotions which Nora affirms to have experienced are based on her dynamic structure of goals, which demonstrates her commitment to planning session with clear objectives that she wants to achieve. When this happens, the resultant emotions are happiness, feeling good and satisfaction, because the session has worked according to these approaches. If the focus is also on the agent (herself), the resulting emotions are of gratification and pride for having achieved them. The online context challenges teachers to adapt the way they teach (Hollebrands & Lee, 2020) and to take stock of their own capabilities, something worth noting.

With respect to the events which Nora deems undesirable, we can see that the online context causes her anxiety and suspicion, as it does not allow her to carry out the class in the way that she is familiar with and that she considers appropriate for facilitating the students' learning. She highlights this when she states that she misses being able to get close to the students to encourage them, guide them in what they are doing, and encourage them to share things with the group and enrich the contributions. In addition, she lacks feedback from the looks on the students' faces when she is in the classroom: what they are understanding and what not, what they want to ask or reply to, what they like, and so on. This forms part of the isolation of the online teacher and the lack of feedback and interactions with the students in the online context (Collazos et al., 2021). Based on these emotions, questions arise not only regarding the need for support in adapting to the online context, but also for emotional support. According to Fernández-Berrocal et al. (2017), training in emotional competences can provide benefits at both a personal and group level.

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Teachers' professional development: a cultural matter. How to describe cultural contexts?

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Teachers' professional development in collaborative contexts is a growing trend in Mathematics Education research. Particularly, Japanese Lesson Study has seen a great focus on its dissemination around the world. Research shows that Japanese culture is one of the main reasons that makes Lesson Study effective: understanding Lesson Study means understanding the cultural context in which it originated. We attempt to describe the Japanese and the Italian cultural contexts. Since there exists no consensus on what is essential to analyse in order to understand a cultural context, we present two approaches to this description, and consider some advantages and shortcomings. We hope to sprout discussion on the possibility to create guidelines for describing cultural contexts, shared by the community of researchers in Mathematics Education: awareness of beliefs, identity and practice is a sensitive element for successful mathematics teacher professional development.

Keywords: Cultural context, culture, Japan, Lesson Study, Italy.

Introduction

Over the last 20 years, teachers' professional development (TPD) in collaborative contexts has received ever-growing attention from the community of researchers in Mathematics Education (Robutti *et al.*, 2016), and a recent survey by Bakker and colleagues (2021) confirmed the trend. Among the many different collaborative methodologies for TPD in Mathematics and Mathematics Education, Lesson Study has seen a great deal of research focused on its dissemination (i.e.: Huang *et al.*, 2019; Huang & Shimizu, 2016; Quaresma *et al.*, 2018). Lesson Study (LS) is a collaborative TPD methodology, part of the Japanese paradidactic infrastructure (Winsløw, 2011) since the 1880s, focused on the co-responsibility in the lesson-planning process of the involved teachers and knowledgeable others (Huang *et al.*, 2019).

LS is also the focus of the authors' doctoral dissertations (in progress), which also aim at introducing LS in the context of Italian TPD. During YESS11, TWG1 dedicated to *teacher education and professional development* saw four out of eleven papers focused on LS in different contexts (Italy, Mozambique, Portugal, and Switzerland). In the discussions around the four papers, one question resulted relevant: what is the *cultural context* in which the research takes place? Indeed, being aware of their cultural context is one of the essential competences of mathematics teachers (and researchers) to gain awareness of their beliefs, their identity and their professional practice and to develop their teaching knowledge: a sensitive element for successful mathematics teacher professional development, and a demand evermore necessary and therefore not negligible (Andrews, 2010). Yet, issues arise when we try to address such demand: in the following, we will attempt to describe the Japanese and Italian cultural and institutional context and discuss such issues.

Literature review

Unsuccessful attempts at translating LS outside of its cultural context (Demir *et al.*, 2012; Fernandez *et al.*, 2003) suggest that, if LS is not introduced in a cultural context with proper consideration to the differences with the Japanese cultural context, it might be rejected by the institutions. Ebaeguin & Stephens (2014) suggest to address the cultural compatibility of LS. A number of scholars proposed different theoretical lenses to analyse why LS is so widespread in Japan (i.e.: Krainer, 2011; Lewis, 2016). We can suppose that LS exists because of the Japanese culture, and Japanese culture is one of the main reasons why LS is effective: LS is a cultural activity (Stigler & Hiebert, 2016). The question arises if maintaining the efficacy of LS across different cultural contexts is feasible.

Despite the rising awareness on the importance of studying cultural contexts and identities to contextualize global trends in Mathematics Education (Bakker *et al.*, 2021), the majority of reports on LS around the world seems to depict LS as an isolated practice in the Japanese panorama of TPD practices (Miyakawa & Winsløw, 2019) and seemingly ignores that the Japanese definition of LS is not as clear cut as the American one (Miyakawa & Winsløw, 2013). This suggests that "to develop a deeper understanding of Lesson Study in a post-modern global world, there is a need to seek views beyond those presented from an American perspective" (White & Lim, 2008, p. 915).

Understanding LS means understanding the context in which it originated. At the same time, to introduce LS in a new context, it is essential to know the TPD practices already in existence (Miyakawa & Winsløw, 2013). Yet, there exist no consensus on what is essential to analyse in order to understand a cultural context. What is *culture*? This paper has two aims: to provide arguments to the importance of understanding the cultural contexts involved in the research, and to provide a currently-missing description of the Italian TPD context in the English language. We provide a tentative analysis of the Japanese and the Italian cultural and institutional contexts to guide future studies on LS in Italy, and we also hope to sprout discussion on the possibility to create guidelines for describing cultural contexts that might be shared by the community of researchers in Mathematics Education.

Theoretical Framework

Culture may be described as "any aspect of the ideas, communications, or behaviours of a group of people which give them a distinctive identity and which is used to organise their internal sense of cohesion and membership" (Scollon & Scollon, 1995, p. 127) or as "[t]he system of shared beliefs, values, customs, behaviours, and artefacts that the members of society use to cope with their world and with one another, and that are transmitted […] through learning" (Bates & Plog, in Freimuth, 2006, p. 2). Anthropologists have not reached a shared definition (Spencer-Oatey, 2012), and proposing one would be outside of our expertise. In fact, our aim is not to propose our own definition but to observe how existing approaches and definitions may interact with learning and teaching processes in Mathematics, particularly in TPD. It is a facet of our doctoral research, especially within a semiotic context (Manolino, 2021). Here we rely on a *popular* understanding of what *culture* is, as the definition is not central to this paper. In the following, we propose two different approaches to the definition and description of *culture* and *cultural contexts*: the first one is synthetic, the second one is descriptive. We hope to show some advantages and shortcomings of each of them, which should provide a mean to engage in this discussion.

Hofstede's Dimensions of Culture

Hofstede defines culture as "the collective mental programming of the people in an environment. Culture is not a characteristic of individuals; it encompasses a number of people who were conditioned by the same education and life experience" (in de Mooij, 2010, p. 48). Hofstede identifies basic value orientations of a certain national culture. These values "are broad preferences for a certain state of affairs (e.g., preferring equality over hierarchy) […] transmitted by the environment […] shaped by the time we hit 10-12 years of age" (https://news.hofstede-insights.com/news/what-do-we-mean-by-culture described in five dimensions, scored over 100 points, and represent:

- *Power Distance* Index: “the extent to which less powerful members of a society accept and expect that power is distributed unequally” (de Mooij, 2010, p. 75). The higher the score, the more hierarchical a society is.
- *Individualism* vs Collectivism: “[…] people looking after themselves and their immediate family only, versus people belonging to in-groups that look after them in exchange for loyalty” (de Mooij, 2010, p. 77). Higher scores indicate individualistic values.
- *Masculinity* vs Femininity: “The dominant values in a masculine society are achievement and success; the dominant values in a feminine society are caring for others and quality of life” (de Mooij, 2010, p. 79). Lower score indicates a feminine society. Please note that this label is problematic as it reinforces harmful gender stereotypes, and in the following we will use the alternative “*Tough* vs Tender”.
- *Uncertainty Avoidance* Index: “[…] the extent to which people feel threatened by uncertainty and ambiguity and try to avoid these situations” (de Mooij, 2010, p. 82). The higher the score, the less open to changes is a society.
- *Long-Term* vs Short-Term Orientation: “[…] the extent to which a society exhibits pragmatic future-oriented perspective rather than a conventional historic or short-term point of view” (de Mooij, 2010, p. 85). Lower scores point to a society that prefers short-term planning.

One peculiarity of Hofstede’s dimensions is that “[t]he country scores on the dimensions are relative, in that we are all human and simultaneously we are all unique. In other words, culture can only be used meaningfully by comparison” (https://hi.hofstede-insights.com/national-culture).

Levels of Co-Determination

We start from “the notion of [teaching] “practice” as a link between culture […] and the larger cultural contexts” (Hatano & Inagaki, 1998, p. 80). In the Anthropological Theory of the Didactic, practices are described in terms of praxeologies: the know-how (*praxis*) and the know-why (*logos* - the discourses that justify the know-how) related to a task. Chevallard (1985) suggests that teachers’ praxeologies are shaped by a plurality of agents (politicians, scholars…) and historical or institutional conditions that defines the boundaries of what teachers can or cannot do, their *noosphere* (the sphere of those who thinks). Chevallard (2002) pictures the complex relations of the factors influencing teachers’ praxeologies, which are influenced not only by the teachers’ decision, but at a higher level by the society in which the teachers and students are immersed, as shown in Figure 1.

Upper levels	*Lower levels*
Humanity	↓↑
↓↑	Discipline
Civilisation	↓↑
↓↑	Domain
Society	↓↑
↓↑	Sector
School	↓↑
↓↑	Theme
Pedagogy	↓↑
↓↑	Question

Figure 1: Scale of levels of co-determination (Florensa *et al.*, 2018, p. 5)

Similarly, we suggest that professional development practices are influences by the context in which teachers and their educators are immersed. It is important to notice that Chevallard's framework does not use the term *culture*.

Context Analysis

Hofstede's Cultural Dimensions

Ebaeguin and Stephens (2014) suggest that comparing Hofstede's scores might be a starting point for studying the introduction of LS in Australia, as they connect the efficacy of LS in Japan to the Japanese scores. The scores for Japan and Italy according to Hofstede's cultural dimensions can be freely collected from the website https://www.hofstede-insights.com/product/compare-countries/ and are shown in Table 1. The labels are simplified because of space constraints.

Table 1: Hofstede's score for cultural dimension for Italy and Japan

	Power Distance	**Individualism**	**Tough**	**Uncertainty Avoidance**	**Long-Term**
Italy	50	76	70	75	61
Japan	54	46	95	92	88

Within Hofstede's description, the two cultures appear different in almost all categories: according to these descriptors, Japanese culture appear less individualistic, more prone to success (and therefore more competitive), less open to "thinking outside of the box", and keener to long-term planning. There are some similarities, as Japanese and Italian cultures seem to have a shared approach to hierarchy.

Levels of Co-Determination

Using Chevallard's didactic transposition lens and moving within the co-determination levels, we attempt to provide a description of the Japanese and Italian institutional contexts. Since many sources are available on the Japanese context, the description will be briefer. The description of the Italian context, on the contrary, will be as detailed as the format allows, inspired to that proposed for Japan by Miyakawa and Winsløw (2019).

Japan is an East-Asian country, influenced by countries of "Confucian Heritage Culture" (Mason, 2014) such as China and Korea. These countries generally share some cultural values that are reflected in the school system, and can be considered part of the Japanese system of school-related beliefs:

> a high regard for education […]; […] the cultivation of the self; a strong work ethic […]; a belief […] that success depends more on effort than on innate capacity […]; respect for teachers […] (Mason, 2014, p. 2).

In Japan, the national curriculum is detailed and rigid. Textbooks are essential for lesson planning. Long-term planning is centralized at the prefectural or school level, so teachers' attention is focused on learning units and lessons. Classes are homogeneous by level: strict entrance tests are usually required for accessing high schools, while students with disabilities attend special schools. Japanese teachers spend the working day at school, where they have their personal workspace in a room shared with the whole teaching staff: in this space, they prepare lessons and discuss with their colleagues. In-service professional development is compulsory and takes place during working hours (Bartolini Bussi & Ramploud, 2018). LS is only part of Japanese TPD activities, which have many common features: in particular, the practice of open classes with observers is common (Miyakawa & Winsløw, 2013). Participating in optional TPD activities increases teachers' chances of career advancement (Miyakawa & Winsløw, 2019).

The Italian school system is centralized. Recent reforms (2010 and 2012 respectively for secondary and primary school) stressed the importance of inclusiveness (law 133 and 169/2008). Italian school is structured around the concept of equity, and special schools do not exist: all students are given the same opportunities to reach the same goal, plus aids if needed. The Ministry of Education provides the *Indicazioni Nazionali* (National guidelines), which contain contents and aims for each subject, and its number of hours in a year. These contents are not prescriptive, but at the end of the 8th and 13th grades there are two national exams. Each teacher has the responsibility of the didactical plan for their classes, also according to the *Piano Triennale dell'Offerta Formativa* (Three-year Educational Plan – describing the cultural-pedagogical inspiration and the curricular, extracurricular, didactic and organisational design of the proposed activities). The contents of this document are specific of each school and decided by the collegiality of teachers and school staff.

Freedom of teaching, understood as professional autonomy in carrying out teaching activities and free cultural expression of the teacher, is guaranteed as a constitutional right: Article 33 of the Constitution states "Art and science are free and free is their teaching". Institutionally, the duration of the lesson is 60 minutes. The teacher can have up to three consecutive lessons in the same class, without interruptions. During the lesson, the teacher is usually the only adult figure in the class. The Italian teacher works at school from one to six hours a day, dedicated to classroom lessons. The planning of individual lessons is not part of the working hours, nor there are places in the school dedicated to this activity: the teacher's paradidactic activity takes place in personal and private time and space. There are no compulsory contents or practices for TPD, they are chosen by teachers according to their own needs. In-service TPD is compulsory (law 107/2015), there is no minimum number of hours per year, and must be carried out outside working hours. Teachers' career advancement is based exclusively on seniority, although some economic incentives are given to those that take relevant roles in the school organization (Blandino, 2008; Capperucci, 2008).

On paper, teachers have numerous occasions for improving their professionalism. The Ministry[1] attests more than 500 agencies offering TPD opportunities. Universities, academic associations, teachers' associations, and educational companies which fulfil quality standards defined by the Ministry, are registered in a national database and can publish their TPD proposals on a digital platform (S.O.F.I.A.). The in-service professional development "system" is conceived as a "lifelong learning environment" for teachers and is intended as a "network of opportunities for professional growth and development for teachers" (law 107/2015). At national level, proposals come from the national education centre, academic associations, teachers' associations, educational companies. At regional level, regional school offices intervene by supporting, managing, and publicising the proposals. At local level, experienced teachers also offer courses in their school, sometime opens to teachers in the surrounding area. No official account is given on how many teachers participate in TPD. Yet, the impression is that this vastity of opportunities does not correspond to a high-quality offer: the Ministry states that the quality of TPD programmes is compromised by the general "low quality of models and methodologies" (law 107/2015) suggesting that teachers might be easily lost

[1] https://www.miur.gov.it/accreditamento-enti-e-qualificazione-associazioni

and caught in low quality programmes. The Ministry does not provide guidance to orientate in this labyrinth.

Some Reflections

At the end of this section, we can ask ourselves: is this description complete? Did we miss any essential point? Did we provide too much information, and made our description useless? Is this description reliable? Hofstede's synthetic data, for example, provides a quick overlook on the differences between two cultures and invite to carefully consider LS introduction in Italy. Yet, this description is problematic as it eliminates complexity, and the risk of overgeneralizing is high. Furthermore, these scores are open to interpretation. The similar scores for PDI might suggest that Italy and Japan have a shared view on hierarchy, but we propose a different interpretation: while Japanese invites consciousness of one's hierarchical position in any social setting and act accordingly (Ebaeguin & Stephens, 2014), Italian culture dislike control and formal supervision. An analysis based on Hofstede's cultural dimensions, in support of our assertions, can be found in Giordanengo's Master Degree dissertation (2020)[2].

Discussion

It emerges that the Japanese and the Italian cultural and institutional contexts share some similarities and come with a number of differences. Hofstede's cultural dimensions show divergent basic values. However, to consider Hofstede's analysis *complete enough* to understand the similarities and differences between Japan and Italy would be preposterous, and similar numbers can still be interpreted with very different founding values. A detailed look suggests similarities between the two educational contexts, yet striking differences in the institutional and paradidactic organization. We believe this is sufficient to justify the importance of a cultural approach when practices from a cultural context are brought in different contexts. A tentative description of the Italian institutional (school) context was provided as a reference for future studies. Italian researchers are invited to amend this description, which is certainly lacking details.

This paper answers no research questions but considering our attempt to respond to the need to provide a description of the Italian institutional context in which students, teachers, and researchers work every day, leaves us with some questions. Are we satisfied with the result? No, we are not: the description misses many details, and we are not sure that what we provided is enough to really understand the context. How is Mathematics as a school subject considered at a cultural level? How is the teacher role considered in each society? Many questions are left unanswered, yet we often hear from reviewers that we should focus on describing just some aspects. How detailed can these descriptions be? Too little or too many information will lead to the same result: little understanding of the cultural context. Can we really achieve this *correct* kind of detail? Again, the answer, in our opinion, is no. The gap is embedded in the notion of cultural context and in any possible analysis of it. A number of scholars (e.g., Lotman, 1990) have declared the impossibility of a full knowledge of culture, as we are embodied in it and in what François Jullien - sinologist - calls the *unthoughts* (for a broader understanding, see Mellone *et al*., 2019). To be aware of these unthoughts may not be

[2]https://sia.unito.it/studenti/intesi/Ricerca_tesi_libera/ricerca_tesi_dettaglio.asp?id_upload=192959&cdl_tesi=&cdl=&matricola=781420

enough anymore. What is incumbent on us is to frame our research accordingly, as to provide careful attention to their influence on teaching and learning processes in mathematics.

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Math for All: Exploring the impact of a professional development program to improve mathematics learning for students with disabilities

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Math for All (MFA) is an intensive professional development (PD) program for in-service teachers. It consists of five one-day workshops and classroom-based assignments, providing a total of 40 hours of PD. MFA engages teams of general and special education teachers in adapting math lessons collaboratively to help all students, including those with disabilities, achieve high-quality learning outcomes in mathematics. A cluster randomized controlled trial (RCT) was used to test the efficacy of MFA; it included 32 schools, 98 4th and 5th grade teachers, and approximately 1,500 4th and 5th grade students. MFA had statistically significant, positive effects on teachers' self-reports of their preparedness and comfort with teaching mathematics to students with disabilities. A school-level analysis found a moderate MFA effect on student achievement. Quasi-experimental analyses of a subgroup of teachers showed initial evidence of MFA impacts on their classroom practices.

Keywords: Mathematics teacher professional development, mathematics teacher beliefs, mathematics teacher self-efficacy, mathematics teaching practice, mathematics achievement.

Introduction

This paper presents the results of a randomized controlled trial (RCT) that tested the efficacy of a professional development (PD) program called Math for All (MFA). MFA is an intensive PD program designed to help general and special education teachers in Grades K–5 to personalize rigorous mathematics instruction for a wide range of learners, including students who are low performing, and students with disabilities. MFA consists of five full-day PD sessions and related assignments (a total of 40 hours of PD) carried out at regular intervals throughout the school year. The program is designed to have a direct impact on teachers' knowledge, beliefs, and classroom practice. The PD introduces teachers to a neurodevelopmental framework[1] (Barringer et al., 2010) as a lens for better

[1] This framework describes eight constructs related to learning processes (i.e., attention, temporal-sequential ordering, spatial-ordering, memory, language, neuromotor function, social cognition, and higher order cognition). Those who use the framework are encouraged to think through how these constructs interact when student learn, and to adapt mathematics lessons based on individual students' neurodevelopmental learning profiles.

understanding individual students' strengths and challenges and the demands of mathematical activities. The PD also engages teachers in in-depth analyses of mathematics lessons, including examination of their mathematical goals, and different instructional strategies and teaching practices that support the attainment of these goals while attuning to individual students' strengths and needs. MFA was developed by Bank Street College and EDC with funding from the National Science Foundation and is published by Corwin Press (Moeller et al., 2012; 2013). MFA incorporates several components that RCTs or quasi-experimental studies (QEDs) have shown to be effective for supporting elementary school teachers' professional learning and for improving achievement of struggling students, such as teacher collaboration for instructional planning and peer coaching (e.g., Stevens & Slavin, 1995), formative assessment and progress monitoring of students (Gersten et al., 2009), and lesson study (Lewis & Perry, 2017).

MFA is not tied to any specific K–5 mathematics curriculum. Rather, it helps teachers to use and adapt their existing mathematics curriculum to make it more accessible to a wide range of learners. With funding from the Institute of Education Sciences at the U.S. Department of Education researchers carried out an RCT in collaboration with a large urban school district in a midwestern state in the U.S. In this paper is we report findings relating to three main research questions, which probed for the impact of MFA on (a) teachers' comfort and preparedness to teach mathematics to students with disabilities, (b) teachers' classroom practice, and (c) student performance on a standardized mathematics achievement test.

Theoretical framework and related literature

A number of factors influence student achievement (as measured by performance on standardized achievement tests), with teacher quality being the most powerful (e.g., Nye et al., 2004; O'Dwyer et al., 2010). Various teacher characteristics such as experience, education background, dispositions (beliefs and motivations), as well as their knowledge (content knowledge, pedagogical content knowledge, general pedagogical knowledge), have been shown to impact student outcomes to varying degrees (e.g., Clark et al., 2014). Classroom practice is another factor that has been linked to student achievement (e.g., Clements et al., 2013). Research has helped to identify key features and principles of instructional practices that are associated with higher student achievement in mathematics, such as strategies for teaching students who struggle in mathematics (Gersten et al., 2009).

However, major questions remain with respect to how PD can play a role in improving teacher quality, practice, and student achievement, given the mixed findings often generated. Recent reviews of research on teacher PD (e.g., Darling-Hammond et al., 2017; Gersten et al., 2014; Yoon et al., 2007) attest to the paucity of rigorous evidence that links PD to improved student outcomes in mathematics and other subject areas. Moreover, little is known about the specific aspects of teacher quality that PD can most effectively target. Although there is general consensus that teachers must have mathematics content and pedagogical content knowledge to effectively teach mathematics, and many PD efforts target these teacher qualities, evidence that attests to the effectiveness of content-focused PD has been difficult to come by. Three recent studies (e.g., Garet et al., 2016; Jayanthi et al., 2017; Jacob et al., 2017) found only limited evidence of the impact of content-focused PD on teachers' mathematical and pedagogical content knowledge and instructional practices, and no effects on student outcomes.

In assessing the efficacy of MFA, we were particularly interested in understanding its impact on teachers' dispositions. Teacher dispositions are related to teachers' professional background and experiences, knowledge, and teaching contexts, and to characteristics of the students they teach (Clark et al., 2014). Research has demonstrated that teachers who have negative self-efficacy about mathematics (math anxiety) can have negative effects on the mathematics achievement of their students (Beilock et al., 2010; Ramirez et al., 2018). Because teachers draw on cognitive and affective resources during instruction (e.g., Anderson et al., 2018), teacher dispositions constitute an important outcome to measure when assessing the impact of professional development.

To assess teacher dispositions, we constructed two eleven-item scales that measure teacher comfort and preparedness with various practices that have been associated with differentiated mathematics teaching practices (see Table 1). These scales have been used in multiple studies of MFA (Duncan, et al., 2022) and have demonstrated high internal consistency with Cronbach alphas ranging from .85 to .95.

Table 1: Comfort and preparedness scales

Question stem: How prepared/comfortable do you feel about the following?
a. Teaching standards-based math to students with disabilities.
b. Identifying the math strengths of students with disabilities.
c. Identifying the math needs of students with disabilities.
d. Understanding the mathematics of the lessons I teach.
e. Analyzing the demands of mathematical tasks on students.
f. Determining the goals of the math lessons I teach.
g. Understanding learning trajectories in mathematics (how the math I teach relates to what students learned before and what they will learn later).
h. Selecting specific strategies to address the strengths of students with disabilities in math.
i. Selecting specific strategies to address the needs of students with disabilities in math.
j. Adapting math lessons for students with disabilities to help them meet standards-based goals.
k. Collaborating with my colleagues when planning math lessons.

Note. Items are rated on 1-5 Likert scales, anchored by 1=not at all prepared to 5=very prepared, or 1=not at all comfortable to 5=very comfortable

We hypothesized that improved comfort with and preparedness for teaching mathematics to students with disabilities will result in high-quality classroom practices, which are differentiated based on individual students' strengths and needs without undermining the rigor of the mathematics to be taught. This in turn would lead to improved student mathematics achievement (as measured by performance on standardized achievement tests), so we expected coordinated improvements in teachers' dispositions, classroom practices, and student achievement.

Methods

An RCT of MFA was conducted from 2015 to 2017 to help build the evidence base around the impact of PD interventions. Schools were randomized by a statistician blinded to study condition into either the MFA PD treatment group or business-as-usual (BAU) control group. The sample included 32 schools from a large, midwestern urban school district in the U.S., 98 4th and 5th grade general and special education teachers, and approximately 1,500 4th and 5th grade students. This study focused on estimating MFA impacts on teacher outcomes after one year of PD because this was the point when the maximum MFA-BAU contrast was expected. For student-level outcomes, a two-year study was originally planned but findings presented here describe outcomes after Year 1 because there were challenges in maintaining the sample across both years, largely because of student mobility and difficulties with collecting parent consent to use achievement data collected by the school district. Key Year 1 findings are supported by a strong design and allow for solid causal inference.

Research Questions and Outcome Measures

The study's first research question was: *Does participation in MFA PD, compared to business-as-usual (BAU) experiences of a control group, improve teachers' comfort and preparedness to teach mathematics to diverse students (including those with disabilities) after the completion of the PD?* Separate measures of teachers' self-reported comfort-level and preparedness were used as dependent variables to address this research question. Two researcher-developed 11-item scales were used (see Table 1), and corresponding Cronbach alphas were .886 and.950. The scales were included in a larger teacher survey that was administered at the beginning and end of the school year.

Research Question #2. Does participation in MFA PD, compared to the BAU experiences of a control group, result in improved mathematics classroom practice after the completion of the PD? A subsample of 40 classrooms were observed at the beginning and end of the school year using the Classroom Assessment Scoring System (CLASS). The CLASS, a widely used and psychometrically sound observation approach (Pianta et al., 2012), was used to generate dependent variable data for this question. Unfortunately, only 40 teachers agreed to be observed and this undermines the benefit of randomization. Therefore, related analyses were conducted using a quasi-experimental design wherein the strength of causal inference is predicated on showing that teachers across the two study conditions were similar on baseline assessments of their classroom teaching practice.

Research Question #3. Does the use of an MFA approach in the classroom result in improved student achievement in mathematics after one year of intervention exposure? The NWEA MAP assessment, used by the school district in which the study was carried out, was the measure used to assess student mathematics achievement. Coefficient alphas for this measure's related subtests range from .92 to .96 and test-retest reliabilities range from .77 to .94; there is also strong evidence of the measure's construct and concurrent validity (see Malone al., 2020 for details).

Data Analyses

Impact analyses for Research Questions 1 and 2 entailed using a two-level hierarchical linear model with teachers clustered by schools, and a term for assessing the treatment impact at level two (i.e., schools, the unit of randomization). Impact analyses for Research Question 3 were conducted using three strategies: (1) a school-level analysis; (2) a student-level hierarchical analysis that accounted for student clustering within schools; and (3) a hierarchical student-level analysis that included grade level as a moderator. The first strategy was used because of the difficulties with attrition. This school-level impact model entailed using each school's mean achievement at post-test as a dependent variable to assess MFA's impact on student achievement at grades 4 and 5. This analysis included all 32 study schools and does not have cluster-level missing data. The second two strategies both accounted for student clustering in schools and included a term for assessing the treatment impact at the school level. All mean contrasts presented here adjusted for baseline differences. These baseline differences were observed using the same measures that produced dependent variable data. Missing data (and by extension Year 1 attrition) were addressed by using multiple imputation procedures, which we consider to be our primary analyses. We did however re-run impact models using listwise procedures (i.e., no imputed data) to perform sensitivity checks (see Enders, 2010). For Resarch Question 3, school-level impact analyses did not formally have missing data; furthermore, Grade 4 subsample analysis yielded inconsistent results across imputed and listwise analyses, so we present related findings from both approaches. All required assumptions for impact modeling were met.

Key findings

Research Question #1

The pattern of results was the same for both scales: the MFA group reported lower levels of preparedness and comfort at the pretest, compared to BAU teachers, but there was a steep increase from fall to spring. The opposite pattern was observed for the BAU group. Results were statistically significant. Effect sizes using the Hedges' g statistic were $g = .54$ ($p < .05$; $M_{diff} = .803$; $SD_{pooled} = 1.48$) for preparedness and $g = .67$ for comfort ($p < .05$; $M_{diff} = 1.08$; $SD_{pooled} = 1.621$). To summarize, MFA teachers increased their senses of preparedness and comfort in teaching students with disabilities, as compared to BAU teachers.

Research Question #2

Unfortunately, there was a large baseline difference favoring the MFA group, which undermined causal inference, and Bonferroni corrections for multiple comparisons rendered results from this small sample not statistically significant. Analyses showed that at the posttest (and after adjusting for pretest levels), MFA teachers scored higher in the domains of Emotional Support ($g = .98$; $p > .05$; $M_{diff} = 1.115$; $SD_{pooled} = 1.118$), Instructional Support ($g = .69$; $p > .05$; $M_{diff} = .526$; $SD_{pooled} = .765$), Classroom Organization (.78; $p > .05$; $M_{diff} = .733$; $SD_{pooled} = .94$), and Student Engagement (.54; $p > .05$; $M_{diff} = .435$; $SD_{pooled} = .8$). These contrasts should be interpreted with caution, but they do suggest MFA had a positive impact on teachers' classroom practices.

Research Question #3

The resulting g from the first school-level analytic strategy was .33 ($p > .05$; $M_{diff} = 1.82$; $SD_{pooled} = 5.45$), favoring MFA schools. This finding was however not statistically significant, which likely stems from the analysis being underpowered given there were only 32 school-level means. The student-level analyses mirror the pattern shown in the cluster-level analysis; that is, while the results favor the treatment group ($g = .11$; $p > .05$; $M_{diff} = 1.54$; $SD_{pooled} = 14.46$), the differences were, again, not statistically significant. When grade level was examined as a moderator, different patterns between grade 4 and grade 5 students were found. In grade 4, students whose teachers participated in the MFA PD had higher posttest scores than students whose teachers were in the BAU group and results based on analyses with imputed datasets were statistically significant (g = .26; $p < .05$; M_{diff} = 3.62; SD_{pooled} = 13.9). However, impact analyses that did not entail use of imputed data did not allow for rejecting a null hypothesis (g = .20; $p > .05$; M_{diff} = 2.77; SD_{pooled} = 13.6). In grade 5, there were small mean differences between the MFA and BAU groups. The overall pattern of findings suggests MFA PD might have had a positive impact on student achievement, but student attrition prevent conclusive findings.

Discussion

This study yielded evidence that MFA had a positive impact on teacher's self-reported sense of comfort and preparedness with respect to teaching students with diverse learning needs. While the evidence that MFA impacted teacher classroom practice and student achievement is less strong, it is still compelling. As we seek to better understand the impacts of PD on teacher and student outcomes, it is important to "open the black box" and flesh out the mechanisms by which PD can affect teacher practice, which in turn, affects student achievement (cf., Clarke & Hollingsworth, 2002; Goldsmith et al., 2014). The data presented here converges with other recent studies that have demonstrated that teacher dispositions may be key mediators to consider in our models of teacher PD (e.g., Miele, et al., 2019; Schoen & LaVenia, 2019).

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Developing an identity as a mathematics teacher through group work

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A concern in teacher education is to tighten the connection between coursework and practice through work with core practices. In this study, we investigated student teachers' learning processes when working in groups in mathematics teacher education, where the tasks the student teachers worked with were centered around core practices. Taking a social view of learning, we describe student teachers' learning to teach mathematics through developing an identity as a mathematics teacher. Group discussions were audiotaped, transcribed, and analyzed using a grounded approach. Our findings suggest that group work centered around core practices gives student teachers good opportunity to negotiate meaning of core practices and develop an identity as mathematics teachers.

Keywords: Mathematics education, communities of practice, elementary school teachers, group activities, educational practices.

Background

The development of a mathematics teacher identity is critical for learning to teach (Goodnough, 2010). da Ponte and Chapman (2008) suggest that investigation and reflection on practice play important roles in developing a mathematics teacher identity, and several researchers have investigated opportunities for developing identity during teacher education (Lutovac & Kaasila, 2018). Essien (2014) found that during instruction in teacher education, student teachers develop an identity as mathematics learners, not as mathematics teachers. Learning mathematics and learning to teach mathematics is different in that the latter draws upon a broad range of experiences and knowledge, and teacher education programs should provide student teachers with coherent experiences to support their learning to teach mathematics. Hence, the contexts in which the student teachers develop their mathematics teacher identity through the teacher education program need to be tightly interwoven (Van Zoest & Bohl, 2005).

A concern in teacher education internationally and in Norway is that there is a gap between the teacher education coursework and school practice (Hammerness, 2013). The gap is due to a lack of coherence between activities in teacher education coursework and school practice, and teacher education programs need to build bridges between the different contexts. To bridge coursework and practice, Grossman et al. (2009) suggest that teacher education programs should be organized around the central practices of teaching, called core practices. Core practices are something a teacher often does, which enhance the students' opportunities to learn and preserve the complexity of teaching at the same time as novices can begin to master them (Grossman et al., 2009). Examples of core practices are eliciting and responding to students' ideas, leading a whole-class discussion towards a mathematical goal, or attending to students' understanding and helping them progress. Teacher education programs centered around core practices will help student teachers develop knowledge, skills, and identity, which can narrow the gap between coursework and school practice.

A typical learning activity in mathematics teacher education is group work. The goal is often that the student teachers get the opportunity to discuss and develop their mathematical thinking and their knowledge of students' mathematical thinking (Crespo, 2006). Tasks in group work can be organized around core practices, such as asking students teachers to make sense of their students' work (Crespo, 2006; Kazemi & Franke, 2004) or planning and co-teaching together (Haniak-Cockerham, 2019). More research on implementing work with core practices in teacher education coursework is needed (McDonald et al., 2013). In this study, we investigate the learning processes when student teachers participate in group work organized around core practices.

Taking a social view on learning, we consider identity to be learning as becoming (Wenger, 1998). Our research question is: *how does some student teachers' identity develop as they work with tasks centered around core practices of mathematics teaching?*

Theoretical framework

Wenger (1998) stated that learning is participation in communities of practice. A community of practice is defined by mutual engagement, joint enterprise, and shared repertoire. Student teachers learn to teach in several communities of practice during their education, for example, the coursework in mathematics teacher education, the school in which they have school placement, and the classroom in which they have school placement. When student teachers participate in each of these communities, they become more central members of the communities, and their learning to teach mathematics is a nexus of their learning trajectories in all these communities. Van Zoest and Bohl (2005) argue that a theoretical framework that considers the broad range of student teachers' experiences and contexts is necessary when studying learning to teach.

In studying student teachers' learning to teach, *identity* is a useful theoretical construct (Van Zoest & Bohl, 2005). Wenger (1998) introduced identity to shift the focus to the individual, but from a social perspective. Following Wenger (1998), identity is more than the everyday use of identity as a sense of self. The construct includes knowledge, beliefs, our perceptions of others, and others' perception of us as we participate in communities of practice (Van Zoest & Bohl, 2005). As student teachers engage in practices related to mathematics teaching, their identity is developed through their modes of belonging in these practices, and we can study student teachers' learning beyond the scope of only one community.

Identity is developed through participation and non-participation in three modes of belonging: engagement, imagination, and alignment (Wenger, 1998). First, participation in engagement means to contribute to negotiation of meaning and adopt others' contributions. In engagement, members gain ownership over meaning through how they make use of, control, and adopt the meanings they negotiate (Wenger, 1998, p. 200). Non-participation is to not contribute to negotiations or to have one's ideas ignored. Second, participation in imagination is to imagine oneself across time and space, experiencing the meanings of other communities as one's own. Non-participation in imagination is to have limited access to other communities' practices, making negotiation impossible. Third, participation in alignment is to coordinate actions and efforts with the practices and meanings of another community, while non-participation is when directions are strict, leading to inflexible and vulnerable coordination. The modes of belonging strongly influence both one's identity and the community one participates in (Goodnough, 2010). To operationalize mathematics teacher identity, we will draw on the practices of mathematics teaching.

Wenger's (1998) framework describes learning in general, so we draw on core practices of mathematics teaching (Grossman et al., 2009) to describe mathematics teacher learning as modes of belonging. We operationalize the development of mathematics teacher identity as modes of belonging in the core practices learning about students' (mathematical) understanding and orchestrating (mathematical) discussions. *Engagement* in these core practices is to mutually participate in understanding and fine-tuning how one can learn about students' understanding through attending to their work, responding to their work, asking questions, and eliciting their understanding. *Imagination* in core practices is to understand students' work and plan questions through imagining oneself as a teacher in the classroom. In coursework, the core practices student teachers participate in belong to an imagined classroom community. *Alignment* in the core practices is to align efforts with the valued enterprise. In our mathematics teacher education, reform-oriented mathematics education is communicated by the teacher educators. Student teachers participate in alignment through questioning and responding in line with the reform-oriented view of mathematics. Through modes of belonging, student teachers develop a shared understanding of the core practices, and since practice and identity influence each other (Lutovac & Kaasila, 2018), participation in core practices influences student teachers' identity development.

Method

We collected data from two groups of participants: in-service teachers (ISTs) participating in a one-year teacher development program and pre-service teachers (PSTs) in their third year of teacher education. The ISTs were primary school teachers working in the 1st to 7th grades. From 2019 to 2021, group discussions in four lessons were audio-recorded and later transcribed and coded. The group discussions centered around different activities that involve working with core practices.

The tasks in the group work were centered around four core practices: attending to student work, eliciting students' ideas, responding to student work, and asking questions for whole-class discussions. First, the student teachers (ten ISTs and sixteen PSTs) practiced attending through making sense of students' written work with a multiplication task, a fraction task, and through making sense of students' utterings in a transcription of a dialogue about area measurement. Further, the student teachers practiced eliciting students' ideas and responding when they were planning questions to get insight into their thinking and support further thinking in the fraction and measurement tasks. Last, they practiced planning questions for a whole-class discussion in which connecting students' ideas was the goal in the multiplication task. The group discussions on the multiplication, fraction, and measurement tasks lasted approximately 50, 20, and 30 minutes, respectively.

We took a grounded theory approach to the analysis (Charmaz, 2006), meaning that questions arose from the codes, minimizing the risk of any pre-existing assumptions affecting our analyses. The two researchers first coded the discussions statement by statement, led by the question "what are they saying?". Comparing and grouping led to an initial set of codes, and in the next cycle, we coded the discussions based on actions, leading to a set of codes of actions student teachers do when they work with core practices in groups. We noticed that what the student teachers said and did depended on the context they were discussing. Further, we chose to analyze our data through Wenger's (1998) notion of identity, taking the multiple contexts into account. The data was coded again through participation and non-participation in the different modes of belonging. The researchers coded one of the transcriptions together before coding the rest of the data material separately. Dialogue between the researchers was maintained throughout the coding process to secure similar coding and increase reliability.

Results

We organize our results in two sections, participation and non-participation. We found participation to be more prominent than non-participation, and through excerpts of our data, we will show how the different modes of belonging contributed to the student teachers' identity as *mathematics* teachers.

Participation

Throughout our data, we observed participation in a combination of engagement and imagination. The following discussion between ISTs where they discuss students' strategies for solving the multiplication task 13·27 is an example of this.

Tiril:	Sort of, added them.
Kaia:	Yes, eight and four.
Tiril:	Eight and four is this one.
Oda:	But I don't understand. Why did she multiply 27 by two first?
Lisa:	In order to be able to double it.
Tiril:	Because now she has, in a way, she is supposed to have 27 thirteen times. Now she has 27 two times, which is 54, and that is, if she takes 54 twice, then she will have 108.
Oda:	Yes, four times.
Tiril:	And this is eight times.
Oda:	And then she has five left.
Tiril:	Yes, and then she has, she has taken it eight times, right? Plus four times.
Oda:	Ok, like that
Tiril:	So, she has taken not two, but one
Oda:	Thank you, now I understand.

In the first part of the excerpt, the ISTs participate in engagement, working with the core practice of attending to students' work by discussing every step of a student strategy in detail, building on each other's statements, and listening to each other. Meaning about students' multiplication strategies and their ideas is constructed. Further, in the last half of the excerpt, when they discuss the strategy, they actively refer to the student and her work and talk about her as if she was a real student. Through work with the representation of the student's work, they imagine how the student has been thinking and participate in negotiation of meaning about attending to students' work through imagination.

Further, we identified participation in a combination of alignment and imagination. In the following excerpt, PSTs discuss how they can continue to help students develop their ideas about equivalent fractions based on some students' written work.

Anna:	(…) "How can you continue the discussion if a student answers the following?" Maybe to get them to explain what they have been thinking. How did you get this? What did you do?
Nina:	Yes, I would follow up with "how have you been thinking to reach this answer?"
Anna:	Yes, and put into words what they have done.

Nina: Yes.

Anna: (…) "What can you, as a teacher, ask the students in order to build understanding for equivalence and common denominator?" That's what we have to help Martin with, who does not have [common denominator in] his numerical expression.

Nina: We can ask Jenna, what is, or in another way, but something about her answer, which is 1 and 8/16. She must have understood that 8/16 is equal to 1/2. (…) We could, for example, use a number line, mark 1/2, and then divide it into 16 pieces, and check that 8/16 is at the same place as 1/2. Then they can maybe realize that 1/2 is the same as 8/16. That they are equivalent, isn't that what it means to be equivalent?

In the first four statements, Anna and Nina are planning questions they can ask the student to emphasize their thinking process, aligning with the view of reform-oriented mathematics teaching that has been communicated in the teacher education program. However, the questions they are planning are quite general and could be asked in almost any kind of mathematics teaching. In contrast, in the last two statements, Anna brings in one of the students and her interpretation of his understanding, which Nina builds upon by bringing in the other student and using her answer as a starting point for showing equivalence using a number line. When imagining the students and how they can elicit and respond to students' thinking, the PSTs are also negotiating meaning through imagination, and their questions are directly connected to the teaching situation in question.

Further, several discussions were guided by the use of a framework or instructions in the task. Below, the ISTs are discussing a dialogue between students measuring the area of a blackboard using sheets of paper.

Hedda: (…) Are we using all these points? Or… the eight, those from Lehrer?

Oda: Yes, that's what we'll do.

Hedda: The first, they found out with some help from the teacher that they couldn't write letters. Isn't that the first point? To realize that unit of measure has the same property as the object.

Oda: No, isn't this two? Where are you?

Hedda: He got some help from the teacher to figure out that it would be difficult to measure using the letters.

Else: Yes, but that is the same as number four?

Oda: Tiling… No identical units.

Here, the ISTs are engaging in the core practice of attending to students' understanding, trying to label student actions in the dialogue using Lehrer's (2003) framework. In the third utterance, Hedda points to what the students do and tries to understand what aspect of measurement they are working with by connecting the student actions to the framework. Their attempt to apply the framework to the dialogue leads them into a discussion about the meaning of the different aspects and the students' understanding, and together they figure out which of Lehrer's (2003) aspects is the relevant one. When they are engaging in attending to students' understanding, the framework is supporting their work.

Non-participation

As mentioned, participation was most prominent in our data, but there were also occurrences of non-participation in different modes of belonging. In the excerpt below, the ISTs work with students' written solutions to 13·27 and plan questions for the following classroom discussion, where connecting students' ideas is the goal.

Else: Why is it 90 plus 50 here? (…) 140. Nothing about why she picked those numbers. Maybe it is to figure out 295 plus 54? Then it is just 90 plus 50, and then maybe she added 200.

Anne: And then 9.

Else: And the ones. I don't have the slightest clue what I would say, I would call in sick that day.

Bettina: I find this really hard.

Else: I don't understand what…

Anne: I think it looks like her strategy is quite good. Per Christian, on the other hand, can't continue to add forever. He can't start in the seventh grade and add, for example, 13 times 69.

Bettina: That is a very non-effective strategy. (…) What we want is for him to learn a more effective strategy.

Anne: Yes, maybe through the use of the area model, and later learn this strategy. (…) How can we facilitate for the students to share their thoughts? That is easy, they can just come in front of the class and explain. Which talk moves will we use? Well, confirm their strategies, and maybe let someone else repeat.

Bettina: Yes, that is what we have talked about today regarding conversations.

Anne: Yes. I think these questions are easy. But this planning sheet, I don't think I will use it. It is not useful for me.

They are building on their engagement in making sense of the student strategies they have chosen to emphasize in the discussion when they try to formulate questions for the whole-class discussion, but they do not conclude on any questions they can ask. When Else says that she would call in sick that day and the others respond that they find it difficult as well, non-participation in engagement is shaping their identity. Further, they try to formulate some questions through the use of talk moves as a framework, but the questions they are formulating are not helping them connect students' ideas. The ISTs apply the talk moves inflexibly, not taking the teaching situation at hand into consideration, and their identity is shaped through non-participation in alignment. In the last statement, Anne creates a distance between herself and the learning material in the group work, and since she is unwilling to negotiate meaning, her identity is shaped through non-participation in engagement.

Further non-participation in imagination shaped student teachers' identity when they felt that they did not have enough information about a situation, as illustrated below.

Anna: Yes, we might ask those questions, but as I've said, I find it hard when I haven't seen what they have drawn or how they came up with these fractions.

Anna's statement indicate that she would have been able to negotiate meaning if she had been the teacher and had access to the students' work, but here she lacks information and is therefore not able to imagine the students' work and her responses to it.

Discussion

Our analysis of the group discussions provided insight into how participation and non-participation shaped the student teachers' identity as mathematics teachers. Student teachers' identity was developed through engagement in the core practices the tasks were centered around, through imagination of the teaching situation, and alignment with the practices for mathematics teaching communicated in the teacher education programs. The student teachers' participation in engagement could be supported by frameworks for understanding students' ideas, as in the discussion about the measurement task. However, as we can see in the discussion between Anne, Else, and Bettina, teacher educators need to ensure that the student teachers are not adopting frameworks strictly and inflexibly. Further, student teachers' participation in imagination was supported by a representation of the classroom situation, either written work or transcriptions of dialogues. Our results are in contrast to those of Essien (2014), who found that student teachers developed a mathematical, not mathematics teacher, identity in coursework. One explanation for differing results can be that the coursework in Essien's (2014) study was not centered around core practices. From our analysis, it is evident that a combination of tasks centered around core practices, representations of the teaching situation, and frameworks for understanding student ideas support student teachers in developing a mathematics teacher identity.

Wenger (1998, p. 183) stated that most of what we do involves a combination of different modes of belonging, which is also evident from our analyses, where the student teachers negotiate meaning through more than one mode of belonging. The different modes of belonging working together promote student teachers' identity development, because they counterbalance each other (Goodnough, 2010). For example, imagination is helping the student teachers in attending to students' ideas when engagement becomes too narrow in the discussion about the multiplication strategy. Van Zoest and Bohl (2005) suggested that the different communities in which student teachers learn to teach should be strongly intertwined to help the student teacher develop a mathematics teacher identity. Work centered around core practices is a way of intertwining the school community and the teacher education community (Grossman et al., 2009), and the student teachers had opportunities to develop their mathematics teacher identity in several modes of belonging.

When we were planning this study, we chose to collect data from both PSTs' and ISTs' coursework, because we were expecting a difference in how they would work with core practices. Despite our expectations, our analyses did not reveal any differences in the two groups of teacher students' participation in core practices. Since identity captures beliefs, experiences, and knowledge (Van Zoest & Bohl, 2005), one could expect that PSTs' and ISTs' identity development would be different. However, the ISTs in our study are participating in a teacher development program for teachers who do not have any prior education from mathematics teacher education. The ISTs' mathematics teacher identity might therefore not include much prior experience or knowledge, making their identity more similar to the PSTs' identity. Further, in our study we are not giving a description of the student teachers' identity before and after group work, because we would need longitudinal data, and possibly interview data, to do so. Rather, we describe the processes in which the student teachers develop their identity, finding that group work around core practices gives both PSTs and ISTs opportunities to participate in different modes of belonging.

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Secondary teachers' conceptions of infinity in different context

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This study investigates how secondary school mathematics teachers think about infinity when they deal with this in different context. The research tool was two tasks based on hypothetical scenarios. The first one concerns the calculation of an infinite sum and the second is a version of the Ping-Pong Ball conundrum. Two math teachers evaluated the hypothetical students' reasoning and expressed their views in writing and during interviews. From the findings emerged that the different context of the tasks, formal and non-formal context, influenced the teacher's approach to infinity. The perspective through which the participants approached the infinite processes of the tasks, as well as the duration of these determined their views.

Keywords: Teachers conceptions, infinite sum, infinity, paradox.

Introduction

The concept of infinity as a learning item has been widely studied by mathematics education researchers (Montes, Carrillo & Ribeiro, 2014). Montes et al. (2014) highlighted that the process of developing the cognition of infinity and the differing conceptions of infinity were the focal point of the research. However, they consider that a study on the conception, knowledge, and practice of secondary teachers, especially in-service, about infinity is needed. Dubinsky, Weller, McDonald & Brown (2005a) noted that "Philosophers, mathematicians, mathematical historians, students, and mathematics education researchers, among others, have struggled to resolve various paradoxes, dichotomies and issues regarding conceptions of infinity" (p.336). Kolar and Čadež (2012) referred that the abstract nature of infinity, the understanding of it as an endless process or as a set of an infinite number of elements as well as the understanding of paradoxes are the difficulties that the individuals have to manage.

The paradoxes can be used as a lens to understanding infinity and as means to identify specific difficulties of the actual infinity (Mamolo, 2014). It is noted that these paradoxes are based on paradoxical assumptions, and they are inspired by the attributes of the mathematical infinity and the overturns of fundamental preconceptions (Mamolo & Zazkis, 2008). Kolar and Čadež (2012) focusing on primary teacher students, pointed out that an individual's understanding of infinity can be influenced by the context and type of the given task.

This paper aims to investigate how the context influences secondary mathematics teachers' conception of infinity through their engagement in two tasks with different context.

Theoretical Background

The conception of infinity is related to the dichotomy between potential and actual infinity. In particular, the potential infinity is considered as an infinite – endless process, and the actual infinity can be perceived as a completed process (Dubinsky et al., 2005a). In this context, Dubinsky, Weller, McDonald & Brown (2005a; 2005b) focused on understanding the aspects of infinity using the APOS

Theory. The potential infinity is seen as a process of infinite repeating steps and the actual infinity is the mental object obtained through the encapsulation of that process. Thus, they highlighted that infinity can be perceived as an infinite process, as a single operation that can be carried out and finished as well as a completed totality (Dubinsky et al., 2005a). According to them, it is important to notice that the potential and actual infinity represent two different cognitive conceptions that are related to mental mechanisms. Dubinsky, Arnon, and Weller (2013) studying the preservice teachers' understanding of the 0,999… =1 introduced an intermediate stage in APOS Theory between Process and Object, which they called Totality (TOT). Thus, the completed idea for the infinity can be constructed by the interiorization moving from action to process, detemporalization to progress from process to totality, and encapsulation to progress from totality to object (Dubinsky et al., 2013).

Moreover, Dubinsky, Weller, Stenger and Vidakovic (2008) and Mamolo and Zazkis (2008) studied the understanding of infinity using paradoxes, in which procedures with infinite steps could be completed in a finite time. As Dubinsky et al. (2008) found, when resolving the Tennis Ball paradox, university students focused on the existence of an incomplete process, or they claimed that infinite balls would be in the two bins at the end of the process. Mamolo and Zazkis (2008) studied high school mathematics teachers who participated in a master program and undergraduate students of liberal arts and social sciences, the perceptions of infinity using the "Hilbert's Hotel" paradox and "Ping-Pong Ball" conundrum. They noted that the students' answers focused on the practical impossibility of paradoxes. Thus, they concluded that "the students who acknowledged the gap between their intuitive and formal understandings of infinity may have taken an important first step toward encapsulating infinity as an object" (Mamolo & Zazkis, 2008, p. 180).

Methodology

In this paper, we present a part of border research focused on secondary education mathematics teachers' conception of infinity. This research is a multiple case study focusing on ten in-service teachers, with various teaching experience. They were informed about the aim of the research and they agreed to participate in this. They had a degree in mathematics and during this research, they were postgraduate students in mathematics education. In this paper, we analyse the data of two of them, Alice and Sergio. These cases were chosen since they approached the infinite processes under different perspectives. Specifically, perspectives based on the actual, potential, a combination of the potential and actual conceptions, and finite reasoning of an infinite process have emerged.

The research data were the teachers' written answers in two tasks and semi-structural interviews. The aim of the interviews was to insight into teachers' thinking. The interviews were recorded with teachers' consent and transcripted. The presented extracts in the results are a translation from Greek, which is the language that interviews were conducted, into English.

The research tool was two tasks regarding the conceptualization of infinity. They are based on hypothetical scenarios grounded on previous research findings and experience concerning learning and teaching of series. The tasks aimed to engage the teachers with students' misconceptions. Thus, we are not focused exclusively on the correctness of the responses but on whether the teachers can identify students' errors and their causes, as well as teachers' reasoning about them (Biza, Nardi & Zachariades, 2007).

1st Task

Two math students were asked to calculate the infinite sum $\frac{1}{9}+\frac{1}{81}+...$ The students thought for a while and continued with the following dialogue:

Stud. A: This is the infinite geometrical series $\sum_1^{\infty}\frac{1}{9^n}$, which converges due to $|\frac{1}{9}| < 1$. Am I right;

Stud. B: Yes. So, the infinite sum is converging to $\frac{1}{8}$.

Stud. A: No, it is equal to $\frac{1}{8}$.

Stud. B: It cannot be equal to $\frac{1}{8}$ since as we add terms the infinite sum is continually increasing. So it will converge on $\frac{1}{8}$.

Stud. A: It is continually increasing, but when the process will be completed, it will be $\frac{1}{8}$.

Stud. B: In an infinite sum the process is infinite and endless. Thus, this cannot be equal to a number. It is continually approaching the number and it tends to this.

a) Do you consider that one of the two answers is correct? Justify your answer. b) If you think all the answers are incorrect, which one do you think is the correct answer? Justify your view. c) For the ones that you think are incorrect, which are the misunderstandings that led to this answer.

The first task provides a context based on the formal mathematical knowledge about series. However, it is inspired by the dilemma of series' (as an infinitely repeating process) equality or convergence to a number. According to Spivak (2008, p.389) the expression "the series $\sum_{n=1}^{\infty} a_n$ does or does not converge" used to refer to infinite series is somewhat peculiar because the symbol $\sum_{n=1}^{\infty} a_n$ denotes at best a number (so it can't converge) and nothing at all unless (a_n) is summable. Thus, the given answers are based on the fact that the used terminology is conflicting since we refer that a series converges despite that it is equal to a number.

2nd Task

Three math students were given the following problem: "Imagine you do the following mental experiment with a duration of just 1 minute. It is assumed that you have an infinite set of numbered balls 1, 2, 3. . . and put them in a box as the following procedure demands. Firstly, you put the first 10 balls into the box, and then remove the number 1 ball in 30 seconds. Then, in half of the remaining time, you put the balls 11 to 20 into the box and remove the number 2 ball. Next, in half the remaining time (and with faster movements), you put the balls 21 to 30 into the box and remove ball number 3, and this process continues. After 60 seconds, at the end of the experiment, how many balls will there be in the box? ". The three students thought the problem and the next dialogue followed:

Stud. A: I think that at the end of the experiment there will be infinity balls in the box since those we insert each time are more than those we remove. In particular, at each step of this process, we put in the box 10 balls and we remove one, so we add 9 balls. Due to the infinite steps of the process after 60 seconds we will have infinity balls in the box.

Stud. B: This is accurate but does not mean that at the end there will be infinity balls in the box. I think that the box will be empty.

Stud. A: I do not think that will be empty. How do you think about it?

Stud. B: I think that firstly we will get the number 1 ball, after the number 2 and finally we will get all the balls.

Stud. C: Is it because the experiment has infinite steps that will not be completed?

a) Do you consider that one of the three answers is correct? Justify your answer. b) If you think all the answers are incorrect, which one do you think is the correct answer? Justify your view. c) For the ones that you think are incorrect, which are the misunderstandings that led to this answer.

The second task is based on a version of the Ping-Pong Ball conundrum (Mamolo & Zazkis, 2008; Mamolo, 2014), promoting a non-formal context focusing on "realistic" processes. The conundrum is referred to an infinitely repeating process and its ending due to the existence of a finite time. For its solution, they are needed the formal mathematical knowledge on series and the coordination of the infinite processes (inserting balls, removing balls, splitting the time). The key point for this coordination is the order of removing balls so there will be a specific time interval in which each of the inserting balls will be removed.

The analysis of data was carried out in two phases. In the first phase, we analysed teachers' written responses identifying keywords in their reasoning in order to design properly the upcoming interviews. In the second phase, we analysed the transcriptions of the interviews, reading them line by line, identifying keywords, and characterising an argument according to the conception that was based on.

Results

In the following, we analyse the data from all the phases of the analysis of Alice and Sergio.

The case of Alice

Alice in her written response to the first task agreed with Student A that the series is equal to the 1/8. She calculated the infinite sum $\sum_{n=1}^{\infty}(\frac{1}{9})^n$ by using the formula of the infinitely decreasing geometric progression and she noted that "the infinity sum indeed is continuously increasing but in the end, it ends". She also mentioned the infinite sums 0.3 + 0.03 + 0.0003 + ... = $\frac{1}{3}$ and 0.9 + 0.09 + 0.009 + ... = 1 as most direct and intuitive examples.

During the interview, she distinguished the process from the outcome of the process saying:

> "The process is different from the result. The process will give a result… If we deal the process as a whole, if we are able to see it as a package, as a whole, the whole process… the whole sum, if we consider it as a whole then it will be 1/8."

The use of terms "package" and "the whole process" indicates that she understands the process as complete. This shows her ability to perceive infinity as actual infinity.

Then, commenting on Student B's view of the infinite sum, she mentioned:

"Student B sees this endless process that is continuously increasing, and he just cannot see it from outside. He cannot come out from it, but he is inside… the result is the one that I see from outside, the infinite sum but from outside."

The teacher used the terms "inside" and "outside" to describe the views from which an infinite process can be approached. She claims that the internal view of the infinite sum results in seeing it as an endless process, while the external view helps you to see the result of the process. She then went back to Student A and mentioned:

"The use of future tense in Student A's argumentation is the point that I focused on…. So he indeed considers that it will be completed sometime… we do not know how "time" will take, but we know that whatever time it takes this [infinite sum] will be 1/8. It is like we don't care about the addition. He sees it from outside."

In the above excerpt, Alice connects the external view of an infinite process with its completion time and argues that in order to have an external view of the infinite sum we have not to care about the time required for it.

Regarding the second task, Alice in her written answer she rejected all students' answers and claimed that there will be a finite number of balls in the box at the end of the experiment.

I disagree with all the students. I consider that the experiment is going to be completed since the time will end. I think that in the box will be a finite number of balls, different than 0 (since the balls are increased continuously so the sum is increasing), and there will not be an infinite number of balls (since the time is going to end despite that the process giving us an increasing number of balls. The time is ending because: $1+\frac{1}{2}+\frac{1}{4}+\ \frac{1}{8}+\cdots=\sum(\frac{1}{2})^n\ =\ \frac{1}{1-\frac{1}{2}}=2.$), so neither infinite balls either zero balls.

In the interview, she first expressed her initial thoughts after reading the task:

"[This task] reminded me the Zenon's paradox, and Hilbert's Hotel paradox. ... And then the time is introduced and confused me."

She recalled two well-known paradoxes as tasks that have similarities with the given one. Nevertheless, she mentioned that the existence of finite time in this task confused her. So, the connection of the infinite process with time seemed to create a conflict to Alice.

Then, Alice reiterated the view she had expressed in her written answer focusing on the realistic aspect of the mental experiment:

"I think that there is a natural number of balls different than 0 in the box. It will not be empty… since we have a sum of positive numbers continuously. Despite that, we remove balls, we add more, so the sum is always positive. And there will not be infinite balls because it is referred [in the task] that the process ends due to time. Just I mentioned I saw it realistic… Sometime the time will run out."

Alice seems to be influenced by the fact that it is explicitly stated in the task that time is fixed and this is a key for her view. This is how she deals task 2 as an outside observer:

"I saw it [the experiment] realistically… So, it says that "time will end" and hence I disagree with all three students… I saw it from the outside, it will end, and the time is changed to zero. I still consider the same about the concept of infinity [as in the first task]. I just decided whether I will be in or outside of the process. I decided to see it outside…. I think that the balls are an infinite set. We have infinite balls, but we are going to use a finite number of them."

She seemed to consider that the process of inserting and removing balls will stop at the end of the time and so finite balls will have been used. Summarizing, it has emerged that Alice demonstrates an actual conception in the first task and a conception based on finite reasoning in the second one.

The case of Sergio

In his written answer regarding the first ask, Sergio disagrees with both students. He calculated the partial sum of the series but he started from k=0. He wrote $\sum_{k=0}^{n} \frac{1}{9^{\kappa}} = \frac{1-\left(\frac{1}{9}\right)^{n+1}}{1-\frac{1}{9}}$ and $\sum_{\kappa=0}^{\infty} \frac{1}{9^{\kappa}} = \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{1}{9^{\kappa}}\right) = \frac{9}{8}$. He stated "this is a geometric sequence with α = $\frac{1}{9}$, where | α | <1, and therefore the sequence is convergent". Then, for Student's B answer he wrote "Incorrect calculation, probably due to carelessness or not remembering the formula correctly" and for Student A:

> He considers that the process will be completed and that the infinite sum is equal to 1/8, which shows that he has not understood that the infinite sum is the limit of a sequence of partial sums.

In the interview, it was initially made clear to Sergio that the sum starts from k = 1 thus the result is 1/8. Then, referring to Student A, he said "…it bothered me that on the one hand, he said converge and then he said equal.". When asked if he agrees with Student B, the next dialogue followed:

Sergio: …you have troubled me. I say it converges…in the end sums converge to the value 1/8. It is convergence. It is not equal. This is not another representation.
Researcher: OK. What does the series do?
Sergio: The series, you mean the infinite sum, is the limit of the partial sums.
Researcher: So the series is equal or converges to 1/8?
Sergio: It is equal to the value that converges… OK, it is equal to 1/8 … I don't know, I have mixed feelings, I fall into contradictions...

Then, when asked if he still disagrees with both students, Sergio answered:

"I disagree with both of them regarding the completion of the process … The one said that "when it is completed" but it will never be completed, and the other one that he cannot decide because it [the process] will not be completed. … We can admit that if we had infinite time this would be equal to 1/8…. If you stop somewhere ... then you are close to 1/8, but you did not reach it …However, if you do not stop then is going to be 1/8."

From the above, we see that Sergio is not sure if the series converges or is equal to 1/8. He also puts the dimension of time and considers that the infinite sum will be equal to1/8 in infinite time.

In his written response for the second task, he recalled Zeno's paradox and he wrote "…I "tend" to agree to Student A" but he also writes "...but what happens, in the end, I do not think we can decide.".

In the interview he said:

> “I disagree on whether it [the experiment] will end. Thus, I do not know if we can talk about the end of the experiment…. We have two times. The time experienced by the one who is in the experiment and the one who observes it. He who is in the experiment will always have half of the remaining time. It will not stop… the one who is inside in the experiment will tend to have infinite balls because he is constantly holding more and more in his hands [he means in the box] but I cannot say for sure that when it [the experiment] will end because it will not end.”

In the above extract, Sergio highlighted the fact that the infinite sum of time intervals will not be completed even though the experiment will end in a given time. In addition, he deemed that the process of inserting and removing balls cannot have a result since the process is endless. At the end of the interview, when he asked which student he finally agrees with, he agreed with Student C. In conclusion, it has emerged that Sergio demonstrates a combination of potential and actual conceptions in the first task and potential conception in the second one.

Discussion

From the preceding analysis, the secondary teachers’ conception of infinity has emerged. As the literature stated, the teachers were able to perceive the infinity either as potential or as actual infinity (Dubinsky et al., 2005a). However, it is revealed that an individual’s conception may be at an intermediate level, between potential and actual infinity, as Dubinsky et al. (2013) stated. Both teachers demonstrated different conceptions of infinity in the two tasks. Alice demonstrated an actual conception of infinity in the first task and a conception based on finite reasoning of an infinite process in the second one. On the other hand, Sergio demonstrated a combination of the potential and actual conceptions in the first task and a potential in the second one. In particular, in the first task, Alice seemed to have an actual conception of infinity since she approached the infinite process as a completed set. In addition, even though the task did not provoke it, she distinguished the views that the infinite process can be approached as the participant in it or as the observer of it. Her decision to be an observer led her to the actual conception. However, Sergio’s conception is between potential and actual infinity. He referred to the external and internal view on the infinite process, as Alice did, but he considered himself as a participant. Therefore, the endless conception of infinity has appeared. Even though he faced a conflict on the used terminology about series, his prior formal mathematical knowledge seemed to influence him to consider that an infinite process can have an outcome. Thus, this intermediate conception has emerged. We note that in the first task, the teachers introduced the aspect of time in their reasoning, even though its context does not promote it, and they expressed different views on it. Alice noticed that time could not influence the extraction of an outcome because sometime this will be completed, reinforcing her actual conception. On the contrary, Sergio considered that only in infinite time the series will be completed, presenting a potential conception.

Regarding the second task, the “realistic” and controversial context of the paradox seemed to influence the teachers’ conceptions of infinity. In particular, both teachers could not correctly coordinate the infinite processes, and they approached the experiment time differently. Alice focused on the finite time of the experiment, and she considered that the process of inserting and removing balls is not completed but will stop when the time is up. Therefore, she approached the infinite process under finite reasoning. On the contrary, Sergio focused on the infinite process itself, so he opposed that the experiment ends and the extraction of an outcome (Mamolo & Zazkis, 2008), presenting a potential conception

The above findings illustrate that the context of this task influences secondary mathematics teachers' conceptions of infinity. This type of task seemed to allow teachers' misconceptions to emerge and could be used in mathematics teachers' education to enhance their knowledge.

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Nurturing Pre-service Teacher's Awareness About Students' Mathematical Thinking Through Lesson Study

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The current study aims to reveal the development of the pre-service teacher's awareness on students' strengths and weaknesses in learning the limit concept through lesson study. We focused on a participant's development within instrumental case study. The findings showed that the two phases of lesson study including planning and enacting contributed to the development of the pre-service teacher's awareness on students' strengths and weakness in learning the concept of limit through different aspects.

Keywords: Pre-service teacher education, specialized knowledge, knowledge of features of learning mathematics, lesson study

Introduction

The common idea of many alternative views - how teachers reveal their knowledge rather than what they know - is one of the reasons that makes mathematics teacher knowledge specialized (Scheiner et al., 2019). This specialized knowledge includes mathematical and pedagogical content knowledge as well as reflecting this knowledge on practice. In this structure, one of the most important and comprehensive knowledge can be considered as understanding the nature of learning mathematics which includes students' mathematical thinking, their interaction with content and their strengths and weaknesses in learning the concept (Carrillo-Yañez et al., 2018). The literature has revealed that mathematics teaching, which is carried out in order for students to make sense of the concept and provide a correct conception, is a very complex process (Cengiz, Kline, & Grant, 2011). Therefore, teachers need to attend students' mathematical thinking and make instructional decisions considering students' thinking (Sherin, Jacobs, & Philips, 2011).

On the other hand, the concept of limit can be considered as a cornerstone and one of the comprehensive concepts in mathematics to conceptualize related mathematical concepts. Besides its central position in mathematics (Cornu, 1991), the concept comprises many topics all of which are based on a common basis, either explicitly or implicitly. One of these topics is indeterminate forms (e.g., $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^{∞}, 1^{∞}) and accordingly the difference between indeterminate-undefined forms in limit. In addition to the difficulties experienced by the students regarding the limit concept itself, students have difficulties in knowing the distinction between the notions of undefined and indeterminate, and they often tend to use these terms interchangeably (Jaffar & Dinyal, 2011). Moreover, the limited literature about these notions indicated that most of the students calculate limits in the indeterminate forms and use L'hospital rule without questioning the reason behind the forms and related calculations (Hulliet & Mutemba, 2000; Arce, Conejo, & Ortega, 2016). Considering these difficulties, teachers need to be aware of students' mathematical thinking, strengths and weaknesses in teaching the concept and its related notions to provide an effective learning environment. Teachers can gain this awareness through the combination of knowledge of content and students, more teaching experience with different level of students, and interaction with their colleagues. Since these notions are quite complex even for pre-service teachers (Hulliet & Mutemba, 2000), this awareness could be acquired while being a pre-service teacher before performing the teaching profession.

One of the ways to nurture the awareness of pre-service teachers for learning students' strengths and weaknesses (Lee, 2019) is lesson study development model which is a cycling process that a group of teachers/pre-service teachers work collaboratively (Stigler & Hiebert, 1999). It includes four main phases: (1) investigation: setting a learning goal in consideration that students have difficulty in understanding, (2) planning: building a research lesson plan in collaboration, (3) research lesson: conducting the lesson plan in a real classroom where one of the group members teach the concept while others record students' reactions and take notes of their thinking processes, and (4) reflecting and discussing on how effective the lesson was in facilitating student learning (Stigler & Hiebert, 1999). This phase might be followed by discussions on how to improve lesson plan, and the cycle might be applied again to revise missing points. Bearing all these in mind, the current study aimed to develop pre-service teachers' knowledge of students' mathematical thinking for the notions of indeterminate-undefined forms through lesson study.

Theoretical Framework: Mathematics Teachers' Specialized Knowledge

As a part of a longitudinal project, we adopted the model of Mathematics Teachers Specialized Knowledge which includes two main domains as mathematical knowledge and pedagogical content knowledge encircled around the belief towards mathematics and teaching mathematics. There are three sub-domains for each knowledge domain: knowledge of topics (KoT), knowledge of structure of mathematics (KSM) and knowledge of practices in mathematics (KPM), which are under the domain of mathematical knowledge. On the other hand, knowledge of features of learning (KFLM), knowledge of mathematics teaching (KMT) and knowledge of mathematics learning standards (KMLS) are under the domain of pedagogical content knowledge. For an effective mathematics teaching, in addition to mathematical knowledge, having knowledge about the students' interaction with the content and how they learn is actually one of the basic requirements of being a good mathematics teacher. Therefore, we paid attention to KFLM which is concerned with the areas of knowledge about mathematical content, the process of learning the content, and the characteristics of this process, and how students interact with the mathematical content in this learning process (Carrillo-Yañez et al., 2018). It should be indicated that the focus of KFLM is not solely students, rather the focus of KFLM is the interaction between the content and students (Carrillo-Yañez et al., 2018). There are four indicators for KFLM including learning theories related to the content, strengths and weaknesses in students' learning relation with the content, the students' interaction with the content, and the student's motivation, expectations and interests with regard to mathematics. In this paper, we focused on the pre-service teacher's awareness of students' strengths and weaknesses in the context of learning indeterminate-undefined forms.

Bearing the lesson study, theoretical framework and the aim of study in mind, we addressed this research question: (1) What is the nature of pre-service teachers' knowledge of features of learning mathematics in the learning indeterminate-undefined forms in the context of limit concept during the lesson study process? (a) How does lesson study contribute to the nurturing of the pre-service teacher's awareness about students' strengths and weaknesses in learning indeterminate-undefined forms in the context of limit concept?

Method

The study was an instrumental case study (Mills, Durepos, & Wiebe, 2010) in which we used lesson study as an instrument to provide insight into development of the pre-service teacher's awareness on students' strengths and weaknesses. To deepen the analytic stance, we focused on one of the three pre-service teachers, Mila, who was chosen purposefully for several reasons, one of which was the fact that she didn't have any experience on making a lesson plan or teaching the concept of limit in addition to her eagerness to learn the concept and to hear the students' voices in learning mathematics.

The study was carried out in the last semester of the undergraduate mathematics teacher education program, as the pre-service teachers take the fundamental mathematics and mathematics education courses until their last semester in the program. The procedure of lesson study was conducted in a similar way with the description in the introduction section. To nurture the pre-service teacher's awareness about students' mathematical thinking, based on the suggestions of the literature on improving teachers' awareness of students' mathematical thinking (Lee, 2019), the phases of the lesson study were designed with some additional steps such as promoting the group to think about interrelationships among important mathematical ideas and students' mathematical thinking.

Data Collection and Analysis

The data were generated through mainly observation of lesson study process including group discussions in planning and reflecting phases and teaching episodes in research lessons which were conducted in Turkish in two lesson study cycles. In addition, the field notes and lesson plans were used to support the claims generated from the data. The first author was the knowledgeable other in the lesson study process. We obtained the data through audio- and video-recording during the data collection process. After all the data was transcribed, we started the data analysis with partitioning of the snapshots of the pre-service teacher's interthinking and exploratory talk (Littleton & Mercer, 2013), which includes contributions of students' mathematical thinking process considering their strengths and weaknesses within the collaborative process of lesson study.

We used thematic analysis with deductive approach in which we determined the codes and the theme before analyzing the data. We used the indicator of KFLM-awareness about students' strengths and weaknesses- as a theme in analyzing data. The codes were determined as (i) lack of awareness- the general expressions about students' mathematical thinking or without consideration of students' strengths and weaknesses, and (ii) being aware- interpretative comments on students' mathematical thinking and the relationship between students' learning and the content in planning. In a similar vein, the codes for enacting were determined as (i) lack of awareness-acting without consideration with students' strengths and weaknesses, and (ii) being aware- acting with considering students' mathematical thinking, using their strengths to overcome their difficulties. In this way, we dealt with the nurturing process from lack of awareness to being aware in the journey of the pre-service teacher.

Findings

We deal with the nurturing process of lesson study in two parts: *planning* (determining lesson goals and planning of the research lessons), and *enacting* (implementing the research lesson and writing reflections).

Planning

The topic of indeterminate-undefined forms was the subject that Mila herself had difficulty in understanding, and furthermore she developed her mathematical knowledge on this subject during the lesson study process. To put it right from the start, Mila actually developed her knowledge of students' strengths and weaknesses based on her own learning process in the past.

We observed in the pre-interview that Mila, similar to other group members, had some confusions about the difference between indeterminate and undefined forms. Therefore, her awareness about students' confusions between these two notions was gained (or developed) in accordance with the awareness of her own lack of knowledge. The following excerpt showed a starting point of the discussion on these two notions in the planning phase:

Mila: For example, let's imagine that $\frac{0}{0}$ equals x. Multiply the ins and outs. x can take more than one value.
Participant 1: Ah yes.
…
Participant 2: When you spoke of indeterminate, something came to my mind. I realized that it was undefined. I realized it after reading the article: Undefined and indeterminate are two different things.
Mila: Honestly, I didn't know either, I think students might get confused.
...
Researcher: Then what do you say about the number divided by zero?
Participant 2: Undefined.
Participant 1: Professor, I think first of all let's start with the difference between undefined and indeterminate? Let's start, as in the way Mila said.

In the excerpt above, we did not pay attention to the mathematical correctness of the expression. Rather, we focused on how the pre-service teacher considered students' mathematical thinking while constructing the content in planning. The development was observed through development of her mathematical knowledge.

In addition, the observation of other group members' lessons provided her to see beyond her thoughts about students' strengths and weaknesses in learning the concept of limit. The observation of the Participant 2's research lesson which was just before Mila and then watching it on video enabled her to make more mathematical contributions to students' learning about mathematics during the planning phase.

Researcher: You can also explain it in that way when you are explaining. You will start with indeterminate forms. There was a question the students asked Participant 2 as "why infinity divided by infinity does not equal to 1". Participant 2 will ask you if it's a sufficient explanation.
Participant 2: I said that if there is only one number, there is infinity divided by infinity which equals to the number of a, then infinity equals to 'a times infinity'. Well, do I know this a, but it may be 1, it may be 2, it may be 1000, I asked whether I could say something definite for a, they said no, so I said there would be indeterminate.

Mila: This is not the answer of the student's question "why not 1", instead it looks like the answer to the question of "why does this indicate indeterminate"!

Participant 1: Let’s say, here is already undefined! Again, infinity divided by infinity must be one for the inside and outside product.

[*Researcher intervened here*]

Participant 2: I should have said infinity can be described as an increase; it is not an infinite number.

Mila: We need to be particularly careful about the infinity; a bit as an adjective! So, we can start from there and talk about indeterminates.

Researcher: So, you can start here!

Participant 2: For example, they asked why 1 to the power of infinity and zero times infinity indicated indeterminate.

Mila: They are right about the 1 to the power of infinity, we should definitely mention it!

It can be understood that Mila's ability to express more mathematically what aspects of students can be strengthened and which weaknesses can be eliminated shows the change that occurred during the planning phase.

Enacting

In implementing the third lesson plan aiming to grasp the concept of continuity by establishing mathematical relations and to use it together with limit in mathematical applications, Mila started her lesson based on her experiences. In other words, she knew that it would happen because she got to know the students from the narrations of her previous friend and told them about their learning only out of curiosity.

Mila: Now I guess you guys were pretty curious about indeterminate.

Student 1: Yeah!

Mila: Yes, I've already had such a feeling, so I want to examine these indeterminates together with you. First, let's talk about the $\frac{0}{0}$ and $\frac{\infty}{\infty}$ uncertainty. What do you think might be the reason for this indeterminate?

Students: [Grumbling sounds of what they don't know]

Mila: So, I'm going to write you two functions and ask you to find the limit at the point I gave you.

In the excerpt given above, Mila tried to make them notice the different results of the same limit forms which caused the indeterminateness. By wanting them to calculate the limits of the given points of the functions, it can be understood that Mila considered students’ strengths about the concept of limit. Moreover, it should be indicated here that this is an example about Mila’s awareness about students’ expectations about mathematics.

The example given above is an example from a planned situation in implementing the lesson plan. However, our main expectation was to observe the pre-service teacher’s awareness and actions in unplanned and unexpected situations. It would show how the pre-service teacher use her awareness in her teaching. In the enacting phase of the first cycle, there were two situations she faced with in this way. In the continuation of the indeterminate form of $\frac{0}{0}$ and $\frac{\infty}{\infty}$, she observed that students tended to make mistakes while exemplifying or demonstrating indeterminates based on the question frequently asked by students in the implementation of the second lesson plan.

Mila: If I ask you a question now (She turned to her presentation but realized that the question she wanted was not there). Yes, I didn't write it here... (Closing the presentation screen) Forget about it then. I will write the question myself. Let's say

I take $f(x)$ to the $g(x)$ as limit x goes to infinity ($\lim_{x\to\infty} f(x)^{g(x)}$) What if $f(x)$ was 1 for me as the limit goes to x. If limit x goes to infinity and $g(x)$ is infinity for me, what will the result be? So, when I think about this $f(x)$ to the $g(x)$ structure…?

Student 1: 1 to infinity (1^{∞}).

Mila: 1 to infinity becomes infinity. Because I know from the limit rules that I can distribute this limit and the limit $f(x)$ to the $g(x)$ would be 1 to the infinite for me.

Student 2: I think it should be 1 too.

Student 3: I've always wondered about that too.

Mila: You were wondering, right? I heard it. There is actually a reason for this. What we said is that the main reason for the indeterminates is that we get different results in the same form of the limit. In fact, nothing different from the ones here (she showed what she wrote about $\frac{\infty}{\infty}$) appears for 1^{∞}. Now I will write you two examples.

Up to now, the excerpt showed that she tried to use students' strong sides to teach them the logic of indeterminate forms. In addition, it can be said for this introduction that she considered the students' interest (emotional aspects of learning the concept) which was their curiosity about why the forms of 1^{∞} is not equal to 1. In the continuation of the excerpt, she wrote the functions: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$ and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2} = \infty$. While she aimed to use their strengths, students got confused about why they look for the same results for different functions. The following excerpt showed her awareness about the aspects related to students when she answered students' questions.

Mila: So that's why 1 to the power of infinity is indefinite.

Student 1: But that's not 1 to the power of infinity, right? We said 1 to the infinitely indefinite thing, isn't it something different? e is here.

Mila: Hmm, is it confusing that it's equal to e?

Student 1: No, there is a number called n.

Mila: Yes.

Student 1: 1 is not infinity, I mean, I don't think they are the same thing!

Mila: There is a number called n, is your question related to it?

Student 1: n goes to infinity or exactly 1 to the infinity is not equal to this.

Student 3: He means something (talking about his friend) different in two functions. As if the two functions are different, it's logical that we find different results anyway, isn't it?

Mila: Hmm I got it! But I'm telling you this. So, let's look at the equation I got over here, okay (it shows the resulting limit e)? When I look over there the limit n goes to infinity and that inner side is equal to 1 for me. Therefore, when I overwrite it here, I get the 1 to the infinity form. Here I got 1 to the infinity, and what happens when I get the same form of other functions? Here I am writing the same thing again (showing the second function). Here, my inner side became 1 and my upper side became infinity. In other words, they seem to be different functions, but since we do not perceive infinity as a number, we say that it is increasing gradually, but we do not know how much it increases, so this is the reason why it creates indeterminate.

Student 1: I get it!

Student 2,5,7: Yeah, I understood perfectly!

In the face of this unexpected situation, the pre-service teacher drew an unplanned path for herself by using the strengths of the students and at the same time establishing mathematical connections. It showed that gaining awareness of students' understanding is closely related to her knowledge of practices in mathematics which can be explained as awareness of mathematical reasoning on how to explore mathematics by seeing connections. At the beginning of lesson study process, Mila commonly preferred to reason by means of counterexample not considering students’ learning whenever it was asked how she would show one’s correctness or incorrectness. Developing her awareness of students’ learning mathematics provided her to use students’ strengths for demonstrating the concept in teaching. The fact that the pre-service teacher is aware of the example she gives to create a mathematical knowledge is closely related to the students' awareness of using calculation, which is one of students’ strengths.

As a result, the lesson study process contributed the development of the pre-service teacher’s awareness on students’ strengths and weaknesses in learning the notions through various aspects. The *planning* phase provided her to discuss on students’ learning and develop her awareness through different perspectives in the discussion with her colleagues. The *enacting* phase supported her development by enabling her to see what she had discussed and planned in a real classroom through students’ interaction with mathematical content.

Discussion

The findings showed that the lesson study process promoted the pre-service teacher’s knowledge for gaining awareness of students’ strengths and weaknesses in learning the topics of indeterminate-undefined forms. In planning, at the beginning of the lesson study process, Mila used her own experiences related to learning the concept. However, such views can be considered as limited, in other words, lack of awareness, for effective mathematics teaching, because attentions of both pre-service and in-service teachers to students’ mathematical thinking and learning process in interaction with mathematical content provide them to see beyond what they think about mathematical knowledge and enable them to make instructional decisions (Sherin, Jacobs, & Philips, 2011). As the literature supported (Guner & Akyuz, 2020), working collaboratively on students’ learning in planning might promote to open their eyes to observe students’ strengths and weaknesses in interaction with mathematical content. Accordingly, the development of Mila’s awareness enabled her to make the instructional decision to create her own path by being aware of the strengths of the students rather than applying the lesson plan as it is in the enacting phase. The indicator of KFLM does not only include knowing where students have difficulties or strengths; rather it covers to enact this knowledge combining with content knowledge in making instructional decisions (Carrillo-Yañez et al., 2018). Considering that Mila took instructional decisions to help students overcome difficulties by leveraging their strengths, it can be concluded that the enacting phase of lesson study presented a great opportunity to develop this side of knowledge of students’ learning mathematics.

The study focused on one pre-service teacher within classroom discussion during the lesson study process. In future research, the focus could be given on more participants in classroom discussions. Furthermore, the study contributed to the limited literature about the pre-service teacher’s knowledge of indeterminate-undefined forms in the context of limit concept. In addition, as a part of the nature of the limit concept, there is a powerful relation between the development of mathematical knowledge (KPM, in this study) and increasing awareness of students’ learning mathematics. Therefore, the future studies which examine other notions within the nature of the limit concept considering other indicators of MTSK can shed light on the literature on mathematics teacher knowledge.

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Examining assumptions about the need for teachers to transform subject matter into pedagogical forms accessible to students

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This paper explores fundamental assumptions about the notion of transforming subject matter, which is widely regarded as a central practice of teacher work and a crucial feature of teacher knowledge. First, the notion of transforming subject matter and the ways in which it has been taken up in discourses on teacher knowledge are discussed. Second, fundamental but mostly implicit assumptions are explored and challenged, including the individual teacher as the locus of transformation, the possessor of the knowledge in question and the gatekeeper who provides students access to subject matter content. Finally, these widely held assumptions are problematised against the background of French and German traditions of didactics. These traditions see the ability to transform subject matter not as a characteristic of the individual teacher, but of social and cultural systems that are institutionally contextualised and oriented towards normative conceptions of education.

Keywords: Didactic transposition, elementarisation, pedagogical content knowledge, pedagogical transformation, subject matter content.

Introduction

It would be fair to say that Shulman's (1986, 1987) work on teacher knowledge has been a major driving force in promoting teaching as a profession, particularly in the English-speaking educational research community. Shulman (1986, 1987) provided the impetus for the view of teaching as a profession by asserting that teachers have a specialised knowledge base that differs from other professionals. Of particular importance was Shulman's (1986) introduction of the construct of pedagogical content knowledge, which refers to a specialised kind of content knowledge that goes beyond subject matter knowledge per se and encompasses subject matter knowledge for teaching. Pedagogical content knowledge, according to Shulman (1986), includes "the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that makes it comprehensible to others" (p. 9).

It is these ways of representing and formulating of the subject matter content – the transformation of subject matter in ways comprehensible to students – that Shulman (1987) conceived as the core task of teaching and the defining feature of pedagogical content knowledge. Teachers must transform subject content into pedagogical forms, such as examples, representations, and instructional tasks, which make the content accessible to pupils. Guided by this orientation, Shulman (1987) defined pedagogical content knowledge as "the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organised, represented, and adapted to diverse interest and abilities of learners, and presented for instruction" (p. 8).

While the construct of pedagogical content knowledge has received wide attention and has been the subject of numerous examinations (e.g. Abell 2008; Bromme 1995; Cochran et al. 1993; Depaepe et al. 2013; Gess-Newsome & Lederman 1999; Hashweh 2013; McEwan & Bull 1991), the underlying notion that teachers necessarily transform subject content into pedagogical forms has, for the most part, been taken-for-granted. Indeed, the notion of transforming subject matter has often been used implicitly instead of explicitly thematised in educational research, and thus, has hardly been the subject of debate and hence remained beyond the scrutiny of critical reflection (see Deng 2007).

This paper examines the notion of transforming subject matter by identifying and questioning theoretical underpinnings that have not been discussed in any substantial way. Such examination is particularly relevant in the field of teacher knowledge, where the vast majority of the literature has reproduced and reinforced the basic assumptions in Shulman's path-defining contributions.

On the notion of transforming subject matter and its context

The notion of transforming subject matter came to light in Shulman and his colleagues' research program *Knowledge Growth in Teaching* (e.g. Grossman et al. 1989; Shulman 1986, 1987; Wilson et al. 1987), which studied the interaction of content knowledge and pedagogical development among novice school teachers of different disciplines. The primary focus of this research program was on how novice schoolteachers adapt their content knowledge of an academic discipline so that it becomes suitable for classroom teaching. The fundamental assumption was that subject matter contents "must be transformed in some manner if they are to be taught" (Shulman 1987, p. 16). It is this transformation of subject matter into pedagogical forms accessible for students that has been taken as the central intellectual task of teaching and that became the defining principle for pedagogical content knowledge. In arguing for a knowledge base of teaching, Shulman (1987) claimed that

> the key to distinguishing the knowledge base of teaching lies at the intersection of content and pedagogy, in the capacity of a teacher to *transform* the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students. (p. 15, italics added)

The notion of transforming subject matter has sparked wide interest in the English-speaking education research community after Shulman (1986, 1987) introduced it as the defining premise for pedagogical content knowledge. The notion of transforming subject matter content has been described and interpreted in multiple ways. For instance, Gudmundsdottir (1991) described this transformation process as a 'reorganisation' of content knowledge that derives from a disciplinary orientation, and Grossman et al. (1989) defined it as a 'translation' of subject matter knowledge into instructional representations. Marks (1990), on the other hand, portrayed it as a process of 'reinterpretation' that means, "the content is examined for its structure and significance, then transformed as necessary to make it comprehensible and compelling to a particular group of learners" (p. 7).

Common to these considerations is the view that the transformation of content knowledge into a form of knowledge "that is appropriate for students and specific to the task of teaching" (Grossman et al. 1989, p. 32) is seen as a means and an end in itself. However, such considerations can hardly be regarded as new from the point of view of several European traditions of didactics, especially the French and German traditions of didactics. Questions of preparing subject content are described in these traditions yet as part of didactic theories and broader educational considerations, not as characteristics of teacher cognition. For example, in the French tradition of didactics (*didactique*), it has long been known that disciplinary subject content cannot be directly adopted as content for teaching. The notion of didactic transposition (*transposition didactique*) refers to the processes that take place from the moment it is decided that some scholarly body of knowledge should be taught, to the moment this body of knowledge is taught in educational settings such as school classrooms and learned by a group of students (Chevallard 1985). The taught knowledge, however, is not a reproduction of scholarly knowledge, but originates from it and is shaped by various institutional and cultural forces that may vary in time and space. In the German tradition of didactics (*Didaktik*), on the other hand, the preparation of subject matter for teaching was not only about making it accessible to students, but also about expanding its potential for the personal and cultural formation and maturation of students. An important approach in this tradition is the didactic analysis (Klafki 1958), in which subject matter is examined for its exemplary significance and its present and future relevance for students, in addition to its accessibility.

Though different in their suggested practices, these traditions share the idea that disciplinary subject matter needs to be structurally modified in ways teachable by the teacher and learnable by the students.

On the fundamental assumptions about the notion of transforming subject matter

For Shulman (1986, 1987), the bridging from the disciplinary subject matter teachers acquired in college or university to the content they teach in school is the central issue that novice teachers have to face. The key to this bridging is the transformation of the disciplinary subject matter knowledge a teacher possesses into pedagogical forms accessible to students – a transformation assumed to be engineered by the individual teacher. The teacher's own content knowledge of the academic discipline is the matter or substance of transformation, and the teacher's orientation to the structure of the discipline and the structure of students' minds (including their prior knowledge) is the foundation for the restructuring of content knowledge for pedagogical purposes.

On this view, the subject matter content knowledge taught in school is a pedagogical and personal revision of the disciplinary content knowledge of the teacher. How well this pedagogically revised content knowledge fits or connects with students' prior knowledge, then, determines whether students have access to the knowledge at stake. Consider, as a case in point, Shulman and Quinlan's (1996) retrospection on the research program *Knowledge Growth in Teaching*:

> The central feature of this research program was the argument that excellent teachers transform their own content knowledge into pedagogical representations that connect with prior knowledge and dispositions of learners. The effectiveness of these representations depends on their fidelity to the essential feature of the subject matter and to the prior knowledge of the learners. The capacity to teach […] is highly dependent on […] how well one understands ways of transforming the subject matter into pedagogically powerful representations. (p. 409)

On this view, the individual teacher is the locus of transformation. It is the teacher who possesses the content knowledge at stake and who controls the selection, sequencing, and pacing of what knowledge is learned and how it is learned. Such assumptions rely on a teacher-direct view of the teaching-learning process, in which learning is directed and modelled by the teacher. Figure 1 illustrates the simplicity of which this view operates from within the didactic triangle, focusing the attention on the teacher–subject-matter edge of the didactic triangle.

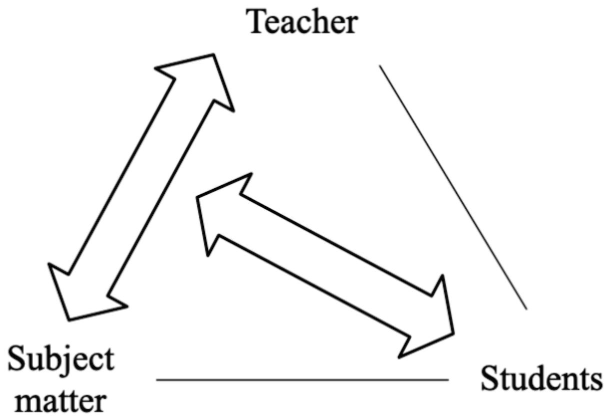

Figure 1: Shulman's view of the transformation of subject matter

For Shulman, it is the individual teacher who *structures* the subject matter content; *represents* it in the form of analogies, demonstrations, examples, metaphors, and so forth; *adapts* these representations to students' general characteristics; and *tailors* the adaptations to those specific individuals or group of students to whom the subject matter will be taught in the classroom (Shulman 1987, pp. 16–17). Through structuring, representing, adapting, and tailoring of subject matter, the teacher identifies and creates ways of representing and reformulating content knowledge that makes it comprehensible for students.

According to this view, the decision of what knowledge (and how knowledge) is acquired is made through the authority of the individual teacher, as the possessor or owner of the knowledge at stake. That is, the teacher controls what is learned and holds the criteria for what forms of knowledge are valued. This view, however, is problematic as the choice of which forms of knowledge are valued becomes a question of whose forms of knowledge are valued, leaving unexamined the *problématique* that any issue of 'what knowledge' is indeed an issue of 'whose knowledge' (see Moore 2009). Knowledge, consequently, is reduced to those who hold it or gain access to it, and those who have it or get it are in power. The teacher is positioned in the locus of power – as the knowledgeable who grants students access to the content knowledge in question. What students have access to are the pedagogically and personally revised forms of the content knowledge the teacher possesses. This understanding is problematic as it prevents realisation that the knowledge students have access to might not necessarily be 'powerful knowledge', which is deemed critical for responsible citizenship in a society (Young & Muller 2013). It might instead be 'knowledge of the powerful', which reinforces the interest of those previously educated within a system built on that understanding.

What the French and German traditions of didactics set apart from the notion of transforming subject matter

The notion of transforming subject matter reflects certain premises of the French and German traditions of didactics. One of the commonalities of these approaches and traditions is that they consider the content knowledge of an academic discipline to be an essential point of departure for transformation or transposition. Another commonality is the idea that such content knowledge is to be transformed or transposed in some ways to be accessible for students. These traditions differ fundamentally, however, in regard to their theoretical underpinnings. First, in Shulman view, the individual teacher is the locus of transformation, and transformation is seen as an internal process that takes place in the mind of the teacher. In the French and German traditions, however, didactic transposition is institutionally contextualised and culturally shaped and directed towards normative conceptions of education.

Second, the unit of analysis, in Shulman approach, is the individual teacher and her or his mental processes, and the capacity of transformation is a key characteristic of teacher expertise. How well a teacher transforms her or his personal content knowledge into forms that fit or connect with students' prior knowledge and dispositions defines whether students have access to the knowledge in question. Access for students is, thus, a pedagogical and psychological matter. In the French and German tradition, however, the unit of analysis goes well beyond the individual teacher. It includes how social, cultural, and political contexts shape and frame the work of teachers. Knowledge is institutionally contextualised and actualises in social practices and cultural activities, such as teaching. The issue of student access is as much a social, cultural, and political one as it is an epistemological, cognitive, and didactic one. In the following, the theoretical principles and foundations of the French and German traditions and their implications are discussed in more detail.

Transformation of subject matter content knowledge, in Shulman's view, is something engineered by the individual teacher. Chevallard (1985), the initiator of the French tradition of didactic transposition, however, portrayed transposition as socially and culturally produced. Didactic transposition describes the inevitable processes of change by which scholarly knowledge as it is produced by scholars, for instance, is transformed through various negotiations to knowledge to be taught that is socially considered as important and, as such, officially prescribed by the curriculum. Over different elaborations according to various circumstances, this knowledge is then transformed to knowledge as it is actually taught by teachers in their classrooms, and eventually to knowledge as it is actually learned by students. The theory of didactic transposition accounts for the various constraints the diverse agents in the transposition process are subject to, and it attempts to reveal the 'transparency illusion' of those who think of transformation of subject matter as something deliberately chosen (see Bosch & Gascón 2006).

The activity of transposition does not belong to any individual but involves groups of people who interact with one another, including disciplinary experts, education researchers, curriculum developers, and teachers, among others. These groups of people belong to what is called the *noosphère*, the sphere of those who think about education, an intermediary between the education system and society (Chevallard 1985).

The capacity of transformation is, in this view, a property not of a single teacher, but of social and cultural systems that enable the development of subject matter knowledge in social institutions and the organised and institutionalised preparation of subject matter for students. This tradition acknowledges institutions at the source of knowledge and highlights the fact that what is taught at school originates in other institutions and is a result of complex processes of negotiations among different actors involved in the process of didactic transposition (Chevallard 1992).

The process of didactic transposition, thus, underlines the 'institutional relativity of knowledge' and situates didactic phenomena at an institutional level, beyond individual characteristics of the institutions' subjects (Bosch & Gascón 2006). What is taught at school has to be legitimised by external entities that guarantee the social value and epistemological significance, as well as the cultural relevance of the subject matter content (Chevallard 1992).

Similarly, subject matter content, in the German tradition of didactics, is selected based on normative criteria according to its educational value for promoting and actualising *Bildung* – that is, the personal and cultural formation and maturation of students (von Humboldt 1795/1960). Via didactic analysis, the role, meaning and importance of the subject content in contributing to general educational aims are to be explored and questioned (Klafki 1958). The selection of subject matter content is, thus, a selection of aims, goals, and values of education in a given society. What content knowledge is valued and what is worth teaching are not left to be decided by the individual teacher but are based on formal criteria developed in the interplay of cultural-historical traditions and current societal needs.

For subject matter content to become educative, it needs to be 'elementarised' (*elementarisiert*) that is, concentrated, intensified, or abstracted to what has fundamental relevance for students (Klafki 1954). Elementarising does not simply mean a simplification of subject matter content, but a magnification of educational content (*Bildungsinhalt*) for opening or unlocking the educational substance (*Bildungsgehalt*) of the subject matter content.

Conclusion

The introduction of pedagogical content knowledge in the English-speaking community has re-introduced the centrality of subject matter content into the teaching equation, a centrality well-known and well-established in several European traditions of didactics. This paper in no way undermines Shulman's efforts in bringing the subject matter to the forefront in discourses of teaching, teacher knowledge, and teacher education. Instead, this paper has examined fundamental assumptions underlying the notion of transforming subject matter as introduced by Shulman in light of the French and German traditions of didactics. The fundamental differences between Shulman's lines of thinking and these European traditions of didactics are not in the focus, in the instructional process itself, but in their theoretical principles and foundations.

Transforming subject matter content, for Shulman, has been seen as a central characteristic of an effective teacher, a process that takes place in the teacher's mind. It is the individual teacher that is the locus of transformation, the possessor of the content knowledge at stake, and the gatekeeper for granting students access to the subject matter content. Such an understanding has largely derived from cognitivism and the individualisation of the teacher in engineering the transformation. That is, questions of core practices of teacher work and issues of access have been mostly psychologised, with little or no account of broader social, cultural, and political aspects of education. In the process of individualising the transformation process, any serious sense of social structures and the cultural and political forces that shape and form the transformation of subject matter for educational purposes have generally been disregarded.

The French and German traditions of didactics, on the other hand, are based on social and cultural aspects of education, and on tradition and history. Processes of transposition or elementarisation in these traditions are socially and culturally constructed, institutionally contextualised and directed towards normative conceptions of education. The capacity of transformation of subject matter content, thus, is not a property of an individual teacher, but of social and cultural systems that organise ways of preparing subject matter content for students, in which certain kinds and forms of knowing are valued based on history and tradition, as well as on societal needs and cultural practices. Rather than an end state, the didactic transposition and elementarisation are continuously driven by unrelenting negotiations – balancing different interests and concerns of a given society.

The unit of analysis, thus, goes well beyond the individual teacher and includes the cultural and institutional contexts that shape how transformation unfolds in any given community and institution. Such considerations reveal that ways of transformation always reproduce social arrangements of power and privilege that need to be carefully examined and questioned. This may be of particular importance as the reliance on the individual teacher as the locus of transformation and the possessor of knowledge in Shulman's account perpetuates an educational system that favours the interests and concerns of those who already control intellectual and thus curricular legitimacy.

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Enhancement of research excellence in Mathematics Teacher Knowledge: collaborative designing of lessons and learning progressions

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Keywords: Pre-service mathematics teachers, design capacity, reasoning and proof.

Introduction

The title of this poster is a name of a Horizon 2020 project with the acronym MaTeK. The project belongs to the Twinning action, which stands for research infrastructure building and institutional networking. The general goal of MaTeK project is to strengthen the research performance and develop the excellence of the Department of Mathematics Education (DME) at the Faculty of Mathematics, Physics and Informatics of Comenius University in Bratislava, Slovakia (UK BA), in the field of mathematics teacher knowledge enhancement. Teacher knowledge is the prerequisite of the education enterprise, and student knowledge development its objective. For researchers, it is essential to understand what kinds of knowledge mathematics teachers develop, and how they use their expertise in teaching in order to help students to develop deep knowledge in mathematics. One of the ways how to address this issue is to work with pre-service mathematics teachers on lesson and learning progression design capacity (Pepin et al., 2017). The specific goal of MaTeK is to provide the DME with opportunities to work with and learn from partners and exchange best practices in the field of mathematics lesson and learning progression design, with focus on the topic of reasoning and proof (R&P). R&P is an important strand of mathematical proficiency (e.g., Kilpatrick et al., 2001), and research has shown that pre-service teachers are often not adequately prepared to cultivate opportunities for students to engage with reasoning and proof (e.g., Stylianides & Stylianides, 2009).

MaTeK is twinning the departments of Mathematics Education of five European universities: 1) Comenius University in Bratislava, Slovakia (UK BA), coordinating/applicant institution; 2) Charles University, Prague, Czech Republic (CU); 3) University of Palermo, Italy (UNIPA); 4) Norwegian University of Science and Technology, Trondheim, Norway (NTNU); and 5) Middle East Technical University, Ankara, Turkey (METU).

Research plan and activities

The general goal of MaTeK will be achieved through a series of twinning schemes, including workshops, seminars, summer schools, staff exchanges, and conference attendance, that will all be closely related to a well-defined common research study, with the following research question: *How can pre-service teachers' design capacity in terms of reasoning and proof be enhanced?*

The project has started in January 2021 and, within its duration (until December 2023), will follow a Design Research approach (McKenney & Reeves 2019), with the following three research phases: 1) Exploratory study (context and needs analysis); 2) Lesson and Learning Progression design & Intervention; and 3) Enactment & re-design. Our main target group in phases 2 and 3 is pre-service mathematics teachers for grades 5-10 at each MaTeK institutions. Their understanding of the mathematical learning progressions and lesson plans design involves important aspects of pedagogical content knowledge (Carlson & Daehler, 2019).

Proposal

The poster will present in more detail the design of the study, including the planned research phases, as well as the theoretical frames used (e.g., design capacity (Pepin et al., 2017); learning progression (Fonger et al., 2018); different modes of reasoning (Stacey & Vincent, 2009) and expected outcomes. In addition, selected preliminary results related to the context and needs analysis will be presented, e.g., a framework for textbook analysis regarding R&P, and a survey on in-service mathematics teachers' use of resources for designing and conducting lesson plans.

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Possible approaches for the pre-service mathematics teacher's preparation to apply the digital technology in their own teaching

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The paper presents a study of the development of TPACK of Preservice Mathematics Teachers (PMTs). Two interconnected components of TPACK, the use of digital tools (DT) as instruments for solving problems and for creating digital materials, are dealt with from the perspective of their role in teacher education and mutual relationships. The paper aims to provide examples of how different teaching approaches could develop the knowledge and skills of PMTs in such a way that PMTs will be able to use DT properly in their own teaching.

Keywords: TPACK development, PMTs education, digital technology in PMTs preparation, searching and evaluation of existing materials.

Introduction

The use of DT in mathematics education at all levels has been a broadly studied topic already for several decades (see e.g. Ruthven, 2007). In 2011, Jančařík and Novotná see one of the reasons for a slow integration of DT into (not only) mathematics education in comparison with the robust development of technological tools for education and the low level of teachers' experience with using it. The rate of this integration increases markedly slowly when compared with the speed of development of technology. Jančařík and Novotná (2011) named as one of the reasons for the situation insufficient teachers' preparation for this integration. Successful use of DT in mathematics classrooms requires that the teacher is able to use it efficiently when solving problems. This is only one of the required competencies. Without additional pedagogical knowledge, e.g. designing lesson plans, selecting suitable materials available (e.g. on the internet), modifying and creating materials intended for the taught group of pupils, evaluating their quality and needed improvements, the use of DT in classrooms remains less efficient.

The research presented in this article is based on a long-lasting collaboration of two groups of researchers/teacher educators at two faculties, the Faculty of Mathematics, Physics, and Informatics of Comenius University in Bratislava and the Faculty of Education of Charles University in Prague. The history of both countries, Slovakia and the Czech Republic, is connected in their past histories, even from the Great Moravia period. The educational systems in the current two independent countries develop separately, but they have many similarities and we found it interesting to look at differences and similarities after almost 30 years of separate history. One of the common topics is an integration of DT into (lower and upper) secondary teacher education. Needing to understand if "insufficient teachers' preparation" is one of the reasons why we analyzed the integration of DT in different preservice secondary mathematics teachers' courses. Teacher knowledge should not only concern the ability to work with DT or solve mathematical tasks. The further important teacher' abilities are proper implementation of DT into the educational process, which includes lesson plan preparation, design and re-design of teaching resources, choosing appropriate and available educational materials, and the design of appropriate assessment and evaluation tools. All these abilities are partial domains of model TPACK that are briefly addressed in the next part of the paper. The collaboration of the two research teams opened new perspectives to handling several questions dealt with by both groups individually. The perspectives stemming from this collaboration can be clustered into two major groups.

The first group (represented here by Case 1) is concerned with the use of DT as a scaffolding and "technical" support when learning mathematics, especially when introducing a new concept and solving problems. This group is linked with improving future teachers' skills in using DT. The second group (represented here by Case 2) is concerned with the use of ready-made digital resources available on the internet and their own production for use in classrooms.

Current research also identified issues such as "limitations of trained staff and the need for practitioners to troubleshoot issues" (Buteau et al., 2010, pp. 58–59). The research question is: What components and forms of the implementation of DT into teacher education should be incorporated into PMTs training and what is the recommended order of their implementation?

Theoretical framework

The framework for TPACK was introduced by Mishra and Koehler (2006). This framework clarifies the kinds of knowledge required by a teacher in order to ensure the effective integration of technology into their teaching. It can be seen as an enlargement of Shulman's (1986) PCK model with technology as an additional domain.

There are three main competencies in the TPACK model defined by Mishra and Koehler (2006): Technological knowledge (TK), Content knowledge (CK), and Pedagogical knowledge (PK). Technological, pedagogical, and content knowledge (TPACK) is one of four competencies, Technological-content knowledge (TCK), Technological-pedagogical knowledge (TPK), Pedagogical-content knowledge (PCK), and Technological, pedagogical and content knowledge (TPACK), that address how these three main domains interact (see also Fig. 1). TPACK was proposed as the interconnection and intersection of TK, CK, and PK.

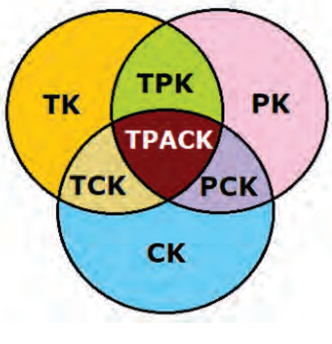

Figure 1: TPACK framework (based on the image at tpack.org)

TPACK, according to Koehler et al. (2014, p. 102), “refers to knowledge about the complex relations among technology, pedagogy, and content that enable teachers to develop appropriate and context-specific teaching strategies”. The development of TPACK could be done in three possible ways as identified by Koehler et al. (2014). The first is from PCK to TPACK, the second from TPK to TPACK, and the third, developing PCK and TPACK simultaneously. In our research, we also used the fourth way from TCK to TPACK, which is not mentioned in Koehler et al. (2014). We used it in our previous studies (a summary of those studies is in Slavíčková, 2021). The layering of theoretical frameworks can help us to better understand the studied phenomenon. We used such an approach in Slavíčková (2021) when layering Mishra and Koehler’s framework (Mishra & Koehler, 2006) with Ball’s “egg” framework (Ball et al., 2002).

In this paper, we focus on two from seven components of teachers’ TPACK: the use of DT in mathematical courses (Case 1) and the development of the ability to critically evaluate and modify materials available on the internet (Case 2).

Methodology

The collaboration of the two teams is ongoing which is why we can present at this point only the results of research anchored in one course at each group level and methodological instruments linked to them. In both cases, a qualitative research design was used.

The Bachelor level mathematics teacher education at both universities focuses on the CK. In Case 1, we focused on PMTs' willingness to use different types of DT in Calculus lessons. The participants were 26 PMTs in the second year of their bachelor study program. We offered them several digital tools like Graphic Calculus, Derive, GeoGebra; they had the freedom to use other devices as well. They were asked to use them not only at home preparation but also in the lessons. The situation in 2020 (and 2021) was more manageable using DT due to the online form of classes.

In the Master’s level mathematics teacher education at both universities, attention is paid to the PMTs’ ability to select and use appropriate digital materials available in a ready-made form and modify them for the conditions of their own teaching. On this level, choosing DT is not guided and it is up to the PMTs judgment to choose the proper one for their lesson preparation. They also learn to design their own original materials. For collecting data about PMTs’ coping with this important activity (Case 2), the course of CLIL (Content and Language Integrated Learning) was selected as representative in our study.

PMT’s worked outside the CLIL course contact lessons. They were asked to choose a material available on the Internet proposed for use in a mathematics CLIL lesson, evaluate it and propose improvements. and justify their proposals. In the research described in Case 2, we worked with 11 students who were in the first year of the Master’s level study. They were used to use DT in the way described in Case 1. Their essays were analyzed and compared. We focused on the selected topics, the nature of criticized issues pointed out and proposed modifications. All analyzed materials were in English. The analyses of essays were accompanied by discussions of the authors of the essays with the researchers.

Case 1: Development of TCK

As an example of TCK development in PMTs preparation, we choose a Calculus course with the integration of DT at Comenius University in Bratislava. The reason is that Calculus is one of the compulsory courses in PMTs preparation. There were two reasons for the implementation of DT into teaching CK. First, we wanted to demonstrate that DT could help PMTs visualize the situation and construct abstract knowledge. Second, we hypothesized that if students experience teaching new concepts by using DT, they will be more open to using it in their future teaching careers.

Research description

PMTs used digital environments for modeling different situations in mathematics or real-life connected tasks. We present two examples of such problems

Problem 1: *An account starts with 1 EUR and pays 100 % interest per year. If the interest is credited once, at the end of the year, the value of the account at year-end will be 2 EUR. What happens if the interest is computed and credited more frequently during the year? How can we make 3 EUR at the end of the year?*

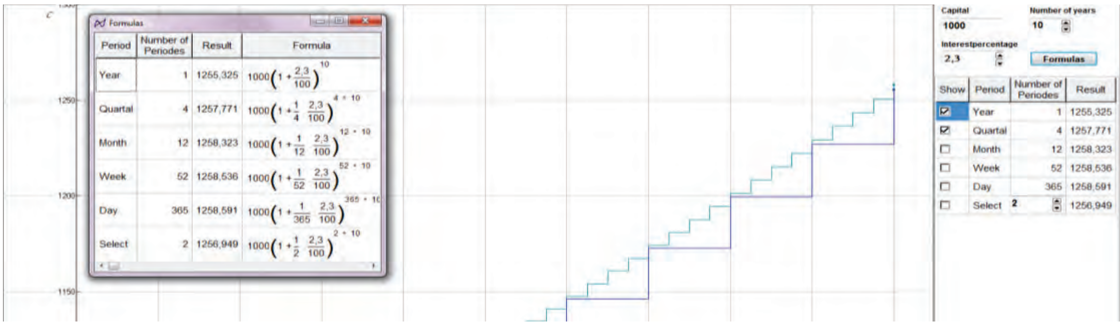

Figure 2: Using Graphic Calculus for modeling the situation

As can be observed in Figure 2, a comparison of compound interest calculated every quarter (bigger steps), and compound interest calculated every month (smaller steps) can bring small differences in the final amount of our money at the end of a year. Making these calculations more often does not change the output a lot, the changes are on the positions of thousands or lower. Not all PMTs noticed this phenomenon and the answer, it will never be 3 EUR, was surprising to them. This activity led to the definition of the Euler number in a limit form.

Problem 2: *Using a preferable DT, find out (a) properties of given sequences, (b) categorize them into several categories so every sequence in that category fulfill the condition characteristic for that category, (c) provide example of at least 2 sequences for each determined category.*

$$\left\{\frac{1}{n}\right\}, \left\{1-\frac{5}{2n}\right\}, \left\{3+\frac{2}{5n}\right\}, \left\{\frac{n+5}{n^2+7}\right\}, \{(-1)^n\}, \{\cos(\pi n)\}, \{7.(-1)^n\}, \left\{(-1)^n\frac{n+5}{n^2+7}\right\}, \{2^n\},$$
$$\{n+5\}, \{9n+7\}, \left\{\frac{n^2+5}{n+7}\right\}, \{7-n^2\}$$

Will your categories work for other sequences as well? Justify your answer.

When working with this problem, we followed the spiral scheme Manipulating–Getting-a-sense-of–Articulating–Manipulating–… described by Mason and De Geest (2010). We merged some types by articulating and further manipulating the original 7 categories of sequences (increasing, decreasing, lower bound, upper bound, bounded, no bounds, periodic). We started activities related to the intuitive understanding of a limit of a sequence. The role of DT was crucial here since the possibility of visualization, zoom, and change the interval for making observation and getting a sense of noticed phenomenon, helped students predict the situation for a "larger *n*".

Introductory problems or tasks were introduced also for the limit of a function, derivatives of a function, the definition of Riemann integral, etc. Different GeoGebra applets and software mentioned above were used during the lessons for the discussions and for home preparation.

Results

There are two main results we observed during the teaching period. Firstly, PMTs who actively used DT were having fruitful discussions during the lessons. We observed that mostly those who have a high level of TK were in that group of PMTs. Additionally, nowadays this group of PMTs in their Master's level (2 years after the intervention) is more open to integrating DT into their lesson plans. Unfortunately, PMTs mostly focused on repeating existing knowledge, and most of the tasks they re-design were strongly procedural.

PMTs proved high flexibility in the 2nd semester. The guidance from the teacher was not as significant as in the 1st semester of Calculus. PMTs started looking for different types of software that would help them to manage the abstract concepts they had to learn.

Discussion of Case 1 results

In the curriculum, PMTs in their 2nd year of study at the university had only one subject focusing on the development of the three main domains identified by Mishra and Koehler (2006): TK, CK, PK. The course of calculus was the first in which TCK started to be developed. From the results, we can conclude that PMTs development of TCK was in general sufficient. Following our previous results (Slavíčková, 2021), PMTs with lover DT skills were less willing to use DT. This phenomenon changed in 2020 when the COVID-19 situation pushed us all to use them in everyday life.

The reason for procedural-oriented outcomes of PMTs could be our colleagues' teaching when they stress procedures or algorithms for solving typical tasks. PMTs could conclude that mathematics aims to manage plenty of procedures. Single intervention is not sufficient here, and more collaborative work is needed. Therefore, cooperation with other colleagues preparing lessons focusing on using DT, in general, is crucial.

Case 2: Further development of TPACK

As an example of a suitable activity in a course, for future mathematics teachers aimed to develop their ability to choose and modify existing materials available on the internet, we choose the course of CLIL (Content and Language Integrated Learning) at the Faculty of Education of Charles University.

Coyle, Hood, and Marsh (2010, p. 3) characterize CLIL as follows: "CLIL is an educational approach in which varies language-supportive methodologies are used which lead to a dual-focused form of instruction where attention is given both to the language and the content." CLIL teaching units have two educational goals of the same importance (language and content). Teaching and materials have to pay attention to both. Both educational goals are interconnected even if the teacher or students focus on one of them. The dual-focused nature of CLIL offers a rich resource of ready-made digital teaching materials. They can focus on the non-linguistic subject, language or, in the ideal case, on both at the same time. An example of such dual-focused material is presented e.g. in (Hofmannová & Novotná, 2007).

Evaluations of materials

Students focused on both, mathematics and language. From the language perspective, they paid attention mostly to the used vocabulary and mathematical terminology, and suitability of the used language level from the perspective of the target group of pupils. They proposed reformulations of parts where they considered the language too complicated (including the length of paragraphs etc.). In one case they evaluated the used language as outdated and recommended to change it to the contemporary forms. They recommended modifications of the problem settings aiming to have the texts more similar to the Czech environment.

They also mentioned the differences in notation (e.g., in writing dates or mathematical notations). In this case, they did not propose to use the Czech way but to practice the English one continuously so that pupils get used to using it.

They underlined the suitability of the use of problems with more than one correct result and solving procedure. As to the tasks, they evaluated dividing complex tasks into simpler ones as more suitable. They also paid attention to the quality and correctness of figures and the level of difficulty of tasks.

They mentioned also the layout of the material, mainly the comprehensibility of the text and aims of the activities, and proposed improvements. In cases where the answers to the tasks were pre-prepared, the students proposed deleting it and letting pupils formulate the answers themselves.

The students asked for complex materials where the teacher can find all information needed as help for the teacher. One student worked with a whole book and recommended it as an inspiration for the teacher's preparation of lesson plans.

Discussion of Case 2 results

The students who participated in the CLIL course did not have the courses of didactics of mathematics completed. Nevertheless, the majority of them were able to choose materials suitable for use in their CLIL classrooms and propose meaningful modifications. They presented their pieces of work in CLIL course lessons and discussed deeply the quality and usefulness of materials. They used their experiences from the courses of Didactics of mathematics. Their experiences from the use of DT in mathematics courses helped them to overcome obstacles based on the implementation of DT.

The participating students were aware of the importance of similar activities. Let us cite from one student's comment: "*Given how much internet is infested by materials of bad quality, it was nearly impossible to find materials meeting my requirements …*"

Conclusion

The two analyses allowed us to estimate the role of PMTs' mastering the use of various DT as instruments for learning mathematics and solving problems themselves on the one hand and working with digital materials during their teaching at schools.

Answering out our research question, *what components and forms of the implementation of DT into teacher education should be incorporated into PMTs training and what is the recommended order of their implementation*, we identified several aspects which should be considered. Firstly, PMTs should start with the development of TCK and TPK sooner than on the Master level. Secondly, using different topic areas and making connections between the main domains of TPACK can help PMTs gain better insight into the issues of implementation of DT into their teaching. Thirdly, closer communication among PMTs' educators is needed, especially those focusing on three main domains of the TPACK model, in other words, educators responsible for mathematics preparation, technology preparation, and pedagogy preparation. All of them should communicate with each other and with a specialist in mathematics education to be concise and help develop TPACK. Fourthly, our observations and analyses of materials produced by PMTs indicate that PMTs' level of TPACK is sufficient for their successful implementation of DT in their classrooms once they enter the practice. They are able to work with ready-made materials and modify them in a creative way. When doing so, they combine competencies gained in the use of DT in their own learning of mathematics and solving problems, as well as general and subject didactics.

Based on our research, recommendations concerning implementing different activities connected to using DT in PMTs preparation can be drawn. It turns out that starting with TCK development is a good starting point (Slavíčková, 2021). Then we could continue with pedagogy-oriented activities (like in Case 2), re-designing the given (or found) materials to create an own material by using DT. When PCK is developed, it can be quite late to start with TPACK. As Mishra and Koehler (2006) identified, once the teachers are familiar with processes and lesson design without using DT, it is difficult to change their mindset.

The number of participating PMTs and courses was small, and the presented results cannot be generalized. Still, they indicate the importance of both components of including DT in (not only) mathematics teacher education. This study opened new questions, e.g., how well-equipped are our PMTs for their real teaching? How can we measure that? What are the indicators of a well-prepared teacher? This will be the focus of our subsequent research.

Cooperation between our universities continues. We are preparing further interventions and comparisons of our PMTs' results e.g. when creating lesson plans or in their flexibility in adapting their teaching to different environments (mainly by using digital tools).

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Mathematical identity transformations

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This paper examines the case of Ruby, *a prospective mathematics teacher, and explores the transformation of her mathematical identity as she moved through her studies from grade school through university. I utilize the conceptual framework of multiple "I-positions" from Dialogical Self Theory to reflect on the changes in Ruby's mathematical identity through analyzing her retrospective self-narrative. Based on this case study, I determine that the dynamics of I-positioning, involving emergent coalitions of I-positions or the emergence of a single I-position of greater authority, may play an important role in the formation of a positive or negative mathematical identity. I also find that significant events such as changing circumstances, new reflections, decisions, and actions may act to precipitate the reconfiguration of I-positions, and thereby bring about the transformation of one's self-understanding as an (in)competent student of mathematics.*

Keywords: Mathematical identity, mathematics learners, prospective mathematics teachers, dialogical self theory.

Introduction

Identity has been explored through various theoretical frameworks, (Beauchamp & Thomas, 2009), several of which are featured in mathematics education research on the concept, e.g., Holland et al. (1998), Sfard and Prusak (2005), and Wenger (1998). Inspired by Lev Vygotsky, Mihkail Bakhtin and Pierre Bourdieu, Dorothy Holland et al. (1998) conceive of identities as resulting from processes in which people are both agents and subjects of simultaneously culturally constructed and socially imposed worlds. Following on such conceptualizations, Boaler and Selling (2017) consider mathematical identity to be informed by the "ways in which students think about themselves in relation to mathematics, and the extent to which they have developed a commitment to, are engaged in, and see value in mathematics and in themselves as learners of mathematics" (p. 82).

Similarly, Wenger's sees identity as developing through "negotiated experiences of self" through dialogue in communities of practice (1998, p. 150). He understands identity formation as a *learning trajectory* in which "the past and the future [are] in the very process of negotiating the present" (p. 155)—a theoretical approach that emphasizes identity continuity and malleability. Following on this conception, Rø (2019) characterizes teacher identity development as a result of identification and negotiation in the process of participation in and at the boundaries of various communities of practice. Finally, a substantial number of identity studies in mathematics education have adopted, following on the work of Paul Ricoeur (1992), the conception of identity as based on narrative (e.g., Sfard and Prusak, 2005 (Kaasila, 2007; Lutovac, S., & Kaasila, 2014, 2018).

Earlier studies that analyzed the identity narratives of mathematics learners and pre- and in-service teachers have described identity changes but have not provided sufficient information about the mechanisms that account for them. This study attempts to add this information through analyzing the retrospective self-narrative of one pre-service mathematics teacher. The study does not focus on her current self-narrative as a pre-service teacher, but rather traces shifts in her mathematical identity as she moved through her studies from grade school through university and until she enrolled in a teacher education program. As such, I aim to analyze transformations in one pre-service teacher's mathematical identity and the process of its realization, and to do so in the context of Dialogical Self Theory, which provides us with an epistemic and analytic lens through which to consider those processes.

Theoretical Framework

Dialogical Self Theory

Hubert Herman's and Els Hermans-Jansen's notion of identity formation as a process of telling and retelling one's self- narrative in dialogue with oneself or another person(s) is congruent with the conceptualizations mentioned earlier (1995). Dialogical Self Theory (DST) (Hermans & Hermans-Konopka, 2010) begins with the assumption of selfhood as multi-voiced and intrinsically dialogical, composed of a dynamic multiplicity of varied internal narratives--"I-positions"—representing multiple perspectives of the self. Each I-position has a voice and is in dialogue with other I-positions, all of which reflect different relationships with independent others. The self is comprised of both internal and external I-positions. Internal positions reflect aspects of the individual's identity (e.g. Korean, prospective mathematics teacher, immigrant, daughter, friend, etc.). In addition, any current set of I-positions includes, not only those existing in the present, but may incorporate past positions as well--for example "I- as once a struggling math student"—as well as external I-positions that represent the attitudes and example of people seen as significant in one's life—a parent, teacher, good friend or boss, for example (Hermans & Hermans-Konopka, 2010). Past and present I-positions are likely to speak in voices that are in conflict, and to present ideas that contradict each other, which puts the self in a position of struggling to reconcile the different views in a coherent narrative.

For DST, the self is both stable and dynamic, always undergoing reorganization of positionings, and making meaning of experiences old and new. In fact, in Herman's and Hemans-Jansen's words "the self is an organized process of meaning construction." (1995, p.14). As part of this process, we construct our self-narratives, which we tell and retell. In these narratives, we give special significance to some events over others. Some special events might disrupt or challenge an established self-narrative. These have been described by Carla Cunha et al. (2012) as Innovative Moments (IMs)—which may be, for example, new actions, thoughts, feelings, intentions. The IMs are seen as voices that are potentially disruptive to the dominant self-narrative and represent a possibility for the emergence of new I-positions, and for a self-narrative reconstruction. The transformation of self-narratives involving IMs's emergence and expansion has been used in DST-based psychotherapy (Gonçalves et al., 2009).

According to Cunha et al. (2012), in successful cases of psychotherapy there is a noted emergence of three types of IMs: action(s) through which the person challenges the dominant self-narrative; new reflections that run counter to person's previous ways of thinking, and protest; action or thought that refuses the current self-narrative. They "announce" a potential emergence of new positions, which may bring forth a different dynamic between I-positions, as well as new dominant I-position(s). The appearance of these IMs is followed by emerging “reconceptualized IMs”—new actions, reflections and protests, and chances for reorganization.

Narrative Inquiry

Mathematical identity is here understood in terms of the narratives persons tell in order to situate themselves not only in relationship to mathematics and their mathematical lives (cf. Kaasila, 2007a; Connelly & Clandinin, 2000; Sfard & Prusak, 2005), but also in relation to significant persons, groups, or cultures (Hermans& Hermans-Konopka, 2010). I consider the notion of the “extended self” (Hermans & Hermans-Konopka, 2010), to be a further development of George Herbert Mead’s concept of the “generalized other” and Lev Vygotsky’s (1962) internalization of cultural and individual voices. All three approaches suggest that mathematical identity includes not only one’s view of oneself as a mathematics learner in the context of one’s classroom and the work with one’s peers, but also the internalized voices of one’s peers, teachers, parents, and other significant others. As such, in analyzing the following case study, I use a multi-voiced dialogical conception of mathematical identity, which acknowledges its extension to include real or imagined voices (views) of those others. Mathematical self-identity is plural and made up of multiple I-positions in continual dialogue (Hermans& Hermans-Konopka 2010).

Here I examine the case of Ruby, a young Korean American undergraduate student completing her coursework in mathematics education and explore her shifting mathematical identity. I reflect on the changes in her retrospective self-narrative, tracing her trajectory as a mathematics learner as she moved through her studies from grade school through university. I use DST and the conception of IMs to identify changes in the relations of authority between internal and external I-positions, reflect on the emergence and dynamics of a coalition of positions, and their significance in the realization of the ongoing reconstruction of her mathematical identity.

Methodology

This study adopts a case study methodology in order to explore and report the details (Creswell, 2005; Stake, 1995) of one prospective teacher’s mathematical identity and its changes over a period of roughly six years. The subject, Ruby (pseudonym), was a twenty-one-year-old Korean American full-time undergraduate mathematics education student in her third year of preparation as a prospective mathematics teacher in a public university in the northeast US. She was a student in an introductory mathematics education course that I was teaching at the time. She was an intelligent and diligent student; her assignments were always on time, and she was detailed-oriented and thorough. Ruby had finished 10th grade in Korea, after which she had relocated with her entire family to the US, and she finished the last two grades of high school in a large urban public school.

Data collection occurred over the course of one academic year during Ruby’ completion of the introductory mathematics education course. Three semi-structured interviews were conducted throughout – two of them administered consecutively within a week, and the third roughly four months later. Each interview was introduced by an invitation to talk freely about her experiences as a mathematics learner in middle, high school, and university. The interviewer asked follow up and clarification questions. The interviews were transcribed and comprised the main data source for this study. Additional informal discussions helped me develop an in-depth understanding of issues

(Creswell, 2012) related to the study of the changes in Ruby's mathematical identity. I used open coding (Creswell, 2006; 2012) to identify and explore the themes that emerged from the data.

Adopting methods from narrative analysis (Polkinghorne, 1995), I sought to identify coherent themes running through the interviews. First, I divided the interview transcripts into episodes reflecting four major narrative shifts related to Ruby's middle and high school experiences in Korea, high school and university in the US. Each episode was further divided into segments, each of which was associated with a specific I-position—for example, I as trying hard in mathematics (internal I-position), or I-as always failing math tests (internal I-position), and I as being encouraged and helped by my mother (an external I position representing the internalized voice of Ruby's mother). Based on a comparison of the segments within and across interviews, I developed codes for the I-positions relevant to the episodes in the self-narrative. Furthermore, I reviewed all identified I-positions, selected the ones that involved self-reflection and observing one or more of the other I-positions, and coded them as meta-positions.

The positions in each narrative episode were analyzed in terms of configurations: were there many different I-positions speaking in single "voices" or were they forming a coalition of voices that spoke as a "choir," and how were these configurations related to Ruby's self-narrative and her positive or negative self-identification as a mathematical learner? For each episode, I searched for evidence of relatedness between Ruby's I-positions, and for the existence of power differences as reflected in the relative dominance of some. I also looked for significant events as identified in Ruby's self-narrative, and for changes in her I-positions around the time of these events. Guided by Cunha et al.'s (2012) conception of Innovative Moments (IMs) I noted new reflections and actions on Ruby's part, and examined them in relation to new I-positions that had emerged.

Findings

Ruby's Self Narrative

"I wasn't good at math back then"

When Ruby was in middle school, she liked math. She described herself as "being interested" and as "enjoying the math classes." She liked all subjects and thought that she was good at them. Things changed when she moved to high school. Ruby said that there were lots of tests, and she was not performing well on them. There were always students who had better test scores. "I don't know what I was doing, but I always had errors [on the tests]." Her high school math teacher was "very strict and stern." Ruby thought he believed that she was not putting enough efforts and being "lazy", although she said that she was trying hard. She felt he was "angry and disapproving of her." He had told her that "she didn't have the mathematical mind." To my question whether her parents were helping her, she described her mother as encouraging and supportive and trying to help with homework. Ruby "dreaded" her math classes and was afraid of failure. When I asked her about this period, she started describing it with "I wasn't good at math back then [in Korea]"

"Not loving math, but **OK"**

Things had changed somewhat when she moved to US. Her last two years of high school were not easy—she was trying to catch up with English--but she suddenly found that she was doing much better in her math classes. She "knew lots of the math stuff" they studied, especially during her first year in the US. To my question whether she felt successful, she replied that she was doing much better than before [in Korea], but that she didn't think at the time that she would study math in the future. She had A's at the time, but at the interview she described herself as "not loving math, but OK"

"I had decided that I can do it"

When she finished high school, Ruby applied to a public university close to her hometown and started her first year as an undeclared major. She had to find out what she wanted to study, but at this point she knew "it wouldn't be math." However, Calculus was a mandatory course and she registered to take it, among other courses. She wasn't sure about it and was afraid of failing it. Then she discovered that she liked the course. It seems like it was the instructor at first and the course structure. "The instructor was amazing," Ruby told me, and she "liked everything about this class"—the instructor and students, the group work and the assignments. "I loved it [the course]," she told me with enthusiasm. Ruby felt that she was doing very well, and she began to think that she might continue to study mathematics and look into related programs. In the following semester she decided to take the next Calculus course with the same instructor. She felt successful, and this is when she began thinking that she might want to become a math teacher. Her mother advised her that it would be a good profession for her. "By the end of this second semester," Ruby said, "I had decided that I can do it [could be successful in becoming a math teacher] and applied to the mathematics teacher program."

The interviews with Ruby revealed the various I-positions she took while she was a middle and high school student in Korea, a student in high school in US, after she moved to the US, and as a university student. Table 1 below shows some of these, identified on the basis of the interviews conducted with her.

Table 1. Ruby's I-positions

Periods	Some of Ruby's I-positions as a mathematics student
Middle school	I as … interested in mathematics (internal) enjoying mathematics (internal) good in all subjects (external, my teachers)
High school, in Korea	I as… trying hard in mathematics (internal) always failing math tests (internal) being encouraged and helped (external, my mother) being lazy and not studying mathematics hard enough (external, my math teacher) making my math teacher angry (external) not having the support/approval of my teacher (internal) dreading math (internal)

	not having the mathematical mind (external, my math teacher) not being good at math (internal, metaposition)
High School in US	I as… knowing lots of [math] stuff (metaposition) not loving math, but ok (metaposition) having As in math (internal) doing much better than before [in Korea] (metaposition) not planning to study further math (internal)
University in US	I as… a university student (internal) exploring possible study paths (internal) not sure about taking a Calculus class, and afraid [of failure] (internal) successful Calculus student (metaposition) loving math (metaposition) someone who might be successful in becoming a math teacher (internal) a future math teacher (internal)

These I-positions seem to have informed her mathematical self, and the way she understood herself as a mathematics learner and one (in)capable of learning mathematics. Table 1 shows a dynamic multiplicity of perspectives--I-positions--representing the multiple perspective of the self in the different contexts in which she found herself, e.g. middle school, high school in Korea, high school in the US, and university. These I-positions have informed her changing self-narrative (Hermans & Hermans-Jansen, 1995) and reflect different relationships with independent others or herself.

In middle school, the I-positions identified in the interviews, “I as enjoying math”, “I as interested in mathematics,” and “I as being a good student in all subjects” supported each other in creating a coalition of I-positions, whose unified voices supported a self-narrative of a good and capable mathematics student. However, when Ruby transitioned to her Korean high school, her self-narrative dramatically changed. Her high school math teacher became a powerful voice and a mirror through which she saw herself in the way she perceived her teacher saw her—not making enough effort, “lazy,” “not having a mathematical mind.” This powerful external I-position had become dominant, overpowering the voices of her other I-positions, including the supportive external I-position of her mother. It led to her shifting her self-narrative and repositioning to new I-positions as “failing math,” and “dreading math.” The voice of her math teacher had taken on special significance in the multi-voiced self, and had challenged, overpowered, and overridden her previous self-narrative.

However, yet another change in circumstances forced further repositioning. Ruby moved to the USA with her family. She found herself in a 10th grade math class, where she discovered that she knew quite a lot of the mathematics content under discussion. The situational change had in fact delivered her from the power of her Korean math teacher’s voice. A new coalition of voices—her mother’s supportive one, and her own metapositions--“I knew lots of the stuff they studied,” and ‘I was doing much better than before”-- helped her move away from some of her previous external I-positions, e.g.

"I- as not good at math." Ruby changed her self-narrative again and described herself at this time as someone who was "not loving math, but ok." At this point, she was not seeing herself as someone who would study math further, and perhaps choose it as a career path.

However, another reconfiguration of new and old I-positions seems to have precipitated yet another shift in her self-narrative when she went to the university. She took a Calculus course with an instructor she liked, in a learning environment she found empowering. Both shifts in her self-narrative seem to have been instigated by innovative moments (IMs)— new reflections of herself as a math learner and her proactive step of registering for a Calculus class, which challenged previous self-valuations (Cunha, et al., 2012). These led to her solidifying new I-positions as a successful Calculus student and as loving math. Table 2 below, identifies some IMs that seem to have contributed to shifting Ruby's self-narrative and her mathematical identity. Her decision to explore a possible university study trajectory and her coping with her ambivalence about the Calculus course led to new reflections and new actions on her part. Her metapositions at this point seem to have been a powerful factor in helping her de-position the "I as not planning to study further math." Over the course of fewer than two semesters, Ruby moved to a new I position as "someone who might be successful in becoming a math teacher," which was, in fact a position of rebellion against the depositioned one.

Table 2. Innovative moments (IMs) in Ruby's positioning

Period	Ruby's I-positions	Emerging Innovative Moments (IMs)
High School in US	I as… *knowing lots of [math] stuff (metaposition)* *not loving math, but ok (metaposition)* having As in math (internal) *doing much better than before [in Korea] (metaposition)* not planning to study further math (internal)	IM reflection IM ambivalence IM action IM reflection
University	I as… a university student (internal) exploring possible study paths (internal) not sure about taking a Calculus class, and afraid [of failure] (internal) *successful Calculus student (metaposition)* *loving math (metaposition)* someone who might be successful in becoming a math teacher (internal) a future math teacher (internal)	IM new action IM new action IM ambivalence IM new reflection IM new reflection IM rebellion/protest IM rebellion/protest

Emerging circumstances, new reflections, proactive decisions, and confronting old I-positions led Ruby to challenge her previous self-narrative and, eventually, to re-write it. Ruby decided to become a math teacher. She applied and was accepted into her university's mathematics education program, graduated successfully, and currently works as a math teacher.

Conclusion

This paper has examined the case of Ruby, a young Korean American undergraduate mathematics education student, and explored her mathematical identity-transformation narrative as she moved through her studies from grade school through university. I have drawn on Dialogical Self Theory and the concept of Innovative Moments to reflect on the changes in Ruby's mathematical identity through analyzing her retrospective self-narrative. I have determined that the dynamics of I-positioning—emergent coalitions of I-positions or a single I-position of greater authority -- may play an important role in the formation of a positive or negative mathematical identity. I also found that significant events such as changing circumstances, new reflections, decisions, and actions may act to precipitate the reconfiguration of I-positions and lead to identity transformation. As such, the study raises questions about the role of reflection in the reconstruction of self-narratives, as well as what classroom methodologies might best facilitate such reflection. In order to nurture students' mathematical identities, new approaches need to be explored that help and encourage students to challenge established self-narratives in reference to their mathematical identity. Mathematics has become a forbidding gatekeeper for many economic, educational, and political opportunities for students, many of whom have developed self-narratives that act to prevent them from identifying themselves as capable math learners. As such, disrupting such self-narratives and working proactively to reconstruct negative mathematical identities represents an important educational task.

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Creation in mathematics education Teachers' beliefs and traditions in Hungary

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Keywords: Belief, culture, mathematical creation.

Introduction

The concept of creation flows along the history of European culture starting from the biblical Genesis creation ("God said"), through the creator divine Word (Word = God) which is known from the Gospel of John. To the history of the artistic *creation* belongs the myth of Prometheus, or some works of Shakespeare. According to the conceptualisation of Sturm und Drang, in the *Genieperiode*, the creator is a genius, the self-contained artist is legalised to create a whole world with his own artistic power through breaking the rules. The same power appears in János Bolyai's famous line in connection with mathematical creation, which goes like this: "*from nothing I have created a new different world.*" (Prékopa, 2006, 17)

Research

We use the method of cultural context analysis (Lehtonen 2000). In our research we examine the cultural-historical sources of *creation* in mathematics, and the cultural presence of this idea in the mathematics education in Hungary. We use the sentence of Bolyai to show, how developed the idea of creation from the 19th century to a core element of the Hungarian didactical tradition in the 20th century.

We are convinced that this tradition contributed to the history of mathematics education in Hungary on the one hand, to the education of gifted pupils, and on the other hand, to the movement of *complex mathematics education* in the 20th century. T. Varga's complex mathematics education experiment belongs to the guided discovery method in ME and partially also to the inquiry based ME. Creativity, happiness, autonomous thinking, the freedom of doing failure were the key-elements of Varga's method, and *belief*, creation must be the part of mathematics also in the school. Of course, it does not mean the creation of a new mathematical world, but the understanding of it within the environment of school mathematics, mainly among the frames of conceptualisation taking place during games which were based on experience. (Halmos & Varga, 1978, 229).

In Bolyai's sentence – according to the concept of Romanticism – only the chosen ones and geniuses are blessed with the ability of creation who are separated from the public. By the 20th century, the concept of creation was deprived of this romantic genius-cult and one of the key points of Tamás Varga's reform was that he did not think in elite education but he intended to make mathematics and the possibility of experiencing mathematical creation within the frames of public education available for everyone (C. Neményi, 2013).

Conclusions

1) With philological methods we can proof, that Bolyai's sentence was inspired by a Hungarian poem or by a poetical biography.

2) This fact belongs to the history of the connection between creation and mathematics. With all its antecedents and effects it can be proof for the fact that the *need for creation* as a *belief* is a key part of the Hungarian tradition. This *belief* is worth being taken into consideration when a current phenomenon of the teaching of mathematics is to be interpreted. As the need for mathematical creation is not only present in the works of the quoted authors, but it is also present in those collaborations where primary and secondary school teacher training takes place. This way, primary and secondary school teachers encounter the *need for creation* in the context of school mathematics either indirectly or directly, but with high probability.

3) The relation of arts and mathematics has a remarkable tradition in the Hungarian concept of mathematics. (See Rózsa Péter's popular titled *Playing with Infinity* (Péter R., 1961), or István Lénárt's work and the so called *Lénárt Sphere*, which is suitable for visualising and teaching spherical and hyperbolic geometry in the junior section of elementary school (Lénárt, 2003); or the *Experience workshop global steam network* by Kristóf Fenyvesi / https://experienceworkshop.org/?lang=en .)

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An exploratory study on the difficulty perceived by primary school teachers on a mathematics INVALSI item

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This paper deals with the knowledge that teachers demonstrate when they interpret pupils' difficulties in the teaching and learning of whole numbers and decimals. This is part of a project aimed at investigating the link between standardized assessment in mathematics and the teaching-learning processes at the primary school level, in which data from a large sample of Italian primary school teachers were quantitively analyzed. In the current paper, we consider the first analysis together with a second questionnaire administered to a sample of 46 to enlarge the qualitative information about their perception of the difficulties. The answers from the second questionnaire were analyzed using the Mathematics Teachers' Specialized Knowledge model (MTSK) as a descriptive and interpretative tool. The results highlight the teachers' lack of awareness of the difficulties and reveal the need for specific training.

Keywords: MTSK model, mathematics teacher educator, teaching-learning process, INVALSI national assessment.

Introduction

Our starting point is that standardized assessment data can be considered as a tool that teachers can use not only for the design and implementation of meaningful teaching and learning activities, but also for providing students with detailed information on their learning (Wiliam, 2010). This also applies to the Italian National Standardized Assessment (by the INVALSI Institute), through which teachers can thus build reflective and metacognitive paths, functional to a real teaching by skills. Our goal is to analyze teachers' perception of difficulty and their beliefs about the INVALSI mathematics tests. In particular, in this study, we want to determine what specialized knowledge and beliefs can be inferred from teachers' answers to a questionnaire about the difficulty of a specific INVALSI item.

Theoretical framework

The current research belongs to an interdisciplinary research project, which was initiated in 2017 by the "Gruppo INVALSI", composed of disciplinary experts and pedagogues, within the Observatory of Didactics and Disciplinary Knowledge established by SIRD (Italian Society for Educational Research). These researchers were interested in exploring, on the one hand, the proximity or distance between the functions and contents of the INVALSI items and, on the other hand, the beliefs and statements about teachers' teaching practices. To this end, they shared an interest in constructing a questionnaire for investigating the perceptions of primary school mathematics teachers with respect to the INVALSI tests.

The questionnaire concerned the teaching of mathematics (how teachers interpreted seven INVALSI items of grades 5 and 6, and their results, and how teachers perceived misconceptions, errors and levels of difficulty), aspects of general education (what beliefs and attitudes teachers have and how they pour them into teaching practices) and questions regarding teachers' opinions on the INVALSI assessment program, the didactic usefulness of INVALSI items didactic practices related to INVALSI items, and the attitude towards the ideology of natural gifts (Ciani & Vannini, 2017), together with professional training and personal data. This questionnaire was administered to N_1=526 teachers (Faggiano et al., 2021).

Among the various elements to be analyzed, there were undoubtedly factors related to teachers' perceptions and opinions that can facilitate or inhibit the didactic impact of the INVALSI assessment. Main research data showed that there was a metadidactic conflict (Arzarello & Ferretti, 2021) between what teachers believe and their classroom practices. Figure 1 shows the different research variables framing the main study. This framework has its roots in the INVALSI and OECD-PISA ones (INVALSI, 2018; OECD, 2019).

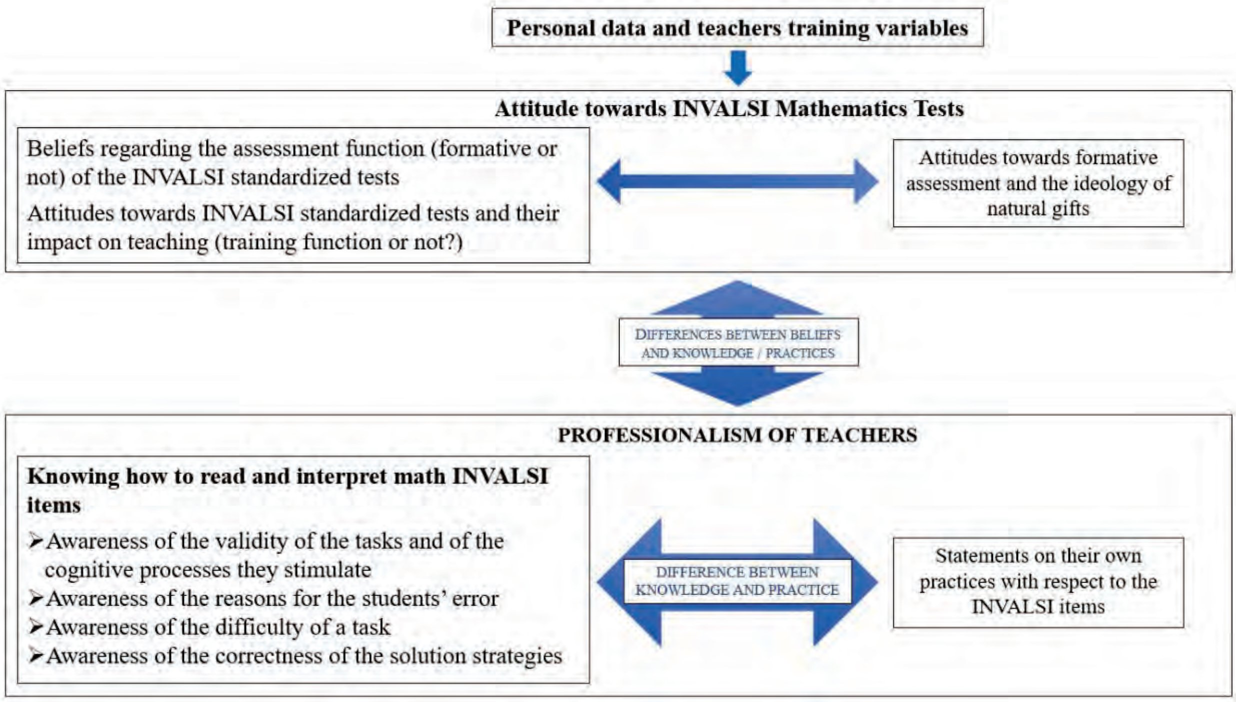

Figure 1: Framework of the study

As a part of this main study, in the current paper our goal is to analyze the knowledge, teaching experiences and beliefs of primary school teachers in reading and interpreting the INVALSI questions and data in the field of mathematics.

Among the different models in the research literature characterizing mathematics teacher's knowledge, we have chosen the MTSK model (Carrillo et al., 2012; Carrillo-Yáñez et al., 2018), due to the intimate relationship it establishes between the specialized knowledge and the beliefs. The MTSK model considers, following the line of Shulman (1986), two major domains of knowledge: Mathematical Knowledge (MK) and Pedagogical Content Knowledge (PCK). Within the MK domain, the Knowledge of Topics (KoT) refers to knowledge of mathematics as a discipline, and also includes school mathematics. The MK also includes the Knowledge of the Structures of Mathematics (KSM) subdomain, which defines that the teacher can know how the subject matter relates to other mathematical topics or concepts, and, finally, the Knowledge of Practices in Mathematics (KPM), which defines that the teacher knows the mathematical task underlying the mathematical activity, or knows the forms of argumentation, validation, and proof. The PCK includes three subdomains: The Knowledge of Mathematics Teaching (KMT), that contemplates the teacher's knowledge about how to transform the mathematical content to make it understandable to others; the Knowledge of Features of Learning Mathematics (KFLM) is the teacher's knowledge of how the mathematical content is learned, as well as the difficulties or strengths that students may encounter when learning a given content; and, finally, the Knowledge of Mathematics Learning Standards (KMLS), which is defined as the teacher's knowledge of what should be learned at each stage of schooling.

Moreover, the MTSK model includes beliefs on mathematics, and its teaching and learning that are considered as elements that permeate the teacher's knowledge in the different subdomains, with that, we seek to construct increasingly accurate images that allow us to interpret the teacher's knowledge and the aspects that influence it.

Instrument

The analysis of these data from the main study prompted us to plan to carry out qualitative study. Therefore, we identified one of these INVALSI items from main questionnaire to use it as an anchor and we administered it to a small group of primary school teachers (N_2= 46) by adding an open question that allowed teachers to provide a reason for choosing the difficulty items using. The data thus collected were analyzed using the MTSK model (Carrillo-Yáñez et al., 2018). Therefore, the instrument consisted of a first quantitative survey, followed by an open-ended question, and this quantitative-qualitative analysis conforms a sequential explanatory mixed method research design (Creswell & Plano Clark, 2017).

Results

Quantitative study

We show the selected INVALSI item with its nationwide results in 2009 (see Figure 2), obtaining only 33% of correct answers. This means that it turned out to be a very difficult item for Italian students in the last year of primary school.

D10. To which number does “12 tens, 7 tenths, 2 thousandths” correspond?

A.	12.702	44.5%
B.	120.702	33%
C.	12.72	18.6%
D.	120.72	3.3%

Figure 2: Item 10, Grade 5 Mathematics INVALSI test (2009) with percentage of students’ answers (in red those relating to the wrong answers and in green the one relating to the correct answer). The translation of the original item is made by the authors

The purpose of this INVALSI item was to verify the ability to manage the conversion between different representative registers relating to the writing of numbers (Duval, 1993). As it can be seen in Figure 2, 44.5% of Italian 5th graders chose option ‘A’. Those who chose option A did not correctly transform the 12 tens but recognized the correct position of tenths and thousandths. 18.6% of students chose option ‘C’ where both conversion errors are present.

In the main study, without informing the participants about the correct answer rate given by the students, the following question was posed to the teachers: “On a 1 (very easy) to 10 (very difficult) ranking, how difficult do you think the item is for 5th grade students?”

Table 1: “How difficult do you think the question [in Figure 2] is for 5th grade students?” (N_1=526)

	Valid percentage	Cumulative percentage
1 (very easy)	14.5	14.5
2	27.7	42.2
3	16.7	58.9
4	7.6	66.5
5	13.0	79.5
6	5.6	85.1
7	7.2	92.2
8	5.8	98.1
9	1.4	99.4
10 (very difficult)	0.6	100.0

This item was considered by 86.2% of the teachers participating in the main survey as particularly suitable for evaluating students’ learning and, above all, 87.6% said they used similar items regularly in their assessment tests. This result confirmed what was found in an early work on this data (Arzarello & Ferretti, 2021): teachers’ perception of students’ difficulties does not correspond to the INVALSI national data.

Qualitative study

In order to better investigate the phenomenon, we administered the open-ended question about the difficulty of the item to N_2=46 primary school teachers, by asking them to justify their answer on the perceived difficulty of the item. Results for this sample are aligned with the first one. In fact, 71.7% of teachers participating in the qualitative questionnaire considered the question to be easy. To analyze the justifications provided by teachers, we decided to use the classification of the MTSK model, due to the relationship that the model establishes between knowledge and beliefs and conceptions. The teachers' answers were often attributable to different categorizations present in the model; in the following table we try to provide an overview of the choices made and the most significant answers.

Table 2: Example of match between the categories of the model and the answers of the teachers

Teachers' answer (translated by authors)	Categories of the MTSK model
At the end of the 5th grade the pupils are fully capable of composing and breaking down integers and decimals	KMLS + KoT
Because it is in the program and feasible	KMLS + Belief
I think it is quite difficult because, despite being within the reach of a fifth-grade boy or girl, it is necessary to reflect on the INVALSI context which puts pressure and often influences the pupils.	Belief against external assessment

Now let's see what the percentages of the categories were with reference to the perceived difficulty.

Table 3: Number of categories identified in our sample

Perceived difficulty	Categories of the MTSK model	Number of responses that match with the category	Some examples of teachers' answer (translated by authors)
From 1 to 5 (33 out of 46, that is 71.7% of the respondents)	Knowledge of features of learning mathematics (KFLM)	6	Most pupils don't make mistakes
	Knowledge of mathematics learning standards (KMLS)	15	At the end of the 5th grade, they should have a clear understanding of the concept of digit place value, so it should be easy to identify the correct answer.
	Knowledge of topics (KoT)	15	If you work on decomposition, it is not difficult
	Knowledge of practices in mathematics (KPM)	1	It is not very difficult some pupils are used to reasoning
	Knowledge of mathematics teaching (KMT)	3	In classroom activities I have always proposed exercises of this type.

	Belief	7	KMT = he/she talks about what he/she does Belief = repeating similar exercises as a learning method
From 6 to 10 (13 out of 46, that is 28.2% of the respondents)	Knowledge of features of learning mathematics (KFLM)	10	12 tens are difficult for some to transform
	Knowledge of topics (KoT)	3	It is a difficult question on at least two points. The first concerns the non-canonical representation of the number 120, precisely 12 tens. Teachers do not always work enough on these aspects. The second point concerns the presence of 0 at the cents position and if the topic has not been acquired with certainty, students can easily fall into error.
	Knowledge of mathematics teaching (KMT)	2	When explained and visualized with structured material it shouldn't be difficult.
	Belief	1	I think it is quite difficult because, despite being within the reach of a fifth-grade boy or girl, it is necessary to reflect on the INVALSI context which puts pressure and often influences the pupils.

It is interesting to note that beliefs were present exclusively among those who perceived the question to be simple. In fact, the only belief present in the answers of those who perceived the question to be difficult was the one reported in Table 3, and we can see how in reality the teacher judges the question to be simple ("being within the reach of a fifth-grade boy or girl") but he/she considered the complexity came from "the INVALSI context which puts pressure and often influences the pupils".

Discussion and conclusions

The analysis of the answers evidenced that most of the teachers mastered the whole and decimal numbers, as well as their composition and decomposition (Table 3). Reading the answers with evidence of KoT, it can be stated that the knowledge of the positional value of the digits was considered by teachers to be essential for the knowledge of the decimal numbering system, since, in the canonical decomposition, the number 120 can be represented as 12 tens.

In the answers evidencing KMLS there were interesting elements to be interpreted. For example, when a teacher stated that this topic must have been learned in the third year, and that, therefore, this topic must already be known well in advance. Another interesting answer about KMLS was given by a teacher who stated that this topic is processed in the fourth year, but is resumed in the fifth year, thus, it is understood that he/she knew that it is a progressive content which requires several academic courses to ensure it. Other teachers said that, being part of the program, students need to know that content: this shows they knew that at the end of the fifth-grade students should be perfectly able of composing and decomposing whole and decimal numbers (this is relative to KoT). These differences may be related with the fact that teachers can handle at the same time different points of view about mathematics and its teaching and learning (Rodríguez-Muñiz et al., 2021). It seems necessary to underline that, in total, there were 24 answers related to KMLS, KoT, or both, among those who considered the question to be simple. We wonder if the way in which the question was asked pushed teachers to focus only on content and learning standards (KoT and KMLS). It seems to us that some teachers focused on content and standards due to a belief: they assumed the curriculum is well designed, and, so, if this is a standard it MUST be easy.

On the other hand, some answers showed evidence of KFLM, that is, the focus on how students think and construct the mathematical knowledge about this content, rather than on the curricular design (where the topic is, what the standard is). Moreover, they knew those features in the process they went through to understand it, as well as some of the learners' strategies when interacting with the content. They also knew some of the difficulties and obstacles associated with this content. It did not seem casual that most of these teachers perceived the question to be difficult.

The results proved to be significant in understanding the effective impact of the INVALSI tests on classroom teaching and the real need for training accompaniment that should be designed to support teachers in their daily management of teaching-learning processes of mathematics in primary school. Therefore, the most significant result is that the questionnaire highlighted a discrepancy between teachers' beliefs about the INVALSI tests and their statements relating to teaching practices. In addition, the use of the MTSK model to deepen our understanding of the mathematics teacher's specific knowledge has allowed us to understand in more detail the characteristics that teachers express when asked about the degree of perceived difficulty.

The analysis of the results is still ongoing and intends to deepen, in particular, the links between how teachers perceive the difficulties of students in INVALSI questions; how teachers interpret students' responses and errors; how much teachers find the INVALSI questions useful and how they use them in teaching practice. We believe that this information can be extremely useful in the design pre-service training for primary school teachers.

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Interpreting the Knowledge Mobilised by Prospective Teachers for Addressing in Learning Polyhedra

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This study provides an interpretation and categorisation of the knowledge mobilised by six preservice primary teachers regarding potential difficulties facing pupils and their modes of reasoning when studying polyhedra. It analyses the collective reflections the prospective teachers make as they carry out a training activity based on the transcription of a recording of a lesson on visualising polyhedra. The extract features a teacher working with six pupils in fifth grade of primary school (10-year-olds) giving their reasoning about how to count the number of edges on a cube. The results show that the prospective teachers are aware of various characteristics of learning about polyhedra, which are underpinned by personal theories (beliefs and conceptions), which are consistent with formal theories in the research community. Nevertheless, there is evidence that some of the preservice teachers need to reinforce their knowledge of the mathematics of polyhedra.

Keywords: Preservice teacher education, visualization, solid geometry, teacher knowledge, primary school.

Introduction

Researchers into Mathematics Education recognise teaching geometric concepts such as polyhedra as an effective way to promote visualisation ability and spatial thinking (Idris, 2005). Various authors consider knowledge of the topic indispensable for prospective teachers (Gutiérrez, 1996; Mulligan, et al., 2020; Sinclair, et al., 2011; Kinach & Coulson, 2014), given that they need to be appropriately trained in order to incorporate aspects of visual space into their mathematics lessons and so develop the ability in their students. There is a tendency to believe that visualisation of geometric figures falls within the scope of people's intuition. However, research shows that visualisation is a complex process, is not purely innate, and can be learnt and developed over time (Nagy-Kondor, 2014).

In this respect, it is appropriate for teacher educators to develop tasks aimed at helping future primary teachers to construct their knowledge of polyhedra and improve their skills of visualisation. Such tasks should deepen prospective teachers' mathematical knowledge of polyhedra, and at the same time develop their awareness of the kinds of difficulties their students might face in learning about the topic, and how they are likely to interact with the topic (Montes, et al., 2019; Policastro, et al., 2019). Drawing on the work of Coles (2019), we designed a training task involving the analysis of transcriptions from actual classroom episodes. The aim was to give the pre-service teachers (henceforth PSTs) some insight into the actual practice of teaching, and in some respects require them to put themselves in the shoes of the learner. This paper reports on the results of one part of the overall task with the PSTs. It aims to answer to the research question: What knowledge is mobilised by the PSTs in carrying out a task based on extracts from a lesson on polyhedra?

Theoretical framework

Below we give an account of the two sources for the theoretical underpinnings of this study.

Mathematical Teachers' Specialized Knowledge

Our framework for situating the PSTs' knowledge is the model of Specialized Knowledge of Mathematical Teachers - MTSK, (Carrillo et al., 2018). This model is concerned with the knowledge that is the basis of a teacher's actions. It is made up of two domains: mathematical knowledge (MK) and pedagogical content knowledge (PCK). The authors of the model divide the two previous domains into six subdomains: Knowledge of Topics (KOT), Knowledge of the Structure of Mathematics (KSM), Knowledge of Practices in Mathematics (KPM), Knowledge of Mathematics Teaching (KMT), Knowledge of Features of Learning Mathematics (KFLM) and Knowledge of Mathematics Learning Standards (KMLS).

In the training task shown in this article we will only focus on two subdomains of MTSK, which are KOT and KFLM. We chose these categories and their indicators to select the transcript of video, pose interesting questions for the training of the future teacher and to be able to categorize the answers of the PST. This decision is based on the interest that the university's training plan had, the methodology worked to date and the style of the teacher trainer. The MTSK is the glasses that helps us to be clear about the aspects that we want our future teachers to learn. Although we recognize that it is necessary to delve into literature that details such knowledge, i.e., literature that details the difficulties of learning polyhedrons and the development of visualization skills.

Knowledge of visualisation abilities and difficulties in learning about polyhedra

Our review identified some of the most common difficulties students face when they learn about solid shapes, including: the complexity of identifying the defining characteristics of polyhedra, understanding the planar representations of three-dimensional objects, determining the hidden elements of a planar representation, and developing and manipulating mental images of 3D objects (Bishop, 1989; Del Grande, 1990; Gutiérrez, 1996; Presmeg, 1986).

Research by Widder et al (2019) and Fujita et al. (2020) confirm that visualization difficulties are still latent in classroom reality. Furthermore, they convince us that whenever there is a planar representation of three-dimensional objects there will be visualization difficulties. We share with these authors the curiosity to understand the problem from the same 3d shape, the cube, an apparently simple geometric body, but one that is not free from visualization difficulties.

We identify with the perspective of Widder et al (2019), who describe a 2-D sketch in spatial geometry as "a cluttered scene comprising geometrical features that consist mostly of points and edges with no clear outer distinction between helpful and misleading attributes that could guide the learner to focus on helpful visual aspects and disregard misleading ones" (p.500). On the other hand, Fujita (2020) in a similar scenario affirms that this visualization difficulty not only needs "visualization skills" and comments on the need to harmonize visualization skills with a high knowledge of the domain, specifically the properties of mathematical content. These are two of the visualization skills proposed by Gutiérrez (1996) that we want to highlight, which are important to learning polyhedra:

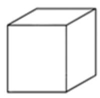
Mental rotation: The ability to produce dynamic mental images and to visualize a configuration in movement.

Perceptual constancy: The ability to recognize that some properties of an object (real or in a mental image) are independent of size, colour, texture, or position, and to remain unconfused when an object or picture is perceived in different orientations. (p.10).

Methodological design

The framework of the study is research-based design (Wake, 2018). The aim is design of training tasks which facilitate the mobilisation of PSTs' specialised knowledge in terms of the MTSK model. The designed task here requires PSTs to read the transcript of a recording of a real lesson (Sherin et al. 2011), which shows the interaction of a primary school teacher with 6 children in the 5th grade of primary education (10 years) who do an activity about how many edges a cube has. The video of this teacher was chosen from a repository of the research group of the University of Huelva, because she is characterized by her good practices, but above all because she empowers her students to verbalize their reasoning, ways of thinking and difficulties when learning polyhedra.

Juan:	How many edges are there? I've counted four from where the vertices meet.
José:	I've counted twelve. Two on the right, plus two on the left, plus two underneath and two on top.
Maestra:	But what you've said doesn't give you twelve.
Álvaro:	I counted twelve counting one by one.
Rocío:	I counted four on top, four underneath, two on the left and two on the right.
Pedro:	José made a mistake because he didn't include two on the top and two on the bottom.
María	Me, I only counted the ones at the top and on the bottom…so that's eight. I forgot the sides.
Maestra:	So, we can say four on top, four underneath, and four on the sides, which is three times four, so, three times four equals twelve.



a. What is the reason for Juan's answer? Draw an illustration.
b. What is the reason for José's answer? Draw an illustration.
c. What is the difficulty for José? What does Pedro note in order to correct him?
d. What is the importance of María's answer?
e. What is the difference between the way Rocío counts the edges and the teacher?

Figure 1. Training Task

It is more common to implement training tasks that use videos of real teachers' classes. However, in this context we decided to use the transcript (a text) because we wanted to ensure that future teachers would focus only on issues related to mathematical knowledge, the way the subjects reasoned and the difficulties that arose in each situation. With a video you can also, but there is more possibility of being distracted and paying attention to questions of general pedagogy.

The questions of the task were aimed at mobilizing the specialized knowledge of the future teacher. For example, in question c the intention is to mobilize the KFLM (the difficulties primary pupils encounter in the topic of polyhedra) is directly sought. Although in question a and b they also have the opportunity to do so. In questions a and b, the main intention is to mobilize the KFLM (the ways they interact with the content, theories of learning about geometric shapes, and knowledge of the interests and expectations of the pupils regarding this topic).The mathematical knowledge that a future teacher would need is implicit in all the questions, however, we decided to establish question d, to raise the importance of the knowledge of the definitions, properties, registers of representation, procedures and phenomenology associated with the topic of polyhedra. And how mathematical knowledge helps future teachers to understand the difficulties of their students. The question e has the same intention as a and b, but adds an extra, which is the interaction of two subjects to the same content (teacher and student). Show the future teacher the possibility of two paths and how they differ or have similarity.

The results reported here derive from the implementation of task with prospective teachers, who are in the final year of the degree in Primary Education at the University of Huelva. To carry out an organized analysis, we consider it appropriate to record the information in a table with the following structure (Table 1).

Table 1. Structure to organize information analysis

Nº	PSTs	Unit	Interpretation	Knowledge, MTSK

The analysis was carried out under a qualitative methodology with an interpretive approach, specifically content analysis was used and the use of the MTSK model as an analytical tool (Krippendorf, 2018; Yin, 2015; Carrillo et al., 2018).

Results

First, we present the results of the knowledge mobilised by the PSTs in response to the difficulties associated with learning about polyhedra which corresponds to the KFLM sub-domain. We then consider their mathematical knowledge of cubes associated with KoT. Finally, we analyse PSTs' knowledge of the ways pupils might interact with content (cubes), also associated with KFLM.

Knowledge of the difficulties associated with learning about polyhedra

Difficulties in interpreting planar representations of 3D objects

All of the PSTs recognised that there were difficulties in interpreting the plane representations of solids. For example, they identified that some pupils have considered just one of the faces of the cube, in other words, a square. This is seen in detail when PST1, PST2 and PST5 explain the reason for Juan's answer, as seen in Figure 2, which is the PSTs' illustration of Juan's reasoning.

PST1: He's thinking of a square with four vertices four sides. He's treating the sides as if they were edges.

PST2: He's only counted the edges of the front face, which is the one he can see.

PST5: Juan's answer tells us that he can't perceive or understand the three dimensions. He only counts the edges of the front face.

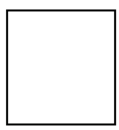


Figure 2. The PSTs' model of visual representation of Juan's difficulty

Another difficulty concerning the representation of a 3D object on a plane is to identify hidden elements. We can see that the PSTs know this difficulty, for example, when PST5 said “how José has counted the edges, we can see that he has not taken into account the hidden edges”. If we follow PST5’s reasoning and look at Figure 3, there are in fact nine edges visible and not 8 as José affirms. PST5 does not question the missing edge.

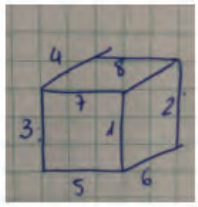


Figure 3. “Failure to count the hidden edges”. PST5 interpretation of José answer

Difficulty in the development of visualization abilities: “Perceptual constancy” and “Mental rotation”

Four out of the six PSTs know the significance of the position and orientation of an object for recognising its properties. PST1, PST3, PST4 and PST6 are of the opinion that José considered only the edges of the front and back faces of the cube when he counted the edges, leaving out the edges running along the sides, which are indispensable for forming the remaining faces; in other words, they think that he excluded the edges which give the cube depth in its visual representation.

PST1: José hasn’t got a clear conception of the definition of an edge, as they are straight segments linking the vertices, as you can see in the drawing, four of the edges that make it a polyhedron are missing.

PST3: He’s missed out the part represented by the dotted lines.

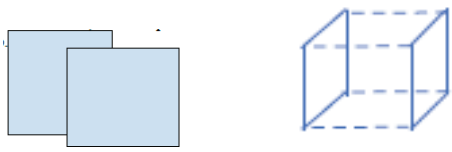


Figure 4. Visual representations by PST1 (left) and PST3 (right)

On the other hand, PST6 knows that José’s answer could be due to a difficulty in mentally manipulating the image, suggesting that: “he hasn’t got the cube fully formed in his mind, it is hard for him to turn it around”.

Difficulty in counting and communicating the processes

The PSTs recognize that pupils have difficulties other than visualization skills such as counting and communicating processes.

PST2 defends José’s answer, suggesting that at first, he gave the right answer, but then subsequently failed in his explanation of his procedure. However, PST2 does not explain a possible cause for the failure in José’s communicative skills. PST5 considers that the pupil might know the answer, but has difficulty in explaining and communicating the procedure by which he got there. PST6 also believes that it might be the consequence of forgetting elements when it came to counting, suggesting: “he forgets to count the edges which give it depth.”

Mathematical knowledge of cubes (definitions, properties, registers of representation, and procedures)

Teachers should undoubtedly have a wide range of knowledge (more extensive than their pupils) so that they can interpret their pupils' interventions, and can understand the origin of their difficulties and the various processes involved in learning. In the results shown up to now it has been possible to see items of mathematical knowledge, some use of formal language, and the identification of elements and ideas associated with how the representation of a cube is constructed, among others. Some excerpts from the PSTs' written answers also provide a direct illustration of their mathematical knowledge regarding the sufficient conditions for constructing and defining a polyhedron. For example, when PST1 and PST4 mention the importance of Maria's contribution in to the lesson, they make it clear that two isolated faces cannot form a polyhedron, while in contrast, the intersection of various polygonal regions (faces) do make a polyhedron.

PST1: The important point is that if they don't count the edges on the sides, it does not make polyhedron.

PST4: The important point is that if they don't count the faces on the sides, they can't construct a polyhedron.

Despite the above, it can equally be seen that the PSTs are not exempt from misunderstandings or being unaware of issues in mathematics topics. An example of this occurs with PST2 when he tries to differentiate the reasoning of María and the teacher (see Figure 5).

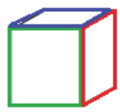


Figure 5. Illustration accompanying PST2's answer

PST2 fails to give the correct interpretation of how the teacher visualises the cube, and how she decomposes the cube into three parts which share the property of having four edges.

By way of contrast, we present the PST4's written answer and accompanying illustration (see Figure 5), who shows knowledge of the representation of a polyhedron, characteristics of a polyhedron and visualisation abilities (KoT) which enables them to arrive at an appropriate explanation of the teacher's and Rocío's reasoning.

PST4: Rocío and the teacher coincide in the way they count the number of edges the top and bottom faces of the cube have, but diverge when it comes to counting the faces on the side. The teacher says that there is a total of four edges, while Rocío divides these four edges into two on the left and two on the right.

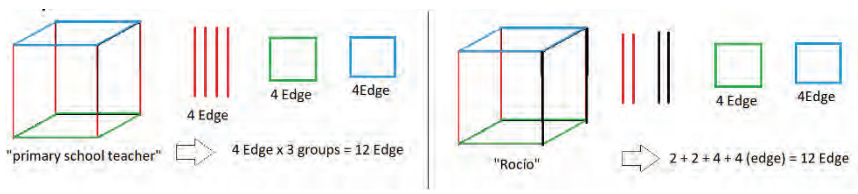


Figure 6. Illustration accompanying PST4's answer

Knowledge of the interaction of the student with the content (cubes)

Based on Figure 6 and the remark made by PST4 in the section above, we can claim there is evidence that the PSTs establish differences in the reasoning of the participants, as in case of the teacher and Rocío, for example. The observations of PST1 and PST3 confirm this, illustrating their knowledge about what strategies the pupils and teacher employ on different occasions for structuring a cube.

PST1: Rocío specifies where each edge is located, that is, on top, underneath, right and left. However, the teacher says straight off that the sides have four edges without specifying the location.

PST3: The way Rocío does it is simpler so that her classmates can understand it, but the teacher groups together the two edges on the left and the two on the right. So, she has three groups with four edges each, which she can multiply together.

PST1 and PST3 show how two participants can interact with a content item in different ways. PST1 notes that Rocío uses previous knowledge of directions (up, down, left and right) to indicate which parts she was talking about. PST3 recognizes that Rocío and the teacher have differences in terms of the way they count the edges of the faces on the sides.

Conclusions

Carrying out this training task has enabled us to identify that the prospective primary teachers know some learning pupils' difficulties, how they reason (KFLM), and their mathematical knowledge (KoT). The performance of the task foregrounds that the PSTs are developing both mathematical and pedagogical content knowledge (Montes et al., 2019).

In the results of PST1, PST2 and PST5 we evidenced the knowledge about the difficulty involved in representing 3d objects in plane (Fujita et al. 2020). Another knowledge that emerges is that the interpretation of the planar representations of 3d objects is linked to the identification of their elements, discriminating the useful elements from the misleading ones (for example, elements that are superimposed or those that are hidden) (Widder et al., 2019).

In the case of PST1, PST3, PST4 and PST6 recognize the elements of a geometric body when it is in a known orientation and position. For example, a cube from a cavalry perspective (Larios, 2003). Finally, PST6 is the one who makes explicit the need to make a mental image and manipulate it (Gutiérrez, 1996). In the planning phase, it was expected that the PSTs would be able express their knowledge in terms of formal theories, because voluntary readings were recommended for the course. It seems that the PSTs base their knowledge of the teaching-learning process on personal theories (beliefs and conceptions), which might be subject to their experience as learners. Despite this, their statements are consistent with the literature (e.g., Gutiérrez, 1996).

Finally, we consider it necessary to review the task and consider the following factors: the importance of drawing, the different types of representations of a cube (other perspectives, decomposition of the cube, in 3D using software, etc) and changes in representation registers.

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Teacher Knowledge in the Context of Articulating Arithmetic Operations and Place Value

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Place value plays a pivotal role in understanding numbers and arithmetic operations that universally cut across the primary and secondary curriculum. This study investigates the nature of 40 qualified teachers' knowledge in the context of articulating the role of place value in understanding arithmetic operations. The participants have 3-45 years of teaching experience who currently teach grades 1-8 (ages 6-14) in public or private schools in Turkey. The data consists of the videotapes of the semi-structured interviews in which we asked the teachers to analyse five different scenarios consisting of alternative student solutions regarding arithmetic operations. Our initial findings suggest that teachers who operate with (partial) specialised content knowledge or without it analyse the given alternative solutions qualitatively differently. The analysis also reveals that the teachers rely mostly on common content knowledge that has little or no connections to a solid place value understanding.

Keywords: Place value, specialised content knowledge, mathematics teacher knowledge, arithmetic operations

Introduction

Place value plays a pivotal role in understanding number structure and arithmetic operations (Nuerk et al., 2015) that universally cut across the primary and secondary curriculum. Arithmetic and symbolic calculations require a solid understanding of place value. However, numbers and arithmetic operations are mostly treated procedurally in schools and the teaching that fosters rote learning is a common practice among teachers (Pesek & Kirshner, 2000). As a result, we see students treating numbers as a combination of concatenated single digits (Fuson et al., 1997) or having difficulty understanding the meaning of each digit in a number (Kamii, 1986). Furthermore, understanding number requires one to form groups and think about groups (or groups of groups, etc.) as single entities or composite units (Hiebert & Wearne, 1996), which is also problematic among students (Thanheiser, 2015). Such difficulties affect students' understanding of the procedures in making sense of arithmetic operations (Verschaffel et al., 2007) and algorithms (Kamii & Dominic, 1998).

Even teachers struggle to clarify their rationale for algorithms as they primarily operate from procedural aspects (Ma, 1999). They have difficulty articulating number concepts (Thanheiser et al., 2013) and place value (Southwell & Penglase, 2005). Primary school teachers are not even aware of "the impact of place value understanding on the learning of mathematics" (Houdement & Tempier, 2019, p.36). Such a crucial area requires teachers to have the necessary knowledge and understanding to teach conceptually. The research literature piles up in articulating student understandings or misunderstandings (e.g., McClain, Cobb, & Bowers, 1998), whereas it falls short in delineating teacher knowledge and the nature of that knowledge. Several studies focused on the understandings of trainee teachers (e.g., Lo, Grant & Flowers, 2008; McClain, 2003), but our experience suggests that qualified teachers do not have a solid knowledge of concepts, as mentioned earlier. Knowing more about the source of teacher difficulties and the nature of their knowledge specific to teaching would inform educators in developing well-designed professional development programmes and give them opportunities to improve the quality of instruction in schools. As Thames and Ball (2010, p.228) pointed out, identifying the mathematical demands of mathematics teaching "allows us to identify the mathematical knowledge needed for teaching", which will help us improve the quality of instruction in schools. Therefore, the current study investigates the nature of qualified teachers' knowledge specific to teaching place value as it "should be at the core of teachers' education" (Houdement & Tempier, 2019, p.36). We pursue the following specific research question: 'What does it mean to operate with/without specialised content knowledge in the context of place value and arithmetic operations?'

Theoretical Framework

It is hard to argue against the criticality of teacher knowledge for effective teaching. Shulman (1987) characterised teacher knowledge as a combination of two major domains, subject matter knowledge (SMK) and pedagogical content knowledge (PCK). Researchers who followed the footsteps of Shulman (e.g., Grossman, 1990; Fennema & Franke, 1992; Ball, Thames & Phelps, 2008) have refined these domains over the last few decades. Recently, Deborah Ball and her colleagues (2008) defined SMK as a combination of three subcategories: common content knowledge (CCK), specialised content knowledge (SCK) and horizon knowledge. In this framework, they described CCK as "mathematical knowledge and skill used in settings other than teaching" (e.g., simple calculations, solving mathematical problems correctly) (Ball et al., 2008, 399). SCK is "the mathematical knowledge and skill unique to teaching" to respond to everyday tasks of teaching (e.g., mathematical knowledge required to analyse alternative student solutions, sizing up the nature of a nonfamiliar error), and horizon knowledge is "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p.403). The framework also considers pedagogical content knowledge as a combination of knowledge of content and students (KCS), knowledge of content and curriculum (KCC) and knowledge of content and teaching (KCT). KCS is the "knowledge that combines knowing about students and knowing about mathematics" (e.g., student thinking, common student errors), and KCT is an amalgam of knowing about teaching and knowing about mathematics (e.g., different instructional models of place value and their effective deployment) (Ball et al., 2008, p.401). Finally, KCC is about combining knowledge of content and the curriculum.

To distinguish between these knowledge types, Ball et al. (2008, p.404) gave the following examples for the task of "selecting a numerical example to investigate students' understanding of decimal numbers". In this case, they consider "ordering a list of decimals" as CCK, "generating a list to be ordered that would reveal key mathematical issues" as SCK, "recognizing which decimals would cause students the most difficulty" as KCS, and "deciding what to do about their difficulties" as KCT (Ball et al., 2008, p.404). In this example, knowing what grade level is appropriate for teaching decimal ordering would be labelled as KCC. The development of decimals throughout the mathematics curriculum would be an example of horizon knowledge.

We adopted the Ball et al. (2008) framework to analyse teacher knowledge. We specifically zoomed in on how teachers operate with SCK or in the absence of SCK to pursue the aforementioned research question.

Method

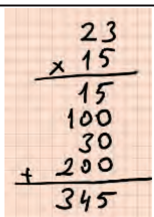
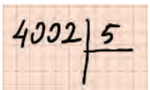
Participants

Participants were 40 teachers, 35 of whom worked as primary school teachers in public or private schools (teaching ages 6-11). Four worked as mathematics teachers in middle schools (teaching age 11-15) by the time this study was conducted. These teachers got their degrees from 20 different institutions from different regions of Turkey. Their teaching experiences range in years: 6 of them 1-5 years, 10 of them 6-10 years, 7 of them 11-15 years, 5 of them 16-20 years, 2 of them 21-25 years, 2 of them 26-30 years, 8 of them more than 30 years.

Data Collection Instrument

The data comes from one-on-one semi-structured interviews. We recruited these teachers using convenience sampling on volunteers by considering their varying teaching experiences. We videotaped the interviews without revealing their identities. We gave the participants previously piloted scenarios consisting of alternative student solutions during the interviews and asked them to analyse these scenarios. Each scenario, applied in separate pages, focused on a single arithmetic operation requiring the teachers to draw on SCK regarding place value. Note that we did not train these teachers about SCK or any other knowledge component. SCK is the mathematical knowledge required to respond to everyday tasks of teaching mathematics. For example, the mathematical knowledge required to analyse students' alternative solutions falls under the umbrella of SCK. Each scenario provided teachers with an alternative student solution requiring them to draw on their mathematical knowledge of place value, and arithmetic operations to some extent. Last two of these scenarios are illustrated in Table 1.

Table 1: Last two scenarios given to teachers during the interviews

	**4.** What would you say about the method used by Nazlı, who did the multiplication as illustrated on the left? If you were the teacher in this situation, how would you respond to this student's work once she made this explanation?
	**5.** Emre, who starts to do the following division (4002÷5), objects using the following reasoning: "When solving such problems, we first ask 'how many 5's are in 4?' Here, there is no 5 within 4. Well, isn't 4 actually 4000? And isn't it the case that there are 800 fives within 4000? Why don't we w rite 800 in the quotient section then?" How would you respond to this student's work as his teacher?

The scenarios allowed us to evaluate the degree of teachers' operating with SCK, the nature of SCK and the teacher knowledge in the context of place value and multidigit arithmetic. This paper will only provide a brief analysis of teacher responses for Scenario 4 and Scenario 5. A mathematical analysis of these scenarios is provided below.

Scenario 4 requires a solid mathematical knowledge of the multiplication algorithm as it is applied in four steps rather than two, making it an alternative solution. In the given solution, 23×15 is calculated using distributive law as follows (also see Figure 1):

$$23\times15 = (20+3)\times(5+10) = (5\times3)+(5\times20)+(10\times3)+(10\times20) = 15+100+30+200 =345$$

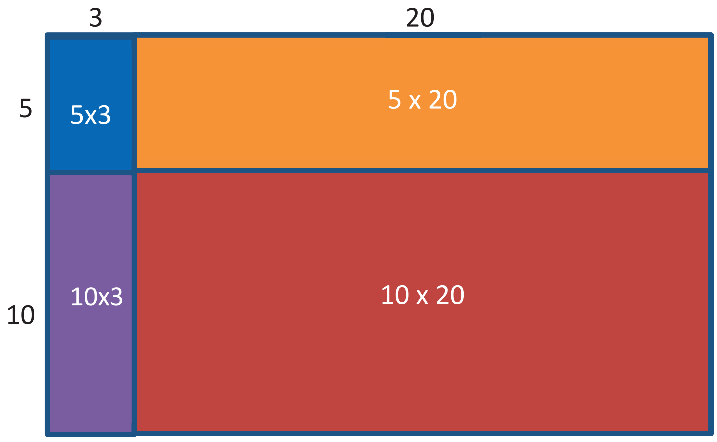


Figure 1: Analysis of 23×15 using distributive law geometrically

The multiplication algorithm includes the calculation of '(5×3)+(5×20)' as a first step yielding 115 and '(10×3)+(10×20)' as a second step resulting in 230 – in fact, procedurally, 230 is written as 23 by aligning 2 with 'Hundreds' column, 3 with 'Tens' column and ignoring 0 in the 'Ones' column. In contrast, the given alternative solution treats these two steps as a combination of four seemingly isolated steps, requiring teachers to analyse the reduction of four steps to two. Therefore, in responding to this alternative solution, teachers need to draw on the following mathematical knowledge: the role of place value in making sense of numbers (e.g., thinking about 23 as a combination of partitions 20 and 3), the connection between distributive law and multiplication algorithm (e.g., thinking about why and how to distribute 5 and 10 onto 20 and 3), and the relationship between four steps described above and two-step algorithm (e.g., how 15+100+30+200 is reduced to 115 and 230 (or reduced to 23, one place shifted to the left)).

Scenario 5 also provides an alternative solution for 4002÷5 by investigating whether the first question to ask in approaching such a problem is 'how many 5s are in 4' (quotative division) as usually taught in schools. Here, "4" within "4002" can mean its face value (4) and place value (4000). To respond to such an alternative solution, teachers need to know that "4 thousand" is to be shared among five parties (partitive division) as thousands, and since this is not possible, it needs to be converted to "40 Hundreds" to be shared as hundreds among five parties. This distribution results in "8 Hundreds", giving the result of the division as 800 with a remainder of 2, or '800R2'. Because of these reasons, the question of 'how many 5's are in 4?' is not mathematically appropriate, and the long division algorithm is to be interpreted by teachers with partitive division meaning (an amount being equally shared among several groups and determining group size).

Data Analysis Procedure

Data analysis is carried out in the form of content analysis and is still ongoing. In analysing the data, we mainly focus on the kind of knowledge the teachers draw on (e.g., SCK, CCK) and its nature. First, we go through all the videotapes and identify instances in which teachers do (not) refer to SCK and the rationale behind their analyses. As we go through the data and as we gain more insight into how teachers operate, we begin to provide a fine-grained description of the nature of SCK for arithmetic operations and place value. After the initial run of the data, we check whether there are similarities among teacher interpretations and how they refer to different knowledge types. We will finally develop categories that characterise the nature of teacher knowledge regarding the place value concept and arithmetic operations. In this paper, we only share some preliminary findings.

Results

The data has given us certain clues about how weak and detached from the conceptual ties the teachers perceive the place value concept and how they drew on their knowledge. In addition, the data led us to believe that inquiring about the validity of the given alternative solution or even having a suspicion about the validity of the given solution help teachers operate more closely with SCK than other knowledge categories. Because of space limitations, we give two examples of this finding below.

I. Evaluation of multiplication scenario: When working on Scenario #4, the teachers considered the alternative solution as a brand-new way of doing multiplication that they had not seen before and pursued one or more of the following methods or arguments:

1. The teachers initially ignored the core of the given alternative solution and labelled it as "wrong" without a thorough analysis. They compared the alternative solution to the routine/procedural application of the multiplication algorithm they already knew (CCK) and used it in their practice. A typical teacher (out of those 12 teachers) reaction to this scenario was, "*If the student comes to me with something like this, I will tell that it can't be done this way, and then I will delete this method*". In this sense, teachers did not focus on the nature of the given solution as they operated with CCK. They were not even suspicious until they solved the given multiplication problem by referring to CCK and realised that their result matched the given result in the alternative solution.

2. The teachers solved the problems in the given scenarios independently of the given alternative student analysis by operating from CCK in a very procedural way – a typical teacher reaction (out of 14 teachers) was, "*Now, I need to find the result of this calculation first*". For example, in multiplying two two-digit numbers (23×15), their initial reaction was to follow the standard multiplication algorithm: applying the 'Ones' column of the multiplier to both 'Ones' and then 'Tens' column of the multiplicand (5×3 and 5×2 without considering 2 as 20), and then doing the same for 'Tens' column of the multiplier and multiplicand (1×3 and 1×2 without considering 1 as 10 and 2 as 20). This way of operating is about the use of CCK others also use in settings other than teaching (mathematics). This way of operating might be convenient to teachers, and therefore they did not feel the need to start with an analysis of the given alternative solution. Finding the matching results fed a suspicion for teachers about the validity of the given alternative solution. Once they became suspicious, they began to analyse the given solution superficially by analysing each given step, which also led them to pursue the question, "*would it be right*?" rather than sticking to their initial rigid observation, "*it is wrong*". What seemed to help teachers move away from operating with CCK and getting close to the use of SCK was the change of perspective (Ball & Bass, 2009) from the "*it is wrong*" argument to "*would it be right*?" argument.

3. Twenty-one teachers referred to one of the other knowledge types (e.g., KCT, KCC, KCS) as if they wanted to fill the void of SCK with those components. For example, teachers referring to

KCT did it by referring to what they would typically do in their teaching and the procedural aspects of the algorithm they would highlight. (e.g., always starting from 'Ones', moving to 'Tens').

II. Evaluation of division scenario. When it comes to Scenario #5 (about division), all the teachers except very few could not sufficiently articulate the dilemma between "*There is no 5 within 4*" and "*there are 800 fives within 4000*" on a conceptual basis by associating the issue with place value. They all, except three, referred to the long division algorithm and ignored the dilemma mentioned earlier. More specifically, a typical teacher's (from among the 37 teachers) initial analysis was:

Teacher:	I need to explain them that we need to work with the face value, we need to have them accept this. … I can take out 4 Turkish Liras [*referring to 1TL coins*] from my pocket and ask, can you give me 5 Turkish Liras from here, can you give or not?

When further probed about the meaning of leftmost two digits, 40, within the number "4002", these teachers were not able to talk about the place value and its role either. Such an approach focused on the face value instead of place value resulting in dependency on CCK or KCT rather than on SCK.

Only three teachers out of 40, when probed to mentally or physically use base-ten blocks, modelled the given division process with the help of base-ten blocks and explained this issue by mentioning that 4 thousand blocks cannot be shared fairly among 5 parties since there would not be enough thousands for each party. This is illustrated for one of those teachers in the below dialogue.

Interviewer:	When we think about the materials here [*referring to base 10 blocks*], what does it mean to say there are no 5 within 4?
Teacher:	Since these are 'Thousands' as wholes, I cannot divide them into 5 people. I need to convert them to 'Hundreds' to be distributed to everyone as 8 'Hundreds'. The remaining 2 would be mine.
Interviewer:	All right, here we say there are 8 fives within 40. Is there a connection between the division algorithm and the application you made with base-ten blocks?
Teacher:	[…] Since I cannot distribute 'Thousands' as 'Thousands', I distribute them as 'Hundreds' [*circling 40 within 4002 on paper*]. So, in a way, I convert this [*referring to 4 thousand*] into 'Hundreds'.
Interviewer:	What does that 40 represent?
Teacher:	Let that 40 represent 40 'Hundreds'. Then 5 times 8 makes 40 – 40 'Hundreds'. I mean, when we say algorithm, if we sift through it, it is 40 [*groups of]* 'Hundreds'.
[…]	
Interviewer:	What does that 8 represent [*referring to the 8 in quotient*]?
Teacher:	8? It represents 8 'Hundreds' per person.
Interviewer:	What does it mean to put zero [*next to 8*] with base-ten blocks?
Teacher:	I do not have any 'Tens'. [*adds 0 next to 80*] I am distributing 'Tens', but I give you zero 'Tens'. I will also share 2, but since 2 is not enough for 5 people, I give you zero 'Ones'. In other words, we confirm that it is '8 Hundreds'.

In this dialogue, the teacher used several components to analyse the given alternative student solution. We consider these components as part of his SCK. These are:

- Investigation of the conditions for which the given alternative solution is correct,
- Referring to place value and its role in division using base-ten blocks,
- Operating from the sharing meaning of division.

This teacher did make a sufficient analysis of the given alternative solution by using all these components together. However, such reasoning was not apparent in this teacher's explanations until the base-ten blocks were shown. In this sense, manipulatives played a particular role in triggering this participant's thinking to carefully evaluate the act of sharing '4 Thousand' units among 5 parties. Base-ten blocks allowed this teacher to question the validity of the given argument. The other 37 teachers, when probed, could not resolve this conundrum, and they continued to operate with CCK and/or KCT even with base-ten blocks. Therefore, the three SCK components need to be abstracted as a totality to sufficiently analyse a given alternative student solution.

Conclusion

Our initial findings suggest that teachers who operate with (partial) SCK or without SCK analyse the given alternative solutions qualitatively differently. The ones operating without SCK are the ones who are under the complete influence of CCK (e.g., procedural application of multiplication or division algorithms) and who analyse the given alternative solutions by purely comparing them to the algorithms. Teachers who operate with partial SCK try to fill the void by bringing in examples from other teacher knowledge types as excuses such as KCS (e.g., this will be quite difficult for students), KCT (e.g., I normally teach it this way) or KCC (e.g., this is not appropriate for lower grades). The teachers operating with SCK approach the given alternative solutions by thinking about the question, 'what makes this solution valuable/reasonable?' and then refer to the role of place value as the core of the given arithmetic operations of multiplication and division.

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TWG21: Assessment in mathematics education

Introduction to the papers of TWG21: Assessment in mathematics education

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Introduction

CERME12 followed the COVID-19 pandemic that hit the world. During this time much of what we did within our classrooms and outside was re-thought to allow for social distancing and remote learning. This of course initiated important changes in the assessment of mathematics. In the CERME event held online in 2021, we drew six areas of interest for our group – some of which reflect the need for remote teaching and learning. These included the use of technology in the assessment of mathematics. For CERME12 we received 20 papers and four posters which covered four themes of the call: The assessment of specific mathematics competencies; Technology – computer aided (formative or summative) assessment; Teacher assessment with focus on teacher education; Teacher assessment in the light of transferring assessment methods to methods with technology.

Thematic clusters

Here we group the papers in the thematic clusters to reflect on their links and findings. We then conclude with some reflections for the coming CERME13.

The assessment of specific mathematics competencies: Ramo et al. investigate university students' preferences when choosing between self-assessment and traditional exams. They found that many students selected self-assessment because of the possibility of self-regulation, for affective reasons and for practical reasons (e.g., time-malmanagement). The authors conclude by indicating this methodology as supporting agency and self-regulation. Ferrara and Pozio investigate routes into algebraic thinking by examining the responses to one item of the Italian national text concerning the area and the perimeter of an isosceles trapezoid. They found four distinct routes that students adopt and relate those to proficiency levels. The levels of proficiency increase as the routes become more advanced, which is consistent to the assessment model. Morselli and Robotti describe a learning sequence inspired by the principles of Universal Design for Learning (Rose & Meyer, 2006) on conjecturing and proving for grade 7 students. Promoting appropriate formative assessment strategies, they found that such a sequence can be effective in terms of proof understanding. Færch investigates the potential of tasks in a dynamic online environment for primary school students to foster symbol competency. Okamoto addresses the theme of creativity, proposing a new approach to evaluate the fluency component in Guilford's (1973) model for creativity by means of Fermi problems. His quantitative study involved 291 Japanese junior high school students. Results show a strong correlation between fluency in general, creativity, and fluency measured by the richness of aspects in solving a Fermi problem. However, further research is needed to examine the validity and reliability of a test to measure creativity using Fermi problems. Rieu et al. investigate prospective teachers' diagnosis of students' misconceptions. More specifically, they investigate the presence of confirmation bias in teachers' detection of misconceptions in decimal fractions. The study identifies normative-accurate and confirmatory-biased judgment processes. A follow-up study, investigating relevant information based on alternative hypotheses to obtain an accurate diagnosis, shows it to be an efficient judgment strategy and as such, it should be incorporated into teacher education. Winnberg studies high-stakes summative tests through the lens of connectivity. The framework of connectivity (Gueudet et al., 2018) was initially developed for the analysis of digital resources such as e-textbooks, and Winnberg proposes to adapt it to the analysis of digital tasks. In the paper, two tasks are analyzed and a final discussion on the viability of the use of the framework is presented. Giberti and Passarella address the issue of real tasks in standardized assessment. They propose a criterion to identify "real" tasks in standardized tests, which they employ to analyze items from the Italian national standardized assessment. Data analysis shows that such tasks are widespread in national assessment. A further analysis suggests that such tasks are not particularly difficult for students, but they are discriminative: students with higher ability pass these tasks easily, while students with lower ability struggle. Saccoletto aims to investigate teachers' assessment in argumentation tasks. To this aim, she plans to propose to teachers a selection of students' answers to be discussed.

Technology – computer aided (formative or summative) assessment: Moons and Vandervieren investigate a semi-automated grading approach of handwritten questions which uses atomic feedback. Assessors using this approach appreciated being able to see the effects of their grading choices on the outcome of the assessment. This ongoing work will also shed light on the possibly added value of this semi-automated grading approach in comparison to traditional grading. Min Chia and Zhang describe secondary mathematics teachers' views on online assessment during the first waves of the COVID-19 pandemic. Based on survey responses of 92 teachers and interviews with three teachers they found that teachers mainly used the same assessment methods during the pandemic, and online teaching, as they had done before the pandemic. Online assessment was for them mainly assessment for and of learning. Brunstrom et al. reported on a pilot study focusing on the (re)design of a digitized task environment in which both a dynamic mathematics software and a computer-aided assessment system are implemented. They discuss their research with 256 first year engineering students in calculus and related the characteristics of the explanations and formulae students provided in their responses to the tasks.

Teacher assessment with focus on teacher education: Kaplan-Can et al. investigate the characteristics of tasks produced by trainee teachers aimed at eliciting cognitive demanding algebra tasks. They found that trainee teachers - after suitable training - can generate such tasks but most of the tasks involved algebraic manipulation and few involved mathematical modelling. Andersson and Erixon present the results of an ongoing study that investigates four mathematics teachers' use of formative assessment. They aim to identify aspects of formative assessment that are important for beginning mathematics teachers (BMTs) and the differences between participants' use of formative assessment in practice. Preliminary findings show that BMTs use different ways to present lesson goals and they refer to written tests to elicit evidence of learning. Moreover, students rarely function as agents in the formative assessment processes. Eichholz et al. present a study concerning the evaluation of the effectiveness and acceptance of the education concepts in the qualification program developed for the project. They aim to compare the implementation of two variations of formative assessment: teachers learn to develop diagnostical questions and support activities and curriculum embedded, or teachers use ready diagnostic and support tasks. They expect a positive effect in favour of the first approach and assume a moderator effect of the association between high pedagogical content knowledge and a positive self-efficacy of the teachers on the second approach. Kjensli et al. report on a project in a course for prospective teachers, where the teacher educator modelled feedback based on formative assessment principles. The study analyses prospective teachers' responses when they are challenged to reflect from a teacher's perspective on how to use different models to compare fractions in a primary classroom. The findings indicate that prospective teachers tend to use feedback to move forwards in their teacher perspective. Barana et al. aim to understand the evolution of participants' knowledge and perception of automatic formative assessment throughout a STEM teacher education course. Analysing responses to a questionnaire they found that teachers became more aware of knowledgeable of this type of assessment, over the course, and that this appears to result in a more student-centered approach to formative assessment. Saksvik-Raanes and Solstad investigate digital items that were designed to measure arithmetic competence as a component of the foundational number sense framework for five- and six-year-old children. Using Rasch analysis of the performance of 302 Norwegian children they found that items' difficulty levels were strongly influenced by the type of problems and the magnitude of the answer in it. Supplemented by a qualitative analysis of students' strategies they conclude that there were more factors at play, but that this work could enable us to study in more detail how children model and use strategies to solve mathematical problems.

Teacher assessment in the light of transferring assessment methods to methods with technology: Klothou et al. investigate how secondary school teachers assess students' written texts in mathematics and the resources they draw on while assessing these texts. Findings show the complexity of the factors that influence the outcomes of assessment in mathematics. Teachers grade students' written texts differently. Teachers' beliefs about the nature of the subject and the nature of mathematics matter, either alone or in combination with expectations about 'communicating', emerged as the prevailing resources teachers draw on while assessing students' written texts. Holm Gundersen and Kohanová present the results of a qualitative study focusing on characterisations of enactment of formative assessment during mathematical conversations by Norwegian primary school teachers. Two second-grade teachers were observed during mathematical conversations with their students in a teaching session regarding various strategies for addition. The paper suggests a model that characterises the formative assessment enacted during a mathematical conversation from a teacher's perspective. Faggiano et al. present initial findings from a survey, administered to 421 Italian in-service primary teachers, on the beliefs regarding the knowledge and skills investigated by the national standardized assessment tests, their proximity to didactic practices in mathematics and the role they assume within the school context. The paper discusses the way teachers interpret data coming from standardized assessment and if/how they use them in their teaching practice. Findings show a meta-didactic conflict generated by teachers' difficulties in interpreting the standardised tests and in using them coherently with the framework on which the tests were designed. Kiss et al. analyse Hungarian students' oral explanations based on their written work, looking for additional information about students' thinking processes. Additionally, the study examines the quality of verbal communication. Findings indicate that the teacher obtains a more detailed picture of students' current level of development via oral explanations. Grapin and Sayac describe a comparison between two forms of assessment at the end of primary school: paper pencil and tablet-based assessment. They considered different approaches to digitalizing assessment, either using the opportunities a digital environment provides, or migrating the paper pencil assessment onto a screen. They opted for the former. Students appeared to perform better on the paper-pencil based assessment than on the tablet-based assessment, although more of the possibilities the tablet provides could be used in further studies. Geszler describes the outlines of her PhD study, in which she investigates the opportunities for digitalization of the final examinations of mathematics in Hungary. To this aim she investigated at the practice in four other countries - Finland, Germany, Georgia, and Denmark - and compared them on several aspects. She concludes that in Hungary there is a will to work towards computer-based adaptive testing, allowing for creativity, analyzing, and mathematical modelling can exist, which could have an active impact on education itself.

Conclusions

Our group again represented work engaging with a variety of methodologies and research questions – confirming that assessment is a big umbrella theme that encompasses a great number of approaches and foci such as the assessment of mathematical competencies and specific topics. However, from the summaries presented, there are two main issues that emerge clearly: the increased use of technology for assessment (both for formative as for summative purposes) which goes beyond the consequences of the pandemic, and the attention to the education of new teachers with respect to assessment practices. These two themes are not un-related: the way in which we assess mathematics is changing, new technology that was introduced during the pandemic out of necessity is now adopted widely, and teachers (at all levels) may be trained in how to use the technology appropriately and what are the affordances of technologies in assessment. Moreover – a productive use of technology must also be adopted – therefore we saw many studies addressing affordances and drawbacks of technology in assessment. We believe that the use of technology will be again a big theme in our next meeting in CERME13 in 2023 - Hungary.

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Formative assessment
– in the hands of beginning mathematics teachers

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Formative assessment is a key competence of professional teachers, yet complex and demanding to develop. Thus, beginning mathematics teachers (BMTs) cannot be expected to have fully developed this competence. In an ongoing study we examine and depict similarities and differences between four BMTs' use of formative assessment in practice. We will use data from video-recorded observations to analyze to what extent and in what ways the BMTs use five key strategies in formative assessment practice and the main idea of this practice, that is, to use information about students' learning to adjust teaching and learning to the students' needs. The study will provide insight into crucial aspects of formative assessment that BMTs need to learn about and reflect on during and after teacher education. Implications for teacher education are discussed.

Keywords: Formative assessment, teacher education, mathematics teachers, beginning teachers.

Introduction

In this paper we focus on mathematics teachers' use of formative assessment in their early careers, and in particular on aspects of formative assessment especially important for these teachers to be vigilant about and reflect on during and after their teacher education (TE). Formative assessment (also called "assessment for learning") refers to a classroom practice in which assessment is used to identify students' learning needs so that teaching and learning can be adapted accordingly (Black & Wiliam, 2009). The teacher as well as the students can be agents in the processes involved in this practice. The empirical evidence of increased student learning in all school years and subjects (e.g., Black & William, 1998; Hattie, 2009) – including mathematics (Palm et al., 2017) – has led to great attention to formative assessment in education research, policy, and practice (DeLuca et al., 2016). Additional theoretical argumentations further motivate the use of formative assessment in mathematics education (e.g., Schoenfeld, 2020).

The increased interest of using assessment to increase the learning of the students has broadened the concept of assessment beyond using assessment for marking and grading (DeLuca & Bellara, 2013), and assessment competence has become a key competence of professional teachers (Xu & Brown, 2016). Consequently, teachers need to learn about formative assessment in their teacher education (DeLuca & Johnson, 2017; Shepard et al., 2005). They need to learn about why using formative assessment and how to put it in practice. More specific, a use of high-quality formative assessment requires teachers to: understand the purposes of and principles on which this classroom practice is based; have ability to incorporate this classroom practice into their teaching; and to use a critical reflection on aspects of quality relating to their understanding of that practice (Xu & Brown, 2016; Young & Kim, 2010). Because the implementation of high-quality formative assessment is complex and challenging (Black & Wiliam, 2009) beginning teachers cannot be expected to have reached full competence for such implementation. Rather, their formative assessment competence will develop under the right conditions (De Luca & Johnson, 2017). However, more studies are needed at different levels and with different foci in order gain knowledge about how to design the right conditions and effective support for teachers' development of assessment competence (DeLuca & Johnson, 2017). In this paper we focus on qualitative aspects of formative assessment classroom practice.

More specific, we use the framework by Wiliam and Thompson (2008) to characterize beginning mathematics teachers' (BMTs) formative assessment practices and shed light on aspects that are important for those teachers to be aware of and reflect on during and after TE. With the involvement of the teacher as well as the students in various assessment and learning processes, this framework reflects the complexity of putting formative assessment into practice. The framework consists of the main idea of using information about students' learning to make decisions about how to adjust teaching and learning to meet students' needs, along with five key strategies (KS) guiding the implementation of this assessment in practice:

KS 1. Clarifying, sharing, and understanding learning intentions and the criteria for success

KS 2. Engineering effective classroom discussions, questions, and tasks that elicit evidence of learning

KS 3. Providing feedback that moves learners forward

KS 4. Activating students as instructional resources for one another

KS 5. Activating students as the owners of their own learning

A clear learning goal with specification of what is counted as criteria of success is crucial in all assessment. In formative assessment the sharing of learning goals and success criteria (KS1) is essential for the feedback processes between the agents (teacher and students) involved in the formative assessment processes. A clear learning goal facilitates eliciting the relevant information about students' learning needs (KS2) and providing adapted instruction, that include teacher feedback (KS3), that move students' learning forward. In addition, a clear learning goal and success criteria are decisive for peer assessment and peer feedback (KS4), and for self-assessment with subsequent adjustments (i.e. self-regulated learning, KS5). Thus, this framework can be used to analyze and characterize qualitative aspects for each KS, as well as the integrated use of them to fulfil the main idea of formative assessment.

In a previous case study, we used the above-mentioned framework to characterize one mathematics teacher's development of formative assessment competency during and after TE. We recognized that the teacher during this time made incremental changes, however crucial for the formative assessment processes in the classroom. That is, the changes enhanced the possibilities for the teacher to gain insight into students' thinking, which is crucial for decisions-making about what feedback to provide the students and/or about adaptions of other learning conditions to meet the needs of the students. This insight about how a seemly small change makes a big difference in formative assessment processes made us interested in comparing BMTs' formative assessment practices. An assumption is that comparing the way the BMTs use formative assessment will gain insight into aspects that are important for BMTs to learn about concerning the purpose and complexity of the use of formative assessment. Such aspects will be significant for early career mathematics teachers to evaluate their understanding and need of development. The insights will also be useful in teacher education when preparing BMTs for the development of formative assessment competence. Moreover, researchers may be inspired to study what variables that affect putting these aspects into practice, circumstances that help mathematics teachers implement those formative assessment aspects, and obstacles the teachers need to overcome to be able to use high quality formative assessment.

Aim and research questions

In the study we examine and depict the similarities and differences in four BMTs' use of formative assessment in their classrooms. The aim of the study is to identify aspects of formative assessment that are important for mathematics teachers to become aware of and reflect on. The study is guided by the following questions: To what extent and in what ways are the five key strategies used in the BMTs' classroom practice?; What are the similarities and differences between the four classroom practices?; and What do those similarities and differences mean from the view of the purposes and principles of formative assessment?

Method

Informants and data

The data used in the study comes from a project called TRACE[1] in which student teachers in their last year of mathematics teacher education at two Swedish universities were asked to volunteer as participants. The informants received information about the study and ethical aspects, and written consent was obtained before data collection began. Due to the 2020 Covid19 pandemic restrictions the data collection was interrupted. In this study we use data (video-recorded classroom observations) from the four informants that were possible to trace more than one year after their graduation. These informants participated in the same teacher program at the same university in Sweden. They graduated with a teaching degree, grades 7-9, with mathematics as specialization. The teaching being observed is also from mathematics lessons in grades 7-9. Three lessons per informant are observed. Gry and Elvin (fictive names) graduated in January 2017. They were observed in October 2018. Tina and Anton (fictive names) graduated in February 2018. They were observed in September 2019.

Analysis

In the analysis we use the framework by Wiliam and Thompson (2008, see above), and draw on previous research experiences of developing and using an analytical tool based on this framework (Andersson et al., 2017; Andersson & Erixon, in press). This means that we use a previously developed tool at start, yet prepared to adapt or complement the coding manual whenever needed. The main codes are: transparency of the learning goals and success criteria (KS1); elicitation of information about student learning (KS2); teacher feedback (KS3); feedback between students (KS4); and students' regulation of their own learning (KS5). For each main code a number of initial subcodes

[1] TRACE project founded by The Swedish Research Council, project/grant number [017-03614]

are set. During initial coding these subcodes are customized to the data so that all lesson activities related to formative assessment are identified and coded. A selection of subcodes is presented in Table 1. These are exemplified in the preliminary findings. We limited the presentation of the process of categorization due to the limited space in this paper. Our analysis is qualitative, but we count some frequencies in our data. This is not because we will try to generalize our result, rather the reason for examining the frequencies is to understand qualitative differences. For example, we compare the extent the informants use different feedback types, by counting numbers and calculating their use as if the lessons were equally long.

Table 1: Subcodes for each main code

Main Code	Subcodes	Main Code	Subcodes
KS1	Explicit formulated learning goal	KS3	Confirming the answer as correct or acceptable
	Goal of doing		Repeating or reformulating what the student said
KS2	Questions about how and why		Explaining (often by simplifying or deepening)
	Questions that require a short answer, yes/no		Initiating mathematical reflection on the solution process
	Assessment material		
KS4	Nothing has been found so far	KS5	Nothing has been found so far

As a last step of the analysis, we will examine the similarities and differences from the view of the purposes and principles of formative assessment. In this step we will consider both the purpose and principles for each KS and their integrated use. This step has not yet started systematically, but we provide one example in the preliminary findings below. Both authors are engaged in data analysis to ensure reliability.

Preliminary findings

Below we present some preliminary findings, with a focus on the similarities and differences in the BMTs' use of the five key strategies. The findings will later be complemented with narratives to contextualize the formative assessment activities each teacher uses. Moreover, an overview of similarities and differences will be presented along with concrete examples. Finally, the findings from analyzing the similarities and differences from the purposes and principles of formative assessment will be presented (one example regarding teacher questions is now included below).

The transparency of the learning goal (KS1)

We have identified three ways of presenting the goal of the lesson: As a learning goal, what students are going to work with, and what pages or tasks to finish. Gry referred about twice as often to the learning goal as the other teachers. For example, she presented the goal as follows:

Gry: Do you know what a polynomial is? We are going to talk about what a polynomial is.

Gry also referred to the mathematical content when she presented the goal of the lesson as something the students are going to work with:

Gry: There are polynomials of degree one, and we have worked with first degree polynomials recently. We are soon going to work with second degree polynomials.

Elvin, however, in most cases presents the goal of the lesson in terms of what tasks to complete, as a goal of doing:

Elvin: Solve the tasks in chapter 1.6.

Elicitation of evidence of learning (KS2)

We were not able to confirm that the evidence of learning that the teachers elicit actually was used to adjust the teaching and/or learning in the classroom. Thus, we treat the KS2-activities as potentially being formative assessment activities. As example, all teachers referred to written tests. A difference was found regarding their use of questions, in terms of what types of questions they used.

Anton and Tina asked a lot of questions that only required short answers. In comparison with Elvin they asked such questions about seven times as often as he did. Gry asked questions about the mathematical content about twice as often as Elvin. About 62% of questions Gry asked could be answered with a yes or a no:

Gry: This became very theoretical, right?

For Gry, the remaining questions (38%) were about what and how, and the students needed to explain to answer them:

Gry: Why is this a polynomial of degree one?

In addition to these activities we could identify that Elvin and Tina used "exit-tickets" to assess what the students have learned during the lesson. Tina introduced the exit-ticket to her students as follows:

Tina: Before you go just so I can see what I have done and what you have learned […] the question is What have you learned? What do you need to develop based on this lesson?

Teacher feedback (KS3)

Also regarding feedback there were differences between the teachers. Elvin used feedback much more often than Gry and Tina did. Four types of feedback dominated. Two of them regard the correctness of student response: feedback confirming the answer as correct or acceptable, and feedback repeating or reformulating what the students said. The other two were: feedback to explain, and feedback to initiate mathematical reflection on the solution process. Elvin and Tina used the last two types of feedback when elaborating on tasks and/or examples that the students found hard to understand. Gry stood out, she did not use the last type of feedback at all. Below quotes will exemplify the different types of feedback:

Example of feedback to confirm the answer as correct:

Student: Well, the square root of 2600.
Elvin: Yes.

Example of feedback to repeat or reformulate:

Student: Can you round to 51?
Elvin: A question from [the name of the student], can you round to 51?

Example of feedback to explain:

Elvin: She converts to centimeters and then she divided by the growth per month, the number of centimeters divided by how much it grows per month.

Example of feedback to initiate mathematical reflection on the solution process:

Elvin: Then we have factorized, can you factorize further?

Moreover, Elvin stood out in another respect. He used a fifth type of feedback, that is, answering questions from the students. He used such feedback about six times as often as the other BMTs:

Student: Can you show another one, so that you have to think a little differently. Like this 327, or any number that you cannot factorize prime numbers?

Erik: Yes, some numbers cannot be factorized in prime numbers, for example 11.

Feedback between students (KS4)

Both Tina and Elvin encouraged the students to work in smaller groups or in pairs so that they could discuss and help each other with the tasks. Nevertheless, there were no examples of how the teachers actually supported their students to be able to assess and provide feedback to each other.

Students' regulation of their own learning (KS5)

Nothing has been found so far.

From the view of the purposes and principles of formative assessment – one example

The questions the BMTs used during the lessons show qualitative difference from the perspective of providing the teacher with useful information to adjust learning activities and providing feedback. The questions that can be answered with a yes or a no are less useful than questions were the students need to explain their thinking. The BMTs used such question to different extent.

Discussion

In this study we aim at identifying aspects of formative assessment that are important for mathematics teachers to be vigilant of and reflect on during and after their TE. Compared to the previous study – where we focused on the development of formative assessment competence of mathematics teachers in early career – this study has a stronger focus on what it means and what it takes to implement high quality formative assessment (see Xu & Brown, 2016; Young & Kim, 2010). When all findings are in place, we will be able to make conclusions about what aspects of formative assessment that were identified as crucial for beginning mathematics teachers. We will then discuss how formative assessment was used by the BMTs; the similarities and differences; what those findings means from the perspective of the purposes and principles of formative assessment; and implications for teacher education. The discussion below is restricted to the preliminary findings.

In all classrooms, the students rarely function as agents in formative assessment processes. This is true for their function as resources for each other (KS4), as well as, for regulating their own learning (KS5). This crucial aspect is also connected to clarifying and sharing learning intentions and criteria of success (KS1). For students to be involved in formative assessment processes, they need an idea of the learning goal and what constitutes progress in that learning. Moreover, the goal of a lesson is often communicated in terms of what pages or tasks to finish. Thus, another crucial aspect is to not take for granted the students' understanding of learning intentions and success-criteria, and maybe even to examine the students' perception of those intentions and criteria.

The differences between the informants reveal other crucial aspects. The teachers' different ways of using questions have different potential for the teacher to receive sufficient information about student learning needs to adjust their teaching and their feedback. Elvin and Tina also stood out by using exit-tickets at the end of their lesson. Regarding feedback, the use of feedback to inform whether the student's thinking is right or wrong can be relevant, but the other types of feedback that were used have higher potential for helping the students to move forward in their learning. The informants' use of feedback to initiate reflection on the solution process and Elvin's feedback answering students' questions, both reveal a crucial aspect about how to get the students involved in feedback interactions. This aspect also includes that the students can be involved in a high-quality way – exemplified by Elvin's student, who asks a specified question that both reveals his learning needs and gives the teacher the opportunity to give effective feedback.

All together, we identified six crucial aspects: Do the students function as resources for each other (i) and themselves (ii)? Does the learning goal surpass the goal of doing and how do the students perceive the learning goal and criteria of success (iii)? Is the information about student learning sufficient (iv), what feedback did actually support students' learning (v); and in what ways are the students involved in feedback interactions (vi)?. We expect to find additional crucial aspects, including aspects that concern the integrated use of the key strategies.

Implications

Implications for TE regard aspects are related to requirements to high-quality formative assessment practice. First, BMTs need to be aware of the importance of having information about students' learning needs and consequences of not having such information, as well as what potential is inherent (or not) in different feedback types (i.e. understand the purposes of and principles on which this classroom practice is based). Second, they need to know what to do when they experience not having enough information and how to decide whether their feedback was helpful and what to do if their feedback was unsuccessful in helping the student (i.e. have ability to incorporate this classroom practice into their teaching). Furthermore, the BMTs need to use a critical reflection on these aspects of quality relating to their understanding of formative assessment practice.

Limitation

In this study the informants participated in the same program at the same university. Possibly, the differences in classroom practice would have been even larger if the informants came from different programs and universities. Another concern is that we will not be able to identify all kind of activities related to formative assessment when only using video-recordings of lessons. We will only be able to identify direct observable activities and phenomena. Regarding both limitations, we believe that – building on the findings from the present study – future studies can find complementing crucial aspects that BMTs need to learn about and reflect on concerning their understanding and need of development. We argue that studying the classroom practice using "in practice data" is a good start.

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Evolution of teachers’ perception of Automatic Formative Assessment during a training course

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Formative assessment is a practice that provides ongoing feedback that instructors can adopt to improve their teaching and students can benefit from to improve their learning. Teachers and instructors need training, otherwise formative assessment and immediate feedback could also produce negative results. This paper aims at understanding the evolution of participants’ knowledge and perception of Automatic Formative Assessment throughout a teacher training course. The course was open to STEM teachers of all grades. The analysis considers teachers’ responses to questionnaires with both close-ended and open-ended questions. Results show the empowerment and more awareness of teachers’ application of Automatic Formative Assessment; in particular, teachers developed a more student-centered approach to formative assessment. This experience helped us list a set of recommendations and advice for those who approach Automatic Formative Assessment.

Keywords: Automatic assessment, formative assessment, mathematics teaching, teacher training.

Introduction

Several studies show that formative assessment (FA), when used following proper learning models, is correlated to higher students’ achievements (Black & Wiliam, 1998). Surprisingly, it has been shown that FA and feedback practice could also produce negative results, especially when they are not used appropriately (Kluger & DeNisi, 1996). Many studies point out that a high number of Mathematics teachers lack pedagogical and content knowledge about FA, so they have difficulties in implementing it successfully (Herman et al., 2015; McGatha et al., 2009). Therefore, teacher training is a key element for an effective use of FA in teachers’ daily practice, even more when integrating technologies into assessment. The recent health emergency has required a digitalization of the assessment practices, and this has been one of the most critical aspects of distance education (OECD, 2020). The urgency of re-skilling has been felt by many teachers in the last two years.

This paper deals with an in-service teacher training course on Automatic Formative Assessment (AFA) for STEM teachers, which was held online in Spring 2021. The course was aimed at developing competences in the use of an Automatic Assessment System (AAS) particularly suitable for STEM, and in the design of questions for the FA according to a particular pedagogical model. The goal of this paper is to understand how the teachers’ knowledge and perception of AFA evolved throughout the course. We investigate this issue through a qualitative and quantitative analysis of teachers’ answers to two questionnaires. In the following paragraphs, we will discuss the theoretical framework on AFA and teacher training in Mathematics; then, we will illustrate the structure of the course and the analysis methodology; afterwards, we will present and discuss the results gained and interpret them in light of the theoretical framework; in conclusion, we will provide some hints for the creation and management of activities with AFA, collected through the teachers' experiences.

Theoretical framework

Automatic Formative Assessment in Mathematics

For this study, we accept Black and Wiliam's definition and conceptualization of FA:

> Practice in a classroom is formative to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited. (Black & Wiliam, 2009, p.5)

This definition entails the collection of data about students' understanding, and the use of such data to change the learning path. Their conceptual framework includes five key strategies through which formative practices can be enacted by three agents (students, peers and teachers):

> (S1) clarifying and sharing learning intentions and criteria for success; (S2) engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding; (S3) providing feedback that moves learners forward; (S4) activating students as instructional resources; (S5) activating students as the owners of their own learning. (Black & Wiliam, 2009, p.4).

In previous work, we have defined AFA as the use of FA in a Digital Learning Environment through the automatic elaboration of students' answers and provision of feedback, where FA is intended as in the Black and Wiliam's definition (Barana et al., 2021). We have developed and experimented a model for the design of activities with AFA (Barana & Marchisio, 2019), relying on an AAS such as Möbius Assessment, whose engine is empowered by advanced mathematical capabilities. By exploiting programming languages or mathematical packages, similar AASs allow to build interactive tasks based on algorithms where answers, feedback and values are calculated over random parameters and can be shown with different representations. Thus, new solutions for computer-based items can be conceived, including dynamic explorations, animations, and symbolic manipulations, which offer students experiences of mathematical construction and conceptual understanding (Sangwin, 2015). According to our model, AFA activities should: (a) be always available in a Digital Learning Environment, without limitations in data, time and number of attempts; (b) be algorithm-based, so that random values, parameters, formulas and graphs make questions, and their answers change at every attempt; (c) be open-ended; the AAS's mathematical engine assures that open mathematical answers are graded independently of the form in which they are provided; (d) provide students with immediate feedback while they are still focused on the task; (e) provide students with interactive feedback just after giving an incorrect answer. It has the form of a step-by-step guided resolution that interactively shows a possible solving process; (f) be contextualized in real-life, thus contributing to the creation of meanings through the association of abstract concepts to concrete experience.

Teacher Training on Automatic Formative Assessment

In the literature, there are not many studies on teacher training on AFA. Roschelle et al. (2016) showed that combining the use of an AAS and teacher training on how to use assessment information to shape teaching helped improve learning results. The teacher training was mainly based on workshops and coaching. Haj-Yahya and Olsher (2020), in a study involving pre-service teachers, showed that training on FA and on the use of an AAS helped them improve their skills in observing, interpreting and responding to students' work. According to Schütze and colleagues (2017), it is often hard for teachers to put into practice the knowledge gained in teacher development programs. When dealing with teacher training on the use of new technologies, it is common to refer to the TPACK model (Voogt & McKenney, 2017), which aims at integrating technology, pedagogy, and content knowledge for improving teaching skills. Several studies have pointed out that including active learning and job-embedded practices is more effective for improving teaching (McGatha et al., 2009). Moreover, embedding the course in a practice community where teachers can share knowledge and experiences helps them not to feel like isolated learners, empowers their active participation, and boosts the results (Lave, 1991). This aspect is particularly relevant in online courses, especially during pandemic times. Establishing solid AFA practices in the daily teaching is not easy; often, it implies a revolution of one's habits. Supporting teachers in the application of what they have learnt during the course becomes crucial. We based the development of our training course on the following principles:

- providing teachers with technological knowledge and skills about the use and management of an AAS for Science and Mathematics;
- providing teachers with pedagogical knowledge and skills on formative assessment and on the design of tasks for AFA according to our previously cited model;
- embedding the training course in a virtual practice community, where teachers could share materials, experiences, idea, difficulties and find support in their colleagues;
- supporting teachers in the creation of their AFA activities and in their experimentation in the classroom, so that they can feel sustained in their first impact with the change of practices and encouraged to go on in the revolution of their teaching style;
- including teachers, after the training course, in a continuous training program and in a practice community with other expert teachers, so that they can be supported in their future activities.

Methods

Our research about AFA is mainly focused on the training course "Automatic formative assessment in STEM disciplines", delivered 15 hours in synchronous mode and 7 hours in asynchronous mode, where 15 school teachers, 11 of whom teaches Mathematics, learned how to use the AAS Möbius Assessment. Teachers also received methodological training for designing AFA activities according to the model we presented in the Theoretical Framework. In this course, participants had to carry out e-tivities to become confident with the digital environment and to actively enter the community, such as creating items and sharing them in a common database. At the beginning of the course, each teacher had to choose a topic on which to create activities and define learning objectives, and then they identified questions that made the achievement of the objectives stand out. Moreover, they designed feedback that allowed students to bridge the gap between current performance and desired performance and began implementing it with interactive feedback. The last and most important step of the course was to administer the activities they prepared all over the course to their students. The teachers received training and support from tutors, who presented examples of questions and activities and described how to monitor student progress in order to guide, review and modify teaching activities. Tutors actively helped teachers in the delivery of the experimental activities with their students, too. During the course, teachers were encouraged to share their experiences and difficulties through a forum monitored by the tutors, and to share their materials in a common database. After the course, teachers had the chance to enter the wider community of practice of the national “Problem Posing and Solving” Project, which has been adopting similar methodologies since 2012 (Fissore et al., 2020). The course was open to STEM teachers of all grades, from primary to upper secondary education. Also some teachers of humanistic subjects enrolled and followed the course. In total, 24 teachers enrolled in the course. However, we restricted the analyses only to the STEM teachers who reached completion in all the e-tivities and participated in at least 70% of the synchronous meetings.

Teachers had to respond to an initial and a final questionnaire, which represent the main source of data for our analysis. The initial questionnaire asked teachers their previous idea, perception, and usage of AFA, the application of certain methodologies with their classes and their adoption of digital tools. All teachers who attended the first meeting submitted the initial questionnaire. On the other hand, only the teachers who completed the course submitted the final questionnaire. From the final questionnaire we could gather their perception of AFA after the first implementation in the classroom and cast a glance at their future proposal to adopt what they had learned during the course with their students. The questionnaires consist in a mix of close-ended questions (mainly Likert scales) and open-ended questions, thus data collected are both qualitative and quantitative. Concerning quantitative data, we considered the main descriptive statistics, while for qualitative responses we classified the wide range of responses into a smaller number of categories, identified on the basis of the theoretical framework, in order to analyze them: this categorization depends on the researcher's interpretation of the respondents’ responses, thus we compared two independent evaluations.

Results

In the initial questionnaire we asked teachers to give their own definition of FA. We granted them the possibility to answer in an open format, which resulted in a lot of different answers regarding various aspects. The reference to the five key strategies of (Black & Wiliam, 2009) triggered our classification of teachers’ answers in this framework, since we were able to refer them to one or more strategies. All the strategies but one were referenced. S1 was referred in two answers, here we provide an example: “an essential and necessary tool, for both teachers and students, to learn and to communicate”, which highlighted the sharing capabilities of the methodology, and its learning-centered purpose, being communication central for making clear intentions and criteria. S2 was implied in 9 answers out of 15; here we report the most exemplifying: “to understand the learning of the students”, shedding light on the search for evidence of student understanding. S3 is related to two answers, one of them is: “the act of communicating to the student if, and how much, they have acquired the important competences in the context of the course, and how they can improve in the future”; this communication can be regarded by all means as feedback, which allows the learner to move forward. Finally, S5 is mentioned in only one answer: “it has to allow students, in case of errors, to understand and remediate them”: they are invited to master their learning. No answer referenced S4, probably because to be an instructional resource for one another is a process between peers, and therefore probably not the first one a teacher thinks about, albeit the importance of peer instruction and evaluation mediated by teachers is undisputed. Note that some of the answers cannot be properly considered as definitions, regardless of being in accordance with the literature or not: for example, the answer we put in relation with S5 is indeed a feature rather than a true description of the methodology. In three very short answers we could not find any reference to the 5 strategies. The strategy that had been referred to more frequently was, with a large gap, S2: this outcome highlighted how most of the teachers before our course saw FA as a tool centered on themselves, rather than on the students. We also compared the teachers’ answers with the definition of FA (from Black and William, 2009), which considers data collection and their use to improve teaching and learning. Data collection was referred to 10 times, their use to improve teaching and learning 5 times. Teachers also considered the fact that assessment should occur during the course, 6 times explicitly, plus some other answers where it is implied (for example, where FA is described as “a method for detecting students’ knowledge and competences”). We also looked for references to the agents included in the teachers’ definitions (teachers, students, or peers). No answer referred to peers, consistently with the absence of S4; while two answers referenced both teachers and students, the former ones appeared 10 times, while the latter ones only 4. Note that teachers largely included considerations about students’ learning, but mostly without giving them an active role. This is in line with the absence of S4 and confirms a teacher-centric approach to FA before attending the course.

Let us now consider the evolution of the answers relative to the actual use of FA by teachers. Before our course, several of them already made use of it, at least in some of its traits, but it was only after our course that most of them really became aware of its potential. Indeed, before the course, only 2 teachers declared using automatic assessment for FA regularly (most of them, only seldom), while 7 used it for summative purposes with regularity. For example, a teacher stated that she had used formative assessment by assigning students various activities, such as questionnaires, tests, exercises, with a successive discussion on aspects like the difficulties they had faced, the errors they committed, the solving processes. After the course, she recognized that she had acquired more cognizance relatively to the importance of FA, by specifically referring to real time interactive feedback as a tool for giving key support. Another teacher used grids with indicators. However, letting students become aware of how the assessment works or giving them parallels between the evaluation and the formative goals (essentially, S1), is only a first step towards a true obtainment of the strategies’ outcomes. Indeed, answering after our course, she specifically referred to the creation of interactive feedback, with hints also aimed at helping disadvantaged students, in order to let each of them focus on understanding the solving process, rather than on the mere result of the problem. In this context, the use of variable data, thanks to the algorithmic capabilities of the AAS, was recognized as further enhancing the formative scope of a question. A teacher proficiently used automatic means with gamification features, which could be useful to engage students and to foster their interest in using more specific tools for formative assessment, as he noted. Finally, it is noteworthy to take into account one answer before the course, which considered tests, homework checks, actively involving students during lessons, and classroom simulations, in which they were allowed to help one another. The latter feature is important because it references S4, which had been excluded from the answers relative to the direct question about how to define formative assessment. Unfortunately, the teacher giving this answer followed our course only partially, and she did not respond to the final questionnaire, so we are not able to track possible progresses in that direction. These qualitative considerations can be integrated with some quantitative data. In the two questionnaires, we asked teachers some specific questions involving the strategies, in terms of attention devoted, perceived importance, and application capabilities with formative assessment. Specifically, before the course, we asked them how much, in their didactic activity, they paid attention to reaching the goals explicated by the strategies; note that the teachers were generally unaware, at the time, of the relation between the concept of FA and the strategies, as shown by several definitions they proposed. In a Likert scale from 1 to 5, where 1 means "nothing at all" and 5 means "very much", the averages are depicted in row (b1) of Table 1.

Table 1: Average scores of questions relative to strategies, before and after the course

Before (b1, b2) and after (a1)	S1	S2	S3	S4	S5
(b1) Attention paid to strategies and goals	3.5	3.7	3.7	2.9	3.5
(b2) Perceived importance of strategies	3.9	4.1	4.3	3.9	4.5
(a1) Help in applying strategies from AFA	3.5	3.9	4.2	3.1	4.1

It emerges again how, while the other strategies were generally present in the teachers' practices, with an average considerably higher than the midpoint 3, S4 underwent a weaker trend. A similar question regarded again the strategies, but in terms of how much teachers perceived them as important for learning. Two aspects emerged from the row (b2) of Table 1: the first one is the higher marks, likely because it is easier to think about a strategy than actually apply it; the second one is how S4 is still the weak link, although the differences between the other ones are less marked. Finally, after the course, we asked teachers how much they thought AFA would be able to help them in applying the strategies. It can be noted that the averages of row (a1) in Table 1, in comparison with those relative to the attention paid before the course, namely row (b1), increased in the score for all strategies except S1 ranging from +0.2 to +0.6. It can mean, as a general remark from the scores, that teachers recognized the importance of these strategies and of their goals and outcomes throughout the course.

We now give an example, in Figure 1, of a question designed by a teacher towards the end of our course. It has been proposed as an algorithmic question, which means that random parameters allow the exercise to be repeated with different values, and with interactive feedback in case of a wrong answer. The coefficients are random integers in a proper range, generated every time the question is accessed. If the correct answer to the first section is given at the first or at the second attempt, the exercise ends, and the student obtains a full score. If none of the two attempts results in the correct answer, as in Figure 1, subsequent sections open, guiding the student towards the solution by asking questions relative to intermediate steps, with the partial score that can be seen as an indicator of the student still needing help and refinement of their preparation (see the strategies). Note that in formative assessment scores are not as central as in summative assessment, but they can be of some help anyway along the other feedback, for example in moving students forward and letting them aware of their own learning, that are S3 and S5.

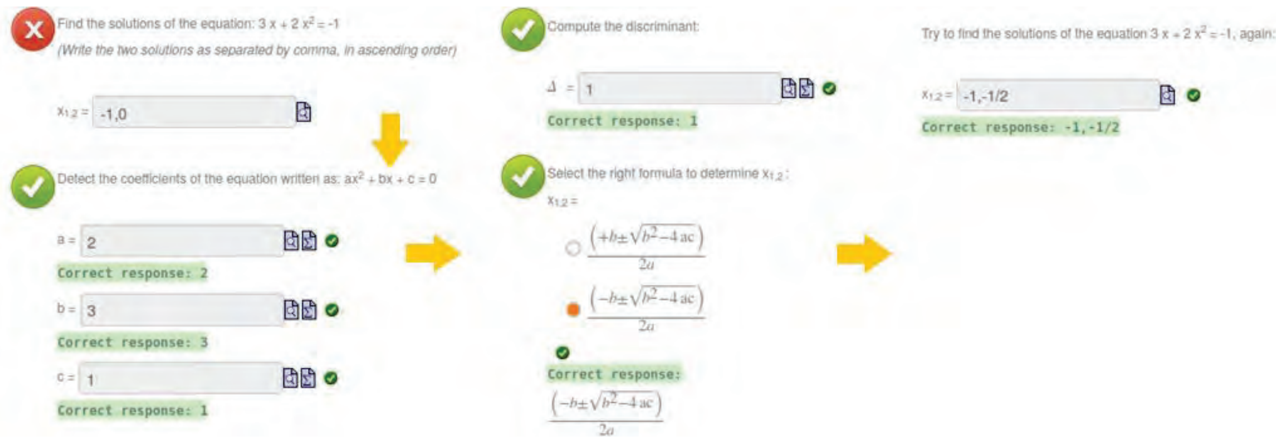

Figure 1: A question designed by a teacher (*sections up to down, then left to right*)

Discussion and conclusions

AFA is a relatively new area of research and there is a need for teacher professional development on the use of it, as well as of research about the impact of training on teachers' practices with AFA. This study presents some limitations; first of all, the low number of teachers involved, which hinders generalization of the results. However, thanks to this study, we were able to observe the evolution of the teachers' knowledge and perception of AFA during a training course. If at the beginning of the course teachers tended to use automatic assessment as distinct from FA, throughout the course they learned to integrate the two methodologies, taking advantage of the potential of the digital technologies to empower the formative practices. We also observed that teachers mostly had a teacher-centered perspective of FA at the beginning of the course. They were used to paying more attention to proposing task in order to collect evidence of students' understanding rather than activating students in covering the gap between actual and desired performance. Their answers could suggest that the course may have given them the necessary tools to give students ownership of their learning during assessment. This resulted both from their answers to the final questionnaire, and from the increase in the scores given to the item relative to S5 after the course, which is not extremely strong (+0.6), but is the highest one. According to the teachers, the provision of feedback to move the learner forward is particularly enhanced by AFA, when implemented according to our model. The training modalities helped teachers to gain experience in the use of AFA, encouraging them to carry on this path. They perceived the usefulness of the course especially in a context of hybrid or face-to-face teaching. Thus, they participated actively and enthusiastically, despite the online training modality, also because the trainers tried to stimulate confrontation and discussion during the synchronous meetings.

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Designing for a combined use of a dynamic mathematics software environment and a computer-aided assessment system

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This paper reports on a pilot study with the focus on (re)design of a digitized task environment utilizing two types of technology – a dynamic mathematics software and a computer-aided assessment system. The data consist of responses from 256 first year engineering students, taking their first Calculus course, on two different types of task. The results are discussed in relation to (re)design of tasks as well as possible feedback design options to enable a formative assessment approach.

Keywords: Asymptotes, dynamic mathematics software, computer-aided assessment, task design, feedback.

Introduction

Central in introductory Calculus courses – a critical course for first year engineering students – is the concept of functions. For example, Oehrtman et al. (2008) highlight a weak function understanding as one of the reasons why many students fail in their first undergraduate mathematics courses. Oehrtman et al. (2008), in discussing important features of students' function understanding, advocate a focus on promoting "…rich conceptions and powerful reasoning abilities…" (p. 27), and not merely "…symbolic manipulations and procedural techniques…" (p.28).

As one way to foster students' conceptual understanding in mathematics, the literature suggest utilizing dynamic mathematics software (DMS) environments for student collaboration on inquiry-based tasks, particularly in relation to functions (e.g. Brunström & Fahlgren, 2015; Jaworski & Matthews, 2011). Today many mathematics courses in higher education utilize computer-aided assessment (CAA) systems (e.g. Rønning, 2017), in which it is possible to embed DMS environments (Sangwin, 2013). However, there are few studies that have investigated the integration of these two types of technology (Luz & Yerushalmy, 2019). Indeed, designing tasks that utilize the affordances provided by a DMS environment and that also can be automatically assessed by a CAA system adds to the already established complexity of designing tasks for interactive learning environments that promote mathematical understanding (Joubert, 2017).

This paper reports some results from a pilot study conducted during autumn 2020 with the aim of trialling different types of task designed for a combined use of DMS and CAA, and to get a deeper understanding of student strategies when performing these tasks. Findings from the pilot study will inform the (re)design of tasks as well as the development of possible types of automated feedback to increase first year engineering students' engagement and conceptual understanding of functions.

Theoretical framing

Besides topic-specific theories, related to mathematical functions, the task design was guided by theories associated with student generated examples and theories on feedback.

Functions and graphs

It is well established that for students to comprehend the concept of functions it is essential to be able to move flexibly between different representations, such as formula and graph (Leinhardt et al., 1990). Moreover, according to Duval (1999), representation and visualization are at the core of mathematical understanding. He distinguishes between two cognitive operations; processing and conversion, where the former concerns mathematical processes made within the same representation, such as algebraic manipulations, and the latter means changing between different representations. While many students can learn processing, it is the conversion, e.g. to translate a formula into a graph or vice versa, that many students find challenging (Duval, 1999). In particular, it is the translation from graph to formula that is the most challenging (Leinhardt et al., 1990). Furthermore, this kind of task, referred to as ‘translation task’ in this paper, is suitable for a CAA system since it can recognize any correct form of the function formula.

Student generated examples (SGEs)

Prompting students to generate examples that fulfil certain conditions has been proposed as a way to engage students actively in their development of conceptual mathematical understanding (e.g. Watson & Mason, 2002). This idea has been adapted to CAA systems since it allows for automatic assessment of higher-order mathematical skills (Sangwin, 2003). In this paper we use the notion ‘SGE task’ when referring to tasks using this idea. To further challenge students’ thinking Yerushalmy et al. (2017) suggest asking students for several examples, which differ as much as possible. Moreover, Yerushalmy et al. emphasize the importance of designing feedback on students’ responses on such tasks to support their mathematical reasoning processes (2017).

Feedback

So far, CAA systems have mainly been used for assessing basic mathematical procedural skills. It is a challenge to design tasks for a CAA system that address higher-order skills in mathematics, and to design feedback that goes beyond categorizing a final answer as being right or wrong (Rønning, 2017). In the wider literature, this type of feedback is referred to as ‘elaborated feedback’ (e.g. Shute, 2008). In a meta study investigating the effects of computer-based feedback on students’ learning outcomes, van der Kleij et al. (2015) found that elaborated feedback was more effective than verificative types of feedback, especially for higher-order skills. In particular, the eight studies related to mathematics pointed in this direction.

When interacting with a DMS environment, the other type of technology reported in this paper, students are provided instant feedback on their action. It is this feedback that makes it possible to use a DMS environment as an arena for exploration, conjecturing, verification, and reflection. However, this feedback does not provide explicit suggestions on how to proceed, and thus the benefit of the feedback depends on students themselves being able to interpret the results of their actions in the DMS environment (Olsson, 2018). By embedding DMS tasks in a CAA system and utilizing the affordances provided by the two types of technology, we endeavor to enhance the provision of feedback – the goal of our upcoming project.

Method

Research context

The pilot study took place at a Swedish university, involving 256 first year engineering students taking a first course in Calculus. The course assignment included small group activities, in the form of task sequences focusing on function understanding, designed for a combined use of a DMS environment (*GeoGebra*) and a CAA system (*Möbius*). However, there were also tasks with individual elements for each group member requiring an individual answer, to ensure active involvement by all students.

This paper will examine patterns of student response to two related tasks concerning rational functions, and specifically the relationship between asymptotes and function formula. In the light of these patterns, we will offer thoughts on the types of elaborated feedback that could be beneficial, and also provided by an automated assessment system.

The tasks

Task 5 (see Figure 1) is an example of a 'translation task' intended to be solved in groups. Students are expected to realize that it must be a rational function with one horizontal and two vertical asymptotes. Then, they are supposed to utilize the vertical asymptotes to construct the (factorized) denominator, and the horizontal asymptote to conclude that the numerator should be of degree two with the coefficient 2 in front of the x^2 term. However, there is a need for further information to arrive at a final formula, i.e. two points on the graph. The reason behind asking for an explanation (Task 5b) was twofold; both to promote student reasoning, and to provide insight into student strategies when solving the task.

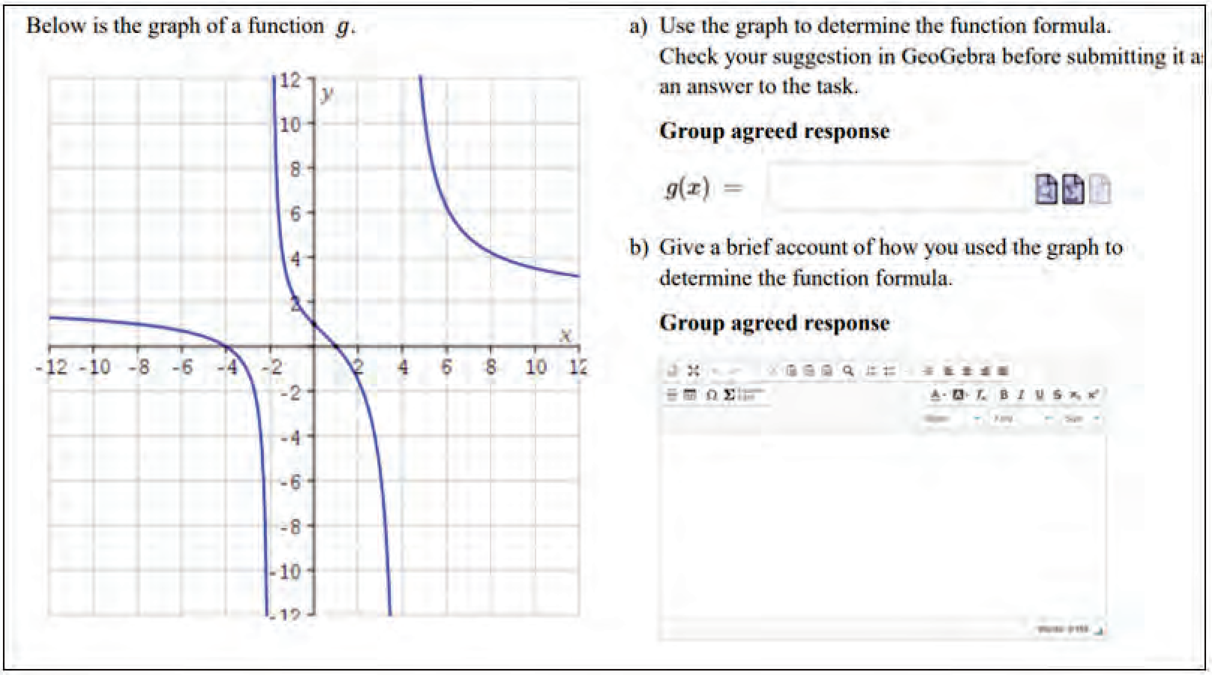

Figure 1: Task 5 as it is presented in Möbius

Task 7 (see Figure 2) is an example of a 'SGE task' in which students received different values of the asymptotes, and were supposed to provide individual answers. This task closely relates to Task 5 in that it involves two vertical asymptotes and one horizontal asymptote. In performing this task, students are supposed to consolidate the key ideas addressed in these tasks.

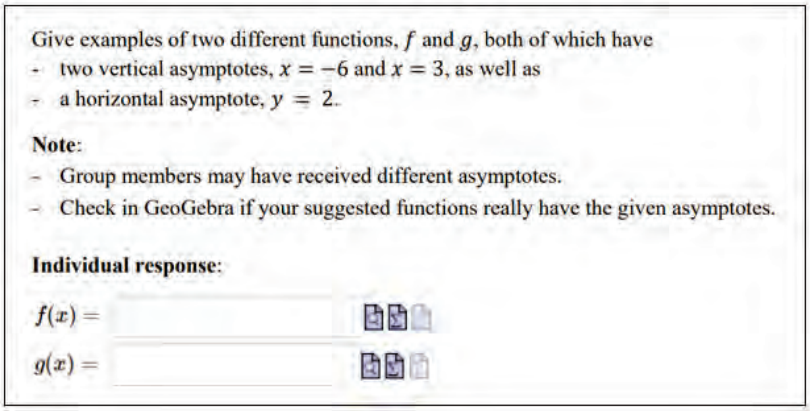

Figure 2: Task 7 as it is presented in Möbius

Data collection and analysis

The data used to analyse student responses to these tasks consists of their answers submitted to the CAA system. The data analysis process involved several stages. To provide an overview of the data material, we made a preliminary analysis based on about a quarter of the submitted responses. This overview of students' various responses to the tasks gave insights into interesting instances. For example, when students provided unexpected responses, i.e. not in line with the expected solution strategy, or when it was a wide range of student responses. In addition, this preliminary analysis generated initial codes to be further developed and used in the next stage of the analysis process. At this stage, all responses to the tasks were analysed and coded. Next, the initial codes were organized into categories to discern general patterns in the data material (Saldaña, 2013). The categories obtained for these tasks are introduced in the tables in the Result Section.

Results

Table 1 provides an overview of the group responses, in terms of function formulas, to Task 5a.

Table 1: Overview of the responses provided in Task 5a[1]

Code	Description	Function formula	Frequence
Formula 1	Single quotient	$g(x) = \frac{2(x+4)(x-1)}{(x+2)(x-4)}$	47 (46,5%)
Formula 2	Partial fraction, reduced quotients, and the constant term 2 (i.e. the horizontal asymptote)	$g(x) = \frac{2}{x+2} + \frac{8}{x-4} + 2$	17 (16,8%)
Formula 3	Reduced quotient, and the constant term 2 (i.e. the horizontal asymptote)	$g(x) = \frac{10x+8}{(x+2)(x-4)} + 2$	22 (21,8%)
Other	Not categorized		9 (8,9%)
No answer			6 (5,9%)
Total			101 (100%)

[1] In some responses, the numerator and/or the denominator is/are not factorised.

Mainly three types of formula were observed. Almost half (47/101) of the groups used Formula 1. Notable is that as many as 39 of the groups gave a Formula 2 (17) or a Formula 3 (22) response. These types of formula have not been present either in the textbook or in lectures when treating asymptotes in the course. Examples presented at lectures concerning horizontal asymptotes has been in the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions of the same degree (except when the horizontal asymptote is $y = 0$). Among the 9 responses not categorized, there are 4 incorrect answers.

The group responses to Task 5b provide some information about students' thinking behind their answer in Task 5a, since they were encouraged to explain how they arrived at a particular function formula. Group responses were inspected and compared to identify a set of elements of explanation which could be used to summarise the content of any response. This made it possible to code the responses in terms of explanation elements, i.e. descriptions of what students referred to.

In Table 2, an overview of the explanation elements provided by the student groups is given. The rightmost column shows the total number of each explanation element among the group responses. Almost all groups (82/86) explicitly refer to the vertical asymptotes in their explanation. However, all the categorized formulas (1 to 3) indicate that all 86 groups, that provided a function formula, utilized the vertical asymptotes. Similarly, the constant term in Formula 2 ($g(x) = \frac{2}{x+2} + \frac{8}{x-4} + 2$) and Formula 3 ($g(x) = \frac{10x+8}{(x+2)(x-4)} + 2$) indicate that these groups have utilized the horizontal asymptote, even though eight of these (39) groups did not mention this in their explanation.

Table 2: Overview of the explanation elements referred to in Task 5b

Type of formula / Explanation elements	Formula 1	Formula 2	Formula 3	Total
Vertical asymptotes	46 (97,9%)	16 (94,1%)	20 (90,9%)	82 (95,3%)
Horizontal asymptote	16 (34,0%)	16 (94,1%)	15 (68,2%)	47 (54,7%)
Zeros	44 (93,6%)	0	0	44 (51,2%)
One further point	22 (46,8%)	3 (17,6%)	0	25 (44,6%)
GeoGebra	5 (10,6%)	3 (17,6%)	6 (27,3%)	14 (16,3%)
Equation system	1 (2,1%)	7 (41,2%)	10 (45,5%)	18 (20,9%)
Total	47	17	22	

A closer look at the Formula 1 responses reveals that the predominant (46/47) characteristic of the associated explanations was to refer to the vertical asymptotes (to form the denominator of a single quotient expression as the product of the corresponding linear factors). The predominant (44/47) approach was then to refer to the zeros of the function (to form a numerator for the quotient expression as the product of the corresponding linear factors). One further step is needed to complete the quotient expression. Only 16 groups (out of 47) used the approach that the task design hoped to elicit, i.e. to use the horizontal asymptote to establish a limiting value for the quotient expression as $x \to \infty$. Thus, the remaining 31 groups did not refer to the horizontal asymptote as an explanation (see Table 2) for the factor 2 (of the numerator). Among these 31 groups, 22 refer to one further point, 5 to GeoGebra, and one group to a system of equations. Hence, we can conclude that at least 28 groups (out of 47) did not utilize the horizontal asymptote.

In Task 7, each student response consists of two examples of function formulas. However, since few students (6/256) provided a different type of formula in their second example, only the first example is reported in this paper. Table 3 provides an overview of the types of formula discerned (based on the numerical values in the example in Figure 2) as well as the total number of responses belonging to each category (the rightmost column). Moreover, Table 3 shows the correspondence between these answers and the group answer on Task 5a.

Table 3: Overview of each individual response to Task 7 in relation to their group answer in Task 5a[2]

Formula Task 5a / Formula Task 7	Formula 1	Formula 2	Formula 3	Not categorized	No answer	Total
$f(x) = \frac{2x^2 + ax + b}{(x+6)(x-3)}$	26 (20,8%)	2 (4,3%)	3 (5,4%)	1 (5,6%)	5 (50,0%)	37 (14,5%)
$f(x) = \frac{a}{x+6} + \frac{b}{x-3} + 2$	16 (13,1%)	30 (63,8%)	8 (14,3%)	1 (5,6%)	2 (20,0%)	57 (22,3%)
$f(x) = \frac{ax+b}{(x+6)(x-3)} + 2$	75 (61,5%)	14 (29,8%)	43 (76,8%)	15 (83,3%)	2 (20,0%)	149 (58,2%)
$f(x) = \frac{x^2 + ax + b}{(x+6)(x-3)} + 1$	4 (3,3%)	1 (2,1%)	0	0	0	5 (2,0%)
No answer	4 (3,3%)	0	2 (3,6%)	1 (5,6%)	1 (10,0%)	8 (3,1%)
Total	125	47	56	18	10	256 (100%)

The predominant (149/256) type of formula used was a reduced quotient and the horizontal asymptote as a constant term ($f(x) = \frac{ax+b}{(x+6)(x-3)} + 2$). Notably, quite a few students (37/256) expressed the formula as a single quotient, i.e. in the form $\frac{p(x)}{q(x)}$.

As there were no zeros given in Task 7, it is primarily the group answers Formula 2 and Formula 3 in Task 5a that closely relate to the answers in Task 7; Formula 2 ($f(x) = \frac{2}{x+2} + \frac{8}{x-4} + 2$) corresponds to the second type of formula (in the leftmost column) and Formula 3 ($f(x) = \frac{10x+8}{(x+2)(x-4)} + 2$) corresponds to the third type of formula. Most of these students use the same type of formula in their individual response, 63,8% and 76,8% respectively, as in their group answer. However, Table 3 shows that as many as 29,8% of the students who used Formula 2 in Task 5a switched to the third type of formula ($f(x) = \frac{ax+b}{(x+6)(x-3)} + 2$) in Task 7. Moreover, few students that responded Formula 2 (4,3%) or 3 (5,4%) in Task 5a provided an answer in the form $\frac{p(x)}{q(x)}$ in Task 7. Overall, Task 7 worked

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[2] In the student responses, different numerical values are used instead of *a* and *b*.

well in that most of the students seemed to realize how they could use all asymptotes to produce a function formula.

Discussion

As stated in the introduction, this pilot study will inform the (re)design of tasks as well as the development of possible types of automated feedback. In this section, we elaborate on this issue in relation to the findings.

Since the intention with the ‘translation task’ (Task 5) was to encourage students to reflect on the relation between a function graph with asymptotes and its formula, the high number of Formula 1 responses (to Task 5a) without reference to the horizontal asymptote in the explanation was unexpected and undesirable. One way to tackle this issue might be to use a graph without evident zeros. However, the possibility to use different approaches based on various graph features may promote instructive student discussions. Another way could be to indicate the asymptotes in the graph. However, since the identification of asymptotic behavior in a graph is central in understanding rational functions, this might simplify the task too much.

Yet another way to tackle this issue is to develop automated and adapted feedback, which in turn require a redesign of Task 5b. Instead of asking for an explanation, ask students to declare the explanation elements used by choosing among various suggested options. Depending on their response, they will receive different elaborated feedback (Shute, 2008). For example, if they not have used the horizontal asymptote, they will be asked to solve a new task in which they are (explicitly) asked to utilize the horizontal asymptote.

Also, when considering that almost 10% of the student groups failed to provide a correct answer to Task 5a, there is a need for developing formative instant feedback for these students. One suggestion is to offer the students a second chance to solve the task after they have been watching a short video introducing the key ideas addressed in the task.

The ‘SGE task’ (Task 7) worked well, and revealed various student strategies. However, almost all students provided the same type of formula in both their examples. Since we think that it is instructive for students to realize that there are various ways of thinking, which results in different types of formula, it would have been great if the CAA system could recognize the type of formula used by a student. So, for example, if a student uses a formula of the following type: $f(x) = \frac{ax+b}{(x+6)(x-3)} + 2$ (in both examples), the elaborated feedback (Shute, 2008) could be something like: “Great, the answers are correct. However, another correct answer could be: $f(x) = \frac{2x^2}{(x+6)(x-3)}$. How do you think a student who came up with this answer has been reasoning? Now, use this strategy to provide an example of a function with the following asymptotes…”

The study is limited by the lack of information on how students utilized the DMS environment to check their conjectured function formulas before submitting them as answers to the tasks. To be able to further develop the task design, particularly the formative feedback from the CAA system, we need to better understand the reasoning behind students’ various responses. This requires empirical data in terms of screen recordings (including audio) from students ‘ongoing work. Consequently, we suggest this as a natural progression of this pilot study.

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Assessment of/for/as online learning: Mathematics teachers' views on online assessment during the COVID-19 pandemic

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Assessment is an important aspect of mathematics teaching and learning. This paper aims to explore Hong Kong mathematics teachers' views on online assessment during the COVID-19 pandemic. The data analysed is part of a larger project comprising an online survey and semi-structured interviews. This paper presents data from the survey responses of ninety-two teachers and from interviews of three teachers. Thematic data analysis is conducted, focusing on teachers' views on assessment during online learning. Findings show that the forms of assessment that Hong Kong mathematics teachers conducted during the pandemic were similar to those from before the pandemic. They view online assessment mainly as assessment for and of online learning. This paper also discusses suggestions for online mathematics assessment.

Keywords: Online teaching, assessment, mathematics teachers' views.

Background

Assessment plays a crucial role in supporting teaching and learning at school (Black & Wiliam, 1998, 2011). Harlen (2007) denoted assessment as "the process of gathering, interpreting and using evidence to make judgments about students' achievements in education" (p. 11). The students' achievements include their knowledge, attitudes, or skills. This process may occur inside or outside the classroom. Assessments can be formal (e.g., an examination or test) or informal (e.g., a conversation between teacher and student), external or internal. Moreover, according to their different functions, assessments can be categorised as summative or formative. Summative assessments refer to those that measure what students have already learned. Teachers may design them for several purposes, such as grading and selecting students, or evaluating experiences regarding a teaching unit. In contrast, formative assessment primarily focuses on informing learning and teaching by providing feedback to students and teachers (Curriculum Development Council [CDC], 2017; Hodge & van den Heuvel-Panhuizen, 2014). Regardless of the forms of assessment, teachers play an important role in the process of assessment (Black & Wiliam, 2011). During the COVID-19 pandemic, all the learning activities are conducted by online mode, including the assessments. Teachers' role became even more crucial during the pandemic. How to conduct online assessment may become challenges for many teachers.

Although assessment is an integral part of curriculum planning, pedagogy, and assessment cycles, there often exists "a major gap between the nature of the planned and implemented curriculum" (Morris, 1995, p. 43). For example, in Hong Kong, the formal curriculum stresses the formative role of assessment by teachers; however, when the curriculum is implemented, internal assessments are frequently based on those used in public examinations (Morris, 1995). During the online teaching, it is necessary to know whether such gaps still exist and how they have been changed. Online assessment refers to any form of assessment conducted during or after online learning, for example, online quizzes, online submissions of homework, online question sessions, and other online interactions. Although some teachers have previously used e-learning tools (such as Kahoot!, GeoGebra, and Google Forms) to teach in traditional classrooms, conducting online assessment is a new challenge for most. Teachers' views on assessment can influence their teaching practices, especially in a new context, such as using online assessment during online teaching (Kaiser et al., 2017; Mirian & Zulnaidi, 2020). By using e-learning methods, teachers can give rapid and accurate feedback to students, which contributes positively to students' motivation for learning (Becta, 2003). However, many barriers and challenges also exist for teachers when conducting online teaching (Jones, 2004; Dhawan, 2020). For example, students in online courses may also feel isolated and disconnected from the instructor and other learners (Choy et al., 2002). The lack of student-teacher interaction in online classes is another primary concern (Mayes, 2011).

Because of the COVID-19 pandemic, teaching online has become a new normal approach worldwide. Given the significance of online assessment in mathematics classrooms, there is a need to examine how technology influences the forms of assessment teachers conduct and their experiences during online teaching. Hence, this paper aims to explore mathematics teachers' views on online assessment during the pandemic. Specifically, this paper investigates four research questions:

1. What kinds of assessment did mathematics teachers use during online learning?
2. What did teachers think of conducting online assessment during online learning?
3. What types of constraints did teachers face in conducting assessment during online learning?
4. How did teachers deal with the constraints on online assessment?

Assessment of/for/as learning in mathematics

Assessment in mathematics can be both summative and formative. Summative assessment focuses more on providing a comprehensive and summary description of students' performance and progress in learning. On the other hand, formative assessment focuses more on diagnosing students' strengths and weaknesses in learning, providing feedback, and reviewing learning and teaching strategies (CDC, 2017). In general, assessment can serve the following three purposes: assessment *of* learning, assessment *for* learning, and assessment *as* learning.

Of these three purposes, assessment *of* learning aims to provide learners with results about what they have achieved in their learning within a certain period (Harlen, 2007). Assessment methods for this purpose include testing at the end of a class session, as well as formal examinations in the middle or at the end of the academic year. Assessment can sometimes also take the form of homework, whereby the teacher gives grades to learners. Next, according to Mok (2011) and Harlen (2007), assessment *for* learning uses assessment methods to provide learners with feedback to assist their future learning. Such methods focus on monitoring and providing feedback to learners during mathematics classes. They can take the form of question-and-answer (Q&A) sessions or classwork. Finally, assessment as learning uses assessment methods to develop self-monitoring (Earl, 2013) and self-directed learning (Mok, 2011) among learners. For such assessments, learners assume an active role in their learning process, unlike for the other two purposes, for which the teacher initiates the learning.

In existing literatures, the above concepts mainly apply to how teachers conduct assessment in physical learning contexts (see Black & Wiliam, 2011; Earl, 2013; Harlen, 2007; Mok, 2011). Conceptual frameworks for analysing teachers' views on assessment during online teaching and learning remain limited. Hence, we are proposing assessment of/for/as online learning as a framework to analyse teachers' views on online assessment during the pandemic. In this paper, online assessments are not limited to strictly online forms, such as live Q&A sessions or online quizzes during online learning, but also include forms that involve a physical component, such as answering questions from textbooks or workbooks to be submitted after online learning.

Methodology

This study examines data extracted from a large comparative study between Hong Kong and Italy, designed as an exploratory study to investigate mathematics teachers' views on online teaching and learning. Data presented in this paper is original and not have been published previously. The study comprised an online survey and semi-structured interviews following the survey.

Participants

Participants were recruited by random sampling through emails and social media. A survey link was distributed online through social media. For this study, assessment-related responses from 92 Hong Kong mathematics teachers were selected, as shown in Table 1. Three interview cases in which "assessment" was mentioned were also selected to support the findings from the survey.

Table 1: Background of selected teacher participants

School level	Teaching experience (years)				Total
	<5	5–10	11–15	>15	
Primary	9	7	5	18	39
Secondary	31	10	4	8	53
Total	40	17	9	26	92

Instrument

A team of researchers from Hong Kong and Italy designed the online survey instrument, which comprised 13 items. Four items captured background information, while the other nine open-ended questions related to teachers' views on online teaching and learning. For instance, item 9 read: "During distance teaching, do you think that your way of teaching mathematics somehow changed? Please explain." Teacher participants were required to state whether any changes in their teaching practices occurred during the pandemic and to provide explanations for them. This paper presents the data extracted from the nine open-ended questions, comprising responses related to assessment during the pandemic.

Data analysis

Data were imported into the analytical software NVivo 12 Pro to conduct thematic analyses on both survey and interview data. For both sets of data, the original interview transcripts in Chinese were used to prevent important information from being lost during translation. Translations are only done for data presented in the results.

Themes related to assessment, such as online assignments, classwork, homework, tests, and evaluations, were first identified from the survey data. Then, similar themes were combined and merged into four major assessment methods: classwork or homework, live Q&A, tests or quizzes, and evaluations. Meanwhile, data related to constraints on and unsuccessful implementations of assessment were coded under the node constraints. Thereafter, based on the proposed framework regarding views on assessment, the data was further coded according to the purposes of "assessment *for* online learning", "assessment *of* online learning", and "assessment *as* online learning". The numbers and percentages of teachers in each coding category were then recorded and calculated. For the interview data, themes that expressed positive and negative views were grouped separately to provide more information about teachers' views.

Results

Online survey

There were 92 teachers who mentioned "assessment" during distance learning in online survey. Table 2 shows that teachers used four main types of assessment during distance learning and the constraints related to the forms of assessment used. 42.39% of teachers used classwork or homework to assess student's learning. For example, one teacher mentioned, "[I] gave students assignments through Google Classroom, marked them and returned them to students." About 6.52% of teachers mentioned they posed questions during online teaching sessions to assess students' understanding. For example, another teacher responded that "[I] need to ask questions frequently to know their learning progress." 16.30% of teachers revealed they had conducted online tests or quizzes to assess their students' understanding, using online platforms such as Kahoot! and Quizizz. A third teacher mentioned that they dealt with distance learning by using "some online interactive platforms, for example, Kahoot! and Nearpod are used to get feedback from students." Lastly, 11.96% of teachers mentioned that they used assessment for general evaluation, without specifying in which form, such as "[I] used Google Meet to follow up student's learning progress too."

Table 2: Forms of assessment used and constraints faced by Hong Kong mathematics teachers during the pandemic

	Classwork/homework (%)	Live Q&A (%)	Test/quiz (%)	General evaluation (%)
Forms of assessment	42.39	6.52	16.30	11.96
Constraints	23.91	7.61	10.87	44.57

There was 44.57% of the 92 teachers disclosed their unsuccessfulness or difficulties in conducting assessment during distance learning for general evaluation (Table 2). As one teacher mentioned, "[it was] more difficult to assess student's level of understanding and did not manage to conduct assessment for learning effectively." 23.91% of teachers reported constraints in assigning, monitoring, marking, or collecting classwork or homework. For example, they were unsuccessful in "monitoring students' progress in doing classwork". Meanwhile, 10.87% of teachers said they struggled to conduct a test to assess their students' learning progress. For example, another teacher said that "a test could not be held". Lastly, 7.61% of teachers reported difficulties in conducting assessments during online live teaching; a teacher stated, "[I] did not manage to ask different types of questions to students to assess their learning progress".

There were 61 teachers who provided explicit descriptions about their views on online assessment, as shown in Table 3. Of these 61 respondents, 39.34% viewed assessment as being *for* learning in general. Most of these respondents gave general answers about assessment purposes, such as monitoring students' progress. 37.70% of teachers view assessment as being *for* online learning through classwork or homework. For example, one teacher mentioned, "[I] used Google Forms for homework" to deal with online teaching and learning during the pandemic, reflecting the purpose of assessment *for* online learning. Lastly, 24.59% of teachers viewed assessment as being *of* online learning through tests or quizzes. As reported in their survey responses, they manage to "[conduct] evaluation of learning progress [by using] Quizizz App", reflecting the purpose of assessment *of* online learning.

Table 3: Teachers' views on online assessment during the pandemic

Views on assessment	Classwork/ homework (%)	Live Q&A (%)	Test /quiz (%)	Evaluation (%)
Assessment for online learning	37.70	16.39	9.84	39.34
Assessment of online learning	6.56	0	24.59	4.92
Assessment as online learning	11.48	1.64	1.64	1.64

Online interviews

Data analysis of the online teacher interviews revealed that overall, Hong Kong mathematics teachers had differing views about the conduct and outcome of assessment. This section describes three cases selected from the interview data to represent teachers with negative, positive, and mixed views, respectively. The negative views recorded such as assessment being difficult, problematic, and ineffective, positive views included technology tools have made the assessment easier and mixed views with both positive and negative aspects.

Janet, a secondary mathematics teacher with less than five years of teaching experience, hold a negative view about online assessment. She thought online assessment was difficult to be conducted. She changed from grading students' homework using Google Classroom to assigning homework and conducting live teaching through online platforms such as Zoom and Kahoot! to assess students' learning. She commented:

> It was very difficult to assess [students' learning]. This was because they may not complete their homework alone. [Then], last time in school, a short test could be conducted [during the lesson]. [But] now [the test] cannot be conducted because it [would be] unfair to [other students] if we conducted [the test] with them.

Janet's responses reflected her emphasis on assessment *for* online learning to assess students' learning during live teaching and through assigning homework. However, constraints existed for the effectiveness of online assessment through homework, in the sense that certain students might not complete their homework independently (assessment *as* online learning). Besides, she had concerns about the validity and fairness of conducting tests (assessment *of* online learning).

Conversely, Nick, a secondary mathematics teacher with less than five years of teaching experience, viewed using technology in assessment positively. He used Zoom for live Q&A to help students who had problems with homework and Google Forms to assess students' learning during distance learning (assessment *for* learning).

> In my opinion, since we have to use technology [now], [then] we have to fully utilise it: why use [the technology] just because we have to? … From the online teaching and learning perspective, objective questions are easier [for students to handle] … In my view, students' self-discipline is very important in the online [teaching and learning] mode.

Nick focused on live Q&A to assist students in their learning, whereby students had to take initiative to ask questions during live teaching sessions (assessment *as* online learning). He then used multiple-choice questions instead of written questions to evaluate his students (assessment *of* online learning) during the pandemic. He changed his assessment methods to enable students to answer questions. Furthermore, he emphasised the importance of assessment *as* online learning during online teaching and learning.

Lastly, Marry, a primary mathematics teacher with over 15 years of teaching experience, held mixed views on online assessment. During distance learning, she used the online platform Power Lesson to upload video recordings and assign exercises to students. She also assigned more multiple-choice questions to students online (assessment *of* learning). However, due to requests from parents, she started to incorporate live Q&A via Zoom (assessment *for* learning) into existing assessment methods:

> You will feel like you are talking alone. I do not know whether they learned something or not, whether they completed [their work], I do not know too. So, I feel the difficulty is I have to treat them as [if] they [had] never [learned it] before.

For Marry, her initial approach to assessment reflected her view of assessment *as* online learning, whereby students completed assigned exercises by themselves after watching her video recordings. Her comments on not receiving responses from students during live teaching sessions reflected constraints in assessment *for* learning. During the interview, she further commented that it was problematic that students did not cooperate during group work (assessment *as* learning). Then, she could not request students to write their answer on the whiteboard (board work) and review their work during live teaching (assessment *for* learning).

Discussion

The present study aims to explore Hong Kong mathematics teachers' views on online assessment during the COVID-19 pandemic. Findings show that the online assessments that teachers conducted during online learning were similar to the face-to-face assessments they conducted during in-person learning. These assessments would be conducted during class time, such as through classwork and Q&A sessions. Then, homework or tests were assigned or conducted at the end of class. Mok (2019) reported similar features in the data collected during 2016: normal face-to-face assessments in Hong Kong mathematics classes took the form of individual classwork, with the teacher walking between desks to monitor students' learning progress. The finding shows most Hong Kong mathematics teachers viewed assessment as being *for* online learning and *of* online learning during the pandemic. Teachers paid more attention to what students learned from the class and valued their performance. This may be due to cultural influences, such as the Chinese culture and the culture of examinations (Wong et al., 2012). Teachers also acknowledged the importance of students' autonomy in online assessments, which reflected the purpose of assessment *as* online learning.

The constraints that Hong Kong mathematics teachers reported primarily concerned the effectiveness of online assessment, as well as students' autonomy in completing homework and participation in live teaching sessions. The constraints that were reported, especially regarding students' autonomy were common challenges that teachers worldwide faced during online learning (Dhawan, 2020). To overcome these constraints, most teachers who viewed assessment as being *for* online learning tried to conduct live teaching and ask more questions during teaching. Teachers who viewed assessment as being *of* learning used more multiple-choice questions to assess students online instead of subjective or problem-solving questions. Nevertheless, teachers who viewed assessment *as* learning posted the answers to classwork or homework online for students to check their work by themselves, or hosted live teaching sessions for students to ask questions.

Conclusion

This study examined current mathematics teachers' views on online assessment during the pandemic. It provided insights into how technology influences the forms of assessment that teachers use and their experiences during online teaching. First, the findings revealed that online assessments conducted during the pandemic were similar to in-person assessments conducted before the pandemic. This implies that mathematics teachers' existing conceptions of teaching may be very strong and not easily changed, even when the teaching environment has changed. Second, teachers' views on online assessment and its constraints in various forms revealed that effective strategies to shape students' autonomy in online learning are needed. More teacher training must be given to teachers in terms of how to conduct online assessment. Information technology is no longer a supplement to teaching and learning, but has become the new norm in a post-pandemic era. The teaching modes that schoolteachers use therefore need to be more flexible to meet the changing times. In terms of our study, we acknowledge that our sample size is not large enough to generalise our findings. Future studies on this topic can be conducted by adopting different methods, such as large-scale questionnaires. Data from students could also provide another perspective on the effects of online assessment.

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Formative assessment in early mathematics teaching – a comparison of two different approaches

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Keywords: Early mathematics teaching, teachers' professional development, formative assessment.

Introduction

Especially with regard to inclusive education, teachers must recognize the potentials and preconditions of students during the learning process to build on these for successful teaching (e.g. Moser Opitz & Nührenbörger, 2015). However, the formative use of diagnostic information to support decisions is an unfamiliar concept for many teachers (e.g. Zeuch et al., 2017). Studies on the implementation of formative assessment (e.g. van Geel et al., 2016) suggest that structured and curriculum-based measures have positive effects. However, meta-analyses show that there is considerable need for research to understand the conditions of successful implementation (e.g. Kingston & Nash, 2011).

"Formative Assessment in Inclusive Early Mathematics Teaching" is a design-oriented research project in Germany. An in-service program was developed with the focus on diagnosis and individual support. The implementation of two variations of formative assessment – planned for interaction (PI) and curriculum embedded (CE) (e.g. Shavelson et al., 2008) – will be compared. In the approach "Födima-PI" the teachers learn to develop diagnostical questions and support activities, in "Födima-CE", they use prepared diagnostic and support tasks.

Research design

The project explores how teachers and multipliers can be effectively qualified for professional support-oriented diagnostics and how successful transfer processes into practice can be supported (e.g. Gräsel, 2010). In 2021/22 134 teachers from 37 schools will participate in the project. The research goals concern the evaluation of the acceptance and professional effectiveness of the training concepts and the qualification program. In the first year, the project focuses the following research questions:

1 How do teachers evaluate the conceptual framework?
2 What are the effects with regard to attitudes, pedagogical content knowledge and performance?
3 How effective are the concepts with regard to the development of basic mathematical skills, motivational variables and the self-concept of students?

A *pre-post-follow-up-test design* includes standardized questionnaires and interviews to gain information about the participants' *self-efficacies*, *pedagogical content knowledge* and their *attitudes* towards formative assessment before and after their participation, to evaluate the effectiveness of the program. Also, the *acceptance* of the developed in-service program will be surveyed.

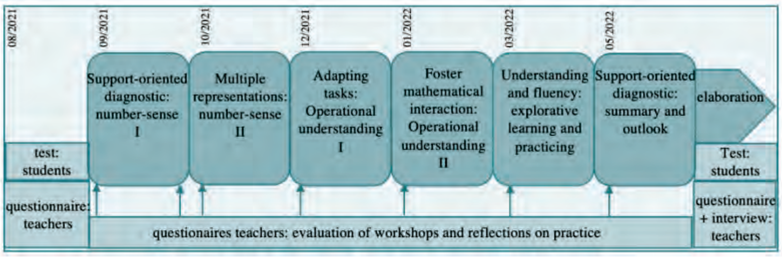

Figure 1: Födima year 1

We expect a positive effect in favor of the Födima-CE approach due to its somewhat tighter structuring and we assume a moderator effect in the form that high pedagogical content knowledge and a positive self-efficacy of the teachers are associated with better effects for the Födima-PI approach.

After evaluation, the two variations will be elaborated into a qualification program for facilitators. There will be further research related to this.

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Development and evaluation of primary school students' mathematical competencies via online assignment portals

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Development and evaluation of primary school students' mathematical competencies via online assignment portals

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Keywords: Formative assessment, mathematics, competency, competency-oriented education.

Mathematical competencies (Niss & Højgaard, 2019) have been developed and implemented in the Danish curriculum through the last two decades. At the same time, online assignment portals have gained considerable ground in the Danish elementary school. Denmark's largest portal alone, matematikfessor.dk, has 75% of the Danish primary and lower secondary schools as regular subscribers, and an average of DKK 1.5 million assignments are answered daily (EduLab, 2021). When Denmark was locked down on March 11, 2020, the number of daily completed tasks rose to approximately 6,000,000. During the lockdown, the number of unique daily logins rose from approximately 45,000 to 130,000 (Elkjær & Jankvist, 2021). Assignments in such portals, however, are mainly used for training students' basic skills. In the Danish context, a report from 2016 concludes that as many as 78% of the tasks that students in Danish primary and lower secondary schools encounter in mathematics teaching can be categorized as training tasks (Bremholm, Hansen & Slot, 2016).

The purpose of this ongoing project is to investigate to what extent it is possible to develop tasks for online assignment portals, so that students in the intermediate stage of primary school have the opportunity to develop mathematical representation competency as defined by Niss and Højgaard (2019). The tasks are to be designed in a way that supports the teachers' formative assessment of their students' mathematical representation competency. As the portal works now, the teachers receive information about the number of correct responses provided by each student. In addition to these information's, the aim is to inform teachers of the level of competency for each of their students.

The methodology of design-based research (Cobb, Jackson & Sharpe 2017) is used for both continuous development and adaptation of the competency-oriented tasks. The tasks are to be designed in clusters in a way that allows for student learning, while the student's answers inform the assessment tool without the student taking an actual test. The assessment tool is formative (Black & William, 2009) as knowledge of the students' representation competency within fractions will both inform the further work in the online assignment portal and support the teacher's formative assessment through a description of the student's competency level in terms of radius of action, degree of coverage and technical level (Niss & Jensen, 2019). A minimum of three design iterations are expected to be completed during the project.

During the first iteration, development of competency-oriented tasks for online assignment portals will be based on findings from a systematic literature review regarding task design with a focus on online assignment portals and mathematical competencies, respectively. Existing knowledge of the interfaces of these areas must result in a framework for how competency-oriented tasks are to be designed for an online portal.

During the second iteration, coding categories for the assessment tool are identified based on three dimensions of possessing and developing a competency: degree of coverage; radius of action; and technical level (Niss & Højgaard, 2019). Degree of coverage regards the extent to which the various aspects are covered. A person who can both interpret and understand relations between different representations and has knowledge about information loss and gain when shifting between representations has a higher degree of coverage- than a person who can only do the first.

Radius of action is the spectrum of contexts and situations in which the competency can be activated. The degree to which the student is possessing representation competency is to be measured by the extent to which the student can solve tasks in routine and/or new situations; in relation to internal mathematical situations and/or application-oriented situations; use fractions in different situations. The technical level deals with conceptually and technically advanced conditions and tools the person can activate the competence towards. The student's responses are used to identify development opportunities so that the student's next set of tasks can help further development the competencies radius of action.

During the third iteration, both task design and the assessment tool will be tested on a larger number of students to further inform both the task design and the assessment tool.

Matematikfessor.dk makes their online assignment portal available for testing of tasks and evaluation tool.

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How do primary teachers interpret and use standardized assessment: the case of the crochet placemats

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This paper presents the first findings from a survey, administered to 421 Italian in-service primary teachers, on their beliefs regarding the knowledge and skills investigated by the national standardized assessment (INVALSI) tests, their proximity to didactic practices in Mathematics and the role they assume within the school context. The case presented in this paper is discussed in order to investigate the way teachers interpret data coming from standardized assessment and if/how they use them in their teaching practice. Findings show an overspread meta-didactic conflict generated by teachers' difficulties in interpreting INVALSI tests and in using them coherently with the framework on which the tests have been designed.

Keywords: Standardized assessment, meta-didactic conflict, primary teachers.

Rationale and theoretical framework

At international level, the strictly link between standardized assessment and mathematics education is increasingly emerging (De Lange, 2007). Standardized assessment has the main purpose of system evaluation and its repercussions within social, political and educational fields, is increasingly taking hold, both at the level of the school system, and at the level of impact on classroom practices (Looney, 2011). There is a growing interest in broadening standardized assessment beyond the evaluation of school systems to mathematics education research as a new methodological tool (De Lange, 2007; Meinck, Neuschmidt, & Taneva, 2017). In Italy, the INVALSI national agency is responsible of the standardized mathematics assessment (INVALSI tests) that occurred, for the first and second cycle of education, since 2008. The INVALSI tests are administered at a census nationally and the results, elaborated on a valid statistical sample, are returned to the schools and publicly discussed every year. The theoretical framework of the INVALSI tests is in line with the main research results in mathematics education and with the National Curriculum Guidelines (INVALSI, 2018). This ensures that the macro-phenomena highlighted in the standardised assessment can be framed with the main research lenses and that they highlight new characterisations or phenomena (Bolondi, Ferretti, & Santi, 2019, Ferretti, & Bolondi, 2019, Ferretti, & Gambini, 2018). Several researches show how the results of the standardized assessment can be used in a formative perspective within didactic practices and as a tool within teachers' professional development paths (i.e., Doig, 2006; Di Martino, & Baccaglini-Frank, 2017; Ferretti, Gambini, & Santi, 2020). The research project within which this study moves, is in tune with these strengths of thought and aims to identify training needs of Italian teachers and to propose guidelines for the improvement of teaching practices, regarding the use of mathematics INVALSI tests. In particular, we investigate the presence of any conflicts, in the sense of Sfard (2008), that might exist between the "language of standardised assessments" and mathematics teachers' interpretation of it (Arzarello, & Ferretti, 2021). As we will explain in the following section, one of the main aims of this research is to understand the role and meanings that teachers attribute to the INVALSI tests, investigating how they interpret, consider and use the INVALSI tests and their results.

The interdisciplinary research project and our research questions

Herein, we analyse the first results of an interdisciplinary research project aimed at investigating the link between the INVALSI tests in Mathematics with the teaching and learning processes of Mathematics, in particular with teaching practices. The research project is conducted by the INVALSI Disciplinary Didactics Group of the Italian Society of Research in Didactic S.I.R.D., composed by mathematics education researchers and pedagogists. The interdisciplinary collaboration consisted in the design and in the administration of a tool for detecting teachers' attitudes towards INVALSI agency with its aims and working methods, as well as its tests in Mathematics and their repercussions on teaching practices. We are interested to understand what are the “tools” possessed and used by the teachers: to read and interpret INVALSI tests and their results; to identify possible effects of the INVALSI tests on their Mathematics teaching practices. The analysis of data coming from the results of the INVALSI standardized assessment is not enough to identify training needs in this direction at a national level within schools and to propose guidelines for the improvement of practices regarding the use of INVALSI tests. Rather, to reach these goals, we intend to develop a meta level analysis focusing on the way teachers interpret and use these data in their teaching practices. To do that, a survey was designed and administered to investigate primary teachers' beliefs regarding the knowledge and skills investigated by the INVALSI tests, their proximity to didactic practices in Mathematics and the role they assume within the school context.

In this paper we analyse data concerning the Mathematics Education section of the survey. The section focuses on 7 tasks of the INVALSI tests, chosen because considered suitable to highlight different didactic macro phenomena. With respect to each of the 7 tasks, teachers are asked to answer questions aiming to investigate: their ability to read INVALSI data and identify the reasons of students' mistakes; how suitable an INVALSI task is considered for assessing students learning and how commonly it is used in assessment practices. The final part of the Mathematics Education section of the survey contains some transversal questions that aim at investigating how the mathematical contents and the skills detected with the INVALSI tests are more or less close to the daily personal didactic practices and perceived as consistent/inconsistent with the National Curricula Guidelines. All these variables represent the focus of the survey and are the object of the analysis proposed in this study. In what follows we will exemplify our study presenting and analysing teachers' responses to some of the questions of the survey. More specifically, we intend here to answer the following research questions:

Q1: to what extent teachers' interpretation of the INVALSI tests and their results is coherent with the framework on which the tests have been designed?

Q2: to what extent the mixed (qualitative and quantitative) investigation based on our survey results to be a tool to answer Q1?

The crochet placemats task

This paper focuses on some questions of our survey, which refer to the INVALSI task of Figure 1.

To make 4 crochet placemats, grandmother uses 6 balls of cotton.
a) How many balls of cotton of the same type she has to use to make 20 placemats?
Answer:............
b) Write how you did to find the answer.

Figure 1: Task 11, Mathematics INVALSI test Grade 05, s.y. 2012-13

Teachers were given a set of students' answers to the task, consisting in both the numerical result and the solution strategy. For each of the proposed student answers, teachers were asked to choose its degree of correctness (*Completely Incorrect, Partially Incorrect, Mainly Correct, Completely Correct*). These 4 possible options are given to the teachers without an explanation of the difference between *Partially Incorrect* and *Mainly Correct:* to the teacher's choice of *Partially Incorrect,* we attributed the feeling that what is important for the teachers in that answer of the student is the fact that it is incorrect (even if only partially), rather *Mainly Correct* is the answer that we expected to be chosen by the teachers which consider the answer of the student to be correct (even if not completely).

The proposed student answers are:

1) 30, I calculated 20+6+4
2) 24, I multiplied 6 balls by 4 placemats, and I got 24
3) 120, I have multiplied 6 balls by 20 placemats
4) 30, Since for 4 placemats we need 6 balls of cotton (so 2 more), for 20 placemats we just need to do plus 10
5) 30, The grandmother uses a ball and a half to prepare a placemat, so I did this = 20 x 1.5 = 30
6) 30, To do 20 placemats I have to do 5 times 4 placemats, so I need 30 balls
7) 24, I have multiplied 20 placemats by 1 ball and a half, and I got 24.

As we can notice, option 1) reports a correct numerical answer supported by a wrong strategy; on the contrary, option 7) reports a correct strategy with a wrong numerical result. Only the options 4), 5) and 6) are correct. Despite the solution strategies reported in these three options are all correct, they are characterized by different approaches. Options 5) and 6) mainly follow a multiplicative model of reasonings, whereas option 4) makes use of a mixed additive-multiplicative model (Arzarello, 2018).

Results

The survey, that has been offered to be voluntarily filled on-line, involved 421 Italian in service primary teachers: most of them teach since more than ten years.

Table 1 shows the valid percent of the teachers' responses inherent to the wrong student answers proposed in the survey. A first look at the responses given by the teachers shows that most of them (more than 80%) recognised the first three options to be incorrect (Table 1). In particular, it seems to be reasonable that the correct numerical result (30) of the student answer, even in presence of a wrong strategy, can justify the fact that the number of teachers, who marked option 1) as *Completely Incorrect* is slightly lower with respect to those who considered *Completely Incorrect* options 2) and 3). Conversely, student answer 7) was considered *Incorrect* (*Completely* or *Partially*) by the 68.9% of the teachers (Table 1), even in presence of a correct strategy. A deeper analysis of data reveals that the 90% of the teachers which evaluated option 1) to be *Correct* (*Completely* or *Mainly*), considered *Completely Incorrect* option 7), and at the same time, the 99% of those who evaluated option 7) to be *Correct* (*Completely* or *Mainly*), considered *Incorrect* (*Completely* or *Partially*) option 1). This inverse correlation between teachers' responses given to student answers 1) and 7) is also confirmed by the factorial analysis we will present below (see Table 2).

Table 1: Valid percent of teachers' responses

Solution	Valid Percent			
	Completely incorrect	Partially incorrect	Mainly correct	Completely correct
1) *30, I calculated 20+6+4*	81.9	14.9	2.6	0.6
2) *24, I multiplied 6 balls by 4 placemats, and I got 24*	84.9	11.9	2	1.2
3) *120, I have multiplied 6 balls by 20 placemats*	85.5	12.5	1.2	0.8
4) *30, Since for 4 placemats we need 6 balls of yarn (so 2 more), for 20 placemats we just need to do plus 10*	23	32	34.8	10.2
5) *30, The grandmother uses a ball and a half to prepare a placemat, so I did this = 20 x 1.5 = 30*	4.7	5.2	18.8	71.3
6) *30, To do 20 placemats I have to do 5 times 4 placemats, so I need 30 balls*	6.9	8.3	21.4	63.4
7) *24, I have multiplied 20 placemats by 1 ball and a half, and I got 24*	46.3	22.6	29.7	1.4

Concerning the other three given answers, number 5) is mostly considered *Completely Correct*. A high percentage of teachers also recognised solution strategy 6) to be *Completely Correct*, while the teachers' responses inherent to solution strategy 4) reveal to be much noisier than numbers 5) and 6).

Also for the three correct student answers, some interesting results can be found observing data more in details. First of all, it seems worth of note that among the 192 teachers (45.6% of the total) which considered both the options 5) and 6) to be *Completely Correct*, only 29 also saw option 4) as *Completely Correct*. That is only the 6.9% of the teachers considered all the three student answers 4), 5 and 6) to be *Completely Correct*. Among the teachers who considered *Completely Correct* both options 5) and 6), we have also found out the number of those who saw option 4) as at least *Partially Incorrect*: they are 82 (the 19.5% of the total). This means that the 42.7% of those teachers, who considered *Completely Correct* answers 5) and 6), judged *Incorrect* (*Completely* or *Partially*) the mixed solution strategy presented in answer 4). Conversely, focusing on the total number of teachers

(232), who considered *Incorrect* (*Completely* or *Partially*) option 4), we can observe that 82 of them (that is the 35.3%) judged *Completely Correct* student answers 5) and 6).

The bivariate factor analysis (Suhr, 2006), using the Varimax method (SPSS software), allows the identification of two factors that saturate 44% of the total variance (Table 2). These results are interesting: in fact, option 1) and option 7) are in the same component with opposite signs; option 3) and option 2), both reporting wrong multiplicative model of reasoning, are in the same component; the first component also contains the two correct options 5) and 4) with opposite signs with respect to options 3) and 2); also the two most recognised correct strategies in options 5) and 6) belong to the the same component. However, some options reveal low factorial coefficients, so highlighting the presence of some critical aspects. These critical aspects are worthwhile of a further investigation, which we are now pursuing using qualitative methods of analysis.

Table 2: Varimax Rotated Factor Matrix with Kaiser normalization applied to teachers' responses

	Components	
3) *120, I have multiplied 6 balls by 20 placemats*	.721	
2) *24, I multiplied 6 balls by 4 placemats, and I got 24*	.715	
5) *30, The grandmother uses a ball and a half to prepare a placemat, so I did this = 20 x 1.5 = 30*	- .567	.508
4) *30, Since for 4 placemats we need 6 balls of yarn (so 2 more), for 20 placemats we just need to do plus 10*	- .424	
7) *24, I have multiplied 20 placemats by 1 ball and a half, and I got 24*		.751
1) *30, I calculated 20+6+4*		- .490
6) *30, To do 20 placemats I have to do 5 times 4 placemats, so I need 30 balls*		.449

Teachers were also asked to answer two other questions about their awareness of the validity of the task in order to assess students' learning and the extent of the use they make of similar tasks in their assessment test. Diagrams in Figure 2 show the valid percent of teachers' answers to these questions.

As those above, also these data show some unexpected correlation. We will discuss this issue in the next Section.

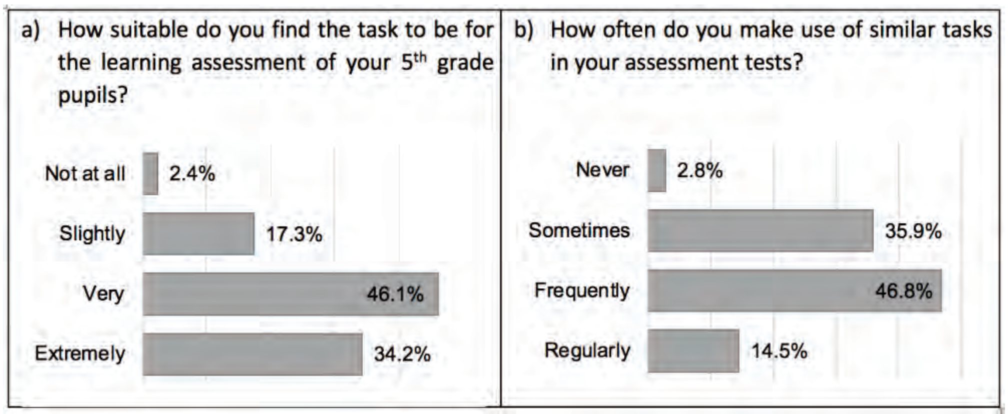

Figure 2: Teachers' responses to the questions about the validity of the task to assess students' learning and the use of similar questions in their assessment test

Discussion

We have already pointed out some contradictory aspects within teachers' answers to our survey. Figure 2 is emblematic for that. In fact, with respect to the two questions in it, we can see that, for the 80.3% of the teachers, the crochet placemats task is (*Very* or *Extremely*) suitable for the students' learning assessment, while the 61.3% of the teachers declared to make use (*Frequently* or *Regularly*) of similar questions in their assessment test. A first possible interpretation of this discrepancy might be connected with the perception of a higher difficulty embedded in the assessment of argumentative and problem solving skills, especially using open questions: although teachers recognize that the crochet placemats task is suitable to assess students' learning, in their usual assessment tests they prefer to assign more procedural exercises. However, we believe that the discrepancy highlighted by our data can be also considered in the light of the mentioned teachers' difficulty in recognising the correctness of “unusual” solution strategies.

We can interpret all these results basing on the concept of *meta-didactical conflict*, elaborated by Arzarello and Ferretti (2021). It exploits an analogy between the answers given by teachers in surveys like that illustrated here and the “incommensurable discourses” described by Sfard (2008). She shows how, in classroom interactions, interlocutors many times share the same words but with a different, incommensurable, meaning, of which they are not aware: a conflict is so generated. Something very similar we found when we analysed the answers to our questionnaire: here we found incommensurable languages. As the name we adopted suggests, this conflict consists of three components and is meta-didactic. We have defined it as meta-didactic because it is inherent in discourses about didactic processes such as assessment, students' skills and mistakes.

In this study we have highlighted how a twofold conflict emerges in many teachers' answers, because they interpret: (i) the difficulties of students in the INVALSI tests in a way that is completely different from what unquestionably appears from the data of the survey; (ii) the rationale of the INVALSI tests in a contradictory way (e.g., in the way how they couple the dyads suitable/not suitable Vs most used/ not used in the examples with respect to what appears in the survey data).

To summarize our findings, we can say that teachers' answers to our survey show an overspread meta-didactic conflict. We consider worth of note, that even if teachers declare to consider the task highly suitable to assess students learning, they do not say to make an extensive use of this kind of task and, above all, in most of the cases, they are not always able to recognise all the suggested correct solution strategies. These observations can give a negative answer to our research question Q1. Furthermore, the case of the crochet placemats task is an example of how our survey resulted to be a useful tool in order to investigate teachers' interpretation of INVALSI data and their use in the teaching practices. In this sense, our result can give an answer also to research question Q2.

Conclusion

In the paper we have presented the first findings from a survey aimed at investigating the way primary school mathematics teachers read, interpret and use INVALSI Italian standardized assessment. According to the INVALSI theoretical framework (INVALSI, 2018), the crochet placemats task has been designed with the aim to evaluate if fifth grade students were able to solve a given problem of direct proportionality and to describe their solution strategies. At a meta level, in order to evaluate the correctness of the solution to the task, teachers have been requested also to recognise the correctness of the strategies employed by the students. This needs a deep understanding of the underlying conceptual field of the multiplicative structures. Analysing teachers' responses to the survey, we have observed a basic meta-didactical conflict. The necessity of overcoming this conflict can suggest suitable designs for professional development programs for in-service teachers in order that they develop a correct culture of assessment and base their teaching practices on a correct interpretation of the data in national surveys. If we want to overcome the risk that standardised assessment is limited to classifying students, schools and nations, we need to develop the dialogue between standardised assessment and mathematics education. To fully recognise the potential and educational goals of standardised assessment, there is a need to develop and disseminate effective theoretical tools for interpreting the quantitative data and the macro phenomena emerged.

Moreover, our data provide several elements to be further analysed, both quantitatively and qualitatively, to shed light to critical aspects connected with teachers' perception of the usefulness of standardized assessment in their professional activities, that is the main general aim of our interdisciplinary research project.

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Incorrect responses to the national assessment of mathematics: Gaining insights into mathematical proficiency at middle school

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In this paper, we analyse the incorrect responses of grade 8 students to mathematical items of the national assessment of mathematics. We frame our study starting from an item which requires to write the perimeter formula for an isosceles trapezoid as a function of one variable. Focus is put on common features of groups of incorrect responses, which we look at as routes to algebraic thinking in this context and which reveal partially correct responses. Studying the mutual relationships between the routes and the students' proficiency levels reported via the assessment scores, we gain insights into the students' mathematical proficiency and the nature of their errors. Expansion of our route-based investigation to other items still highlights the inclusion of partially correct answers, which might entail a wider range of acceptable responses and more complex marking involving partial credit scoring.

Keywords: Assessment of mathematics, error, route, proficiency level, partial credit.

Introduction

In this paper, we first investigate the incorrect responses to a mathematical item by grade 8 students in a national assessment of mathematics. The item asks to write the formula for the perimeter of a simple figure as a function of one variable and had a very high percentage of incorrect responses. It has to do with the algebraic understanding of formulas, as part of the mathematical literacy required at the end of middle school. Specific focus on writing a formula is crucial, since formulas evoke problematic pairs of mathematical thinking, such as conceptual and procedural, relational and operational, and general and particular (e.g. Sfard, 1991; Arcavi, 1994). Each pair emphasises a tension between the semantic and the syntax of formulas, which usually remain bounded to applying static calculation instead of being conceived as versatile means to establish relationships, despite their role in transitioning from arithmetic to algebra and in mathematical modelling. A formula further requires the usage of letters as variables, of which students often reveal static images, as unknown values, or labels, with difficulties to conceptualise quantities in a problem context, how they can be related, how they vary together (Bush & Karp, 2013; Carlson et al., 2015).

We know that, despite their role in reasoning algebraically or functionally, formulas did not receive extensive attention in mathematics education research, especially in relation to student literacy. In considering the specific item mentioned above, we aim to investigate how formulas are (or can be) interpreted by middle school students and the corresponding difficulties in seeing them as algebraic relations or functions. To address this concern, we take a sampled group of grade 8 students, who took part in a national standardized assessment test, and we focus on their incorrect responses to the item to better interpret their approaches to the solution. We further an initial analysis of these answers, which showed types of emerging errors, related to known misconceptions or procedural forms of mathematical thinking (Pozio & Bolondi, 2019), to better see them in terms of conceptualisation processes about the mathematical task rather than as reasoning products.

We then briefly expand our considerations to other items, based again on central but different aspects of mathematics thinking and learning at middle school in our country. We overall frame this study with reference to the content of the initial item (algebraic knowledge) to introduce the reader to the idea of *route* as a methodologically relevant means to study the ways students can face mathematical items and produce incorrect responses, and to focus on their mathematical literacy.

Theoretical highlights

Arcavi (1994) offers the idea of symbol sense to discuss the versatile nature of algebra and its use as a tool to understand, express, and communicate generalizations, to reveal structure, and to establish connections. Symbol sense is difficult to achieve, especially in the early grades as it involves letter usage and understanding of the roles and multiple meanings of variables (Bush & Karp, 2013; Küchemann, 1978; Philipp, 1992). Küchemann and Philipp analysed misconceptions related to letter usage and observed that variables can be treated as labels or objects, constants, unknowns, generalised numbers, varying quantities or variables, parameters, or abstract symbols. Philipp focused on the difficulties students have with variables as related to an inability to recognize the role of the variable, and Küchemann studied how certain ways of using letters were typically perceived as less demanding than others. Following Küchemann, students understand the meaning of the use of symbols in algebra only when they can work with letters as variables. Usiskin (1988) pointed out misconceptions about a view of variables as simple labels and reported failure to grasp variables as varying quantities rather than missing values. Kieran (2007) argued that the meaning of a variable varies depending on the context, an obstacle when learners face problems involving this notion. Discontinuities with arithmetic are also implicated in work with variables, which uses number from a more structural point of view (Stacey & MacGregor, 2000; Warren et al., 2016). According to Stacey and MacGregor, difficulties in formulating and solving algebra word problems have often to do with a compulsion to calculate, which rests with arithmetic problem solving and accounts for many misconceptions of algebraic thinking at the early grades. These authors identified different “*routes*” to a solution of simple problems on equations by grade 10 learners. They associated non-algebraic routes to students using arithmetic reasoning or trial and error; superficially algebraic routes to students writing equations to solve the problem as formulas describing sequences of calculations; algebraic routes to equation writing and solving. Capraro and Joffrion (2006) further claimed that writing equations from word problems is difficult for middle grade students, whether concerned with misconceptions or literal translation.

These studies shed light on the fact that the inclination to misconceptions may be related to mainly procedural or operational approaches. Formulas, as algebraic statements that value the interplay of procedural and conceptual (relational), are instead a powerful means to mobilise symbol sense. Interestingly, the context of measurement teaching and learning shows confusion between area and perimeter, especially for rectangles, indicating weakness in grasping both formulas (Smith & Barrett, 2017). Similarly, even middle school learners still struggle to distinguish area from the length of the region’s boundary (Chappell & Thompson, 1999).

In respect to this body of literature, our first aim in this paper is to gain insight into how formulas are conceived by middle school students and to better understand difficulties and misconceptions related to formulas in context. To this aim, we focus on types of errors as critical to meaning making, and we look at them as generative of specific approaches to the solution of a mathematical item, that is, of "routes" to the solution, borrowing from Stacey and MacGregor's study. This helps us see ways that middle school students still struggle with seeing formulas as means to establish relationships. A second aim of the study is to shed light on the relevance of routes for the analysis of students' incorrect responses and related mathematical literacy. In the end, attention is therefore shifted to other items of the national assessment in which high percentages of students performed incorrectly.

Context and participants

The context for this paper is the national standardized computer-based assessment of mathematics administered in 2018 to grade 8 Italian students at the end of the school year, as part of the evaluation of their middle school-based education. The assessment is administered by the National Institute for the Evaluation of the Education System (Istituto Nazionale per la Valutazione del Sistema di Istruzione e di Formazione–INVALSI, in Italian). While the participation in the national computer-based test (CBT) is mandatory for all grade 8 students, a sample group is selected for the evaluation process. CBT uses a wide number of items (item bank) from which to extract clusters of items for creating multiple forms of the test. A peculiarity of the CBT is the development of empirically based proficiency levels to report student progress. Accordingly, students' mathematics proficiency and item difficulty are located along a shared scale, which is divided into five levels, 1 to 5, lowest to highest (therefore, a level refers both to student proficiency and to item difficulty). This approach allows describing what students can do with mathematical tasks at different levels of difficulty.

In this paper, we particularly draw on one mathematical item of the 2018 assessment of mathematics at grade 8. The item requires students to write the perimeter formula for a simple geometrical figure as described in the next section. Out of the 574506 students who completed the assessment test, 31300 constituted the sampled group. Of these, 4543 students answered the item under consideration. Some 2147 students out of our 4543 respondents gave an incorrect answer (47,2%), only 1561 students answered the task correctly (34,4%), and there were 835 missing answers (18,4%). Successful students were for the majority at a high proficiency level (4 and 5), while the 71% of students at low proficiency (levels 1 and 2) do not even attempt to answer.

A mathematical item about algebraic knowledge

The mathematical item, which is the initial point of this paper, is an open-constructed response item that involves an isosceles trapezoid given by a picture, as shown in Figure 1. Relational information for which the long base is twice the short base is presented in a textual stimulus at first (line 1). The picture additionally provides a number (5) and a letter (b) respectively for the leg's length and the short base's length. The task follows (line 2) asking the students to write a formula to express the trapezoid's perimeter "as a function of b". Apart from knowledge about simple calculation of the perimeter, the task therefore focuses on reasoning functionally on the formula, implying the use of the letter b to capture relationships beyond coordination among the different semiotic sets.

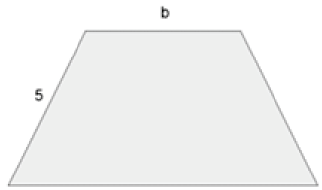
Domanda **Nel trapezio isoscele in figura la base maggiore è doppia della minore.** (1)  **Scrivi una formula che esprima il perimetro *p* del trapezio in funzione di *b*.** (2) *Digita la risposta alla domanda.* (3) **Risposta:** p =	1. In the isosceles trapezoid in the figure, the long base is twice the short base. 2. Write a formula that expresses the trapezoid's perimeter p as a function of b. 3. Type your answer to the question.

Figure 1: A mathematical item about algebraic knowledge

Data and method

We are interested in investigating the incorrect responses given to the item to better understand its complexity and the ways in which middle school learners approach it. For the analysis, we consider those incorrect responses (1759) which concern the use of letters to write the requested formula, leaving out only numerical values or expressions, and word answers as single letters. The data we use are quantitative and qualitative. Quantitative data are percentages of correct, incorrect, and missing answers, the students' proficiency levels and the item difficulty. Qualitative data are the written responses, which were accessible thanks to the expert correction that INVALSI uses for open constructed-response items.

Data are finally combined to study the relationship between the routes and the proficiency levels of the students. Our method can be seen as a qualitative method in principle. The analysis was developed by first looking at the incorrect responses to search for common features, for example reference to a single semiotic set, like natural language or numbers, and attention to the relationships implicated in the item. We then grouped the answers according to the identified features into those that we interpret as routes and sub-routes to the solution. The same method was applied to study incorrect responses to other items centred on different but crucial aspects of mathematical literacy at middle school. These items also share with the original one high percentages of missing and incorrect answers.

Findings

We recognised three major approaches to the formula item, essentially: there have been students who have tried to write the formula making explicit reference to the relationship between the bases (*Relational route*); students who have only partially referred to the given elements to write the perimeter (*Partial route*); who, on the other hand, has written an area formula (*Habitual route*). We also found another widespread approach, that of students whose answers present letters but seem to escape decipherable orientations, while still relating to elements in the item (*Murky route*). The routes are captured by the examples in Tables 1 to 4 as follows (the number of responses of a certain kind is in parentheses at the top).

Table 1: Examples of answers in the Relational route and its five sub-groups (228)

***R*1** (62)	***R*2** (32)	***R*3** (4)	***R*4** (66)	***R*5** (64)
Using a letter in place of number 5	Using only numbers for the formula	Using mainly words for the formula	Reversing the relationship	Using wrong typing for operation signs
*bx*3+2*l*	[6+(6*2)]+(5*2)	Short base plus short base times two plus ten	*B*+1/2*B*+ (5*2)	5*2+*b*+*b***b*

The students whose responses belong to the Relational route (Table 1) are essentially capable to orient themselves to the information presented across the different semiotic sets and the various degrees of relationship in the item. Beyond writing a relationship between the bases, in fact, these students know that they must find the perimeter of an isosceles trapezoid and consider all its elements, gaining information from the text and the picture. The route is made of 5 sub-routes (*R*1 to *R*5, Table 1).

Table 2: Examples of answers in the Partial route and its four sub-groups (676)

***P*1** (229)	***P*2** (276)	***P*3** (145)	***P*4** (26)
Using letter *B* for the long base	Multiplying *b* by 2 instead of 3	Seeing the perimeter as sum of any 4 sides	Writing the procedure in words
B+*b*+5+5	(5+5)+*b**2	*B*+*b*+*l*+*l*	Long base + short base + 2*5

The students who undertake the Partial route, which includes 4 sub-routes (*P*1 to *P*4, Table 2), are capable to understand the request of a formula even if they do not refer it to the entire context, which demands connection with various degrees of relationship. The responses in this route essentially share particular focus on writing a formula for the isosceles trapezoid's perimeter, while forgetting some of the elements in the task, like the relationship between the bases (*P*1) and the specific length of the leg (*P*3). *P*1 and *P*3 exemplify how letters *B* and *l* respectively are used as labels (Usiskin, 1988).

Table 3: Examples of answers in the Habitual route and its five sub-groups (293)

***H*1** (112)	***H*2** (35)	***H*3** (31)	***H*4** (28)	***H*5** (87)
Using letter *B* for the long base	Taking the relationship between the bases	Writing the area of a triangle	Using words to write an area formula	Using letter *h* in some way
(*b*+*B*)*xh*/2	[*b*+(*b**2)]**h*/2	*b***h*/2	*b* plus *B* times 5 all divided by 2	[(*B*+*b*)**h*]

The Habitual route contains the 5 sub-routes *H*1 to *H*5 (Table 3) and comprises answers that concern the writing of an area formula rather than a perimeter formula (the known confusion introduced above) and, therefore, are far from solving the item correctly. We can realise this by observing the use of letter *h*, which habitually labels the height of a geometric figure, multiplied–in most cases–by a base or by the sum of two bases. The students in this route seem to associate the trapezoid and the formula with the task of finding the area of the trapezoid as students learn to measure the area of specific figures, not the perimeter, by means of rote procedures. While some consider the relationship between the bases (*H*2), many show a vague awareness of an adequate area formula (*H*3, *H*5).

Table 4: Examples of answers in the Murky route and its 3 sub-groups (562)

***M*1** (202)	***M*2** (180)	***M*3** (180)
Taking essentially *b* and 5	Taking essentially *b* and 2	Taking *b* or other letters
5*2+2**b*+*B*-*b*	*b**2+*b**2+*b*+*b*	*l*+*l*=*b*+*B*

The Murky route, with its 3 sub-groups (*M*1 to *M*3, Table 4), basically shows attempts to examine information for producing an answer that still uses letters, revealing some reasoning albeit quite rough. Most of the answers are relatively oriented to partial elements of the item, in many cases they exclude the legs' length or the relationship between the bases. We often assumed that these students have little if any clear idea of which elements to use for finding the perimeter of the trapezoid. It may also be that the request of writing the perimeter in function of *b* had implied specific reasoning on how to use letter *b*, and other letters, in a formula, or particular focus on one or the other base.

Conclusive discussion

For the sake of space, we cannot elaborate on the routes further. We shift attention to the relationship between the routes and the students' proficiency levels, turning to graphical representations of the distribution of the routes across the five levels and of the levels across the four routes. Figure 2 shows how percentages of students in each route are divided into the five proficiency levels (the first graph).

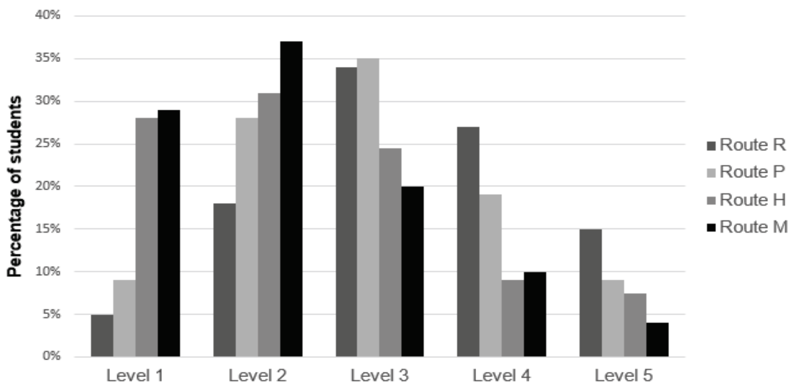


Figure 2: Distribution of routes across proficiency levels

A general trend emerges in the routes. In particular, the 29% of students at level 1 of proficiency and the 37% of students at level 2 belong to the Murky route (*M*), while students with medium proficiency (level 3) are present in both the Relational (*R*) and Partial (*P*) routes, with a percentage of about the 35% (shortly, the 70% of students who show medium proficiency are in the first two routes). Turning to high proficiency, we see that level 4 and 5 students are present at most for the 27% and the 15% respectively in *R*. We essentially see an increase of percentages of students with low proficiency as we pass from routes *R* to *M*, and a decrease for students at medium-high proficiency level. This trend is consistent with the assessment model, for which the closer an answer is to a correct one, or the more it includes steps toward the solution (partially correct answer), like reference to the relationship between the bases, the higher is the probability that the answer is given by a high proficiency student.

That less than 10% of students in *P* are at level 1 or 5, while the rest is at levels 2 to 4, also means that middle level students are basically capable to write a generic formula for the perimeter of a quadrilateral but may have troubles to write a more complex relationship between the given data. Most students in *H* are within the ranges of low proficiency, showing confusion between perimeter and area and, possibly, weakness in grasping both formulas, as outlined by Smith and Barrett (2017). Briefly speaking, the routes inform about the relative distance from solving the item correctly. The routes contain partially correct answers (especially *R*), which often satisfy the question intent and therefore might be considered for grading in the assessment. An additional aspect is the presence of all levels in all the routes. We can conclude that students make errors which reveal much about their algebraic proficiency in context, especially regarding the interpretation of formulas and the use of letters, which are still often conceived respectively as mere procedures or rules and labels or names.

After this first analysis, we moved to use our route-based approach to look at how other items of the national assessment in grade 8 were incorrectly answered and gain more insights into mathematical proficiency at middle school. Proportional reasoning and the relationship between simple geometric figures' area and perimeter are the core mathematical contents involved in these items. The incorrect responses are 1154 out of 2231 for the first item, and 6504 out of 8997 for the second item (300 and 777 missing answers respectively). Examined through routes, the answers highlight a multiplicity of approaches to single solutions, the impossibility of unambiguous interpretations in certain cases, and the inclusion of responses that are partially correct while satisfying the question intent. Again, the routes inform us about students' kinds of errors and misconceptions (and their mathematics literacy), which deserve specific attention from the educational point of view. The routes also help us reason about the importance of the question intent and the need for a wider range of acceptable responses and more complex marking by partial credit scoring, in line with the PISA and TIMSS assessment.

A general, conclusive observation relates to the fact that the sample we analyse is representative of the entire population of middle school students in grade 8 in our country. These students come from different classes of different schools spread across the national territory. Therefore, studying their incorrect answers is meaningful to the extent to which it allows us to better understand ways that specific aspects of mathematics thinking and learning are practiced at middle school in our country and conceptions that learners develop from them.

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Opportunities for digitalization of the final examination of Math

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Keywords: Final examination, digitalization, mathematics test.

Some research compares the final exam systems of compulsory education (between 16-18 years; Kákonyi, 2015), but no attempts have been made towards a comprehensive study about the level of digitalization of the mathematics tests. This poster highlights the main characterizations of the Math exam in four countries (Finland, Denmark, Georgia, and Germany) that made steps towards digital assessment. The long-term research purpose is using this international comparison to influence the practice in Hungary towards more digitalization.

In Finland, Mathematics test is not compulsory. It is organized on two different levels of difficulty, and any scientific, graphic, or symbolic calculator can be used that doesn't have a data transfer feature. Traditional test papers are no longer available as Finland achieved complete digitalization by 2019. The digital examination was not an adoption of the paper-based test onto a computer, but instead required the transformation of types of tasks. The list of measurable skills also changed, making it possible to detect the efficiency of information processing, critical thinking, and real-world problem solving by multiple-choice and open-ended questions (The Finnish Digital Matriculation Examination, 2021).

Mathematics is a compulsory subject in Denmark's school-leaving exams, which terminates lower secondary education. Computers have been involved in final exams in certain subjects since 1994. At first, only the paper-based examination booklet was adapted to digital environment and computers' text editor function dominated. During the test, examinees use their own laptops without an internet connection. The paper-based test's open-ended questions can be answered on paper or in a Word document in a digital environment. These documents then must be delivered on a USB-drive and printed. The final aim is the renewal of the complete process, to be able to measure creativity, critical thinking, and the ability to work in teams and to solve real-life problems (The Danish Ministry of Children and Education, 2019).

Georgia's national exams apply computer-based adaptive testing, where the test items are selected sequentially according to the correctness of a student's answers on previous items. The main reason behind digital testing is the prohibitive cost of a national paper-based examination system, but this also requires investments in school ICT infrastructure. Although the delivery of the test has become electronic, the answer sheet remained paper-based, even in 2020. The exam of Mathematics is optional, and the most controversial part of the system is the lack of measuring a wide range of competencies due to its multiple-choice test format (Trucano, 2015).

In Germany there is no unified school-leaving examination, as every federal state has its own Abitur. The mathematics test can be substituted with another subject among the mathematical-scientific fields. The exercises cover probability theory and statistics, have many arithmetic components, and require the knowledge of mathematical justifications and evidence. There have been some attempts to achieve centralized final exams. In 2017 a joint task pool was launched to improve the comparability of the Math exams in different federal states. Closed format tasks characterize the exam questions, which also show the influence of IT tools. Graphic calculators and Computer Algebra Systems, like GeoGebra's CAS, are legal aids. In Grade 12, all students must decide whether they want to take the CAS or the classic Abitur in Math. All solutions must be documented on paper (Csapodi, Filler, 2017).

In Hungary, there is a centrally standardized school-leaving exam, called *érettségi*. The same requirements apply all over the country. Mathematics is a compulsory subject, but candidates can choose the level of the exam; the intermediate level measures basic knowledge, while the higher level requires deeper analytic skills, and the questions are open-ended. Scientific calculators that cannot store and display textual information are allowed. The current tests are considered objective and valid, and therefore adequate for measurement, so it is not our intention to make changes in the types of items, as they did in Georgia. Within the framework of my PhD research, I'm working on developing tasks that can be set in the digital environment and focusing on the measurement of problem-solving skills.

In conclusion, there are numerous ways to involve a computer in the final exam of Math; from embedding the paper-based tasks without any modification to a digital environment, to computer-based adaptive testing and to problem-solving tasks, where creativity, skill of analyzing and assessing information objectively, and ability to mathematical modelling dominate. We would target the latter in Hungary, considering that final exams always have an active impact on education itself, so all the small changes must be made bit by bit, monitoring their effectiveness and providing constant feedback.

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Real tasks in Italian mathematics standardized assessment

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In this research we propose a criterion to classify real mathematics tasks in a standardized assessment. Specifically, we used this criterion to categorize the tasks of six, grade 5, standardized tests administered in Italian schools from 2009 to 2014 and we analyzed the psychometric features of these items. Results show a consistent presence of real tasks in Italian standardized assessment, indicating that one of the competences that these tests are intended to assess is the ability to develop and apply mathematical thinking to solve problems from everyday situations, as desired by the European Council and PISA. Moreover, the items classified as real are not necessarily difficult for students, but they are discriminative. The analysis at an item level of this kind of tasks could provide further information about how students face real tasks in a standardized assessment

Keywords: Standardized assessment, rasch model, real task, mathematical competence.

Introduction

In recent years, the growing attention given to national and international standardized assessment is promoting new reflections on how these tests are constructed and which information emerge from their results (Giberti & Maffia, 2020). The theoretical framework of each survey is fundamental to understand the aim following which the tasks are constructed and the mathematics that the test is going to assess. The most famous and discussed international survey is PISA, the Programme for International Student Assessment, promoted by the Organization for Economic Co-operation and Development (OECD). In PISA theoretical framework, Mathematical literacy is defined as “an individual’s capacity to reason mathematically and to formulate, employ, and interpret mathematics to solve problems in a variety of real-world contexts” (OECD, 2019); then all PISA mathematics problems are set in a context, which can be personal, occupational, societal, or scientific.

This contribution reports first results of an ongoing research project whose overall aim consists in studying how tests of Italian standardized assessment (administered by the Italian National Institute for the Evaluation of Educational Systems – INVALSI - every year at grade 2, 5, 8, 10 and 13) are built and what mathematical competencies and types of reasoning are aimed to be assessed. Specifically, the aim of this paper is to analyze INVALSI tests, proposing a criterion based on Palm’s framework (Palm, 2006) to evaluate the presence and quality of real tasks. Furthermore, the psychometrical features of those tasks previously identified as real will be analyzed.

Theoretical Framework

In the Italian context, the National Indications for the First Cycle of Education (MIUR, 2012), emphasize how mathematical knowledge should offer skills for perceiving, interpreting, and linking artifacts and daily-life events. Students are required to analyze situations and to translate them into mathematical terms and choose the actions to be performed to produce a solution to the problem.

These indications are reflected also in the European context. Indeed, the European Council (Recommendations of the European Parliament for lifelong learning (2018/C189/01)) recently defined as *mathematical competence* the ability to develop and apply mathematical thinking and the ability to use different representations (formulas, constructs, models, …) to solve problems starting from everyday situations. Despite the importance of including real-life experiences in mathematics classrooms, there is a strong discontinuity between in- and out-of- school mathematical experiences (Passarella, 2021). One of the possible causes of this fracture is the stereotyped nature of the problems proposed by textbooks and teachers, which, rather than serving as an interface between mathematics and reality, promote in students an exclusion of realistic considerations (Verschaffel et al., 2020). Instead, real and less stereotyped problems must be inserted in the school practice, in order to create a bridge between mathematics classroom activities and everyday-life experiences (Passarella, 2021; NCTM, 2000). Given the importance of real problems for mathematics classrooms, the following question emerges: when a mathematical task might be defined as *real*? Several studies delt with the concordance between mathematical school tasks and the corresponding out-of-school situations: Wiggings (1993) with the introduction of the term authenticity; Niss (1992) and the mathematical literacy framework of the OECD's Program for Student Assessment, PISA (OECD, 1999); Realistic Mathematics Education, that considers realistic and rich contexts as starting points for mathematization processes (Van den Heuvel-Panhuizen & Drijvers, 2014). In this contribution, we will consider the framework proposed by Palm (2006), that was developed to depict those aspects of real-life situations that should be considered important in their simulations (Figure 1).

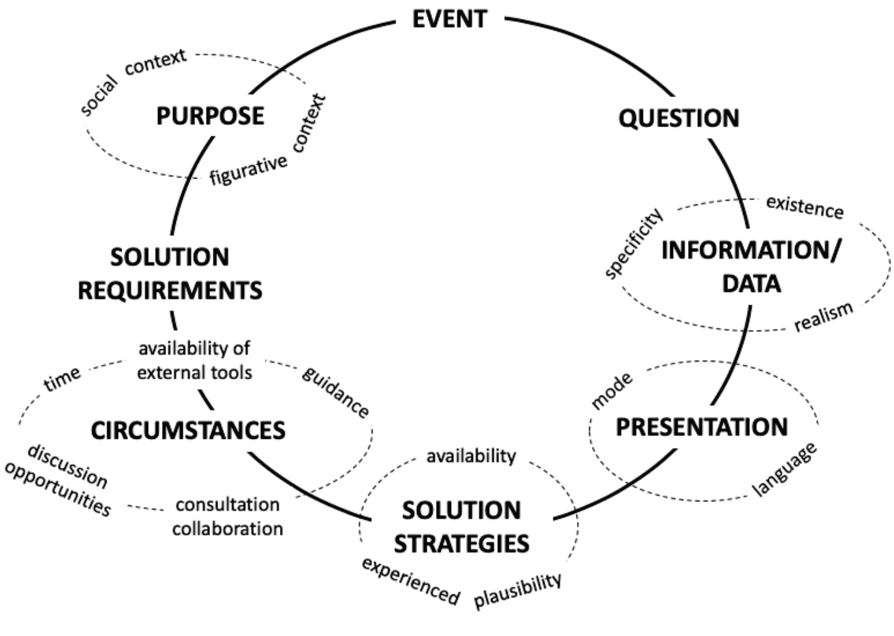


Figure 1: Palm's framework for simulations of real-life situations

A detailed description of these aspects can be found in Palm (2006). In Table 1 we describe those aspects that will be considered in the rest of the work, reporting for each of them a key question, in order to start reasoning on the concordance between mathematical school tasks and corresponding situations in real-life beyond the mathematics classroom. The framework proposed by Palm can enhance discussions amongst teachers and researchers on the characteristics of tasks that simulate real-world situations, and for the design and analysis of real tasks for mathematics classroom activities (Palm, 2006). Starting from Palm's framework, in this work we are proposing a criterion to evaluate the presence of real tasks in the Italian national tests in mathematics. Furthermore, using this criterion, we identify real task and analyze their psychometrical features to understand the behavior of the students facing these items.

Table 1: Key questions for some aspects from Palm's framework

Event	Could the event described in the school task take place in real-life?
Question	Does the questions in the school task might be posed in a corresponding real-life event?
Existence	Do the information available in the school task exist in a corresponding real-life event?
Realism	Are the values given in the school task close to values in a corresponding real-life event?
Specificity	Can specifications of the school task context be compared to a reasonably extent to the corresponding out-of-school situation?
Presentation Mode	Is the problem communicated orally or in a written form? Are the information presented in diagrams, tables, graphs, …?
Language	Does the language used in the school task not negatively affect the possibilities for students to use the same mathematics as they would have used in a corresponding real-life event?

Research Questions

The first aim of this contribution is to evaluate the presence of real tasks in the Italian national tests in mathematics (INVALSI tests), in order to investigate if INVALSI tests are intended to assess students' ability to develop and apply mathematical thinking to solve problems starting from everyday situations, as desired by the European Council and PISA. Consequently, our first research question is:

RQ1: Are there real tasks in the standardized INVALSI tests?

If the answer to the first research question is yes, considering that previous studies have highlighted that there is a strong discontinuity between in- and out-of- school mathematical experiences (Passarella, 2021), we would investigate the psychometrical features of the tasks classifies as real, in order to start understanding how students face these tasks.

Accordingly, our second research question is:

RQ2: If yes, which are the psychometrical features of these tasks? Are there some recurrent features that can be interpreted as a specific behavior of students facing a task of this type?

In this paper we will answer to this second research question considering the general psychometrical features provided by the Rasch Model, considering the whole set of items and providing an example of analysis at an item level.

Data and Methods

INVALSI tests are national standardized assessment administered every year in different school grades from primary to upper secondary school. In all grades INVALSI tests investigate students' competencies in mathematics and Italian language but, in upper grades, also an English test was included in the survey. In this work we decided to focus on grade 5 mathematics tests and compare the results in different years: from 2009 to 2014. INVALSI test are administered, for each grade, to the entire student's population. In this research our analysis is based on the probabilistic national sample drown from the entire population which is also considered for all the statistical analysis of the INVALSI Institute. For each test the sample includes approximately 30 000 students[1] and sample data are completely void of the effect of cheating because the test administration is conducted by an INVALSI inspector who guarantees the correctness and fairness of the process. All mathematics tests include different items in terms of typology (Multiple choice, Open-ended, Justification Open-ended, True/False), and mathematical content categories (Numbers, Data and Probability, Relationships and Functions, Space and Shape).

To answer to our first research question, we propose a criterion (Figure 2), based on Palm's framework, to classify mathematical tasks as to be real or not. In this direction, to evaluate the presence of real tasks in the INVALSI tests, we started categorizing each task of every INVALSI test in terms of its feasibility. Feasibility, in accordance with Palm (2006) is made by two aspects: *event* and *question*. Event refers to the possibility that the event described in the school task can take, or not, place in real-life. Question refers to the possibility that the posed question might be posed, or not, in a corresponding real-life event. Those tasks that positively reflected both the aspects of event and question had been categorized as real. Then, for those real tasks, their quality was evaluated. Quality refers to the degree by which a task is similar to a corresponding real-life event. Specifically, quality is defined considering five aspects from Palm's framework, namely *existence*, *realism*, *specificity*, *presentation*, *language*. For each aspect a score was given: 0 if that aspect had no correspondence in a real-life event; 1 if that aspect had a partial correspondence in a real-life event; 2 if that aspect had complete correspondence in a real-life event. In conclusion, to every real task was associated its level of quality by calculating the mean among the scoring (0, 1 or 2) given to its indicators (*existence, realism, specificity, presentation, language*).

To answer to our second research question, we analyzed data from each test using the Rasch model (Rasch, 1960) which is the one adopted also by INVALSI statistical team. The Rasch model is a simple logistic model belonging to Item Response Theory and it provides a joint estimate of the difficulty of the items of a test and of the ability of each student by placing them on the same scale from −4 to +4. Specifically, in this study we considered a parameter measuring the difficulty for each

[1] More precisely the sample was composed by 43586 students in 2009, 35567 in 2010, 31564 in 2011, 30870 in 2012, 24774 in 2013 and 25349 in 2014.

item (Delta) and a parameter measuring the fit of the item (Weighted). Other information could be identified observing specific graphs called Distractor Plots, output of the Rasch Model. The Distractor plot of an item allows the comparison between the Item Characteristic Curve (ICC, which expresses the probability of responding correctly to a specific item depending on the difficulty of the item itself and the ability of the respondent) and the empirical trend of each possible answer, as function of students' ability. Moreover, the software used to implement the Rasch Model (ConQuest) gives a parameter, belonging to the Classical Test Theory, which highlights how much each item discriminates between students with high ability levels and students with lower ones. All the items classified as real were then classified on the bases of difficulty (Delta), fit of the item with the Rasch model (Weighted), and discrimination. This analysis from a psychometrical perspective is performed at an item level: a single mathematical task might be composed by 1 or more items, and each item is identified by a request. Furthermore, we also considered the Distractor plot of each of these items and started investigating possible features such as under-discrimination/over-discrimination, particular distractor trends (for example humped performance trend - Ferretti, Lemmo & Giberti, 2018) but also guessing effect[2] (correct answers given regardless of being certain about its trueness).

FEASIBILITY		
	Yes	No
Event	The event described in the school task can take place in real-life	The event described in the school task can not take place in real-life
Question	The question posed in the school task might be posed in a corresponding real-life event	The question posed in the school task might not be posed in a corresponding real-life event

(Yes; Yes)

QUALITY			
	0	1	2
Existence	Data and information available in the school task do not correspond to the ones available in a corresponding real-life event	Data and information available in the school task partially correspond to the ones available in a corresponding real-life event	Data and information available in the school task correspond to the ones available in a corresponding real-life event
Realism	Values given in the school task are not close to values in a corresponding real-life event	Values given in the school task are partially close to values in a corresponding real-life event	Values given in the school task are close to values in a corresponding real-life event
Specificity	Specifications of the school task are not analogous to the ones in a corresponding real-life event	Specifications of the school task are partially analogous to the ones in a corresponding real-life event	Specifications of the school task are analogous to the ones in a corresponding real-life event
Presentation	The task is not presented and formulated in a manner analogous to reality	The task is partially presented and formulated in a manner analogous to reality	The task is presented and formulated in a manner analogous to reality
Language	The language is not analogous to the language that might be used in a corresponding real-life event	The language is partially analogous to the language that might be used in a corresponding real-life event	The language is analogous to the language that might be used in a corresponding real-life event

Figure 2: Classification of mathematical tasks

Results

In Table 2 results from the classification of tasks as real and in terms of their quality are reported. The distributions of real tasks vary between 24% and 45%. The quality of the real tasks was relatively high in every test. These results give some first suggestions concerning the theoretical framework on which the test is constructed and the type of mathematics that the test is going to assess. Indeed, the presence of a considerable number of real tasks, all of them of medium-high quality, shows how INVALSI tests stressed the importance to give students opportunities to face with real-life problems, in accordance with the definition of mathematical competence and the Italian National Guidelines.

[2] Indeed if we consider multiple choice tasks with 4 possible answers the probability for a student to answer correctly randomly is 25% but we have information regarding the intensity of this phenomenon considering the intersection between the Rasch model and the y axes of the distractor plot.

Table 2: Results from the classification of real tasks

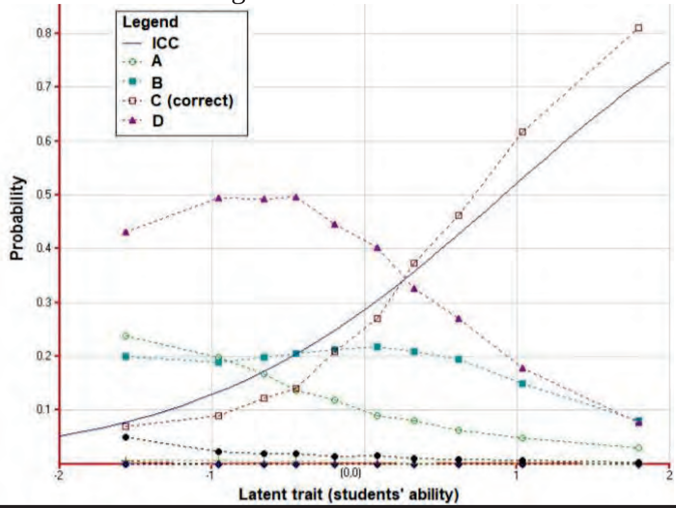
Year	Distribution	Total average of real tasks quality	SD
2008-09	24% (7/29)	1.4	0.3
2009-10	28% (9/32)	1.5	0.5
2010-11	33% (10/30)	1.7	0.3
2011-12	33% (11/33)	1.6	0.4
2012-13	31% (11/35)	1.7	0.3
2013-14	45% (13/29)	1.5	0.5

Then we classified on the bases of their psychometric features the 61 tasks (corresponding to 96 items in total) identified as real tasks. Considering items' difficulty (Delta parameter) we found that realistic tasks seem not to be a source of difficulty for students: the 35% of the items are easy with a delta smaller than -1; the 33% of the items shows a medium-low level of difficulty (-1<delta<0); the 27% of the items shows a medium-high level of difficulty (0<delta<1); and finally, we found only 4 difficult items (delta >1). The difficulty average is -0.61 highlighting that generally students are not particularly in trouble with real tasks. Furthermore, the item fit is considered good (values ranging from 0.8 and 1.2) for all the items except one, and the vast majority (84%) of the items identified have an optimal fit with values ranging between 0.9 and 1.1. Therefore, we can state that in the items identified the real context does not create a misfit with the model. The items identified as real are almost all discriminative items: the 81% have a good discrimination index (>0.3) and more than the 50% of the items have an optimal capacity to discriminate between students with lower ability levels and students with higher ones.

The items identified cover all item typology and all content domains but almost half of them belongs to the domain Data and Probability and only 5 items belong to the domain Space and Shapes, even if many real situations could become the stimulus for geometrical real problems.

In this research we started also to consider distractor plots of the identified real tasks, to detect possible recurring students' behavior facing a real task as function of their mathematical ability. This analysis will be the focus of further studies, but we report here an example of item analysis to explain its potentialities to investigate this issue. The task reported in table 3 was administered in the INVALSI grade 5 test in 2010. Accordingly to our proposed criterion, the task was classified as real with a quality of 1.2 (existence=1, realism=2, specificity=1, presentation=1, language=1). From a psychometrical perspective the item is medium-difficult with a difficulty index equal to 0.91 and the 32% of correct answers. The weighted index (0.9) highlights a good fit of the item. However, the distractor plot shows a little underdiscrimination of the item: the model overestimates the percentage of correct answers for students with medium-low ability and it underestimates it for higher ones. This characteristic might be linked to the influence of the real context that could give more difficulties for students of medium low ability levels due to the fact that they are not used, in the Italian school routine, to face numerous real problems. Moreover, also the trend of the incorrect answers is particularly interesting: distractor A decreases with students ability, this option might be chosen by students that find the correct answer “No” but they don’t link this answer to the correct argumentation; distractor B is constant for almost all ability levels; finally distractor D has an humped performance trend and is more attractive for students with medium-low ability levels. The guessing effect (probability for students of lower ability levels of choosing the correct answer randomly) is low as highlighted by the distractor plot.

Table 3: Example of a real INVALSI task (Task D14 grade 5 INVALSI test administered in 2010)

<table>
<tr><th>Task</th><th>Statistical data</th></tr>
<tr><td>Sandro has 20 dm of string to close four packages he needs to ship. For each package he needs 60 cm of string. Will he be able to close the four packages?
A. No, because 60 is greater than 20
B. Yes, because 20 dm are more than 6 dm
C. No, because 240 cm are more than 20 dm
D. Yes, because decimeters are larger than centimeters</td><td rowspan="3">Delta = 0.91 Weighted = 0.90 Discrimination = 0.51

</td></tr>
<tr><th>Our classification</th></tr>
<tr><td>Real task
Quality: 1.2</td></tr>
</table>

Conclusions

In this paper we reported the first results of an ongoing research project whose overall aim consists in studying what mathematical competencies and types of reasoning are intended to be assessed in the Italian standardized tests. Specifically, we focused on the presence of real tasks in INVALSI tests and on how students face with this kind of tasks. In order to find answers to our research questions, we firstly proposed a criterion, based on Palm’s framework (Palm, 2006) to classify real tasks. Results show that there is a consistent presence of real tasks in INVALSI tests. This fact suggests that one of the competences that these tests are intended to assess is the ability to develop and apply mathematical thinking to solve problems from everyday situations, as desired by the European Council and PISA. Concerning the psychometrical features, from a preliminary analysis of real task at an item level we can observe that, despite the limited use of real problems in didactical practices in Italy (Passarella, 2021), students do not reveal difficulties in these items, and most of them have a low or medium low difficulty level (Delta). This lack of work on real problems does not also influence the fit of these items except in a few cases. Indeed, real items are coherent with the latent trait measured by the whole test. An interesting outcome is that most of the real tasks are discriminative: students with high ability pass brilliantly these tasks, while students with low ability struggle more. Finally, we observe that in INVALSI test there is only a limited number of real tasks in geometry (Space and Shape), while this content domain could be one of the main interesting to promote this kind of tasks.

Future research is needed, particularly on: extending the analysis to tests up to 2019-20 and to the other grades of schooling, since in this study the focus was only on the fifth-grade; deepening the analysis of students' behaviors in facing real problems with some interviews to students and focus groups with teachers; investigating which other mathematical competencies are assessed in INVALSI tests; conducting a comparison with the PISA results in order to uncover and discuss differences or similarities between the two standardized tests.

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From paper-pencil to tablet-based assessment: a comparative study at the end of primary school

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Assessments are increasingly being designed on a digital artefact (computer or tablet), while in France the equipment rate in primary schools is still low. Some computer-based assessment tasks use specific software functionalities (the dynamic aspect of a geometric figure for example), while others "migrate" from paper and pencil (PP) to digital artefact without using specific functionalities of dedicated software. In this research, we are interested in the validity of assessment tasks designed on tablet (especially when students do not usually use a tablet in the classroom and/or in assessment situations) and in the effects of migration (from PP to tablet) on student performance and procedures from one medium to another.

Keywords: Assessment, validity, comparative analysis, tablet-based assessment.

Introduction

Most standardized large-scale assessments like PISA or TIMSS have ever migrated (or are going to migrate) from paper-pencil (PP) to digital artefact (DA), like computer or tablet. Several reasons could be quoted for justifying such evolution and designing digital tests like increasing efficiency and consequently reducing costs or giving instant feedbacks, especially in the case of formative assessments (Threlfall, 2007). In France, since 2019, at the end of elementary school (Grade 5), a representative sample of students takes with tablets (and not with paper-pencil, as they did before), a national test, which aims to assess mathematics skills and knowledge and to observe their evolution at six-year intervals.

All French elementary schools have been equipped with computers and Internet access, but students haven’t regular access to these digital environments: on average, there are 7.8 students per computer at elementary school (no data is produced about tablets) (Ministère de l’Education Nationale, 2018). These observations made us wonder about the validity of computer or tablet-based tests: if students are not used to doing mathematics with such artefacts, we can ask if the test assess really what it has to assess (mathematic skills and knowledge) or does it assess other competencies, like digital skills. More specially, is the test itself valid (especially when students do not usually use a tablet in the classroom and/or in assessment situations)? is longitudinal comparison performance (in the case of large-scale assessments) relevant? are mathematical processes to solve tasks the same when a similar problem is on PP or DA?

Previous research and theoretical background

In their review of assessment in mathematics education, Nortvedt and Buchholtz (2018) point to the development of digital assessments with parallel advances in psychometric models and the development of adaptive assessments. At the same time, they also highlight the limitations of such assessments, particularly because they often only assess knowledge on simple tasks (and do not allow for the assessment of students' problem-solving skills). The authors conclude that "technology can also limit what is assessed" (p. 560). In general, the design of digital assessments raises validity issues, and we will explain how we address them. We do not review in this text all the research about digital assessments, but we limit our subject to the comparison between paper-pencil and digital-based assessments.

Comparative research between paper-pencil-based assessments and digital ones

Numerous studies, especially the Anglo-Saxon ones, aim to compare success scores between PP and DA were carried out at different school levels and on various disciplines. First, we can observe that most of these studies are based on data analysis without considering students' procedures (see, for example, Hamhuis & al., 2020). Second, their results are divergent, and they do not allow us to conclude, on the fact that a medium (DA or PP) would promote or not student success (Lemmo & Mariotti, 2017). For example, in France, from the results of two large-scale assessments, Bessoneau, Arzoumanian, and Pastor (2015) identified two variables that particularly influence the success in a mathematical item depending on the medium used: the structure of the item (length of texts, number of documents, etc.) and the type of tasks proposed. In particular, the items presenting a syntactic and linguistic complexity relative to the statement are better succeeded on PP whereas, in the case of an item requesting a direct taking of information (in a table or on a graph), success is better on a DA. In addition, problems requiring several steps of resolution are more successful on PP.

About such performance comparisons, we share Lemmo and Mariotti's (2017) point of view: "task comparability cannot be measured only in terms of students' outcomes, but it is also established by the comparison between the solving strategies that they use" (p. 3541).

About validity and legitimacy, Threlfall et al. (2007) explore in their research several aspects of "what may be lost and gained by undertaking mathematics assessment on the computer" (p. 336) and their conclusion, based on student's performance is :

> It is not only that translating paper and pencil items into the computer format sometimes undermines their validity as assessments, it is also that some paper and pencil items are less valid as assessments than their computer equivalents would be. (p. 335)

Let us now explain how we study the validity of a test from a didactic point of view while considering the modality of assessing (PP or DA).

Validity and instrumental genesis

For studying the validity of assessment tasks, we have developed in previous research a methodology with two complementary approaches (Grapin, 2016): one epistemological and didactical and another psycho-didactical. The first approach provides us with evidence of validity based on the a priori analysis: for each task, we list, among other things, the different solving procedures, the possible errors, and their origins, but also the complexity of the tasks (Sayac & Grapin, 2015). This analysis allows us to study whether the solving of the task mobilizes the mathematical knowledge that we want it to assess. The psycho-didactical approach is focused on student activity, i.e., that they develop when carrying out the task; in our case, this includes their mathematical activity (their solving strategies), but it takes also into account the process of instrumental genesis (Rabardel, 1995). Moreover, “instrumental genesis is not the same for all students; it depends on their relationships with both mathematics and computer technologies” (Defouad 2000, as cited in Trouche (2005)). We hypothesize that students who use these artefacts routinely in the private sphere will have easier use of them in the school sphere; we also assume that students who have been able to use these artefacts in classroom situations will have a different instrumental genesis than others. So, to study the psycho-didactic validity of assessment tasks with a DA, it is therefore essential to determine students’ use of the artefact (whether at school or home) and to observe how they solve the task with this artefact.

For this comparative research, we also have to determine how and when two tasks could be considered equivalent. Riplay (2009) distinguishes migratory and transformative approaches to switch from PP assessment media to a digital one. The migratory approach consists of the transition of PP task to DA without any modification; in the transformative approach, the original PP-based tests are transformed with the integration of specific functionalities of the artefact. The migration of tasks also requires considering the functionalities of the software or the application used for the test.

Research goals

We have chosen to focus our research on the comparison between PP and tablet-based assessment because one of the national large-scale assessments in France at the end of primary school migrated from PP to tablet in 2019. This raised questions about the comparability of results from one year to the next, but above all for us, several questions about validity. In this paper, we focus on two aspects of our research: the way we designed the test and analyzed its validity (1) and the analysis of the first results considered instrumental genesis (2).

(1) We have designed the test to compare the students’ strategies in the case of migratory and transformative approaches: even if only some knowledge related to general tablet functionalities (virtual keyboard, drag and drop, virtual eraser in the draft) are needed to write or to provide the answer, we have considered that there was a change to the task (we will give examples in the following sessions). Under these conditions, is, a priori, the student's mathematical activity the same when solving a problem on a tablet or with PP? what are the differences in terms of mathematical strategies? And finally, are the same knowledge assessed?

(2) Since French schools are still poorly equipped digitally, we assume that students who regularly use tablets at home may have more advanced instrumental knowledge than others; since the time required must be short in an assessment’s context, it is therefore possible that tests on a DA generate academic inequalities linked to the artefact itself. Is that so? How can the use (regular or not) of tablets impact students' scores and procedures? What are the implications for task validity?

The first question will be principally dealt with in the following part (methodology and design of the tests), and the second, in the part dedicated to results.

Methodology and design of the tests

To study more specifically the students’ activity in a tablet assessment situation and to be able to compare it with that on PP, we conducted in June 2021 a study at the end of Grade 5 with 80 French pupils from priority education schools and from “ordinary” schools (choosing two types of schools will enable us to observe whether inequalities are generated by the artefact itself).

Administration of tests and survey

All the students took the same two tests (PP and tablet). Each test consists of solving 23 tasks whose required knowledge of whole numbers and decimals, arithmetic, and problem-solving. The paper-pencil-based test took place during a regular classroom math session. The tablet-based test took place after PP one with the two researchers in the classroom. We observed how students were using the tablet and the difficulties they might encounter, depending on the tablet’s functionalities involved in certain items. During this observation phase, we focus on the student's instrumental genesis and study whether it interferes with the student's mathematical activity. We don't ask the students individually about their procedures because the a priori analysis of the tasks enables us to infer their strategies from the proposed answers.

Moreover, we ask students if they have a tablet at home and how often they use it (at home and in the classroom). The answers to this short questionnaire will also make it possible to judge the validity of the tasks and to ensure that they do not generate inequalities between students (especially those who have a tablet at home and those who do not).

Design of both tests

The choice of the didactic variables’ values made it possible to design mathematical equivalent tasks; we describe, in this paragraph, with examples, how we have designed such tasks in migratory and transformative approaches.

First, we have chosen, as equivalent tasks in a migratory approach, multiple-choice questions, as the example below (Table 1).

PP task	**Tablet task**
What is the result of 45,83 × 10 ? *Tick the correct answer.* □ 450,830 □ 450,83 □ 45,830 □ 458,3	Quel est le résultat de 23,85 x 10? *Coche la bonne réponse.* ☐ 230,850 ☐ 230,85 ☐ 23,850 ☐ 238,5

Table 1: Example of a QCM in PP and tablet environments

We can observe, in this first example, that the two tasks (PP and tablet) involve the same mathematical knowledge with the same level of complexity; the wrong answers correspond to the same type of errors both on the PP task (PPT) and on the tablet task (TT).

When the transition of an item from the PP to tablet involved the use of the virtual keyboard or drag and drop, for example, we considered these modifications to be in a transformative approach. Nevertheless, we were careful to design mathematically similar tasks by choosing appropriate values for didactic variables, but let us explain, using the following example, how this type of transition can impact student answers. For assessing knowledge about writing numbers and units, we design the two following tasks (Table 2):

PP task		**Tablet task**	
Connect each number on the left with its equal number on the right.		Move the left or right label so that the number on the left is equal to the number on the right	
		Fais correspondre chacun des nombres de gauche avec celui qui lui est égal à droite.	
372 centièmes	0,372	483 centièmes	0,483
372 dixièmes	3 720	483 dixièmes	4830
372 unités	37,2	483 unités	48,3
372 dizaines	3,72	483 dizaines	4,83
372 millièmes	372	483 millièmes	483

Table 2: Example of a task using tablet functionalities

In this example, in the case of the TT, students can forget numbers or make mistakes but he or they cannot rely on one label on the right with two others on the left (unlike with the PPT). The treatment of the answers, especially in this case, is also simplified with the tablet.

We also want to study how students solve arithmetic problems, especially to observe how they use a draft on a tablet: on PP, they can easily make a diagram, write the operation, and use paper as a draft. With the tablet, we had provided a draft zone, but students need to understand pictograms (Fig. 1) for being able to draw, erase, organize their calculations.

Figure 1: Pictograms for using draft zone

Two types of problems were designed: a division problem (text of problem 1 - PPT: *"9 students of 6 years old must share 1,557 masks. How many masks will each student have?"*) and a number problem (text of problem 2 - PPT: *"Six 4-year-old students must share 6,000 sheets of paper. How many sheets of paper will each student have?"*). Students cannot use a calculator either in PP or with the tablet. For solving problem 1, they have to use a draft for calculating 1557:9 but for problem 2 they can mentally answer. With problem 1, we'd like to study how students use the draft of the tablet, and with problem 2, we'd like particularly to observe whether the tablet promotes mental calculation procedures.

Results

Results for all students

For all 80 students, the average success score in PP is 16 correct answers (out of 23 tasks) and 14.5 on the tablet. Now let's look at the difference in success question by question: we studied the number of correct answers per question, and we calculated the difference between the number of correct answers of PPT and the number of correct answers of TT (Figure 2). We observe for example that 4 students out of 80 did better on question 21 (QCM – the task is given in table 1) on PP than on a tablet.

On the whole test, only 2 tasks (Q1 and Q19) of 23 are better performed on a tablet than in PP, but only by 1 or 2 more students.

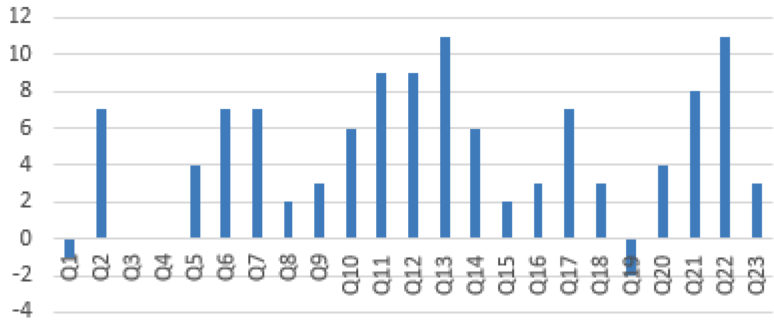

Figure 2: Difference between the success score in PP and on the tablet per question

It's not surprising that the division problem (problem 1 – Q13, quoted before) is more successful on PP than on a tablet, but we must study exactly what are the errors and procedures in PP and on a tablet for better understanding this result. We have the same analysis to do with other questions, especially when the difference between performance on PP and with the tablet is important. We'll then be able to determine the validity of such assessment tasks, whether PPT or TT.

Per student, we observe also that the difference in success score on the 23 questions between PP and tablet can vary between -8 and +14. 16 students have the same score in the two tests but 7 students have a score difference of 4 points (Figure 3).

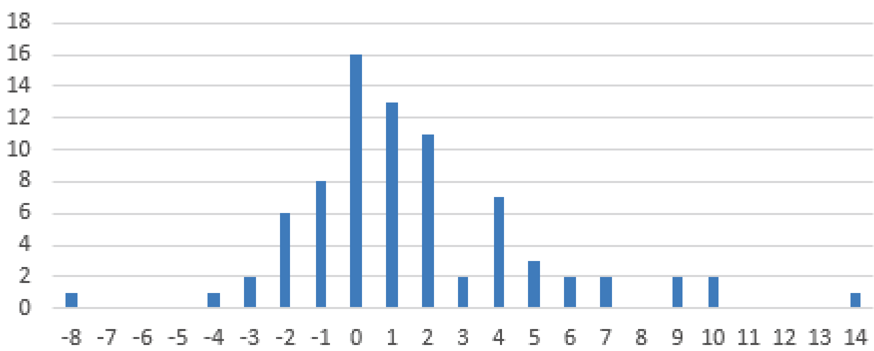

Figure 3: Number of students by the difference in success scores between PP and tablet

Results for students according to their use of the tablet

During the observation phase, we noticed that most students use the tablet by themselves, without help; only instructions on how to write the comma, use the “drag and drop” and erase in the draft have been given by the researchers. The answers of students confirm that 80% of the students regularly use a tablet at home (more than once a week) and only 17 students use a tablet less than once a month.

For these 17 students, we observe that the average success score both on PP (16,5) and tablet (14,5) is better than the average score of all students. We are currently further studying the responses and procedures of these students. We can observe for example, that, for problem 1 (division problem), which requires a little more advanced use of the tablet and the draft area, 4 students who had correctly solved the problem in PP were mistaken on the tablet (either because they did not answer anything, or because they made a mistake in the division or did not finish it). We cannot give general conclusions from this example, but during the presentation, we’ll present the detailed results, and we’ll try to show the relationship between the regularity of tablet use, the students’ mathematical activity, and their performance.

Conclusion

To complete these first results, we will reproduce this experimentation on a larger sample of students, taking care to change the order of the two modalities (PP then tablet *vs* tablet then PP). We also wish to integrate tasks that use other functionalities of the tablet (such as the zoom to place a number on a graduated line or the integrated calculator to perform operations in problem-solving).

The question of the validity of the assessment tasks is raised on the DA, as on PP. DA offers several possibilities for designing new types of tests especially diagnostic and formative ones with automatic feedbacks (for example, Sirejacob & al., 2019), but if the designers of these tests do not consider the specificities of the DA, the validity of the test is not guaranteed: students' mathematical knowledge is not correctly assessed and all the feedbacks are not suitable. Our research aims to identify points of vigilance about the validity of tablet-based assessment and the comparison between PP and tablet performance and strategy.

This research also allows us to better study *a priori* the complexity of a task on a tablet (Sayac, 2018; Sayac and Grapin, 2015) by adding a specific dimension related to the instrumental genesis and the functionalities of the support involved in the resolution of the task.

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Zooming in mathematical conversations in the light of the formative assessment

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This paper presents the results of a qualitative study focusing on characterisation of enactment of formative assessment during mathematical conversations by Norwegian primary school teachers. Two second-grade teachers working in a primary school in a large city in Norway were observed during mathematical conversations with their pupils in station teaching regarding various strategies for addition. We suggest a model that characterises the formative assessment enacted during a mathematical conversation from a teacher's perspective.

Keywords: Formative assessment, mathematical conversations, learning of mathematics.

Introduction

Assessment is vital to the education process and it has been on the political education agenda in many countries for several years, also in Norway. In 2010 the Norwegian Directory of Education and Training started a national programme The Assessment for Learning (AfL). The involved schools worked over a period of 16 months towards an overall goal, which was "to improve formative assessment practices in the classroom by developing distinct criteria to clarify how to reach curriculum goals" (Hopfenbeck et al., 2013, p. 28). This AfL initiative was a continuation of a previous programme (Improved Assessment Practice) and as Smith (2016) stated that "despite multiple initiatives, problems with implementation /of AfL/ have remained, and the changes in classroom practice have not gone as expected" (p. 182). Several Scandinavian researchers studied how teachers develop individual AfL literacy usually within an intervention (e.g., Engelsen & Smith, 2014; Andersson & Palm, 2017). However, little is known how Norwegian teachers from schools involved in AfL programme, are practicing formative assessment nowadays.

Assessment plays an important role also in the new Norwegian curriculum (LK20), which was launched in August 2020. The competence goals in mathematics are built around six core elements and assessment is described in a special paragraph for each grade (Utdanningsdirektoratet, 2019). As stressed in LK20, assessment should help promote learning and to develop competence in mathematics. Teacher should ensure good conditions for students' participation; provide guidance and adapt teaching so that students can use the guidance to enhance their learning. The role of both the teacher and the student in the assessment process is undoubtedly essential and although student's peers are not mentioned, we see signs of formative assessment (FA) here (Cowie & Bell, 1999), despite using the word "undervegsvurdering", which literally translates as "assessment along the way". In LK20 (Utdanningsdirektoratet, 2019) a particular emphasis is also given to oral skills when students should create meaning through conversation in and about mathematics. One way how to engage students into communication in mathematics is to lead mathematical conversations (MCs), which are "not only very good methods for teachers to elicit evidence of students' understanding and misunderstandings in order to inform the next steps in learning and teaching, they are in themselves powerful learning activities" (Swaffield, 2011, p. 443). The main purpose of a MC is to support and promote students' learning through a discussion in which students can clarify their own thinking and learn from others (Chapin et al., 2009).

In this context, and also in line with Bennet's (2011) criticism of FA related to its lack of conceptual understanding and exemplification in specific subject areas, we consider pertinent to shed a light on MCs in terms of FA. Thus, we seek to answer the following research question: *How can the enactment of formative assessment during a mathematical conversation be characterized for Norwegian primary school teachers who participated in the Assessment for Learning programme?* The study presented here was part of the main study, in which we, in addition, also tried to understand how the teachers who participated in the AfL programme, perceive FA in mathematics.

Theoretical framework

In the research field of mathematics education, several theoretical frameworks and models are designed to explain FA (e.g., Cowie & Bell, 1999; Wiliam & Thompson, 2007). Although the term *formative assessment* does not represent a well-defined set of artefacts or practices (Bennet, 2011), the following conceptualization by Cowie and Bell (1999) captures the meaning of many definitions found in the literature: FA is "the process used by teachers and students to recognise and respond to student learning in order to enhance that learning, during the learning" (p.101).

Wiliam and Thompson (2007) operationalized FA in a form where the three key processes of teaching and learning and the three agents in the classroom (teacher, peer and learner), are interconnected within the five key strategies. Cowie and Bell (1999) introduced a model (Figure 1) to describe and explain the nature of the formative assessment process in science education.

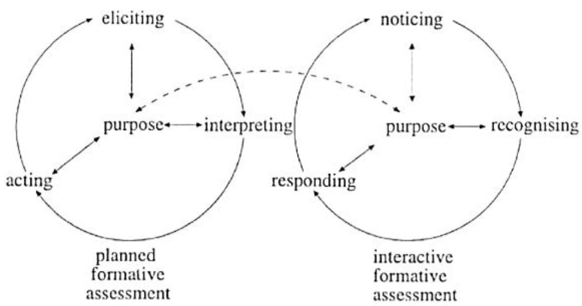

Figure 1: Model of formative assessment (Cowie & Bell, 1999, p. 113)

The model was developed from a consideration of the data collected through a research project investigating FA in science classrooms in New Zealand. It consists of two kinds of FA: *planned* and *interactive* (PFA/IFA). PFA involves *eliciting* assessment information using specific planned assessment activities, *interpreting* and *acting* on the information. The purpose for doing the assessment strongly influences the other three aspects of PFA process. IFA takes place in teacher-student(s) interactions that arise from the learning activity and are thus not planned. IFA, besides the purpose, involves also three aspects, *noticing* in the context of the learning activities, *recognizing* significance of what was noticed for the development of the student's understandings and immediate *responding*. It is usually used with individual students or small groups. Teachers switch between PFA and IFA as the purpose changes. The purpose of the PFA is to obtain information from the whole class about progress in learning as specified in the curriculum to inform the teaching. The purpose of the IFA is to mediate in the learning of individual students with respect to science, personal and social learning.

Although the second key strategy of Wiliam and Thompson (2007) mentions effective classroom discussions, the model of Cowie and Bell (1999) enables us to zoom into single episodes of MCs and study, how teachers support students' learning during the learning.

Methods

Research Context, Data Collection and Data Analysis

The data that constitutes the empirical base for this study is an observation of two second-grade teachers and their pupils during one mathematics lessons in late autumn 2018, as well as interviews with these teachers before and after the observed lesson. The interviews constitute the main source of data for other part of the earlier mentioned main study. For this study they play a supportive role, for example, regarding clarification of the learning goals of observed lessons, and additional information about typical way of performing FA, which helps us to answer our research question.

The chosen primary school in a large city in Norway was selected based on its participation in the AfL programme, as well as on its accessibility to the first author. Following criteria were used to select two teachers to participate in this study: (i) has worked as a teacher in the school for the last 8 years (participation in AfL); (ii) educated as a mathematics teacher (professional skills), (iii) teaches mathematics at the present time. Together with the headmaster of the school and the selection criteria, it was decided which teachers were asked to participate in the study. The names of pupils and teachers are altered; the teachers are called Jorunn and Hilde. In collaboration with Jorunn and Hilde and due to the purpose of the study, it was decided that station teaching was appropriate to observe. Station teaching is teaching divided into several learning activities. The teacher began the lesson by explaining the five learning activities and where in the classroom the assignments were located. Then the pupils were divided into five groups of 3-4 pupils, and the teacher decided which group to start with the different activities. The pupils worked on the same learning activity for 20 minutes. Switching from one activity to another was performed and controlled by the teacher. The content of the different learning stations was independent of each other but selected based on the teacher's thoughts on the balance between work and play. The observed station is called teacher-controlled station as the teacher participates in the pupils' learning work at that station, and none of the remaining four stations.

The learning activity the teachers chose for this station included a clue task (Figure 2), and it was for the first time Jorunn and Hilde used this type of task in a MC that focuses on addition strategies. The task consists of a text assignment and four clues. The teacher's guide to the clue task (Brataas, 2018) specifies that the teacher's role is to introduce the task, guide pupils, and lead a summary with the intention of having pupils present their solution suggestions. In other words, the learning activity is structured in a way that allows the teacher to gather information about the pupils' learning.

> The teddy bear tries to find out how many soft toys they are in Felix's room. Some are small, some are large, but all have a permanent position in the room. Can you help the teddy bear to find out how many soft toys there are?
>
> 1a. On the shelf above the bed are four cute penguins and two lurking foxes.
>
> 1b. On the bedside table, there is the teddy bear and four other soft toys waiting for Felix to come home.
>
> 1c. In the windowsill, there are twice as many soft toys as it is on the shelf above the bed.
>
> 1d. On the bookshelf, there are three times as many soft and scary dinosaurs as there are soft toys on the bedside table.

Figure 2: Clue task (Brataas, 2018, p. 2)

Both teachers used the same task in the teacher-controlled station, and both had the same structure during the lessons. The data in this study, therefore, consists of ten transcripts of MCs between the teachers and different pupils groups at the teacher-controlled station, five with Hilde (H1-H5) and five with Jorunn (J1-J5).

The analysis of the data was driven by deductive thematic analysis (Braun & Clarke, 2006) by using the four aspects (purpose; elicit/notice; interpret/recognize; act/respond) of FA from the Cowie and Bell model (1999) as codes. In the analysis we have chosen to disregard connection with PFA or IFA because our aim is not to focus on the type of FA, but its characterization during a MC.

Analysis

In this paper we present three episodes, as examples of enactment of FA during a MC. The first episode J1 is a conversation between Jorunn and a first group consisting of three pupils. Jorunn presented goals for the activity and equipment that was available on the table and gave a general description of the learning activity.

9	Jorunn:	And I have sort of a goal here, about what we are going to look at, it is about – can you three cooperate, we will look at this. Also, we will look whether you manage to double [the pupils answer yes along the way, nod and pay attention]
10	R:	What was double again?
11	Jorunn:	Yes, what was double?
12	pupils:	It's the same as plus.
13	Jorunn:	Yes, but can you give an example?
14	R:	Oh! Is it taking the same thing again?
15	Jorunn:	To take the same again, you knew it yourself! Very good. Also, there is to check if you guys had a little fun when we are done [the goal]. That's what I want us to do. Are we ready?

At the beginning of the activity, one of the pupils asked for an explanation of the term doubling (line 10). This episode may be an example of how Jorunn *noticed* and *recognized* the pupil’s input as a valuable contribution to the development of pupils’ understanding of the concept of doubling.

Her *response* was to give the pupil room to think, she repeated the question and also encouraged peers to help by asking for examples.

Four pupils participated in episode J3. As the pupils began working on the clue 1c, Jorunn elicited information about their learning. Large proportions of the conversation contain exchanges of meaning related to the concept of doubling.

150	Jorunn:	Yes, how many were there on the shelf above the bed then?
151	Z:	six, [counts further] seven, eight, nine, ten, eleven, twelve
152	Jorunn:	Yes, is anyone using a different strategy?
		[short break, accepting that pupil Z is counting further and actually calculating the number of toys in the windowsill]
153	Jorunn:	Do you think it was difficult and why did you write six plus five?
154	C:	I should actually have written six there [point to five on the sheet, the pupil C wrote 6 + 5]
155	Jorunn:	Okay, you can erase it. [pupil C erases it]
		[short break]
156	Jorunn:	But what strategy do you use to calculate six plus six then?
157	X:	six plus six
158	C:	ehm, I have a good one. Because .... eee sixes can be split into three
159	Z:	six
160	Jorunn:	mmm it can
161	C:	It can also be split into five, it can be split into five and five and two.
162	Jorunn:	Five and five and two yes, it works. Mmm very good.
163	Z:	six and six is
164	X:	if we have one, two, three, four,
165	Jorunn:	Was anyone using the doubling strategy then?
166	X:	five, five, six

In the episode J3, we see that Jorunn has deliberately tried to *elicit* information about the pupils' learning, but she received little response from them. It is possible to interpret it as an example of Jorunn adhering to the plan to elicit information about the pupils' doubling strategies (as expressed in the interview before the lesson), i.e. the planned formative assessment. But at the same time, it is necessary to be critical of what information Jorunn actually received and how she acted.

In the episode H3, four boys participated in the learning activity, and the episode is taken from the last part of the conversation, which is linked to the last clue 1d (in the Figure 2).

260	Hilde:	Then there's one clue left, boys, are we ready? [the boys nod] In the bookshelf there are three times as many soft and scary dinosaurs as there are soft toys on the bedside table. Wow…
261	Q:	three like that
262	Hilde:	if these are the soft toys [pointing to the pile they made with blocks to represent the soft toys] that are on the bedside table, and you should have

three times as much. [gasps a little] What does it mean when it's three times as much?

263 Q: three times as much

Pupil Q repeated the term with wonder in his voice (line 263), and further in the conversation several pupils tried to explain what they thought.

276 Q: So then I take three again, one, two, three, then I've come to nine.

277 Hilde: It sounds very clever, but it's not three, it's five. And five three times. [pupils think, and continue working for 2-3 minutes]

278 Q: five [holding up the hand] then to these I have ... I've done it two, ... fifteen!

279 Hilde: Fifteen! Oh, we have to make the last fifteen, you guys are ready?

Hilde confirmed the pupils' work before she told them that they had to start with the amount of five and not three (line 277). She praised the pupil's thinking and repeated the information about the amount with another wording 'five three times'. The *response* from the teacher led the pupils to continue to work. After a few minutes of work, the pupil Q addressed Hilde and explained 'five three times' by showing the amount of five with his fingers on his hand. The teacher responded by repeating the number and urged the pupils to add fifteen blocks to the pile they had made for the number of soft toys in the room. This episode is an example of Hilde's response when pupils worked to explore the concept of three times as much. The teacher let the pupils work on the assignment without interfering with the work process. If the teacher made a conscious choice by allowing pupils to explore the concept with the wrong amount, the choice can be seen in the context of interpreting information about the pupils. It is possible to interpret the episode as an example of Hilde *recognizing* the thinking of the pupils and giving the pupils confirmation of thinking before correcting the amount.

The interviews revealed that both teachers thought of MCs as a way of conducting formative assessment in their mathematics teaching, and especially with young students. It is their most typical and frequent way of conducting FA. They stressed the importance of getting feedback immediately:

> You have to do it /assess/ while they /pupils/ are in the process. ... we can always put a smiley face in the book or something like that, but it kind of does not become what is [pause for thoughts] … yes, in relation to pupils, because they are so much here and now. (Hilde)

> I would say that my practice has changed from being an "after-assessor" to being "along the way". So that I do not spend a lot of time on corrections afterward anymore, because, in relation to primary school, I think it has no sense. They /pupils/ are done. And when they are done, they are done. They do not look at what they did, so I try to put as much weight as possible along the way so they get the feedback as soon as possible. (Jorunn)

Discussion

In this paper we focused on characterization of enactment of FA during MCs. Our findings revealed that both teachers were continuously collecting evidence of pupils' learning by listening, asking questions, revising the learning goal according to elicited information, interpreting pupils' thoughts, rephrasing their questions, providing time for pupils to think, acting, etc. In other words, they modified teaching in relation to how pupils responded to the learning activity. FA enacted during a MC seems to be a dynamic process, in which its four aspects were interrelated, both in PFA and IFA. FA evolves like a spiral through these aspects, but at the same time back and forth through the aspects. Moreover, there is not always a straight forward process with one action following another, as rendered in Figure 3, which shows the circular characteristics of FA enacted during a MC.

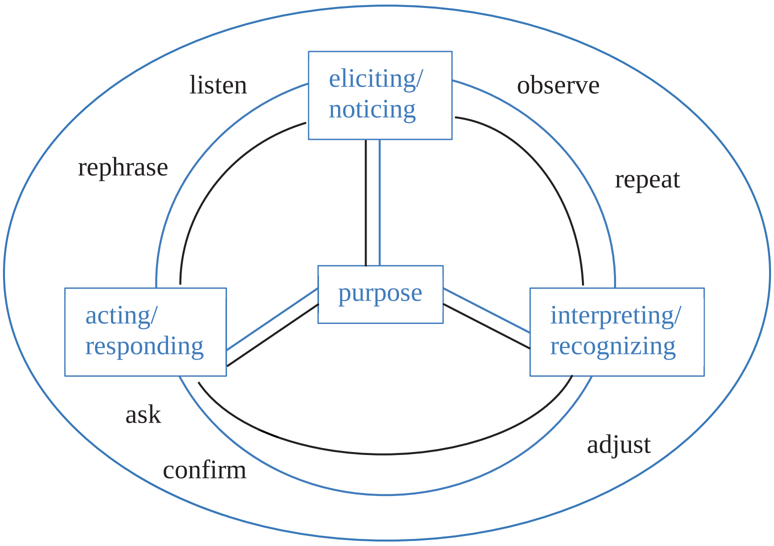

Figure 3: Our model suggestion of FA enacted during a MC, from a teacher's perspective

In Bell and Cowie's study (2002), science teachers indicated that eliciting and noticing were easier to do in the classroom than taking action and responding. Based on our observations, we concur with this finding, as the FA is "a complex, skilled task and it relies on the teacher's pedagogical knowledges" (Bell & Cowie, 2002, p. 92). Bell and Cowie (2002) suggest that "any future teacher development would need to focus on taking action and responding" (p. 94), as this determines whether the assessment is, in fact, formative or not. Bennett (2011) suggests that "to realize maximum benefit from formative assessment, new development should focus on conceptualizing well-specified approaches built around process and methodology rooted within specific content domains" (p. 5). Our study elucidates a MC as a specific way of conducting FA, which is in line with Ruiz-Primo (2011), who argues that assessment situations can occur in almost any learning activity if the teacher is aware of the student's learning. Faced with the new curriculum LK20, with an emphasis on speaking more mathematics, this study stands as an example of how two teachers assess "along the way" ("underveisvurderer") and/or act out FA during MCs. MCs are not FA per se. Our study contributes to the existing knowledge on teachers' FA practices when MC is the dominating teaching strategy.

Notes and acknowledgments:

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Investigating the characteristics of algebra tasks generated by preservice mathematics teachers

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This study was conducted to investigate the characteristics of algebra tasks that middle school preservice mathematics teachers developed at the end of a three-week training related to examining and categorizing algebra items in the previous high school entrance examinations (2018-2020). Twenty-nine third-year middle school preservice mathematics teachers participated in the study. The data of the study included 12 algebra tasks generated by preservice teachers at the end of the training and their characteristics. Findings showed that preservice teachers were able to develop cognitively demanding algebra tasks. Most of the tasks aimed to assess students' knowledge in geometry and measurement learning areas besides algebra. Lastly, most of the tasks were related to manipulating symbols, while four tasks focused on modeling problems using equations or algebraic expressions.

Keywords: Task development, preservice teachers, cognitively demanding algebra tasks, high stakes assessment.

Introduction

Assessment can be categorized based on its purposes. Formative assessment or assessment for learning is used purposefully for learning (Laud & Patel, 2013). It provides students and the teacher with a rich stream of information that can be used to adjust instruction to meet students' needs and enhance their learning (Wiliam, 2007). On the other hand, other potential actors such as universities, policymakers, and administrators need summative data since they cannot deal with the vast quantity of evidence collected through formative assessment (Burkhardt & Schoenfeld, 2018). Hence, summative assessment or assessment of learning is crucial to measure students' level of accomplishment, especially in countries where the result of high-stake exams may cause a big change in students' lives.

The results of students' performance in high stake exams directly impact teachers and students in some countries like the UK and the USA (Burkhardt & Schoenfeld, 2018), including Turkey. The types of tasks in the exams and the valued competencies influence teachers' actions in the classroom (Barnes et al., 2000) and therefore students' learning. In some cases, most classrooms' learning activities were reformed and became parallel to the task structures covered in the exams (Burkhardt & Schoenfeld, 2018).

In the case of Turkey, the High School Entrance Exam (HSE) system was changed in 2018. One of the fundamental changes made in the HSE was the structure of the items covered in the exam. Before the change, HSE measured basic skills at the level of knowledge, comprehension, and application. Now, high-level skills such as making interpretation and inference and analytical thinking are measured (Biber et al., 2018). Reports informing the results of the HSE exams held in 2018 and 2019 showed that mathematics tests had the lowest rate of correct response of all the subjects (Ministry of National Education [MoNE], 2018a; 2019). One of the reasons for this situation might be that mathematics teachers were unprepared for the cognitively demanding assessment tasks and could not find sufficient resources to use in their classrooms (Biber et al., 2018). Since tasks in the mathematics textbooks are not compatible with the HSE items and the resources teachers use in their lessons are insufficient, preservice mathematics teachers' awareness about the cognitively demanding HSE items needs to be improved. In addition, whether preservice teachers can generate cognitively demanding tasks is worthy of investigation since these tasks can be used as teaching tools as well as assessment tools.

This study aimed to investigate the characteristics of algebra tasks that middle school preservice mathematics teachers (PMTs) developed after participating in a three-week training related to examining and categorizing algebra questions in the previous HSE examinations (2018-2020).

Algebra is a bridge between mathematics and other branches of science (Erbaş et al., 2009). Research points out the importance of developing students' algebraic thinking starting from kindergarten (Stephens et al., 2017), and one of the ways this could be achieved is through professional development (Kieran et al., 2016). Although there are studies focused on investigating the questions that PMTs pose during diagnostic algebraic thinking interviews (e.g., van den Kieboom, 2014), limited research exists on examining PMTs' generation of cognitively demanding algebra tasks.

Kaput (2008) focusing on arithmetic and algebra problems proposed a framework that included two core aspects for algebraic thinking. The two core aspects of algebra were "(A): algebra as the systematic symbolizing of generalizations of regularities and constraints" and "(B): algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbol systems" (Kaput, 2008, p. 11). Kaput stressed that while both aspects of algebra are significant, school algebra generally focuses on Core Aspect B, more specifically reasoning and actions on generalizations. While in Core Aspect A, students are encouraged to notice regularities, generalize and represent those generalizations. In this study, Kaput's (2008) framework was used to categorize PMTs' algebra tasks with respect to the core aspects of algebra.

Measuring students' ability requires the classification of levels of thinking. Bloom's revised taxonomy provides a measurement tool for thinking and classifies thinking into six cognitive levels of complexity: remembering, understanding, applying, analyzing, evaluating, and creating (Anderson et al., 2001). In a similar way, Smith and Stein (1998) proposed four categories of cognitive demand; (i) memorization; (ii) procedures without connection (PW/oC); (iii) procedures with connection (PWC); and (iv) doing mathematics in order to help teachers select and create cognitively demanding tasks to increase students' ability to think and reason. They identified the first two categories as low-level demands while the last two categories as high-level demands. We used Bloom's revised taxonomy and Smith and Stein's (1998) categorization to categorize the cognitive levels of PMTs' algebra tasks in the study.

Methods

Basic qualitative research method (Merriam, 2009) was employed in this study in order to reveal the characteristics of algebra tasks generated by PMTs.

Participants and Study Context

The study participants were 29 third-year middle school preservice mathematics teachers who were enrolled in a four-year middle grades (grades 5-8) mathematics teacher education program at a public university in Ankara, Turkey. The program mainly offers introductory education and mathematics courses in the first two years. The departmental courses mostly start in the third year of the program. They include courses that focus on developing pedagogical content knowledge, such as the Methods of Teaching Mathematics in Middle Schools I and II. In their fourth and last year in the program, PMTs take School Experience and Practice Teaching courses. Related to the context of this study, a semester before this study was conducted, PMTs were enrolled in the Assessment of Learning in Science and Mathematics course. Throughout this course, PMTs were introduced to the different types of assessment, including formative, summative, and diagnostic assessment types. They were also asked to develop different assessment instruments, including multiple-choice, short response, true-choice, and open-ended with their rubrics.

Data Collection

The study's data were drawn from a study[*] that aimed to investigate middle school PMTs' conceptions of algebra, their awareness of the characteristics of algebra items in HSE, and their improvement in generating algebra tasks at the end of the training. The training was carried out in the Methods of Teaching Mathematics in Middle Schools II course, which focused on teaching "proportional and algebraic thinking", "statistics", and "probability", respectively. The Methods course was offered in the Spring Semester of 2020-2021 academic year by the first and second authors, and the researchers implemented the training of this study. The PMTs attended the course through online education via Zoom. The training took place as part of the algebraic thinking weeks.

The PMTs were asked to read the chapter on algebraic thinking by Van de Walle et al. (2013) and several other articles and a book chapter that were mainly intended for in-service and preservice teachers. These readings were discussed in class, and related activities were conducted in small groups. Usually, whole-class discussions took place after the small group discussions. At the beginning of the semester, two groups were assigned to design lesson plans focusing on algebraic thinking, choosing an objective from the curriculum. These plans were implemented through micro-teaching, and the groups received oral feedback from the instructors and their classmates. Then the training for this study started, which lasted for three weeks (12 class hours). PMTs were asked to analyze 2018, 2019, and 2020 HSE algebra items first individually and note the characteristics of the questions (objectives/contents addressed by the problems, and cognitive levels according to Bloom's revised taxonomy and Smith and Stein (1998), and their justifications). Then they discussed their analysis in the same groups of 4 or 5 each week. After small group discussions, whole-class discussions took place, and some sample items the researchers chose were discussed together. At the end of the training, groups were asked to generate two cognitively demanding open-ended algebra tasks in the same groups (6 groups and 12 tasks in total). They were also asked to write the related

[*]The research is funded by the Middle East Technical University Research Fund GAP-501-2021-10644.

objectives/contents from the curriculum, expected student responses (both correct and incorrect), and cognitive levels according to Bloom's revised taxonomy and Smith and Stein’s (1998) categorization.

Data Analysis

For the scope of this paper, we focused on 12 algebra tasks generated by PMTs at the end of the training and their characteristics. The data were analyzed through content analysis. Cognitive levels of the tasks were categorized based on both Smith and Stein’s (1998) framework and Bloom’s revised taxonomy (Anderson et al., 2001). We used Kaput’s (2008) framework to determine which core aspect (A or B) each task focused on. Kaput’s (2008) two core aspects and the cognitive levels of the tasks were determined separately by the researchers and later discussed to reach a consensus. Learning areas and contents were accessible from the national middle school mathematics curriculum (MoNE, 2018b).

Findings

Findings showed that all the tasks developed by PMTs were cognitively demanding algebra tasks according to Smith and Stein’s (1998) categorization and Bloom’s revised taxonomy. We classified all tasks developed by PMTs as PWC. Furthermore, according to Bloom's revised taxonomy, we classified PMTs’ tasks at two levels: nine tasks at the analyzing level, three tasks at the applying level. When we compared our and PMTs’ decisions about the levels, we found medium to a high inter-rater agreement. Specifically, out of 12 tasks, we found the level of the eight tasks the same using Blooms’ revised taxonomy (about 67% agreement); the disagreement was mostly between the levels of applying and analyzing. The agreement increased to about 92% for Smith and Stein’s categorization.

Most of the algebra tasks generated by the PMTs aimed to assess students’ knowledge in geometry and measurement learning areas. PMTs used geometric shapes to assess students’ ability in doing operations with algebraic expressions in almost all these tasks. More specifically, in these tasks, students were expected to measure the area/perimeter/length of a single or complex shape by making addition, subtraction, or multiplication with algebraic expressions. On the other hand, the remaining tasks were prepared to assess students’ understanding in only the algebra learning area. (See Table 1 for all 12 tasks)

To exemplify, in Task 11 (T11, see Table 1) generated by Group 6, students were required to use a proportional relationship to find the price of 1 L of Brand A and Brand B juice as algebraic expressions and solve first order inequalities with one unknown to reach the answer. This task assesses students’ knowledge only in the algebra learning area. We classified the task’s level of cognitive demand as PWC. Students need to engage with conceptual ideas, including using the information from the table provided, setting up an inequality to compare the prices of the juice brands for the same amount to complete the task and explain their reasoning. We also categorized it at the analyzing level considering Bloom’s revised taxonomy since it requires relating parts to one another and an overall structure (Anderson et al., 2001) and making connections between different contents, including integers, ratios, and inequalities.

Table 1: Characteristics of the tasks developed by PMTs

Groups	Ts	Smith & Stein	Bloom's Revised Taxonomy	A brief description of the contexts addressed by the tasks
G1	T1*	PWC	Analyzing	This task asks students to calculate the basal metabolic rate to decide on the number of calories people need to take to not gain weight and to propose a diet to have less/more calories.
	T2	PWC	Applying	This task asks students to find the shortest distance, which a goat can use to climb the top of the mountain using a right triangle.
G2	T3	PWC	Analyzing	This task asks students to divide a square field with one side $2x + 4$ cm into two equal parts to plant cotton to one part and corn to the other. Students are required to decide where two identical 360-degree rotatable fountains are set up to water the corn field's maximum area and find the least area in an algebraic expression where water cannot reach.
	T4	PWC	Analyzing	In this task, students are expected to form a T shape by using all tangram pieces in the square-shaped tangram board with an area of $16x^2 - 8x + 4$ cm^2 and write the algebraic expression for the height of the T shape.
G3	T5	PWC	Analyzing	Given a scenario, students are asked to create an Atatürk corner using rectangular materials whose areas and short side lengths were given algebraically. They are also asked to express the area of Atatürk's picture algebraically.
	T6*	PWC	Applying	This task gives a situation where the discount is applied for the amount of the tickets purchased and asks students to write inequalities that express the given situations. It also asks students to find the profit if two groups of students buy the tickets together instead of separately.
G4	T7	PWC	Analyzing	The task gives information that there are two gardens whose perimeters are equal. The area of one is $9x^2 + 18x + 9$ cm^2, while the other's area is 35 cm^2 and asks the sum of the values x can take. (Each side is a natural number).
	T8	PWC	Applying	In this task, a bus route is given on a map, and two different ticket types (students and adults) are defined. Ticket prices are given as algebraic expressions. Students are expected to find the cost of an adult ticket to go to a city in the route, examining two traveling situations and solving it.
G5	T9	PWC	Analyzing	This task asks students to create squares from cardboard with an area of $4x^2 - 8x + 4$ cm^2 and regular triangles and pentagons whose one side is half of the length of one side of the square. Students are expected to place these geometric shapes 3 cm apart on a rope with $15x + 45$ cm length and find how many geometric shapes are used.
	T10	PWC	Analyzing	This task gives a situation where the water pipes are laid in a square garden with a side of $3x + 6$ m. The length of pipes to be laid adjacent to the garden walls is $2x + 4$ m. Pipes narrow by half after 3 m. The narrowed water pipes pass over each other. Students are required to find the area in the garden where the water pipe is not laid.
G6	T11	PWC	Analyzing	In this task, two types of juice brands (Brand A 200 ml and Brand B 500 ml) are defined, and the price of each is given as algebraic expressions, $4x + 6$ and $5x + 40$ for Brands A and B, respectively. Students are expected to find how many Turkish Liras (TL) Ayşe paid for 1 L of juice at most if she bought Brand A juice and made a profit and to explain their reasoning. (The money she pays is an integer.)
	T12	PWC	Applying	In this task, students are given the information that the length of one side of a house with a square base is $2x + 2$ m, the area of one side of the house is two times the floor area, and the area of one window of the house is one-eighth of the side area. They are expected to find the area of the exterior of the house to be painted.

**Two groups developed questions that had multiple parts. For those questions, the highest level of cognitive demand or taxonomy level was noted as the levels of the questions.*

Task 7, (T7, see Table 1), created by Group 4, required to use algebra, geometry and measurement knowledge. Students were expected to factorize the algebraic expression to find the perimeter of the square and solve the resulting first-order equation with one unknown to find the sum of the unknown (x) values. We classified the task's level of cognitive demand as PWC since the solution requires cognitive effort, including setting up an equation to find the values for x. Students cannot follow procedures mindlessly, and they need to make connections between different learning areas, algebra, geometry, and measurement. We also classified the task as analyzing according to Bloom's revised taxonomy since the task requires analytical skills. Specifically, students were asked to find the possible values for x, using the information that the perimeters of the shapes were equal given the areas and factoring 35 to find the length of one side of the rectangle.

When we examined the tasks developed by the groups according to the framework put forward by Kaput (2008), most of the tasks were closely related to Core Aspect B, which was explicitly about manipulating symbols. Specifically, while four out of 12 tasks were categorized closely related to Core Aspect A, the rest were found more closely related to Core Aspect B. This finding could be because, based on our analysis, PMTs aimed to assess students' abilities to make operations with algebraic expressions in most of the algebra tasks generated. To exemplify, T11 was a task categorized more closely with Core Aspect A. Students are expected to use algebraic expressions to set up an inequality to model the problem context in the task. In comparison, we found T7 primarily concerned about solving equations to find the possible values for x. The other tasks related to Core Aspect A also involved modeling the problem context using equations or algebraic expressions.

Discussion

Mathematical tasks are at the center of students' learning since tasks give messages to the learners about what mathematics is and what doing mathematics includes (National Council of Teachers of Mathematics [NCTM], 1991). The tasks that mathematics teachers select, adapt, or develop and implement with their students have paramount importance since they also influence the level of students' learning. Hence mathematics teachers need to be aware of levels of demands to generate cognitively demanding tasks. On the other hand, some studies revealed that although higher-order thinking skills are crucial for education, what teachers understand from this and how they apply it in their instruction are unclear (Schulz & FitzPatrick, 2016). Turkish mathematics teachers can be an example of this situation since they had difficulty in finding resources for cognitively demanding tasks (Biber et al., 2018). This study showed that the algebra tasks PMTs developed were cognitively demanding. We classified all tasks generated as PWC. They were also at the analyzing (9 tasks) and applying (3 tasks) levels according to Bloom's revised taxonomy. This finding might indicate that PMTs were able to generate cognitively demanding algebra tasks. Even if PMTs created these tasks as possible HSE items, they might also be selective in the tasks they bring to the class and give place to the cognitively demanding tasks in the classrooms.

This study also showed that PMTs focused on geometry and measurement learning areas while generating cognitively demanding algebra tasks. They mainly preferred to assess students' learning in making operations with algebraic expressions by using geometric shapes. The same trend was also seen in HSE items. When we investigated the past HSE items, we observed that if the items were prepared to assess students’ algebra, geometry, and measurement knowledge, students were expected to do operations with algebraic expressions by using geometric shapes. Since PMTs investigated the HSE items of the last three years during the training, they might have been influenced by these items and tended to prepare similar items.

Regarding the analysis of core aspects of algebra, the findings showed that most of the generated tasks focused on calculation and solving instead of relating and representing. Kieran (2004) suggested several things to develop students’ algebraic thinking. These include “a focus on both representing and solving a problem rather than on merely solving it” and “a focus on relations and not merely on the calculation of a numerical answer” (p. 141). Although these tasks were developed for HSE, what is being asked in the examinations influences what teachers and students value in their classrooms (Barnes et al., 2000). Therefore, it is essential that PMTs also try to generate tasks that focus on generalizing, relating, representing, and solving, and calculating. We suspect that this finding might also be due to the tendency of the past HSE items that PMTs examined in the training. Although some tasks were related to Core Aspect A, we identified the majority associated with Core Aspect B.

As part of this study, PMTs only examined 2018, 2019, and 2020 HSE algebra items. Examining only the past HSE items might be a limitation that influenced the characteristics of algebra tasks generated by PMTs. Hence using different task sources can be recommended for future studies. Investigating the HSE items holistically, without differentiating the learning area, can also be suggested to see the general characteristics of items that are not specific to the learning area. Furthermore, since demanding tasks used in the classrooms positively affect students' learning, encouraging PMTs to develop cognitively demanding tasks in algebra and different learning areas can also be suggested.

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Written test with oral explanation during the pandemic

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In Hungary, assessment in school mathematics has always relied heavily on students' written work. Ensuring that only the student's own knowledge is manifested during the exam is a real challenge in distance learning. We combined the written test with verbal explanation by students. After writing down the solutions and sending them to the teacher, students also recorded short audio of each task. In this paper, we analyze students' oral explanations based on their written work, looking for additional information about students' thinking processes, furthermore, examine the quality of verbal communication.

Keywords: Curriculum based assessment, metacognition, online education, mathematical problem solving.

Introduction

During the pandemic, teachers experienced the differences between face-to-face and online instruction (Doucet et al., 2020). They realized worldwide that online schooling requires collecting and reviewing old teaching methods adaptable to the new situation or even searching for new approaches. It is true for the ways of assessment of students learning as well. In Hungary, assessment in school mathematics has always relied heavily on students' written work. Written responses to mathematical tasks are often the basis of summative and formative assessments. However, ensuring that only the student's own knowledge is manifested during the exam is a real challenge in distance learning. The written examination was not appropriate anymore, while the oral was too time-consuming and unusual for senior high school students. It was our starting point when we combined the written test with verbal explanation by students. After writing down the solutions and sending them to the teacher, students also recorded short audio of each task.

Taking not only aspects of teaching and learning but also researching into consideration, the so-called “loud test” seems to be a valuable tool to detect students' metacognitive activities as well. After the solution of the task is written down, the laud explanation forces the learner to deal with the looking back phase in problem-solving (Polya, 1945), which is the most neglected phase, as much research shows (Cai & Brook, 2006).

In this paper, we analyse students' oral explanations based on their written solution, looking for additional information about students' thinking process, furthermore, examine the quality of verbal communication. Qualitative analysis is done on a per-pupil basis. We formulated the following research questions:

1. Does the oral explanation contain more information about the learner's knowledge and thinking process than can be read from the written solution?

2. What types of metacognitive activities can be observed in the verbal explanations?
3. What characterizes senior high school students' professional language communication?

Theoretical background

In the educational process, we can distinguish three types of assessments: (1) assessment before instruction (diagnostic), (2) assessment during instruction (formative), and (3) assessment after instruction (summative). Formative and summative assessments are often characterized as assessments for learning and assessment of learning (Chigonga, 2020). It's clear that there is a significant overlap between assessment for and of learning. We designed summative assessment tasks to measure students' expected standards; however, we also aimed to get information about students' thoughts and possible misconceptions. It means that the assessment process was summative and formative at the same time.

The challenges to written assessments, especially that of problem-solving, are several. Most test situations require students to produce an extended written account explaining their problem-solving process and proposed solution. It is problematic because considerable skill is needed to make a clear and comprehensive description of the problem-solving process, a skill that students may or may not have (Monaghan et al., 2009). From this aspect, it seems that an oral explanation of the problem-solving process may be easier for students, as they do not have to transform their thoughts into a mathematically correct written form. Morgan and Watson (2002) argued that all mathematics assessment is interpretive in nature, so it also shows subjective features. Regarding the assessment of students' problem-solving ability, Teledahl concluded that ”The fact that there are different ways to interpret what the students have written further strengthens the conclusion that using this writing, to assess other mathematical abilities, may be problematic. ” (2017, p. 3602)

Huxham et al. (2012) note five main advantages to oral assessments: they (1) develop oral communication skills; (2) are authentic; (3) can be seen to be more inclusive; (4) can be a powerful way of evaluating understanding and (5) are more difficult to cheat in. Furthermore, in an oral exam, a small knowledge gap can be bridged with the teacher's help so that the student can solve a task that he/she would not be able to do in writing. Our “loud test” is similar to the oral exam in more aspects (see (1), (4), (5)), but the students have no opportunity to receive ongoing confirmation or help from the teacher. However, oral exams have not only advantages but also disadvantages. Anxiety and fairness are the most common concern (Iannone & Simpson, 2012). Since our “loud test” was not a typical face-to-face oral exam, we believe that anxiety did not significantly affect the results. As the evaluation did not coincide with the oral presentation, the teacher had the opportunity for a thoughtful assessment, so there was little harm to fairness.

Professional language communication is an essential element of both written and oral exams. The professional language of mathematics comprises technical terms (like geometric sequences or logarithms) and specific, often very concise, sentence structures. “In mathematics classes, we face the challenge of developing individual language use from orality towards literacy, in the direction of learning to speak and write mathematically.” (Marei, 2019, p. 1657)

Asking students to think aloud is naturally connected to metacognition because the situation itself contributes to rethinking the problem-solving process. According to Flavell et al. (2002), metacognition refers to people's knowledge of their own information processing skills, knowledge about the nature of cognitive tasks, and strategies for coping with such tasks. Moreover, it also includes executive skills related to monitoring, self-regulation, and evaluating one's own cognitive activities. Metacognition happens when students analyze tasks, set goals, implement strategies and reflect on their own learning (Spencer, 2018). Metacognitive experiences refer to a person's awareness and feelings elicited in a problem-solving situation (Schneider & Artelt, 2010). Libet's (2002) research suggests that roughly half-second time units can be associated with conscious decisions. From a pedagogical point of view, this means that if we make a decision during learning, it becomes conscious only a little bit later. It also follows that the speed of procedural metacognitive processes may prevent the individual from reporting them orally simultaneously as the process. However, the possibility of later conscious access and verbal reporting remains (Csíkos, 2017). According to our research setting, the oral explanation of the students took place later in time than the description of their thoughts.

Methodology

The target class consists of 29 high school students (Grade 12), 27 of whom participated in the research. They have four lessons per week taught by one of the authors of this article. The research was done in the school year of 2020/2021 autumn in a small town in Hungary. The schools were open from September 1 to November 11 of 2020, then online education started. The summative assessment fell on November 16, so most of the material was taught face-to-face. The topic of the assessment was geometric series. The first task was considered a routine task. Two terms of a geometric series were given, the first term, the quotient, and the sum of the first ten terms had to be calculated. The answer was two different geometric series. The second and third tasks were real-world problems, and the second was similar to the third but simpler. In this paper, the third task is highlighted:

> *The number of wild koalas in Australia is getting smaller and smaller. Surveys show that the number of koalas is decreasing by 9 % every year. In 2009, 43000 wild koalas were counted on the continent. Considering the same decrease, find the number of years after which the number of koalas in Australia falls under 60% of the 2009 data.*

Because of the school closure, the students responded to the written test at home. They also made audio recordings explaining how the tasks were solved (“Summarize and explain your proposed solution orally in 1-2 minutes.”). The time limit was 45 minutes to solve three tasks, scan the papers, record the oral explanations with mobile phones, and send them to the teacher. The students got scores for the written solution and the recording.

The recordings were investigated in the same way:

1. We identified the coding units; namely, we found those parts of the recordings which contain extra information, i.e., not just repeating or summarizing what is on the paper. (A recording may include more than one such unit.)
2. Using the content analysis method, these units were coded first according to three criteria: The extra information refers to A) one step of the solution, B) the entire solving process, C) the interpretation of the result obtained. For example, we coded the unit as A, if the student justifies one of the steps in the solution, B, if he/she explains the chosen strategy or model, or C if he/she notes

that his/her result is consistent with the problem situation or modifies it accordingly. These codes are closely related to the following metacognitive activities: A) monitoring, self-regulation, B) strategy, C) reflection, evaluation. (Not all extra information counts as metacognitive activity, but it creates the possibility of it.)

For each unit, a second code evaluates the quality of the oral explanation in terms of the mathematical language register: 1) Correct; 2) Sloppy (everyday language, but the mathematical content is clear); 3) Incorrect+ (incorrect terminology, but the mathematical content is recognizable); 4) Incorrect– (incorrect terminology, no recognizable mathematical content); 5) No (There is no mathematical content in the unit).

The recordings were analyzed independently by the two authors. In case of disagreement, the authors' consensus fixed the units and their codes. Figure 1 illustrates our coding system.

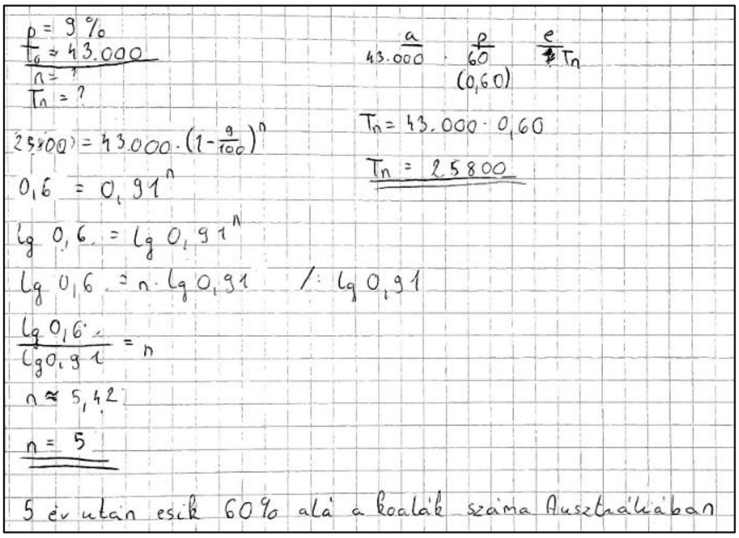

Figure 1: The written solution of Student S13

The associated sound recording with the codes, where the numbered units are written in italics:

"*In the 3rd task, we only know the rate of interest, the percentage, and the total value.* (S13-Unit 1, B, "Sloppy") and from these items, I calculated the value of T_n with percentage calculating. Then I got the result which I immediately substituted into the formula. *And here, in the parenthesis, there is a minus because there is a decrease.* (S13-U2, B, "Sloppy") I got the result 0,6 by dividing 25800 with 43000. The $0{,}91^n$ is the task in the parenthesis. *We brought in the base 10 logarithm because, with the use of it, we can get the result of n.* (S13-U3, A, "Sloppy") Then I arranged the things and calculated. I would rather say that I arranged the equation so that I could get the value of n. *I got a result of 5.42. Although the digit after the decimal point is less than 5, we have to lower the result because the task also includes a decrease.* (S13-U4, C, "Incorrect+") I also answered in a whole sentence because it was a word problem." (Student S13)

Findings and discussion

27 students wrote the test; each consisted of 3 tasks. Out of a total of 81 written solutions, 3 were not accompanied by audio recordings so that we could examine 78 audio recordings. There were 40 recordings, which contained extra information. These belonged to 19 students out of the 27. In their

audio recordings, 4, 9, and 6 students provided additional information to one, two, and three tasks, respectively. The number of coded units associated with the first task is the lowest, 18 units, while for Task 2 and 3, these numbers are 30 and 31. This result confirms that Task 1 was indeed a routine task for our students, in which the way to solve it was conventional, so the students did not feel the need to explain their answers in more detail. Task 2 and Task 3 were real word problems in which finding and matching a model was part of the solution strategy. Obviously, these problems required more thinking, and they provided more opportunities to express background information in the recordings. The coding results for the types of extra information (Table 1) are also related to this idea. In the routine task, the students did not interpret the answer at all, while in the case of world problems, the number of units where students tried to explain their way of thinking or interpret the result obtained is higher.

Table 1: The number of units according to what the extra information refers to

	A	B	C	Total
Task 1	10	8	0	18
Task 2	3	18	9	30
Task 3	10	13	8	31

The coded units were also examined according to the types of extra information and the number of students. Analysis of the audio recordings supports our hypothesis that students also perform metacognitive activities during the oral explanations. Table 2 shows some examples of this in connection with Task 3.

Table 2: The types of extra information and the number of students

Codes	Number of Ss	Examples in connection with Task 3
A: explanation of a certain step of the solution	14	“I brought in the logarithm because n was known. Because we can write the unknown n in front of the logarithm.” (S9) (self-regulation) “Then I got an exponential equation, but I don't have the same base, so I brought in the base 10 logarithm. This way, I got that lg1075 = n*lg1,09.” (S4) (self-regulation, monitoring)
B: explanation regarding the entire solving process	16	“In Task 3, I also calculated with geometric sequence.” (S27) (strategy) “The value of q is 0,91 because 9% can be replaced by 0,91.” (S25) (strategy)
C: the interpretation of the result obtained	11	“The answer is 6 years because I got 5,41641 and it is more than 5 years, so 5 years are not enough, that's why the result is 6 years.” (S25) (evaluation)

We can say that almost half of the class reached the goal of the recordings. Justifying certain steps, explaining the key elements of the solution process, and interpreting the result provides a great opportunity for some metacognitive activities.

Figure 2 presents the students' distribution regarding their scores on the written test and the number of coded units. The total score was 21; the red line in the figure shows the class average of 10.4. The blue points indicate the 19 students. The average score of the students who add extra information is 10.6 (deviation: 4.4), while the average score of the rest of the students is 9.9 (deviation: 5.0).

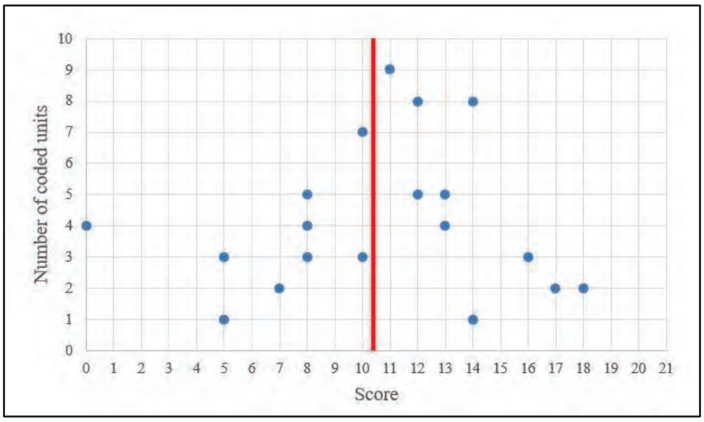

Figure 2: The scores and the number of coded units by students

The willingness to explain in detail does not appear to be closely related to the points achieved. The students whose points were close to the average provided the most extra information. Students with higher scores typically did not supplement their written solution with additional oral information, even if it was quite sketchy. We also found a lot of interesting and instructive information in the recordings of the students with weaker performance. The example below shows how S3 realized that his solution was wrong. He tried to modify but also incorrectly.

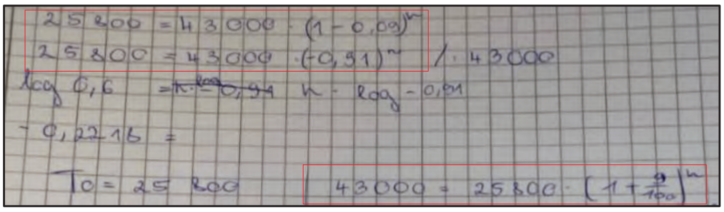

Figure 3: The written solution of Student S3

The explanation of S3: *"First, I did it with a minus, but it's sure didn't turn out well."* (C - reflection) *Then I made it with a positive and came up with 6 years, which I don't know … I don't think it is good either."* (C - reflection)

The quality of the professional language of the examined coded units in all three tasks is summarized in Table 3. The verbal expression of the coded units was mainly mathematically correct; there were no "Incorrect–" performances. It probably means that the students only dared to say things they were sure about to a certain extent. It is important because, in this way, they are reinforcing and deepening their right ideas, not their wrong ones. The laxity of the students encourages the teacher to further corrections, and some "Incorrect +" units bring the mistakes and misconceptions to the surface. These things remain unknown in the written tests, but they are considered mistakes in the oral explanation.

Table 3: The quality of the professional language

	Correct	Sloppy	Incorrect+	Incorrect-	No	Total
Number of units	26	28	15	0	10	79

The first example highlights the inaccurate formulation of one of the identities of exponentiation. The other points out the incomplete knowledge regarding the concept of even root power and the term of absolute value:

– Student S4 refers to one of the exponential identities as “$q^6/q^4=q^2$ because when dividing, we subtract the exponential value [instead of subtracting the exponents]”.
– Student S12 explained what he wrote ($q^4 = |16|$, then $q = 2$ and $q = -2$) as follows: “The value of q is 4 [instead of the exponent of q is 4)], and it is an even root power, that's why we have two quotients. We had to use the absolute value of 16 [instead of the absolute value of 2], so we got 2 and -2.”

Conclusion

We detected some positive impact of the “loud test”-method not only on learning but also on teaching. On the one hand, it was useful from the students’ point of view because students discovered errors in their written work during the recording that they modified, although not always correctly. On the other hand, at least half of the class prepared sound recordings that helped the teacher better understand their thought related to problem-solving and provided a more detailed error analysis, contributing to better future teaching (the effect of formative assessment). Furthermore, the “loud test” seemed to be a valuable tool to detect students' metacognitive activities. Loud explanations force learners to deal with the looking back phase in problem-solving, the most neglected step described by Polya (1945). However, the success was influenced by its quality and the student's content knowledge as well. The need to consciously apply the solution steps and use the appropriate professional language has also become apparent. In summary, we can state that the teacher gets a more detailed picture of students' current level of development, although the correction of students' works takes more time.

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Norwegian primary teacher education: Prospective teachers' responses to short written feedback

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This paper reports on a project in a mathematics course for prospective teachers, where the teacher educator modelled feedback based on formative assessment principles. We analyse and discuss the prospective teachers' responses when they are challenged to reflect from a teacher's perspective on how to use different models to compare fractions in a primary classroom setting. We find that the prospective teachers tended to use the feedback to move forwards in their teacher perspective, while some of them reflected on their uncertainty of how to use the models in a classroom. We argue that the prospective teachers got the opportunity to reflect on their own learning process.

Keywords: Formative assessment, teacher education, prospective teachers, feedback.

Introduction

Research on written feedback as formative assessment for mathematics teachers' education is sparse, however there is some work emerging. For example, Buchholtz et al. (2018) examined what learning opportunities could be identified when combining formative and summative assessment of prospective teachers' professional competence. Their findings showed that a significantly higher number of learning opportunities were perceived when these two forms of assessment were combined. In another study, Kastberg et al. (2020) found that mathematics teacher educators' written feedback could be described as effective when it gave prospective teachers the opportunity to build on their own answers; however, Kastberg et al. saw little evidence of feedback intended to help prospective teachers self-regulate their own learning.

Mathematics courses in teacher education aim to prepare prospective teachers to teach mathematics and provide experiences on which they can base their later pedagogical practices. Research suggests that prospective teachers benefit from experiencing for themselves the didactics the teacher educator intends for them to learn, especially if they receive an explanation of why certain practices are of value in classrooms (Rojas et al. 2021). Summative assessment is still the prevalent assessment method for teacher education courses (Mumm et al. 2015), which means that prospective teachers receive little prompt individual feedback during their day-to-day teaching and learning. This is also the case at our university, where, in our mathematics courses, students have one exam at the end of each semester (for a total of three exams over 1.5 years) and two or three longer written assignments (mostly in groups) each semester, for which they receive feedback. One way of giving students more continuous feedback is through formative assessment, which combines participants' awareness of learning goals, academic and process-oriented guidance such that it contributes to a community of learning (Black et al., 2003). Written feedback gives teacher educators and prospective teachers the opportunity to revisit the comments over time to reflect and discuss important points in their work (Kastberg et al., 2020). In this project we wanted to explore how short written feedback could be used in a mathematics education course. The research question discussed in this paper is: what characterises prospective teachers' answers to short written feedback intended to raise awareness of use of different fraction models for primary school teaching?

Theory

The concept of formative assessment is used in the literature with different interpretations (Mumm et al. 2015). In this study we use Black and Wiliam's (2009) framework which emphasise five aspects of formative assessment: clarifying and understanding learning goals, effective classroom learning activities, giving feedback that points forwards, using students in peer learning and getting students to own their own learning process. An assessment is formative if used as a guide to what the participants should learn, what they already know and where to go next (Wiliam, 2007). Short-written feedback can contribute to this process of learning, and Swaffield (2011) highlights for assessment to be fruitful, it is important that it is directed towards learning that occurs in activities that are taking place there and then. Furthermore, feedback has greatest effect when aimed at a particular task and have concrete suggestions for improvements (Hattie & Timperley, 2007). In a classroom, formative assessment can occur in two different ways: spontaneously or planned (Dixson & Worrell, 2016). Spontaneous formative assessment is characterised, for example, by the teacher taking hold of academic moments that appear in the teaching and allowing the participants to contribute with academic justifications and examples on the spur of the moment. Planned formative assessments allow teachers to become aware of participants' current competences, and such information can help facilitate and adapt further learning and teaching (Wylie et al., 2009).

Lunenberg et al. (2007) argue that teacher educators can model teaching in four different ways: implicit modelling, explicit modelling, explicit modelling by facilitating to classroom practice and connecting their own teaching with theory on how to teach. Implicit modelling is when the teacher educator uses themselves as good examples of how to teach. However, without an explicit discussion of why their teaching is a model for good teaching practices, there is a risk of prospective teachers not recognising the transfer value because they do not recognise the connection to practice and theory. Prospective mathematics teachers need to know not only the right answers to questions but also how to teach mathematics in appropriate ways to pupils. This can be achieved by teacher educators modelling, for example, how to use different models for teaching fractions. Olanoff et al.'s (2014) research summary showed that while student teachers often knew how to procedurally calculate with fractions, they struggled with knowing why the procedures worked and how to use number sense to understand fractions. The work developed in this study draws from the outlined theory, emphasising how formative assessment, using short written feedback contribute to guiding prospective teachers in their learning of teaching mathematics. Based on Hattie and Timperley's (2007) framework of effective feedback, which includes questions such as "where am I going", "how am I going" and "where to next", we see the goal of this study as an opportunity to identify prospective teachers use of such feedback.

Research design and context

Prospective teachers intending to teach in grades 1-7 (ages 6-12) in Norway have a mandatory mathematics course (equivalent to 30 ETCS), which is a blend of learning mathematics and learning how to teach mathematics. Pupils in Norwegian primary schools do not receive grades, so their own future assessments will be mostly formative. In this semester they studied 10 ECTS in mathematics (in addition to 10 ECTS in Norwegian and 10 ECTS in pedagogy). The course theme in this period was fractions; with the use of different fraction models (i.e., area model, set model, and place on a number line) and how to use visualisations and representations for pupils' understanding of fractions.

Each lesson started with an introduction to the intended goal, and at the end of the lesson, the prospective teachers were asked planned question(s) based on the lesson's teaching goal, in line with Swaffield's (2011) point that the assessment should be directed towards the activities at that point in time. The prospective teachers got about 10 minutes to independently answer these questions in a notebook and handed it in before leaving. The teacher educator would afterwards write short written feedback to their answers, in the form of questions and reflection notes. The notebooks were given back at the start of the next lesson. The prospective teachers got the opportunity to reflect and make written changes according to the feedback. The idea was to encourage them to evaluate their own work, challenge and explore their choices and teacher knowledge so that the gap between what the prospective teachers understood and what the teacher educator wanted them to understand decreased.

The teacher educator (one of the authors) was a teacher with more than 10 years teaching experience from both lower secondary school and teacher education. She was experienced in using formative feedback, primarily in lower secondary school, and familiar with Black and Wiliam's (2009) framework. During the project the authors of the paper discussed how to give feedback to the answers. All the prospective teachers in the class agreed to participate in this study; however, not everybody was present for each lesson. Data were collected during a five-week period, within a weekly session of three hours. However, in this paper we only analyse one cycle of feedback and answers after one lesson, in which the teaching had focused on how to visualise and compare fractions using different models. The stated learning goal of this was: ''you should be able to use different models for fractions and assess when the different models are suitable.'' The questions the prospective teachers answered at the end of the lesson was ''which of these fractions is larger, $\frac{9}{4}$ or $\frac{7}{6}$? Answer with as many visualisations as you know'' and the same question, with comparing the fractions $\frac{3}{7}$ and $\frac{4}{9}$.

Data analysis

The data analysed in this paper come from 30 prospective teachers who had answered the questions after the lesson and responded to written feedback. The feedback from the teacher educator was in the form of short questions and comments and varied from short encouraging answers, such as ''good'' to questions, such as, ''could you show this with more visual models?'' and ''could you show this more accurately?'' This included responses intended to encourage the prospective teachers to think about how to use their knowledge in a classroom setting, such as ''how could you show this for a 5th grade?'' and ''how could you do this if the pupil did not know how to find a common denominator?''

The coding and thematic analysis were done by all three authors. We read all feedback and answers individually. We discussed what we had seen and decided to code the feedback given by the teacher educator as ''mathematical feedback'', ''reflective teacher questions'', ''encouraging comments'' and ''other''. These categories were not mutually exclusive, some of the comments and questions were shared between content knowledge of fractions and how to teach fractions, while other questions and comments were focused on either content knowledge or teaching knowledge. The coding of the teacher feedback was afterwards done individually by the first and third author, and then compared and discussed until we reached consensus. In this paper we report on the analyses of the prospective teachers' responses to the questions and comments that gave them the opportunity to reflect and comment on the use of their knowledge in a mathematics classroom. We focused our analysis on the notebooks of prospective teachers who had been given what we coded as a reflective teacher question (22) and had in some way answered this question (16). These answers were then individually divided by the authors according to the prospective teachers' revisions and comments 1) with a focus on the mathematics content, 2) with a focus on how to teach, and 3) those that were either wrong or not in accordance with the feedback. Here we again compared and discussed for consensus in the grouping. These three categories enabled us to compare the prospective teachers' responses. Here, we report on three characteristic examples of the prospective teachers' answers.

Results

The analysis of the prospective teachers' answers to questions about how the knowledge could be implemented in a classroom showed that they answered quite differently to similar questions. The answers showed that the prospective teachers were unsure of how to use their knowledge in a teacher context, and some, in their reflections, also showed this uncertainty.

The first example is from a prospective teacher who answered the feedback ''which model do you think is best suited for explaining to a 5th grade'' which they had gotten in response to their answer to original question, ''which of these fractions is larger $\frac{3}{7}$ and $\frac{4}{9}$? Answer with as many visualisations as you know.'' There was also a comment on a more mathematical issue, where the teacher educator had drawn an arrow between the number lines and asked, ''what was important here?''

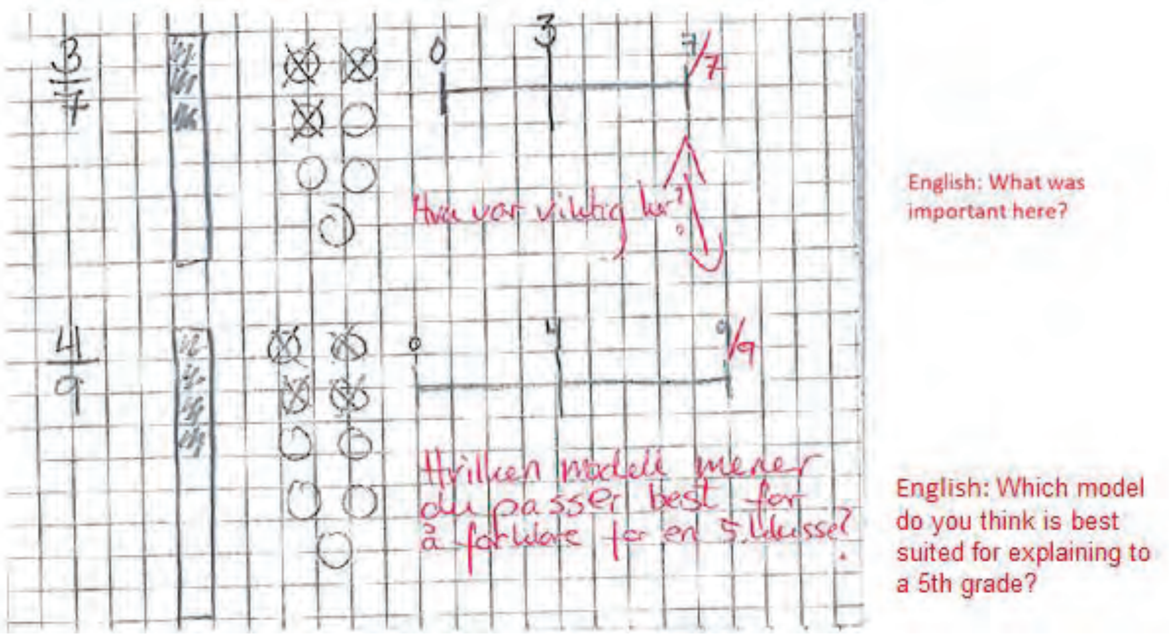

Figure 1: Prospective teacher's original answer, with feedback from teacher educator in red

In the prospective teacher's original answer, they used an area model, a set model, and a number line to visually represent the fractions $\frac{3}{7}$ and $\frac{4}{9}$. There was no written answer to which fraction is larger, and it was not clear from the answer how the prospective teacher's intended to use these to figure out which fraction is larger. They answered the feedback from the teacher educator and wrote

> To be best for a 5th grade it is important that the distance on the number line is the same for each value, and maybe it should have been marked [with] the numbers across…? Or maybe the most important [is] to show that 1 is $\frac{7}{7}$ so that it does not get confusing that $\frac{7}{7}$ is the value of one whole and not the value 7.

In their answer, the prospective teacher incorporated both the mathematical issue about a number line with 7 or $\frac{7}{7}$ and discussed what could be done when teaching pupils. They used the feedback to reflect on what could be important in a classroom, but they signalled uncertainty with their use of ''…? Or''.

Similarly, another prospective teacher had written in their original answer that their own drawing was inaccurate. The teacher educator challenged the prospective teacher to again think about how to compare $\frac{3}{7}$ and $\frac{4}{9}$ for pupils.

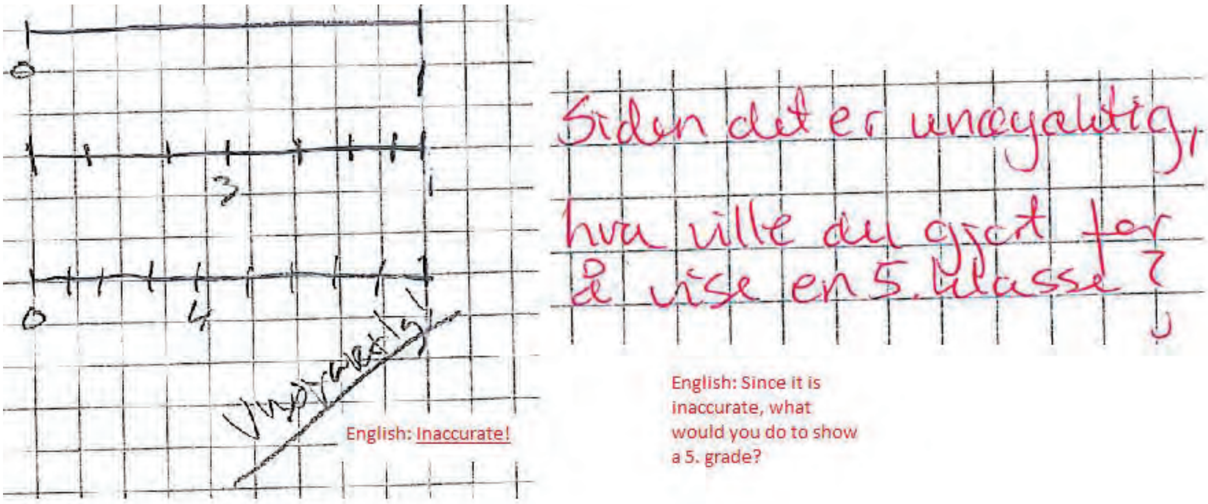

Figure 2: Prospective teacher's original answer (left) and teacher educator's comment (right)

This example shows how the prospective teacher used the feedback (see Figure 2), ''since it is inaccurate, what would you do to show a 5th grade?'', to answer, ''for explaining for a fifth grade I would focus on the explanation of parts and whole, and from there discuss these two magnitudes. I think. Difficult to answer this now.'' The answer shows that the prospective teacher's awareness that they could not fully answer the feedback and that they were still in a learning process.

The third example is from a prospective teacher who had gotten feedback on their answer to the question, ''which of these fractions is larger, $\frac{9}{4}$ or $\frac{7}{6}$? Answer with as many visualisations as you know.'' As seen in Figure 3 (left-hand side), in their original answer that they show knowledge about the different representations (i.e., circular and rectangular area models and number lines), but they did not show how these models could be used to compare fractions. Here, the teacher educator gave the feedback. ''Ok! How can you show this clearly for a 5th Grade which is larger?''

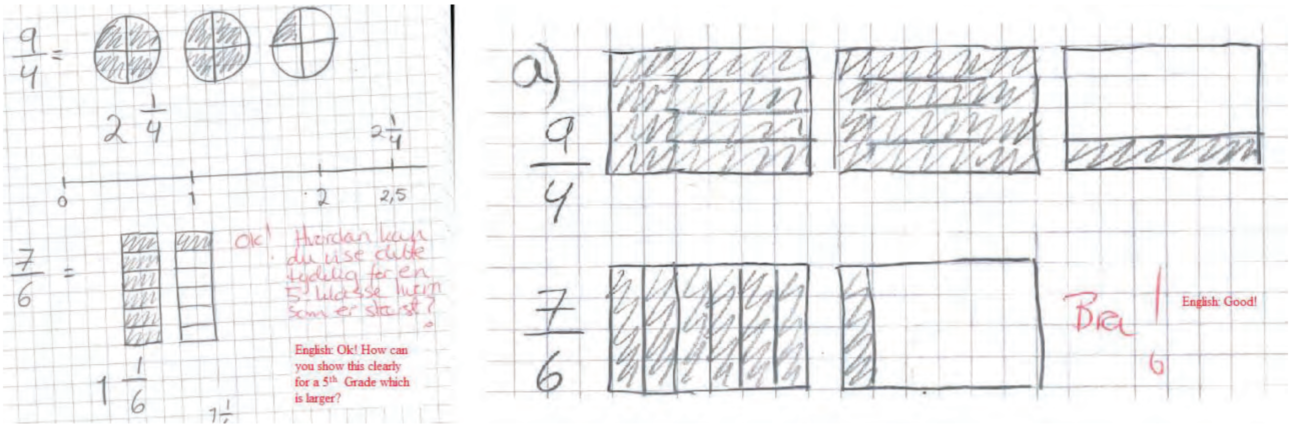

Figure 3: Prospective teacher's original area models (left) and revised area models (right)

The prospective teacher's revised answer (Figure 3, right-hand side) shows the fractions sketched using a rectangular area model, but here, they chooses the same shape and size for a unit fraction. Although there are no notes to accompany the drawing, the drawing is precise. It shows $\frac{1}{4}$ and $\frac{1}{6}$, and the rectangles are drawn above each other, which makes a comparison easy. However, in this question, it may have been obvious to the prospective teacher that $\frac{9}{4}$ was bigger than $\frac{7}{6}$, since $\frac{9}{4}$ is more than two wholes, and $\frac{7}{6}$ is just above one whole. Therefore, the shape of the area models may not have mattered to the prospective teacher in this case.

Discussion and concluding remarks

In this project, the teacher educator used implicit modelling for an assessment practice that the prospective teachers could use in their own classrooms. We saw that the teacher educator used different kinds of comments in the feedback, which ranged from encouraging remarks to mathematical comments, and comments directed at giving the prospective teachers a nudge towards thinking of their future classroom teaching. The three answers shown in this paper are examples of the prospective teachers' reflections after they had revived short written feedback intended to point forwards and help them take a teacher perspective. The prospective teachers, by answering the questions and receiving tailored comments, got the opportunity to reflect on how the fraction models could be used in teaching mathematics and experienced an assessment practice, which they also could use in their own teaching. This is in line with Rojas et al.'s. (2021) suggestion on how to improve learning processes in mathematics teacher education.

It seemed like that when first asked about comparing fractions in different ways, the prospective teachers answered with a focus on showing that they could draw different models. This may be a consequence of the question they were asked, which consist of two parts, both comparing the fractions and showing different visualisations. It may be that the prospective teachers focused on showing their knowledge of different fraction models and disregarded the point that they should compare the fractions. In the lesson the teacher educator had both showed how to visualise different fraction models and showed how to use them for comparison. With the original questions, the teacher educator's intention was that the prospective teachers should model ways that could be used in a classroom, so the questions in this case had two functions. The teacher educator could use the questions and answers with feedback to help the prospective teachers turn their attention towards 1) using the fraction models as teaching tools and 2) realise that there was a gap between the prospective teachers' answers and the teacher educator's intentions. For formative assessment it is important that learning goals are understood, and here the formative assessment gave both the teacher educator and prospective teachers the opportunity to adjust their understanding of the goal of the lesson. Such common understanding can contribute to a community of learning (Black et al., 2003).

The feedback they received gave the prospective teachers opportunities to assess and reflect on their previous answers. They showed that they were aware they were still in a learning process, and some of their answers communicated to the teacher educator their own uncertainty about how to respond to the feedback. We argue that the prospective teachers were in a process towards owning their own learning, which is an important aspect of a formative assessment (Black & Wiliam, 2009). The prospective teachers get little individual feedback during their mathematics courses in our institution. With such short-written feedback, they got the opportunity to reflect on if they could answer the questions related to the lesson. The feedback can use in their learning both immediately when they received the feedback and later while revising the course content on their own. We also found that the prospective teachers used this feedback loop to establish a rapport with their teacher educator and indicate where they were unsure about something, which was useful for both.

From a teacher educator's perspective, the formative assessment gives the teacher educator an opportunity to identify where the prospective teachers are struggling and provide feedback to help promote learning. This can be done for individuals as comments and by lifting problematic areas into the teaching of the whole class. In retrospect, we recognise that more thorough feedback could have been given in the answers. For example, this could be to point out more misunderstandings and give more specific positive remarks. However, because the project was intended to be carried out with teacher educators' busy schedule, it is important to reflect on which and how many comments are given.

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Differentiations and resources in assessing students' written texts in mathematics

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Assessment plays a very important role in the educational process in the subject of mathematics. This study investigates the way 33 secondary school teachers who teach mathematics assessed students' written texts in mathematics as well as the resources they draw on when they assess such texts. The research tool is a questionnaire based on three authentic students' written texts, graded by the mathematicians and the grading justification that was given. Data analysis revealed strong differentiations in grading. The resources on which the mathematicians draw on to justify the grading relate mainly to their beliefs about the nature of mathematics and their expectations of 'communicating' mathematical knowledge.

Keywords: Assessment in mathematics, resources, grading.

Introduction

Assessment has been disputed for ages despite its importance in the educational process as it is considered to legitimize social inequalities reducing them to inequalities of individual abilities. Research in mathematics education distinguishes three types of assessment of students' work: diagnostic, formative and summative assessment (van de Walle et al., 2017). Summative assessment has begun to be challenged at an institutional level while the usefulness of formative assessment has been highlighted.

In recent years, research has focused on teacher's pedagogical discourse regarding the assessment of students' work, i.e. the set of knowledge he/she uses and the practices he/she adopts when participating in the assessment process, and offers a framework that allows the identification of the concepts attributed by the teacher (Tang et al., 2012). While assessing, the teacher draws on resources, that is, structures of accumulated knowledge, which include language, representations, values and beliefs, to produce or interpret texts. Morgan and Watson (2002), considering the purpose of assessment to be the evaluation of students' achievements, highlight the complexity of the process, as teachers' judgments are influenced by various factors that 'force' them to draw on different resources every time. In this sense, it is to be expected to observe variations in assessment, which are due, among other things, to the assumption that there is not necessarily a 'relationship' between the text produced by a student and the meanings that the teacher, as a reader of this text, creates (Morgan & Sfard, 2016).

Earl (2006) adds students' perceptions of their capabilities. As a result, the feedback provided motivate the development of their self-image/self-concept. In addition the teacher's expected cognitive skills for the students, sometimes positively or negatively at times, affect the objectivity that assessment in mathematics, needs to promote. Zhao et al. (2016), in their study, refer to the assessment criteria adopted in mathematics curricula as important. Finally, Buchholtz and Nortvedt (2018) include in their assessment resources the different expectations and beliefs of teachers about students' 'cognitive-learning achievements' in relation to gender or the 'categorization' of students into 'majority and minority students'.

The focus of the present study is on the interpretative nature of the assessment of students' written mathematical texts, as well as the resources linked with these interpretations. The aim of the study is to show these resources used by teachers when assessing students' written mathematical texts.

The assessment of students' written texts in mathematics: theoretical framework

Morgan (2000) investigated the relationship between the mathematical knowledge 'shown' in students' texts and the mathematical text assessed by assessors. By identifying the features of the texts that were important for the teachers and the resources they used for assessing, Morgan (2000) categorized the resources and concluded that identical student's texts receive different ratings from teachers and she has identified the resources that teachers used to understand and assess students' written texts as follows: (a) a teacher's personal knowledge (PK) of the subject of mathematics and the relevant curriculum, combined with the affective aspects that are evident as a teacher's personal 'story' in relation to mathematics is written out, (b) teachers' beliefs about the nature of mathematics (NM) and the ways in which it is linked to assessment, (c) teachers' expectations of the ways in which the individual involved in the educational process can 'communicate' mathematical knowledge (ECM), (d) teachers' experience and expectations of their students and the classroom as a whole (EEC), (e) teachers' experience and their impressions and expectations of students individually (EES), (f) teachers' cultural background and language skills acquired (LC).

In the study of Morgan and Watson (2002), informal assessments appeared to be influenced by a variety of factors that had little to do with students' mathematical achievement, while the use of different resources, such as expectations about the nature of mathematics or personal mathematical understanding, including experiences, knowledge, beliefs and priorities can lead to very different judgments for individual students. Klothou and Sakonidis' research (2011) attempted to study the pedagogical discourse that primary teachers develop about students' mathematical achievement. The analysis revealed that different teachers can interpret the same or similar students' texts in many different ways, either by considering different elements as important or by assigning different value to similar features. Teachers' assessments are mainly fuelled by the resource about beliefs of the nature of mathematics and its connection to assessment, and tend to be informal in nature and clearly lack certain criteria. Assessment is perceived in different ways by teachers, with the main influencing factors being the curriculum, parents' perceptions of students' grades, cognitive criteria and personal relationships with students. Stagnation in the field of assessment was also evident, as grading is based on formal exercises and closed-ended questions.

The study

Traditional forms of assessment are now challenged by (a) the belief that mathematical knowledge is complex, so traditional forms of assessment are almost impossible to 'measure' it successfully, (b) the strong sense that even the most objective tests contain and record cultural biases, (c) the recognition of the dominant influence of the assessment system on the curriculum (Klothou & Sakonidis, 2015).

Given the importance of assessment in the cognitive, social and emotional development of the student, research findings point out that the same student's text can be read and assessed very differently by different teachers or even by the same teacher in different circumstances and moments depending on the resources that each one uses, take on greater significance. The different experiences, expectations, personal 'stories' of each teacher, as well as the epistemological consistency of each teacher, makes the meaning he/she attributes to and extracts from each text different.

In the Greek educational system, the importance attached to students' written mathematical texts is undeniable, however there hasn't been sufficient research in the field on secondary education. Focusing on secondary education and based on authentic written mathematical texts of students, we try to open a discussion by posing the following research questions.

1st research question: Are there differences in the assessment of the same written mathematical texts of high school students by mathematicians in secondary education?

2nd research question: What resources do mathematicians in secondary education draw on when grading an unknown student's written mathematical text?

The sample consisted of 33 mathematicians who taught mathematics in public schools and were selected by convenient sampling. Specifically, the sample consisted of 14 women and 19 men, from different places of Northern Greece. 16 of them have a postgraduate degree while 6 are in the process of obtaining one. Three of them have a doctoral degree and another 3 are in the process of obtaining it. 2 of them have 0-5 years of teaching experience, 4 have 6-10 years of teaching experience, 4 have 11-15 years of teaching experience, 11 have 16-20 years of teaching experience, 5 have 21-25 years of teaching experience and 7 have more than 25 years of teaching experience.

The research tool of the study was a questionnaire, which consisted of (a) 18 questions about their beliefs about the nature, learning, teaching and assessment of mathematics and (b) three authentic student' texts, which were graded by the teachers as well as the reasons for the choice of grade that was given. The present study investigates the second part of the questionnaire in order to study the resources adopted by the teachers in their assessment process in mathematics. This questionnaire was also used in a relevant related study with primary school teachers in order to compare possible differences in assessment between primary and secondary teachers.

This paper presents the results regarding the grading and the justification of the three authentic students' mathematical texts that was given by the mathematicians of the sample.

Mathematicians were asked to grade three authentic students' writing assignments in mathematics that involved: the first, solving a first-grade equation, the second, solving a problem and the third, solving a geometry exercise. The first of the three written texts was chosen because it is typical in written examinations in the Greek educational system for students in the second grade and it combines algebraic and arithmetic knowledge. The second one was chosen as a typical example of mathematization of a problem. The third one is a geometry problem including the drawing of a geometrical shape. All three written texts included parts that were answered wrongly as well as parts that were answered correctly because we wanted to see how teachers assess when they have dilemmas.

The questions to the mathematicians and the students' texts are presented below:

Question to the mathematicians: "Study a 14 year-old student's response to an exercise given on a written exam at the end of a school year. Score from 0 to 4 points and justify the mark given (decimal values are also acceptable)".

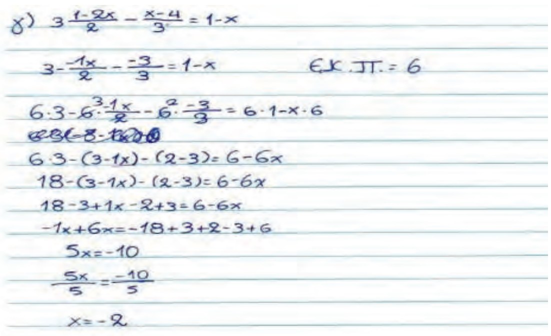
γ) $3\frac{1-2x}{2} - \frac{x-4}{3} = 1-x$
$3 - \frac{1x}{2} - \frac{-3}{3} = 1-x$ ΕΚΠ. = 6
$6\cdot 3 - 6\cdot\frac{1x}{2} - 6\cdot\frac{-3}{3} = 6\cdot 1 - x\cdot 6$
$6\cdot 3-(3-1x)-(2-3)=6-6x$
$18-(3-1x)-(2-3)=6-6x$
$18-3+1x-2+3=6-6x$
$-1x+6x=-18+3+2-3+6$
$5x=-10$
$\frac{5x}{5}=\frac{-10}{5}$
$x=-2$

Figure 1: Student's text 1

Question to the mathematicians: "Below is the response of a 15 year-old student to a test. The problem was as follows: Demis, Kostas and Maria have a total of 52 euros. Demis has three times as much money as Kostas and Maria has 1/3 of Kostas' money. (a) If Kostas has x euros, then how can we symbolize the money that Demis and Maria have? (b) Find how much money has each. Mark from 0 to 3.5 points for this text and justify the grade you have given (decimal values are also acceptable)".

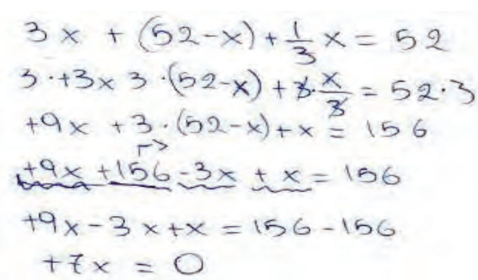
$3x + (52-x) + \frac{1}{3}x = 52$
$3\cdot +3x \; 3\cdot(52-x) + 3\cdot\frac{x}{3} = 52\cdot 3$
$+9x + 3\cdot(52-x) + x = 156$
$+9x + 156 - 3x + x = 156$
$+9x - 3x + x = 156 - 156$
$+7x = 0$

Figure 2: Student's text 2

Question to the mathematicians: "Below is a response to a test from a 15 year-old student. The exercise was as follows: Given an isosceles triangle ΑΒΓ (ΑΒ = ΑΓ) and its bisectors ΒΔ and ΓΕ. After drawing the shape, prove that (a) triangles ΒΔΓ and ΓΕΒ are equal, (b) ΑΔ=ΑΕ. Grade this text from 0 to 4.5 points and give reasons for the grade you gave (decimal values are acceptable)".

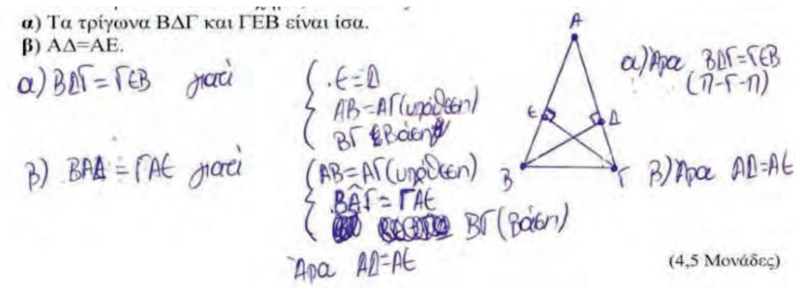

Figure 3: Student's text 3

Techniques of Grounded Theory combined with Morgan's (2002) categorization of resources were used in order to analyze the data. Careful readings of the data were carried out to identify the resources to which the teachers referred when assessing students' texts.

The abbreviations used for the resources on which the mathematicians drew on are: NM: teachers' beliefs about the nature of mathematics and the ways in which these are linked to assessment; ECM: teachers' expectations of the ways in which students can 'communicate' mathematical knowledge; NM+ECM: a combination of NM and ECM. The resources 'teachers' personal knowledge of mathematics and curriculum combined with the affective aspects of their personal 'story' in relation to mathematics (PK), 'experience and expectations for pupils and classes in general' (EEC), 'experience, impressions and expectations for specific pupils' (EES) and 'language skills and cultural background' (LC) appeared a few times, always in combination with the resources NM and/or ECM, were grouped under the name COMB.

Results

1st research question: The mathematicians' grades were converted into the scale of 0 to 10. The average of the three scores per teacher was also calculated. The grades are shown in Table 1, grouped within a range of 2 points.

Table 1: Descriptive statistics of teachers' scores

Grading	1st text	2nd text	3rd text	Average of the 3 written texts per mathematician
[0-2)	2 (6%)	4 (12.1%)	7 (21.2%)	3 (9.1%)
[2-4)	9 (27.3%)	3 (9.1%)	16 (48.5%)	10 (30.3%)
[4-6)	12 (36.4%)	14 (42.4%)	7 (21.2%)	14 (42.4%)
[6-8)	9 (27.3%)	8 (24.3%)	3 (9.1%)	6 (18.2%)
[8-10]	1 (3%)	4 (12.1%)	0 (0%)	0 (0%)
Total	33 (100%)	33 (100%)	33 (100%)	33 (100%)

Average	4.54	5.32	3.3	4.39
Standard Deviation	2	2.39	1.82	1.64

From Table 1 it is clear that there were strong differences in all 3 texts. The 2nd text was graded the highest and the 3rd text was graded the lowest. In the 1st text more than 1/3 of the teachers (33.3%) graded below 4, while 1/3 of the teachers (30.3%) graded more than 6, in the 2nd text about 1/5 of the teachers (21.2%) graded below 4, while more than 1/3 (36.4%) graded above 6; in the 3rd text 1/5 of the teachers (21.2%) graded below 2, while 1/3 of the teachers (30.3%) graded more than 4. In the average below 4, slightly less than half (39.4%) graded below 4, while less than 1/5 (18.2%) graded above 6.

2nd research question: Teachers were asked to justify their grades and specifically to mention any necessary information regarding their thoughts, opinions so that we could understand how they graded each text. The analysis was carried out by carefully reading the justifications and placing them in one or more resource categories.

Below there are some characteristic extracts from the discourse mathematicians used in the categorization. For the category NM: “most part of the equation is correct as well as the result”, “the student knows enough about the concepts, he can create an equation”, “the student did not understand that the answer is not ‘realistic’. For the ECM category: “the student does not state the criteria”, “the student does not justify the equations”, “the student does not use symbols”, “the student does not solve the problem questions”. For the categories NM and ECM the same mathematician stated: “the student does not write all the elements in the shape, ignores basic solving practices, has not understood the meaning, does not know basic things of geometry, does not adequately justify what he reports”, “the initial thought is correct, he did not clearly answer the questions and got the wrong result”. For the COMB category: “the student acquired the necessary procedures for solving”, “the steps for solving are correct, some misinterpretation in the operations, I could have given a slightly higher grade, if I judged that the student made a serious effort and I wanted to encourage him”.

Table 2 presents the results of the analysis on the resources used by the mathematicians.

Table 2: Frequency of mathematicians by resource

Resources	1st text		2nd text		3rd text	
	N	%	N	%	N	%
NM	23	69.7	15	45.5	6	18.2
ECM	0	0.0	2	6.1	2	6.1
NM+ECM	1	3.0	14	42.4	19	57.6
COMB	9	27.3	2	6.1	6	18.2

Table 2 shows that, for the first text, the majority of the mathematicians (69.7%) used resources related to the nature of mathematics, assessing understanding of the solution process and fluency in operations. Slightly less than one-third of the mathematicians (27.3%), in addition to beliefs about the nature of mathematics, drew on pedagogical-emotional resources, such as expectations for the classroom or for a particular student, either in the positive direction ("I give him/her a graceful grade for trying to get to the end of a process he/she does not understand") or in the negative direction ("mistakes that may have been allowed in earlier grades, but not now").

For the second text, the results are different. Almost half of the mathematician (45.5%) focused on whether the student knew how to mathematise the verbal description and then solve the equation. The other half (42.4%) additionally assessed how the student 'communicated' this knowledge and considered the absence of an answer to question (a) to be significantly negative. In relation to the third text, more than half of the mathematicians (57.6%) drew on their beliefs about the nature of mathematics and also on their expectations about how mathematical knowledge was 'communicated'. Symbolism, having justification and using one question to solve the next were highlighted as desirable elements. For the last one, one mathematician even stated that he does not take the "trivial solution" into account at all. Less than 1/5 of the mathematicians (18.2%) drew only on their beliefs about the nature of mathematics and 6.2% only on the way the student 'communicated' his knowledge. Almost 1/5 drew on other resources and one mathematician commented on the part of the answer that the student had erased. Finally, the mathematicians rated the construction of the shape from 0 to 1.5 when the perfect total grade was 4.5.

Discussion and concluding remarks

1st research question: Teachers graded the three students' texts very differently, which is consistent with the findings of other studies (Morgan, 2002) that the same text can be graded very differently by teachers who teach mathematics because it depends on the resources they use and thus the meaning that each assessor gives to the written text is not unique.

2st research question: Almost all mathematicians drew on resources relevant to their beliefs about the nature of the subject matter. In the two texts, 2 and 3, the mathematicians also referred very frequently to what they would like a mathematical text to include when the students communicate their knowledge. In a similar study concerning primary education (Klothou & Sakonidis, 2011), in all cases the teachers drew primarily on resources related to their expectations of how mathematical knowledge can be 'communicated' and then on resources related to their beliefs about the nature of mathematics. In this study, beliefs about the nature of mathematics, either alone or in combination with expectations about 'communicating', emerged as the dominant resources on which teachers primarily drew. Finally, the results of this study also show that the resource pattern used in primary schools, namely that of the expectation of 'communication', does not appear. The assessment of each text was recorded with its own characteristics, significantly different from the others.

Despite its limitations, this study essentially aimed at investigating the assessment process in mathematics in secondary education. It highlighted the complexity of factors that influence the outcome of student's assessment in mathematics and are equally related to the subject matter itself as well as to socio-emotional aspects of those involved, leading them to use resources that are in fact unique to each individual.

This interpretative nature of assessment, combined with the importance of its results for student' development and the criticism regarding its contribution to the reproduction of social inequalities, create a field that needs further study in order to bring to the fore the weaknesses and strengths of existing assessment frameworks. Such knowledge being shared in the educational community is a step towards the search for and adoption of frameworks that value the learning achievements and potential of each individual student.

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Handwritten math exams with multiple assessors: researching the added value of semi-automated assessment with atomic feedback

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Digital exams often fail in assessing all required mathematical skills. Therefore, it is advised that large-scale exams still feature some handwritten open answer questions. However, assessing those handwritten questions with multiple assessors is often a daunting task in terms of grading reliability and feedback. This paper presents a grading approach using semi-automated assessment with atomic feedback. Exam designers preset atomic feedback items with partial grades; next, assessors should just tick the items relevant to a student's answer, even allowing 'blind grading' where the underlying grades are not shown to the assessors. The approach might lead to a smoother and more reliable correction process in which feedback can be communicated to students and not solely grades. The experiment took place during a large-scale math exam organized by the Flemish Exam Commission, and this paper includes preliminary results of assessors' and students' impressions.

Keywords: Assessment, computer-assisted assessment, state examinations, feedback, inter-rater reliability.

Introduction

Regardless of all the practical advantages digital exams offer, Hoogland and Tout (2018) warn that digital questions focus on lower-order goals (e.g., procedural skills). They argue that handwritten questions are better suited to assess vital higher-order goals (e.g., problem-solving skills). Lemmo (2020) highlights substantial differences in students' thinking processes when the same question is asked digitally or paper-based. Bokhove & Drijvers (2010) point out that handwritten questions allow students to express themselves more freely. For all these reasons, Threlfall et al. (2007) advise deciding for each question individually whether the digital or handwritten mode is appropriate, leading to exams that are a mixture of both.

One major issue with handwritten questions is to find ways to assess them efficiently and reliably. Indeed, when the correction work is distributed among several assessors, guaranteeing grading reliability (Billington & Meadows, 2005) and consistent feedback (Baird et al., 2004) is challenging. Most exam designers try to ensure reliability by pre-developing a solution key with grading instructions for assessors (Ahmed & Polit, 2011).

General idea

In this paper, we introduce a novel approach to assess handwritten students' solutions with multiple assessors in a *semi-automated*[1] (SA) way: students solve these questions the classical way by writing on a sheet of paper. Next, these sheets are scanned, and assessors use the SA-system to correct the

[1] In the rest of this paper, we use the abbreviation **SA** to refer to our semi-automated assessment approach with reusable feedback.

solutions on a computer. Exam designers provide a solution key for each question consisting of different feedback items, written in an *atomic way* (see below), anticipating the most common mistakes. These feedback items can be linked to partial points for grading. When correcting a student's solution, the assessors have to select the appropriate feedback items, so the same feedback items are reused repeatedly. If a certain solution approach by a student is not covered in the available feedback items, an assessor can add a new feedback item. This new item is immediately available to all other assessors, leading to a *dynamic solution key* that expands as more and more corrections are made. When all assessors finish their job, the system produces individual reports for all students, including the grades and feedback.

In the following subsections, we discuss atomic feedback, the link with adaptive grading, and the idea of 'blind' grading.

Atomic feedback

Classic written feedback has traditionally consisted of long pieces of written text (Winstone et al., 2017). With its long sentences describing all of the errors in a student's work, classic written feedback is intrinsically not very reusable, as it is too explicitly targeted toward specific students. To overcome this difficulty and maximize the reusability of feedback, one of the key ideas underlying the proposed SA system is that it allows exam designers and assessors to write *atomic feedback* (see Figure 1). To write atomic feedback, one has to (1) identify the possible independent errors occurring and (2) write separate feedback items for each error, independent of each other. These atomic feedback items form a point-by-point list covering all items that might be relevant to a student's solution. The list can be hierarchical to *cluster* items that belong together (see Figure 1).

We have intensively studied atomic feedback in the first study of this PhD-project in the context of individual math teachers giving feedback to their students (Moons et al., 2020). The results of this study (Moons et al., in press) indicated that atomic feedback is significantly more reused than non-atomic feedback, so we found formal requirements to write feedback that can be reused. Teachers also tend to give more feedback when writing and reusing atomic feedback instead of saving time. However, since the atomic items are shared across multiple assessors in this second study, an additional criterium for atomicness is added: (3) a knowledgeable assessor must be able to determine unambiguously whether an item applies to a student's answer or not. As such, each item implicitly represents a yes/no question. Related atomic feedback items and intermediate steps in a solution key can share the same color to visually clear their connection (see Figure 1).

Adaptive grading

To obtain grades, exam designers can associate atomic feedback items with partial points to be added (green items in Figure 1) or subtracted. It is also possible to associate items with a threshold (e.g., 'if this feedback item is ticked, no points', red items in Figure 1).

The point-by-point list of atomic feedback items ultimately forms a series of implicit yes/no questions to determine the students' grade. Dependencies between items can be set, so items can be shown, disabled, or changed whenever a previous item is ticked, implying that assessors must follow the point-by-point list from top to bottom. This adaptive grading approach resembles a flow chart that automatically determines the grade, but – by ticking the items that are relevant to a student's answer – might at the same lead to several other envisioned benefits: (1) a deep insight into how the grade was obtained for both the student (feedback) as well as the exam committee and (2) a straightforward way to do correction work with multiple assessors as personal interpretations are avoided as much as possible (inter-rater reliability).

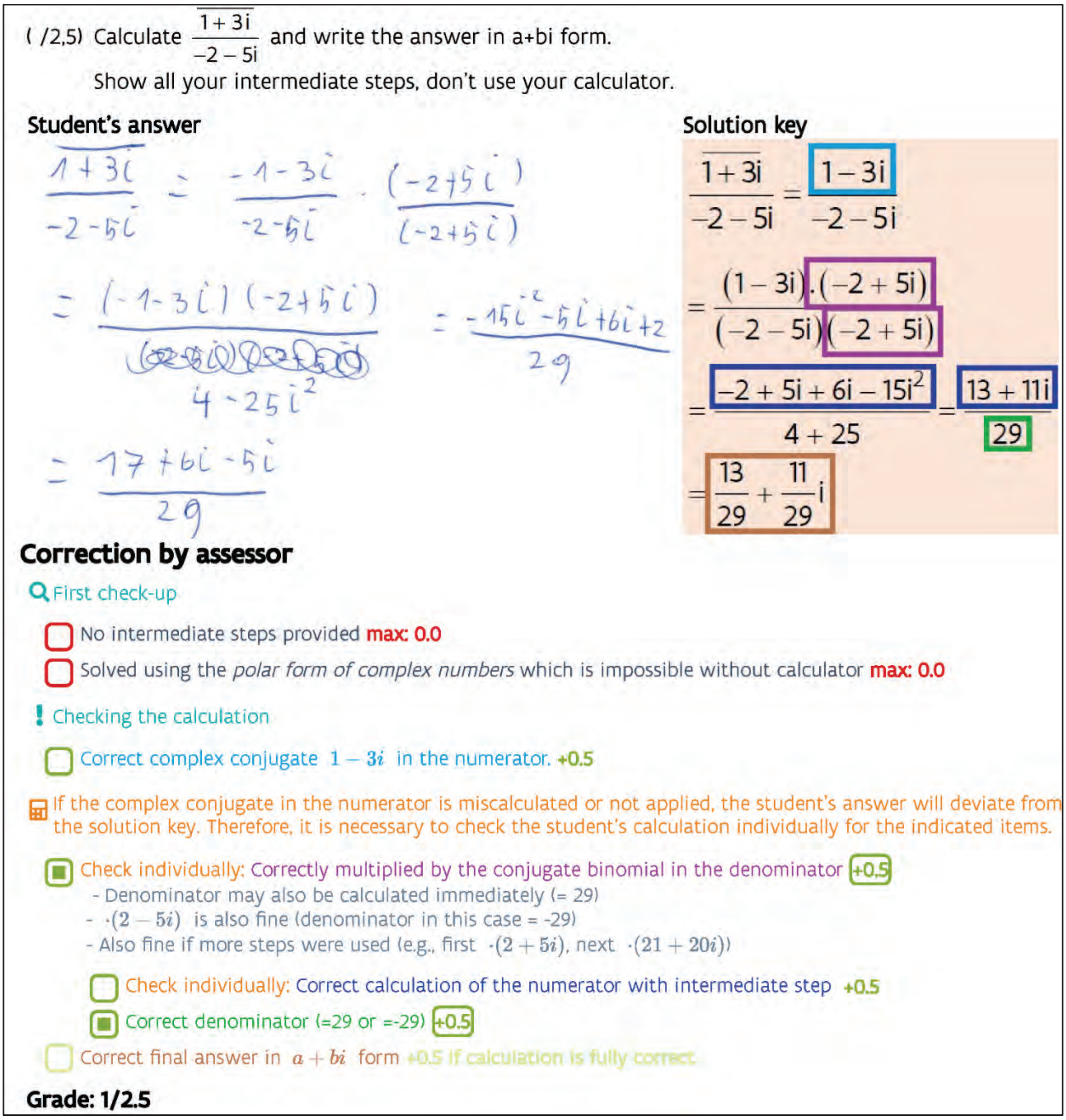

Figure 1: An example of adaptive SA grading with atomic feedback

In Figure 1, an example of the SA approach is given. The students' answer survives the 'First check-up' items; checking one of them would otherwise disable all of the following items. As the item 'Correct complex conjugate 1-3i' is unticked, the computer knows that a mistake happened; however, assessors should continue their assessment of the answer by taking into account that the students' steps will now deviate from the solution key for some items; these items are indicated by 'Check individually.' All the orange content would have disappeared when the item 'Correct complex conjugate 1-3i' had been ticked. The item 'Correct final answer in a + bi form' only gets enabled when all previous green items are ticked. The two ticked items each add 0.5 points to the grade, leading to a total of 1 out of 2.5.

Blind grading

Imagine that all references to points/grades disappear in Figure 1. This leads to the experimental idea of '*blind grading*' where the assessor chooses the appropriate feedback items without seeing the associated scores. The system still calculates the grades, but these are invisible to the assessors. The envisioned advantage of this grading approach is that assessors only need to focus on the content of a student's answer; any emotional barrier to select a feedback item disappears, possibly leading to higher grading reliability. Indeed, Ahmed & Pollit (2011) already indicated that deviations from a traditional solution key often occur when assessors disagree with the obtained grade. A possible disadvantage is that assessors might be afraid of being too lenient or too harsh. They lose an important frame of reference since they cannot compare if the calculated grade matches their sense of fairness.

The opposite mode of *blind grading* will be called '*visible grading*' in the rest of the paper; this is the standard mode where assessors can see the associated points for every feedback item and the calculated total grade (see Figure 1). Note that blind grading should not be confused by anonymous grading (Hanna & Leigh, 2012); in anonymous grading, assessors do not see the student's names to avoid certain biases (e.g., gender, ethnicity,...).

Research questions

After introducing the general idea and the key concepts, we present the first two research questions associated with preliminary investigations on assessors' and students' impressions of SA grading.

- (RQ1) How did assessors appreciate the SA system with blind/visible grading regarding perceived usefulness and ease of use?
- (RQ2) How did the students perceive their personal atomic feedback with grades?

Methods & Materials

The experiment is being executed in association with the Examination Commission of the Flemish government. Flanders is the Dutch-speaking part of Belgium. Flanders is a region without any central exams (Bolondi et al., 2019): every secondary school decides autonomously on the assessment of students. Consequently, the Examination Commission does not organize national exams for all Flemish students but organizes large-scale exams for everyone who cannot, for whatever reason, graduate in the regular school system. This way, students who successfully pass all their exams at the Examination Commission can still obtain a secondary education diploma. Students participating in these exams prepare autonomously or use a private tutor/school. The Examination Commission only provides clear guidelines for students on the content of the exams, carries out all the exams, and awards diplomas, but does not provide any teaching activities to students. We received ethical clearance from the ethical committee of the faculty of social sciences from the University of Antwerp.

Materials

Development of the ‘group’ SA-system

We developed an adaptation of the SA tool (described in Moons et al., in press) ready for handwritten assignments with a group of assessors. The tool is integrated as an advanced grading method in Moodle, an open-source e-learning platform. As Moodle is a framework offering many readily available components (such as a grade book, log in and uploading assignments,…), it guarantees rapid application development. The group assessment tool contains all the features explained in the introduction of this paper.

Mathematics exam

The mathematics exam for this experiment was developed by the exam designers of the Flemish Examination Commission in the way they always develop exams. Their solution key was turned to atomic feedback items for SA grading in close cooperation with us. The exam was one of the two math exams for the advanced mathematics track of Flemish secondary education and features complex numbers, matrices, space geometry, discrete mathematics, statistics, and probability. Interestingly, the exam is already a mixture of fully automated and handwritten questions: 46% of the exam grades are obtained with digital questions. Our experiment will only focus on the 54% part that consists of 10 paper-based questions with an open-answer format.

Survey based on the TAM model for assessors

We developed a short, validated survey based on the Technology Acceptance Model (Davis, 1989) to measure how assessors experienced the SA system’s usefulness and ease of use. The survey distinguished between the SA system using visible grading and the SA system using blind grading.

Survey based on feedback perceptions for students

We constructed a questionnaire loosely based on Weaver (2006) to measure how students perceive the personal atomic feedback they received.

Participants

60 students participated in the math exam linked to this study. The grading work was distributed among the 3 exam designers (employees responsible for the math exams of the Flemish Examination Commission) and 7 external assessors. These external assessors are mathematics teachers across Flanders who do this as a side job.

Methods

The Examination Commission designed the exam in August 2021. Next, their correction key was transformed to atomic feedback for SA grading in close cooperation with us. In October, all assessors received training with the SA system using a demo exam. The students took the exam on the 29th of October, 2021. All their answers were scanned and made available in the SA system. Every assessor had one month to correct the exams with the SA system. All student’s exams were distributed among the assessors. Especially for this experiment, all assessors got 30 randomly selected exams on top that were corrected by all (assessors were not aware of this). Half of the assessors corrected the even question blind, the other half the odd questions. Correctors filled in the survey based on the TAM model when they finished their assessment work. At the end of November 2021, all exams were corrected, and a personalized survey was sent out to the students. In this survey, students got access to their solutions to the questions, together with the provided atomic feedback (see Figure 1). The survey probed students' understanding of their feedback, how they liked the atomic form of the feedback, and the usability of getting feedback along with grades. After completing the survey, the students were invited for an in-depth interview on the same topic as the survey.

In February 2022, all assessors will re-correct the 30 exams corrected by all, but this time in the traditional way of the Examination Commission. In this traditional way, they must just communicate a grade for each question based on a paper-based solution key. This re-assessment will give deep insights into the inter-rater reliability of (visible/blind) SA grading versus traditional grading.

Results & Discussion

The results of (RQ1) on the assessors' views measured using the TAM model are shown in Table 1. The scales are measured on a 7-point Likert scale.

Table 1: Results of the TAM model by the assessors for both visible as well as blind SA grading

Scales	**Visible SA grading** M±SD	**Blind SA grading** M±SD
1. Perceived Usefulness	5.7 ± 0.7	4.6 ± 1.5
2. Perceived Ease of Use	5.4 ± 1.0	4.5 ± 1.4
3. Anxiety	2.5 ± 1.1	3.6 ± 1.7
4. Attitude Towards Using	6.1 ± 0.8	4.4 ± 1.7
5. Behavioral Intention to Use	5.6 ± 1.2	4.4 ±1.7

Table 1 shows that assessors have a strong attitude towards using visible SA grading, meaning that they like working with the visible SA grading system. Assessors rated their anxiety for visible SA grading as low and gave a high rating to the perceived usefulness, perceived ease of use, and the behavioral intention to use for visible SA grading. Blind SA grading was less appreciated on all scales. All assessors (100%) indicated they preferred visible over blind grading. Reasons given include the lack of control (71.4%), an alienated feeling (57.1%), and fear of missing items to be ticked (42.8%).

For (RQ2) on the student's perceptions of their received personal feedback (see Figure 1 for an example), the corresponding survey items are listed in Table 2. Of the 60 students who took the exam, 36 students participated in this online survey (60%). Results are expressed on a 7-point Likert scale and indicate that they would greatly appreciate if the Examination Commission would adopt this approach. Students feel that they understand their atomic feedback, learn from it, and see the connection with the obtained grades. It is important to remember that these results are entirely based on self-reporting, and other qualitative techniques (which are being carried out) are necessary to check if students indeed understand the given feedback.

Table 2: Overview of the students' survey items corresponding to their personal feedback

Students' survey item	M±SD
1. My feedback was too uninformative or brief to be helpful	3.6 ± 1.9
2. My feedback encouraged me to improve	4.7 ± 1.7
3. I will make even better exams based on my personal feedback	4.9 ± 1.6
4. This personal feedback helps me to reflect on what I have learned	5.0 ± 1.3
5. My feedback indicated clearly how my scores were calculated	5.5 ± 1.1
6. I understand most of my feedback	5.3 ± 1.4
7. It would be great if the Examination Commission always gave this type of feedback	6.3 ± 0.7
8. I feel demoralized or angry after reading my feedback	2.8 ± 1.8
9. The relationship between the feedback and the score is clear	5.2 ± 1.2

Conclusion

This paper introduced preliminary results of the second study of this PhD-project, investigating the possible added value of semi-automated assessment with atomic feedback when multiple assessors have to correct the same mathematics exam. The first results indicate that assessors rate visible SA grading highly but are less keen on using blind SA grading. On the other hand, students seem happy with the atomic feedback SA grading produces. Nevertheless, there are still many facets to this research study that have not been highlighted: the inter-rater reliability (comparison between blind SA, visible SA, and traditional grading), measurements for assessor reliability, the effect of the dynamic solution key,… are still uncultivated territory in the exciting universe of SA grading in mathematics education.

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Looking back for understanding: formative assessment strategies and inclusive activities for conjecturing and proving

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In this contribution we discuss a teaching and learning sequence on conjecturing and proving in grade 7. We rely on the Universal Design for Learning principles to design inclusive educational activities and we also promote the activation of formative assessment strategies, so as to create an educational path where the teacher monitors and, if necessary, refines the learning trajectory of each student of the class. Focusing on one formative assessment activity, we discuss the effectiveness of the sequence in terms of proof understanding and inclusion.

Keywords: Conjecturing and proving, formative assessment, inclusion.

Introduction and background

In this contribution we discuss the design and implementation of a teaching and learning sequence aimed at fostering students' first encounter with conjecturing and proving in lower secondary school (grade 7). The sequence is designed considering the principles of Universal Design for Learning, so as to realise an inclusive activity, and adopting formative assessment strategies.

In order to frame our teaching and learning sequence, firstly we refer to the specific mathematical process at issue, that is **conjecturing and proving**. There is a wide body of research on the teaching and learning of proof, see the work of Stylianides et al. (2016) for an overview. Here we focus on the first approach to proof, encompassing both understanding what a proof is and in learning how to prove (Balacheff, 1987). As De Villiers (1990) points out, it is crucial to make students aware of the different functions that proof has in mathematical activity: verification/conviction, explanation, systematization, discovery, communication. Lin et al. (2012) present a series of principles for task design aimed at promoting conjecturing, proving, and the transition between conjecture and proof. In relation to conjecturing, it is important to provide students with an opportunity to engage in: C1) observing specific cases and generalizing; C2) constructing new knowledge based on prior knowledge; C3) transforming prior knowledge into a new statement; C4) reflecting on the conjecturing process and on the produced conjectures. Concerning the transition from conjecture to proof, the teacher should propose tasks that raise students' need to prove. Moreover, the teacher should establish "social norms that guide the acceptance or rejection of participants' mathematical arguments" (p. 317), emphasizing that the acceptance /rejection is based on the logical structure of the argument and not on the authority of the instructor. In relation to proving, it is important to guide students: P1) to express in different modes of argument representation (verbal arguments, symbolic notations, etc.); P2) to understand that "different modes of argumentation are appropriate for different types of statements" (p. 318); P3) to create and share their own proofs and to evaluate proofs produced by the teacher, thus "changing roles"; and P4) to become aware of the problem of sufficient and necessary proof.

In particular for algebraic proof, Boero (2001) describes the fundamental cycle of *formalization*, *transformation* and *interpretation.* Performing a proof by algebraic language encompasses the following crucial issues: the choice of the formalization, that must be correct but also goal-oriented; the validity and usefulness of the transformations; the correct and purposeful interpretation of algebraic expressions in a given context of use.

As previously outlined, our aim is to design an inclusive teaching and learning sequence. **Universal Design for Learning** (UDL) is a multifaceted theoretical framework of learning that conceives teaching and learning as a dynamic system that must face the needs of all the learners (Rose & Meyer, 2006). CAST (Center for Applied Special Technology), a non-profit education research and development organization, created the Universal Design for Learning framework and the UDL Guidelines (https://udlguidelines.cast.org/more/research-evidence), offering a set of concrete suggestions that can be applied to any discipline or domain to ensure that all learners can access and participate in meaningful, challenging learning opportunities. UDL principles are deeply rooted in the foundational works of Vygotsky or Bruner. For example, through Vygotsky's theory UDL emphasizes one of its key points of curricula—the importance of graduated “scaffolds”.

The first UDL principle (UDL1) focuses on **providing multiple means of engagement**: indeed, besides recognizing the necessity of recruiting students’ interest, one must know that not all learners will find the same activities or information equally relevant or valuable. To reach this aim, the UDL framework suggests to:

- UDL1.1 Vary activities and sources of information;
- UDL1.2 Design activities so that learning outcomes are authentic, communicate to real audiences, and reflect a purpose that is clear to the participants;
- UDL1.3 Provide tasks that allow for active participation, exploration and experimentation;
- UDL1.4 Invite personal response, evaluation and self-reflection to content and activities;
- UDL1.5 Include activities that foster the use of imagination to solve novel and relevant problems, or make sense of complex ideas in creative ways.

The second UDL principle (UDL2) focuses on **providing multiple means of representation**. As far as representation of math objects is concerned, this principle suggests to provide options for perception in terms of alternative options of the registers of representation (algebraic language, for instance) through which the mathematical object can be represented. This allows students to improve comprehension about information at disposal, activating background knowledge, highlighting big ideas and new ideas to answer tasks and to guide information processing and design strategies of solution. We note that this is also in line with research in mathematics education, addressing the use of physical and digital artifacts for students with low achievement in mathematics and students with Mathematical Learning Difficulties (Robotti et al., 2015).

Providing **multiple means of action and expression** is the third UDL principle (UDL3). Physical action on some of these mathematical objects’ representations can support mathematics thinking. Similarly, the action supports communication (thus, not just writing but also gestures, or moving objects). This enhances the management of information, resources and ideas in order to develop mathematical thinking, supports the executive functions which are essential in guiding appropriate goal setting and progress monitoring.

Such a focus on goal setting and progress monitoring establishes a natural link with **formative assessment**, conceived as a method of teaching in which "evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited" (Black & Wiliam 2009, p. 7). A key element of formative assessment is feedback, that is any "information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding" (Hattie & Temperley, 2007, p.81). Wiliam and Thompson (2007) provide a description of five main formative assessment strategies: (FA1) clarifying and sharing learning intentions and criteria for success; (FA2) engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding; (FA3) providing feedback that moves learners forward; (FA4) activating students as instructional resources for one another; (FA5) activating students as the owners of their own learning. Such strategies may be activated by the teacher, but also by the peers and by the student himself. Indeed, formative assessment should in principle lead to auto-regulation.

Method

From a methodological point of view, the sequence is the result of cycles of design, enactment, analysis and redesign, according to the design-based approach (DBCR, 2003). Following this approach, we take as a starting point specific theoretical claims concerning the teaching and learning process (guidelines for conjecturing and proving, UDL principles, formative assessment strategies) and we aim to understand "the relationships among theory, designed artifacts, and practice" (DBCR, 2003, p. 6), also considering the design of the teaching and learning sequence as an outcome of the research in itself. Moreover, the research is characterized by a strong interaction and collaboration between researchers and teachers, who take part in the design, implementation and a posteriori analysis. The teaching and learning sequence we present is the result of cycles of design, enactment, analysis and redesign that started in 2012 and involved four teachers of a lower secondary school in the North of Italy. At present, five cycles were performed in grade 7 (pupils' age: 12-13). During the implementation we collected teacher's notes, observer's notes, video recordings of the class discussions and written productions of the students. In this contribution we mainly rely on students' written productions.

In the subsequent paragraphs we illustrate the design of the sequence, guided by our theoretical tools, and we focus on the last formative assessment activity (Step 6), with the aim of exploring two research questions. The first question concerns the efficacy of the designed sequence to promote a first encounter with conjecturing and proving. The second question concerns the efficiency of the sequence in terms of inclusion.

The sequence

The teaching and learning sequence concern isoperimetric rectangles. At the core of the sequence is the conjecture and explanation of the fact that, among all the rectangles with fixed perimeter, the square has the maximum area. In the designed sequence the students, starting from the empirical evidence, grasp the idea of variation of the area according to the length of the sides. Students are asked to conjecture about the maximum area. Afterwards, they are guided by the teacher to feel the need for a general explanation and to appreciate the power of algebra in leading to a proof. The task sequence is organized in the following steps.

Step 1: Students address individually an explorative task in paper and pencil: "*Draw four rectangles with a perimeter of 20 cm*". Afterwards, students are asked to work in small groups: "*Compare the methods you used to draw the rectangles and synthesize*".

Step 2: Students work in groups and cut cardboard to create a set of isoperimetric rectangles.

Step 3: Students work in groups, aided by paper and pencil and cardboard, on the following questions: "*Do you think all the rectangles have the same area? If not, what is the rectangle with the biggest area?*". After each of the steps 1, 2, 3 the teacher promotes mathematical discussions.

Step 4: Once established that the square is the rectangle with the biggest area, the teacher guides the students to prove it. The proof, carried out in algebraic language, is presented at the blackboard, with the teacher involving the students via open questions. When presenting the proof, the teacher refers to the previous steps and highlights the crucial steps of the proving process. For instance, the teacher underlines that using algebra allows one to generalize from a specific rectangle to the generic rectangle, that is a rectangle with the same perimeter as the square.

Step 5: Each student receives a sheet containing the written proof of the property, and is asked to fill some open sentences concerning the proving process (for instance: "I put the rectangle over the square in order to…" ; "I use letters because….."). After step 5, the teacher collects written answers from the students and promotes a discussion amongst them.

Step 6: Each student is asked to answer to the following open question: "*Looking back at the previous steps, you may note that we worked on the problem of isoperimetric rectangles by means of different approaches: we used paper and pencil, cardboard, we drew a rectangle over a square on the blackboard, we used letters. What does each approach tell you? Do they make you understand the same thing? Were they equally easy to follow and understand?*".

The sequence is conceived according to the principles by Lin et al. (2012): students observe specific cases and generalize, so as to formulate a conjecture (C1); concerning the transition to proof, the teacher proposes different modes of argument representation (P1), paying special attention to the link between geometric and algebraic representations. During the guided proof and the subsequent individual reconstruction, students are led to reflect on the cycle of algebra, from the formalization (using letters to express relations) to the transformation and interpretation of algebraic expressions.

The sequence is also conceived according to the UDL principles: the steps involve different registers of representation, hence many channels of access to information (UDL2). Throughout the sequence, each student may address the problem on the basis of the privileged channel. For instance, the student may do some conjecture on the biggest area on the basis of the drawing, or on the basis of the manipulation of cardboards. In line with the UDL principles, modes of representation alternate so as to scaffold the solving process of any student. Working on different modes of representation (UDL3) is a support for reasoning, and also promotes motivation (UDL1). Each student can ground his/her reasoning on the mode of representation that is most suitable for him/her. This fact, giving the students autonomy in the choice of the way of tackling the problem, promotes sustaining effort and persistence and provides multiple means of engagement (UDL1).

From the perspective of formative assessment, we may note that during the designed sequence many formative strategies are activated. In steps 1 and 3, the student is asked to explain respectively the procedure for drawing isoperimetric rectangles and the conjecture. Explaining makes the student responsible for his/her learning (FA5). During the group work and all the class discussions, students act as instructional resources for their classmates (FA4). During the guided proof of the statement (step 4), the teacher explains the learning objectives of the activity (FA1). During steps 5 and 6, students are encouraged by the task itself to become responsible for their own learning (FA5); the teacher may gather information on the learning process, and use it to provide individual feedback (FA3).

Analysis

We focus on Step 6, which is a veritable formative assessment activity, where students are asked to give meaning to all the sequence, and also to reflect on their understanding and possible difficulties. Besides being responsible for their own learning (FA5), students also prove to be aware of the learning intentions of the activity (FA1). Moreover, encouraging students to make a self-reflection on the experienced learning sequence is in line with UDL1.4.

We selected some excerpts that provide evidence of the efficacy of the sequence in terms of first encounter to proof, and also in terms of inclusion.

> **Camilla**: In the first approach I understood well what is the meaning of isoperimetric rectangle. In the second approach I understood well which was the rectangle with the biggest area because overlapping the cardboards one group created a square which is a special rectangle. We understood that [the square] has the biggest area. In the third approach we specified better why the square has the biggest area.

Camilla describes the journey from discovery to explanation. Moreover, Camilla ascribes to each approach a specific role in terms of construction of meaning: the first one (figural) makes the students understand the problem and the relations at issue, the second approach (kinesthetic) leads to the conjecture and its perceptive verification, the third approach (verbal, non-visual, symbolic) allows to generalize and reach an explanation.

> **Erika**: The first approach was not very useful to me because, since we used a particular measure, I did not know whether what I understood could be applied to any rectangle. Moreover, it was not very useful because just drawing you could not see anything special and if you noticed something, you could hardly see it. The second method was very useful because we all had the idea of overlapping them to see which was the one with the biggest area and we understood it was the square. And also thanks to a sort of "ladder" with the square as a starting point and each step was a rectangle with longer basis and shorter height in comparison with the side of the square. But in this way you don't understand why the square is the rectangle with the biggest area. The third

method was the most important because it gave motivation to the fact that the square is the rectangle with the biggest area.

Erika proposes meta-level reflections on the function of each approach towards meaning construction: for instance, she points out that the drawing is too specific and static and doesn't allow seeing invariants. On the contrary, the dynamic actions on concrete figures in cardboards allowed her to see invariants and relations. Erika is aware of the fact that the cardboard figures give a perceptual evidence for the conjecture ("*also thanks to a sort of "ladder" with the square as a starting point*") but do not provide a general explanation ("*in this way you don't understand why the square is the rectangle with the biggest area*"). Erika also recognizes the value of the algebraic language as a generalizing tool ("*The third method was the most important because it gave an explanation*"). Erika also explicates a link between action on cardboard figures and algebraic transformation. We point out that Erika judges the approaches in terms of "usefulness", thus expressing her personal preference and functionality in relation to the objective of conjecturing and proving. Erika seems to be fully aware of the learning intentions and criteria for success of the activity (FA1).

Gaia: I had more difficulties understanding the last activity because with the cardboards and without letters it is easier… you can move figures, you cut the pieces that are left, you add to what is missing… but the concept is not as accurate as the one with the letters.

Gaia points out that the concrete representation on cardboards allows action (thus promoting conjecture) and communication to the classmates. At the same time, the concrete representation does not hinder the necessity of moving to another representation (algebra) in order to generalize.

Beatrice: we did many approaches, but the easiest was the one where we had to draw four rectangles with the same perimeter but drawn in different ways; the approach of cutting cardboards was not difficult, the only problem was to draw rectangles that were equal to those on the cardboard; the method on the blackboard seemed to me more difficult to understand. On the blackboard there were a square and a rectangle overlapped, they had the same perimeter and, by means of calculation, we had to explain why the area of the square is bigger than the area of the rectangle. They all say the same thing, but in different ways, for example they want to make understand that rectangles that are isoperimetric to the square are infinite, but with drawing and the cutting you understand less because you cannot draw infinite ones, whilst with the mind and numbers you can go on to infinity. For me, a student in difficulty should try the first two methods, but a student not in difficulty should try the third one. With the third method I had difficulty because I could not understand well, while with the first two methods I understood the concept of area, but I could not immediately grasp the idea of infinity but it was not possible to do it. The first two approaches are also more amusing because you can compare your ideas with the ones of your classmates and if you don't understand your classmates can help you, while if you are alone you have to understand by yourself, which is more difficult. […] Not everybody understands letters and figures and calculations, but with the drawing and easier explanations you understand more.

Beatrice recognises that the sequence was organised in terms of evolution of generality. She is also aware at metacognitive level of the fact that the algebraic approach, although valuable in terms of generality, is more demanding in terms of cognitive load. She adds considerations in terms of engagement and points out the different methods are suitable for different students.

> **Ivan**: They are all useful and each of us can use their own method but they all take the same result, so there is not a wrong method among them. They all take to the result and each of us may use their own.

Ivan, in a very synthetic way, seems to confirm that the designed sequence achieved its main goals: provide all students with multiple modes of representation, occasions of action and motivations to address the problem, and construct the meaning of algebraic proof.

Discussion and conclusions

The students’ reflections in step 6, conceived as a formative assessment occasion, prove to us that the designed sequence promoted an inclusive approach to conjecturing and proving. We observe that students were able to report the evolution from exploration, to conjecture, to proof. Moreover, they were generally aware of the need for a general explanation and appreciated the proving power of algebraic language.

From the point of view of inclusion, we saw the effectiveness of the three principles of UDL “in action”. A crucial issue is that in the teaching and learning sequence each register has its own status (and students prove to be aware of this), but for each student a register may play a specific function with respect to the learning objective. Some students construct the meaning by means of the kinesthetic register, other students by means of the algebraic one. Some students are completely aware of the generalising power of algebra, other students appreciate the necessity of algebra after having dealt with the dynamism of cardboard figures. In general, cardboard figures are efficient in activating the reasoning that leads to the conjecture, because the dynamic work on the figures allows one to identify geometric invariants. Interestingly, students themselves are aware of the two dimensions of the registers (status in reference to the designed teaching sequence, function in relation to the personal learning experience throughout the sequence). This suggests that providing multiple registers of representation enriches and makes the teaching and learning sequence really inclusive, without losing the content-related objectives of the activity (approach to proof). Moreover, we observed that students link their appreciation of one register to the fact that working in that register fosters understanding.

In the previous analysis we used the formative assessment activity (Step 6) to discuss the “inclusiveness” of the sequence. Conversely, we argue that the formative assessment activity performed in Step 6 was a way of fostering engagement and self-reflection. UDL principles suggest to promote also self-reflection on the performed activities (UDL1), and this clearly coherent with FA1 and FA5 that are activated in Step 6. Thus, we may argue that the formative assessment acted as a means of inclusion.

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Can fluency as a factor of creativity be measured simply by means of a Fermi problem, and what influence does academic performance in mathematics have on this?

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The assessment of creativity has been studied much in the field of psychology. However, there is little research on easy-to-use assessment methods for schools in mathematics education. The goal of this study was to analyze the correlation between fluency measured by a Fermi problem and fluency in general creativity. Furthermore, it is to examine the possibility that Fermi problems can be used as a simpler way to measure fluency in mathematics education. As a result of surveying Japanese junior high school students (n = 291) and analyzing the results, a strong positive correlation was found. In addition, the results of the multiple regression analysis showed that fluency measured by a Fermi problem is a relatively strong factor when fluency in general creativity was the dependent variable, even when controlling for academic performance in mathematics and age.

Keywords: Creativity, fluency, fermi problem, mathematical modelling.

Introduction

Fostering creativity is required around the world. In the United States, for example, the educational goal is to develop 21st century skills. One major skill is creative thinking (Piirto, 2011). In Europe, the goal is to develop key competencies. Creativity is said to be built into all key competencies. Other countries are also trying to develop skills and competencies like those listed above. Fostering creativity is also a goal of mathematics education in Japan. The Japanese National Curriculum Standards (2017) guidelines propose that creativity should be fostered in each subject, nevertheless, clear guidance does not exist. Moreover, it is also not explicitly stated how creativity would be assessed. Because of this situation, a method should be developed that can be handled in mathematics education and can be used in schools to assess creativity in a simple way. Previous studies have shown that Fermi problems require creative thinking, and they seem to be useful to utilize as teaching materials to foster creativity. Additionally, it is hypothesized that there is a strong relationship between creativity and the richness of aspects considered in a Fermi problem. It would be possible to measure creativity by means of Fermi problems. The present study has two purposes. The first is to investigate the correlation between fluency measured by the richness of aspects solving a Fermi problem and fluency in creativity in general. The second is to examine the possibility that Fermi problem can be used as a simpler way to measure creativity in mathematics education.

Theoretical Frame

Fermi Problem

“Fermi Problems” were named after the Italian nuclear physicist Enrico Fermi. He had a special way of raising problems and probably the most famous of them is “How many piano tuners are there in the city of Chicago?”. Morrison described Fermi problems as “That is the estimation of rough but quantitative answers to unexpected questions about many aspects of the natural world" (Morrison, 1963, p. 627).

Fermi problems have been used in many areas of mathematics learning such as mathematical modelling. For example, Ärlebäck (2009) used of Fermi problems in a study conducted with high school students and Andrea (2005) used Fermi problems with pupils at the primary stage of education.

Fermi problems have also been found to be valuable as a learning material for mathematics. According to Silver:

> The development of students' creative fluency is also likely to be encouraged through the classroom use of illstructured, open-ended problems that are stated in a manner that permits the generation of multiple specific goals and possibly multiple correct solutions, depending upon one's interpretation. For example, consider the following "Fermi-style" problem […] (Silver, 1997, pp. 77).

If the relationship between Fermi problems and creativity becomes more clear, the value of dealing with Fermi problems in mathematics education will also be explained. Furthermore, it is also hypothesized that it would be possible to measure creativity by means of Fermi problems.

Creativity

Previous studies have expressed various definitions of creativity. Treffinger (2011) reviewed the literature for definitions of creativity up to 2011 and collected more than 100 references which discusses creativity from different perspectives. The present study focuses on fluency, as one of the elements of creativity. Guilford is a pioneer in the study of creativity since the 1950s. Guilford (1954) hypothesized that creativity has the following factors: sensitivity to problem, fluency, flexibility, originality, penetration, analysis, synthesis and redefinition. In the course of his research, Guilford refined these creativity factors and defined the factors while testing his hypothesis. In Guilford's study, fluency is defined as "The ability to think of many ideas; many possible solutions to a problem" (Guilford, 1973, pp. 2). Tests have also been developed to measure these factors. For example, based on research by Guilford, Torrance developed tests to measure the creativity factor. In Torrance's study, the measure of fluency is "In all tasks, fluency is defined as the total number of relevant responses, relevancy being defined in terms of the task assigned" (Torrance, 1963, pp. 9). Even in the 2000s, the research on creativity tests continued. Kim's (2006) critical examination of the Torrance Creativity Test and his research on the effective use of the Torrance Creativity Test and showed that fluency is one of the elements of creativity.

Fermi problems are seen as a form of mathematical modelling therefore the present study focuses on creativity in mathematical modelling. Mathematical modelling is the process of translation between the real world and mathematics in both directions (Blum & Borromeo Ferri, 2009) and some there some studies have investigated the association between mathematical modelling and creativity. Dan and Xie state that "We evaluated 33 engineering students in a class and obtained the distributions of the students' mathematical modelling skills and their creative thinking levels. The data from the experiments show that there is a strong positive correlation between these two kinds of competencies" (2011, pp. 457). Thus, it was assumed that mathematical modelling seems to have a positive impact on creativity. Wessels' (2014) study also discusses the relation between creativity and mathematical modelling. Wessels identifies the following four elements as a measure of the association between mathematical modelling and creativity:

- Fluency that refers to the generation of different solutions.
- Flexibility that entails the change of shift that takes place in the emphasis, direction, or approach of creative problem solvers.
- Novelty that refers to the level of originality in the development of new and unique solutions.
- Usefulness that is grounded on the relevance, adaptability and reusability of solutions in other real-world situations.

Wessels states that "A framework with four criteria for the identification of creativity was successfully used to evaluate levels of creativity in the solutions to the MEAs (model-eliciting activities)" (2014, pp. 1). Lu and Kaiser (2021) also state the relationship between mathematical modelling and creativity. They defines the three elements of creativity in the modelling cycle as Usefulness, Fluency, and Originality. They suggest that when assessing modelling competency, it is better to include the perspective of Usefulness among the three elements in the assessment items.

Thus, previous studies have discussed creativity and mathematical modelling. However, only a few studies have discussed the connection between fluency in general creativity, fluency measured by a Fermi problem and academic achievement in mathematics in detail. Additionally, previous studies have shown that assessing creativity in mathematics education is time-consuming. Therefore, a simpler way to measure creativity will need to be developed.

The present Study

In this paper, two research questions are considered.

1. Is there a correlation between fluency in the Fermi problems and fluency in creativity in general?
2. If there is a correlated, can the Fermi problem measure fluency in general creativity without being influenced by academic performance in mathematics, age, or gender?

Therefore, the purpose of this study is to determine whether there is the relationship between fluency in general and fluency measured by the richness of aspects in solving a Fermi problem. Additionally, it is investigated whether the Fermi problem can be a simpler measure of fluency in general creativity, when controlling for factors such as academic performance in mathematics, age and gender.

Participants and Procedure

A total of 291 Tarui Municipality Fuwa Junior High School (public junior high school) students participated in the survey (100 seventh graders, 169 eighth graders, 22 ninth graders). The academic level at this school is slightly lower than the national average. First, students received a mathematical performance test with a time limit of 50 minutes. Then they took the test for creative thinking and the Fermi problem test, which were both 10 minutes. Mathematical performance tests such as the one used in this study are part of the school curriculum and take place at the end of every school year. The students took the test in a relaxed state, as it was explained to them that the results of the test for creative thinking and the Fermi problem test would not affect their school grades in mathematics or any other subject at all.

Mathematical Performance Test

This is a test given regularly to measure the academic performance in mathematics and the author of the study was not involved in its design because the test was prepared by an educational publisher. In seventh grade students are tested on positive and negative number, equation, proportional and inversely proportional, figure problems, such as surface area and volume. In eighth grade, in addition to the aforementioned items, they are also tested on polynomial calculation, linear functions and proof problems for congruent figures. In ninth grade simultaneous equations and proof problems of similarity are added. These problems are asked in approximately the same proportion.

Test for Creative Thinking (TCT)

The Takano's TCT (1989) to measure fluency in general creativity was used in the present study. The TCT is based on Guilford's TCT (1959) and was adapted by Takano for Japanese students. The questions are as follows. A picture of an empty can was shown, and the students were asked, "What are the possible uses for the objects in this picture? Please think of as many as you can." The intention of this question is to figure out how many ideas the students can write. The number of possible uses of the empty can that the participants could think of was defined as fluency. For example, if a student gave two responses, "I use it as a vase" and "I use it as a tool for drawing circles", the fluency is 2. If no answer is given, the score is 0. In the following there are some examples of students' responses to the question: using it as a container, using it as a musical instrument, using the lid of the can as a cutter, using it to play bowling, using as a penholder, putting a stone in it, then using it as a weapon etc.

Fermi Problem Test

The Fermi problem used in the present study is "How many liters of water does one person use in a year?" In this Fermi problem, fluency is defined as "the richness of aspects solving a Fermi problem." This definition is different from definition of Wessels (2014) or Lu and Kaiser (2021). For ease of the evaluation in school, the number of ideas that can be evaluated more clearly and simply was used as the definition. For example, if a student considered "the amount of drinking water" and "the amount of water used in the shower" as the elements needed to solve the problem, the fluency is 2. If a response was "I use 10 liters in the morning, 20 liters in the afternoon, and 30 liters at night." Then the day was divided into three parts in chronological order the score was three. If there was no number to assume or only the answer, it was determined to be 0. After considering many problems in my preliminary research and discussing them with some pedagogy professors, it this problem was adopted.

Results

The statistical analysis was conducted to investigate the first research question, "How much of a correlation is there between fluency in the Fermi problem and fluency in creativity in general?" The correlation between fluency measured by the richness of aspects solving a Fermi problem (fluency of a Fermi problem), fluency in general creativity by TCT (fluency by TCT) and academic performance in mathematics was examined. Correlations with mathematical performance were also investigated, as previous research suggested that creativity and the Fermi problem have a strong relationship with mathematical performance. As can be seen (Table 1), there is a strong positive correlation between fluency of a Fermi problem and fluency by TCT. There is also a weak positive correlation between academic performance in mathematics and fluency by TCT. Similarly, there is a weak correlation between academic performance in mathematics and fluency of a Fermi problem.

Table 1: Correlation between Fluency by TCT, Fluency of a Fermi problem, and Math-Performance

	Fluency by TCT	**Fluency of a Fermi problem**	**Math-Performance**
Fluency by TCT	1		
Fluency of a Fermi problem	0,614**	1	
Math-Performance	0,247**	0,278**	1

*p<0,05;**p<0,01 Math-Performance (Mathematical performance test scores)

A multiple regression analysis was then conducted to further investigate the connection between the fluency in general creativity and the fluency in a Fermi problem, which was investigated in the present study. Fluency in general creativity was used as the dependent variable to determine the extent to which fluency as measured by the Fermi problem is a construct of fluency in general creativity. As can be seen (Table 2) two models were created for analysis. No multicollinearity was found in the two models. Model 1 adds age and gender as well as math-performance as control variables. Model 2 uses only math-performance as a control variable. Both models also show significant differences. In addition, there is a significant difference fluency of a Fermi problem and academic performance in mathematics in both models.

Table 2: Results of multiple regression analysis with fluency by TCT as the dependent variable

	Regression coefficient	**Standard error**	**Standardized regression coefficient**
Model 1: Dependent variable is fluency by TCT and 4 variables (Adjusted R^2= 0.297,F(4,286)= 31.58,p<0.001)			
Fluency of a Fermi problem	0.435***	0.045	0.492
Math-Performance	0.012*	0.005	0.124
Age	0.383*	0.195	0.099
gender	-0.275	0.229	-0.061
Model 2: Dependent variable is fluency by TCT and 2 variables (Adjusted R^2= 0.290,F(2,288)= 60.29,p<0.001)			
Fluency of a Fermi problem	0.444***	0.045	0.503
Math-Performance	0.011*	0.005	0.115

*p<0,05;**p<0,01; ***p<0,001 Math-Performance (Mathematical performance test scores)

Discussion and Conclusion

Analysis of the data showed a strong correlation between fluency in general creativity and fluency measured by the richness of aspects in solving a Fermi problem. The result of the multiple regression analysis in Table 2 also show that fluency of a Fermi problem is a relatively strong factor when creativity in general creativity was the dependent variable, even when controlling for academic performance in mathematics and age. Thus, it is predicted that the Fermi problem can measure fluency in general creativity without much effect from factors of age and academic performance in mathematics. Previous studies of the relationship between mathematical modelling and creativity have defined fluency as the number of different solutions or the number of different models considered. In addition, the mathematical modelling problems used in these studies were modelling problems of relatively high challenging level (Wessels, 2014; Lu & Kaiser, 2021). If the problem is a mathematical modelling problem at a challenging level, it is possible that it may take longer time to solve the problem, or that students seem to be satisfied if they can find one solution or model for the problem. This makes it difficult to measure fluency appropriately. To avoid such difficulties, the present study assessed fluency from a different perspective. By defining fluency as "the richness of aspects of solving a Fermi problem", it is possible to evaluate fluency more openly in terms of the items students have created and thought about in order to solve a problem, even if they cannot create a complete single solution or model. Additionally, Fermi Problems require to consider a large quantities of scenarios quickly (Ärlebäck, 2009). In fact, the Fermi problem used in this study showed that most of the students were able to answer or came up with ideas to solve the problem within 10 minutes. Thus, the Fermi problem can be handled in a short time. Therefore, the assessment using the Fermi problem, as in the present study, is simpler than the assessment used in previous studies. Hence, this suggests that Fermi problems can be simply used as one of the fluency assessments in school.

Focusing on the relationship between academic performance in mathematics and fluency, there is a weak correlation. There are two possible reasons for the weak correlation. First, Japanese junior high school students are not familiar to modelling problems like the Fermi problem. Therefore, it is predicted that even students who have a high academic performance in mathematics were not able to achieve a high level of fluency. Second, the issue seems to be the nature of mathematics problems. Just presenting ideas is not enough to solve a mathematical problem. It is necessary not only to come up with ideas for solutions, but also to be able to change the solution method to a better one to handle them properly, and to calculate without making mistakes. These reasons are considered to be the cause of the low correlation. In the present study a low relationship between the fluency and academic performance in mathematics was found, but one cannot deny the possibility that there is a strong relationship when looking from another perspective.

There are some limitations to the present study. First, in the present study, the reliability and validity of the test to measure fluency with the Fermi problem was not verified. Therefore, it is not possible to state that fluency in general creativity can be completely measured by means of this Fermi problem. Second, students dealt with "How many liters of water does one person use in a year?" In this case, results showed the relationship between fluency in general creativity and fluency measured by the richness of aspects solving a Fermi problem. However, it was not possible to shown whether there is the relationship between fluency in general creativity and all other Fermi Problems. It could be the case that there are some Fermi problems that limit fluency in the first place. Furthermore, the survey was only conducted with Japanese students in one school. It is possible that other countries will have different results. Moreover, the present study was only focused on fluency, which is one of the factors of creativity. It cannot be conclude from the results of the present study that Fermi problems is strongly related to all factors of creativity.

Although there are some limitations, it was suggested that the existing definition of fluency in mathematical modelling could be reconsidered, and that Fermi problems could be used to measure fluency in a simpler way. In the future, it will be examined the validity and reliability of a test to measure creativity using the Fermi problem. Additionally, the relationship between general creativity factors other than fluency and Fermi problems will be investigated to complement and extend the results of present study.

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Fostering self-regulated learning by increasing student agency in assessment – student perceptions

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This study investigated students' perceptions of different assessment methods in an undergraduate mathematics course. Each student could choose from two summative methods, exam and self-assessment, for their course grade. Embracing this form of agency, most the students chose self-assessment. A content-based analysis of the reasons students gave for their choice produced a variety of expressions of self-regulation. Many students recognised the value of self-assessment for their continuous learning and were motivated by it. Others acted from affective reasons, enjoying or agonising about one or the other method. Still others aimed to control their behaviour by challenging themselves or managing their workload. Choices were also made from purely practical reasons. The results give teachers valuable information on what students consider important in assessment. They also point to the usefulness of the study design for studying student agency and self-regulation.

Keywords: Undergraduate mathematics, self-regulation, agency in assessment, students' perceptions, self-assessment.

Introduction

There is a growing consensus among pedagogical researchers and policy-makers that university students should be involved in assessment processes as agents, not just passive assessees (e.g., Falchikov, 2004). This would improve students' engagement with the learning goals and assessment criteria, as well as their self-regulation skills, including monitoring and reflecting on their own learning (e.g., Adie et al., 2018). In this study, we enhanced student agency in assessment by giving them the possibility to choose between a traditional and a novel, student-centred assessment method for their course grade. By analysing the reasons students give for their choice, we aim to obtain in-depth information about students' self-regulation, as well as their perceptions of novel summative assessment methods in mathematics.

Student agency is usually restricted to formative assessment, which takes place during learning and usually does not involve high-stakes grading. Students may, for example, be involved in self-assessment or peer review processes. On the other hand, Nieminen and Tuohilampi (2020) suggest that student agency in summative assessment may in some contexts offer benefits that are not accessible through formative assessment. They reason that some students may not see the benefit of formative assessment to themselves but consider it instead to be conducted for someone else. However, involving students in summative assessment presents many challenges, for example regarding validity and fairness. Also, summative assessment has traditionally been in control of the teachers, and students may end up resisting any change that they consider a risk for their grades.

Hence, it is important to study students' reasoning around assessment choices in order to find ways to promote student agency also in summative assessment.

Assessment in mathematics is very traditional through all levels of education. In the UK, the most common assessment method in university mathematics is the closed book exam, and there has been no significant change in this in the last 10 years (Iannone & Simpson, 2021). Also, in schools assessment in mathematics is often traditional. In Finnish schools, the most common assessment method in mathematics is the exam, and alternative assessment methods are not used as frequently as in other subjects (Atjonen et al., 2019).

This study is the first step in a larger project concerning self-regulation and student agency in assessment. We let students in an undergraduate mathematics course choose the summative assessment method from two options: exam and self-assessment. The students made the choice in the beginning of the course but were able to change their choice before the assessment took place. The pedagogical motivation for this setting was to enhance students' agency by giving them power over their own assessment. We also presumed that students' perceptions of the assessment methods would be more authentic if they had to commit themselves to a particular method (as opposed to asking them about assessment that they were not engaged in). Our research questions are the following:

Research question 1: How do mathematics students perceive a novel summative assessment method compared to a traditional one?

Research question 2: What characteristics of self-regulation are expressed in the justifications students give for their choice of assessment method?

Self-regulation and student agency in assessment

Self-regulation refers to monitoring, controlling, directing and evaluating one's own thoughts and behaviour in order to achieve desired outcomes (e.g., Zimmerman & Schunk, 2001). Self-regulation of learning is considered an important skill in higher education and life-long learning (e.g., Zimmerman, 1990; Heikkilä & Lonka, 2006). Pintrich (2000) introduced a model of self-regulation that identifies four areas of regulation: cognition/metacognition, motivation/affect, behaviour, and context. For example, regulation of motivation may include assessing the value of a task and one's capacity of succeeding, and regulation of context may include changing the task or leaving it.

In order to cultivate self-regulation skills, students must achieve *agency*, in other words they must be able to make decisions and act autonomically towards a desired goal. One way of increasing student agency is to involve students in assessment. Assessment has traditionally been in control of the teachers, but recently, student agency in assessment – in particular, formative assessment – has been emphasised in international policy (Adie et al., 2018). Adie et al. (2018) proposed a definition of student agency in assessment based on Emirbayer and Mische (1998): agentic students (1) make choices and take action in assessment (2) within the boundaries of different contexts, environments and timeframes (3) thereby reproducing or transforming traditional assessment structures.

Self-assessment is a typical way of transferring control to students. Panadero et al. (2018) note that self-assessment has long been studied as a link between formative assessment and self-regulated learning. They point out that although self-assessment is theorised to support self-regulation, empirical evidence is mixed. On the other hand, Nieminen and Tuohilampi (2020) compared student perceptions of their agency in formative and summative self-assessment contexts. They observed that only those students who were randomly selected to perform summative self-assessment for their course grades, perceived the self-assessment useful for their future studies or working life. Compared to students assessed with an exam, they also expressed a feeling of 'studying for themselves'.

Students' perceptions of assessment

It is essential to understand students' perceptions of assessment in order to enact the positive impact of assessment on learning (Conlon, 2006; Guo & Yan, 2019). Past studies have shown that students tend to have mixed perceptions of assessment, and the purposes of assessment (formative vs. summative) might greatly influence their perceptions (Brown & Harris, 2016; McMillan, 2016). With regard to self-assessment, there is no consensus among students either. Some believe self-assessment to be useful, while others doubt its effectiveness in improving their learning (Hung, 2019; Lew et al., 2010). The learning environment can have a significant influence on students' perceptions of assessment (Gijbels et al., 2008) and self-assessment (Hill, 2016). Hill (2016) found that students seldom conducted self-assessment without explicit encouragement. However, once given the opportunity to practice, their attitude towards self-assessment became more positive and they were willing to continue to carry out self-assessment in their future learning.

Undergraduate mathematics students' perceptions of assessment have been found to be different from other fields. In their review, Struyven et al. (2005) found that higher education students find alternative assessment methods to be fairer and to lead to deeper learning than traditional methods. However, the review does not include any studies on mathematics students. Indeed, Iannone and Simpson (2015) found that in the UK, mathematics students prefer traditional assessment methods such as closed book exams. Iannone and Simpson (2017) have also compared mathematics and education students' perceptions of assessment in two universities in the UK. Students in both groups preferred assessment methods that, in their view, discriminated with respect to academic ability. However, their perceptions concerning discrimination differed. Education students preferred projects and dissertations, and mathematics students preferred closed book exams. It can be concluded that disciplinary factors need to be considered when studying students' perceptions of assessment.

Context and method

The context of this study was an undergraduate mathematics course taught at a research-intensive Finnish university. The topic of the course was linear algebra, and it was one of the first mathematics courses students take. The course lasted for 7 weeks and was worth 5 credits (ECTS). Typical major subjects among the students who took the course were mathematics (including teacher education), computer science, economics and statistics.

The course was taught using an inquiry-based teaching method (see Rämö et al., 2020). During the course, all students took part in formative self-assessment exercises which followed the DISA self-assessment model (Häsä et al., 2019). In the exercises, the students assessed their competencies using a detailed rubric written by the teacher and received automated feedback on their assessments.

For the summative assessment that took place at the end of the course, students could choose from two options: exam and self-assessment. In the former case, the grade was determined by the exam together with bonus points received from weekly tasks. In the latter case, the students self-graded their course according to the DISA model. In this model, a student's self-assessment is checked against the tasks they had completed during the course. The teacher can step in if the self-assessment is not in line with the student's weekly performance.

In the beginning of the course, students were explained the teaching arrangements of the course. As self-assessment was assumed to be a new assessment method to many students, and all students had to complete formative self-assessment exercises during the course, the teacher explained what the benefits of self-assessment are considered to be. After this, the students had to choose the summative assessment method, but they were told that they could change the assessment method before the end of the course if they wanted to. The students were asked to give a reason for their choice.

The participants in this study are 333 students who participated in the linear algebra course and gave consent to use their answers in the study. The data consists of two datasets. Firstly, it contains the reasons students gave for their choice of assessment method in the second week of the course ("On what basis did you make your choice?"). Secondly, it contains reasons students gave for changing their choice in the last week of the course ("Justify carefully why you wish to change your choice."). Students' answers were gathered via the Moodle platform that was used for teaching the course. In the beginning of the course, 83% (n=275) of the students chose self-assessment and 17% (n=58) examination. There were 18 participants who changed their choice: 14 from examination to self-assessment and 4 from self-assessment to examination.

The data was analysed using abductive content analysis, in which identified categories are related to theoretical concepts but not directly based on them (Timmermans & Tavory, 2012), and it was coded with an Atlas-ti programme for qualitative data analysis. The justifications varied in length from a few words to several sentences and each justification could include several reasons for the choice. For the students' choices in the beginning of the course, altogether 16 codes were identified, forming five main categories: self-assessment enhancing learning; examination preventing learning; external reasons for self-assessment; negative aspects of self-assessment and examination enhancing learning.

Results

The most common reason to choose self-assessment was that **self-assessment enhances learning**. Half of the justifications endorsing self-assessment were related to this category. Participants for instance argued that *they learned more by self-assessment*:

> I feel that I invest in the tasks more when I know that the self-assessment is based on them. And when I do tasks with thought, I learn the course content better.

They also stated that *self-assessment was a more efficient and beneficial way of learning*:

> In my view, self-assessment supports deeper learning to a greater extent, because often when you study for an exam, learning may remain superficial.

> I choose self-assessment, because I believe that it challenges me to study things properly during the course but also gives me information on how I have understood things and what is my level of mastery.

Self-assessment was *seen to balance the workload during the course*:

> Moreover, you can do the final assessment a bit earlier which makes it possible to focus earlier on other final assessments and Christmas holidays.

In addition, self-assessment was reflected to *be a more holistic and a long-term way of learning compared with examinations*:

> I want to learn to better interpret my learning in practice and I believe I'll get a more realistic outline of what I learned in the course by means of a long-term follow-up process than through an exam at the end of the course.

Some participants also mentioned that self-assessment caused less stress than examination: "Self-assessment is less stressful."

More than one quarter of the participants endorsing self-assessment as a first choice, considered that **examination prevented their learning**, not just because it can be stressful, but because it focused on only the limited aspects of what have been learned:

> In self-assessment I must really evaluate the work I've done, whereas the exam seems to be just one performance in which you can succeed (pass) even when you don't completely comprehend things.

Approximately 23% of the justifications for choosing self-assessment were related to **external reasons**, mostly referring to timing issues, and not taking a stand on effects on learning: "It is better for my timetable."

Participants endorsing examination for the first choice reasoned most often that **examination enhances their learning** by challenging and motivating them to learn.

> I choose the exam because I regard it as a more challenging alternative to myself.

> I feel the exam is a motivating and pleasant way of assessing mastery.

Examination was also considered to accomplish long-lasting learning outcomes and to be a good way to validate what one has learned.

> The exam creates extra motivation to permanently learn the content. It is also, in my opinion, a good way to validate one's level of mastery.

Negative aspects of self-assessment were also reasons to choose examination as the assessment method. Some participants had had negative experiences of self-assessment before and found it laborious and difficult: "Self-assessment is repulsive and difficult" or they were reluctant to try new experiences: "I'm not open to new experiences."

Those students who changed their initial choice from examination to self-assessment (n=14) did it often based on practical reasons, such as a more flexible timetable. Some students changed their mind because during the course they found that self-assessment was beneficial for them or they wanted to try something new. Those four students who changed their choice from self-assessment to examination argued for instance that they felt they could perform better in the examination than by doing self-assessment.

Discussion

In our study, most students chose self-assessment instead of exam. This is in line with several previous studies in which students have been found to prefer novel assessment methods over traditional ones (Struyven et al., 2005). However, it contradicts previous studies in which mathematics students in the UK have been found to prefer exams (Iannone & Simpson, 2015, 2017). The cultural context of these studies may explain the different results to some extent. In Finland, assessment is not high-stakes: exams can be retaken easily and single course grades do not affect students' studies or future careers. In the UK, assessment is often high-stakes, and this may lead students to view alternative assessment as risky. Finally, in the case of our study, the teacher informed the students about the benefits of self-assessment, which may have affected students' choices.

In previous studies, mathematics undergraduate students have been found to prefer assessment methods that they perceive to discriminate according to academic ability (Iannone & Simpson, 2017). We found similar perceptions in our study when students described how a certain assessment method enhances or prevents learning. What is notable is that some students linked the discriminating ability with exams but others with self-assessment.

Our study design allowed students to achieve agency in assessment in several ways. Practising self-assessment in a formative setting is itself a typical way to include students in assessment (Panadero et al., 2018). Also, some students opted for summative self-assessment, which has been reported to enhance future-driven agency and ownership of learning more than formative self-assessment (Nieminen & Tuohilampi, 2020). Thirdly, students were given power to choose the assessment method themselves.

Students' reasons for choosing the assessment method give information on how they regulate their learning in Pintrich's model for self-regulation (2000). Students who chose the exam expressed regulation of affect or behaviour: they enjoyed exams or found self-assessment disagreeable, or they felt that exam would challenge them to study harder. Students who chose self-assessment expressed similar regulation: self-assessment was more flexible in terms of scheduling (behaviour) or less stressful (affect). However, they also mentioned aspects of motivation and cognition, stating that they would learn more or in a more holistic way, and even metacognition, claiming that self-assessment would enable them to monitor and learn about their own learning.

From the teacher's point of view, it is valuable to understand not only what kind of assessment students prefer, but also what they would consider important given the chance to choose a method themselves. This knowledge will help in designing new, student-centred assessment structures. Allowing students to make the choice also revealed detailed information about what they focus on when regulating their own learning in an authentic situation. However, since it was the teacher of the course who collected the students' choices and justifications, some students may have embellished their reasoning, even though they knew that their answers would not affect their grades. This must be taken into account when drawing conclusions from the results.

The current study design offers several possibilities for future research concerning student agency in assessment. We will continue by analysing students' justifications further and by linking them to quantitative data on self-regulation and approaches to learning. As another direction that could be based on a similar design, we recommend studying the effects of increased agency in assessment on students' academic achievement or well-being.

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Judgment bias in diagnosing misconceptions with decimal fractions

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A prerequisite for remediating student misconceptions is their accurate diagnosis by the teacher. However, studies on judgment accuracy show that teachers differ substantially regarding their judgments even though little is known about the reasons so far. The present study investigates whether teachers' diagnosis of misconceptions with decimal fractions is subject to judgment biases. For this purpose, we propose a cognitive model for diagnosing misconceptions based on the process of hypothesis testing. The study results show that the formulation of alternative hypothesis and the processing of relevant information are predictors of high judgment accuracy when diagnosing misconceptions. Furthermore, normative and confirmatory biased judgment processes could be distinguished. Implications for teacher education are discussed.

Keywords: Assessment, misconceptions, confirmation bias, cognitive processes.

Judgment processes in diagnosing misconceptions

The term misconception suggests that students commit a systematic error due to a naïve theory. From a student's point of view, it is rather a strategy based on a hitherto reasonable idea. In mathematics education, various student misconceptions are well researched (cf. Confrey & Kazak, 2006), such as the assumption that "multiplication makes bigger, division makes smaller" which is correct with natural numbers and is thus often overgeneralised to fractions. From a teachers' perspective, misconceptions are constructs that emerged as a result of the learners' experiences in different contexts but that no longer function correctly when transferred to another area of knowledge (Fujii, 2014). In order to create adaptive learning opportunities, learner misconceptions must first be diagnosed. Then the teacher can trigger a cognitive conflict and resolve it by introducing the actual mathematical concept (Corno, 2008). Thus, teachers' judgment accuracy when diagnosing misconceptions – that is to determine it precisely - seems crucial for student learning. However, various studies have shown that teachers' judgment accuracy on student performance varies widely (mean effect size between teachers' judgments of students' academic achievement and students' actual academic achievement of r = 0.63 in the meta-analysis by Südkamp, Kaiser, & Möller, 2012).

In their framework, Loibl et al. (2020) conceptualise diagnostic judgments in pedagogical contexts as a teacher's inference about learners (e.g., their abilities) or materials (e.g., task difficulty) based on the information that is explicitly or implicitly present in a diagnostic situation. This definition locates diagnostic judgments within the larger field of social judgment and cognitive information processing and allows investigating the genesis of (correct and incorrect) diagnostic judgments. In this line of research, recent studies focus their research interest on the judgment processes and examine which information teachers actually gather and process to form their judgment (e.g., Rieu et al., 2022).

The judgment process can be influenced by personal expectations (often leading to erroneous diagnoses), which have already been documented in the area of ethnicity, socioeconomic status, and gender (McKown & Weinstein, 2003; Rubie-Davies, Hattie, & Hamilton, 2006; Südkamp et al., 2012). Biases like the confirmation bias – the tendency to selectively choose and process information supporting the initial hypothesis – influence the judgment process when the diagnostic judgments are based on hypothesis testing (Herppich et al., 2018; Oswald & Grosjean, 2004; Westhoff & Kluck, 2014).

The current study

One diagnostic situation in which the teacher's cognitive processes can be modelled as hypothesis testing (Trope & Liberman, 1996), is the detection of misconceptions in decimal fraction (e.g., *longer-is-larger, shorter-is-larger*, Stacey, 2005). In this process, an erroneously solved task often cannot be clearly assigned to one single misconception, but only the structured processing of several tasks and its solutions by the student allow the precise diagnosis of the misconception.

We assume that cognitive biases occur in this knowledge-based process and that these biases systematically favour the confirmation of a hypothesis and make its rejection unlikely (overestimation of the a priori probability of the hypothesis, selective gathering of hypothesis-confirming information, hypothesis-consistent interpretation of ambiguous information according to Schulz-Hardt & Köhnken, 2000, figure 1).

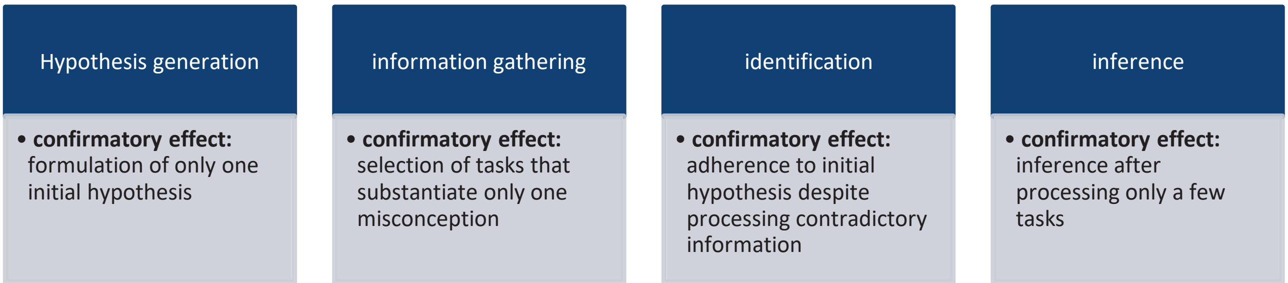

Figure 1: Hypothesis testing process and confirmatory effects in the domain of diagnosis misconceptions

The present study defines diagnostic judgments as information processing and analyses the causes of diagnostic errors. The collection of external indicators such as the formulation of the initial hypothesis and the number and type of information processed allow conclusions to be drawn about the genesis of diagnostic judgments (Loibl et al., 2020).

Specifically, it is assumed that in the ambiguous diagnostic situation of detecting misconceptions in decimal fraction comparison, confirmatory biases occur and prevent an accurate diagnosis. To this end, the following research question is investigated:

> *Is the diagnostic process of misconceptions in decimal fractions subject to confirmation biases?*

Based on the modelled judgment process and the theoretical assumptions, it is assumed that

① the generation of the initial hypothesis reveals if the (prospective) teachers recognise the ambiguous situation when diagnosing misconceptions in the area of decimal fractions based on one erroneously solved task and that

② the amount and type of processed information indicate possible confirmatory biases when gathering and identifying information. The processed information influences the accuracy of judgments when diagnosing misconceptions in the area of decimal fractions.

Method

To test these assumptions, prospective mathematics teachers (N = 79, average age = 21,7 years, 85% were female) at the beginning of their studies were confronted with one erroneously solved task by a virtual student in the domain of decimal fraction comparison (cf. figure 2). The first task and its incorrect solution represent an ambiguous diagnostic situation, as several misconceptions can be responsible for the student error. The diagnostic goal for the participants was to clearly determine the existing misconception of the presented student who consistently solves tasks according to a precise misconception. For this purpose, the necessary specific PCK concerning the misconceptions was visible to the participants and in total, 7 standardised cases had to be diagnosed.

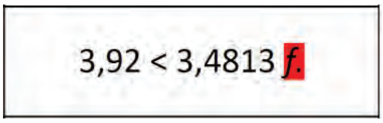

Figure 2: erroneously solved task in the domain of decimal fraction comparison

In an online survey, the prospective teachers first formulated an initial hypothesis about the misconception based on the erroneous student solution to one task. Afterwards they could choose further tasks which were solved by the student after their selection. That is, after clicking on it, teachers saw the solution to this task generated according to the misconception. Finally, the teachers submitted their final diagnosis.

The tasks offered for selection were shown in groups of four. The tasks differed with to the relevance of their information for the diagnosis. Tasks with relevant information for diagnosis allowed to distinguish between two possible misconceptions in the ambiguous diagnostic situation (i.e., these tasks are typically solved correctly with one misconception, but not with the other). Tasks with irrelevant information for diagnosis are either solved correctly by all learners despite the presence of a misconception or do not provide any additional information to the incorrectly solved tasks presented at the beginning.

A digital questionnaire was used to collect the initial hypothesis (single hypothesis or alternative hypotheses) and the number and type (diagnostically relevant or irrelevant) of information the participating persons used to get to their final diagnosis.

Results

Due to floor effects, two case diagnoses were excluded from the calculation. The judgment accuracy for the diagnosis of 5 cases over all participants was 63.3% (SD = 0.48) which designs the ratio of correct diagnoses of the presented students' misconception.

In a first step, the predictors for high judgment accuracy for the diagnosis of misconceptions in the area of decimal fractions are to be determined. Based on the theoretical modelling of the diagnostic process as hypothesis testing, the formulation of the initial hypothesis, the amount of information processed and the proportion of relevant diagnostic tasks are examined. Table 1 gives an overview of the average values of the three predictors.

Table 1: Representation of the average values of the assumed predictors for judgment accuracy according to accurate or incorrect diagnoses

	accurate diagnoses (n = 250)	incorrect diagnoses (n = 145)
average number of alternative initial hypotheses (SD)	0.26 (0.44)	0.15 (0.36)
average number of processed further tasks (SD)	2.73 (2.28)	2.94 (2.73)
average proportion of processed further relevant diagnostic tasks (SD)	0.68 (0.31)	0.54 (0.36)

The influence of the type of initial hypothesis (single hypothesis or alternative hypotheses) on the accuracy of the judgment was compared for accurate and incorrect diagnoses. The one-factor ANOVA indicates that significantly more accurate judgments are given after the formulation of an alternative hypotheses ($F(394) = 6.333$, $p = .012$, $d = 0.263$).

In addition, it was hypothesised that the amount of information processed, i.e. the number of tasks selected to see further solutions of the student, would also have an impact on judgment accuracy. To calculate this influence, the average number of processed tasks was compared for accurate and incorrect diagnoses. Group comparison using an ANOVA indicates no significant difference between accurate and incorrect judgments ($F(394) = 0.647$, $p = .422$, $d = 0.084$).

As a final predictor of judgment accuracy, we examined whether the type of information selected had an impact on judgment accuracy. For this purpose, the proportion of processed tasks that provide relevant diagnostic information was examined. The one-factor ANOVA indicates that accurate judgments, compared to incorrect judgments, are obtained by processing a significantly higher proportion of diagnostic tasks ($F(394) = 18.025$, $p \leq .001$, $d = 0.444$).

In a second step, the study aims to differentiate between different categories of information processing. Based on the finding that one of the predictors of high diagnostic accuracy is the formulation of an alternative hypothesis, different categories of judgment processes can be distinguished. First, after formulating an alternative hypothesis, a correct or incorrect diagnosis can be made. Secondly, a correct initial hypothesis can lead to correct or incorrect diagnoses. Thirdly, a correct or incorrect diagnosis can also be made after an incorrect initial hypothesis. Altogether, 6 different categories of judgment can be distinguished.

Out of the 6 possible judgment categories, two seems of special interest: On the one hand, the process that maps the ambiguity of the situation and subsequently leads to a correct result. Founded on the model assumptions, the ambiguous diagnostic situation should start with the formulation of alternative hypotheses and the given information (the tasks to select) must be processed as hints to correct diagnoses (normative judgment process). On the other hand, and taking into account the research interest, the process that starts from a single confirmatory misguided hypothesis and despite contrary information leads to an incorrect diagnosis (confirmatory-biased judgment process). Table 2 provides the descriptive overview on these two judgment categories concerning the number of processed information and the proportion of relevant diagnostic information.

Table 2: Descriptive overview of the information processing of the two judgment categories correct diagnoses based on alternative initial hypothesis and incorrect confirmatory diagnoses

	N	average number of processed information (SD)	average proportion of relevant diagnostic information (SD)
Normative judgment process: correct diagnoses based on alternative initial hypothesis	65	2.51 (2.02)	0.72 (0.28)
Confirmatory-biased judgment process: incorrect confirmatory diagnoses	84	2.48 (2.05)	0.47 (0.36)

The comparison using a one-factor ANOVA shows that the two judgment processes do not differ in the amount of information processed ($F(148) = 0.009$, $p = .926$, $d = 0.016$). Significant differences are shown, however, in the proportion of relevant diagnostic information processed and in the certainty of the final diagnosis: correct judgments based on an alternative hypothesis are made using a greater proportion of relevant information ($F(148) = 1.734$, $p \leq .001$, $d = 0.219$).

Discussion

Accurate diagnosis of student misconceptions by the teacher is crucial for building resilient conceptions in learners (Bradshaw & Templin, 2014). The research interest of the present study focuses on the emergence of such judgments via the investigation of the underlying cognitive processes (Loibl et al., 2020; Rieu et al., 2020). For such a complex judgment situation, it is assumed that biases take place and have a negative impact on judgment accuracy (Oswald & Grosjean, 2004).

The present study defines diagnostic judgments of misconceptions in decimal fractions as information processing in form of hypothesis testing and examines the impact of the created initial hypothesis, the number and type of processed information on the final diagnosis. It is expected that the information processing of the participating persons indicates the presence of confirmatory biased diagnoses.

In a first step, the formulation of alternative hypothesis and the processing of relevant information could be identified as predictors for the accuracy of judgments in the assessment of misconceptions in the area of decimal fractions. Due to the fact that all participating students had the necessary specific PCK about the misconceptions, these results show that the ambiguity of the diagnostic situation must be perceived and subsequently appropriate strategies must be used to cope with it. These findings complement previous studies on knowledge-guided information processing in diagnostic situations (Ostermann et al., 2018; Rieu et al., 2022) and person-dependent diagnostic sensitivity as a disposition (Kron, Sommerhoff, Achtner, & Ufer, 2021).

In a second step, categorising and contrasting the normative process with the confirmatory process highlights the differences between the two approaches concerning the type of information processed to obtain accurate judgments. These results indicate that an information-integrating strategy leads more often to a correct diagnosis (Böhmer, Hörstermann, Gräsel, Krolak-Schwerdt, & Glock, 2015; Fiske & Neuberg, 1990).

Despite several limitations, the results of the present study allow first insights into the judgment processes of prospective teachers when diagnosing misconceptions in the area of decimal fractions. The categorisation carried out on the basis of the type of initial hypothesis and further information processing permits an initial distinction between normative-accurate and confirmatory-biased judgment processes. The normative procedure, which processes relevant information based on alternative hypotheses in order to obtain an accurate diagnosis, should be incorporated into teacher training as a judgment strategy in complex situations to achieve a higher diagnostic accuracy and thus increase the adaptivity of teaching.

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Initial phase of a study on assessment and argumentation: students' answers analysis and selection

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Keywords:Argumentation, assessment, answers selection.

Introduction and Theoretical framework

This poster presents the first phase of an on-going study. The aim is to investigate features that are considered by Italian secondary school teachers when assessing the answers to tasks that require a justification. A small group of about ten teachers with different years of experience will be involved and asked to assess some students' responses during in-depth interviews. In this work, we focus specifically on a reflection about students' answers selection. We assume that the students' text that will be considered by teachers could highly influence the characteristics that will emerge and consequently impact on the whole study. Our aim is not to select *correct* answers, but to highlight texts differences that could help us to point out some specific aspects of teachers' points of view. To this aim we consider two theoretical constructs. Drawing from Habermas' concept of rationality (Habermas, 2003/1999), as adapted in Morselli and Boero (2009), we mainly consider three aspects. The epistemic aspect, which regards statements validation in respect to accepted premises and ways of reasoning; the teleological aspect, related to the problem-solving strategies and choices; and the communicative aspect, concerning both the expression of the reasoning and the adhesion to mathematical culture standards. In addition, we consider the three assessment criteria of argumentation conceived during a European Project focused on Formative Assessment by the Italian research group (Cusi et al. 2019). These are: correctness, related to the absence or presence of mistakes in the result, in the resolution process, or in the theoretical recalls; completeness, inherent to the presence of sufficient information leading to the conclusion; and clearness, concerning the way of expression and the reader's ease of understanding. To better characterize the data, we expand the criteria of correctness including the idea of *pertinence* of the mathematical knowledge used in the task resolution. These three criteria have not been used as an analysis instrument, rather as a tool to trigger classroom discussion about argumentation. We tried to include these three criteria with the idea that they could be more easily shared and discussed during teachers' interviews.

Methodology and Results Discussion

Figure 1 represents the chosen task. It was administered during the 2018- Italian Standard National Test (SNV) to 10^{th} grade Italian students, and it is related with the process "Arguing and Proving" in the SNV framework. We refer to the work of Garuti and Martignone (2019) for more details about content and influences of SNV framework. We selected this task as it offers the opportunity of working with a great variety of answers proposed by the Italian students. The qualitative analysis is conducted on 500 responses randomly selected from all the 15 233 answers. Answers are coded in relation to the three aspects of Habermas' concept, each of them being characterised with respect to different components emerging from the study of students' productions.

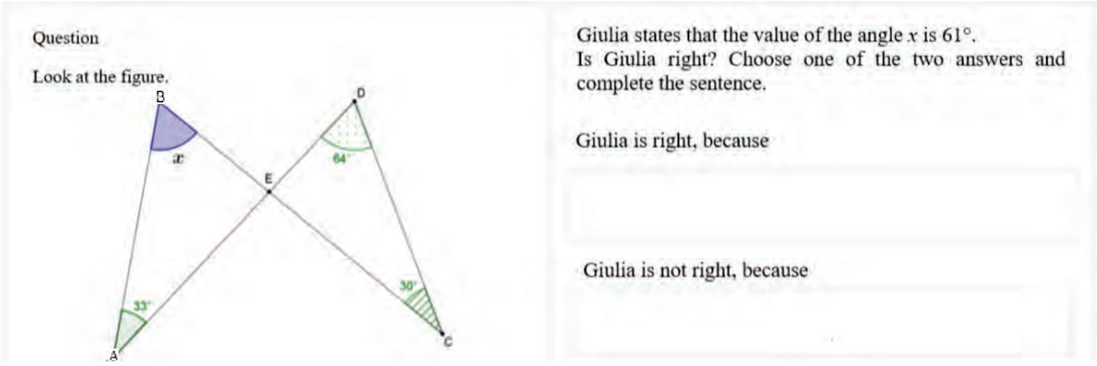

Figure 1: Task from SNV 2018, grade 10 (translation by the author)

Analysis led to the characterisation and selection of different responses. In respect to the teleological aspect, answers are characterised by the strategies used by students, as for example the choice to use a protractor, computational strategies, or to base on visual perception; the ways of implementing the chosen strategies (considering, when possible, the *correctness*) and the way of representing the steps of the procedures or strategies used (*clearness* and *completeness*). The epistemic aspect is shaped by the presence of specific theoretical justification (also referring to the *completeness* of the answers), and the *correctness* of the explicit theoretical references (taking into account also wrong premises that lead to the correct answer, as in the case of students considering the sum of the measures of the interior angles of a triangle being equal to 120°). As far as the communicative aspect is concerned, the different answers show different awareness of the mathematical culture standard, and attention to audience comprehension (considering *clearness* and *completeness*). To sum up, using these two theoretical concepts allows us to characterise the different answers, the selection of which could be a good starting point to discuss with teacher about possible evidence for acceptable answers.

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Factors that influence the difficulty level of digital arithmetic assessment items for first-grade students

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Digital technologies enable new possibilities for the assessment of mathematical competence. When designing an assessment, it is essential to know how different design elements affect both the item difficulty and the strategies used by the children. In this paper, we investigate digital items that were designed to measure arithmetic competence as a component of the Foundational Number Sense (FoNS) framework for five- and six-year-old children. A Rasch analysis of the performance of 302 Norwegian children showed that the type of arithmetic problem and the magnitude of the answer strongly affected an item's difficulty level. Our qualitative observations indicated that certain additional design elements of the items might have influenced both the items' difficulty and the children's solution strategies. From a mixed methods perspective, we discuss the potential of different design elements to better assess children's understanding of numbers.

Keywords: Assessment, digital technology, numbers sense, arithmetic competence, primary school.

Introduction

Digital technologies bring both constraints and affordances to assessment in mathematics education (Threlfall et al., 2007). When assessing young children's mathematical competence, using a digital medium can alleviate the effect of irrelevant demands, such as reading or writing skills. At the same time, we might add elements in the design process of a digital item that could affect the assessment in unintended ways. Carefully designing digital assessment items might enable us to improve assessments and tell us more about the children's solution processes (Saksvik-Raanes & Solstad, 2021). To realise the full potential of digital assessments, we need to know more about how children perceive the different design elements of digital items and what strategies they use to solve them.

In this paper, we investigate digital arithmetic items for five- and six-year-olds from a mixed methods perspective and pose the following research question: Which design elements influence the level of difficulty of digital arithmetic assessment items, and how do the design elements influence the strategies that first-grade children use to solve these items?

Frameworks

Arithmetic competence as a part of the FoNS model

The Foundational Number Sense (FoNS) model describes the number-related skills that require instruction (Andrews & Sayers, 2015). In FoNS, the number sense concept is defined as multi-layered, flexible and relational. The FoNS model divides the number-related skills into eight interrelated categories: number identification, systematic counting, number–quantity relationships, quantity discrimination, representing numbers, estimation, simple arithmetic competence, and awareness of number patterns. In this paper, we focus on simple arithmetic competence, which is described as a child's ability to manipulate small sets through addition or subtraction.

Item design

The arithmetic items presented in this paper were designed based on four main categories: change and combine (sum items) as well as compare and equalise (difference items) (Carpenter & Moser, 1984). Previous studies have shown that children in kindergarten are highly capable of solving such items through modelling the situations in the problem (Carpenter et al., 1993). Sum items are composed in two ways. Either with one initial quantity and an action that causes a change (join), or with two initial quantities that may be considered separately or as a part of a whole. Difference items involve comparing two quantities to determine the difference between them. Equalise problems include an additional action that is to be performed to make the two sets equal.

In addition to problem type, the difficulty of the items was expected to depend on three further design elements. Some items used pictorial representations of numbers, and other items used symbolic representations of numbers. The items involved different numerical values between one and twenty. We balanced the number of items involving small (< 10) and large (≥ 10) numbers and items having ordered and unordered response buttons (see Figure 3).

Methods

Participants and procedure

Fifteen arithmetic items were solved by 302 first-grade children who were a part of a larger study that investigated 368 children's number sense using digital assessment tools. To select participants for the project, we invited about 50 elementary schools in and around Trondheim municipality in Norway to participate in the project. Eight of the interested schools were chosen to participate and all the 1st grade children in these schools carried out the assessment. The children were five and six years old.

A researcher visited the schools over a period of two months at the beginning of the school year. Groups of six to eight children carried out the assessment on separate tablet computers. The participants were seated in such a manner that they would not be disturbed by each other's screens or sounds. All children were given the same instructions before they started the assessment and were free to finish it at any time. Pre-recorded voice instructions were given for each item. The arithmetic items appeared at the end of the full assessment. There was no time limit for the items, but the time taken for each item was recorded. The children typically spent between 15 and 25 minutes on the full assessment, of which about one-fifth comprised arithmetic items.

Qualitative data from individual interviews conducted with 19 first- grade children solving the arithmetic items, were collected independently as a part of a master's degree project (Schjølberg, 2021). The goal of the interviews included in the master's project was to get an overview of the children's strategies. One of the strategies applied by one of the students who participated in the master's project is included in this paper. The interviews were carried out at about the same time as the main data collection.

All the described studies have been approved by the Norwegian Centre for Research Data, and the necessary guidelines related to depersonalisation and parental consent have been followed.

Items

The 15 arithmetic items were designed to investigate the different aspects of the children's arithmetic competence that could influence item difficulty. Four items involved the difference between two numbers. Two of these 'difference items' were compare problems, and two were equalise problems (Carpenter & Moser, 1984). The compare items involved small numbers (< 10), while the equalise problems involved large numbers (≥ 10).

Eleven items asked for the sum of two numbers. Eight of these 'sum items' included the systematic variation of three design elements: (i) small (< 10) or large (≥ 10) answer, (ii) symbolic or pictorial representation of the problem and (iii) ordered or unordered response buttons (see Figure 3).

A priori, we expected the difference items to be more difficult than the sum items, the items with large numbers to be more difficult than those with small numbers, the items with symbolic representations of numbers to be more difficult than those with pictorial representations and the items with unordered response buttons to be more difficult than those with ordered response buttons.

Analysis

All items were scored dichotomously, meaning that the children received one point for a correct answer and zero points for a wrong answer. Rasch measurement was used for the quantitative analysis of the children's responses using the Winsteps software (Linacre, 2017). The Rasch model is a probabilistic measurement model that provides interval-scale measures of item difficulty and person skill on the same measurement scale in units of logits (Wright, 1977). The probability that person v scores 1 point on item i depends on the difference between the skill of person v, β_v and the difficulty of item i, δ_i according to

$$\boldsymbol{P}\,\{\boldsymbol{X_{vi}} = \boldsymbol{1}|\boldsymbol{\beta_v}, \boldsymbol{\delta_i}\} = \frac{\boldsymbol{e}^{(\boldsymbol{\beta v} - \boldsymbol{\delta i})}}{\boldsymbol{1} + \boldsymbol{e}^{(\boldsymbol{\beta v} - \boldsymbol{\delta i})}}$$

Winsteps implements the joint maximum likelihood estimation (JMLE) algorithm to estimate the parameters of this model.

The excerpt from the individual interviews demonstrates how some children used the available resources on the screen to find the right answer to the problems. The qualitative data was analysed using a thematic analysis (Bryman, 2016).

Results and discussion

Task type

From Figure 1, we see that the type of task strongly influenced the difficulty of the items. As expected, the four difference items were also the four most difficult arithmetic items (Figure 1, orange markers). An independent samples t-test between the four difference items and four comparable summation items (symbolic representations involving large and small numbers and ordered and unordered response buttons) showed that this difference was significant ($p = 0.026$; $df = 6$).

Surprisingly, within the difference category, both compare items had higher difficulty than the two equalise items. The compare items were more difficult despite involving small numbers, while the equalise items involved large numbers and came with more complex voice instructions.

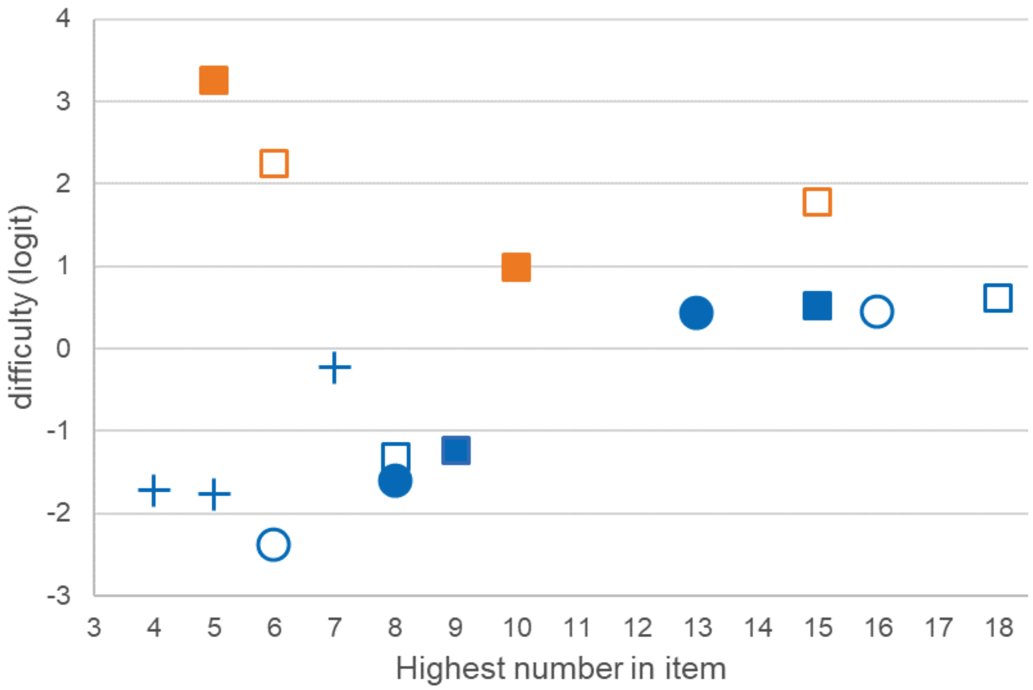

Figure 1: Item difficulty ordered by the highest number involved in the item

The items are categorised as (i) sum item (blue markers) and difference item (orange markers), (ii) pictorial representation (circular markers) and symbolic representation (square markers), and (iii) ordered response buttons (open markers), unordered response buttons (filled markers) and no response buttons (plus markers)

The compare and equalise items were visually identical (Figure 2), and three design elements differentiated them: the magnitude of the answer, the order of the response buttons and the voice instruction given. The answer was less than 10 for both compare items, while the answer was greater than 10 for both equalise items. The following voice instruction was given for the compare items: “How many more marbles are there in the blue box?”. For the equalise problems, the following voice instruction was given: “There should be an equal number of marbles in each box. How many more should the red box have?”.

In the design process, the difference items were challenging to create in a way that would enable all children to understand the given voice instructions. We wanted to keep the instructions as simple as possible to adapt to the attention span of the target group. At the same time, the compare and equalise problems represent two semantically different problems. The equalise problems involve one more step than the compare problems, as an action is performed on one of the two groups when comparing the quantities. We therefore expected that the equalise items would be more difficult. However, Figure 1 shows that the compare items were the most difficult.

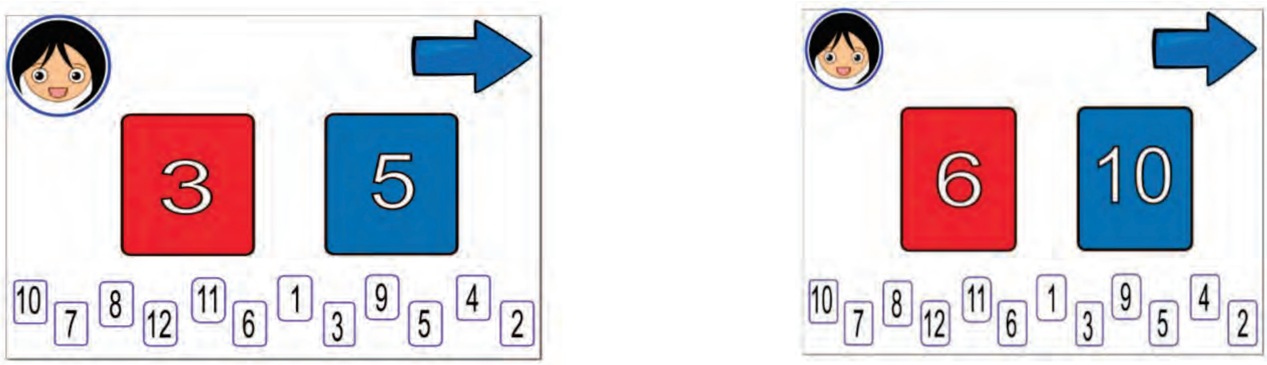

Figure 2: Two difference items

Left: item 13 (compare). Right: item 15 (equalise)

One reason why the compare items appear to be more difficult could be that the added action in the instructions for the equalise items make them more concrete, and this might have aided the children's comprehension of the items. Carpenter et al. (1993) found the kindergarteners in their study to be highly competent in solving compare problems through modelling. It is also possible that some children ignored the "more" word in the instructions for the compare tasks and interpreted it to mean "how many marbles are there in the blue box?". These results indicate that simplified instructions in word problems may lead to more misunderstandings and reduce the child's possibilities for modelling the situation.

The level of abstraction in the illustrations of these four items might also have contributed to their relatively high difficulty compared to the sum items. The children's previous experiences could also have played a role in determining the level of difficulty, as it seems that they were more familiar with the language-related problems that involved addition than subtraction.

Taken together, these results underline the importance of carefully investigating the various design elements when developing digital assessment items.

Magnitude of the answer

For the sum items, we found that difficulty was strongly correlated with the magnitude of the answer of an item. The Pearson correlation between difficulty and answer magnitude was $r = 0.88$ ($p < 0.001$) for all 11 sum items and $r = 0.96$ ($p < 0.001$) for the eight sum items that had a shared problem structure (Figure 1). In particular, the four sum items with a large answer were 2.1 logits more difficult than the corresponding four sum items with a small answer on average. An independent samples t-test showed that this difference was significant ($p < 0.001$; df = 6).

Number representations

A pictorial representation of a number is often thought to be easier to understand than its more abstract, symbolic representation. However, in the group of the eight sum items that shared a problem structure, we found no significant difference in the difficulty between the four items with symbolic representations (blue squares in Figure 1) and the four corresponding items with pictorial representations (blue circles in Figure 1) ($p = 0.65$; df = 6; independent samples t-test). One reason for this might be that the response buttons were written in the symbolic representation. Thus, knowing the correct answer only verbally would not be sufficient to provide a correct response. Indeed, from the qualitative data, we observed that some children knew how to verbally count to 20 without recognising the corresponding written numerals (see the next section).

Order of the response buttons

Based on pilot studies, we had the a priori expectation that unordered response buttons would increase the difficulty of the items because they do not easily allow children to rely on verbal counting strategies. However, at least at first glance, the structure of the response buttons did not seem to strongly influence the difficulty of the items (Figure 1; open vs filled markers). An independent samples t-test between the four sum items with ordered response buttons and the four sum items with unordered response buttons was not significant ($p = 0.88$; df = 6).

On closer inspection, the four sum items with large answers were found to be of similar difficulty (Figure 1) even though the two items with ordered response buttons had larger answers than the two items with unordered response buttons. It is therefore possible that the ordered response buttons made the two tasks with the largest answers easier to solve. The latter interpretation is substantiated by qualitative analyses of the children's solution strategies. One example is item 10, which involved numbers that some children were not very familiar with. After the voice instruction "What is sixteen and two altogether?", the child was to choose the correct answer from the response buttons at the bottom of the screen. The qualitative observations gathered during the data collection led us to carry out a small qualitative interview study on the children's solution strategies.

Qualitative observations of Agnes's strategies

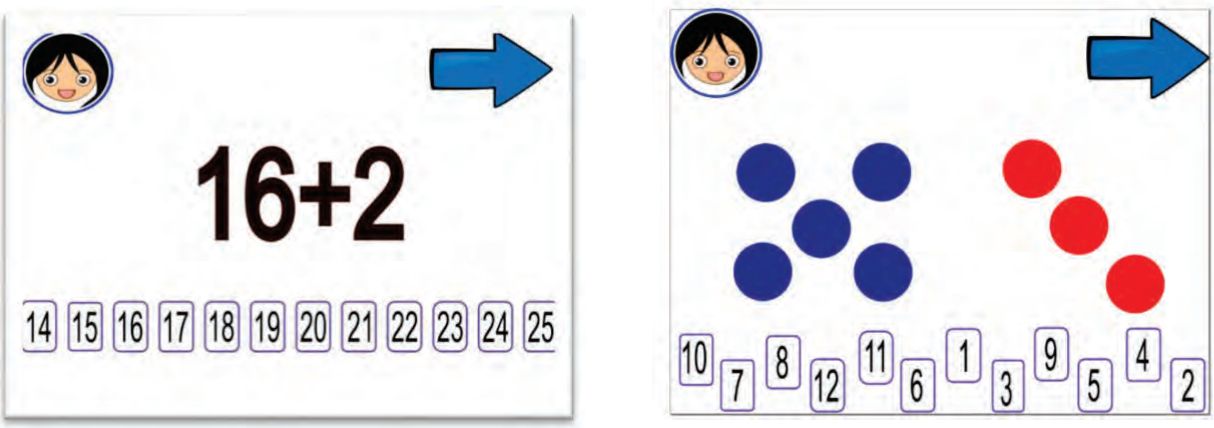

Figure 3: Sum items with systematic variation in the design elements

Left: item 10 with large answer, symbolic representations and ordered response buttons. Right: item 5 with small answer, pictorial representations, and unordered response buttons

In item 10, one of the children, Agnes, used the number alternatives on the screen to find the right answer, but she did not know what numeral she ended up with:

> Researcher: What is 16 and 2 altogether?
> (..)
> Agnes: It is... 16 and 2…
> (..)
> Agnes: Wait… and then we go 1-2.
> (Agnes points to 16 and makes two jumps with her finger on the numerals to the right)
> (..)
> Researcher: What are you thinking?
> Agnes: That one.
> (Points to 18)
> Researcher: Do you know what number that is?
> Agnes: 1-2-3-4-4-5-6-7-8-9-10-11-12-13-14-15-16-17-18. Eighteen!
> (Counting to 14 and then pointing to the numerals on the screen)

Agnes used the buttons to find the numeral that displayed the answer when she was unable to recollect the numerals after 14. In the design process, we did not expect the children to use the number alternatives in this way. These observations also emphasise the importance of investigating the available resources and how these could affect children's solution strategies.

To obtain a more fine-grained analysis of the role of ordered and unordered response numerals, we need to investigate an instrument in which items with ordered and unordered response buttons are designed with identical arithmetic problems.

Conclusions

To investigate the factors that influence the level of difficulty in digital assessment items for arithmetic, we have looked at the role of problem type, representations, numerical values and differently ordered response buttons. We have also considered how children may use the ordered response buttons to find the correct answer for an item.

The strongest determinant of item difficulty was the type of problem: difference items were more difficult than sum items, and compare items were more difficult than equalise items. The second strongest determinant of item difficulty was the numerical value of the item's answer. Whether the problem was presented in a symbolic or pictorial form did not affect item difficulty. Finally, although we could not conclusively determine the influence of ordered or unordered response buttons, our data indicates that ordered response buttons allow children who have not yet acquired mastery over large numerals to use these buttons as a number sequence that helps them solve the problem. Including both kinds of response buttons might help distinguish between the children's knowledge of large numerals and their reasoning regarding the number sequence or with a number line.

While digital technologies continue to influence the assessment of students' mathematical competence with its new possibilities, it is also important to consider the technical and methodological challenges involved in this development (Nortvedt & Buchholtz, 2018). There are many aspects to consider when investigating the various elements that affect children's solution processes when interacting with digital technologies. To ensure the validity of such assessments, it is important that future research looks more into how the various possibilities that digital technologies enable can both improve and hinder students' performance. One way forward could be to compare items that have different design elements but similar answers. Looking more directly at a greater variety of both word problems and symbolic items can allow us to determine how the different contents affects the difficulty of the items. With the use of digital technology, we could also look more closely into young children's competence for solving digital word problems. For instance, one could record the children's solution process while they are introduced to a variety of word problems with more elaborate instructions and pictorial representations. The use of different digital aids, with a larger degree of interactivity, could enable us to study in more detail how the children model and use different strategies to solve the problems.

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Analysis of digital assessment in mathematics through the lens of connectivity

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The focus of this paper is to investigate the framework of connectivity in relation to analyses of summative digital assessment in mathematics. As an example, two digital items in mathematics are analyzed according to the framework of connectivity. The results imply that the framework, along with some instrumental adaptations, can be used to guide item analysis. In future, the framework could likely guide analyses of digital national tests in mathematics. A suggestion is to broaden the scope of this research to a macro level, which might focus on assessment platforms, and possible connections to other digital resources.

Keywords: Assessment, mathematics, connectivity, item analysis, technology.

Introduction

The growing use of technology in society and accordingly in schools (e.g. the use of e-textbooks[1]) can motivate digital assessment because technology is already present in students' learning processes and arguably assessment needs to keep up with this development (Beller, 2013; Bennett, 2002). Other arguments for digital assessment is the possibility to reduce teacher assessment related work-load and improve accessibility for students with special educational needs (Skolverket, 2021). Many researchers (see for example Beller, 2013; DePascale et al., 2016; Sireci & Zenisky, 2006) have motivated for digital assessment due to new opportunities for construct broadening which makes it possible to assess aspects which are not assessable in paper-based tests. For example, a spreadsheet program can be connected to a digital test item and thus assess knowledge and understanding in a more realistic way.

However, there are several issues regarding digital assessment, especially in mathematics. These challenges can be related to practical problems such as the transfer of information from screen to paper and vice versa, not having access to digital scribbling spaces and not being able to naturally navigate between items in a digital test (Russell et al., 2003). Problems can also be related to limitations in automatic assessment systems which may lead to artificial and non-authentic items in mathematics (Drijvers, 2018). Challenges of digital assessment can furthermore be related to validity and construct-irrelevant variance (Messick, 1989). In the context of mathematics and assessment this can refer to the risk of assessing computer skills instead of mathematics. These questions are especially relevant for high-stakes summative tests.

Background

In Sweden, such high-stakes summative tests are known as the national tests, because they are administered nationwide to specific year groups on an annual basis. The national tests are to be digital,

[1] Pepin et al. (2016) define an e-textbook as "an evolving structured set of digital resources, dedicated to teaching, initially designed by different types of authors, but open for re-design by teachers, both individually and collectively. (p.644)

or partly digital, in a few years' time. A decision has not yet been made as to whether the national tests in mathematics will be completely digital or performed as a combination of both computer and paper-based tests (Skolverket, 2021). Research is ongoing and during forthcoming field trials will study student responses to different digital item formats and digital items in an online assessment platform will be investigated. The PRIM[2] group at Stockholm University researches and develops the national tests in mathematics for primary school (grade 3, 6 and 9) and the first mathematics course at upper secondary school (grade 10) in Sweden. At the end of school year 9 the students sit a national test in mathematics, comprised of written parts (with multiple choice, shorter constructed response, and longer constructed response questions) and an oral part. The national test is assessed based upon scoring guidelines, and the results on the parts are aggregated to produce a score which corresponds to a grade. The students' result on the national test must then be given a special consideration, together with all other available results, in teacher's grading.

For the national tests in Sweden a systematic framework exists (Skolverket, 2017), the purpose of which is to ensure high quality national tests and high reliability regarding the use and consequences of the test results in relation to the purpose of the test, which is fair and equal grading. Importantly, the framework applies to the current paper-based national tests, and emphasizes that if relevant factors are changed, e.g. the mode of administration or the technology used to administer and assess items, evidence of validity must be reconsidered (Skolverket, 2017).

Consequently, in order to develop digital national tests of high quality, new analytical frameworks are needed. It is also reasonable that these frameworks should account for the increasing use of digital resources in education. The framework of connectivity developed by Gueudet et al. (2018) could meet such requirements. This framework has been developed for analysis of digital resources and is gaining recognition in the mathematics education research community. The concept of connectivity has been used to study how students use digital resources, such as online exercises and teaching materials (Sabra, 2019). With regards to assessment, I have not been able to find research applying the theoretical lens of connectivity. Despite this possible shortcoming, I argue that it is a suitable candidate for guiding analyses of digital assessments in mathematics, due to the incorporation of new learning theories that recognize modifications of learning processes in times of digitalization. However, the framework is developed for e-textbook analysis and in order to apply this framework to the context of digital assessment, some adaptations might be required. Therefore, the idea framing this paper is to investigate whether the framework of connectivity can be used in order to analyze digital items in mathematics. The ultimate goal, but beyond the scope of this paper, is to develop the framework to guide holistic examinations of digital national tests in mathematics. Thus, the paper is framed by the following research question:

How can the framework of connectivity be adapted in order to guide analyses of digital assessment in mathematics?

This question is highly relevant because digital assessment in mathematics (at least beyond the scope of multiple choice questions) is a relatively new phenomenon in educational systems. This new mode

[2] PRIM-group is a research and test developing unit, within the Department of Mathematics and Science Education at Stockholm University, where the author of this paper is affiliated.

of assessment calls for innovative analytical tools in order to develop and analyze the forthcoming digital national tests. The next section introduces the theoretical concept of connectivity by grounding the concept in modern theories about learning and understanding.

Theoretical framework

Since teaching, learning and assessment in many school systems is happening with and through digital resources (Drijvers, 2018), an analysis of these digital resources calls for new analytical frames (Gueudet et al., 2018). A traditional way of performing textbook analysis is to use quantitative measures. Researchers in mathematics education have also performed textbook analyses categorizing the content at macro and micro level[3]. An analysis at macro level may suggest a horizontal/holistic analysis with attention to general features of the textbook, whereas an analysis at micro level may propose a vertical analysis where specific mathematical content or competencies are considered.

Upon introducing the concept of connectivity Gueudet et al. (2018) ground the concept in Hiebert and Carpenters (1992) framework about understanding; that is, in order to understand and develop mathematical knowledge it is key to be able to make connections between structures of mental representations. The number and the strength of these connections is assumed to determine the level of understanding. This theory about understanding is thereafter connected to the epistemological position *connectivism*, introduced by Siemens (2005) as a new learning theory in which learning is not just seen as an internal activity, but also as actionable external knowledge, for example within a database or an organization. One argument presented in connectivism is that some learning theories (e.g. cognitivism, behaviorism, and constructivism) were developed at a time when learning was not affected by technology, and as such are unable to fully explain how learning takes place in a modern digital world. Within connectivism, learning is essentially about the connections which enable learning to take place. These connections can be understood practically as connections between people, as well cognitively as connections opened up by a digital resource.

This paradigm shift in how learning processes are viewed in a digital age is therefore motivating Gueudet et al. (2018) to study e-books looking

> for connections in, between, and across individuals' cognitive/learning tasks and activities, and how e-textbooks may support those (*micro* level)
>
> for "connected" learning between and across groups of individuals, teachers, or students (*macro* level) (p. 543)

The different ways of making connections within a digital resource is referred to as connectivity at micro level (internal) and the different ways of making connections from/to a digital resource is referred to as connectivity at macro level (external). The concept of connectivity of a digital mathematics resource is defined as the "connecting potential for a given user (student or teacher) both practically as well as cognitively." (Gueudet et al., 2018, p. 545)

[3] In Gueudet et al. (2018) the notion of textbook analysis at macro and micro level is explained in detail with examples of studies, and research questions when investigating textbook analysis at macro and micro level.

Relevant for this paper is the view that digital mathematics items presented in an assessment platform are digital resources for students. The framework of connectivity is developed for analysis of digital resources such as e-textbooks, and since an e-textbook in mathematics contains several items this view seems reasonable. Another reason for this is the possibility to integrate digital resources into an item (e.g. the spreadsheet in the item presented in Figure 1) which might offer students a connection between a software and a mathematical problem. Since this paper is focused on mathematics items presented in an assessment platform, only connectivity at the micro level is considered in the analysis.

The next section introduces two digital items in mathematics that were developed based upon curriculum descriptions of digital technology. The two items are thereafter analyzed through the theoretical lens of connectivity, at micro level. The adaptions needed in the framework of connectivity are subsequently presented in the results section.

Methodology

The items in this paper were developed in spring 2021 together with a year 9 test developer. The focus was content related to digital technology and the use of digital tools, according to the mathematics curriculum for years 7–9. For example, the curriculum states that “All pupils should be given the opportunity to develop their ability to use digital technology.” (Skolverket[4], 2018, p. 8)

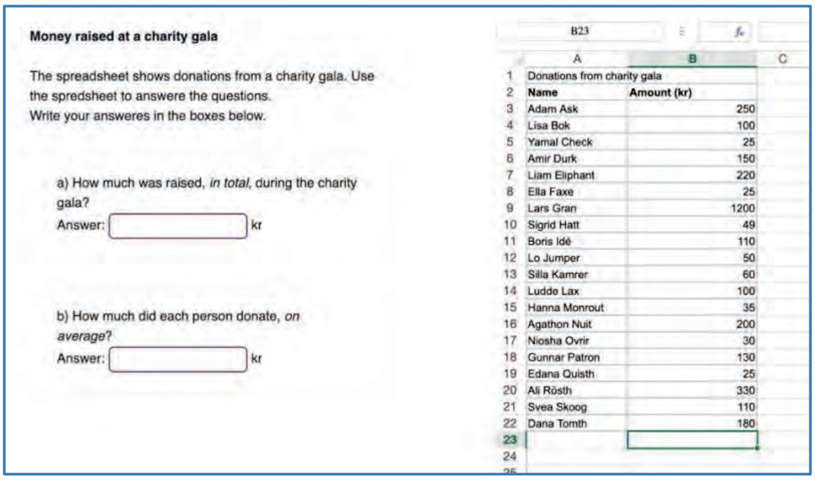

Figure 1: Money raised at a charity gala

The first item, Money raised at a charity gala (see Figure 1), has been translated from Swedish and presented in Figure 1 as it appears in the assessment platform used in Sweden. This item contains a spreadsheet showing donations at a charity gala. In the spreadsheet the students can interact with the spreadsheet, e.g. highlight cells; calculate the sum and mean. Students submit their responses by filling in the two boxes on the left of the screen.

[4] Skolverket is the Swedish National Agency for Education

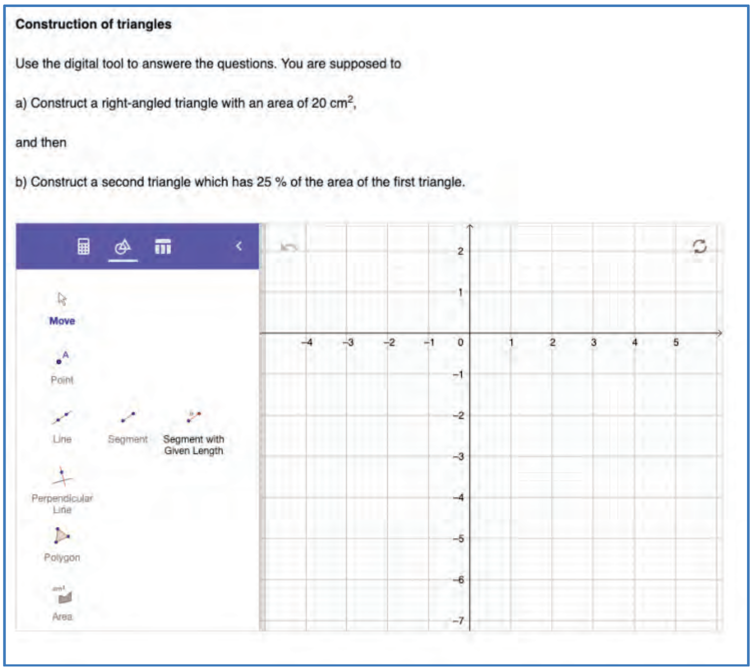

Figure 2: Construction of triangles

The second item, Construction of triangles (see Figure 2), contains a GeoGebra (Hohenwarter, 2021) component where the students can draw lines or line segments between points in a coordinate system, and calculate the area of a polygon. This item was developed based upon specifications in the curriculum about geometry and digital tools. Skolverket specifies in the core content for mathematics, years 7–9 that as part of teaching for geometry "Depiction and construction of geometrical objects, both with and without digital tools…" must be carried out in class, as well as "…Scales for reducing and increasing two and three dimensional objects." (2018, p. 59)

These two items were analyzed through the theoretical lens of connectivity at a micro level (Gueudet et al., 2018). The analysis was performed by examining the items using the micro level analysis grid presented in Gueudet et al. (2018, p. 548), see Table 1. The examination was carried out by searching for connections according to each of the aspects in the analysis grid. Since only two items were analyzed, a specific concept was not considered in the analysis, unlike how the analysis was performed in Gueudet et al. (2018).

Table 1: Micro level grid (for a given set of concepts)

Connections to/in terms of
Previous knowledge (learned in previous chapters in the same textbook)
Further knowledge (role of the chapter considered in further chapters)
Different concepts
Different moments of appropriation of the same concept
Different topic areas within mathematics
Authentic situations, real-life problems, problems referring to other disciplines
Different semiotic representations
Different software and calculator
Different strategies for the same exercise
Variations of the same exercise: same exercise with different values, different exercises around the same question etc.
Different students' needs: presence of exercises for high-achieving students, for students who have difficulties etc.
Assessment procedures and storage of results

Results

The analysis revealed that the following categories of connections in the micro level grid cannot be used: Previous knowledge, future knowledge, different moments of appropriation of the same concept, different topic areas within mathematics, variations of the same exercise, and assessment procedures and storage of results. These categories are not applicable due to differences in design and purpose of an e-textbook as opposed to a digital assessment in mathematics.

The item Money raised at a charity gala offers connections to different concepts, and to a real-life situation where software is used for a specific task. There are also connections between different methods to solve the item. The student can use a calculator tool to calculate the sum and the mean. Moreover, the integrated spreadsheet program will offer the possibility to highlight cells and read the sum and mean, or calculate the sum and mean with integrated functions in the calculating program.

The item Construction of triangles offers connections to a dynamic geometric representation. With the dynamic geometry program students are able to construct polygons and measure the area with the area tool, or calculate the area with the integrated calculator. The item also allows students to work with different methods, e.g. the area of a scaled object is equal to the squared scale factor.

The analysis further reveals that both these items are missing connections to the needs of different students. For instance it is not possible to listen to the text of the items, and there is no accompanying instruction video about how to use the digital tools included in the items. Thus the framework of connectivity helps by focusing on connecting potential, or the absence of connecting potential, when reviewing and developing items.

Discussion

These results illustrate that the framework of connectivity at micro level can be used in order to find connections. However, when analyzing the two items some adaptations of the framework were needed. Thus the lens of connectivity was not adapted in a theoretical way, rather the adaptations are related to the number of categories in the analytical tool (see results), and also to the method of analysis. Relating to method of analysis, when Gueudet et al. (2018) describe the analysis at micro level, they are searching for connections for a given mathematical concept (e.g. functions) between different parts of the e-textbook. These connections can also be related to how the concept is presented differently in different parts of the e-textbook, and can also be related to the progression of a concept. When analyzing isolated digital items in mathematics such comparisons are more limited due to the summative nature of testing, and in this case due to the fact that only two items (dealing with different mathematical concepts) were analyzed. In order to analyze the two presented items it was thus not possible to consider a single mathematical concept, and therefore some of the categories in the analytical tool were not applicable. Not being able to use the full range of the micro level tool is a limitation of the presented study and a recommendation for future research is to start with a large item-pool and choose items for a given concept or a mathematical theme, e.g. functions. In this way, it will likely be possible to look for connections to different moments of appropriation of the same concept, and also for variations of similar items connected to a specific concept. Another limitation of the presented study is the inability to contrast the results with other studies using the theoretical framework of connectivity within the scope of assessment.

A question for future consideration is how the framework could be used generally to analyze assessment in mathematics, and more specifically to guide analyses of digital national tests in mathematics. Such studies could also investigate the potential to consider a mathematical theme in the analysis, and furthermore to explore new categories that might be included in the analytical tool. A suggestion is to explore whether mathematical competencies (e.g. mathematical reasoning, problem solving) could guide these analyses along with a given mathematical theme. Another implication for future research is to analyze a digital resource (e.g. an assessment platform) through the lens of connectivity, at a macro level. Such an analysis may focus the potential of linking to and between users and resources/tools outside an assessment platform. This may include the potential to connect users with users (teachers, students), as well as users with designers (teachers, assessors, test developers, test designers) and the platform's interaction with other external resources, e.g. programs and teacher digital resource systems.

In the analytical framework of connectivity the possibility to make connections is seen as a critical feature of a digital resource, and important for learning and understanding (Gueudet et al., 2018). The importance to connect concepts is articulated in the syllabus (Skolverket, 2018) for the compulsory school, in the general aim for mathematics: "Teaching in mathematics should essentially give pupils the opportunities to develop their ability to: […] use and analyse mathematical concepts and their interrelationships" (p.56). The opportunities to develop mathematical abilities might be achieved through digital resources and the relationship between concepts might be described in terms of connectivity.

In an era of digitalization, connectivity also extends beyond digital resources and can be viewed as a critical feature of a teacher's working environment (Gueudet et al., 2018). Furthermore, this also relates to assessment. Since technology already is a part of learning processes in a digital age (Beller, 2013), it is fair that this technology, with its potential to make connections, also becomes a part of assessment. When digital resources play an even more important role in assessing knowledge and understanding, theoretical perspectives such as connectivism and theoretical frameworks such as connectivity can bring new insights and may offer novel possibilities to explore assessment beyond the narrow scope of this paper.

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TWG22: Curricular resources and task design in mathematics education

Introduction to the papers of TWG 22: curricular resources and task design in mathematics education

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Keywords: curricular resources, task design, textbooks, curriculum implementation

Scope and focus of the working group

At the macro-level, teachers and students work with mathematics curriculum resources, both digital and traditional, inside and outside the classroom. Individually or collectively, teachers select, (re)design, modify, and interact with such resources for lesson preparation, student assessment, and the planning of their courses. These resources (e.g., educative curriculum materials) are the focus of professional development sessions, where mathematics teachers, often with educational researchers, design and transform curriculum resources, including blended materials, and in the process develop design capacity and valuable knowledge for teaching.

At the micro-level, curriculum resources contain mostly tasks derived from textbooks or other sources. The representation of these tasks in resources, their sequencing, and the teachers' actions during their enactment can limit or broaden the cognitive demand the tasks impose and affect students' views of the subject matter. Thus, they can influence the opportunities afforded to students to make mathematical connections, and to develop mathematical concepts, skills, or habits of mind. The literature indicates that tasks play a key role in effective teaching. There has been an upsurge in publications on various aspects of task design (e.g., task features that can help generate certain forms of mathematical activity); methods of task analysis (e.g., analyses of the learning affordances of certain kinds of tasks); and principles for task implementation in conventional and digital learning environments (e.g., factors affecting the fidelity of implementation of tasks in the classroom). Students can also be involved in task design activities to foster their reflections about what they know, understand, and do.

We summarize the papers presented at the conference according to the following predefined themes:

- Empirical research on teachers' and students' interactions with curriculum materials, resources, and tasks, related competencies (e.g., pedagogic design capacity), and influences on this interaction;
- Theoretical foundations and methodologies of task analysis helpful for task design and the design of curriculum resources;

- Studies on the use of carefully designed curriculum materials, resources, and tasks to support the implementation of particular learning goals and to enhance mathematical competence;
- Collaboration between teachers, between teachers and researchers, and possibly also students for designing tasks and resources and for analyzing their implementation;
- Affordances and constraints of digital and conventional tasks and resources.

Issues addressed during the sessions

An important issue in TWG22 was empirical research on teachers' and students' interactions with curriculum materials, resources, and tasks, related competencies (e.g., pedagogic design capacity), and a better understanding of how tasks and resources influence practice. At CERME12, three papers focused on students and their interactions with (digital) curriculum resources:

- Annalisa Cusi and Agnese Ilaria Telloni: Engaging students as designers of digital curriculum resources: Focus on their praxeologies and new awareness.
- Amal Kadan-Tabaja and Michal Yerushalmy: Online feedback designed to support self-reflection while solving fraction example-eliciting tasks.
- Carlos Quiroz, Saba Gerami, and Vilma Mesa: Student utilization schemes of questioning devices in undergraduate mathematics dynamic textbooks.

In these three papers, students assumed the roles of designers and users of digital curriculum resources in addition to those of learners. The papers reflected a combined focus on students with digital rather than traditional curriculum resources, mainly investigating the affordances of the former and how they are used or how they influence the learning process. All three papers related to attempts to foster students' reflection about the study content and learning process, which is regarded as an important aspect in learning.

Quiroz et al. investigated students' use of questioning devices — a new special feature of interactive textbooks, which seeks to engage students in thinking about the content and allows them to type responses to the questions in the textbooks. These answers are immediately shared with the teachers, who can use the information to adjust their teaching. Quiroz et al. identified three classes of situations, in which these questioning devices were used, and one utilization scheme for each situation. The findings reveal that students used the questioning devices for their intended purpose, but also for other purposes, exerting their agency in using curriculum resources for their needs. The term "questioning device" reflects the authors' and teachers' perspective and does not account for the students' perspective on this feature. For students, it is an answering rather than a questioning device. This leads to the question of how students' needs are considered in the design of digital curriculum resources. In an additional poster, Kanwar and Mesa introduced viewing patterns that were mapped to students' use of questioning devices.

Insight into how students think about the affordances of digital curriculum resources in mediating mathematical content was provided in the paper by Cusi and Telloni. Students were asked to design GeoGebra applets related to the content they had learned in a course supporting the transition from secondary school to university. Although students did not intend to become teachers or acquire pedagogical or didactic knowledge about the subject matter, they showed ability to reflect on the design principles they implemented in the design process and how this influenced their learning. Involving students in the design process of digital curriculum resources is a novel didactic approach afforded by digital resources. On the one hand, it has the potential to provide further insight into students' learning of mathematics. On the other, it seems to be a promising approach to deepening students' understanding of mathematical content. However, open questions remain regarding the timing and the framing of the design activities: Should students design while learning the content or after they have learned it? What happens with the designed resources? What is the goal of designing that is presented to the students? Should students design for other students, for themselves, or for the instructor?

Kadan-Tabaja and Yerushalmy presented an example of an interactive task with related feedback aimed at fostering students' self-reflection and meta-cognitive skills. The special design of the feedback, which combines automated evaluation of students' solutions with opportunities to reflect on the solutions, seems to be a promising approach for future feedback design.

Two papers theorized student interactions that could manifest through concrete design features of mathematical tasks, problem posing, and working backwards, and suggested finer definitions and characterizations for given student interactions:

- Ling Zhang, Andreas Stylianides, and Gabriel Stylianides: Problematizing the notion of problem posing expertise.
- Daniela Assmus and Torsten Fritzlar: Working backwards revisited: Some theoretical considerations.

Zhang et al. compared problem posing of master's and sixth grade students to demonstrate the challenge of characterizing expertise. Using a data-driven approach, they identified expert problem posers based on participants' problem posing characteristics, such as the number of problems posed, their complexity, and clarity. They found that performance was not aligned with the participants' previous mathematical experience or backgrounds. Their results suggest that defining an expert should be specific to the expertise assessed, considering additional aspects beyond mere mathematical background.

Assmus and Fritzlar demonstrated a range of tasks designed to promote the working backwards heuristic. Although this seems to be a single problem-solving strategy, the tasks analyzed revealed differences stemming from the design of the tasks. Focusing on characteristics such as operations and order, careful analysis shows that task design that incorporates different subsets of these characteristics requires distinct solving processes. These differences can considerably influence problem demands and students' problem-solving processes. In a related poster, Assmus and Forster demonstrated different designs of working backwards problems, showing that distinct solving processes were related to different proportions of elementary school students who had solved them correctly.

Two papers addressed contextual and cultural influences on teachers' and students' interactions with curriculum resources, and urged rethinking the conceptualizations of resources and textbooks:

- Dubravka Glasnović Gracin: Rethinking resource conceptualization in times of pandemic and earthquakes: What is important for (mathematics) education?
- Hendrik Van Steenbrugge: Rethinking the notion of textbooks as mediators between the official curriculum and classroom practice.

Glasnović Gracin analyzed how the relevance of resources varies depending on the context. Van Steenbrugge sought to identify how culture becomes apparent in curriculum resources beyond the influence of the curriculum, analyzing textbooks and teacher guides from different countries. Although different in their research object and methods: case study of two teachers (Glasnović Gracin) vs. document analysis of textbooks and teacher guides (Van Steenbrugge), both studies showed the interrelatedness of curriculum resources with the social and cultural context in which they are developed and used. This raises the issue of how to read and interpret studies about curriculum resources from different social and cultural contexts, and suggests that results from one context cannot easily be transferred to another.

An additional perspective on resources was apparent in papers that studied the affordances and constraints of digital vs. conventional tasks and resources. Three papers at CERME 12 revealed different research perspectives.

- Lisnet Mwadzaangati and Mercy Kazima: An examination of mathematical affordances available in grade 2 teachers' guide and learners' textbook on addition of whole numbers.
- Ayla Carvalho and Rúbia Amaral-Schio: Characterizing the presence of activities using GeoGebra in Brazil's mathematics textbooks.
- Malin Norberg: Students' expressions about working successfully with mathematics textbooks: Multimodality and socio-mathematical norms in early years.

Mwadzaangati and Kazima examined the learning affordances provided for a given learning goal. They analyzed the learning affordances of a teachers' guide for teachers in relation to the addition of whole numbers for grade two students relying on variation theory. The study examined the teacher guides used by teachers in Malawi to choose tasks for implementation in their classrooms. The paper adopted two perspectives, discursive and cognitive, by connecting mathematics discourse in instructional aspects with the cognitive ones of what is said and presented. The tasks in the textbooks were analyzed from a cognitive point of view, based on the additive structure of the tasks and their possible variations (Marton & Pang, 2006). The findings point to the low quality of Malawian mathematics textbooks on the addition of whole numbers suggesting that this might be one of the causes of persistent low performance of learners in mathematics (simple strategy). A general question is: How can the results presented influence textbook design or the classroom?

Carvalho and Amaral-Schio reviewed four printed collections of textbooks (16 textbooks from sixth to ninth grade) used in Brazil to characterize and count the textbook activities using GeoGebra, the main dynamic geometry system tool used in Brazil, and to bring to the fore their potential for exploration to engage students in mathematical discovery. The study shows that GeoGebra was used only for geometry activities in all textbooks. Few activities have been found, and even fewer with the potential of exploration. Yet the results do not represent the actual use of GeoGebra in classrooms, raising the question of the distance between the learning affordances of a task and effective learning.

Norberg reported on an ongoing study what was considered successful for 18 7-8-year-old students working with mathematics textbooks. The paper raised the question of socio-mathematical norms related to students' work, and what counts as mathematically sophisticated, accepted, different, efficient, given that norms regulate students' learning. The findings show a tension between using aids and being considered successful in mathematics, which could affect students' possibilities for mathematical learning. They raise questions about what is accepted vs. what is desirable. For instance, using mathematical symbols has been shown to be a way of appearing successful in math, whereas other modes of communication (fingers, gestures, images, diagrams, etc.) are not considered as successful, even if they are efficient. It may be important to consider learning goals in this study as well. And how is it possible to infer mathematical norms from observations? How is it possible to separate the effect of the didactic contract from norms when observing students? Methodologically, what was the influence of the researcher during data collection? In an additional poster, Tutuncu and Hodgen offered a method to analyze the potential of educative materials to offer productive formative assessment by combining analysis of formative assessment practices with the guidance provided to enact these techniques.

Two other issues of TWG22 concern the theoretical foundations and methodologies of task analysis for task and curriculum resource design, and collaboration between teachers, between teachers and researchers, and possibly also students for designing tasks and resources, and for analyzing their implementation. At CERME12, two papers focused on analysis of the selection and characterization of tasks by prospective teachers and teacher collaborative decision-making in task characterization.

- Cengiz Alacaci: Prospective elementary teachers' selection of mathematical tasks.
- Andreas Bergwall, Elisabet Mellroth, Torbjörn, and Johan Nordin: Teachers' characterizations of challenging introductory and enrichment tasks.

Alacaci's exploratory paper viewed mathematical tasks as powerful tools to develop mathematical ideas in the classroom and useful in teacher education. One assumption was that understanding task perceptions of prospective elementary teachers could help predict their eventual modification and appropriation for classroom use, which can affect teachers' practice. It also raised the issue of "good" mathematical tasks — good for what purpose? One of the consequences of the experiment was that prospective teachers were encouraged to think critically about tasks. Bergwall et al. investigated the collaborative characterization by eight teachers of tasks suitable for introduction or enrichment, and presented several dilemmas. Among the results was the observation that introductory tasks should have an easy entry level and not require pre-knowledge of the upcoming concept, while an enrichment task should require relatively deep conceptual pre-knowledge. Teachers' verbalization of task characteristics was one outcome, but not all tasks met all criteria. The teachers were involved in the writing of this paper.

Both papers raise some theoretical and some methodological issues about the choice of the theoretical framework, as well as the selection, characterization, and classification of tasks by teachers with regards to context, teachers' goals, and implicit characteristics, such as teachers' tacit knowledge (Herbst et al., 2011; Herbst & Chazan, 2011). This highlights the challenge in future attempts to generalize from these studies. In an additional poster, Gustafsson, van Bommel, and Liljekvist suggested pursuing the analysis of teachers' discussions in communities of practice to explore the various facets of mathematical knowledge for teaching that guide these discussions.

A central issue for TWG22 is the use of carefully designed curriculum materials, resources, and tasks to support the implementation of particular learning goals and to enhance mathematical competences. Thus, it is important to investigate how to design tasks and items to provide students with opportunities to make mathematical connections and develop mathematical concepts, skills, and habits of mind. The characteristics of task design can influence the processes that characterize students' interaction with the tasks or items themselves. Among these, reading processes can play an important role because they can direct and determine students' problem-solving processes. At CERME12, two papers focused on how the design of specific tasks and resources influences students' reading processes:

- Valentin Böswald and Stanislaw Schukajlow: Reading comprehension and modelling problems: Does it matter where the question is placed?
- Anneli Dyrvold and Ida Bergvall: The role of dynamic elements in digital teaching platforms: An investigation of students' reading behaviour

Both papers investigate the role played by specific factors in influencing students' reading processes. Böswald and Schukajlow presented a theoretical paper aimed at reflecting on the ways in which the position of the question within the text of a modeling problem can determine students' reading processes. They suggested that placing the question before the text in modelling problems can make the goal clearer for readers, supporting them in distinguishing between relevant and irrelevant information. Dyrvold and Bergvall investigated the influence of the choice of dynamic elements in digital items on students' reading behavior in a digital multimodal environment. Their analysis identified various types of challenges that students may face in working with dynamic elements. In particular, they stressed the potential of dynamic elements to evoke deep engagement in interaction and the risk of misunderstandings or omissions in relation to these elements.

The discussion of these two papers highlighted important aspects in the choice of a task design aimed at supporting students' reading processes. The two main aspects under discussion were: (a) the interrelation between enhancing efficient reading and enhancing students' comprehension and reasoning; (b) the role played by metacognitive processes in guiding students' reading processes when interacting with tasks with certain characteristics; and (c) the effect of students' age and of the focus on different mathematical contents on how the design of tasks and resources influence students' reading processes. Another common issue addressed by both papers was methodological: the use of eye-tracking technology to investigate students' reading processes when interacting with the designed tasks and resources. Some of the questions that arose concerned distinguishing the effects of task design on students' reading behaviors from the effects of other contextual or external factors, and the possible use of artificial intelligence in developing a categorization of reading behaviors.

In two additional posters, other design principles were suggested to achieve specific aims: Vytautas argued that problems that do not explicitly state the concepts needed to solve the problem can be defined as epistemologically potent. These tasks can still be approached by students, because the unknown concepts are not mentioned in the wording of the problem, leading to a meaningful learning process. Stenberg, Haavold, and Sriraman suggested employing pathologies to create uncertainty in order to catalyze creativity for mathematics students.

Another important issue for TWG22 was the design of various types of materials, tasks, and resources to be used in given learning environments. The choice of materials, the sequencing of tasks, and the actions performed by the teacher can affect students' learning processes. The role of the design of resources, materials, and learning environments was a common theme of the following papers:

- Henrik Stigberg: Digital Fabrication for Mathematics Education: A Critical Review of the Field
- Johanna Zöchbauer, Markus Hohenwarter, and Zsolt Lavicza: Improving the GeoGebra classroom tool to better accommodate online educational resource development based on the SAMR model
- Sofía Paz-Rodríguez, Armando Cuevas-Vallejo, and José Orozco-Santiago: A hypothetical learning trajectory for linear combination of vectors in R2

Stigberg proposed a critical review of digital fabrication for creating manipulatives in mathematics education research, stressing the role played by these technologies in enabling teachers to create context-sensitive manipulatives for teaching activities. Zöchbauer et al. investigated how the GeoGebra classroom tool can be improved to better accommodate a set of online educational resources to be used in combination with tangible tools. They used the case study approach to examine the implementation of these resources, highlighting the need for improvements, for example, making the sharing of resources with students easier and faster or enabling the teacher to have an overview of the number of students who answered the questions posed to them. The investigation presented by Paz-Rodríguez et al. enabled them to stress some limitations in the approach they adopted, which focused on the design of technology-mediated tasks, following a hypothetical learning trajectory aimed at supporting university students' conceptualization of linear combination. The analysis of the results of a teaching experiment in an online linear algebra class enabled the authors to identify elements that could support a redesign of this hypothetical learning trajectory.

Among the issues the discussion in these three papers raised, we mention the role played by frameworks and theoretical models to support both the design process (as in the papers by Zöchbauer et al. and Paz-Rodríguez et al.) and the research on this design (as documented in the paper by Stigberg). In particular, reference to the models guiding the design process is a common aspect in both the research presented by Zöchbauer et al., where the SAMR model (Puentedura, 2006) supported the design of an environment aimed at fostering the integration of digital media into the classroom, and the research presented by Paz-Rodríguez et al., where the C&P principles (Cuevas & Pluvinage, 2003) provided a set of criteria aimed at supporting the task design process.

The discussion on these papers also raised some reflections about another fundamental issue related to the design of learning environments, in particular when the focus of the design is also on the role of tangible tools or physical objects, considered as products of digital design (like in the paper by Stigberg) or as tools to be combined with digital ones (like in the paper by Zöchbauer). The paper by Stigberg raised the issue of teachers' learning when they make, share, and use manipulatives, suggesting the need to adopt communities of practice as a framework for understanding teachers' learning of digital fabrication for mathematics education. The paper by Zöchbauer et al. introduced initial reflections on the role played by the teacher's orchestration in combining the use of digital and tangible tools. They noted that teachers should make the connection between digital and tangible tools clearer if they want students to work effectively with a combination of these tools.

Possible directions for future research

TWG22 topics at CERME12 shared many research themes with CERME11 (e. g. different theoretical perspectives leading task design and analysis) and and also added novel themse as mentioned above. When discussing these studies TWG22 discussions often took the opportunity to discuss possible ways to enhance and deepen the topics studied in TWG22. The following directions for future research emerged from the discussions:

(a) Developments of means to communicate over curriculum materials and the wide perspective that takes into consideration different agents, open new opportunities to focus on the role of students in all stages stemming from task and curriculum design and implementation;

(b) The different theories employed in various studies underline the neeed for finer definitions and moving toward more precise and fine-grained analysis of processes and practices, which are theorized only in general forms;

(c) There is place for reports focusing on issues of scalability: many reports focused on a micro-process of design or implementation; large-scale studies or design considerations for scaling-up could contribute to a wider research perspective;

(d) While many conference research reports focus on short processes, long-term and longitudinal studies are needed for broader perspective. We acknowledge that long-term and longitudinal studies are probably challenging in the format of a conference research report; therefore intermediate stage reports are also welcome;

(e) More Research reports are needed to deepen the investigation of the teachers' role in their design and co-design interaction with resources and environments, in the context of professional development programs or research projects.

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* all the papers mentioned in this document are within the references lists of the papers presented at TWG22, except for the one in the list.

Prospective elementary teachers' selection of mathematical tasks

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Mathematical tasks are powerful tools to develop mathematical ideas in classrooms and are useful in teacher education linking teacher decision-making, student work, and curricular themes. One of the core elements of teacher development is to be able to choose and use "good" mathematical tasks. Understanding task perceptions of prospective teachers can help predict their eventual modification and appropriation for classroom use. The current study investigated the features that prospective elementary teachers (PT) attend while selecting "good" mathematical tasks. Employing a modified repertory grid technique, we identified attributes of desirable tasks as viewed by PTs. We then present interpretation of findings for mathematical task design and for teacher education.

Keywords: Documentational approach to didactics, teacher education, mathematical tasks, task design.

Introduction

Due to recent reform efforts in mathematics education, many innovative mathematical tasks for classroom use have been available from commercial resources, professional organizations, or released public accountability tests such as NAEP (e.g., Mohr et al., 2019). This necessitated sharper skills for in-service and prospective teachers for selecting "good" mathematical tasks.

A mathematical task is a problem devoted to developing or assessing a particular mathematical idea (Stein et al., 1996). They are effective tools for communicating curricular visions of mathematics education. Therefore, designing, analyzing, appropriating tasks, and talking about leading classrooms to effectively implement them have become common practice in teacher education programs (Tekkumru-Kisa et al., 2020).

Given the potential role mathematical tasks can play in mathematics classrooms, considerable research has been done on the topic. Stein et al., for example, (1996) looked into the role of mathematical tasks in realizing curricular goals. They observed that mathematical tasks typically move through stages from *tasks as published* in the original curricular resources, to *tasks as set up* by the teacher, and to *tasks as implemented* in classrooms. It was possible to engage students with higher forms of mathematical thinking when assigned mathematical tasks with high cognitive demand but assigning students these tasks did not ensure their high levels of reasoning. Tasks often decline in elicited student thinking due to *inappropriateness* of the task or other factors such as prevalent norms in classrooms, issues of classroom management, or use of time (Stein et al., 1996; Tekkumru-Kisa et al., 2020). We believe that a core skill to be developed in teacher education should be to understand the factors that underlie a better match of tasks with student characteristics and develop the ability to modify the tasks to better match with the needs of students.

Theory

Taking a resource perspective, we consider mathematical tasks as a type of curriculum resource. Hence, we chose *documentational approach to didactics* (DAD) as a framework in this study

(Gueudet & Trouche, 2009). The framework provided a perspective to understand how teachers choose and interact with mathematical tasks. Pepin (2018) for example, proposed that looking into how teachers (and PTs) relate with mathematical tasks "as is" is not very informative, but we need to explore how they actually work with these tasks. The first step to use a task is to select them. By inference from the DAD framework, we think that PTs actively construct meanings and learn from choosing, transforming, and revising mathematical tasks. Further, as an extension of DAD perspective, the dialectic process of *instrumentation* (how a mathematical task influences a PTs didactic activity) and *instrumentalization* (how the PTs shape mathematical tasks for possible use in classroom) can capture the PTs' interaction with mathematical tasks (Gueudet & Trouche, 2009). Before instrumentation and instrumentalization take place, a PT needs to choose a mathematical task from among alternatives and consider using it in lesson planning. Then the question is, what attributes do PTs attend to choose a mathematical task from among alternatives? Answering this question is important to understand teacher-task interaction *vis-a-vis* mathematical tasks. Only the tasks that are selected or fall below the perceptual sieve can be modified and eventually used. It would be reasonable to expect that didactical affordances of a task (i.e., *instrumentation*) or modifications needed to be made on a task once it is selected (that is, *instrumentalization*) should be shaped by the PTs perception of the original task. So, we need to know more about how a (prospective) teacher think while selecting tasks before the teacher enters into a serious relationship with the task. DAD offers powerful insights into what happens after a task falls into the "basket" of use for a teacher. The goal of the present study is however to uncover what happens before it is selected. So, we hope that our findings will complement and further the insights offered by the DAD about teacher reasoning in the context of task use for instruction.

Methodology

To explore the task features PTs attend while they select tasks for classroom use, we employed a modified repertory grid technique for data collection. The technique is based on the assumption that a person's thinking is directed by the way in which s/he interprets events or objects through his/her personal constructs (Pope & Denicolo, 2016).

We compiled a set of six mathematical tasks organized into groups of two triads. The tasks in each triad were selected to address a common key concept at the same grade level. The tasks were selected from published sources (see Table 1) and they were adapted to the study.

To make the comparisons meaningful, we formed the task triads addressing the same key mathematical concept. However, we wanted to include tasks that represented variation in context; with abstract, illustrative, and applied contexts. This was to see whether PTs prefer tasks with certain contextual structures. However, we did not want to vary the tasks in cognitive demand level, as this may bias participant's selection towards tasks with higher cognitive level. So, all of our tasks had high cognitive demand (either "procedures with connections" or "doing mathematics") according to Smith and Stein's (1998) scheme. This was ensured by independent assessment of the tasks by two mathematics educators prior to the study.

The participants of the study were 14 PTs enrolled in a course "problem-based mathematics education" at master's level. Student teachers are given weekly problems to solve in small groups in

Table 1: Mathematical tasks used in data collection and their key mathematical concepts

Triad	Name of task	Key concept	Grade level	Source
	The pocket problem	Reading and		NCES (1993, p. 126)
1	The siblings problem	interpreting data	4	Small (2020, p. 186)
	The mystery graphs	from graphs		Zawojewski (1996, p. 376)
	The matchsticks pr.	Representing		Rivera & Becker (2008, p. 66)
2	The string task	linear functions	8	Posamentier & Krulik (2009, p. 79)
	The sumo wrestler			(Please see footnote below)[1]

this course that exemplify an inquiry vision of teaching and learning mathematics, analyze them for the curricular ideas and problem-solving strategies embedded, and offer and discuss classroom strategies for lessons centered around these problems. The course was offered in the fourth year of a combined 5-years teacher education program for elementary teachers at a Norwegian university.

In line with the repertory grid technique, data collection was completed in two steps. The first step required comparison of tasks in triads and the second involved individual surveys. The set of tasks were given in print to participants during class time. They were asked to discuss in small groups and solve the tasks. Then for each triad, the groups were invited to write any similarities or differences they could think of between the pairwise combinations of the three tasks (the constructs) from the perspective of their possible use in classroom. Once they complete writing the constructs, they were asked to indicate three tasks that "they like best for classroom use" from among the six. Following the session, the researcher prepared an all-inclusive but non-repeating list of the constructs elicited from the groups before the next class[2].

After the first data collection session, the researcher organized the constructs into rows of a table and the names of tasks into columns. In this session of data collection, individuals were given a print copy of the table (similar to table 3) and were asked if the constructs in the table were clear. The purpose was to make sure that all the constructs were understandable to participants. Then they were invited to rate each task for the constructs using a Likert-type scale from 1 to 5; 1 standing for "not true at all" and 5 "very true" and the values in between varying accordingly. Ratings were completed individually and took approximately 30 minutes.

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[1] Link for this task: https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:forms-of-linear-equations/x2f8bb11595b61c86:writing-slope-intercept-equations/e/constructing-linear-functions-word-problems

[2] Repertory grid technique typically requires construct elicitation and ratings from individuals. Here, for the purpose of eliciting richer constructs, we asked the PTs to work in small groups to compare the tasks in step 1. But they rated the tasks in step 2 individually. Hence, we called our version of the data collection a "modified" repertory grid technique.

Results

Table 2 gives the tasks "liked" by the participants from among the given set and the number of participants who selected these tasks. The "string task" was liked by all PTs. The "matchsticks" and the "sibling" problems were the next two most popular tasks and were selected by nine and six participants respectively. The "mystery graphs" problem was least preferred with only 3 participants.

Table 2: The tasks liked by the participants

	Tasks	# of participants who selected the task
1.	The pocket problem	5
2.	The sibling problem	6
3.	The mystery graphs	3
4.	The matchsticks problem	9
5.	The string task	14
6.	The sumo wrestler	5

To understand the attributes that might have led the PTs choice of the tasks, list of constructs and the average ratings of the tasks for the 14 participants are computed and given below in table 3. For ease of comparison, ratings were given separately for the top three tasks based on PTs choices and the remaining three tasks. We present the constructs based on which the 6 tasks received high ($M > 3.5$) and low ($M < 2.5$) average ratings. High ratings are color coded in yellow, and low ratings are coded in brown for ease of seeing patterns in ratings. Because some constructs were worded negatively, we wanted to examine the tasks that received low and high average ratings so that we can identify both the negatively and positively viewed task features for the preferred tasks. Consequently, a yellow rating in table 3 does not always mean a desirable task attribute.

In our interpretation of findings, we give reference to constructs with construct-numbers presented in table 3. Below, we report by the common themes we think constructs are related to, for task attributes.

Use of visual elements

Using visual elements effectively seems to be a common feature among the favored tasks (see construct 1), with the most favored tasks having the highest rating based on this criterion.

Mathematical power

The task's potential to create a deep mathematical discussion in class (construct 3) and to require deep understanding and reflection (construct 18) were two factors that seemed to underlie favorable tasks. The only exception was the "sibling problem" to this observation. This task was assessed to

Table 3: Elicited constructs for comparison with average ratings of tasks

The constructs / task number in table 2	**5**	**4**	**2**	**1**	**6**	**3**

1	This task uses visual elements effectively.	4.6	4.2	3.9		3.3	1.7	3.1
2	The context is interesting in this task.	3.2	3.1	4.1		4.0	3.6	4.2
3	This task can create a deeper discussion in class.	4.5	4.2	3.8		3.9	3.5	3.9
4	This task requires developing a formula to solve it.	4.7	5.0	1.0		1.0	4.9	1.1
5	This task is structured in steps and is easier to follow.	4.3	2.3	3.3		3.2	2.4	2.3
6	This is an easy and clear task to do for pupils.	3.4	2.7	3.9		3.2	3.0	1.5
7	You need to interpret textual information to solve this task correctly.	4.1	2.6	3.3		4.8	4.5	5.0
8	This would be a good task to introduce a topic.	2.9	3.0	3.8		2.6	2.6	1.9
9	This task is more demanding in thinking and reasoning.	4.2	3.9	1.9		3.4	3.4	4.6
10	The story may be overwhelming for the kids in this task.	2.8	1.9	1.3		2.4	2.8	4.4
11	This task may be misinterpreted by the pupils.	2.5	2.7	2.0		3.5	3.4	4.4
12	This task may be difficult for individual pupils and hence better for a class discussion.	3.9	3.3	2.4		3.9	3.5	4.1
13	The wording is a bit difficult in this task.	2.2	1.9	1.6		2.0	2.2	4.5
14	This task requires just reading off information and is not a real problem-solving task.	1.6	1.4	4.8		2.4	2.4	1.9
15	There is too much text here.	1.6	1.1	1.3		1.5	1.8	4.4
16	This is more like a "pure mathematics" task.	3.2	4.4	3.4		2.5	3.9	2.1
17	This task is not very original, I have seen tasks like this one before.	2.8	3.4	4.2		2.4	3.1	1.9
18	This task requires deep understanding and reflection.	3.8	3.8	1.4		2.6	2.8	3.5
19	This task has a realistic and applied context.	2.6	3.2	4.3		4.1	3.9	4.1
20	There is a simplicity that I like in this task.	3.8	3.7	3.9		3.1	3.9	1.9

have the potential to create good mathematical discussion in class, but not requiring deep understanding or reflection by itself. PTs might have thought that a deep discussion opportunity can potentially be created by teachers while using this task.

Simplicity

Simplicity was another prominent attribute for preferred tasks (construct 20). Preferred tasks did not have too much text to read and interpret for solution (construct 15). The story of the task being overwhelming seemed also to be a distinct feature of the *least* preferred task (line 10) which may also lead to difficulty and misinterpretation in reading the text (constructs 11 and 13). Another aspect of complexity in textual information, namely a high level of need for interpretation (and hence requiring higher reading ability) was not seen favorably among PTs in mathematical tasks (construct 7).

Level of challenge

A high level of challenge and demand in mathematical reasoning seem to be important but not sufficient (and in one case, not even necessary) for the tasks to be seen favorably (constructs 9, 12 and 14). The highest cognitive challenge was seen in the "mystery graphs" task, but it did not receive high ratings for preference (for probably lack of simplicity). However, the two most favored tasks (the "string task" and the "matchsticks problem") had the next two highest ratings in demand for mathematical thinking and reasoning. Conversely, easiness of a task was not a common theme in the desired or not-so-desired tasks (construct 6). It looks like the *right level* of cognitive challenge (not too much, not too little) seems to be an attribute deemed desirable by PTs.

Other features of the tasks

Other features of tasks did not seem to be strongly associated with task preference. Among these are the features related to the context of tasks being interesting, applied or pure (constructs 2, 16 and 19), and tasks being structured in steps (construct 5).

Conclusions and discussion

In our view, the results paint a positive picture of prospective elementary teachers' ability to distinguish among tasks. In particular, PTs seem to have a sophisticated view of mathematical tasks. This may be due to prior course experiences on task design and analysis. However, this needs to be further investigated, perhaps with a bigger sample size. Effective use of visual elements was common to all highly rated tasks. Mathematical power embedded in the task that can support a good classroom discussion, nurture deep understanding of mathematical ideas, and provide opportunity for pupil reflection were also considered highly for their perception. However, high mathematical power did not necessarily mean complicated design, too many words, or a cluttered story line in their eyes. Hence, simplicity was another prominent feature of desirable tasks according to PTs. In other words, tasks should have not only the power, but also the economy and clarity and hence simplicity in set up, story line and wording. An optimum level of difficulty or challenge was considered highly for desirable tasks. A high level of difficulty or easiness by itself were not associated with "good" tasks. Selected tasks included both not so demanding as well as reasonably demanding tasks which pointed to an optimum level of challenge.

How do we interpret these findings from the DAD perspective? As PTs look for mathematical tasks for instructional planning, their *instrumentation* process could be guided by such powerful notions as mathematical power, visual elements, simplicity and level of challenge for pupils. These same notions can guide the *instrumentalization* (that is, task modification) for actual classroom use as well. When PTs are considering a task, they may be already thinking about how they will use it (see for example construct 8 in Table 3). Our study provides new insights into DAD and how teachers' perception of a task might be guiding task appropriation before use.

We also find it interesting that mathematical power and simplicity came out as prominent task features. In fact, their presence together reminded us Birkoff's (1956) model of aesthetics in mathematical products (as quoted in Dreyfus & Eisenberg, 1986). Birkoff offered a formulaic model of aesthetics; $M = O/C$, where M stands for *aesthetics*, O, a measure of *order* or mathematical structure, and C for *complexity*[3]. The formula suggests that aesthetic value of a mathematical object is a *proportion*, counterbalanced by two factors, *order* and *complexity*. In other words, aesthetics may be directly proportional to the mathematical structure embedded in it, while complexity being inversely proportional, and aesthetics may change with differing values of order and complexity. It looks like, even though the PTs did not use the term "aesthetics" in this study, their search for power and simplicity at the same time, might imply a desire for aesthetics in mathematical tasks. We thought this would connect with Sinclair and Crespo's (2006) point on aesthetics in a new light in mathematical task design.

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Influences of text design on problem solving processes and results in working-backwards problems

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Working backwards is an essential problem-solving strategy that can be applied already at primary school age. For this, very specific tasks are often used with a missing initial value, given operations that are applied in a fixed order to this initial value, and a given final value in which the described process culminates (see P1 and P2 in figure 1).

Problem 1 (P1)	Problem 2 (P2)
Maja sells trading cards at the flea market. The first child buys half of the trading cards and then one more. The second child takes half of the rest and then two more cards. After that there are 6 cards left. How many cards did Maja have at the beginning?	Maja sells trading cards at the flea market. The first child buys half of the trading cards. The second child then buys one card. Then the third child takes half of the rest. The fourth child buys two more cards at the end. After that there are 6 cards left. How many cards did Maja have at the beginning?

Figure 1: Used problems

To solve problems like this by working backwards, it is necessary to reverse the operations described and to execute them in reverse order starting from the final value. Thus, reversible thinking skills are required. According to Krutetskii (1978), these are developed in germinal form at primary school age (in mathematically gifted children), but many children still show difficulties in processing such tasks. In particular, reversing the order of operations proves challenging, as has been shown in various studies with children in grades 2 to 6 with similar problems to P1 (Rott 2013, Aßmus 2010). It can be assumed that children at this age are not (yet) aware of the effects of different orders when linking additive and multiplicative operations.

However, another or further cause could lie in the textual design of problems like P1. Buying half and one (or two) additional card(s) is possibly interpreted as one action, especially because the operations in the text are related to the same recipient. Thus, the two sub-operations are not considered separately, making their order irrelevant to the students.

This aspect is investigated in the study of working backwards described here considering the following research questions: To what extent do third and fourth graders' solution rates differ when solving a problem with operations performed by different people (e.g. problem 2) versus a problem with linked operations performed by the same person (e.g. problem 1)? To what extent do the results differ with respect to consideration of the order of operations?

The study

Design: As part of a written entrance test for a university project for mathematically gifted primary school children, Problem 1 was used in 2013/14 and Problem 2 in 2015-18 (see figure 1). The test was administered under standardized conditions (individual work, comparable instructions, identical working time). Problem 1 was given to 156 children (grade 3: 120, grade 4: 36), Problem 2 to 283 children (grade 3: 200, grade 4: 83).

The mathematical model is identical for both problems: $((6 + 2) \cdot 2) + 1) \cdot 2 = 34$. In P1, however, two operations each are tied to one actor, whereas in P2 they are performed by different children and thus described as separate actions. As a further difference, in P2 each action is noted in a new line as well as the course of the process is made clear by supporting temporal adverbs.

The differences in the solution rates as well as the noted calculation steps are examined with a special focus on the correct use of the operation order. Based on qualitative content analysis, the answers were independently categorized several times and coded using MAXQDA.

Results: P2 was solved by a larger proportion than P1 (grade 3: P1: 5.8%, P2: 25.5%, grade 4: P1: 11.1%, P2: 34.9%). Furthermore, there were differences in procedures with correct reversal of operations but incorrect order (grade 3: P1: 14.2%, P2: 6.0%, grade 4: P1: 16.7%, P2: 9.6%). Here, problem 1 in grade 3 was usually handled by the calculation $(6 \cdot 2 + 2) \cdot 2 + 1 = 29$. This did not occur at all for problem 2. Also paired with other errors, the proportion of processes with incorrectly reversed order was reduced in grade 4 (grade 3: P1: 10.8%; P2: 9.0%, grade 4: P1: 13.9%, P2: 3.6%).

Overall, it can be stated: The text problem with an unknown initial state remains very challenging for the students of both grades. Basically, however, the study showed that the linguistic presentation of the task text and its modification has a clear influence on the problem-solving process and the children's success in solving the task. In particular, errors based on the use of an incorrect order of operations occurred in lower proportions.

Further studies: To narrow down which variables where causal for the higher solution rates, a third version was tested (Grade 3: N = 66, Grade 4: N = 39) in which two operations were presented in one line and one sentence, as in P1. Otherwise, the task corresponded to P2. Solution rates decreased only slightly in grade 3 (P3: 22,7%) compared to P2, and actually increased in grade 4 (P3: 43,6%). We therefore suspect that separating the operations was more important for the success i than consistently noting the individual actions line by line and sentence by sentence.

We are not interested in simplifying problems mathematically to the point of making a problem accessible to as many students as possible, since this often involves trivializing the problem. However, purely linguistic difficulties should be overcome with a formulation that allows as many as possible to adequately understand the problem in the first place. Qualitativ research will follow.

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Working backwards revisited – some theoretical considerations

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A look at scientific and pedagogical literature on mathematical problem solving reveals a wide range of different problems for which the use of the heuristic strategy of working backwards is considered useful by the respective authors. A comparison of these problems shows that the amount and type of information given, and thus the requirements and processes involved in working backwards, are quite different. This purely theoretical article proposes a systematization of the problems and processes that can provide a valuable basis for future studies and for design and revision of teaching materials. On the one hand, two different facets of working backwards and, on the other hand, three types of problems are distinguished and illustrated by examples.

Keywords: Problem solving, heuristics, working backwards.

On the importance of working backwards

Worldwide, problem solving is considered as an important process standard for school mathematics in all grades. This includes the goal of enabling students to acquire heuristic strategies and to apply or adapt them to new mathematical problems.

In mathematics, but also in "real life", working backwards is regarded as a particularly important heuristic strategy (e.g. Zimmermann, 2003). Pólya, who recommends it as a general approach to problem solving unless there is a specific reason to do otherwise (Pólya, 1981), describes working backwards as follows:

> … we assume what is required to be done as already done (what is sought as already found, what we have to prove as true). We inquire from what antecedent the desired result could be derived; then we inquire again what could be the antecedent of that antecedent, and so on, until passing from antecedent to antecedent, we come eventually upon something already known or admittedly true. (Pólya, 1971, p. 142)

Thus, working backwards is characterized by trying to get from the sought to the given (problems to find) or from the statement to prove to the assumptions (problems to prove). This temporarily changes the focus when working on the problem: starting from the unknown or unproven aim, the step-by-step construction of a previously unknown solution path occurs (phase of analysis), which afterwards should enable the problem solver to proceed in the reverse direction from the given to the sought (phase of synthesis). This view is widely shared. However, in many teaching materials, mathematics didactic textbooks, and research papers, problems or problem solutions are assigned to this strategy that do not fit this theoretical view. For example, Posamentier and Krulik (2015, p. 30) see the following as "a typical problem that lends itself nicely to the working-backwards strategy":

> *Problem 1:* David just came back from playing four games of baseball cards. He now has 45 cards in his package of cards. When I asked how he did, he told me he had lost one-half of his cards in

> the first game. After the second game, he had 12 times what he had before. In the third game he won 9 cards. The fourth game was a tie so no cards changed hands. How many cards did he start with? (cf. Posamentier & Krulik, 2015)

We agree with the authors insofar as the problem text describes a process whose final state is known (45 cards) and whose initial state is searched for. If one tries to determine this initial state by applying the reverse process steps in reverse order to the final state, however, one starts *from given data*. Thus, this approach contradicts Pólya's classical view of working backwards as *starting from the unknown*. Because this discrepancy is so widespread, it seems necessary to us to attempt a theoretical clarification of the working-backwards strategy in educational contexts. This could also contribute to a better understanding and evaluation of corresponding tasks and how students work on them.

We start our theoretical considerations with a more detailed analysis of two sample tasks and thereby elaborate processes that are commonly understood as working backwards. Based on this, we deduce two facets of working backwards: *working backwards in the classical sense* and *working backwards as reversing a process.* The article ends with a systematization of problem types for working backwards and a discussion of very specific cases and possible implications for further research or applicability in everyday school settings.

Two facets of working backwards

In order to elaborate different facets of working backwards, we compare two problems that are commonly associated with this heuristic strategy:

> *Problem 2:* Construct to a point outside a circle the tangents of the circle through this point.

This problem requires the execution of a construction. If the construction procedure is not known, it can be determined by working backwards as described by Pólya: One thinks of the tangents as already determined, thus the desired final state of the problem as already achieved. By analyzing it (possibly based on a sketch), conditions are determined backwards step by step, which must be fulfilled in order to be able to execute the construction forwards later on. The aim one starts from is what is sought in the problem.

Generalizing, it can be said that the final state of the problem (F), which forms the starting point of the working process, is (only) assumed to exist and to be constructible. Additionally, the individual operations from the initial (I) to the final state are not given. One has to investigate on one's own, which properties the assumed solution has and from which antecedent the desired aim or partial aim can be reached forward. This search process sometimes also includes aberrations or detours. To summarize: In problems of this type, *planning* starts from the aim which is the *unknown solution of the given problem*, working backwards *towards* the things "within one's power" (given data, assumptions, …).

> *Problem 3:* "A certain man entered a certain pleasure garden through 7 doors, and he took from there a number of apples; when he wished to leave he had to give the first doorkeeper half of all the apples and one more; to the second doorkeeper he gave half of the remaining apples and one more. He gave to the other 5 doorkeepers similarly, and there was one apple left for him. It is sought how many apples there were that he collected" (Fibonacci, 2002, p. 397)

In Problem 3, starting from an unknown initial value, a process as a chain of operations and the resulting final value are described. This process can be reversed to determine the searched initial value, here by applying the reversed operations in reverse order to the final value. One starts from the „aim" of the described process which, however, does not correspond to what is sought in the problem.

Generalizing again, unlike problems of the above type, the final state as starting point is concretely given here. Also concerning the knownness of the steps to be executed, differences become apparent because the problem text already contains all information about the operations to be executed from the initial to the final value. The "path" is thus given, for a solution it must be gone in reverse direction. Therefore, reversal processes play an essential role. If these are mastered, search processes are not necessary. To summarize: In problem of this type, *doing* starts from the aim which is the *given final state of the described process*, working backwards *using* the things "within one's power".

The approach described for Problem 2 fits with the traditional view of working backwards by Pólya and others as described above. Despite the outlined differences, tasks such as Problem 3 or Problem 1 are very often used to address working backwards in classroom situations. The two different approaches do not seem to be commonly distinguished or they are perceived as matching each other; possibly the approach to Problem 3 is assumed as preparatory to Problem 2. A possible explanation for the latter could be found in the didactic approach of complexity reduction in the introduction to the working-backwards strategy: The idea of working backwards should first be experienced as starting from a given final state with given operations, in the hope that students can later transfer this approach to problems with unknown final states and partial aims or operations to be found on one's own. Whether and to what extent such a transfer will be really attained, a corresponding heuristic program is really successful, has not been investigated so far, to our knowledge.
Due to the commonality of starting at an aim, we also understand both described approaches as working backwards. However, the differences that have been highlighted indicate that two different facets of working backwards should be distinguished, the first of which we would like to name *working backwards in the classical sense* and the second *working backwards as reversing a process*.

A review of mathematics didactic research and teaching literature reveals a large spectrum of different problems for working backwards, which can be assigned primarily to the second facet. In the following, a systematization is proposed with the aim to work out problem-type dependent requirements related to working backwards.

Problem types allowing working backwards

For a classification of different problem types for working backwards it can be distinguished on the one hand whether the initial state of a problem or a described process is determined by the problem itself, on the other hand whether the operations from the initial to the final state are given in the problem. In our view, three types of problems can be distinguished on the basis of the specification of the operations:

A. The operations and their order, and thus the entire process, are known / are described in the problem definition.

B. The types of operations are known; their respective number and order is not known.

C. The operations are not known.

Problems of types A fit to working backwards as reversing a process, problems of type C to working backwards in the classical sense. Problems of type B leave scope for interpretation.

Figure 1 gives an overview of the problem types we have identified that allow working backwards. It was obtained on the basis of an extensive review of research and teaching literature in German and English. Nevertheless, it does not claim to be exhaustive, which is indicated by the ellipsis points.

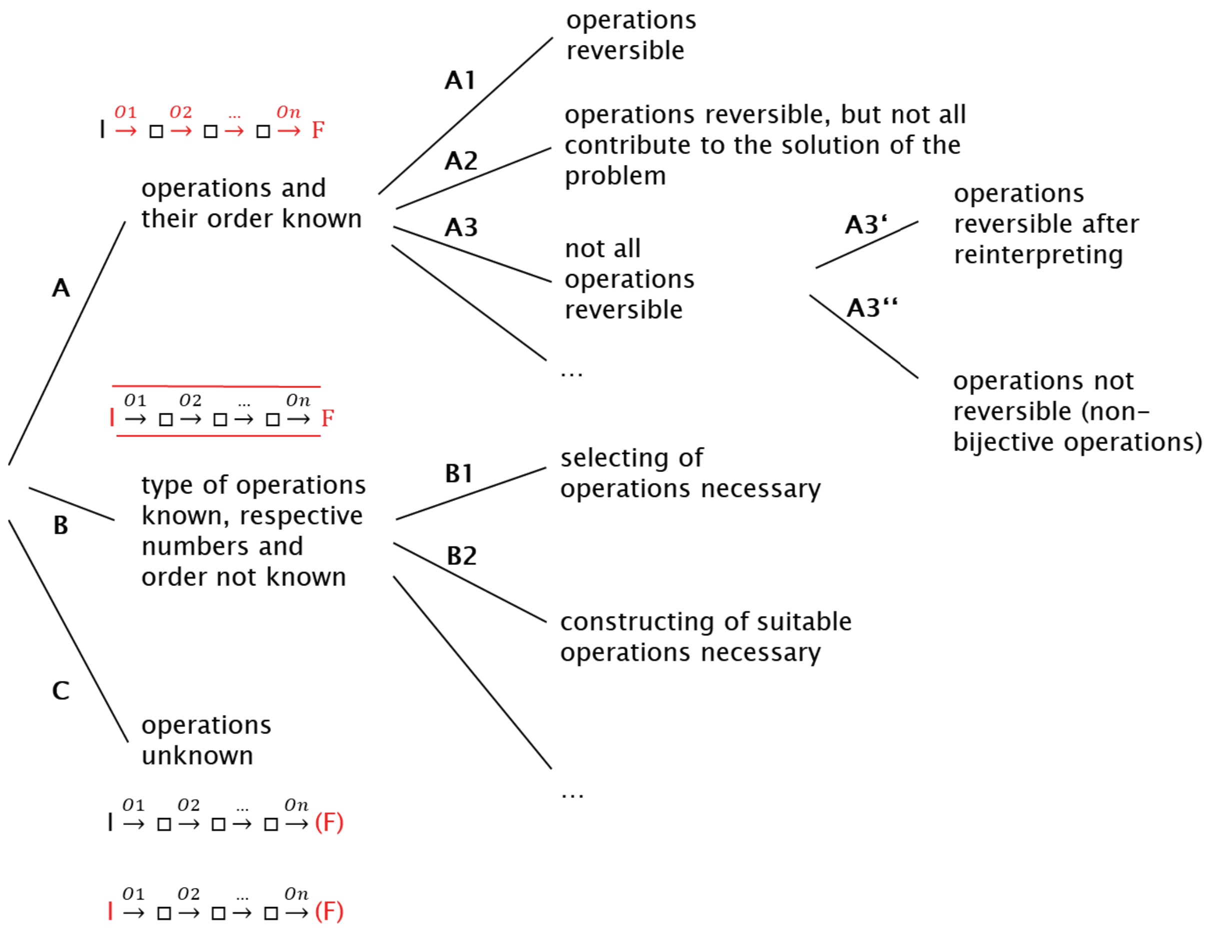

Figure 1: Problem types allowing working backwards

(Highlighted in red are the known elements of each problem type.)

Type A: This includes problems in which operations are applied to an initial value in order to achieve a final value. The initial value is unknown, the operations and the resulting final value are given (often in text form). The (number and) order of the operations is fixed by the problem. As a result, also the intermediate values arising when working backwards are determined. Problem 3 belongs to this problem type.

Subtypes of this problem type can be formed according to the reversibility of the operations: While subtype A1 includes those of the above problems where all operations are directly and uniquely reversible (Problem 3), this is not the case for problems of subtype A3 for at least one step. This occurs, for example, in the case of non-bijective operations (Problem 4), but also in problems as

Problem 5, where (at least) one change described in the text (taking away a third) involves the composition of several mathematical operations (determining the third and subtracting it from the previous number) which is why its direct inversion is not possible. It may be possible to generate a reversible operation (multiplying with 2/3) by reinterpreting the given one.

Problem 4: I think of a number, square the number, subtract 37, multiply the result by its successor, and get 156.

Problem 5: „Three tired and hungry men went to sleep with a bag of apples. One man woke up, ate 1/3 of the apples, then went back to sleep. Later a second man woke up and ate 1/3 of the remaining apples, then went back to sleep. Finally, the third man woke up and ate 1/3 of the remaining apples. When he was finished there were 8 apples left. How many apples were in the bag originally“? (Watson, 1988, p. 20)

In problems of subtype A2 given operations are uniquely reversible one by one, but not all values resulting in this way lead to the correct solution of the problem. Thus, not every reversed operation contributes to the solution of the problem. An example is the number machine (Problem 6): An inversion of the operation "minus 1" is always possible, however, starting from an odd number it delivers an even number as intermediate result, which would have to be halved on input into the number machine (and not decreased by 1). An unexamined reversal of all possible operations thus leads to partially incorrect results in this problem.

Problem 6: The number machine works according to the following rules: If you enter an even number, the number is halved. If you enter an odd number, the number is reduced by 1. Once you have converted a number, you can enter the new number back into the number machine. You can do this until you reach 0. For example, if you start with the number 6, you get the sequence $6 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 0$. Since you need four steps to reach 0 from 6, 6 is a "four-step number". Find all the four-/, five-/, six-step numbers. How many are there in each case? (cf. Fritzlar et al., 2006)

A special form A* of this type represents problems, with which the order of the operations is known, but the continuous adherence to this is not necessary for success or single operations can be ignored. The first is the case, for example, if several additive or several multiplicative operations follow one another, which are interchangeable due to commutativity. An example is Problem 7. The order in which the 15 and the 20 balls are removed is not relevant for the result as long as these are the last two operations. For working backwards, only the inversion of the operations is necessary here. Therefore, problem 7 can be assigned to subtype A3*.

Problem 7: "The Wolverines baseball team opened a new box of baseballs for today's game. They sent 1/3 of their baseballs to be rubbed with special mud to take the gloss off. They gave 15 baseballs to their star outfielder to autograph. The batboy took 20 baseballs for batting practice. They had only 15 baseballs left. How many baseballs were in the box at the start?" (Portnov-Neeman & Amit, 2016, p. 87)

In problem 1 in the introduction, the last game has no influence on the number of cards, so this step can be ignored. Therefore, it is a problem of type A1*.

Type B: In this problem type, the operations are not explicitly given, but only known in their nature. From a larger "range of operations" the target-leading reversible operations are to be selected (subtype B1) or e.g. by structure recognition to be constructed (subtype B2). The operations must be selected, for example, in the classical jug-pouring problems (Problem 8). The type of operations (filling the cans, decanting the water, possibly emptying the cans) is known, but the exact order and the number of steps must still be constructed in the problem-solving process. When working backwards, the concrete decanting actions have to be selected, which lead to possible intermediate aims (full can, empty can) when running through the sequence of actions forwards, until the initial state (all cans empty) is reached.

> *Problem 8:* „Lauren has an 11-liter can and a 5-liter can. How can she measure out exactly 7 liters of water?" (Posamentier & Krulik, 2015, p. 31)

The so-called NIM games can be assigned to problem type B2, as illustrated by Problem 9. Although possible operations are given in the problem (adding natural numbers that are smaller than 5), it is not possible to use the working-backwards strategy directly. Only by constructing of suitable operations, with which winning positions can be reached, working backwards becomes possible. In Problem 9, such winning positions are the numbers 15, 10, 5 and 0, which result starting from the target number 20 by stepwise subtraction of the sum of the smallest and largest possible number to add (1 and 4). Here, also the initial value (the number 0) is known in advance.

> *Problem 9:* Two players take turns adding numbers between 1 and 4 to the total reached so far. The player who reaches 20 wins.

Since in problem 8 operations are partly implicit and especially their reversibility has to be considered, from our point of view search rather than selection processes play an important role in working backwards. Therefore, we assign it to working backwards in the classical sense, just like problem 9, in which suitable operations have to be constructed.

Type C: In problems of this type, no explicit hints are given about the operations to be used. The path between initial and final state must therefore be found completely independently. This type includes problems which can be assigned to working backwards in the classical sense, especially proof or construction problems like Problem 2. The initial state can be known or unknown.

Some special cases

A classification of tasks is usually somewhat fuzzy. In the following, we will briefly discuss some special cases that are interesting from our perspective.

> *Problem 10:* After going to the store and buying milk for $2, Tony had $3 in his wallet. How much money Tony must have had before he bought the milk? (cf. Katz et al., 2016)

On the one hand, this task could be assigned to type A1 in purely formal terms. On the other hand, it can be objected that in the task only a single operation is described, which can be reversed immediately. However, if one understands working backwards as a heuristic strategy, this is a global (not local or even one-step) approach to a challenging problem (not to a simple task). Therefore, the second facet of working backwards occurs at most as a germ in this task.

Problem 11: A farmer wants to cross a river and take with him a wolf, a goat, and a cabbage. But the boat can only fit himself plus either the wolf, the goat, or the cabbage. If left unattended together, the goat would eat the cabbage, or the wolf would eat the goat. How can the farmer bring the wolf, the goat, and the cabbage across the river?

Katz et al. (2016) consider this problem, which has been known since the 9th century (Folkerts, 1978), to be particularly suitable for working backwards. However, there is no difference between the initial and final state of this transport problem (except the river bank) and all operations are uniquely reversible. Thus, working backward results in exactly the same requirements as working forward; the working-backwards strategy is not beneficial here.

This also addresses the question of when (beyond Pólya's general recommendation) working backwards is appropriate. According to Sewerin (1979), this is the case when the final state of the problem is clear, when operations are known that lead to or can be applied to the final state, and especially when the initial state does not allow or suggest a goal-directed approach. Indeed, in this case, working forward, if at all possible, would be associated with large uncertainties and the risk of "dead ends." Schreiber (2011) therefore considers working backwards as a heurism of reduction.

Problem 12: Four students in the class weighed themselves. Cobi was 15 kilograms lighter than Adi and Jenya was seven kilograms heavier than Gaby. They also noted, that Gaby was twice as heavy as Cobi. If Jenya weighed 71 kilograms what was Adi's weight? (cf. Portnov-Neeman & Amit, 2016)

Given relations can be arranged in a chain, which starts at the searched element and ends at a given element. In similar problems, such a chain of relations is sometimes already explicitly given. Starting from this, the problem can be (further) processed analogously to problems of type A. Because problems of this kind seem to us quite rare and artificial, we would like to include them as special cases in the presented classification. If, as in problem 12, a suitable chain of relation is not given, but can be generated by sorting given specifications, the problem seems to us to be located in the transition from problem type A to problem type B. If a chain is already explicitly given, it would be a special case of type A.

Possible consequences

All of the problems collected can be found in this or a very similar form in papers that aim to investigate or promote working backwards as a heuristic strategy. However, the differences identified here, which can have a considerable influence on problem demands and students' problem-solving processes, have not been addressed so far, which seems to us to be quite important for the validity of the studies and programs. In addition, we believe that it could be worthwhile to systematically investigate whether an intensive work on the second facet actually facilitates the acquisition of the classical working-backwards strategy. Of course, the elaborated theoretical foundation could also be used for mathematics didactic design research, specifically for the design and revision of problems for working backwards. The precise knowledge of the requirements of problems for working backwards are, of course, also important for the mathematics teaching in school and for supporting students in the acquisition of heuristic experiences and competencies.

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Teachers’ characterizations of challenging introductory and enrichment tasks

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Developing tasks for use in mixed-ability classrooms presents teachers with several dilemmas. By making one such dilemma an explicit object of inquiry, this study aims to capture characteristics for challenging tasks suitable for introduction or enrichment. It is based on eight teachers’ collaborative and retrospective analysis of challenging tasks developed in a combined research and school development project. Among the results are the observation that introductory tasks should have an easy entry level and not require pre-knowledge of the upcoming concept, while an enrichment task should require relatively deep conceptual pre-knowledge. It is suggested that attention to seemingly contradictory features of introductory and enrichment tasks can fuel collaborative learning processes so that they include several important aspects of tasks aimed at challenging all students. Teachers’ verbalization of task characteristics is one outcome of such a process.

Keywords: Task design, mathematical enrichment, mathematical introduction, upper secondary school.

Introduction

Mathematics tasks serve many purposes in mathematics classrooms. A suitably designed task has the potential to provide all students in a mixed-ability classroom – those with learning difficulties as well as those with high abilities – with opportunities to develop their conceptual knowledge. To do this, the task should challenge all students. It should offer them opportunities to struggle with important mathematical ideas (Hiebert & Grouws, 2007), require them to put effort into their work, and involve some level of confusion (Bobis et al., 2021). In the mixed-ability classroom, tasks with a “low floor” (i.e., an easy entry level), or with enabling prompts, can help engage students who otherwise have difficulties with challenging tasks. When the same task also has a “high ceiling”, an open end, or extending prompts, students with high ability in mathematics can also be challenged (Bobis et al., 2021). However, the guidance included in the task formulation must not turn into funneling – that is, leading the student around the difficulties and avoiding the struggle (Bauersfeld, 1998) – as this would place effective learning at risk. Tasks designed to let students create their own solutions lead to better conceptual learning, compared to when a task instructs students to apply ready-made methods (Russo et al., 2020; Sullivan et al., 2015).

In the professional development project that we report on in this paper, teachers have repeatedly decided whether a task is suitable for introduction or enrichment, whether it offers sufficient guidance without funneling, or whether it is specific enough while also offering opportunities for general reasoning (Mellroth et al., 2021). In this particular paper, we focus on the distinction between introductory and enrichment tasks. We argue that this distinction is beneficial when designing tasks for a mixed-ability classroom, and aim to capture some useful characteristics for these types of tasks and elaborate on teachers' notions of introductory and enrichment tasks. More precisely, we aim to answer the following research question: What characteristics do experienced upper secondary teachers attribute to challenging mathematics tasks suitable for introduction and enrichment, respectively?

Literature review and theoretical considerations

Teachers themselves emphasize that they need challenging tasks, especially when introducing a new concept and when they want to help students deepen their knowledge (Mellroth et al., 2021). However, textbooks tend to offer few such tasks (Jäder et al., 2020). Finding tasks that are appropriate is difficult and time-consuming (Mellroth, 2018) and involves a wide range of didactic considerations (Bergwall & Mellroth, 2021), as tasks often needs to be selected and (re)designed to fit discrepancies to learning goals (Jäder, 2019). But teachers who participate in long-term professional development on developing challenging tasks for all students become competent in differentiating tasks in order to provide each student with appropriate challenges (Mellroth, 2018; Mellroth et al., 2021)

Teachers' collaborative efforts to find, develop, or adapt challenging tasks for use in their classrooms can be conceptualized as a collaborative learning process (Mellroth et al., 2021) situated in an activity system (Engeström, 1987). In this perspective, tensions and contradictions within and between different elements of the activity system are what drive the learning processes. One way such contradictions can manifest is as dilemmas (Engeström & Sannino, 2011). In the context of differentiating mathematics instruction for a mixed-ability classroom, teachers are faced with the dilemma that, on the one hand, all students must be able to participate in joint classroom activities, and on the other that they need to be offered support and challenges tuned to their individual needs. This dilemma can take many different forms.

The classification of tasks as introductory or enrichment tasks was introduced by the project's teachers in response to such a dilemma. However, the teachers' criteria for those two task categories were not a priori made explicit, and it was not clear if they represented opposing ends on a continuous scale, a dichotomy, or just two properties that a task can have in any combination.

Method

We will answer the research question by reporting on findings from a combined research and school development project in which eight Swedish upper secondary mathematics teachers develop challenging tasks and two researchers study their collaborative learning processes. The focus will be on the teachers' retrospective analysis of tasks they have classified as either introductory or enrichment tasks. The paper has been co-written by the researchers and two of the teachers in the project.

The context and the school development project

Swedish upper secondary school encompasses Grades 10 to 12 (students aged 16–19 years). Students choose from a variety of theoretical and vocational programs. The project's teachers teach mathematics within a technology program, a theoretical program with a high density of mathematics, science, and technology, aimed to prepare students for tertiary education in STEM subjects. Five mathematics courses, building on each other, are offered within the program. For practical reasons, the teachers in the project formed two subgroups, one for those currently teaching the first two courses (here referred to as Group A) and one for those teaching the other three (Group B).

The school development project is a 2.5-year (Aug. 2019–Dec. 2021) project on collaborative learning in mathematics teaching. For collaborative learning to be successful, it is important that participants focus on developing some aspect of their practice that they themselves perceive as problematic. Early in the project the teachers decided to develop a collection of tasks, a problem bank. They felt that mathematics textbooks lacked tasks that can offer challenges to both students with difficulties and those who are highly able in mathematics (cf. Jäder et al., 2020). Their mutually agreed-upon aim for the problem bank was that it should include challenging tasks suitable for introducing new mathematical concepts (introductory tasks), and tasks that could be used to help students develop in-depth knowledge about one or several mathematical concepts (enrichment tasks). Thus, the decision was made early on to distinguish between introductory and enrichment tasks.

During the first two years of the project, the researchers and teachers read and discussed research-based literature on task design (e.g., Sheffield, 2003), differentiated instruction (e.g., Tomlinson, 2016), and high ability in mathematics (e.g., Szabo, 2017). Collaboratively, the teachers merged research findings with their own teaching experiences and developed (or adapted) 13 tasks for use in their own mixed-ability classrooms. The teachers adjusted the tasks to fulfil criteria of rich learning tasks (Sheffield, 2003), for example that (1) everyone should be able to start working with the task, (2) it should be possible to solve the task in several ways, (3) the task should be engaging, and (4) the task should offer an open end. They also tested a majority of the tasks (the COVID pandemic made classroom testing difficult), and then re-analyzed and revised some of them. Therefore, the teachers can be considered competent and experienced in task design as well as collaborative forms of educational development.

During the development and testing phase, the researchers took on the passive role of observers. To project meetings, the teachers brought tasks they found promising to develop to be challenging for all students. In the continued development process, the promising tasks were analyzed using a task analysis guide developed within the project. The criteria for rich learning tasks mentioned above formed part of the guide. Information was collected digitally, and was intended to be included in the problem bank as support for its future users. One item in the guide concerned whether a task was suitable for introduction or enrichment. Of the 13 developed tasks, one was classified as a pure introductory task, six as pure enrichment tasks, and three as both introductory and enrichment tasks. Three tasks were classified as neither introductory nor enrichment tasks, and are therefore out of the scope of this paper.

Data selection and analytic procedure

The results presented in this paper are based on the teachers' retrospective analysis of a subset of the tasks they classified as introductory or enrichment tasks (or both). This new task analysis was conducted during the last meeting of the project's second year. Prior to the meeting, the teachers voted on which tasks they found to be of most interest to analyze according to their suitability for introduction and enrichment, respectively. At the meeting, and based on the votes, the two groups singled out one task each from each category. Group A chose Colored Cube for introduction and The Ant's Walk for enrichment, while Group B chose Ferris Wheel for introduction and Disease Spread for enrichment. Of these four tasks, Ferris Wheel was the only one which had been classified as suitable for both introduction and enrichment. In the next step of the analysis the teachers focused on its use as an introductory task only. English translations of the four tasks are presented in Figures 1–3. Due to space limitations, the descriptions have been somewhat shortened.

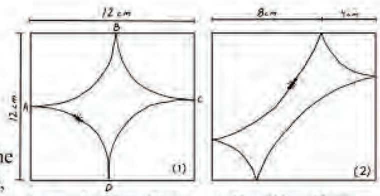
Colored cube

A 4x4x4 cube is colored on all sides.

How many small cubes…
- will have colored surfaces?
- will be completely uncolored?
- will have three colored surfaces?
- will have two colored surfaces?

- What are the solutions for a *nxnxn* cube?
- Are there any limitations?

The Ant's walk



Four circular arcs with the same radius have their middle points on the corners of a square, see (1). An ant walks along the arcs from A to B to C to D to A. How far has the ant walked?

The situation changes to (2) two circular arcs with radius 8 cm and two with radius 4. Show that the path has the same length as in (1).

The circular arcs can have many different radii in a square with the side 12 cm. Show that the length of the path is always the same.

What are the possible radii, if the ant is not allowed to cross a path it already has walked.

Figure 1: The two tasks chosen for retrospective analysis by Group A

Ferries Wheel

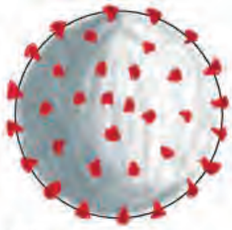
The diameter of a Ferries Wheel is 60m. Its center is placed 40m above the ground. One lap takes 4min. Assume that the wheel begins to rotate when a passenger is at the bottom.

1. Sketch a graph of the passenger's height above the ground at different times.
2. A mathematical model for the movement is given by $h = a + \cos(ct)$. Use the attached GeoGebra file and investigate how a, b and c affect the model. Determine a, b and c for your graph.
3. Sketch a graph for a passenger's position in the x-direction at different times. The movement in x can be described by $x = a + b\sin(ct)$. Determine a, b and c.
4. The movement in the x-direction can be described by $x = a + b\sin(ct)$. Determine a, b and c.
5. Describe the movement in part 2 with a sine function. You may need another constant.
6. Derive the equation of a circle by using the functions found in part 2 and 4.

Figure 2: The introductory task chosen for retrospective analysis by Group B

Disease Spread

Students are asked to implement the model and investigate how changing the constants in the model effects the spread of infection.



The population in a society is divided in three groups:

a) *Susceptible* (S): The proportion who have not yet been infected.

b) *Infected* (I): The proportion who are ill.

c) *Recovered* (R): The proportion who have recovered and become immune.

In this model, everyone who has recovered will be immune.

The SIR model for how the disease spreads can be described with the following differential equations:

$$\frac{dS}{dt} = -\beta IS \qquad \frac{dI}{dt} = -\beta IS - \gamma I \qquad \frac{dR}{dt} = \gamma I$$

β describes how contagious the disease is. γ describes the rate of recovery.

In this model the proportion of susceptible population decreases proportionally to the product of S and I. Those who leave Group S go to Group I. The proportion of those who leave Group I is proportional to I. Those who leave Group I go to Group R.

Examples of values for a disease are $\beta = 0.5$, $\gamma = 0.14$, $S(0) = 1$, $I(0) = 1 \cdot 10^{-6}$, $R(0) = 0$.

One way to compare different diseases is to find the *R*-number definded as $R = \frac{\beta}{\gamma}$.

Figure 3: The enrichment task chosen for retrospective analysis by Group B

All eight teachers, four from each group, participated in the analysis. We agreed on the following procedure: Each group would analyze their introductory task and enrichment task in parallel and answer two questions for each task. For the introductory task: a) What, above all, makes the task suitable as an introductory task? b) What, above all, makes it less suitable as an enrichment task? For the enrichment task: a) What, above all, makes the task suitable as an enrichment task? b) What, above all, makes it less suitable as an introductory task?

The time for analysis amounted to 45 minutes, approximately split as follows: 10 minutes for individual reflection, 20 minutes to compare and discuss individual reflections within the group and come to an agreement, 15 minutes to write a summary of the group's analysis. The discussions were audio-recorded for future reference, but the results presented below are based on the written summaries only.

From our perspective, those summaries are the results of the task analysis. However, for the presentation in the *Results* section, they have been translated to English and rearranged to have similar dispositions. This means that to some extent a content analysis has been conducted. All participating teachers have had opportunities to contribute to and influence the final formulations.

Results

First, as some tasks were categorized as neither introductory nor enrichment tasks, this categorization is not exhaustive. Neither is it a dichotomy, as some tasks were classified as suitable for both introduction and enrichment. Therefore, we conclude that from the teachers' point of view, a task can have both these properties in varying degree and in any combination. To cast further light on this, we now present the summaries of what it is that makes the four tasks suitable/not suitable as introductory and enrichment tasks, respectively. We start with Group A's analysis of Colored Cube and The Ant's Walk, and then Group B's analysis of Ferris Wheel and Disease Spread. Comparisons, similarities, and recurring themes are touched on in the *Discussion* section.

Group A considered Colored Cube to be primarily an introductory task. They found it suitable for this use because it is visual, has an easy start, and has a step-by-step increase in difficulty. It also offers opportunities for review later, and has a cliffhanger. Colored Cube satisfies the criteria for a rich learning task, and offers opportunities for the use of different solution strategies (here the group referred to Frank Lester). Group A found the task less suitable as an enrichment task because its first subtasks are too easy. They suggested that, for use as an enrichment task, the first four or five subtasks should be omitted, and the students should be asked to head directly for the general case.

The Ant's Walk was categorized as an enrichment task. Group A's main reason for this was that the task requires the student to produce a general, algebraic solution. The reason why they considered the task less suitable as an introductory task was that the step between providing a numerical solution and a general solution is too big.

Group B found Ferris Wheel to be a good introductory task because it refers to an everyday situation that students are familiar with, which helps turn the abstract theory into something concrete and tangible. Students can handle the problem even though they are unfamiliar with the underlying mathematical concepts/theory. In addition, they get a taste of what will be treated later within the trigonometry topic. The reason why Group B found this task less suitable as an enrichment task was that it would be too easy for someone who has understood the concepts/theory of trigonometry. Also, the second (and more difficult) part of the task is too similar to its first part.

Finally, Disease Spread was considered a suitable enrichment task because it requires considerable pre-knowledge about differential equations. Students need to understand that they cannot solve the system of nonlinear differential equations analytically but instead need to invoke digital tools. The task is relevant (in light of the COVID pandemic) and interesting, and lets the students see mathematics in a complex context. The results offer opportunities for interesting discussions, and the task can easily be modified and extended. The high demand on pre-knowledge, the inclusion of differential equations on a (for upper secondary school) high level, and the use of digital tools were also reasons why Group B found Disease Spread to be less suitable as an introductory task.

Discussion

In this paper we have asked what characterizes a challenging task that is suitable for introduction and enrichment, respectively. We have answered this question from the viewpoint of an experienced group of upper secondary teachers who have participated in a school development project and designed, analyzed, tested, and revised challenging tasks for use in mixed-ability mathematics classrooms. During their work, the teachers have in various ways encountered the dilemma that, on the one hand, every student must be able to work with the same task and, on the other, the task must offer challenges for all students. This dilemma can be conceptualized as a manifestation of a contradiction between agreed-upon aims of the tasks (Engeström & Sannino, 2011). As contradictions in an activity system fuel collaborative learning processes (Engeström & Sannino, 2011), such dilemmas should not be avoided but rather made visible and an explicit object of inquiry. The analysis of the four tasks presented in this paper is the result of such inquiry. The teachers' verbalization of the respective characteristics of introductory and enrichment tasks can be seen as an outcome of a collaborative learning process.

We therefore believe that our study and its results can contribute to mathematics teaching practice and educational research in different ways. Here, we highlight three. The first is that the results highlight characteristics for introductory and enrichment tasks that can guide teachers in designing, assessing, or selecting material for classroom enactment. Even though the results are hardly surprising, they point to important dimensions along which a task needs to be assessed in order to determine whether it is suitable for introduction or enrichment, with the two kinds of tasks often representing opposite ends of the scale. An introductory task should have an easy start and be visual and concrete, while an enrichment task should not have too easy a start and should aim for general solutions. In introductory tasks, the gap between subtasks must not be too big, and in enrichment tasks not too small. In introductory tasks one should not head for general, algebraic solutions too quickly, while in enrichment tasks one can go directly to general solutions. Introductory tasks must not require pre-knowledge, while enrichment tasks should do just that. One can be tempted to conclude that introductory tasks are those with a "low floor" or with enabling prompts, while enrichment tasks are those with a "high ceiling" or with extending prompts. However, for effective learning, all students should be offered opportunities to struggle with mathematical ideas (Hiebert & Grouws, 2007). To offer appropriate challenges to students with difficulties as well as those with high abilities in mathematics, both introductory and enrichment tasks should be designed to have both a "low floor" and a "high ceiling" (Bobis et al., 2021). We therefore suggest another interpretation: Introductory and enrichment tasks offer different kinds of learning challenges. These differences require that, when designing introductory tasks, extra focus should be placed on ensuring a "low floor", while the design of enrichment tasks requires greater focus on providing a "high ceiling".

A second contribution is that our results show how attending to the question of whether a task is an introductory or an enrichment task also can serve as a catalyst for discussions of other important aspects of task design for differentiated mathematics instruction. The act of designing tasks suitable for introduction and enrichment will present teachers with other dilemmas, such as whether a task provides enough guidance without funneling (Bauersfeld, 1998), and whether it is specific enough without depriving students of opportunities for general reasoning. Thus, for the collaborative learning process, attending to the question of whether a task is an introductory or an enrichment task might be more important than deciding on absolute criteria for such tasks.

A third, and methodological, contribution to the research community is the method used to clarify an outcome of a collaborative learning process: by conducting a retrospective analysis related to a previously discovered dilemma – in this case, the dilemma of designing introductory and enrichment tasks that are challenging for all students in the mixed-ability classroom.

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Reading comprehension and modelling problems: Does it matter where the question is placed?

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Text comprehension is a key aspect to consider when designing modelling problems. One important feature of mathematical problems is where the question is placed in the text. We present a theoretical background on text comprehension and modelling problems, and we discuss the pros and cons of placing the question before the text rather than placing it after the text. A review of the research revealed that placing the question before the text is more likely to result in improved comprehension. Further, we propose consequences for task design and future research.

Keywords: Reading comprehension, eye movements, modelling, mathematical materials.

Introduction

Prior research has indicated that mathematical problems are very important for learning mathematics. For example, in the TIMSS study, 80% of the time spent in mathematics classes was spent working on mathematical problems (Hiebert et al., 2003). Thus, in order to appropriately foster students' learning, thorough research must be conducted on task design. Niss et al. (2007) distinguished mathematical problems on the basis of their connection to reality, resulting in three different types of mathematical problems: modelling problems, word problems, and intramathematical problems. Whereas the third type have no connection to reality at all and can therefore be solved by applying heuristics and mathematical procedures, word problems are moderately connected to reality, meaning that the intended use of mathematics is embedded in an extramathematical context. Word problems differ from modelling problems in the strength of their connection to reality and in the cognitive processes necessary to solve the problem (Niss et al., 2007). Modelling problems require the problem solver to make assumptions about missing data and to structure and simplify the given context, whereas word problems are already prestructured and simplified. Usually, modelling problems are presented as texts. Thus, reading comprehension and the construction of an adequate mental representation of the real-world situation are essential for solving modelling problems. Based on this mental representation, the solution plan can be developed and carried out. As Leiss et al. (2019) showed, text comprehension was a significant predictor of students' performance in solving modelling problems. Thus, there is a need for research on text comprehension and task design in modelling. Initial attempts to conduct research on the design of modelling problems were carried out in the past by posing open modelling problems with missing information (e.g., Fermi problems) or by asking students to develop multiple solutions to the problem or to draw a picture that represents the situation (for an overview, see Schukajlow, Kaiser, and Stillman, 2021). However, to date, not much is known about how task design affects the reading comprehension and solving of modelling problems. One variation in task design involves so-called *question-placement effects* (Thevenot et al., 2007). The theory of question-placement effects suggests that comprehension should benefit from reading the question before reading the text of the mathematical problem. The aims of this theoretical paper are (1) to summarize and link theoretical and empirical findings from research in the areas of reading comprehension and modelling, (2) to ground the influence of questions and their placement on comprehension in mathematical modelling problems, and (3) to present consequences for research and task design.

Theoretical background

Text comprehension

Van Dijk and Kintsch (1983) distinguished between three levels of text comprehension that are built on each other but differ primarily in their complexity: the surface code, the text base, and the situation model. The surface code of a text consists of the exact wording and syntax of what is read, and thus, it is a pure representation of what has been read. In most cases, readers retain only the text surface of the last sentence or part of a sentence they read. Comprehension based on the surface code alone would be indicated through verbatim reproduction of the given text.

The text base reduces the text surface to the semantic content of the text – the wording and syntax are therefore no longer exactly the same as the surface code. However, the meaning of the text remains, represented in so-called *propositions* (van Dijk & Kintsch, 1983). This level illustrates a mental representation of the semantic content of the text via the process of building coherence locally, that is, the relationships between the subjects, objects, and so forth, and the predicates (e.g., traits or actions) given in the surface code in a sentence or between adjacent sentences. Successful comprehension on the level of the text base means the text base leads to an understanding of the basic features of the text, but deeper comprehension of the content of the text is not sufficient.

The highest level of comprehension is the situation model (van Dijk & Kintsch, 1983). It contains the content or context of the text and therefore no longer represents the explicit text but rather what the text is about. This mental model is constructed from the text base by drawing inferences and is augmented by prior knowledge and even external sources, such as the content of other texts or pictures. Building coherence across the entire text (i.e., global coherence) is crucial for constructing this mental model. The situation model may then be used to detect inconsistencies presented in external information and to evaluate the memory contents (Radvansky & Copeland, 2004). In the past, research on reading comprehension has been carried out by using different research approaches, including analyses of eye movements.

Eye movements during reading

The comprehension process is based on eye movements. Eye movements during reading can be classified into two different types: fixations and saccades. The former is “the state when the eye remains still over a period of time” (Holmqvist et al., 2011, p. 21). During reading, fixations occur when the eye temporarily stops moving on a specific word in order to process its meaning. Between two fixations (e.g., while switching from one word two another during reading), rapid eye movements occur, the so-called *saccades*. In reading research, a saccade that opposes the typical reading direction (e.g., *left to right* in Latin typeface) is called *regression*. Clifton et al. (2016) observed that the processes of word recognition and text comprehension (i.e., the construction of a situation model) are especially likely to have a strong influence on eye movements during reading.

Text comprehension in mathematical modelling

According to research by Reusser (1990) and Leiss et al. (2010), a situation model must be constructed to solve complex word problems and modelling problems. The real-world situation is represented in the so-called *situation model*, and it serves as the foundation for further steps in the solution processes. Blum and Leiß (2007) acknowledged this necessity and included the situation model in their modelling cycle, which has often been used to describe the process of solving modelling problems in the past. By comprehending the text, students construct an idiosyncratic mental representation of the real-world situation. This process can be described as a task-oriented structuring of individual knowledge. Importantly, the situation model contains more than just extramathematical knowledge (i.e., prior knowledge and prior experience). Rather, over the course of constructing a viable situation model of the problem, the student must identify a mathematically significant gap in the real-world situation in order to be able to complete the remaining steps in the modelling cycle on the basis of this situation model (Reusser, 1990). While constructing a situation model, the mathematically relevant information related to the question needs to be represented, and relevant information needs to be distinguished from irrelevant information regarding the modelling problem's question. On average, students spend around 40% of the time necessary to solve modelling problems on the comprehension process (Leiss et al., 2019). Therefore, there is a need to investigate how different task designs affect students' efficiency during this process. The process of comprehending the modelling problem is then followed by the process of structuring and simplifying the given information and thus constructing the so-called *real model*. Based on this deliberately structured mental model, mathematization is subsequently used to construct a mathematical model, which will allow the student to work mathematically within the mathematical model in order to generate mathematical results. These results then have to be interpreted on the basis of the real-world situation, followed by the deliberate validation of these results.

Hegarty et al. (1995) proposed two approaches used by successful and less successful word problem solvers: the direct-translation strategy and the problem model strategy. Problem solvers who utilize this first approach tend to base their solution plan on the selection of keywords and numbers given in the text of a problem. This rather superficial approach (for an overview, see Verschaffel et al., 2020) then leads to an incorrect solution more often when compared with the second approach the authors identified: the problem model strategy. Problem solvers who utilize this second approach deliberately construct a situation model of the situation described in the text and are therefore able to detect inconsistencies between their situation model and the given information by applying their solution process, a practice that is necessary for correctly validating the solution. In line with these results, Strohmaier et al. (2020) identified eye movements corresponding with these two approaches, plus a third approach that characterized readers who struggled while solving word problems. According to Strohmaier et al., utilizing the direct-translation strategy manifested in "a very linear and intense reading pattern with long mean fixation durations, short saccades and few regressions"(p. 10). On the other hand, the problem model strategy manifested in a shorter mean fixation duration and a high frequency of regressions. The third pattern Strohmaier et al. (2020) identified, which was linked to struggling readers, consisted of shorter saccades, higher fixation counts, as well as a higher regression count, resulting in overall longer reading times.

Effects of the placement of the question on comprehension in mathematical modelling problems

The reading goal is important for comprehension processes. In complex word problem solving, the reading goal is set by the question that accompanies the text of the task. While solving mathematical problems with relationships with the real world, the reading goal is to generate an accurate and deep understanding of the situation in order to answer the question. Usually, the question points to a mathematically relevant gap in the situation, which can be closed by using mathematics as a tool (Reusser, 1990), that is, by constructing a mathematical model and applying mathematical procedures. Because of the importance of questions for reading comprehension processes and for solving problems, the placement of the question needs to be evaluated critically with respect to task design. Depending on whether the question is placed before or after the text describing the real-world situation, comprehension processes can differ significantly. Placing the question before the text (i.e., reading the question first) should enhance the comprehension process because the reading goal is made more specific. Additionally, the question may serve as a tool for organizing and structuring the reading material, thus helping a student focus their attention on the relevant information. Therefore, decisions about the relevance of information given in the text should be made more precisely and quickly and, in conclusion, more efficiently. Furthermore, constructing coherence between the title and the question may enable initial inferences to be drawn and may therefore generate initial comprehension. The question and the title should facilitate the comprehension process, which should help a student solve a modelling problem similar to the ways a title facilitates a reader's understanding of a narrative text. This hypothesis was derived from numerous studies that have shown the effect of placing questions before the text on the retrieval of goal-relevant information (cf. Carpenter et al., 2018). As a result, the situation model should be more adequate (i.e., goal-related comprehension should be improved). The question could also be placed after the descriptive text. By placing the question after the descriptive text, the automated comprehension process would usually result in reading the text first and the question afterwards. Therefore, readers would have to construct hypotheses about the relevance of information for the superordinate reading goal, that is, solving the (at this point still unknown) problem. Placing the question after the text would result in the problem solver having to revisit the text after reading the question or adapting their situation model accordingly by shifting the relevance attributed to the information, ultimately leading the problem solver to learn from the text.

When discussing question-placement effects, some considerations have to be kept in mind. First, a question-placement effect on comprehension might only occur if the question cannot be inferred immediately and unambiguously from reading the text at hand. This is because, if the question is already clear to the reader, although they have not read it yet, the reading goal can be adjusted accordingly, thus organizing the process of distinguishing between goal-relevant and goal-irrelevant information. Second, the benefits of placing the question before the text are assumed to be specific to goal-relevant information (Carpenter et al., 2018). Therefore, the use of questions placed before the text might impair a student's ability to learn from the text. Third, the beneficial effects of the placement of the question in mathematical modelling problems should depend on the problem solver's mathematical competence. This is the case because the mathematically significant gap in the real-world situation needs to be identified, that is, the problem solver needs to construct a basic mathematical model that is based on the question in order to distinguish relevant from irrelevant information. The construction of the mathematical model requires adequate mathematical competence, and it was found to be affected by mathematical performance in the past (Schukajlow, Blomberg, et al., 2021).

Research on question-placement effects in mathematics education is scarce. Thevenot et al. (2007) showed that performance in solving arithmetic word problems was significantly better if the question was placed before the text. Their results opposed the results found by Arter and Clinton (1974), who did not find significant effects of placing the question before the text on error counts in a study on irrelevant information in arithmetic word problems. However, effects on the time needed to solve the problems were found, thus influencing the efficiency (the ratio of accuracy and time).

We would like to propose two prototypical comprehension processes regarding the two possible question placements in modelling problems by illustrating these induced effects with the sample problem called "Buddenturm" (Figure 1). We suggest that problems used for research in the field of question-placement effects should contain a title, a question, and a brief text that describes the situation. According to Thevenot et al. (2007), the descriptive text needs to be constructed in such a way that the problem's actual question cannot be inferred immediately and unambiguously. Otherwise, effects of question placement might be impaired as stated above. To do so, from our point of view, each problem should contain additional numerical information that is not relevant to answering the question. In the "Buddenturm" problem, the annual data, the length of the city wall, and the tower's former height are examples of irrelevant information.

Buddenturm

Question: How much area was refurbished in 2002?

The so-called "Buddenturm" is the only remaining fortified tower from the former city wall of Münster and was built in 1150. A few remaining rudiments of the 8-10-meter-high and 4-kilometer-long city wall can still be seen in two parts of the round tower. The Buddenturm is 30 meters high in total and has a 5-meter-high conical roof. In 2002, the outer wall of the 12.5-meter-wide tower was renovated along with the remains of the city wall by applying a new layer of limestone. This repair is only one of the few that the Buddenturm has experienced in its almost 900-year history; for example, it was still 40 meters high until 1945.

Figure 1: "Buddenturm" modelling problem with the question placed before the descriptive text

Assuming a linear reading pattern, the title of the problem is read first and does not depend on the placement of the question. By recognizing the name of the tower or by decoding the word *Turm* (in English: *tower*), readers can activate prior knowledge and integrate it into their initial situation model. After that, if the question is placed *before* the text (as is the case in Figure 1), readers will then read the question. By integrating the question and its content into the situation model, the reader may be able to draw inferences from it, such as that the tower is antique or at least in need of refurbishment. In conjunction with the information that the size of an area should be calculated to solve the problem, readers can enrich their situation model even further. After that, while reading the descriptive text, relevant and irrelevant information can be identified as such, simplifying the process of structuring the situation model for mathematization later on. In summary, the comprehension process initiated by placing the question before the text can be processed on a deeper level from the very beginning, rather than the problem solver focusing on generating a rough understanding of the situation. In turn, if the question is placed *after* the text, all information from the descriptive text needs to be integrated into the situation model initially, resulting in an understanding of the overall situation. To some extent, this is the result of the descriptive text being structured in such a way that the question (and thus, the reading goal) cannot be inferred unambiguously while solving modelling problems. Therefore, the ability to distinguish between relevant and irrelevant information during reading might be impaired. Although readers might anticipate a particular question, they cannot evaluate the relevance of information with certainty. This is because the actual question has not yet been integrated into the situation model. After reading the question and integrating it into the situation model, the situation model can be (re)structured and simplified for mathematization later on. Rereading the descriptive text might be necessary during this phase so that the problem solver can adequately represent all the relevant information.

These two prototypical comprehension processes might differ across the range of novices to experts as well as interindividually, as the processes depend strongly on prior knowledge. Experts in understanding modelling problems might structure their comprehension process in a specific way so that they read the title of the problem first but then immediately read the question, effectively simulating the question being placed before the text and thus possibly benefitting from the early integration of the question in their situation model.

Consequences for task design and future research

There are two possible conditions for the placement of the question, both already being used by teachers and textbooks: *before the text* and *after the text*. As described above, both conditions may have a positive influence on parts of students' cognition while also impairing other parts. The latter may challenge the problem solver by making it harder to initially comprehend the modelling problem, especially if prior knowledge is poor. But the reader will ultimately learn from the text, that is, the reader's ability to learn about the content of the text by solving the problem might be improved (e.g., learning that the tower's height in Figure 1 has changed across the centuries). The former may impair long-term learning about the content of the text but may in turn improve text comprehension, ultimately leading to an improved solution process, which in turn may result in improved modelling competence. However, both possible conditions and the extent of their influence have yet to be studied thoroughly. Thevenot et al. (2007) examined question-placement effects for arithmetic word problems, which were designed for fourth-graders. For modelling problems, research needs to be done on the influence of the placement of the question on text comprehension and the efficiency of text comprehension. Eye-tracking technology can be utilized productively to identify how the placement of the question influences students' comprehension of mathematical modelling problems and how students attempt to comprehend them. If a question-placement effect can be demonstrated through carefully designed studies, research on possible effects on modelling performance through effects on text comprehension may be conducted. However, the question-placement effect and its extent should furthermore depend on students' cognitive characteristics, such as working memory capacity, mathematical competence, and reading competence and on the problems' characteristics (e.g., the difficulty) (Thevenot et al., 2007) or the type of problem (e.g., "dressed up" word problems or intramathematical problems). As prior research was conducted on word problems, we can mostly speculate about how question placement might affect how students solve modelling problems.

Conclusion

In this theoretical paper, we discussed the theoretical background for question-placement effects in the reading comprehension of mathematical modelling problems and derived why placing the question before the text that describes the real-world situation should be beneficial for learners' comprehension processes, ultimately benefitting the solution processes. In short, placing the question before the text should help the reader better distinguish between relevant and irrelevant information by having a specified reading goal and constructing an adequate initial situation model just from the title and the question about the modelling problem.

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Characterizing the presence of activities using GeoGebra in Brazil's mathematics textbooks

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The Brazilian normative presents the obligation of using dynamic geometry software in the textbooks. However, each methodological approach allows many different applications. This paper aims to characterize the presence of activities using GeoGebra in Brazil's mathematics textbooks. To assemble these activities is essential to construct and explore visual aspects associated with experiments, as found in the literature. Analyzing four textbook collections of math for middle school, this paper found 36 activities. Through a methodological procedure created from the theoretical framework, the activities' characteristics show that textbooks must offer enhancing activities, not borderlines, and not use technologies as a pretext to be approved. So, there's space to change their approach in textbooks. Besides that, the proposed procedure altered perceiving an activity in a textbook, allowing future analyses.

Keywords: Dynamic geometry software, visualization, mathematical discovery.

Introduction

The textbook is one of the most used resources by teachers, which means it can influence an educator directly in a classroom, consequently, teaching and learning process. The importance of this resource is viewed through a Brazilian government program named National Textbook and Didactic Material Program – PNLD (Programa Nacional do Livro e do Material Didático), recognized as one of the most extensive national distribution textbooks programs in the world (Brasil, 2020a).

This program aims to offer didactic materials, including textbooks, for free, to every student studying in a public school. It is also important to highlight that the books are reused to assist the students for four years. In this period, it can only demand repositions or complementation (Brasil, 2020a). Only in 2020, more than 170 million textbooks were distributed, with an estimated total investment of BRL 1,4 billion, equivalent to approximately EUR 280 million, using the quotation in the given period (Brasil, 2020b).

All this process involving PNLD includes many stages, among which can be highlighted the writing of textbooks by the authors, the choosing of approved textbooks by the schools, the acquisition by the government of these books, the later production, and the distribution of the material having its final use inside the classroom. Thus, analyzing some of these PNLD textbooks allows us to understand Brazilian education, a relevant research field.

In this process, a crucial stage is the writing of textbooks by authors. Once, in this way, it is possible to observe the adopted approach of an object of knowledge by each one and the decision about how digital technologies will be used in their process. That's why textbooks are ingrained with the authors' conception about mathematics and the teaching and learning of math, explicitly or not (Borba & Villarreal, 2005).

As long as the contribution of the technology towards the changes in the social configuration is undeniable, which is also reflected in the learning space, it is necessary to think about integrating textbooks and digital technologies. For that reason, since 2014, the PNLD's public notice, dedicated to middle schools' textbooks[1], was the first of the Program that contemplated the inclusion of digital technologies.

Furthermore, PNLD 2020, bringing middle schools' textbooks written and published in 2018, was the first that had the main goal to contemplate the National Common Basic Curriculum – BNCC (Base Nacional Comum Curricular) (Brasil, 2018). It presents 11 of 121 math skills[2] addressing digital technologies, in which eight of them denote dynamic geometry software. Therefore, it is considered that it is clear the obligatoriness of the digital technologies' presence in textbooks available in Brazil currently.

However, these digital technologies in textbooks do not ensure that the authors present proposals using their potentialities or technologies just with a pretext to the same suggestions. So, an essential part is to think about this presence and these activities.

In light of the previous, this research considers it essential to reflect on: how the presence of activities using GeoGebra can be characterized in Brazil's mathematics textbooks? It opts here for this particular software, GeoGebra, because it is mentioned by three of four collections analyzed in this paper. All these collections present specific pages dedicated to a pedagogical task to be realized by students using this dynamic geometry software. Such studies will be called activities in this paper.

Theoretical framework

There is an essential differentiation between drawing and construction in the dynamic geometry paradigm, where movement is intrinsic to its processes. The first one is related to the non-resistance to the dragging proof; if a picture is dragged and doesn't keep its fundamental properties, the image is just a drawing (Laborde, 1998), as a picture of a geometrical object. The construction creates illustrations that permanently preserve its fundamental properties, even when one of its moving elements is dragged. Thus, the geometric figure resists the dragging proof (Borba et al., 2020).

Assuming this paradigm, it is understandable that the construction is essential to activities using GeoGebra to make scenarios that can enable mathematical investigation (Borba et al., 2020). Besides that, it may encourage the student to comprehend the properties that the geometric figures present

[1] Each year, this process occurs with a specific stage of school education: childhood education, primary school, middle school and high school. So, these textbooks are used during four years (before 2020, the period of use was three years).

[2] BNCC is a normative document that defines the set of essential learning that all students must develop throughout the stages of basic education. In Mathematics, there are 121 skills in total for the four years of middle school and every textbook has to follow it. If do not, they are not approved.

and the relations between the parties through visual feedback that the software provides (Laborde, 1998).

This idea of movement can be explored with the visualization since, based on the dynamism explored from the construction of geometric pictures, the possibility of manipulation opens up along with the visualization of objects. This means elaborating on a mathematics activity based in GeoGebra should explore the visual aspect once the software offers quick feedback. This feedback requires an interpretation by the students, and because of it, the visualization has a crucial role in dynamic geometry (Laborde, 1998).

Zimmermann and Cunningham (1991, p. 3) define mathematical visualization as "the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding." Therefore, visualization is associated with understanding a problem as well as discovering something, as seen by the authors. At the same time, it requires an active role by whom is visualizing.

In this point of view, after constructing some geometric object, the activity must have a goal to visualize this construction. It can be raising a hypothesis, getting convinced of some conjecture, or confirming previously emerged ideas. This interaction with the constructed object is necessary, and this is possible because of the manipulation that GeoGebra offers.

Associated with the idea of visualization with discovery, it is the experiment. According to Borba and Villarreal (2005), an experiment is executed to discover something unknown, check the truth of a hypothesis with the view to accept or reject it or afford examples of known fact. These authors point out that the first idea of the experiment goal is discovering something new, which means it is related to mathematical discovery.

This idea is primordial on the production of mathematical senses, as Borba et al. (2020) point out. They express that discovering patterns or singularities between the representation of math objects (or components of these representations) propels the production of mathematical senses (Borba et al., 2020). Thus, there is an "empirical" dimension that involves both thinking and learning mathematics. In this process to the occurrence of mathematic learning, Borba et al. (2020) argue that the technological resources became leading figures for that predominant experimental and visual feature. Therefore, it can perceive the importance of experimental dimension gains in GeoGebra and the visual dimension already approached.

Therefore, it is relevant to note that Borba and Villarreal (2005) point out that an experimental approach in mathematics education, using technologies, provides:

"The possibility of testing a conjecture using a significant number of examples and the chance of repeating the experiments, due to quick feedback given by computers; the chance of getting different types of representations of a given situation more easily; a way of learning mathematics that is resonant with modeling as a pedagogical approach." (Borba & Villarreal, 2005, pp. 75–76)

So, it's relevant to seek comprehension for integrating visualization, exploration, and mathematics learning in activities using GeoGebra in textbooks. Whence, it is essential to characterize the activities through these approaches that could potentialize learning math.

Methodology

Four middle school math textbook collections were analyzed out of 11 approved in PNLD 2020 to perform this qualitative research. The collections will be named Collection A to D in this paper. Each of them consists of four books, from sixth to ninth grade (students here are between 11 and 15 years old), totalizing 16 textbooks analyzed. Of the more than 10 million math textbooks distributed in 2020 to Brazilian public schools, approximately 70% are from these four collections, highlighting them as representative collections for the analysis, as shown in Table 1. All these 16 books that were analyzed are teacher's books[3].

After choosing the collections, the most distributed books from different publishers, the researcher searched for activities using GeoGebra and identified the chapters where they were. These chapters were analyzed, purposing to identify how the activities using GeoGebra were distributed through the textbooks, according to grades and contents of geometry.

The theoretical framework's reading developed a methodological procedure to characterize the presence of activities using GeoGebra in mathematics textbooks, following Figure 1. Every task found was classified by it.

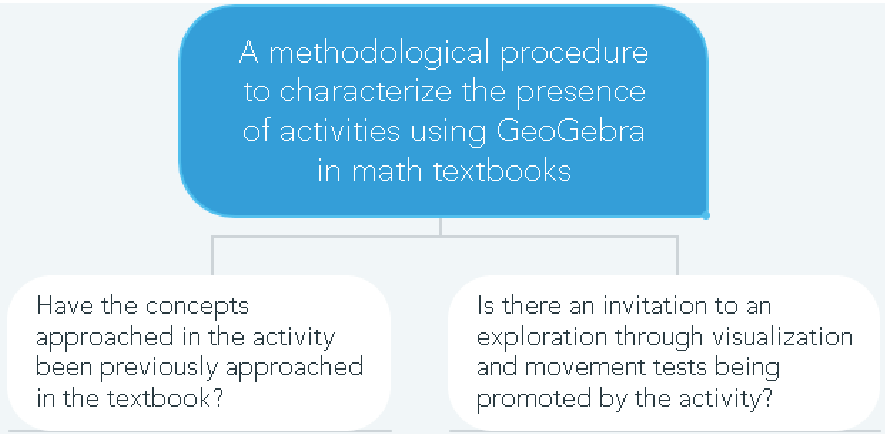

Figure 1: Methodological procedure developed by the researcher

To classify the activities by the methodological procedure, it was observed the following: 1) if the content worked in the activity was approached previously in the same textbook, or it is the first time that the author presented the content to students through GeoGebra; 2) after constructing a geometric object, if there is an invitation to drag something with a goal, it means, an exploration through visualization and movement or there is not this invitation and, so, the student construct a geometric object and does not have an exploration with it.

From two questions in Figure 1, therefore, it can be classifying the activities in four groups: activities that: 1) work concepts already approached in the textbooks and have an invitation to an exploration; 2) work concepts already approached in the textbooks and without an invitation to an exploration; 3)

[3] In Brazil, every public-school teacher receives a textbook containing student textbook and also some tips, instructions and answers for all activities of it (in another color, to do this differentiation, as Figure 2 shows in red).

work concepts are not approached in the textbooks previously and have an invitation to an exploration; 4) work concepts are not approached in the textbooks previously and without an invitation to exploration.

Giving attention to the third group, it is possible to see that activities from that have an approach promoting mathematical discovery. This assumption is based on the fact that in this case, the experimenters can discover something new that they did not know before, and test and visualize this result with quick feedback, following the theoretical framework.

Discussion of results

Analyzing the Collections A, B, C, and D, the results are shown in Table 1. Collection A, which represents almost half of textbooks coverage in Brazil, has just one activity using Geogebra per year, totalizing four activities.

Table 1: Data of 4 collections analyzed

Collection	Coverage in Brazil in 2020	Quantity of activities using GeoGebra
A	49,8%	4
B	10,1%	8
C	6,1%	15
D	3,4%	9
Total	**69,4%**	**36**

Focusing on those two questions presented in Figure 1, the results are shown in Table 2. It is possible to see that most activities have an invitation to exploration (first column); at the same time, works concepts are already approached (first line). Seven activities do not invite the student to explore, which shows they do not exploit the potentialities of GeoGebra of manipulation.

Table 2: 36 activities classified in four groups

	THERE IS AN INVITATION TO AN EXPLORATION	THERE IS NOT AN INVITATION TO AN EXPLORATION
CONCEPTS PRESENTED HAVE ALREADY BEEN APPROACHED	Collection A: 3 Collection B: 6 Collection C: 2 Collection D: 5 Total: 16 (44,4%)	Collection A: 0 Collection B: 1 Collection C: 1 Collection D: 3 Total: 5 (13,9%)
CONCEPTS PRESENTED HAVE NOT BEEN APPROACHED PREVIOUSLY	Collection A: 1 Collection B: 0 Collection C: 12 Collection D: 0 Total: 13 (36,1%).	Collection A: 0 Collection B: 1 Collection C: 0 Collection D: 1 Total: 2 (5,6%)

Also, there are five activities working concepts already approached in the textbook and having not an invitation to explore, as Figure 2 shows an example.

Previously, the chapter where this activity is localized has already approached the relations between angles in two parallel lines cut by a transversal. There is an activity on the same page with the same goal. The only difference is this first is to be done with pencil, paper, and a protractor. Furthermore, students are not invited to drag any constructed points to help convince them about corresponding or supplementary angles. So, this activity doesn't bring a potential use of GeoGebra, and a student could do this same task without any digital technology.

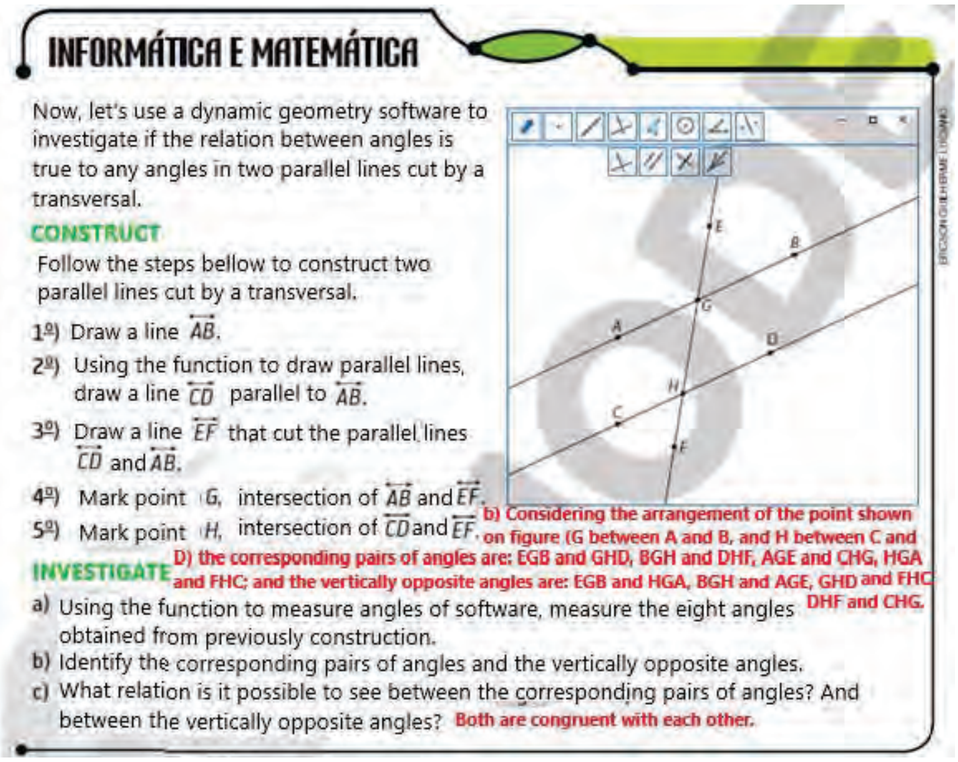

Figure 2: Activity using GeoGebra from Collection C – seventh grade

After this example of a borderline activity, backing to Table 2, in the third group, it is seen that Collections B and D do not present any activity promoting mathematical discovery. Otherwise, Collection C shows 12 out of 15 activities promoting mathematical discovery. For that reason, an example from this collection, in particular, will be explored.

The first activity from the sixth-grade textbook uses GeoGebra to represent parallel, concurrent, and perpendicular lines. It begins by requesting for the experimenter to construct parallel and perpendicular lines to be compared subsequently. For such, the student shall use three non-colinear points, constructing a ray and two segments, as Figure 3 shows. By measuring the angles and segments that appear in the image, the "investigate" part in the textbook shows up, as can be seen in the exact figure. This activity occurs at the end of the chapter so that the theory of all these points mentioned has already been approached. However, what is proposed is an investigation that has not been approached yet, that is, the conjecture that the shortest distance between a point and a straight line is the segment that unites them, forming 90°.

After the request to construct the parallel and concurrent lines in this activity, the student is requested to move the point E to equal AE to the measure of CD (letter b). For this, point E will have to be moved until CEA forms 90º. When continuing the activity (letter c), the student, from visualization and movement, using experimental procedures, is invited to elaborate the mathematical conjecture that the shortest distance between a point and a line corresponds to the measure of the segment that joins this point to the straight line forming a right angle (letter d).

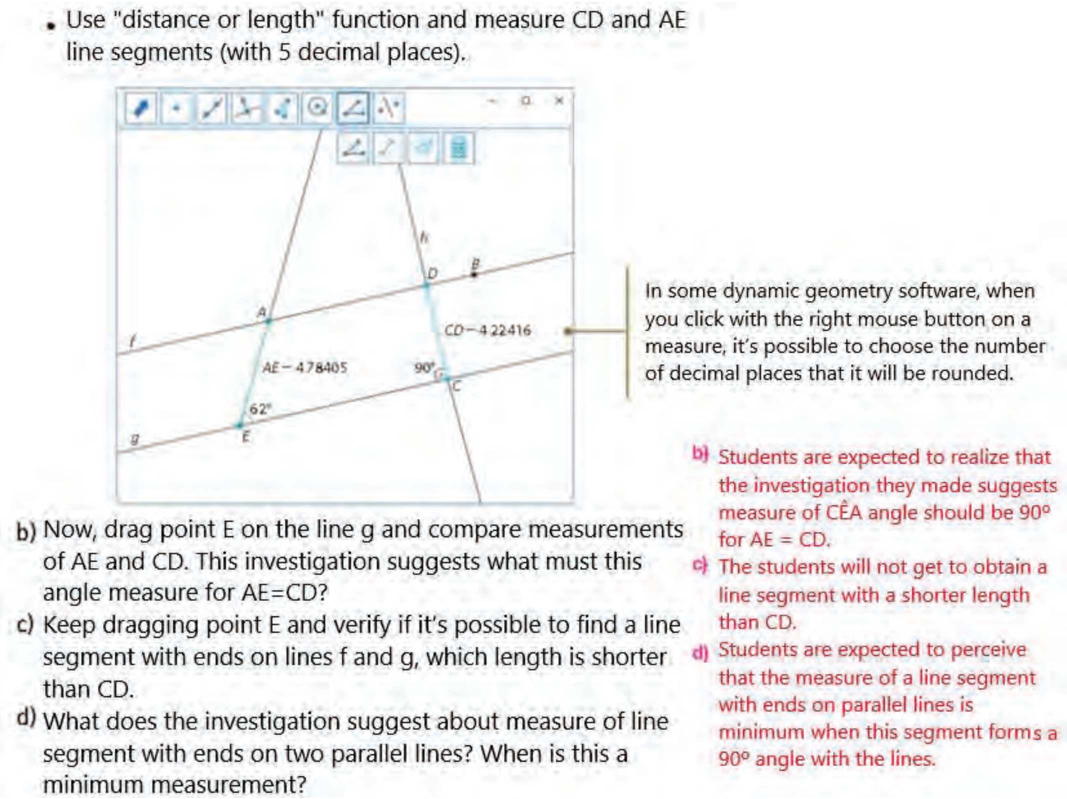

Figure 3: Final part of 1st activity from collection C promoting mathematical discovery – sixth grade

This way, it can define that the activity has a mathematical discovery approach (Borba & Villarreal, 2005). It happens because experimental procedures are used to generate a conjecture; there is the discovery of mathematical results previously unknown to the student and the possibility of several tests by dragging point E. It occurs while the students verify the measurement of the segment AE and the angle CEA compared to the DCE angle, which means they are invited to visualize and movement. This kind of activity can enhance students' mathematical learning once that it gives depth and meaning to understanding, as Hohenwarter et al. (2008) point out

"Instead of giving students an answer to a problem they didn't have in the first place, such explorations allow a more meaningful introduction of the abstract concept […] as a solution to a problem students experienced themselves." (Hohenwarter et al., 2008, p. 3)

Conclusion

Half of the Brazilian students, who use Collection A, can see one activity per year using this software, which is low. Giving focus on Collection C, which brings a more relevant number of activities using GeoGebra, it is the author's choices for an approach that has a goal to promote mathematical discovery, as can be seen from the methodological procedure (12 out of 15), what does not happen on Collection B and D once. This emphasizes the gap in the use of activities with dynamic geometry software in Brazil, making clear that there is space to introduce more activities using GeoGebra in the textbook and change their approach.

This paper was aimed to characterize the presence of activities using GeoGebra in Brazil's mathematics textbooks through two questions from the methodological procedure. From this, we could see how some activities (7 out of 36) don't invite students to explore and hence don't utilize the potentialities from GeoGebra and its dynamism. It's necessary that textbooks offer enhancing activities, not borderlines, and not use technologies as a pretext just to be approved by PNLD, for example.

We sustain that it is possible to potentialize the mathematics learning process through mathematical discovery once the students are invited to explore and search for their own results without a ready answer. That means they don't know where to arrive in the beginning. A textbook mustn't supply some answers before these activities using GeoGebra. In this way, students are more involved, as Borba and Villarreal (2005) observe, challenging them to discover something new by visualizing a problem and testing with quick feedback. And there is space to appropriate this approach better for making activities using GeoGebra in mathematics textbooks from Brazil, looking for promoting more meaningful math learning for students.

It is also noted that methodological procedure created from theoretical framework contributed to verify if an activity has an approach based on visualization and exploration and if it promotes mathematical discovery, changing to perceive an activity in a textbook, which shows us a fruitful field of development to future analyses.

Acknowledgment

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Engaging students as designers of digital curriculum resources: focus on their praxeologies and new awareness

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In this paper we present the results of a pilot study focused on an educational programme aimed at involving upper secondary students in the design of digital curriculum resources (DCR) using the GeoGebra software. We analyse the reflections proposed by the students-designers during semi-structured interviews developed at the end of the educational programme. As a result of this analysis, we propose a characterization of students-designers' praxeologies in relation to the task of DCR-design. This characterization highlights their awareness both on the characteristics of the DCR that supports students' learning and on the role of the design process in fostering the designers' learning.

Keywords: Digital curriculum resources, students-designers, praxeologies, awareness.

Introduction

In the last decade an increasing interest in studying the role played by digital tools in supporting the process of task-design has emerged (see, for instance, Leung & Baccaglini-Frank, 2017). Here, we focus on digital curriculum resource-design to broaden the set of possible products of the design-process: not only tasks, but also revision guides for specific topics, digital materials to introduce new mathematical topics, interactive e-books. We refer to Pepin et al.'s (2017) definition of digital curriculum resources (DCR) as "organised systems of digital resources in electronic formats that articulate a scope and sequence of curricular content" (p.3).

Many studies in mathematics education have been focused on the effects of involving teachers in the process of task/resource-design on both their professional development and the effectiveness of the designed instructional materials (Jones & Pepin, 2016). Few studies have, instead, focused on students as co-designers (Diamantidis, Kynigos & Papadopoulos, 2019) or designers of digital tasks/resources (Alessio et al., 2021). Our research is set in this mainstream of studies. In this paper, we present the results from a pilot study aimed at involving upper secondary school students as designers of DCR.

Theoretical framework

The first component of our theoretical framework, the tetrahedron model (Albano, Faggiano & Rossi, 2018), is considered to interpret the dynamics that characterize the process of DCR-design. This model takes into account the relationships between the Teacher (T), the Student (S), the Mathematics (M) and the Designer (D) of digital materials, and considers technology as a mediatory sphere embedded in the tetrahedron with vertices T, S, M and D.

The second component of our theoretical framework is introduced to analyse how the process of DCR-design affect students' development of specific competencies and awareness about their learning. The design of DCR could be conceived, in fact, as a specific task for students. For this reason, a product of the process of DCR-design is the students' development of specific praxeologies associated to this task. The notion of praxeology has been introduced by Chevallard (1991). It could be structured in two main levels (García et al., 2006): the *know-how level*, which includes the task, or a family of tasks, and the techniques used to face the task; the *knowledge level*, which includes the "discourses" developed to justify or frame the techniques for the task. The discourses on the techniques, as stressed by Artigue (2002), could focus on both their *pragmatic* value, connected to their productive potential in terms of efficiency, costs, validity, and on their *epistemic* value, more difficult to be grasped, connected to the way in which the techniques could contribute to the understanding of the objects they involve.

Context of the study

The students involved in our study (7 upper secondary school students, grade 12) participated to an educational programme set up by the Department of Industrial Engineering and Mathematics of the University of Ancona. The programme, aimed at supporting the transition from secondary school to university, was articulated in 10 meetings (50 hours in total) during which participants have been guided in deepening their knowledge of specific mathematical topics through the use of GeoGebra as a tool for DCR-design. The first 2 meetings were devoted to introducing mathematical topics that participants had not faced at school, namely complex numbers and limits. The introduction of these topics was supported through the use and analysis of specific applets aiming at different goals (explanation, exploration, visualization, remediation, self-assessment). In this way, students experienced specific functionalities of the GeoGebra software, such as check boxes, input fields, sliders, drag and drop. In this initial phase, in order to make participants become familiar with the software, they were also guided in the creation of their first GeoGebra applet concerning the exploration of the function y=sin(x).

In the second phase (8 meetings), participants (in the following, SDs, acronym for students-designers) were asked to design and implement DCR through the creation of GeoGebra applets, choosing the mathematical topic on which they preferred to focus. SDs, who worked in small groups, were suggested to consider the difficulties students could face and to exploit the potentialities of the software to create DCR aimed at supporting them in overcoming these difficulties. This activity, which was the central one in the educational programme, was carried out in presence of tutors (university researchers), available to provide SDs with technical hints concerning software aspects, if necessary. Specific criteria for the DCR-design were not shared with SDs in order to enable them to identify their own criteria.

If we refer to the tetrahedron model to interpret SDs' activity within our educational programme, we can highlight specific new dynamics. In fact, while, usually, students are located in the S vertex of the tetrahedron and are mainly involved in the dynamics that characterize the S-M-T or S-M-D faces, in our programme students are located at the D vertex (as SDs) and the focus is on the D-M-T face. T represents the tutors, who has assigned to D the task of designing DCR, using digital tools to support the learning of the mathematical content M. Since the resource produced at the end of DCR-design will be, in turn, a task for other students, the meta-reflections developed by the students-designers enable them to become strongly involved also in the interactions that characterize the S-M-D face of the same tetrahedron (where S represents the hypothetical students for whom the DCR are designed).

Research aims and methodology

The main aims of this research are: (1) to characterize the praxeologies, developed by the SDs involved in our study, in relation to the task of DCR-design using the GeoGebra software; (2) to investigate how SDs interpret their experience of being both students and designers.

The data collected were: (a) the digital resources designed by SDs, (b) SDs' answers to a final written questionnaire on the process of DCR-design, and (c) a final audio-recorded semi-structured interview.

In this paper, we focus on the analysis of SDs' interviews. To characterize their praxeologies (first research aim), during the interviews, SDs were asked to describe the process of DCR-design (know-how level) and to justify the choices they made during this process (know-why level). In particular, they were asked to justify their choices by focusing on: the didactical objectives of their DCR (to review a topic, to assess students, to support recovering…); the potentialities and constraints of the GeoGebra software as a tool to support the design of their DCR (pragmatic value of the adopted techniques); the difficulties related to the mathematical content on which the DCR is focused and the ways in which the DCR-design could support the students' learning of this content (epistemic value of the adopted techniques). To investigate SDs' interpretation of their experience of being both students and designers, at the end of the interviews, they were also asked to assess their experience, by reflecting on its usefulness (or not).

The transcripts from the interviews were coded independently by the two authors, who identified specific excerpts useful to characterize both the know-how and know-why levels of SDs' praxeologies (research aim 1) and the ways in which SDs interpret their experience (research aim 2). Afterwards, codes were discussed in order to come to agreement.

Data Analysis

Characterization of students-designers' praxeologies related to DCR-design

Because of space limitations, we will focus on two SDs – Marco and Anna – and on their discourses about their design of a specific kind of DCR: a revision guide for a mathematical topic. We chose to compare the praxeologies of these two SDs because of their different previous experiences with GeoGebra and the different choices they made in relation to the mathematical topic on which to focus in the DCR-design. Anna had never used GeoGebra previously at school, while Marco attends to a school focused on digital technologies' use. Moreover, while Anna chose to focus on a mathematical content she already studied at school (goniometric inequalities), Marco chose a topic he studied for the first time within the educational programme (the comparison of infinities). During the interviews, Anna and Marco effectively re-construct their design, enabling us to characterize their praxeologies, related to the *common task* of designing DCR to support students in the revision of specific mathematical contents. Here we focus on common elements of these praxeologies.

In the excerpt of Anna's interview on which we focus here, she reconstructs the design of a DCR aimed at supporting students' revision of goniometric inequalities. In particular, the applet she created

(Figures 1A and 1B) is aimed at making students reflect on different possible representations of the solutions of the inequalities $cosx > a$ and $cosx < a$.

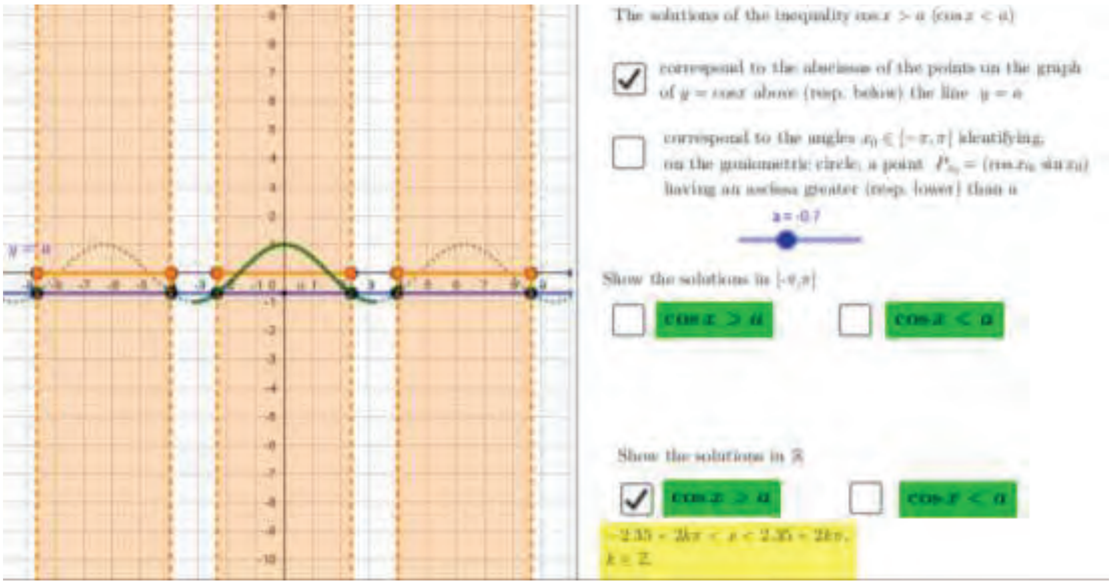

Figure 1A: The DCR on which Anna's interview is focused; the solutions of a goniometric inequality are represented as abscissas of points on the graph of the goniometric function $y = cos\ x$

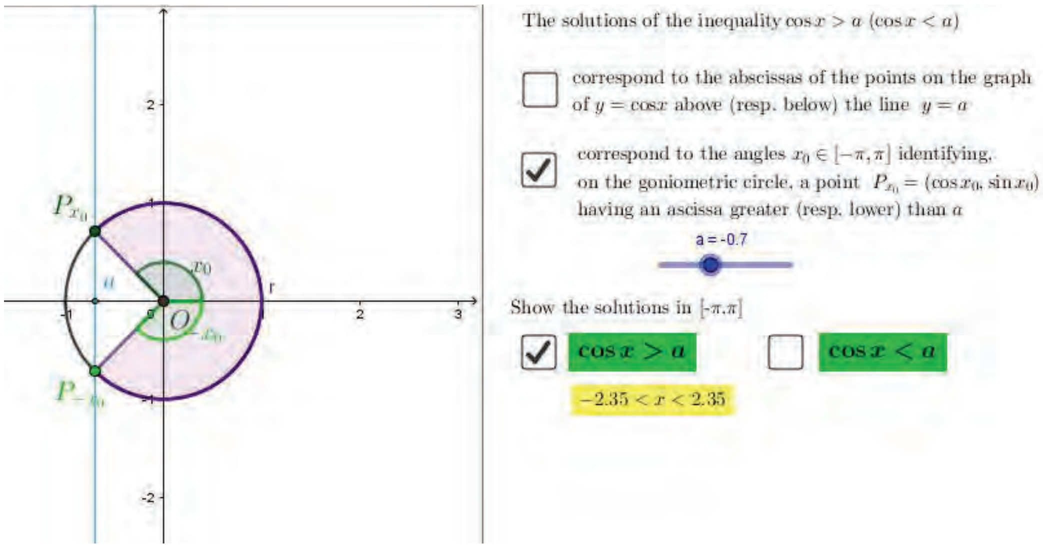

Figure 1B: The DCR on which Anna's interview is focused; the solutions of a goniometric inequality are represented as points on the goniometric circle corresponding to specific angles

In his interview, Marco focuses on his design of a DCR to support students' revision of the comparison of infinities in the study of limits. He created an applet (Figure 2) aimed at making students graphically compare the "ways" in which the functions $y = \log_a x$, $y = x^p$, $y = a^x$ and $y = x^x$ tend to infinity when the x variable tend to + infinity.

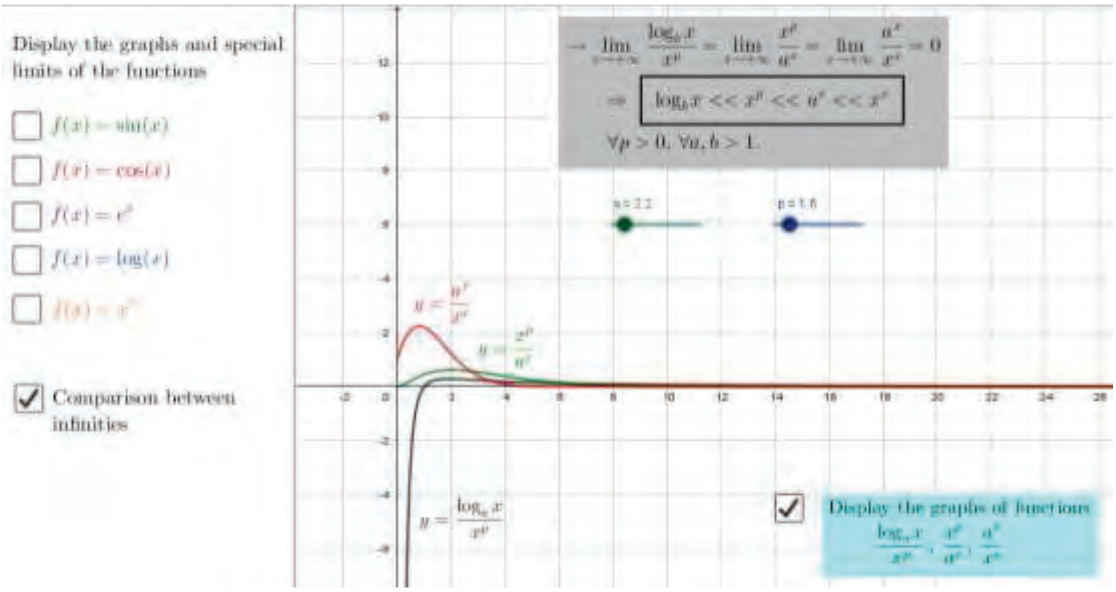

Figure 2: The DCR on which Marco's interview is focused

Even if they focused on different mathematical topics, the two students adopted similar *techniques* (*know-how level*) to face the task of creating revision guides. In their reflections, in fact, they stress on the importance of creating digital resources that enable students to: (a) choose the objects to display on their screens (different kinds of representations, graphs of specific functions, symbolic formulas…) and the order in which these objects are displayed; (b) see multiple representations at the same time and interact with them; (c) observe the effects of the variation and covarion of specific parameters using the sliders.

The use of these techniques is evident in both the DCR designed by Anna and Marco. Both students explicitly refer to these techniques in their interviews. For example, referring to *techniques (a) and (b)*, Anna declares that she has designed her DCR to enable students to choose the representation of the solutions of a goniometric inequality and also whether visualizing these solutions only in a range or across the whole set of real numbers (as in the Figures 1A and 1B). Marco refers to the same techniques when he stresses that, thanks to the design of his DCR, students can display different graphs at the same time and choose if visualizing the graphs of the functions $y = \log_a x$, $y = x^p$, $y = a^x$ and $y = x^x$ or the graphs of their ratios, that is of the functions $y = \frac{\log_a x}{x^p}$, $y = \frac{x^p}{a^x}$, $y = \frac{a^x}{x^x}$.

Both Anna and Marco inserted sliders in their DCR (*technique c*). Anna inserted a slider to enable students to explore different possible situations, according to the value of a within $cosx > a$ and $cosx < a$. Marco inserted two different sliders to enable students to visualize how the graphs of the functions dynamically vary according to the values of the parameters *a* and *p*.

When they justify these techniques (*knowledge level*), Anna and Marco focus both on the description of practical aspects connected to the DCR-design process (*pragmatic* aspects) and to the reflection on how these techniques could contribute in creating DCR that really support students in the learning process (*epistemic* aspects).

Both Anna and Marco stress that making students choose what to display (*technique a*) enables them to manage the information on which to focus. Anna, for example, focuses on pragmatic aspects when she declares:

> "[*this applet*] enables students to make graphs appear and disappear when they want, instead of visualizing them all together. In this way, there are not too many information, but only the information I require, so I can manage them".

In relation to the possibility of deciding the order in which to display information, she adds a reflection that shifts the focus on epistemic aspects, connecting the chosen technique to difficulties she experienced when studying goniometric inequalities:

> "It seems better to focus first on [*the solutions in*] an interval, so as to gradually introduce the periodicity. In the classroom, also, one of the main difficulties is to write the solutions with periodicity".

Marco also reflects on the epistemic value of technique *a*, when he explains how the teacher could use the DCR he created:

> "This exploits the potentialities of the software; [*the DCR*] has been designed for didactic purposes: [*the teacher*] shows the file and can show, for example, the cosine function first, then the parabola and this allows to give a further meaning to the concept of limit; the software allows you to show these things in the order you prefer and then allows you to explain".

Anna and Marco focus also on the potentialities related to making students work with multiple representations (*technique b*). Anna, for example, declares that her choice to adopt technique *b* in the design of her DCR was aimed "to give, to the users, the possibility to display all the representations of the solutions of goniometric inequalities". While Anna's reflection mainly refers to the pragmatic value of technique b, Marco focuses on the epistemic value of the same technique, by stressing that making students work with different representations enable them to give different meanings to mathematical concepts: "Give a new meaning, seeing limits, truly see them! It is something that this software enables to do."

When reflecting on *technique c*, Marco focuses on the role played by sliders as tools that support students' understanding, by enabling them to work with dynamical figures (pragmatic level) that represent classes of mathematical objects (epistemic level):

> "There are [*in my DCR*] classes of functions depending on sliders. This applet has been conceived with a didactical aim, so that a student who faces this topic for the first time, playing with sliders, can see how limits and classes of infinities change. It seems to me that it is like 'putting your hands in it', something more than seeing it in a 'sterile' way, within a textbook. This [*technique*] enables [*students*] to better understand what they see".

Anna justifies her choice of inserting a slider in her DCR by focusing on both pragmatic and epistemic aspects, since she observes that using sliders, differently from working with paper and pencil, enables students explore a wide range of possible situations (pragmatic level), interacting, at the same time, with different problems belonging to the same class of problems (epistemic level):

> "For the user, the sliders offer the opportunity to visualize and try different possibilities also in various points of the circumference and of the graph; if I had done it on a sheet [*paper and pencil*], I would have assigned a unique value to a [*the parameter in the inequality* $\cos x > a$], so there would have been less chance of comparison."

SDs' interpretation of their experience as both students and designers

When Anna and Marco are asked to assess their experience as designers of DCR, they both interpret the DCR-design as a particular problem-solving activity that enables the designer to reflect on aspects that usually are taken for granted. In this way, they highlight to be aware that the meta-reflections on the interactions that characterize the S-M-D face of the tetrahedron, which they developed thanks to the experience of DCR-design, have influenced their relationship, as students, with the mathematical content at stake. According to them, these meta-reflections gave to the designer the opportunity to autonomously discover, while facing the difficulties connected to the design process, key-aspects of knowledge construction. Marco, for example, declares:

> "It has been instructive, for me, to create tasks on these topics. In creating tasks, I learnt these topics. It has been a sort of problem solving where the problem is to create a problem. … At a didactical level, this work [*the DCR-design*] was very useful: putting your hands inside it, having to think about tasks … it seems to me that it is one of the best ways of learning a topic."

Thanks to their reflections on the dynamics characterizing the S-M-D face of the tetrahedron, Anna and Marco also have the opportunity to compare their experience as SDs with previous experiences as students. In particular, they contrast their learning through DCR-design, described as a "learning by discovery", with the learning realized through textbooks, described as a "pre-packaged learning", as testified by this reflection, proposed by Anna:

> "Creating resources from scratch was a completely new experience for me; I was used to textbooks, where exercises, and also theory, were ready-made. I found myself 'at the opposite side' with respect to those who study. It represented for me a new and more complete form of revising [*contents*]".

Final discussion

In this paper we have analysed the process of DCR-design in which a group of secondary school students have been involved during an educational programme carried out within a university context. The analysis we have performed, focusing on the reflections developed by SDs during the semi-structured interviews we carried out, has enabled us to reflect on the effects of DCR-design at two levels: (1) the level of students as designers; (2) the level of students as learners.

As regards level (1), the investigation of SDs' praxeologies has highlighted their capability of justifying their choice of adopting specific techniques in their DCR-design by clearly referring to both the efficiency of these techniques with respect to those adopted within a paper and pencil environment (*pragmatic value*) and the learning that each technique could support (*epistemic value*).

As regards level (2), the semi-structured interviews we have conducted have boosted SDs' reflections on the ways in which acting as designers could support the designers' learning itself, showing the effectiveness of activities aimed at fostering students' reflections on the dynamics that characterize the S-M-D face of the tetrahedron.

As a further development of this research, we are deepening our investigation of SDs' praxeologies related to DCR-design by combining the analysis of a-posteriori interviews with the audio and video-recording of the DCR-design process. We are also working on the re-design of the teaching methodology that characterized the educational programme to promote opportunities for SDs to share their reflections about their double role of designers and learners with their school mates, involving also their teachers in this process.

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The role of dynamic elements in digital teaching platforms - an investigation of students’ reading behaviour

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The use of digital teaching materials in mathematics education has gained ground since the first introductions of various hard- and software. A distinguishing feature for digital teaching materials is the possibility to offer interactive and dynamic elements. In this study, eye-tracking is used to explore students’ reading behaviour when working with mathematics items in a digital environment. In particular, the focus is laid on how students read depending on the extent to which the items offer dynamic elements. Analysis of data from the eye-tracking in combination with students’ responses in the interviews provide a broad picture of different types of challenges that students may face in working with dynamic elements. The results also reveal that commonly used dynamic elements as films or feedback on given answers are valuable because users emphasize them as useful and informative.

Keywords: Mathematics, eye tracking, feedback, stimulated recall.

Introduction

New ways to visualise and opportunities to interact with mathematics provide learning opportunities that have proven valuable, for example in digital books that incorporate dynamic geometry applets (Radović et al., 2020) and in tasks with the ability to give feedback (Stevenson, 2017). In this study we use eye tracking to analyse students’ reading behaviour when working with mathematics items with five types of dynamic functions. Increased knowledge about what dynamic functions means for a student working with mathematics is valuable in relation to not only digital teaching platforms, but also other types of digital teaching materials.

Background

Dynamic functions in digital teaching material provide opportunities for learning not possible when printed materials are used. For example, Pohl and Schacht (2019) stresses the need for further research about new digital textbook elements. They contribute an empirical study of student use of textbook elements for the learning of mathematics, highlighting that these elements provoke new types of activities indicating that mathematical hypotheses may also be generated in new ways. In the contemporary rush to digitize there is also a need for awareness of potential shortcomings of digital materials. For example, writing mathematical equations may be harder than on paper and there is a risk of extended use of multiple-choice questions, something that can limit the readers opportunities to actively construct their understanding. Analyses of students’ interaction with digital teaching materials have revealed that digital materials contribute in determining how students engage in problem solving by reinforcing iterative strategies such as trial and error behaviours (Lantz-Andersson et al., 2009). Such strategies may impede learning if used instead of problem solving based on analyses of the task, which means that the manner in which students interact with digital materials is important. Digital teaching materials are for example argued for based on an intention to let students experience dynamic functions and to benefit from automatic feedback from the material. The possibilities to interact with dynamic media, enables an enhanced expressivity and the possibility that the technology functions as a collaborator rather than merely a tool or mediator, which means the mathematical activities become more participatory (Moreno-Armella et al., 2008). The expanded space of interaction in digital resources in contrast to printed (Pepin et al., 2017) implies a potential to play an important role in students' learning. Research also reveals increased learning when digital materials are used to achieve knowledge acquisition through active learning, for example in an interactive response system (Wang, 2020) or using a computerized training method to learn mathematics (Taleb & Hassanzadeh, 2015). An investigation of the effect of various types of feedback offered in digital textbooks revealed a low effectivity of the feedback regarding whether a correct solution was given after receiving the feedback (Rezat, 2019). The feedback did not lead to the desired results even if it was given stepwise in relation to further trials, and the author suggests further development of feedback in digital textbooks.

These previous studies give rise to the question about how students experience work with dynamic mathematics items and receive different types of feedback. In this study we make a contribution by using eye tracking to investigate students' encounters with digital items offering different types of feedback.

Eye tracking, theory and practice

A fundamental assumption when using eye tracking methodology is the eye-mind hypothesis; what you pay attention to and think about is associated with where you place your gaze (see Hoffman, 1998). This assumption has been questioned, and results also reveal that the gaze and the attentional focus sometimes diverge (Schindler et al., 2016). There are however convincing results about the correlation between the gaze and thought (Andrá et al., 2015). What is possible to measure is the students' gaze and how much, how often and for how long the gaze lingers on different elements in the text. Because a prerequisite for decoding and comprehension, the two components of reading according to Gough and Tunmer (1986), is to place the gaze on the text, the data is considered to reflect parts of the reading process. Based on the eye-mind hypothesis, the gaze is here interpreted as reading the multimodal and dynamic text elements in the analysed items.

In mathematics education research, eye tracking has proved to be particularly beneficial for studying processes, not outcome, and for research including aspects of visualisation and mental representation (Strohmaier et al., 2020) and because these descriptions hold for the current study the method choice is reasonable. The current study takes its starting point from a previous study revealing a limited use of dynamic and interactive elements in digital teaching platforms in mathe-matics (Dyrvold, submitted) which raised a question whether such elements should be used more. In the current study, eye tracking enables a comparison of reading behaviour depending on the offered dynamic and interactivity in mathematics items. Heat maps (visualising fixations) offers snapshots of students' reading that are easy to interpret, and therefore valuable in stimulated recall interviews.

Aim

The aim of the study is to achieve an in-depth understanding of reading behaviour in a digital multimodal environment: teaching platforms in mathematics. To enable analyses of which role variations in interactivity and dynamics plays for reading behaviour, mathematics items with elements that differ in those aspects were designed. These different element types are described in the Method section. Two research questions are posed; i) what do different element types mean for how a text is read?, and ii) how do students experience reading digital materials with different element types?.

Method

Three grade nine students (15 years old) with much gaming experience have participated in this study. The students were selected using a convenience sampling and were familiar with the interviewer. The students' previous mathematical achievements were unknown. Each student worked with five items in a digital environment, each containing one mathematics task, and their work was monitored using eye-tracking equipment. The items were designed to touch on areas of mathematics that are new to the students. The apt difficulty of the items was outlined based on a review of Swedish grade nine textbooks. At a later stage tentative items were further discussed and adjusted in cooperation with two teachers who are also experienced textbooks authors. All items had the same structure, consisting of three parts, an introductory text, a question and some essential theoretical content (hereafter called Theory), each part constituting an area of interest (AOI). The eye-tracking analysis was built on how the student's gaze moved between these AOIs (visualised as the four grey areas in Figure 1).

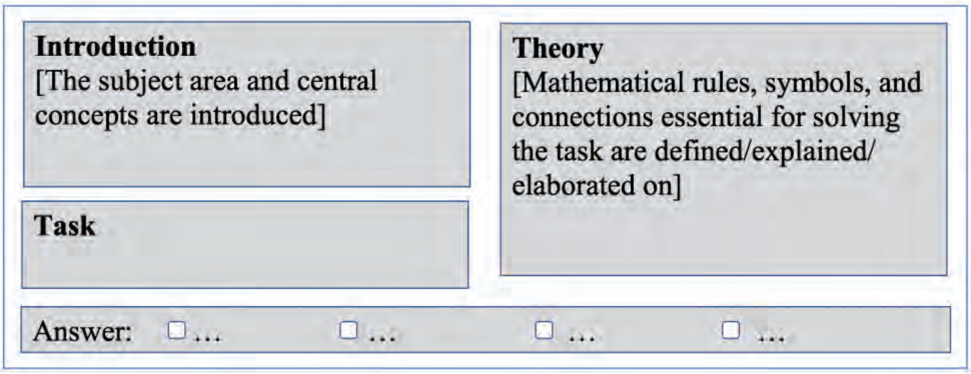

Figure 1: Basic design of the items and visualization of areas of interest (AOIs)

The intention was that the task should not be able to solve based only on the introduction; rather the Theory would be needed. The Theory is essential in the analysis because the type of elements (i.e. how dynamic and interactive) utilized in this section is altered between the items. With access to the information in the Theory, the intention was that the tasks could be solved without using paper and pencil. The Theory in each item was designed with a dynamic function, from one of five element types (ET) (Dyrvold, submitted). The element types with their characteristics are as follows: ET1, static presence: static content, e.g. similar to typed text in print; ET2, opted presence: static content opened by a click, e.g. a definition; ET3, dynamic presence: continuously dynamic but not interactive, e.g. film; ET4, dynamic feedback: dynamic and (instantly) interactive, e.g. to choose in a list and receive feedback; ET5, continuous dynamic feedback: continuously dynamic and interactive, e.g. by providing feedback when dragging.

The students were interviewed with stimulated recall directly after working with the items following a semi structured and pre-formulated interview guide, based on three main themes. The first theme took its point of departure in the question “What did you think about the items in general?” in order to capture the students’ general impression of the items and the various dynamic functions. The second theme concerned the question “Is there anything special that you thought of when you worked with the different types of Theory?” (Interviewer exemplifies if needed). After that, heat maps from the students’ work were shown to the students (visualizing main fixation points) and they were asked questions on their reading of the items and the five element types for example: “How did you think when you worked with this item”, “You looked at X for a long time/several times, do you remember how you thought or why you looked at just that?”, “When you see how you read the Theory, do you remember anything special?”, and finally “What are your thoughts about the various dynamic functions in the items?”. The interview leader was well known to the students, the interviews were recorded and transcribed verbatim, and then summarized using short notes. Analysing these two kinds of data together contribute to fulfil the aim of the study because the different data nuance and enrich each other.

Eye tracking analysis

The students’ readings were analysed based on the following analysis questions: (1) How appealing do the students perceive the different element types? (TFF – time to first fixation on the ET); (2) How do the students use the different element types? (Fixation duration (FD) on ET and total number of fixations (TNF) on the ET); (3) How do the students read the explanations in the Theory due to element types? (Fixation duration on introduction and on task in relation to ET ((I+T)/ET)

Time data from the eye tracking and short notes from the interviews with each of the three students were compiled and qualitatively analysed to gain insight into the relation between reading behaviour and experience of reading the text. The current study is to be followed by a larger quantitative study and results in relation to the current study will also be used to refine the method in the larger study.

Result

The results reveal both differences and similarities in how students perceive and read mathematics offered using different element types. There are not very distinct differences between how students interact with the five element types but if the element types are clustered according to level of interactivity and dynamics, some patterns occur. The results reveal both differences and similarities in how students perceive and read mathematics offered using different element types. There are not very distinct differences between how students interact with the five element types but if the element types are clustered according to level of interactivity and dynamics, some patterns occur.

Table 1: Overview of collected data about the element types for each participant

	Item 1:ET1			Item 2:ET 2			Item 3:ET 3			Item 4:ET 4			Item 5:ET 5		
	S1	S2	S3	S1	S2	S3	S1	S2	S3	S1	S2	S3	S1	S2	S3
TFF: ET	0.6	1.6	0.3	89.6	0.1	37.8	1.2	1	6.4	11.3	21.1	19	20.1	20.7	4.3
TNF: ET	29	52	59	107	81	69	158	205	122	49	56	31	145	100	156
AFD: ET	8.1	21.7	19	33.3	29.5	19.64	63.5	91.2	47.7	22.2	29.3	10.5	66.5	50.6	61.1
AFD: I	34	13.6	20.3	48.7	26.9	34.1	8.7	20.7	12.7	15	19.7	15.6	10.7	11.1	8.5

AFD: T	7.7	7.9	8.6	33.3	47.5	22	37.5	44.1	57.1	54.7	124.8	32.5	23	20	14.1
[AFD]: (I+T)/ET	5.17	0.99	1.52	2.46	2.52	2.85	0.73	0.71	1.46	3.14	4.94	4.56	0.51	0.61	0.37

Measures are given in seconds except TNF, which is given in number of counts

Results in relation to the two research questions about how the items are read and how students experience the reading are presented in themes, where each theme relates to both research questions.

Theme 1. There is a pronounced difference in how the students read an item with only static content (ET1) in contrast to items with dynamic and interactive content (ET2-5).

Less fixations and less time is spent on the Theory that is purely static. One student (S1) refers to the heat maps with ET1 saying “I read it [Theory] several times” and “The concepts were difficult and therefore I looked at the image [introduction] several times” indicating an experience of having engaged deeply with the text. Eye tracking data reveal that items with ET2-5 do generally reveal more fixations and totally longer summarized fixation time on the AOIs. The students’ experience from reading these items and reflections on the heat maps reveal several explanations to this pattern. Most prominent is utterances about joy and usefulness: “I liked the opportunity to drag” (S2) (see Figure 2), ”I thought it [receiving feedback] was good … because it shows that you understand” (S3).

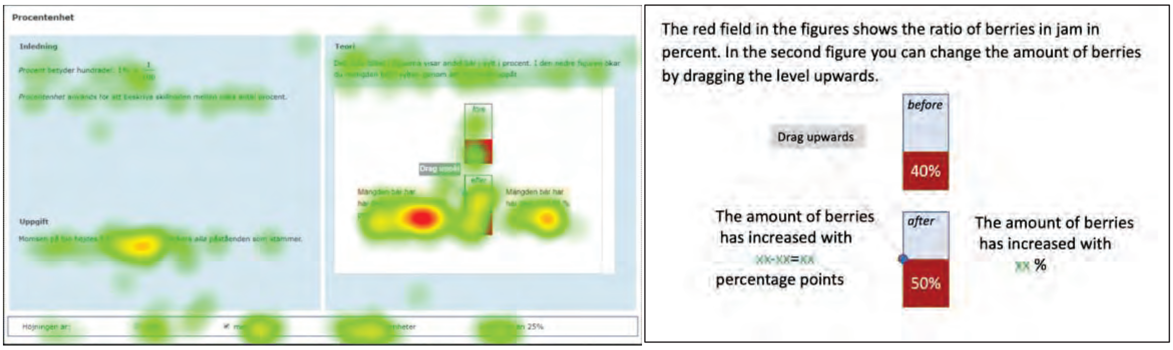

Figure 2: Heat map showing S2 dragging in ET5 and translated Theory part.

In the dynamic element (Figure 2) the green values on both sides on the rectangle changes dynamically when the level is dragged upwards. The translation of the introduction and task reads as follows: Per cent means hundredth 1%=1/100. Percentage points is used to describe differences between different numbers of per centage. Task: The VAT on cinema tickets increased from 6% to 25%. Tick all correct alternatives (19%; more than 50%; 19 percentage points; less than 25%).

Theme 2. Dynamic elements have a potential to evoke deep engagement in interaction, but there is also a risk of misunderstandings or omission in relation to the elements.

The eye tracking data reveal that the students fixate their gaze on Theory as a static page (ET1) or film (ET3) immediately (almost in every case within two seconds). In the Swedish context, these element types are something the students are familiar with. In the item with ET2, on the other hand, one student (S1) left the Theory unread for quite a time because the student did not realize that more content was offered behind a click. Another student (S2) referred to the same content as clearly presented and “fun” to be able to open. In the item with ET5 the invitation to interact with the material is by one student perceived as part of the task to solve, not Theory, “Because it was possible to drag, it made me think that it was part of the task” (S1), something that made the student a bit confused.

Theme 3. Some element types demand investment in time from the reader, but time spent on the element does not assure that the reader is engaged in a learning situation.

The eye-tracking data do rather clearly show that the students engage the most with the Theory presented as a film (ET3) or a continuously dynamic activity (ET5). If the reader does what is requested these elements have an inherent demand of persistence. A film may not make sense if not watched to the end, and a dynamic activity may leave questions lingering if not completed. In this study the participants invested the required time, and when that is the case, the content offered has a large potential to provide a fruitful learning situation. On the other hand, if for example a film is omitted, this learning situation is lost. The student with most fixations on the film (S2) watched the film twice and explained clearly the benefits with a film “It is easier to understand when someone is reading. Then you get help with symbols that are read”.

In addition to these results there were some differences between the participants’ reading and reflections about the items that call for attention because they highlight that there is no such thing as the best teaching material. The usefulness is to a large extent dependent on who the reader is and if a sound base material is offered, it provides a good fundament for a learning situation. The teacher is essential and discussions and other activities can contribute further to the individual use of digital teaching platforms. For example, one student (S3) explains that he missed the information “drag” when he read the item with ET5. He experienced the item as too difficult and did mainly struggle to get a grip on what was requested in the item. In this particular case, a discussion with a peer or a teacher could have clarified the intention with the item.

Discussion

This study is designed as a response to the current development of digital teaching materials. The current study contributes to this issue with insights about how dynamic and interactive elements can be used to boost students’ gain from interaction with digital materials. The results highlight students’ engagement with items that are dynamic and interactive but also some potential challenges. The analysis reveals both enthusiasm and engagement in the dynamic ETs, also for a minor addition of questions which return ‘correct/false’ directly, in the Theory (ET4). The positive result in relation to ET4 was not expected, since it is such a trivial addition to a static text. However, the same element type caused some quandaries because the questions signal to the students that this is the task to solve, not a part of an explanation.

Previous studies highlight that what students choose to engage with and how, in digital materials, is a factor worth taking into account when evaluating such materials (Lantz-Andersson et al., 2009; Bartelet et al., 2016), and the attraction to element types that offer interaction is notably on this issue. The students in the current study invested in the dynamic element types, also when the student “didn’t understand anything”, and such engagement must enhance the chance of learning. The persistence with which the students engaged with ET3 and ET5 was also surprising. The participating students are used to rapid activities in a digital milieu, which may cause restlessness, something that did not seem to affect the reading negatively. A lingering hypothesis is that previous rich digital experience may have fostered an ability to skim and to use split vision in a fruitful way. Such an ability could explain that students with many fixations and a high fixation duration on the ET also had high values for the AOI on introduction and on the task. The question remains to investigate further.

In this study the combination of eye tracking data and students' reflections on their reading behaviour constitute rich data useful for in-depth analyses. The number of quantitative measures may falsely signal an intention to generalise about relations between reading behaviour and particular element types, which is not our aim. These measures are, in the current study, meaningful in relation to the interview data, because reading behaviour differs between readers and also depending on the intention with the reading. It is also important to be aware of the difference between the five items, both regarding representations and mathematical complexity. This means the reading experience may differ more regarding other aspects than the ETs between the items, and accordingly comparisons between the reading of ET1-5 must here be done very carefully. A limitation in the current study is also that the items are designed to suit the set up with quantitative analyses using eye tracking equipment and therefore the dynamics and interactivity is only found in the Theory.

Visualisations such as heat maps are often criticised for being just 'eye candy' but when used to replace read aloud protocols this kind of data is very useful because it aids the students in recalling their solution process. The interviewed students were, aided by the heat maps, able to give rich descriptions of how they experienced the items depending on element type in the Theory. A refinement of the interview guide will however be made in relation to the heat maps because the students were very prone to describe what they saw, not their thoughts when they read or reasons behind the reading pattern. The results are applicable not only in relation to digital teaching platforms but for all types of digital teaching materials in relation to the extent they incorporate dynamic and interactive elements and functions.

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Rethinking resource conceptualization in times of pandemic and earthquakes: What is important for (mathematics) education?

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The conceptualization of resources provided by Professor Jill Adler distinguishes between basic resources, human, material, and social and cultural resources. In 2020, a year of the COVID-19 pandemic and lockdowns, Croatia dealt with another huge challenge: several big earthquakes struck the Zagreb area and its surroundings. These catastrophes had an effect on education in general as well as within mathematics education. The study presented in this paper refers to a teachers' view on the stabilities and changes in basic, material and non-material resources in a period of time marked by the pandemic and earthquakes. The participants highlighted internet connection and physical interaction as important resources for (mathematics) education. The study results indicate the dynamic nature of the resources within the resource systematization; their significance changes under different circumstances.

Keywords: Resources, mathematics education, pandemic, earthquake.

Introduction

Resources surely play an important role in mathematics education. Teachers and students use and adapt a wide range of resources for teaching and learning mathematics. Although the research was focused to a large extent on material resources such as textbooks (Fan et al., 2013), non-material resources such as human resources also proved to be important in learning and implementing innovations (Black & Atkin, 1996). These various types of resources, material and non-material, raised the need for their conceptualization within mathematics education (Adler, 2000).

The research on resources has so far encompassed different cases from different countries (Trouche et al., 2019). In this paper, the focus is on the issue of how the resources for (mathematics) education change in times of natural catastrophes. The context that gave rise to this issue lies in the circumstances of the pandemic in 2020 and the several big earthquakes that struck Croatia in the same year. According to the United Nations report (United Nations, 2020), the COVID-19 pandemic "has created the largest disruption of education systems in history" (p. 2), encompassing closures of schools and other learning institutions. On 16 March 2020, the Croatian government mandated the closure of schools and introduced remote teaching with provided solutions for the use of various ICT tools (Ministry of Science and Education, 2020). These solutions ranged from television schooling to online chat groups and virtual classroom platforms (Ristić Dedić & Jokić, 2020).

Soon after this first lockdown, on 22 March 2020, Zagreb was struck by a 5.5 magnitude earthquake which damaged many buildings in the Croatian capital, including schools and faculties. It was followed by numerous after-shocks, and then on 29 December 2020 an earthquake with a magnitude of 6.6 struck Petrinja, 45 km from Zagreb, causing thousands to lose their homes and the destruction of many buildings, including schools. These circumstances meant additional changes had to be made to educational practices in Croatia, including the use of resources important for teaching and learning mathematics. For example, the school building itself is considered as a basic resource (Adler, 2000). But what if the school is destroyed or the students and teachers cannot enter it because of restrictions due to the pandemic? In these circumstances the school building loses its significance as a basic resource and is replaced by other resources, such as internet connection and computer.

The abovementioned catastrophes from 2020 raised issues on how they affected the use of resources for teaching and learning mathematics in compulsory education. The study presented in this paper therefore raises the following question: What material and non-material resources for mathematics education were used or highlighted as important by teachers in times of pandemic and earthquakes? How were they used?

Theoretical framework

Adler (2000) conceptualized resources as an important issue for mathematics teacher education, indicating that resources refer to anything likely to re-source the practice of mathematics teachers. The author categorized resources as basic, human, material, and social and cultural (Adler, 2000). Basic resources are "necessary for the maintenance of schooling" (p. 209), such as buildings, water, electricity, but also to basic teacher qualifications. Human resources refer to teachers and their knowledge-base, parents, colleagues and collegiality. Material resources include school mathematics materials (textbooks, educative software, etc.), technologies (e.g., computers, chalkboards and tablets), mathematical objects (e.g., proofs, lumber lines) and everyday objects (e.g., money, newspapers). Social and cultural resources refer to language (e.g., verbalisation, communication) and time (e.g., timetable).

This classification, which encompasses a wide spectrum of resources, from basic, to material and social, raises the question of their stability in different circumstances. Adler (2000) claims that "the effectiveness of resources for mathematical learning lies in their use, that is, in the classroom teaching and learning context" (p. 205). In this study, the aim was to find out what different kinds of resources the participating teachers used and considered important for teaching mathematics during 2020, and in what way.

Methodology

In order to examine how the use of resources for mathematics education change in times of natural catastrophes, a qualitative approach was chosen. The case study focused on compulsory education (grades 1-8) in the Zagreb area. The study involved two teachers from two schools in Zagreb, one of primary grades (in this text named as Teacher Zg1) and one of middle grades (named as Teacher Zg2). These teachers were chosen because both their schools were damaged in the Zagreb earthquake in March 2020 and they both have 25 years of teaching experience and therefore they can express and compare the phenomena of schooling before and during the new circumstances. Primary grade teachers in Croatia teach six different subjects to their class in grades 1-4, one of the subjects is mathematics (4 lessons per week). In middle school (grades 5-8), mathematics is also taught 4 lessons per week. The study encompassed a qualitative approach, using semi-structured interviews. The intention was to get a better in-depth insight into the participants' views and teaching experiences in 2020. The interview questions and outlines follow the participants' use of resources during 2020 regarding Adler's (2000) conceptualization of resources: Explain how (mathematics) education was organized in 2020 in your school. Did it undergo any changes after the earthquake? The semi-structured interview further contained outlines on basic resources (What resources were crucial for teaching in 2020, i.e. resources without which schooling wouldn't be possible); human resources (What knowledge and competencies did you need to have? What was the role of parents? What about the collegiality of teachers – did this change in any way? What about the collegiality of students?); material resources (Describe the technology issues for mathematics teaching in 2020. How did you use textbooks and other school materials?), and social and cultural resources (Describe the educational issues connected to language and communication in mathematics education in 2020. How does time management in 2020 compare to the previous period?)

The interviews were conducted during 2021. The interview transcriptions were analysed according to the qualitative content analysis methodology, using codes that are in line with Adler's categories.

Results

The findings showed different resources important for mathematics education in 2020. In this section, the results are organized according to the categories provided in the methodology section.

Basic resources

Both teachers highlighted internet connection and computers (or tablet, laptop, cell phone) as basic resources for teaching and learning in 2020. Still, there was a problem with these resources because these were not equally available to each student.

> Teacher Zg1: They (students) had to have a computer and internet at home. (...) I said, if it is possible, somehow, not to use a mobile phone, because it was difficult for them. It was no good, they couldn't see well on Zoom.
>
> Teacher Zg2: They all got tablets, this generation got them within the curricular reform process. (...) Internet... That's another story, many of them really did not have internet access so they used SIM cards provided with the tablets, which needed to be loaded. But the parents did not know how to do it ... it was difficult.

These results imply that the new circumstances may lead to changes in basic resources, showing their dynamic and changeable nature. As seen in the interview excerpts given above, the school building was no longer considered a basic resource; the key basic resources in 2020 were internet access and computers.

Human resources

Human resources (teacher, parents, cooperation with colleagues) are also found to be important educational resources in these new circumstances. Still, teachers had to possess the additional competencies necessary for remote teaching. Besides digital competencies, participants highlighted adaptability and resourcefulness.

> Teacher Zg1: Well ... digital competencies, I mean, IT literacy, resourcefulness. (It required) a different, I mean really different, approach.
>
> Teacher Zg2: I thought I'd manage it, because we mathematicians can usually handle the technology a bit better than other teachers, especially primary teachers. (...) In fact, we handled it on our own.

Both participants found support in cooperation with their colleagues, particularly in the first weeks of the lockdown.

> Teacher Zg1: When we got the announcement about online schooling, we had a meeting in school. Our IT teacher asked if anyone wanted to create a class web page. We all (colleagues) looked at each other, and she showed us how it should look. (...) We were given support (by the IT teacher) and really we managed the technical issues together at first. After that, we managed at home on our own.

Also, teachers' informal groups were helpful, such as Facebook or WhatsApp target groups.

> Teacher Zg1: I must say that we were on Facebook, I was in the teachers group (...) teachers were giving video lectures on how to work. These were teachers who had already had experience with some projects. But that was own-initiative, not connected to the Ministry. (...) About the Ministry, here, I can't say that we got any support.
>
> Teacher Zg2: Besides being left school-less, we also didn't have anywhere we could actually meet, and that was not ideal. But the solution was our joint WhatsApp group... where we asked each other things...

Assistance from human resources could be therefore divided into two parts by both participants: the first approximately two weeks, when they strongly relied on colleagues and IT teachers, and the period after that, when the participants mainly used their own resources and learned on their own. In the first time period, both participants used non-simultaneous teaching, using videos and other materials that they had prepared or found. After that, both participants realized that lessons in real time with students were a better solution. These findings highlight the dynamic nature of the resources used over time.

Parents are also important human resources in the process of education, and the cooperation of parents, teachers and students is found to be crucial in these new circumstances.

> Teacher Zg1: Yes, (this cooperation) was more intensive in the first one to two weeks, until the parents figured it out. They were very upset. (...) Until we overcame the technical issues, we were in touch really a couple times a day.

Unlike the primary teacher, the middle school participant complained that the remote teaching decreased the quality of communication with parents.

> Teacher Zg2: Parents' engagement was surely greater (than before the lockdown) ... but for me the bigger problem was that the online teaching significantly decreased the quality of communication with parents. (...) I found that really hard, and it continued into the next school year. Well, I didn't see the parents for such a long time, we talked over the phone, but that was not good. Everything was so distant...

The results on human resources presented in the study highlighted not only the dynamic nature of resources over time (the use of different resources before and after the lockdown), but also the dynamics across the resources. For example, the change in basic resources (internet access and computers in remote teaching) led to changes in human resources (e.g., the need for digital competence in teachers and parents, and different types of communication between teachers, students and parents).

Material resources

Material resources, as perhaps the most visible type of resource for teaching and learning mathematics, also underwent changes. As computers and internet access became basic resources, school mathematics materials mainly turned to ICT, making the importance of teachers' digital competence even more apparent. The results given within human resources showed that in the first two weeks the participants tried to provide teaching which is not given in real time. In this period, different material resources were used (videos, educational web pages with various educational links and texts). In the second phase, the participants realized that e-teaching in real time was better for them and for their students.

Teacher Zg2: During the first days I was personally preparing materials which were sent to pupils. These were textual materials, mostly with links to various videos ... I did not make videos. I know that many of my colleagues did, but I thought that there was no need to because there are so many existing materials. (...) But, afterwards, with time, I moved on to live meetings... I have a laptop with a touch screen (...) From that moment it was like I was reborn.

Teacher Zg1: Firstly, I used to record (videos) until 1 or 2 in the morning... recording my voice with the presentation simultaneously. But then I realized that it's better to make a presentation, long division and mathematical content that they find difficult ... to make a presentation with animations and then to go through it together with the pupils.

During the second phase, the participants also used e-textbooks, presentation tools, quiz applications and other educational software.

Teacher Zg1: And Wordwall... every day I gave them a task at the end of the lesson, a quiz. (...) We also used Matific, it's great, and it proved to be very good for us. We promptly had access to the results and the ranking list.

Teacher Zg2: Platforms provided by Profil (the publisher), online textbook, GeoGebra of course, because there is a lot of material provided in GeoGebra ... (...) Online textbooks helped a lot.

These two phases highlight both of the aspects of the dynamic nature of resources: over time and across the resource types (different phases required different human and material resources).

Even in these new circumstances, it can be said that the participants tried to conduct online lessons which are structurally similar to their lessons before the lockdown. Zoom lessons in real time encompassed following the teacher's actions via the screen, and the students had to use their tangible textbooks, notebooks and pencils, as they would do in the classroom. Also, they used everyday objects for learning mathematics.

Teacher Zg1: They (students) always readily had all materials, manipulatives, textbooks, as well as notebooks. (...) Children had to prepare for mathematics lessons, we set that up, some candies or pencils, so that they can do division. In that way, we started from concrete objects and did a lesson motivation. We improvised and tried to make them to come to new knowledge through discovery.

Teacher Zg2: They used their notebooks a lot. (...) They had their textbooks available, and I would open the online textbook, and I sketched on it if necessary, some illustrations, some drawings, especially if we were doing geometry.

The Ministry of Science and Education provided TV-school every day on national TV and on two other TV stations. In this way, the lessons for each subject and each grade 1-12 were covered. These lessons were recorded and could be watched at any time on the Ministry website. Nevertheless, the two participants in this study did not utilise them in their teaching.

Teacher Zg1: But mainly the mathematics (provided by TV lessons) did not correspond to my plan and programme. Therefore, in fact, I did everything myself.

The other participant did not mention TV schooling at all.

Social and cultural resources

Remote teaching also brought modifications in the usage and importance of some non-material resources for teaching and learning mathematics, such as time or verbalization and communication. When it comes to communication, remote teaching made clear the difference between uni-directional and bi-directional communication with students. Uni-directional communication was shown through the web pages on which participants put links and materials for their students. Bi-directional communication with students was conducted through WhatsApp groups, MS Teams or Zoom live meetings. Both participants highlighted the role of bi-directional communication as an important resource for learning. They stressed calling on students to answer a question or make a response as being particularly important in mathematics education.

> Teacher Zg1: So that each pupil participates somehow, so that I can call on each of them, even if I only saw them on the screen.
>
> Teacher Zg2: So, in this way (with Zoom meetings) actually, I think, we "saved" the school year. I would ask a student to tell me what I should write down (on the touch screen).

Remote teaching implies students separated from their teachers, in recent times mainly using computers. Still, the participants highlighted direct student-teacher contact as the non-material resource which they missed the most. For them, it was important for social, psychological and pedagogical reasons, particularly in the early grades.

> Teacher Zg1: Absolutely I missed it (direct contact). Touch as well, teamwork ... all that interaction was somehow lost in the digital environment. (...) Exactly, the contact with children. Well, education is a live process. (...) Nothing can compare with live contact teaching, no way. In other subjects, too, but in mathematics especially. Because that contact with children ... to see in their eyes if they understand or not, discovery learning, I think that it can't be realized that much through Zoom on the screen. And somehow, they can't concentrate so well in that way.

Language is also an important resource for mathematics education. Teacher Zg2 claimed that in particular textual tasks were neglected in her remote teaching in 2020.

> Teacher Zg2: In online education, I could not include many textual tasks, because the students did not manage to master even the simple routine tasks. Because everything was different, slow... a lot of waiting for a connection (...) everything was slower.

This quote brings us to another important non-material resource: time. The participants complained that remote teaching takes a lot of time, both for preparing lessons and during lessons.

> Teacher Zg1: Since we (teachers) were not skilled, we always needed more time. I would, for example, record a video lesson 20 times, and I still wasn't satisfied with it. (...) It took a lot of time, and I think it was not as effective as the live teaching.
>
> Teacher Zg2: Still, searching for a suitable video or illustration took a lot of time. So, in the first days I was sitting in front of the computer from the morning until the evening...
>
> Teacher Zg1: Everything (during the lesson) was so slow, the tempo was much slower. (...) The students did not manage to do the tasks. Exercises, revisions – these parts we missed the most. We needed more time because everything was so much slower.

These quotes also reveal the dynamics and interconnectedness of the different types of resources: as the basic and material resources switched to ICT, and teachers, parents and students were "not skilled" (human resources), the whole process of education needed more time (socio-cultural resource).

Interestingly, it seems that the study highlighted another important non-material resource in remote teaching – pedagogic intentions during the lesson. Pedagogic intentions (such as discovery learning, teamwork, calling someone out to the board) were a challenge in remote teaching.

> Teacher Zg1: We couldn't manage the teamwork. (...) And so, it was not comparable with contact teaching when everyone has it on their desks ... when I see them all and have complete control, I can lead them through their discovery learning.

Discussion and conclusions

The study presented in this paper showed the dynamic nature of the resources within their systematization (Adler, 2000), because their significance changed under different circumstances, particularly in terms of remote teaching. The school building was no longer considered to be a basic resource; the most important resources for the maintenance of schooling in 2020 were internet access and computers. Teachers' knowledge-base (human resource), including new digital skills and resourcefulness, were highlighted as very important by both participants. Material resources were mainly leaning on IT (internet mathematics materials, videos, e-textbooks, educational software, Zoom lessons in real-time, etc.). Social and cultural resources highlighted by the participants were bi-directional communication and time. The two teachers most missed some important pedagogical intentions, such as discovery learning and teamwork.

Although recent research has focused on the teachers' experiences from contact to remote teaching (e.g., Azari & Fajiri, 2021; Mullen et al., 2021), the study presented in this paper considers this transfer from the standpoint of resources as the focus. Such a view revealed that the dynamic nature of resources was manifested in two aspects: dynamics over time, and dynamics across the different types of resources. Dynamics over time refers to the use of different resources before and during remote teaching, within the categories introduced by Adler (2000). Dynamics across the resource types refers to how the changes necessitated in the basic resources brought about changes in the other resources, i.e. the human, material and socio-cultural resources. Adler (2000) highlights that "mathematics teacher education needs to focus more attention on resources, on what they are and how they work as an extension of the teacher in school mathematics practice" (p. 205). This study may bring new insights into the individual teachers' experiences with resources in times of pandemic and earthquakes.

The findings showed that the participants tried to align remote teaching to their "old habits" in teaching, trying to keep the teaching structure and resources as they were before the lockdown. This result indicates that the transition from face-to-face contact to effective remote teaching is a sensitive process that requires more attention and teacher education. Mullen et al. (2021) highlight "that this process does not always respect the transformational nature of the transition, and that the recontextualized use of essentially unaltered traditional pedagogies might not be the best response" (p. 4). It particularly stresses mathematics teaching because it requires corrective feedback, multi-modality, and specific devices suitable for writing mathematics (Mullen et al., 2021). On this issue, general and specific support from authorities is very important because "teachers need competence in managing learning that is different from before the COVID-19 pandemic" (Azhari & Fajiri, 2021, p. 13).

The study presented in this paper is a qualitative study, which had no intention of generalizations. However, future research may encompass a larger quantitative study with more participants from the Zagreb area. The comparison of findings in Zagreb and other regions or countries may contribute to isolating the issues related particularly to using resources in times of earthquakes, unrelated to the pandemic. Further, Adler's (2000) conceptualization of resources refers to the perspective of mathematics teacher education. It would be interesting to look at the students' perspective in using resources as well (Glasnović Gracin & Jukić Matić, 2021).

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Investigating curriculum resources and mathematics knowledge for teaching in teacher planning discussions

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Keywords: Mathematics teachers, curriculum resources, MKT.

The research topic

This poster reports on the design of a study that will be conducted in the fall of 2021, with the aim to create more knowledge on in-service mathematics teachers' use of curriculum resources. Our research questions are: what types of resources are used, in what way, and for what reasons? The data in the study consist of audio-recorded collaborative teacher planning discussions in Sweden where, just like in other countries, new teaching resources (digital and analogue) have made their way into practice the last decade. It is of interest to explore how these resources are used together with existing resources, particularly as the Swedish curriculum offers and leaves teachers with large freedom in choosing their teaching resources.

The Design Capacity for Enactment (DCE) framework (Brown, 2009) describing resources use and Mathematics Knowledge for Teaching (MKT) (Ball et al., 2008) will tentatively be the frameworks used for analyzing these discussions.

Theoretical background

Teachers participate in several different communities of practice (Lave & Wenger, 1991). One such community can be collaborative planning, where teachers together discuss matters of teaching in-between enacting lessons, either casually or more organized with specific common goals.

While planning lessons (individually or collaboratively) teachers draw upon different curriculum resources as well as their experience. When it comes to teachers' use of these resources in the practice of planning for and enacting teaching, Brown (2009) has proposed the DCE framework that connects curriculum and teacher resources, as well as different types of use, such as offloading, adapting, or improvising.

Brown's model emphasizes different types of resources. Each resource can incorporate different aspects relevant for teaching mathematics. These aspects can be labeled using the teacher knowledge domains in the MKT framework (Ball et al., 2008): *Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), Horizon Knowledge (HK), Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT)* and *Knowledge of Content and Curriculum (KCC).*

In short, CCK can be described as the mathematical knowledge needed to teach, for example correct mathematical concepts and language, or to know if a calculation or statement is correct or not. SCK helps to connect, deepen and analyze different mathematical concepts, used in teaching. KCS relates to students through, for example, knowledge on common mistakes or thinking in relation to a mathematical content. KCS can help identify student thinking from spotting a mistake, and KCT in its turn, helps in choosing a suitable response to the student, or to guide the teaching of that content in general. HK is about knowledge on how mathematical concepts are related and connected. Parts of HK are closely intertwined with KCC, which corresponds to knowledge about the conditions for teaching mathematics in school, such as how the curriculum defines the mathematical progression through its sequencing, or the subjects' connections to other subjects.

Method

A previous project examining mathematics teachers' communities of practice, has generated recorded data from upper-secondary school mathematics teacher discussions. Around ten of these discussions from three different teacher groups with about five teachers in each, ranging upwards 60 minutes in length each, concern the planning of lessons covering a variance of upper-secondary school mathematical content. These planning discussions will be transcribed and analyzed through content analysis, with a deductive approach. The analysis of the discussions aims to describe the practice of teachers in terms of types of knowledge used within this planning context, using the domains of the MKT framework (Ball et al., 2008) for the description of types of knowledge and aspects of Browns' DCE framework (Brown, 2009) for the types of resource used.

Possible results and implications

Based on a preliminary analysis, an asymmetry regarding what domains of the MKT framework that are being visited during these planning discussions, is expected to be found. Further, an asymmetry is expected to be found in the types of curriculum resources that are being used. There is also a possibility to investigate whether certain types of resources associates with certain domains of MKT. This can help to create new knowledge on what kind of support teachers need from curriculum resources, both in content and style. The poster will set out the design for the study and following the analysis conducted during the fall 2021, some of the results will be possible to include at the CERME12 conference.

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Online feedback designed to support self-reflection while solving fraction example-eliciting tasks

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Feedback as a personal metacognitive tool can be effective only if learners respond to it by self-reflection on their learning process. A central challenge of the online assessment platforms is to develop ways that guide students to self-reflection. We report on a study involving a dedicated design pattern of a pair of complementary example-eliciting tasks designed based on the logic-of-yes and the logic-of-not principles. Each task, presented as an interactive diagram, requires constructing examples of fractions and classifying them using a given set of characteristics. The set was given to students in a personalized report and used to automatically analyze their work. We explored whether and how the design supports 5th-grade students' learning the topic of fractions using online resources. We analyzed task-based protocols of one pair of tasks, aiming to identify connections between the design patterns and students' self-reflection, performance, and metacognitive skills.

Keywords: Feedback, self-reflection, example-eliciting tasks, metacognitive, task design.

Introduction and theoretical background

Feedback is considered as one of the most powerful ways of supporting learning processes (Hattie & Timperley, 2007) and developing metacognitive skills, but it should not an effective tool if students fail to consider it or do not engage with the information they receive. Shute (2008) pointed out that elaborated feedback, which addresses the topic, response, particular errors, and examples, and provides guidance toward a correct answer, appears to enhance students' learning more than other types of feedback. Shute's review indicated that presenting too much information may result in superficial learning and invoke cognitive overload. Even a comprehensive design of assessment and how it is being reported cannot support the learner or lead to self-assessment unless metacognitive reflection skills of planning, monitoring, and evaluating follow (Ruchniewicz & Barzel, 2019). In this study, we distinguish between two meanings of feedback. Feedback as an object, which refers to the contents of feedback itself (the information), and the feedback process, which describes the student's interactions with the task and the feedback information.

Ruchniewicz and Barzel (2019) investigated feedback as part of autonomous learning and explained how it turns into a process, in which learners make sense of information about their work to improve learning strategies, metacognitive skills, and future performance. Following their study, we designed tasks that require constructing examples with an interactive diagram and characterizing the construction using a given set of statements that the student is asked to reflect upon as part of solving the task with feedback. In our study, the feedback was automatic, and the set of statements used to automatically analyze the submitted constructions appeared in the personalized reports given to students. Arzarello and Sabena (2011) proposed two types of inquiry logics to be considered when designing tasks to help students in inquiry-based knowledge acquisition: the "logic of yes" (LoY), which leads students to empirically test their examples, and the "logic of not" (LoN), guiding students to an indirect validation of an example by attempting to argue the impossibility to find a counterexample. These designs should lead students to explore, observe facts, and ask and answer questions to discover connections between them (Arzarello & Sabena, 2011). Following these inquiry logics, students in our study were requested to reflect on their examples before submitting them.

Fractions are a central topic in the mathematical curriculum. Learning the concept of fractions poses a significant challenge to students and has been for a long time a focus of research of the mathematics education community. Arnon, Nesher, and Nirenburg (2001) studied and reported on fifth graders who learned with the Shemesh software, which was designed to promote conceptual learning of equivalent fractions, offering concrete representations of the fraction and the operations performed on it. The current study, based on a similar representation, uses the Seeing the Entire Picture (STEP) platform. STEP is a formative assessment environment that involves the student with automatic information analysis of rich example-eliciting tasks (EETs). EETs require students to formulate an example and present it by submitting one or several constructions based on an interactive diagram.

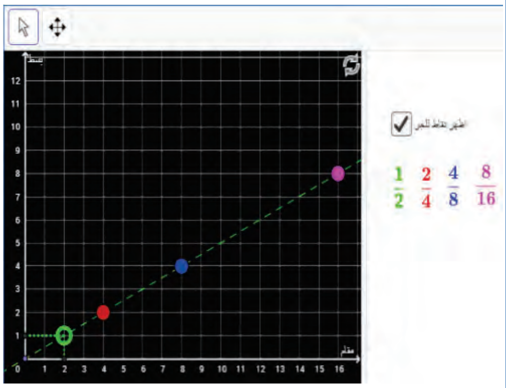
Recently, online platforms have been offering ways to obtain rich and varied evidence that enriches learning assessment. Traditionally, automatic assessment provided judgmental reaction, verified the correctness of information, and often suggested tasks for students to perform to achieve better results (Scalise & Gifford, 2006). With STEP, the automatic analysis generates information about the correctness of the task, but its main objective is to analyze the work according to other mathematical and pedagogical characteristics, which were defined when the task was designed and were individualized to report back to the student. These include methods of work, misconceptions or common mistakes, and the diversity of the example space (Olsher, Yerushalmy, & Chazan, 2016). The platform automatically points out whether or not the characteristics are present in the student's submissions. We refer to the interactions of the student with the activity – the task requirements and the automatic report – as the "online personalized feedback" process. The novelty of this study lies in the interaction of the online personalized feedback process with the task design to lead students to self-reflection and examine their performance in three areas: (a) the correctness of the answer, indicating whether the example meets task requirements and conditions; (b) the metacognitive skills: planning problem-solving goals, strategies, and allocating resources; monitoring what has been achieved to set goals of comprehension and performance; and evaluating the assessment of given problems or understanding of mathematical concepts, for example, through self-reflection (Ruchniewicz & Barzel, 2019); and (c) the richness of the personal example space, as the collection of examples to which an individual has access at any moment (Liz, Dreyfus, Mason, Tsamir, Watson, & Zaslavsky, 2006, p. 133).

We ask whether the interaction of the online personalized feedback process with the task design results in self-reflection and better performance on the part of the students. We studied the potential of task design that explicitly requires students to conduct a reflection on their work. This reflection should be made explicit by analyzing (a) student`s checking and marking all the useful listed characteristics that they have encountered already in the previous automatic report; and (b) student`s submitting of answers that meet as many (logic-of-yes) or as few (logic-of-not) possible characteristics while working on two consecutive EETs on the topic of fractions. Our research question was: Is the special design of a complementary pair of EETs reflected in the students' self-reflection, performance, and metacognitive skills, and if yes how?

Methodology

Research setting. To explore the effect of task design on student performance, we designed tasks for which the feedback process is part of student self-reflection, and conducted an empirical study of the way students engage with such tasks. Two students, aged 12 years, participated in this study. They shared one computer while solving the tasks. In the instructions we provided to student before they start working on the tasks, we encouraged conversation and a think-aloud process. We transcribed the students' segments of discussion and analyzed them qualitatively to find evidence of connection between the engagement with the task design principles and the students' self-reflection, performance, and metacognitive skills. The students had learned fractions according to the national curriculum in a regular classroom. They were presented with a sequence activity of four tasks and given 45 minutes to complete these tasks delivered on STEP. Following each task, five sets of mathematical characteristics were used to automatically analyze the work, and the feedback was given to the students as automatic report. For the second and third tasks, students were asked to characterize their example using this set of characteristics.

Construction part



Automatic report

The requirement of the task

1.The fractions that you chose are equivalent X

The characteristics of the submission

2.The visual representation line crosses all points at the same time

3.You chose a fraction that is less than one

4.One fraction is an expansion or reduction of the green fraction

5. The numerator and the denominator of one fraction are larger additively by the same number than the numerator and the denominator of the other fractions

Figure 1: Diagram for the first task **Figure 2: Automatic report for the example on the left**

Diagnostic and summative tasks. The interactive diagram (Figure 1) that the student used to solve each task displayed points in the coordinate system representing a fraction. The numerator was represented by the number that appears on the vertical axis, and the denominator by the number that appears on the horizontal axis. Students were asked to choose a fraction by dragging first the green point, then the other three points to represent equivalent fractions to the one chosen. The requirement was to construct three examples as different as possible of equivalent fractions. The criteria of difference between the submissions were based on a comparison of the characteristics of each. When all fractions were equivalent to the fraction that the student chose, a line demonstrating the "equivalent fractions" appeared automatically. The diagnostic task (Task 1) and the summative task (Task 4) were identical and were assigned to evaluate the effect of the personal report and the work on Tasks 2 and 3. Both were intended to assess the characteristics of the students' submissions when they created equivalent fractions. The criteria for assessing changes between the diagnostic and summative tasks were based on a comparison of the automatically assessed characteristics, assuming that the change in the example space indicates changes in the students' concept of equivalent fractions.

Online personal feedback process refers to the interactions of the student with the task requirements and the automatic report. The report was automatically generated in response to submissions. It included five characteristics (Figure 2), one corrective characteristic (characteristic 1) and four characteristics of the submission labeled 2-5: characteristic 2 had the potential to explain the visual representation of the examples; characteristic 3 had the potential to explain the mathematical characteristic of the example; characteristic 4 had the potential to explain the method for finding equivalent fractions; and characteristic 5 had the potential to explain misconceptions or common mistakes in the course of finding equivalent fractions. Figure 2 shows the analysis of a correct submission (characteristic 1) which indicated whether the student's submission was answering the requirement of the task (green checkmark) or not (red X sign), three mathematical characteristics that were identified by STEP in the students' submission, these were automatically highlighted in yellow (characteristic 2, 3, 4) and one that was not present in the example (characteristic 5). These characteristics were designed to reflect the curricular foci of the task and the basis for the student's engagement and self-reflection process in the LoY and the LoN tasks.

LoY and LoN intermediate tasks (Figure 3). To help students self-assess their performance while reflecting on and regulating their learning, the task was designed to include a construction and a reflection part. The construction part required students to choose two equivalent fractions by dragging two points. The reflection part required students to characterize the construction using a given set of characteristics. This set had been used to automatically analyze the work and was given to the student as a personalized report. STEP enables students to compare their self-reflection with the automatic report. Each task placed different constraints on the statements that should be reflected in the construction, indicating either a LoY or a LoN. The LoY (Task 2) was formulated as follows: "Choose two equivalent fractions by dragging the red and the green points. To the right of the interactive diagram, five statements can help you characterize the two fractions you have created. Check each characteristic that is present in your example before submitting it. Try to submit two fractions in a way that your submission should comply with as many statements as possible. Submit three such examples, as different as possible from each other." The maximum number of characteristics that the students could check was four because characteristics 3 and 5 were mutually exclusive. In the LoN task (Task 3), the students were asked to submit examples that comply with as few statements as possible, the minimum number of characteristics being two (characteristics 1 and 2). In these tasks the representation line did not appear automatically. It was a tool that students could choose to use or not, when the fractions were not equivalent there would appear two distinct lines- green and red, the lines were united when the fractions were equivalent. The using of the representation line could help students connected it with the correctness of their answers.

Data sources and analysis. The data needed to answer the research question were based on the automatic information analysis by STEP, observations of students working together on the tasks, their initial submission, the processing of the automatic feedback, and their final submission. Their discussion was video recorded. We analyzed the students' interactions while working on the tasks and their response to the online personalized feedback process. After the discussion was transcribed, we analyzed the segments and correlated the analysis with the STEP data to find an indication of the principles of task design being reflected in their work and meta-cognitive skills.

Results and preliminary remarks

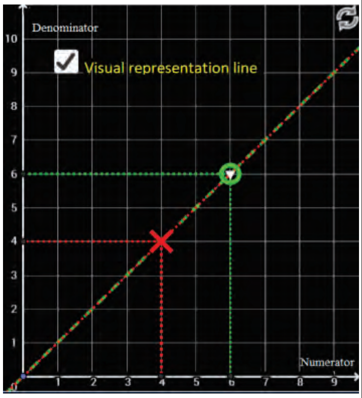
Construction part	Reflection part	Automatic report
$\frac{6}{6}$ $\frac{4}{4}$ 		**The requirement of the task**
	☑ 1. The fractions that you chose are equivalent	1.The fractions that you chose are equivalent ✓
		The characteristics of the submission
	☑ 2. The visual representation line crosses all points at the same time	2.The visual representation line crosses all points at the same time
	☐ 3. You chose a fraction that is less than one	3.You chose a fraction that is less than one
	☑ 4. One fraction is an expansion or reduction of the green fraction	4.One fraction is an expansion or reduction of the green fraction
	☑ 5. The numerator and the denominator of one fraction are larger additively by the same number than the numerator and the denominator of the other fraction	5. The numerator and the denominator of one fraction are larger additively by the same number than the numerator and the denominator of the other fraction

Figure 3: Construction, reflection and report for the example 4/4=6/6

Below we present empirical evidence of the students' self-reflection process as they were working on the tasks. Limar and Sana worked on the first task. Limar chose fractions that are equal to one (2/2=3/3=4/4=5/5), (9/9=8/8=7/7=6/6), (6/6=8/8=9/9=10/10). Limar noticed that the presentation line that appeared indicated that their answers were correct. For each example, the STEP automatic report indicated characteristics 1, 2, and 5. The students were interested only in the correctness of their submissions (especially Limar, who led the working on this task. The pair quickly moved to the LoY task. Limar chose the fractions: (2/2=4/4), (4/4=8/8), (4/4=6/6), and checkmarked statements 1, 2, 4, and 5 for the first submission (2/2=4/4) after reading every statement. The students chose the same characteristics for the second and the third submissions, without reading the statements. They evidently thought that having chosen fractions that were all equal to one, the submissions should have the same characteristics. After receiving the report, they examined the statements they had checkmarked.

26	Sana:	We chose all fractions equal to one. Why in this submission (means 4/4=6/6, Figure 3) statement 4 [which they had checkmarked] is not highlighted [by the online feedback]? [Reading statement 4] OK, fractions should be an expansion or reduction of one of the other; in this submission there is no integer that we multiply by 4 and we get 6. We shouldn't have checked this statement.

The students compared their checkmarked statements in the reflection part of the LoY task with the highlighted statements in the online feedback because they noticed that the online feedback did not match their self-reflection, which made them check their answer again and changed their mind about the answers. Because they were interested only in the correctness as indicated in the feedback and skipped the other characteristics, they did not notice that the statements in the online feedback were the same statements that they had received for the LoY task, otherwise they may have been more specific while checking their statements. Figure 3 shows an example of the students vs. submission and the corresponding online feedback.

When the students started to work on the LoN task, Limar chose the fraction (3/3=6/6). Segment 29-34 shows their working on the LoN task.

29	Sana:	But if we do as we did in the previous task [she drags the points to 10/10=7/7], this will give us three characteristics, and we want as few statements as possible. Here, there is no expansion or reduction, there is no integer that we multiply by 7 and we get 10. [Checkmarks statements 1, 2, and 5 and submits]. The question is, can we choose an example that has fewer than three? [Meaning checkmarked statements]. [Drags the green point and chooses 1/2] This is half, this is less than one [points to statement 3], it's not good.
30	Limar:	Yes, it’s not good. Try to do 2/1 [dragging the green point]. This is equal to the number 2. But how can we find a fraction that is not an expansion or reduction to this fraction? Usually, when the fraction is not equal to one, I do expansion to find the equivalent fraction.
31	Sana:	Wait [writes in the notebook] 2/1 expanding by 2 is equal to 4/2, if we expand again by 2, we get 8/4. No, it’ll always be an expansion because when we expand by 2 we get an even number. Maybe we should expand by an odd number, 3 or 5 for example?
32	Limar:	[Writes in her notebook (2x3)/(1x3)=6/3 and (2x5)/(1x5)=10/5)].
33	Sana:	[Points at the fraction that she wrote in the notebook and the fractions that Limar wrote] Yes, here are two fractions 6/3=4/2. There is no integer number that we multiply by 4 and get 6. The same with 4 and 10.
34	Limar:	Let’s take 6/3=4/2 [They check the presentation line that the two points are there]. Great! We have only two checkmarked characteristics.

The students tried to have as few characteristics as possible, the fewest being two, characteristics 1 and 2. They had to check the statements and to find examples that met the requirements of the LoN task. It was clear to them that they should *not* choose a fraction that was less than one. They used the expansion to find equivalent fractions, and used even and odd numbers to find fractions that are not expansions or reductions. They succeed in finding such fractions, although their explanation was not entirely correct. For every example the students used the representation line to check whether it crossed the points because it indicated for them the correctness of the answer. The students then went on to the last task, and Limar chose (12/12=13/13=16/16=19/19) (segment 36-37).

36	Sana:	Let’s choose also fractions that are not equal to one. [Drags the green point to 1/2, expanding by 2, 3, and 4 in her notebook and dragging the points to 1/2=2/4=3/6=4/8, then she submits].
37	Limar:	[Laughing] I want to choose easy fractions. Here we shouldn’t checkmark any statements, for me, the main thing is the correctness of the answer [drags and chooses 20/20=21/21=23/23=25/25 and submits].

Sana tried to create a new type of fraction that they did not have in previous submissions. This fraction related to statement 3, which they were sure not to checkmark in the LoN task. Despite Sana’s suggestion to choose fractions that are not equal to one, Limar chose to submit fractions that equal precisely one, as she did in the first and second tasks. She justified her choice by having to meet the demands of the task. She decided to choose easy correct answers, especially given that the task did not require checkmarking any statements, as she stated. Thus, the statements in the LoY and the LoN tasks served as a tool for self-reflection, otherwise the mathematical discourse about the tasks concerned only the correctness of the submissions. For Limar, 20/20=21/21=23/23=25/25 and 12/12=13/13=16/16=19/19 were correct answers, and she may have seen them as different examples although both are equal to one.

Discussion

In this study, we designed a pair of complementary example-eliciting tasks following the LoY and LoN principles. Each task was given as an interactive diagram and required constructing examples of fractions and characterizing the construction using a given set of characteristics. The set was used to automatically analyze the work and was given to the student as a personalized feedback. We explored whether and how the tasks design elicited the students' metacognitive skills and led to learning the topic of fractions.

The empirical results show aspects of work on these specially designed tasks and suggest that self-reflection has occurred in response to these designed features. Three skills of metacognition were apparent in the work on the tasks and on the online feedback: *planning*, *monitoring*, and *evaluating.* **Planning**. The students planned their chosen type of fraction in response to the requirements of the LoY and LoN tasks, which motivated the student think not only about the correctness (as they did in the first task), but also about other mathematical characteristics in their examples, or to identify characteristics that should not be present in their submissions. The students chose fractions that were "not easy" for them to choose, and generated more varied examples in their submissions than they did in the first task (unlike in the last task where they paid little attention to the feedback because they were sure that they could answer correctly). Thus, task design helped students enrich their example space. To solve the LoN task, the students discovered a new strategy for finding two fractions that were not expansions and shared their previous knowledge about the concept of even and odd numbers regarding statement 5 in the task. **Monitoring.** The students used the online feedback of the platform to compare their self-reflection in the LoY and the LoN tasks. The option of making comparison part of the task design enabled students to identify features that characterized their answers, and they tried to understand the meaning of every characteristic and the relation between them. One of the relations was the connection between the correctness and the representation line; the students chose their examples, then checked their correctness based on the representation line. All this led them to **evaluate** their learning process, rethink their choices and concept image, and change them. The task design also led the students to modify and adjust their learning process.

The findings are consistent with the literature that found a large potential for task design that combines feedback with the students' metacognitive reflection and self-assessment in advancing their performance and metacognitive skills (Ruchniewicz & Barzel, 2019, p. 55; Carless, Bridges, Chan, & Glofcheski, 2017). This is especially true for designs based on LoY and LoN principles (Arzarello & Sabena, 2011): LoY activates the students' inductive approach by finding an example that supports the statements, whereas LoN energizes students' thoughts as they seek a counterexample to the statements (Soldano, 2017). We combined the two logic principles with statements from an automatic feedback that students received in response to their submissions. This task design developed students' metacognitive skills, and enabled them to engage with the online personalized feedback process and compare their self-reflection with automatic feedback, leading to a change in their performance. These findings may serve as a basis for further research in the field of task design and self-reflection. The study was limited by the small number of participants and should be reproduced with larger groups.

Acknowledgment

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Mapping student real-time viewing of dynamic textbooks to their utilization schemes of questioning devices

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Keywords: Real-time textbook viewing data, questioning devices in textbooks, utilization schemes, undergraduate students, knowledge graphs, mathematics textbooks.

We seek to map real-time viewing data of dynamic textbooks with student utilization schemes of questioning devices, an interactive feature that collects student responses to questions that promote reflection. An instrumental approach (Rabardel, 2021) suggests that knowledge of students' utilization schemes can provide useful information to textbook designers and authors (Quiroz et al., 2021). This information supports instructors' reflection on how students utilize textbook features designed to engage them with the content, so that they can adapt instruction. However, identifying utilization schemes requires intensive qualitative analysis of data collected via observations, interviews, or students' self-reports. As these analyses are time consuming, finding alternative ways of reliably identifying utilization schemes in a short period of time when many students are using a particular feature is beneficial. We present exploratory work that uses real-time viewing data of features in dynamic textbooks to identify viewing patterns that could be reliably mapped to student utilization schemes, possibly automatically. The textbooks we study are written in PreTeXt (https://pretextbook.org/) a markup language that precisely identifies every element in a textbook; when opened in a browser, it is possible to collect student responses to questioning devices and information about when individual users view specific textbook elements and for how long. These textbooks allowed us to gather large amounts of data from many students (over 400) who used the same textbooks across many states. The textbooks are used in calculus, linear algebra, or abstract algebra courses.

We provide here only a brief description of how we used knowledge graphs (Hamilton, et al., 2017) with tracking data (date and time of viewing, the sections that were viewed, and the total time in minutes spent viewing the section) to map viewing patterns of questioning devices to their utilization schemes using about 4,000 viewings of questioning devices collected over five semesters from 492 consenting students from 27 teachers in the U.S. Knowledge graphs are graphical representations of information that capture interactions between individuals and entities. The viewing data are represented in a network of nodes and edges to establish the occurrence of consistent and repetitive viewing behavior across the same student or several unique students. Each node contains the name of the section viewed and the amount of time spent in minutes viewing that section. Directed edges denote a jump to the next section, undirected edges denote possible concurrent viewing sections within a minute of viewing during the viewing period. Each knowledge graph is unique to each day of viewing for a particular student. An algorithm created to automate this process generates combinations of nodes and edges to create unique knowledge graphs facilitating the identification of common patterns that occur while viewing the textbook. The evidence needed to support our study is hypothesized to be of three types (1) Knowledge graphs for different students following the same utilization scheme should be consistent; (2) Knowledge graphs for the same student following the same utilization scheme should be consistent; and (3) Each utilization scheme should lead to a network of knowledge graphs representative of a community of unique students with similar viewing patterns. We used the three utilization schemes identified by Quiroz et al. (2022): Familiarizing with Content Before Class (US1), Studying by Practicing and Doing Homework (US2), and Self-Evaluating and Understanding of Content (US3). With this technique we were able to identify three types of knowledge graphs with unique features that map them to each utilization scheme. We show a case, the first network of knowledge graphs represented in Figure 1, which always follows a linear viewing trajectory. The student views the sections in the textbook in the order they appear. This type of knowledge graph was mapped to the first utilization scheme, familiarizing with content. The other two networks of knowledge graphs were mapped to the other two utilization schemes (not shown for space reasons). The accuracy of the algorithm is high, 91%,86%, and 89% for US1, US2 and US3 respectively.

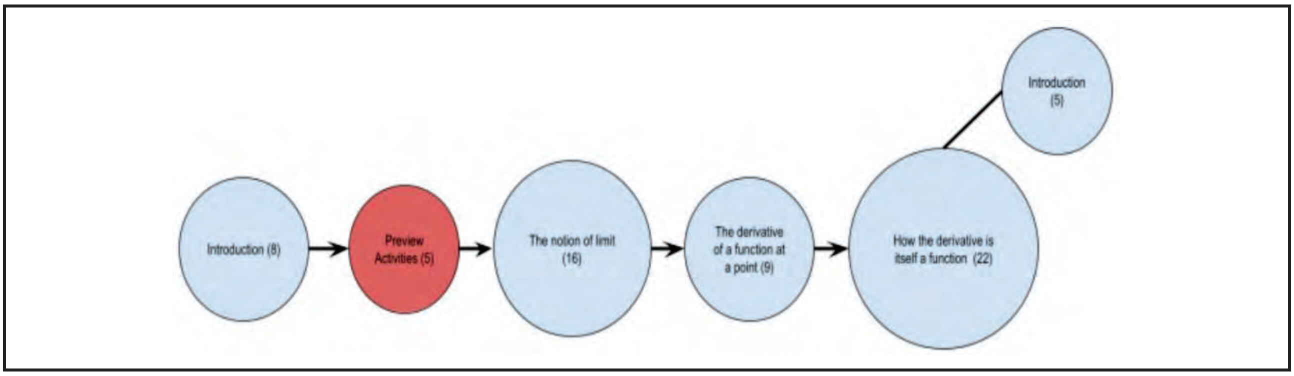

Figure 1: Representative knowledge graphs for viewing data mapped to US1: Calculus student 50 (Teacher 35); 29% of students, 91% accuracy

We will study the nature of the networks for the 18% of unmapped viewings, to see what they reveal about use of questioning devices. This problem mostly occurs when students view the textbook multiple times on the same day, deploying different utilization schemes each time, and thus the algorithm has difficulty making the classification.

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Towards a coherent mathematical story: epistemologically potent problems

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Keywords: Mathematical story, coherence.

Research topic

Instructional coherence has been identified as a significant factor that can improve students' conceptual understanding of mathematics (Hiebert et al., 2003; Chen and Li, 2010). It allows students to make connections among the ideas that are presented, see causal and logical linksbetween them and sustain interest and engagement in a topic (Chen and Li, 2010; Richman et al., 2019; Dietiker et al., 2020).The concept of instructional coherence has been linked to literary stories by noting that good lessons of mathematics should possess features of good stories (Chen and Li, 2010). Dietiker and her colleagues have identified that if mathematics teaching is interpreted as a literary story, coherence of the story does contribute to the perceived interestingness of it and engagement with it (Richman et al., 2019; Dietiker et al., 2020).

However, little is known about how to purposefully construct coherent stories of mathematics. Hence the research question of this poster is: *given a set of mathematical concepts, what are the first steps one can take in order to create a coherent mathematical story that would introduce the chosen concepts?*

Theoretical framework

Dietiker (2015) has developed a theoretical framework to view a mathematical text as a story. In the framework, the notion of mathematical plot plays a central role. In order to study mathematical plot, one has to consider what questions are present in the reader's mind at a given point in a story. Some questions are quickly answered by the story, while others may be open for a long time. Having questions that are kept open for a long time is a key property for a story to be coherent (Richman et al., 2019).

The coherence of the story is defined as "the extent to which the events and mathematical ideas of the mathematical story (...) are connected to each other for a reader" (Richman et al., 2019, p. 4) and a way to test if a mathematical story is coherent is to look for overarching questions that are engaged throughout the length of the story and frequent progress on these questions is being made. If such a question exists and the rest of the questions considered in the story allow for progress to be made on the main question, then one can expect that the mathematical story is coherent.

Epistemologically potent problems: a backbone for a coherent mathematical story

Consider the following problem: "Alice has 2/3 of a baguette. If she eats 3/5 of that, how much of the whole baguette would that be?" Notice that in the wording of the problem, the concept of fraction multiplication is not mentioned, yet the conventional way to solve such a problem is to multiply 2/3 by 3/5. So this problem has an interesting property – it does not mention fraction multiplication, yet it requires this concept to be solved.

Problems with this property have the potential to show to students why mathematical concepts (fraction multiplication in this case) are needed, from what problems they arise. Hence I call such problems epistemologically potent (Harel, 2013). Such problems cannot be easily solved by students as they have not been exposed to the concepts that are needed to solve the problem, yet can be approached by them, because the unknown concepts are not mentioned in the problem's wording.

If a teacher were to construct a lesson or a series of lessons built on the analysis of an epistemologically potent problem, then due to the fact that the solution to the problem requires a mathematical concept that is currently not known to the students, many questions will need to be considered and answered before solving the initial problem. Since all the questions would arise from considering the same initial problem, they would all be related. Hence the instruction based on the analysis and consideration of an epistemologically potent problem is likely to be coherent.

Implications

Epistemologically potent problems can serve as a good starting point when considering how to construct a coherent mathematical story, since such problems are, by definition, accessible to students, yet require the use of concepts that are yet to be discovered by the same students, hence allowing for a continued exploration and progression on the problem.

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An examination of mathematical affordances available in grade 2 teachers' guide and learners' textbook on addition of whole numbers

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This paper present findings from a textbook analysis which examined the structure and patterns of variation of addition examples presented in a grade 2 mathematics teachers' guide (TG) and learners' textbook (LT). Classification scheme for addition problems was used to analyse the structure of the examples, and Mathematics Discourse in Instruction framework for Textbooks was used to analyse their patterns of variation. The findings revealed that the example sets in the TG and LT afford development of additive reasoning as they contain different patterns of variation that might lead to higher levels of generality if teachers focus learners' attention to these. The examples however constrain development of additive reasoning as they comprise of only one structure with both addends given and require calculating the result. To decide a better professional development for teachers, it is necessary to study how teachers use these resources to plan and deliver mathematics lessons.

Keywords: Examples, mathematics teachers' guide, mathematics leaners' textbook.

Background

Malawian learners, on average, have been performing poorly in mathematics in both national and regional assessments (Ravishankar et. al., 2016). Despite Malawi's overall improvement on national primary school examination pass rates, performance in mathematics is still very low (Ministry of Education Science and Technology [MoEST], 2020). At international level, the Southern African Consortium for Monitoring Education Quality (SACMEQ) results show that Malawian learners perform extremely low on number and operation in mathematics (Milner et al., 2011). This is very worrisome considering that number concept and operations define numeracy, which is an indicator of educational quality (Ravishankar et al., 2016). The problem of learners' low performance in mathematics in Malawi has persisted even after government's implementation of several educational reforms to improve education quality. Some of these reforms include revising of primary education curriculum and curriculum materials such as teacher guides (TG) and learners' textbooks (LT) for all subjects. Despite the crucial role that mathematics TGs and LTs play in the planning, teaching and learning of mathematics in developing countries (Leshota, 2020), little research has been conducted to examine how the content of the revised TGs and LTs afford or constrain learners understanding of mathematics. In mathematics education, "mathematics textbooks play a particularly prominent role in guiding teachers on specific materials to teach (Chang & Salalahi, 2017, p. 236). Thus many mathematics teachers use textbooks such as TGs and LTs to decide the type of tasks to implement in their classrooms and how to engage students in such tasks (Leshota, 2020; Stylianides, 2014).

The findings presented in this paper are part of an ongoing study which aims at examining the mathematical affordances presented in Malawian primary mathematics textbooks. In this paper, we pursue the following research question; what mathematical opportunities are made available in addition of whole number examples presented in mathematics grade 2 TG and LT? For every primary education subject in each grade in Malawi, there in one TG and one LT. The TG contain instructions for teachers in terms of the tasks, examples, and resources to be used for teaching while the LT contains learners worked and exercise examples. In most cases, these two books are the only curriculum resources used by Malawian primary school teachers to plan and teach their lessons. Therefore this study is useful in informing policy during revision of the books as well as informing educators on necessary professional development needs for primary school mathematics teachers not only in Malawi but also in other countries. As noted in literature, mathematics textbooks such as TGs are the mostly used resource in mathematics teaching and learning not only in developing countries but in developed countries (Stylianides, 2014). We specifically focus on the quality of addition examples by examining their structure as we agree with Olteanu (2018) that mathematics teaching and learning is mainly done using examples. Suggesting that the quality of mathematical examples presented in textbooks determine the quality of mathematics teaching in the classroom (Leshota, 2020; Ronda & Adler, 2016).

Analytical frameworks

Two frameworks were used to analyse the examples in the TG and LT: the classification scheme for addition problems developed by Carpenter, Fennema, Franke, Levi and Empson (2015), and the Mathematics Discourse in Instructional framework for Textbook analysis (MDITx) developed by Ronda and Adler (2016). Carpenter et al. (2015) describe 2 classes of addition problems (which are called examples in MDITx) according to the kinds of action or relationships described in the problems; join and part-part-whole problems. “Join problems involve a direct or implied action in which a set is increased by a particular amount and part-part-whole problems involve adding or subtracting 2 disjoint subsets” (Carpenter et al., 2015, p. 8). These two types of addition problems can contain different structures in which either the result/whole is unknown but the start or change/parts are known, or one start/change/part is unknown, but the result/whole is known. When children engage with addition problems that require them to find the result/whole, they use a joining all basic counting strategies. In joining all (counting all) strategy, children use objects to first count each addend separately (numbers of object being added), put the objects together and counting them all again to find the sum (Carpenter et al., 2015). For children to use complicated counting strategies like joining to and trial and error, they need to be given either a join problem in which the start or change is unknown or a part-part-whole addition problems in which either the first or second part is unknown (Carpenter, et al., 2015). In joining to counting strategy, children do not count each start/change/part, but they start counting on from a predetermined number like either the given start/change/part (Carpenter, et al., 2015). As such joining to counting strategy is complex counting strategy and it enhances conceptual understanding and additive reasoning. Carpenter et al. (2015) therefore suggest that to increase number sense when learning addition of numbers, children should be given a variety of addition problems that offer them opportunities to use both simple and complicated counting strategies. We therefore used this classification scheme to examine the addition examples in the TG and LT to find out if they are varied in a way that promote learners’ number sense.

The MDITx framework describes the quality of mathematics made available to learn in a textbook (Ronda & Adler, 2016). MDITx comprises of five key elements aimed and achieving generality and structure; object of learning, examples, tasks, naming/word use and legitimations. The object of learning is what learners need to know and be able to do at the end of the lesson, as such, it is the goal of the lesson. Opportunities for learning mathematics are either afforded or constrained by the way author(s) use examples, tasks, words and legitimations (Ronda & Adler, 2016). Due to space limitations, we only give a brief description of what examples entail in MDITx as these are in focus in this paper. Examples are a particular case of a larger class used for drawing reasoning and generalisations (Ronda & Adler, 2016). They are what teachers and learners mainly work on during mathematics instruction. Carpenter et al. (2015) refer to examples as problems. MDITx draws from key principles of Variation Theory which emphasize on paying attention to variation amidst invariance when selecting examples (Marton & Pang, 2006). This means that textbooks must contain examples which are deliberately sequenced to enable learners to understand a particular object of learning in a coherent manner through noticing aspects that remain the same and those that change. Ronda and Adler (2016) therefore suggests that to analyse and determine variation in an example set, three categories of variation must be used. These are; contrast (C) (when differences are noticed), generalization (G) when similarity is noticed and fusion (F) (when at least 2 different objects of learning are in focus). Thus they describe a set of three progressive indicators for analysing and coding example spaces in a textbook lesson as follows: Level 1, if only one pattern of variation is used throughout the textbook lesson, Level 2, if two different patterns of variation are used in the textbook lesson, and Level 3, if all three patterns of variation are used. A fourth code called NONE is used to code example spaces in which no pattern of variation is detected. We used these descriptions to code the example sets during data analysis.

Methodology

Analysis of the structure of addition problems in grade 2 mathematics TG and LT (Kachisa, Mphando, Mwale, Soko, & Toto, 2012a; 2012b) involved examining what is given and what is required to be calculated in each problem using Carpenter et al.'s (2015) classification scheme. Thus, we examined whether a problem contained both addend and required calculating the sum, or whether it contained the sum and one addend and required calculating the other addend. To examine the variation in the examples, we regarded examples under each activity in both the TG and the LT as an example space and examined the type of variation available using MDITx framework by Ronda and Adler (2016). Where one type of variation was used throughout, we coded the example space as level 1. If two types of variation were used, we coded the example space as Level 2, and Level 3 if all three types of variation were used.

Results

The findings show that Unit 2 of both the TG and LT contain three main activities with specific object of learning for each activity. In the TG, each activity has several tasks and each task is accompanied with a set of examples. We begin by presenting findings on the nature of the structure of addition examples/problems, followed by findings on the nature of variation afforded by the examples using Figures 1, 2 and 3.

The structure of the addition problems

As it can be seen in Figures 1, 2 and 3, both the TG and LT have provided addition problems containing both addends and requiring finding of sums. Only activity 1 of the TG contain word problems as well as non-word problems, but the other activities only contain word problems. All LT activities do not contain word problems. The word problems in activity 1 of the TG belong to two types of addition problems. Problem 2a *Chifundo has 2 mangoes and Paul gives her 3 mangoes, how many mangoes does Chifundo have altogether?* is a join problem because as it implies action of giving mangoes, hence causing an increase in Chifundo's total number of mangoes. The problem has initial number of mangoes that Chifundo had (2 mangoes) and then the change or increase she is given by Paul (3 mangoes), and the requirement is to find resulting amount (altogether). This problem can be represented as $2 + 3 = \square$. Addition problems 2b and 4 are part-part-whole problems because they do not imply action but require finding the sum of two disjoint sets of objects like sweets, sticks and stones. For example, problem 2b *Mphatso has 9 sweets and Tamanda has 4 sweets, how many sweets do they have altogether?* The structure of this part-part-whole problem is similar to that of problem 2a and can also be presented as $9 + 4 = \square$. As such learners might count the two sets of sweets separately and then count the total by starting from first set and continuing with the other set. The same structure of providing two addends that require finding sum is also observed in problem 4 and the other non-word problems in activity in all activities in both the TG and LT. This implies that both the TG and the LT do not contain addition problems with a structure that contains the sum but require finding of one of the addends like $9 + \square = 13$ or $\square + 4 = 13$.

Variation afforded by the examples

The examples under each task were regarded as an example set while all examples under each activity were regarded as an example space for a particular object of learning. Each activity in the LT mainly contains one task with several examples under each activity and these were also regarded as an example space. Table 1 presents a summary of the findings on the nature of variation afforded by the example set under each task.

Table 1: Nature of variation of examples

Activity 1 : Adding numbers horizontally	**Activity 2: Adding numbers vertically**	**Activity 3:Mastering addition facts**
Example set 1 (TG): Level 1 (C).	Example Set 1 (TG): Level 0	Example set 1 (TG):Level 3 (G, C, F)
Example set 2 (TG): Level 1 (C)	Example Set 2 (TG): Level 2 (C)	Example space from LT: Level 3 (G, C, F)
Example Set 3 (TG): Level 0	Example space from LT: Level 3 (G, C, F)	

Example Set 4 (TG): Level 1 (C)		
Example Set 5 (TG): Level 1 (C)		
Example space from LT: Level 3 (G, C, F)		

As Table 1 shows, in Activities 1 and 2 in the TG, one example set is in level 0 because it only had one example so it was not possible to generate pattern of variation. The other example sets in Activity 1 and 2 of the TG are in level 1 as they only enhance contrasting pattern of variation. Only examples of activity 3 in the TG enhance all patterns of variation. The LT examples for all activities enhance all patterns of variation. I clarify these findings using Figures 1 and 2.

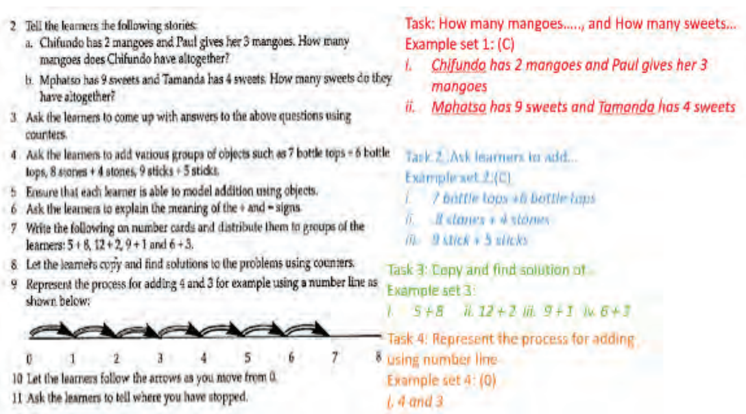

Figure 1: Activity 1 from TG (Kachisa et al., 2012a, 13)

Figure 2: Activity 1 from LT (Kachisa et al., 2012b, 23)

As it can be noticed in Figures 1, 2, the number of addends and the position of the = sign remain the same (invariant) in all examples, what changes are the values of the addends. As shown in Figure 1, the examples from each example set contain different addends that generate different sums. We therefore coded each example set as contrast (C) as at it might enable generalising that different sets of addends generate different sums, hence level 1 of generalisation. In the LT, the example space under activity 1 is in Level 3 of generalisation because it contains examples which enhance all patterns of variation as shown in Figure 2. Several examples can be combined from the example space in Figure 2 to form example sets that might enhance generalisation through noticing different patterns of variation. Examples 1-2, 2-4, and 16-18 can be used to generalise that different sets of addends can generate similar sums. As such we coded these example sets as G. Since these examples also enhance development of number facts, we also coded them as F. Similar patterns of variation and fusion can also be drawn through combination of examples 9, 11 and 14, as well as examples 10, 12 and 15. However, if the learners solve the examples in a manner that is presented in the LT, then examples 6-15 would enhance generalisation through contrast (C) pattern of variation as they addends and sums are different. As such we coded this part of example space as C. Since the example space contain examples which might help understanding of two objects of learning such as adding numbers and number bases, then we also coded it F.

Discussion of the results

In this paper, we examined the mathematical opportunities made available in grade 2 TG and LT by analysing the structure and nature of variation of examples/problems on addition. As the findings have revealed, the TG has provided more Level 1 example sets that enhance noticing of only one pattern of variation than Level 2 and Level 3 example sets that enhance noticing more than one pattern of variation. All level 1 example sets in the TG are enhanced contrast pattern of variation, meaning that the TG has provided more opportunities for only noticing how different set of addends can generate different sums, but not to notice how the different sets of addends can generate similar sums. Contrast is a first step or low level type of generalisation and is supposed to be followed by similarity to enable children to move into deeper levels of generalization and understanding through identification of more patterns and justifications (Watson & Mason, 2006). This suggests that to enhance development of additive reasoning, the TG is supposed to deliberately contain examples that afford them opportunities to recognise aspects that change within aspects that do not change (Ronda & Adler, 2016; Watson & Mason, 2006). Provision of example sets that help learners to see not only contrast but also similarity signals a move to higher level of generality (Ronda & Adler, 2016) and it can engage learners with understanding of mathematical structure (Watson & Mason, 2006). The findings however show that the examples in the TG activity of number bases might enhance learners' understanding of the commutative property that the change in the order of the addends does not lead to change in the sum of the addends (Carpenter et al., 2015; Hunter, 2010). Understanding of the commutative principle helps learners to develop generalisations that help develop algebraic reasoning which is the difficult part of later mathematics (Hunter, 2010). Activity 3 of the TG and other example sets in the LT might also enhance learners' development of additive reasoning through enhancing understanding of number facts (Carpenter et al., 2015). Although the LT contains example sets that enhance noticing of both similarity and contrast patterns of generalisation, the instructions from the TG only require the teacher to ask the learners to do the tasks but not to focus on noticing any pattern of variation. This implies that depending on their knowledge, some teachers may help the learners to pay attention to these patterns of variation while others may not. As Brown (2009) noted, teacher knowledge greatly influences how teachers adapt textbook content.

As the findings reveal, the TG and LT have only provided joint and part-part-whole addition problems which contain two addends and require leaners to find the sum, but they do not provide problems requiring finding an addend. According to Carpenter et al. (2015), when children are given join problems to find the result, and part-part-whole problems to find the whole, they only use simple counting strategies like joining all modelling strategy and counting on modelling strategy. Thus children need to be given problems whose either start, change, or part are unknown in order for them to use more sophisticated modelling and counting strategies like joining to, separating from, separating to, counting on to and counting (Carpenter et al., 2015). Providing different unknowns in addition problems enable children enhance their additive reasoning through promotion of use of different counting strategies (Carpenter et al., (2015). The findings imply that both the TG and LT place more demand on the teacher especially during planning of lessons for delivering the curriculum (Brown, 2009). Thus implying that learners' full development of additive reasoning relies on teachers' ability to notice the shortfalls of the examples in both the TG and LT and deciding how to adapt them to increase their level of generalization and enhancement of development of additive reasoning. One of the factors that influence teachers' ability to recognise what the textbook affords and constraints is teacher knowledge (Brown, 2009; Leshota, 2020). Considering that like most Sub-Saharan countries, teacher knowledge is one of the challenges constraining Malawi from achieving the SDG 4 of improving education quality (MoEST, 2020), then it is unlikely that most Malawian teachers might notice the gaps in these mathematics textbooks. These findings confirm Milner et al.'s (2011) suggestion that the low quality of Malawian mathematics textbook content might be one of the causes of persistence of learners' low performance in mathematics. We concur with Leshota (2020) that countries with high textbook compliance policies need to ensure that they develop high quality textbooks and train their teachers on how to use these materials when planning lessons for delivering curriculum.

Conclusion

This study investigated mathematical opportunities available in addition examples/problems for addition topic in Malawian TG and LT of grade 2. The findings revealed that both the TG and LT contain addition examples comprising of only one structure whereby both addends are given and a sum is to be calculated, hence limiting learners' development of additive reasoning through use of complex counting strategies that develop when learners practice calculating an addend when given sum and one of the addends. Regarding variation of examples, the findings revealed that the TG has more example sets that can enhance achieving low levels of generality and few example sets that can enable achieving higher levels of generality. The LT contains more example sets that can enhance noticing of more than one pattern of variation, hence capable of enabling learners to move to higher level of generality of addition. However, the teacher might not focus learners' attention to notice the different variations because the TG which is the main resource that the teacher uses to plan lessons does not contain any instruction about noticing similarity or difference. Thus the structure and patterns of variation of the addition examples available in the TG and LT place much demand on the teacher as they require much adaptation to enhance additive reasoning through changing of structure and variation. This might be possible if teachers have adequate content knowledge to adapt these materials. These findings suggest that there is a need for professional development of teachers on how they can use these resources as suggested by other researchers. However, for an effective professional development of teachers, it is necessary to investigate how teachers use these resources to plan and teach mathematics lessons to find out if they do notice the gaps in the examples and make necessary adaptations.

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Students' expressions about working successfully with mathematics textbooks

– multimodality and sociomathematical norms in early years

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This paper reports on an ongoing study about students' expressions about mathematics and what is considered successful when working with mathematics textbooks. 18 Year 1 students (7–8 years) working with mathematics textbooks were analysed according to how they used different communication resources, or modes, and sociomathematical norms. The results showed sociomathematical norms about (1) aids that are supportive for solving the exercises in the textbooks, (2) clues that are given in the task design and (3) aspects considered successful by the students. It is stated that there is a tension between using aids and being considered successful in mathematics, which could affect students' possibilities for mathematical learning.

Keywords: Early years, mathematics textbooks, multimodality, sociomathematical norms.

Introduction

Mathematics is for everyone. Therefore, mathematics teaching should be designed so that students' different ways of expressing mathematics are utilized. Mathematics is multimodal (O'Halloran, 2005), which means that different resources for communications, or modes (Kress, 2010), is essential in doing and expressing mathematics. All students' mathematics learning is supported by focusing on the multimodal nature of mathematics and the possibility of expressing mathematics in many ways, with different modes such as images, gestures, speech, and writing. However, if one communication resource, mathematical symbols, already in year 1 has a unique position (Norberg, 2020), students, if possible, choose to avoid other forms of modes, mathematics teaching risks suiting only the students who are already confident with the mathematical symbols.

Students' possibilities to express mathematics in mathematics teaching are affected by the norms in the classroom, what is considered accepted and what is considered desirable. Mathematics textbooks are common learning tools used by more than 75% of students in compulsory schools worldwide (Mullis et al., 2012). The use of textbooks means activities with mutual participation between textbooks and students (Rezat & Sträßer, 2012). In a theory-oriented paper using a socio-didactical tetrahedron Rezat and Sträßer (2012) stated that social and sociomathematical norms are considered essential in students work with mathematics textbooks. Students use textbooks differently (see, for instance, Norberg, 2020; Rezat, 2013). Ewing (2006) stated that students who find working with mathematics textbooks hard are "blamed for their inability to learn" (p. 14). Also, students' individual work with mathematics textbooks is an unexplored research field (e.g., Österholm & Bergqvist, 2013) Therefore, studying sociomathematical norms, normative aspects specific to mathematical activities when students work with mathematics textbooks can create essential knowledge about mathematical learning. Multimodality entails an approach in which all communication resources are utilized, which further strengthens the study's relevance.

This paper presents preliminary results from an ongoing study built on a reanalysis of data from my PhD project (Norberg, 2020) about students' meaning making when working with mathematics textbooks in Year 1 (7–8 years). The study builds further on students' expressions on what is considered successful or not when working with mathematics textbooks. The study is also a pre-study for a proposed research project. The paper aims to deepen the understanding of students' meaning making when working with mathematics textbooks regarding sociomathematical norms and the use of different modes, clarified by following research questions: *What modes are used in the students' meaning making?* And *What sociomathematical norms related to the use of different modes are conveyed by the students?* I will share my findings so far, and I hope this will be a starting point for discussions that can increase my knowledge of sociomathematical norms linked to students' work with mathematics textbooks. In the long run, a study in this area could provide essential knowledge about what shapes conditions for young students' mathematical learning.

Background

The study reported in this paper derives from the research field of mathematics education and students' experience of working with mathematics textbooks emphasising two different areas, mathematics as a multimodal subject and sociomathematical norms. Besides that, there is also a particular focus on younger students' experiences.

In an introduction of the papers presented in TWG22 at CERME10 and CERME11 (Pepin et al., 2019), the authors stated that one of the directions for future research discussed was to understand better what impact tasks have on students. That is focused on in this paper. Knowing what sociomathematical norms exist when students work with their textbook's better conditions for students' mathematical learning can be created. Another possible direction for further research discussed was to include students, and their learning experiences when designing tasks. The results from this study could contribute to such knowledge.

The subject of mathematics has a long history of using several different modes to create mathematical representations. O'Halloran (2005) describes mathematics as multisemiotic and argues that the strength of the subject of mathematics lies in the fact that the different modes together give more information than the sum of what each mode can contribute. Yackel and Cobb (1996) found in a visit to a classroom for year 2 (7–8 years) that it was central that the students discovered that mathematics could be expressed in different ways. Although mathematics is a multimodal subject, a former study showed that mathematical symbols have a special position and that it is the desirable mode to use to represent mathematics (Norberg, 2020). A conclusion drawn was that striving not to use the images can lead to the students' meaning making not being directed towards the mathematics content that the exercise is designed to offer (Norberg, 2020). This motivates the multimodal approach used in the study reported in this paper.

Sociomathematical norms are related to students' work with mathematics and concern what counts as mathematically sophisticated, accepted, different, efficient, and not (Yackel & Cobb, 1996). Many students find mathematics difficult and have poor self-confidence regarding their mathematical knowledge, also at earlier stages (Larkin & Jorgensen, 2016). This relates to sociomathematical norms (Yackel & Cobb, 1996) which form conditions for both students and teachers and regulate students' learning. This is supported by Rezat and Sträßer (2012) who concluded that students and teachers act accordingly to their norms in mathematics teaching. In a study of year 1 students (6–7 years) and teachers with data from videotaped lessons during a school year, Perry et al. (2011) concluded that students at this early stage can be engaged successfully in learning mathematics and that teachers play a big part in this. Students are thereby socialized into mathematics teaching culture. Wester et al. (2015) found differences between the students' and the teacher's views of norms and the students participating (14–15 years) expressed that manipulatives should be used when you need to learn something new but that it is better not to use manipulatives for your calculation. All in all, research on students working with textbooks is sparse, sociomathematical norms are important for students meaning making, and a multimodal approach provides a tool for studying all modes used in the communication between student and textbook.

Theoretical approaches

Sociomathematical norms are defined as "normative aspects of mathematical discussions that are specific to students' mathematical activity" (Yackel & Cobb, 1996, p. 458) and derive from a constructive perspective, like multimodal social semiotics. Sociomathematical norms differs from social norms which are general norms about being a student. *Multimodal social semiotics* (see, e.g., Kress, 2010) assumes that all communication is made through various communication resources, or modes. Yackel and Cobb (1996) distinguish between sociomathematical norms that are considered mathematically sophisticated and those considered accepted. The perspective provides tools that can be used both to support researchers to make rational descriptions of mathematical representations, and help structure interpretations of representative functions (Morgan, 2006). Sociomathematical norms contribute with tools to detect which norms are constructed in the classroom. Multimodal social semiotics enables the possibility to analyse all communication in the classrooms, in all modes used.

Central to the project is the student's *meaning making* activities. Students' working with mathematics textbooks is understood as meaning making and focuses on meaning making as a multimodal activity (Kress, 2010). Meaning making is defined as an activity where the individual tries to understand the world around her and happens continuously, everywhere (Kress, 2010). Meaning making takes place in a social, cultural, and historical context and describes the individual's focus. *Multimodality* refers to the perspective's assumption that all communication occurs in different resources for communication, *modes* (Kress, 2010), such as images, gestures, speech, and writing. In multimodality, meaning is attributed to all modes. All modes carry different potentials for meaning making (Jewitt et al., 2016; Kress, 2010), which means that you cannot give the exact same information in, for instance, writing mode and image mode. *Modes* are culturally and socially shaped resources for meaning making and are used in communication and representation (Kress, 2010).

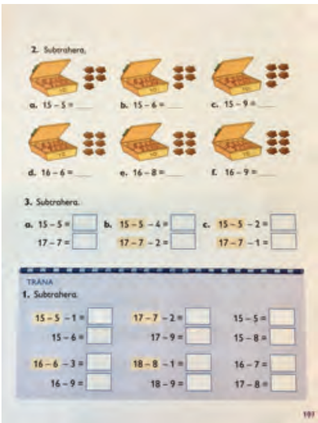
Methodology

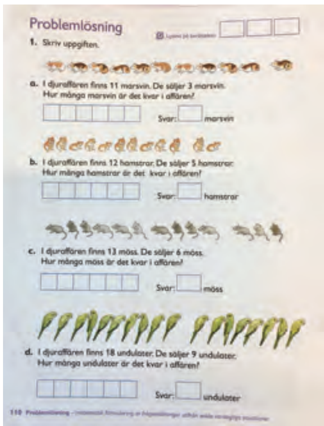
Different data types were collected: textbook pages, video transcripts of students working individually with these pages, and the students' solutions to the exercises. Video transcripts and representations from 18 Year 1 (ages 7–8) students were collected. The school is located in a medium-large city in the centre of Sweden and chosen from a convenience sample. All students in the class whose caregivers had both completed the consent form participated in the study. Before the data collection, I spent a week with the class to get to know the students so that the video recordings would not feel uncomfortable to them. Here, my former work as a compulsory-school teacher was valuable.

The aim was to study the students' work in an as realistic situation as possible. Of course, the most natural situation would have been studying the students in the classroom during mathematics lessons; but, since much of the work with the mathematics textbook consists of working alone in silence, that approach would not yield much insight into the students' meaning making. Therefore, I studied the meaning making of one student at a time in a room adjacent to the classroom, which made it possible to ask questions to the students during and especially directly after their work.

The video material consists of 450 minutes of film – approximately 25 minutes per student with a tablet in May 2017. The student and I sat at a table, with the tablet placed obliquely above us. In order to influence as little as possible, the student started working on the exercise by herself (or himself). Then, I asked questions to understand how the student was making meaning from the exercise, such as, "Can you tell me how you did this?", or "I saw that you did something with this image here, can you tell me about that?".

The textbook series used in this study is widely used in Sweden: *Favorit matematik* (Favourite Mathematics) (Ristola et al., 2012a, 2012b). The pages were chosen based on the results of a quantitative study (Norberg, 2021) using the following criteria: the exercises had to address subtraction as an arithmetic operation, be commonly used in the textbooks, and show breadth based on the different modes were used. The pages were copied in colour and given one at a time. Examples are shown below, with the publisher's permission (see Figures 1–2). *Exercise* is defined as tasks of the same kind. For example, three exercises are pictured in Figure 1.





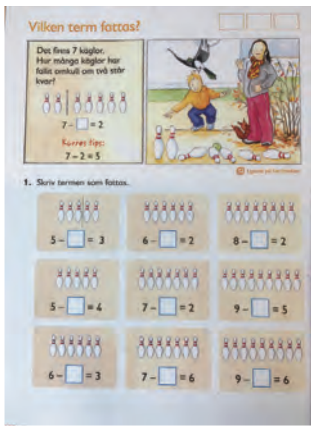


Figure 1: Exercise 1, Ristola et al. (2012b). p. 107. Illustrator: Rajamäki, M.

Figure 2: Exercise 2, Ristola et al. (2012b). p. 110. Illustrator: Rajamäki, M.

Figure 3: Exercise 3, Ristola et al. (2012a). p. 150. Illustrator: Rajamäki, M.

Framework for Analysis

The analysis stems from the research questions and, the categories developed inductively. Due to the multimodal approach, the video transcripts were first transcribed under three separate columns following different modes: speech, images, and body language. This answers to the first research question. In the image column, the students' use of images was documented and instances of a student drawing a picture to support her calculation. The body language column communication made with the body was documented, such as pointing, using finger-counting, shaking the head, e.g. The textbook pages and the students' solutions to the exercises were used as support for the analysis, for instance, for discovering which task the student focused on or how she/he represented the answer on the paper. In the next step, I was looking for expressions for how the student used the mathematics textbook, both things emphasised as good working methods (considered mathematically sophisticated, Yackel & Cobb, 1996) and those emphasised as less good (considered acceptable, Yackel & Cobb, 1996). This was understood as sociomathematical norms regarding the use of the mathematics book. After that, the transcripts were coded by reading through the material several times and highlighted using a colour code. Then condensed meanings were summarised in a matrix, from which various categories emerged. This answers to the second research question.

Preliminary results

Students expressed norms about *Aids supporting the calculation*, *Clues in the task designs supporting the calculation* and *What is considered as successful when working with mathematics textbooks* when working with the textbooks. The multimodal assumption is noticeable in the results as communication in all modes (for instance, images, gestures, mathematical symbols) are made visible.

Aids supporting the calculation

The norms that emerged in terms of aids supporting the calculation was:

(1) use *finger-counting,* which showed in two different ways in the data. One way was to hide the fingers under the table, which indicate that the student should preferably not use the fingers. However, another way was to do the finger calculation above the table, in front of us, indicating that other students did not have the understanding that finger-counting should be made without showing. This could indicate that there might be different norms according to different students in the classroom, this must be studied further.

(2) *draw an image,* which was used for support even though there were images in the exercises.

The student draws an image in order to support her calculation (Figure 1).
Me: Is it possible to use the images in the exercise to do the calculations?
Student: Yes, I think so, but I didn't see how to.

(3) *ask the teacher or a friend,* some students suggested one of these options, and some suggested both options connected to how to work with the mathematics textbook. This could refer to social norms instead of sociomathematical norms and needs to be studied more deeply to know whether this norm is connected to mathematics teaching or teaching in general. There can be differences between subjects when it comes to asking for support.

(4) to use a *number line*, which refers to a number line that the students place before them during mathematics teaching.

Clues in the task designs supporting the calculation

Norms about clues in the task-design that hold information about how to solve the tasks was expressed in the data. The students expressed clues from:

(1) *to read the instruction and/or guide boxes,* one expression was that you only read as far as you need to understand.

> "I read to here (points at the middle of the sentence instruction the exercise), and then I understood what to do."

This category could also refer to social norms and needs to be studied more to know whether this is connected to mathematics teaching or teaching in general, due to differences between subjects.

(2) the number of empty pre-drawn boxes indicate how long the answer should be

> The student solves the task with the guinea pigs (Figure 2) but frowns when he realises that one of the boxes in the answer remains empty. He then erases what he had written.
>
> Me: Why did you erase? Do you think it was wrong?
>
> Student: No but, it shouldn't be like that. It shouldn't be empty boxes… (He thinks for a while). Ahh, one number in *each* box, *now* I know.
>
> So, when he wrote "11" not in one box but in two separate boxes, the written calculation used all the boxes, and the student was satisfied.

(3) *dashed numbers* indicate that they should be filled in.

(4) *not all tasks in an exercise can have the same answer*

> A student is figuring out the answer of a task in an exercise (Figure 3). The first answers were the same, and the student says: It cannot always be two so I think it is eight this time.

(5) *tasks at the beginning of an exercise are easier than the rest.* Students expressed that the first tasks are easier to solve and that it gets harder further down in the exercise.

What is considered successful when working with mathematics textbooks

The data also reveals norms for how a successful student working with the mathematics textbook acts, norms considered mathematically sophisticated, which are:

(1) *Solving all tasks on the page without skipping some of them*. This was expressed when some of the students questioned when I told them to solve some of the tasks and not the whole page.

(2) *Quickly understand how to solve the exercises*. It was expressed as good if the student understood what to do with the exercise immediately, perhaps without reading the instructions. So, working fast is something that is considered successful.

(3) *Solving the tasks with mental calculation* is something that is expressed as good. This is often followed by answering in mathematical symbols on an empty line.

(4) *Not using the images*. Students expressed that it is desirable to solve the tasks without using the images for support.

> Me: Can you use the image to do the calculation?

Student: Yes, but you don't need that if you are good at math.

This approach was also shown in action when some students used the image to solve the exercise. When they worked with the exercise, I saw them using the images, but when I asked how they went along with the exercise, they did not mention the image. First, when they were specifically asked if they used the image, the students said they did.

Concluding remarks

The results showed norms about aids and clues in the task designs that support students' calculation, and norms about what is considered successful when working with mathematics textbooks. The aids mentioned were finger-counting, drawing an image, to ask a friend or the teacher or to use a number line. These are all considered resources and modes supporting the calculation. The clues given in the task design were to read the instruction and/or guide boxes, that the number of empty pre-drawn boxes indicate how long the answer should be, dashed numbers, that not all tasks in an exercise can have the same answer and that tasks at the beginning of an exercise are more accessible than the rest. Things considered as successful when working with mathematics textbooks were to solve all tasks on the page without skipping some of them, quickly understand how to solve the exercises, solve the tasks with mental calculation and not to use the images. This approach was considered mathematically sophisticated (Yackel & Cobb, 1996) by the students.

In further research, it would be interesting to study sociomathematical norms linked to digital textbooks. Partly because there is potential for more modes to be used and partly because students can interact with the textbook. Also, it is sometimes possible to manipulate the tasks during the work with the digital textbook and get feedback during or after the work. More knowledge about this could provide essential knowledge supportive for students' learning in mathematics.

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A Hypothetical Learning Trajectory for linear combination of vectors in $\mathbb{R}^2$

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This paper presents the results of the first cycle of an ongoing design research. A Hypothetical Learning Trajectory (HLT) is proposed to introduce the concept of linear combination in a first linear algebra course. To evaluate the HLT, we conducted a teaching experiment with engineering students in a Mexican public university. In the analysis phase, the HLT was compared with the Real Learning Trajectory (RLT) to evaluate how university students' understanding progresses when they work with tasks designed under the practical action projects. We use emergent modelling as a theoretical framework for describing students' level of understanding achieved. Results showed some limitations of the HLT to achieve a formal level of understanding.

Keywords: Design research, emergent modelling, hypothetical learning trajectory, linear combination, university mathematics.

Introduction

Linear algebra is one of the fundamental and required courses for many science and engineering students. However, several researchers have reported that linear algebra remains a difficult course for students (Dorier et al., 2000; Stewart & Andrews-Larson, 2018). For example, Dorier et al. (2000) point out that students' difficulties with linear algebra are a consequence of abstraction, the abundance of new words, symbols, definitions, theorems, and the lack of connection with previous knowledge.

Consequently, since the late 1980s, several studies have been carried out on teaching and learning linear algebra concepts. Trigueros and Wawro (2019) report that linear algebra topics most frequently studied are span, linear independence, linear transformations, eigenvectors, and eigenvalues. In particular, the existing documentation on the concept of linear combination is scarce. Nevertheless, some researchers (Cárcamo et al., 2021; Stewart & Thomas, 2007) point out the importance of its learning to construct and understand other linear algebra concepts such as linear dependence, linear independence, spanning set and span.

Concerning studies on the linear combination, Parraguez and Uzuriaga (2014) present a genetic decomposition of the concept under the APOS (Action, Process, Object and Schema) theory. Mutambara and Bansilal (2021) examined the errors and misconceptions that mathematics teachers have when solving problems based on the linear combination and found that the most frequent errors are the foundational type, relating to misinterpretations of solutions to systems of equations. Furthermore, other studies present teaching and learning proposals, such as the one conducted by Wawro et al. (2012), who propose an instructional sequence known as The Magic Carpet Ride based on the Realistic Mathematics Education (RME) theory to study span and linear independence. This sequence begins with a problem that introduces the formal notation and language of scalar multiplication, linear combinations, and systems of equations. In general, the results found in the literature served as the basis for the design of our HLT tasks.

Theoretical framework

In this study, we use the work of Simon (1995) to construct a HLT with tasks based on the practical action projects principles (Cuevas & Pluvinage, 2003). *Hypothetical learning trajectories* are a structure that describes the process by which the teacher develops a classroom activity plan. A HLT is made up of three components: learning goals, learning activities, and the hypothetical learning or hypothesis of students' learning process. This last component is hypothetical because the students' RLT is unknown (Simon, 1995) . RLT is defined by Leikin and Dinur (2003) as the learning trajectory carried out in the classroom, including real events and procedure.

Although the HLT constitutes the starting point for task design, it does not indicate how a task should be designed. In this sense, we use the principles proposed by Cuevas and Pluvinage (2003), or C&P, as principles for designing the tasks presented in this study. These principles are decomposition into partial operations or mathematical concept analysis, prior to the design, to determine concepts necessary to achieve understanding; the action or the need to include dosed problems that promote student's active participation; starting a task with a problem in context; implementation of inverse operations; articulation of representation registers if the concept allows it.

On the other hand, emergent modelling is one of the three RME instructional design heuristics which describes how a series of models may support the mathematical advancement of students. Instead of telling students how to interpret a given model, an incremental process is sought in which models are constructed (Gravemeijer, 2020). The students are expected to develop formal mathematics through their informal mathematical activities. This heuristic is compatible with our approach to designing tasks for learning the concept of linear combination. In this respect, after a teaching experiment, we use Gravemeijer's (1999) levels of emergent modelling as an analytic framework to assess students' levels of understanding achieved with HLT. Gravemeijer distinguishes four levels of activity:

1. *Situational level (activity in the task setting).* At this level, interpretations and solutions depend on understanding of how to act in the setting (often out of school settings).
2. *Referential level*. In this level, models-of refer to activity in the setting described in the task. Consequently, models are grounded in students' understanding of real settings and are integral to explanations in which students describe how they interpreted and solved tasks centering on the starting point settings.
3. *General level*. This level begins to emerge when students reason about the mathematical relations involved. Therefore, it emerges as students' reasoning loses its dependency on situation-specific imagery. In this sense, models-for serve more as a means of mathematical reasoning than as a way of symbolizing mathematical activity based on environments.
4. *Formal mathematical level.* This level will often coincide with the use of formal notations. It is achieved when students no longer need the support of models for mathematical activity.

This study is guided by the following research question: How do technology-mediated tasks organized in a HLT support the conceptualization of linear combination in university students? This paper presents a HLT for linear combination and aims to examine how university students' understanding of the concept of linear combination progresses when working with tasks mediated by digital technology and designed under C&P principles.

Methodology

The study presented in this paper was developed under the Design-Based Research (DBR) methodology consisting of the following phases: preparation and design, teaching experiment, and retrospective analysis (Bakker & van Eerde, 2015).

Preparation and design phase

In this phase, a first HLT for the concept of linear combination in $\mathbb{R}^2$ was designed and developed (see Table 1). This HLT consists of four tasks that start with a problem in context about a carnival game we call "*Atrapa el Oso*" (Catch the bear), in which students must manipulate two robotic arms to grab a teddy bear. For each task, virtual interactive didactic scenarios (VIDS) were developed, accompanied by exploration and guided learning sheets (EGLS). These VIDS simulate the game: a board on which are placed a teddy bear and two robotic arms (green and red) whose movement depends on the vectors $\mathbf{k_1u}$ and $\mathbf{k_2v}$, respectively, as shown in Figure 1. The tools of the VIDS are four buttons (A, B, C, and D) that change the bear's position on the board, two sliders (k_1 and k_2) which are scalars that multiply vectors **u** and **v** to control the movement of the robotic arms ($\mathbf{k_1u}$ and $\mathbf{k_2v}$) and two buttons to grab or release the teddy bear. In task 4, the sliders or scalars values are hidden to be found by the students.

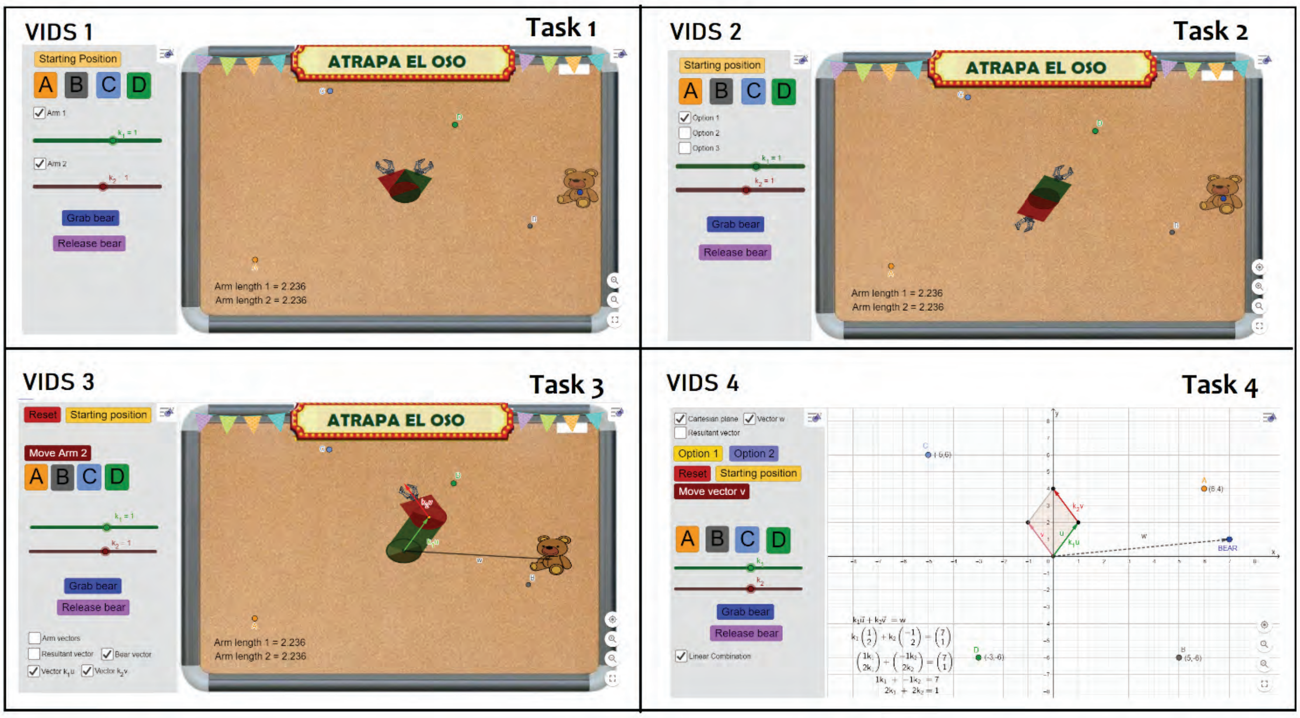

Figure 1: VIDS used in each HLT task

Table 1: HLT for the linear combination of vectors in $\mathbb{R}^2$

Learning goals	Learning activities	Hypothetical learning process
Task 1: Investigate which positions on the board it is possible to grab the bear with the robotic arms. Goal: Identify the motion limitations of robotic arms by analyzing the length and rotation of the links (magnitude and direction of vectors).	Students interact with VIDS 1. They change the bear's position and manipulate the sliders (k_1 and k_2) to identify in which of the four positions it is possible to grab the bear with each proposed robotic arm: a green robotic arm (k_1u with $\mathbf{u} = \binom{1}{2}$) and a red robotic arm (k_2v with $\mathbf{v} = \binom{-1}{2}$).	Students will identify arms' characteristics such as lengths and directions of movement: the green arm moves in a northeasterly and southwesterly and the red arm in a northwesterly and southeasterly direction. Therefore, it is possible to grab the bear in D with the green arm and C with the red arm.
Task 2: Investigate whether it is possible to grab the bear in position A by combining the movements of the robotic arms. Goal: Identify the need to bring the robotic arms together to grab the bear at any position on the board (addition of non-collinear vectors).	Students interact with VIDS 2. They identify the robotic arms characteristics in three options: $\mathbf{u} = \binom{1}{2}$ and $\mathbf{v} = \binom{-1}{-2}$; $\mathbf{u} = \binom{1}{-2}$ and $\mathbf{v} = \binom{-1}{2}$; and $\mathbf{u} = \binom{1}{2}$ and $\mathbf{v} = \binom{-1}{2}$. Then, they discuss whether it is possible to grab the bear in position A by combining the movement of the robotic arms.	Students develop a vector addition representation, i.e., they recognize that combining the movements of the arms consists of placing the red arm's origin at the green arm end. This representation makes it possible to identify that it's possible to grab the bear in position A only with option 3.
Task 3: Investigate if it is possible to grab the bear in positions A, B, C, and D with two degrees of freedom (2 DOF) robotic arm. Goal: Development of vector addition and identification of k_1 and k_2 as scalars.	Students interact with VIDS 3. They can link the robotic arms to form a 2 DOF robotic arm to identify the positions in which this new arm can grab the bear. Finally, they discuss the function of the sliders relating the k_1 and k_2 values to the magnitude and direction of vectors $\mathbf{k_1u}$ and $\mathbf{k_2v}$.	Students deduce that the 2 DOF arm can be modeled as a vector addition represented graphically as arrows. They also identify the movement of the arms as a scalar multiplication where k_1 and k_2 are the scalars.
Task 4: Model the problem of grabbing the bear in different positions on the board. What values must k_1 and k_2 have to grab the bear? Goal: Construct the linear combination expression modeling the bear grabbing problem.	Students interact with VIDS 4. They identify an expression that models the bear grabbing problem, at positions A, B, C, and D, with robotic arms (first with non-collinear vectors and then with collinear vectors). Finally, students solve systems of two equations to find k_1 and k_2.	Students model the bear grabbing problem with the equation $k_1\mathbf{u}+k_2\mathbf{v}=\mathbf{w}$ and identify the change produced by the sliders as scalar multiplication of vectors. They can also define whether $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Teaching experiment

This study reports the results of the first cycle of design research on understanding the concept of linear combination as part of an ongoing doctoral study. A teaching experiment was performed during the SARS-Cov-2 pandemic using the Zoom platform in a linear algebra class to investigate how the HLT helped students learn the concept of linear combination. Participants were 38 second-year engineering students at a Mexican public university who had already learned the concepts of linear equations, matrices, vectors, scalar multiplication, and vector addition. Participants had not previously studied the concept of linear combination. The experiment consisted of 4 sessions in which students worked in 10 groups of three or four students and two whole-class discussion sessions. Students were presented with the HLT tasks via links to access EGLS and VIDS. During the group interaction, each group performed the following activities: sharing the computer screen with the VIDS, reading the EGLS, discussing tasks, and recording the computer screen interaction. One of the authors was the class teacher. The data collected consisted of student EGLS, video recordings of group interactions, and whole-class discussions.

Analysis and results

The students' EGLS were analyzed in the retrospective analysis phase, and video recordings of each working group were transcribed. Subsequently, the research group met to examine the students' RLT, identify their actual learning and define the level of understanding achieved on each task. The following sections will show an approximation of the level achieved by a group (students S1, S2, S3, and S4). This group was chosen for the richness of their discussion and interpretation of tasks.

Task 1 results

The RLT shows that students achieved the situational level in task 1. Initially, the students acted in the task setting to identify the positions in which they could grab the bear with the given robotic arms. To do this, they used the sliders to determine the movement characteristics of each arm. Although S1 used the terms 'magnitude' and 'direction' (an informal way of talking about the properties of vectors) during the group discussion, he used them to describe the movement of the robotic arms.

S1: Is it possible to grab the bear in position D with arm 1?
S3: There, it will be possible (S3 answers before interacting with VIDS 1).
S4: If you could place the bear in any position on the board, at what position on the board, could you place the bear to grab it with this arm? And why?
S1: Well, it would be in position D, wouldn't it? Because the direction cannot be moved in the arm, only the magnitude can be extended.

Task 2 results

The RLT shows that students achieved the referential level in task 2. Students began to reason about the mathematical relationships involved in the starting setting when they discussed the following question: *If you could combine the movement of these arms, might you grab the bear in position A?* From the group discussion, it was revealed that the idea of combining the movement had led students to the concept of vector addition.

S1: I believe it is possible. If we take the vector k_1 and then add it to k_2 but in a smaller magnitude, then yes, it would be possible, wouldn't it?
S3: Only by addition, correct?

S1: Aha! That is if we leave k_1 like that and decrease the magnitude of k_2. Then, it would be possible if we put k_1 at the end to k_2.

S4: But can't we change the origin?

S1: Umm, no, but we can do like the addition, that is, they must keep their direction, but we can do like vector addition because it says: If you could combine the movement of these arms? I mean, I imagine that by combining, I want to think that we can put k_2 at the end of k_1.

S3: Something like what I am going to write down, that at the end of k_1, you put k_2, and here you have the vector, right? (S3 uses the zoom tools to draw an arrow representing the vector addition as shown in Figure 2).

Figure 2: S3 drawing of vector addition

Task 3 results

The RLT shows that students remained at the referential level when solving task 3. The students used the sliders to identify the positions in which they could grab the bear with a 2-DOF robotic arm, and they recognized this model as vector addition. However, the focus of the discussion remained on the starting setting. Although students were expected to identify the movement of each arm as the scalar multiplication of vectors, they didn't achieve this objective. The students knew that the sliders controlled the movement of the robotic arms, but they could not relate the observed changes to the scalar multiplication where k_1 and k_2 values are scalars.

Task 4 results

The RLT shows that students achieved the general level in task 4 without getting the formal level. This task set the stage to introduce the definition of linear combination as follows: 'A linear combination of vectors is the sum of two or more vectors multiplied by a scalar'. Students' reasoning lost dependence on situation-specific imagery. The focus of discussions shifted from evaluating ways to grab the bear whit the robotic arms to determining whether a vector could be expressed as a linear combination of vectors. This reasoning can be observed when students discuss the following question: *Suppose the bear is in the position denoted by the vector* $\mathbf{w} = \begin{pmatrix} -5 \\ 6 \end{pmatrix}$. *Is it possible to express* **w** *as a linear combination of the vectors* $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$*?*

S1: Let's say v is from here to here, right? And then this would be from here to here. Can it be a linear combination w? (E1 draws the blue arrows shown in Figure 3).

S4: No, I don't think so.

S1: I don't think so, too, because if we do a vector addition, this one could be pointing this way (E1 draws another arrow as shown in Figure 4), and if we narrow it down, it would also be pointing this way, but, no, wait! I think this one can be.

S3: I think it could be because of the vector you put.
S1: The v, right? Can you put it in the negative?
S3: Yes!
S1: And then add up, yes, yes, it is true.
S3: If we make this vector multiply by a negative scalar, like here, maybe it will work.

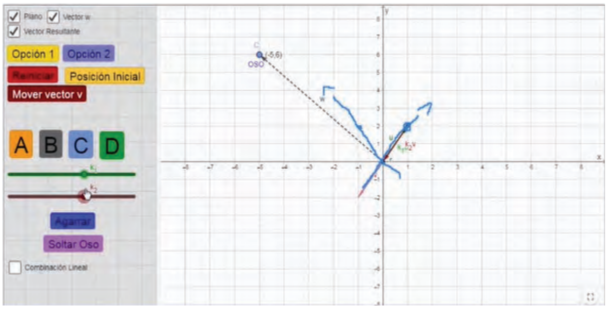

Figure 3: S1 drawing of vectors u and v

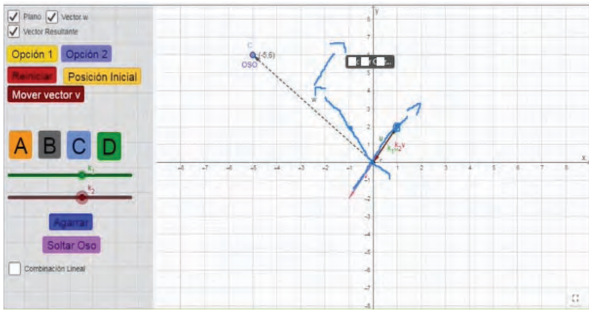

Figure 4: S1 drawing of linear combination

Preliminary conclusions

This paper presented the results of the first cycle of an ongoing design research, in which a HLT for the concept of linear combination was developed. Tasks were designed under the C&P principles and digital technology. Emergent modelling was used as an analytic framework to identify the levels of understanding students achieved with the proposed HLT. The results of the teaching experiment show that tasks based on the context of the game "Catch the bear" supported students in developing the concept of linear combination at the situational, referential, and general levels of understanding. However, they did not help in developing the formal level. In general, we observed that the interaction with the VIDS promotes dependence on models in the mathematical activity concerning the linear combination of vectors. Therefore, in the second cycle of design research, we should incorporate tasks that promote the development of formal mathematics. Since we present the case of four students, this is a first approximation, and an analysis of all groups is necessary.

On the other hand, to develop a formal level of understanding, there must be a balance between the conceptual and procedural. In this respect, results also showed that students have problems solving systems of equations, which are essential to developing the procedural part of linear combination. Therefore, we agree with Mutambara and Bansilal (2021) about promoting access to technological resources to reduce the time spent doing many computations by hand and to spend more time in developing conceptual understanding.

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Student utilization schemes of questioning devices in undergraduate mathematics dynamic textbooks

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We describe how students utilize questioning devices, an interactive feature designed into university textbooks to invite them to read the textbook before the class in which the content will be discussed. This feature has been added to three open-source and open-access dynamic textbooks intended for calculus, linear algebra, and abstract algebra. Using self-reports from 392 students in 23 courses and an instrumental approach, we identified three classes of situations in which the feature was used and three corresponding utilization schemes: Familiarizing with Content Before Class, Studying by Practicing and Doing Homework, and Self-Evaluating Understanding of Content. We suggest further areas of inquiry.

Keywords: Questioning devices in textbooks, instrumental approach, undergraduate students, mathematics textbooks, utilization scheme.

Mathematics digital textbooks offer opportunities for students to engage with mathematics in ways that go beyond the mere act of reading the textbook. In this paper, we focus on what we refer to as dynamic textbooks—a special type of digital textbooks that are open-source and open-access and have embedded interactive features. Because textbook features shape students' opportunities to engage with mathematics, how they are designed to engage students matters (e.g., level of interactivity, presentation, and order of features). In writing these textbooks, authors need to make many design decisions with students in mind as the main textbook users; as such they may not fully anticipate all the ways in which students will use textbooks features or the conditions that motivate their use. In the absence of this understanding, author-designers may add features that in the end might not be as useful for the students. In this study, we contribute by investigating student use of questioning devices that were added to three undergraduate dynamic mathematics textbooks in linear algebra, abstract algebra, and calculus. Specifically, we answer the following research question: What are the students' utilization schemes of these questioning devices?

We build on prior work (Mesa et al., 2021) in which we defined questioning devices as a form of questioning embedded in textbooks that seek to engage students in thinking about the content on their own before class. Differently from questions used in classrooms or asked in exercises, this other form of questioning in textbooks seeks to establish a personal relationship between the reader, the author, and the mathematics (Weinberg & Wiesner, 2011). In the textbooks used in this study readers can type responses to those questions directly into their textbooks, immediately sharing their thoughts with their teacher prior to a lesson. Teachers then become aware of how students might be thinking about the material and adjust their lessons accordingly. In Mesa et al. (2021), we found instructors' utilization schemes of these questioning devices. Because instructors' schemes can be related to students' utilization schemes, in this paper we complement our investigation by turning our attention to students. Because we have reviewed the literature on questioning devices in detail in Mesa et al. (2021), in this paper we provide an overview of the literature that focuses on students' use of digital and interactive textbooks.

Literature Review

Instructors indicate that reading the textbook is a very important skill relevant not only for mathematics majors (Mesa & Griffiths, 2012) but for all students because it supports mathematics learning. Judging from the extant research on student reading practices in university mathematics (e.g., Wiesner et al., 2020), knowing whether and how students read their textbooks is a key question for teachers and researchers. Research on university student use of digital textbooks are typically case studies at single universities or classrooms and describe use of interactive components (e.g., Johnston & Ferguson, 2020); relation between student grades and time spent reading (e.g., Allred & Murphy, 2019); skills, emotion, and participation and interaction (e.g., Bikowski & Casal, 2018); differences in student motivation and perceptions of learning when using interactive textbooks (Kondratieva, 2018); or use of digital resources in connection to rules of didactical contract (Gueudet & Pepin, 2018). These researchers indicate that undergraduate students see digital textbooks as resources that motivate and facilitate learning and have documented different levels of reading, different uses of features, and different expectations regarding the utility of interactive features. The differences seem to be related to students' prior knowledge and to personal goals regarding their courses (e.g., passing a course). While these studies suggest that students might not use their textbooks as intended, they do not indicate how students interact with specific interactive features.

Theoretical Framing

In Rabardel's instrumental approach (Rabardel, 2002) an instrument is "a whole incorporating an artifact (a human-made or a physical object) and one or more utilization schemes" (p. 65). A utilization scheme encompasses stable and structured manners that a user has for operating an artifact that includes observable behaviors and non-observable goals and rationales. Following Vergnaud (1998) we define schemes as the "invariant organization of behavior for a certain class of situations" (p. 229). This organization includes goals and expectations; rules of action (which can be seen as the "generative part of the schemes"); operational invariants (propositions that are held to be true by the subject when they act, that are used "to infer, from the available and relevant information, appropriate goals and rules); and possibilities of inferences (p. 229). Vergnaud views "situations and schemes [as] essential in the development of knowledge" (p. 237). Instrumental genesis s the process through which a user transforms an artifact into an instrument by creating utilization schemes associated with that artifact. Because pre-existing artifacts are instrumentalized by a user, different subjects can produce different instruments for the same artifact. Instrument production, carried out by the subject, encompasses two subprocesses, instrumentalization (the "emergence and evolution" of the artifact component of the instrument) and instrumentation ("the emergence and evolution of utilization schemes and instrument-mediated action: their constitution, their functioning, their evolution by adaptation, combination coordination, inclusion and reciprocal assimilation, the assimilation of new artifacts to already constituted schemes, etc." Rabardel, 2002, p. 103). The notion of utilization schemes is central to instrumentation. To document student instrumentation of an artifact—

questioning devices in dynamic textbooks—we identified the utilization schemes students reported when using questioning devices available in three textbooks designed to increase their interaction with the content.

Methods

This research is part of a larger mixed-method study that seeks to understand how teachers and students in undergraduate levels utilize three open-access open-source dynamic textbooks: *Active Calculus* (hereafter AC, Boelkins, 2019), *First Course in Linear Algebra* (hereafter FCLA, Beezer, 2019), and *Abstract Algebra: Theory and Applications* (hereafter AATA, Judson, 2019). The three textbooks are written in PreTeXt (https://pretextbook.org/), a markup language that allows the textbooks to be rendered in various formats (e.g., HTML, PDF, Braille, bounded). With PreTeXt, authors can infuse the textbook with interactive features, such as live computation cells with Python (Sage cells), immediate solution feedback, and collection of student responses to questioning devices for instructor use. The three textbooks were chosen to represent different student audiences. In the United States, calculus is a course that is taken by most students seeking an undergraduate degree in science, technology, engineering, mathematics, business, economics, and health. Linear algebra is a gateway course for mathematics and computing related majors, and in some cases for biology. Abstract algebra is a course usually taken only by mathematics majors (which include future secondary teachers). The questioning devices are called Preview Activities in the calculus textbook and Reading Questions in the linear algebra and abstract algebra textbooks. For this study, we analyzed data from 392 consenting students from 23 teachers across the United States who participated for a semester from Spring 2019 to Spring 2021 (187 used AC, 145 used FCLA, and 60 used AATA). Students (147 female, 142 male, 3 non-binary or trans, 100 no information) had an average age of 20 years, an average GPA of 3.43, and obtained an average grade of 82.58 in their respective courses.

We used data from five student logs (biweekly surveys with 8 to 19 closed- and open-ended questions about student textbook use). We used responses to two questions (see Table 1) intended to identify why and how students used the questioning devices (Q9, and Q5 are open-ended)[1].

Table 1: Log questions used in this study and number of responses by textbook

Question	FCLA	AATA	AC	Total
Log 1 Q9: Your textbooks include Reading Questions/Preview Activities in some sections. How have you used these?	130	41	145	316
Log 2 Q5: What are your priorities (objectives, goals) while using Preview Activities/Reading Questions? Explain how you use them.	86	32	125	243

Analysis

We used a constant comparative analysis with the answers to the two open-ended questions (Q9 and Q5) to identify classes of situations, goals, and rules of actions. The analysis proceeded in four steps.

[1] Across all the textbooks less than 10% of the students stated that they never used these features, indicating high use across the sample.

First, we identified three classes of situations in which students reported using questioning devices: *Before Class Preparation, Study,* and *Self-Evaluation. Before Class Preparation* included situations in which students expressed, they used questioning devices to get ready for an incoming class. *Study* included situations in which the students expressed using questioning devices for practicing, doing homework, and studying for exams. *Self-Evaluation* included situations in which students said they used the questioning devices for testing their own understanding of the content. Using these categories, we classified each of the 559 responses (316, Log 1 Q9 + 243, Log 2 Q5) under one of these situations or as "No Class" if the response did not convey enough information to identify a situation (e.g., "yes"). In the second step, we inspected responses within each class to identify goals. We identified and coded eight goals that students associated with the classes of situations: three related to *Before Class Preparation*, four with *Study*, and one with *Self-Evaluation*. In the third step, with the responses organized by class of situation, we identified 10 rules of action. In the fourth step, the first and second author identified the operational invariants, looking for explanations for why students were using the questioning devices. The most frequent reason was that students used the questioning devices because they were required by their teacher (61 out of 511). Other operational invariants appeared but were less frequent.

The first authors coded all the responses; the other two authors independently coded classes of situations, goals, and rules of actions for 10% of the responses. The average Cohen's Kappa for coding for situation, goals, and actions were 0.78, 0.71, and 0.63. These three values are within the range of [0.61 - 0.80], which correspond with a substantial agreement.

Findings: Students' Instrumentation of Questioning Devices

We identified three utilization schemes of the questioning devices, one per class of situations, *Familiarizing with Content Before Class, Studying by Practicing and Doing Homework,* and *Self-Evaluating Understanding of Content* (see Figure 1). While we characterized the utilization schemes separately to make sense of students' use of the questions devices, students often mentioned multiple goals, rules of actions, and operational invariants. Moreover, although we identified all three utilization schemes across students using the three textbooks in calculus, abstract algebra, and linear algebra, not all goals were mentioned across these groups of students.

In the first utilization scheme, familiarizing with content before class, students said they use questioning devices to get acquainted with the upcoming material in the course:

> They have been assigned to us before classes. I look over (*rule of action: skimming*) the preview questions [questioning device], try to solve the activity (*rule of action: answering*) and it gives me an idea of what we will be working on in class that day (*goal*). (30.07, Spring19, AC)[2]

In this excerpt, a clear goal is stated: to get "an idea" of what will be taught in class. We identified two more goals under this class of situations in the corpus: to better prepare for class and to connect

[2] We identify quotes of a student using the convention "XX.YY, term, textbook" where XX refers to the instructor and YY to the student of that instructor.

previous material with new content. To accomplish these goals, four actions were mentioned: skimming and answering the questioning devices, reading the chapter associated with the questioning devices, and writing down questions they could ask in class. To justify these actions, students collectively offered four reasons: the questioning devices were assigned (e.g., “My professor has it mandatory to answer the reading questions for each section,” 36.20, Fall20, FCLA), the questioning devices addressed important content (e.g., “I usually remember the material in a reading question as what is most important for me to know,” 46.05, Spring21, FCLA), the questioning devices helped them with understanding the new material (e.g., “These are very helpful in introducing the concepts of the section,” 35.70, Fall20, AC), or because their instructors used the questioning devices to teach in class (e.g., “We have used these in our class to give our initial answers to the reading, and then our professor gives us feedback on our responses,” 23.20, Fall20, AATA).

Utilization scheme 1: Familiarizing with content before class		
Goals	Rules of Action	Operational Invariants (rationale)
1. To prepare for class 2. To connect previous materials with new ones 3. To find out what will be taught	- Skimming questioning devices - Answering questioning devices - Reading the chapter associated to questioning devices - Preparing questions for the class	- I answer the questioning devices because my teacher told me to - I noticed that the questioning devices address important content - I noticed that the questioning devices help me understand the new material - My teacher uses them in class to teach
Utilization scheme 2: Studying		
4. To practice 5. To do homework 6. To prepare for assessments 7. To find out what is important in the chapter/section	- Skimming questioning devices - Answering questioning devices - Checking their answers to the questioning devices - Taking notes of parts of the questioning devices and their solutions	- I answer the questioning devices because my teacher told me to - The questions devices help me learn - I noticed that the assessments have very similar problems and questions as in the questioning devices - I noticed that the questioning devices address important content
Utilization scheme 3: Self-evaluating		
8. To check own understanding of the content	- Attempting the questioning devices before reading the chapter - Attempting the questioning devices after reading the chapter - Re-reading the chapter if unable to answer the questioning devices - Comparing answers to the questioning devices with answers provided by textbook, classmates, teacher, or other resources	- It is important to me to make sure that what I know is correct

Figure 1: Students’ utilization schemes of questioning devices

In the second utilization scheme, students said they use questioning devices to study after the material was introduced in class:

> When I have a similar homework problem to work out, I will use the Reading Questions (*rule of action: answering*) for extra practice and to gain better understand[ing] (*goal: practice*; *operation invariant: because I learn by answering questioning devices*). (45.15, Spring21, FCLA)

This student indicates using the questioning devices as “extra practice” when doing homework because doing so will help gain a “better understanding” of the material. We identified four goals associated with this scheme: to practice (when questioning devices are not assigned as homework), to do homework (when questioning devices are assigned as homework), to prepare for assessments (e.g., “Used them as reference to how class questions on exams and quizzes will look like,” 45.53, Spring21, FCLA), and to find out what is important to study in the chapter or section associated with the questioning devices (e.g., “If these are not assigned as homework, I use these questions as reinforcement of the main/big ideas from the chapter,” 33.13, Spring20, AATA). We identified four student actions: skimming and answering questioning devices, checking answers to the questioning devices, and taking notes on the questioning devices and their solutions. We identified four operational invariants that students reported as justifying these actions: Students were required by their instructors to work on the questioning devices during class or at home (e.g., “We use them in class and as homework,” 30.23, Spring20, AC); using the questioning devices benefited their own learning of the material (e.g., “I take advantage of this completely in order to be successful. i [*sic*] enjoy how you can reflect and absorb the material this way,” 44.03, Spring21, FCLA), problems on their examinations were similar to those in the questioning devices (e.g., “I complete the preview activities for a grade but I think that they help a lot in the initial learning process because it lets students know what questions on a test might look like,” 35.57, Fall20, AC), and ideas showcased in the questioning devices were important ideas in the chapter or section (e.g., “I find them [reading questions] helpful because they make sure I understand important parts of the section,” 44.06, Spring21, FCLA).

In the third utilization scheme students said they use the questioning devices for self-evaluation of their understanding of the material:

> I mostly use this to check for my understanding of the chapter. If i [*sic*] get to the reading questions and do not understand them I know I have to reread the chapter again. (37.08, Fall19, FCLA)

This quote represents the essence of the scheme, that students take the questioning devices as a kind of arbiter of the content of the chapter; not knowing the answers triggers a process of reading other parts of the textbook. Three rules of action were associated with this scheme: attempting the questioning devices to evaluate their understanding before reading a chapter (e.g., “Do them to see where my baseline is for this chapter,” 35.65, Fall20, AC); attempting the questioning devices after reading the chapter to check ability to answer them; when unable, students re-read the chapter; and compare their answers to the questioning devices to other resources, such as the solutions provided by their textbook, their classmates, or their instructors (e.g., “Do them and then go over them with classmates," 35.81, Fall20, AC). The operational invariant is inferred as the need to know that their knowledge is correct; this is done either by contrasting responses with the content of the reading questions or with external resources, such as peers or the instructor.

Discussion and Conclusion

These findings are important for textbook designers, as well as instructors using the textbooks, because they confirm that the questioning devices are being used for their intended purpose and more. The three utilization schemes help us understand what motivates students to use questioning devices and how they use them. For two schemes, (*Familiarizing with Content Before Class*, *Studying by Practicing and Doing Homework*) one main motivator was that teachers assigned the questioning devices to the students. This highlights the important role that teachers have in influencing student use of textbooks, supporting prior findings that indicate that when asked, students will use the textbooks as teachers suggest (Mali & Mesa, 2018). However, that was not the only motivator for students to use the questioning devices; students, by *motu proprio* used the questioning devices for familiarizing themselves with the content, studying by practicing, and more interestingly for evaluating their own understanding of the material. Because the third scheme does not seem to be motivated by the teacher assigning the questioning devices, we wonder whether this particular scheme would emerge without exposure to the questioning devices initiated by the teachers.

The findings show that although designer-authors and instructors may have specific ideas of how a feature should be used, students find ways to exert their agency and decide how to use them. In the case of questioning devices, these features were designed to be completed before a lesson; therefore, they may not be the best fit for students when it comes to studying and self-assessment. Although we encourage students to use their textbooks for their needs, we believe that by learning about the intended purposes and embedded opportunities of textbooks' features students can make better use of their textbooks tailored for specific situations. For example, the WeBWorK exercises embedded within the AC textbook provide immediate feedback to students while questioning devices do not; thus, we expect that these features are better suited for self-assessment than questioning devices.

This study was done to complement our prior work on teacher utilization schemes of questioning devices identified in Mesa et al. (2021), namely (1) *Instructors complete the questioning devices for lesson pre-planning,* (2) *Instructors require students to do the questioning devices for the purpose of lesson planning,* (3) *Instructors use the questioning devices for the purpose of instruction,* and (4) *Instructors require students to complete the questioning devices for the purpose of assessment.* We anticipate that the four teacher utilization schemes are related to the student utilization schemes of the questioning devices we found here. As a next phase in our work, we will determine the connection between these utilization schemes and investigate differences by textbook.

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Uncertainty as catalyst for creativity in students' mathematical problem-solving process

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The role of creativity in mathematical competence and problem-solving

Among the fundamentals of mathematical competence is problem-solving abilities, which in the context of school mathematics often is related to mathematical creativity. Sriraman (2005) defines mathematical creativity as novel or insightful solutions to a problem, as well as the formulation of new questions which allows "old" problems being viewed differently, where the use of imagination is a key element. A certain level of mathematical ability is necessary for creative mathematical thinking to manifest, but mathematical ability can also lead to fixations and hinder creativity (Sriraman & Haavold, 2017). Contrasting fixations, cognitive flexibility is often viewed as essential for gaining creative insight (Haavold & Sriraman, 2021). Thus, creativity seems to play an essential role in mathematical competence and problem-solving, and therefore an important question to be answered is how creativity can be catalysed in mathematics education. The focus of this research proposal is to give an outline to an investigation of whether and how uncertainty can catalyse creativity in mathematics students' problem-solving process.

Uncertainty and its relation to creativity in mathematical problem-solving

Sriraman (2021) defines uncertainty as a state of ambiguity and curiosity, as well as the sense of being in doubt, also labeled "creative doubt". Sriraman (2005) argues that creativity requires exposure to uncertainty and experience with the struggle of creating mathematics and solving problems. Creative problem-solving is further associated with cognitive flexibility, which involves the ability to switch between mental sets and strategies when dealing with uncertainty (Haavold & Sriraman, 2021). Sriraman (2021) found that uncertainty plays an important role as a catalyst for mathematical creativity in the work of professional mathematicians, but uncertainty is otherwise a little researched concept in the field of mathematics. Therefore, an interesting perspective would be to investigate whether the relation between uncertainty and creativity which Sriraman (2021) suggests, also exists in the work of students of mathematics. In the following paragraphs, pathologies as a potential task design for the purpose of catalysation is explained.

Mathematical pathologies as a facilitator of uncertainty and creativity

Sriraman and Dickman (2017) argues that mathematical pathologies can foster creativity through addressing misconceptions and false beliefs. The authors define pathologies as "[…] examples that are specifically designed to violate properties that are perceived as valid. The term 'pathological' is also specifically used in mathematics to refer to objects "cooked up" to provide interesting examples of counterintuitive behavior" (Sriraman & Dickman, 2017, p. 137). Examples that are directed at violating valid properties create a space where domain limiting barriers must be overcome and this is where creativity plays an important role. A pathology can take form as a counterexample, but it's not a necessity. Further, a pathology is not the same as a misconception because the intention of pathologies is not to point out the wrong- or rightness of a mathematical concept. The intention is to foster creativity in the face of uncertainty. Lastly, a pathology is a task-design, while a misconception is something within the mind of a student (Sriraman & Dickman, 2017). In the following methodology of the research proposal, pathologies are proposed as generators of uncertainty in the process of catalysing creativity in students' mathematical problem-solving.

Methodology

To investigate whether and how uncertainty can catalyse creativity for mathematics students, we tend to employ pathologies in task-based interviews with students. An example of a task that can be given is presented in Figure 1. This task addresses incorrect fraction cancelation which by chance still leads to the correct answer. Students can be asked to investigate if there exist other fractions with this property. While solving the tasks, the students should think aloud so that insight to their problem-solving process can be accessed and analyzed. The focus of the analyzes will be mapping of how fixations occur and what leads to creative, flexible ways of viewing the problem when dealing with uncertainty generated by pathologies**.** Both video and sound, in addition to collecting their written answers, are useful data collection tools. The results can indicate whether and how uncertainty provided by mathematical pathologies can catalyse creative solutions to a problem in the work of mathematics students. The hypothesis is that pathologies as task design can catalyse creativity by generating uncertainty, also for students of mathematics.

$$\frac{19}{95} = \frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5} \qquad \frac{26}{65} = \frac{2\cancel{6}}{\cancel{6}5} = \frac{2}{5}$$

Figure 1: Two- digit anomalous fractions (Sriraman & Dickman, 2017, p. 140)

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Digital Fabrication for Mathematics Education: A Critical Review of the Field

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This article presents a critical review on digital fabrication for creating manipulatives in mathematics education research. It provides an overview of the research field from two perspectives: 1) explored digital fabrication technologies and reified mathematical concepts, and 2) applied pedagogical theories and research methodologies. Results show that 3D printing is the most common technology for creating manipulatives reifying mathematical concepts within geometry, algebra, and functions, as well as fractions. In the typical research project, a qualitative paradigm is chosen, and learning is investigated from a constructionistic perspective. The review reveals a small but trending research community and concludes with two specific opportunities for consolidating research on digital fabrication for mathematics education.

Keywords: Critical review, digital fabrication, manipulatives, mathematics education.

Introduction

This article presents a critical review of previous research on digital fabrication for mathematics education. Digital fabrication (DF) is “the process of translating a digital design developed on a computer into a physical object” (Berry et al., 2010, p. 168). DF technologies such as 3D-printers, laser cutters or vinyl cutters, have become affordable and can be found at Makerspaces and FabLabs[1] around the world. These tools enable people to create professionally looking items rapidly and at a relatively low cost. Blickstein (2013) argues that DF and making can play a major role in education, “bringing powerful ideas, literacies, and expressive tools to children” (p. 2). DF technologies might support mathematics teachers through the creation of manipulatives. Manipulatives or concrete material are physical objects that are used to reify abstract concepts in mathematics education. Previous research highlights, that students benefit from long-term use of manipulatives in mathematics education. Both students’ achievements in and motivation for mathematics increase when manipulatives are used (Pires et al., 2019). However, manipulatives are not self-explanatory and to draw maximum benefit from students’ use of manipulatives, teachers must continuously situate their activities and the physical objects based on students’ previous experiences, as well as their teaching context (Thompson, 1992). Stylianou (2010) has found that teachers have knowledge gaps of how different representations, such as manipulatives, are translated into mathematical concepts. DF technologies could enable teachers to create context-sensitive manipulatives for teaching activities, situated in the classroom and enhance teachers’ knowledge on manipulatives to improve their teaching. There is prior research (Ford & Minshall, 2019; Hielscher & Smith, 2014; Papavlasopoulou et al., 2017) reviewing DF education in general. To my knowledge, there does not exist a review focusing on mathematics education and how DF can be used to create manipulatives,

[1] https://fabfoundation.org/global-community/

which is the objective in this article. I performed a critical review of previous research in the field to answer the following research question: *What characterizes research on digital fabrication for creating manipulatives in mathematics education?*

Method

The aim of this review is to explore how DF has been used in mathematics education research so far, focusing on DF technologies as well as mathematical concepts and manipulatives on one hand and used theories and research methodologies on the other. I applied a three-step iterative critical analysis (Çorlu et al., 2017; Stigberg, 2017): finding appropriate papers including selection of databases and search terms; eliminating irrelevant papers based on a set of exclusion criteria; analyzing appropriate papers based on key terms answering the research questions. Grant and Booth (2009) describe that "a critical review goes beyond mere description of identified articles and includes a degree of analysis and conceptual innovation" (p. 93). In this review, I will analyze identified articles, highlight research gaps and provide opportunities for further research in DF for mathematics education.

Step 1: Retrieving Publications

The database used as primary source for finding literature is Ebsco, including Education Source, Education Research Complete, Academic Search Primier, ERIC, CINAHL, MathSCINet via EBSCOhost and MEDLINE. To broaden the search and ensure to include as many relevant publications as possible, I chose to perform a search on Google Scholar as well.

From the research topic and research questions, I derived the following keywords: "digital fabrication", "3D-printing" or "laser-cutting", in combination with "mathematics education" and "manipulatives" or "concrete materials". 3D-printing and laser-cutting are popular DF technologies and often used as representatives or synonyms of DF and therefore highly relevant as keywords. A search for vinyl-cutter, another DF technologies gave no hits and was excluded from the final search query. In mathematics curricula, both terms manipulatives and concrete material are used to describe physical objects that reify a mathematical concept and were included in the search. Finally, I used '*' special character in the search query to allow different forms of all keywords. I did not define a timeframe since I was interested in finding all relevant literature. The final queries used in the search can be found at: shorturl.at/eqxP4. The search queries were used for a full text search in both Ebsco and Google Scholar. The literature search resulted in 73 hits from Ebsco and 277 hits from Google Scholar. After removing duplicates, the total number of found articles is 254.

Step 2: Appropriate papers

In this step, the papers that were not appropriate for the review were eliminated based on the following exclusion criteria: not peer-reviewed (36), non-English work (10), not empirical (purely theoretical papers or review articles) (66), do not concern mathematics education or manipulatives not investigated (159), studies on STEM projects not specifically highlight mathematics education or creating manipulatives (81), or 3D-pen as technology (5). Several publications corresponded to more than one exclusion criteria. For a complete list of inclusion/exclusion criteria, see link: shorturl.at/aioyU. The elimination process resulted in a final list of 17 papers to be included in the analysis. The included literature can be found in **Table *1***.

Table 1: Overview of included articles

Code	Article
A	Paul, S. (2018). 3D Printed Manipulatives in a Multivariable Calculus Classroom. Primus: Problems, *Resources & Issues in Mathematics Undergraduate Studies, 28*(9), 821–834. https://doi.org/10.1080/10511970.2018.1445675
B	Wan, Anna and Jessica Ivy. "Adding a New Dimension to Teaching Mathematics Educators." *Handbook of Research on TPACK in the Digital Age*, edited by Margaret L. Niess, et al., IGI Global, 2019, pp. 390-412. https://doi.org/10.4018/978-1-5225-7001-1.ch018
C	Ulbrich, E., Lieban, D., Lavicza, Z., Vagova, R., Handl, J., & Andjic, and B. (2020). *Come to STEAM. We have cookies!* 297–304.
D	Fernández, E., Davidson, J., & Pomponio, E. (2021). *Dare to Care: The Impacts of a Caring Pedagogy on Mathematical Making, Teaching, and Learning*.
E	Junthong, N., Netpradit, S., & Boonlue, S. (2018). *Design and Development of Teaching Tools in Dimensional Geometry for Visually Impaired Students Using Object Models from 3D Printing*. 7. https://doi.org/10.17758/HEAIG2.H0418464
F	Greenstein et al. (2020). Exploring the interwoven discourses associated with learning to teach mathematics in a making context Greenstein, S., Jeannotte, D., Fernández, E., Davidson, J., Pomponio, E., & Akuom, D. (2020). Exploring the interwoven discourses associated with learning to teach mathematics in a making context. *Conference Papers Psychology of Mathematics & Education of North America*, 840–844. https:/doi.org/10.51272/pmena.42.2020.
G	Corum, K., & Garofalo, J. (2016). Learning about Surface Area through a Digital Fabrication-Augmented Unit. *Journal of Computers in Mathematics and Science Teaching*, *35*(1), 33–59.
H	Greenstein, S., & Seventko, J. (2017). *Mathematical Making in Teacher Preparation: What Knowledge Is Brought to Bear?* North American Chapter of the International Group for the Psychology of Mathematics Education.
I	Akuom, D., & Greenstein, S. (2021). *Prospective Teachers' Design Decisions, Rationales, and Resources: Re/claiming Teacher Agency Through Mathematical Making*.
J	Greenstein, S., & Olmanson, J. (2017). Reconceptualizing Pedagogical and Curricular Knowledge Development Through Making. *Design Journal*, *4*, 10.
K	Greenstein, S., Fernández, E., & Davidson, J. (2019). Revealing teacher knowledge through making: A case study of two prospective mathematics teachers. *Conference Papers -- Psychology of Mathematics & Education of North America*, 1151–1156.
L	Hallowell, D. A. (2020). *Spatial Reasoning in Elementary School Children's Geometry Insight: A Neo-Piagetian Developmental Proposal*.
M	Mohamed, M. M., Paoletti, T., Vishnubhotla, M., Greenstein, S., & Lim, S. S. (2020). Supporting students' meanings for quadratics: Integrating RME, quantitative reasoning and designing for abstraction. *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 193–201. https:/doi.org/10.51272/pmena.42.2020

N Ceragioli, F., & Spreafico, M. L. (2020). Tangible Tools in Mathematics for Engineering Students: Experimental Activity at Politecnico di Torino. *Digital Experiences in Mathematics Education*, *6*(2), 244–256. https://doi.org/10.1007/s40751-020-00063-7

O Davidson, J., Fernández, E., & Greenstein, S. (2019). Teachers making manipulatives to promote pedagogical change. *Conference Papers -- Psychology of Mathematics & Education of North America*, 1359–1360. eue.

P Dilling, F., & Witzke, I. (2020). The Use of 3D-Printing Technology in Calculus Education: Concept Formation Processes of the Concept of Derivative with Printed Graphs of Functions. *Digital Experiences in Mathematics Education*, *6*(3), 320–339. https://doi.org/10.1007/s40751-020-00062-8

Q Corum, K., & Garofalo, J. (2016). Learning about Surface Area through a Digital Fabrication-Augmented Unit. *Journal of Computers in Mathematics and Science Teaching*, *35*(1), 33–59.

Step 3: Analyzing the papers

Quantitative and qualitative data was extracted from the articles and gathered in a database. The analysis of each paper included general bibliographic information, as well as type of DF technologies, reified mathematical concepts, applied theories, research design and methodology.

Results

There is limited research on DF technologies for creating manipulatives in mathematics education (17). Most of the initially found publications (254) present research on STEM (81) or do not focus mathematical content or manipulatives (159). Often research presents interdisciplinary projects to facilitate students design thinking and innovation skills. Furthermore, searching on Google Scholar gave several articles that were not peer reviewed (36) such as teaching material or blog entries. Research on 3D pens was excluded based on the definition of DF as computer-controlled tools (Gershenfeld, 2012). Most of the research on DF technologies included in this review is published 2018 or later (14) indicating a growing interest in DF technologies for education. Furthermore, the research community around DF in mathematics education is limited to a small number of research groups. For example, Greenstein published 7 out of the 17 articles (F, H, I, J, K, M, O). A major research project on DF technologies in mathematics education is a project concerning pre-service teachers called: Prospective Teachers Making for Mathematical Learning (F, I, K). In the following, I have analysed the 17 articles from two perspectives: Firstly, focusing on DF technologies and created artefacts reifying mathematical concepts. Secondly, focusing on applied theories, research design and methodologies.

Digital fabrication technologies

3D printers are the most used DF technology for creating manipulatives (15). Two articles, based on the same project, explore the use of a die cutter as DF technology (G, Q). Four articles provide a rationale for their choice of DF technology. 3D printers are starting to become cheap and accessible (A, B), enable teachers and students to create artifacts with little effort and in reasonable time (C, P) and the possibility to reproduce existing artefacts (P). The reviewed articles present different software choices for modelling 3D objects: TinkerCad (C, J, N), OpenScad (C, P), SketchUp (E) and AutoDesk 123D Design (L). FabLab Model Maker was used to design 3D models for the die cutter technology (G, Q). Nine articles do not provide information about a specific 3D modelling software. Wan and

Ivy (2019) mention Thingiverse[2], an online platform where makers can share their work, as one possible resource for finding 3D models for mathematics education.

Artifacts reifying mathematical concepts

The reviewed articles present three core mathematical concepts: geometry, algebra and functions, and fractions. Geometrical objects such as rectangular prisms and cubes (C, D, G, Q), cones (J, N), prisms representing triangles (M), tessellation with pattern blocks (D), general geometric properties (L) and 3D printed geoboards for representing area and volume (E). Created objects reifying concepts of algebra and functions such as: coordinate system (J), 3D printed representations of graphs of functions with one variable (P) and two variables where students investigate contours, partial derivates, gradient vector field, and restrictions to the curve (A), and models of the integral as area under a graph (J). Mohamed et al. (2020) presents different 3D printed triangular prism to investigate students' reasoning about quadratic changes. 3D printed objects reifying fraction as value (F, I) and fraction of time using circle segments (K, O).

Opportunity 1: Exploring alternative technologies, added mathematical concepts, and resources for creating and sharing manipulatives

The results expose that previous research on DF in mathematics education is sparse. There is a need to explore different DF technologies, alternative manipulatives reifying more mathematical concepts, and investigating the use of available online resources. Most of the research has been done using 3D printers, limiting the type of manipulatives that can be created through additive manufacturing. Laser cutter or vinyl cutter enable subtractive manufacturing and open a new design space for manipulatives. So far, laser cutter technologies are expensive, but one could argue that they will become more commonplace in schools like 3D printers today. Manipulatives are created for geometry, algebra, and functions, as well as fractions. One interesting research path is investigating how manipulatives can reify other concepts such as arithmetic, decimal system, combinatorics, or probability, all are part of the mathematics curriculum. Research has focused on how students and teachers can use software to create 3D models that can be printed. There are plenty of online resources available for teachers to choose from e.g., Thingiverse. Understanding how manipulatives can be shared by teachers, taking part in the maker culture is missing. From prior professional development projects, I often met teachers requested pre-made material or lesson plans they can adopt to their own teaching. They express a lack of time to develop their own material on the one hand, but they prefer to be creative and develop their own teaching materials on the other hand.

Applied theories

Three theoretical characteristics have emerged: 1) constructivism and constructionism as theoretical underpinnings, 2) design theories for creating a learning context, and 3) theories for analysing teachers' and students' knowledge. Firstly, many of the reviewed articles refer to constructivism and constructionism as rationale when creating manipulatives (9). Papert's constructionism (Papert & Harel, 1991) is based on Piaget's constructivism (Piaget, 2013), but emphasizes that learning is

[2] https://thingiverse.com

happening when we create sharable things (Ackermann, 2001). Secondly, the research group around Greenstein frames the process of creating manipulatives as learning by design (H,I,K). The approach presented by Koehler and Mishra (2005) provides an "opportunity to consider the interplay between the evolving artifact and the application of teacher knowledge domains in the artifact's development" (Greenstein et al., 2019, p. 1152). Thirdly, teachers' technological, pedagogical, and content knowledge (TPACK) is explored in three articles (B,H,K). Dilling and Witzke (2020) apply theory of subjective domain of experience when analysing students' knowledge. According to this theoretical framework, students experience is linked to the specific learning context and their knowledge needs to be described according to their situational link, including previous experiences. Greenstein et al. (2020) is an article that investigates learning from a sociocultural perspective, using theory of commognition (Sfard, 2007) to analyse students' changes in discourses. In their article, they specifically analyse, students' narratives about; mathematical objects, participants of the discourse, learning about mathematics, and design decisions. An overview can be found at: shorturl.at/mnsN3.

Research methodologies

The included articles present different cases how DF was used in a teaching context. Most commonly, researchers report on their own teaching experiences in higher education (10), on interventions applying manipulatives in a classroom setting for K-12 students (5), or from a workshop or course for in-service teachers (2). Most articles (14) use purely qualitative methods in their research design based on observation, video recordings, interviews, and hand-ins. Three articles (B,H,L) use mixed methods to collect data including surveys, and pre- and post-tests. Research on teaching experiences in higher education analyse cases based on selected students and their work (D,F,I,J,K,M,O), data from students' self-reports (B,H), or description of the course and used manipulatives (A,N). Research projects involving K-12 students investigate interventions with manipulatives developed by researchers in a classroom setting (E,L), and students' learning outcomes when creating manipulatives (G,P,Q). Ulbrich et al. (2020) report on their experiences from a workshop series with 200 in-service teachers. They conclude that it is essential to learn more "about a teacher's needs and expectations using technologies" (Ulbrich et al. 2020, p. 303). Greenstein & Olmanson, (2017) provide a case of a DF course for in-service teachers including examples of created artefacts. Seven articles did not use a framework for analysing their results.

Opportunity 2: Adopting communities of practice as a framework for understanding in-service teachers learning of digital fabrication for mathematics education

The typical research project investigates DF for pre-service teachers applying constructionism (Papert & Harel, 1991) as theoretically underpinning describing learning from an individual perspective. Constructionism provides an acknowledged framework for understanding learning when creating and sharing manipulatives. However, for professional development projects with in-service teachers, a sociocultural perspective on learning is preferable, because it offers a "more effective means to, understand and implement an educational partnership for work-place learning" (Spouse, 2001, p. 512). The review found two research projects exploring how in-service teachers could use DF to create their own manipulatives and only one research project applies a sociocultural perspective. To consolidate research on DF for mathematics education both perspectives (individual and sociocultural) need to be investigated appropriately for both pre-and in-service teachers.

Communities of practice (CoP) deploys a situated learning perspective with sociocultural elements and is widely accepted in education research and work-place learning (Hammersley, 2005). CoP could provide a theoretical lens on how learning emerges when pre- and in-service teachers engage in authentic learning experiences, such as making, sharing, and using available manipulatives to develop their professional identity in their community (Wenger, 1999). CoP has been proposed as a suitable research paradigm for the mixed methods approach, which has been found in three articles included in this review (Denscombe, 2008).

Conclusion

This paper presents a critical review of 17 research articles on DF for creating manipulatives in mathematics education. Previous research is concentrated at stray research clusters. The typical DF research project explores manipulatives for reified mathematical concepts in geometry, algebra and functions, and fractions, using 3D printing and is published 2018 or later. Qualitative methods or mixed methods are predominant for data collection. The review reveals three main theoretical perspectives. Firstly, constructionism, with roots in constructivism (Papert & Harel, 1991), as underpinning theories. Secondly, design theories, such as Learning by Design (Koehler & Mishra, 2005), and thirdly, theories for analysing teachers' and students' knowledge. Finally, the paper provides two specific opportunities for consolidating research on DF for mathematics education: 1) Exploring alternative technologies, added mathematical concepts, and resources for creating and sharing manipulatives, and 2) Adopting communities of practice as a framework for understanding in-service teachers learning of DF for mathematics education.

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An Analysis of Educative Curriculum Materials for Formative Assessment

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Keywords: Formative assessment, educative curriculum materials.

Background

Black and Wiliam's comprehensive review (1998) that includes more than 250 studies worldwide suggests the close relationship between achieving learning outcomes and teaching practices associated with formative assessment. However, despite considerable research and development work, the implementation of formative assessment remains a problem. A key issue is that variety has been observed among the definitions and exemplars of formative assessment. Although these definitions and exemplars do not contradict each other, they highlight different elements. Formative assessment includes the elements such as identifying learning intentions, monitoring students' learning, regulating learning, identifying the actors of learning, and using appropriate teaching strategies and tools. All these elements should be considered holistically for productive practices. Moreover, it should be noted that the features of these elements can vary from discipline to discipline (and sometimes even between topics within the same discipline). Teachers need exemplars in order to understand the principles of formative assessment authentically and enact them in practice. Well-designed educative curriculum materials can be used to communicate these principles to teachers. That is to say, educative curriculum materials aim to improve teachers' learning alongside students' learning (Davis & Krajcik, 2005). In this research, the aims are to identify educative exemplars that can communicate the aspects of formative assessment to teachers and suggest principles that can guide future design. Using multiplicative reasoning as a critical case, the existing curriculum materials are analysed to achieve these research aims.

Theoretical Framework

Two frameworks guide the material analysis. First, formative assessment strategies suggested by Wiliam and Thompson (2007) are used in order to identify formative assessment techniques. These strategies include five aspects: understanding and sharing learning intentions, eliciting students' learning, feedback, using students for each others' learning and self-regulation. This framework is grounded in the idea that the function of formative assessment is to regulate learning. This regulation is examined in three ways: proactive regulation (before the lesson), interactive regulation (during the lesson) and retroactive regulation (making the decision for the next lessons or different contexts). Second, the educative features of the materials are analysed in terms of including the guidance to enact these techniques and the rationale of these techniques (Quebec-Fuentes & Ma, 2018).

Methods

The materials to be analysed were selected purposively from curriculum materials designed for early secondary mathematics teaching. First, these materials are chosen based on whether they include topics relevant to multiplicative reasoning such as ratio, proportion and geometric similarity. Second, the materials that include a separate teacher guide are chosen. Third, the materials that include formative assessment features explicitly or implicitly are chosen. When sampling the materials, whether the materials include formative assessment explicitly or implicitly has been decided according to how the designers advertise these materials. As a result, five groups of multiplicative reasoning lessons are chosen from the projects Mathematics Assessment Project (MAP), Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS), CornerStone Maths, Mathematics Formative Assessment Systems (MFAS) and from a commercial resource widely used by teachers in England, White Rose Maths. While MAP, ICCAMS and MFAS lessons are advertised as formative assessment resources, CornerStone Maths and White Rose Maths involve formative assessment as one element of teaching. The main reason to choose a variety of resources is to access rich formative assessment techniques and educative features. It is not aimed to compare the materials from different resources and rank their overall quality. Instead, the educative features in the data will be analysed to inform future design. In order to achieve research aims, reflexive thematic analysis is conducted (Braun and Clarke, 2006).

Progress after CERME 12 conference

At the CERME 12 conference, the poster that outlines the methodological decisions of this research was presented. Following the discussions at the conference, the data was revisited and the suggested initial themes in the poster were revised. As a result, in terms of educative support, five themes are identified: learning goals, students' thinking and misconceptions, students' participation, differentiation and assessment norms. These five themes are found as a result of a largely semantic coding that mainly explored explicit educative support in the materials. In the next step of the data analysis, we prefer a more latent coding orientation in order to reveal implicit educative support hidden in the materials regarding these five themes.

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Rethinking the notion of textbooks as mediators between the official curriculum and classroom practice

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Textbooks are frequently understood to mediate between the official curriculum and the classroom from an alignment perspective. Such an understanding of textbooks has advanced our knowledge of the multifaced nature of curriculum implementation, though it runs short on explaining the origins of aspects in textbooks that aren't typically covered in the official curriculum. As an alternative, I propose to understand textbooks as a locally constituted hybrid. Such a position does not a priori approach textbooks from their alignment with the official curriculum and classroom practice, but might take a network of social practices in relation to textbooks into consideration. I believe that this helps to inquire about the origins of the social aspects of teaching and learning mathematics typically covered in textbooks, but usually not covered in great detail in official curricula. I also believe that such an understanding helps to foreground textbook studies as a full-fledged research field.

Keywords: Mathematics textbooks, culture, social relations.

Introduction

Mathematics textbooks are commonly situated in between the official curriculum and the actual classroom practice, often with an assumption that content and values travel from the official curriculum through the textbook into the classroom (e.g., Pepin, Gueudet, & Trouche, 2013; Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002). Although the specific labeling differs between studies and frameworks, our field of study differentiates between the official curriculum (standards, national or regional frameworks, and sometimes even high-stake tests), written curriculum (the textbooks), teacher-intended curriculum (teachers' ideas for instruction), enacted curriculum (as instruction unfolds in the classroom), and attained curriculum (students' outcomes) (e.g., Remillard & Heck, 2014; Valverde et al., 2002). Underlying this conceptualization is an idea of alignment: an alignment between the official curriculum and the textbooks, the textbooks and the teachers' ideas, the teacher ideas for instruction and the actual instruction, and the actual instruction and students' outcomes. Such an understanding of curriculum and of textbooks has helped to advance the field's understanding of how curriculum possibly impacts student learning, and of the multifaced nature of curriculum implementation.

The curriculum implementation process is primarily framed from a content-related perspective. Yet, as much as teaching and learning mathematics is about mathematical content, it is a social activity, making it relevant to also inquire about the nature of the social relations between the teacher, students, resources, and mathematical content (Love & Pimm, 1996; Pepin & Haggarty, 2001; Rezat & Sträßer, 2012). Whereas official curricula typically don't contain much detail about the configuration of these social relations (Boesen et al., 2014; Valverde et al., 2002), mathematics textbooks do. Thus, the idea of alignment between the official and written curriculum runs short in understanding where the configurations of social relations in textbooks come from. To advance the field's related understanding, I propose to change a predominant understanding of how textbooks are culture. In this paper's remainder, I will first exemplify the predominant view of alignment in influential curriculum studies. I will then describe two views of culture in mathematics education, arguing that one particular view has dominated the field, and to understand textbooks as a locally constituted hybrid. Next, I will apply such an understanding to interpret an example textbook study of configured social relations and present a network alternative to the predominant alignment view.

An alignment view in influential curriculum frameworks and studies

In this section, I attempt to illustrate how an alignment view in thinking about curriculum has dominated the field. I do so in relation to the highly influential framework adopted in the TIMSS textbook studies and in relation to a group of cross-contextual studies that explicitly aimed to uncover part of the cultural dimension of teaching and learning mathematics.

The TIMSS textbook studies

The TIMSS studies center on a model of educational opportunities in which textbooks play a mediating role between systemic intentions and classroom instruction. The grounding of the model of educational opportunities, Valverde and colleagues (2002) describe, is a tripartite model of curriculum: the intended curriculum (the curriculum as system goals), the implemented curriculum (the curriculum as instruction), and the attained curriculum (the curriculum as student achievement). Stressing the mediating role of textbooks, Valverde and colleagues (2002) position the textbook as the potentially implemented curriculum between the intended and implemented curriculum. The language underlying the TIMSS textbook studies reflects a view that understands content and values to travel from the intended (or official) curriculum, through the textbook, into the classroom:

> The model envisions content standards, frameworks, programs of study and the like as primary defining elements of potential educational experiences. […] Even so, another characterization is essential for the model. This is a portrayal of the most important instruments intended to translate these goals into prescriptions or suggestions for specific opportunities to be created in classrooms. Thus, an additional important component of the TIMSS measurement strategy included a look at how textbooks provide templates for student actions in the classroom. […] How textbooks are related to the intended curriculum and the particular vision that they promote regarding what students are expected to learn are among the fundamental features of educational systems. […] The ability to link these features to other outcomes – such as instructional practices and student achievement is another strength of this approach. (Valverde et al., 2002, pp. 8–9)

The abovementioned description stresses the primary role of the intended or official curriculum, the role of textbooks as translators of the official curriculum, and the linking of textbooks to classroom practice and student achievement. In fact, the title of this study "According to the book. Using TIMSS to investigate the translation of policy into practice through the world of textbooks" (Valverde et al., 2002) stresses exactly this idea of linearity.

Other TIMSS textbook studies (Schmidt et al., 2001; Schmidt, McKnight, Valverde, Houang, & Wiley, 1997) have added empirical grounds to such a linear notion of alignment. Reflecting the textbook research field's focus on mathematical content (Fan, Zhu, & Miao, 2013; Herbel-Eisenmann, 2007), these two studies prioritized the content over the social dimension of teaching and learning mathematics. Also, building further on the TIMSS framework, Remillard and Heck (2014) have further specified the official curriculum. Underlying their model is also the notion of alignment from the official to the operational curriculum, partially mediated by textbooks. Remillard and Heck also draw an arrow from the operational to the official curriculum, yet this is one of the few arrows that don't get concretized or explained in great detail.

Cross-cultural textbook studies

Pepin and colleagues have studied French, German, English, and Norwegian textbooks in terms of their cultural underpinnings (Pepin et al., 2013; Pepin & Haggarty, 2001). This influential body of cross-cultural work found that the textbooks differed in terms of the particular mathematical focus and the sequencing of activities. French textbooks, for instance, were clearly sequenced according to a learning trajectory consisting of small investigations, essential content, and practice whereas Norwegian textbooks contained a mix of practical activities followed by exercises. These scholars understood the observed differences in terms of underlying beliefs about rationality (as part of encyclopaedism) in France and doing math in Norway. Traces of these underlying beliefs were also observed in official curriculum documents and teachers' actual teaching, which led Pepin et al. (2013) to conclude that these underlying ideas travel from the official curriculum, through the textbooks, to the classroom practice. Interestingly, Pepin et al. (2013) also mention that the French and Norwegian textbooks under study were designed by teachers and teacher educators, which calls to complicate the idea that underlying ideas travel from the official curriculum, through the textbooks, to the classroom practice. Teachers and teacher educators represent classroom and teacher education practice, each bringing in their particular interest (e.g., Apple, 1992). The design of textbooks by teachers and teacher educators is a frequent practice in multiple educational systems across the globe. It seems that multiple paths toward the textbook, or even a network is at stake. Therefore, I propose to reconsider textbooks as culture. That is at focus in the following section.

Two views of culture

Reflecting on research on instructional practices to support equitable learning opportunities in mathematics classrooms, Hodge and Cobb (2016) come to identify two views of culture in relation to teaching and learning mathematics. The vast majority of studies, Hodge and Cobb argue, is to be situated in the Cultural Alignment Orientation. Related studies understand culture as a network of relatively stable practices of a bounded community. Research then departs in students' out of school practices to work toward aligned classroom practices that offer meaningful opportunities to learn. The Classroom Participation Orientation, on the other hand, considers culture as a network of local hybrid practices that people jointly constitute. In this orientation, research tends to originate in classroom practices that offer meaningful opportunities and considers adjustments and supports to ensure participation of all groups of students. Hodge and Cobb argue that the latter orientation considers cultural alignment as one of the several resources for more equitable learning opportunities, but not as the default option as is the case for the first orientation, and hence as more promising in terms of providing instructional guidance.

Relating Hodge and Cobb's identified views of culture to textbook analysis, I believe it is fair to say that most of the (cross-cultural) textbook studies are to be situated in a cultural alignment orientation.

As illustrated in the previous section, influential studies point to the alignment of textbooks with the official curriculum and with classroom practices (Pepin et al., 2013; Pepin & Haggarty, 2001; Schmidt et al., 2001; Schmidt et al., 1997; Valverde et al., 2002). Instead, I propose to understand textbooks and the culture in textbooks as a locally constituted hybrid. Such a perspective does not a priori focus on textbooks in terms of their alignment toward the classroom and the official curriculum, but tries to better understand how textbooks embody culture and how actors representing multiple social practices – hence the hybrid – are engaged in jointly constituting – hence the local – textbooks as cultural artefacts. As Apple wrote about textbooks about 30 years ago, "[Textbooks] are the simultaneous results of political, economic, and cultural activities, battles, and compromises. They are conceived, designed, and authored by real people with real interests" (Apple, 1992, p. 4). In the following section, I will describe an example textbook study that focuses on the social dimension of teaching and learning mathematics – for which the cultural alignment orientation falls short in explaining where the observed configurations of social relations come from. I will then try to illustrate how the notion of textbooks as a locally constituted hybrid one might help to inquire in a more powerful way about the origins of these configurations.

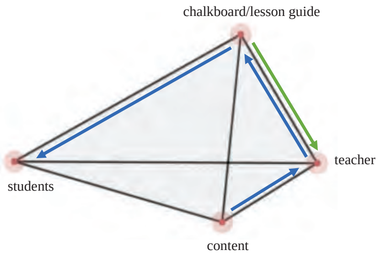
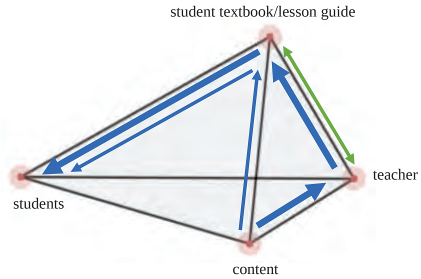
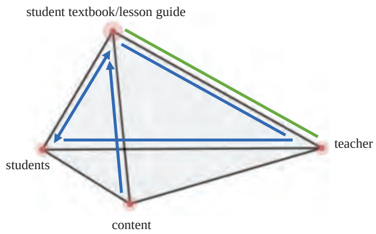
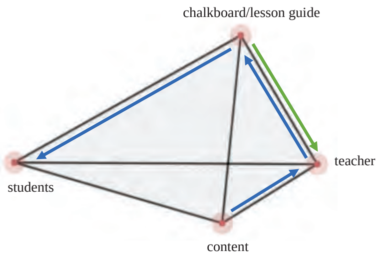
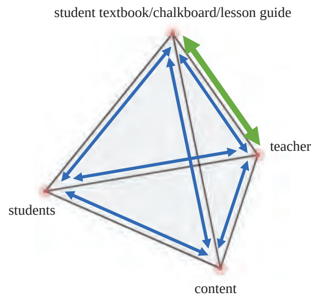
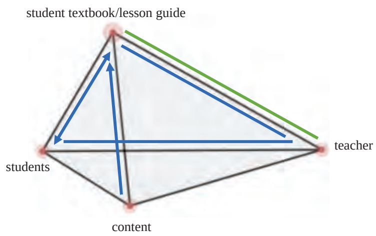
The textbook as a locally constituted hybrid

Intended configurations of social relations in textbooks from Sweden, USA, and Flanders

In a related study, I have analyzed textbooks from Sweden, the USA, and Flanders (Van Steenbrugge & Remillard, in preparation). We focused on lesson guides, or the section where textbook authors explicitly communicate with teachers about the author-intended lessons. Bezemer and Kress (2015) describe that writing is no longer the main mode of communication to facilitate students' engagement with textbooks, and we soon experienced that the same applies to the sections that talk explicitly to teachers. The selected textbooks from all three educational contexts appeared to communicate consistently through written text, layout, and images, which helped to identify particular configurations of social relations between the teacher, students, the central artefacts at stake, and the mathematical content (See Table 1).

We started from Rezat and Sträßer's (2012) tetrahedron model to visualize the social relations between teacher, students, and artefact, and their position relative to the mathematical content, but further specified the model in three ways. We differentiated between the social relations during lesson enactment (blue lines) and lesson preparation (green lines) because lesson guides both communicate about the intended lessons and are usually read by teachers during lesson preparation. We also specified the central artefacts at stake: student textbook and/or chalkboard during lesson enactment, and lesson guide during lesson preparation. Finally, we characterized the relations in terms of agency, intimacy, and prominence. Arrows signal who has agency over whom. In some cases, agency was shared between two nodes, which we indicated by means of a double-sided arrow. In other cases, none of the nodes seemed to have authority over one another, which we indicated by means of a line without arrows. Variations of distance between nodes is used to indicate that some social relations between nodes were more intimate compared to other nodes. We marked more prominent relations from less prominent ones by means of thicker lines. The figures appear to be quite consistent within each context, but differ considerably across the three contexts.

Table 1: Configured social relations in elementary mathematics textbooks from Flanders, USA, and Sweden (from Van Steenbrugge & Remillard, in preparation)

<table>
<tr><th>Flanders</th><th>USA</th><th>Sweden</th></tr>
<tr><td>Kompas (KP)
</td><td>Math in Focus (MiF)
</td><td>Matte Direkt (MD)
</td></tr>
<tr><td>Nieuwe Tal-rijk (NT)
</td><td>Everyday Mathematics (EM)
</td><td>Matte Eldorado (ME)
</td></tr>
</table>

These representations of the particular privileging of configured social relations relate to particular ideas about what it means to be a student learning and a teacher teaching mathematics. These ideas seem to differ primarily from context to context, with some more within-context variation in the USA, where multiple groups advocate for certain views in relation to teaching and learning mathematics. In Flanders, for instance, student access to mathematics seems to be mediated by the teacher and the chalkboard. During class, the teacher, ultimately, is the authority of what counts as mathematical correctness. This stands in sharp contrast to the particular configuration of social relations in EM, from which it seems that mathematical authority is a shared responsibility. One of the aspects that differentiates the Swedish textbooks from the others is their configuration of the teacher-related relations. The blue lines without arrows between teacher and student, and teacher and textbook indicate that the teacher has a responsive, rather than an active role during the lesson. The important message here is that the configurations of social relations in textbooks, and the underlying particular ideas of what it means to be a mathematics student and teacher, seem to differ across contexts. This points at context-specific culturally valued norms as to what it means to teach and learn mathematics. That is the starting point for the next section.

Where does the configuration of social relations come from?

The previous section suggests that textbooks from different educational contexts privilige particular configurations of social relations, which have roots in underlying meanings of being a student and teacher. These differences call to nuance a linear conception as to how culture finds it ways into the classroom and the assumed transmissive role of textbooks in this matter (e.g., Pepin et al., 2013; Valverde et al., 2002). Official curricula do not contain such detailed descriptions of social relations as the ones represented in Table 1 (Boesen et al., 2014; Valverde et al., 2002). So where do these configurations of social relations in textbooks come from? Even if it is likely that the enacted lessons differ from the pictures that come out of our analysis (Stein, Remillard, & Smith, 2007), it is worth thinking about where the observed cross-cultural differences come from. In discussing that particular question, I propose to consider textbooks as a locally constituted practice. Thus, rather than primarily understanding textbooks in function of the official curriculum and classroom practice, it is worth focusing on the social practice of textbook design in trying to better understand how cultures of mathematics teaching and learning are constituted in textbooks.

From a locally constituted hybrid perspective on textbooks, the observed privileging of configurations of social relations represent mathematics education culture amongst the actors involved in textbook production. In many cases, textbook authoring includes participation among teachers, teacher educators, researchers, and other stakeholders such as the inspectorates. Each brings in a specific interest, even if not explicitly named. The textbooks under study are published by commercial publishers, which also brings an economic aspect into the picture. Also, families or other influential communities such as the labor market practices influence textbook production and adoption (e.g., Apple, 1992). The view of textbooks as a locally constituted hybrid helps to uncover potential sources other than the official curriculum. Explicitly studying this hybrid and locally constituted nature of textbooks therefore seems to have the potential of adding to our understanding of the cultural and political significance of textbooks.

How could an alternative to the predominant view of alignment look like?

I think that Valero’s (2010) conceptualization of mathematics education as a network of social practices is helpful in this regard. Such a conceptualization stresses that the social practices outside of the classroom, such as the textbook design practice, are as significant practices of mathematics education as the actual classroom practices. Figure 1 is an attempt to capture some of the relevant social practices in relation to textbooks and textbook design. There are connections with the official curriculum and classroom, but this is not the main flow. There are also connections to other social practices such as teacher education, teaching, high stakes testing, mathematics education research, and digitalization. These social practices are also connected to one another. High stakes testing, for instance, has connections to the official curriculum and the classroom. Digitalization has connections to the mathematics education research field, the classroom, and families. As indicated by the dark blue lines in Figure 1, the configurations of the social relations, then, can come from a negotiation (locally constituted) between teacher educators, teachers, and researchers involved in the design of textbooks, where each draws on her/his particular background (the hybrid). Nowadays, and especially with the COVID-19 pandemic as a catalysator, digital textbooks and platforms are increasingly present in the learning and teaching of mathematics, which also potentially impacts the configurations of social relations (e.g., Ruthven, 2018). Furthermore, the constitution of the network and the prominence of social practices and connections between these practices can accommodate to the particular educational context. In Sweden and the USA, for instance, significant curriculum interpretation resides at the individual teacher level and in high stakes testing, respectively (Van

Steenbrugge et al., 2019). This makes the official curriculum – teachers – classroom connections particular prominent in the Swedish network compared to the official curriculum – high stakes testing – classroom connections in the USA (See the orange and red lines in Figure 1).

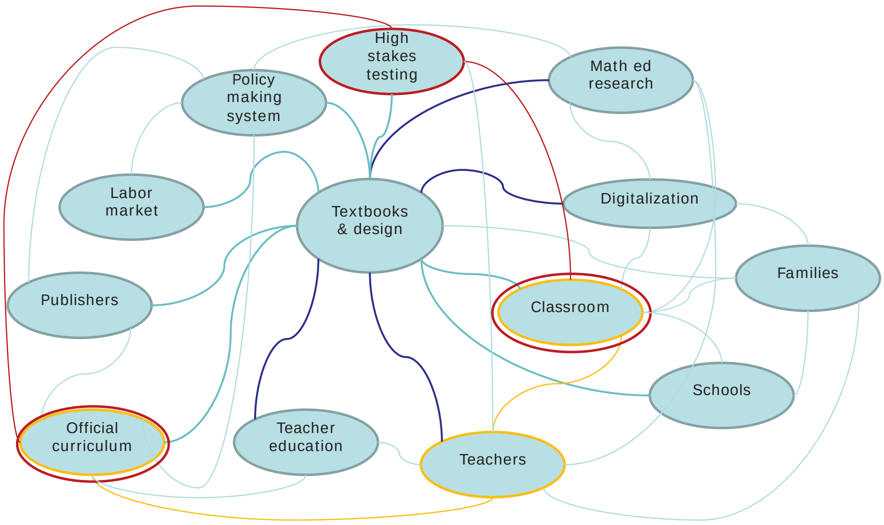


Figure 1: Textbooks in a network of social practices

As one can see in Figure 1, the network allows to take distance from one primary form of alignment. Rather than considering other factors as potentially influencing alignment, it understands these factors as constitutive elements of the network of social practices that relate to textbook design. It also points towards understanding textbook studies as a research field on its own, not just as auxiliary to studying classroom practices and student performance.

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Problematizing the notion of problem posing expertise

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Problem posing, which we view as a form of task design performed by the learner, is an important mathematical activity. Several studies have investigated differences between expert and novice problem posers, but no clear definition of problem posing expertise has generally been agreed upon. In a study involving 66 masters and 60 sixth-grade students we identified expert problem posers based on their performance on problem posing tasks (with or without numbers/context) rather than as in prior research presupposing who expert problem posers were based on their mathematical experience or backgrounds. The results showed that masters students had a significantly lower proportion of the Top-level (expert) problem posers and a significantly higher proportion of the Bottom-level (novice) problem posers than sixth graders, demonstrating that presupposing problem posing expertise based on mathematical experience or backgrounds would have been misleading.

Keywords: Mathematical problem posing, experts, novices, task design.

Introduction

Problem posing has recently attracted much attention among researchers and educators, including curriculum standards (e.g., NCTM, 2000). As a form of a curricular opportunity (embedded in curriculum resources) in which teachers and students can engage, problem posing is an important mathematical activity like the well-known problem solving. Several types of problem posing tasks have been used in research and teaching practice (Lee, 2021). The aim of having problem posing tasks in curriculum resources could be not only for assessing students' mathematical understanding or mathematical learning, but also for supporting students to become competent problem posers and problem solvers (Cai et al., 2015). One research strand in this area views problem posing as a goal of mathematics instruction and focuses on how one develops problem posing capacity (Cai & Leikin, 2020), showing that students and teachers are capable to pose problems (Cai et al., 2015) and that training to problem pose is feasible (Cai et al., 2015; Koichu & Kontorovich, 2013). However, little attention has been paid thus far to the meaning of problem posing expertise. This is problematic because without a good understanding of what constitutes an expert problem poser, efforts to develop problem posing capacity among students and teachers are lacking important conceptual foundation.

In this study, we aimed to identify expert problem posers by examining our participants' performance on problem posing tasks rather than as in prior research presupposing who expert and novice problem posers were based on their mathematical experience or backgrounds. Specifically, we addressed the following research question: *How can we distinguish between expert and novice problem posers based on their performance on problem posing tasks, and how do the resulting groups of expert and novice problem posers compare with how these groups would normally be defined based on participants' mathematical experience or backgrounds?*

Theoretical Considerations

Experts in Mathematical Problem Posing

Several researchers explored differences between expert and novice problem posers by assuming that problem posing expertise was connected to a particular attribute of participants' mathematical experience or backgrounds – such as their problem solving experience (Pelczer & Gamboa, 2009), teaching experience (Voica & Pelczer, 2009), or mathematical maturity (Zhang et al, under review) – but the results showed that their "expert" problem posers were not always performing better than the "novices." For example, Voica and Pelczer (2009) compared problems posed by pre-service and in-service teachers, considering the former group to be novices in problem posing and the latter group to be experts. They found that their perceived expert group did not perform better than their perceived novice group, presumably because in-service teachers' pedagogical knowledge and classroom experience constrained their views of the problems that could be posed. Also, our prior research (Zhang et al, under review) had examined problem posing by comparing mathematically more mature students (masters students majoring in pure mathematics or mathematics education) and mathematically less mature students (sixth-grade students). The surprising finding that the more mature participants did not outperform the less mature participants in problem posing further prompted us to problematize the notion of problem posing expertise and its presumed association with participants' prior mathematical experience or backgrounds.

Kontorovich and his colleagues conducted a series of studies to explore the possible characteristics of the problem posers for mathematical competitions, arguing that those who systematically create problems for high-level mathematical competitions may be considered as expert problem posers (Kontorovich, 2020). While this notion of expertise is still linked to participants' mathematical backgrounds, problem posers for mathematical competitions may be less controversial as an expert group than in-service teachers or masters students due to mathematical competitions often being reputed as treasures of "elegant," "intriguing," and "surprising" problems that reach the students after thorough committee discussions (Koichu & Andzans, 2019).

To conclude, no clear definition has generally been agreed upon for who problem-posing experts are though prior studies have provided some useful insights into possible characteristics of these experts. The participants of those studies identified as expert problem posers were presupposed to be experts based on their prior mathematical experience or backgrounds. Yet this approach to identifying experts yielded some surprising findings such as novice problem posers outperforming presumed experts.

Criteria for Identifying Expert Problem Posers

Most of the research on problem posing has involved the assessment of the types, quality, and quantity of the posed problems, often inferring from these the participants' problem posing ability. Silver and Cai (2005) proposed three criteria that might be selectively applied to most problem-posing tasks when used in assessment settings: quantity, originality, and complexity. Several researchers have adapted these criteria to assess participants' problem posing ability (Cai et al., 2020). In what follows, we discuss four common criteria used in prior studies for assessing the problems been posed. In the study we report herein, we used these criteria to operationally measure problem posing performance and thus as criteria for identifying expert problem posers.

The first criterion is *the number of mathematical problems posed* (Leung & Silver, 1997; Silver & Cai, 2005). Silver and Cai (1996) found that nearly 30% of the problems posed by middle school students were either nonmathematical problems or simply nonproblem statements (even though the directions clearly asked for mathematical problems). Crespo and Sinclair (2008) hypothesized that students' difficulties generating mathematical problems might relate to a lack of opportunity to explore a problem situation adequately before and during the posing process. Therefore, the number of mathematical problems been posed might reflect how adequately the posers explore the problem situation. The number of problems posed was also used as a measure of fluency in studies that assessed creativity using problem-posing as a test tool (Bicer et al., 2020). The more mathematical problems being posed, the more adequately posers might have explored the problem situation and the higher their anticipated level of mathematical fluency.

The second criterion is *the number of posed problems that are solvable* (Cai et al., 2015; Leung & Silver, 1997; Zhang et al., under review). Even though students and teachers were found to be able to pose mathematical problems, the posed problems were not always solvable or relevant (Silver & Cai, 1996), which suggests that the participants might not have selected enough elements or organized adaptive relations to construct the problems. The more solvable mathematical problems being posed, the more adaptive elements and relationships posers might have selected to formulate the problems or the better their understanding of the problem-posing tasks.

The third criterion is *the complexity of the posed problems*. Some research measured this criterion by the number of steps for solving the posed problem (Cai et al., 2020; Leung & Silver, 1997). According to the concept of problem space proposed by Milinkovic (2015), any problem can be described in terms of its context, elements, and the relationships between elements. Zhang et al. (under review) considered the sum of relationships and elements in the constructed problem space to reflect the complexity of the posed problems. The larger the sum of relationships and elements constructed in the problem, the more complicated a problem space has been formulated by the poser.

The fourth criterion is *the clarity of the posed problems*. According to NCTM (2000), using mathematical vocabulary, notation, and structure to clearly represent ideas, describe relationships, and model situations are important for students' ability to communicate mathematically. Zhang et al. (under review) found that nearly a third of the problems posed by students were expressed unclearly or partially clearly. The greater the clarity of the posed problem, the better the ability of the poser to communicate in problem posing.

Methodology

Participants

The participants were 66 masters students and 60 sixth-graders from 11 classes of a primary school, all in China. The masters students were recruited via an advertisement that we posed to call for volunteers among masters students majoring in pure mathematics or mathematics education in a university among the top 100 Chinese universities. Regarding the sixth-graders, 5 or 6 students from each class of the selected school, which was affiliated with the aforementioned university (ranked in the top 10 of the local city), volunteered to participate in this study. Each participant signed an informed agreement letter prior to the study. The sample was appropriate for our study because, according to the conventional way of defining problem posing expertise based on participants' experience or backgrounds, the former group would be the experts and the latter the novices. Whether these expert/novice groups would be confirmed empirically was the subject of the research.

Data Collection

The study was conducted in a soundproof and uniform light laboratory with a voice recording. The participants were given a description of the research procedure along with brief instructions that they were expected to think aloud while completing the Problem-Posing Tasks (PPTs). Then they underwent a training phase to get familiar with problem posing via two simple tasks. Two kinds of test combination (1A-2B-3A-4B and 1B-2A-3B-4A) were provided on a daily rotating cycle. The participants chose their available time to get one of the test combinations, described in Table 1, and were tested individually. We considered two types of tasks (with/without number; with/without context) since several researchers (English, 1998; Leung & Silver,1997) have predicted that task format affects subjects' problem posing performance. The target tasks were chosen from the PPT-number test comprising translated versions of tasks used by Leung and Silver (1997), which included the task situations of House Purchase and Pool Maintenance, and the PPT-context test comprising modified versions of tasks used by Cai et al. (2020), which included the task situations of Driving Home and Sporting Goods. No time limits were set on participants' work with the tasks.

Table 1: Test items and distribution of participants

		PPT-number				PPT-context			
		House Purchase		**Pool Maintenance**		**Driving Home**		**Sporting Goods**	
		1A[1]	1B	2A	2B	3A	3B	4A	4B
Masters Students	P_1=30	✓			✓	✓			✓
	P_2=36		✓	✓			✓	✓	
Sixth Graders	N_1=30	✓			✓	✓			✓
	N_2=30		✓	✓			✓	✓	

[1]*Format A in the PPT-number test is the task with numbers, Format B in the PPT-number test is the task without numbers; Format A in the PPT-context test is the task with context, Format B in the PPT-context is the task without context.*

Data Analysis

We re-analyzed the data in our prior research (Zhang et al., under review), guided by our new research question. The responses of the 126 participants who completed the PPTs were recorded and coded with respect to six criteria as shown in Table 3. These included the four main criteria for identifying expert problem posers that we discussed earlier and two further sub-criteria in which "the complexity of the posed problems" was divided into "the largest sum of relations" and "the largest sum of elements."

Principal Components Analysis (PCA) was conducted according to the four main criteria to select the Top 15% (19 participants, experts) and the Bottom 15% (19 participants, novices) from the overall sample of 126. We chose 15% so as to be selective but also to have enough students in each group to conduct meaningful statistical analyses. For the correlation assumption test, the correlation between each index (criterion) ranged from 0.507 to 0.975 (>0.30), which meant that there was a linear correlation between each index; the KMO (Kaiser- Meyer- Olkin) value of 0.773 indicated that the sampling was adequate (a value between 0.7 and 0.8 was taken to be "middling"); the Bartlett's test of sphericity with an associated p value of <.001 indicated that we could proceed to use PCA to analyze our data. The result of total variance explained showed that only the first component had eigenvalues over 1.00 (that is, 3.267) and that together they explained 81.669% of the total variability in the data. This led us to the conclusion that one factor solution was adequate.

Results

Top 15% and Bottom 15% participants' distribution

We sorted the participants according to the main factor extracted by PCA, and we selected the top 15% (19/126) participants and the bottom 15% participants (19/126). The general information of both groups is shown in Table 2. Surprisingly, in the Top 15% group nearly 80% of the participants (N=15) were sixth graders; only 4 participants were masters students. A Z-Test showed that there is a significant difference (at 95% confidence interval) between masters students and sixth graders in the Top 15% group. In the Bottom 15% group, nearly 70% of the participants (N=13) were masters students; only 6 participants were sixth graders. There is a significant difference (at 80% confidence interval) between masters students and sixth graders in the Bottom 15% group. Table 2 also shows the distribution of participants in the two categories of tests.

Table 2: Top 15% and Bottom 15% participants' distribution

	Categories of students			**Categories of tests**[2]	
	Master students	Six graders	Z-Test [1]	1A-2B-3A-4B	1B-2A-3B-4A
Top 15% (total=**19**)	4	15	2.234(95%)	9	10
Bottom 15% (total=**19**)	13	6	1.492(80%)	13	6

[1]95% (80%) refers to significant difference at 95% (80%) confidence interval (2-tailed). [2]Specific information about the categories of tests was shown in Table 1.

Top 15% and Bottom 15% participants' performance in each of the problem posing criteria

We selected the top-level and bottom-level students by ranking the *combined score* on the main factor, which was extracted by PCA. However, the top-level students' performance in *each criterion* was not necessarily better than the bottom-level students' performance. This made meaningful the comparison of the two groups across each criterion separately, which we present in Table 3.

Regardless of the task format (with or without context/numbers), the results of a multi-factor analysis of variance indicated that the mean performance of the Top 15% participants was higher than that of the Bottom 15% participants on all criteria. In Table 3 we present the findings for the two PPT-context tasks; the findings for the two PPT-number tasks were similar but are not reported here due to space constraints. Specifically, for the Driving Home task, the Top 15% participants posed more mathematical problems (F=83.97***,η_p^2=.712), posed more solvable problems (F=102.4***,η_p^2=.751), posed problems with a larger sum of relationships and elements (more complicated problems) (F=63.36***,η_p^2=.651), and posed more clearly-expressed problems (F=90.28***,η_p^2=.726).

Table 3: Top 15% and Bottom 15% participants' *performance* on the PPT format with/without *context*

Criteria		**Driving Home**			**Sporting Goods**		
		With context	Without context	*Group effect*	With context	Without context	*Group effect*
Number of math problems posed	Top 15%	7.67(2.9)	5.30(1.7)	F=83.97***	7.70(4.9)	3.11(1.1)	*F=23.74****
	Bottom 15%	1.77(0.7)	0.50(0.5)	η_p^2=.712	2.00(0.9)	0.31(0.5)	η_p^2=.411
	Task format effect	*F=9.70**; η_p^2=.222*			*F=12.95**; η_p^2=.276*		
Number of solvable problems posed	Top 15%	7.33(2.4)	5.20(1.8)	F=102.4***	5.30(2.3)	3.11(1.1)	*F=55.97****
	Bottom 15%	1.69(0.8)	0.17(0.4)	η_p^2=.751	1.33(0.8)	0.23(0.4)	η_p^2=.622
	Task format effect	*F=12.03**; η_p^2=.261*			*F=12.93**; η_p^2=.276*		
Complexity of the posed problems	Top 15%	10.33(1.3)	7.00(0.0)	*F=63.36****	10.00(2.4)	7.00(0.0)	*F=37.18****
	Bottom 15%	5.23(2.7)	1.17(2.9)	η_p^2=.651	4.83(3.4)	1.62(3.1)	η_p^2=.522
	Task format effect	*F=28.99***; η_p^2=.460*			*F=12.91**; η_p^2=.275*		
- The largest sum of relationships	Top 15%	3.78(0.4)	2.00(0.0)	*F=52.53****	3.00(0.8)	2.00(0.0)	*F=37.35****
	Bottom 15%	2.00(1.0)	0.33(0.8)	η_p^2=.607	1.33(1.0)	0.46(0.9)	η_p^2=.524
	Task format effect	*F=52.53***; η_p^2=.607*			*F=12.74**; η_p^2=.273*		
- The largest sum of elements	Top 15%	6.56(0.9)	5.00(0.0)	*F=62.78****	7.00(1.6)	5.00(0.0)	*F=36.96****
	Bottom 15%	3.38(1.8)	0.83(2.0)	η_p^2=.649	3.50(2.3)	1.15(2.2)	η_p^2=.521
	Task format effect	*F=19.67***; η_p^2=.366*			*F=12.94**; η_p^2=.276*		
Clarity of the posed problems	Top 15%	7.56(2.8)	5.20(1.8)	*F=90.28****	7.50(4.6)	3.11(1.1)	*F=26.45****
	Bottom 15%	1.77(0.7)	0.17(0.4)	η_p^2=.726	1.83(1.0)	0.23(0.4)	η_p^2=.438
	Task format effect	*F=12.08**; η_p^2=.262*			*F=13.00**; η_p^2=.277*		

*[1] According to Cohen (1988)'s partial eta squared, 0.01 is considered a small effect, 0.06 is considered a medium effect, and 0.14 is considered a large effect. [2] ***p < .001, **p < .01, *p < .05.*

Discussion

Prior research tended to presuppose problem posing expertise based on participants' mathematical experience or backgrounds (Koichu & Kontorovich, 2013; Kontorovich, 2020; Pelczer & Gamboa, 2009; Voica & Pelczer, 2000). In this paper, we problematized whether this way of identifying expert problem posers is indeed valid. Specifically, we followed a data driven approach to identify expert problem posers based on participants' problem posing performance rather than presupposing who the expert problem posers were based on participants' prior mathematical experience or backgrounds. Had we followed prior practice, we would have considered the masters students to be the experts and the sixth graders to be the novices in problem posing. However, our PCA showed that the masters students had a significantly lower proportion of Top-level problem posers and a significantly higher proportion of Bottom-level problem posers than the sixth graders. This result indicated that it was reasonable to identify expert problem posers by their problem posing performance rather than by their mathematical experience or backgrounds. Also, we compared the Top-level and Bottom-level groups across each of the four criteria used in prior studies for assessing problem posing performance; in PCA the 126 participants were ranked by an extracted main factor with a combined score. The fact that experts outperformed novices in each theory-driven criterion complements the PCA findings in suggesting that the identification of problem posing expertise by the data driven approach as used in this study was indeed appropriate.

From interviews with masters students in our prior study (Zhang et al., under review), we can get some insights as to why the masters students as a group did not do as well as one might have expected in the problem posing tasks. When asked about what their biggest challenge was when posing the problems, the masters students indicated that the cues motivating them to construct new problems were those they had seen in their primary school textbook or relevant reading materials. They did not attempt to use higher level mathematical knowledge. Also, the masters students were more used to working on problem solving tasks rather than problem posing tasks. Conversely, over the last decades primary school students have had increasing opportunities to pose problems in their classes. This is supported by Cai et al.'s (2017) investigation of the problem posing tasks in Chinese textbooks from the 1990s to the 2010s, which found that the number of problem posing tasks significantly increased over the years. The aforementioned indicates that participants' problem posing performance is influenced by many factors, and so it is not reasonable to assume in advance that one group of participants will be the expert group simply from their level of mathematical knowledge/backgrounds.

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Improving the GeoGebra Classroom tool to better accommodate online educational resource development based on the SAMR model

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This paper describes an exploratory case study within a project aiming to develop a novel connected classroom technology (CCT) for offering new possibilities for teaching with technologies. It outlines a lesson in which three-dimensional coordinates were taught through an online resource created based on the SAMR (Substitution, Augmentation, Modification, Redefinition) model and the GeoGebra Classroom tool. Moreover, this paper presents findings of our study focusing on the interconnections of tangible and digital tools as well as on tasks that offer whole-class activities and student interactions. Findings of our study indicate that creating online resources based on the SAMR model, collaborative work between developers of tools and task designers could be valuable. In addition, when creating online resources offering advanced teaching possibilities, developers of these tools and task designers must collaborate and interact with teachers who are using them.

Keywords: Class Activities, connected classroom technology, digital tools, tangible tools, educational resources.

Introduction

This research project aims to develop a new, mathematically suited, tool offering novel opportunities for mathematics teaching to support teachers as well as to assist students' learning. Our objective is to develop a tool that allows a variety of possible applications in teaching and supporting learning. The idea is to use powerful mathematics software and further develop an existing online tool into a connected classroom technology (CCT) by implementing new features and functionalities (see Zöchbauer & Hohenwarter, 2020; Zöchbauer et al., 2021). The technology, in this case, is GeoGebra Classroom, an online tool for teachers' and students' devices that are connected online and can be used for teachers and students to interact with each other. GeoGebra Classroom can be used for interactive teaching approaches in various settings and can be applied to all levels of education. Moreover, we have great opportunities to work with developers to adapt this tool based on our research findings.

"Connected classroom technology refers to a networked system of personal computers or handheld devices specifically designed to be used in a classroom for interactive teaching and learning" (Irving, 2006, p. 16). This technology offers a broad range of innovative features such as facilitating communication between teachers and students, displaying student responses in real-time, and allowing rapid aggregation of student work by teachers (Irving, 2006; Shirley et al., 2011; Wright et al., 2018). Furthermore, CCT enables most students to contribute directly to interactive activities and to play a more active role in classroom discussions (Shirley et al., 2011; Wright et al., 2018). GeoGebra Classroom could be also used in the classroom for formative assessment enabling teachers to utilize a wide variety of teaching methods. According to the Organisation for Economic Co-operation and Development OECD (2005), "formative assessment refers to frequent, interactive assessments of student progress and understanding. Teachers are then able to adjust teaching approaches to better meet identified learning needs” (OECD, 2005, p. 13).

Within this project, we conduct studies in various STEM lessons for evaluating potential uses of GeoGebra Classroom for mathematics teaching and learning. This paper describes our most recent research, where the first author was using tangible and digital tools in a lesson by conducting an exploratory case study with a mixed-methods approach. Moreover, the first author created online resources that offer a unique experience thanks to new technology. For creating such online resources the SAMR (Substitution, Augmentation, Modification, Redefinition) model of Puentedura (2006) was used.

Regarding the combination of tangible and digital tools for the learning and teaching of mathematics, there has been a growing interest, which is also due to a slow evolution of teachers’ practices in integrating digital tools in mathematics teaching (Soury-Lavergne, 2021). Nemirovsky and Sinclair (2020) “distinguish digital tools as being computer-screen centred and tangible tools as being for holding, grabbing, and manual transport” (Nemirovsky & Sinclair, 2020, p. 107). By using digital tools as well as tangible tools there is a high potential for bodily engagement (Nemirovsky & Sinclair, 2020).

In this case study, the first author used online resources with GeoGebra Classroom as the digital tool (see Figure 2) and some Dick System Maths Cubes together with a handmade 3D coordinate system (see Figure 1) as tangible tools. For working with the digital resource, all students used an iPad.

This case study is part of a larger project, where the GeoGebra Classroom tool is evaluated with mixed methods, where we utilized the combination of educational research with usability and user experience methodologies. In this case study, we focus on the usage of the online resource, which is adapted according to the SAMR model. Moreover, we explore the potential of GeoGebra Classroom for whole-class activities and collaboration as well as for group discussions. This case study shows an intermediate step in the development of the online tool and how to improve it. Consequently, the research question for this case study is: How can the GeoGebra Classroom tool be improved to better accommodate online educational resources developed based on the SAMR model?

Theoretical Framework

For creating online resources, we looked at different models and chose Puentedura's SAMR model (Puentedura, 2006). This model provides a tool for integrating digital media into the classroom. Moreover, it describes how teaching and learning are transformed using technology and explains how the design and processing of tasks in the classroom can be improved (bildung.digital, 2021). The four technological levels of use of the SAMR model are *Substitution*, *Augmentation*, *Modification*, and *Redefinition*.

At the lowest level, *Substitution*, there is only a transfer of analog worksheets to digital worksheets happening without functional improvement. An enhancement or a simple functional improvement of the tasks happens in the second level, *Augmentation*. The first two levels are considered *Enhancement* steps, whereas a *Transformation* takes place in the last two levels. In level *Modification*, the tasks will be rephrased in a way, so that digital support is essential for solving the tasks and where the advantages of the technology will be made explicit. In the top-level *Redefinition*, the tasks will be created in a way that solving them without technology is not possible (bildung.digital, 2021; Puentedura, 2006).

Setting and Task Design

The case study involved a class of 7th-grade secondary school students (age 12-13), learning about the coordinates in a 3D cartesian coordinate system. We have chosen this group of students as they used GeoGebra Classroom and GeoGebra resources before and therefore we do not have to focus on learning the system but could focus on the usage. In a classroom setting, all students received the Dick System Maths Cubes together with a handmade 3D coordinate system (see Figure 1) as tangible tools and an iPad for working with the digital resource (see Figure 2). The students received the devices at the beginning of the lesson from the school equipment. In the class, 21 students participated in the lesson, but only two-thirds of them filled in the questionnaires afterward.

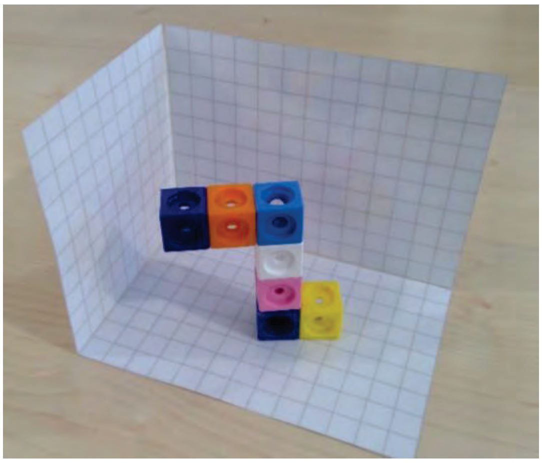

Figure 1: Dick System Maths Cubes

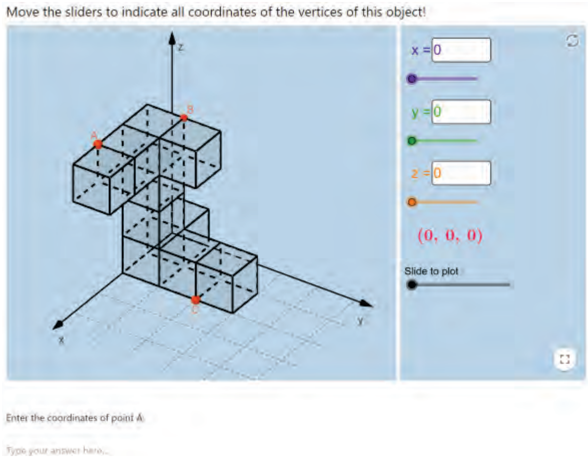

Figure 2: Online Resource

The resource was created by the first author, and the tasks were formulated according to the SAMR model. When creating the tasks, we tried to follow the steps of the SAMR model until we reached the top-level *Redefinition*. For the students, the resource was created in German, a translation can be found here: https://www.geogebra.org/m/tuzwbwez.

Firstly, texts were inserted into the online resource as they would have been on a paper and pencil worksheet (level *Substitution*). Secondly, questions were inserted in the online resource as multiple-choice questions (MCQ) or open questions (level *Augmentation*). Then, the 3D object was not inserted as an image, but as an object in the GeoGebra 3D applet, where additional features are possible. Students can move or rotate the object so that they can observe it from different perspectives. Moreover, some sliders are added, where students can move them to show and move a point in the coordinate system (level *Modification*).

As *Redefinition*, a whole-class activity was inserted as the last activity in the online resource. For this activity, the teacher collected all responses of students reasonably quickly and used them for further questions. As a first task, students entered coordinates of a point that was located a) inside the object, b) outside the object, or c) on the surface of the object. Afterward, the teacher selects one of the answers, reads it aloud and the other students had to choose where this point was located by selecting the correct answer of the MCQ. Before the teacher was choosing another example, all students selected “New”. This activity is considered a redefinition because, within seconds, the teacher can select random students’ examples and observe students' responses to these new tasks by using the MCQ feature. This way the teacher generates a lot of new tasks and can assess students’ understanding quickly and conveniently.

Methods

To answer the research question, we have selected an exploratory case study. In a case study, a temporary phenomenon is being investigated in-depth and in a specific real-world context (Yin, 2014). In this research, the first author used the current version of the GeoGebra Classroom tool together with some tangible tools in a lesson sequence and collected different kinds of data. A mixed-methods approach was used, and qualitative and quantitative data were analyzed with different instruments.

The whole lesson sequence was video- and audio-recorded with cameras and the teacher’s device was screen recorded. Moreover, we automatically collected student products in GeoGebra Classroom and additionally the teacher’s reflective notes that were made after the lesson. After the lesson sequence, students received an online questionnaire, which consisted of some information about the students, a German translation of the System Usability Scale (SUS) by Brooke (1996), some open questions about their experience with GeoGebra Classroom, and the German version of the User Experience Questionnaire (UEQ) by Laugwitz et al. (2008). The SUS is a simple usability scale giving a global view of subjective assessments of usability (Brooke, 1996), whereas the main goal of the UEQ is a fast and direct measurement of user experience (Laugwitz et al., 2008).

In this case study, the Content Structuring Analysis (Mayring, 2014, 2020) was used, where, as a first step, the categories were defined. According to Mayring (2020), the qualitative content analysis approach is category-based, where the categories refer to aspects within the text and the content analysis procedure is research question oriented. Therefore, concerning the research question, we defined the categories regarding the four technological levels of use: *Substitution, Augmentation, Modification,* and *Redefinition*. Content structuring in particular means to “filter out from the material specific content dimensions and to summarize this material for each content dimension” (Mayring, 2014, p. 104), and this was accomplished in the next step. Then, we extracted all coded material for each category and summarized it according to Mayring (2014). In each category, all issues are considered, where the usage of GeoGebra Classroom with the resource did not work as expected or where obstacles or interruptions happened during the lesson. Moreover, we collected all issues, where an improvement, either from the teacher or the online resource could be made.

Results

In the first category *(1) Substitution,* all issues are summarized that are referring to the transfer of analog worksheets to online resources only. In the beginning, students needed some minutes to access the online resource and asked three times in total for the link, although it was presented with the QR code on the projector screen above the board in the classroom. Even if the teacher saves time for printing and copying the worksheets in advance, during the lesson, it took nearly five minutes of the 30 minutes lesson sequence to hand out all the other things like iPad, cubes, and coordinate systems.

One thing that was mentioned by students in the questionnaire three times and also seven times during the lesson, was that there were some lags during the lesson caused by the usage of the GeoGebra Classroom tool. Additionally, in the recorded video and teachers' reflective notes, it was shown that not all students used tangible tools for solving the given tasks. Some of them only worked with digital tools and not with tangible ones.

Category *(2) Augmentation* summarizes all issues, where a functional improvement was visible. The teacher could observe in real-time all responses of the students and check several times on the dashboard if the students finished the task on the online resource before discussing it. But during the work with the MCQ, some issues occurred. When discussing the results of the MCQ, the teacher never knew if all students answered this question when opening the chart with all students' responses. Although the teacher could observe in real-time all changes of the selected answers, and see which one has selected which option, there was no information if every student had already answered the question. In the first attempt, the teacher counts all numbers that were shown, but if one student selected two possible solutions, then this method did not work. One thing that worked well was that the teacher showed the different notations of the coordinates (see Figure 3) and explained to them the correct mathematical notation of coordinates.

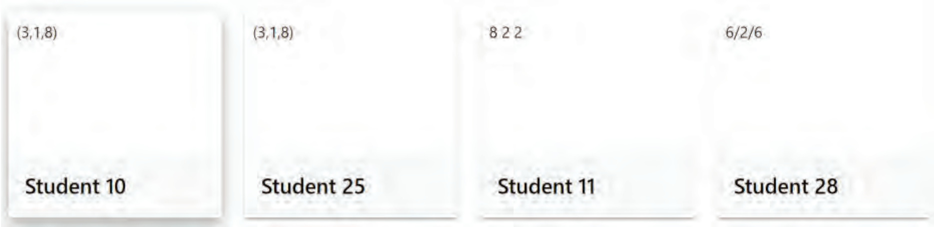

Figure 3: Example of Student Responses regarding Coordinates Notation

All issues regarding the inserted GeoGebra 3D applet (tasks with a significant redesign) are collected in the category *(3) Modification*. The online resource that was used in this lesson, was initially created to be used on a computer with the mouse, but, during the lesson, the students worked with iPads. Therefore, additional hints of using the GeoGebra 3D applet without a mouse would have been helpful. For example, in some cases, the object rotated all the time and students did not know how to stop it. The teacher was interrupted several times to help the kids with this problem. Another issue that occurred was that students could accidentally drag some of the cubes' edges in the 3D applet and therefore change the object dramatically. It was not possible to reset the object to the original one, as the reset button was not available for the 3D view, and refreshing the page did not help either. In this case, these students had to leave GeoGebra Classroom and the online resource and rejoin as new users. Although their work was saved on the teacher dashboard, the students could not access their previous work anymore.

The last category *(4) Redefinition* shows all issues regarding the whole-class activity, a previously inconceivable task. Nevertheless, for the work with the whole-class activity, where several rounds were played, the teacher requested in the reflective notes to just reset all the MCQ solutions instead of choosing the fourth answer "NEW" and to get a notification, when or if all students answered the MCQ. Because during this short whole-class activity it always took some time, until all students switched from their answer to "NEW" to start with the new example. Moreover, during orchestrating the discussion, the teacher was interrupted two times to help students with technical issues.

The results of the SUS and UEQ will be compared within this project and evaluated in the next phase of the project. Furthermore, the analysis of the relation between the SUS and UEQ data, and the qualitative lesson observation is still in progress and will be presented later.

Discussion

Sinclair and Robutti (2020) describe that the use of digital technology has the two main functions supporting organizing students' work and supporting new ways of doing and representing mathematics. In our study, it was shown that the GeoGebra Classroom tool has the potential to be seen as supporting both functions, whereas in both cases an improvement is necessary.

This case study aimed at investigating how the GeoGebra Classroom tool can be used for online education resources based on the SAMR model of Puentedura (2006). As it was presented in the results section, there are currently several issues that need to be addressed. To answer the research question, we came up with ample amounts of suggestions on how the GeoGebra Classroom tool can be improved to better accommodate online educational resources developed with the SAMR model.

First, sharing the resource with the students should be made easier and faster. The lags which were also mentioned in the questionnaire by the students should be fixed. Nevertheless, it is not clear whether this problem is caused by the GeoGebra Classroom tool itself, the WI-FI connection in the school, or the devices that the students used.

As it was mentioned by Shirley et al. (2011) and Wright et al. (2018), students play a more active role in classroom discussions and the technology enables most students to contribute directly to these interactive activities, such phenomena were also shown during the lesson. Students worked with the online resource and participated actively in the last activity, the whole-class activity. However, when working with the MCQ, it would be great to have an overview of how many students just answered an MCQ instead of always going back to the overview page or counting them. If this was made possible, the teacher could foster students to play even a more active role in the whole-class activity and could easily assess students' understanding quickly. Moreover, a reset button for the MCQ would be helpful when asking a lot of similar questions with the same answer options. Furthermore, in some cases, the option to only allow one possible choice of the MCQ would be helpful.

Besides, there occurred even more issues during the lesson sequence, that are not due to the new tool, but to the creation of the online resource. Thus, before further evaluating the tool, the online resource needs to be enhanced as well. Regarding the 3D applet, editors must always ascertain that all objects are fixed and cannot be dragged by students accidentally. Moreover, a reset button for every view in the GeoGebra applet should be inserted or activated. Furthermore, teachers need to tell students in advance how various features work, such as the rotating feature, and how the students can stop the rotation. Therefore, maybe some additional hints of using the 3D applet would have been helpful for the students and to avoid interruptions. Hence, it was shown that the *Transformation* levels *Modification* and *Redefinition* were not completely fulfilled in this online resource. Some students could not benefit from the 3D object, as there occurred several problems and therefore the 3D applet with the sliders was useless and the resource did not reach the *Modification* level. To have an online resource, where all stages of the SAMR model are fulfilled, the resource and the system need to be further developed. Moreover, teachers as well need some guidance to work with similar classroom activity tasks.

Additionally, if teachers want students to work with a combination of digital and tangible tools, they should make that connection clearer. Only handing out tangible tools and just giving oral tasks is not enough. Teachers could present this assignment as the first task on the online resources to advise them to build the given object by themselves or verbally repeat this several times and check again if really all students use these tangible tools. Another idea could be to use the *Pause* button of GeoGebra Classroom to interrupt the work of students. When activating the *Pause* feature, students cannot continue working on their tasks and the teacher can get the students' attention and give them important information or additional explanations.

This case study is part of a bigger project and the results of the SUS and UEQ will be compared within this project and evaluated. Once the tool is further developed, the research will be repeated with another case study in a similar setting, but with an enhanced tool and online resource. As the preliminary findings already show, the user experience of the tool is also dependent on the creation and the design of the tasks in online resources.

To sum up, tasks designers, developers, and teachers should be aware of several issues related to educational online resources that are in the *Transformation* level as well as in the *Enhancement* level of the SAMR model of Puentedura (2006). As we can learn from this case study, when creating online resources according to the SAMR model, there are more collaborators than the teacher involved. To have online resources that could offer new possibilities of teaching, developers of new tools and task designers need to work together and with the help of teachers, who are in the classrooms every day and can observe and confirm if the developments are successful.

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TWG23: Implementation of research findings in mathematics education

Introduction to the papers of TWG23: implementation of research findings in mathematics education

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This introduction offers an overview of the eighteen contributions (15 papers and three posters) to the TWG23 at CERME12. The three thematic discussions that took place in this Thematic Working Group are addressed, namely: the role of change, matters of scaling, and the conception of stakeholders.

Keywords: Implementation research in mathematics education, change, scale, stakeholders.

Introduction

The 12th Congress of the European Society for Research in Mathematics Education (CERME12) was first postponed and then held virtually, in February 2-5 2022, due to the COVID-19 pandemic. Before the conference, a virtual pre-conference event was organized in February 2021. This event included two discussion sessions dedicated to Thematic Working Group activities, and particularly to planning the activity of the Thematic Working Group 23 (TWG23) "Implementation of research findings in mathematics education". This introductory report gives an overview of the TWG23 at CERME12.

Contributions to the TWG23 at CERME12

The TWG23 at CERME12 was led by Mario Sánchez Aguilar, Boris Koichu and Morten Misfeldt, along with Rikke Maagaard Gregersen (until August 2021) and Linda Marie Ahl (from September 2021). The TWG23 received 18 contributions consisting of 15 papers and three posters. The authors of the contributions came from Austria, Denmark, Finland, Greece, Israel, Italy, Luxembourg, Mexico, Norway, Sweden, and the UK. The contributions were organized into thematic categories:

- Implementation of problem-solving and problem-posing approaches
- Implementation of teaching models and teachers' perspectives on implementation
- Conditions for sustainable implementations
- Diagnostics tasks, instructional sequences, and curriculum design
- Implementation of programming, computational thinking, and other digital technologies

Due to the overarching nature of the poster by Konrad Krainer, it was selected to be the first contribution to be presented in the TWG23. This poster presentation served to set the scene in the group by means of providing the TWG23 participants with conceptual categories (technical rationality, reflective rationality, and societal rationality) that were then often referred to during the conference for identifying and contrasting different approaches to the implementation of innovations and implementability of research.

The first thematic category "Implementation of problem-solving and problem-posing approaches" included contributions providing methodological and theoretical tools for the implementation of mathematical problem-solving and problem-posing approaches. The paper by Nafsika Patsiala and Ioannis Papadopoulos presented an instrument to record and examine whether students develop — through their experiences with problem-posing — the habit of mind named "seeking and using structure". The paper by Jason Cooper and Boris Koichu introduced the notion of "problem-solving implementation chain" for analyzing the evolution of a problem-solving activity as it passes from the proponents to teachers and finally to students.

The second thematic category grouped two types of contributions. One type of contribution consisted of the studies that addressed the implementation of specific teaching approaches. This was the case of the research by Morten Blomhøj and collaborators, who introduced a three-phased didactical model to facilitate the implementation of an inquiry-based approach to mathematics teaching. Another research in this group of contributions was presented by Ola Helenius, who offered a theoretical discussion on large-scale implementation of a research-based teaching model for elementary school arithmetic. The other type of contributions within this category included studies focusing on teachers' actions and perspectives on implementation processes. This focus was evident in the study by Maria Kirstine Østergaard and Uffe Thomas Jankvist, who used theoretical constructs from implementation research (IR) to identify elements of a mathematics teacher's practice and thereby identified factors that seemed to influence the implementation of teaching units aimed at fostering students' reflections on the nature of mathematics as a discipline. Other papers included in this category were by Åsmund Lillevik Gjære, who examined one Norwegian teacher's enactment of an innovative system for mathematics teaching called developmental education in mathematics; and by Alessandra Boscolo, who reported the perspective of teachers about the implementation of active, bodily experienced mathematics learning activities.

The contributions grouped in the third thematic category brought to the fore the discussion of the sustainability of the implementation of innovations. The paper by Johan Prytz and colleagues paid particular attention to the issue of sustainability of an innovation in mathematics education, and the potential role of textbooks in sustaining the innovation. Another contribution within this category was the work presented by Mario Sánchez Aguilar and Apolo Castaneda, who pointed out the importance of distinguishing between politics of enactment and implementation as the first step in integrating the analysis of political sustainability into IR.

In the fourth category, a group of contributions addressed issues of the implementation of diagnostics tasks, instructional sequences, and curriculum design. One contribution included in this category was the work presented by Morten Elkjær and Jeremy Hodgen. These scholars formulated an implementation process model for designing and implementing tasks for formative feedback in an online learning environment. Another contribution in this category was the literature review developed by Linda Marie Ahl and colleagues. This review dealt with the implementation of instructional sequences aimed to enhance students' learning of mathematical concepts or competencies. The review identified which competencies are targeted in the chosen sample of studies and what characterizes the implementation of the instructional sequences. Also included in this category was the contribution of Ellen Jameson and collaborators, who pointed out challenges to implementing mathematics education research through the processes and products of curriculum design. The work reported by Tuula Koljonen and colleagues examined the feasibility of using the Documentational Approach to Didactics to gauge the fidelity and characteristics of teachers' implementation of scripted teaching sequences for the teaching of arithmetic in primary school.

The fifth category included contributions focused on the implementation of digital technologies, mainly those related to computer programming. One case is the work by Raimundo Elicer and collaborators, who addressed the role of educational task design in implementation research. In collaboration with a 4th grade school teacher, the authors designed a geometric task from a hypothetical learning trajectory that required students to draw on their knowledge of mathematics and Programming and Computational Thinking (PCT). Another presentation was based on the paper by Andreas Lindenskov Tamborg and colleagues, who reported on the development of a survey tool to investigate how PCT is implemented in Denmark, Sweden, and England. Within this category was also the poster called "Comparing the implementation of programming and computational thinking in Denmark, Sweden and England" by Morten Misfeldt and collaborators (this poster was one of three winners of the first ever ERME Poster Award). Finally, another poster included in this category was by Ben Pierre Haas and collaborators, where they gave an overview of how technologies such as augmented reality, 3D printing, and tutoring systems could be employed by different users for teaching and learning STEAM-based educational ecosystems.

Thematic discussions

The TWG23 program included three thematic discussions. The themes were selected on the basis of previous discussions that had emerged as central for IR. Namely, they had been identified based on the papers published in the special issue on IR of ZDM – Mathematics Education (Koichu et al., 2021) as well as based on the papers published in the first two issues of Implementation and Replication Studies in Mathematics Education (IRME).

The role of change

The first theme was "intended change". The background for this theme was that it has proved difficult to measure the success of implementing innovations. For instance, are we reaching the intended change in a given implementation? And how can this question be addressed? The question for the first thematic discussion was:

> How can we work with articulating and evaluating intended change? What are the pros and cons in relation to constructions such as program theory (theory of change) and realistic evaluation?

Matters of scaling

Large-scale implementations are complex endeavors, often intending to reach a large number of classrooms. If an idea is proven to work well in a certain setting, we are interested in disseminating it into other settings. As phrased by Artigue (2021, p. 22): "...implementation research, even if it can take advantage of studies of limited scope, must be able to meet, as a genre, the scaling-up challenge". The question guiding the second thematic discussion was:

> What are the possible roles of small-scale vs. large-scale studies in implementation research? For example, in relation to different phases such as planning, testing, enacting, and evaluating.

The conception of stakeholders

No large-scale implementation can be successful without the scaffolding of stakeholders at different levels. Krainer (2021, p. 1175) asked: "Who are the relevant stakeholders whose voices should be heard when discussing implementation? What is the role of policymakers, administration experts, researchers, and practitioners with regard to defining and solving problems that occur in practice?" With insight into the absolute necessity of having a strategy for working with stakeholders, the theme for the third thematic discussion was:

> How do we work with stakeholders? This includes conceptualizing their roles, designing/framing their participation, and evaluating the impact of their involvement in relation to three levels: administration/policymakers, researchers, and practitioners.

Concluding remarks

Concerning "change", the TWG23 participants agreed that the tension between intended change and achieved change in an implementation project is a delicate question of interest for our research field. As for "scale" and scaling, the participants agreed on the need for both small-scale and large-scale studies in mathematics education IR. Moreover, it was clear that there is a need for further discussions on the conception and definition of "stakeholders". The overall outcome of the TWG23 thematic discussions was that none of the three themes addressed can be fully explored in isolation from the others.

Future research directions emerging from the TWG23 discussions at CERME12 are the following: we need to discuss how a theory of change can be used to design, understand and evaluate implementations. It is necessary to further explore how small-scale and large-scale studies can provide the relevant information for different parts of an implementation process. There is a need for progressing our knowledge on how the concept of stakeholders can be used to refine different types of analysis of implementation projects. These discussions will continue at CERME13 in Budapest in 2023, and also on the pages of IRME.

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Political challenges for the implementation of research knowledge as part of educational reforms and mathematics textbooks

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This paper reports an exploratory study of political factors that may influence the implementation of research knowledge in the formulation of educational reforms and mathematics textbooks in Mexico. The study is based on the analysis of an in-depth interview with a key informant, who has extensive experience as a textbook author and as an advisor in the Ministry of Education of Mexico. Two instances are identified in which there is an effort to implement research knowledge from the field of mathematics education in the study programs or the mathematics textbooks. The factors that appear to have hindered the implementation of such research knowledge are also identified.

Keywords: Implementation research, educational policy, mathematics textbook research.

Introduction

The implementation of any educational innovation is shaped by political factors. There are studies in the field of mathematics education that illustrate aspects of the interaction between policy and the implementation of innovations in mathematics instruction (Krainer, 2021; Prytz, 2021). These studies highlight the importance of having a better understanding of the political factors that influence the implementation of innovations in mathematics education. Lester (2018) notes that politics is centrally important to implementation for at least two reasons: First, it creates the legal authority, funding, and other capacities that are needed to implement a program effectively. Second, politics does not stop when a law has been enacted. It continues throughout the implementation process.

The importance of recognizing the political in implementation research has been pointed out by some educational researchers. McDonnell and Weatherford (2016) mention that there is little evidence in educational research about the political dynamics of implementation. However, it is also acknowledged that a problem with this empirical focus is that much of the political action takes place far from the public eye, making its analysis complicated. Thus, the study of the political factors that may shape the implementation of educational innovations tends to be elusive.

This paper reports on an exploratory study that focuses on identifying political factors that have the potential to shape the content of educational reforms, in particular the content of mathematics textbooks that are officially authorized for use in the lower secondary education system in Mexico. This study is based on the analysis of an interview with an experienced mathematics education researcher, who has extensive experience as a mathematics textbook author but also has an academic advisor to the Ministry of Education of Mexico (in the following called the ministry).

Background and study aim

One of the main concerns of the TWG23 has been: how can we bring into practice the accumulated research knowledge produced within the field of mathematics education? (Jankvist et al., 2017). Addressing this concern implies acknowledging the existence of factors that influence the enactment of any educational innovation, but it also entails recognizing that some of these factors may be political in nature. The studies by Krainer (2021) and Prytz (2021) illustrate aspects of the intricate interrelationship between the implementation of innovations and policy.

Krainer (2021) refers to the IMST (Innovations in Mathematics and Science Teaching), which is a project triggered by Austria's modest performance in the TIMSS advanced mathematics and physics achievement test of 1995, and with the aim of fostering innovations and improving teaching and learning at secondary schools in Austria. Through the analysis of this project Krainer asserts that in the implementation of large initiatives such as the IMST project "*policy*, *research*, and *practice* need to be regarded as influential and closely interrelated communities regarding implementation." (2021, p. 1185, emphasis in the original). In turn, Prytz's (2021) historical investigation provides evidence on how the role of researchers and the procedures that they followed for preparing three Swedish major development projects in mathematics education—the New Math project (1960–1975), the PUMP project (1970–1980), and the Boost for Mathematics project in (2012–2016)—, co-varied with the shift from centralization to decentralization that happened gradually in school governance policy from the mid-1970s to the 2010s.

The study reported in this paper addresses TWG23's expressed concern: 'how can we bring into practice the accumulated research knowledge produced within the field of mathematics education?', but placing special emphasis on the political factors that may influence the implementation of such research knowledge. Thus, the aim of this research study is *to identify possible political factors that influence the implementation of research knowledge produced within the field of mathematics education, as manifested in educational reforms and mathematics textbooks in Mexico*.

The empirical data for this study consists of an in-depth interview with a mathematics education researcher with a history of collaboration with the ministry as a textbook author and advisor. To make sense of the information contained in the informant's narration, some notions related to the politics of implementation were used. These notions are introduced in the following section.

Conceptual framework

The Oxford Advanced Learner's Dictionary define *politics* as "the activities involved in getting and using power in public life, and being able to influence decisions that affect a country or a society". This definition conforms to the conception of 'politics' used in this study. However, we are aware of the existence of early efforts to formulate a notion of 'implementation politics':

> Implementation politics is, I believe, a special kind of politics. It is a form of politics in which the very existence of an already defined policy mandate, legally and legitimately authorized in some prior political process, affects the strategy and tactics of the struggle. (Bardach, 1977, p. 37)

Political scientists such as the previously cited Eugene Bardach (1977) and Pressman and Wildavsky (1973) are pioneers in the study of implementation as part of policy processes. In the field of educational research, McDonnell and Weatherford (2016) take these early works as a reference to argue that politics permeates the implementation process, but they also warn about the importance of distinguishing between the politics of enactment and the politics of implementation, as a first step in integrating the analysis of political sustainability into implementation research.

The *politics of enactment* refer to politics that come into play during the process in which a policy or reform is configured and becomes official. The *politics of implementation* refer to the process of translating a policy or reform into educational practice. McDonnell and Weatherford (2016) pinpoint three cross-sectional characteristics to identify political factors "that influence whether a policy option gets on decision makers' agendas and is eventually enacted" (p. 235), namely: (1) the time frames and (2) the decision venues for policy design and enactment as compared with those for implementation; and (3) the interest-based coalitions active during each phase. Table 1 shows a comparison of how these three characteristics can be manifested in the politics of enactment and in the politics of implementation.

Table 1: Comparison of how the time frame, decision venues and interest-based coalitions can be manifested in the politics of enactment and in the politics of implementation. Taken from McDonnell and Weatherford (2016, p. 235)

Comparing the Politics of Enactment With the Politics of Implementation

	Enactment	Implementation
Time frame	• Episodic; proposals on and off the policy agenda, sometimes over long periods • Proposals modified as policy problems and political coalitions change • Once on a decision agenda, proposal passed or rejected quickly	• Continuous process, typically over several years or longer • Variation comes from different interpretations, not changes in formal provisions • Long time frame for implementation may be bad politics
Decision venues	• Contained within one or two venues, with clearly defined decision rules • May be visible only to policy elites • Information available from a limited number of well-known sources	• Multiple venues, with considerable variation • Many small decisions by implementers aggregate to become agency policy • Information typically diffuse and anecdotal • Localized, varied, and unpredictable policy feedback
Interest-based coalitions	• National interest groups dominate the process • Enactment coalitions and resulting policy create incentives for whether winners and losers remain mobilized postenactment	• State and local affiliates and grassroots groups often have incentive to mobilize • Groups active during enactment may modify issue position • Interests may support or oppose the policy by attaching other policies and political issues to it

The notions contained in this section were used to give meaning and structure to the narration provided by the key informant who participated in this exploratory study. The characteristics of this key informant and other components of the research method are discussed in more detail in the next section.

Method

The exploratory research reported in this manuscript uses as source of information an in-depth interview with a key informant with extensive experience as an author of mathematics textbooks and as an educational advisor. In the following sections we provide details about this key informant and the interview conducted.

The key informant

Key informant refers to the person with whom an interview about a particular organization, social program, problem, or interest group is conducted. According to Fetterman (2008), key informants "are individuals who are articulate and knowledgeable about their community. They are often cultural brokers straddling two cultures" (p. 477). The importance of brokers in the development of national mathematics curricula has been acknowledged, as they are individuals who "act as conduits for introducing elements of one practice [research] into another [mathematics teaching]" (Potari et al., 2019). They are insiders that typically provide information through in-depth interviews and informal conversation, and their experience is important to assess their quality as a source of information.

The key informant participating in this study is a broker. He is an experienced mathematics education researcher who has 20 years of experience as a mathematics textbook author. He has also been part of disciplinary commissions invited by the ministry, which provide advice on the disciplinary content that should be included in the national educational reforms and the study programs. For 36 years he has held a researcher position in a mathematics education research department in Mexico City. The key informant was emphatic in clarifying that he knows first-hand most of the information provided in the interview, however, he also acknowledges that there is part of the information that he knows second-hand through third parties directly involved in the process.

The interview and its analysis

As mentioned before, an in-depth interview was used as a research instrument for this study. It was carried out on September 3, 2021 via Zoom, and lasted 1 hour and 10 minutes. The two authors of this paper participated in the interview. There was no script for the interview, but rather an informal chat between the three people involved. Before starting the interview, an attempt was made to communicate to the informant the purpose and meaning of the research study. The interview was recorded for later analysis.

A tape-based analysis (Onwuegbuzie et al., 2009) was applied to the interview recording. Here the researchers first became familiar with the data—by listening to the interview repeatedly—in order to identify the parts of the interview that provide information about the political factors that may influence the implementation of research knowledge produced within the field of mathematics education in mathematics textbooks. This procedure was applied independently by each of the authors of the study. Subsequently, a meeting was organized between the researchers to exchange their views on which moments of the interview provided relevant information for the study. These key moments of the interview were identified and agreed upon through a discussion among the researchers. The results of this analysis are presented in the next section.

Results

The interview with the key informant was rich in information. Through the analysis of his account, it was possible to identify a structure that interrelates different political actors involved in the design of the study programs for primary and lower secondary education in Mexico, which are official documents from which the mathematics textbooks are derived. The analysis of the interview also allowed us to identify two instances in which there is an effort to implement research knowledge from the field of mathematics education in the study programs or the mathematics textbooks. The factors that appear to have hindered the implementation of such research knowledge were also identified.

Next, the structure that interrelates different political actors involved in the design of the study programs for secondary education in Mexico is presented, which will be used as a reference to illustrate the two instances in which there is an unsuccessful effort to implement research knowledge from the field of mathematics education.

The political structure underlying the development and establishment of study programs

When planning an educational reform, the ministry is in charge of producing the *study programs* for secondary education. Study programs are guiding documents aimed mainly at teachers from all over Mexico, which contain the topics that must be covered for each discipline—including mathematics—, as well as the teaching approaches that must be used to communicate them. These study programs are the foundation for developing the textbooks. For the ministry to approve the national use of a mathematics textbook, it must adhere to the contents and procedures expressed in the mathematics study program.

According to the key informant, there are different groups of stakeholders involved in the configuration of the study programs. There are *disciplinary commissions*, which are groups of specialists from different areas of knowledge who act as counselors to the ministry about the contents and approaches that should be reflected in the study programs. There are also the *teachers' associations*, who through workshops and consultation forums obtain previews of the potential contents of the study programs, and express their opinions to the ministry about them. In turn, the ministry considers these opinions to define the study programs.

When the study programs are approved they are published and made available to the public[1]. It is then that *textbook writers*—sponsored by publishing houses—can read, interpret and translate the ideas of the study programs into their textbooks. These textbooks must be evaluated and approved by *expert evaluators*, which are scholars hired by the ministry to verify that the textbooks adhere to the authorized study programs.

Unsuccessful efforts to implement research knowledge—and hindering factors

The analysis of the interview led to the identification of two instances where there are unsuccessful efforts to implement research knowledge from the field of mathematics education. These instances are named: (1) it is important to include probability as part of the mathematics education of primary school students, and (2) it is necessary to distinguish between a 'problem' and an 'exercise' when adopting a problem-solving approach. The following sections illustrate what each of these instances consisted of, and what are the factors that seem to hinder their materialization.

(1) *It is important to include probability as part of the mathematical education of primary school students*. The informant recalls an educational reform in 2009, in which the ministry decided to remove the teaching of probability from the study program for primary education—indeed, the 2009

[1] The current study programs are available at https://www.planyprogramasdestudio.sep.gob.mx (information in Spanish).

study program for primary education in Mexico does not include the teaching of probability (see Secretaría de Educación Publica, 2009). However, research suggest an intuitive approach to the teaching of probability where children start from their intuitive ideas related to chance and probability. There are also recommendations related to understanding and applying basic concepts of probability for children in Grades 3–5 (see Batanero et al., 2016). Our key informant—at that time a member of the disciplinary commission for mathematics—was part of the opposition to this measure. He considered it inappropriate and in opposition to world trends in probability teaching.

Informant: When I was participating [in the disciplinary commission] I said that this could not be, that this was an issue that did not correspond to what all countries did. Rather, all countries incorporate probability from the beginning of primary school, and we did not have probability throughout primary school! […] In fact, I invited Carmen Batanero to give some talks there at the ministry and I had discussed this issue with her. And she mentioned that it did not seem like a good idea that probability was not in elementary school.

Despite the opposition, the teaching of probability was not included into the study programs. The informant identifies teacher associations as a possible obstacle in the incorporation of probability teaching in primary school. According to the informant, the teaching of probability was postponed to the middle school level on the argument that primary school teachers have many difficulties in understanding probability and therefore were not well prepared to teach the topic.

Informant: What was the justification for this? X told me [he mentions a ministry official] that it is because the teachers did not understand probability. They had complained that they did not know what to do, and that is why it was removed. I think that was a very strong decision that it was due to the teachers' complaints in the different workshops.

Instance (1) serves as an illustration of the politics of enactment that can take place during the configuration of a study program or curriculum. It illustrates how decision venues are reduced—only some selected individuals have access to the configuration of the study program—, but it also shows the existence of interest-based coalitions (such as mathematics teachers and their associations) that can hinder the enactment and implementation of research knowledge due to various reasons, such as the lack of adequate mathematical knowledge for the teaching of certain topics.

(2) *It is necessary to distinguish between a 'problem' and an 'exercise' when adopting a problem-solving approach*. In the educational reform of 2011, the study program for lower secondary education indicated a problem-solving approach to the teaching of mathematics. The analysis of the interview suggests that the lack of clear guidelines on what it means to adopt a problem-solving approach in a mathematics textbook, is a factor that seems to minimize the fidelity and homogeneity of implementation of a problem-solving approach to mathematics textbooks.

According to the informant, the study program contains only general guidelines on how instruction should be, and these guidelines are primarily aimed at teachers. The lack of clear and specific guidelines for the preparation of textbooks opens a space for different interpretations of the study program. Writers should interpret the study program and translate it into a textbook, hoping that their interpretation matches that of the expert evaluators (for their book to be approved by the ministry) and that of the mathematics teachers (for them to purchase the book).

An illustration of this is our informant's interest in having his textbooks clearly distinguish between an 'exercise' and a 'problem', as suggested by the fundamentals of problem-solving in mathematics education (e.g., Schoenfeld, 1985). However, implementing this distinction in his textbooks may have negative consequences. For instance, the informant thinks that his textbooks are not popular among mathematics teachers, precisely because they include actual problems and not exercises, which makes his textbooks more difficult for teachers and students to use.

Informant:	I, for example, think that my textbooks are difficult for students and teachers […] Because I try to pass on mathematical messages like the ones that are promoted... For instance problem solving. I try to make the problems actual problems in the sense that research says, that is, they are situations that the student will not know how to solve from the start, and that they may have to make an effort to solve it. Well, just that idea makes the textbook that I write different from other textbooks where for them [the other authors] a problem is an exercise.
Interviewer:	It is easier for the teacher to use this second kind of book than yours.
Informant:	Exactly, exactly.

Instance (2) illustrates aspects of the politics of implementation as experienced by a textbook author when trying to implement research ideas as part of the contents of a mathematics textbook. The textbook author who participated in this study claimed to receive diffuse and non-explicit information about the specific contents of the textbooks, which requires interpretation. However, the author's interpretation of the specific contents may be in opposition to the interpretation of the expert evaluators who have an influential opinion on which books are authorized by the ministry.

Conclusion

We have tried to illustrate the potential of the concepts of "politics of enactment" and "politics of implementation" to illuminate the intricate relationships between the implementation of innovations in mathematics education and politics. These theoretical notions bring to the fore the issue of political sustainability into implementation research. In particular, this theoretical framework allowed identifying and situating two political factors that may influence the implementation of research knowledge from mathematics education: (1) interest-based coalitions —such as mathematics teachers and their associations, and (2) issues of (mis)communication within decision venues. It would be necessary to analyze other implementation experiences to corroborate the potential of this conceptual framework for the uncovering of the intricate relationships between implementation initiatives and politics.

We are aware that the empirical method used in this study—which is based on an in-depth interview of a single key informant—may have some reliability issues. However, it is possible to enhance the rigor and reliability of this method. One way to do this is by increasing the number of key informants, and trying to corroborate and triangulate the consistency of their testimonies and interpretations. Another possible strategy is to use the "respondent validation" technique where the interviewee is asked to assess whether the researchers are accurately interpreting their experiences that were the focus of the study.

Our intention is to continue and expand the study reported here, interviewing more key informants (brokers) and implementing strategies to enhance the rigor and reliability of the research method.

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Implementation research on instructional sequences focusing on mathematical concepts and competencies: Results from a review

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To get an overview of how many and which studies in the literature specifically address the implementation of innovations from mathematics education research, we have conducted a systematic literature review. We report on the subset of 19 studies from the review, dealing with the implementation of instructional sequences aiming to enhance students' learning of mathematical concepts or competencies. The research question is: Which mathematical concepts and/or competencies are in play, and what characterizes the implementation of the instructional sequences? Results show that spatial reasoning, patterns, and structure gained the most interest, addressed in 6 studies. The other studies are relatively evenly spread over the concepts: algebra, arithmetic, calculus, number theory and proof, and the competencies: proportional reasoning and problem-solving. Seven studies, with long-term goals, describe a design for scaling the implementation.

Keywords: Implementation research, mathematical concepts, mathematical competencies, goals.

Introduction

That implementation research (IR) in mathematics education research (MER) has gained momentum during the past few years is beyond any doubt. Since 2017, a thematic working group (TWG 23) at CERME has been dedicated to the topic. In 2021, a new journal—*Implementation and Replication Studies in Mathematics Education* (IRME)—was launched by the well-established Dutch publishing house, Brill. Finally, also in 2021, a special issue of ZDM was dedicated to the topic of implementation research in mathematics education. Ongoing discussions in relation to IR in MER concern, for example, the use of theoretical constructs from outside the field of MER (e.g. health science, economics, etc.) versus those available inside of MER; what we should take implementability to mean in relation to IR in MER; to what extent IR should mainly address large scale studies; etc. (Jankvist et al., 2021). Yet, it seems to us that to engage in these discussions on a more enlightened basis, a natural starting point is to get an overview of both how many and which studies in the MER literature specifically address "implementation". We have taken on this task by conducting a systematic literature review.

Taking on this task involved a few delicate considerations on our behalf, since most of the reported research studies in our field, at least from an inclusive standpoint, may be considered as studies addressing some kind of implementation. Two criteria were decided upon and enforced in order to avoid a too large number of papers to consider. Firstly, we limited the review to include papers that clearly stated dealing with some kind of implementation. Secondly, we limited the review to only consider studies published in the top twenty quality-ranked MER journals following the recent journal categorization by Williams and Leatham (2017). The literature search was carried out on February 4-5, 2021, and initially resulted in 1,093 papers, which through a screening process (see the following section) were reduced to 98 papers. In this paper, we focus on a smaller subset, consisting of 19 papers, dealing with the implementation *of instructional sequences to enhance students' learning* on specific mathematical concepts, or competencies. The research question is: *Which mathematical concepts and/or competencies are in play, and what characterizes the implementation of the instructional sequences?*

Review methodology

We conducted the literature searches in ERIC (EBSCO) searching for manuscripts with implement* in the title and/or abstract, journal by journal of our top 20 samples (Williams & Leatham, 2017). The advantage of doing the entire search in one database is that it is easy to collect the results in one folder. To ensure that no article had been overlooked, we repeated the search implement* in the title and/or abstract on each journal's website. We found 1,093 peer-reviewed articles fitting the inclusion criteria. We used the software *Covidence* to manage our literature review.

Each paper was screened by two reviewers. The screening was made in two steps. First, we screened the title and abstract. In cases where we were hesitant, e.g., because the abstract did not provide sufficient information, we chose to forward the paper to full-text screening. In the full-text screening, papers were included if they followed Century and Cassata's (2016) definition of IR:

> ... as systematic inquiry regarding innovations enacted in controlled settings or in ordinary practice, the factors that influence innovation enactment, and the relationships between innovations, influential factors, and outcomes. (Century & Cassata, 2016, p. 170)

As evident from this quote, another central term in IR is that of *innovation*. Innovation refers to the practical implementation of ideas resulting from research that involve a change (e.g., in behavior or practice) for the individuals enacting them. (Century & Cassata, 2016).

Of a total of 139 papers, 97 remained after the full-text screening (see table 1). These were categorized in terms of, Curriculum reform (31); Curriculum materials (22); Professional development projects (25); and finally, Mathematical concepts, competencies, and instructional sequences, (19). The latter is the focus of this paper. We define an instructional sequence as one, or more, cohesive series of lessons that address a concept (e.g., fractions) or a competency (e.g., problem posing and solving). There are no clear cuts between the categories. An instructional sequence may stem from a new curriculum material, that is implemented through a PD project, due to curriculum reform. Decisive for how the categorization is done is the focus of the paper.

The data extraction from these papers included general information on the author(s), title, purpose statement(s), country where the study was conducted, research question(s), methods, target group, and results. The specific information about the implementations included what kind of innovation from mathematics education the study concerned, specific or general goals in the short- or long term, phase of implementation studied, stakeholders responsible for the implementation, and identified factors of influence for the outcomes of the implementation.

Table 1: Process for inclusion of studies

	Inclusion Criteria	Exclusion Criteria	Excluded
Title and abstract screening 1093 studies imported	Implement* in abstract or title	Innovations that do not stem from results from ME Studies that do not involve a change in behavior or practice Papers not belonging to the sample of 20 journals	955
Full-text screening 139 studies imported 97 remained for data extraction	The studies should fit the definition of implementation research and one or more criteria for doing implementation research (Century & Cassata, 2016)	Does not fit the definition of implementation research. The innovation is not from ME. Not in the sample of 20 journals. Full text is not available. Duplicate	21 8 5 4 3

IR theoretical constructs applied

While most MER studies aim at investigating and improving student learning, implementation studies often consider a dimension of scaling (Coburn, 2003). Scaling, in turn, is entirely dependent on the timing of scaffolding (Helenius, n.d.) between three major groups of *stakeholders*, namely agents from *practice*, *research*, and *policy* (Krainer, 2021). Agents from *practice* include all teachers who carry out teaching but also principals and other staff who are responsible for the teaching that is carried out in an educational context. Agents from *research* refer to both actively involved researchers and the overall production of reported research findings from the research community that informs implementation projects. Agents from *policy* refer to all agents with the power to spread innovations over school districts, make decisions on the direction and budgeting for educational efforts, as well as the administrative superstructure required to implement political decisions.

Our interpretation of *goals* draws on Krainer's (2021) description of four different kinds of goals for IR: 1) Concrete and short-term goals; 2) Concrete and long-term goals; 3) General and long-term goals, and 4) General and short-term goals. While the *general-concrete* dichotomization is introduced by Krainer, we find it more linguistically natural to instead speak in terms of the opposites, *general* versus *specific*. Therefore, when we operationalize Krainer's goals in our analysis we use *specific* instead of concrete. For the dimensions short-term and long-term, we conceptualize the longevity aspect in terms of scale-up possibilities for the innovation (Coburn, 2003).

Specific and short-term goals: In this category we put papers addressing a specific limited goal to enhance teaching and learning. The innovation concerns a mathematical concept or competency. When we classify something as belonging to short-term goals, we consider the intended life-cycle of the innovation. As a consequence, implementations without a scaling plan that are studied with a longitudinal research methodology will be classified as short-term. Our definition of scaling follows Coburn's (2003) notions of depth, sustainability, spread, and shift in reform ownership. *Depth* refers to change in classroom practice that goes beyond a shift in teaching resources and the introduction of specific activities. Coburn argues that scaling includes a shift in teachers' beliefs, norms for communication, and pedagogical practices. *Sustainability* concerns the scaffolding tools that are left to maintain the vitality of the innovation after the support of the reform leaders are withdrawn from the organization. When Coburn considers *spread,* she, in addition to scaling to other schools and classrooms, also includes spread within the organization. Finally, Coburn adds the dimension of a *shift in reform ownership* to the notion of scale. When reform is launched, the ideas and activities are owned by the creators of the reform. According to Coburn, the authority to scale the implementation needs to shift to the districts, schools, and teachers. Only then can scaling in depth, sustainability, and spread be maintained. If none of these scaling dimensions are discussed, or implied, the paper is considered to be a short-term implementation. *Specific and long-term goals,* on the other hand, discuss at least some dimension of scaling.

General and long-term goals refer to innovations that intend to change the practice of mathematics teaching in general, as opposed to a focus on changing the teaching of a specific concept, subject, or competency. For example, as a result of alarms from international tests, politicians may plan for increasing the mathematics teachers' general content knowledge at scale and/or state-wide curriculum reforms, to be implemented with long-term goals. At the other end of the spectrum, we find locally introduced projects within organizations that aim to fundamentally change teaching locally. Thus, with *general and short-term goals,* we refer to non-content or non-competency-specific innovations. For example, limited periods where a new model of the organization of classroom teaching in mathematics is tried without an existing plan for scaling.

Results

We summarize the answer to our question: *Which mathematical areas of concepts and/or competencies are in play, and what characterizes the implementation of the instructional sequences?* in terms of specific- or general-, and short-term or long-term goals in table 2 below.

Table 2: Results

Concepts/competencies	Author(s)	Target population(s)	Characteristics of goals
Algebra (2)	(Adiredja et al., 2020) (Tsai & Chang, 2009)	Undergraduate Grade 8	Specific and short-term Specific and short-term
Arithmetic (3)	(Savard & Polotskaia, 2017) (Tyminski et al., 2014) (Polotskaia & Savard, 2018)	Grades 1-4 Pre-service teachers Grade 1-2	Specific and short-term Specific and short-term Specific and short-term
Calculus (1)	(Carter et al., 2016)	Undergraduate	Specific and long-term
Fractions (2)	(Osana & Royea, 2011) (Thanheiser et al., 2016)	Pre-service teachers Pre-service teachers	Specific and short-term Specific and short-term
Number Theory (1)	(Strømskag, 2017)	Pre-service teachers	Specific and long-term
Problem-solving (1)	(Leung, 2013)	Elementary in-service teachers	Specific and short-term
Proofs (1)	(Stylianides & Stylianides, 2009)	Pre-service teachers	Specific and short-term
Proportional reasoning (2)	(Howe et al., 2011) (Wright, 2014)	Grade 7 Grade 8	Specific and long-term Specific and short-term
Spatial reasoning and patterns (6)	(Papic et al., 2011) (Mulligan et al., 2018) (Mulligan, Oslington, et al., 2020) (Mulligan, Woolcott, et al., 2020) (Patahuddin et al., 2020) (Pollitt et al., 2020)	Pre-school Grades 3-5 Pre-school Grades 3-4 Inservice teachers Pre-school in-service teachers	Specific and long-term Specific and long-term Specific and long-term Specific and long-term Specific and short-term Specific and short-term

Concluding discussion

The first thing that we notice from table 2 is that the targeted concepts and competencies are well spread, although with one exception, namely spatial reasoning. The reason that spatial reasoning and patterns have gained greater interest than other concepts and competencies is that the same research group, working on a large-scale implementation, authored four of the six studies on spatial reasoning and patterns (Mulligan et al., 2018; Mulligan, Oslington, et al., 2020; Mulligan, Woolcott, et al., 2020; Papic et al., 2011). Second, concerning goals, we notice that all studies are specific; a result driven by the delimitation of the category's focus on some concept or some competence. Third, only seven studies address implementation where scaling is a part of the project. Of these seven studies, four belong to the large-scale project, mentioned above, about patterns and structure in Australia (Mulligan et al., 2018; Mulligan, Oslington, et al., 2020; Mulligan, Woolcott, et al., 2020; Papic et al., 2011). The Mulligan et al. studies describe (briefly) plans for *depth*, *sustainability*, *spread,* and *shift in reform ownership*. The intervention program in Papic et al. (2011) reported on *spread* through replication and adaptions to other preschools. One paper belongs to a large-scale project on proportional reasoning in the UK (Howe et al., 2011), where *sustainability* is supported by free web-based modules with lesson plans. One paper is a theoretical paper drawing a picture of how an instructional design for teaching number-theory could be *spread*, within the organization, and also to other schools and classrooms, without the drawback of considering limitations in the organization (Strømskag, 2017). Finally, one paper has a plan for *spread* within the organization but no plan for scaling to other schools and classrooms (Carter et al., 2016).

Reviewing the 20 top-ranked journals ended up in only 19 papers addressing specific instructional sequences on concepts and/or competencies. How can that be? We hypothesize that the discourse surrounding implementation research is somewhat new in MER. The papers in this sample are from the year 2009 to a peak in 2020, with five papers. Further, given that we limited the categorization addressed here to specific instructional sequences on concepts and/or competencies, all goals in this subset are classified as specific. Taken together, that can constitute a problem if we want to understand how large-scale projects work. This is because stakeholder groups of policymakers, in particular those on a school-district level, seldom operate with just specific goals. On the contrary, curriculum reforms are often justified by general goals. We believe that the picture will change when we deepen our study to the other identified categories, i.e., curriculum reform, curriculum materials, and PD projects. Regarding short-term and long-term goals, only papers belonging to large-scale projects, where stakeholders from all levels are involved, explicitly discuss scaling. Depending on the goals, different requirements are placed on the involvement of different stakeholders. Stakeholders with the power to make general decisions are necessary for scaling in school districts. Projects aiming at long-term implementation thus need a plan—and a theory—for how to involve stakeholders at different levels at the 'right' points in time.

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Inquiry-based mathematics teaching in practice: a case of a three-phased didactical model

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During the latest decades, inquiry-based teaching (IBMT) has become one of the top issues at the agenda for educational politics. IBMT is seen as having a potential for enhancing the students' motivation for and appreciation of mathematics as a field of activity and as a tool for understanding the world. IBMT can be conceptualized and operationalized in different ways. In this paper, we focus on a three-phased didactical model for IBMT, which can frame the students' inquiry in and with mathematics, and support the teachers' planning and implementation of an inquiry based activity. More specifically, we present a case study of how the use of the didactical model can facilitate the implementation of IBMT, and what are the challenges that remain and need to be addressed.

Keywords: Inquiry based learning, professional development, case studies.

Introduction

During the latest decades, inquiry-based teaching has migrated from science education into mathematics education, and inquiry-based mathematics teaching (IBMT) has become one of the top issues at the agenda for educational politics (Artigue & Blomhøj, 2013). Inquiry-based approaches have also found their way into mathematics curricula documents in Norway as well as in other countries. IBMT is seen as having a potential for enhancing the students' motivation for and appreciation of mathematics as a field of activity and as a tool for investigating the world. In this context, at The Arctic University of Norway in Tromsø (UiT), researchers designed a four-year research and development project SUM (Sammenheng gjennom Undersøkende Matematikk-undervisning) funded by The Norwegian Research Council (FINNUT) for 2017- 2021. SUM is a four-year research and developmental project, in which researchers and mathematics teachers collaborate with the overall aim of contributing to coherence in children's and students' motivation for, activities in, and learning of mathematics throughout the entire educational system. The objective of SUM is the implementation of research findings related to IBMT as a means for better coherence across transitions in the educational system. In a previous CERME paper, we reported on some preliminary findings related to how the design of the project itself affected the implementation of IBMT in educational settings (Haavold & Blomhøj, 2019). One of the main findings was that the use of a three-phased didactical model used in the project was highlighted as an important enabler for implementing IBMT in the classroom. Most teachers participating in the project had stated that the model provided them with a more structured approach for designing IBMT for their practice. The model reduced a large and complex task to several smaller less complex steps of planning. Furthermore, the teachers said that the model helped them to be aware of and maintaining their own and their students' different roles at each phase of the model. However, the teachers also found it quite uncomfortable to give up control, and allow the students to investigate and explore more freely.

In this paper, we focus on one particular case and present and analyze an IBMT lesson with regard to how this model provides a frame for the students' work and for the teachers' planning and implementation of IBMT. More specifically, we attempt to answer the following research questions: *How can the use of a particular three-phased didactical model facilitate the implementation of IBMT, and what challenges regarding the students' learning through IBMT can be identified?*

The keywords facilitate and challenges can be reframed within determinant frameworks of implementation theory, as enablers and barriers, or more generally as influential factors, that have an impact on implementation outcomes (Nilsen, 2015).

Inquiry based mathematics teaching

In SUM, IBMT is conceptualized with reference to Artigue and Blomhøj (2013). Here the historical threads are retracted back to John Dewey (1859-1952), who developed an educational theory with inquiry as the main driver for humans' development of knowledge. Learning by doing has become parole for this theory, focusing on students' research-like activities. An essential point in Dewey's theory is, however, that knowledge is produced in an interplay between inquiry in problem situations and related reflections – denoted reflective inquiry (Dewey, 1938). Therefore, IBMT should engage students in relevant inquiry activities and challenge them to reflect on their experiences in order to support their learning of key mathematical concepts and methods. Artigue and Blomhøj (2013) discus how IBMT can be understood and characterized in relation some theoretical frameworks such as problem-based learning, problem-solving, modelling, realistic mathematics education and the theory of didactical situations. Seen from the perspective of practice, IBMT has considerable overlaps with all these frameworks, and to some extent, IBMT can be seen as an overarching umbrella. From a theoretical point of view, however, the foundation of IBMT in Dewey's philosophy emphasizes the importance of anchoring the students' learning in memorable experiences and related reflections.

At seminars in the SUM project with the teachers, common factors in how IBMT can be operationalized across the different educational levels have been discussed and illustrated with concrete examples of different types of inquiry activities. Moreover, as common starting point the teachers were introduced to a three-phased didactical model for IBMT (Blomhøj, 2016, chap. 6). This model is summarized in figure 1. In the first phase, the scene should be set for the students' inquiry activities. Multiple approaches is possible, essentially the students should be motivated to investigating a phenomenon or a problem, which make sense for them. In the second phase, the focus is on the students' investigative work. Here the students should have sufficient time, freedom and support for their inquiry activities. The crucial challenge for the teacher is here to help and support just as much as needed without depriving the students the essential mathematical challenges and the related learning opportunities. In the third phase, students' experiences, results and reflections are systematized and shared in the class. The teacher can draw on the students' presentations or products and pinpoint essential elements for the class. Alternatively, the teacher can organize the results from the students' work in dialogue with the class, providing explicit and concrete anchoring to the students' work. Combinations of these formats are of course also possible. The shared experiences and results can serve as a basis for reflections in class. It is the task of the teachers to identify key points in the students' work and to link them to content and learning objectives in the curriculum.

A three-phased model for inquiry based mathematics teaching

1. Setting the scene for the students activities
- creation of questions, amazement or challenges
- establishing the didactic environment for the work
- dissemination of the temporal and practical framework
- clarification of product requirements and success criteria / form of assessment

2. Students' independent inquiry oriented activities
- sufficient time, freedom and support for students' work
- support and challenge through dialogue, cf. the principle of minimal guidance
- preparation through construction of possible dialogues

3. Shared reflections and learning
- experiences, results, and reflections from the activity are systematized and shared
- mathematical points are drawn from the shared results and reflections
- points and results are linked to institutionalized knowledge, e.g. the curriculum

Figure 1: Three-phased didactic model for IBMT

Methods

In this paper, we focus our analyses on an IBMT-lesson implemented in upper secondary school by one of the teachers (Ann) participating in the project. Ann has three years of experience teaching upper secondary and a masters' degree in mathematics education with a particular focus on secondary mathematics education. The lesson took place in a first-year upper secondary class consisting of 25 students aged 15-16 in a small to medium-sized school outside of a city in Northern Norway. Based on a survey directed at the students in the SUM project at the beginning of the school year, it seemed like the students in this class had mostly little to some experience with IBMT from their previous teaching. During interviews and in a questionnaire handed out early in the project, Ann had expressed interest for and curiosity about IBMT, but she had also said that it was difficult to find the time and the resources for designing and implementing IBMT in her classes. Furthermore, she was unsure that students' learned "what they were supposed to learn" in IBMT lessons. Ann had therefore decided, sometime during the third year of the project, that she wanted to use a so-called taxicab-geometry as a starting point for a 90 minute IBMT lesson, as it offered some "structure and clear learning targets related to proportionality and linear growth" as she put it. Prior to the lesson, Ann had been exposed to the three-phased model during the first two years of the SUM project, and had recently began using it as a tool for planning IBMT lessons in her regular practice.

Taxicab geometry is a form of non-Euclidian geometry, which can be thought of as taxicabs roaming a city with streets forming a lattice of unit-square blocks. The taxicabs can only move horizontally and vertically and U-turns are not allowed. The difference between the taxicab system and Euclidian geometry is that the usual distance function is replaced by a new metric in which the distance between two points is the sum of the absolute differences of their Cartesian coordinates, hence the distance between points in the lattice can be counted, see figure 2.

Before the lesson, Ann had prepared the following system of tasks:

1. *Can you draw roundtrips that start and end at point A with lengths 8, 9, 12, and 17?*
2. *Can you mark all points, which have the same distance to point A and B?*
3. *Can you mark all points that have distance three to point A? How many points like this are there? Can you name this pattern of points?*
4. *Make a formula for the number of points with distance r to a specific point.*
5. *Make a formula for the number of points with distance less than r to a specific point.*

Figure 2: Points A and B with distance 5

In line with best practice regarding case studies, we used multiple data sources (Yin, 2014). We collected data from semi-structured observations of the lessons, short interviews with the teacher before and after the lesson, sound recordings of the teacher and groups of students during the lesson, and focus group interviews with students after the lesson. The main purpose of the analysis was to identify how the three-phased model provides a framework for the teachers' work in the classroom. Although there are no routine procedures for analyzing case studies, the analysis must logically piece together the available data into broader themes that capture essential aspects (Yin, 2014). Here, we attempted to identify and explain important episodes during the lesson that could help us understand both the challenges and opportunities the teacher experienced during each of the three phases of IBMT. First, we identified interesting episodes from the case based on our observational notes and sound recordings. We then cross-checked these episodes with data from our interviews before and after the lesson, in order to better understand why these episodes occurred. Finally, we discuss the potential for the students' learning, and possibly missed opportunities during these episodes.

Results

The lesson began with Ann handing out the tasks and an explanation of the rules of the taxicab geometry to the students and telling them to sit in groups of 2-3. She then explained to the class that "this lesson is about thinking, exploring and conjecturing" and that the students had to "investigate and come up with solutions themselves". After letting the students look at the tasks and rules for a minute, she explained the rules of taxicab geometry to the whole class and she demonstrated an example of finding the distance between two points in taxicab geometry. She then told the students that they could start working on the handed out tasks. For the next five minutes, the classroom was quiet and the students did not write anything down in their own notebooks. Ann then asked the whole class to pay attention. She showed the students an example of a roundtrip of distance 8 from point A, before challenging them to find other roundtrips with distance 8, and then 9, 12, and 17.

Observations in phase 1 thus show that the students' did not immediately go to work on the tasks after Ann's short introduction and explanation. This indicates that phase 1 was not without its challenges for the teacher and students. Observation alone cannot tell us why the students sat quietly at their desks, seemingly unwilling or unable to work on the tasks. However, interviews with the teacher and groups of students after the lesson point to at least some factors.

First, the students said that the tasks were difficult, and they were unsure of what to do. Asked why they thought the tasks were difficult, they explained that the tasks were new and unusual. One student, for instance said that; “the tasks were difficult…to understand. I did not know what to do. They were new and I have not seen this before.” The students also mentioned that the tasks lacked clear instructions, and they had to read the tasks several times before realizing that they had to solve them on their own. Second, the teacher said she expected the students to begin working on the tasks after her short introduction, and that she was surprised they sat quietly at their desks. After seeing this, Ann told us she thought the students needed a specific example to show the students how to get started. Third, we also noticed from our interviews an apparent disparity between the teacher and students regarding the purpose of the lesson. In the interviews, Ann mentioned that the lesson was built around the concepts of linear and quadratic functions and direct proportionality, as those were the key subjects in the curriculum “they were working on right now”. For instance, task 4 according to Ann, should “build a better understanding of direct proportionality and linear functions”, and in task 5 the students are challenged to develop a quadratic function. In the interviews, the students, on the other hand, expressed confusion about what the lesson was really about. As one student said, “I do not know what this has to do with the other stuff we have been working on lately”.

Based on these observations, it would seem as the students did not lack motivation or interest, but rather that their expectations caused some miscommunication at the beginning of the lesson. From the teacher’s perspective, both the purpose and intention of the tasks were clear. However, from the students’ perspective, the tasks were difficult; as they did not knew how to get started on them and because they did not fully understood the learning purpose of the tasks. The issues mentioned here are not specific to this particular lesson. They are relevant for IBMT in more general. Dissemination of the temporal and practical framework and clarification of product requirements, success criteria and forms of assessment are general key issues in the first phase of IBMT. In other words, students and the teacher need to have a similar understanding of the task; what are the students expected to do, and what types of intellectual products should they produce. Brousseau and Warfield (2020) refer to this as a situation of devolution, where the students accept the challenge of an engaging and instructive mathematical situation. Similarly, Stein et al. (2008) have highlighted the importance of anticipating how students might interpret and attempt to tackle mathematical problems when implementing cognitively challenging tasks.

In the second phase, the focus is on the students’ investigative work. We observed several episodes that indicate challenges for both the teacher and the students during this phase. One issue we repeatedly noticed was that groups of students would often sit quietly, raise their hands and wait for the teacher to help them. For instance, at one point, only about five minutes into the second phase, students in seven groups raised their hands and asked the teacher for help. Only after the teacher had provided some form of guidance did the students proceed with their work. The following sequence illustrate this. Two boys raise their hands, and Ann approaches them:

Ann: How are things going? Did you get an answer to task 4?

Boy1: No, we did not understand it.

Boy2: We gave up.

Ann: No, tell me. What is it you do not understand?

The boys then show the teacher what they have done so far, and the teacher notices that they have marked points outside the intersections between the horizontal and vertical lines in the grid. The teacher asks the boys if it is possible to mark points outside the intersections.

Boy2: Aha, now I get it. The points have to be here (points to an intersection). Then it is easy. We have 4, 8, 12 points.

Boy1: I get it. We just multiply 1 by 4, then 2 by 4, then 3 by 4 and so on.

Boy2: This is brilliant! What a genius way of doing it.

Boy1: Very clever!

During the interview after the lesson, the teacher mentioned this episode with an explanation for why it occurred. According to Ann, this was an example of something she had noticed almost every time she had tried out IBMT lessons. The students are often passive during the investigative phase, and they quickly give up and ask for help from the teacher – even though they have not tried any strategies or approaches themselves. According to Ann, this is something that applies to almost all students in her class; “The students enjoy these IBMT lessons, but they give up quickly. They do not seem to want to think for themselves. They want me to tell them what to do”.

Another point worth mentioning about this episode is that the students’ solved the task by generalizing the empirical pattern of the number of points when the distance increased by one. Neither the teacher nor the students ever asked themselves why this pattern occurred. A key activity in both IBMT and mathematics in general (e.g. Barbeau & Hanna, 2008) is proving and reasoning deductively, and this was an excellent opportunity for the teacher to ask the students why this pattern appeared or if this pattern would continue when the distance became larger. The reason we bring up this point is that this was something we have noticed in several other IBMT lessons as well. Both teachers and students often seem to be satisfied with empirically based solutions. However, challenging the students’ schemas of reasoning and proof from empirical to more deductive schemas as defined by Harrel & Sowder (2007) could be seen as one of the potentials of IBMT.

In the third phase, at the end of the lesson, and after a short break, Ann told the class that they would now look at the tasks together. She asked the students if they have found any answers to the tasks, and then worked her way through each of the five tasks. For each task, she asked the students if they have an answer for her. The incorrect answers from the students were mostly sidestepped, and instead, she wrote down the correct answers from the students on the board. During this phase, Ann was very focused on connecting the students’ work to the topic of proportional growth and linear functions, as that was the topic they were currently working on. For instance, when she asked the students for solutions to task four, she stressed the importance of connecting the answers to concepts like proportionality, linear functions, and straight lines. After the lesson, Ann said in the interview, that one of the main goals of the lesson was that the students would get a better understanding of “the relationship between tables, graphs and equations of linear functions”. Ann went on to explain that “ideally, the students’ would have been more active and more responsible for the third phase”. However, because she thought the students were a bit passive and quiet, she felt the need to make sure that the lesson ended with clear solutions to the tasks and a clear connection to the curriculum.

The interviews with the students after the lesson help us explain why the students were “passive and quiet”. As one student said, “this is just how we do it. At the end of the lesson, the teacher goes through the tasks and explain the solutions”. Another student added; “I did not quite understand the purpose of the lesson, so I prefer that teacher explain it to us”. Here we notice two things. First, it seems as if there are an expectation and habit of the lessons ending with the teacher explaining solutions and the resuming purpose of the lesson to the students. Second, it seems that the students were unsure of the purpose of the lesson in terms of the intended learning, even after the activity, and therefore they wanted the teacher to explain it to them. This observation shows that for the students, the activity constitutes a break of the didactic contract (Brousseau & Warfield, 2020).

Conclusion

In this paper, we analyzed a case of IBMT with particular emphasis on the role and function of the three-phased model as a framework for the activities in the classroom. Although there are idiosyncrasies tied to the taxicab case, it illustrates several characteristics observed across many IBMT lessons in the SUM project. Thus, the points made concerning the role and function of the model in the case are to a large degree concretizations of general trends observed across many teachers, grade levels, and subjects in the SUM-project.

The taxicab lesson adhered clearly to the three-phased model. As noted by (Haavold & Blomhøj, 2019), a general observation from the SUM project is that the teachers have adopted the model as a didactical tool for planning, conducting and reflecting on their IBMT lessons. The teachers carefully separate their lessons into an introduction, an investigative phase, and a plenary summary. This might seem like an obvious and minor effect. However, in data collected at the start of the project, teachers expressed reluctance to let students investigate on their own without much guidance from the teacher (Haavold & Blomhøj, 2019). Thus, the careful use of the model is an important change of practice for many teachers. The taxicab lesson also illustrates certain challenges teachers face developing their IBMT practices. The students appeared relatively passive during the investigative phase, and according to Ann, this was something she frequently noticed when trying out IBMT lessons. In part, this may be due to the students being relatively unfamiliar with inquiry activities. Implementing IBMT lessons may necessitate renegotiating the classroom didactical contract (Brousseau & Warfield, 2020), and the teachers in the SUM project have indeed expressed views that the students need to be gradually accustomed to inquiry activities. The taxicab case illustrates this, as Ann specifically implemented a very structured IBMT lesson closely tied to the goals in the curriculum they were currently working on. Nevertheless, students’ independent work during phase two was still somewhat of a challenge. Concerning the three-phased model, it is clear that merely allocating some time for students’ independent inquiry (phase two) is not sufficient. Teachers also need to prepare for supporting students during their independent work, for example through anticipating how students may interpret the task and how they might approach solving it (cf. Stein et al., 2008).

Additionally, the taxicab lesson illustrates that phase three can be somewhat of a challenge for the teachers. Although the teacher linked the lesson and work to the curriculum, i.e. institutionalized mathematical knowledge, it is clear that the third phase lacked a systematization of and reflection on the students' solutions, as well as student engagement and active participation. Instead, the teacher mostly highlighted correct solutions without contrasting them with other solutions. The teacher mentioned in the interview that the students were passive, so she felt the need to take control and link the lesson to the curriculum. However, her aspiration to link the taxicab activity to specifically linear functions and proportionality may have contributed to hinder a productive discussion of the taxicab activity in itself and other key mathematical ideas.

To summarize, our analysis of the taxicab case illustrates how the three-phased model provides a frame for the teachers' planning and implementation of inquiry activities. In addition, it highlights the need for finding ways of supporting the teachers in planning and executing phase two and three of IBMT lessons. This plays directly into the dual nature of SUM: On the one hand, SUM as a research project aims to elucidate the extent to which teachers develop their competence for using investigative approaches in their teaching. On the other hand, SUM as a development project aims to support the teachers in developing a practice for IBMT. In that respect, the insights we gain from observing the teachers' implementation of IBMT in their practice in turn shapes how we as researchers collaborate with teachers in SUM. Hereby, SUM explores both the potentials and the limitations in developing practices of IBMT in collaboration between teachers and researchers.

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Active, bodily experience mathematics learning activities. Looking at implementation from the teachers' perspective

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Despite both the research in cognitive psychology and neuroscience, and in mathematics education has increasingly highlighted the relevance of active, bodily experience in mathematics learning, often school teaching seems to be far removed from these perspectives, still largely transmissive. The research presented in the paper is part of an ongoing study, carried out in two different cultural contexts (Italy and Australia), aims to explore the reasons for that distance, by investigating the perspective of primary and secondary school teachers on these teaching proposals and their implementation in the classroom. In addition to the direct involvement of teachers, the designed research includes interviews with experts in the field of mathematics education. After presenting the investigative perspective on the implementation of active, bodily experience learning activities proposed in the research project, an explanatory example will illustrate how the analysis of experts interviews could contribute to identifying determinants of implementation.

Keywords: Mathematics activities, enactive pedagogy, embodied cognition, manipulatives-based learning, teachers' beliefs.

Introduction

The role of the body and movement in the exploration and construction of mathematical concepts is a central topic in much of the relevant literature in mathematics education. The roots of this tradition can be found as far back as the early 1900s, in the well-known Italian contributions of Maria Montessori and Emma Castelnuovo, as well as in Jean Piaget, John Dewey, and Jerome S. Bruner's theoretical works. More recently, cognitive psychology and neuroscience have highlighted the profound interrelationship of perception-action and conceptualization in learning processes. This research has been key to the development of the embodied cognition theory (Lakoff & Nunez, 2000; Varela et al., 1991). In recent decades, the role of students' active, bodily experience in the exploration and construction of mathematical concepts has assumed increasing relevance in the research in mathematics education. As pointed out by Drijvers in the ERME-11 plenary (2019), the growing interest in research from this perspective is evidenced, for example, by the fact that two special issues of *Educational Studies in Mathematics*, have been devoted to the embodiment in mathematics education. Furthermore, several theories that focus on perceptual-motor involvement in the teaching-learning process of mathematics have been proposed. Although conceptualized differently, examples include *enactivist pedagogy* (Abrahamnson et al., 2022), *inclusive materialism* (de Freitas & Sinclair, 2014), and *multimodal approaches* (Radford et al., 2017). On the other hand, the European tradition of the so-called *experimental approach*, and the important recent contributions from these perspectives in research in mathematics education, have found a foothold in most national policies, which encourage teaching strategies in which students are actively involved through the use of manipulatives and tools (e.g. Bartolini Bussi et al., 2010). In Italian national policies, for instance, we could find it in the references to the *Mathematical Laboratory*, elaborated in the intended curriculum, both in the programmatic document *Materiali UMI-CIIM Mathematica 2003,* and in the institutional documents National Guidelines (MIUR, 2012) and National Guidelines and New Scenarios (MIUR, 2018). However, this growing interest in research and developments at a theoretical level has not been matched by an equally wide resonance in classroom practice, which is often far removed from this perspective. Indeed, everyday teaching practice is often inconsistent with this, still largely based on a purely transmissive approach (OECD, 2009, pag.98; OECD, 2016, pag.65) and tend to focus on the implementation of clarity-of-instruction (e.g., procedural) rather than cognitive activation practices (OECD, 2019).

This gap between research findings and indications and the uneven proposal of these activities in classrooms warrants a research interest in active, bodily experience learning activities implementation. In our research, we are particularly interested in exploring what is the current implementation of these activities at school, since, as highlighted by Century and Cassata (2016):

> Documenting implementation as conducted enables researchers to understand the ways that innovations are operationalized in practice, the influential factors that affect the practice (…), the different patterns of practice, or “configurations” (…), and in some studies, the relationship between these patterns and outcomes. (p. 190)

This paper will present an investigative perspective that looks at active, bodily experience learning activities, that we identify, somewhat improperly, as an innovation or research finding, investigating teachers' perceptions in this regard. We also aim to show how we want to use the analysis of interviews with selected experts in mathematics education to create a conceptual framework that, along with insights from a literature review, will allow us to interpret teachers' survey findings. These interviews are designed to investigate the experts' views on what the core elements and expected outcomes of these activities are, and to identify the factors believed determinant in and for their implementation. In the following lines, we will provide a brief justification of the proposed perspective of inquiry, illustrating the main conceptual statements assumed in the study.

The object of implementation: an all-embracing construct

The variety of theories mentioned in the paragraph before, develop theoretical constructs that are the result of the specific philosophical, psychological and pedagogical perspectives that determine them. Looking at the implementation with an exploratory study aimed to understand teachers’ perspectives, the research cannot move within a specific theoretical framework but needs to be based on a negotiation of meanings. Due to this fact, we need to individualize a comprehensive construct as *the object* of our research on implementation, which acts as an umbrella for the multitude of theoretical proposals developed, and which should be clear and easy-accessible for teachers.

We will use the terminology *active, bodily experience mathematics learning activities* to refer to activities designed in the perspective of enactive-embodied learning or, more generally, to activities in which students are actively engaged in exploring mathematical concepts using manipulatives, tools (virtual or physical), or the whole-body movement. Descriptions and appropriate commonly known examples will introduce the definition. We are attempting to clarify the characteristics of the construct under analysis, with an explorative study conducted with experts in the field of Mathematics Education, selected both for their experience as theoretical developers in this research field or in professional development courses with teachers.

An old matter, a current issue

Although the proposal of active, bodily experience learning activities in schools has such a long and widely debated tradition, the presence of this perspective in teaching practice is uneven. Furthermore, these theoretical perspectives, encouraged by experimental findings, are recently developing a wide range of innovative educational artifacts and proposals. Through a new point of view with which to look at the implementation of these approaches in schools, this study is shaped as research on the implementation of research findings (Century & Cassata, 2016).

In the research, we look at implementation from the perspective of teachers, assuming that they can give us precious insights on the current implementation as conducted in classrooms. These include, for instance, the teacher's beliefs and experience which plays a central role in educational change (Coburn & Talbert, 2006; Peterson, 2013), and the perception of the activities studied, that has a drastic impact on their implementation (Domitrovich et al., 2008; Ruiz-Primo, 2006), as highlighted in the case of the introduction of manipulatives by Golafshani (2013) and Vizzi (2013).

Research Aims

As briefly outlined in the introduction, many research findings and theoretical results have been developed over years concerning the importance of actively engaging students in hands-on activities and the role played by perceptions and movements in mathematics teaching and learning processes. Nevertheless, in our study, we want to go beyond the different theoretical perspectives, with the intention of investigating teachers' perspectives regarding the proposal and implementation of active, bodily experience learning activities, "to describe the extent and nature of innovation use in practice, including adaptations and omission of core components, and explore the contextual factors that support or inhibit innovation use" (Century & Cassata, 2016, p. 190). The research presented in this article first aims to present the perspective adopted to explore the implementation of learning activities of this type in the study. Secondly, it aims to illustrate how the analysis of interviews with experts in mathematics education, on these research findings and their implementation, can be used to identify the core elements and outcomes considered essential parts of this *innovation* and the key characteristics that determine their implementation.

Research framework and state of the art

The presented research is part of an ongoing doctoral research project investigating the proposal and implementation of active, bodily experience learning activities in classrooms. The research project aims, firstly, to offer an overview of the literature on theoretical perspectives and empirical research focused on the involvement of the body and movement in learning mathematics activities consistent with an experimental, hands-on approach; secondly, to highlight the presence/absence of official national and international guidelines, educational policies and curricular documents in this direction. As a third goal, the research aims to investigate the perspective of mathematics teachers, both primary and secondary, with respect to the implementation in the classroom of active learning strategies, which involve the body and movement of students and to identify possible hindering and facilitating factors, such as teachers' beliefs.

The research project is an exploratory mixed-method study, that includes desk research, designed to achieve the first two goals. To reach the third goal, we have designed an explorative study that aims to investigate primary and secondary mathematics school teachers' perspectives regarding the proposal and implementation of these activities. This stage of the research includes:

- semi-structured online interviews with experts in Mathematics Education aimed at documenting experts' views on active, bodily experience activities to identify a conceptual framework on the main issues outlined in the teachers' survey.

- an online questionnaire to reach a convenience sample of primary and secondary mathematics teachers, working in diverse schools or localities, using a web-based instrument combining rating items, multiple-choice items, two vignette-items, and a few short open-ended questions.

- individual or focus group semi-structured online interviews with a smaller number of teachers, grouped according to school levels, complementing the survey. This phase aims at deepening some topics for which the questionnaire might not yield sufficient information.

The explorative study is being carried out in two very different cultural contexts, Italy and Australia. The involvement of such different contexts may offer the opportunity to observe some latent and implicit variables that may not emerge by conducting the investigation immersed in a single educational and cultural system.

Research design and methodology

The instrument

To collect experts' opinions on the implementation of active, bodily experience learning activities in classrooms, we are carrying out individual semi-structured interviews via Zoom, approximately one hour-long, with a small number of experts in mathematics education, both in Italy and in Australia. The interview prompts are designed to get the experts' views on key aspects of implementing active, bodily experience mathematics learning activities at school, especially in relation to teaching practice. The first goal of the interviews is to collect the experts' opinions on the proper terminology to define the activities under investigation in a clear and easily accessible way for teachers. This exploratory phase should also provide a set of examples that might be commonly known and recognized by teachers, at different school levels. Furthermore, with the interviews we aim to shape a conceptual framework of experts' views around the main questions underpinned the survey on teachers' perspectives:

I. Whether and why is it important to implement active, bodily experience mathematics learning activities at school?
II. What are the beliefs that should guide teachers in proposing these activities?
III. Which levels of awareness and knowledge should accompany teaching when implementing these activities? (e.g., in terms of teaching strategies, assessment, etc.)
IV. Which characteristics concerning the implementation of these activities at school determine their teaching effectiveness?

V. What are the main limitations of the use of these activities in daily teaching practice? What are factors that hinder/favour the implementation of these activities at school?

Interviews are transcribed and analysed according to the thematic content analysis method, using open, inductive coding in the first instance and then refining the results with a focused, axial coding that will lead to the construction of concept maps. A tool to represent narrative material will be used to analyse these qualitative data and to compare and contrast individual responses to each question. Analysis of the narrative material will produce a conceptual framework that highlights the overall and variety of experts' opinions on core elements and expected outcomes of these activities are, and that identify the factors believed determinant in and for their implementation.

Participants

In Italy, we have selected and interviewed 8 experts in mathematics education. The experts were contacted via email, and their participation in the study is voluntary. They are 7 accomplished academics and a teacher-researcher, who have a wide range of different research interests: mathematics difficulties and the use of representations, teachers' beliefs and problem-solving, teacher education, semiotic mediation, proof and argumentation, cultural transposition, multimodal approaches and gestures, Montessori method education. Overall, all of them have long experience in teachers' professional development courses and empirical research in classrooms and they are familiar with the topic.

The scheduled interviews in Australia are 6 and we are currently conducting them. The experts, contacted via email and voluntarily involved, are all academics, in some cases former teachers, and they were selected on the basis of their research interests and experience in the field of teachers' professional development.

A preview of how the analysis of Italian interviews is being carried out is presented below. The interim analysis is partial and has not yet been subjected to triangulation.

An illustrative example of Data Analysis: Question 1

In this paragraph, a brief presentation of the conceptual framework identified by the experts around question 1 of the previous list will be provided, showing the main themes that emerged in their answers. The map shown in Figure 1 is a simplification for illustrative purposes of this conceptual framework. The construction of the map started with the analysis of a first interview. The themes that emerged analyzing the transcript of the interview were categorized with an inductive analysis, and a first concept map referring to the individual interview is created starting from these categories. Around the key concept, consisting of the topic of the question, we positioned the themes that emerged in the expert's answer as nodes in a graph, and, using directional arrows, we represented the network of internal relations between the various themes. Next to the arrows, labels were juxtaposed in order to specify the nature of the links identified between the emerged themes. A similar analysis was carried out for the other interviews; the opinions, that emerged from these, added new thematic nodes or strengthened the links already identified. The analysis process was not linear and evolved through the renegotiation of the emerging categories, depending on continuous re-readings of the narrative material. The map, in Figure 1, presents the set of themes that emerged and the totality of the links identified by the analysis of all the interviews conducted. The themes and relations referred to each single contribution are identifiable, due to the assignment of an identifying colour for the arrows which indicate the links narrowed by the same expert. In this way, we were also able to represent the recurrence with which a theme emerges in the various contributions analysed.

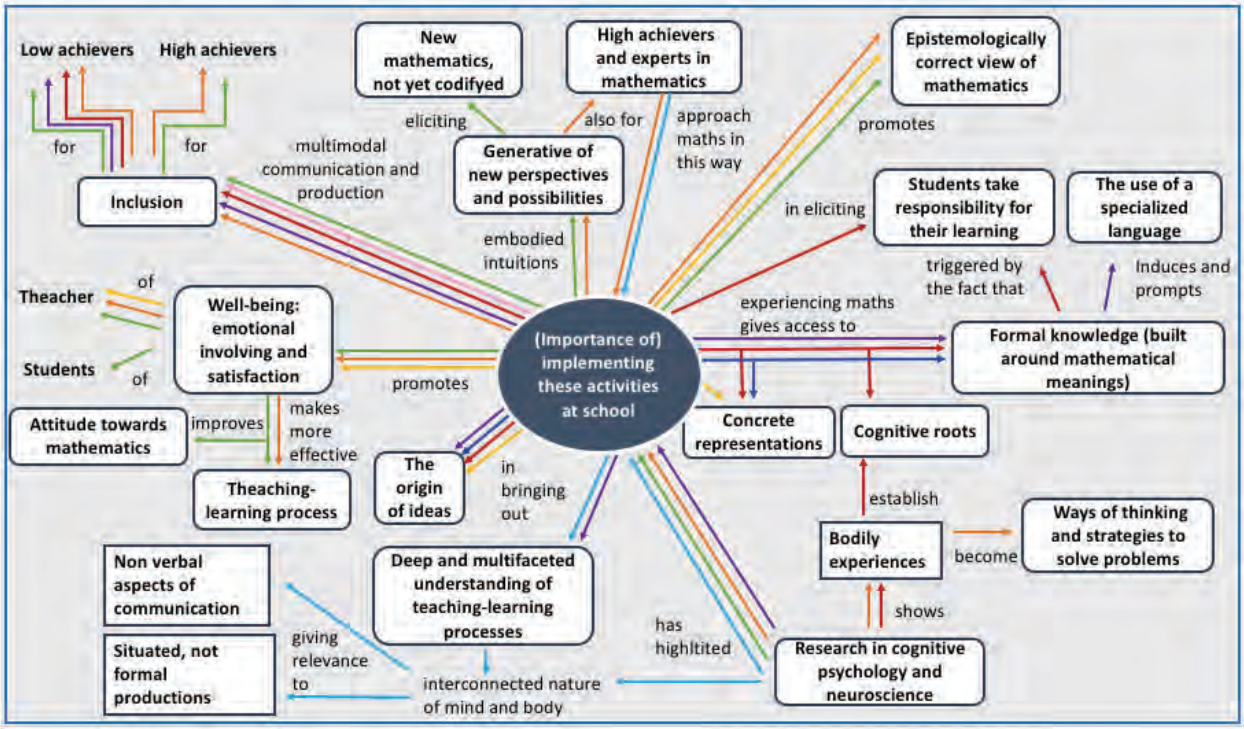

Figure 1: Interim concept map of experts' framework around the question: "Do you think it is important to implement such activities in school? Why?"

Briefly discussing the partial and provisional results analysed so far, some strong and shared themes clearly emerge. First, the activities under consideration are considered to be more inclusive, as they use more channels for accessing and producing information than the traditional verbal one. While the focus is mainly on the difficulties developed in mathematics, it is also stressed that such inclusion is to be considered in a broader sense, also with regard to high achievers. References to experimental cognitive studies are also frequent, testifying to the validity of these proposals for the construction of mathematical meanings, insofar as they offer cognitive roots, for instance in motor schemes, on which to anchor the manipulation of mental and abstract concepts. Moreover, these activities are considered capable of promoting situations of well-being, centred on involvement and satisfaction, both for students and for teachers. Furthermore, two experts have highlighted that in the proposal of these activities, which take into account the profound interweaving of perceptual-motor and conceptual aspects that occur in learning processes, more complete interpretations of the teaching-learning process of the discipline emerge. From a more disciplinary point of view, there is a broad agreement that these activities are able to put students in direct contact with the origin of mathematical concepts, which find their basis in the human experience of exploring the world. Experimenting with mathematics, constantly moving from the concrete to the abstract, leads to the achievement of a meaningful understanding of formal knowledge. In addition, the intuition that passes through the body is hypothesized by two experts as generating new possibilities and perspectives for the discipline itself, in the creation of new knowledge. It is also emphasized that even professional mathematicians often use bodily artifacts and intuition to think about mathematics. Therefore, by proposing these activities, we enable students to develop a more epistemologically correct view of mathematics as a dynamic discipline that proceeds in an exploratory and creative way, advancing by trial and error, employing all available cognitive resources to solve meaningful problems, and only then becomes formally structured.

Conclusion: Limitations and further directions

In this paper, we have presented the state of the art of research project. Structured analyses will be conducted on the collected narrative materials, of which we have presented only an exemplar preview. However, even from such a cursory presentation of preliminary analysis of the interviews, in the provided example emerged insights about, for instance, the overall and variety of outcomes that the researchers believe can be achieved by implementing this type of activity.

From the analysis of the Australian interviews that we are currently conducting, we will see if, overall, the two groups of experts, referring to a different cultural context, identify similar or different core elements of the object under study and key variables that play a central role in the implementation of these activities. Furthermore, from the data collected with the questionnaire addressed to teachers, and subsequent follow-up interviews, a comparison will be made to see whether and to what extent teachers' perspectives are aligned with the directions that emerged from the analysis of the experts' interviews, also considering the differences emerging from the two contexts involved.

The research presented is part of a doctoral study aimed to contribute to a hoped-for narrowing of the gap between research findings and their implementation in schools by highlighting specific needs emerging from the survey of teacher perspectives. The research findings could inform the design and development of innovation, indicating directions that could be taken to address these emerging needs, and providing support in practice settings.

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Problem-solving implementation chain: from intended to experienced

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The importance of mathematical problem solving (PS) has long been recognized, yet its implementation in classrooms remains a challenge. In this theoretical essay we put forth the notion of problem-solving implementation chain as a sequence of intended, planned/enacted and experienced activity, shaped by researchers, teachers and students respectively, where the nature of the activity and its aims may change at the links of the chain. We propose this notion as an analytical framework for investigating implementation of PS resources, and illustrate this framework in a case of middle-school teachers enacting a particular problem they encountered in professional development. We show how adaptation along the implementation chain can be viewed not merely as loss of fidelity, but rather as an opportunity for researchers, teachers and students to refine their perspectives on PS activity, and to learn through the implementation process.

Keywords: Implementation, fidelity, adaptation, boundary-crossing, documentational approach to didactics.

Introduction

Problem solving (PS) was put on the mathematics education agenda decades ago (Pólya, 1945/1973; NCTM, 2000; Schoenfeld, 1985), and remains one of the spotlights of mathematics education research to this day (Felmer et al., 2019; Lester & Cai, 2016; Schoenfeld, 2013). It is considered both as a self-contained goal and as the means of accomplishing other significant objectives of mathematics education (Schroeder & Lester, 1989). In addition, PS is considered one of the central activities in mathematics as a living science, and thus it has been hoped that it would also become a central activity in mathematics education (Mamona-Downs & Downs, 2005; Lester & Cai, 2016). However, implementation of PS in the school reality is relatively rare, and even when implemented, activities that designers of instructional resources intend as PS are liable to be enacted as drill and practice, a kind of activity that still tends to be dominant in mathematics classrooms (Lester & Cai, 2016; Felmer et al., 2019). Lester (2013) attributes “this unfortunate state of affairs” (p. 251) to the fact that research on mathematical problem solving remains largely a-theoretical. Schoenfeld (2013) suggests that the current challenge is to specify the theoretical architecture of PS activity, in order to explain “how decision making [on PS implementation] occurs within that architecture” (ibid., p. 17) and to theorize on “how ideas grow and can be shared in interaction” (ibid., p. 20). The conceptual framework of *problem solving implementation chain* (PS-IC) aims to address these lacunas. Briefly, we consider PS-IC as a sequence of actions and interactions beginning with the development of a problem-solving resource by researchers, which teachers then engage with in professional development (PD), and finally teachers and students make use of in classrooms.

Problem-solving implementation chain – a conceptual framework

Current research (e.g., Aguilar et al., 2019, Koichu et al., 2021) conceptualizes implementation of innovation, such as incorporation of PS resources in a classroom that normally features drill and practice, as a change-oriented process that takes place between proponents of an innovation (often education researchers) and its adapters (often mathematics teachers) who adapt the resource while taking agency over it. Taking inspiration from Stein et al. (2007), who distinguished between intended, enacted and experienced curriculum, we propose the term *intended PS* to denote the proponent's vision of enactment for a particular problem, *planned PS* to denote the vision of the teacher, *enacted PS* to denote what actually takes place in the classroom from the perspective of the enacting teachers, and *experienced PS* to denote what takes place from the perspective of students. Thus, PS-IC can be operationalized as a dynamic sequence of intended–planned–enacted–experienced PS perspectives that are shaped by interactions between the stakeholders involved.

From the perspective of the resource proponent, the intended PS is implicit in the problem, and may also be communicated orally or in writing, for example in a teacher guide. There is a duality in teachers' interactions with resources that is conceptualized in the documentational approach to didactics (DAD, Trouche et al., 2020) as *instrumental genesis*. An *instrument* is taken to be an artifact (e.g. a problem) along with schemes for its use. On the one hand, teachers are invited to appropriate the intentions of the designer in a process called *instrumentation*. On the other hand, "the conceptions and preferences of the user change the ways in which he or she uses the artifact, and may even lead to changing or customizing it" (Drijvers & Trouche, 2008, p. 368) in a process called *instrumentalization*. We consider intended and planned PS as *documents* (Trouche et al., 2020), in the sense of a particular resource (problem) along with an elaborated *scheme of usage*, which includes the *aim* of the activity (e.g., to engage students with particular mathematical competencies), *rules of action* (e.g., orchestrating a classroom discussion based on students' ideas), *operational invariants* for PS activity in general (e.g., students should first "get stuck" and then attempt to invent a solution over a period of 20-30 minutes), and *possibilities of adaptation* (e.g., what to do if some students do not make any progress on their own). The process in which teachers develop such documents is called *documentational genesis* (Trouche et al., 2020).

The use of a new resource is not dictated by its proponent, but neither is it completely in the hands of the adapters (as is often tacitly assumed in DAD). In PD, researchers and teachers may engage in a kind of co-documentation, which may give rise to a new hybridized scheme of usage. A similar process may take place through interactions between teachers and students. Thus, to understand why and how PS documents evolve, we propose to study the "links" in the chain – interactions between researchers and teachers at the link between intended and planned, and between teachers and students at the link between enacted and experienced. Cooper et al. (2020) have conceptualized the kind of expansive learning that can take place between researchers and teachers in PD as *boundary crossing* (Akkerman & Bakker, 2011), and have demonstrated various learning mechanisms that may take place when these communities interact, such as *coordinating* perspectives in attempting to translate the discourse of others (e.g. what do you mean by *problem*?), *reflecting* on each other's practice (e.g. researchers considering the practical rationality of teachers' actions), and *transforming* practice through hybridization. Robutti et al. (2020) have investigated the role of mathematical tasks in such learning as *boundary objects*.

Thus, the notion of implementation chain provides a conceptual framework to address two classes of research questions: 1. What are the differences between PS at various phases of implementation, from intended to experienced? 2. How and why do these differences develop through interaction between researchers, teachers and students?

Illustrative analysis of a PS implementation chain

We now illustrate how the notion of PS-IC can support research. For this we rely on a particular problem, drawing both on empirical data (what took place) and on thought experiment (what could have taken place). The context of this example is a PD program that aims to encourage and support PS in

"Factoring" exam grades

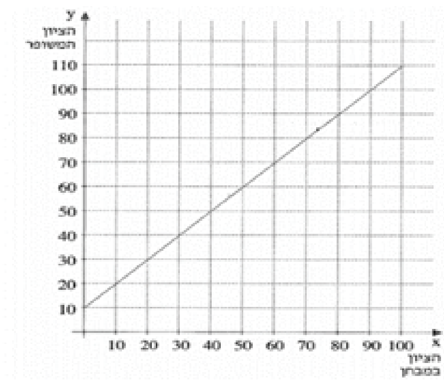
Grades in a math test were rather low. The teacher considered 4 proposals for raising the grades:

Proposal B

Original grade	Factored grade
20	24
40	48
60	72
80	96
100	120

Continuing according to the same linear rule

Proposal A



Proposal D

x – the original grade
y – the factored grade
$y = 10\sqrt{x}$

Proposal C

Add to the original grade half the difference between 100 and the grade.

1. Think about a grade you received recently, and about how the 4 proposals would influence your factored grade.
2. Av said: "My grade was 81, and one of the proposals would increase my grade significantly more than the others.

   Debby said: "it isn't fair for students who failed the test to receive such a significant increase, while those who passed receive a smaller increase.

Which proposal did each of them have in mind? Explain your reasoning!

3. The teacher decided to accept the following conditions:

   a) The factored grade should not be less than the original grade.

   b) The factored grade should be between 0 and 100.

Which proposals should the teacher consider?

The teacher eventually adopted proposal D. Try to explain what her reasons might have been.

Figure 1: The problem

middle-school mathematics classes. Challenging problems were designed and piloted by a development team comprising researchers, teachers and teacher educators. Then, in a 30-hour PD program, teachers solved problems, discussed and planned classroom enactments, enacted problems in their classrooms, submitted reflective reports on the activity, and discussed their enactments. Both authors observed the PD meetings, and we rely on our observations and on the reflective reports. We elaborate on a particular problem (Figure 1), which was enacted in 6 classrooms. Our analysis has five focal points: The intended, enacted and experienced schemes of enactment, and interactions between researchers and teachers and between teachers and students.

Intended enactment scheme

The scheme of enactment as intended by the proponent – the didactic aims of PS activity, rules and principles of its enactment, and its potential adaptation – are captured across a collection of documents, including teacher guides for each of the developed problems. We lay out some of the main aims and principles, and demonstrate them in the context of the problem under consideration.

Aims: Mathematical problem solving is both a goal in its own right and a means toward deepening mathematical understanding. As a goal, students should learn to solve problems in authentic contexts, applying a variety of generic and mathematical competencies. Seeking a fair rule for factoring grades is challenging due to the need to explicate a mathematical model of fairness, translate data between 4 different representations of functions (verbal, symbolic, numeric, graphic) and to come up with an innovative strategy. As a means, it can contribute to students' flexible translation between representations of functions, and foster an appreciation of the importance of this competency.

Rules of action (local and invariant): Teachers should allow students to struggle in their attempts to make sense of the problem and to find a solution strategy. Telling students what to do (e.g. "calculate the result of factoring the grades 50, 75, 100 in each of the proposals") can demote PS to mere drilling of procedures. The PS environment should be *choice-affluent* (Koichu, 2018), in the sense that students can choose whether to try to solve the problem individually, critique or contribute to a peer's solution, make sense of or explain a proposed solution, etc.

Adaptation: Teachers' enactment of PS should not be prescribed. There are many different valid rules of action, and teachers should be aware of many of them, and have the agency to make informed decisions on such factors as class organization (individual, pairs or small groups, whole class), time allocation, etc., adapting the activity to their local context. Accordingly, the teacher guide avoids prescription of rules of action, and instead highlight affordances of different possible modes of enactment. When students get "stuck", teachers should provide guidance that alleviates frustration without eliminating the challenge. For example, the teacher guide suggests that students might be provided all 4 functions in a single representation (Figure 2), possibly without specifying which graph corresponds to which proposal, and without specifying how this hint might prove to be helpful.

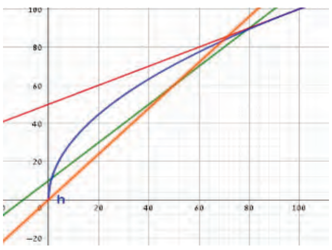


Figure 2: A possible hint

Scheme as planned and enacted

Teachers' planned schemes were inferred through their participation in the PD, where PS activities were discussed, and their enactment schemes were inferred through analysis of a structured 4-page detailed, semi-structured report that they submitted following classroom enactment of the problem.

Aims: Teachers' discourse on PS was dominated by two main concerns: what are their students capable of (e.g. "it's too difficult"), and what is institutionally possible (e.g. "I have no time for lengthy activities that are not aligned with the curriculum"). As a result, some of the teachers did not have the patience to engage with problems deeply enough to appreciate their didactic potential. Yet in the particular problem under consideration, three teachers noted curricular aims – to summarize and assess the topic of functions, or to practice the topic after formal instruction was completed, noting that it provided opportunities for students to deepen their understanding. Some teachers mentioned aims that were not directly related to the curriculum, including practicing collaboration in group work and encouraging mathematical reasoning.

Adaptation: Three teachers modified the problem. One merely omitted the last section to leave time for a significant discussion of sections 1 and 2, having decided not dedicate more than a single 45-minute lesson to the problem. Two others attempted to make the problem more relevant for their students. One did this by rephrasing section 3 in the first person ("which proposal should I consider"?). Another teacher took this idea of relevance one step further. She enacted the problem after returning a graded test, and instructed each student to calculate the result of each proposal on their own grade and to submit the results on a google form that she had prepared. The authentic data that was collected – the result of applying 4 different factors to 27 distinct grades (in total 108 calculations) – was then made available to all students as a resource for working on the problem. While this denied students the initial struggle, it added an element of personal relevance.

Rules of action: There was much variation in the way the six teachers enacted the problem. The total time that students worked on the problem ranged between 30 and 90 minutes. Two teachers began the activity with a whole class introduction, while the others immediately launched students' work on the problem. All teachers held a whole-class discussion at the end of the activity. Work that led up to the discussion was in two cases individual, and in 4 cases was in pairs or small-groups. All of these schemes fall within the range of the intended scheme. Yet all the teachers appeared to be rather quick to defuse students' frustration by providing hints or guidance as soon as they began to struggle.

Emerging invariants of PS enactment: Teachers' reflection on the activity provided a glimpse on their emerging invariants for PS activity (i.e. how they are likely to enact their next PS activity). One teacher valued the fact that her students were challenged, though she reported that she helped them during their individual work on the problem. It remains to be seen how she will come to balance challenge with her tendency not to let students remain "stuck". Two other teachers stressed the need to provide ample time for students to work and reason individually, suggesting that selecting short problems (or shortening longer one) may become a PS invariant for them. Two teachers were coming to adopt an invariant of small-group work, affording quiet students an opportunity to be heard.

Experienced PS

Based on the PS as enacted, it seems unlikely that students would have the opportunity to experience PS as intended, mainly because teachers were quick to assist struggling students. Nevertheless, one teacher indicated that here students were successful "after many attempts", suggesting that they were allowed to struggle, and some teachers were surprised by their students' capabilities (e.g. to represent functions on Desmos), suggesting that they may in future pose problems that they consider difficult.

Intended-planned link (teacher-researcher interactions)

Both the problem and the reports can be considered boundary objects in the following sense: Teachers planned lessons around problems designed by researchers, seeking pedagogical affordances they were not previously aware of, and reflecting on the activity attending to aspects that researchers considered significant. Researchers attempted to make sense of teachers' actions in PD discussion and through their reports. Beyond the obvious opportunities for learning through coordination and reflection on each other's perspectives, we bring some examples of learning through transformation of practice: One teacher's omission of the final, most challenging section of the problem may appear at first to "cripple" it by lowering the challenge. Yet making explicit the practical constraint of fitting the activity into a single 45-minute lesson has encouraged us to rethink how this can be reconciled with our views on PS, and has given birth to a kind of hybrid activity – a principle for designing "condensed" problems that fit into a single lesson without compromising crucial aspects of the PS activity. We are also coming to appreciate the role of teachers in motivating their students' PS activity by creating local relevance. This relevance is multi-faceted, and includes curricular relevance (carefully timing the enactment to serve a didactical purpose at a well-defined point in the curriculum), personal relevance (improving one's grade on a test) and social relevance (co-generating data to solve a problem, engaging in group work).

Enacted-experienced link (teacher-student interactions)

We are encouraging teachers to elicit feedback from their students to gain access to their experiences. We do not yet have data on this practice, yet can hypothesize on its potential to negotiate shared meanings and aims for PS activity (e.g. "it's not about getting the right answer") and to develop practices that will provide all students with opportunities to engage in mathematical reasoning.

Discussion

This paper proposes the analytical framework of *PS implementation chain* (PS-IC) for analyzing the evolution of problem solving activity as it passes from the proponent (intended), to teachers (planned/enacted), and finally to students (experienced). In combining DAD (Trouche et al., 2020), and the notion of boundary-crossing (Akkerman & Bakker, 2011) we propose a framework that can describe aspects of PS evolution, along with mechanisms and rationalities for this evolution.

Some might consider divergence from the intended activity as loss of fidelity (e.g., Brown et al., 2009), and as such undesirable. In contrast, we hold that local adaptation of the activity and its aims is desirable and even necessary. While the proponent of an educational resource can attempt to convey the rationale for some proposed scheme of enactment, it is ultimately the teacher who is the expert on the contextual parameters of local implementation (students, curriculum, institutional norms and expectations, etc.) and who should decide on the aims and rules of enactment that are appropriate for a particular setting. Nevertheless, we can easily imagine how PS can diverge from the activity as intended beyond recognition as what we would consider PS, particularly if resources migrate between very different educational systems. Our analysis suggests how this fidelity-adaptation dichotomy can be reconciled. Interactions at the IC links around the enactment of PS, conceived as learning through boundary-crossing, may instigate researchers to generalize their intended scheme to accommodate a broader set of educational contexts, or even revise their schemes – recognizing new aims and affordances of new rules of action, and modifying invariants of the intended PS. Thus, fidelity may be maintained not by keeping the enacted scheme close to the intended, but by continually realigning the intended with the enacted.

We briefly note the implications of our theoretical perspective for upscaling. One may hope, through the accumulation of data and experience, to provide an "ultimate" set of problems and teacher-guides. However, the theoretical perspective we have put forth denies the existence of such a set. Rather, it is the very notion of PS-IC that needs to be made visible to teachers at scale. A community of teachers, sharing their documentational genesis (resources, practices and insights) can achieve this.

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Revising a programming task in geometry through the lens of design-based implementation research

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The paper addresses the role of educational task design in implementation research. Its point of departure is the first revision of a task developed for 4th grade in collaboration with a Danish school teacher. The authors informed the task from a hypothetical learning trajectory that requires students to draw on their knowledge both of mathematics and programming and computational thinking. From a task-design standpoint, the teacher's experience proves necessary to adapt the task to local reality. The collaborative process illustrates broader implementation issues, such as problems and roles of different stakeholders, innovation adaptation and capacity building. The lessons are consistent with the emergence of design-based implementation research as a more comprehensive model.

Keywords: Computational thinking, design-based implementation research, hypothetical learning trajectories, programming, task design.

Introduction

The mathematics education community has recently gained a renewed interest in programming and computational thinking (PCT) as related subject matter areas. Although there is consensus on the potential synergies between mathematics and PCT, exploiting them has proven to be a difficult task (Misfeldt, Szabo & Helenius, 2019). Research in mathematics education has taken several approaches to address this difficulty. For example, by considering how to train and prepare mathematics teachers to teach PCT (e.g., Kilhamn & Bråting, 2019), coming up with ways to conceptualise its relation with mathematics (e.g., Benton et al., 2017; Weintrop et al., 2016), and trying to understand better students' learning processes and dispositions in computational thinking-driven mathematics classrooms (Pérez, 2018).

More locally, Denmark is planning to implement technology comprehension as a new subject in K-9 education comprising four competency areas: digital empowerment, digital design and design processes, computational thinking, and technological knowledge and skills—including programming (Smith et al., 2020). The Danish Ministry of Education is conducting a pilot project at 46 schools is seeking to gain experience with implementing it (1) as a subject in its own right and (2) as an element integrated into existing subjects, here among mathematics. In both cases, the ministry has published a tentative curriculum with added technology comprehension learning goals and prototypes of teaching resources.

This paper takes an implementation perspective on the challenges of intertwining PCT with mathematics teaching, anchored in the Danish school context. The broader aim of the research project is to support mathematics teachers in connecting PCT and mathematics in their everyday practice through the development of carefully designed tasks. By engaging in an educational design process, we draw on this approach to iteratively design, implement, and refine activities developed to fit the mathematics teachers' needs and support students' learning.

We anchor the content of this paper at a school that is part of the aforementioned pilot project, where we collaborate with a mathematics supervisor to refine a task developed to integrate PCT and mathematics in grade 4. In this context, matters of implementing a new subject and of task design interweave.

Design-based implementation research

Implementation research in mathematics education seeks to close the gap between research and practice (Jankvist et al., 2021), including the characterisation of a particular innovation and the factors conditioning its implementation (Century & Cassata, 2016). We aim to illuminate the characteristics of the new subject as embedded in the mathematics curriculum as a factor. In particular, we narrow down the innovation to a particular task.

One emergent model acknowledging the role of educational design in the context of implementation is design-based implementation research (DBIR). As noted by Fishman et al. (2013), DBIR is different from conventional design research, which only focuses on student learning, in that it points out "how the deployment of new tools (e.g., curricula, technologies) can bring to light new needs for coordination across different system levels and for capacity building" (p. 144). DBIR is then characterised by four principles "(1) a focus on persistent problems of practice from multiple stakeholders' perspectives; (2) a commitment to iterative, collaborative design; (3) concern with developing theory and knowledge related to both classroom learning and implementation through systematic inquiry; and (4) a concern with developing capacity for sustaining change in systems." (Fishman et al., 2013, pp. 136–137). Based on this framing, we ask the following research question: *What can we learn about the implementation of PCT in mathematics education by designing a single task with a teacher from a DBIR perspective?* The second and third principles pertain to standard design research (Bakker & van Eerde, 2015). In this paper, we focus on the first and fourth principles to organise our discussion, which is anchored in the early stages of designing a programming and geometrical task.

An essential conceptual tool for developing the design and learning from iterations is that of a Hypothetical Learning Trajectory (HLT). In the context of one or two lessons, an HLT is "the consideration of the learning goal, the learning activities, and the thinking and learning in which students might engage" (Simon, 1995, p. 133). Artigue (2021) highlights HLTs as a critical theoretical resource from design-based research to connect research and practice. An HLT seeks to make explicit what will happen when the design is brought into the hands of the practitioners concerning the real-world problem it aims to address. It describes the process that ought to take place during the intervention with the design. An HLT is informed by theory and should be sufficiently exhaustive so that its assumptions can be challenged by a real-world enactment of the design (Cobb & Gravemeijer, 2008). The HLT enables one to expose unforeseen interplays between intention, design, and reality, thereby challenging and qualifying our theoretical assumptions. After an intervention, one is thus able to consider whether it led to the desired outcome or not and point expressly to where our understanding of the design was confirmed or challenged—and what needs to be modified (Doorman et al., 2013). In the following, we describe the HLT informing our task design.

Designing a programming and geometrical task

Before the interaction with the expert mathematics teacher, the authors developed an a priori HLT addressing what we anticipated to be the main challenges in implementing PCT in mathematics. We envisioned that to ensure that students engaged in learning processes involving both mathematics and PCT, it was necessary to develop a task that required them to activate knowledge from the domains of both geometry and programming. Moreover, we believed that creating such a task would have a positive side effect for mathematics teachers in that the relevance of integrating PCT in mathematics would be clear. The learning goal is to deduce and test a general expression for the internal angle of any regular polygon. To solve the programming task, students should do the mathematical action of conjecturing a general expression for the internal angle of an *n*-gon. In turn, to test out such an expression, they must find a way of programming it in *Scratch*. A sample solution is depicted in scratch.mit.edu/projects/541978601.

The learning activities, depicted in Table 1, scaffold students' work by drawing on the *use-modify-create* principle from programming education (Lee et al., 2011), which states that it is easier for students to start with a pre-made script and adapt it to solve variations of a problem before elaborating new code. We also envisioned a progression in the task based on increasing difficulty by starting from a square, then a triangle, hexagon, pentagon and finally generalised polygons.

Table 1: Original hypothetical learning trajectory

Learning activities		Hypothetical learning process
Use	a *Scratch* code that draws one segment.	Become familiarised with basic *pen up*, *down*, *move* blocks.
	a *Scratch* code that turns the pen 90°. Explore what happens when run repeatedly.	Become familiarised with *turn*(degrees) block. Realise that four repetitions suffice to draw a square.
Modify	the code to draw a square with one run using a repeat loop.	Intuitive inclusion of a *repeat*(4)-block.
	the code so that it draws an equilateral triangle.	Modify to *repeat*(3) and *turn*(60°), revising to turn (120°) distinguishing between internal and external angles.
	the code so that it draws a regular hexagon.	Modify to *repeat*(6) and *turn*(60°), after previous experience or trial and error.
	the code so that it draws a regular pentagon.	A trial-and-error strategy will be slow or unsuccessful, prompting to find the 72° external angle with pen and paper.
Create	a code that draws any regular *n*-sided polygon. Teacher introduces *variable* block.	After registering previous cases, students find collectively a pattern that can be generalised for internal/external angles.

The design decisions and hypothetical learning processes were rooted in the literature on embedding PCT in mathematics classrooms. One of the mathematical courses from technology comprehension available in tekforsøget.dk/forlob involves programming *BeeBots*[1] to draw polygons in first grade. The robots, however, can only turn 90°, limiting the tasks to squares and rectangles. Drawing

[1] BeeBot is a robot designed for use by young children, which can be used for teaching sequencing, estimation, problem-solving and more.

polygons is also part of the *ScratchMath* project's "Beetle geometry" module (see, e.g., Benton et al., 2017). Their design proposes exploring four increasingly difficult polygons, figuring the turning angle: square, rectangle, equilateral triangle, and regular hexagon.

Furthermore, Herheim and Johnsen-Haines (2020) explored two seventh-grade students' productive struggles while drawing a pentagon. From here, some of the expected obstacles are mistaking internal by external angles and a trial-and-error strategy that will not suffice to reach a solution, thus calling for reasoning and calculating the exact 72° angle. Accordingly, students should overcome sequential challenges. First, students would use a loop to draw a square in one run. Secondly, they will sort out the internal angle for a triangle and hexagon and adapt their code. Next, they will require a pen-and-paper solution for the pentagon, as opposed to trial-and-error. Finally, they would generalise the latter case, introducing a variable for the number of sides.

Upon designing the task described above, the two first authors interviewed an expert mathematics teacher who acted as a supervisor at the school on which the task is to be implemented. The purpose of the interview was two-fold: (1) to test and reach a mutual understanding of the core implementation challenges to address and (2) to refine the task according to these challenges. The interview was semi-structured and 60 minutes long. The teacher agreed to have the meeting recorded for research purposes. Below, we describe the insights regarding our initial HLT that this interview brought.

Revisions of the task based on teacher interview

The teacher agreed on the learning goal, but the sequence of activities and hypothetical learning processes are revised in Table 2. Based on her input, three changes are worth mentioning: starting with a blank script, as opposed to using a working code, further use of pen-and-paper, and sequential order of polygons according to the number of sides.

Table 2: Revised HLT based on teacher's input

Learning activities	Hypothetical learning processes
What are polygons? Write your findings in *Book Creator*.	Students collectively activate their previous knowledge on polygons and their elements (sides, angles, vertices).
Create a *Scratch* code that draws an equilateral triangle. Create a *Scratch* code that draws a square. Create a *Scratch* code that draws a regular pentagon. Create a *Scratch* code that draws a hexagon.	Students engage in a trial-and-error strategy, registering in *Book Creator* the parameters used for *repeat* and *turn* blocks in increasing order. These are tools for the next activity.
Create a code that draws a figure of your choice: a flag, a labyrinth, a logo, or your own artwork.	This motivation is set up in the introduction but executed in the end. It gives students purpose to explore polygons' properties and how to depict them in *Scratch*.

The teacher challenged the pedagogy of *use-modify-create* by proposing that students should start creating their own code. The idea of starting with a given script is that it avoids frustration and makes the first step easy.

Teacher: It's easy to do it. But … instead of giving the code from the start, the first thing is to let the children make it themselves. Maybe they can do it; maybe they can't. But they need to understand what are we doing now. Instead of just having the code,

> “how can we do this?” “What is the problem?” “What do we need to know?” “Which kind of code do we need to do to code a triangle?”

Overall, her take is that more relevant than a smooth start in terms of difficulty, the students ought to perceive the posed problem as their own. Moreover, enabling struggle (Pérez, 2018; Herheim & Johnsen-Haines, 2020) and debugging (Weintrop et al., 2016) is consistent with the literature on computational thinking in mathematics classrooms.

Since an essential part of our HLT was requiring students to activate both mathematical and programming knowledge, all mathematical operations in our task were conducted in *Scratch*. We envisioned that the students would occasionally enter a flawed code, which the *Scratch* drawing would make clear (a flawed figure), and that the students iteratively would correct the code to draw the desired one. While the teacher acknowledged and agreed with the importance of this iterative approach, she anticipated that the students would not by themselves engage systematically in a new iteration if they were not encouraged to document their working process. Therefore, she suggested supplementing the task with pen and paper for students to log their attempts and use these logs actively in the next iteration.

> Teacher: I call it [to] scribble. They have to take notes of what they did. After all, the idea is not that they randomly should enter numbers from 1–200. What they enter should be based on their thinking. They have to learn to be systematic, you know.

In the original design, pen-and-paper was reduced to a minimum and based on the idea of computational thinking as a thought process that becomes concretised by programming in a programming environment. In technology comprehension, programming is considered a technological skill detached from computational thinking (Smith et al., 2020). However, the teacher had a different approach:

> Teacher: I think the children start, and then we need to engage [them] in the thinking along the way. They are not sitting and planning.

The task ultimately aims at educating students to draw an n-sided polygon, for which they require to find a general expression for its internal angle. Though the literature supports the scaffolding in our initial design (e.g., Benton et al., 2017), the teacher suggests arranging the sequence in increasing order. She draws on the Swedish researcher Olof Magne’s juxtaposition of arithmetic operations in teaching:

> Teacher: And I have worked with that. Instead of just starting with addition, then I took it all [i.e. all arithmetical operations], and the children didn’t have the same problems as they usually have. They learnt to make divisions in the first grade because we learnt it all. That’s part of my mindset.

Overall, these proposed changes to the task respond to letting students *own* the problem and *explore* possible solutions systematically instead of solving a fixed sequence of exercises and offering scaffolded solutions.

Discussion

As described previously, DBIR consists of four main principles, namely persistent problems from multiple stakeholders, iterative and collaborative design, theory development, and building capacity for sustainable change. Although we believe the study reported in this paper has accommodated all of these principles, the teacher interview led to refinements of the task that we anchor in the first and fourth.

Discussing the task gave perspective towards different stakeholders and their presumed roles. In particular, the new version of the task will be more open, thus engaging students to solve a problem perceived as their own. Moreover, we envision tackling the issue of teacher training in programming by seeing students coding themselves:

> Teacher: They (…) think that [the] responsibility for teaching *Scratch* in on their shoulders; they don't take it. But then they see that the children can manage *Scratch* and they have (…) seen what the children can do, and have seen the children's eyes, and they have felt the good spirit about working with math and *Scratch*, then they are convinced.

Tasks supporting PCT and mathematics integration should accommodate both aims of ensuring students' mathematical learning processes *and* anticipated or perceived challenges from teachers' perspectives. The design process reported in this study corroborates that these ends are compatible by meaningfully relating the two subject matters and acknowledging that teachers do not feel sufficiently proficient in PCT to teach it. On the same line, it is worth mentioning that the teacher did not question the HLT's learning goal, as it addresses one of the main issues of technology comprehension tasks, namely their connection to the mathematics subject matter:

> Teacher: The teachers know math, and they know the didactics around that. But they don't know the didactics around technology comprehension. (…) Suddenly it was something they had to work with (…) and then, they saw the math and they thought the math wasn't... eh... very good.

The case of this first design iteration illustrates how one can anticipate adaptation and reduce the distance between what is supposed to be and actually being implemented. For example, the initial design included a working code assuming that students had not been introduced to the pen environment on *Scratch*. However, the teacher challenged this claim.

Although previous research has found *use-modify-create* to be an appropriate approach to scaffold students who are not accustomed to coding, the teacher had experience and knowledge of the importance of showing students that they are able to develop code on their own from very early on. Besides informing refinement of the task, this insight indicates that while theory on the didactics of PCT can inspire initial task design, it does not necessarily align with what is likely to work in a specific context. That is not to say that one particular teacher's opinion must be generalised at face value. In turn, these conflicts can be tried out as hypotheses for theory development, in its humble sense (diSessa & Cobb, 2004). Furthermore, from a DBIR perspective, the quest for designing resources that build capacity over time must account for the diversity of pedagogical choices.

Conclusion

In this paper, we have investigated what we can learn about the implementation of PCT in mathematics education by designing a single task with a teacher from a DBIR perspective. Revisions to the task were made by interviewing a teacher involved in the future process of implementing such a sequence. The new version promotes further ownership on behalf of the teacher by making her take part in the design process and revise implicit and explicit assumptions. Further, working on a particular HLT goes beyond local task design. Fidelity is a recurrent theme in implementation research (Century & Cassata, 2016), likely because the design of innovations may conflict with teachers' established and preferred practices. Involving the teacher in the initial design phase allowed us to anticipate such conflicts and to alter the innovation design in accordance with her reality. Our future collaborative endeavour with the teacher will teach us more about whether it leads to high or low degrees and innovation enactment adaptation.

The openness to practitioners' needs can make theory development more grounded (Bakker & van Eerde, 2015), but it leaves at least three options to move forward. In a humble sense, the educational design enables developing a local theory of how a task could successfully implement innovations that integrate mathematics and PCT at a specific school. A second alternative is to inform the conjecturing of hypotheses to try out at a larger scale. For example, one may validate or dismiss the pedagogical choice for *use-modify-create*. DBIR, however, offers a broader perspective. For capacity building's sake, a variety of possible approaches can be provided to practitioners, who may pick and adapt tasks according to their priorities. In that sense, theory development ought to be at a higher level than a particular HLT. Instead, resources that sustain change for embedding PCT in mathematics may refer to general domain-crossing themes or scenarios, such as drawing geometrical figures with programming.

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The components of Vergnaud's notion of scheme as frame for designing diagnostic tasks in the area of linear equation solving

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This paper formulates an implementation process model for designing and implementing tasks for formative feedback in an online learning environment for mathematics classrooms using Vergnaud's notion of scheme. In Denmark, the online environment matematikfessor.dk is used by more than 70% of Danish k-9 students. Using variations on one task, we illustrate how Vergnaud's notion of scheme can be used to aid teachers in assessing students' understanding of problems involving linear equations. Specifically, we focus on the intentional, computational and generative aspects of students' schemes.

Keywords: Implementation, diagnostic tasks, task design, online environment, linear equations.

Introduction

There is a great deal of evidence that formative assessment can have a positive impact on learning (Black & Wiliam, 1998). Attempts to promote formative assessment have often resulted in teachers having substantial difficulties implementing these ideas (Bennett, 2011). A key issue is that teachers face challenges inferring students' learning needs based on those students performance on tasks (Thompson & Thompson, 1994). Several programs have been launched historically in the attempt to document specific difficulties in learning mathematics. These projects have contributed to the understanding of many misconceptions and other obstacles to learning mathematics (Rhine et al., 2018). Tasks of a diagnostic character have in many cases been a part of these projects. There is an extensive literature on how the tasks performed, why they were chosen and why the tasks are believed to be sensible choices for exploring certain areas of difficulty in learning mathematics (Küchemann, 1981). However, few papers systematically address the considerations or design principles underlying the construction of these tasks in order to provide relevant feedback to teachers.

In this paper, our focus is on formulating an *implementation process model* to guide the process of translating research into practice (Nilsen, 2015). In this case, the model is directed at the design of formative tasks within online learning environments (OLEs) that implement Vergnaud's (2009) notion of scheme to enable teachers to better interpret, and respond to, learners' errors. We do this in the specific learning contexts of the concepts of linear equations and the equals sign, and draw on research on learners' errors and difficulties with these concepts (Rhine et al., 2018). OLEs can play a significant role when implementing research-based knowledge into the classroom, since OLEs can provide feedback directly to the end users, namely the teachers (Dyssegaard et al., 2017). In addition, these environments can produce substantial assessment data almost immediately with relatively little effort from teachers. However, a key restriction, and challenge to the implementation of Vergnaud's scheme, is that learners' responses to tasks are restricted due to input and marking constraints and thus cannot be fully open-ended.

In this paper, we propose the following research question: *How can the notion of scheme guide the design of diagnostic tasks for implementation in OLEs about known difficulties with the concept of linear equations and the equals sign, in order to enable teachers to better interpret learners' difficulties?*

The context: *matematikfessor.dk, a*n OLE for mathematics

In Denmark, as in many other systems, OLEs are increasingly used by teachers and students. *Matematikfessor.dk*, the environment discussed in this paper, has been running for over 10 years. More than 500,000 students in primary and lower secondary schools have access to the environment and, on a typical day, 45,000 unique students use the variety of tasks offered by the site, and collectively answer 1,500,000 tasks. OLEs like *matematikfessor.dk* therefore have access to a large amount of data and can potentially provide valuable feedback to teachers about the difficulties that students encounter in learning mathematics. As such OLEs become more ubiquitous, it is crucial to develop an explicit understanding of the process of designing tasks for implementation in online environments in order to enable teachers in preparing better teaching.

Theoretical constructs

Difficulties in learning mathematics related to linear equations

We adopt the view of Jankvist and Niss (2015) that identifies genuine difficulties in learning mathematics as, *"those seemingly unsurmountable obstacles and impediments – stumbling blocks – which some students encounter in their attempt to learn the subject."*(Jankvist & Niss, 2015, p. 260). One kind of stumbling block that many students experience in the beginning of lower secondary school or when attempting to learn the solving of more 'abstract' (Vlassis, 2002) linear equations is the role and interpretation of the equals sign (Kieran, 1981). Jones et al. (2012) argues that in order to achieve a better understanding of the role of the equals sign, one must learn to substitute one representation for another equal representation. Many strategies for solving equations exist and are all sensible tied to different tasks and/or situations. However, at some point even linear equations can become abstract or complicated to an extent where only one strategy truly remains. Many of the strategies such as 'guess-and-check' or 'working backwards' do not necessarily require a deep understanding of the role or interpretation of the equals sign (Linsell, 2009). However, in order to apply more advanced equations solving strategies students must become more flexible in their understanding of the equals sign (Kieran, 1981).

The notion of scheme and its role in activity

In Vergnaud's (2009) work, we become familiar with how the scheme as a concept works as an organizer of action or activity when faced with a situation or a class of situations:

> [Schemes] describe ordinary ways of doing, for situations already mastered, and give hints on how to tackle new situations. Schemes are adaptable resources: they assimilate new situations by accommodating to them. Therefore, the definition of schemes must contain ready-made rules, tricks and procedures that have been shaped by already mastered situations (p.88)

Such a situation or class of situations could be exemplified as working with algebraic expressions or engaging in solving linear equations. If we accept that schemes are organizers of the activity of an individual, we can create assumptions about students' schemes by observing their action and activity in desired situations. Ahl and Helenius (2018) claim that this is why schemes are both didactically as well as analytically more interesting than the idea of conceptual understanding.

Vergnaud (2009) defines a scheme as having four aspects:

> The **intentional aspect** involves a goal or several goals that can be developed in subgoals and anticipations. The **generative aspect** involves rules to generate activity, namely the sequences of actions, information gathering, and controls. The **epistemic aspect** involves operational invariants, namely concepts-in-action and theorems-in-action. Their main function is to pick up and select the relevant information and infer from it goals and rules. The **computational aspect** involves possibilities of inference. They are essential to understand that thinking is made up of an intense activity of computation, even in apparently simple situations; even more in new situations. We need to generate goals, subgoals and rules, also properties and relationships that are not observable.
>
> The main points I needed to stress in this definition are the generative property of schemes, and the fact that they contain conceptual components, without which they would be unable to adapt activity to the variety of cases a subject usually meets. (p. 88)

Essential to the schemes from Vergnaud's perspective are the operational invariants (the epistemic aspect of schemes), consisting of *concepts-in-action* and *theorems-in-action*. A *concept-in-action* Vergnaud describes as *"an object, a predicate, or a category that is held to be relevant."*(Vergnaud, 1988, p. 168). In every mathematical action, we choose certain objects, predicates or categories of such that are believed to be relevant in the current situation or setting. A *theorem-in-action* as a proposition held to be true. When we engage in a mathematical situation, we believe certain "theorems" to be true or false, about the objects relevant to the situation. Vergnaud states that there is a dialectic connection between theorems and concepts, and this emerges from the fact that more advanced mathematical concepts originate from theorems and vice versa. Nonetheless, is it important to distinguish the cognitive function of the operational invariants in this very specific manner. Concepts-in-action are individually available concepts in a for the enactor relevant representation to the situation. Vergnaud's (1988) interpretation of the representation is similar to what others call a conception or a concept image (Tall & Vinner, 1981). Concepts-in-action bear no value of logical truth, just relevance to the situation. Theorems-in-action are by nature true or false. These entities are sentences (or propositions) that provide the concepts with the possibility of inferences to take place. The *rules of action* are not to be confused with the *theorems-in-action*. The function of the *rules of action* (the generative aspect of the scheme) is to be appropriate and efficient, but they rely implicit on theorems-in-action (Vergnaud, 1997). Vergnaud (2009) emphasizes that schemes are efficient organizers of activity by nature, and, should they also become effective, the scheme can be considered an algorithm. He further clarifies that schemes do not have all the characteristics of algorithms. The effectiveness of algorithms lets them find a solution to a task in a finite number of steps (if a solution is possible).

Task design and implementation considerations

In this section, we present the task design and the reasoning behind it. The design will focus on the role and interpretation of the equals sign as presented in Jones et al. (2012). The formulation is guided by the four components of the scheme (Vergnaud, 2009). An added discussion of the considerations of the implementation in an online environment precedes each task formulation. The overarching design principle idea comes from Ahl and Helenius (2018) who present a situation where a student is asked to calculate the average speed. The student invokes a scheme seemingly capable of handling average speed to some extent but ends up invoking and working with a scheme that incorrectly interprets average speed using an alternative (and incorrect) scheme involving a different average, namely arithmetic mean. Inspired by this, we formulate a set of tasks that invoke two potential paths to solution. The aim is to setup situations where two different schemes might be in play at the same time but the answer to the task should reflect which of the schemes that got the upper hand. We also reformulate the task in different ways to attempt to address the different aspects of the scheme.

In *matematikfessor.dk*, the tasks must meet certain criteria regarding structure and user input types. The input types are restricted to inputting either a number or a multiple-choice selection. Each task must have a unique 'right answer' and must be presented in such a way to make immediate feedback possible. Our task design is intended to disrupt the 'typical' didactical contract (Brousseau, 2006) in that that 'right' answer highlights aspects of the student's scheme.

In the following, we demonstrate the task design in examples focused on the intentional, generative and computational aspects of the scheme. We do not focus explicitly on the epistemic aspect of the scheme, because this aspect, focused on the operational invariants, is such an essential part of the scheme that this aspect and its elements inform or are present in the other three components. For example, one would simply not be able establish goals without having at least partial access to a concept relevant to the situation, a concept-in-action. The focus will be on evaluating whether students invoke schemes capable of handling substitution based on equality or rather a scheme suitable for solving equations in order to solve the task and thus provide teachers with information on their students' schemes.

For the task design, we have chosen the following setup, in order to attempt to possibly invoke two schemes at the same time. We have chosen the following sequence of tasks to illustrate this design principle. Given the special constraints, we have chosen to only provide a single sequence of tasks, all adaptations of one item. Using a scheme for handling and exchanging equal terms should be the 'easier' path to the goal of solving the following task that serves as an outset for the variations in the following sections. An expert equation solver would choose the most efficient scheme to solve the task - "I know x has a specific value and I could easily calculate it, but I don't need to for this task'.

What number should go in the empty space?

$$\begin{aligned} 3x \quad &= 11 \\ 3x + 4 &= \_\_ \end{aligned}$$

If students are able to insert 11 instead of the term $3x$ in the second equation (swap $3x$ and 11), our assumption is that the task is then easily solved by a secondary school student. However, if students are more inclined to determine the value of *x*, because that is what is ‘expected’ when faced with linear equations, then they might have a difficult experience, because the numbers are chosen in such a way that 3 is not a divisor of 11 (and *x* has a rational value).

Setting ‘Goals and anticipations’ (the intentional part of the scheme)

In this category, we present the task in a formulation where setting a goal for, or an anticipation of the task form of the solution. In many cases, one expects that, when confronted with a task containing a linear equation. finding the unknown value is crucial. We propose that when focusing on the goals and anticipation part of the scheme, the task could be formulated like the following;

Is it necessary to know the value of *x* in order to fill in the empty space?

$$\begin{aligned} 3x \quad &= 11 \\ 3x + 4 &= \_\_ \end{aligned}$$

Goals and rules are set and established based on the concepts-in-action and the theorems-in-action. Whether a student regards the equality between $3x$ and 11, and the fact that the term $3x$ is present in both equations as relevant information (concepts-in-action), could provide about whether s/he is capable of swapping the terms 11 and $3x$ in the two equations. Should a student not choose the substitutional link between the two equations to be relevant we expect that s/he would argue that they would like to know the value of *x* in order to fill in the empty space. One might argue that the goal of the task is blurred by the new formulation, since there is an empty line ‘begging’ for a number to be put on it, but the task is answered by a simple yes or no. However, for the purpose of assessment for learning, teachers might learn about their students’ schemes, with this formulation as opposed to just receiving a correct or incorrect answer from students filling in the empty space.

Establishing ‘Rules of action’ (the generative part of the scheme)

In this category, we attempt to uncover the rule, or strategy, for solving the task the student’s would apply. As mentioned in the above category, the operational invariants help set the goal or apply rules. When forming a strategy for solving a task, theorems-in-action might be more in focus. Therefore, we formulate the task not in terms of concepts relevance but rather in theorems that lead to rules. Now we confirm that the equation $3x = 11$ is in fact important in order to achieve filling in the empty space. We propose a task where the formulation hints at what ‘path to the goal’ a student chooses, an equation solving strategy or a substitution of equal terms strategy.

How is $3x = 11$ important in order to fill in the empty space?

$$\begin{aligned} 3x \quad &= 11 \\ 3x + 4 &= \_\_ \end{aligned}$$

1. Because $3x$ and 11 can be swapped in both lines
2. Because I can calculate the value of *x*
3. It is not important

In this formulation, the teacher will get a slightly different view on what path a student is willing to choose. A further formulation of the task focusing on *rules to generate action* could look like;

How would you attempt to find out that number go in the empty space?

1. I would find the value of x
2. I would swap $3x$ for 11
3. I don't know

If the scheme(s) the student is drawing upon are only able to apply the rule of action to determine the value of the unknown, the teacher is provided with valuable information. In this way, a teacher gets a different perspective on basically the same task but in a different formulation and with a different focus or aim.

Generating space for 'Possible inferences' (the computational aspect of the scheme)

In this last example, we attempt to address the computational part of the scheme with another formulation of the initial task. In order to address whether a student make inferences about the element $3x$ when they compare it to a similar looking task, but where the scheme for solving equations should be rendered useless, because $3x$ has been replaced by a blue box.

Is it the same number that go in the empty space in both tasks?

$$\begin{array}{lcl} 3x & = & 11 \\ 3x+4 & = & \_\_ \end{array} \qquad\qquad \begin{array}{lcl} \boxtimes & = & 11 \\ \boxtimes + 4 & = & \_\_ \end{array}$$

If a student does in fact not see the similarities between the two tasks, but uses a scheme for solving equations working with the left most task and however is able to swap the blue box and the number 11 in the right most task, the student might become suspicious. One might expect the student to wonder why this is the case or why the tasks performs differently yet so similarly. A different formulation could therefore be;

Is it surprising the same number go in the empty space in both tasks?

Is it for the same reasons that the same number go in the empty space in both tasks?

The reason for wording the task this way is establish a cognitive conflict with students that are in fact surprised that 15 is the correct solution for both empty spaces. We attempt to create a link to a scheme we consider similar to the scheme capable of swapping mathematical equal terms by introducing the task with the blue box.

Concluding remarks

For this paper, we have explored considerations and the reasoning behind an implementation process model to guide the diagnostic task design in an OLE. This exercise was to implement Vergnaud's scheme in order to enable or improve teachers' opportunities to interpret learners' errors within the constraints of the OLE. We based the task around different formulations or variations of an item using the components of the scheme. We have shown such slight reformulations of a task, guided by the components of the scheme, have the potential to reveal different aspects of students' understanding related to linear equations and the equals sign. The tasks could be presented to the students in the order presented in this paper or they could be presented each to one third of the class for a subsequent discussion. The strength of the online environment is that teachers, as well as students, can get easy access to feedback. We believe that because the tasks call for two specific schemes as generators in the situation, students would get a unique opportunity to be shown why it is in fact not necessary to know the value of the unknown.

We have used Vergnaud's intentional, generative and computational aspects of the scheme to demonstrate how an 'exemplar' item can be designed to distinguish between the different schemes that students use to tackle a task. The intentional aspect relates to *what* needs to be done to solve the task or *what* is expected of the student in this situation. The generative aspect relates to *how* the expectation are met or *how* progression is to be made in the situation. Finally, the computational aspect relates to *why* the desired goal is achieved or *why* new connections to other schemes or concepts make sense or might be established. We have touched briefly on omission of the epistemic part of the scheme and will leave a deeper explanation for further work. The *what*, *how* and *why* might potentially contribute to shared or agreed upon *theory of change* (Jankvist et al., 2021) and thereby strengthening the implementability of the tasks by sharing the idea of respecting the *what*, *how* and *why* with teachers using matematikfessor.dk (Jankvist et al., 2021).

Teachers can be provided clear feedback on what different scheme their students are most willing to invoke when solving these tasks. In the implementation of these items, teachers are not provided a guide to interpret the answer their students provided to the items. Teachers become aware of a possible need to teach substitution of equal terms. We have selected a task that illustrates the approach of the design frame for varying the task respecting the *what*, *how* and *why*. The next step in the process is to implement this framework with a range of tasks for relevant online feedback to teachers using *matematikfessor.dk*. Knowledge sharing and feedback from teachers using the tasks in their classroom will pave the way for a sensible implementation of a range of sequences of tasks.

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Depth as a key issue for implementing DEM: The case of a teacher

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This case study examines one Norwegian teacher's enactment of an innovative system for mathematics teaching called developmental education in mathematics (DEM). The findings show that despite appropriate textbooks, high motivation, a belief that its principles are effective for mathematics teaching and learning, and indications of a shift of ownership of DEM, the teacher did not follow the fundamental principle of appropriate mathematical challenges for the students. Based on the findings, this paper identifies challenges regarding qualitative aspects, such as depth, of the implementation of the DEM project, while suggesting a path forward for a type of scaling-up process that does not necessarily include spreading to the largest possible number of schools.

Keywords: Developmental education in mathematics, Zankov, Vygotsky, scale of implementation, depth of implementation

Introduction and aim

In Norway, some teachers have had success using a system for teaching elementary mathematics called *developmental education in mathematics* (DEM). DEM consists of mathematics textbooks adapted from Russia and a didactical theory developed by Russian psychologist Leonid V. Zankov (1901–1977), who was a student of Lev S. Vygotsky. Inspired by the excellent results of the pilot class (Melhus, 2015), around 100 schools across Norway have now adopted DEM.

However, a recurring issue in mathematics education is turning small-scale successes into improvements of practice on a larger scale (e.g., Jankvist et al., 2021). A main problem of the DEM project is a lack of systematic knowledge about the teaching practices of the various schools since much of the effort so far has focused on curriculum development and dissemination. In addition, some schools simply use the textbooks without consulting DEM facilitators, leading to an even wider knowledge gap. In this sense, DEM remains an innovation at an early stage despite the history of the first teacher going as far back as 2009 (Gjære & Blank, 2019). To begin to address this knowledge gap, a PhD project seeks to characterize both the potential for DEM to support students' mathematical development and the challenges that some teachers face along the way.

The aim of this paper is to analyze one teacher's enactment of DEM to answer the following research question: How is the main principle of teaching at an optimal level of difficulty realized in a 4th Grade mathematics classroom? The findings will form a basis for discussing more general challenges pertaining to scale, specifically the *depth* of implementation (Coburn, 2003) of the DEM system.

A short introduction to DEM

DEM builds on the didactical theory developed by L. V. Zankov. Its use in mathematics education in Norway depends on a series of textbooks written in the 1990s under the guidance of Iren Arginskaya, a mathematician and member of Zankov's research group. These books follow Zankov's principles and have been translated and adapted for Norwegian schools. The main goal of DEM is not only to increase the mathematical abilities of the students, but more so to stimulate their general development (Melhus, 2015; Zankov, 1977). The didactical principles of DEM are as follows (Zankov, 1977):

1. Teaching at a high (optimal) level of difficulty
2. The leading role of theoretical knowledge
3. Proceeding at a rapid pace
4. Promoting students' awareness of the learning process
5. Systematic development of each student in the classroom

The five principles form a whole; they are interconnected and augment each other (Zankov, 1977). Nevertheless, this paper focuses on the first principle. Zankov's system has been called "implementing the zone of proximal development" (Guseva & Solomonovich, 2017), since this concept lies at its core and its realization is fundamental and necessary to promote students' development. Zankov (1977) built on Vygotsky when he wrote that the ZPD

> is identified by noting the kinds of problems that the child is unable to cope with himself, but can solve with the aid of grownups, in collaborative activity, or through imitation. But what a child can do in cooperation with someone else today, he will be able to do alone tomorrow (p. 18).

This can be contrasted with problems students can do by themselves, in their actual zone of development. Zankov (1977) underlined the importance of the students' emotional engagement needed to spend the intellectual effort to cooperate with others and overcome difficult problems. DEM teachers said that they saw "challenge-as-fun" as a central characteristic of DEM and something that had changed their views about teaching and learning mathematics (Gjære & Blank, 2019).

Framing the study within a discussion of *scale* in implementation research

An increasing number of schools now use the DEM textbooks. However, a discussion of the scale of an implementation cannot rely on numbers alone. Addressing this issue, Coburn (2003) suggested four major dimensions for assessing both quantitative and qualitative aspects of *scale* of educational implementations: Depth, sustainability, spread, and a shift in ownership of the innovation (Figure 1).

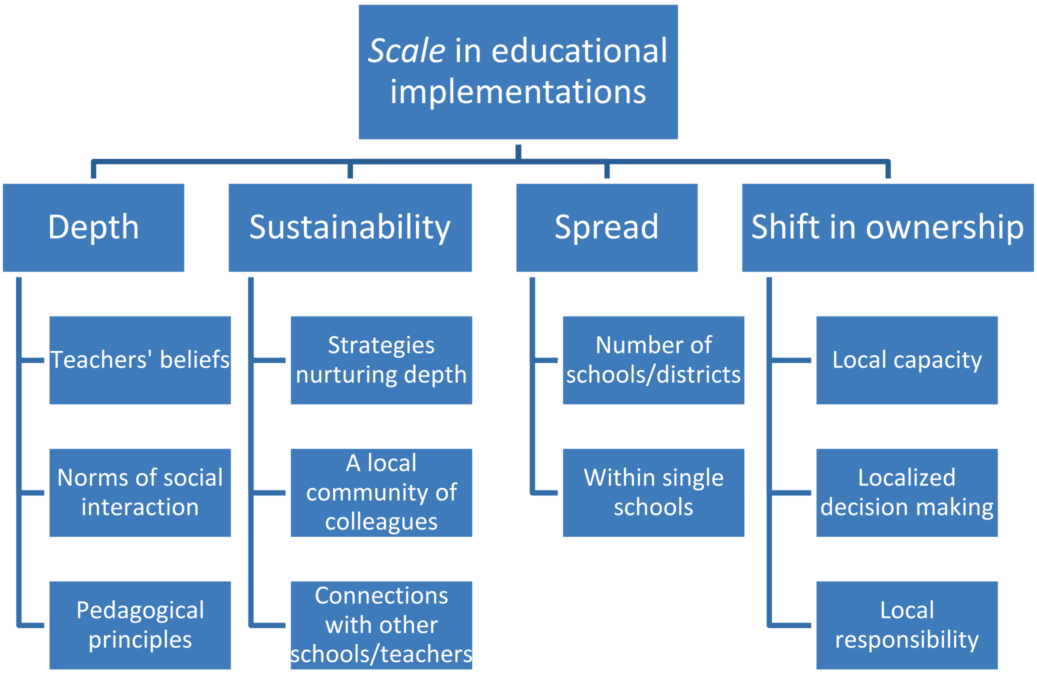

Figure 1: A conceptualization of scale of implementation, adapted from Coburn (2003) by the author of this paper.

The explicit goal of the DEM project is not to spread to as many schools as possible but to offer an alternative for teachers who are interested and find the system to suit their students' needs (Gjære & Blank, 2019). Thus, attention shifts to the other aspects of scale, namely, depth, sustainability, and a shift in ownership of the DEM system. These will form the foundation of the discussion in this paper.

Case description and method

This case study draws its data from a wider set of classroom videos, where four experienced DEM teachers participated. They were all "early movers", meaning that they were among the first to use the DEM textbooks in Norway. The 4th Grade teacher in this study works at a school that functions as a "model school" for DEM, where those who are interested can come and observe DEM lessons and talk with the teachers there. She has also participated in dissemination activities, such as writing about DEM in a journal for mathematics teachers and giving presentations about the positive experiences at her school. The same teachers also participated in focus group interviews before and after classroom videos were recorded. These interviews were analyzed separately and indicated that implementing DEM had changed their views on teaching and learning mathematics (Gjære & Blank, 2019), including the teacher in this case study. The teacher was asked to plan her lessons as usual and not think about the video cameras.

According to Eun (2019), the concept of the ZPD encapsulates Vygotsky's theory of learning and development and can profitably be used as a lens to analyze various aspects of teaching-learning situations. In this paper, the concept of the ZPD serves as an analytical frame to interpret the way the teacher engaged with the students to challenge them mathematically. While all three of this teacher's lessons were analyzed in whole in the preparation for this paper, the data presentation has been shortened to include only one task from each lesson due to space limitations.

Findings

Episode from Lesson 1: Solving an equation

The equation to be solved was $(3n + 10) : 8 = 35$. A possible solution method, introduced in the 4th Grade textbook, is based on doing opposite operations: If $a : b = c$ and a is unknown, then you can find a by multiplying c by b. Here, $(3n + 10)$ takes the role of a and you can find its value by multiplying 35 by 8. However, the teacher introduced the activity by reminding the students that they had to do the same operations on both sides of the equal sign, a slightly different and more general approach. Student 17 suggested beginning by multiplying 8 by 35, which corresponds to the textbook method, but had difficulties explaining further. Student 8 was asked to elaborate on Student 17's response:

206	S8	(*comes up to the board*) Well, you could say that one (*points at the left-hand side of the equation*) equals that one (*points at the right-hand side*).
207	Teacher:	Aha! Let me see if I understand you two correctly. It is divided by eight, times eight, really, on both sides (*she transforms the equation, see the second line in Figure 2).*
208	S8:	Um, yeah.
209	Teacher:	Is that your thinking, Student 17?
210	S17:	Yeah
211	Teacher:	That you multiply by eight on both sides. Yes!

S8 addressed the main concept of equality (utterance 206) but did not explicitly connect it to solving the equation. The teacher, however, went directly to the conclusion and wrote the transformation of the equation herself (207). The students' responses (208 and 210) were not convincing. This pattern persisted during the whole solution process, with the teacher leading and writing, and students only supplying short answers along the way (see Figure 2).

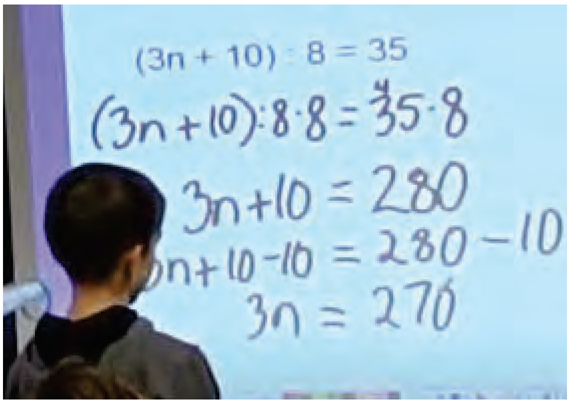

Figure 2: Student 8 watches as the teacher helps him write the solution of the equation.

This activity did not follow the DEM principle of teaching at an optimal level of difficulty. Zankov (1977) was clear that the students themselves must make the effort to solve challenging problems to develop their abilities. The beginning of this episode suggests that the task could suit the students' ZPD nicely since S17 and S8 both contribute with mathematically productive statements about the equation although their reasoning is incomplete. In utterance 207, however, the teacher completed a reasoning step for them, based on her own solution method and not the textbook method. This took away the challenge for the students to solve the equation themselves, resulting in a lack of mathematical activity.

Episode from Lesson 2: Finding the volumes of right rectangular prisms

This activity was mostly a whole-class discussion about how to find the volumes of right rectangular prisms. During the discussion, some students expressed frustration by moving about on their chairs, sighing audibly, or answering in odd voices. Two prisms, 1 and 2, were pictured on the board, along with a table to fill out with length, width, height, and volume of each prism.

277	Teacher:	OK! Let's use the formula to fill out this table and calculate the volume of these two figures. What then, can we say about the <u>length</u> of figure 1? (*She pauses. The students are unrestful and only a few have their hands up*) The length of figure 1? The length of figure 1, people. S11.
278	S11:	Um, it's five, I think. Yeah.
279	Teacher:	Yes, five (*writes "5" under Length, figure 1, in the table*)
280	S11:	But I found out what the whole was, too.
281	Teacher:	The length is five. What about the width? (*pause*) S17?
282	S17:	Two.
283:	Teacher:	Two. (*writes "2" in the table*) and the height? S6?
284:	S6	THREE! (*answers in an odd voice*)
285:	Teacher:	And then the volume is?
286	Students	Twenty-four / thirty (*both numbers are heard*)
287:	Teacher:	Five…? Five time two is…? (*speaks very slowly and clearly*)
288:	Students:	Ten!
290	Teacher:	Ten times three?
291	Students	THIRTY! (*shouting*)

Note that S11 was ready to provide an answer in utterance 280. This, along with some signs of unrest, suggests that the students found the progression too slow. When given a worksheet on the same topic, some students expressed a lack of challenge:

307	Teacher:	You are to find the volume of these prisms. But you see, they aren't quite filled up with cubic centimeters. Can you still find the <u>length</u>, the <u>width,</u> and the <u>height</u> of these prisms?
308	A student:	Um, yeah.
309	Another:	°That's easy° (*heard whispering to his desk mate*)

The whole activity, with different examples of prisms, took around 17 minutes, 10 of which were whole-class discussion. This activity also lacked the kind of challenge that characterizes a "Zankov's lesson": The students found calculating the volumes of these prisms easy and they could do it by themselves, meaning that the activity was within their actual zone of development, and the extended whole-class discussion reduced the pace of progression.

Episode from Lesson 3: A numerical pattern

The task was: "Find the pattern of the sequence and write the next number: 2, 5, 11, 23, 47, 95, …". The students first discussed in pairs for a couple of minutes before presenting their results, and three different ways of describing the pattern were presented (Figure 3).

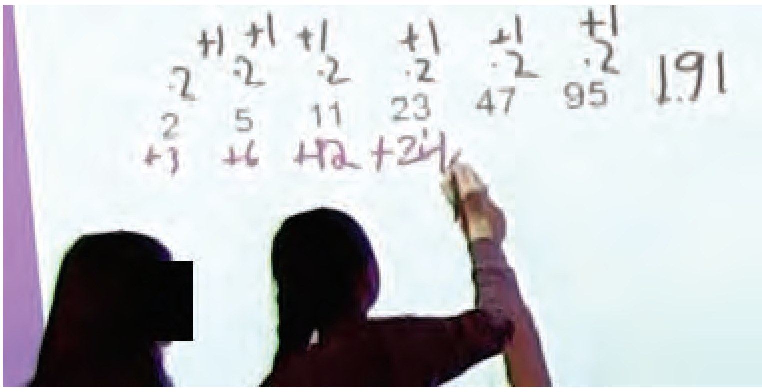

Figure 3: Students describing the number pattern.

The students' three ways to describe the pattern were: To get from a number to the next, multiply by 2 and add 1 (written ·2+1 in Figure 3); that the "add on"-numbers double for each step (+3, +6, +12, …); and finally, to get from a number to the next, add one more than the number itself (e.g., to get from 5 to 11, you add 6, which is 5 + 1). The whole task was quickly done; from thinking time in pairs to students having presented three different pattern descriptions it took about 6 minutes. The teacher did not interfere with the students' thinking. She took the role as a discussion moderator while checking if the other students understood or agreed (they used hand signs to indicate this).

Notably, the teacher allowed her students a lot more room to discuss and present ideas in Lesson 3 than in the other two lessons. This can be gleaned from the description of the pattern activity above. However, she also refrained from interfering and did not push the students further, e.g., by directing their attention toward relationships between the three pattern descriptions. The students did the task on their own and presented their results without further explorations, and the activity therefore remained within their actual zone of development. While they were successful in solving the task, the teacher did not seize the opportunity to challenge them further. However, both this and the other two lessons demonstrated another important aspect of DEM, namely, cultivating a sympathetic and respectful community of learners (Zankov, 1977). For instance, the teacher called for an appreciative applause for the presenting students after this episode, which she also did several times during the three lessons.

Discussion

In general, the analysis showed that the teacher did not follow the didactical principle of optimal difficulty in the three observed lessons. In one sense, this finding could be interpreted as a lack of *fidelity* in the sense of "the extent to which an innovation in enacted according to its intended model" (Century & Cassata, 2016, p. 171). However, the word "fidelity" usually has meanings associated with *loyalty* or *faithfulness* (Cambridge Dictionary, 2022), so in this sense, a lack of fidelity could imply a lack of loyalty to DEM, which is clearly not the case here. On the contrary, this teacher spoke highly of DEM and showed indications of having given it a central place in her teaching practice.

According to Coburn (2003), a *shift of ownership* of the innovation is necessary to achieve a lasting impact on teaching practice at scale. At the school of this teacher, the staff have formed a local DEM community, they have decided to let DEM take a central place in the school system and influence other school subjects, and they (including the teacher in this case) have taken part in dissemination activities; all of this across an extended period of time with only limited support from university facilitators. These are indicators not only of taking ownership of DEM, but also of sustainability and spread within the school (see Figure 1, p. 3). This means that the school is well positioned to continue using DEM independently from university facilitators. However, as the findings above show, there are still challenges to work out.

Realizing pedagogical principles of the system in practice relates to what Coburn (2003) refers to as *depth* of implementation scale. In her conceptualization of scale, depth is both a key dimension in its own right and an underpinning of the other dimension. In this case, there were in fact indicators of depth of the implementation for this teacher since she reported to have changed her beliefs about mathematics education (Gjære & Blank, 2019). Also, the norms of social interaction in her mathematics classroom corresponded with the sympathetic community of learners suggested by Zankov (1977), although it is not clear how much this has changed in her practice as she was not observed prior to DEM. However, the main goal of DEM is to stimulate students' development (both general and mathematical) by engaging them in solving challenging problems and encouraging them to be persistent, analytical, investigative, self-reflecting, critical and creative. There is also the question of whether whole-class problem-solving discussions are suited for realizing the principle of optimal difficulty, since one must expect diversity among students within a class. Considering these issues, it becomes clear that depth, and in particular the realization of the principles of the system, must take center stage in any discussion of scale of the DEM project.

Concluding remarks

The discussion of this case study demonstrates the usefulness of Coburn's (2003) conceptualization of scale to bring an alternative and more varied perspective on scaling-up processes for innovation projects that do not aim for the greatest possible spread. *Scale* in educational innovations often implies spreading to many schools. The DEM project, however, is concerned with providing an alternative system for teaching elementary mathematics that could possibly improve the practice of those teachers who are interested and motivated. For assessing the scale of such projects, qualitative dimensions such as depth, sustainability and a shift in ownership becomes even more important. This could also raise the general question of whether spreading to many schools should always be implicit in scaling-up processes of innovations in mathematics education.

Challenges ahead for the DEM project include supporting a greater depth for the various DEM schools as well as sustainability and a shift in ownership of the project for the involved schools.

Addressing issues of depth requires a more nuanced understanding of the difficulties with realizing the didactical principles like those that the teacher of the case study experienced. Such research efforts could be combined with either initiating the formation of "satellite communities" in the various municipalities where DEM is used or reaching out to contact already existing communities, with the purpose of improving the scale of the DEM project across all four dimensions of Coburn's (2003) conceptualization.

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STEAM-based learning ecosystems involving Digital Toolkits, Tutoring Systems, 3D Printing and Mathematical Trails

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Keywords: STEM Education, Teacher education programs, Educational technology.

During the past 5 years, we have been experimenting with a wide range of educational technologies in elementary schools and higher education with a focus on STEAM (Science-Technology-Engineering-Arts-Mathematics) integration and exploring role of mathematics within STEAM. These projects included STEAM integrated approaches for teacher training, special needs education with Augmented Reality and 3D Printing, remote teaching and automated tutoring systems. Over this period we observed a shift towards technology-based teaching and learning in education, and we aimed to identify how educational ecosystems with a variety of technologies such as Augmented Reality, 3D Printing, or tutoring systems could provide increased accessibility and opportunities for STEAM-based educational approaches. In this poster, we will give an overview of how the above-mentioned technologies could be employed by different users for teaching and learning STEAM-based educational ecosystem. Thus, in our research, we evaluated effects on students' learning in our educational projects. These projects involved automated tutoring systems, mathematical modelling of real-world objects with CAD Software, Dynamic Mathematics Software and 3D Printing devices, and outdoor mathematical trails with GPS-supported software and a number of Erasmus+ projects contributed to our studies Each study originate from the projects mentioned above was embedded within a joint research framework, nurtured by the results and participants' feedback from our studies. Building on this framework we focused on identifying how students, teachers, pre-service teachers and parents could access learning tasks and settings with different educational technologies and experience new opportunities in STEAM learning. From complementary findings of these studies we identified the importance of an interconnection of the different tasks and technologies, which can contribute to a creative, learning ecology (Szabó et al., 2021) of mathematics learning. Each kind of technology supports different approaches to learning and teaching, and in the combination of various technologies offers a wider accessibility to skills and knowledge for both teachers and students. Hence, in this poster, we present how the different educational technologies utilizing our findings could be used in an educational ecosystem, supporting with technology-based creative approaches.

Methods

Tasks and uses of technologies were based on the interplay of Blums and Leiß modelling theory (Blum & Leiß, 2007), problem-solving approaches (Liljedahl et al., 2016), Dienes' principles on learning (Dienes, 1960) and STEAM pedagogical frameworks (Haas, 2021). The tasks we elaborated in these projects are linked to school curricula and based on reliable methodologies implemented in schools and higher education training (e.g.: active discoveries, peer learning, student-centered learning). We followed recent studies on creative ecologies of everyday learning to identify structures for a system implementation (e.g.: Szabó et al., 2021). In every study, we followed a design-based research methodology (Lee & Hannafin, 2016). However, we adapted the methodology slightly for the different studies, due to the used technologies, environment, or possibilities and the restrictions imposed by the COVID-19 pandemic. Moreover, we used quantitative, qualitative and mixed-methods triangulation in the different studies. We will present methods separately for each study in the poster. We obtained results indicating that those educational technologies with real-world connections are likely to engage students, parents and teachers in new motivating and creative ways.

Discussion

Through addressing open-ended problems using guided support with decreasing feedback and scaffolding, our learners and teachers were able to develop their process skills and content skills in STEAM disciplines at their own pace. These task designs can be implemented in creating new learning settings and each technology supports students in their learning and teachers or parents in their teachings. Students could use the automated tutoring system in class or at home to train process skills with scaffoldings and learn to apply these skills in active mathematical modelling approaches with real-world information, situations, objects, or places. Thus, over time it could be a valuable asset to support students individually or in groups or even the entire class with the automated tutoring system within traditional courses. With the dynamic geometry software, CAD software and 3D printing, students learned new strategies and skills. The solving behaviour in geometric tasks gained in structure and the visual-spatial ability was developed (Haas, 2021). Further, students gained confidence and experience through a play-based approach to tasks with enjoyment. Finally, with GPS-supported software, we were able to transfer learning directly into students' real world in combination with Augmented Reality, 3D printing, and Dynamic Geometry Software.

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Designing stakeholder scaffolding for implementing a teaching model at scale

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The challenge of implementing pedagogical innovations at scale is discussed by looking at the set of stakeholders. A large-scale multi-municipality implementation of a highly specific research-based teaching model for elementary school arithmetic in Sweden is used as an exemplary case. To create a lens through which the stakeholder structure can be discussed, the concept of stakeholder scaffolding is introduced. The main message is that what is to become a stakeholder in an implementation program should not only be decided through making an inventory of possible stakeholders and deciding which ones to engage. It is also a matter of designing new groups with given roles and responsibilities in the implementation program.

Keywords: Professional development, design research, large scale, arithmetic, implementation.

Introduction

In organization theory, the concept of stakeholder refers to "any group or individual who can affect or is affected by the achievement of the firm's objectives'' (Freeman, 2010, p. 25). When the stakeholder term is used in connection with implementation research in the field of education it is rarely explicitly defined. Certainly, when an educational innovation is implemented, there is not often a firm with particular objectives at the center. Instead, it makes sense to view many implementation efforts as coordinated by networks of parties. Therefore, I will look at multi-stakeholder network theory (Roloff, 2008) for guidance and, define a stakeholder as *any group or individual who can affect or is affected by the approach to the educational issue addressed by the network of collaborating parties*.

By *stakeholder scaffolding*, I shall mean the intended interaction patterns between identified or constructed stakeholder groups and how these intended interaction patterns are designed to support the implementation process of a pedagogical innovation. In this sense, the stakeholder scaffolding is an aspect of the program theory of the implementation project. I use the theoretical construct *program theory* in the same way as Chen, that is, "as a set of explicit or implicit assumptions by stakeholders about what action is required to solve a social, educational or health problem and why the problem will respond to this action" (Chen, 2012, p. 17). In Chen's conceptualization, a program theory has two components, the change model and the action model. The *change model* specifies how activities in the program are supposed to affect certain determinants in the system that are to be changed in order to reach specified goals or outcomes. The *action model* is "a systematic plan for arranging staff, resources, settings, and support organizations to reach a target group and deliver intervention services" (Chen, 2012, p. 18). Stakeholder scaffolding, as I will describe the concept, is hence a part of the action model. The purpose of the concept of stakeholder scaffolding is to provide a lens that helps researchers and program designers to focus on the specific roles different stakeholder groups can assume in relation to the action model.

The aim of the present paper is to discuss stakeholder scaffolding in relation to the scaling of educational interventions. The theoretical basis for my discussion builds on Coburn's (2003) characterization of scale as a four-dimensional construct concerning depth, sustainability, spread, and change of ownership. Further, the issue of scaling will be discussed from a design perspective where I do not only discuss which potential stakeholders a particular educational system might already involve, but also how to deliberately design stakeholder groups with the purpose of achieving scalability.

The discussion will be carried out by exemplifying the stakeholder scaffolding in a large-scale multi-municipality implementation of a highly specific research-based teaching model for elementary school arithmetic in Sweden. This means that this is not an empirical article but descriptive and theoretical. Because of space limitations, I will discuss Coburns construct directly in relation to how the implementers have tried to take the four components of scale into account in the implementation program design. For readers that prefer an introduction to Coburn's (2003) conceptualization of scale, the original article is a very worthwhile read as are many of the shorter summaries (e.g., section 5 in Roesken-Winter et al., 2015). I will now go on to describe the implementation program, before moving on to discussing it relative to Coburn's four scaling dimensions.

Thinking, reasoning, and reckoning (TRR) 1-3

The pedagogical program I will describe has two roots. On one hand, it builds on a specific teaching model communicated through descriptions of teaching sequences in the form of very detailed teacher guides. On the other hand, it builds on a professional development (PD) program that is a collaboration between the National Center for Mathematics education (NCM) in Sweden, a multitude of municipalities, and Sveriges Kommuner och Regioner (SKR). SKR is a national organization of municipalities and regions that stands free from the national political system. In what follows, the Swedish acronym TRR will be used to designate both the teaching model and the collaborative program for implementing it. Before describing the structure of TRR as an implementation program, I will briefly describe the TRR teaching model.

TRR for grades 1-3 comprises four themes for each grade where each theme is a collection of five teaching cycles. A cycle covers three lessons. The teacher guides include sequenced descriptions of which representations to use, which terminology and expressions that are supposed to be used by the teacher, which questions to use when probing pupils reasoning, suggestions for how to sequence qualitatively different types of pupil reasoning and so on. Each cycle focuses on a particular mathematical phenomenon from one of the conceptual fields number, additive structures, or multiplicative structures (Säfström et al., 2019). A cycle has six phases: chanting, introductory whole class discussion, pair work, follow-up discussion, individual documentation, and follow-up discussion of the documentations. For each phase, the social arrangements and expectations of teachers' and students' roles are different. The TRR teaching model builds on a randomized control trial evaluated design research project for grade 0 in Sweden (Sterner et al., 2020; Sterner & Helenius, 2015).

The core of TRR is that participating teachers use the TRR cycles as a basis in their teaching. Each year, about half of the teaching time will then be directed by the TRR model. The rest of the time teachers can plan as they please. Participating teachers are formed in professional development (PD) groups of at least three teachers, teaching the same grade and teaching the same cycle at the same time. Before the first cycle in a new theme, between cycles and after the last cycle of the theme, the PD groups meet and discuss the upcoming cycle and the experience of teaching the previous cycle guided by structured PD-questions from a booklet under the guidance of a local facilitator appointed by the municipality.

I will now move on to discuss TRR from a top-down perspective. The initiative is joint between NCM and SKR and there is a joint steering group but no clear ownership of the project as a whole. TRR is run in (so far) yearly rounds, each engaging around five municipalities working together in so-called working networks of normally five municipalities. After a municipality has agreed to participate and signed a formal agreement with SKR, they appoint a process leader, which is normally a mathematics teacher that already has certain special leadership assignments in the municipality and a networking group consisting of the highest responsible politician for school matters in the municipality, the highest non-political leadership, two principals, two teachers, and the process leader. What I have described so far can be called the institutional scaffolding of TRR and is illustrated in Figure 1.

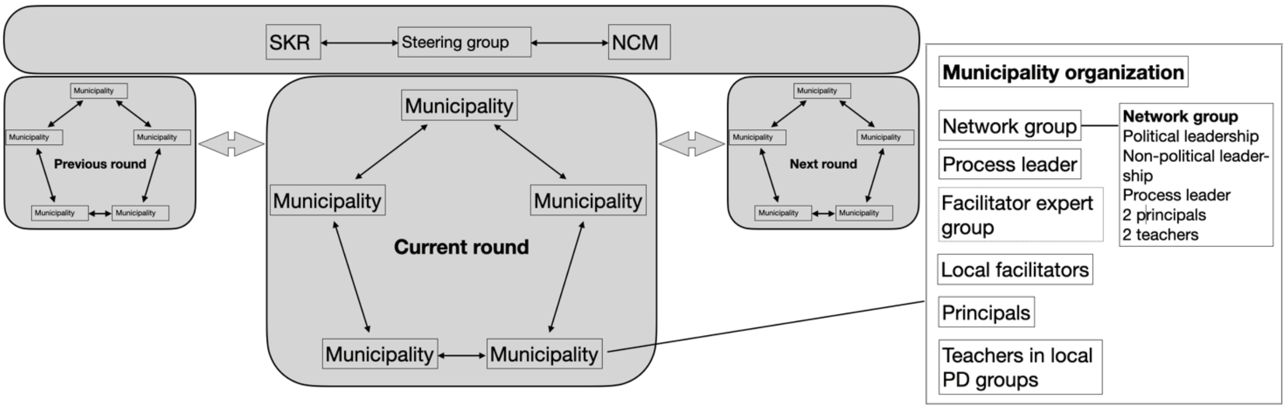

Figure 1: The institutional stakeholder scaffolding of TRR

The temporal scaffolding, which is how activities are distributed over time, is illustrated in Figure 2. In Sweden, a school year begins in the autumn. Therefore, in the temporal description of TRR what we describe as, for example, year 1 (Y1) is the year that starts in the autumn when the teachers of an attending municipality start to use the TRR teaching model at scale as well as the associated PD-program. Y2 is hence the second year of TRR usage and Y0 is the preparatory year before.

The first thing that happens after the municipality has signed the contract to participate is to appoint the process leader and the network group. All process leaders in the current round then meet for a process leader conference (PLC). This is a first or a rather tight series of PLCs that during the pandemic has been held almost every month. Slightly after, the network conferences (NWC) are repeated two times a year. To ensure knowledge transfer between the different rounds, around half of the PLC and NWCs in Y0 and into Y1 are joint with the previous round and later joint with the next round. Both PLCs and NWCs are two-day sessions.

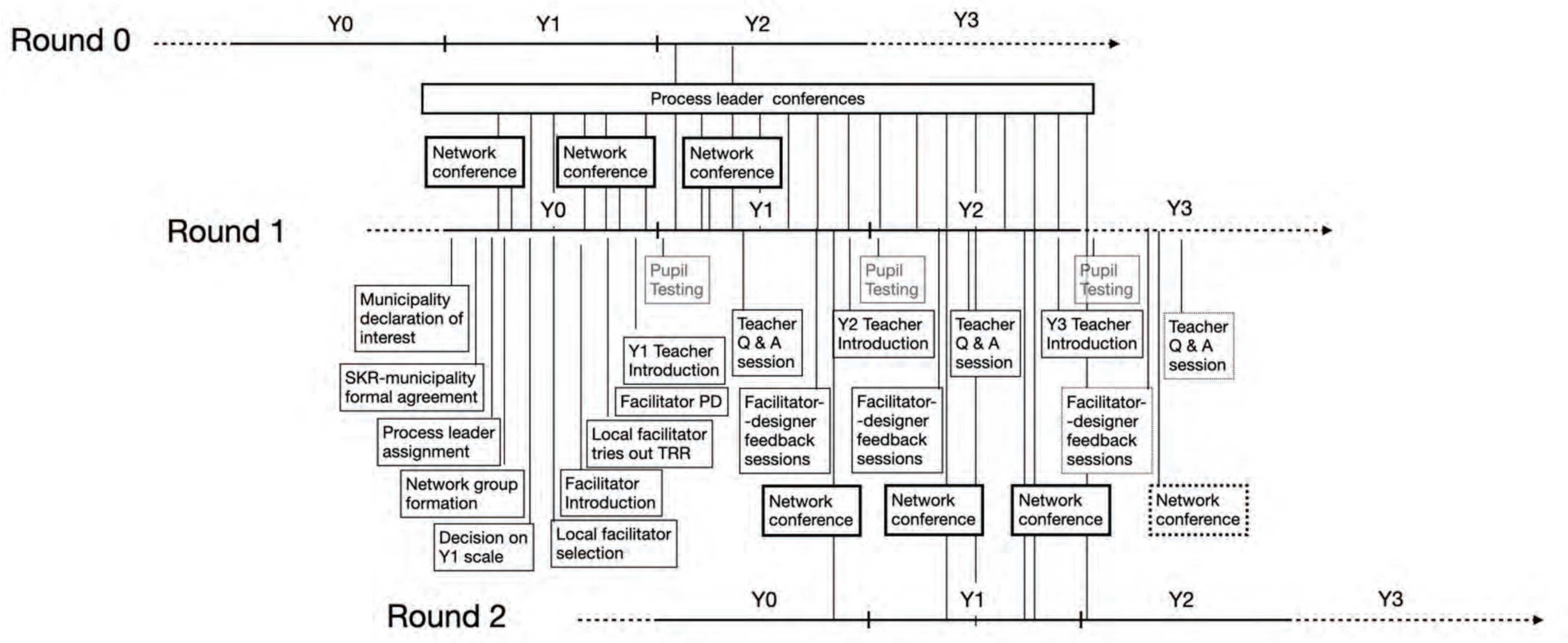

Figure 2: **The temporal stakeholder scaffolding of TRR**

An important decision that municipalities have to take early is at what scale to run the program. We have examples where municipalities have joined with all their teachers and schools in all grades 1-3, some that join with all grades in a few schools, some that join with all teachers in grade 1, and one municipality that only joined with one school. An early decision on the scale is important because the number of teachers and how they are distributed between grades and schools determine the number of local facilitators and we want the local facilitators that are to work with PD-groups when Y1 starts to start preparing, by being introduced to TRR in Y0 and start to use some cycles in the classes they currently teach. Municipalities are then responsible to book introductory sessions with NCM for the teachers to start in Y1. Some want two such sessions and some stick with one. In subsequent years, the individual municipalities can also decide to book Q&A sessions with NCM for both facilitators and teachers. These are normally local events. In the temporal diagram in Figure 2, I have also indicated where pupil testing can be, but that can vary between municipalities depending on their scaling plan and how to set up a control group design.

Formally, the project runs three years for each municipality, but in practice, it has so far continued with less and less PLCs and NWCs. Municipalities in Y3 and beyond have so far continued to book introductory sessions for new teachers and Q&A sessions for working teachers with NCM.

Four dimensions of scale and how it applies to TRR

I shall now discuss the stakeholder scaffolding in TRR described above, from the perspective of scale as Coburn's (2003) four-factor construct: depth, sustainability, spread, and shift in ownership. It should be noted that the stakeholder groups I describe below are intended to be examples of possible roles for stakeholders in relation to scaling. I make no claims that the particular stakeholder scaffolding of TRR is particularly good or that it is always well functioning.

Depth

Coburn's conceptualization of depth concerns "change that goes beyond surface structures or procedures [...] to alter teachers' beliefs, norms of social interaction, and pedagogical principles as enacted in the curriculum" (Coburn, 2003, p. 4). The focus in Coburn's description is on students' and teachers' roles in the classroom, like for example the patterns of communicational interaction. As described, TRR gives very strong guidance for teachers to follow particular interaction patterns, as well as particular ways of treating the mathematical content. Obviously, not all TRR intentions are followed. The implementers recognize that small adaptations from teachers affect how the content is portrayed to the pupils (Koljonen et al., 2022). Therefore, highly prescriptive teaching designs still need deliberate stakeholder scaffolding to support depth. In TRR, the local PD-groups are one such structure intended to create a support for interpreting the intentions of each cycle as well as following up the classroom implementation and also induce a form of peer pressure. The local facilitators are a stakeholder structure designed to support the PD groups. Forming the facilitators as a group with a particular role is a key element in the stakeholder scaffolding. It is our experience from previous large-scale PD-programs (Helenius, 2021) then letting one group of teachers, the facilitators, work ahead and giving them some additional PD as well as officially designating them as a facilitator group tends to shift of identity from a receiver of the PD to a deliverer of the PD which enforces loyalty to the program intentions and supports them in their role relative the PD groups.

Sustainability

Sustainability concerns the chances that deep changes stay in effect in classrooms, schools, and municipalities where they have been established. Referring to a wealth of literature, Coburn describes sustainability as dependent on four factors, which all reinforce a professional community of colleagues in the school that supports depth of change of which we will deal with two here. The first of these is the need for knowledgeable and supportive school leadership. In TRR, all levels of school leadership from principals and up are specifically engaged. The principal level is represented in the network group, which is the most complex of the designed stakeholder groups. It is also a common arrangement that the PL is co-opted into the municipality principal group, typically led by the municipality school leader, which further allows the process leader to influence and engage principals. The second factor I will mention is the connection with other schools and teachers engaged with a similar reform. Since TRR is a municipality driven there is plenty of such interaction within municipalities.

Spread

Spread may be the most obvious factor of scaling, as it involves the fundamental factor of replicating the implementation of some reform or innovation with more students, teachers, schools, or districts. Coburn makes a point of considering spread also within different levels. Spread within classrooms could for example involve pedagogical innovations proposed in one subject spreading to a teacher's teaching in other subjects. Spread within a district or municipality could involve certain norms related to a pedagogical teaching innovation spreading to local school policy (Coburn, 2003).

The stakeholder scaffolding of TRR has several elements designed to support spread to other schools and classrooms. The scaling that occurs to new municipalities with new rounds obviously leads to scaling, and so far all participating municipalities have decided to continue from Y1 to Y2, and beyond and have included more schools and teachers except in the cases where all schools were already engaged.

Shift in Ownership

The aspect of scale that Coburn perhaps puts the most emphasis is on a shift in ownership. It is worth noticing that the metaphor of shift of ownership presupposes an initial owner, an item to be owned, and a new owner. In Coburn's description, it is some reform program that is the item and the initial owner is some actor external to teachers, schools, and districts. With such a shift, knowledge of the reform can become local and a theory-based practice can arise so that the reform can become self-generative (Coburn, 2003). With TRR, the stakeholder scaffolding sets up a quite advanced structure for shared ownership right from the start, that gradually can shift towards municipalities and teachers. Municipalities have to themselves decide the scale with which to enter the program, depending on how much they believe in the program and in their own ability to administer it and later decide how to scale it up. Recall, the arrangement that teachers should meet regularly in PD-groups and be able to teach the same cycles in parallel puts quite high administrative demands on all levels of leadership. It is questions like this that are discussed in the early PLCs and NWCs. Because municipalities are responsible for scaling also makes it important that teachers remain positive and even when feeling like the teaching is very difficult keep feeling that they have support from the organization. It is so far our experience that facilitators, but also many teachers, identify with the program and help to spread the word to teachers in the municipality that are yet not engaged. A final note on ownership is the several of the municipalities, without consulting SKR or NCM, have produced their own professional videos on TRR, to share with parents, the general public, and other teachers.

Discussion

Designing models for implementing pedagogical innovations at scale is a vital concern in contemporary implementation research. In the present paper, I took a closer look at the stakeholder issue related to scale and coined the concept of stakeholder scaffolding, which I presented as a part of the action model of the program theory (Chen, 2012). Chen already identifies particular types of stakeholders when schematically describing an action model, the implementing organization, peer organizations and community partners, and the program implementers. It is worth noting that Chen is discussing a wide section of innovations spanning over the social, educational, and health sectors. This means that the specificity when discussing the roles of the stakeholder groups he mentions is low. In the case of TRR, it is for example not very easy to separate what is the implementing organizations from the peer organizations. But a point Chen is making when pointing out different types of stakeholders as part of the action model is that types and roles of the stakeholders *should* be planned in advance and *be* part of the action model. Going back to the description I gave drawing on Roloff (2008), it states that a stakeholder can be *any group or individual who can affect or is affected by the approach to the educational issue addressed by the network of collaborating parties*. A main message in the present article is that what is to become a stakeholder in an implementation program should not only be decided by reviewing groups that can be affected by the program and can affect it, like researchers, policymakers, principals and teachers. Instead, the view I am advocating is that stakeholder groups can be deliberately created as crossover groups between such levels as part of the design of a program.

In TRR, one such designed stakeholder group is the group of process leaders in each round. By identifying such a group and giving them opportunities to work together, their roles in the respective municipalities will be strengthened. The process leader group is formed through the multitude of process leader conferences, which also ensures that the process leaders are the most informed persons which further strengthens their authority and agency within municipalities. The arrangement with process leaders coworking with the peer group from a previous round, and eventually, with the peer group from the successor round further strengthens their agency. Other examples of designed stakeholder groups are the network groups in the municipalities with the role of giving coherent support for TRR across all institutional levels in the municipality.

In summary, TRR F-3 is an example of a teaching model that is very demanding for teachers that are to use it in their classrooms. Despite the highly scripted form of the TRR teacher guides, we believe it is virtually impossible for an individual teacher to operationalize TRR in their classroom. It can only be done in deliberate collaborative settings with a high degree of organizational support. In short, it can only be done by ensuring that all of Coburns four aspects of scale are accounted for in the implementation program. Designing and implementing a well-thought-out stakeholder scaffolding is hence an important part of the program design. When researching effects of large-scale implementation programs, I see it is a good idea to put special focus on if the stakeholder scaffolding worked as intended or not.

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Implementing research in the practice of curriculum designers: Barriers and an example approach

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Barriers to educational designers' engagement with research, including barriers to access, interpretation of implications, implementation, and interpretation of outcomes, can limit the translation of research into practice through educational design. Our aim in this paper is to define four challenges to implementing research through the processes and products of curriculum design and to explore theoretically-grounded principles for addressing these challenges through an example of a design tool, the Cambridge Mathematics Framework. To illustrate the principles underlying this tool, we give a brief concrete example of its use in a curriculum design implementation.

Keywords: Curriculum, concept mapping, design, evaluation

Introduction

The benefits of engaging with research in educational design have long been recognised in a range of approaches which place value on developing explicit, well-supported reasoning to inform design decisions, and on iterative refinement using data from implementation in authentic contexts (McKenney & Reeves, 2012). Lowering barriers to implementing research through educational design may allow these benefits to be realised more often and more fully. In this paper, we aim to characterise four challenges to implementing research through the processes and products of design and explore the question of how these challenges might be addressed. Using the example of a design tool, the Cambridge Mathematics Framework (CMF), we present principles and underlying theoretical influences for addressing these challenges with respect to research implementation and evaluation methods, and briefly illustrate its use in a curriculum design implementation. In the following section, we begin by describing research implementation challenges.

Challenges to research implementation through processes and products of design

In our examination of factors which might potentially affect educational designers' ability or opportunity to implement research in design, we have identified four challenges.

1. There are factors which limit engagement with research. The resource-intensive nature of research implementation presents a challenge to engagement with educational research (van Schaik et al., 2018). Even in national-level curriculum design, time and resource constraints have been noted which can limit the amount and depth of engagement with research (Jameson et al., 2021). Smaller-scale design efforts may be more narrowly focused but may bring proportionally fewer resources to bear on research integration, or on evaluating the influences of research-informed features when designs are implemented.

2. Design is an intermediary construct. We hold that any designed resource or experience is an artifact which serves as an intermediary between the theories and intentions of the designer and those of teachers and students, because designers typically do not enter classrooms to directly implement and explain the resources they have created. The distance in time, space and perspectives between researchers, designers, teachers and students often leads to differences between what is stated by researchers, intended by designers, enacted by teachers and experienced by students (Barab, 2014). Koichu and Pinto (2019) call implementation a “multiparty enterprise” and the CERME11 TWG23 adapted a definition of implementation to include the notion of distance between communities of resource proponents and communities of resource adapters (Artigue, 2021). With respect to curriculum design, Remillard and Heck (2014) describe this distance within their curriculum model, an update to Stein et al.’s (2007) model of curriculum phases, which the participants of TWG23 in CERME11 adapted in a model of research implementation (Aguilar et al., 2019). The new research implementation model highlights the transformations that research implications undergo as they are first used to structure objectives (intended implications), are translated into classroom experiences (enacted implications), and result finally in particular outcomes (attained implications).

3. Research is one influence among many on outcomes. There is a dilution of research implications which is not made explicit in the CERME11 model (Aguilar et al., 2019). Design itself (the “attributes of the innovation”) is only one of many factors which may contribute to the final impact of innovations on learning (Century & Cassata, 2016, p. 186).That is, the further down the chain from published research, the more room there is for other factors to influence what happens and the less research influences may be contributing to observed outcomes, even taking differences in intentions, background and interpretation between researchers, teachers and students out of the picture. Therefore, the enacted and attained implications may be impossible to separate from the individual instances of enactment and subsequent learning outcomes as a whole, which may take their specific forms for a variety of reasons. A third challenge, then, is one for implementation research: How can we analyse the contributions of research influences, particularly when research is being translated into practices which occur at some distance from the classroom (as with some forms of educational design practice including curriculum design)?

4. Research implications are not always straightforward. The final challenge we will highlight here is that research in mathematics education does not inherently add up to a coherent whole. It is a heterogeneous body of work reflecting a wide range of perspectives, assumptions, goals and practices, serving numerous wider agendas shaped by research communities rather than by the requests of teachers or administrators (Burkhardt & Schoenfeld, 2003). Knowledge from research likewise differs in nature and focus from pedagogical knowledge (McIntyre, 2005). Implications for teaching and design are therefore seldom straightforward. Moreover, while there are strong reasons for researchers and educators alike to consider implications from very different sources together as a coherent set, the diversity of the field can make this a complex undertaking involving detailed understanding of a wide range of relevant theories and experimental designs (Prediger et al., 2008). Despite this, it is important for designers to be able to access research influences at this level of detail, to navigate a variety of theoretical frames appropriate to different aspects of design problems, and to use available data accordingly (Kieran et al., 2015).

There is no one-size-fits-all solution; however, next we present one approach developed to help designers overcome these challenges when developing mathematics curricula.

An approach to addressing challenges of implementing research through design

In this section we provide as much detail of the components, use, and theoretical basis of the CMF as space permits. The CMF is a conceptual mapping tool which designers can use to explore, analyse and accommodate research implications for the interdependence of mathematical ideas when discussing and justifying the writing and ordering of curriculum content. It consists of the following key components: (1) a dynamic network of selected mathematical ideas in the domain of school mathematics and connections between them, which we have developed based on thematic review, synthesis and interpretation of the mathematics education research literature, along with connections to research sources; (2) tools for exploring, analysing and visualising the network and the research base; (3) Research Summary documents which explain the research justification for specific groups of mathematical ideas and relationships and which undergo external review; (4) descriptions of how students engage with these ideas in practice, in a form designed for teachers and designers to relate to their own experience; and (5) external content such as curriculum statements or mathematical tasks which can be mapped on to the network of mathematical ideas and analysed in terms of content and ordering. Ideas in the network are not ordered in an absolute sense, by year or age, but by how ideas contribute to one another. This ordering is not linear but allows for many paths from one idea to another which a designer may find useful to consider. Importantly, it is a conceptual ordering and not meant to imply teaching order or student trajectories, as students often need to go back and forth as they develop experiences working with a set of ideas. Finally, (6) the CMF can be used to produce concrete artifacts for analysis and discussion in the form of maps in which designers have gathered relevant collections of waypoints and linked them to their own content.

In a typical curriculum development scenario, a designer using the CMF might start by searching for mathematical ideas relevant to the section of the curriculum they're working with and display them in the order they depend on one another, from left to right, together with curriculum content on the screen as a map. Then curriculum content can be linked, or "mapped onto" to the mathematical ideas, which helps designers to explore possibilities within the map as a whole for reordering, adding, subtracting or changing curriculum content. When changes are made to the curriculum content on the basis of the CMF, the research justification provided in the CMF for the mathematical ideas and connections can serve as a reference for designers in documenting and communicating their justifications to stakeholders.

We structure the theoretical underpinnings of the CMF around several key design assumptions; in this paper we describe three which are most relevant to research implementation. We have drawn on literature in multiple areas: domain coherence in mathematics, conceptual mapping, and shared representations as boundary objects. Our first design assumption is that support for research-informed perspectives on domain coherence in the design of curricula or resources can contribute to their value for teaching and learning by allowing designers to consider a wider range of implications and develop curricula and resources with greater domain coherence as a result. This assumption is related to challenges 1 and 4, since it involves providing support (i.e. making it more straightforward, thorough or feasible) for using research perspectives in design decisions and provides aid to designers in synthesizing research from multiple sources and perspectives. It is based on strands of research involving the conceptual structure of mathematics learning and notions of coherence. Thurston (1990) describes this structure as being "like a scaffolding, with many interconnected supports"; he conceives of it as tall, because concepts build on one another, broad because of the multitude of interconnections necessary to provide foundations for building, and connected, with the connections helping students and mathematicians to compress sets of ideas to make use of them more efficiently (Thurston, 1990, p. 2). Tall (2013) suggests that this structure is built through categorization (the "recognition of essential properties"), encapsulation ("repeating actions" that "can be manipulated as mental objects") and definition, using "language to formulate specific concepts as a basis for reasoning and proof" (Tall, 2013, p. 51). Research on learning in different topic areas of mathematics often has implications for the particular mathematical ideas and connections of importance, and we have applied Tall's principles for learners to our expression of conceptual structures implied by research for designers. On this basis we view designs with high domain coherence as those which aim to leverage conceptual dependencies that are well-aligned with those reported in research. This definition is in agreement with the idea of "learner-centered curricular coherence as…an organizational means to promote a high likelihood that each learner traverses one of many possible paths to understand target disciplinary ideas […] by building on and continuously broadening and modifying their ideas and experiences." (Confrey et al., 2017, p. 719) and has also been informed by perspectives emphasizing the logic underlying sequences of ideas (Schmidt et al., 2005; Stark, 1986).

The second design assumption is that our goals from the first assumption can be aided by a mapping approach. This assumption is related to challenges 1 and 2, as mapping can make detailed relationships explicit and easier both to work with and to discuss. The affordances of conceptual mapping for analysing and communicating relationships between mathematical ideas have been demonstrated in research on learning trajectories and related constructs (Confrey, 2019). These affordances are also well characterised more generally in studies of knowledge management (Eppler, 2004).

The third design assumption is that the maps and mediating documents of the CMF can serve as shared artifacts which may facilitate discussions about design both within and between communities of practice, and that doing so may help design intents to be realised more fully by teachers or adapted for them more effectively by eliciting feedback. This assumption is again related to the need for communication in challenge 2. For this we draw on Remillard and Heck's (2014) characterization of curriculum and resource implementation as taking place in a system of communities between which designs are translated and possibly transformed as they are documented by designers, enacted by teachers and received by students. This has led us to shape aspects of the CMF according to the properties of shared knowledge representations, especially boundary objects as proposed by Star & Griesemer (1989); in particular the qualities of being adaptable yet structurally consistent and of allowing members of different communities to recognise information from their own perspectives and connect this to others' through a common space or artifact.

Taken together, these design assumptions are intended to address challenges 1, 2 and 4 by helping designers to deal with issues of coherence and the complexity of the research base and by providing a shared frame of reference for design discussions and support within and between communities. Challenge 3, however, is the issue of how research influences in design contribute to design process and teaching and learning outcomes. Much of this challenge involves implementation research rather than the design of the CMF itself.

The CMF is currently being implemented in a number of different curriculum and resource design contexts at different scales. In all cases so far, the CMF as a design tool is being implemented with designers, whose designs are subsequently implemented in learning contexts. The contributions of the CMF are therefore directly related to design outcomes and somewhat indirectly related to teaching and learning outcomes, as any design influences would be. We are developing a preliminary evaluation framework to helps us to analyse the direct and indirect contributions of our research interpretations in the CMF across a variety of authentic implementation contexts. Such contexts can be expected to have uncontrolled external factors (Barab, 2014) and multiple influences determining outcomes (Stern et al., 2012), making contribution analysis a useful approach for evaluating impact and gaining insight into specifically how our approach might be improved.

This framework helps us to characterise implementations according to several factors. First, we have identified four categories into which our evaluation goals might fall (adapted from Stern et al., 2012): Attribution: how much influence did the CMF have on the outcomes?; Contribution: Did the desired outcomes occur?; Mechanism: By what means did the CMF contribute?; and Translation: What circumstances are conducive to achieving intended outcomes using the CMF? The first two categories involve the value and relevance of the CMF; the last two may help us to learn about broader contexts in which use of the CMF is more likely to be successful. In addition to goals, this framework will help us to position an implementation according to meaningful duration, conceptual range, systemic scope, and distance of the CMF from measured outcomes, and will help us to frame outcome indicators and measures accordingly. Below, we briefly discuss some design outcomes from a recent implementation of the CMF which further illustrate this approach.

A brief description of some illustrative implementation outcomes

Our most recent case study took place as we were beginning to develop the evaluation framework. In the summer of 2020, the CMF was used in work on Statistics and Probability strands of the Australian Curriculum as part of the review currently being carried out by the Australian Curriculum, Assessment and Reporting Authority (ACARA). The ACARA team worked with the CMF and the Cambridge Mathematics (CM) team and described design outcomes from this work through face-to-face meetings and semi-structured diaries. Below we give a very brief description of the context and summarise the outcomes we have synthesized from meetings and diaries with respect to the four research implementation challenges; this synthesis contributed to identifying these challenges.

1. There are factors which limit engagement with research; 4. Research implications are not always straightforward. The ACARA team reported that while they spent just as much time thinking about research, using the CMF they were able to use this time to bring much more research from sources which were new to them into their discussions. They felt in some ways that there would always be some research that they would not have time to fully engage with, but that the way it was presented in particular topic areas in the CMF helped them to address specific questions and introduce new ones. They made use of a variety of CMF features in order to access research; they began with Research Summaries and CMF content curated for them according to their existing curriculum statements, and branched out by using search features and following connections to look for other aspects of topics to consider. In the course of this, they reported being unable to find some things; we realised that while some of what they were looking for was implicit in some CMF content, it was not explicit enough for the ACARA designers to recognise, and they had to spend more time looking for it than we intended. This helped us to revise not only the presentation of that content but to examine other areas of the CMF for similar issues.

The ACARA team also reported engaging with research implications in a mix of forms. They looked back and forth from research summary text to maps to descriptions of map elements. They were able to find some useful ideas through their own queries. The flexibility of the multi-representational approach in the CMF meant that to the extent required the ACARA team was able to develop and follow their own lines of inquiry beyond the starting points curated by CMF writers, with a few exceptions mentioned in 1. Moreover, the Research Summary starting points contributed substantively to several of the discussions we observed.

2. Design is an intermediary construct. We observed the ACARA team using CMF maps and Research Summaries in discussions with us, and they felt the information they found helped them to provide useful research justifications for their decisions to ACARA colleagues and teachers, who reviewed initial changes in reference groups, by illustrating (in a different form) some of the connections highlighted in the CMF. These justifications were positively received and the ACARA team felt they had been effective. The ACARA team also was able to highlight specific changes to the wording of curriculum statements which they made to bring in influences they found in the CMF and make them more explicit for teachers. See 3. for examples.

3. Research is one influence among many on outcomes. The ACARA team was able to clearly identify specific changes in their internal discussions due to use of the CMF, some of which led to curriculum changes. Two examples are provided below:

- In one set of statements about data, they changed language to explicitly reference the shape of distributions. Distribution and shape were major themes in the research they engaged with through the CMF, and they wanted teachers and students to be aware of the shape of distributions when interpreting data. Across Years 4 and 5, for example, the words describe and interpret have been expanded to acquiring, validating and representing types of data.
- They referred to research encountered in the CMF to develop a progression in statistical investigation through Years 3–10. This progression moves from guided statistical investigation involving categorical or discrete data to planning and carrying out statistical investigations, analysing data, communicating findings, developing questions, incorporating summary statistics, sampling, and working with multiple data sets and with bivariate data.

Conclusions

In our illustration, when using the CMF designers were able to engage productively with a larger number of research implications than they had before in a similar amount of time, and encountered ideas that led to specific adjustments in their approach, as well as ideas which validated many of the directions they had been going in the review. However, while they felt this enhanced their work, it did not save them time as there is always more research available than time to engage with it; the CMF simply created a higher limit to engagement. They were able to build and communicate research justifications including themes from the CMF, and they were able to directly attribute some of their design decisions to the CMF. In many cases the research synthesis aided the ACARA designers as the CM writers intended, but some cases in which this did not take place were identified for further work. These outcomes from our work in progress on evaluation suggest that the CMF might be a reasonable example of one approach to addressing the research implementation challenges we have described.

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The documentational approach to didactics as a gauge of fidelity of a teaching model implementation

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We examine the feasibility of using the Documentational Approach to Didactics to gauge the fidelity and character of teachers' implementation of scripted teaching sequences for the teaching of arithmetic in primary school. By analyzing a video-recorded lesson, we found that the teacher adapted the scripted teaching sequence by merging it with other resources such as self-made worksheets and an interactive whiteboard. The teacher also made adaptations to the prescribed organization of the teaching. Through the analysis, we gain insight into the nature of the adaptions and find that Documental Approach to Didactics can be used as a lens to distinguish between surface adaptions, that do not interfere with the program theory and adaptions that violate core ideas of the innovation. The results provide a background for a discussion of pro-fidelity versus pro-adaption point of view in implementation research.

Keywords: Documentational approach to didactics, fidelity, implementation, mathematics, education

Introduction

The extent to which an educational innovation implemented in classrooms reaches its desired outcomes depends on how teachers adapt the innovation in their practice. The extent to which teachers faithfully follow the instructions and intentions of an intervention can be discussed in terms of fidelity. The rationale for measuring fidelity and striving for high fidelity is to allow analytical separation of the quality of an educational innovation from the quality of the implementation of it, as well as to provide a reference for scale-up or replication research involving the intervention. Century and Cassata (2016) call the view that more fidelity is better the pro-fidelity perspective. They argue that to understand the nature of implementations of pedagogical innovations rather than just the effects of them, the pro-fidelity perspective ought to be complemented with a pro-adaptation perspective encompassing a research focus on "why adaptations are made, where implementation challenges come from, and how the nature of an adaptation matter for outcomes" (p. 200).

It has been noted by researchers that innovations that are general, non-specific, and involve little prescribed structure might need more adaptation (Cohen & Ball, 1990) and hence would benefit from a pro-adaptation research design (Century & Cassata, 2016). Highly structured innovations that are specifically prescribed could on the other hand benefit from a pro-fidelity research design (O'Donnell, 2008). The study presented here reports a case from a context where the innovation is a highly prescribed curriculum based on a specific teaching model, but as we will show, a pro-adaptation perspective still has merits.

The context of the study is a large-scale intervention, spanning 14 municipalities and over a thousand teachers, where the outcome on the student level is being evaluated with a control group, pre-post-test design. Yet, despite an intricate implementation design involving collegial collaboration to stimulate high fidelity, the project team was aware that adaptations would be made, and that fidelity would vary. To evaluate the intervention and gain worthwhile feedback, we were therefore in search of a theory base for researching teachers' adaptations of resources. This methodological paper reports on the use of the Documentational Approach to Didactics, DAD, (Trouche et al., 2020) for this purpose. DAD has previously been used to characterize teachers' use of and blending digital online resources. However, we see the potential to use DAD to characterize teachers' implementation of a highly prescriptive curriculum resource. Therefore, the present paper aims to examine the feasibility of using DAD to characterize the fidelity and adaptation of a highly prescriptive curriculum resource.

Theory

The theoretical framework DAD can be used for studying teachers' goal-oriented work through their interactions with resources used in their teaching (Trouche et al., 2020). An objective with DAD is to achieve a comprehensive understanding of teachers' work through elaborating what it can mean "to use" a resource. To understand the relevance of the term documentational, it helps to know the etymology of the word document. The word originates from Old French where it, in the late 15th century came to mean "teaching, instruction". The concept is also linked to the Latin "documentum" which refers to "lesson" and "to teach".

DAD builds on French didactics' theory and the tradition of the instrumental approach. The instrumental approach takes as a point of departure, normally, a digital artifact that is to be used by teachers or pupils. The fundamental observation is that when an artifact becomes incorporated in an individual's handling and thinking scheme, there are two processes involved: instrumentation and instrumentalization. The instrumentation process is when the affordances of the artifact shape the persons' practice. The instrumentalization process is when an individual uses her knowledge and dispositions to use the artifact for fulfilling her needs and goals. Combined, the instrumentation and instrumentalization processes interact to create the instrumental genesis, the transformation where the artifact becomes an instrument for the individual (Artigue & Trouche, 2021). The instrumental genesis, in turn, involves an interaction in which the user and the potentialities of the artifact (the physical tool) are reciprocally constituted in action (Artigue, 2002).

In the documentational approach, the role of the artifact is replaced by the role of one or several resources that a teacher might use or be influenced by in her teaching. The concept of resource encompasses any artifact with the potential to promote mediating teaching activities, including textbooks, curricular guidelines, digital resources, student worksheets, or work with a colleague. In DAD the teacher's role is to be a designer who, guided by a goal, transforms resources into documents in the process of interacting with a resource or a set of resources. This process is called documental genesis, in line with how instrumental genesis is described above (Trouche et al., 2020). This process ends with a document that is the product of a teacher's resource design work in each situation. The totality of a teacher's developed documents is called a document system.

A document is thus a combination of when a resource is appropriated and reshaped by a teacher, in a way that reflects their professional experience with the use of resources. *Document = Resources + schemes of usage*. A teacher's resource system is developed and formed by all the resources the teacher uses for a given activity (Trouche et al., 2020). The resource systems will be different for different teachers, even though they use the same resources since the resulting documents are dependent on the teacher's characters and prior knowledge. Thus, the process of documental genesis is ongoing since teachers' utilization schemes, how teachers adapt and are adapted by the resources they use change with experience (Vergnaud, 1998).

Method

We have compared the enacted actions of a teacher involved in a large-scale implementation program, with the scripted actions that carry the core ideas of the program, the program determinants. To set the stage for this study, we will elaborate on the implementation program by explicating elements of its program theory. When we use the concept *program theory*, we refer to Chen (2012), where we refer readers for further elaboration. Briefly, a program theory is an explication of a program's goals and how the program is supposed to achieve its effects. The change model of a program theory is the description of the "leverage or mechanism upon which [the program] can develop a treatment or intervention to meet a need. That leverage or mechanism is variously called the determinant or the intervening variable" (Chen, 2012, p. 18).

The teaching model builds on a randomized control trial evaluated design research project for grade 0 in Sweden (Sterner et al., 2020; Sterner & Helenius, 2015). The program has a comprehensive organization and involves municipality collaboration, teacher professional development, and local facilitator and leadership organization (Helenius, n.d.). In what follows, we will designate both the teaching model and the program for implementing it by its Swedish acronym TRR. In its current form, TRR is implemented in a program run in 14 municipalities in Sweden with over 1000 teachers. The TRR teaching is arranged in cycles composed of six phases, each comprising a particular pedagogical arrangement (see Sterner et al., 2020 for a description). Participating teachers complete 20 cycles that cover half of the mathematics teaching over a school year. A cycle normally covers three lessons and is communicated by a resource book, with detailed descriptions of how to carry out teaching. The detailed descriptions carry the program determinants and view as the desirable outcome in instruction. Of the large selection of possible determinants to distinguish in TRR, we present four of relevance in the sequence of teaching we will analyze.

Determinants of TRR resource book

Determinant 1 (D1). Building on the pupil's productions and reasoning. While the tasks, representations, and language to use are explicitly described in the TRR resource book, the main role of the teacher is to get the pupils to reason about their work and engage pupils in reasoning about each other's work. To aid the teacher, the material contains advice on how to sequence student presentations, depending on different approaches taken to the task and on particular types of questions or prompts to use when engaging pupils to talk about their work.

Determinant 2 (D2). The problem-solving strategy. In TRR, we endorse a particular strategy for approaching word problems: Formulate in your own words, use one of the endorsed representations for the situation type represented by the problem, write a mathematical expression, and find out the answer. A normative choice in TRR is that we prefer if pupils write the mathematical expression as close to the mathematical situation that is represented in the problem as possible. In this case, we prefer the situation: *The rabbit jumps five times farther than the bear* to be represented as rabbit's jump = 5 · bear jump, regardless of if the task continues: *The bear jumps 3 meters, how far does the rabbit jump?* or *The rabbit jumps 15 meters, how far does the bear jump?*

Determinant 3 (D3). The multiplicative model. In TRR we use three different models to build the multiplication concept on, none of which is repeated addition. One of them is rectangular arrays, initially built by cubes and later cut out from square paper. In the second teaching cycle on multiplication, the class constructs what is referred to as the big multiplication table which is a 10-by-10 matrix of rectangles, where the entry (i, j) displays the i times j rectangle with the product $i \cdot j$ written on it (Säfström et al., 2019).

Determinant 4 (D4). The chanting phase of the teaching cycle. The first phase in a teaching cycle is always chanting. The chanting signals that a mathematics lesson starts and connects to the mathematics to be treated in the teaching cycle. The chanting phase is not supposed to contain teacher explanations. But despite that, it might contain mathematical information not apparent to the pupils. In the cycle under scrutiny here, the chanting comprises variants of If 3 times something is 12 then something is 4. The resource book instructs the teacher to have the big multiplication table visible and pointing at the 3-row, while uttering three, moving the hand along the row and stopping at the rectangle with 12 on it when uttering twelve, and then moving the hand to the top column and stop at 4 when uttering four. The chanting is hence supposed to be accompanied by a gestural indication of how to solve an open multiplication problem of the type $a \cdot \_ = b$ by using the big multiplication table.

The data for the study presented here comes from a 49 minutes iPad video recording of one lesson with 8-year-old children (grade 2). The teacher has a formal teacher education for grades 1-7 and 24 years of teaching experience. At the time of the data collection, he had been involved in the TRR program for three years but had not taught the cycle discussed here before. The parts showing the teacher's use of resources were transcribed. Only the excerpts that are presented in this paper were translated into English. In collaboration, all three authors conducted a thematic analysis of the teacher's actions. With the theoretical lens of DAD, we describe the teacher's document system, that is his schemes of usage of the two resources, the TRR resource book and the whiteboard. Then we compare his document with the determinants D1-D4 from TRR's change model.

Results

Two resources, the TRR resource book and an interactive whiteboard with an attached document camera, are in play in our excerpts. D1-D4 are references to the TRRs determinants presented in the method.

As per TRR resource-book instructions, the chanting of the type: *If 1 times something is 2 then something is 2* is displayed on the whiteboard, which in the teacher's case is an interactive whiteboard. For each chanting, the teacher draws the corresponding rectangle next to the chant, e.g., *if 3 times something is 12, then something is 4*, the 3 by 4 rectangle with 12 printed on it is drawn. Next to the rectangle, the teacher has also written the open multiplication statement 3 $\cdot$ _ =12 on the interactive whiteboard.

The sequence starts 01:26 minutes into the lesson and is close to two minutes long. In line with the teaching model, the teacher leads a collective chanting phase (D4).

Teacher: We now start with the chanting, so read with me. If 1 times something is 2 then something is 2 (points at the chanting text). So, 1 times something is equal to 2 (points at the open statement) then it is 2 of course (points at the rectangle) /.../ If 7 times something is 42 then something is 6 (points at the chanting text). Here (points at the rectangle) we can see sixes, 7 pieces, 6 pieces of sevens, and 7 pieces of sixes, so ... hm, we stop here, and we move on to the presentations.

The teacher uses the whiteboard through instrumentalization, where using the whiteboard is a part of his utilization scheme. The use of the resource, the interactive whiteboard, forces instructions in the TRR resource book to be instrumented through the interactive whiteboard, leading to a transfer of the instructions. In this transfer, the chanting instructions themselves pass through unchanged. But, the inclusion of the arithmetic expression in the form of an open multiplication problem is an addition that does not conflict with the determinants. The omitting of the use of the large multiplication table (D3, D4), in favor of a single rectangle image, circumvents the idea of the 10-by-10 matrix as a support for pupils' reasoning. As a result, the gestural indication (D4) of how open multiplication problems can be solved using the big multiplication table (D3) will make the pupils less prepared for treating the word problem in the cycle, according to the preferred strategy (D2).

The second excerpt, starting at 03:10, displays a whole class discussion aiming to build on pupils' reasoning (D1). The teacher starts by asking a pair of pupils to come forward to discuss their work, from the lesson before. The pupils do not bring any work with them, because the teacher has beforehand created a slide for the whiteboard. He simultaneously displays the problem formulation at the top, and the pupils work, followed by three representations: the number line, a rectangular array, as well as the instructions from the TRR resource book (D2, D3). The teacher says:

Teacher: Can you explain what you have done? Tell us what you have done, and start with the number line. So, can you tell us a little about it?

Presenting pupil: We know that Pi has 30 cucumbers [the teachers encourage them with mmm] and if Pi has 5 then Nitty has ehh… Nitty had...I do not know...you…

The TRR resource book encourages open questions. Can you describe in your own words what you have done? But, the teacher's call to start with the number line deprives the pupils of the initiative. The pupils end up having difficulties telling what they have done, and how they reached their answers. The teacher moves on to leading questions, to guide the pupils:

Teacher: Hm, aha...help...if Pi has 5 (points at the number line), so how many did Pi have?

The pupils that have approached the whiteboard to describe how they worked with the problem (D1, D2) turn their attention to the whiteboard and say 5.

Teacher: Yes, and how do you know? It says here (pointing at the task formulation on the whiteboard)

The initiative has transferred from the pupils that should have been asked to describe their work in their own words, leading to a loss of fidelity to D1 and D2.

What follows next is that the teacher takes over the initiative and explanations more and more. The teacher uses the picture of the pupils' work on the interactive whiteboard, which enables him to draw and write on the picture when explaining. When it is time for writing mathematical expressions, the teacher leads the discussion, even more. At this stage, the teacher also involves the rest of the class (D1). The excerpt below is 10:43 minutes into the lesson

Teacher:	What kind of expression fits this one (points at a rectangle, $3 \cdot 6$). Now I want you in the audience to think as well. What fits that rectangle? What is the multiplication for this (points at the rectangular array)? You too are audience. Now you are an active audience (addressing a pupil) and so you have to think. Ehh
Teacher:	(points at the class) ... shshh ... (the teacher asks a pupil in the class for an answer)
Pupil in class:	3 times 6
Teacher:	Yes, why is that? (writes on the board $3 \cdot 6 =$)
Pupil in class:	It's 3 pieces of sixes
Teacher:	Because it is 3 pieces of sixes, sure. And how much is 3 pieces of sixes? (asks the presenting pupil)
Presenting pupil:	18
Teacher:	You remembered, great! Thank you for your presentation.

The teacher successfully led the pupils to reason about the properties of multiplication with the rectangles (D3). However, the fidelity to using pupils' productions and reasoning (D1) is low.

High fidelity can be traced in the teacher's document, the instrumentalization of the whiteboard and the choice to place the pupils' solutions projected on the board instead of having pupils posing their solutions on the static board. As a consequence, the teacher saves time, and the class can easily participate in the discussion since the written solutions are visible to everyone in the class (D1). Through the instrumentation of TRR, the teacher includes the rectangular array (D3) and open multiplication statements (D2) into pupils' worksheets and presentations.

Discussion

To examine a method for gauging fidelity, we have reported a single case study of a teacher with several years of experience in implementing TRR - a pedagogical innovation in the form of scripted teaching sequences. We examined the teaching by focusing on the concept of the document system from the theory Documental Approach to Didactics, DAD. The analysis showed adaptions of the highly prescriptive curriculum resource in the teacher's documents. Some adaptions retained fidelity to the determinates in the program theory's change model, while others led to less fidelity relative to the determinants. The two resources, the TRR resource book, and the interactive whiteboard shaped the document system. It is expected that when a teacher already has an established document including a particular resource (here: whiteboard) and a new resource (here: TRR resource book) is to be incorporated and joined with the previous resources, an adaptation takes place. This adaptation results in a new document including the use of both resources.

The teachers' utilization scheme involving the interactive whiteboard shaped the teaching. The teacher's instrumentalization of the interactive whiteboard on one hand strengthened the instrumentation of certain TRR determinants, for example enabling swift and easy presentation of pupil's work (D1). On the other hand, the usage of the whiteboard drives a tendency to reshape TRR instructions away from the instructions in the TRR resource book. One example is when the reference to the big multiplication table is replaced by the reference to individual rectangles that symbolize a particular multiplication. In principle, the transformation of the TRR chanting instructions (D4) to the whiteboard format could have been done as faithful instrumentation but ended up involving an adaptation of the TRR instruction, essentially breaking D4 as well as D2, by not presenting the big multiplication table (D3) as a resource in the problem-solving strategy (D2). The instrumentalization of the whiteboard can be said to affect the instrumentalization of the TRR resource book so that the resulting document system breaks some of the TRR determinants. The other example from our result section can be understood in similar ways.

A pure pro-fidelity stance could perhaps indicate a need to complement the printed TRR instructions with ready-made interactive whiteboard resources. Some teachers have asked us, developers, for such supporting resources. Despite our choice to present TRR in a highly prescriptive form, we are however worried that more specific resources could lure the teachers into a surface adaption of TRR, omitting the determinants. Instead, in line with Century and Cassata (2016), we think a pro-adaptation perspective is vital. The DAD framework provides a good basis for examining the nature of adaptations. In the presented case, we could for example conclude that some adaptions made were related to the mixing of resources. Since we know it is a relatively common practice to use interactive whiteboards, we can suspect that similar adaptions might be made by other teachers too. As developers, this is useful information since it might for example call for revisions of the teacher guides concerning how careful we describe the core features of certain figures or practices in the material.

A downside of the DAD-based analysis reported in this paper is that it is too extensive to carry out on a large scale. But even examining some distinct cases can inform us on why and how adaptations are made, as in this case. Through DAD analyses we can gain insight into the nature of the adaptions. We get a tool to distinguish between surface adaptions that do not interfere with the determinates in the change model from the program theory (Chen, 2012) and adaptions that violate core ideas of the innovation. The insights from the analysis give important feedback for future development of the professional development program that goes with the implementation, as well as the structured teaching instructions it builds on.

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From ‘technical rationality’ and ‘reflective rationality’ to ‘societal rationality’

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Keywords: Implementation, practice, research, policy, societal rationality.

‘Implementation’ as a challenge for policy, practice, and research

This contribution builds on Koichu, Aguilar, and Misfeldt (2021), who regard implementation as “an ecological disruption to a particular mathematics education system, through the gradual endorsement of innovation in conjunction with an action plan” (p. 986), and follows Willke (2005), who regards observation as noticing a “meaningful difference”, and intervention as “effecting” (generating, making) the “meaningful difference”, thus ‘implementing’ steps towards the desired situation.

Based on four examples, Krainer (2021) argues that the implementation of research, and its implementability, are dependent not only on researchers and on practitioners, but also on policymakers, too, in particular if scaling up is regarded as important. Writing a paper for a special issue on implementation-related research in mathematics education (see Koichu, Aguilar, & Misfeldt, 2021), reflecting on the long-lasting Austrian initiative IMST (which mainly aims at scaling up innovations in mathematics, computer science, science, and technology teaching), inspired the author to theoretical considerations on the interplay between policy, practice, and research in the IMST context. He describes two contrasting approaches related to implementation and implementability of research, namely ‘technical rationality’ and ‘reflective rationality’, and develops a third approach named ‘societal rationality’.

In the following, these three approaches are sketched for further discussion in the TWG.

‘Technical rationality’, ‘reflective rationality’, and ‘societal rationality’

‘Technical rationality’ was introduced by Schön (1983) and follows three basic assumptions:

- There are general solutions to practical problems.
- These solutions can be developed outside practical situations (in research or administrative centres).
- The solutions can be translated into practitioners’ actions by means of publications, training, administrative orders, etc.

In contrast to technical rationality, reflective rationality (e.g., Altrichter et al., 2008, p. 270), building on the notion of “reflective practitioner” (Schön, 1983), follows three very different assumptions:

- Complex practical problems require particular solutions.
- These solutions can be developed only inside the context in which the problem arises and in which the practitioner is a crucial and determining element.

- The solutions can only rarely be successfully applied to other contexts, but they can be made accessible to other practitioners as hypotheses to be tested in practice.

Comparing the strengths and weaknesses of these two approaches leads to the insight that the ‘meaningful difference’ (Willke, 2005) lies between the ‘general’ (main focus of ‘technical rationality’) and the ‘particular’ (main focus of ‘reflective rationality’). However, they should not be regarded as opposite, but as complementing each other. When implementations aim at spreading to a larger number of people, the perspectives of practice, research, and policy need to be included. All these stakeholders should be jointly co-responsible for a successful implementation. Therefore, a third approach, building on the strengths of the two mentioned approaches and on the societal dimension of this process, Krainer (2021) proposes ‘societal rationality’ as a third approach, following three alternative assumptions:

- Practical problems require an adequate link between general and particular solutions. The more complex the problem, the more important the particular.
- The solutions gain in quality if all concerned (including policy, research and practice) are involved in the problem definition and in the solution and evaluation process.
- The solutions can at best be partially applied to other contexts. Concrete examples, critical reflections, theoretical considerations, empirical findings, general guidelines, specific or general quality criteria can be used to adapt solutions context-sensitive.

Reflections on these approaches may lead to new questions and insights on teacher educators’ identity and professional growth (Krainer, Even, Park Rogers, & Berry, 2021).

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Comparing the implementation of programming and computational thinking in Denmark, Sweden and England

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Keywords: Programming, computational thinking, implementation, mathematical competencies

Background

The mathematical competencies that students need in the 21st century contain a sizeable digital element and include programming and computational thinking (PCT). Many countries are attempting to implement PCT in compulsory education (Bocconi et al., 2016), calling for a range of national and local decisions about the relation between mathematics and technology teaching. These decisions regard content, the group of teachers responsible, and the school subjects adopting PCT.

Different countries are investing massively in pursuing different paths relating PCT and digital mathematics. Still, there is currently no solid knowledge foundation on which decisions about these matters can rely. In some cases, PCT is considered part of mathematics. Others view it as part of an integrated science topic, a transdisciplinary element in all topics, or a subject in its own right.

The Danish government is experimenting with a two-tier strategy where PCT is both a subject or specialization and an approach integrated into subjects. In Sweden, programming has been a compulsory part of the mathematics curriculum from grades 1-9 to since August 2018. Here, PCT is strongly associated with algebraic thinking and the concept of algorithm. In the UK, programming has been part of the curriculum since 2013 as an individual topic, with an explicit dedication to algorithmic and computational thinking. Being a first-mover, the UK has solid experiences with the various challenges of implementing programming into compulsory school, including mathematics.

Approach

The so-called second wave of focus on PCT in compulsory education can be traced back to Wing (2006), who described computational thinking (CT) as decomposition, data representation and pattern recognition, abstractions, and algorithms. Building on Wing's definition, educational research has attempted to clarify and activate CT as a set of teachable competencies. This endeavour is often done by highlighting the relations to mathematical competencies such as abstraction, problem-solving, modelling and algorithm building.

To study the relation between mathematics and programming, we apply the Danish competencies framework (Niss & Højgaard, 2011) augmented with the notion of "Mathematical Digital Competencies (MDC)" (Geraniou & Jankvist, 2019). MDC involve "being aware of which digital tools to apply within different mathematical situations and contexts, and being aware of the different tools' capabilities and limitations" as well as "being able to use digital technology reflectively in problem-solving and when learning mathematics" (p. 43). It also entails "being able to engage in a techno-mathematical discourse" (p. 42), which we envision becomes crucial concerning PCT.

Besides giving theoretical meaning to PCT and MDC, we acknowledge the need to study implementation aspects, such as the lack of PCT-trained teachers and curricular compatibility. We relate to the work in implementation research in mathematics education (Koichu et al. 2021; Jankvist et al. 2021) to frame the inclusion of PCT as an educational innovation set to be put into practice.

Method and Plan

The project combines a comparative study of how PCT is integrated with mathematics teaching at the compulsory level in three countries with design experiments and the development of exemplars. To develop synergies between mathematics teaching and PCT in the Danish school system, we are in the process of comparing approaches and collecting resources from the three countries (mid-2020 to mid-2022). This work is described in Elicer and Tamborg (forthcoming). Based on these findings, we are planning design-based interventions to verify and refine our understanding of potential synergies between PCT and mathematics (early 2021 to early 2023). The refined teaching sequences and model of synergies will be developed as educational resources in the last part of the project (2023 and 2024).

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When researcher and teacher talk past each other: an IR analysis

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This paper discusses influential factors in an implementation of a longitudinal innovation based on results from research on beliefs, especially reflection, in mathematics education. As part of this innovation, the researcher constantly found herself to be "talking past" the involved expert teacher, who was responsible for implementing the innovation in the classroom. In particular, three influential factors appeared to play a central role in the case presented in the paper: characteristics of the particular end-user; attributes of the innovation; and not least implementation support strategies.

Keywords: Implementation research; influential factors; reflection; beliefs; probability.

Introduction

In a longitudinal study aiming to develop middle school students' beliefs about mathematics, the first author experienced that despite good intentions, long preparations, careful planning, and coaching of the involved teacher, the "reality" of what was implemented in class was far from the agreed-upon initially designed activities. We hypothesize that constructs from IR may shed light on this. Hence, our research question may be phrased as: *Can theoretical constructs from IR provide explanations as to why the researcher and teacher of our longitudinal study managed to talk past each other for a period of two years on the topic of 'reflections'? And if so, then in which respects, and to what extent?* With our answer, we hope to provide some illustration that IR may have something to offer the explanation of phenomena present in qualitative studies in mathematics education research. This paper thus addresses influential factors in relation to the implementation of designed teaching units to foster reflections with students concerning the nature of mathematics as a discipline.

IR theoretical constructs

When attempting, as in our longitudinal study, to change educational practices, the complexity of the setting fosters a variety of factors influencing the process, depending on both the characteristics of the innovation and the level of change (e.g. individual or organizational level). On an individual level—as in the case in this paper—a change in practice, e.g. for a teacher, can be both challenging and psychologically threatening, as well as cause doubts and uncertainties (Century & Cassata, 2016), even when the intended change might not seem difficult to its promoter. Other influential aspects might be the character of the innovation, environmental factors, support strategies, or time. Century and Cassata (2016) present a list of factors that might influence the implementation of an innovation. Here, we address and connect three of these in the analysis of the implementation in our selected case: *characteristics of end-users*; *attributes of innovation;* and *implementation of support strategies*.

The *characteristics of individual end-users* of an innovation (who in this case are the teachers, as we investigate the process of implementation of teaching principles) can potentially play a crucial role in an implementation process, not least in innovations that give room for the users' interpretations and adaptations. Prior knowledge, individual competency, professional identity, and feeling of agency are all examples of factors that might influence the implementation. In this regard, Rogers

(2003) mentions different types of knowledge connected to the implementation process, e.g. *how-to knowledge*, (practical knowledge about how to apply the innovation) and *principles-knowledge* (understanding the thoughts behind the innovation). Century and Cassata divide the characteristics of the end-users in two: (a) those related to the innovation, "e.g. level of understanding, expertise, prior experience, beliefs, values, attitudes, motivation, or self-efficacy" (p. 185); and (b) those existing independently of the innovation, "e.g. willingness to try new things, organizational skills, classroom management style, or views about teaching and learning in general" (p. 185).

Attributes of the innovation include both objective characteristics of the innovation (e.g. number of components or design features) and subjective user perceptions (e.g. relevance or ease of use). Some innovations are rather explicit, specifying the innovation in detail, which does not leave room for the users' adaptations. Others are more ambiguous and thereby more dependent on the interpretation and realization of the users.

Support strategies can be essential in an implementation process, and should according to Century and Cassata (2016) ideally be included in an innovation, based on underlying theories. Support to the users in their process of change can appear in various formats, e.g. professional development, strategic planning, or evaluative processes.

Setting the scene—the overall project and the central element of 'reflection'

Our two-year longitudinal study began in 2019 in two Danish 6th grade classes. The study used a Design-Based Research approach, involving two mathematics teachers in designing certain teaching principles, as well as in the implementation, evaluation, and adjustment of these principles in iterative cycles. The hypothesis behind the study was that a longitudinal change of focus in the teaching of mathematics can contribute to a change in the students' beliefs about mathematics—specifically that an increased focus on (1) the application of mathematics; (2) the historical development of mathematics; and (3) the nature of mathematics as a subject (Niss & Højgaard, 2019), can influence their beliefs about mathematics as a discipline. Hence, the main focus of the overall study is the development of students' beliefs, and not as such the teachers' professional development. However, as the present study addresses issues related to the implementation process within the overall study, the teachers thus become the focus here.

Philipp (2007) defines beliefs as "lenses through which one looks when interpreting the world" (p. 258). Because of the stability and psychological importance of beliefs, changing students' beliefs can be both difficult and time-consuming (Green, 1971). A central element in this process is *reflection*. Beliefs that are developed based on experiences or reason can be said to be *evidentially* held. In contrast, *non-evidentially* held beliefs are either transferred from others (e.g. teachers, parents, stereotypes, etc.) or derived from already existing beliefs, and these tend to be more difficult to change with reason. Non-evidentially held beliefs can often be illustrated as convictions of the sort that are impossible to argue against. Hence, when educating students, attention must be paid to providing examples and opportunities for experiences on which they can base and develop their beliefs to ensure that these are evidentially held. However, if such beliefs are to last, the provided evidence must necessarily be followed by reflection. Relations between beliefs are established when reflections are considered and assessed (Green, 1971), and these relations are what include and maintain beliefs in a cluster, hence making them more stable. "Reflection" thus played a central part in the study.

Even though the designed teaching principles were adjusted along the way, four main principles were consistent during the two years of intervention. That is, all teaching modules included: (1) concrete examples of the application and/or the historical development of mathematics; (2) mathematical problems and methods; (3) dialogue about the application, the development, and/or the nature of mathematics; and (4) *individual and/or shared reflection.* As Gregersen et al. (2019) point out, teachers' *principles-knowledge* can increase their experience of an innovation's relevance. The participating teachers were thus introduced to the importance of reflection in the initial phase of the study. The connection between changing beliefs and students' reflection was communicated in two ways: First, in a document stating the purpose of the intervention and the role of reflection when aiming to change or develop students' beliefs. The word "reflection" was highlighted and mentioned five times in the document to illustrate its centrality. Second, the document was discussed in a subsequent meeting between teachers and researcher, where the importance of reflection was further emphasized. In the following, we present an illustrative case of how one teacher attempts to realize the intended teaching principles, with a special focus on *reflection.* The case revolves around a lesson concerning probability that was planned in collaboration between researcher and teacher. We investigate three phases of the implementation process: planning, implementation, and evaluation. Thereby, we are able to compare the intention, the realization, and the teacher's considerations behind any deviations from the intention. The data include sound recordings of planning and debriefing sessions, as well as video recordings and field notes from the lesson. Furthermore, we include excerpts from a meeting six months prior to the lesson to illustrate the teacher's general considerations about students' reflections. In our analysis of the data, we apply the above-mentioned theoretical constructs from IR, thus seeking an explanation to the apparent miscommunication between researcher and teacher regarding the concept of reflection within the overall study.

An illustrative case of one teacher

Six months into the intervention, the teacher in our case explained how the teaching principle related to reflection had caused her to be "more systematic about individual and shared reflection, and what it can be used for". Although she had 20 years of teaching experience and was the school's mathematics counselor, she still felt that the principles, in general, made her more aware of her teaching choices. When the other participating teacher described her difficulties with implementing reflection, the teacher of this case even argued why reflection is important, and how she motivated her students to reflect:

> I have spent time in the class talking about short-term memory and remembering to bring it [the learned content] back to the working memory—you need to think about it and bring it back. And if you don't do that several times, the brain will toss it. (…) So there is a reason that we do it [reflect].

Hence, the teacher appeared to be conscious of the importance of reflection and its role in the project. However, she also gave a small hint of doubt, expressing that she was not sure that she was always able to transfer her intentions "all the way into the classroom". The following case, which confirms this doubt, took place six months later (a year into the intervention).

Planning: To ensure that the purpose of the intervention was met, the researcher participated in the planning of at least one lesson within every teaching sequence. In a sequence about probability, the topic of a 90-minute lesson was chosen to be the historical development of the field, exemplified in

Pascal and Fermat's approaches to solving the question of distributing stakes in an unfinished game of chance, as presented in a simplified version by Berlinghoff and Gouvea (2004). The problem concerns a game of flipping a coin for two players. Each player stakes €10 and tosses the coin in turn. If heads, the player tossing the coin gets a point; if tails, the other player receives a point. The winner is the first to reach three points. However, the game is interrupted, when the score is 2-1 in favor of the player about to toss the coin, and the distribution of the €20 stakes is to be decided. The planning of the lesson was based on Chapter 21 of Berlinghoff and Gouvea (2004, p. 207-214): "What's in a Game? The Start of Probability Theory". This chapter describes the story and the mathematical theory behind the problem as well as the methods used for solving it. Since both teachers had expressed some doubts about how they could implement the teaching principle concerning reflection, it was agreed that the students' considerations, suggestions, reflections, and discussions should be the focus of the lesson. Space for reflection would be given in the students' discussions of each other's solutions, and the relation to the solutions of Pascal and Fermat. Our teacher expressed her intention to "let the students consider a solution themselves", with the purpose of allowing them to "experience frustration and give their contributions". She also suggested that part of the students' reflections could regard the validity of the methods presented by Pascal and Fermat. She wanted to engage the students in the role of experts to act as mathematicians by showing them that some of their considerations and conjunctions could be compared to those of Pascal and Fermat. After an exchange of ideas, the lesson was planned to include five activities, which are listed below. Each of these activities was thoroughly discussed, both in regards to the content and the purpose in relation to the goals of the lesson. Several activities relied on the mentioned book chapter, and thus the teachers should make themselves familiar with its main points, especially regarding the mathematical methods involved.

Implementation: Below, the five activities of the 90-minute lesson are described in terms of the content of each activity, the intention decided by the teachers and the researcher in the planning process (purpose), and the actual realized implementation (reality).

<u>Activity 1:</u> Presentation of game, the distribution problem, and Pascal and Fermat. Told as a story.
<u>Purpose:</u> Engaging the students by telling a story, and inviting them to "play along". Introducing the historical persons involved, and inviting the students to consider a distribution of the stakes.
<u>Reality:</u> The teacher told a story about a rich, French, 17th-century nobleman, who liked to gamble. Concept of 'stakes' and rules of the game was explained. Pascal and Fermat were never mentioned.

<u>Activity 2:</u> In pairs, the students play the game and consider the distribution of stakes. Coins are available. The pairs present their suggestions. Subsequently, pairs with deviating solutions are put together in groups of four, who discuss their suggestions and try to agree on a shared solution.
<u>Purpose:</u> Becoming familiar with the game and the problem. Expressing immediate suggestions for a solution, which might involve discussion and argumentation. Considering possible scenarios of the outcome of the game, leading to mathematical considerations. Making the students reason and argue by pairing groups with different solutions.
<u>Reality:</u> Handed a wooden coin and eight pieces of paper (money bills working as stakes) on which they could write the number 10, the nine pairs of students played the game four times. After a while, the teacher stopped the games and asked the class if they believed this game to be fair. Several students complained that their coin always landed on the same side. However, the teacher neither engaged in a dialogue about the concept of fairness, nor what such a bias would mean in regards to

the game. Instead, the students were asked to play again and stop when one player had 2 points and the other 1—and then discuss how the stakes should be distributed if the game could not be completed. The students did, and the teacher circled between them asking guiding questions: “How is your distribution fair?”; “Would you both be satisfied with that solution?”. She noticed that most pairs either assigned all the stakes to the player with 2 points, or shared the stakes evenly, and she stopped the students. When asking each pair for their solution, this tendency was affirmed. Even though the teacher asked questions that might make the students reflect on probability (e.g. “Who has the largest chance of winning?”), they never engaged in any mathematical consideration. The teacher asked them to play again, this time stopping when the points were even. Expectedly, they found the distribution easier now that the probability of winning was equal for both players, which was pointed out by the teacher. She now encouraged the students to have a “serious discussion” about the distribution in the case of 2 points versus 1, making them aware of the possibility to exchange the €10 bills. The pairs were sent on a 2-minute “walk-and-talk”, which should result in an agreed solution to put on the whiteboard. Thereby, the original intention of putting pairs together with different solutions was never realized. Neither was the intention of having the students prepare a mathematical argument for their solution.

Activity 3: Presentation of solutions, the arguments behind them, and the strategies to reach them.
Purpose: Having the students explain their suggestion for a certain solution, and the method for reaching it. Comparing different solutions and different methods and arguments. Including arguments and strategies behind the solutions to emphasize that reasoning and methods are important aspects of mathematics, which are needed to make well-informed and accurate decisions.
Reality: The students put their solutions on the whiteboard. Out of nine pairs, four suggested that the €20 were distributed evenly, four suggested a 15-5 distribution, and one suggested that ⅔ of the money would go to the player with 2 points. The teacher merged the last suggestion with the 15-5 solution and asked the class to explain why this solution was fair. One student answered: “Because the player with the most points should have the most money.” The teacher asked for an argument behind the 10-10 solution, and another student answered: “Because they have equal chances of winning”. The teacher questioned this by drawing the students’ attention to the minimum number of tosses needed for each player to win. Another student exclaimed: “The chances are not equal, because one player has a 75% chance of winning, and the other has 25%”. Unfortunately, the teacher did not elaborate on this rather clever observation, although it appeared an excellent opportunity to engage in mathematical considerations. Neither the mathematical argumentation nor the strategies to reach a solution were discussed.

Activity 4: Shared classroom reflection: can we all agree on a solution? Which methods did we use? Were some solutions, methods, or arguments better than others? Which methods did Pascal and Fermat use? Did they resemble our methods?
Purpose: Comparing and discussing the solutions and arguments from the presentation. Offering the students an opportunity to reflect on their process and the mathematical ideas behind the different solutions as well as the validity of a mathematical argument. Placing the problem and their ideas in both a historical and a mathematical perspective by comparing the methods used by Pascal and Fermat. Illustrating the character of mathematical methods, and that the students too can engage in problems that mathematicians struggled with.

Reality: This activity was not realized in any way.

Activity 5: Follow-up on the historical development of probability theory.
Purpose: The work of Pascal and Fermat is considered one of the initiators of probability theory, soon followed by theories about e.g. the Law of Large Numbers, expected outcome, and analytical theory of probabilities. All of which are now applied in many fields such as medicine, insurance, business, law, etc. Making the students aware of this development and the importance of the field that sprung from the problem that they had just worked on, inserts mathematics in a context that exceeds school and illustrates its role in the world.
Reality: The teacher returned to the story of the French nobleman, who actually met this exact problem and asked "some mathematicians". Pascal and Fermat were still not mentioned by name. Their methods and solutions were only mentioned as follows:

> What they came up with was actually some of what you suggested. Their solution is 15-5. Because there is a difference in the players' chances of winning. They reached their solutions in a slightly different way as you reached yours differently. And they were great mathematicians (...). And you were also able to do this. And this was the beginning of the kind of mathematics that deals with probability.

Hence, the intentions of comparing the solutions and methods of both each other and of Pascal and Fermat were never realized, despite being the main goal of the lesson. The mathematical content was neither presented nor discussed, and the historical significance was only mentioned in bypassing. Most unfortunate was that the students were not offered the intended opportunities for reflection.

Evaluation: After the lesson, the teacher expressed that the class was not used to working in such an "unstructured manner", and neither was she (not specified any further, though). She also admitted that she did not thoroughly read the chapter on probability. This may be the reason for the lack of mathematical content in the lesson. However, after a suggestion from the researcher, the teacher presented a mathematical argument for the 15-5 solution in a lesson four days later, by studying the possible scenarios if the game had been continued. In the subsequent debriefing, the teacher described her criteria for success in the lesson: "that the students are able to say 'we have conducted an investigation, and it seems that…'". Even though all the pairs did reach a conclusion during the lesson, the teacher appeared somewhat unsatisfied. She appreciated having a story to start from but felt that it was difficult to get the students to use mathematical argumentation. Furthermore, she regretted not making the students aware of the possibility of exchanging the bills earlier: "It was not until I told them that the bills could be exchanged that they started thinking about it. If they had received ten €1 coins instead, I wonder what might have happened." She never addressed the skipped activities of reflection, nor the lack of the historical dimension. Her considerations—and the researcher's observations—formed the basis for a discussion on how the students could be supported in their argumentation so that it could be more mathematically founded.

Analysis and discussion of case in terms of IR

Characteristics of end-user: There appeared to be a discrepancy between the teacher's characteristics in relation to the innovation and those existing independently of the innovation. The teacher was an expert teacher, with a strong professional identity. She often advised her colleagues on mathematics teaching and learning. Concerning the innovation, she was highly motivated and perceived the innovation as relevant, both for her teaching and in her professional development. Her statements in the planning phase indicated that she was very aware of the innovation's intention and that she had a clear idea of how this intention was to be implemented. Our case, however, reveals that even though she possessed this *principles-knowledge* of the innovation, she might not have had adequate prior experience with the intended teaching approach, and thus her *how-to knowledge* was insufficient. Furthermore, her comments during the evaluation phase may be a sign that her identity as an expert teacher was threatened by the uncertainties and doubts that she experienced during the lesson. The fact that she did not mention the skipped activities of the lesson plan, which primarily involved reflection, seemed to indicate that she was not as aware of this purpose, contrary to what she expressed during the planning. Yet, it could also be a sign of denial of a feeling of failure or inadequacy.

Attributes of the innovation: As the innovation was based on a small number of somewhat general principles (cf. above), the level of explicitness was quite low. Hence, the implementation of the principles was highly dependent on the interpretation and realization of the teacher, requiring a high level of *how-to knowledge*. This meant that the final and determining decisions in the classroom were in the hands of the teacher, and thus became the realized innovation. Despite the shared planning of the presented lesson (supporting the teacher's *how-to knowledge*), and the regular discussions regarding the centrality of *reflection* for the development of beliefs (*principles-knowledge*), the teacher still decided to leave out the activities offering possibilities for the students to actually reflect. Furthermore, to promote the teacher's feeling of agency in the innovation, the allocation of contributions was that the researcher would primarily function as a theoretical expert and the teacher as an expert on practice. Consequently, the teacher was responsible for the detailed planning and preparation of lessons. On the one hand, this enabled her to adapt the teaching to her individual approach and to the students. On the other hand, the researcher had even less control of the actual implementation of the innovation, and the risk of non-intended realization increased (significantly).

Implementation support strategies: As suggested by Century and Cassata (2016), several theoretically based formats of support were included to assist the teacher in the implementation process. For example, the 'whys and hows' related to the concept of reflection were thoroughly discussed to enhance the teacher's *principles-knowledge*—a strategy that seemed to benefit their experience of relevance, as seen in Gregersen et al. (2019). These support strategies were further developed during the study. Central to the cooperation between researcher and teacher was the shared planning and evaluation sessions. In our case, the planning not only included discussions and clarification of focus and main purpose of the lesson (*principles-knowledge*), but also a description of the lesson's activities and their individual purpose (*how-to knowledge*). This kind of detailed planning had not previously been conducted in cooperation with the researcher, but an increased awareness of the challenges connected to the implementation had led to this initiative from the researcher, which was welcomed by the teacher. Likewise, the described case became the cause of further adjustments of the support strategy, eventually including shared preparation and considerations of potential student responses and appropriate teacher reactions.

When these three influential factors are compared and connected, interesting issues related to the implementation process are revealed. Firstly, the attributes of the innovation define the teacher as an expert on practice, thus making her responsible for the realization of the intention. However, the characteristics of this specific teacher make this realization unpredictable. In addition, the support strategies intended to account for this problem are complicated by the allocation of expertise between the teacher and the researcher. A possible dilemma occurs when deviations from the intention that are observed in the teaching are to be addressed in the evaluation. It is, on the one hand, essential to the success of the innovation that the intention be realized. On the other hand, the communication between researcher and teacher must remain respectful of their respective areas of expertise while at the same time supporting and benefiting future cooperation and innovation. Addressing problematic issues related to practice thus becomes a difficult act of balance. In this case, the evaluation session led to adjustments in the support strategy that increased the explicitness of the innovation, thereby changing its attributes and affecting the level of the users' autonomy. Studying the influential factors of this case clearly illustrates that the attributes and overall goal of this innovation may to some extent be incompatible. The goal of developing the students' beliefs through teaching principles may demand a *change* that is too ambitious in terms of *the culture of practice*—the influence of which was at first hand underestimated by the researcher. For example, this case shows how the implementation of opportunities for reflection is hindered by a gap between the intentions of the researcher, the apparent intentions of the teacher, and what is practically possible within the context of the culture of practice and the teacher's *how-to knowledge*. Returning to the research question, it is clear that IR constructs do have something to offer our qualitative case study in terms of explanatory power to the lack of mutual understanding between researcher and teacher. The IR analysis made it clear that when the innovation is not specific or explicit enough, then the implementation of it is proportionally dependent on the characteristics of the end-users. This should be dealt with both in the innovation design and in the support strategy.

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Developing an instrument to connect problem-posing strategies and mathematical habits of mind

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Successful problem posing is heavily based on the ability to see and use the structure of the given problem. Therefore, the students must develop the mathematical habit of mind of 'seeking and using structure'. Up to now, these two areas of problem posing and mathematical habits of mind are examined separately in the relevant research literature. In this paper, we present an instrument that attempts to bridge these two areas. The instrument monitors the student's problem-posing strategies and assigns a numerical weight in the solver's choices according to how powerful they can be considered from the mathematical point of view. Finally, some suggestions for the instrument's implementability in the research, mathematics teachers' education, and classroom settings are made.

Keywords: Problem posing, habits of mind, structure

Introduction

Problem posing is gradually gaining more attention in both the research community and in school daily practice in mathematics. Since one can find more than one definition in problem posing (Papadopoulos et al., in press) we choose in this paper to align with Silver's (1994, pp. 19) definition who sees problem posing as "both the generation of new problems and the re-formulation of given problems". Cai et al. (2015, pp. 4) mentioned that "despite the interest in integrating mathematical problem posing into classroom practice, our knowledge remains relatively limited about the cognitive processes involved when solvers generate their own problems, the instructional strategies that can effectively promote productive problem posing, and the effectiveness of engaging students in problem-posing activities". So, to go deeper into how students generate their own problems, we shift our attention to the mathematical habit of mind (HoM) called *Seeking and Using Structure* (Goldenberg et al., 2015) since the essence of the successful problem posing is the ability to recognize and use the structure of the given problem. Despite Silver's (2013) acknowledgment that it is worth measuring the development of certain mathematical dispositions or habits of mind, these two areas of mathematics (e.g., problem posing and habits of mind) have not been connected yet. To meet this challenge, we attempt (as a part of a broader study) to develop an instrument to monitor the students' development of the habit of mind called *seeking and using the structure* through their effort to pose new problems. We aim to show its future implementability in the research, teachers' education, and classroom settings.

Literature review

The theoretical underpinning of the design of this instrument lies in the realms of both the notions of *structure* and *mathematical habits of mind*. Understanding the structure of mathematical problems is considered critical in problem solving and posing. According to Mamona-Downs & Downs (2005) "structure can be thought of as a unity […] but it also offers access to its 'parts'[…]. Hence, a structure has simultaneously a global aspect and an analytic one" (p. 390). Schoenfeld & Herrmann (1982) claim that problems can be perceived by the solvers on the basis of *surface structure* which refers to the items described in the problems themselves or of *deep structure* which refers to the mathematical principles necessary for a solution. A correct perception might lead to a straightforward solution whereas an incorrect one may send the solver on a "wild goose chase". Moreover, it is supposed that the solver does not have easy access to a procedure for solving the problem but does have an adequate background to make progress on it.

Problem posing activities are cognitively demanding tasks (Cai & Hwang, 2002) requiring students to think differently than in problem-solving to improve their understanding by reflecting on the deeper structure and goal of the task. In relation to the structure of the problem-posing situations, Stoyanova and Ellerton (1996) describe three problem situations: free, semi-structured, and structured. Moreover, Ellerton (2013) in her Active Learning Framework (ALF) aiming to incorporate problem posing in primary school mathematics, claims that students are actively involved in posing problems with similar structure to given ones but in a different context. Kwek (2015) tried to identify patterns in students' mathematical learning and thinking during classroom-based problem-posing tasks. The findings gave evidence that the students' ability to identify the mathematical structure of the problem was a vital cognitive factor.

In our work, we define mathematical habits of mind as considering mathematical problems in very specific ways that look like the ways mathematicians use. It is about a way of thinking -almost a way of seeing a particular situation- that comes so readily to mind that one does not have to rummage in the mental toolbox to find it (Goldenberg et al., 2015). One habit particularly important for problem posing is the habit of *seeking and using structure*. In the context of solving and generating problems in early mathematics, the notion of mathematical structure concerns relationships between quantities, group properties of operations (e.g., associative and/or commutative operation), relationships between the operations (e.g., distributive property of one operation over another), relationships across the quantities (e.g., transitivity of equality) (Warren, 2005). Therefore, the development of the habit of *seeking and using structure* helps the students to see the logic and coherence in every new situation they encounter. Both, the identification of the information in a given problem that would be suitable to generate a new problem and posing questions that fit a particular situation are strong signs that students can see and use the problem's structure. The *seeking and using structure* habit has been recently investigated in other settings. Papadopoulos (2019) examined the potential contribution of puzzle-like activities in exhibiting this habit in the context of early algebraic thinking in primary school students.

Given that this habit is considered vital for successful problem posing and is developed through an accumulated experience with problem-posing activities we attempted to develop an instrument that would capture the relevant students' ability.

An implementable research instrument

According to our knowledge, the only instrument related to habits of mind is the one developed by Matsuura and his colleagues (2011) to measure secondary teachers' mathematical habits of mind in problem solving. It contains items rooted in familiar secondary mathematics and is for use for research purposes only. Inspired by this tool, we tried to think and design an instrument related to problem posing that would be based on the different problem-posing strategies used by the students. The application of such strategies is considered an indication of *seeking and using the structure* of a given problem since problem-posing strategies are heavily based on the use of the structure of the given problem. The whole effort of designing this instrument followed a series of certain steps.

Step A – Collecting strategies from the existing literature

The first step was to follow the literature to collect the suggested problem-posing strategies. This step resulted in the following collection:

Reversing known and unknown information: This strategy (also known as the 'symmetry' strategy) is one of the most common and frequently met strategies in mathematics textbooks. The answer to the original problem would now be part of the given data and some of the given information would be the unknown (Grundmeier, 2015).

Change the context: The structure of the original problem is kept but the original context is changed. Ellerton (2013) said that the process of designing a different context for a problem that has a similar structure to the given problem proved to be challenging for most students.

Change numbers: The students change the numerical data of the original problem to create a new one keeping the structure unaltered. Depending on the nature of the change, the new problem could be either similar or interestingly different from the original one.

Change the question: The data of the original problem remain the same, but the question is removed and replaced with a different question that could fit the given data (also known as the 'goal manipulation' strategy).

The answer is a method: The original question remains the same, but the numerical data are left off. The potential solvers will have to find how they'd solve the problem if the numbers were known, which means to find the method to solve the problem (Goldenberg et al., 2015).

"Frontless" problems: Only the question is retained. In that way, these questions are in many senses as close to real-life as we can get. So, the question comes first, and then we have to figure out the information and the method we need (Goldenberg et al., 2015).

Missing middle problems: The original question is kept but some numbers are left off and the solvers are asked to find what additional information would be required to answer the question (Goldenberg et al., 2015).

"Tailless" problems: In this case, the question at the tail end is omitted. The goal is for the students to derive what they can from the given information. It looks similar to the *Change the question* strategy. However, in the former, the question of the initial problem has already been substituted and the aim for the student is to solve the new problem. In the latter, the problem is left open and the aim for the solver is to think: what information is here and what can I do with it? (Goldenberg et al., 2015).

What-if-not: This is the most known strategy in problem posing (Brown & Walters, 1983). According to this, we form a list with all the problem's attributes and then we start negating each one of them using the what-if-not questions which means to ask what would happen if these attributes were different. Each negation results in a new problem. This strategy allows students to discuss a wide range of ideas and dive into the structure of the problem.

What-if-yes: A variation of the "What-if-not" strategy is put forward by Leikin and Grossman (2013) and Grundmeier (2015) based on adding attributes and/or properties to the given problem instead of removing or negating them.

Step B – Forming the instrument

In the second step, we contacted experts in the area to evaluate these strategies in relation to how powerful from the mathematical point of view can be considered. There were three options for each strategy (A, B, and C, from the most powerful to the less one). Eight experts from the USA (2), Germany (2), Hungary (2), France (1), and Greece (1) participated, and their answers were combined to form the final categorization in Table 1. Each problem posed by students can now be labeled as A, B, and C according to the strategy used. In case no strategy evidence is present at the students' answer the proposed problems is labeled in category D.

Measuring the *seeking and using structure* HoM	**Category**
The answer is a method	A
What-if-not	
What-if-yes	
Change the context	
"Frontless" problems	B
Missing middle problems	
"Tailless" word problems	
Change the question	
Change numbers	
Reversing known and unknown information	C
Change numbers	
No evidence	D

Table 1: Problem posing strategies

The *Change numbers* strategy is included in two groups according to whether this change results in a similar problem or another open or more mathematically interesting problem.

Step C – Instrument's exemplification

The third step of the process was to exemplify the instrument using a specific given problem. The following problem will be used to generate new problems using the above-mentioned strategies: *Every day I save 50 c. to buy a book which costs 8.50 €. How many days would it take to save that amount of money?*

Below we present how this problem could be reformulated or a new problem could be generated using the above-mentioned strategies.

Reversing known and unknown information: The answer to the initial problem is 17 days ($8.50 \div 0.5 = 17$). This information will be part of the given data so that the given and the goal will be swapped. The new problem is: *For the next 17 days I will save 50 c. per day to buy a book. How much does the book cost?* (Type C).

Change numbers: Depending on the way this strategy is applied it might be possible to have different problems of different weight. The new problem could be a simple one, in which the student simply mimics the original one by changing numbers (example a). But there are cases (example b), in which the new problem still resembles the initial one, but the new numbers may lead to interesting situations.

a. *Every day I save 20 c. to buy a book which costs 6.20€. How many days would it take to save that amount of money?* (Type C)
b. *Every day I save 1€ to buy a book which costs 8.5€. How many days would it take to save that amount of money?* (Type B)

This second example leads to an interesting situation. How long does it take me? 8.5 days? Or 9 days?

Change the question: The challenge here is to find another question that fits the already given data. Example: *Every day I save 50c. to buy a book that cost 8.5€. What is the smallest number of coins I need to buy the book?* (Type B)

"Tailless" word problems: The question at the tail end is omitted and the problem is open for statements or questions. For example: *Every day I save 50 c. to buy a book which costs 8.50€. What can you figure out from this information?* The absence of the question also helps kids notice that more than one sensible question could be asked. For example: "Will I have enough money in two weeks?" (Type B)

Missing middle problems: The original question will be the same, but some numbers will be left off. The missing number is part of the required information that is not given. This number is not the answer to the question the problem poses but it is necessary to solve the problem. Example: *Every day I save some money to buy a book which costs 8.50€. How many days would it take to save the amount of money I need?* (Type B)

"Frontless" problems: Only the question is retained. Clearly, the question can't be answered without information, and we must then figure out what information we need and what method we must use to answer. An example: *There's a book I want to buy. How many days will it take me to save up the money I need? What information do I need to know in order to figure this out?* (Type B)

Change the context: This strategy is indicative of a deep understanding of the problem's structure. The core idea remains the same, but it is transferred in another setting (e.g., from coins and their relations to numbers and halves): *How many halves do I need to get 8.5?* (Type A)

The answer is a method: The required answer is a description of how the problem would be solved if there weren't numbers. Example: *Every day I save the same amount of money to buy a book. How can I figure out how many days it will take me to get the amount of money I need?* (Type)

What-if-not: One attribute of the given problem is that a certain amount of money per day is saved. So, **what if** that amount was **not** saved per day but, for example, every three days? The new problem becomes: *Every three days I save 50 c. to buy a book which costs 8.50€. How many days would it take to save that amount of money?* (Type A)

What-if-yes: An additional information is added which has an impact on the solution process. For example: *Every day I save 50c and I share this amount fairly with my brother. How many days would it take me to buy a book which costs 8.50€?* (Type A)

Step D – Use in a real case study

The final step for the instrument is to use it in a real case study. There is a pilot study running this year. The aim is to gradually familiarize the students with this range of strategies in problem posing to see later whether they can use some or all of them in different problem-posing situations. The sessions start with problem-solving activities and then the students are asked to generate new ones inspired by the solved problem. To be successful they need to see the structure of the problem and apply those of the problem-posing strategies that fit the given data. Their problems will be collected and categorized according to the strategy they used. A score will be assigned to each student according to the produced problems and the type of strategies used. This score assigned does not determine the extent to which the habit of *seeking and using structure* has been developed by the specific student. It rather serves as a way to monitor the efforts done by the student to notice whether advanced problem-posing strategies are used over time.

Concluding remarks

This paper aimed to present an instrument we developed to record and examine whether students develop over time and through an accumulated experience on problem posing the habit of mind named *seeking and using structure*. The instrument works on the basis of the problem-posing strategies the students use. The more they use advanced problem-posing strategies (e.g., types A and B), the more they seem to have developed the *seeking and using structure* HoM. We see the potential implementation of this instrument in the sense of 'if and how the innovation achieves desired outcomes for the target population' (Koichu et al., 2021, p. 978). This problem-posing intervention is the innovation that constitutes the object of implementation (Century & Cassata, 2016) aiming to develop structure-sense to students. The instrument will help to determine whether the expected results are produced in the target population. So, from this point of view, researchers' understanding of the issue of problem-posing strategies that effectively promote students' structure-sense which finally result in productive problem posing (Cai et al., 2015) will be deepened. Further insights into the relationship between problem posing and the habit of seeking and using the structure of the given problems will be gained. The instrument can also be useful in the context of educating future mathematics teachers. They need to be familiar with problem-posing strategies, recognize the structure of a problem, and use it to formulate new problems. This starts from the fact that future mathematics teachers very often-and in relation to problem posing- emphasize the result instead of the process (Klinshtern et al., 2015). However, it is the latter that is closely related to the notion of structure. So, it is important to shift their attention from the result to the process. The instrument will be useful to monitor the potential development of the *seeking and using the structure* HoM. Finally, the instrument might also be useful for in-service teachers. Following the whole classroom or each one of their students across the school year using this instrument, they will have the chance to identify any progress in the development of the habit of *seeking and using structure*. The variation in scores over time will facilitate the understanding of the students' crucial ability to see and use the structure of a given problem situation.

What is the most important however is that this instrument is bridging two areas, problem posing and mathematical habits of mind, that are still unconnected. Our long-term aim is to use this instrument to follow a primary school classroom in a year-long intervention in the context of a problem-posing intervention.

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What is a successful implementation in mathematics education? On sustainable innovations and the role of textbooks

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This paper concerns what a successful implementation of an innovation in mathematics education can be and how that can be achieved. Focus is on sustainability of an innovation and the role of textbooks. We use two historical Swedish development projects in mathematics education for the discussion. The material is official reports and governmental documents concerning the projects.

Keywords: Implementation, sustainability of an innovation, textbooks.

Introduction

This paper is part of the project *Implementation research as an emerging field of mathematics education*. The project's aim is to create a theoretical framework for implementation research (IR) in the field of mathematics education research (MER), in particular for research on large-scale development projects. The overall project examines which existing IR theories and which parts of these are applicable to implementation and development projects related to mathematics. To verify this, we test the theories and related concepts that we find relevant through comparisons of five development projects.

This paper concerns two of those projects—New Math and PUMP[1]—and the aim is to understand how the concept of *sustainability*, a key IR concept, is applicable in MER. This concept is essential for how to conceive what a successful implementation of an innovation is. The analysis is focused on textbooks, since that is a characteristic of school subjects. Textbooks also involve a type of stakeholders—publishing companies— that we do not find in many other subfields of IR. Our research question concerning the New Math and PUMP projects is: *What was the role of textbooks in the implementation process, and how were textbooks related to efforts of sustaining or maintaining an innovation?*

Previous research

In their overview of IR on large-scale innovation, Century & Cassata (2016) identify a number of factors that influence whether an implementation of an educational innovation is a success or a failure. In what respects an implementation is a success can be understood in different ways. One way is to consider outcomes in terms of student results or changed behaviours of teachers or students. However, there are two other aspects of success, which the outcome perspective is

[1] PUMP = *Processanalyser av Undervisning i Matematik/Psykolingvistik* (Process analyzes of Teaching in Mathematics/Psycholinguistics)

depending on. One aspect is *fidelity* to the innovation or the reform program. That is, to succeed in getting a great number of teachers to use the innovation as planned. The rationale is that teachers need to apply the innovation as planned, if it is to make sense to talk about positive or negative effects on outcomes. The second aspect is to succeed in getting the teachers to apply the innovation for a long time, or at best, forever. In that perspective a successful innovation endures. The rationale here is that positive effects are pointless if the innovation does not endure.

These two other aspects or perspectives on success can to some degree be conflicting. One way to obtain endurance is to have innovations that allow for adaptions to local and changing circumstances over time. Such innovations are then considered sustainable. However, adaption is in conflict with fidelity. In a fidelity perspective, endurance is then a matter of maintenance in order to preserve the innovation. The weakness of that perspective is that the context of an innovation can and do change, which means that adaptions may be necessary.

This paper concerns factors and conditions that may contribute to innovations in mathematics education (ME) becoming sustainable or maintainable for longer periods of time.

The overview of Century & Cassata (2016) is efficient as it guides us to essential issues in IR, for instance the ideas concerning sustainability. However, it is not possible to discern if there are certain factors or conditions that are more or less relevant to achieve sustainable or maintainable innovations in different areas. Neither is it clear in what respect textbooks and publishing companies constitute factors or conditions that facilitate or inhibit successful implementation of an innovation. A similar problem we find also in the publications Century & Cassata (2016) refer to. Regardless of field—for instance mathematics education (Clements el al., 2015), kindergarten (Lieber et al., 2009), positive behaviour support (McIntosh et al., 2013), sex education (Rijsdijk et al., 2014), and science education (Century & Levy, 2002)—the researchers apply concepts and theories from IR as if they were applicable in all fields. Some of them do address the role of textbooks or other teaching materials to obtain sustainability (Century & Levy, 2002; Lieber et al., 2009; Rijsdijk et al. 2014), but others do not.

In some cases, such as positive behaviour support (McIntosh et al., 2013), it seems natural not to include textbooks and other teaching materials as a factor for obtaining sustainability; the teachers in the McIntosh et al.'s (2013) study were supposed to follow a certain program for positive behaviour support and it did not concern the teaching of school subjects. In the other extreme, we find mathematics education and science education. These are contexts where textbooks have existed for a very long time and should not be considered non-essential parts of the teaching practices. However, only Century & Levy (2002) address the role of textbooks, in science education, not Clements el al. (2015) in their paper on mathematics education.

Our contribution to previous research is about deepening the understanding of how textbooks can be managed in different ways in development projects to sustain or maintain innovations. In particular, we are interested in how project managers or reformers tackled the publishing companies. These companies must be considered stakeholders in educational reform processes as their existence rests on teachers and schools buying their products. Moreover, we should not assume that the purpose of an innovation coincides with the interests of publishing companies.

Theory and method

Our analysis of strategies for obtaining sustainable or maintainable innovations is based on Coburn's (2003) theory of scaling up an educational reform, which in our view also includes implementation of innovations. According to Coburn (2003), scale comprises four interrelated dimensions: depth, sustainability, spread, and reform ownership.

Depth concerns in what respect teachers change beliefs, norms of social interaction, and pedagogical principles. This is in contrast to so-called superficial changes such as changes in materials, classroom organization, or the addition of specific activities.

Sustainability concerns time and schools' ability to make innovative changes to remain in the teaching practice. This often means allocating tools and resources (e.g., financial, staff, and administration) for that end. In our analysis, we also use the concept *maintenance* in order to capture different ways to make an innovation endure, see previous section.

Spread is what traditionally is associated with scaling up. The implementation of an innovation is scaled up when an increasing number of classrooms and schools get involved.

Reform *ownership* is a matter of external reformer handing over control to districts, schools, and teachers. Or more precisely "creating conditions to shift authority and knowledge of the reform from external actors to teachers, schools, and districts".

By our study, we want to supplement Coburn's (2003) theory by relating textbooks and other teaching materials to the four dimensions just mentioned. We argue, from a theoretical point of view, that textbook and teaching material can be involved in the four dimensions in a substantial way. As to *depth*, textbooks and teaching material are designed according to some pedagogical principle; explanations and exercises are not developed and organised at random. Thus, using a textbook is then a matter of applying that principle. Innovations brought by textbooks are then sustained or maintained by publishing companies, not just school authorities. As to *spread*, that is the raison d'être for commercial publishing companies. Nobody has to remind them of that. And a textbook can give more or less *ownership* to teachers.

An important point is that school authorities, which often initiate and drive reforms, and publishing companies, can have different interests. And those interests may be in conflict with each other. Our assumption is that if you try to scale up the implementation of an innovation and make it sustainable, the chance of success is affected by the extent to which you work with or against publishing companies. And if you are not working with them, it can be a good idea to have a strategy of managing potential conflicts and fending of companies.

In our analysis, two development projects are compared. For each project we identify a strategy for managing textbooks and publishing companies and each strategy is tied to an aim of sustaining or maintaining innovations. The analysis includes Coburn's (2003) other three dimensions of depth, spread, and ownership. The materials are official reports and governmental documents concerning the development projects and to minor extent communication with people involved in the projects. The material has been treated as narrative sources concerning what intentions people had and what happened during the development and implementation of innovations. None of the sources contain an explicit strategy for managing textbooks and potential conflicts with publishing companies. But some sources do concern textbooks, quite a lot and very explicitly in fact, and we have then sought to identify how the treatment of textbooks gave the reformers an advantage over publishing companies. As to the New Math project, we rely on findings presented in already published studies. But the studies are also based on official reports and governmental documents.

New Math project (1960–1975) and the role of textbooks

If we consider the plans for the New Math reform in Sweden, it contained a lot of innovations that altogether were deep. By far, it was not a matter of adding sections of set theory in some school years. On the contrary, it was a matter of providing new principles for structuring and teaching all school courses in mathematics (1–12). Set theory was supposed to constitute a foundation on which the other school topics should rest. In teaching, concepts should be introduced and explained by means of concepts, expressions, and illustrations related to set theory. In this way, coherence between all topics should be created. This would also facilitate a teaching focused on understanding rather than just procedures, which was in line with the theory of cognition, learning and mathematics that guided the reform. But, apart from set theory, New Math also brought other innovative concepts, for instance from vector geometry, trigonometry, and functions (Prytz, 2018).

The spread was supposed to be total in the sense that the innovations concerned all school years 1 through 12 and all Swedish schools. The way to achieve this was to implement the innovations in connection with the national curriculum reform of 1969, which then brought a radically new course program for mathematics.

To maintain the innovations, textbooks were an essential component. Much of the development phase (1961–1968) concerned textbook development, which was financed and driven by central school authorities. The overarching aim was to develop textbooks that could fit the radically new curriculum. The idea seems to have been to provide the publishing companies with an extensive example of what a new type of textbook should look like. In practice, many publishing companies managed the conversion by hiring people involved in the New Math project. However, compliance with the new curriculum was secured by a mandatory textbook review. And if we consider the content of the textbooks published in connection with the curriculum reform of 1969, the compliance with the new course program in mathematics was indeed good. Since the textbooks review was a matter of controlling the fidelity with the innovations, we find it relevant to talk about maintaining rather sustaining innovations when it comes to New Math (Prytz, 2018).

As to transfer of ownership, teachers were given in-service training concerning the New Math, which is a clear example of providing teachers knowledge of the innovation. However, the focus was on mathematical content rather than teaching methods. And in comparison, e.g. to the Boost for Mathematics project, it was brief. And do not forget the national textbook review that limited the possibilities to deviate from the curriculum. Thus, the central school authorities will and ability to transfer ownership do not appear to have been great (Prytz, 2018).

Regarding textbooks and the strategy to maintain the innovations, the New Math project's attitude versus publishing companies was of a brutal kind. In the devolvement phase during the 1960s, all people involved authoring the experimental textbooks, except one, had a background as textbook author for school years 1 to 12. So, there were no ties to publishing companies in that respect. The trials and testing were done in a scientific context with researchers in charge and almost without involvement of publishing companies. Thus, the companies had little influence on the process that led to the example they had to follow. On top of that, there was the textbook review (Prytz, 2018).

If we then consider the demise of New Math in Sweden, this brutish attitude towards the publishing companies appears to have been functional. A few years after the 1969 curriculum reform, the central school authorities decided not to drive important components of New Math, not least the parts concerning teaching principles. However, the mathematical course program did not change. And in 1974, the textbook review became optional. After that, publishing companies began to issue more traditional textbooks, in parallel with New Math textbooks (Prytz, 2018).

This shows how willing and able the publishing companies were to act quickly and produce textbooks that were in conflict with the innovative components of New Math. Most likely they aimed at making a profit of teachers' dissatisfaction with New Math. To what extent teachers were dissatisfied is hard to estimate, but critique was aired well before the reform. This indicates how important the mandatory textbook review was to prevent publishing companies from interfering with the original plans for the implementation.

PUMP project (1970–1985) and the role of textbooks

In comparison to the innovations of the New Math project, the PUMP innovations can appear to have had little depth. They concerned just arithmetic in school years 1 through 6 and the central innovative component was an assessment material. Moreover, no particular pedagogical principle for teaching was prescribed. On the other hand, the assessment material and its underpinnings were very carefully crafted, tried, and tested. And they prescribed a detailed sequence in which exercises in arithmetic should appear in teaching. There were also good arguments for this sequence since the material had been tried and tested empirically. In addition, there was a cognitive theory about working memory to further support the sequencing of the content (Kilborn, 1979). Our point here is that this type of sequencing concerns the very basics of teaching. How and what a teacher communicate with the students is depending on how the content, for instance exercises, are sequenced (cf. Bernstein, 1974). So, in that respect the PUMP material had depth, but it was another type of depth than in the New Math material. In brief one could say that the New Math gave principles and examples for the sequencing of the content and principles for teaching; PUMP gave the sequence for the content.

The spread of the PUMP material was very modest in comparison to the New Math reform. There was no policy demanding all teachers to use the material (Kilborn, 1979). But the fact that a developed version of the PUMP material is still in use today indicates that the material had spread.

This endurance suggests that the efforts to sustain the material were successful. Different aspects of how that was done will be studied in our project. In this paper, we consider more closely how the PUMP people could avoid threats to their sequencing of the content from publishing companies.

Here it is important to notice that textbooks in essence constitute a sequencing of the content. So, there is a potential conflict.

For the PUMP people, the fight had two fronts: on the one hand the publishing companies, on the other, the New Math curriculum. And they launched massive critique in both directions. It was massive in the sense that they had a solid and detailed assessment material; this material was used in classroom studies to detect what type of difficulties students had in arithmetic, many of the difficulties appeared to be connected to textbooks and the curriculum; and the material was finally used to study the structure of textbooks. The analysis of the textbooks was thorough. The PUMP people mapped the amount of exercises the student encountered each week for a whole school year. And the level of difficulty of each exercise was estimated by means of the assessment material. In this way, they could estimate in what respect the progression was too steep or too flat. Often the textbooks raised the level of difficulty very fast. The PUMP people tied these problems to the New Math curriculum which they considered too unclear in terms of progression. They also found some directives misleading (Kilborn et al., 1977).

According to one of the leading persons in PUMP (personal communication), their textbook critique had great impact on the publishing companies. There was even an expression for it: textbooks were “pumped” before they went to the market. However, this claim needs further studies of the textbooks to be corroborated.

As to transfer of ownership, the PUMP assessment material was a readymade tool for the teachers to use. It was not something a teacher could own in the sense that they could gain knowledge about it and modify it. That was not the intention. However, the PUMP people argued that the material would make teachers less dependent of publishing companies and their textbooks. The idea was that teachers could use the PUMP material to plan their teaching and choose textbooks that fitted their plans. For that end, the material could also be used by teachers to evaluate textbooks (Kilborn et al. 1977). So, here we see another type of ownership. This is also the reason why we find it relevant to talk about sustaining rather maintaining innovations when it comes to PUMP.

Conclusions

We have applied Coburn’s (2003) theory of scaling up an educational reform to characterize the implementation of innovations in two historical Swedish development projects in mathematics education: the New Math project (1960–1975) and the PUMP project (1970–1985). This theory comprises four dimensions: depth, sustainability, spread, and reform ownership. In addition, we have made a distinction between maintaining and sustaining an innovation. We have also analysed the role of textbooks in strategies for sustaining or maintaining innovations in the projects. Our characterisation of the projects is summarised in Table 1 below.

Our analysis of strategies for sustaining or maintaining innovation has also involved the reformers attempts to manage publishing companies—the producers of textbooks—and potential conflicts of interest. In both projects, this management was done in a purposeful manner. In the case of New Math, textbook development was the centre piece in the development phase. A process publishing companies had very little influence over. And in the implementation phase, the companies were forced—through a national curriculum and a mandatory textbook review—to follow the examples set by the textbooks that were the results of the development phase. In the PUMP project, we can observe another strategy. Textbook development was not part of the PUMP project, but it included a comprehensive textbook review, based on the detailed assessment material (a key innovation) that had been developed and empirically tested. This material laid out a very detailed sequence in which exercises should appear in teaching. The review resulted in harsh critique of the textbooks. We have indications that this criticism had a great impact on the publishers. But further research is needed.

Table 1. Characterization of the New Math and PUMP projects

Project	*Maintain or Sustain*	*Spread*	*Depth*	*Ownership to teachers*
New Math	Maintaining innovations, mainly through formal curriculum and textbook control	Whole country through curriculum and textbook control. Concerned all schoolyears and all mathematical topics.	Through general principles and examples that sequenced the content	Limited in-service training of teachers.
PUMP	Sustaining innovations by giving a tool (an assessment material) for developing teaching methods and evaluating textbooks	Modest due to voluntary use of assessment material. And it concerned school years 1–6 and arithmetic.	Through a concrete and detailed sequencing of the content	Teachers should use assessment material for pedagogical develop-ment. Not possible to modify the material

We also have evidence of these strategies being necessary in the sense that publishing companies pursued aims other than those of the reformers. This became visible later on in the implementation phase of New Math, when the central school authorities made the textbook review optional in 1974 and refrained from driving key innovative elements of the New Math reform, even though the New Math curriculum was still in effect. The companies reacted quickly and were able to start production of more traditional textbooks. This is an example of how publishing companies respond to market force and how it can undermine the maintenance of an innovation. If there is a demand for a certain type of textbooks, the publishers tries to meet demand without losing the market share that rests on product recognition. Consequently, publishers balance on a fragile thread between the diametrical poles of change and tradition, when considering which path yields the most profit.

Regarding contribution to previous research, in particular Century & Cassata (2016) and Coburn (2003) which concern IR in general, we stress the theoretical relevance of our findings. In a previous section we have explained that existing IR models does not in greater detail concern the role of textbooks and publishing companies in processes of sustaining or maintaining innovations. We argue that this is something to consider when developing IR models specific to mathematics education and large scale implementation projects. Our findings indicate that the characteristics of the process of implementing innovations on a larger scale (sustainability/maintainability, spread, depth, and ownership) can be related to strategies for managing textbooks and publishing companies in the implementation process. Our findings also indicate that success in sustaining/maintaining innovation and achieving spread may be depending on such strategies.

These indications point at more general issues of innovation, education, and market forces. Issues we find extra relevant for mathematics education research since not only textbook companies but also technology companies have influenced school mathematics for a very long time.

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Towards designing a comparative survey for implementing PCT in Danish, Swedish, and English K-9 mathematics education

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This paper reports on the development of a survey tool to investigate how programming and computational thinking (PCT) is implemented in Denmark, Sweden, and England. The survey is targeted at mathematics teachers and aims to understand their enactment of PCT in their field and their perceived attributes of the innovation. Developing this kind of survey is difficult as implementation strategies differ significantly. This paper argues that one way to build a foundation for comparison is to inform its focus by utilizing 1) implementation theory and 2) innovation-specific theory.

Keywords: Computational thinking, design-based implementation research, implementing PCT in mathematics.

Introduction

During the last decade, an increasing number of countries have implemented curriculum revisions to include elements of programming and/or computational thinking (CT) in K-9 schools. These countries have adopted different approaches to this implementation process (Bocconi et al., 2016), which involve variations in the nature of the implemented innovation, the offered support strategies, the characteristics of the end-user, and the organizational and environmental factors surrounding the implementation process. While PCT as a compulsory school subject is not always explicitly linked to the mathematics curriculum, both researchers and practitioners in mathematics education acknowledge that there are potential synergies when integrating PCT within the subject. Despite this, previous research indicates that the establishment of meaningful synergies is far from a trivial endeavor (Misfeldt et al., 2019). At this point in time, mathematics teachers from many countries are likely to have encountered elements of PCT in their teaching; more than 50% of all math teachers in The International Computer and Information Literacy Studies' (ICIL) measurement of computational thinking reported an emphasis on computational-thinking-related tasks in their subject (Fraillon et al., 2020). ICIL's framework investigated information and communication technologies in schools across the subjects, including both explicit and implicit aspects, and it found similarities in the content, resources, methods to support learning, and priorities, despite rather different formulations of plans and curricula (Fraillon et al., 2020). Each of the nations did however focus more on computer and information literacy than on computational thinking. From an implementation perspective, this leaves us in a situation where it is still unclear how and to what extent experiences of teaching PCT and mathematics differ when situated in divergent contexts This would be valuable in gaining an understanding of the implications of various implementation strategies for mathematics teachers' practices. Still, the differences that make comparisons interesting represent a challenge when developing a research design. Although many nations have implemented an innovation, which can be labeled under the broad umbrella term of CT, there are substantial divergences in what aspects of the concept are emphasized. Moreover, national implementation strategies regarding PCT are often comprehensive and organized differently. These differences make it difficult to develop a design that is both broad enough to build a foundation for comparison and still sufficiently sensitive towards country-specific contexts. Previous research has studied teachers' conceptions of PCT in national contexts (Misfeldt et al. 2019) and across countries (Manila et al. 2014). However, this research mainly focused on mathematics teachers' ideas about PCT, while Manila et al. (2014) studied educators in all subjects. Additionally, none of these studies explicitly concentrated on implementation. This paper reports on our work of developing a questionnaire that enables us to find variations in how PCT and mathematics are enacted and experienced by teachers in three broadly different contexts. We chose Denmark, England, and Sweden for comparison as they have adopted distinct strategies for implementing PCT. Here, we will discuss how to design an implementation study that is sensitive to country-specific variations, creates meaningful survey items for educators in all nations to ensure reliability and validity (Wikman, 2006), and can thus provide a meaningful foundation for comparison. In this paper, we intend to spark a discussion about what properties we should expect from comparative implementation studies in relation to mathematics education. To engage in this debate, our starting point will have its basis in our reflections relating to developing a survey for mathematics teachers in Denmark, Sweden, and England on how they 1) teach mathematics and PCT and 2) what they see as the main difficulties and potentials of doing so. From an implementation perspective, our survey thereby specifically concentrates on what Century and Cassata (2016) refer to as the end-users' innovation enactments and their perceived attributes of the innovation itself. The overall aim is to address the following research question: *How can we develop a survey to compare mathematics teachers' enactments of PCT in mathematics education and their perceived attributes of the innovation in different national contexts?*

We begin the paper by outlining the situation in the three countries. Next, we describe the issues we encountered when developing a survey that can be both applied to the different situations in the countries and provide a meaningful foundation for comparison and the theoretical sources that we drew on to address these difficulties. Finally, we present how this was translated into a survey design and the limitations and strengths of this approach in terms of reliability and validity.

The situation in the three countries

As described in the introduction, Denmark, Sweden, and England have adopted different approaches to implementing curriculum revisions regarding PCT. In this section, we describe the nature of the approaches in the three countries and summarize their main differences from the point of view of implementation as well as investigate the relationship between PCT and mathematics.

Sweden

In 2018, the Swedish K-9 curriculum was revised as part of a national strategy to build students' digital competency. The rationale for this strategy was dual. First, the Swedish government highlighted that being digitally competent in the sense of understanding and mastering technology has become a prerequisite for being an active part of a democratic society, making it imperative.

Second, reports from the Statens Medieråd identified substantial differences in the digital habits and competencies of young Swedish people based on variations in gender, ethnicity, and demographic/socio-economic backgrounds. The national government referred to the increased focus on digital competency in compulsory schools as an approach to address this issue (Skolverket, 2018). This increased concentration on digital competency was cemented by revising the curriculum for several subjects in compulsory schools and by adding certain elements to legal documents, which re-establish the purpose of schooling in Sweden. Both initiatives aimed to make the responsibilities of Swedish schools clearer (Skolverket, 2018). This strategy led to revisions of all major subjects in schools (including mathematics, technology, civics, biology, chemistry, physics, Swedish/Swedish as a second language, and handicrafts), which were altered to include digital competency. In the context of mathematics education, this led to an innovation consisting of a revision of the curriculum, meaning programming was introduced as part of problem solving and algebra. The new programming component was thus embedded in the curriculum by rephrasing existing descriptions of the subject to integrate the skill. For example, in the case of algebra levels one to three, the curriculum specifies that students should learn "how unambiguous step-by-step instructions can be constructed, described and followed as a basis for programming" and that they should have knowledge of the "use of symbols in step-by-step instructions". Meanwhile, in problem solving for grade levels seven to nine, the curriculum specifies that students should acquire knowledge of "how algorithms can be created, tested and improved in programming for mathematical problem solving".[1]

Denmark

Denmark is yet to implement computational thinking in the Danish K-9 curriculum. Nevertheless, mathematics does include many aspects of CT-related teaching, especially with regard to problem solving, modelling, and tools/aids. The related goals incorporate the following: "The student can plan and undertake problem-solving-processes" (problem solving), "The student can undertake modelling processes, including with the use of digital simulation" (modelling), and "The student has knowledge of different concrete materials and digital tools" (remedies)[2]. However, these desired competencies have not been formulated as items specifically relating to PCT in the curriculum. From 2018–2021, a new subject entitled technology comprehension (TC) was implemented at 46 schools across the country as an experimental pilot project (Smith et al., 2020). The project included two implementation strategies, namely 1) implementing TC as a subject in its own right and 2) integrating it into other subjects (into mathematics in this instance). In both cases, TC was comprised of four areas of competency: digital empowerment, digital design and design processes, computational thinking, and technological knowledge and skills—including programming (Smith et al., 2020). The Danish Ministry of Education (UVM) also published a tentative curriculum with added TC learning goals (UVM, 2019). Although the new subject might forecast that an actual PCT-oriented subject is on its way into Danish schools, it has not yet been settled on as to how or when it will be fully implemented.

[1] https://www.skolverket.se/publikationsserier/styrdokument/2018/laroplan-for-grundskolan-forskoleklassen-och-fritidshemmet-reviderad-2018

[2] https://emu.dk/sites/default/files/2020-09/GSK_F%C3%A6llesM%C3%A5l_Matematik.pdf

England

England was among the first countries in Europe to absorb programming into the curriculum in 2013/2014 as a mandatory subject in its own right called computing. A key attribute of this innovation is its emphasis on technical and computer science-related content. The curriculum thus states that the aim is to ensure that all pupils, among other things, can understand and apply fundamental principles/concepts of computer science, analyze problems in computational terms, evaluate and apply information technology (including in relation to new or unfamiliar technologies), and analytically solve problems. In this respect, the computing subject in England can be considered a simplified version of what is taught in computer science at a university level. These attributes of the innovation reflect that the computing subject in England is intended to address a challenge that is phrased differently compared to the Danish context. In the former nation, one of the main aims of implementing the new computing subject was to lay the stepping stones for creating a workforce with adequate competencies to teach the next generation of programmers. In England, teachers of the now defunct ICT subject were assigned the responsibility of teaching its new computing counterpart. These educators received no formal training on how to disseminate the new subject to students, and there were no central initiatives to develop teaching materials. According to Larke (2019), the rationale behind not developing such initiatives was to ensure teacher-based autonomy when using and developing the materials they found were adequate to meet the needs of the curriculum. Although the computing subject in England is not related to mathematics, there are several ongoing research projects that explore the potential synergies of integrating programming into it. One of these projects is Scratch Math[3] which has developed a number of teaching materials that integrate mathematics and programming that are accessible to all teachers.

Theoretical background

As evident above, the implementation scenarios in the three countries differ in several ways, making it challenging to design a comparative research design that can adequately examine the three contexts. One particular challenge is that the implementation processes are comprehensive and organized very differently across the countries. To provide a solid foundation for comparison, it is thus important to ensure there is a theoretically delineated focus on the implementation process in which teachers are the appropriate respondents to answer the questions in the survey. To guarantee this, we informed the survey's implementation focus by drawing on research by Century and Cassata (2016) who developed an influential characterization of implementation research in education; they described it as both an inquiry into the innovation itself but also of factors that influence how it is enacted, the relationship between multiple innovations, and the outcomes of it. They outlined five key aspects of major importance for implementation in education, namely the characteristic of the individual user, organizational and environmental factors, implementation over time, implementation support strategies, and attributes of the innovation. Using Century and Cassata's (2016) wording, we were able to define our primary interest in the survey as trying to gain insights into 1) how mathematics teachers enact the innovation and 2) their perceived attributes of the innovation they enact. While

[3] https://www.ucl.ac.uk/ioe/research/projects/ucl-scratchmaths

Century and Cassata (2016) argue for the importance of studying the innovation, they distinguish between its actual (or objective) characteristics and the perceived attributes by the end-users. In the survey, we focused on the latter. Although the innovation (PCT) is described in somewhat loose terms in the three countries concerned, the relationship between PCT in mathematics and PCT is relatively clear. We therefore believe that available policy documents provide sufficient information about the actual characteristics of the innovation. Moreover, the existence of loosely described curriculum innovations makes understanding teachers' enactments of them even more important. We will return to the implications of this choice in the discussion.

A second challenge relates to the fact that the innovation being implemented is similar in that it focuses on CT, yet it is outlined in significantly different manners within the curricula and policy papers across the three countries. Although the innovation in all three nations addresses what could be labeled as aspects of PCT, there are differences in terms of which parts of the broad concept of PCT are emphasized. This is problematic as we intend to quiz mathematics teachers from the three countries on what aspects of PCT and mathematics they combine by providing a number of pre-determined content areas of PCT and mathematics for the respondents to choose from. These content areas are thus at risk of either being too comprehensive (if they were to include all elements of CT from each country) or biased towards the context of only one of the nations. To address this, we informed our questionnaire by employing Weintrop et al.'s (2016) framework for computational-thinking practices in mathematics education, which was developed to provide teachers with guidelines on how they can assimilate CT into their mathematics teaching; it also specifies four main areas of CT: data practices, modelling and simulation practices, computational problem-handling practices, and systems-thinking practices. With regard to which content areas of mathematics teachers combine with PCT, we utilized the 10 subject areas described in the KOM framework that was developed by Niss and Højgaard (2002). They are numbers, arithmetic, algebra, geometry, functions, infinitesimal calculus, probability, statistics, discrete mathematics, and optimization (Niss & Højgaard, 2002). Drawing on these models, we developed questions that ask teachers which PCT/mathematics subject areas they combine in their teaching; at the same time, we were careful not to bias what we asked towards the situation in one of the countries. More generally, by informing our survey with these theoretical frameworks, we aimed to ensure alignment between the survey questions and variations in the innovation across the three contexts and saw to it that the survey includes questions for teachers to answer that provide insights into their enactment of the innovation and their perceived attributes of it. Below, we describe how we operationalized these guiding theoretical principles within concrete survey questions.

Designing a comparative implementation survey

The survey is organized into three main sections: a background section, a section on teachers' enactments of the innovation, and finally a section on their perceived attributes of it. In this paper, we have especially focused on the last two sections, which we describe below.

Items in innovation enactment and perceived attributes of the innovation

The first obstacle when developing survey items is adhering to the need to concentrate on specific aspects of the implementation process, which mathematics teachers can provide comparable answers to (Wikman, 2006). As described, we define this focus as innovation enactment and perceived attributes of the innovation by drawing on Century and Cassata's (2016) research. Even though this provided a focal point for the differing innovations, to ensure the reliability of our survey items and thus the validity of our survey, we still needed to provide a foundation for comparison without being biased towards the context in one of the countries. As previously mentioned, we chose to employ a relevant framework by Weintrop et al. (2016) to ensure that the examined elements of PCT were based on generally accepted models. With regard to the areas of math content, we relied on the 10 subjects from the KOM framework (Niss & Højgaard, 2002).

The issue of potentially biasing the questions towards one of the countries' modes of implementation stems from the major differences in relation to the focus of the innovation. With the Swedish implementation, we would expect a high number of teachers working extensively with computational problem-handling practices due to the heavy emphasis on this in their curricula. On the other hand, with regard to Danish schools, we expect there to be a preoccupation with computational problem-handling practices but also with modelling and simulation practices as it is an explicit learning goal in mathematics (see the section on Denmark). As England has implemented programming as its own subject, it is still rather unclear to what extent teachers recognize their practices as computational thinking and what math-related subjects are connected to this. This is where the practices of Weintrop et al. (2016) become useful. As this CT definition has been developed for science and mathematics, we anticipate that teachers in all three countries will be able to recognize the four practices. When employing the country comparison by ICIL (Fraillon et al., 2020), we also expect there to be some similarities between the practices and context areas, even though the innovations are rather different. Using the more fine-grained sample of 10 math subjects, we are also equipped to enable a comparison of how mathematics teachers enact PCT in their educational activities and to ascertain to what extent the different practices highlighted by Weintrop et al. (2016) are signified. This construction of items not only allows us to look at what CT-related practices educators employ in their teaching but also to investigate what subjects in the math field they are coupled with. This facilitated our investigation into the different practices that are utilized across countries and made it easier to look into the relationship between the innovation and the practices themselves. In the questionnaire, we thus asked teachers if they include programming and computational thinking via each of the four practices described by Weintrop et al. (2016): "One can work with programming and computational thinking in different ways. To what extent is working with data practices part of your teaching?" If teachers reply that they include it to some or a large extent, we then ask which of the ten outlined content areas from the KOM framework that they are coupling the PCT practices with.

After forming the item construction of PCT practices and math subjects, we created a framework for comparing innovation enactments instead of the innovation itself. As an add-on to that, we aimed to compare teachers' perceived attributes of the innovation, both across country-specific innovations and the enactments by teachers. With regard to the perceived attributes of the innovation, we drew on a survey made for Swedish teachers (Misfeldt et al., 2019), yet we excluded questions that were too strongly influenced by the Swedish implementation of PCT into the math subject. The survey addresses teachers' perceived attributes of the innovation by including relevant questions, such as those relating to educators' experiences of the integration of PCT and math (e.g., "To what extent do you agree with the following statement: Teaching programming supplements math teaching to a large extent"), how teachers experience programming as it relates to the enactment of the praxis (e.g., "My students are using their math capabilities when they are programming"), and finally the relationship they perceive between math and programming. In sum, the survey described above allows us to inquire about what aspects of PCT and mathematics educators in the three countries combine in their teaching and their experiences of doing so. We expect that the survey will enable the gathering of valuable insights into the different ways teachers navigate and experience teaching PCT and mathematics in the three contexts.

Discussion and conclusion: Comparative implementation studies in mathematics education

There are variances in how Danish, Swedish and English mathematics educators combine their subject and PCT within their teaching; their experiences of doing so are also contextualized by significantly different implementation strategies. This is exactly why a comparative investigation is interesting. Although our development of the survey tool described here is still at an early stage, we believe that our approach could provide inspiration for how to conduct comparative implementation studies within mathematics education. Our survey design is informed by two main types of theoretical frameworks: one is a theory that is specific to the innovations that are being implemented and the other relates to implementation research. We chose to inform our survey in line with these theoretical resources to address the concrete challenges we encountered. This relates to the need for a delineated focus on certain aspects of the implementation process (as opposed to the entire thing) and finding ways of developing questions for the teachers that were not biased towards one of the three countries; this would ensure the reliability of the survey items and thus the validity of our questionnaire. Furthermore, adhering to these models enabled us to compare similar enactments across contexts and their correlation to perceived attributes of the innovations. By focusing on innovation enactment and the perceived attributes of the innovation, we have to assume to some extent assume that we have an understanding of said innovation and the related implementation strategies in the three countries. This assumption is based on the fact that both the curriculum revisions and the relationship between PCT and mathematics are well-described in policy documents in all three nations. However, the delineated focus on enactment and the perceived attributes of the innovation come at the cost of gaining insight into other, potentially equally important aspects of the implementation processes. This choice mirrors our primary interest in understanding the daily practices of mathematics teachers with regard to what they do and how they experience their practices of combining PCT and mathematics. Fully addressing all aspects of the implementation in detail would require other respondents to partake in the survey (managers, supervisors, municipal staff, etc.) and a much larger population to gain significant results into national variations and stabilities. Since policy documents provide rich descriptions of the implementation processes that are sufficient to pinpoint substantial differences, we believe that understanding mathematics teachers' enactments and perceived attributes of the innovation across these countries can provide important indications as to how such strategies are received by the end-users. Although our survey has not yet been tested, we believe that this work deals with an issue that has not been addressed explicitly in implementation research within mathematics education, specifically comparative implementation research in relation to mathematics education. We hope that the work reported here can spark interest, which will then lead to further engagement in more systematic discussions about this matter.

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TWG24: Representations in mathematics teaching and learning

Introduction to the papers of TWG24: Representation in mathematics teaching and learning

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Introduction

When TWG24, *Representations in Mathematics Teaching and Learning*, was initially opened at CERME10 (Robotti et al., 2017) it included 24 participants from 13 countries with 16 accepted papers and 2 posters; at CERME11 (Baccaglini-Frank et al, 2019) it grew to welcome 31 participants from 16 countries, with 18 accepted papers and 4 accepted posters; and at CERME12 TWG24 enjoyed the online participation of 28 researchers from 10 different countries with 19 papers and 2 posters presented. At CERME12 the structure of the working sessions allocated for discussion of each paper or poster was designed to stimulate interaction and collaboration among participants, even though the conference was held entirely online. Each paper or poster was allocated to one of six working sessions, which typically included three or four papers with theoretical, methodological or thematic similarities. All papers were shared with the participants in advance of the conference, and each presenting author was asked to concentrate especially on providing feedback or questions on the papers presented within his/her session. Presenting authors were asked to prepare a short presentation of their paper, including one slide explicitly addressing the theoretical perspective taken on "representations". The allocated time (20 minutes for papers and 10 minutes for posters) was split equally between author presentation and working group discussion. There was also a workshop session dedicated entirely to working in smaller rotating subgroups on questions designed by the co-leaders, which had emerged from the previous four days' discussions. The final session was devoted to a conclusive discussion chaired by TWG co-leaders with contributions from each of the morning's subgroups.

In this short report of the main themes that were discussed, we divide the themes between more practice-based ones and more theoretical ones.

Practice-based emerging themes

In thinking about representations in practice, important themes emerged related to sharing representational practices in three overlapping zones: 1) across educational systems, 2) within interpersonal learning activities and 3) inside encounters with various mathematical technologies and tools. At the largest scale, we found ourselves asking: *What can we observe about how representations are used across different educational systems? What issues arise when we 'import' something from one education system to another?* This macro-level question was inspired in large part by Palop del Río and Santaengracia's paper exploring the introduction of a concrete approach to the bar-model imported from Singapore educational system into a fifth-grade classroom in Spain. Interested in the bar-model's flexibility in a wide variety of problem scenarios, Palop del Río and Santaengracia sought to test the implicit assumption in Spanish curricula that this representation must be introduced to students in their earliest years of schooling to be an effective tool for thinking.

Although the potentials of (and contested approaches to) the “Singapore bar” remain a paradigmatic example of “importation”, the theme of working across educational systems also surfaced in papers which sought to better understand a variety of under-studied educational activities, asking to what extent these activities can be understood as “systems” in their own right. Angeloni, Wille and Hausch, for example, explicitly challenged the concept of “importation”, arguing that the invention and development of mathematical representations in Austrian Sign Language was a much more complicated affair than “importation” or “translation” might imply. While papers like this one did not directly address the complex politics of national educational systems, their ideas were expanded in the subsequent discussion, highlighting and exploring the emergence of representations in minority or marginalized linguistic, pedagogical, and digital spaces (some more of which are noted below). Challenging views of learning about or with mathematical representations as being static, normative or universalizable experiences, pratice-based evidence inspires us to find new ways of drawing on marginalized mathematical experiences as sources of broader pedagogical insight in their own right.

Several papers focusing on students working collaboratively on mathematical tasks (either with peers or a teacher/researcher) also led our group to focus on the development of interpersonal/interactional representational strategies, thinking about the generation and sharing of representational systems within both individual learning support and whole class contexts. These papers led us to discuss: *How much and in which ways should learner-generated representational strategies be encouraged and incorporated into educational discourse by the teacher (in her classroom or beyond)?* Finesilver, for example, provided a vivid case study of one student, struggling with division problems, personalizing and modifying pictorial representations and metaphors as ways to engage with multiplicative structure, including eventually more abstract tasks. This case demonstrated how idiosyncratic representations can be harnessed to combat exclusion of marginalized learners, in this case neurodiverse experiences. Lisarelli and Poli also reported on a teaching sequence in a class which aimed at developing responsive representational strategies drawing on student-generated imagery as a tool to think with while problem solving. While on the first activity students used a collection of representations, the subsequent whole class discussion helped the teacher navigate towards a consensual representation in the class. Meanwhile Velez, Serrazina and da Ponte aimed to understand exactly how a teacher managed his pupils’ use and interpretation of representations during whole class discussion. Hence their focus was predominantly on the verbal interactions, such as the ways the teacher changed the question type as students’ representations varied.

In thinking about the empirical evidence presented within our group, we also sought to engage with the fundamental question: *What representational practices are demonstrably effective (and less effective) in promoting meaningful mathematical learning? How might this vary in different educational environments?* In our general discussion of the questions highlighted above, many participants talked about expanding from or avoiding overreliance on conventional and ubiquitous representations. They discussed encouraging a mix of speech, gesture, and tool-use as powerful multimodal representational activities in the classroom and beyond. In encouraging and incorporating learner-generated representations, the group discussed balancing freedom and creativity with learning the necessary conventions for participating in the wider mathematical community. The power dynamics of classrooms were also addressed in our efforts to differentiate between representations that are an integral part of thinking and problem solving, as opposed to those produced on request just for pleasing the teacher, or retroactively after having already solved the problem without observable external representational strategy.

Theory-based emerging themes

As for the representations in theory, various themes emerged. We decided to focus the group discussions around the three questions, introduced below:

a) Of the theoretical and analytical frameworks presented, which have synergy? Where are conflicts? What might be fruitful combinations?

b) What hidden assumptions might we have about representations that will be made by or offered to different kinds of learners? How does this affect our choices regarding research participants, methods, and theory? What assumptions need to be uncovered and changed, and how to do this?

c) What are the links between representations in our research data and how we (re)present it to others? What representations serve us well in our professional practices as researchers and how can we develop them further?

Regarding question a), identifying synergy in representational frameworks proved quite challenging, and is deserving of more sustained consideration. However, the variety of analytical systems presented gave rise to many lively discussions considering how complementary aspects might be adapted, combined or developed for research in other contexts or with different types of datasets. Conflicts included some based on familiar divisions, such as between more platonic perspectives, according to which mathematical objects are pure and abstract, accessible only via representations, as opposed to others, in which mathematical objects do not reside in some hyper-reality, but in the discourse itself; there were also competing systems of terminology and classification to navigate. Nevertheless, we consider this diverse and multifaceted – yet interconnected – form of 'rhizomatic' theory-building to be a strength of the field (Deleuze & Guattari, 1980/1987).

In thinking through question b), we recognised that researchers in our group have worked with a wide variety of participants in terms of age, stage of education and level of expertise – from primary education up to PhD students – but also diverse learner groups, not only in terms of the different national educational systems from which we hail, but including e.g. sign language users, one-to-one intervention work with struggling students, and mathematics clubs or communities outside of formal schooling. One example of an assumption that came up was that students perceived as high achieving can easily work with multiple representations, together with the assumption that those perceived as low achieving have trouble transitioning between one representation and another. The result of such thinking is that in many educational settings, the latter students are offered narrower representational experiences, which limits learning opportunities and so disadvantages them even further. This was noted by many of our participants.

An interesting consideration, which emerged from discussion of question c), was about the difficulty of selecting and pairing appropriate and creative analytical tools with representations for communicating data interpretation and results. Choices and decisions about analytical tools, frameworks and procedures may highlight certain aspects of the data, but result in the loss of other aspects, particularly when we then communicate these to others. It was proposed to look to other fields for representational inspiration (e.g. computer science, media studies, dance); however, our group's papers already included interesting examples. Ott and Wille explored patterns of communication in one-on-one teacher-student support, seeking to understand individual learning support through the analysis of how and when communications between students and their instructors moved between two diagrammatic representations of number: the natural numbers and the field of twenty. To do this, the authors developed a visual system for coding the flow of diagrammatic conversation, which allowed them to make general observations about pedagogical patterns in their empirical study. Miragliotta and Lisarelli drew on Sfard's (2008) *realisation trees* in their research. This allowed them to make predictions about the ways in which classroom discourse might take hold of the geometric concept of "the height of a triangle". After analysing the lesson, the realisation tree helped them to map missing strategies and connections, as well as highlight novel ways of thinking about a triangle's height that occurred inside the classroom discussion.

Looking forward

As these questions and examples highlight, issues related to representations – both at the practical and theoretical levels, and in the relationships between practice and theory – seem to be of continued (or perhaps increasing) interest to the educational research community. This includes exploring underlying tensions between the multiple theoretical lenses through which representations can be conceived and studied (Baccaglini-Frank et al., 2022), a theme that was also addressed in the plenary panel at CERME12. Hopefully, both an exciting new fusion of theoretical and practice-based observations will be further discussed in the near future. We welcome more researchers to join our group over the next CERME in Budapest!

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A way of uncovering students' metaphors through scripting

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How can I make use of students' metaphors to look into their understandings of mathematical concepts? This study is aimed at exploring a new use of lesson play scripting as a novel way to investigate written metaphors used by students. Metaphor is a powerful linguistic approach, and I examine students' personal mathematical understanding of fraction concepts through their metaphors. Lesson play refers to scripting for an imagined lesson scenario, and I extend the common use of lesson play for teacher education to a research method applying to students. I implement scripting tasks with 28 Korean primary through eight times scripting classes. The process from preparation to results is described with an example of a student's script, and I discuss the applicability of scripting as a research method.

Keywords: Students' understanding, metaphor, lesson play, scripting, fraction.

Understanding in mathematics.

'Understanding' is regarded as the main goal of mathematics learning, a pre-requisite for further learning, or a way of mathematics teaching (Sierpinska, 1990), and its importance in learning and teaching mathematics has been emphasised. Many types of understanding have been discussed, e.g., conceptual and procedural understanding (Rittle-Johnon & Alibali, 1999), but my fundamental question here is what it means for students to understand mathematical concepts rather than which types of understanding students have. Students actively construct their own knowledge based on their subjective experiences rather than passively accepting information (von Glasersfeld, 1995). Pirie and Kiren (1994) proposed individual understanding as a recursive dynamic process that can be developed and be observed. Tall (2011) regarded understanding as a development process from emerging to forgetting, and to remembering. In line with these perspectives, I am taking understanding mathematical concepts as a constructing process of relationships among concepts in an individual's mental network. Mathematical understanding might be influenced by a particular environment around a student, but the focus of this study is on the manifestation of these influences within an individual.

In terms of the construction of mathematical concepts in embodied minds, De Freitas and Sinclair (2017) pointed out the limitation of the binary approach of abstract and concrete (or mind and body) and proposed the need for rethinking mobility and flexibility in learning mathematical concepts. I regard mathematical concepts as not fixed nor absolute ideas for any individual. Rather, mathematical concepts constitute a network through the constructive progression of an individual's mental network (Olive & Steffe, 2002), and there is no determined end in the network. Our embodied mind creates our language, and we communicate in a consensual domain which refers to a "shared physical context in which interactions occur" (Kravchenko, 2012, p.3). In a consensual domain, human beings interact with environments endlessly and (re)generate networks of interactions in a recursive way. These interactional actions are the basis of linguistic behaviours. Language is not just a tool for transferring individuals' thoughts but a way of living, described as 'languaging' by Maturana (2002, p. 28). Knowledge is generated in language through consensual co-ordinations, and thus languaging is part of how we express our understanding of concepts. As a way of investigating students' understanding of mathematical concepts, I look into metaphors as a powerful linguistic approach for creating, representing, and extending the meanings of concepts. This study focuses on a methodology which gives insight into student metaphors. Davis (2020) presented a summary of the potential different number metaphors used by students, based on teachers' experiences and textbooks; my longer-term aim is to provide a similar list of the range of metaphors students using in dealing with fractions.

Metaphor.

Metaphors naturally exist in our language (Lakoff & Johnson, 2003). Traditional metaphor theory considers metaphors as a role in enriching language. According to the conceptual metaphor theory, however, metaphors are fundamentally regarded as human thoughts. Metaphors are the conceptualisation of one concept in terms of another concept, based on the similarities between the concepts. Abstract concepts are understood via metaphor in terms of more concrete concepts. The conceptualisation is called 'cross-domain mapping' and cross-domain mapping consists of two domains, a target domain and a source domain. The concepts that we want to understand are the target domain (which is a relatively abstract concept), and the concepts in the source domain conceptualise target concepts. For example, arithmetic (target domain) is understood from collecting, constructing, and moving objects (source domain) (Lakoff & Núñez, 2000).

Metaphorical mappings as conceptual structures represent one's mental network between concepts. Lakoff and Núñez (2000) proposed embodied metaphors that are based on mathematical ideas from one's everyday experience and from different mathematical areas, called 'grounding metaphor' and 'linking metaphor'. These metaphors are elaborated not from a single mapping but from integrating many mappings between concepts, and the elaborated mappings develop an emergent structure of an integrated mental network (Fauconnier & Turner, 2008). In this regard, metaphor represents a mental structure in which we live and think and thus can allow exploring students' understandings that underlie their written language.

Fraction.

In this study, I chose a fraction concept for the investigation, inspired by my teaching experience in which I witnessed students struggling with fractions. Fraction is considered the most complex concept addressed before the secondary school. One of the factors contributing to the difficulty of the concept is that a fraction consists of two numbers. For fractional knowledge, students need to consider relations between a numerator and a denominator, rather than the two numbers separately. The two numbers can be regarded as parts and the whole, respectively; in this case the student is using what is referred to as the part-whole scheme. According to the part-whole scheme, a fraction indicates some parts out of a whole. The numerator is regarded as the parts or pieces to consider and the denominator is regarded as the whole. An equipartitioning scheme involves dividing actions that partition a continuous unit into equal-sized segments (Steffe & Olive, 2010). Whereas the partitioning scheme refers to breaking a unit, the iterating scheme refers to duplicating a segment several times to constitute a unit. The iterating scheme suggests that the fraction is freed from parts of the whole and can be often related to understanding the concept of "improper fraction". For 5/4, for example, students iterate 1/4 (a unit fraction) three times to constitute 5/4.

This study is aimed at investigating students' metaphors to understand their understandings of a fraction. Here, I need to engage in the question of how to explore students' uses of metaphors: how can I learn about students' metaphors, to look into their understandings of fraction?

Methodology.

Scripting, lesson play.

I suggest a new way of using lesson play as a research method for investigating each learner's conceptions. Lesson play refers to scriptwriting for an imagined lesson scenario with a dialogue format between a teacher-character and student-characters, which is originally designed for teacher education (Zazkis et al., 2013). Scripting is based on "virtual duoethnography" (Zazkis & Koichu, 2015, p. 166) which is narrative research producing a dialogue with multiple voices of fictional characters, from one person (who can be a researcher or a student), and it is inspired by the dialogue of Lakatos (1976). For scripting, prompts that include mathematical situations are provided, and teachers are asked to write a dialogue script for the ensuing situation of the provided prompts.

Here, however, I implemented scriptwriting to primary students as creating an imagined role-playing scenario. Wille (2017) has conducted imagined dialogues with students to discuss their mathematical thinking processes. Similarly, I provided prompts for mathematical situations to students, and they created a dialogue script for the given prompt. I intended to provide a certain mathematical classroom situation to students in order to understand the mathematical language that they naturally use. Role-play encourages students to participate in a specific situation as a specific person, so that students are able to deeply engage in learning and teachers are able to see their perspectives on the topic (Alkin & Christie, 2002). Scripting is an individual task that can be done by many students simultaneously, and it is less constrained by time and participation than real role-plays. Mathematical writing is encouraged for students to develop their mathematical understanding (Sierpinska, 1990), and the Korean National curriculum emphasises writing under a strong calligraphy culture. In fact, in follow-up interviews with student participants, many students mentioned math writing was easier than math speaking because they have enough time to think instead of having to answer immediately. When making scripts, students might use of their knowledge of mathematical concepts related to the situations of the prompts. Moreover, through the written contexts, a student's view on mathematical concepts can be examined.

Data Collection.

The participants are six grade (11-12-year-olds) Korean students from the same classroom and the number of students is 28. I had eight scripting classes with the students in a primary school. For script writings, I designed eight prompts, based on previous research and my teaching experiences about students' common errors, misconceptions, and difficulties in dealing with the concept of a fraction. If students had similar misconceptions with prompts, I would be able to track the root of their misconceptions. If students had a different understanding of the erroneous situations in prompts, they would describe their thinking to correct the misconceptions.

The fraction concepts dealt with in prompts are neither complicated situations nor beyond the scope of sixth-grade students learning according to the Korean national curriculum so as not to interfere with students writing. The prompts are related to many aspects of a fraction, such as fraction representation, fraction calculation, and fraction comparisons. The following is one of the prompts I used.

A teacher shows two fractions 5/6 and 7/8. Tom suddenly raises his hand.
Tom: Teacher, I have a question.
Teacher: Yes, Tom.
Tom: 5/6 and 7/8 are equal?
Teacher: Why do you think that?

The prompt above implies ‘gap thinking’, with regards to equivalent fractions. Gap thinking, as one of whole number thinking, is considering the gap between a numerator and a denominator by dealing with them separately, rather than considering the ratio between them (Pearn & Stephens, 2004). Gap thinking can lead students to draw either an incorrect answer or a correct answer with erroneous reasonings. Specifically, students can regard fractions having the same gap (e.g., 5/6 and 7/8) as equivalent. In scripts, students may or may not agree with Tom (or both), but my focus is on how the students describe their understanding of the two fractions. While creating the prompt, I anticipated students discussing their understanding or meanings of a fraction. Students work on writing in Korean (their first language), and I analyse the writings in Korean so that I can grasp the meaning and nuance of the contexts more appropriately.

Instruction.

In the first class, I introduced scriptwriting by presenting an example of a prompt and a script and then I asked students to practice writing a script for a prompt that was not related to a fraction. This practice is not only meaningful for students as a practice, but it was also important to me, because through it I could check whether my explanation was enough for students to carry out scripting. In writing, I encouraged students (1) to focus on the dialogue format rather than the contexts of the example script I showed them, (2) to approach the prompt situation with various perspectives (with various characters) so that I could explore their metaphors related to the concept in-depth, and (3) to use visual images such as diagrams, tables, and drawings to support their writing if needed. After the first class, I read students’ first scripts, and on what seemed to be unsuitable (e.g., texts are completely not related to the prompt), I left comments to help students carry out writings.

In the second class, I started with a five-minute ice-breaking time, a five-minute explanation for today’s prompt, and a 20-minute script writing. After scripting, I read the students' writing and left comments. The comments were to encourage their participation, for example, ‘you explained your idea logically’, ‘thank you for your detailed explanation’. Also, I left question comments on sentences where I was not able to understand them well, so that students would write additional explanations for the sentences. Then in the next class, before writing a new script, students read my comments and added answers to my questions.

Findings.

Translating students’ scripts from English to Korean is based on ‘interlinear morphemic gloss’ which is the way of translating in each morpheme or meaning, in order to show grammatical structures and the detailed, direct meanings of the original language (Lehmann, 2004). The translation uses three steps: 1) word by word (or phrase by phrase) translation, 2) direct translation, 3) natural English translation. For example,

	선생님은	전체 몇 조각	에서	몇 조각을	먹으려고	하는 걸까?
Step 1	Teacher	how many pieces of a whole	out of	how many pieces	eat	trying to

Step 2 How many pieces is a teacher trying to eat out of how many pieces of a whole?
Step 3 How many pieces am I trying to eat out of the whole?

The utterance is what a teacher-character said in a student's writing. The teacher-character herself is described as 'teacher', not I, comparing that student-characters themselves are described as 'I'. In Korean classes, it is common for teachers to refer to themselves as teachers instead of 'I', and this student perhaps may have seen many teachers referring to them as a teacher. Or in Korean culture, it is considered rude behaviour for students to call a teacher (adult) 'you' or name, so she might try to avoid it even in scripts. Thus, although the subject is a teacher in the original Korean sentence, it refers to the teacher-character herself, so the subject is translated as I in step 3 translation. The step 2 translation 'how many pieces of a whole' explicitly shows the student's countable numerical understanding of a whole. Though step 2 translation can have ungrammatical parts, it is effective to covey what words the student chose and how s/he describes concepts in Korean. Thus, also due to space limitation, I describe student scripts with only direct translation instead of all the three steps.

I illustrate some parts of Aiden's (the pseudonym of a student) scripts as examples of students' outcomes (scripts) and analysis, with the aim to explore nuances. In Aiden's scripts, I found two grounding metaphors that are related to each other. According to the utterances of the teacher-character Aiden created, she uses an equipartitioning and a part-whole scheme for fractions.

Teacher: How many pieces are there now?
Teacher: But a teacher tries to eat 2 pieces. Then, how many pieces is a teacher trying to eat out of how many pieces of a whole?
Teacher: Thus, a denominator is a total number of something, and a numerator is the number included out of the whole.

While explaining the meaning of a fraction to student-characters, the teacher-character assumes a situation in which she eats pizza with a pizza drawing. By partitioning, pizza as a continuous unit is separated into equal-sized segments. The teacher chooses two pizza slices by shading them and represents a fraction 2/6 as two pizza slices out of the whole pizza divided into six slices, which is a part-whole scheme. At the part-whole representation, I focus on that the parts and the whole are described with countable 'pieces'. She does not compare the size or number of pieces for a fraction but considers the number of pieces (how many) of a numerator and denominator. In her conception, a fraction consists of a numerator out of a denominator, and a numerator refers to the number of pieces as parts and a denominator refers to the number of total pieces in a whole. Aiden's metaphor of a fraction seems to be *pieces* as a grounding metaphor. In other words, she regards a fraction as pieces out of pieces. A denominator is regarded as the number of pieces that partition a unit, and a numerator is the number of pieces chosen. Figure 1 summarises this metaphor.

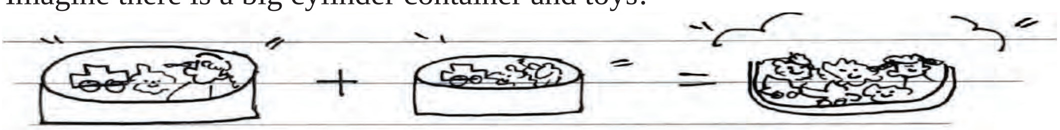
In addition to the pieces metaphor, through the used word 'something' in the last utterance of the teacher, I can see that she thinks of a fraction with objects. A numerator and denominator are objects, which is countable as a cardinal concept.

Source Domain Pieces		*Target domain* Fraction
Pieces out of pieces	→	Fraction
The number of equal-sized pieces that partition a unit	→	Denominator
The number of pieces taken	→	Numerator

Figure 1: Fraction metaphor as pieces

Her countable conception of a numerator and denominator seems to link to her understanding of fraction addition. In her other script, a student-character Jack who reflects her thinking explains the fraction addition 3/5 + 3/5. For the explanation, Jack uses metaphors for a numerator and denominator as a toy and a toy container, respectively.

Jack: Imagine there is a big cylinder container and toys!



When overlapping the same container here, the containers can be overlapped. But toys are the same but do not overlap each other! Thus, the containers are the same, but the number of toys does not overlap each other, so [numerators] become different.

With the toys and container metaphor, Jack describes the principles that denominators are not added to each other but only numerators are added, in the addition of fractions with the same denominator. Jack looks at a numerator and denominator as countable objects (pieces). In her conception, a numerator is the number of objects, and a denominator is the container of the objects. It can be seen as the pieces metaphor, but the difference is that the denominator 5 does not refer to five pieces but one container named 5 itself. Thus, 3/5 means there are three toys in a container named 5, and 3/5 + 3/5 means putting a total of six toys and two containers named 5 together. Since the toy containers as denominators can be completely overlapped, denominators in the fraction addition do not need to be added. On the other hand, the number of toys as a numerator should be added because toys cannot be overlapped. Therefore, fraction addition is combining objects, and Figure 2 summarises this metaphor.

Source Domain Combining objects		*Target domain* Addition of fractions
Put objects together in the same container	→	Addition of fractions with the same denominator
Container of objects	→	Denominator
The number of objects	→	Numerator

Figure 2: Fraction addition metaphor as combining objects

This understanding of a fraction is freed from partitioning. Character-Jack does not split a container into equal-sized parts and does not consider the ratio between a numerator and a denominator. The fraction addition metaphor of Aiden, especially a numerator metaphor, is similar to an arithmetic metaphor of Lakoff and Núñez (2000). They proposed an ‘Arithmetic is object collection’ metaphor: Numbers are collections of objects and arithmetic addition is putting collections together. Similarly, Aiden thinks of a fraction as objects and the fraction addition as putting objects together. The difference between the metaphors of Lakoff and Núñez and Aiden is that their view of numbers is based on a set or collection, while her metaphor of numbers is based on a numerical counting scheme.

Her fraction addition metaphor seems to be related to the multiplication of a fraction and a natural number. In her drawing, the toys in each container are the same, which means that there are two 3/5s, instead of that there are "three toys" and another "three toys". She can use this metaphor for 3/5 × 2. Her conception, however, can have limitations in understanding the addition of fractions having different denominators. When making denominators equal, she can represent denominators as using different containers, but it is unclear how the change of numerators can be described.

Implications.

This study proposes a new use of lesson play as a research method. Through this study, I explored the applicability of lesson play scripting to students. All students in this study finished one scriptwriting in 20 minutes and successfully created their scripts. This enabled me to attain students' data referring to mathematical concepts at a certain mathematical situation and to examine students' conceptual schemes of a fraction. I was able to conduct scripting tasks with many students simultaneously. Through the length of their script over time, I could see that students had participated more actively rather than losing interest in scripting. Although the length of writing is not directly related to the quality of the data, more active participation is more likely to obtain high-quality data through which I can understand students' thoughts. As the class progressed, students became accustomed to mathematical writing by creating various characters.

I am exploring student scripts to find fractional metaphors and will be analysing the work of the other students in my study, as well as Aiden's. I have found more metaphors: e.g., fraction is equipartitioning amount, fraction is iterating, fraction is division, and fraction is a number form. In the future, more findings about fraction metaphors and an in-depth analysis of these metaphors will have to be conducted, and then I would be able to provide a metaphor model that can be used to understand fraction concepts and a framework of metaphor analysis. Lastly, as scripting can be extended as a teaching activity such as writing a scenario for the real performance of mathematics role-plays, I hope to pursue further studies on the use of scripting in mathematics classrooms.

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Signing about elementary algebra in Austrian Sign Language: What signs of the notion of variable can represent

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Languages have a significant impact on mathematics learning, with visual languages and spoken languages in particular differing. The study presented here investigates how one can sign about notions in elementary algebra in different ways in Austrian Sign Language. In particular, the focus is on the question of what sign language signs of the notion of variable can represent and thus how they may impact on the understanding of variables. Distinctions from spoken language are identified. The study is part of a larger investigation into communicating about elementary algebra in sign languages.

Keywords: Language, learning mathematics in sign language, classifiers, elementary algebra.

Introduction

Sign languages are full-fledged natural languages with their own grammars. They are in no way inferior to spoken languages (Beecken et al., 2014) and can express concrete as well as abstract things just as those. Therefore, communication about mathematics is just as possible as in spoken language. There is not only one sign language used around the world but many different ones (Braem, 1995). Their development does not necessarily coincide with that of the respective national spoken languages. Sign languages organize ideas and convey content, meaning and significance in a different way than spoken languages. That means a person who thinks in sign language thinks differently than one who is thinking in spoken language (Grote, 2010). Therefore, the conceptual understanding that is typically oriented toward hearing contexts – i.e., based on the characteristics and features of spoken languages – often does not match the conceptual understanding of students who use sign languages, and thus may not be appropriate for this audience (Krause, 2016, 2019; Krause & Wille, 2021; Wille & Schreiber, 2019). The study presented here is part of a longer series of investigations on the topic "sign language and mathematics education". The specifics of sign languages will be considered with the aim of developing and evaluating approaches and concepts for teaching mathematics in sign language. Language plays an essential role in both teaching and learning mathematics (Fleming, 2007; Thürmann & Vollmer, 2017). Therefore, the research focuses first on sign language, especially on its lexemes and classifiers, two important components of sign languages. The first multiphase investigation concerns specifically talking about mathematical activities in elementary algebra. The mathematical focus is on different aspects of variables (see below): object aspect, substitution aspect and calculus aspect (Malle, 1993). The linguistic focus is on the Austrian Sign Language (ÖGS). The general interest of the study is to investigate how these aspects occur in ÖGS and how mathematical notions and their aspects are represented in this sign language. Knowledge of these two points about sign language is necessary to teach mathematics in this language. In the following, the general question is specified in terms of the object aspect of variables: Which lexemes (signs) are used for the object aspect of variables in ÖGS? In what way are classifiers used for this? What are the differences between sign language and spoken language in this regard? High knowledge of these topics about language is on the one hand necessary for the translation between spoken language and ÖGS regarding mathematical teaching and learning. On the other hand, it provides the basis for the next investigations in mathematics education regarding sign language.

Aspects of variables

The word "variable" corresponds to its etymology: It can appear in different forms, denote different things and it can have different aspects. For example, there are "word variables". These are single words or groups of words that are representative of something else – e.g. of numbers (Akinwunmi, 2012; Küchemann, 1978; Malle, 1993). The "usual" variables used in mathematics are the "letter variables". According to Malle (1993) at least three aspects can be identified for variables: object aspect (the variable is an unknown or unspecified object of thought), substitution aspect (the variable is a placeholder into which numbers may be inserted) and calculus aspect (the variable is a sign without meaning, but which may be operated with according to certain rules) (Malle, 1993; Schoenfeld & Arcavi, 1988; Wille, 2008). If a variable is considered under the object aspect, then the object of thought itself can be different: a figure, a number, a number as a quantity of something etc. Thus, with a single word in spoken language such as "number", one can refer to various things. Therefore, "variable" can denote many different things. When such terms are translated into other languages, phenomena such as diversification may well occur. In translation studies the term "diversification" means the phenomenon in which for one word in the original language there are several words in the target language (Koller, 2011). Vice versa there is a "gap", if for one word there exists no translation. Thus, the first question that arises at this point is:

- *Which sign language's sign or signs are used for the manifold term "variable" (from the point of view of the object aspect)?*

Classifiers in the sign languages

Classifiers have an important function in sign languages. Classifiers (CL) or depicting handshapes are a complex and a highly discussed topic in sign language research. Definitions often diverge. In the presented research a classifier is an element whose meaning is related to the context. It represents "entities" based on their characteristic features and is involved in a morphologically complex structure. With a classifier objects or processes are classified based on common features (e.g. the index finger represents a "person") (Zwitserlood, 2012). Classifiers can be used to refer to the property, position and movement of the signified (Beecken et al., 2014). They follow certain rules of sign language grammar. For instance, if they refer to a noun, they are signed after the sign for that noun. If classifiers refer to a verb, they are signed before the sign for that verb. For the classification of classifiers there are several possibilities concerning semantics or linguistic context. Regarding the mathematical background of the study, first of all, the following classification is made: semantic classifiers (representing a "stylized" shape of the object: e.g., flat hand for cars or tables, claw hand shape for a clock or picture frame, etc.), instrumental classifiers (which denote objects according to how they are handled) and size and shape classifiers (representing size, extent, etc. of an object). The last group of classifiers – if it is considered to be a part of classifiers – can be further divided into static classifiers (to represent the size and shape of the object) and tracing classifiers (by moving one or both hands they outline the shape or size of that object) (Beecken et al., 2014; Zwitserlood, 2012). The use of classifiers for real objects (including humans and animals) is quietly clear. However, when signing about mathematical notions – as about variables – the second question arises:

- *Which classifiers are used when signing about variables with regard to the object aspect?*

The formation of new groups of classifiers related to the mathematical nature of the represented objects is not excluded here. Since mathematical notions in the school context mostly originated from spoken languages the third question investigated is*:*

- *What differences exist between spoken language (German) and ÖGS regarding the object aspect of variables?*

Learning environment

In each session of the learning environment the students get tasks[1] concerning two persons that have to distribute brochures. In the beginning they have one thousand brochures to distribute, but it is unknown how many the one person and how many the other person is going to distribute. Then one person gives a stack with two hundred brochures to the other person. The students are now asked to answer two questions about the amount of the brochures that each person has – before and after the stack of two hundred brochures was moved from one person to the other person.

The students have to perform those tasks in the given order according to the dialogic principle "me-you-we" (Green & Green, 2018; Ruf & Gallin, 1995, 1999). This principle provides for three phases. In the "me-phase" each student deals with the problem alone. The student does not necessarily have to come up with a solution in this phase. Notes or sketches of a solution idea are sufficient – that means: "this is how I do it". In the "you-phase" the students exchange ideas with each other (in pairs of two or in small groups): "how do you do it?". Results and open questions are to be recorded here. In the "we-phase" the results of the "you-phase" are presented, discussed and compared in the plenum in order to arrive at a joint solution and find a convention: "this is how we will do it". The phases "this is how I do it", "how do you do it?" and "this is how we will do it" can be rephrased from the point of view of sign languages as follows: "this is how I sign it", "how do you sign it?" and "this is how we will sign it". However, the last phase cannot always take place in this form, because there are rarely conventions for mathematical terms in ÖGS. The basic idea of the learning environment presented here is that the students become as active as possible – not only mathematically, but first of all in their language, the Austrian Sign Language. The influence of spoken language should be reduced as much as possible, on the one hand with the aim to not disturb the natural communication in ÖGS, on the other hand to not distort the observations of sign language communication. Spoken language could transport elements into ÖGS that are not compatible with the specifics of a sign language and specifically of the ÖGS. Therefore, from the beginning the material was developed in a sign language perspective and not as a translation of a material that was developed in a spoken language. The material contains no text, but only pictures (comics) and QR codes linking to videos

[1] Task card no. 1 at https://me.aau.at/~awille/mathe_in_oegs_variablen_01.html; task card no. 2 at https://me.aau.at/~awille/mathe_in_oegs_variablen_02.html; task card no. 3 at https://me.aau.at/~awille/mathe_in_oegs_variablen_03.html

in ÖGS with explanations and tasks. The students are able to watch these videos as often as they want to on their own smart phones. All videos are signed by a deaf person who also has ÖGS as basic language. A script for the videos is not used in order to avoid any interference errors (that are typical expressions of spoken language but not of ÖGS). Mathematical terms (e.g. variable), for which no signs (in ÖGS) are known, are not mentioned in the videos and in the sessions. These terms are not even finger spelled.

Methods

The research took place in form of two 60 minutes sessions in July and August 2021– with three people in the first and four people in the second session. The participants were deaf adults (age range 30 to 65) and their basic language was ÖGS. Basic language means the language in which a person thinks – his or her "inner" language. Adults were selected for this first part of the study because they were the most competent in ÖGS of those available. They were no more familiar with variables from their school background. Two teachers moderated the sessions. They announced which tasks had to be solved and answered technical questions. The order of the tasks is given as a structured interview. After the "you"-phase, the teachers led the "we"-discussion as an unstructured interview: getting the participants in the discussion and asking further questions (e.g. why is it like you say?). Both sessions were recorded by video and the participants were aware of that. So that is a direct observation with continuous monitoring. The videos were glossed (transcribed with glosses) according to the notation system of (Prillwitz & Wudtke, 1988), but simplified for the aim of the investigation. Glossing means the practice of writing down a sign language text sign-by-sign. Afterwards, the classifiers and signs of the notion of variable were determined. The data were then analyzed according to the principle of content analysis (Kuckartz, 2018).

Findings

With regard to the core of the tasks – the number, i.e., the quantity of brochures that are stacked – five different classifiers occur (Fig. 1). Since the meaning of the classifiers can only be deduced from the context this was also considered. Here are transcribed examples of each that the participants used:

P1: WIEVIEL CL_1-STAPEL FLAVIO WIEVIEL CL_1-STAPEL SANDRA
How many pieces does Flavio have? How many pieces does Sandra have?

P2: ZUSAMMEN EIN-TAUSEND IX CL_2-STAPEL CL_2-STAPEL
In total there are one thousand [stacked brochures].

P1: SANDRA DA EIN-TAUSEND CL_3-STAPEL
Sandra has one thousand [pieces].

P3: ERSTENS DREI CL_4-STAPEL CL_4-STAPEL CL_4-STAPEL
At the beginning there are three stacks.

P1: ZWEI-HUNDERT CL_5-GEBEN
[Flavio] gives 200 [brochures] to [Sandra].

P6: HAUFEN GEBEN WEISS-NICHT CL_1-STAPEL WIEVIEL WEISS-NICHT
From the pile [Flavio gives her] some. But you don't know how many there are in the pile.

For statements about the unknown or unspecified number of brochures on a stack the first classifier was used (Fig. 1a). It is introduced by the “question sign” WIEVIEL (*how many*). In the recordings this classifier occurs when unknown quantity is concerned. The execution of the classifier, moreover, reveals what exactly it refers to: The splayed dominant hand (“sh” hand shape) moves upwards. It represents every single brochure on the stack. Since those brochures are a lot the movement of the hand is smooth. So, the sign was used to denote the unknown or unspecified number of brochures. In particular, the information that the brochures are stacked are by the sign transmitted as well.

Fig. 1a. CL_1-STAPEL (*stack*)

Fig. 1b. CL_2-STAPEL (*stack*)

Fig. 1c. CL_3-STAPEL (*stack*)

Fig. 1d. CL_4-STAPEL (*stack*)

Fig. 1e. CL_4-STAPEL (*stack*)

Fig. 1f.CL_5-STAPEL-GEBEN (*to give a stack*)

Figure 1. Classifiers and a “simple” sign for the representation of a known or unknown quantity of stacked brochures

However, with this classifier one additional information is given simultaneously to the information about the unknown quantity. The dominant hand moves upwards, then downwards and once again a little upwards. This can be a representation of the possible height of the stack depending on what height the signing person imagines. This means that the classifier additionally conveys how large the range of the variable is. Summing up, ÖGS can express with a single sign the following: there are many brochures, they form stacks, we don’t know their height, it can be zero to something, that however will not be “too much”. The other classifiers occur in relation to a known quantity: the classifiers CL_2 and CL_3 in Figs. 1b and 1c represent the one thousand stacked brochures. The difference between the two classifiers is the repetition. So it is with CL_3 clearly represented that there are multiple stacks. With the fourth and the fifth classifier the stacks have the same height as if they would have been divided into three stacks of the same height. The sixth classifier (Fig. 1f) is an instrumental one that together with the verb “to give”. It represents how the stack (of 200 brochures) is given to the other person.

Out of the topic “classifiers”, at the lexical level the sign NOCH-NICHT (*not yet*, see Fig. 2) occurs. It refers many times to a missing (piece of) information. A few examples (from study unit 1 and 2):

P6: NORMAL STÜCK MUSS ERST HAUFEN [pfff]. WIEVIEL STÜCK NOCH-NICHT GEBÄRDEN NOCH-NICHT NOCH-NICHT
Usually, it should be given at first how many pieces there are. However, this has not been said.

P6: WO GEBÄRDEN EIN-TAUSEND WO. NOCH-NICHT GEBÄRDEN NOCH-NICHT
Where has it been said that it was one thousand? Where has it been said? This has not been said?

Figure 2. the sign NOCH-NICHT (*not yet*)

Discussion

Regarding the first question of which sign(s) exist(s) for “variable” under the object aspect in this context, the sign for the first classifier CL_1-STAPEL (*stack*) can be identified (Fig. 1a). This sign can be interpreted as a “sign variable” for the word variable “number” from the spoken language and as a variable itself in terms of the object aspect. Moreover, the sign NOCH-NICHT (*not yet*, Fig. 2) was used to express that an information is unknown.

Regarding the second question of which classifiers are used and how, it can be stated that the classifier CL_1-STAPEL (*stack*) is used to express the indeterminacy of the number of stacked prospects. This classifier differs from the others (for a known quantity) especially in the movement of the dominant hand, which moves once upwards, once downwards and once again upwards, but not the same height as before. This could be related to the fact, that the height (the quantity) is unknown. If the quantity is known, other classifiers are used – and each of them expresses a different detail about the signified: size, how stacks look, if all stacks have the same size etc. However, since a classifier is very closely related to the signified, a strong diversification can be assumed for other contexts as well: Depending on what a variable refers to different signs could be used to define that variable. One reason for this phenomenon could be the nature of the sign languages themselves: they are complex languages that have many slides also on the side of the “factual information” (Schulz von Thun, 2011). Thus, the need for a single equivalent sign for the word variable “quantity” does not seem to exist. Rather, for teaching mathematics in ÖGS, it should be necessary to know how and under which aspect a variable is used in order to render this variable according to the peculiarities and characteristics of sign language. Due to the difference in the movement of the dominant hand, the definition of a new group of classifiers could be possible and meaningful.

In this first part of the study, it was possible to state, on the one hand, possibilities with which signs and with which type of classifiers the object aspect of a variable in ÖGS can be represented. On the other hand, regarding the third question about the differences between spoken language (German) and Austrian Sign Language, it was found that in the sign language communication about the object aspect of the variable a lot of information was given simultaneously: In a single sign such as CL_1-STAPEL (*stack*, Fig. 1a) it is expressed whether the quantity is large or small, what shape or arrangement is meant and in which domain the quantity is located. In contrast, the word "quantity" is not enough to convey all this information in the spoken language. In the ÖGS this is possible in a natural way and can thus describe and represent variables very accurately. This indicates that a translation of (written or spoken) texts – e.g. of teaching material or of teacher explanations – from spoken language into ÖGS should be clearly detached from the source text. It is not only about transferring the texts or the grammar of ÖGS, but also about which information is transferred and how. Typically, for example, in mathematics teaching one could speak about the "quantity of stacks" in this context. In this phrase it is explicitly given that there is a stack and we talk about its quantity. It remains implicit that we do not know the quantity at the moment or in which range this quantity is. With signs, in contrast, more of the implicit information is expressed explicitly. This does not mean that ÖGS is "underrated" compared to spoken language and therefore needs more information and explanations to express complex and abstract content in a meaningful way. Rather, that one must take into account and can use the representational power of sign language. The next step in the study will be to investigate the other aspects of variable and to vary the context.

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Pre-structured learning of heuristic strategies: types and potential of post-reflection tasks in heuristic worked-out examples

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Keywords: Heuristics, problem-solving, worked-out examples, reflection.

Various approaches have already been developed and studied to support students in mathematical problem-solving processes. Most of them aim at supporting students by indirectly or explicitly addressing strategies that are typically helpful for a problem solver (Pólya, 1945) - so-called *heuristic strategies*. These appear in situationally different forms and types (e.g. Schoenfeld, 1985).

Pre-structured learning formats are formats that predominantly initiate learning activities with little (or no) guidance from the teacher. *Heuristic worked-out examples (HWEs)* (Reiss & Renkl, 2002) are one possible pre-structured learning format for supporting secondary school students' problem-solving processes in mathematics classrooms. Bachmann and Müller-Hill (2021) have theoretically analysed the potential of different design elements of HWEs, focusing on promoting learners' development of heuristic strategies as a component of successful problem solving. This analysis underpins the thesis that so-called *post-reflection tasks* can generally initiate learning activities that support the acquisition of heuristic strategies (Bachmann & Müller-Hill, 2021). In this context, the poster contribution addresses the following specific research question: *Which types of post-reflection tasks in heuristic worked-out examples with the potential to acquire an individual heuristic strategy can be identified?*

Sketch of the theoretical framework.

Following general action-oriented learning theories (e.g. Aebli, 1994) and research on mathematical problem solving (e.g. Pólya, 1945), having a heuristic strategy at one's disposal develops out of an individual's concrete problem-solving actions, starting from actions in suitable situations within concrete problem-solving processes to achieve specific (sub-)targets that can be assigned to a heuristic strategy. This development occurs primarily in the form of two complementary processes: internalising and classifying, mediated by language. Based on such a conceptualisation of an individual heuristic strategy, a two-dimensional learning field to acquire heuristic strategies (Figure 1) is developed in Bachmann and Müller-Hill (2021), which refers to four types of learning activities: *performing an action*, *describing an action in one's own words*, *verbalising the target and situation of an action*, and *verbalising the target and situation of an action more generally*.

Applying the theoretical framework to the case of post-reflection tasks.

In Figure 1, we see three types of learning activities that aim to *compare actions within and across problem categories and relate actions across similar situations and similar (sub-)targets*. These activities enable individuals to communicate about problem-solving actions (at a more concrete level) and about heuristic strategies (at a more general and abstract level). Thus, they support the individual construction of an appropriate language and the gathering of experience about concrete problem-solving actions.

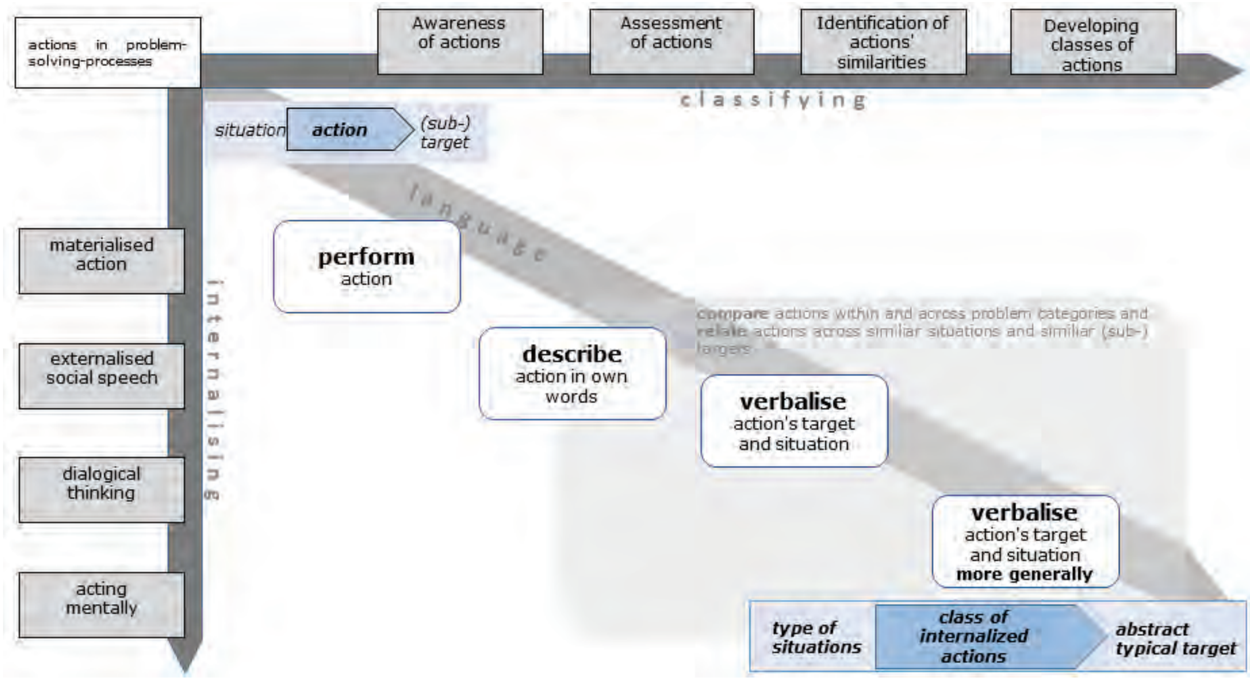

Figure 1: Learning field to acquire an individual heuristic strategy (Bachmann & Müller-Hill, 2021)

Post-reflection tasks focus on such descriptive and verbalisation activities. They are the final task at the end of an HWE. Therefore, it is possible to refer back to the entire solution process presented when working on these tasks. By setting suitable post-reflection tasks, a guided reflection on the solution process and individual actions within it can be initiated for the overall process. Different formulations and presentations of post-reflection tasks can initiate different learning activities of describing and verbalising (Bachmann & Müller-Hill, 2021).

The poster proposes a theoretically based characterisation of types of post-reflection tasks and illustrates concrete examples for each type using different formulations and representations. The possible strengths and weaknesses of the different types and representations derived from the theory form the starting point for future empirical work.

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Second-order covariation: it is all about standpoints

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Second-order covariation is a quite recent theoretical construct in the field of Mathematics Education: it differs from the already existing construct of covariation in placing greater emphasis on the role of parameters, with respect to the other variables, as characteristic of a certain family of functions and as relevant in modelling classes of real phenomena. In this contribution, we address the novelty and importance of this type of reasoning based on a teaching experiment concerning modelling of a thermodynamic situation. Starting from the analysis of three episodes, we highlight some features of this construct, and the emerging interpretation as a change in standpoint that results in different graphical representations suitable to interpret globally the specific mathematical situation.

Keywords: Covariation, variables, parameters, graphs.

Introduction to the research problem.

Covariational reasoning is an essential skill necessary to enter deeply into the processes of mathematical modelling (Thompson, 2011): it can help to model "the change and the relationships with appropriate functions and equations, as well as creating, interpreting and translating among symbolic and graphical representations of relationships" (OECD, 2022). Italian mathematics curricula for secondary schools also address the importance of "represent[ing] the same class of phenomena through different approaches" (MIUR, 2010, p. 337). To pursue this goal, using multiple representations (Ainsworth, 2008) turns out to be of great importance for students in order to fully embrace all the aspects of the mathematical situation under investigation. An example of meaningful graphical representation which lends itself to multiple educational purposes consists of the "psychrometric chart" or Carrier diagram. It describes the thermodynamic parameters of moist air at constant air pressure, so it is a graphical representation of an equation of state. This type of representation requires sophisticated forms of covariational reasoning to grasp how magnitudes are varying and to identify their mutual relationships. To better introduce this chart, we comment on a real chart displayed in Figure 1a and/or on a slightly simplified simulation[1] of it created with Wolfram software, shown in Figure 1b. The dry bulb temperature is shown on the horizontal axis while the vertical axis is the absolute humidity. The chart in its complete form contains many parameters: wet bulb temperature, dew point temperature (the temperature at which the air becomes saturated with water vapor), relative humidity, specific enthalpy, and specific volume; indeed, reading a psychrometric chart is very challenging, but if we just focus on temperature (abscissa), absolute humidity (ordinate), and relative humidity (the parameter), the relation between temperature and absolute humidity is given by an exponential-like function (green curve) and a different curve corresponds to each value of the percentage of relative humidity. Indeed, the Carrier diagram shows the mutual relations between the three variables involved crushed onto a two-dimensional

[1] This simulation is available online at this link: https://demonstrations.wolfram.com/ReadingAPsychrometricChart/.

representation: this flattening makes the reading and the interpretation more complicated than a three-dimensional representation in which the three magnitudes assume the same ontological status of variable. Again: why choose relative humidity as the parameter instead of absolute humidity? How would this choice affect the bidimensional representation?

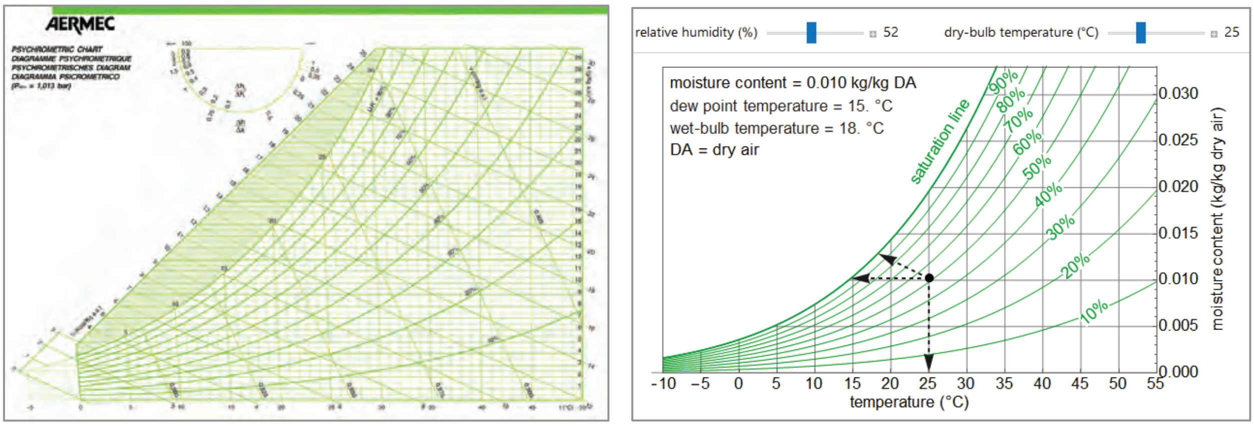

Figure 1: a) An example of psychrometric chart; b) Simulation of a psychrometric chart created with Wolfram software

To answer these questions, the theoretical construct of second-order covariation is introduced. While the construct of covariation focuses on the relation between two or more variables, second-order covariation focuses on the role of characteristic parameters as specific of a certain family of functions. What is defined as being "of second order" depends on the way a specific variable is extrapolated from the assigned scenario and interpreted as a parameter: this determines the standpoint from which the mathematical situation can be represented.

The main purpose of this contribution is to enter the complexity of second-order covariational reasoning required to fully understand the relations between the above magnitudes. Consequently, our research question can be formulated as follows: *how can second-order covariational reasoning be characterized when interpreting representations, like the psychrometric chart, where at least three magnitudes are involved and one of these can be mathematically interpreted as a parameter?*

In this paper, we are going to initially provide a characterization of the construct of second-order covariation referring also to other theoretical contributions in the literature. Then, thanks to the analysis of a few episodes from a teaching experiment involving an Italian secondary school classroom, and based on the interpretation of the psychrometric chart, we are going to explore the covariation emerging in students' reasoning. Finally, we propose a refinement of the construct of second-order covariation which could be interpreted as a change of standpoint suitable to interpret different mathematical representations.

Theoretical framework.

The use of multiple representations strongly supports the learning of mathematical concepts, and among their various functions they provide information that complement each other and/or foster the development of deeper understanding when they support additional information (Ainsworth, 1999). Although multiple representations are beneficial to the learners, they are non-trivial for students to use to relate and identify connections (Ainsworth, 2008). Beyond this, single representations themselves can be challenging: for instance, reading, interpreting, and reasoning about graphs requires grasping the relationship between the values of the magnitudes involved, and so it demands good covariational skills.

Being able to reason covariationally means being able to envision two or more magnitudes that vary simultaneously, as presented in Thompson and Carlson (2017). The last version of this theoretical construct, presented in the same work, and that we are going to refer to as first-order covariation (COV 1), consists of six levels of covariational reasoning ranking from absence of covariational reasoning to smooth continuous covariation. These levels can be interpreted as classes of behaviors or descriptors of a person's ability to engage in covariational reasoning. In this contribution, we consider covariation in a broader epistemological sense, i.e., as the ability to grasp relations of invariance that are more complex than the simple covariation between two magnitudes. Specifically, the construct of second-order covariation (COV 2), recently introduced by Arzarello (2019), is a form of reasoning that consists in suitably envisioning the relationships in which not only variables but also parameters are involved mathematically. In particular, the latter allow to represent classes of real phenomena as families of relations between variables, which are mathematically represented by specific parameters. These determine the peculiarities of the mathematical model. Furthermore, the label “second-order covariation” seems particularly suitable to underline the role played by parameters: indeed, Bloedy-Vinner (2001) already used the expression “second order function” to address those functions whose argument is a parameter and whose output is a function or an equation depending on a specific parameter value.

The necessity of introducing this new order of covariation emerges predominantly from the need of better describing students’ reasoning when dealing with situations of mathematical modelling, such as the law of the inclined plane as presented in Arzarello (2019). However, it also emerges in other studies in literature. For example, Hoofkamp (2011) introduced the term “metavariation” to refer to a variation of the mathematical situation itself and the function as a whole: it is related to the object view of the function and to a qualitative view of the functional dependency and its local and global characteristics. While our approach to second-order covariation mainly focuses on the mathematical and cognitive characterization of this form of covariational reasoning, metavariation emphasizes the instrumentation of second-order covariational processes adopting suitable activities mostly designed with Interactive Geometry Software. This way of reasoning covariationally seems to be more cognitively demanding than the one presented in the already existing taxonomy. Hence, this contribution supports and contributes to the perspective of an enlargement of the theoretical framework that aims at coherently including more complex forms of covariational reasoning and opens up to the possibility that other orders of covariation may exist.

Method.

Participants and task design. The episodes presented and analyzed in this paper are part of a teaching experiment that was held in a 11th grade classroom of 21 students in a scientific-oriented school in Italy. It was conducted at the beginning of the school year 2020/2021 in a mixed modality due to Covid pandemic. Ever since the first year of secondary school the students involved were accustomed to working with different mathematical representations (algebraic, graphical, numerical) and to using Dynamic Geometry Softwares such as GeoGebra. Moreover, in the academic year 2019/2020 the students had already been involved in a teaching experiment concerning the law of the inclined plane, the so-called Galileo experiment (Bagossi, 2021), so they had already experienced covariational reasoning in a context of mathematical modelling. This experimentation had the main purpose of elaborating mathematically on the relationship between temperature and humidity adopting multiple representations and various digital supports. The first task consisted of analyzing the mathematical and physical relationship existing between temperature and humidity through a household experiment during which for an entire day, at regular intervals, students collected the values of temperature and humidity in their rooms with the request to eventually elaborate some hypotheses on a possible relationship between the collected data. In a subsequent lesson, students' hypotheses were discussed, and the teacher commented on a GeoGebra graph displaying some of the data collected. The students, together with their teacher, concluded that the two magnitudes could be related in some way, namely whilst the temperature increases, the humidity decreases, and vice versa. The activity continued with an experiment in class: in a metal pot with some water at room temperature, ice was gradually added until droplets of water appeared on the outside of the pot. At regular intervals of time the students recorded the time spent and the temperature of the water in the pot itself and took note of the temperature when the dew drops appeared (that is the dew point). After a working group session devoted to the understanding of a real psychrometric chart, through a classroom discussion (Discussion 1) led by the teacher, the students investigated the relationships between time, water temperature, room temperature and the reason for the creation of dew drops on the outside of the pot. At the end of the discussion, the teacher provided students with a slightly simplified psychrometric diagram created with GeoGebra (Figure 2a) and a worksheet with a real chart (Figure 1a).

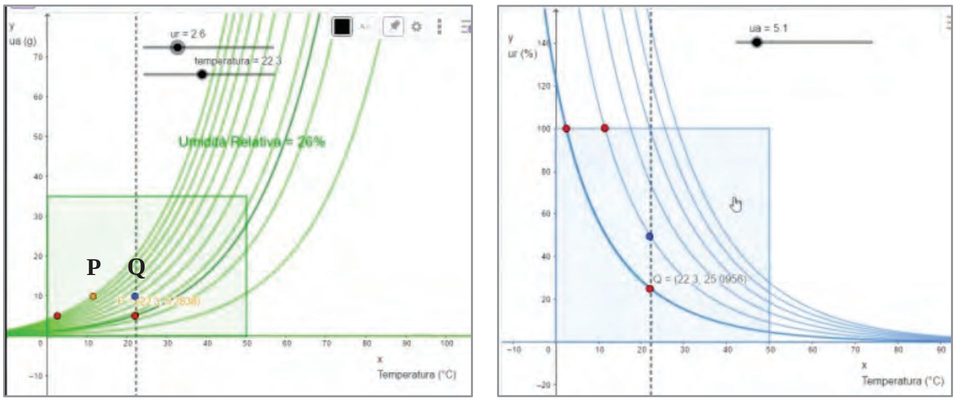

Figure 2: a) GeoGebra applet simulating the psychrometric chart; b) GeoGebra applet describing the relation between relative humidity (y-axis) and temperature (x-axis)

Working in small groups on some guide questions, the students were able to read relative humidity values from the graph representing absolute humidity as a function of temperature and, conversely, to read the values of absolute humidity from a new graph, provided in a second moment, representing relative humidity as a function of temperature (Figure 2b). At the end of the group work, the students discussed with the teacher (Discussion 2) the roles of the magnitudes at stake in the mathematical representations (independent variable, dependent variable, or parameter). We note that the multiple representations involved, such as the real psychrometric chart and the two applets have a complementary function, while the pot experiment with respect to these provides extra information but it will reveal crucial for students to disentangle the meaning of the graphical representations.

Methods of data analysis. The analytic method used in our qualitative study was that of a descriptive coding of the emergent forms of covariational reasoning (Saldana, 2015). A preliminary phase consisted of watching the videos of all the lessons repeatedly in order to identify episodes revealing covariation. During the first cycle of coding, we classified the orders and levels of emerging covariational reasoning describing their features and the quantities involved; then we transcribed the most significant episodes. During the second cycle of coding, we deepened the qualitative description, focusing on the identification of the representations that most influenced the students' reasoning, i.e., the pot experiment or the graphical representations, the use of a qualitative or quantitative narrative, and terms denoting change and movement. Eventually, we revised the coding in light of the qualitative descriptions of the episodes we had elaborated.

Data analysis.

The first episode analyzed here is an excerpt from the 1-hour discussion in presence (Discussion 1) led by the teacher after the working group session on the psychrometric chart. During this episode, students, guided by the teacher, tried to retrace what happened during the experiment with the pot on the psychrometric diagram. The applet reproducing the psychrometric chart was shown on the interactive whiteboard and students had already identified point P representing the point on the saturation line in which the temperature coincides with the temperature of the dew point. Point Q instead has the same ordinate as P and the ambient temperature as abscissa (Figure 2a).

1	Teacher	On the graph, how can I read these passages? We said this is the starting point because we said we do not go from P to Q, but we start from Q. Starting from Q, where did we go?
2	Giorgia	We decreased the temperature hence we moved to the left.
3	Teacher	We decreased the temperature hence we moved to the left. In which way? Did you just decrease the temperature or not? We are during the moment in which you continued to pour and pour [the ice].
4	Emanuele	Only the temperature decreases.
5	Teacher	Only the temperature decreases. And so, on the graph, how do you move?
6	Emanuele	Horizontally.
7	Teacher	Horizontally. We have point Q and we move horizontally to decrease the temperature. Until when do we move horizontally?
8	Emanuele	Until the dew point.
9	Teacher	Until the dew point that is until when we find on which of these green curves?
10	Emanuele	Until that of 100%.

All the episode is centered around a game of displacement between the graphical representation and the experiment facilitated by the mediation of the teacher that constantly asks the students how they would move on the graph, inviting them to relate to the experiment with the GeoGebra applet. This game of displacement results in the interlacing of two different narratives: a qualitative one, used to describe what happened during the experiment that manifests with the use of personal pronouns (e.g., "*we* decreased the temperature" [2]); a quantitative one used to describe how the magnitudes involved are changing (e.g., "Only the temperature decreases" [4]). The students already possess the psychrometric diagram representing the covariation of magnitudes involved and they are moving on this representation: we can recognize the enhancement of a global approach supported by the involved representations.

Discussion 2 was mainly focused on reconstructing the cycle of the pot experiment on the new chart, the one describing the relationship between absolute humidity on the y-axis and temperature on the x-axis. In this second episode, one of the graphs made by the students as homework is shown on the interactive whiteboard (Figure 3). The teacher is commenting on the second step of the experiment the one during which students continued to decrease the temperature in the pot by adding ice cubes and then some drops of water formed on the wall, represented by a horizontal segment, colored in yellow.

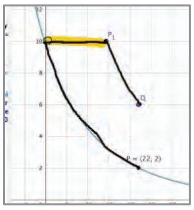

Figure 3: Relative humidity – temperature graph made by one of the students

11	Teacher	What is happening instead on the horizontal trait [of the graph]?
12	Arianna	The relative humidity maintains constant.
13	Teacher	The relative humidity maintains constant.
14	Arianna	And the temperature decreases.
15	Teacher	The temperature decreases. The absolute humidity? Does it decrease or remain constant?
16	Adele	Decreases.
17	Teacher	Decreases. Why?
18	Emanuele	You have the condensation.
19	Teacher	Ok, you have the condensation, and this is what happens practically. But on the graph why? […]
20	Adele	The curve changes.

As in the previous episode, students' linguistic expressions "The relative humidity maintains constant" [12] and "You have the condensation" [18] suggest that the combined and synergic use of the two representations, the chart and the experiment, helped students in reflecting on which magnitudes change and how they change during the different steps of the experiment and the corresponding traits of the chart. Even in this case we can observe the enhancement of a global approach supported by the involved representations.

Finally, this third episode below, coming also from Discussion 2, shows the approach used by the teacher to reflect on the similarities and differences between the two psychrometric charts and it can be intended as a form of conceptualization of the different roles of variables and parameters.

21	Teacher	Are they two different situations/scenarios?
22	Matteo	No.
23	Teacher	No. Why do the two graphs are different if they are not two different situations?
24	Matteo	The value represented on the y-axis is different.
25	Teacher	The value represented on the y-axis is different. If you should make a comparison with something that is not mathematical but concerns real life… We have the same situation/scenario, but the value represented on the y-axis is different… If you should make an analogy…?

26	Giorgia	From the physical point of view, they represent the same thing but from the graphical point of view no… because they are two different values.
27	Teacher	Oh! From the physical point of view, they represent the same thing but from the graphical point of view no because they are two different values. […] Two different situations depending on what?
28	Matteo	A different point of view.

At this point of the discussion, the teacher projects on the interactive whiteboard a new applet showing simultaneously the relationship of absolute humidity versus temperature (Figure 2a) and of relative humidity versus temperature (Figure 2b). The teacher asks to the classroom if the graphs are two different situations or scenarios [21]. Matteo obviously replies no [22], so the teacher asks why the graphs are different if they are not representing different things [23]. When Matteo states that the value represented on the y-axis is different [24], the teacher invites the students to provide a holistic interpretation, an analogy with something non-strictly mathematical [25]. Giorgia suggests that from the physical standpoint the situation is the same: what differs is the graphical representation [26]. Answering a teacher's question [27], Matteo adds that the difference between the two situations depends on "a different point of view" [28]. This interpretation suggests that the different role assumed by variables and parameters does not determine a different situation but a change of standpoint resulting in a different graphical representation.

Discussion.

After a first overall reading of the three episodes presented in the previous section, the question arising spontaneously is: where is covariation? If we think of some typical examples of covariational reasoning such as "A increases, while B decreases", they are surely absent in the excerpts described before. The psychrometric chart at disposal of the students synthetizes in a unique diagram and flattens in two dimensions the relations between three different magnitudes (temperature, absolute humidity, and relative humidity). The students implement forms of reasoning revealing a global approach supported by the adopted representations. In particular, in the physical interpretation of the relations described in the chart, students are deeply supported by the classroom experiment with the pot: all the classroom discussion develops through an interpretation of the diagram with respect to the various steps of the experiment. What the students observe in the last episode is that considering relative humidity as a variable or a parameter does not change the situation, but the perspective with which you look at the situation: the students claim that the relationship is always the same, only expressed in different terms. Second-order covariation is the construct that enables one to read the same mathematical situation from two different standpoints and to recognize a correspondence between the representations: in the real psychrometric chart the parameter is the relative humidity, and this magnitude is the one of second order; in the blue graph (Figure 2b) absolute humidity becomes the parameter and so it is the second order variable determining a different standpoint.

As it is for COV 1, second-order covariation is a cognitive act that cannot be reduced to the reading of a formula choosing to interpret one of the variables involved as a parameter. It is a form of reasoning that consists of envisioning the invariant relationships in a family of functions and this study sheds light on one of its facets that manifests when dealing with graphical representations.

Finally, even if it is not the focus of this paper, we cannot fail to observe the essential role of the teacher in mediating between the multiple representations helping students in relating them.

Moreover, the teacher supports students in better expressing their thoughts and enhances the transition from COV 1 to COV 2, and also from COV 2 to COV 1.

Conclusion.

Second-order covariation requires a complex cognitive engagement: it is a form of covariational reasoning that in activities of mathematical modelling involving multiple representations can be theorized as the ability to read the same mathematical situation from different points of view choosing each time which of the involved variables should be mathematically interpreted as a parameter. The characterization and relevance of second-order covariation still deserve to be deepened through further research.

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From a mathemagic trick to number bases: using cryptography and the software WIMS to stimulate algebraic thinking in primary school

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Keywords: Primary school mathematics, algebraic thinking, educational software.

Introduction

Kieran (2004) suggests that early algebraic thinking is linked to problem solving, modeling, working with generalizable patterns, justifying and proving, making predictions and conjectures, analyzing relationships, and identifying structure. Many studies reveal that algebraic thinking can be developed at an early stage, as long as it is stimulated by appropriate interventions by the teachers (e.g. Blanton & Kaput, 2005). Indeed, the application of student-active methodologies plays a fundamental role in the teaching process, especially in mathematics, as described by Schoenfeld (1992). In this view, our first aim of research is to provide support to improve teachers' practice toward engaging pupils in algebraic activities. We aim to achieve this goal by designing and sharing ready to use materials, which are realizing a *problem-solving approach* and involve modern technologies. The resulting effect on students' learning is our second aim of research: to examine if this type of instruction truly affects the pupils' capacity for justifying and proving.

We present an activity based on the modern cryptography technique of Hamming codes for error correction that is built around a magic trick and is supported by the software WIMS (WWW Interactive Multipurpose Server, e.g., see Cazzola, et al., 2020). The various tasks involved push the students into a learning path that can be described in short as follows:

- Participation in the magic trick
- Question: "How does it work?" (*problem solving, predictions and conjectures*)
- First analysis of the trick (*analyzing relationships and identifying structure)*
- Sharing of observations (*justifying and proving*)
- Conclusion (*generalizations*)
- Bridge to the study of number bases.

The last step is an added value of our activity: students approach a typical algebraic topic associating it to a fun and intriguing game, strengthening their motivation to learn about it. A detailed description of the experience can be found in (Cazzola & Grazian, 2021).

Description of the activity

The game we present is called "7 questions and 1 lie". A volunteer is asked to think about a number from 0 to 15 and to answer 7 yes/no questions about it, with the chance to lie only once. Looking at the answers, the mathemagician will be able to guess not only the mysterious number, but also if there was a lie and the question it corresponds to. Once the pupils' attention is caught, the real work starts, leading the students to find an answer to the obvious question "how does it work?".

The solution involves the use of a particular scheme (Figure 1) containing 7 numbered circles. Each circle corresponds to the question with the associated number and the instruction is to blacken it if the answer is “yes” and to leave it blank otherwise.

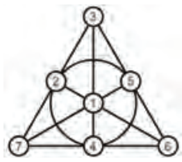

Figure 1: Scheme to solve the “7 question and 1 lie” magic trick

The analysis of the patterns will reveal the solution. Also, if we impose that a blackened circle corresponds to 1 and a blank one corresponds to 0, once the lies are detected and the coloring is fixed, the expression of the mysterious number in base 2 is given by the sequence of 0’s and 1’s given by the first 4 circles. This unusual approach to the study of number bases captures students’ attention and motivates the pupils in discovering more about this algebraic topic. Our activity can be presented using pen and paper only but becomes even more interesting if supported by its digital version, which we elaborated using the software WIMS. The dedicated WIMS page contains the magic trick, a step-by-step solution and a series of games aimed at practicing the description of numbers using different number bases.

Our experience

We presented the activity to four classes of 8-9 years old Italian students, for a total of 78 pupils, in Spring 2021. We conducted classroom observations and collected written students’ feedbacks. During the activity we recognized that pupils were actively engaged and carried out various examples of algebraic reasoning (especially in the form of making and motivating conjectures). This encourages us to further explore this approach and to deepen our investigation on its effects on pupils’ argumentation abilities. Furthermore, we could observe that pupils and teachers reacted enthusiastically to the tasks, strengthening our belief that this kind of proposal can also contribute to the diffusion of problem-solving and student-centered methodologies in primary school education.

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Potential semiotic synergy in *TouchTimes* multiplication

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Abstract: Inspired by the notion of synergy, the sharing of signs between tangible and digital artefacts, we explore signs created by students that emerge through their use of TouchTimes (TT), a multitouch iPad application. We use Peirce to study the tripartite function of the signs (symbol, icon, index) available, as a way to study potential synergies in TT, and how they compare to signs found in other visual models of multiplication.

Keywords: Multiplication, visual representations, TouchTimes, synergy, semiotics, multitouch.

Introduction

Figure 1 shows a drawing made by a grade three (9-year-old) student who had been asked to draw what it would look like to solve the following problem using the multitouch application *TouchTimes* (TT) (Jackiw & Sinclair, 2019): “A bunch of buttons fell on the floor. Nick gathered them in heaps of eight buttons. He made five heaps. How many buttons are there?” We wanted to use this drawing to motivate our interest in exploring the nature and status of visual images used in the teaching and learning of multiplication. The drawing has familiar aspects to it, such as the unit-of-units (of circles) drawn on the right, which can be found in Davydovian (1992) representations of multiplication. It also contains unusual elements, such as the fingers on the left or the two-part structure, with the vertical line down the middle. The left side illustrates the original unit of 8, which is produced in TT by pressing eight fingers on the screen (even though there are ten on the drawing!). And the right side shows the symbolic expression as well as the unit (large lasso) of 5 units of 8 circles.

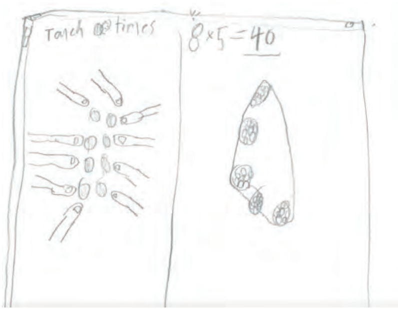

Figure 1: Student drawing of 8 × 5

The complexity of this drawing and its relation to the dynamic, tangible tool prompted us to wonder how it relates to commonly-used visual images of multiplication. Given the multiple ways there are of conceptualising multiplication, and the semiotic differences between them, we are interested in exploring how the use of TT might support the kind of *synergy* described by Mariotti and Montone (2020). We show that the material nature of Peirce’s indexical signs helps explain this synergy.

Visual models and representations of multiplication

Multiplication is a challenging concept for learners and researchers have advocated for the use of visual models or representations (Davydov, 1992). Kosko (2020) uses multiplicative scheme theory to argue that the development of multiplicative reasoning in children moves from counting by ones to conceiving of conceptual multiplicative representations. He argues that visual conceptual multiplicative representations can promote or limit engagement in multiplicative reasoning. However, there are a variety of ways to conceive of multiplication and each representation will stress some elements over others.

Maffia and Mariotti (2018) distinguish two models of whole number multiplication, the repeated-sum model and the array model, which are represented, respectively, in Figure 2a and 2b. They show how different properties of multiplication can be justified using each model. They also acknowledge that while the former model is pervasive in school mathematics, it has limitations, and that many researchers have argued for increased use of the latter model—particularly to support students' understanding of multiplication as a binary operation (in contrast to the unary operation of addition).

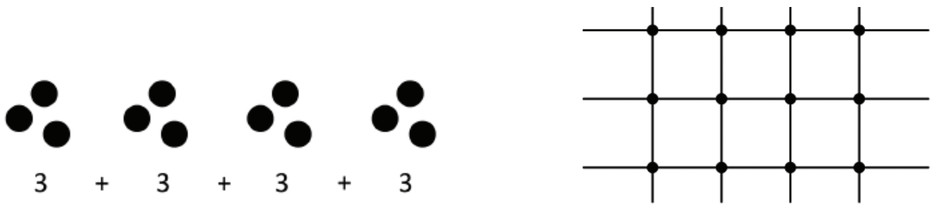

Figure 2: (a) Repeated sum model; (b) Array model (in Maffia & Mariotti (2018))

For example, Clark and Kamii (1996) use the image shown in Figure 3a to contrast additive and multiplicative thinking. While the top representation draws on the repeated sum model, the bottom one highlights the two levels of inclusion involved in multiplicative thinking—both the horizontal one and the vertical one. Looking at the vertical component of the representation, there is a *many-to-one* relation expressed by the three arrows between the bottom row and the middle one; each of the three single points is moved to the next level where it is conceived of as a single unit. Boulet (1998) reformats the Clark and Kamii representation using a Davydovian approach (Figure 3b). Davydov bases multiplication in measurement so that multiplication is related to a transfer of units. According to Davydov (1992), unitization is when "a person purposely alters a unit of count" (p. 10). The horizontal encasing that was visually expressed in Clark and Kamii is absent. This focuses attention to the many-to-one relation, or, in Davydovian terms, unitising: three becomes one.

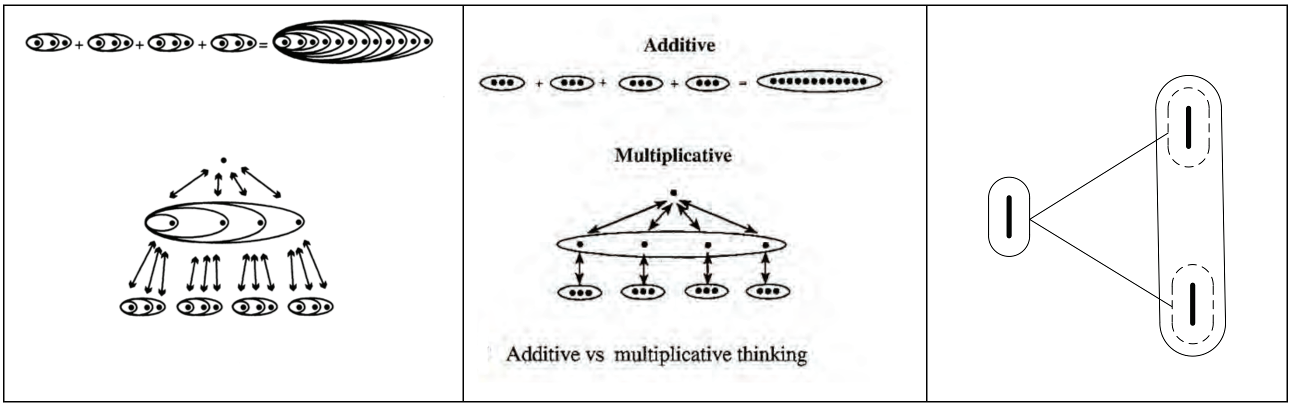

Figure 3: (a) The inclusive model; (b) The unitising model; (c) The splitting model

Confrey's (1994) splitting model (Figure 3c) depicts an original unit, which is then "split" to create duplicate versions of itself simultaneously, which contrasts with the sequential repetition of repeated sum model. The splitting model highlights a *one-to-many* relation in addition to equal grouping.

Finally, Vergnaud (1988) proposes a t-table (Figure 4) as a way of highlighting the four quantities involved in multiplication. He distinguishes two relationships: scalar and functional. The scalar relationship (vertical) is the ratio between two values of a variable: for example, the ratio between the number of cars is 5 : 1, which is the same as the ratio between the number of tires 20 : 4. The functional relationship (horizontal) is the ratio between two values of two distinct variables. For example, the functional relationship between the number of cars and the number of tires is 4.

The number of cars	The number of tires
1	4
5	x

Figure 4: The t-table functional, quarternary model of multiplication

While each of the authors cited above argues for the advantages of their model, we are interested in the question of how these various models might be complementary—if they each emphasise different aspects of multiplications and if student flexibility is valued, then surely exposure to multiple models if desirable. However, little research has focused on how students handle these different models.

Theoretical Framing

We adopt a semiotic approach drawing on the work of Peirce, who argues that understanding is based in signs. He distinguished three types of signs (icon, index, symbol) which differ in terms of the nature of the relationship between the signified and the signifier. Icons operate according to likeness or resemblance between the signifier and the signified. If iconic signs become conventional codes within particular cultures, they may become symbols, which have an arbitrary relationship with that to which they refer, as is the case for most spoken language as well as numbers and signs such as + and –. Figure 2a combines symbols (such as '3') and icons (the discs that resemble objects such as tokens). In Figure 2b, there are iconic elements in that the lines and intersections resemble either a rectangle or a cartesian coordinate system, but for some people, the image operates more like a symbol, if there is no resemblance apprehended. Indices are quite different from the other two in emphasizing the material link between signifier and signified.

Unlike icons and symbols, indexical signs are bound to the context in important ways—they "show something about things, on account of their being physically connected with them" (Peirce, 1894/1998, p. 5). As in the case of smoke billowing from a chimney indicating that the fireplace is in use, the smoke *indexes* the fire. The index is a sign that is materially linked or coupled to "its object". According to Peirce (1932), an index

> refers to its object not so much because of any similarity or analogy with it, (…) as because it is in dynamical (including spatial) connection both with the individual object, on the one hand, and with the senses or memory of the person for whom it serves as a sign, on the other. (2.305)

Verbal indexical signs perform this link to context very effectively, for instance, terms like "this" or "that" or even pronouns like "she", are meaningful only in relation to a given context. The arrows in Figures 3a and 3b act as indices in that they materially couple the three dots to the single dot; they are not similar to the transfer, but they index it. Similarly, the dotted lines in Figure 3c index a transformation of the object from one that has been split (copied) to one that belongs to the product.

Drawing on the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008), Mariotti and Montone (2020) use the notion of *synergy* to refer to the common signs that emerge from students' interactions with different artefacts, which are related to the same mathematical concept. In their study, Mariotti and Montone had students translate and reflect a concave quadrilateral, first with paper and pins which fold and anchor, followed by a similar activity in a dynamic geometry environment. Mariotti and Montone argue that in the sharing of two artefacts, the new meanings emerging from the two artefacts synergistically interweave with each other to "reach the meanings that were intended" (p. 118).

We are interested in the synergy that occurs within the use of one single artefact, TT, in which there are signs that can be associated with different conceptual models of multiplication. This was exemplified in Figure 1, which shows the double unitising of the circles *as well as* the fivefold 'copying' of the 8 circles on the left. We would like to understand how this kind of synergy can arise and hypothesise that this can be done by studying the types of signs (in Peirce's typology) produced in TT. In other words, in order to identify opportunities for synergy, we analyse the TT environment in terms of its different signs, looking to see how they might relate to the different representations described in the previous section.

TouchTimes: A brief introduction

When first opened, TT is divided in half by a vertical line (Figure 5a). Whichever side of the screen is first touched by the finger(s) of the user's hand, results in a different coloured disc (termed a 'pip') appearing beneath each finger and the numeral corresponding to the number of visible pips being displayed at the top of the screen (Figure 5b). The numeral adjusts instantly when fingers are added or removed, whether temporally in sequence or simultaneously, and represents the multiplicand. In order to preserve each pip (that otherwise vanish), the finger(s) must maintain continuous screen contact. When a user taps on the other side of the line with her second hand, bundles of pips (called 'pods') appear with each contact (Figure 5c). The number of pods is the multiplier and each pod contains a duplicate of the pip configuration, matching both the relative locations created by the pips and the colours of those pips (Figure 5d). When pods are created, a second numeral also appears, separated from the first by the multiplication sign ('×'). If the user presses the array icon on the top right corner of the screen, the pips rearrange themselves into rows and the pods into columns.

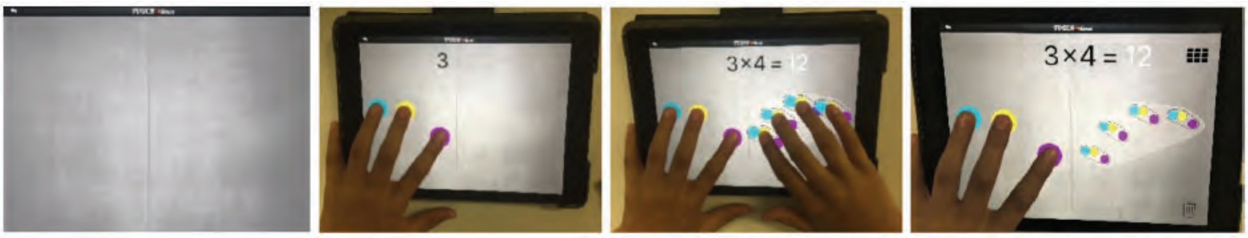

Figure 5 (a) The initial screen of TT; (b) Three fingers placed on the left; (c) Four fingers placed on the right; (d) Lifting the four fingers to see the unit of pods

Although TT provides a tangible, dynamic environment for creating multiplicative relations, and therefore differs significantly form the static representations shown above, it also aims to highlight certain aspects of multiplication and provides multiple opportunities for making different signs. The design of TT draws extensively from Davydov's (1992) model, with the first action being the creation of a unit, and the second action between the creation of a unit of units. Practically, this means a reversal of the repeated sum model since the unit size comes first—and therefore, we would read Figure 5d as 'three, four times' rather than as 'three groups of four', which is how the expression 3 × 4 is described in most North American classrooms. The array icon transforms the unit-of-units display into an array, which turns the repeated sum representation into a unit by unit array.

In addition to emphasizing *unitising*, TT also emphasizes the *splitting* idea in the sense that the 'copies' (literally, multi-plying) are produced simultaneously (pod-fingers touch the screen all-at-once). There is also the one-to-many relation that is emphasized when a pip finger is lifted or pressed since a change in pip on the left produces a change in each and every one of the pods on the right—we refer to this as a *spreading* idea. Finally, multiplication in TT is expressed as a *co-varying* operation between pips and pods (embodied by the use of two hands), with either quantity being dynamically changeable and visibly affecting the other.

In terms of *embedding*, the sequence needed to produce a multiplicative statement is shown in Figure 6. These four steps parallel the quarternary aspect of Vergnaud's description of multiplication (see also Askew, 2018). That is, there are four numbers involved in multiplication. In the first step, there is the determination of the number of pips, this is a variable amount, that is, it can change (at any time). The second step is to see those pips as a unit, or a pod ("1"). This idea is expressed in TT by embedding the pips by a border. The third step determines the number of pods, this is also variable. And the final step is to see these multiple pods as a unit, the product. Again, the four numbers are the number of pips, one pod, the number of pods, and one product.

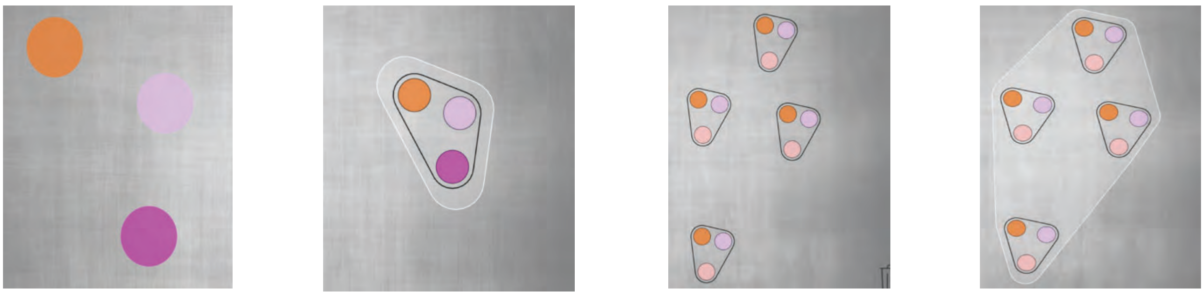

Figure 6 (a) Three pips; (b) A unit of three pips (a pod); (c) Four pods; (d) A unit of four pods (a product)

A semiotic analysis of TT

In terms of the signs at play in TT, there are numerous similarities and differences with the models we have described above, as well as significant polysemy. Some similarities include the use of circular (points, discs) signs to represent objects; the use of encircling to represent sets of objects; and, the combination of number and operation symbols with icons and indices. TT differs significantly from the other representations in three ways: (1) the use of colour; (2) the tangibility—the representations are seen *and* touched; and, (3) the temporality.

Colour is used to emphasise the idea of copying found in the splitting model, when three differently coloured pips created on one side of the screen are replicated on the other. Colour thus enriches the iconicity of 'object to the split'. Groups or units can therefore differ internally, unlike with the circular discs, repeated sum or the dots of the unitizing models. While the number of pods indexes the functional relationships of Vergnaud, colour indexes the scalar relationship (1 : 3). Figure 7 shows two drawings made by grade three (9-year-old) students in response to the button question (see introduction). These students had used TT during one class period and the teacher proposed the button question in order to see whether the students could connect a more traditional multiplication problem with their experiences using TT. We use these drawings simply to exemplify how TT signs might be re-used by learners. In Figure 7a, colour is used to index the split of the original unit into 5 pods.

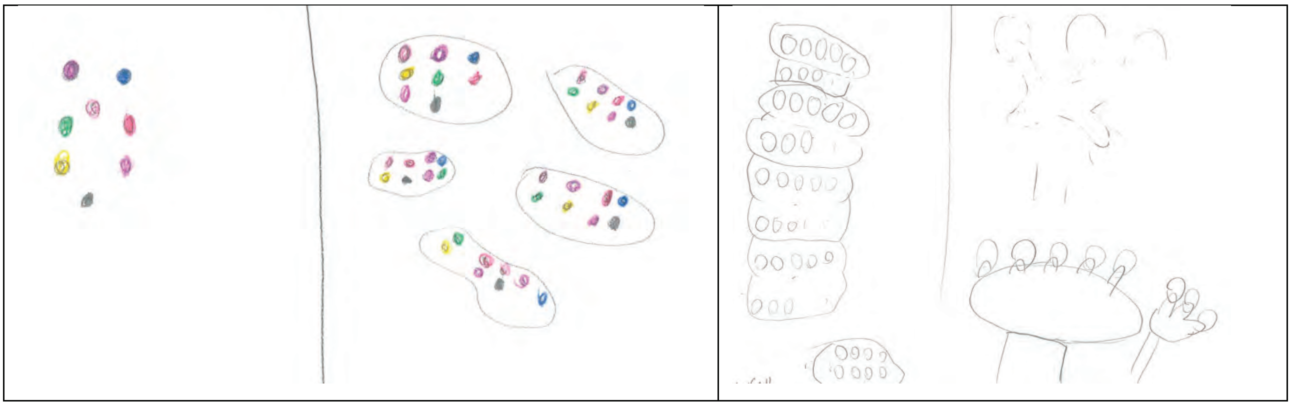

Figure 7 (a) Coloured pips; (b) 5 pods and 8 fingers

Unlike the iconic status of the circles in the repeated sum and the unitizing models, the coloured discs in TT have an indexical quality since they literally index the touch of fingers. In addition, the colour and the shape of the pods serve to index the placement of the pip-fingers. Colour and shape thus perform the same indexical function as the arrows in Figure 3a and 3b, but through different signs. The drawing of the hand in place of the pips on the right side of Figure 7b signifies the indexical relation between the use of one's fingers and the internal make-up of the pods drawn on the left. Similarly, as mentioned above, the splitting model uses a dotted line to index the double nature of the split element as both a copy of the original and as a part of the product. The encircling of the pod on its own and then of the pods altogether seems to have a similar indexical function.

The temporal aspect of TT introduces yet other difference. For example, if one focuses only on the final product, as in the image shown in Figure 5d, the meanings of the signs are different from those produced by the sequence shown in Figures 5a, 5b, 5c. The right-hand side of Figure 5d resembles the repeated sum image shown in Figure 2a and, indeed, could be seen iconically as four groups of three. But in the temporal production of the multiplicative expression, the unit of 3 is produced first, and then copied four times, so that the sign includes both the image on the left *and* the image on the right—it functions indexically as a trace of the finger touches.

There are two other aspects of temporality that functions in tandem and both involve the potential action of fingers. First, the potential to lift or press a pip-finger functions as an index in that it literally "shows something about things", which is that changing the number of pips will change each and every pod—which is the one-to-many idea that is represented in Figure 8a and aligns with Boulet's model. In prior research (Bakos & Pimm, 2020), the repeated lifting and pressing of fingers has been described as "dancing" (because of the simultaneous rhythmic change of all of the pods at once), and is a common mode of interaction for learners. In this dynamic dancing gesture, attention is drawn to a fundamental element of multiplication, that of spreading. Spreading is articulated in the work of Confrey (1994) who first proposed an alternate multiplicative metaphor based on children's facility with the action of splitting. Dancing draws attention to the horizontal embedding of Clark and Kamii, which hints towards the general since the iterative pattern of the embedding can be easily extended. This possibility of horizontal variation emphasises both the horizontality and verticality of multiplication as a reference transformation.

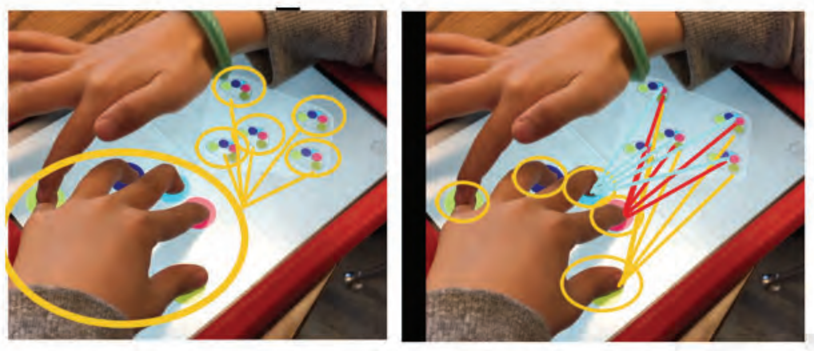

Figure 8: (a) Focus on pods; (b) Focus on pips

The second aspect of temporality is the potential to remove or add any pod (by lifting or pressing a pod-finger) which shows that changing the number of pods will change the product. This is the many-to-one idea represented in Figure 8b, which aligns with Clark and Kamii. Figure 8 highlights an additional polysemic aspect at play in TT.

Our analysis shows the different ways in which TT shares signs with other representations, even though these are quite distinct kinds of artefacts from one another. Focusing on the signs created helps explain how TT might function as an environment that promotes semiotic synergy. In addition, the temporality and tangibility of TT elicit more indexical signs, in lieu of iconic ones, which establishes a dynamic connection between the object and "the senses or memory of the person for whom it serves as a sign", to re-quote Peirce.

Conclusion

In this paper, we have analysed different visual representations of multiplication (which draw on different mathematical models of multiplication) that have been discussed in the research literature, including the repeated-sum model, the array model, the unisiting model and the splitting model. Our goal was to identify the different aspects of multiplication emphasised in these representations so that we could compare them semiotically to *TouchTimes* and also explore the potential for semiotic synergy found within TT. Our analysis revealed that the signs in TT are polysemic, indexical and multiple. While this might lead to conceptual confusion (see Izsák & Beckmann, 2019), we argued that it increased the potential for synergy. The three drawings offered in the paper suggest that students can in fact work with the multiple meanings, and can choose to use some more explicitly than others. The presence of the hands in two of the drawings strengthens our hypothesis that the greater indexicality of TT signs can better support embodied understandings of multiplication.

Acknowledgment

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Domestication of the geometrical eye: unpacking geometry with the GGbot drawing robot

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In this paper, using Radford's Theory of Objectification, we analyze 2 students' learning activity that show how the features of the drawing robot (GGbot) affect student's sensuous cognition, forging their theoretical perception for the learning of geometry.

Keywords: Eye as a theoretician, GGbot, geometry, theory of objectification, sensuous cognition.

Introduction

Digital technologies open new possibilities in the teaching and learning of mathematics. In line with the theory of objectification (TO), we conceive learning as the student's sensuous encounter with systems of thinking and action that have been historically and culturally constituted (Radford, 2021). In this paper, we analyze the role of the drawing robot GGBot (abbreviation of GREATGeometryBot) in the learning of geometry in primary school. We report an exploratory study carried out with grade 3 students performing an activity that involved the use of the GGBot in the solution of geometrical tasks. The aim of the contribution is to understand the impact of the GGBot in transforming the individual's perception (which here we intend being visual, tactile, kinesthetic, and imaginative) in the learning of geometry.

Conceptual framework

The TO, embedded in sociocultural perspectives, stems from a profound intertwining between culture and the individuals' activity. Culture is an intrinsic component of mathematical thinking and learning, and activity is the ontological category of the TO as it realizes the consubstantiality between individuals and their culture. In the stance of the TO, mathematical thinking and learning are not processes confined in the mind but they are intertwined with individuals' social activity. Signs and artefacts play an important role, beyond the role of something that stands for something else or as mediators of activity. In the TO, they are considered an integral part of human thinking and human activity (Radford, 2021). The issue of learning is rooted in the dialectics between the individual and their culture. Learning is a movement pushed by the intrinsic differential between the individual and cultural knowledge. In fact, in attending to knowledge the student has to cope with something that in the beginning is different from him, an alterity that challenges, resists and opposes him. Learning is the process that erases such a difference to make sense of cultural knowledge and transform it into something familiar that allows new forms of action, thinking, imagination and feeling. In order to reduce the distance between the individual and cultural knowledge, activity as a specific human endeavor is required on the part of the student. Radford (2021) conceives learning as a social process of becoming aware of cultural-historical systems of thinking and doing, through our bodily, sensory, and artefactual semiotic activity. We remark that, according to the TO, signs and artifacts are constitutive of the activity that leads students to notice mathematical knowledge. They are bearers of an embodied intelligence and culturally endowed with specific patterns of activity that individuals use in their meaning-making processes and to carry out their actions (Radford, 2021). The outcome of the learning process is the encounter with mathematical (cultural) objects and their transformation into objects of consciousness. From this standpoint, learning has a strong phenomenological nature where noticing occurs in an enlarged notion of mind and consciousness, termed by the TO sensuous cognition, that includes not only ideal and mental features but also embodied ones such as perception, feelings, and kinesthetic activity. In light of the dialectic-materialist approach underpinning the TO, the basic tenet behind the notion of sensuous cognition is that the body, the senses, and the objects of sensation are not a priori entities but mutually transformed by cultural-historical activity entangled with the use of signs and artefacts. The relations of mind and body to the world are historical intertwines with material and ideational culture, and our senses change and develop along with the changes of the cultural-historical dimension (Radford, 2021). From the standpoint of sensuous cognition, human perception is, in the words of Wartofsky (1984, p. 865), “a cultural artefact shaped by our own historically changing practices”. In this regard, perception deploys cultural forms of seeing, touching, hearing etc. that characterize our relation with the world. Within sensuous cognition, the issue of learning is identified with the manner in which perception is transformed into a theoretical cultural form of perception, in progressively noticing and endowing with meaning cultural-historical systems of thinking and doing. How do students change their perception from “spontaneous” forms of attending to objects to a mathematical and theoretical one? To answer this question, we must consider learning as a “domestication of the eye” (Radford, 2021), a long process that allows students, in cultural-historical activity intertwined with the use of signs and artifacts, to transform the eye (and other senses) into sophisticated theoretical organs able to notice and make sense of certain things in mathematical manners - for example recognizing numerosity, algebraic structures, geometric invariants etc. We remark the co-variational nature of sensuous cognition in that learning processes entail a transformation of perception along with the transformation of the perceived cultural object into an object of consciousness. We underline the multimodality entailed with the “domestication of the eye”, both in the various sensorial channels and the richness of signs and artefacts interwoven with cultural-historical activity involved in the transformation of perception (Radford, 2021).

In this framework, we aim to analyze the impact of the GGBot, with its artifactual and semiotic features, which combine the well-known strengths and opportunities offered by the modern visual programming language with those of Papert’s original robotic drawing-turtle (Papert, 1980). The GGBot is composed of two wheels, a marker-holder at each end, where one can insert markers to let GGBot draw (Figure 1a), and SNAP!, the visual programming language used to provide commands to the GGBot. We show some of the available commands in Figure 1b, and we refer to Baccaglini-Frank and colleagues (2020) for more details on how the GGBot works.

a)  b) 

Figure 1: a) view of the GGbot; b) SNAP! Commands list

This contribution focuses on the dialectical movement between primary school students, geometrical knowledge and cultural-historical activity that pivots around the use of GGBot as they learn geometry. More precisely, the present study aims at answering the following research question: how does the GGBot with its related multimodal artifactual and semiotic features (the physical robot, the SNAP! commands, movement, drawing, gesturing, natural language) affect students' sensuous cognition? That is, how does the GGBot "domesticate the eye" (seeing, touching, kinesthetic activity, imagination) for the learning of geometrical figures in primary school?

Methodology, data and analysis

We consider data collected during one session of a sequence of three that was conducted by Anna Baccaglini-Frank (Baccaglini-Frank & Mariotti, in press). Each session lasted a fixed time-frame and involved one class of grade 3 students, the classroom teacher, and the researcher. The students were asked to carry out various types of tasks. First, a technical exploration of the GGBot guided by questions is conducted. Then, students continued working in pairs and they were asked to use SNAP! to code the movement of the GGBot that would lead it to draw with the marker a certain given figure (figure-to-code tasks). The students were involved in another type of task as well, a collective task where they could answer in turn. Starting with a given SNAP! code, they were asked to predict the GGBot's movement and, consequently, foresee the trace that the marker would have left on the paper (code-to-figure tasks). Given the potentialities of predictive tasks in providing insight into the learning process in geometry (Miragliotta, 2020), in this paper, we focus on the code-to-figure tasks. We consider some video recorded sequences where a group of students are predicting the GGBot's drawing outcome of a given code displayed on the projector (Figure 2a). For the reader's convenience, we show in Figure 2b the commands explanation and the expected figure drawn by the GGBot according to the code alongside (what is shown in Figure 2b was not projected nor shared with students during the experiment).

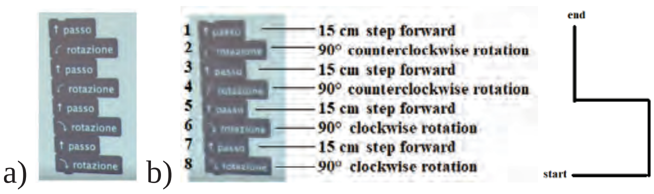
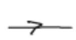
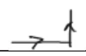
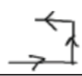


Figure 2: a) SNAP! Commands list (projected); b) Commands explanations and the expected figure

We remark that with the code-to-figure predictive task, students are asked to manage three separate, but connected, passages that entail theoretical forms of seeing, touching, movement and imagination. The first goes from the given code to the prediction of the GGBot movement in their imaginary dimension; the second, from the imagined movement to the prediction of the figure that such movement would trace on the paper; the third, from the imagined trace to its external outcome shared with the use of words, gestures, and drawings. The analysis of the nodes of such a chain of passages allow us to understand how the students' sensuous cognition evolves in the interplays with GGBot features. Our analysis focus is on the mutual transformation of perception and mathematical objects in activities whose outcome is the "domestication of the eye". We consider variables, specific to the task at stake, concerning perception, signs and artifacts, and geometric knowledge: in regard to perception, seeing, touching, movement and imagination; with respect to signs and artifacts, gestures, drawings, natural language and SNAP! commands; in regards to knowledge we focus on the egocentric and allocentric system of references (SoR) – whose involvement in similar activities is well documented in the literature (e.g. Baccaglini-Frank et el., 2014) - the notion of angle and the (mis)matching between the drawing of the figure resulting from the GGBot movement and the one primary school students would have performed using paper, pencil and possibly a ruler. We remark that in the code-to-figure task the GGBot is not physically present as in the figure-to-code ones. Nevertheless, the GGBot, in relation to the previous activities, is present in the imaginative perception as students coordinate gestures, natural language, drawings, and SNAP! commands.

Data and their analysis. The students, the researcher and the teacher are arranged in a circle around a big piece of paper on the floor where there are some drawings of previous activities (familiarization with the robot's functionality and figure-to-code task to draw a square). We focus on 2 students (Angela and Vanessa) and the following tables contain excerpts of the transcription of the video recording significant of the three passages described above (code to imagined movement, imagined movement to imagined figure and imagined figure to external shared the figure). In order to be faithful to the synchronicity in the use of signs and artifacts, we present the data in three columns: one for the gestures and non-verbal signs, the second for the utterances, and the third for the drawings (in Table 1, the drawings are just for the reader's convenience since Angela traced the figure with a finger on the paper). In the transcription: R stands for the researcher, and we numbered the lines using the same number to indicate simultaneity. In the analysis we enumerate both the transcript line (TL) and the SNAP! command line (SL, see Figure 2b).

Angela, who traces the figure with her finger on the paper		
Gestures and non-verbal signs	**Utterances**	**Drawings**
[1] Angela points her index finger on a white area of the paper and then she traces a first segment in the horizontal direction (according to her egocentric SoR) towards her right 	[1] R: like that	[1] 
[2] Angela goes on with a second segment in the vertical direction (according to her egocentric SoR) moving away from herself and articulating the movement in two steps		[2] 
	[3] Angela: ehm	
[4] Angela traces a third segment in the horizontal direction (according to her egocentric SoR) towards her left. This segment is shorter than the first two.		[4] 
[5] Angela traces a fourth segment in the horizontal direction (according to her egocentric SoR) towards her right. This segment is shorter than the first two as well.		[5] 
[6] Angela traces a fifth segment in the vertical direction (according to her egocentric SoR) moving away from herself. Then, she stops moving and gazes at R.		[6] 
[7] Angela gazes at her finger on the paper		

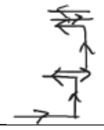
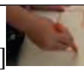
[8] Angela moves horizontally her finger right and left various times	[8] Angela: and like…	[8-9] 
	[9] Angela: and like so and so…	
[10] R points her finger towards Angela	[10] R: So it is an excellent idea. It is a sort of stair, did you see?	

Table 1: Angela

Analysis. Angela traces correctly with her finger the first three segments of the figure (TL 1-4 that correspond to her interpretation of the SL from 1 to 5). When reaching the 6 SL, she is not able to correctly handle the change of direction in the rotation. Therefore, she interprets the change in the direction of the rotation as a reverse direction along the same segment (TL 5). At the 7SL, Angela traces with her finger the vertical segment (TL 6). Then, the last command puzzles her even more and she goes back and forth with her finger (TL 8-9). In her activity, Angela resorts only to gestures and the SNAP! commands. She is able to notice in her imaginative perception the corresponding movement of the GGBot related to the first 3 segments of the figure. Nevertheless, when it comes to the third rotation, her perception is not able to grasp the angle of the figure as it is conveyed by the movement of the GGBot. The coordination of gestures and the SNAP! icons requires a transformation of imaginative perception to notice the angle of the figure in terms of step and rotation of the GGBot and not as the portion of the plane delimited by the two half-lines (i.e. the two sides of the figure). Furthermore, it requires a transformation of imaginative perception able to consider also the connection between egocentric (Angela's) and allocentric (GGBot's) SoR that does not emerge in the use of pencil and paper. Her perception does not encompass the change of direction in the rotation due to both the new way of encountering the angle of the figure as a rotation of the GGBot and the conflict between the egocentric and allocentric SoR. The back-and-forth gesturing along the side of the figure (TL 8-9) and the global absence of structured natural language are tokens of Angela's blurred perception and her struggle in "domesticating the eye" to transform her perception of the angle with respect to the one she learnt before. The process of "domestication of the eye" does not make the necessary leap to handle both the angle of the figure and the SoR, thus missing in the imaginative perception the expected figure drawn by the GGBot corresponding to the SNAP! commands.

Vanessa, who draws the figure on the paper with the marker		
Gestures and non-verbal signs	Utterances	Drawings
	[61] Vanessa: So before she (Vanessa is referring to a classmate's answer in a previous figure-to-code task) did a square, ok? [62] R: ok	
[63] Vanessa draws on the paper a line in the vertical direction (according to her egocentric SoR) moving away from herself.	[63] Vanessa: So, he took a step forward, no?	[63] 

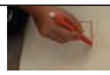
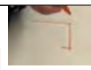
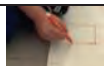
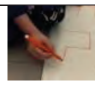
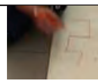
[64] Vanessa draws on the paper a line in the horizontal direction (according to her egocentric SoR) towards her right	[64] Vanessa: a rotation	[64] 
[65] Vanessa separates the marker from the paper and starts to oscillate over the second segment	[65] Vanessa: then another, a ...another step forward, so the rotation…	
[66] Vanessa continues to oscillate repeatedly over the second segment	[66] Vanessa: yes well, the, the step forward	
[67] Vanessa draws a third short segment in the vertical direction (according to her egocentric SoR) towards herself	[67] Vanessa: a rotation	[67] 
[68] Vanessa draws out the line towards herself	[68] Vanessa: and then after the rotation again a step forward	[68] 
[69] Vanessa draws on the paper a line in the horizontal direction (according to her egocentric SoR) towards her right	[69] Vanessa: then he put the rotation the opposite way, and so like this	[69] 
[70] Vanessa draws quickly another two lines	[70] Vanessa: and like this	[70] 

Table 2: Vanessa

Analysis. For Vanessa the conflict starts since the beginning, when she is managing the second SL (TL 64). After she has drawn the first segment, she explicitly links the word rotation with the drawing of the second perpendicular segment. After that, in TL 65, Vanessa should go a step forward with the marker but she is puzzled about where to go due to the previous interpretation of the rotation. Indeed, drawing another step forward in her situation would have resulted in a drawing with a “side doubled” ,i.e., two strokes of the marker (instead of a step, a rotation and then another step). Vanessa’s confusion is highlighted by her oscillating the marker over the second segment and uttering “then another, a ...another step forward, so the rotation…” and “yes well, the, the step forward” (TL 65-66). Vanessa is confused because she lives in a conflict between the two ways in which the notion of angle co-emerges with her sensuous act: the angle as the part of the plane between two half-lines and the angle as the rotation of the GGBot. The gesturing with the marker and the utterances described above, testify such a conflict; she is able to draw the consecutive segments, but, when trying to relate them with the SNAP! commands, she is at odds with what she is doing. In TL 67, Vanessa is managing the second rotation of the 4th SL: she avoids the conflict linking the word rotation with a little portion of a perpendicular segment that she extends synchronously with the words “then after the rotation again a step forward” (TL 68). Vanessa’s use of natural language is always assertive and explicit: she uses words to scan the imagined movement and the resulting trace. Her coordinated use of natural language and the drawing with the marker (TL 67-68) shows in an evident and interesting way the “domestication of the eye” related to the angle and her struggle to erase the differential between her previous form of noticing and the new (GGBot’s) one that is challenging, resisting, and opposing her. We observe Vanessa’s difficulty in fully accomplishing the domestication of her sensuous cognition to “see” angles with the “eyes” of the GGBot. Vanessa handles the coordination of the egocentric and allocentric SoR in the direction of the steps, in fact she always says “forward” as if she were in the SoR of the GGBot. However, she is not able to coordinate the two SoR when it comes to the rotations. She systematically shifts the left with the right and vice versa. Notwithstanding the difficulties in coping with the angle and the two SoR, Vanessa arrives at a drawing consistent, apart from the inversions of left and right rotations, with the SNAP! commands and the ensuing movement of the GGBot. This testifies a first transformation of her geometrical perception to conceive of figures both as theoretically perceived in drawings with pencil and paper and in the entanglement between the SNAP! commands and the ensuing movement of the GGBot in terms of steps and rotations. The coordinated use of the marker to draw and as a pointer, natural language and the SNAP! commands allows a transformation of Vanessa's multimodal perception made of seeing, touching, movement and imagination to encompass new ways of noticing the SoR (egocentric and allocentric), the angle of the figure (resulting from steps and rotations), and the geometric figure (in the interplay between SNAP! commands and the imagined movement of the GGBot). Despite her struggle in coping with new ways of attending to the angle and the SoR, Vanessa's "domestication of the eye" allowed her to connect the two meanings of the figure, the one conveyed via the GGbot and SNAP! and the previous one conveyed via drawings and paper and pencil. Thus, Vanessa testifies in her learning process the mutual transformation of perception and the mathematical object.

Discussion and conclusion

Data show how the encounter of geometrical objects using the GGBot involves a complex intertwining of signs and artifacts (icons, gesturing, natural language, material objects), perception, and geometric knowledge. The analysis of Angela and Vanessa, exposed to code-to-figure tasks, allows us to delineate how the use of GGBot resists and opposes our two students in their process of "domestication of the eye". In previous activities without the GGBot, students' sensuous cognition had been carried out with material objects, rulers, gestures, natural language etc., on which they had direct perceptual and sensorimotor control. Furthermore, perception took place in their egocentric SoR. The introduction of the GGBot strongly transforms students' perception in new cultural and theoretical modes of attending to geometrical objects. Metaphorically speaking, students have to think, perceive, move and "feel" as if they were the GGBot. In the code-to-figure task, pupils do not have direct control on the robot, and they have to establish, in sensuous cognition, a relation between the SNAP! code, the resulting movement of the robot and the geometrical figure that it would have traced. Since the code-to-figure is a predictive task, this happens in their imaginative perception without the physical presence of the robot but forged by the use of gestures, natural language and the sensorimotor activity, inherited by the previous tasks with it. The code-to-figure task suggests to what extent the students' perception has been theoretically domesticated according to the ideal and material characteristics pivoting around the use of the GGBot, to embrace new and richer encounters with geometric knowledge.

Answer to the research question. From a geometrical point of view, above all, students have to handle different SoR (egocentric and allocentric) and angles conceived as a rotation. The predictive code-to-figure task shows that the introduction of the GGBot, with its correlates of signs and artifacts, requires a "domestication of the eye". The transformation of perception is hindered by the conflict between an already theoretically domesticated eye - which encounters a geometric figure in a single SoR, in terms of segments and angles perceived as portions of a plane between two half-lines - and the new GGBot's theoretical eye - which encounters a geometric figure as something constructed in terms of steps and rotations, and the intertwining of the egocentric (student) and allocentric (GGBot)

SoR. In regard to the features of sensuous cognition, on the one hand the introduction of GGBot requires the students to "see" theoretically the figure as a recomposition of the SNAP! commands and the corresponding steps and rotations of the robot perceived visually and kinesthetically. On the other hand, in previous activities with paper and pencil, the students theoretically "see" the geometric figure as successive segments with different orientations perceived visually and kinesthetically as they trace on the paper. Concerning the task under scrutiny, we highlighted, in connection to the deepest conflicts lived by the students, that their "domestication of the eye" does not flow smoothly. The mutual transformation of perception and the geometric objects emerging from the activity with the GGBot establishes a distance between the individual and knowledge, which is perceived as an alterity. The "domestication of the eye" is the long process of learning that allows the student to erase such a distance, and further studies are needed to deepen the process of domestication of the eye in activity involving the GGBot. We hope that our work can suggest possible future research directions.

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Representations as site of the tension between abstract and concrete in mathematical practice: University students at work with a spirograph

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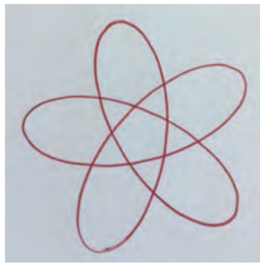
In this paper, we propose to study representations as site of encounter with the lively tension between abstract and concrete in mathematical practice. In so doing, we question visions of practice as a sequence of moves towards abstraction that see mathematical concepts as fixed and disembodied, detached from the physical, and learning in terms of developmental trajectories. Beyond a tendency to reduce abstract and concrete to a dualism conceiving of more of one as less of the other, we are more interested in what learners do and how this mobilises practice as a flow of activity across abstract and concrete. We offer the case of some university students working with a spirograph to experiment with this approach to mathematical practice.

Keywords: Mathematical practice, abstract, concrete, representation, spirograph.

Introduction.

The relevance of representations has been studied in mathematics education research from different points of view. For example, Meira (2002) has looked at mathematical representations as notations-in-use, pointing out that pure referential views of signs and meanings may not give enough emphasis to notational systems as mediational tools that can trigger and sustain mathematical activity. Other studies have investigated how the production of representations by learners does not only support cognition and communication but also transforms the very way that mathematical activity evolves (e.g., Hall, 1996; Nunes, 1997). In this paper, we are interested in this transformative character of the activity within the mathematics classroom, especially when tools are used in practice and allow learners to create and work with representations of mathematical objects. Tools have the potential to transform the teaching and learning of mathematics (see Hoyles, 2018). The ways in which the bodily and the kinaesthetic are involved in interaction with the tool extend the horizon of possibilities that learners come to entertain (Nemirovsky & Ferrara, 2009), and a growing body of research considers notions of embodiment and perceptuo-motor engagement in mathematics to characterise them (Nemirovsky et al., 2013; de Freitas & Sinclair, 2014). Additionally, mathematical activity can be conceptualised as *involving* and *evolving* concepts, showing the role of body and movement in learning mathematics (de Freitas & Ferrara, 2015). We build on these considerations to investigate two main aspects that we see as central in the study of mathematical practice with tools: on the one side, how tools change mathematical practices as tool use constantly reconfigures the activity; on the other side, the ways that tools always demand the body and mobilise mathematics in unexpected ways. We will argue that representations and diagrams are at the core of both aspects as the site of encounter of abstract and concrete in mathematics. We examine these two aspects through the specific case of doing mathematics with a spirograph. The spirograph was invented by a mathematician but it is usually known as a classic toy for children to create diagrams, beautiful drawings that might resemble mandala or recall images of flowers. The mathematics of the spirograph is quite sophisticated and related to the number of radially symmetric curves that can be made using the tool (Ippolito, 1999). Among the many types of spirograph available nowadays, we consider a very simple spirograph that consists of a couple of plastic ring gears with teeth on the inside of their circumferences and three different-sized pinion gears (or gearwheels) with holes in various positions, which are used to insert a pen and draw (Figure 1a). The class of curves that can be traced rotating the pinion gears inside the ring gears enters that of hypocycloids (an example is shown in Figure 1b). In this report, we draw attention to the mathematical practice of a class of university students that have used this spirograph to examine the mathematics out of the obtainable diagrams. Rather than attending to practice in terms of what the students *can achieve*, we shift focus to what the students *do*, and how this engenders and renders what *is produced* in practice, mainly attending to diagrammatic activity.

a



b

Figure 1: (a) The spirograph and (b) an example of hypocycloid drawn with it

Mathematical practice and the tension between abstract and concrete.

Learning mathematics is often conceptualised as a progression or movement from the concrete to the abstract. This progression amounts to a gradual passage from physical acts and senses to the more fixed and disembodied (ideal) mathematical concepts. In this taken-for-granted idea of school mathematics, the concrete is left behind, often delineating a clear cut between the physical and the conceptual and a directed trajectory for learning. Mathematical practice is thus mainly conceived of as the steps and acts of acquiring concepts and attaining abstract knowledge. Despite their prescriptive habit, traditional images for the attainment of abstraction have been questioned in the literature. More complex pictures have been offered: for example, by troubling the divide of concrete and abstract as properties of entities rather than as relationships between persons and things (e.g., Wilensky, 1991), and by introducing diverse kinds of concrete knowledge (Clements, 2000) or ‘hybrid’ notions of situated abstraction (Noss et al., 2002) and abstraction in context (Hershkowitz et al., 2001). These studies have in common a tentative discourse on ways in which something can be abstract or concrete for someone in relation to practice, while challenging the idea of whether mathematical abstractions can always be separated from the context of their construction or application. However, they maintain a vision of teaching and learning processes as tending towards the achievement of abstraction. In a similar vein, the nature of expertise has also been a subject of concern, with expert knowledge in mathematics (and science) as commonly described as formal and abstract. Coles and Sinclair (2018) have more recently critiqued assumptions about developmental paths to abstraction and have rather stressed a relational view of abstract and concrete, which challenges what is meaningful in learning mathematics. Whatever approach we take, the tension between abstract and concrete is pervasive in the doing of mathematics. Following Noss and colleagues (2002), "precisely because mathematical activity is anchored in the artefacts and discourses of the practice, it is recontextualized in ways that sometimes make it difficult to recognize at all" (p. 227). In another work (Nemirovsky et al., 2020), we address the notion of abstraction detaching it from a vision bounded to the dualism abstract-concrete, according to which more of one is simply conceived of as less of the other. Analysing three students at work with a graphing motion sensor, we show how, in the practice with the tool, it is possible to sketch a distinct path for the attainment of abstraction, which involves navigating a surplus of sensible qualities. In this paper, in line with such a non-dualistic perspective on abstraction, we look at the specific work with the representations to shed light onto the ways in which the students (inter)act with mathematical concepts through material activity. In so doing, we want to stress the contextual nature of mathematical practice that emerges from activity within the classroom, instead of reducing it to a sequence of moves toward abstraction. Accordingly, we want to *rethink* mathematical practice as a way of encountering or sustaining the relational dynamics of abstract and concrete in mathematics.

We turn to the French mathematician and philosopher of mathematics G. Châtelet, who has drawn attention to the notion of the virtual as a powerful way of overcoming the so slippery relationship of abstract and concrete and the split of the physical and the mathematical (Châtelet, 1993/2000). Châtelet discusses how the mathematical has a physical aspect to it, by analysing the mathematical practice of scholars in the history of mathematics and physics. The concept of the virtual is that which allows us to reconceive of the mathematical and the physical together, challenging visions that typically associate mathematical thinking to the mind and leave out the body. For Châtelet, the virtual is in the physical world, and its coupling with the actual is at the hearth of the emergence of new mathematical ideas. It pertains to the mobility and indeterminacy at the source of all actions and is mobilised and actualised through activity: the moves, the gestures, the diagrams, and their interplays are those which awake the virtual or potential multiplicities that are implicit in any material surface, and open room for inventiveness in the mathematics classroom (de Freitas & Sinclair, 2014; Sinclair et al., 2013). Briefly speaking, the virtual has to do with all the potential, future alterations of the world, rather than mirroring it as we see it. Borrowing from these ideas, instead of aiming for a clear cut between abstract and concrete in mathematical practice and conceiving of learning as the pursuit of abstraction, we intend to study the *flow*, or movement, of activity across abstract and concrete. We strive to do so in relation to the use of the spirograph, considered as a mathematical instrument (Nemirovsky et al., 2013) that engages kinaesthetically the students' bodies in the creation of mathematical curves. In this context, we look at moves, gestures, diagrams, and their interplays (processes of actualising the virtual) to see how the mathematical activity develops and changes.

Context and method.

The mathematical activity we consider in this paper was carried out in the context of a university mathematics education course attended by a group of 30 university students at the first or second year of their master's degree in mathematics. The activity engaged the class during a regular lesson of the course. The students were divided into groups of 5 people and each group worked for about 2 hours on written tasks to be faced using the spirograph. The tasks aimed at guiding the students to explore the tool and one main idea the students had to play with is that with the spirograph one always draws closed curves. This fact is explainable through elementary concepts of number theory, which allow to model the functioning of the tool depending on the ratio between the numbers of teeth of the chosen ring-wheel combination. With the spirograph used in our activity it is possible to combine six couples (R, W), where R and W indicate the numbers of teeth in the ring gear and in the gearwheel respectively. The mathematical relationships of Table 1 capture the number of 'petals' or 'tips' of the curve created by a specific combination of gears and the number of turns that the wheel covers along the ring to close the curve (lcm: lower common multiple). In Figure 1b we observe 5 petals, drawn with the couple (105, 63): indeed, lcm (105, 63)/63 = 315/63 = 5. Variations on the choice of the hole for the pen modify other characteristics of the curves, like the petals' curvature. For the sake of space, we do not expand our considerations further, but more mathematics can be explored with the spirograph.

Table 1: Some mathematical relationships at play with the spirograph

(R, W)	Number of petals = lcm (R, W)/W	Number of turns = lcm (R, W)/R

The first author was the teacher of the course, and both the authors designed together the tasks and took part in the activity as active observers. Each group was filmed by one author or a research collaborator with a mobile camera for the entire duration of the activity, and all the written productions of the groups were collected. The video and the written protocols constitute the data source of our study. We adopt a micro-ethnographic method (Streeck & Mehus, 2005) to focus our investigation on the students' interactions with the instrument and the corresponding representations.

For this paper, we focus on the work of one group of five students, who have thoroughly explored most of the mathematical ideas, to exemplify the dynamics of abstract and concrete in their activity with the spirograph and how it arises from the students' productive engagement with the instrument and the emerging representations. We point out two main elements which, we believe, speak directly to the lively tension between abstract and concrete in this context: (1) expectations about the 'future' of the curve and (2) interplay between composition and decomposition of gear motions. In the next sections, we first present two episodes and then briefly discuss them.

First episode.

The students (named S1, S2, S3, S4 and S5; three females and two males, Figure 3d) are seated around a table and have at disposal a spirograph, the written worksheet, and blank sheets of paper. We highlight some moments within the first 5 minutes of interaction. Student S3 reads the given description of the spirograph, and the group focuses on the task of describing *what can* be obtained using the tool. The combination of gears (105, 52), randomly selected and used by one student, S2, produces a first diagram (Figure 2a).

The students explore the physical situation: S2 draws moving the pen quite slowly, while his mates carefully gaze closely at him. In several turns, the dialogue develops as follows:

S1: Hm, but it never closes. [Figure 2a]
S2: Perhaps, we don't know.
S3: Well, come on! Go on then (everybody laughs). [S2 accelerates increasing the rotation speed and, after few seconds, removes the pinion gear, stops from drawing, and changes the combination to create another diagram. The student then produces a new diagram, which instead is completed in few turns, using the couple (96, 36). Figure 2b]

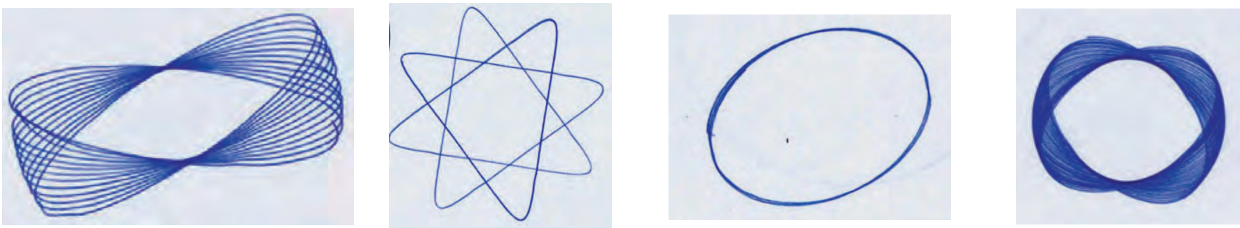
S2: So, it can close.

S3: Try with the most inner [hole], like where the spiral begins, but I don’t know (to S2). [S2 returns to the initial combination (105, 52), but changes the hole: after nearly four complete rotations around the ring gear, he stops. Figure 2c]

S3: Always an ellipse. [S2 changes the hole again, maintaining the same couple of gears]. Anyway, a fixed ellipse always occurs. [Several trials are made, then S2 creates a new diagram. Figure 2d]

S1: Ah, no, we had to go on (laughs).

For 2 more minutes, the students draw diagrams exploring the change of the hole while using the same couple of gears. Then, they start to discuss a collective answer for the task.



a b c d

Figure 2: Four diagrams created by the students within the first 5 minutes of work

Second episode.

After 5 more minutes, the students still discuss how to describe what they have created:

S2: It’s the orbit of a, a satellite, that is, an epicycle (draws circles repeatedly with the pen) isn’t it? That is, it turns around itself, around a…

S3: Yes, after all it’s like the orbit of a planet. [The girls focus again on the written, S2 starts drawing with the spirograph]

S4: What were you telling about the orbit (to S2)?

S2: The orbit of a satellite (mimes inverted commas).

S4: This (points to one of the curves on paper)?

S2: All of them. [S3: Yes]. That is, they rotate around a centre and around themselves, more or less (sketches a new diagram. Figure 3a). That is, for me, we could obtain this thing. [S4 laughs]. Yes! That is, this is rotating around itself and around a centre, and it’s the orbit, in the sense, hm, of the moon around the sun, here there’s the earth (points to the inner circle of Figure 3a, one gear is positioned onto the diagram) and it’s this one, and this one rotates around itself (moves the gear from the diagram and makes it rotate around itself).

S3: Try to go very slow (to S2) and let’s see what happens […] that is, when you turn, let’s go slowly, so that we can see when this [the wheel] has done a turn on itself, and what’s been drawn meanwhile. [S2 marks the point where the pinion gear is tangent to the ring gear. He slowly begins to draw and stops the pen when the mark is touching the ring again. Figure 3b]

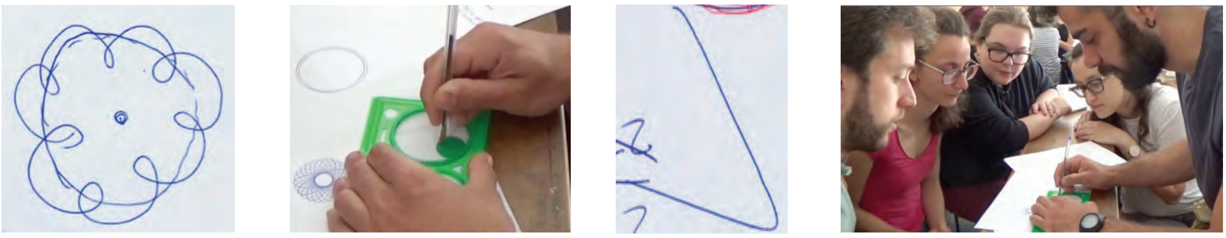
S2: This is a complete turn on itself (points to the drawn curve segment. Figure 3c).

S4: That doesn’t give us any information. [S3 and S1 giggle, while agreeing]

S2: But, if we decompose it (moves the gearwheel to the side), if we had a complete turn on itself (makes the gearwheel turning on itself) without composing it with the other motion (mimes a rotation with the pen over the ring gear), we will have a circumference. If this (points to the wheel) turns on itself…

S3-S4: Yes, but we’ve to consider it must turn around…

S2: Yes, yes (nods), but, I mean, what I was saying is that, well, it turns around itself and does a circle (draws a circle around the gearwheel). Turns around this, a point (points the pen in the middle of the ring gear), well, and it does a circle (draws a circle inside the ring gear). The composition of these two circles gives you, well, of these two motions (points to both the drawn circles).



a b c d

Figure 3: Four moments of the group work in the second episode

Discussion and conclusion.

In the first episode, the students use the spirograph to draw and discuss their expectations about the behaviour of the curves they are (partially or completely) creating. In the initial moment, right after few turns of the pinion gear, the possibility of a 'never-ending going on' intuitively reflects the possibility of non-closure of the curve in the diagram (a delicate, crucial point of the mathematical modelling of the instrument). The trial with a different combination of gears makes a different possibility emerge, that of a closed curve (Figure 2b). The new diagram embeds the end of the drawing movement and suggests it as a possible future also for the first curve, so that the students return to the initial combination of gears but changing the hole for the pen. The recognition of a seemingly elliptical shape (Figure 2c) and the association of a known curve with the "always" possible curve implicates that the students change the hole again, involving the creation of a new curve which do no longer resemble an ellipse. At this point, the possibility for the first curve to close and for the movement to arrive at an end is brought forth again, even beyond the material act of drawing. The iterative nature of the movement that creates the initial diagram, together with the time needed to close it, suggests a possibility for the movement to be repeated unlimitedly. The need of continuing the movement is therefore mapped onto the possible future(s) of the curve. We see how the tension between abstract and concrete is at play in the diagrams as possibilities of movement or possibilities of mathematical creation are encountered by the students and emerge out of the interaction with different combinations of gears or holes. Activity is reconfigured each time these possibilities are changed. The first episode is full of mathematical expectations that emerge from the physical movement, the combination/coordination of the gears and the material activity with them. This deeply explorative phase is characterised by a flow across *what is there* and *what might be*.

In the second episode, once a description must be produced, a new, more explanatory phase begins, in which the students attempt to make sense of the instrument by composing and decomposing its parts and the drawing movements. This happens by shifting attention to the fact that two movements occur simultaneously, each along a circular trajectory: that of the gearwheel along the ring gear, and that of the gearwheel on itself. The students closely observe this behaviour by moving slowly, so that they can decompose the curve "in the making", relating it to the internal gear's rotations. This is done to better unfold the previous reference by S2 to the planet/satellite dynamics, which pointed out that none of the two movements can be thought without the other. We see how the change in pace gives the students a different feeling for the curve, helping them crystallize how movements partake in the creation of the diagram. In line with Châtelet's vision of diagrams as site for contractions and expansions to be realised, in this brief excerpt, the diagrammatic activity with the tool unfolds through the ways in which the students use, reimagine, and rethink the tool, the curve and the composed movements. We have tracked three aspects of the work with the representations: the need of slowing down, the interpretation in terms of planets and satellites, the decomposition of parts that cannot be separated. We can see how these aspects speak directly to the lively dynamics of abstract and concrete that make mathematical practice entangled with the materiality of the tool and inseparable from the fleeting perception of movement. They express the expansions and contractions proposed by Châtelet, capturing the ways in which the students navigate the surplus of sensible qualities that arise in the experience of/with the instrument, and by shifting the activity towards the mathematics of the spirograph, they come to constitute mathematics as experience. Mathematical activity is imbued with possible futures and changes inextricably as it is intertwined with the kinaesthetic and the bodily. The diagrams that emerge are not static characters of the activity but generative of doubts, conjectures, and new meanings.

In this paper, we have offered a non-prescriptive and non-static vision of mathematical practice. We attended to practice as a way to encounter or sustain the tension between abstract and concrete and we focused on its contextual nature and what the students do, rather than what they can achieve. In this dynamic vision, diagrams operate as a locus of practicing and the abstract/concrete relationship. Drawing on Châtelet, we can say that the practice of mathematics is not of mathematics that *can* be practiced, but of mathematics that is *really* in the process of being practiced. Knowledge is of course potential in mathematics, but above all it actualizes itself there. We therefore recognize the complex nature of learning and the way that knowledge emerges from practice and material activity. Diagrams, representations, are intensive sites of actualization(s). Further research is necessary to investigate whether this approach to practice can open the ground for studying the tension between abstract and concrete in other contexts.

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Grouping passengers: A microgenetic case study of a struggling student's representational strategies for quotitive division

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This paper focuses on the arithmetical understandings and behaviours of one fourteen-year old student with a history of very low attainment in mathematics, as she worked on a sequence of scenario-based quotitive division (grouping) tasks with individually-tailored verbal and visuospatial support. The student's independent and co-created visuospatial representations of arithmetical structures, along with verbal comments, were analysed qualitatively using a multimodal microgenetic approach. This paper uses selected illustrative examples to discuss certain arithmetical-representational changes (e.g. employment of pictorial, iconic and symbolic elements), some of the particular difficulties that may be experienced by students with impaired memory for arithmetical relationships and procedures, and potential compensatory strategies. It is intended to stand as a companion piece to the case study presented at the previous CERME TWG24 (Finesilver, 2019).

Keywords: Visuospatial representation, multiplicative thinking, numeracy, low attainment, special education.

Introduction

I encountered Wendy during a larger project investigating low-attaining students' representational strategies for multiplicative structures. Attending a typical inner London secondary school, she turned fourteen during the study; however, her capacities for and approaches to quantitative reasoning were atypical for her age group. Her particular stage of arithmetical thinking (struggling with the move from additive to multiplicative reasoning) has been of particular interest to researchers, and both her strong aversion to division (at least, when presented in traditional symbolic notation) and over-reliance on counting-based strategies, are well-known phenomena. She was very positively disposed to the problem-solving interviews, maintaining excellent focus, and was keen to try new things and experiment with diverse strategies. Furthermore, as she demonstrated good verbal comprehension and would say when she did not understand, I took the opportunity to engage her in some detailed discussions about the tasks she worked on, including how and why her arithmetical strategies worked, which she enjoyed and welcomed. These several factors provided an excellent opportunity for microanalytic case study: to examine this individual's arithmetical-representational strategies in fine detail, and note any changes taking place. Thus, I focused on an arithmetical concept with which the participant was insecure (division), building from activity in which she was comfortable (counting and addition), within scenario tasks that she could work on without peer pressure or artificial time limits. This paper presents and discusses some brief but illuminative excerpts from my work with her.

Theoretical background

There is a strong tradition of research into various aspects of early numeracy, such as counting-based arithmetical strategies, taking place in naturalistic teaching/learning environments. Those which focus on children's own representations of number are often quasi-ethnographic in nature, where (usually very young) children are observed in their mark-making (e.g. Atkinson, 1992), and their representations analysed for ‘emergent’ mathematics. Key to this body of work is that it focuses on children’s own, often non-standard, representational strategies; this is in the pedagogical tradition of “de-centring” (Donaldson, 1978), i.e. to shift from an adult perspective and imagine what a scenario, phrase or object might mean to a child.

While the details and exact terminology can vary, in psychological research paradigms some kind of representational progression is also generally assumed, moving from the most intuitive models of arithmetical relationships with one-to-one correspondence, through pictorial and/or iconic forms that may include one mark standing for a number $\neq 1$ and/or starting to incorporate abstract symbols, to eventual full formal symbolic notation (e.g. in the Enactive-Iconic-Symbolic modes of Bruner (1973)). The development of mathematical concepts has also been linked to increasing awareness of pattern and structure (Mulligan, 2011). The capability to make connections between multiple mathematical representations is important for a learner’s developing conceptual thinking (e.g. Ainsworth, 2006), and in particular representational flexibility for problem solving (Acevedo Nistal et al., 2009).

Nunes and Bryant (1996), among many others stretching all the way back to Piaget, suggest that to understand multiplication/division represents a significant qualitative change in children's thinking (compared to addition/subtraction) – and so is deserving of particular attention. Regarding the increased complexity, Anghileri (1997) points out that a counting strategy in a multiplication or division task requires three distinct counts: the number in each set, the number of sets, and the total number of items. The second of these – enumerating sets rather than units – may be particularly unintuitive for some. Notwithstanding, Carpenter et al.'s (1993) study of kindergarten students (i.e. age 5-6, with <1 year of formal schooling) demonstrated that they could carry out a wider range of division tasks, with greater success, than had formerly been realised – provided the tasks were presented in the form of scenarios which could be directly modelled. Furthermore, they argued that many older students abandon their fundamentally sound problem-solving approaches for the mechanical application of formal arithmetic procedures, and would make fewer errors if they applied some of the intuitive modelling skills of their younger counterparts.

Given this, it is appropriate to combine a subject focus of early division with an analytical focus on informal, nonstandard representational strategies that grow out of intuitive models. Although tasks involving grouping passengers in vehicles are not as common in division research as the sharing tasks featured in the companion paper, Finesilver (2019), they are a highly imaginable scenario for learners, and rely on the fundamental schema of one enclosure with multiple items inside (Nutbrown, 2011).

Much prior research has analysed visuospatial representations via simple taxonomies, organising them into broad categories. Here an alternative approach is used: comparing and contrasting students’ changing representations via a multi-aspect qualitative analytical framework (Finesilver, 2022).

Research questions

1. What arithmetical-representational strategies does the student use in division tasks?
2. What do the strategies tell us about their particular weaknesses and capabilities?
3. How do the arithmetical-representational strategies change over time and interaction?

Methods

The dataset for this study is taken from three (of five) 1:1 problem-solving interviews, each lasting 45 minutes, carried out by the author. Sessions 1-2 focused on other types of multiplication- and division-based activity (e.g. Finesilver, 2019); a significant proportion of this student's time in Sessions 3-5 was spent on the 'grouping' tasks described here. It employs microgenetic methods, which were developed for the study of the transition processes of cognitive development (Siegler & Crowley, 1991). They have been widely used in studies of children's arithmetical strategies and particularly in case studies of individuals with difficulties in mathematics (e.g. Fletcher et al., 1998).

Wendy had been described in a past Educational Psychologist's report as having dyslexia, leading to significant difficulties with numeracy, memory, organisation, and sequencing skills. These descriptions seemed fair at the time I was working with her. However, the report also contained the judgement that she "lacks the ability to retain and process academic subjects that require logical thinking, analysis, sequencing, rationalising and accuracy"; this is an example of the kind of dismissive labelling of certain atypically-progressing students as predetermined future, as well as past, academic failures, which I have criticised (e.g. in Finesilver et al., in press). Her Wechsler Individual Achievement Test (WIAT-II) scores at age 11 had been in the 2^{nd} percentile for numerical operations, and the 0.5^{th} percentile for mathematical reasoning. Like the quote above, these standardised test scores do not at all predict or encompass the nonstandard mathematical reasoning I describe below. Wendy was well aware of the importance placed by the education system on the memorisation of 'times tables', and of her own long-term failure to do so ("even with constant review and revision", according to records), which had left her, unsurprisingly, with a very low opinion of her own mathematical abilities, but she did not radiate overall low self-esteem.

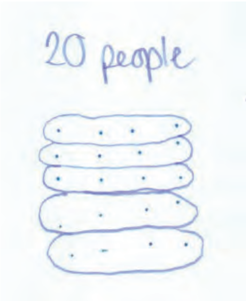
Unlike some other students in the larger study, Wendy's counting and addition were secure and reliable, and due to her growing enthusiasm I had to devise increasingly challenging tasks in situ to adapt to her progress. In each session, Wendy was set a series of quotitive divisions expressed via the scenario of a given number of passengers to be allocated to a given type of vehicle, with the required number of vehicles to be calculated. The quantities used were two- or three-digit dividends and one- or two-digit divisors, chosen in situ, depending on her arithmetical-representational functioning that day and in previous sessions. As Wendy was barely ever able to recall more than a few number relationships either from long-term memory or from previous tasks, I selected the divisors for their ease of repeated addition or step-counting, and re-used certain more familiar number patterns.

All sessions were audio recorded and all markings on paper collated. All markings in purple ink are by the researcher. Each task attempt was qualitatively analysed using a framework for visuospatial arithmetical-representational strategies across multiple semi-independent aspects of display, calculation and interaction. These were used for systematic description of representational elements and forms (*media*, *mode(s)*, *resemblance*, *spatial structuring*, *motion*, *unitcountability*, *enumeration*, *consistency*, *completeness*), evaluation (*strategic soundness*, *execution errors*), and noting teacher-student interactions (*verbal* and *visuospatial prompts*) – see Finesilver (2022). Particular attention was paid to any attempts where change in one or more of the aspects was observed; these were considered microgenetic 'snapshots'.

Selected data

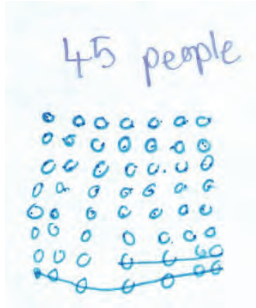
Due to restrictions of space, only a small sample of data may be reproduced in these proceedings; more are included (with verbal transcripts and fuller commentary) in the accompanying presentation. These three subsets of images are selected and grouped to illustrate particular points of interest with respect to arithmetical-representational strategies for quotitive division.

Subset 1: Working with array-structured unitcountable representations

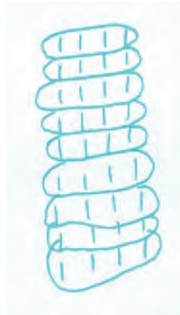


a. 20 people in 4-seater taxis

b. 32 people in 4-seater taxis



c. 45 people in 7-seater taxis

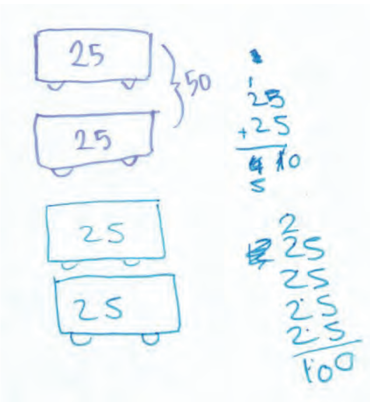


d. 36 people in 4-seater taxis

Figure 1a-d: Selection of unit arrays and array-container blends used in passenger tasks

When we first started work on quotitive division, Wendy struggled to calculate the number of 4-seater taxis required for 20 people (Figure 4a). She drew 20 dots, as always, arrayed neatly in rows, but was then unsure whether rows or columns represented taxis. I demonstrated the option of using containing rings as an additional structuring element. Figures 1b-d show various ways she experimented with spatial structuring in arrays and *array-container blends* (Finesilver, 2017) in subsequent tasks. These are all *unitcountable* in the sense that each represented unit (cube, tally mark, etc.) = 1, and enumeration of the total could be achieved by direct unitary counting (i.e. counting them *in ones*).

Subset 2: Working with number containers for scenario tasks, some with decorative elements



a. 45 then 96 people in 25-seater coaches

b. 343 people in 50-seater coaches

c. 612 people in 200-seater planes

Figure 2a-c: Selection of number containers used in more challenging passenger tasks

When we started working with larger numbers, Wendy was keen to try representational strategies that made sense to her within the scenario but did not involve drawing hundreds of unit marks. I demonstrated the option of building additively to the required total, which she adopted and used successfully in several subsequent tasks, spatially structuring the addends in *number containers* (Finesilver, 2017). The task represented in Figure 2c led to a conversation about the 'left over' 12 people. There was some tension between Wendy's awareness of the conventions of school maths tasks, and concern with the practicalities of the numbers in this scenario and, amused by the thought of a mostly empty plane, insisted the fourth should be "a smaller one" (which she did not draw).

Subset 3: Working with number containers for bare tasks, introducing narrative elements

In my final meeting with Wendy, encouraged by her success with various scenario tasks, I tried setting her a bare division (i.e. no scenario). Her first comment was "Something tells me it's going to have two zeros on the end of it" (indicating a misremembered rote 'trick' for multiplication) followed by blankness. I suggested we might imagine them as 180 people being put in 20-seater coaches: once the task had been 'made real' via a minimal bit of narrative, and again using number containers, she was able to complete this and similar tasks. She would picture a line of parked vehicles and a crowd of tourists grouping themselves tidily to fill them, and was particularly pleased by the option to take some unfriendly symbols of calculation and turn them into a scenario that could be imagined and represented in a way that she could comfortably work with.

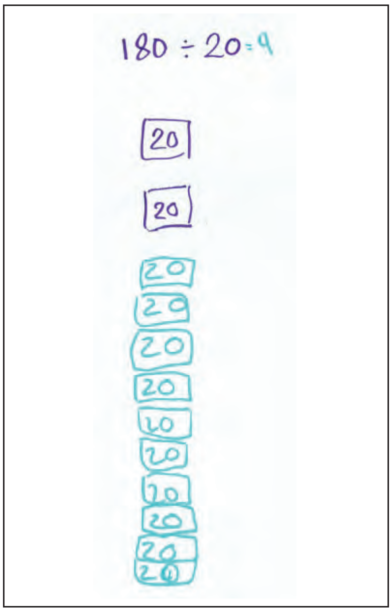


Figure 3: Number containers in a bare task

Discussion

What representational and arithmetical strategies does the student use?

Wendy showed a strong continuous preference for working with pen and paper over the concrete materials which were also available. She employed two main representational-arithmetical strategies, one unitcountable (all units individually represented, arranged in spatially-structured patterns, enumerated by group-counting) and one non-unitcountable (a written number representing each group, organised spatially in one dimension, enumerated by step-counting or addition). The first, which she introduced independently, was used for smaller quantities and the second, which I initially introduced but she readily adopted, for larger – although there would likely be an overlap of magnitudes where either might be chosen.

What do the strategies tell us about their particular weaknesses and capabilities?

Wendy wished to obtain answers she was confident were correct, with what she considered a reasonable expenditure of time and effort, and the strategies she used reflect this. In general her representations were highly mathematically functional, drawing and/or writing the minimum content which enabled task success. I see this as evidence of logical and meta-strategic ability, and a very sensible attempt to work around, and make the most of, her limited computational skills, minimal recall of number facts and relationships, and tendency to lose her place in multi-stage calculations.

Wendy produced 2D array representations quickly, confidently, and accurately. While this initial heavy reliance on unitcountable representations was a weakness, her use of this representation type also highlighted important and useful strengths in creating, recognising, and utilising structured visual patterns. Observation of her interactions with dot arrays demonstrated informal, non-declarative awareness of multiplicative structures, enhanced by the rhythmic nature of her counting. Wendy's flexibility in terms of willingness to experiment with and vary arithmetical-representational aspects can be considered itself a valuable component of metarepresentational competence.

Of particular note is Wendy's continued strong preference for scenario tasks over bare ones, even when they were obviously unrealistic. While her sensible comments about the comparative expense of different sizes of taxi and aeroplane demonstrate a real-world awareness of money and pricing that is often considered an important part of being 'numerate', this tendency to cling to the scenario aspect of the calculations could also indicate an unwillingness to engage in abstraction, and demonstrates a profound need for something extra-mathematical to grasp and cling on to. Numerical relationships, by themselves meaningless, were given meaning when they became numbers of people (etc.). This finding would be no surprise to a teacher of very young children, who has heard many exasperated students ask "Three? But three what?" (for example), but this strength of attachment to imagining numbers of things may be surprising in a secondary school student.

How do the arithmetical-representational strategies change over time and interaction?

The main observed progression was Wendy's capacity for tackling divisions involving increasingly large quantities. Repeated successes and gradual increases maintained the momentum of her confidence, allowing her to forget for a time her dislike/fear of division and negative self-beliefs about her ability to work with larger numbers.

Wendy had started these interviews secure in her purely iconic dot array representations for partitive division (having used them in the previous Sessions 1-2). Although it took some time for her to see that it could be equally useful for quotitive division (and multiplication), once grasped, this was added to her strategic toolkit. Although not addressed directly here, the triple function of unitcountable arrays has implications for deepening the understanding of the relationship between multiplication and division, and the commutative principle. Array-container blends provided a significant bridge between quantities engaged with as iconic units in spatial arrangements, and as numeric symbols. Enclosing equal groups of dots in rings visually transformed the groups so the most salient 'building blocks' were now a set of contained equal groups rather than individual units; visually, these containers became the new 'units'. This was vital for the significant cognitive leap of replacing a container with a number of dots inside it by a container with a number symbol inside it.

Wendy learned to use a building-up, additive strategy for quotitive divisions represented with non-unitcountable number containers. She sometimes chose to continue using drawings alongside symbolic calculation, but inconsistently – this makes their continuing purpose cause for speculation. Wendy did not doodle, and otherwise showed little inclination for decorational drawing, so I suggest that in longer, multi-stage tasks, when she was struggling to focus or experiencing doubt, the drawn element served as reminder and/or for affective support. The tactic of taking an offputting bare task and making it comprehensible (or, perhaps, 'realising' or 'concretising' it) through a familiar scenario was both helpful (in that she was successful in more arithmetically challenging tasks) and appealing.

Concluding comments

A microgenetic level of analysis of this student's arithmetical struggles illuminates certain specific ways of conceptualising and carrying out division-based tasks which may be unexpected and go unrecognised in classrooms. It also demonstrates the possibility of significant improvement which is unlikely to be picked up in standardised diagnostic testing, and has pedagogical implications. Is there evidence that Wendy was not just adopting and developing alternative representational strategies that were more amenable to her than those she had previously encountered, but also moving towards more successful symbolic thinking in arithmetic? Yes. Her mathematical journey was a slow and complicated one, with end point unknown, but this short section of it shows substantive changes.

While there may have been some progress since psychological assessment, Wendy was still performing poorly in her regular school mathematics classroom, and there was a huge discrepancy between these formal judgements made on Wendy's abilities and potential, and her clever maximisation of cognitive resources as captured in microgenetic observation. With the traditional strategies that rely on memorisation of facts and procedures unworkable for her individual pattern of neurodivergent characteristics, she analysed tasks to work out alternative strategies relying instead on sequence, pattern, visualisation, and a realistic self-assessment of the level of arithmetic she could reliably manage (counting and addition). She showed great interest in the numerical structures underlying arithmetical processes, and in suggestions for their use in solving different types of tasks. These self- or co-created strategies of which she had 'ownership' must stand a chance of being remembered better than those which had for most of her school life been ineffectively rote-taught. While Wendy is a single case with an individual pattern of mathematical strengths, weaknesses and strategies, other struggling students will share many of these experiences and traits, although each in their own individual pattern; certainly there are many who are labelled too easily by both teachers and researchers as 'low attaining'. And likewise, their self-belief will likely be low, and their strategic arithmetical-representational creativity dormant, undervalued and unencouraged.

Wendy's actions were not of a child who "lacks the ability to retain and process academic subjects that require logical thinking, analysis, sequencing, rationalising and accuracy". However, it took focused individual investigation and a flexible mixture of informal assessment, tuition elements, and discussion to gain a fair picture of the nature of her mathematical thinking (and this in one small area of the syllabus). If one accepted the statements in her diagnostic report at face value, to expend this time and effort on her would seem irrational. That kind of utilitarian philosophy is one of the factors leading to neurodivergent and learning disabled students being limited from reaching their potential in mathematics. Instead of dismissing learners like Wendy for their failure to perform in the expected ways, their unexpected ways of working mathematically should be valued – they have much to teach us.

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Learning mathematics with media - representing, representations and representation transfer processes

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With digital media, the possibilities of representation have developed fast: in terms of temporal and local availability and in terms of manipulation, dynamization, representation-networking and -synchronicity. They open up an expanded new spectrum of representation-transfer, in which representations, levels of representation and associated processes of representation-transfer must be identified, analyzed as well as researched and evaluated from a mathematic educational perspective. Which cognitive demands are thus placed on learners and which are supported or replaced by media? The goal for research and teaching is a "specific sensitivity for media (use) in (mathematics) education", so that analogue and digital media are analyzed in a potential-oriented way in order to orchestrate them for better teaching and learning.

Keywords: Mathematical concepts, learning with analogue and digital representations, transfer of representations, cognitive demands.

Theoretical Framework.

The learners' own transfer of representation is an essential indicator to develop understanding in the process of learning mathematics (Wittmann, 1981). Representation transfer processes are always required when at least two representations are given. They have to be compared on one or between different levels of representation – the levels acting, iconic, symbolic (see Bruner, 1971). A representation transfer is also required when a new representation has to be constructed based on a given representation. In both cases, given elements of one representation have to be related to given or to be constructed elements of another representation. In these transfer processes, a self-acting, active-discovering engagement with mathematics is central (Krauthausen, 2018; Kühnel, 1954; Winter, 1989, 2016; Wittmann, 1981). A purposeful use of different representations can create learning opportunities to explore relationships between representations and to recognize basic structures (Kuhnke, 2013). The significance of representing, representations, and representation transfer processes is emphasized in national and international curricula and standards as one of the main "general" process-related competencies to be developed in mathematics education (DIPF, 2021; KMK, 2005; Midgett & Eddins, 2021; NCTM, 2000). Representing and representations pursue two basic intentions. In *representing* the focus is *on doing*, on *externalizing one's own thinking* for communication with oneself and with others. Representing for oneself takes place in order to relieve and support one's own thinking processes through what is (visually) represented, in order to orient oneself in one's own thinking and to shape the further thinking process. Between one's own thinking and representing, processes of exploration and analysis are carried out in the context of mathematical discovery and reasoning through processes of ordering, sorting, and structuring (Aebli, 1980; Freudenthal, 1973). Representing for others is done to communicate one's own thoughts through what is (visually) represented. It also helps to explain where words are missing for oneself and for others. The (visual) information can support to get into exchange with others, to communicate about one's own thoughts and to justify findings about relationships and regularities with the help of what is represented (Duval, 2006). *Representations* serve for the process of representing *as a tool to present one's own perceptions and ideas externally*. So, the intention of representations is to document information which is volatile (Huhmann, 2013; Wollring, 2006). According to Bruner (1971), knowledge and thus also representations through which knowledge is expressed can be assigned to the three levels – acting, iconic, symbolic. In summary, these areas, representing, representations, and representation transfer processes lead to comprehension-oriented learning and are an indicator of understanding. In the following we will use the term *TripleR* for these three areas.

With digital media, possibilities of TripleR evolved enormously. Thereby we see a development in terms of temporal and local availabilities as well as in terms of manipulation, dynamization, representation-networking and synchronicity. These new possibilities open up new possibilities of interaction in learning mathematics with media and how teaching-learning processes can be designed (KMK, 2012). Therefore potentials of digital media must first be identified, as well as whether and how these potentials of digital media unfold in the reality of teaching as an added value for learning. Thereby, the question arises, which cognitive demands are supported or replaced by digital media or are still placed on learners by learning mathematics?

Research Desideratum.

Based on the models of knowledge acquisition and representation (Bruner, 1971; Piaget, 1972), further models can be found for identifying and analyzing representation possibilities and representation transfer processes. Lesh et al. (1987) focus in their model on different analogue representation forms and associated representation transfers. The model is extended by a technological form of representation by Johnson (2018), with which digital media, which appeared in many forms e.g., manipulable and moveable pictures, are basically covered. Further distinctions and specific features of representations with digital media as well as learning with these digital representations remain out of consideration. The model of multiple external representations according to Ladel (2009) is based on Bruner's (1971) model and levels of representation - enactive, iconic, symbolic. It focuses on elaborated representation possibilities with digital and analogue media. However, it does not focus on the identification and analysis of representation transfer processes. Overall, the models listed here do not take into account which (cognitive) demands are placed on learners by analogue and digital media during representing, learning with representations, and representation transfer processes, or which are replaced by media. In summary, we see a research desideratum in the model-theoretical identification and analysis of TripleR in learning with analogue and digital media. For teaching and learning we ask – how different media with their different cognitive demands and with possibly different analogue-digital components can be brought into connection with each other with regard to TripleR – and how this can be identified and made visible?

The Representation-Transfer-Spectrum.

Based on the models of Piaget (1972) and Bruner (1971) as well as on the extended new representation possibilities offered by digital media, we have developed an extended model as a representation transfer spectrum as you can see in Figure 1. With this model we want to identify, analyze, and evaluate from a mathematics educational point of view learning in terms of perceiving and acting with analogue and digital media:

1. *Identify*: In which levels are representing, the representations and the representation transfer processes located?
2. *Analyze*: Which cognitive demands are associated with representing, the representations and the representation transfer processes? Which cognitive demands are placed on learners and which are supported or replaced by media?
3. *Evaluate*: Which representing, representations and representation transfer processes are suited from a mathematic educational perspective?

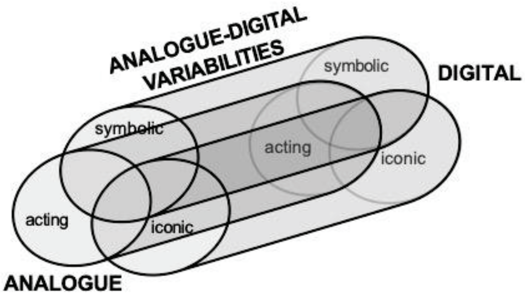


Figure 1: Representation-Transfer-Spectrum (Huhmann & Müller, in press)

Learning objects and associated activities can be located in their representations in the analogue area, digital area or analogue-digital area. Within these areas, a further assignment to the different levels of representation takes place. Between these levels we see no hierarchic arrangement. The focus in this paper is on the reciprocal transfer processes of representations within and between these levels of representation. Build on the levels of representation according to Bruner (1971), the intersections of the levels of representation are new elements of the model. We include these intersections under the term *levels of representation* because representations cannot always be assigned to just one level. If representations contain elements from different levels e.g., iconic elements (depictions) and at the same time symbolic elements (descriptions) (Schnotz & Bannert, 2003), they are to be placed in the corresponding intersection.

Analogue area: The characteristics of the three levels – acting, iconic, symbolic correspond to the known levels from Bruner. If actions are verbally accompanied, verbal expressions are supported by gestures, or symbolic representations are used to act, we identify these as representations of the acting-symbolic level. Representations that contain both depictions and descriptions (e.g. tables, diagrams, function graphs) belong to the iconic-symbolic level. Actions with *inherently unchangeable depictions* are located in the acting-iconic intersection. These can be ordering, sorting and comparing processes of images. Actions such as ordering, sorting and comparing processes with iconic-symbolic images are assigned to the acting-iconic-symbolic intersection.

Digital area: The model extension by the digital area is identical to the analogue area in its structure, but it differs in specific characteristics. On the acting level, there is the fundamental characteristic of the *manipulability of objects of action*. However, these are no longer haptically tangible and movable. Objects are manipulated and moved merely by wiping and tapping movements. Objects that are digitally represented as *inherently unchangeable images* are assigned to the iconic level. Objects that are represented as audio or written text in a symbolic way in the digital area are assigned to the symbolic level. The acting-iconic intersection covers both the user's own digital actions with inherently unchangeable images and representations of digital actions in the form of animations and movies based on images. The acting-symbolic intersection covers the user's own creating, manipulating, and acting with symbolic representations. This involves audio texts and written texts that can be accessed, duplicated, and combined with digital media. The iconic-symbolic intersection covers inherently unchangeable representations that include both depictions and descriptions. The acting-iconic-symbolic intersection covers the user's own actions with manipulable representations as well as representations of digital actions in the form of animations and videos that contain both depictions and descriptions.

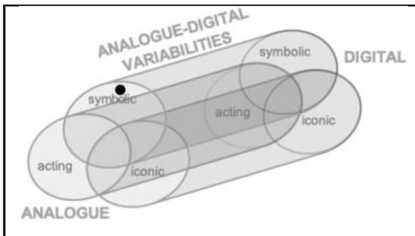
Analogue-Digital area: The analogue-digital area forms a spectrum between the analogue and the digital area. This area is to be explored with regard to the representations of learning objects and activities in terms of their characteristics and possibilities. This involves the identification and analysis of analogue-digital variabilities - in the sense of variable portions closer to the analogue or digital areas as well as with variable focal points in or between the three diameters of the respective areas. In this area, augmented reality and virtual reality applications, among others, are to be considered in a future-oriented manner.

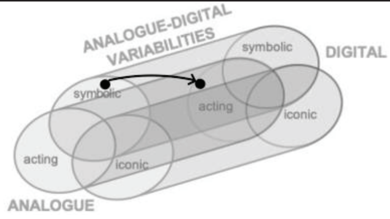
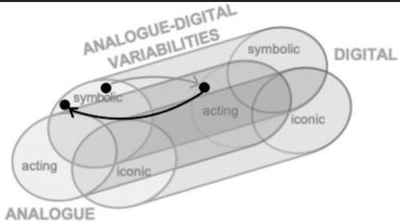
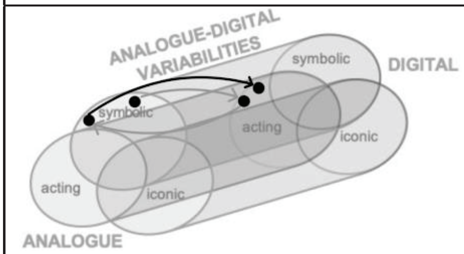
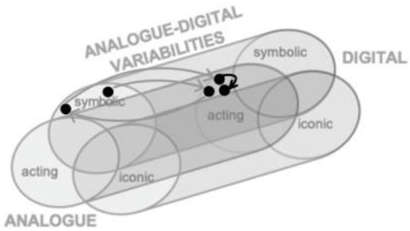
With regard to the digital and analogue-digital area, there is an urgent need for research into these areas, the possible representations of the learning objects, and the suitability of the representation-transfer-spectrum to identify, analyze and evaluate from a mathematic educational perspective.

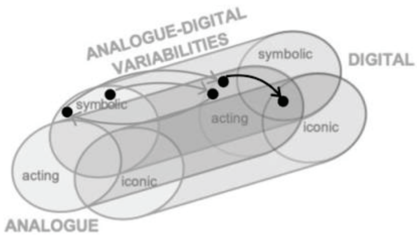
Representation transfer processes become visible in this model by connecting the activities located on the representation levels with arrows, so that the transfer from an initial representation, which is given to learners, to a final representation, which learners are supposed to construct independently or relate both given ones, is recognizable. Representations become visible by dots in the levels.

Example.

The activity *Architect and Bricklayer* (Thöne & Spiegel, 2003) is an example to demonstrate the use of the representation-transfer-spectrum: Learner 1 builds a cube building with cubes and describes it verbally to learner 2. Without seeing the cube building, learner 2 must build a cube building based on the description from learner 1. Afterwards, both cube buildings are compared with each other. The following description of this learning activity includes the implementation of the complementary use of the app *Cubes (Klötzchen)* (Etzold, 2015).

	The task *Build a cube building with 8 cubes* is presented to learners by the teacher on a *symbolic* level verbally or in writing.

<table>
<tr><td>
</td><td>Based on this task, learners build a cube building as a 3D model using the app Cubes. From the originally symbolic presented task, the learners have to perform a transfer of representation to the action level in the digital area.</td></tr>
<tr><td>
</td><td>The cube building build in 3D by using the app is now used by the learners to describe it verbally. At this point, the learners perform a representation transfer from the action level in the digital area to the symbolic level in the analogue area.</td></tr>
<tr><td>
</td><td>This verbal description now serves other learners as a starting point for rebuilding the cube building on their own tablet using the app. Here, too, the learners transfer the representation from the symbolic level in the analogue area to the action level in the digital area.</td></tr>
<tr><td>


</td><td>The two cube buildings of the learners need to be compared with each other. There are different ways to do this.

Both cube buildings are displayed in the 3D view on both tablets and compared with each other. Since both cube buildings can be manipulated in this representation, i.e. single virtual cubes could change places, the process of checking the results takes place at the acting level in the digital area.

It would also be conceivable for the learners to use the construction plan, which is created automatically and synchronously by the app Cubes, to check the results.

For example, both construction plans can be compared with each other. The result verification is now in the intersection acting-iconic-symbolic, in the digital area. However, the following aspect should be emphasized here: The intermodal transfer from the 3D view to the construction plan, from the digital-acting to the digital-acting-iconic-symbolic level, does not have to be performed by the learners themselves, it is taken over by the digital medium.</td></tr>
</table>

Table 1: Transfer of Representation in the Representation-Transfer-Spectrum (Huhmann & Müller, in press)

Findings and perspectives.

The Representation-Transfer-Spectrum is a theoretical model that is in a developmental state and needs further research. According to the example above, we consider it to be suited to identify representations and representation (transfer) processes. The activity *Architect and Bricklayer* served here as an example to demonstrate the use of the Representation-Transfer-Spectrum. First, we identified the TripleRs related to this activity in the model to make visible which forms of representation exist at which levels of representation and which representation transfer processes are demanded. Through this, possibilities of action and affordances, (Gibson, 1977) that means possibilities of action implicit in the medium, become visible. Second, we analyzed which representation transfer processes are performed by the learners and which are taken over by the digital medium. Third, we now would have to evaluate the representations and representation transfer processes from a mathematics educational perspective so that different media for this learning object can be orchestrated for lesson planning.

For *lesson planning*, it may thus serve as a model of thinking and analysis to make design decisions. The overall goal for teaching is a purposeful use and orchestration of different media for learning. From a *research* perspective, the representation transfer spectrum may serve as a model of thinking and analysis for characterizing applications: For applications that have been developed and are in the process of being developed. For exploring applications in terms of how learners use and proceed with them. All in all, it may serve for further development of analogue and digital representation possibilities and design decisions for applications. A main question is which manipulation *possibilities* are implemented in the application and which manipulation *realities* can be identified during individual task processes with the application? With the help of the model, *possibilities* and *realities* of use are to be identified and made visible. Therefore, the goal of this model is to characterize learning objects and their use in the context of digital media with regard to (new) forms of representation, (new) levels of representation and (new) representation transfer processes as well as (new) combination possibilities of representation.

The associated goal for research and teaching is a *"specific sensitivity for media (use) in mathematics education"*, so that analogue and digital media are analyzed in a potential-oriented manner in order to orchestrate them for better teaching and learning. Last but not least, the future-oriented question is which significance the representation-transfer-spectrum can have as a model of thinking and analysis in connection with learning with analogue and digital media, also in other educational disciplines.

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Solving arithmetic word problems: representation as a tool for thinking

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In this paper we report on two tasks that are part of a didactic sequence on arithmetic word problems and we discuss some episodes that happened in one experimental grade 7 class. Our main goals are to gain insights into the role of representation for students in problem solving and to investigate whether a specific task design can contribute to making representation become a tool for thinking. Preliminary results show that students tend to use representation for communicating their answer, but they can be guided in the process of transition towards the use of representation for reasoning about the problem and its resolution.

Keywords: Word problems, mathematical discussion, psychological tool, representation, task design.

Introduction

In this study, we address the question that is at the heart of the ongoing debate and research in Mathematics Education: "How can teachers support learners' representational and meta-representational competences?" and, in order to do so, we focus in particular on representations in the context of word problem solving. The use of representations in problem solving processes has been highlighted as an important issue in the mathematics education literature (e.g., Arcavi, 2003). Representations can take on different roles in problem solving: tools used for finding a solution to the problem; means of validating and convincing of the result obtained; narrative tools used to communicate and present findings to other people. Van Essen and Hamaker (1990) claim that by translating a word problem into a picture, students are brought to pay attention to the relationships in the problem. This is particularly important in our case because we consider the class of word problems, in which the relationships between the quantities involved are known, but not the numerical values of those quantities that have to be found.

There is a huge body of research documenting students' difficulties with arithmetic word problems (e.g., Verschaffel et al., 2000) and algebra word problems (e.g., Clement, 1982). To understand these problems, students have to interpret the semantic context, using their prior knowledge, and construct a representation that allows them to figure out how to search for a solution. For this reason, different forms of representation come into play and the graphic scheme produced by students to visualize and simplify the problem plays a central role in the resolution process, also activating their meta-representational competences (Ott, 2020). There are some contexts and studies where a representation is provided explicitly by the teacher. For example, Singapore has adopted a unique approach to support students in word problem solving: the model method (Ng & Lee, 2009). Students in Singapore are taught to represent known and unknown quantities using rectangular bars that specify quantitative relationships amongst data in word problems. In contrast, we asked ourselves how teachers could support students in becoming aware of the importance of making a representation, so that it is the product of their own cognitive processes, and not something provided explicitly by the teacher. In this paper we present two tasks that are part of a didactic sequence on arithmetic word problem solving through representations and we discuss the methodological and design choices made to promote students' use of *representations for thinking*.

Theoretical framework

In line with the Vygotskian socio-cultural perspective (Vygotskij, 1978), we believe that social interactions and the use of artifacts and signs are of central importance in teaching-learning processes. Social interactions, specific teacher's interventions as in the orchestration of Mathematical Discussion (Bartolini Bussi, 1996), and an accurate task design with proper principles (Gravemeijer & Cobb, 2013), can promote the production of collective signs, the awareness of the meanings of different signs and their evolution towards mathematical meanings. Moreover, thanks to the cultural mediation conducted by the teacher, students acquire the technical tools developed by previous generations and, through the complex process of *internalisation*, a tool can become a *psychological tool* that will shape new meanings (Vygotskij, 1978).

For this study we are interested in the central role that representation can play in problem solving, participating in the construction of mathematical meanings and in the development of cognitive processes. In order for this to happen, however, the semiotic mediation carried out by the teacher is important because it supports the transition from signs (i.e., representation) with a communicative function to signs as psychological tools that are, therefore, generators of intellectual functions (Bartolini Bussi & Mariotti, 2008).

Research questions

The investigation presented in this paper is guided by two closely related research questions: What is the role of representation for students in arithmetic word problem solving processes? How can task design contribute to making representation become a psychological tool for students?

Methods and participants

The activities we report on in this paper are part of a didactic sequence on problem solving, developed within a larger research project in which the authors of the paper were involved. The general aim of this project is designing and experimenting inclusive mathematical activities for middle school, following a design-based approach (Gravemeijer & Cobb, 2013). Specifically, a group of teachers and researchers worked closely together in the design of a didactic sequence that has been implemented in several grade 7 classes in Italy (students aged 11-12), during the school year 2020/21. We considered arithmetic word problems in which the relationships between the given quantities are known but not the numerical values of these quantities, that are sought for. The sequence is divided into 4 phases characterized by the type of relationship existing between data (Table 1).

All the experimentations of the didactic sequence have been conducted by the regular teacher, without the presence of researchers or other teachers in the classroom. The collected data consists of excerpts taken from the students' workbooks and transcripts of short interventions by students during the whole class discussion, which the teacher took note of or, when possible, recorded with a microphone, and then translated into English. Therefore, the discussions were orchestrated by the teacher, who left room for the students by asking a few stimulating questions, according to the task's objectives. These objectives were previously identified with the researchers and teachers involved in the project.

Table 1: Overview of the didactic sequence on arithmetic word problems

Phase	Type of relationship between data in the problems	Example
1.	"Integer multiple" and sum known	The sum of two numbers is 96. One number is three times the other. Find the two numbers.
2.	"Integer multiple" and difference known	A number is five times another number and their difference is 528. Find the two numbers.
3.	"Exceeds by" and sum/difference known	The sum of two numbers is 174. One number is 3 more than twice the other. Find the two numbers.
4.	"Fractional multiple" and sum/difference known	The sum of three numbers is 180. The first is 4/9 of the second and the third is 7/9 of the second. Find the three numbers.

In this paper we focus on the first two tasks in phase 1 of the didactic sequence, that are shown in Figures 1 and 2.

First task

The first task consists of students working individually on the resolution of the problem (Figure 1), and then they are asked to engage in a mathematical discussion orchestrated by the teacher.

Solve the problem and explain your reasoning with the help

of a representation (e.g., drawing, diagram, scheme, ...).

"The sum of two numbers is 96. One number is three times the other. Find the two numbers."

Figure 1: First task

Second task

The task consists of students working individually on the attempt to interpret Bernardo's reasoning and apply it to solve the problem (Figure 2). Then, they are asked to engage in a mathematical discussion orchestrated by the teacher.

"The sum of two numbers is 4710. One number is five times the other. Find the two numbers."

This is a picture of the workbook of a student, Bernardo, who tried to solve the problem but stopped. Help Bernardo to move forward!

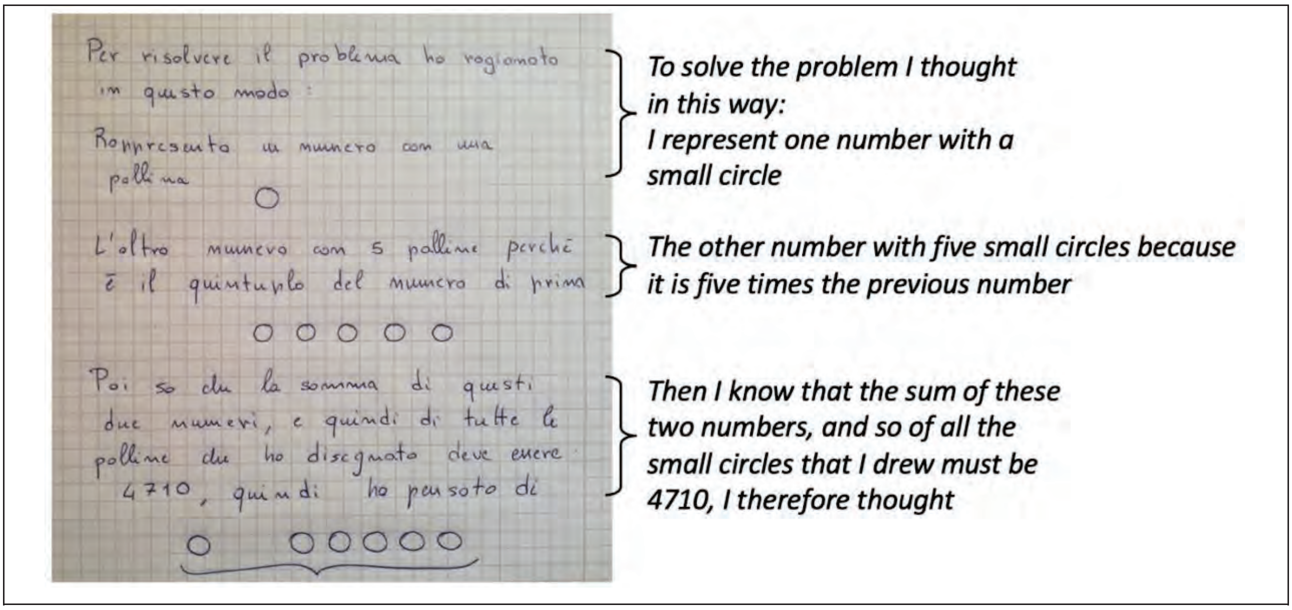

Figure 2: Second task

In the following section we report on the task design made by the group of teachers and researchers involved in the project, both a priori and during the implementation of the didactic sequence, between one lesson and the next, based on students’ reactions. Moreover, we discuss some preliminary results, showing data from one of the experimental classes that has been chosen as a representative example. That class was considered to be representative because it allows us to show the general dynamics of what happened in all the experimental classes, in a quite short time span to be reported on in this paper. Overall, very similar mathematical discussions and students’ reactions came up in the different classes during the didactic sequence.

The class had not previously dealt with this type of arithmetic word problems, not even in geometry, and they had not yet approached equations. Additionally, the teacher had never discussed with the students what the different roles of representation in problem solving might be. Students in this class are used to taking part in mathematical discussions, explaining their reasoning and argumentation. The observation and analysis of signs produced by the students, in their workbook and during mathematical discussions, make it possible to gain insights into the role of representation in their resolution processes of the proposed problems. For answering the second research question, we investigate whether something changes in students’ use of representation in problem solving, in relation to a specific task design.

Discussion of the tasks and preliminary results

The first task (Figure 1) is designed with the aim of observing the different solutions proposed by students, the type of representations they suggest and how they use them. In the task some possible representations are given in brackets because, having never done similar work before, the teacher expected that the term “representation” might not be completely understood by the students and so planned with the researchers to give them some examples of possible signs. After the time for individual work, about 20-30 minutes, students are asked to explain their reasoning to the classmates. In this way, during the discussion the teacher can gather information about the role of representation, whether for students it is a tool for communicating to the teacher the problem answer, that has been found only with calculation, or a tool for thinking about the problem itself.

In the experimental class, we observed that most students made the representation a posteriori (after finding a numerical solution to the problem), following the teacher's request to explain their reasoning or, more explicitly, to use a representation themselves. Indeed, as can be seen in Figure 3, the numerical values of the involved quantities, that is the solution, already appear in the proposed representations. Many students' strategy for solving the problem was to proceed by trial and error, starting with two numbers that partially satisfy the given conditions (e.g., whose sum was 96) and then modifying them so that they also satisfy the other conditions (e.g., one was also three times the other). A few students drew 96 circles, while most calculated the half of 96 but then failed to explain why this strategy worked. These representations are almost never generalized, because they refer to the specific problem and do not represent in general the two given relationships, i.e., "sum" and "three times". Therefore, most of them could not be used for representing a similar arithmetic word problem having the same relationship between data but different numbers. A possible exception is the last screenshot in Figure 3, in which the student used four circles, each accompanied by the letter *n*.

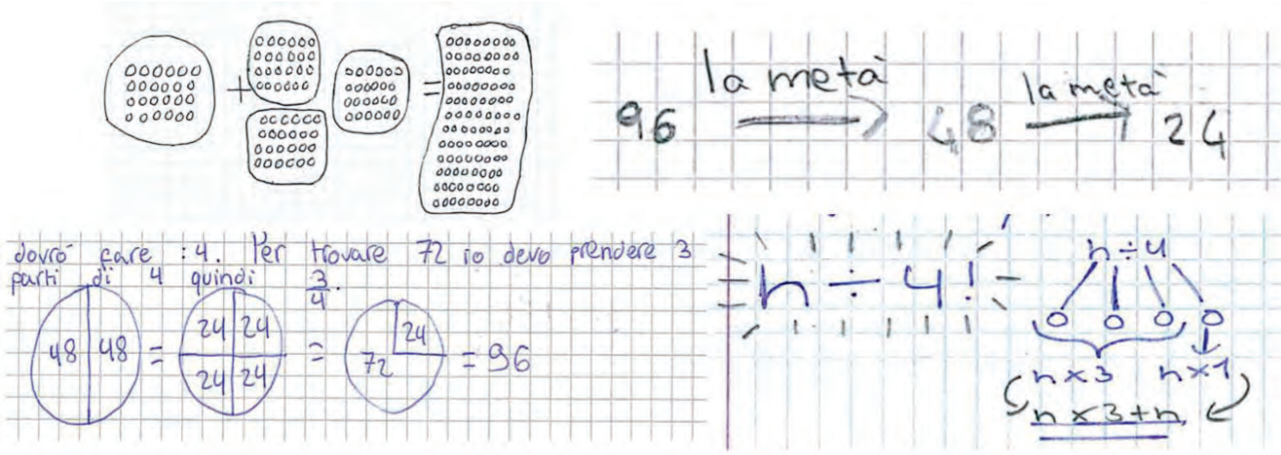

Figure 3: Examples of representations proposed by the students in the first task

In light of what emerged from the discussion of the first task in the experimental class, the group of teachers and researchers realized that these signs were used by students merely as tools for communicating the procedure followed or, simply, for pleasing the teacher's request of making a representation and accomplishing the task. As a result of the teacher becoming aware of this signs' utilization, the second task (Figure 2) is designed to support a transition in the role of representation towards possible tool for supporting students' reasoning in arithmetic word problem solving.

The main idea behind the second task design is to create a dialectic in which the representation is partly created by the students and partly provided by the teacher. Specifically, it is based on a ploy that consists of showing the workbook of a (imaginary) student, Bernardo, who makes a representation using circles for solving an arithmetic word problem that is very similar to that given in the first task. We note that in this class the choice of circles, rather than squares or rectangles or other, came from a student's idea (Figure 3) accepted by the students as the most functional because circles are easy and quick to draw; upon which the teacher worked in continuity.

In this task, to prevent students from trying to solve the arithmetic word problem without paying attention to Bernardo's solution, they are asked to help him (and to do that they need to figure out his reasoning). Indeed, in this type of tasks, the teacher must take care of the devolution of the problem (Zan, 2007): what is the problem that students must solve? The possible options are:

1. Solve the arithmetic word problem, then find the two numbers;
2. Try to understand how Bernardo reasoned;
3. Help Bernardo to complete the solution to the problem, based on the reasoning followed up to that point.

In order for students to put themselves in Bernardo's shoes and enter into his reasoning, which is based on the use of small circles, it is important that they try to exploit this representation. Therefore, in this task students are asked to solve the problem in the third formulation, i.e., to help Bernardo complete the solution. After the time for individual work, about 20-30 minutes, the teacher orchestrates the mathematical discussion, prompting it through questions like: What is this student reasoning? Do you think that the representation with small circles drawn by Bernardo can be useful for finding a solution to the problem? If yes, in what way and if not, why?

In the experimental class, this problem proved to be quite difficult because many students thought they had to divide 4710 by 5 instead of 6. For example, this is an excerpt from a student's workbook:

First of all I found this to be quite a difficult problem, but with Bernardo's scheme it was much easier to solve. When his reasoning stopped the last thing that was on the paper was the diagram with the 6 circles and the curly bracket, which I saw as an addition because the first circle, alone, was the normal number and the other five circles were five times the number so all the six circles with the curly bracket below (which puts them together, i.e. adds them up), and under the curly bracket there will be the number 4710 because it is the result of the sum, as the problem says.

As this student wrote, Bernardo's representation has been helpful to understand why the sum 4710 should be divided by 6, a number not explicitly available in the text of the problem. This was not the only case, for example another student matched each part of the representation with a part of the text of the problem and then completed the solution by exploiting Bernardo's reasoning (Figure 4):

In my opinion, what Bernardo did is very useful for finding the solution to the problem. This is because he translated the sentence "one number is five times the other" by drawing the circles. He represented "a number" with a circle and since the other number is "five times the other" he said that five times the circle is the other number. I wrote the rest of what Bernardo did like this:

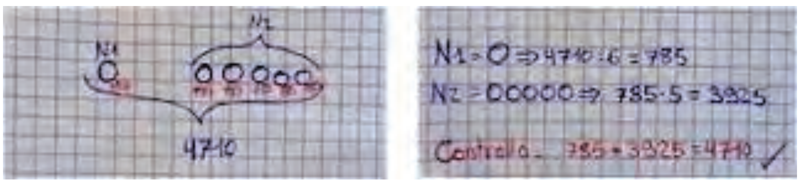

Figure 4: How a student exploited the representation with small circles in the second task

Therefore, students' attempt to follow Bernardo's reasoning, based on the use of circles, to complete the solution to the problem, helped them give meaning to the problem itself. At the end of the whole classroom discussion, they seemed to have grasped the usefulness of the representation proposed by Bernardo that can be used to find out what calculations to do. For example, a student participated in the discussion observing that for them the time for representation usually came after the time for calculation, whereas Bernardo seems to have swapped this order:

In my opinion Bernardo's reasoning was to reason first and then to do the calculation, even though I usually do the calculation first and then the reasoning to explain how I came to solve it.

The teacher highlighted this observation, repeating it aloud, believing that it could be a key element to promote for all students the a priori construction of representations for thinking about the problem.

Conclusion and implications

In this paper we reported on two tasks taken from a didactic sequence on arithmetic word problems and discussed some episodes that happened in one of the experimental classes. The first task is aimed at eliciting different proposals for possible representations from students and their preliminary considerations and ideas on the use of a representation in problem solving processes. In line with the literature on arithmetic word problems, students found some difficulties in solving the problem proposed. We also observed that the representation did not prove to be a useful tool for them to find a solution strategy, but it was used for showing and communicating their answer to the teacher and classmates. Having this in mind, the second task is designed to mediate the process of *internalisation* that may transform the representation into a *psychological tool*. In our experience, this task has been crucial in changing the role of representation for students, because it made them gain awareness of the effectiveness of representation as a possible tool for thinking in problem solving. To achieve this shift towards the representation as a psychological tool for students, the teacher's role in designing the task, within the group of researchers and teachers, and in orchestrating the whole class discussions was fundamental. Indeed, after the second task, in the later phases of the didactic sequence, most of the students started to use the representation with small circles to solve this type of problems, and we think that a number of factors contributed to this success. First of all, the representation with small circles was not provided explicitly by the teacher, but it was the result of a dialectic process. In addition, students were asked to help Bernardo to complete the solution, based on his reasoning that was not centered on calculations. Finally, during the mathematical discussions the teacher repeatedly drew students' attention to the fact that a "good" representation is generalized, i.e., it allows them to solve different problems of the same type. For instance, in our case it could be always possible to use a small circle to represent an unknown quantity, regardless of context, and then representing the other data of the problem through a set of small circles, whose number depends on the relationship given in the text.

A relevant didactical implication of this study is that, in our opinion, it is possible to support learners' representational and meta-representational competences through careful design and methodological choices made by the teacher. The presented didactic sequence on arithmetic word problems is an example of task design in which students acquire the technical tools that, through a process of *internalisation*, become *psychological tools* for them. This is an exploratory study and the results are still at a preliminary and local level, however, we hope that the conclusions we arrived at can inspire the work of teachers and researchers who share with us the assumption that using a representation as a tool for thinking when solving arithmetical problems is of fundamental importance.

Acknowledgment

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Optimization without calculus: What benefits might using physics-based visualizations in the classroom bring?

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Keywords: Optimization, experimentation, physical representations, methods of problem solving.

Introduction

A lot of questions which are at the core of optimization problems are already asked by both primary (age 6 – 15) and secondary (age 15 – 19) school pupils. Nevertheless, it is only quite late (if at all) that the answers to these questions are provided. This is because in the Czech curriculum optimization is taught mainly after derivatives. This sudden appearance of new type of problem causes many obstacles in pupils' understanding. To avoid this, a series of optimization problems and methods, usable at either primary or secondary school, was collected from the history of optimization. One of the methods identified is based around using physical representations.

A series of semi-structured interviews based around this visual method was conducted. The main focus of interviews was to analyze the way pupils react to uncommon and novel methods of problem solving. I sought to find out if physical representations could be suitable for classrooms and to find what advantages and disadvantages such demonstrations might bring for both pupil and teacher.

Theoretical background

Many problems from the field of optimization were solved using parallels with the physical world. For instance, *Heron's problem* was solved by observing the way light beams travel (Rojo & Bloch, 2018) or *Snell's law*, from physics, was used to solve the *brachistochrone problem* by Bernoulli (Tikhomirov, 1986). This approach is not as common today, as it lacks rigor and generality (Polya, 1954). Levi (2009) states that mathematics and physics are so intertwined that one without the other is deprived. Levi (2009) also considers mechanics to be geometry focused on touch and movement. These two elements add extra benefits which geometry otherwise lacks. Using physical experiments allows pupils to perceive the problem with other senses, which leads to multisensory learning.

Methods

For the research, a series of interviews was conducted. Interviews were in form of a dialog between participant and researcher (myself), as I wanted to observe the way the pupil worked with manipulatives. The length of interviews varied from 30 minutes to one hour depending on the pupil's ideas, focus, and will to go on. The research sample consisted of 8 primary and 9 secondary school pupils. Pupils from primary school were selected based upon their positive attitude towards the subject and their skills in mathematics and physics. Pupils from secondary school were volunteers willing to participate in the research. Tasks selected for the interviews were picked based upon two criteria. First, the problems had to be clear enough for both primary and secondary school pupils to understand, and second, the experiments via which the problems were solved had to be real-life performable (not merely thought experiments). The first problem discussed was *Steiner's* *problem*. Pupils were to use the soap films to find the *Toricelli point* and its properties (Courant & Robbins, 1996). A special contraption consisting of two transparent rectangular plates connected to each other by three columns was used. This contraption was submerged into soap water and taken out. Between the plates and the columns, soap films were formed. When looked at from above, we saw a perfect 2D representation of the shortest path with the *Toricelli point* as an intersection of three soap films. The second approach used three weights of the same weight and three pieces of string of arbitrary length. The strings were tied together and the other end was attached to the weights. Three pulleys were needed as well. After we hung the contraption on the system of the pulleys (which formed a triangle) the weights would eventually find a state of equilibrium and would not move anymore. The position of the knot which tied all three strings together represented the *Toricelli point* (Levi, 2009).

Results

It was difficult for pupils to work with manipulatives on their own. They often gave up quickly and had to be guided through the experiment. Their guesses were based solely on visual impressions that the point (intersection of strings/films) is a center of gravity, inscribed or circumscribed circle etc. Pupils mostly abstracted from the physical world and looked at the results as if they were only 2D representations on paper. They saw films and strings as lines and ignored weights, gravity or energy necessary for understanding. After pupils understood the meaning behind the experiments, they were fascinated by the fact that it works. That was an important moment, as the pupils themselves were motivated to look for explanation. Both experiments awoke curiosity in pupils as they asked “why” and “how”. After initial struggle, pupils considered the explanations to be easy and playful. From that I assume this kind of experimentation could be powerful in the classroom as it makes the problems accessible for younger children and motivates them to learn. Several misconceptions in the children’s understanding were revealed as well that might otherwise remain hidden. This suggests the experiments could serve as a diagnostic tool for teachers.

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From figural to theoretical control (and back): a first proposal for framing the interplay between different controls

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Grounding on Fischbein's studies, our work is intended to deeply investigate the systems of control which influence the dialectic between the figural and the conceptual component of figural concepts. We propose a first attempt for characterizing the systems of control by introducing a fine-grained distinction between three dimensions: visual, spatial, and theoretical. This theoretical frame is used for analyzing some excerpts from a case study which involve graduated students in Mathematics, while solving a geometrical task concerning the reciprocal position and the intersection point of cube diagonals. Through the analyses of these excerpts, we show evidence of the three types of control and of their interplay. Finally, theoretical and educational implications of the study are discussed.

Keywords: Figural concepts, systems of control, theoretical control, geometrical reasoning, cube diagonals.

Introduction

Research in mathematics education has developed theoretical constructs, useful for interpreting students' conception of geometric objects and its evolution while learning geometry. A fundamental contribution in this direction was given by the inspiring work of Efraim Fischbein, whose psychological point of view on geometrical objects had a huge impact on many subsequent studies. Starting from a strong assumption on the multifaceted psychological nature of geometrical objects which shares both figural and theoretical aspects, *geometrical reasoning* is interpreted in terms of dialectic between these different aspects and respective *systems of control* (Fischbein, 1993, p. 151). Broadly speaking, the systems of control seem to be involved in recalling and transforming the figural or the theoretical aspects of geometrical objects. They are activated during the geometric reasoning hence they have not to be considered as a specific phase of the problem-solving process. Despite the central role of these controls in geometrical reasoning is recognized (e.g. Mariotti, 1992; Mariotti & Baccaglini-Frank, 2018), a precise description of them and of their possible interplay still remains opaque. Nevertheless, shedding light onto the construct of control will provide a new insight into the solver's geometrical reasoning and consequently inform the design of educational interventions for overcoming student's difficulties.

In this paper, we intend to turn on a spotlight on Fischbein's construct of *control* as a fundamental element of geometrical reasoning. More precisely, we intend to theoretically elaborate on the notion of controls and on their possible synergy. According to this general aim, we propose a frame based on Fischbein and Mariotti's works related to the topics of figural concepts and controls for analyzing different types of control activated by students while solving geometrical tasks.

Theoretical framework and research aim

Fischbein's *Theory of figural concepts*

Building on the *Theory of figural concepts* (Fischbein, 1993), we consider geometrical objects as having a dual nature. More precisely, when people use or refer to geometrical objects, they are thinking in terms of *figural concepts*, that are mental entities which simultaneously possess both *conceptual* and *figural* components. Figural concepts "reflect spatial properties (shape, position, magnitude), and at the same time, possess conceptual qualities - like ideality, abstractness, generality, perfection" (Fischbein, 1993, p. 143). In principle, the two components are strongly intertwined and blended. However, more often the conceptual and the figural components remain under the influence of the respective *system of control*. Indeed, the figural component is influenced by the Gestalt theory of perception or graphical constraints; on the other hand, the conceptual components may be affected by logical fallacies. Following Fischbein, Mariotti (1992) has described the different intervention of the two systems of control as follows.

> The figural control system suggests transforming the drawing, moving (translating, rotating, reflecting, ...) the pieces, changing their places [...] But, only the conceptual control system can affirm the possibility and the correctness of this procedure. (Mariotti, 1992, p. 15)

Moreover, the conceptual control allows the solver to cope with the fact that the same geometrical object can play different roles or have different theoretical properties within the same geometrical configuration (Mariotti, 1995). For example, in a rectangle a diagonal can be conceived also as the hypotenuse of a right triangle, depending on the solver's aims. In this description of the conceptual control both spatial and theoretical elements are involved, but each of them contributes differently to the geometrical reasoning. Indeed, the mastery in managing the spatial aspects allows the solver to see the same object as part of other ones, while the domain of theoretical aspects allows the solver to put these objects in relation to their mathematical description (Mariotti & Baccaglini-Frank, 2018).

Grounding for a characterization of *controls*

In light of the brief literature review discussed above, the construct of *control* echoes the Saada-Robert's (1989) *control function* which concerns organizing knowledge based on the conditions of the given situation and the Vinner's (1997) *control mechanism*, which allows the solvers to dominate, more or less consciously, their spontaneous associative reaction to a given stimulus. However, broadly speaking, although Fischbein has theorized the role that the two systems of control might play in geometrical reasoning, he does not provide an operative definition of them, so that a researcher could observe and study these controls in the geometrical reasoning of an actual solver. A first attempt in this direction was made by Mariotti and Baccaglini-Frank (2018) who provide a definition of a kind of control that is strongly intertwined with the reference mathematical theory within which the geometrical reasoning is carrying on (in this case, the Euclidean geometry). *Theoretical control* is defined as the act of "mentally imposing on a figure theoretical elements that are coherent in the theory of Euclidean geometry" (Mariotti & Baccaglini-Frank, 2018, p. 156).

Moreover, Fischbein considers both shape and position together as parts of the figural component. However, according to the most recent finding in cognitive psychology, there is a distinction to be made in the processing of the *spatial* (e.g., position of an object in space) and *visual* (e.g., color, shape, texture) attributes of an object that is perceived or mentally manipulated. This distinction is corroborated by evidence that two different neuronal pathways are involved in processing spatial and visual information, respectively the "where" and the "what" pathway (Anderson, 2015). For instance, considering a *rectangle* we can refer to the global appearance of the shape which answers to the "what" question about the object, otherwise we can refer to the mutual position of its parts which answers more likely to a "where" question. Furthermore, from a developmental point of view, the spatial aspects play a fundamental role in the personal construction of the conceptual component of a figural concept. Indeed, at the beginning, geometry relies on a natural organization of space that gives a contribution to the conceptual components (Mariotti & Fischbein, 1997). So, within the domain of geometrical reasoning and in order to consider the psychological perspective – in line with the very first Fischbein's declared intention – a solver might exercise a control upon the visual aspects and/or upon the spatial aspects. So, besides *theoretical control*, we will consider and make a distinction between *spatial control* and *visual control*. For example, in the case of the figural concept of rectangle, students might focus on its "rectangularity", since they has experienced the rectangular shape of physical objects (visual control); they might zoom into the shape and focus on the diagonals as part of the rectangle (spatial control); finally, they can recall some other figural concepts that are consistent with the rectangle, such as the right angles or the parallelogram as a more general case of quadrilateral (theoretical control).

In light of the theoretical framework presented here, this exploratory study intends to problematize and shed light onto *systems of control* which intervene in geometrical reasoning in order to pave the way to an operative definition of them which can be shared within the community of researchers in mathematics education.

Methods

The present study is focused on a case study involving 18 master students in mathematics of the University of Turin. The selection of a sample of expert solvers was driven by the hypothesis that these students might possess well harmonized figural concepts. In this paper we focus on the "cube diagonals problem", which was solved by the involved students in groups of three. Each group spent around 30 minutes in solving the task. Data was collected in December 2019 and consists of students' written productions and video-recordings of three groups-work.

Data are analyzed focusing on instances of *visual*, *spatial*, and *theoretical control*. More precisely the control is coded as *visual* when a figural concept is recalled through drawings, gestures or words which describe its global appearance (e.g. showing through a gesture or drawing a triangle; determining the congruence of two triangles according to the figural resemblance of the whole shapes); the control is *spatial* when figural concepts are conceived within a physical reference frame or zoom in/out movements are produced within the considered figural concept (e.g. focusing on right triangles inside the rectangle; among these two, focusing on the triangle "on the right"); the control is *theoretical* when figural concepts are introduced in a way that is coherent with the Euclidean geometry (e.g. justifying the congruence between right triangles inside rectangles by considering the congruence criteria of triangles; identifying a triangle focusing on the three points that generate it).

The cube diagonals problem

The given task is a reformulation of a geometric problem described by Mariotti (2005): "*Determine if the diagonals of a cube meet and if they meet perpendicularly*".

As observed by Mariotti (2005), despite the cube being a familiar polyhedron, students' mistakes in solving this task are common. In order to see whether diagonals are perpendicular or not, it is necessary to conceive them as diagonals of rectangles made by an opposite pair of cube's edges and an opposite pair of cube's diagonal faces. Since diagonals of these rectangles are not perpendicular to each other, the answer is negative. Moreover, in a rectangle, the intersection point of diagonals and their respective midpoints coincide. This further observation supports the unicity of the intersection point of cube diagonals. In the following section we will report the analysis of some excerpts of the video-recording of groups-work. We will refer to them as group A, B, and C.

Data analysis

The first extract is taken from group C. Before starting to solve the problem students share in the group their conjecture: diagonals meet in the "center" of the cube. The word "center" suggests that the figure is conceived within a physical reference frame and that the students are zooming inside the cube, hence that they activate a *spatial control*. Then, students make a drawing of a cube (Figure 1a) and confirm that diagonals meet by activating the *visual control*. Looking at the drawing Ellen says that she is imagining planes made by pairs of cube diagonals. With these words, Ellen starts introducing *theoretical* elements in the figure. The following extract is taken some minutes after her observation:

Ellen: But sorry if you think of these as squares, but placed obliquely, the diagonals are those of the square and they meet and are perpendicular *(mimicking the perimeter of the square and the diagonals with her pen, see Figure 1b)* … That is, if you think of the plane that passes through CE and BG *(gesture with the palm of the hand, see Figure 1c),* these two *(mimicking the diagonals with the pen)* are the diagonals of the square that I construct like this *(with the finger she links the extreme points of the diagonals, mimicking the perimeter of the square)* and meet and are perpendicular.

Richard: It is not a square!

Ellen: You are right, it's a rectangle!

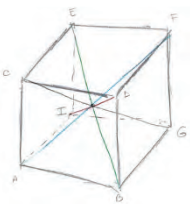
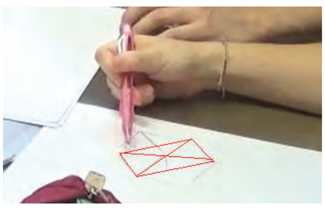
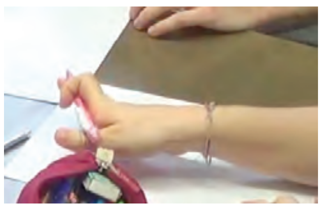
(a)  (b)  (c) 

Figure 1: The drawings and gestures performed by Ellen during the resolution of the given task

Ellen is imaging "squares placed obliquely" which lay on planes that pass through diagonals. The activated control is *spatial* because she conceives planes in a physical reference frame which allows her to see them obliquely and because she made a zoom-in movement from the cube to some of its elements. The activated control is also *theoretical* when the plane is introduced from diagonals CE and BG; Ellen makes a mistake in the identification of the quadrilaterals generated by imaging to link the extreme points of cube diagonals: activating a *visual control* she visualizes squares instead of rectangles. Possibly Ellen's thoughts are driven by the automatic association (Vinner, 1997) which relates the same geometric object in different dimensions (cube in 3D space to square in 2D plane). From this incorrect information, a wrong conclusion is driven ("[cube diagonals] meet and are perpendicular") by activating a control that looks like a *theoretical control* but in fact involves some inconsistent elements.

The following extract shows the discussion on the same geometrical objects (planes inside the cube, type of quadrilateral obtained and relative position between diagonals), taken from group A:

Brian: Eh I'm seeing them... this side here *(pointing to an edge of the cube, see Figure 2a)* and this side here *(pointing the "opposite" edge)* detach a plane *(pointing the diagonals of faces, see Figure 2b)*. A plane where ... this side [measures] 1, this [side] is square root of 2 *(pointing and writing, see Figure 2c)*. It's a rectangle, right?

Red: It looks like a square to me, but in fact...Yes, [sides] detach a rectangle, the diagonals of a rectangle are not orthogonal.

(a) (b) (c)

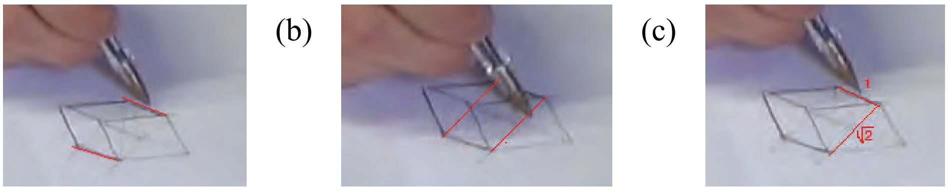


Figure 2: The gestures performed by Brian during the resolution of the given task

In planes recognition, Brian activates a *spatial* and *theoretical control*: the first one allows him to see planes inside the cube with a zoom-in movement ("I'm seeing them"), while the second one allows him to construct them theoretically from cube "sides" ("this side here and this side here detach a plane"). Rectangles inside planes are identified partially by sides' measures, hence activating the *theoretical control* ("his side [measures] 1, this [side] is square root of 2") and partially by observing the shape of the drawing which allows him to discard the parallelogram configuration, hence activating the *visual control*. The concomitance of the *visual* and *theoretical control* might explain the uncertainty expressed by the final question mark on the identification of the quadrilateral nature. Red's words show the predominance of *theoretical control* over the *visual control*: even if visually the quadrilateral "looks like a square", the theoretical elements introduced by Brian allows him to affirm that it is a rectangle. Finally, the conclusion that "the diagonals of a rectangle are not orthogonal" is driven under *theoretical control*.

The following extract is taken from Group B:

Beth: If you cut it in this way (*mimicking the rectangle perimeter with the pencil, see Figure 3a*).

Lucas: Oh, I also thought about these here (*mimicking an X with the pencil, see Figure 3b*).

Beth: These are the diagonals of a rectangle.

Lucas: Yes, exactly they are the diagonals of a rectangle and they meet.

Beth: But not perpendicularly.

Lucas: Diagonals of a rectangle meet perpendicularly (*making the gesture in Figure 3c*), yes always. […] No! (*Lucas draws the rectangle in Figure 3d while Beth draws two intersecting segments on her worksheet*). This drawing shows a counterexample.

Beth and Lucas, activating a *visual* and *spatial control*, introduce a rectangle and its diagonals as figural elements in the drawing (Figure 3a-b). Then, activating a *theoretical control*, Beth identifies the cube diagonals as rectangle diagonals (“These are the diagonals of a rectangle”) and deduces their non-perpendicularity (“But not perpendicularly”). Lucas, activating mainly a *visual control* shown in his arms gestures (Figure 3c), reaches the opposite conclusion (“Diagonals of a rectangle meet perpendicularly”) and then change his mind supported by the produced drawing (Figure 3d).

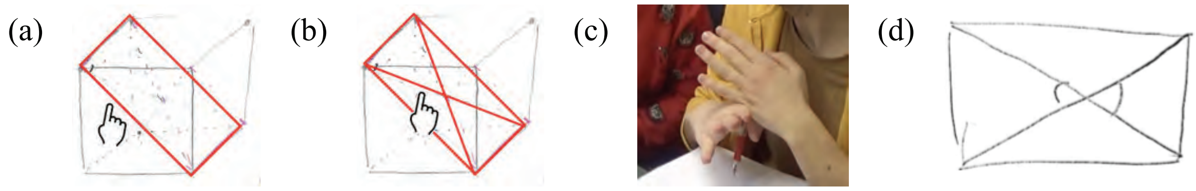


Figure 3: The drawings and gestures performed by Group B during the resolution of the given task

While solving the problem, all the three groups discuss the unicity of the intersection point of diagonals. Group C establishes the unicity of the point activating a *visual* and *spatial control* but failed in showing it with a *theoretical control*. Visually they show in the drawing that all four diagonals meet in the same point and spatially they conceive changing pairs of diagonals in which each pair has a diagonal in common with the previous considered pair. However, since they do not observe that the meeting point is the midpoint of diagonals, they are not able to justify theoretically why the meeting point does not change. Although they try to activate theoretical control by attempting to formalize the relationship between the diagonals, they finally get lost in their reasoning mainly because they have lost the visual and spatial counterparts in this sort of blind formalization.

Instead, Group A and B observe the property firstly activating a *visual* and *spatial control* and then activating the *theoretical control*. The following extract is taken from Group A, and shows the attempts made by students to answer their own question: “Taking the other planes then, their pairs of diagonals of the other two meet at the same point where these two ones meet here?”.

Nick: Yeees... I would say that these two also meet and that the point is the same. Definitely!
Ellen: But yes, it is certainly that because these two *(referring to a pair of diagonals)* meet in the middle, let's say, and also these other two *(referring to another pair of diagonals)* in the middle, both from one sense and from the other, so it is the same point.
Nick: This is the midpoint. Take another plane…
Ellen: Yes, yes...
Nick: You find another point, that is the midpoint then...
Ellen: And therefore it is unique.

The excerpt shows a motion in the activated control from *visual* to *spatial* to *theoretical*. Nick’s first sentence, introduced by the conditional tense (“I would say that...”), suggests that what he states (“the point is the same”) does not come from his theoretical knowledge but could be the result of a visual clue made activating the *visual control*. Moreover, the *spatial control* is intertwined with the *visual* control, since the sameness of the point is reached by zooming in and out between the point and the diagonals (“these two”) and back. The conclusion of his statement with the expression “definitely” seems an anticipation of the *theoretical control* activated afterward. Ellen’s observation “[diagonals] meet in the middle [...] from one sense and from the other” suggests the introduction of different viewpoints on the cube diagonals, hence the activation of a *spatial control*. Also the word “middle” suggests the spatial idea. Nick gives a theoretical status to the different viewpoints, speaking about different planes and to the "middle" point, speaking about "midpoint". Through Nick's words, Ellen's *spatial control* evolves in *theoretical control,* guiding Ellen to the conclusion of midpoint unicity.

Conclusion

In this study we have presented a first attempt for clarifying and operationalizing the controls that might come into play in geometrical reasoning of expert solvers. From the theoretical point of view we have combined the traditional literature in mathematics education on the figural concept (Fischbein, 1993) and the more recent findings in cognitive psychology on the visual and spatial aspects of imagery (Anderson, 2015). In this sense, we propose three different dimensions of control: *visual*, *spatial*, and *theoretical control*. Data analysis shows how this tripartite structure allows us to gain a deep insight into the complex observation of cognitive processes which involve the solver's use of control. In particular, we can observe the different contributions of the spatial and theoretical aspects that were originally encapsulated into the conceptual control.

The analysis shows that, during the discussion of reciprocal positions between cube diagonals, all groups activate *conceptual control* by introducing planes and quadrilaterals with a zoom-in inside the cube conceived within a physical reference. At the same time all groups activate a *figural control* since these planes and quadrilaterals are drawn or described through words and gestures. Distinguishing just between the conceptual and the figural control, all analyzed solutions seem similar to each other, showing harmonized figural and conceptual components of figural concepts involved. However, by introducing *visual, spatial,* and *theoretical control* important differences emerge: Ellen (group C) and Beth's (group B) controls are characterized by a predominance of *visual* and *spatial* over *theoretical*, while Brian's controls (group A) are characterized by a predominance of *theoretical* and *spatial* over *visual*. The predominance of *visual* and *spatial* over *theoretical* leads Ellen to make mistakes in the quadrilateral identification. Differently, the *theoretical control* activated by Brian allows him to conceive the quadrilateral not only from its shape but also from the property of congruence between opposite sides, leading to correct quadrilateral identification. Building on data analyses, we can sketch a preliminary model of the mutual interactions between systems of control: using Fischbein's terminology and imagining the three different dimensions of control in a continuum spectrum from *visual control* to *theoretical control* passing through *spatial control*, the solver's *figural control* might range from the visual to the spatial control and the *conceptual control* might ranges from the spatial to the theoretical control. Ellen and Beth's controls are located mainly at the visual and spatial region of the spectrum (predominance of figural over conceptual) while Brian control is located mainly at the spatial and theoretical region of the spectrum (predominance of conceptual over figural). This continuous spectrum model represents a new hypothesis to be tested and explored in further research.

The finer lens through which we are looking at controls, allow not only to better locate one's control on this spectrum but also to analyze control movements in group dynamics between visual, spatial, and theoretical. For example in group C, Ellen's visual/spatial control moves to a theoretical control thanks to Richard's intervention. In group B, Beth's theoretical control moves to visual thanks to Lucas's intervention. Also in group A we can observe controls' movements in Red's words thanks to Brian's observations. However not always these movements appear in group-works, this is the case of group C while discussing the unicity of the intersection point, remaining anchored in the theoretical control which cannot evolve due to absence of spatial and visual control activation. Moreover, by definition, theoretical control is always consistent with the reference mathematical theory; however, Ellen's first excerpt shows how students might manifest a discontinuous theoretical control, that is a control which might look like a theoretical control, but which in fact contains some inconsistency. Borrowing Vinner's (1997) terminology we might call it *pseudo-theoretical* control. So, further research could investigate what determine the distance or the closeness of the instances of control manifested by an actual solver to the sought-after theoretical control. From an educational point of view, a teacher aware of all the dimensions of control could be more receptive in observing a lack in the theoretical, spatial, or visual components or in their dialectic and consequently can intervene properly supporting students' geometrical reasoning development.

Although we cannot draw any general conclusions because we still have analyzed few data, the excerpts offered seem to support our interpretation of control as a complex structure that can be manifested in different interacting dimensions. We hope that considerations emerging from this paper can draw the attention of our community of researchers on the necessity and opportunity to further elaborate on Fischbein's inspiring notion of control.

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Did you know you can draw a huge number of infinite heights? The students' realization tree of the heights of a triangle

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In this paper we report on the analyses of the mathematical discourse of 7th-grade students about the solution of a task on the recognition of the heights of a triangle. Our aim is to describe through the commognitive lens the realizations that appear in the mathematical discourse of the classroom and to observe possible interactions between them. For this purpose, we construct and compare the expected and the actual students' realization trees, showing the richness of the realizations addressed by the participants in the mathematical discourse.

Keywords: Realization tree, commognition, heights, triangle.

Introduction

A wide literature in mathematics education has documented students' difficulties in drawing and recognizing the heights of a triangle and has highlighted that the most common difficulties are related to obtuse triangles and right triangles (Hershkowitz, 1989; Mariotti, 1995). Among the several reasons for explaining these difficulties, there is the influence of specific representations of the height, that are presented at school, and which become so popular and widespread that they are considered the concept itself by many students (Hershkowitz, 1989). In particular, at least in the Italian context, at the end of primary school students' idea of "height of a polygon" is strongly related to stereotyped representations of it and this is often the result of a univocal definition of height (Sbaragli, 2017).

Based on this scenario, and on the assumption that the notion of height is involved in many important mathematical results that students will face during their learning experiences, this study investigates the effects of a didactical approach for introducing students to the notion of heights of a triangle based on the presentation of multiple characterizations and representations. The effects are described in terms of the features of the height of a triangle, with respect to one side, that have become part of the classroom discourse.

Theoretical framework and research questions

We adopt Sfard's commognitive lens (2008) according to which discourse and its development are analyzed on their own, as main objects of research rather than as means to explore other constructs. We share the underlying assumption of this theory that thinking is not ontologically different from communicating and learning is a way of changing one's discourse. Commognition describes mathematical discourse as involving continuous transitions between *signifiers* that are "words or symbols that function as nouns in utterances" (Sfard, 2008, p. 154) and their *realizations* that are "perceptually accessible object[s] that may be operated upon" for producing narratives about the signifiers (Sfard, 2008, p. 154). These terms are used to emphasize that nothing is "there" to be represented, since mathematical objects are discursive objects. Moreover, some signifiers are more commonly used than others, but there is no one signifier that represents the object. This hierarchical role of realizations can be visually represented through *realization trees* (Sfard, 2008). In this study, we are interested in the classroom discourse on the heights of a triangle. The construction of the students' realization tree allows us to gain insights into its growth and richness as resulted from a didactical sequence, thus, giving information on the formation of the mathematical object "height of a triangle" in students' discourse. Usually when students deal with a new signifier, they use different realizations separately; the conscious reference to the same signifier by using many different realizations is a learning achievement which is the result of a *saming* process (Sfard, 2008).

One of the main elements characterizing the mathematical discourse are *narratives*, i.e. statements (written or spoken) formulated as descriptions of objects, relationships between objects or processes, which are subject to possible approval or rejection. The shared narratives of a classroom are those that are considered true; they may be different from those accepted by the scientific community. Moreover, mathematical discourse involves a consistent use of *routines*, i.e. specific repetitive patterns (Lavie et al., 2019) and of *visual mediators*, i.e. objects of symbolic, iconic, gestural nature to which we refer (Nachlieli & Tabach, 2012). For example, in geometry, the drawing of a figure can be a visual mediator in both paper-and-pencil and dynamic geometry environments.

In light of this theoretical lens, the research questions guiding our investigation are: (a) What realizations of the signifier *height of a triangle* can be identified in students' discourse? (b) Are there interactions between different realizations? If so, when do they occur?

Methodology

The data we present are collected in a 7th-grade class, during a video recorded lesson on the recognition of the heights of a triangle. The students were introduced to the mathematical object *height* through different possible realizations. In particular, two main realizations were proposed by the teacher, who was participating in a larger project aimed at designing and experimenting inclusive mathematical activities at middle school. We describe these two realizations as follows:

T1) *Segment perpendicular to a side, or to its extension, passing through the vertex not contained in that side (opposite).* It was discussed with students that this realization can be realized in two ways: as a segment drawn from the vertex (*T1-a*) or from the side (*T1-b*). The teacher also promoted the construction of heights using both physical (ruler and set square) and digital (*GeoGebra*) artifacts.

T2) *Height of the strip in which the triangle is inscribed.* This realization addresses the height of a triangle in terms of the height of the strip into which the polygon can be inscribed. A strip is determined by two parallel lines: one line contains a side of the triangle and the other the vertex that does not belong to the chosen side.

These two realizations are chosen because *T1* is very familiar among students at the end of primary school and it is largely proposed in school textbooks at different levels; on the contrary, *T2* was intentionally introduced for the first time by the teacher because it lays the foundation for looking at the length of the height as the class of congruent segments and for overcoming the common difficulty consisting in placing the height outside an obtuse triangle.

In this study we focus on a 1-hour lesson, conducted by the second author of the paper with the regular mathematics teacher. Students are given a task (Figure 1), involving an obtuse triangle.

Observe the figure.
Are all the four dotted segments heights of the triangle?
Explain why.

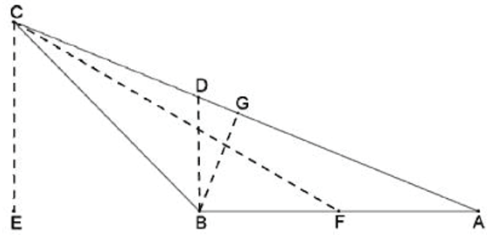

Figure 1: The heights recognition task

The task is designed for triggering the students' discussion on the four dotted segments. In order to answer the question, students have to argue why each of them can or cannot be a height of the given triangle. The explanations will inform us on the realizations that the class has developed until then.

The *a priori* realization tree

Building on Weingarden and colleagues' (2019) elaborations, the theoretical realization tree (Figure 2) is *a priori* constructed according to all the possible realizations of the height that students, who were exposed to specific geometrical activities on the height of a triangle, might refer to.

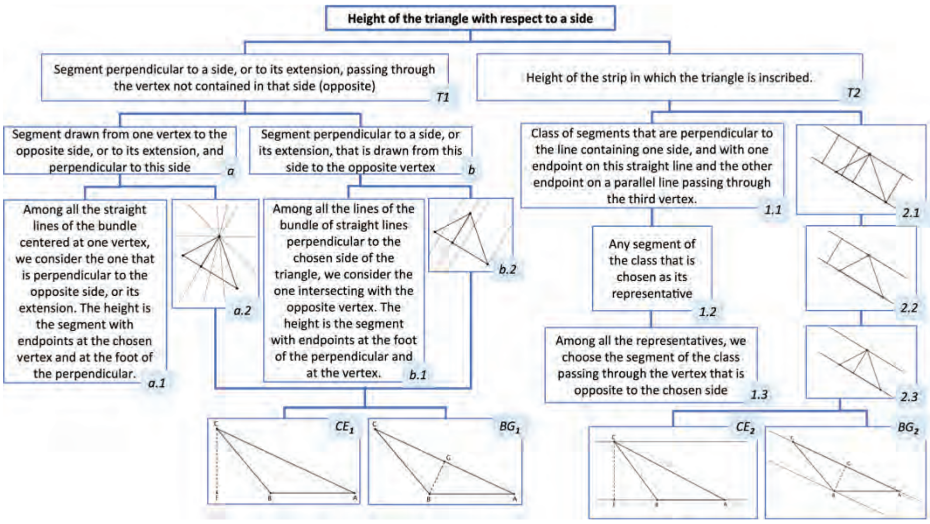

Figure 2: The a priori realization tree of the signifier *height of the triangle with respect to a side*

The tree has the root in the signifier "height of the triangle with respect to a side". From this root, two main branches with T1-realizations (on the left) and T2-realizations (on the right) develop; each branch contains both verbal and visual realizations. At the first level under the root, there are the main realizations that are explicitly used during the geometrical activities.

Focusing on the left side of the tree, the node at the first level splits into two branches (*a* and *b*). At this level the realizations (*a.1, a.2, b.1, b.2*) maintain some traces of the construction rituals. *T1-a* realizations characterize the height by starting from all the straight lines passing through a vertex and choosing the line that perpendicularly intersects the straight line upon which the side of the triangle lies; the second extreme of the height is reached in this way. Within the tree, we reported these realizations by referring to the bundle of straight lines centered in a vertex of the triangle. In these realizations the starting point is the chosen vertex. *T1-b* realizations characterize the height by starting from all the straight lines perpendicular to the side of the triangle and choosing the line that passes through the non-consecutive vertex. Within the tree, we reported these realizations by referring to the bundle of straight lines perpendicular to the chosen side of the triangle. In these realizations the starting point is the chosen side.

Focusing on the right side of the tree, starting from a realization with many heights (*1.1*, *2.1*), following the branch we can reach a realization where only one height passing through a vertex is considered (*1.3*, *2.3*). The coexistence of these different realizations in a student's tree presumes that a *saming* process was accomplished, as for the expert mathematicians.

The leaves at the lower level make explicit the realizations of the heights with respect to the given task. Although they are visualized only as visual realizations, since the given drawing provides the student with a common visual mediator, we use them also for verbal realizations such as "*CE*" or "*BG*" which make sense only coupled with the given drawing.

How the analyses were conducted

Data were cyclically analyzed, passing through several rounds of analyses. The preliminary step is the transcription of the classroom discussion by reporting all the components of students' discourse, i.e. verbal utterances, drawings, gestures. Working on this transcription, the first round is aimed at identifying the instances of realizations – labeled using a progressive number in square brackets – within the students' discourse. For example, an utterance as "*if you put the ruler on BA, you have also the height CE*" is coded as a verbal realization of the height with respect to the side *AB*.

The second round focuses on these instances of realization. The analysis aims at identifying the type of realization that students refer to. In particular, the coding is conducted according to the closeness of students' discourse with one of the descriptions of the height as a T1 or T2 realization. For example, a statement such as "*it is a height because it goes to the opposite vertex*" is coded as a verbal T1-realization of the segment traced from the side to the vertex, and the reference to the movement towards the opposite vertex allows us to label it as a T1-b realization. The closeness is also established according to the inferences of the researcher who knows the endorsed narratives developed by the class during the previous activities. For example, the expression "Paolo's method" is coded as a T2-realization because it was developed by the students for shortly referring to the ritual of drawing the heights inside the strip. This round allows us to qualitatively describe the characteristics of each realization and therefore provides useful information for answering the first research question.

These two rounds of analyses are fundamental for gathering the essential information on the realizations to be exploited for the construction of the actual realization tree. For the subsequent round, we start from an empty structure of the a priori realization tree; we read again the labeled transcription for filling the leaves with the corresponding realizations. For example, since the first instance of realization emerges in Alessio's discourse and it is coded as T1-b, we write "*[1]Alessio"* in the leaf named *b* (Figure 3). The unexpected realizations are included as new leaves with the label *NEW*. After this round of analyses we obtain a new tool for observing at a glance how the realizations were spread during the discussion. Moreover, this round allows us to observe which realizations of the height have become part of the classroom discourse and which not yet. The progressive numbers coupled with the name of the students allow us to keep track of the temporal development of the realizations during the lesson and of the students who participated in the mathematical discourse. The actual realization tree provides us with a tool for observing the possible interactions among different realizations, and therefore for answering the second research question.

Preliminary findings: the realization tree as a tool for analyses

The actual realization tree (Figure 3) highlights that there are some empty leaves for the expected realizations that are not addressed by students and new leaves for unexpected realizations that emerged within the classroom discourse.

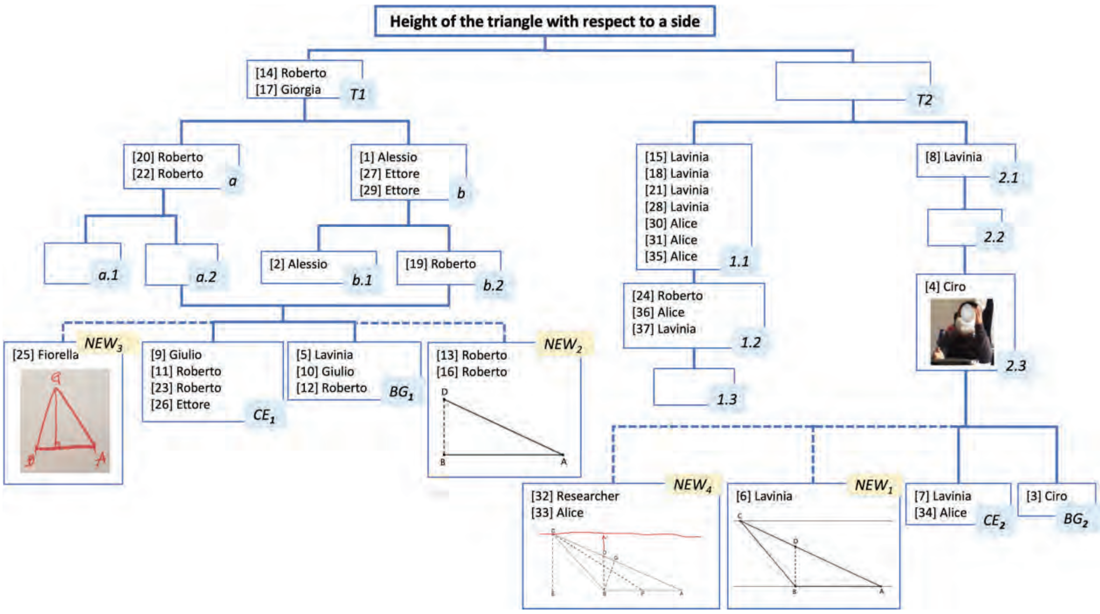

Figure 3: The classroom realization tree of the signifier *height of the triangle with respect to a side*

Comparing the a priori and the actual realization trees

As regards the expected realizations that are not addressed by students, we notice the bundle of straight lines centered in a vertex of the triangle. Although students refer to realizations of height that are drawn from the vertex towards the side of the triangle ([20], [22]), in their discourse we do not find elements that suggest the choice of the perpendicular line among the many passing through the vertex. This aspect can be justified by frequent references to the construction ritual through ruler and set square. Among the T1-realizations, this ritual shifts much of the students' focus toward realizations that implicitly make use of the bundle of parallel lines ([2], [19]); in this way, the perpendicularity is embedded into the use of the set square. Among the T2-realizations, the empty leaf in Figure 3 shows that there are no explicit references to the strip in which the triangle is inscribed. Rather, the students' discourse seems to be linked to the construction ritual, that is Paolo's method, which involves a line parallel to the side in focus.

As regards the unexpected realizations, there are leaves (see *NEW* in Figure 3) which report on the discourse on segments different from *CE* as realizations of the height for the side *AB*, and realizations which refer to triangles different from *ABC*. In the former case, students' discourse is focused on the segment *BD*, after that Lavinia triggers the attention on it ([6]) as a possible T2-realization of the height. Then she adds further details ([15]):

Lavinia: If you put the ruler on the segment AB, okay, are you with me? And then you take the set square. You start drawing some heights and BD is one of those.

Roberto accepts, to some extent, Lavinia's idea of considering *BD* as a height, but he addresses a different triangle ([16]):

Roberto: I think you're looking at a hidden triangle ABD and you think BD is a height.

Claims like this lead us to include another leaf in the actual tree (see *New*$_2$ Figure 3). This is a T1-realization since Roberto's discourse does not contain any reference to the strip.

The last new T2-realization is introduced by Alice who proposes a refined version of Lavinia's realization of the height. Alice starts describing how to draw a height for *AB* which passes through point *B* ([31]); the researcher supports her description by adding new visual elements on the visual mediator, which is shared with the whole classroom. Comparing the new realization with the segment *BD*, Alice explains why *BD* is not a suitable realization of height for *AB* ([33]):

Alice: Therefore, BD cannot be a height, because it has to reach at least... the intersection.

Building on Anita and Lavinia's discourse, students consider the possibility of having more than one height for *AB* and continue arguing on these T2-realizations.

Another unexpected realization is the one proposed by Fiorella. She considers another triangle, *ABG,* and shares her drawing with the classmates (see *New*$_3$ Figure 3). In particular, Fiorella uses this new visual mediator to justify the possibility for *BD* to be accepted as a realization of height, and in line with Roberto's discourse she stresses the need to change the reference triangle. However, she disagrees with Roberto concerning the choice of such a triangle.

Moving from T1 to T2 realizations

By looking at the progressive numbering of the labels in the realization tree (Figure 3) it is possible to observe the distribution of T1 and T2 realizations within the classroom discourse. Generally speaking, each student seems to be tied to a certain type of realization. Indeed, during the entire lesson some students, as Alessio, Roberto, and Giulio, focus on T1-realizations while other students, as Lavinia and Ciro, focus on T2-realizations. Although most of the students remain anchored in a certain type of realization and they do not spontaneously move back and forth between different types of realizations, we can notice that when Roberto moves from the T1 branch to the T2 ([23], [24]) some other students start reporting on T2-realizations in their discourse. Therefore, the global classroom discourse goes through a shift from T1 to T2 realizations. This shift in discourse develops around the segment *BD*. Lavinia claims that *BD* is a height, addressing T2-realizations ([15]). She refers to the use of the set square, describing the construction ritual for drawing "several heights" (T2-realization). She does not speak about moving the set square in order to trace the only segment that intercepts the vertex of the triangle. On the other hand, other students argue that *BD* is not a height, bringing arguments based on T1-realizations. In this respect, the exchange between Lavinia and Roberto is very interesting:

Lavinia: So BA is the segment. Did you know you can draw a huge number of infinite heights?
Roberto: But if they don't go to the opposite vertex...
Lavinia: But that doesn't matter! They don't have to go!

The turning point occurs when Roberto changes perspective and switches from the left to the right side of the tree, that is, when a T2-realization of height enters his discourse ([24]). This shift is shown in the tree: between [23] and [24] Roberto changes side. The corresponding discourse is:

Roberto: Do you know how to make a height? You have to put a ruler and a set square, from the opposite vertex you have to draw a line to the opposite side or to its extension, that's what the EC side is for.

Roberto: If you really want to make a height here [he points at a position on AB], at least you have to use Paolo's method.

The reference to "Paolo's method" seems to be key in shifting the attention of the class towards T2-realizations. Indeed, after Roberto's intervention there are many references to T2-realizations: [28], [30], [31], [32], [33], [34], [35], [36], [37].

At this point Roberto tries to convince Lavinia that in order for the segment *BD* to be a height it lacks an important feature: the point *D* does not intercept the strip.

Roberto: Oh thanks [*sarcastic tone*], but then DB is not a height if it doesn't reach there.

The first realizations of height in his discourse involve a segment with endpoints at one vertex and at the opposite side of the triangle, so his interventions are all placed in the left branch of the realization tree. Then Roberto seems to include in his tree, through a *saming* process, also segments not passing through any vertex of the triangle, and this happens when he starts participating in Lavinia's discourse.

The transition from T1 to T2 realizations seems to play a fundamental role in the classroom discourse. By remaining anchored to only T1-realizations of height, students fail to convince Lavinia that *BD* is not acceptable. When the focus of Roberto's discourse shifts to T2-realizations also the other students become participants in Lavinia's discourse and begin to use the same words and visual mediators. Finally, the classroom discourse finds a synthesis in the realization constructed by the researcher under the students' guidance (see New_4 Figure 3).

Conclusion

The analyses of the classroom discourse allow us to conclude that students spontaneously refer to both T1 and T2 realizations of height. By looking at the realization tree (Figure 3) it is possible to grasp the richness of the classroom discourse, involving different realizations of height: students refer to most of the expected realizations and they also introduce new ones. Each student remains essentially tied to one type of realization; therefore, we do not observe in their discourse interactions between T1 and T2 realizations. However, we can notice how the interaction is promoted by the attempt to match two discourses on two different realizations. This is evident in the discussion between Roberto and Lavinia, who refer to a T1 and a T2-realization of the height, respectively, while they are searching for a shared narrative about *BD*. The interaction becomes necessary for them to participate in the same discourse and Roberto plays a key role in moving from a T1 towards a T2-realization. In this way, he succeeds in finding a common ground with Lavinia and also in involving the whole classroom in the discourse. In particular, the "use of the set square" and "Paolo's method" are shared narratives referring to rituals that have been culturally constructed by the classroom and they are linked to both T1 and T2-realizations.

Comparing the two realization trees, we can claim that students include in their discourse most of the expected realizations. From the educational point of view, this is relevant since the richness of the realization tree seems to be a marker of the effectiveness of an unconventional – but in line with the solid findings of the research in mathematics education – introduction to the height of a polygon which explicitly works on different realizations. From the theoretical point of view, the study presents a first attempt of using realization trees as tools for *a priori* and *a posteriori* analysis of the classroom discourse in the domain of geometry, where the visual mediator (i.e. the drawing) has a specific and prominent role (Mariotti, 1995). In this perspective, the classroom realization tree needs further elaboration to become a more effective tool in terms of reporting on the students' discourse behind the construction of the tree and the role of the teacher's intervention in supporting such a discourse.

Although this is an exploratory study and the findings are quite local, we hope that considerations emerging from this paper can provide educators and researchers with food for thought, in particular concerning the didactical value of promoting a widespread use of many different realizations for the same signifier as a means for triggering the students' participation into mathematical discourse.

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Representing mathematical induction in proving processes

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Using a multimodal semiotic perspective, I investigate the production and use of signs in expert (doctoral) students involved in processes of constructing argumentations and proof by mathematical induction. Focusing on their speech, written inscriptions, and gestures, different categories of signs, related to mathematical induction, are identified and analysed. In this paper I show some paradigmatic examples of these signs.

Keywords: Mathematical induction, proving processes, semiotic bundle, gestures.

Introduction

Semiotic offers an interesting perspective for research in mathematics education, providing a window through which to observe and investigate several teaching-learning processes. Recently the analysis of signs has been enriched including the study of gestures. This has involved different areas of research, as well as the studies on argumentation and proof. Arzarello and Sabena, for instance, bring empirical evidence that "gestures may also play specific roles in providing a logical structure to argumentation" (2014, p. 99). Similarly, Krause observes that the gestures' production "may support the collective act of reasoning […]. It makes traceable how the argument was organized as logical inference" (2015, p.1432). Along the same line, Sabena, registers that the use of gestures "support[s] the students in structuring the entire argumentation at a global level" (2018, p. 556). The study presented in this paper is part of wider research, conducted with a semiotic perspective, on proving by mathematical induction processes of post-graduate, undergraduate and secondary school students.

The proving scheme of Mathematical Induction (MI) is, at the same time, important and useful for a mathematician, interesting from a logical point of view, but also extremely problematic from a didactic perspective. The difficulties involved can be observed across different levels of education, from secondary school students (Fischbein & Engels, 1989), to master's students in mathematics (Carotenuto et al., 2018). A problematic aspect for students is related to its justification, i.e. why, given a predicate P on the natural numbers, we can conclude that $\forall n \in \mathbb{N}$, P(n), by proving the base case P(0) and the inductive step $\forall n \in \mathbb{N}$, P(n)$\rightarrow$P(n+1). Ernest (1984) affirms:

> Many students encountering the method of proof by induction wonder why this rather complex and seemingly arbitrary principle is adopted […] [It] is neither self evident nor a generalisation of previous more elementary experience. (p. 183).

Generally, a non-formal justification is that MI works as a "cascade" of infinite syllogisms (Poincaré, 1906, p. 9): from P(0) and P(0)$\rightarrow$P(1) it follows P(1); from P(1) and P(1)$\rightarrow$P(2) it follows P(2), and so on. Often, to provide an intuitive explanation for MI, teachers and textbooks describe it by using some images (for some examples, see Ernest, 1984). Perhaps one of the most known is the falling dominos analogy: given an infinite line of dominos (i.e. the whole set $\mathbb{N}$), by the fact that the first one is knocked over (i.e. the base case) and the fact that each couple of consecutive dominos is at the right distance so that if one domino falls, it will knock over the consecutive one (i.e. the inductive step), we can conclude that every domino after the first one in the line will fall (i.e. $\forall n \in \mathbb{N}$, P(n)).

Images (or, more generally, signs) like the cascade of syllogisms or the falling dominos, are often used by teachers and textbooks to introduce and describe MI to students. On the other side, similar signs might also be produced and used by subjects involved in proving by induction activities. In this paper, I focus on this second aspect, analysing the signs produced and used by students with experience of proving by induction during the resolution of problems which potentially involve MI.

Theoretical framework

A multimodal semiotic perspective

In order to take into consideration a wide spectrum of signs, in this study I adopt a multimodal semiotic perspective. Arzarello (2006) considers all the different kinds of sings involved in teaching and learning processes (verbal language, mathematical symbols, diagrams, sketches, gazes, gestures, etc.) as an inseparable unit. He defines a *semiotic bundle* as a dynamic structure composed by different *semiotic sets* together with the relationships between them. Within this perspective, a sign may have different components depending on the semiotic sets involved in it. For instance, a subject's spoken utterance with a simultaneous gesture, referring to a certain written inscription, can be seen as a unique sign made by three components (speech, gesture, and inscription).

Linking and iteration signs

Using this multimodal perspective, in a previous study (Antonini & Nannini, 2021), we analysed the semiotic bundle consisting of three semiotic sets (speech, written inscriptions and gestures) in some post-graduate students' processes involved in the generation of a conjecture and of a proof by MI. As a result, we identified two particular categories of signs which seem to play an important role in these processes:

- *Linking signs*: Signs produced or used to refer to two or more entities (objects, mathematical objects, problems, situations, etc.) and to their relationships, where these entities are seen in connection with two consecutive natural numbers.
- *Iteration signs:* Signs that refer to iteration, or that are composed by a repetition (in time or in space) of linking signs, or that refer to a repetition of them.

To give an example of linking and iterations signs we can interpret from a semiotic point of view the above-described image of the falling dominos. The whole image (either pictured or verbally described) is a rather complex sign composed by several other signs. Of those, one is the image of two dominos, one of each falling, potentially falling, or already fallen onto the other one. Out of the analogy this sign represents two propositions, P(n*) and P(n*+1), for which the truth of the first one implies the truth of the second one (i.e., an instance of the inductive step). This is an example of linking sign. Moreover, the whole line of dominos (fallen, falling, or standing), representing the infinite syllogisms obtained by base case and inductive step, is a repetition of linking signs and, globally, can be interpreted as an iteration sign.

In this paper I report on an explorative and qualitative study focusing on linking and iteration signs in post-graduate students' processes involved in problem solving activities. The study investigates the different characteristics that linking and iteration signs can have during the problem resolution process, aiming at a possible classification of them.

Methods

The study is based on interviews in which expert students were asked to solve some problems and then to speak about mathematical induction. Participants were 4 doctoral students in Mathematics. They were interviewed individually for approximately 80 minutes each. They were not aware of the focus of the study. Collected data consist of audio-video recordings and of the written inscriptions produced by the students. In this paper I will refer to the following two problems.

The chessboard problem: *"Consider a $2^n x 2^n$ chessboard. What is the maximum number of squares which can be tiled with L-shaped pieces composed of 3 squares each?"*. The solution, which can be proved by MI on n, is that it is possible to tile the entire $2^n x 2^n$ chessboard except for one square.

The false coin problem: "N identical coins are given. One of these, however, is false and it weighs less than the others. There is a traditional weighing scale at our disposition. What is, in function of N, the minimum number of weighings necessary to determine the false coin?". A partial solution for the problem is that, if $N=3^m$, then with m weightings it is possible to determine the false coin. Again, this can be proved by MI.

Preliminary findings

A classification for linking and iteration signs

With the analysis of the semiotic bundle produced and used by the students during the interviews, it was possible to identify three different categories of linking and iteration signs:

- Linking and iteration signs produced or used to refer to the (mathematical or not) objects described in the problem's text (functions, variables, chessboards, tiles, coins, etc.), to their properties, or to their mutual relationships. For these, I use the term *Ground-level* signs.
- Linking and iteration signs produced or used to refer to the proving scheme of mathematical induction itself, to its logical structure or to the justification of its validity. For these, I use the term *Meta-level* signs.
- Linking and iteration signs that could be seen simultaneously as ground and meta-level signs. This happens when a single component or different components of the same sign refer both to some objects of the problem and to mathematical induction itself. For this intermediate category, I use the term *Hybrid-level* signs.

In the following pages I will present some paradigmatic examples of these categories of signs. The students' names are pseudonyms. In the transcripts, with (*italics*) I describe gestures or inscriptions in the moment when they are made.

Ground-level signs

Lorenzo, in this part of the interview, is dealing with the false coin problem. After reading the text he claims to remember the solution of a similar problem in which nine weights are given, all identical except for one which is lighter. Lorenzo describes the solution of this second problem:

1 Lorenzo: In this case the game was: you split in three groups, then three coins, first group, three coins, second group, three coins, third group *(he writes ‘9’ on the sheet, then he draws three arrows starting from it and pointing to the right, he then writes ‘3’ at the end of each arrow and, finally, ‘1st’, ‘2nd’, and ‘3rd’, Figure 1a)*. Then you say: we take two groups, we weigh them and since there are three possible results, that are one is lighter, the other is lighter, or even, I can determine in which group the lighter one is. Then I iterate the procedure on the others.

Later on, Lorenzo tries to generalise the just described solution for a group of n coins:

2 Lorenzo: So, the reasoning would be… n coins, I split in three groups (*he draws a point and three lines starting from it and pointing down on the sheet*), I select one of them with one weighing, I split in other three groups, (*he draws a second point at the end of a line and then three other lines starting from this new point*), I select one of them with another weighing (*again, he draws a third point at the end of a new line and then three other lines from this. At this point he has drawn a mathematical tree, Figure 1b*).

In the first part of the excerpt (line 1), when describing the solution of the problem for a group of nine coins, Lorenzo produces a sign (the inscription in Figure 1a, together with his speech) which represents how the group of 9 coins is linked to three groups of 3 coins each, namely a linking sign. With this sign, Lorenzo represents the first step of the solution of the problem (to divide the group in three subgroups and to determine which one contains the false coins). Afterwards (line 2), Lorenzo describes a possible strategy to solve the general problem, and to do this he draws an inscription which is composed by the repetition of the previous linking sign, this time without any indication to the number of coins (Figure 1b). This is an iteration sign and allows him to describe the iterative solution of the problem.

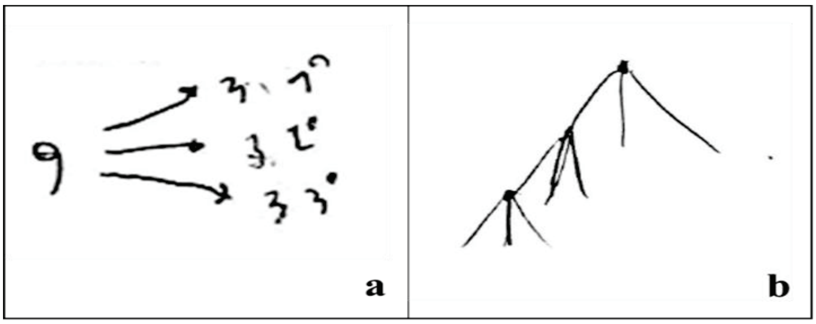

Figure 1: Lorenzo’s inscriptions in line 1, Figure a, and in line 2, Figure b

In the remaining part of the interview related to this problem, Lorenzo explores the mathematical tree that he drew in order to find, in function of n, its height (which corresponds to the number of weightings necessary to determine the false coin). Both the just presented linking and iteration signs are ground-level signs because they refer to different groups of coins and the relationships between them.

Meta-level signs

Guido, at the end of the interview, says to be convinced of the validity of MI as a proving scheme. Then he justifies his answer.

1	Guido:	You do the base case, which is true, and you verify it (*he puts the right hand in front of him at the level of the table, touching his leg with the four fingers, Figure 2a*).
2	Guido:	Then (*he makes two consecutive arc-shaped gestures rotating the right hand in the air keeping thumb and pointing finger at a constant distance and moving the hand from left to right, Figures 2b/c*) the inductive step (*he repeats the previous gestures a second time*) guarantees that it is always true (*he moves rapidly the right hand starting from his leg to an up-right direction in the air, Figures 2d/e*).

Guido's speech alone does not seem to provide a justification for the validity of MI. He only says that after the case base is proved true (line 1), the inductive step assures the truth of the proposition for every natural number (line 2). However, if we look at his gestures, we notice that his discourse is enriched by other semantic elements. The base case is represented by a point on Guido's leg, which he touches with his right hand (Figure 2a). Before saying "the inductive step", Guido makes a rather complex gesture: he moves his right hand in the air from left to right and simultaneously he rotates it twice, keeping thumb and pointing finger at a constant distance, forming two arcs (Figures 2b/c). He then repeats the same gesture while saying "the inductive step".

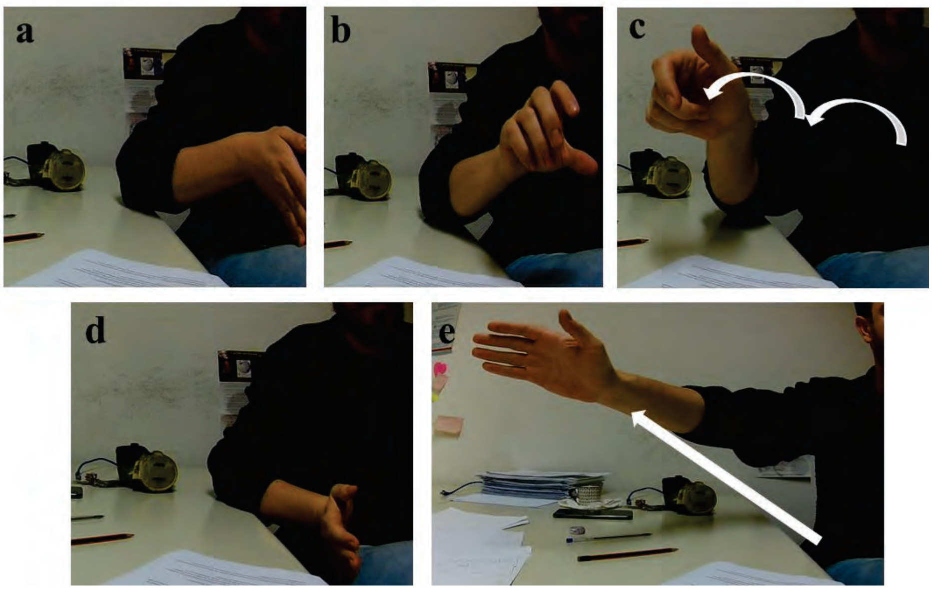

Figure 2: Guido's gestures. In Figure a, it is showed the "case base gesture", in Figures b/c the "inductive step gesture", and in Figures d/e the final gesture of line 2. The white arrows summarise the gestures' trajectories

These gestures are interesting because with them Guido seems to represent the inductive step itself as a series of arcs in the air. Each arc-gesture could indicate, metaphorically, the link between two cases of the proposition to be proved by induction (the implication $P(n)\rightarrow P(n+1)$). These are examples of meta-level linking signs. Finally, while saying "it is always true", Guido performs a new gesture which is a composition of the previous ones, but now more rapid and contracted: He starts touching his leg (as when he was referring to the base case), and then he moves fast to an up-right direction in the air. This time the hand moves in a straight line without shaping any arcs (Figures 2d/e). With this gesture he seems to describe the whole iteration which, starting from the base case and by successively applying the inductive step, allows to conclude the truth of P(n) for all the natural numbers greater than the base. This is an example of a meta-level iteration sign.

Hybrid-level signs

Silvio, in this part of the interview, is dealing with the chessboard problem. Until this moment he has explored the problem, finding that for the chessboards corresponding to n=0, n=1, and n=2, it is possible to create a tessellation which leaves out only one little square and he has conjectured that the same thing is possible for every $2^n x 2^n$ chessboard. Then he has observed that a $2^n x 2^n$ chessboard is composed by four $2^{n-1} x 2^{n-1}$ chessboards. This seems to suggest to him a possible solving strategy:

1	Silvio:	I think that one could do something like by induction (*keeping the pen in his right hand, he rapidly draws some circles in the air, Figures 3a/b).*
2	Silvio:	Because, since in the case zero I have only one little square and it remains out (*with the pen he touches the sheet where he previously wrote* ‘n=0→0 tiles’, *Figure 3c*),
3	Silvio:	then, let’s say, in a sequential way this little square will always remain out (*with the pointing finger of his right hand, he draws some circles in the air whilst moving up his hand, Figures 3d/e*).

In the remaining part of the interview related to this problem, Silvio tries to construct a proof by MI for his conjecture.

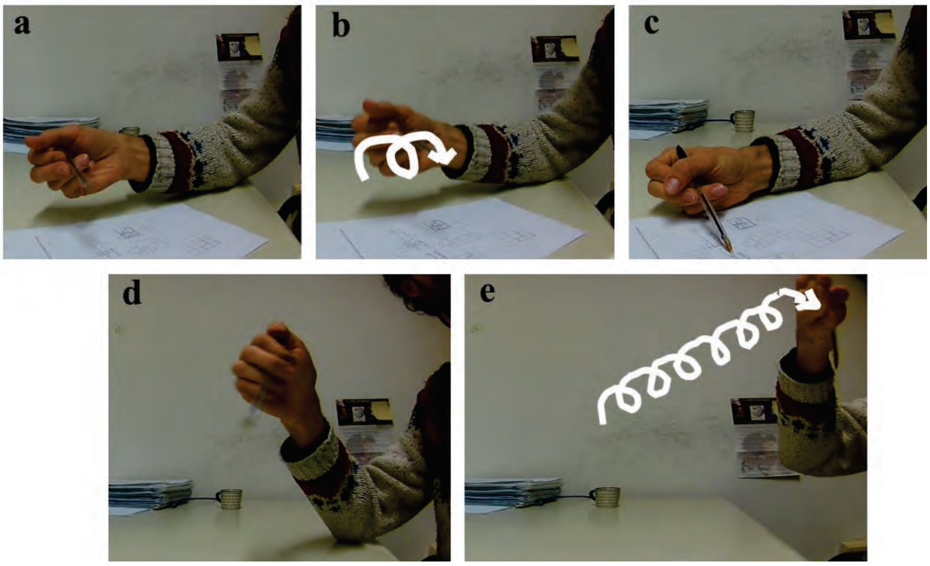

Figure 3: Silvio’s gestures. In Figures a/b it is showed the gesture of line 1, in Figure c the gesture of line 2, and in Figures d/e the gesture of line 3. The white arrows summarise the gestures’ trajectories

The analysis of this excerpt reveals the presence of a hybrid-level sign. Firstly, Silvio claims that a possible solution could be obtained doing “something like by induction” (line 1). Whilst saying this, he makes a gesture which seems to refer to the iterative structure of induction itself (Figures 3a/b). Considering the bundle (his utterance together with the gesture) we can interpret this as a meta-level iteration sign. In lines 2-3, Silvio clarifies what he meant with “something like induction”. He says that the chessboard corresponding to n=0 is composed only by one little square which thus cannot be tiled. Then he says that from this, “in a sequential way”, it could be possible to show that the little square will remain out of the tessellation for all the bigger chessboards as well. In describing this strategy, Silvio is apparently using his previous observation of the fact that it is possible to construct a $2^n x 2^n$ chessboard with four $2^{n-1} x 2^{n-1}$ chessboards. Silvio’s discourse contains elements which refer to some objects of the problem (“this little square will always remain out”), showing that with his speech he is referring to chessboards and tessellations. However, observing the whole semiotic bundle, we can see that he is also referring to the structure of the solving strategy itself. He firstly touches the sheet where the inscription for the 1x1 chessboard is, which he calls the “case zero” (Figure 3c). Then, while saying “in a sequential way”, he draws several circles in the air whilst moving up the hand (Figures 3d/e). Note that now there is not any reference to chessboards or tessellations. This gesture, in fact, seems to repeat the “something like by induction” gesture previously made in line 1, but now it is longer (both in time and space). If we consider the whole movement of his right hand (Figures 3c/d/e), we can see it starting from the sheet, concretely touching with the pen the inscriptions which refer to the chessboards, and then moving up, drawing a sort of helix in the air, as representing the iterative structure of the reasoning by induction itself. This is an iteration signs which starts as a ground-level sign and then becomes a meta-level sign. This is therefore an example of a hybrid-level sign.

Concluding remarks

In the first two excerpts, I showed some examples of linking and iteration signs, both ground and meta-level. These two categories of signs apparently involve different aspects of the problem solving and proving processes. The ground-level signs seem to have an important role in the resolution of the problem, allowing the subject to recognise that it could be solved with an iterative procedure. The meta-level signs, instead, are related to the description of the logical structure of a generic proof by MI. We can in fact observe that the first two excerpts were taken from two very different moments of the interviews. Lorenzo’s example, through which some ground-level signs were shown, is part of his initial exploration of the false coin problem. On the other side, Guido’s excerpt, containing some examples of meta-level signs, refers to the final part of his interview in which, after having solved some problems (some of those by induction), he is explaining why he is convinced of the validity of MI. However, as the third excerpt has shown, it was also possible to observe some meta-level signs in other phases of the interviews, in particular during the exploration of a problem as well. When Silvio produces the meta-level iteration sign, in fact, he is still exploring the chessboard problem, and he has not found a way to tessellate a chessboard using the tessellation of the previous chessboard (a sort of inductive step). Nevertheless, he recognises a parallelism between his possible iterative solution to the problem and the proving scheme of MI. When this happens, he produces a hybrid-level sign: an iteration sign referring both to some problem’s proper elements (ground-level), and to the structure of MI itself (meta-level). The production of these signs highlights a crucial moment in Silvio’s problem solving process. In this moment, in fact, Silvio seems to see in his own argumentation the structure of a proof by MI (he says “something like by induction”). Subsequently, in fact, he decides to prove by MI his conjecture. In other terms, the presence of these signs seems to reveal a continuity between the process of generation of the conjecture and the subsequent construction of a proof for it. It could be interesting to further investigate this last aspect within the framework of the Cognitive Unity (Boero et al.,1996), focusing on the role of hybrid-level and meta-level signs in the transition between the argumentation supporting a conjecture and a subsequent proof by MI. Further research is also necessary to investigate the production and use of linking and iteration signs (either ground, meta, or hybrid-level) in less expert students.

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Diagrammatic activity and communicating about it in individual learning support: Patterns and dealing with errors

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Diagrammatic activity and communicating about it in different representational systems is essential for mathematical discourse. Therefore, it is also of great importance for individual mathematical support. So far, little is known about how teachers conduct such discourses with children. By analyzing the diagrammatic activities and the communication about them with an analyzing method developed for this purpose, patterns of diagrammatic activity and communicating about it can be elaborated. The example of an individual learning support between a preservice teacher and a second grade student who work in the representational systems of natural numbers and the field of twenty is presented. It shows that these patterns can be supportive in dealing with errors.

Keywords: Individual learning support, diagrammatic activity, mathematical discourses, arithmetics.

Individual learning support as increasing participation in mathematical discourse

When learning mathematics, not all students achieve the teaching goals through joint teaching in the class. These students need individual learning support. Individual learning support can be understood as "support of individual […] learners by a person with more expertise in building knowledge or acquiring competences to solve a task or a problem with a view to the learners' future independent mastering of analogous tasks and problems" (Krammer, 2009, p. 89; translated by the authors). This understanding of the term is closely related to Vygosky's concept of the zone of proximal development (Vygotsky, 1978). In this and in concepts based on it (e.g. Rogoff, 1990), the importance of the social and thus of interaction for cognitive development is emphasized. According to Sfard (2001), learning mathematics can thus be understood as a cultural practice. Thus, learners do not encounter mathematics per se, but a cultural practice that is understood as mathematical by members of that culture (see also Tiedemann, 2012). In this sense, learning mathematics means being able to participate increasingly confidently in discourse that is considered mathematical by experts (Sfard, 2001). Interaction with experts is essential for this. In such an interactionist view of learning (Krummheuer, 1992), mathematical support can be understood "as a specific process of interaction […] that is successively established [by adult and child] through a mutually interrelated interpretation and action" (Tiedemann, 2012, p. 49f.; translated by the authors). According to Dörfler (2006), "progressive participation in the social practice of diagrammatic activities" can be seen as typical of mathematical discourse and mathematical cultural practice (p. 213; translated by the authors). Thus, individual learning support in mathematics means that learners gain expertise in diagrammatic activities and can thus participate in mathematical discourse. Tiedemann (2012) was able to identify general patterns of interaction between parents and children in early mathematical education. However, there is a lack of research on what such interaction patterns and routines look like in individual learning support, especially with regard to diagrammatic activities.

Diagrammatic activity and communicating about it

Diagrams, understood in the sense of the American philosopher Charles Sanders Peirce (1839-1914), are seen in this paper as objects of mathematical activity. In this sense, diagrams are signs with a relational character that belong to a representational system (Dörfler, 2006). The representational system determines how diagrams are created and how they are operated or experimented with. For students, each representational system is its own learning content, as is each direction in moving between different representational systems. In what follows, activities with diagrams that are specified by a representational system are referred to as diagrammatic activities (Wille, 2020). Gestures can also be part of a diagram as quasi-materialized inscriptions (Huth, 2020) and likewise be part of diagrammatic activities. It should be noted that a diagram can be interpreted as such if a corresponding representational system is known (Wille, 2020). That is, no diagram is in itself a diagram. It requires interpretation as one. Activities with diagrams such as experimenting or observing can help clarify, structure, and coordinate thought processes (Hoffmann, 2007). In addition, diagrammatic activities can give rise to mathematical concepts and understanding (Dörfler, 2006). In this way, diagrams themselves become the subject of reasoning processes. Another essential part of mathematical activity is communication about it (Dörfler, 2006). Communication about diagrams and diagrammatic activity involves both linguistic and gestural utterances. Communication about diagrammatic activity enables the use of denotations for diagrams that belong to different representational systems, and it also enables interpretations of diagrammatic thinking (Wille, 2020).

For learning processes, it is crucial that students do not just do something, but that their attention is drawn to the crucial aspects through communication (Gaidoschik, 2016). It is often reported that teachers have few professional discourses with the learners and do not pay much attention to the learning processes of the students in the dialogues, but rather often dominate the classroom discussion (Krammer, 2009; Begehr, 2006). Consequently, it is necessary to already qualify preservice teachers to conduct such professional discourses with learners. However, there is a lack of research on how diagrammatic activities and communication about them intertwine in individual learning support between a preservice teacher and a student.

Research interest

The research focuses on situations of individual learning support between a preservice teacher and a student in grade 1 or 2. It addresses the question of how diagrammatic activity and the communication about it intertwine in individual learning support. Since support is most evident when learners have particular difficulties or make an error, special attention is paid to how diagrammatic activity and the communication about it intertwine in such situations.

Setting

At the St. Gallen University of Teacher Education, preservice teachers provide one-on-one support to first and second grade children as part of an elective course. In the spring semester, one pre-service teacher supports one child per week for about 30 minutes. A seminar at the university accompanies the support sessions, which in particular are recorded on video. In the individual learning support, the preservice teachers make use of support activities that were developed in the MALKA project (Wehren-Müller et al., 2018). The goal of individual learning support is to enable students to solve arithmetic tasks without counting strategies, because a solidification of initial counting strategies can lead to difficulties in learning mathematics (Scherer & Moser Opitz, 2010). Due to this, it is necessary for students to develop sustainable ideas about numbers and operations (Häsel-Weide, 2016). To achieve this, learners should be supported in perceiving and determining the cardinality of a quantity through structural subitizing (Schöner & Benz, 2018). This can be practiced especially on structured materials, such as the field of twenty (Häsel-Weide, 2016; Scherer & Moser Opitz, 2010). The representational systems that are essential for structural subitizing in the individual learning support analyzed here are the field of twenty and the natural numbers.

Method

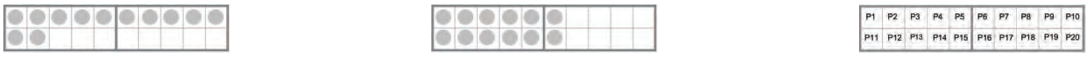
The analysis takes place in several steps. In a first step, an interaction analysis is carried out to reconstruct the interaction processes in detail (Krummheuer & Naujok, 1999). In a second step, the diagrammatic activity and the communication about it is analyzed. For this purpose, an analysis method developed by Wille (2020) for imagined dialogues was adapted for interactions in the two representation systems, the field of twenty and the natural numbers (Ott & Wille, 2021). An analysis sheet for the whole episode is filled in according to the following rules:

- If a *diagram* is used in a turn, a *filled circle* is set in the column of the corresponding representational system. If *communication about* diagrams is used, a *dashed circle line* is set. If both take place, both are noted together. A star is used if an activity cannot be interpreted as a diagram or communicating about it.
- The filled circles or dashed circle lines are connected to each other by *solid lines* if a connection is made by *diagrammatic activities*. The line is *dashed* when the connection is made by *communicating about diagrams*. If both occur, both are noted together.
- If, in a turn, *diagrams of different representational systems correspond* with each other, they are connected by an *arrow*. The direction of the arrow indicates which representational system is used as the starting point.
- If *diagrams that have already occurred once occur again in exactly the same way*, they are connected by *two narrow lines*.
- Communication that cannot be assigned to either one or the other representation system is noted as “others”.

In the following, the results of the analysis of diagrammatic activity and the communication about it of one preservice teacher and one child will be presented. Activities of the preservice teacher are noted in red, activities of the child in blue.

Results

In the following, we will first show some patterns that become visible in the analysis sheet. Afterwards, the handling of errors will be described. The scenes shown in the following are taken from two support episodes between the preservice teacher Mr. Wehrle, who is in the fifth semester of six semesters, and Samira, who is a student at the beginning of the second grade. Mr. Wehrle and Samira work on a structural subitizing task, in which the number of chips in different arrangements on the field of twenty are to be determined (Scherer & Moser Opitz, 2010) (see Figure 1a and 1b). They work with chips and with strips of five on the field of twenty. In the transcript, the field of twenty is numbered as shown in Figure 1c. The transcripts were originally in German and were more detailed.



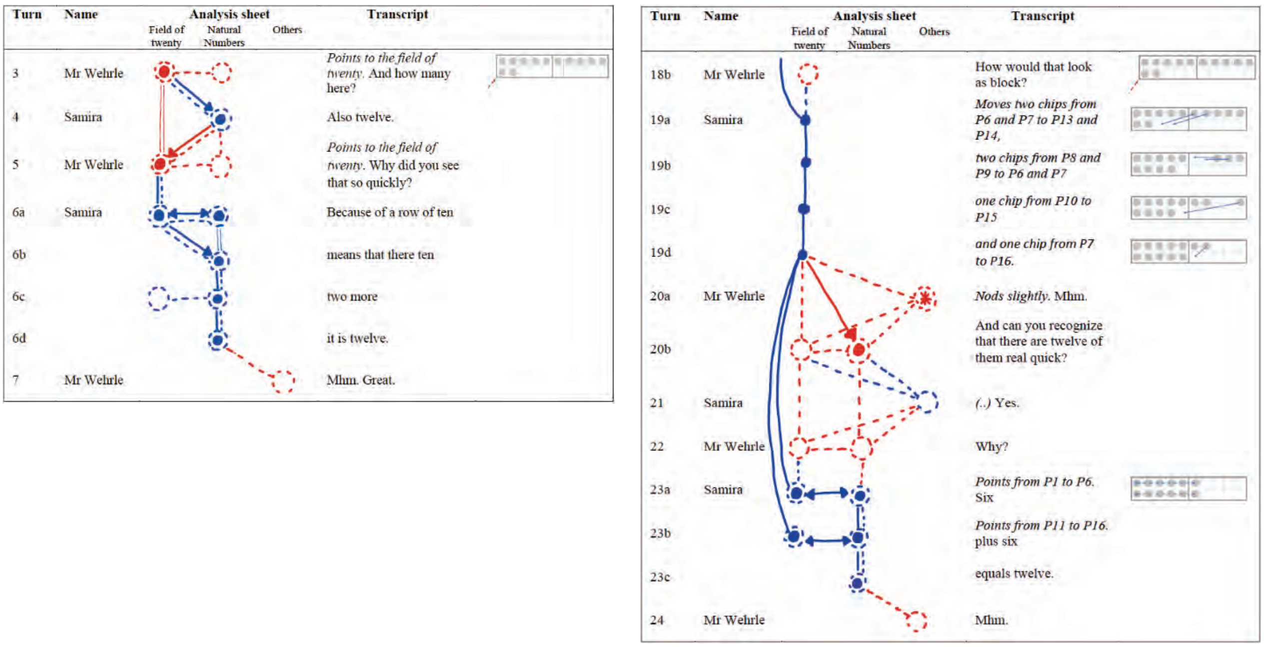
a b c

Figure 1: Field of twenty, (a) row arrangement (b) block arrangement (c) numbering in the transcript

Samira initially has difficulty determining numbers by structural subitizing when they are arranged in a block (see Figure 1b), especially with odd numbers. She becomes increasingly confident in this, but she repeatedly makes errors in determining odd numbers.

Patterns

The analysis sheets show some patterns in the interaction regarding diagrammatic activities and communicating about them. The two selected scenes from episodes 1 and 2 (see Figure 2 and 3) exemplify these patterns.



a b

Figure 2: Analysis sheets and transcripts, (a) episode 1, scene 1 (b) episode 2, scene 3

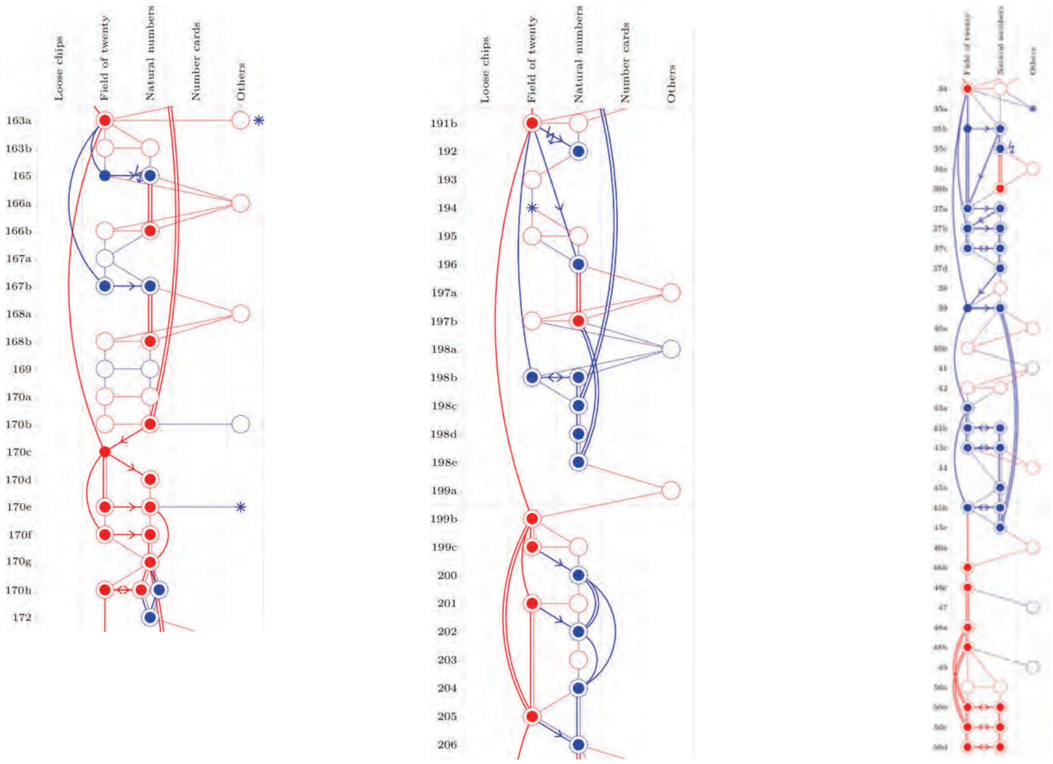
- Mr. Wehrle sets the tasks for Samira often in such a way that both representational systems are linked at least by communicating about a diagram (see Turn 3 and 5 in Figure 2a; Turn 20b in Figure 2b). Samira then also combines both representational systems in her answers (see Turn 6a in Figure 2a; Turn 23a, b in Figure 2b) by using diagrams in both representational systems.
- Mr. Wehrle promts Samira repeatedly to switch back and forth between the representational systems. To this end, he himself switches between the representational systems. He adapts this to the respective situation. If Samira's answer is only in the natural numbers, he switches to the field of twenty (see Turn 4–5 in Figure 2a). If her answer is only in the field of twenty,

he switches to the natural numbers (see Turn 19d–20b in Figur 2b). Therefore, he uses corresponding diagrams to the ones Samira used in the other representational system.

- Diagrams are mainly used by Samira. She also mainly carries out the diagrammatic activities. Mr. Wehrle encourages these according to the principle of minimal help (Aebli, 1998). The impulse to communicate about diagrams usually comes from Mr. Wehrle. He almost always communicates via the diagrams he uses.

Dealing with errors

If a content-related error has occurred during the task processing, a correction is needed. Samira repeatedly makes errors in determining odd numbers that are arranged in a block. In the analysis sheets, errors are indicated with a flash. Figure 3 shows three analysis sheets for error situations.

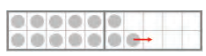


a b c

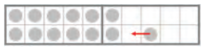
Figure 3: Analysis sheets of error situations, (a) episode 1, scene 13 (b) episode 1, scene 15 (c) episode 2, scene 5

In episode 1, scene 13 (Figure 3a), Samira determines an incorrect number and corrects herself when Mr. Wehrle asks where she sees this number (T. 164–168a). He then asks her, how she got the correct number and Samira reports that she counted (T. 168–170a). Now Mr. Wehrle becomes diagrammatically active himself to correct the error by referring to a known procedure and carrying it out himself. In doing so, he always connects the two sign systems with each other and communicates via each diagram. He thus demonstrates to Samira the possibility of non-counting number determination by way of example. Samira participates by agreeing and taking over the diagrammatic activity at the end:

170b Mr. Wehrle: You had a great trick at twelve earlier.
170c *Pushes the chip from P17 to P18.*

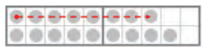
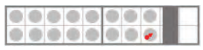


170d At twelve you said
170e here are six. *Points from P1 to P6.* <

170f And here are six. *Points from P11 to P16.* <<
170g Equals twelve.
170h *Pushes the chip back from P18 to P17.*
And then <<< only plus one.

171a Samira: < Yes.
171b << *Nods.*
171c <<< One.
172 Samira: Equals thirteen.

In episode 1, scene 15 (Figure 3b), Samira determines an incorrect number, too. With Mr. Wehrle's advice to look carefully and count the top row, she is able to give the correct number (T. 191–196). When asked how she figured it out, she refers to the previous number and states that there is now one more chip (T. 197b–198e). She is mainly diagrammatically active in the natural numbers. Mr. Wehrle then works with her to determine the number by structural subitizing, always combining both representational systems. He takes the field of twenty as the starting point and encourages Samira to switch between the sign systems. He takes over the diagrammatic activities at the field of twenty:

199b Mr. Wehrle: And have a look. *Puts two chips on P8 and P19.*

199c *Points along the top row of chips.* If you now count this row up here. How many are there?

200 Samira: *(4 sec.)* Eight.
201 Mr. Wehrle: *Covers the chip on P19.* How many are there then in the bottom row up to here? *Points to P18.*

202 Samira: Eight.
203 Mr. Wehrle: Equals
204 Samira: Sixteen.
205 Mr. Wehrle: *Uncovers the chip on P19.* And this one.

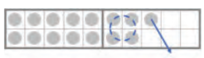
206 Samira: Seventeen.

In episode 2, scene 5 (Figure 3c), again, Samira determines an incorrect number (T. 34–35). When asked why there are this number of chips, Samira now becomes diagrammatically active. She uses both representational systems flexibly, switches back and forth between them and can identify her answer as incorrect and name the correct number:

37a Samira: *Points from P1 to P8.* I counted here, that—it's eight.
37b *Points from P11 to P17.* But there, it can't be eight.

37c *Points to P18.* Because there should be one more.
37d So, it is not sixteen.
38 Mr. Wehrle: *(..)* But?
39 Samira: *(10 sec.) Makes slight nodding movements with the head.* Fifteen.

Mr. Wehrle asks her not to count the number. Thereupon Samira again becomes diagrammatically active. Again, she uses both representational systems flexibly, switching back and forth between them. Now, however, she is proceeding in a very orderly manner:

43a Samira: *(.)* For example. *Pushes the chip from P8 away next to the field of twenty.*

43b Now there are *(.)* seven. *Slight circular movement via P6, P7, P8, P9.*
43c Plus seven.
44 Mr. Wehrle: Mhm.

45a Samira: Equals fourteen,
45b and one more *Pushes the chip back to P8.*
45c equals fifteen

Mr. Wehrle repeats this approach and enriches it with terms such as “doubling” (T. 46–49).

Overall, development can be seen: while Mr. Wehrle initially demonstrates a non-counting strategy, he increasingly takes a step back and challenges Samira to diagrammatic activities. Conversely, Samira is initially strongly focused on the natural numbers. Increasingly, she connects them with the field of twenty and finally becomes independently active in both representational systems in order to correct her error. While Samira initially has to be prompted to communicate about the diagrams and the diagrammatic activity, in scene 5 (episode 2) she does this on her own and uses it to think aloud.

Conclusions and limitations

By analyzing the diagrammatic activity and communicating about it in individual support episodes as a whole, patterns and developments become apparent. In the individual learning support presented here, for example, these patterns can be seen in the fact that Samira was repeatedly encouraged to switch between the two representational systems and to connect them. In this way, Samira can increasingly participate in the mathematical practice of diagrammatic activities (Dörfler, 2006). Mr. Wehrle uses the diagrams of different representional systems, mainly of the field of twenty and the natural numbers, as the subject of the reasoning and communication processes (Dörfler, 2006). In doing so, he encourages Samira to explore and interpret these diagrams. This interaction leads to Samira being able to clarify and structure and her thought processes (Hoffmann, 2007). Especially when dealing with errors, this becomes evident: Samira is finally able to engage in a mathematical discourse for error correction using diagrammatic activity and communicating about. In doing so, she applies elements of the patterns that were previously established between her and Mr. Wehrle.

This study has some limitations up to this point. Since only one situation of individual learning support was initially considered, it is yet unclear whether such a pattern also leads to a similar result in other situations and whether success might be found in other patterns of interaction. In future research, individual learning support of different preservice teachers will be compared in order to obtain insights into teacher training in this area in the long run.

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Bar-model introduction in late primary school

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The Singapore bar-model has recently arrived in many Spanish classrooms, where it has been identified as an effective modelling tool for problem-solving in primary schools. This model introduces a pictorial approach prior to the abstract symbolic resolution. Schools usually change over to this scheme by introducing it in grade 1 (6/7 years old) and progressing yearly with the children. In this study we perform a pre-post analysis of 71 fifth graders. An experimental group of 16 children were introduced to the Singapore bar-model, while the control group, the remaining 55 children, solved the exact same set of problems using their usual symbolic-only approach. Our results show statistically significant improvements in the performance of the experimental and the control groups in the cognitive domains of "knowing" and "applying" as well as in the content domain of geometry.

Keywords: Problem solving, mathematics education, Singapore bar-model, classroom research.

Introduction

Problem-solving is one of the key aspects in mathematical education and maybe one of the most difficult skills to teach. Many students and many teachers struggle with this issue, where solutions come from a complex mixture of creativity and pattern-recognition. Teaching mathematical modelling literature agrees on problem-solving being some cycle (Perrenet & Zwaneveld, 2012) where reality is transformed into a mathematical model whose mathematical solution is found and, later, transformed back into reality.

According to DeWindt-King and Goldin (2003) "a representation is any configuration (of characters, images, concrete objects, etc.) that can denote, symbolize or, otherwise "represent" something else" (p. 2). The representation of the problem data can be a way to emphasize, understand and link the most important information, so that practice can be seen as a useful tool for the students in problem solving (Heagarty & Kozhevnikov, 1999). The scientific community agrees that representing data is useful for improving performance in problem solving, although there is no single way to represent data. The works of Fagnant and Vlassis (2013) and Heagarty and Kozhevnikov, (1999) show several ways, focusing mostly on the schematic representations. Appropriate representation has positive effects on children's performance, which reinforces the importance of working on this content from an early age (Uesaka et al., 2007).

Spanish tradition tends to rely only on the symbolic representation of mathematical models, and other types of representations are hard to find in classrooms and/or textbooks. Among teachers and students, the usual scheme is known as the "*Datos-Operación-Solución*" scheme (which translates as "data-operations-solution"*, DOS,* from now on). Under this scheme, children are asked to organize problem-solving in the three steps shown in practice in Figure 1. First, they write down "Data" and all numerical data that appears in the problem statement, usually with units. Then, children have to perform the needed calculations (*Operations*). Finally, in the *Solution* block, they write down a full sentence stating the result back in the context of the problem.

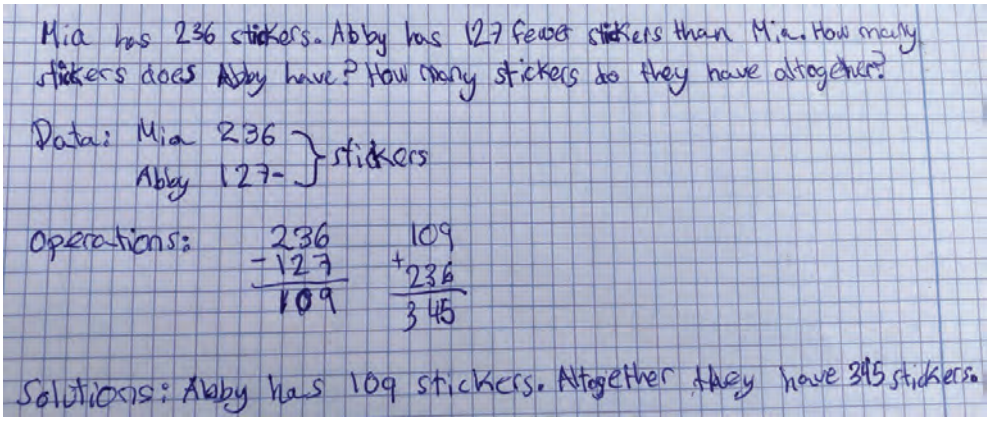


Figure 1: DOS-Scheme implemented by an 11-year-old from a bilingual school (English & Spanish)

It was in the 1980s when Singapore brought in the idea of modeling quantities and their relationships with bars, avoiding the more abstract symbolic-only approach and giving students a pictorial way to represent a problem's mathematical model. According to Blum and Leiss (2005) modelling cycle, shown in Figure 2, BMM allows the students to obtain a model of the situation prior to the mathematizing step. This approach clearly contrasts with the DOS, which moves directly towards the mathematical work without performing any modelling.

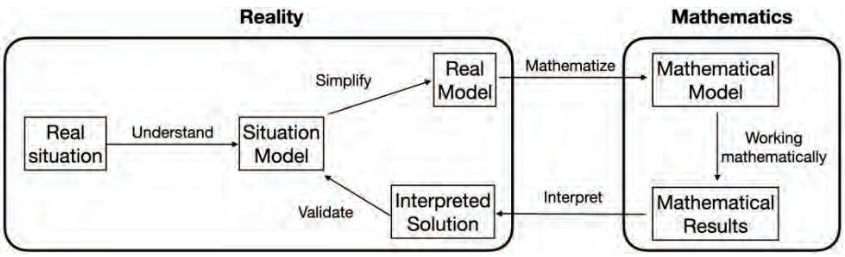


Figure 2: Blum and Leiss modelling cycle

The bar method model (BMM from now on) uses as a fundamental mainstay the concept of the "bar", which is a pictorial diagram that allows the organization of the abstract information, converting it into understandable data (Walker, 2005). Using this method, it is possible to represent data from arithmetic and algebraic problems (word problems), which not only allows us to transform abstract words and concepts into pictorial representations, but also helps to understand and analyze the problem itself (Yeap, 2011). According to Kho (1987) and Kho et al. (2009), once the students have been able to represent the data using the bars, they are much more likely to understand the situation and, therefore, to solve the problem.

Since its introduction in Singapore, the BMM has been a centerpiece in mathematics, beginning in the first grade of primary education. According to Ng and Lee (2009), in the first two years, they usually work through pictorial representations of known objects, to encourage students to have specific references to the data. Subsequently, the concept of the "bar" is introduced, with which they continue to work throughout their school years. Many Spanish schools have started to use Singapore's textbooks. Usually, the method is introduced in grade 1 and progresses with the children as they advance in their schooling years. In those schools, older kids are thus taught with the DOS scheme while their younger schoolmates are introduced to this pictorial representation. This situation leads to the main research question for this study: Can grade 5 students, who have only been taught to use abstract symbolic modelling, also benefit from the introduction of BMM?

Methodology

This research follows a pre-test/post-test design. This study has been carried out in a middle-sized school in Segovia, Spain. All 76 fifth grade students have participated in it, though 5 of them have not been included since they are identified as special-education students. The remaining 71 children are divided into 4 classrooms, each one lead by a different math teacher. Three of the classrooms (55 students) have been assigned as the control group, and the remaining one (16 students) has acted as our experimental group, since their teacher was looking for new approaches to problem-solving methods, given the difficulties their students had on this topic.

The intervention was carried out as follows: For over a month, daily, both experimental and control groups solved the same two problem inventories consisting of a total of 53 word-problems, all able to be solved with the help of BMM. It is important to remark that these children have never been exposed to pictorial models before our interventions. The control group worked with their usual DOS scheme. The word-problem inventory was agreed upon by the four classroom teachers and divided into three sections: additive structure with natural numbers, additive structure with decimal numbers and, finally, multiplicative structures and fractions (that were also represented with bars). With respect to TIMSS' content-domain classification, all the problems fall in the group *numbers*, while regarding cognitive domains, 49% classify as *applying*, 17% as *reasoning*, and 34% as *knowing*.

The first sessions for the experimental group were entirely dedicated to the pictorial representation of the problem statement, introducing BMM by example. In the first two sessions, all questions were removed from the problems' statements to avoid the mechanical use of DOS scheme by the students. Only when students got used to modelling the situation with bars and could pose and answer several questions from their drawn bar-model were the original questions to the problems reintroduced.

To assess the effectiveness of the intervention we used two similar sets of questions released from TIMSS (2011) as pre-test and post-test. The first set had already been configured and tested in the Spanish context in Fraile's PhD (2017) while the second was designed to match this one with a similar weight (shown in Table 1) in both content domains (numbers, data display and geometric shapes and measures) and cognitive domains (applying, reasoning, and knowing).

Table 1: Total number and percentage of the questions in each domain for the pre- and post-tests

	Date	Question content domain			Question cognitive domain		
		Numbers	Data	Geometry	Applying	Reasoning	Knowing
Pre-test	28/04/2020	7 (63%)	2 (18%)	2 (18%)	4 (36%)	4 (36%)	3 (27%)
Post-test	08/06/2020	9 (60%)	4 (27%)	2 (13%)	5 (33%)	5 (33%)	5 (33%)

Among all the questions in the pre-test, three of them threw up unexpected results and were reintroduced to better understand the children's motivations for their answers, redesigning them as diagnostic questions, where incorrect responses allow the teacher to understand where the mistakes may be (Wylie & William, 2006).

Analysis

Students' answers analysis has been carried out using R software. We start by comparing problem solving skills before our intervention between the experimental and control groups, according to their results on the pre-test. The results shown in Table 2 may suggest some bias in favor of the experimental group, but the chi-square test p-value is 0.82, showing no significant differences between the number of correct answers in both groups.

Table 2: Post-test average results in a 0-10 scale for each group in the studied domains. In brackets, pre-test average results and difference between both measurements.

	Student's content domain			Student's cognitive domain		
	Numbers	Data	Geometry	Applying	Reasoning	Knowing
Experimental	8.1 (5.7+2.4)	9.0 (6.9+2.1)	7.2 (6.6+0.7)	8.2 (7.8+0.4)	8.1 (5.6+2.5)	8.0 (4.4+3.6)
Control	5.9 (4.6+1.3)	8.0 (6.9+1.1)	2.6 (4.7-2.2)	5.6 (6.3-0.7)	6.9 (5.3+1.6)	5.4 (2.9+2.5)

The results suggest better problem-solving skills in the experimental group, as well as an improvement of this group after the intervention. Six repeated measurement Anovas were performed to determine statistically significant differences between both groups. These results are shown in Table 3.

Table 3: Anovas' summary for the differences between both groups

Student's content domain			Student's cognitive domain		
Numbers	Data	Geometry	Applying	Reasoning	Knowing
$F(1,69)=8.055$ $p<0.1$ (.)	$F(1,69)=0.737$ $p=0.39$	$F(1,69)=13.72$ $p<0.001$ (***)	$F(1,69)=11.2$ $p<0.01$ (**)	$F(1,69)=1.572$ $p=0.21$	$F(1,69)=10.71$ $p<0.01$ (**)

From the data shown in Table 2, the only significant difference in content domain has been in geometry, with an average difference of 2.9 points between groups due to the worse results in the post-test for the control group. In numbers and in data the differences have been smaller, with 1.1 and 1 average points respectively, only significant to a 90% for the data domain. With respect to cognitive domains, two out of the three categories have had a significant increase for the experimental group showing p-values in the Anova tests below 0.01.

As mentioned before, three questions in the pre-test were introduced in the post-test. Let us start our analysis with the following question (TIMSS Ref.M051091) that falls into the content domain “numbers” and the cognitive domain “knowing”.

Figure 3: Question TIMSS Ref.M051091 that was included (in Spanish) in the pre-test

Which fraction is not equal to the others?

A.1/2 B. 4/8 C. 2/4 D. 2/8

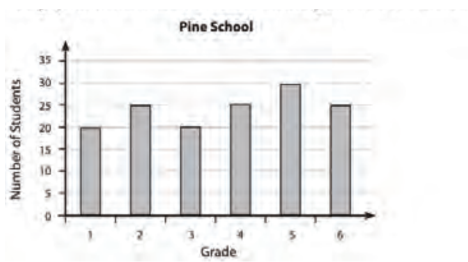
The pre- and post-test results of our fifth graders are shown in Table 4. No significant differences between experimental and control groups have been found in the answers of both groups in the pre-test (Chi-square test p-value=0.18) nor in the proportion of correct answers (Binomial test p-value=0.06). After consultation with teachers and students we understood that “not equal” might be ambiguous for the children. In the post-test we used the unambiguous term “not equivalent” obtaining a non-statistically significant 14% increase in the correct answers. Experimental and control groups differ in favor of the experimental group in the post-test (Chi-square test p-value=0.02; Binomial test p-value=0.004) where 94% of the children in the experimental group can answer correctly.

Table 4: Comparison between pre- and post-test in exp. and cont. groups in M051091 question

	Pre-test					Post-test				
	A	B	C	**D**	Other	A	B	C	**D**	Other
Cont.	21 (38%)	5 (9%)	3 (5%)	**21 (38%)**	5 (9%)	16 (29%)	3 (5%)	3 (5%)	**27 (49%)**	6 (11%)
Exp.	2 (13%)	2 (13%)	0 (0%)	**11 (69%)**	1 (6%)	0 (0%)	1 (6%)	0 (0%)	**15 (94%)**	0 (0%)

The second question that was reevaluated is shown in Figure 4. TIMSS provides the four answers A. 20, B. 25, C. 30 and **D. 35**. Trying to improve its usage as a diagnostic question, we changed the solutions to **A. 35**, B. 7 (number of empty gaps in the graph), C. 180 (6 grades times 30 students) and D. 145 (students represented in the graph). A remark on Fraile’s PhD annotations (Fraile, 2017, p.110) suggested one might further modify in the post-test the answer B. 7 to B’. 30, which was a common mistake, probably due to 30 being the only numerical data in the exercise.

Figure 4: Second question (TIMSS Ref.M051117) that was included (in Spanish) in the pre-test



In the Pine School there is room in each grade for 30 students. How many more students could be in the school?

A. 20

B. 25

C. 30

D. 35

Due to the changes in the test answers, we only analyze the proportion of correct answers. Table 5 shows a higher success rate in the post-test of the experimental group, with 75% of correct answers, with respect to the 44% of the control group but also an increase in the mistakes towards the new option B'.30. The binomial test for the two groups p-value of 0.108, with no statistical significance.

Table 5: Comparison between pre- and post-test in exp. and cont. groups in M051117 question

	Pre-test					Post-test				
	A	B	C	D	Other	**A**	B'	C	D	Other
Cont.	**26 (47%)**	0 (0%)	18 (33%)	6 (11%)	5 (9%)	**24 (44%)**	8 (15%)	11 (20%)	10 (18%)	5 (9%)
Exp.	**8 (50%)**	1 (6%)	6 (38%)	1 (6%)	0 (0%)	**12 (75%)**	2 (13%)	3 (19%)	1 (6%)	0 (0%)

The last question that was reevaluated (see Figure 5) deals also with basic facts about fractions and falls also in the content domain "numbers" and the cognitive domain "knowing". It is an open question where students are asked to write their own answer.

Figure 5: Question TIMSS Ref.M041299 that was included (in Spanish) in the pre-test

Tom ate $\frac{1}{2}$ of a cake, and Jane ate $\frac{1}{4}$ of the cake. How much of the cake did they eat altogether?

TIMSS suggests 2/6 as a common mistake, resulting from adding the two numerators and the two denominators. Our analysis of the pre-test showed that 1/6 is also a common answer. Table 6 shows the results for **3/4** (or equivalent) answers, 2/6, 1/6, other answers and blanks for both tests. No statistically significant differences are found between pre- and post-tests for any of the two groups, but, again, the experimental group obtains a higher rate of correct answers in the post-test compared to the control group, throwing a p-value of 0.064 in the Binomial test.

Table 6: Comparison between pre- and post-test in exp. and cont. groups in M041299 question

	Pre-test					Post-test				
	3/4	2/6	1/6	Blank	Other	3/4	2/6	1/6	Blank	Other
Cont.	15 (27%)	9 (16%)	7 (13%)	3 (5%)	21 (38%)	18 (33%)	8 (15%)	2 (4%)	5 (9%)	22 (40%)
Exp.	6 (38%)	3 (19%)	0 (0%)	1 (6%)	6 (38%)	9 (56%)	2 (13%)	1 (6%)	0 (0%)	4 (25%)

Conclusions

Not surprisingly, both experimental and control groups have benefited on average from the systematic work in problem-solving. A deeper analysis shows that there are three domains in which we have found significant differences between groups at 95% significance-level. With respect to content domain, significant differences have only been found in the geometry block, while the numbers block, throws a p-value only below 0.1. Regarding cognitive domains, "applying" and "knowing" show a statistically significant difference in favor of the experimental group, consistent with the fact that these blocks have been the focus of 83% of the intervention.

The international success average of fourth graders in question M051091 assessing the concept of equivalent fractions is 44%, while Spanish students obtain a much lower rate of 30%. One academic year later, our fifth graders are close to the international average with a slight difference (not statistically significant) in favor of the experimental group and a 45% average success rate. After the intervention, where bars were used to represent fractions, there is a non-statistically significant increase in the results of both groups which now shows a much clearer difference in the experimental group, where only one student misses this question. The differences between groups in the post-test are statistically significant and prove a better grasp of the basic facts of fractions for this group.

The international success average for fourth graders in Question M051117 where they are asked to read and interpret data from a graph is 54%, while Spanish students obtain a lower rate of 50%. In the pre-test there are nearly no differences between the two groups, with C. 180 being the most common wrong answer. The average success for this question is 48% in the pre-test. Option B was replaced by 30 respondents in the post-test, where 14% of the children chose this modified answer (only one chose number 7 in the first version), a clear sign of a better choice for a diagnostic question.

The international success average for fourth graders in Question M041299 where they are asked to add fractions 1/2 and 1/4 is 23% showing, again, Spanish students a much lower rate of 14%. In the pre-test, our fifth graders obtained a better result on average of 30%, with no significant differences between groups. After the intervention, the percentage of correct answers for the control group is 33% and 56% for the experimental group, without statistical significance.

According to the literature, BMM can help children's problem-solving skills when it is taught in the early years. This study suggests that the positive effects of this approach can be attained even as late as 5th grade and that further analysis and interventions should be tested to better understand how it affects the different cognitive domains and to establish with elder children (or even adults) if the opportunity window to improve problem-solving skills with the use of BMM ever closes.

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Solving geometric problems: Squaring in motion using manipulatives, measurement of long-term effects

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Neurophysiological studies have shown that our motor system plays a fundamental role in the scaffolding of our cognitive system. Taking these findings into account, we have designed and produced manipulatives that reproduce geometric figures. Through them, we can amplify our perception of the relationships between different elements. We believe that repeated interactions with the manipulatives generate the ability to imagine similar movements on drawn figures. An example is "squaring in motion", through which a given figure is transformed into an equivalent rectangle. We verified, through a quantitative study involving preadolescents (12-14 years old), the long-term effectiveness of the use of manipulatives on problem solving.

Keywords: Perception, movement, insight, productive thinking, Leonardo da Vinci.

Introduction

Problem solving is a very important aspect of mathematics education. This is already underlined by the NCTM since 2000 and, specifically in Italy, by the 2012 Italian National Guidelines for Instruction concerning the curriculum for pre-school and "first cycle", i.e. 6-14 education. Taking into account the distinction between exercise and problem (Duncker & Lees, 1945), "l'impressione che, proprio in matematica, si chieda ai nostri studenti di attivare essenzialmente processi riproduttivi, risolvendo esercizi, piuttosto che processi produttivi, risolvendo problemi" ("the impression is that, precisely in mathematics, our students are asked essentially to activate reproductive processes, solving exercises, rather than productive processes, solving problems", Di Martino, 2017, p.24 translation by the author). Furthermore, the identification of automated algorithmic processes to solve problems, especially if introduced too early, can clog original sources of insight (Freudenthal, 1983).

In this context we refer to the search for effective strategies to solve problems concerning the equivalence between surfaces. We want to concentrate exclusively on the study of the figures related to them. This is why every problem always has a figure: according to Duval (1995), "the usefulness of geometrical figures in the resolution of a problem of geometry is beyond doubt. They provide an intuitive presentation of all constituent relations of a geometrical situation" (p.143). Therefore, we would like to emphasize how important it is to be able to extract as much information as possible from a geometric figure and, through this, elaborate a solution strategy. In accordance with Kanizsa (1973), we believe that this depends mainly on an appropriate development of geometric perception.

In this paper we propose an innovative way of amplifying geometric perception that has produced immediate results in solving geometric problems and whose effects also appear to be long-lasting.

Theoretical background

Our study is supported by some neurophysiological aspects. One of these concerns the neuronal activations that occur during the observation of an object or picture. There are currently no specific studies on the encoding of geometric figures. However, we can assume that their encoding at a neuronal level is similar to that of words, for which many studies have been conducted. A hypothesized model (Dehaene et al., 2005) verified experimentally by Vinckier et al. (2007) allows us to understand that during word reading, neuronal activation takes place starting from the occipital cortex for the encoding of the most elementary components of the letters, up to the inferior temporal cortex for the encoding of more complex forms like letters and whole words. From studies on macaques (Tamura & Tanaka, 2001), whose brains are very similar to those of humans, we know that at the top of this hierarchy is a single neuron that encodes an entire word. The process does not essentially change if, instead of reading, we are looking at an object. The power of these activations depends on the frequency with which that image is seen and the way it is habitually viewed, i.e., orientation, angle of view, etc. The organization of our perception is undoubtedly influenced by these activations. Frequently seen shapes are more easily recognized than less frequently seen ones. We realize that when we study a complex figure, the shapes we have seen most often are recognized most quickly. As for the others, we must make an extra effort to be able to see beyond what immediately appeared to us. This reorganization of the elements of a figure can lead to sudden *insight*: this is precisely the term coined by Köhler (1921/2018). In this way, *productive thinking* is manifested, using Wertheimer's term (1959), which is necessary to effectively understand the relations contained in a figure of any kind of problem.

To support our study, we wanted to consider another neurophysiological aspect concerning the neuronal activations that occur during interaction with objects such as *artificial manipulatives*, as defined by Bartolini Bussi and Martignone (2020). These are concrete artifacts specifically designed for educational activities. It is well known that the way we grasp an object depends on the series of attempts that we have previously made. The most effective was chosen for our purpose. Through studies on primates (Rizzolatti, Luppino & Matelli, 1998) and humans (Culham et al., 2003), we know that every time we grasp a specific object, first the neurons in the parietal cortex and then those in the premotor cortex are activated to perform a visuomotor transformation. The observed physical properties of an object (shape, orientation, size, etc.) are transformed into the correct hand configuration for grasping it. Moreover, and this is really remarkable, these neurons are not only activated when the individual's hand interacts with the object, but also when s/he simply observes the object (Chao & Martin, 2000) or even remembers the interaction with it (Jeannerod & Decety, 1995).

Finally, two very important complementary studies allow us to highlight the direct interaction between the motor system and perception. The first argues that the previous perception of a bar influences the way we grasp it later (Craighero, 1996). At the same time, grasping a bar actually amplifies its perception (Craighero, 1999). It is as if to say: perception influences action, which in turn influences perception.

Aims of our research and methodology

In this study we have been concerned with finding strategies for solving geometric problems whose text is accompanied by a related figure. We have focused on the concept of equivalence between polygons and curvilinear figures that can be deduced from an appropriate movement that transforms one figure into another. This, as will be shown later, is not a very simple task for students. To overcome this difficulty, we have reproduced geometric figures by means of manipulatives so that, through interaction with them, these transformations can actually be realized.

We know that repeated interactions with objects such as manipulatives stimulate the communication of different neuronal circuits. If we assume that these circuits are also reactivated when studying a geometric figure of the same type as the one simulated by the materials, we hope that the same movements previously performed on the manipulatives will be imagined on the figure. We believe that this amplification of geometric perceptual abilities will motivate students to rely on them more and more when studying a figure. Furthermore, the cortical areas activated using the manipulatives are in addition to those typically involved during the study of geometry. This involvement of a greater number of neuronal circuits should foster the retrieval of mathematical facts and therefore it is also hoped that what is learned through these motor activities will persist much longer than traditional learning.

We wanted to test the validity of these hypotheses by involving lower secondary school students (grade 7, age 12-14) in a quantitative study. Our research aims were:

a) to measure the effectiveness of manipulatives that simulate geometric figures. That is, we aimed to test whether they are able to generate the ability to imagine the same movements when studying a drawn geometric figure. In this research we dealt with geometric problems already provided with the figure. The object of these problems was the calculation of the area of polygonal and curved figures and the verification of equivalences between different figures,
b) to verify whether the effects of using manipulatives persist over time.

The research design used presents some peculiarities both for the type of variables that were to be measured and because the psychometric survey was carried out in schools during curricular hours (see Pasquazi, 2020 for more details). The study was divided into 6 phases: design of the Pre-test (Phase 1), validation of the Pre-test (Phase 2), administration of the Pre-test (Phase 3), the treatments (Phase 4), administration of the second test called Post-test 1 (Phase 5), and the administration of the third test called Post-test 2 (Phase 6).

Initially (Phase 1), we designed a new test consisting of 16 items to be completed in one hour (the time of a lesson) because, given the aims of our research, there was no suitable one. Of course, it was considered that the topics covered in the test items were already known to the students before our study. We show, as an example, a test item planned for this study dedicated to a type of polygon that differs slightly from a known figure (Wertheimer, 1959): *calculate the area of the surface* Q (Figure 1a). The students had to provide two open-ended answers: in *sub-answers* A they had to describe the solving strategy they used to arrive at the solution. The answers were assessed according to the following criteria. A value of 1 was assigned if dynamic aspects resulting from the study of the figure were used to determine a solution strategy. These answers called *dynamic answers*. In the example shown, the solution strategy that exploits the dynamic aspects is provided when equality is established between the projection and the hole, which leads us to imagine moving the projection until it occupies the entire hole. The equality between the protrusion and the hole is undoubtedly evoked by the shape of Q which resembles that of a rectangle, a figure well known to students and therefore "recognized" by our visual system. Otherwise a value of 0 (zero) is assigned if the answer is missing or is present but is not a dynamic answer. In general, for designed items, when the answer is not dynamic, a lot of calculation is required. In the example considered, if we did not recognize the proximity of Q to a rectangle, we would be forced to calculate the areas of the various polygons into which the figure is divided and then add them up. The answers to the questions in the respective items were to be reported in *sub-answers* B: 0 (zero) was given if the answer was wrong or missing, 1 if the answer was correct. For example, sub-answers A in Figures 1b and 1c will be given a value of 1 because it has been recognized that Q is equivalent to a rectangle (at least that is what it looks like in Figure 1b); the score given to sub-answer B, on the other hand, is 0 in the first case (because the calculation of the area of the rectangle is wrong) and 1 in the second.

Subsequently (Phase 2), the designed test was validated by seventy-four students of an institute. By calculating the *point-biserial correlation coefficient* of each item, the *Cronbach alpha coefficient* (α) was obtained (Pelosi & Sandifer, 2003). The latter was used to measure the reliability and coherence of the test. Conventionally, tests with an $\alpha > 0.70$ are considered acceptable.

Once the validation of the test was completed, the experimental phase began, involving eighty-seven students of the same age from two further institutions (different from the one involved in the validation). The students were divided into two homogeneous groups called *experimental group* and *control group*. The experimental phase started with the administration of the validated test, called *Pre-test,* to the students of both groups (Phase 3). The aim of this test was to find out whether or not the students manifested dynamic aspects in their solving approach and what effect this approach had on the correctness of their answers. We emphasize that the administrators, after giving the appropriate instructions, no longer interacted with the students during the test.

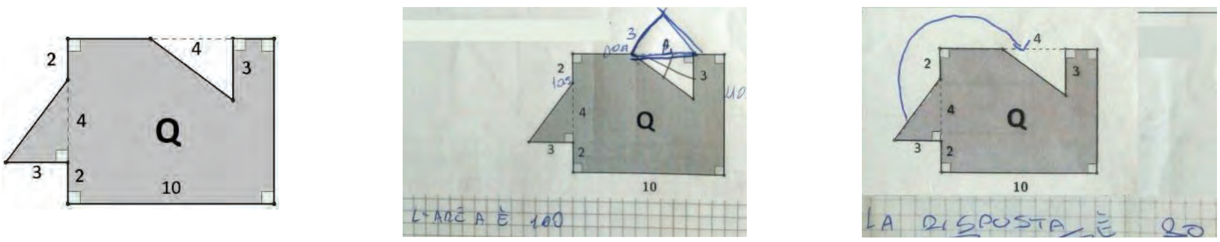

Figure 1: (a) figure associated to an item, (b) and (c) item responses examples

Phase 4 presents a special feature of the research design. Effective treatments were carried out in both the experimental and control group (one meeting per week for a total of three meetings of two hours each). Since there was already an idea of what the results of the Pre-test might be, the treatments aimed at stimulating the perception of dynamic aspects deduced from the observation of the figures. This should have improved the students' performance in answering the test items.

I will show an example of a treatment performed to produce effective resolution strategies related to the test item shown above. Treatment is inspired by the figures of Leonardo da Vinci found in the *Codex Atlanticus*. He performed a lot of squaring of parts of circles and curvilinear figures like the one shown in Figure 2a. His intuitive approach emerges from the captions of the figures, which provide real didactic indications such as the following: "la cosa che si muove acquista tanto di spazio, quanto ella ne lascia" ("what moves acquires as much space as it leaves", folio 505 recto, translation by the author).

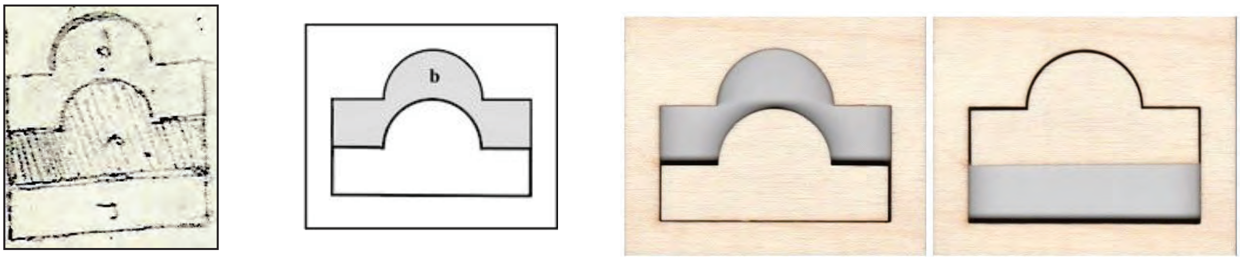

Figure 2: (a) detail of folio 505 recto, *Codex Atlanticus*, (b) figure to an associated problem, (c) manipulatives inspired by the figure of Leonardo

Students in both groups are given the same problem. In the example we want to describe, the task was to measure the area of surface **b** (Figure 2b). Students mostly work in pairs, free to speculate on what they observe, to check, correct themselves if necessary, and then to try a different strategy. Teachers intervened and made suggestions only when students asked for help. Students in both groups were stimulated to formulate and, at the same time, test the validity of their hypotheses. However, the students in the experimental group could use appropriate manipulatives, whereas those in the control group could only draw on their notebooks.

We would now like to briefly describe the manipulatives used by the students in the experimental group. These consist of two elements: a *base* and a *mobile figure* (Figure 2c). By reading the task and looking at the drawing (Figure 2b), students understand how to use the manipulatives appropriately. It is interesting to observe, from the students' actions and statements, the influence of the use of the manipulatives on the perception of geometric properties and how the progressive discovery of these properties guides, in turn, the use of the manipulatives. Initially, the students move the mobile figure at random. Then, as a result of these movements, they understand that all figures obtained from the difference between the base and the moving figure are equivalent. It is evident that the students' eyes light up when mobile figure reaches the opposite position to the starting one. The insight came in: they realize that the difference between the base and the moving figure is now a known surface, i.e. a rectangle. The association between the initial and final difference surface makes the solution obvious. We call *equivalence in motion* this kind of equivalence obtained by movement. Since it is always possible to square a rectangle (Proposition II.14 of Euclid's Elements), it is legitimate to speak of *squaring in motion*.

The answer shown in Figure 1c given by a student in Post-test 2 shows a solution strategy based on squaring in motion that leads to a correct answer. Considering the two right angles on the left in the polygon shown in the Figure 1, we can see that their sides would join if we imagine extending them. This observation contributes to Q evoking a rectangle. The presence of the arrow shows that the student has imagined the movement of the triangle-shaped protrusion in the identical hole. In the Pre-test (Phase 3), the same student had given an unclear and incorrect answer (Figure 1b). The dynamic aspects, if they were present in his mind, had not been effectively exposed.

To test the effectiveness of the treatments, a new test (Phase 5) called *Post-test 1* was immediately administered to the students in both groups. The structure of the Post-test 1 was identical to the previous one, while the questions asked were similar (but not the same). For a separate comparison of the data obtained on Pre-test and Post-test 1 the *Paired Samples t-Test* was used.

In the following six months there was no further interaction with the students involved in the experiment. In order to check whether the effectiveness of the treatments was still detectable at the end of this period, the students were administered *Post-test* 2 (Phase 6), the same as the Pre-test. The administration of a new test long after the previous one is another peculiarity of the research design used in this quantitative study. The data from Post-test 2 were also compared with those from Post-test 1 by means of the Paired Samples t-Test.

Results

In this section we would like to briefly report the main results obtained in the individual phases of the quantitative study. In the validation (Phase 2) of the designed test, for sub-responses A we obtained $\alpha = 0.81$ while, for sub-responses B, $\alpha = 0.88$, thus indicating, in both cases, a very high reliability of the test which motivated us to use it in the next experimental phase.

The analysis of the results of the Pre-test administered in Phase 3 shows, as expected, that there are no substantial differences between the groups. The experimental group obtained an average score of 7.50 (SD 3.76) for sub-answers A and a average score of 7.2 (SD 4.15) for sub-answers B, while control group obtained an average score of 6.12 (SD 4.48) for sub- answers A and an average score of 7.98 (SD 4.50) for sub-answers B. The analysis of the results obtained leads us to conclude that the majority of students in both groups do not prefer to give dynamic answers. Furthermore, the performance concerning the correctness of the answers is also quite unsatisfactory.

In order to verify the effect of the treatments carried out in Phase 4, Post-test 1 was administered a few days later to the students of both groups (Phase 5). The Paired Samples t-Test indicates that the difference between the two averages obtained in Post-test 1 and Pre-test for sub-answers A ($\Delta M = 2.405$, $SD = 3.357$) and for sub-answers B ($\Delta M = 1.595$, $SD = 2.905$), found in the experimental group, are both statistically significant with $t = 4.642$ and $t = 3.558$ respectively and, again for both, p(2-tailed) $< .001$. Similarly, for the control group, the same differences for sub-answers A ($\Delta M = 3.186$, $SD = 3{,}561$) and for sub-answers B ($\Delta M = 1.209$, $SD = 2.739$) were also statistically significant with $t = 5.867$ and $t = 2.895$ respectively and, again for both, p(2 tails) $< .001$. The results therefore indicate that the treatments motivated more students in both groups to provide dynamic answers to the items. As a result, the number of correct answers increased.

After the administration of Post-test 2 (Phase 6), we compared the data obtained between Post-test 2 and Post-test 1 using the Paired Samples t-Test again. In the experimental group alone, the difference between the two averages obtained in Post-test 2 and Post-test 1 for both sub-answers A ($\Delta M = 1.381$, $SD = 4.066$) and sub-answers B ($\Delta M = 1.357$, $SD = 4.119$) were statistically significant with $t = 2.201$ and p(2code) $= .033$ and with $t = 2.135$, and p(2code) $= .039$, respectively.

Thus, the number of students in the experimental group who solve the items by means of dynamic answers increases further. An increase can also be seen in the number of correct answers. In the control group, on the other side, the difference between the two averages obtained in Post-test 2 and Post-test 1 for both sub-answers A ($\Delta M = -0.558$, $SD = 2.914$) and sub-answers B ($\Delta M = 0.116$, $SD = 2.480$) with $t = -1.256$ and $p(2\text{code}) = .22$ and with $t = 0.307$, and $p(2\text{code}) = .76$, respectively, were not statistically significant. This means that there is no substantial difference between sub-answers A and sub-answers B in the two tests for the control group.

Conclusions

In this paper, we have described how to find strategies for solving geometric problems concerning the equivalence of surfaces. We have focused on the ability to provide dynamic answers characterized by the ability to imagine appropriate movements that allow us to transform an irregular figure into an equivalent rectangle. We called this type of transformation "squaring in motion". This strategy has been developed through the use of manipulatives that simulate geometric figures. These allow us to concretely perform continuous movements and visualize equivalences between figures of different shapes.

The effectiveness of the manipulatives presented in this article was verified in a quantitative study involving preadolescents. The latter initially showed that they were unable to imagine movements on the figures of the geometric problems. This had a negative effect on the correctness of the answers. After treatments based on the use of appropriate manipulatives, the students immediately showed an increased ability to solve the problems by providing dynamic answers. Moreover, these abilities were still found six months after the end of the activities. We verified that all these results are statistically significant, telling us that the use of manipulatives can make very important contributions to mathematical learning. However, further investigations on the effectiveness of manipulatives are needed. In particular, we would like to understand why the use of manipulatives did not contribute to the solution strategies for some test items.

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Connecting representations for understanding probabilities requires unfolding concepts in meaning-related language

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It is well known that multiple representations can enhance students' conceptual understanding, but only if (potentially superficial) translation activities are deepened by making the connections explicit. But what exactly does that mean? In our qualitative analysis of a design experiment with two twelfth graders from vocational school on conditional probabilities, we show that explicating connections between multiple representations requires processes of unfolding a highly compacted concept into its concept elements (in our case, part, whole and part-whole-relationship) and articulating them in a decontextualized meaning-related language. By this, we intend to contribute to the research identifying conditions of success for working with multiple representations also for high school students.

Keywords: Connecting multiple representations, explicating connections, unfolding.

Introduction.

It is generally acknowledged that multiple representations can enhance students' conceptual understanding of abstract mathematical concepts (Duval, 2006; Lesh, 1979), but access to multiple representations can be challenging for some students (Ainsworth, 2006; Goldin & Shteingold, 2001). That is why empirical research tried to identify conditions of success under which the use of multiple representations can really contribute to developing conceptual understanding. One identified condition is that representations are not only juxtaposed, but really connected by making the connections explicit (Marshall et al., 2010; Renkl et al., 2013). So far, limited research explored what exactly that means, and how it applies beyond arithmetic and functions, also for probabilities.

Multiple representations have been shown to be relevant also for high school mathematics topics such as conditional probabilities, for which many studies report that visual representations can enhance performance (Böcherer-Linder & Eichler, 2017), e.g. for the conceptual challenge to distinguish conditional probabilities $P(A/B)$ from joint probabilities $P(A \cap B)$ (Binder et al., 2020):

(I) What is the probability that a boy does not play game A?

(II) What is the probability that a teenager is a boy and does not play game A?

An expert can directly assign the correct probability type: conditional and joint probability and translate the questions into a symbolic expression or a visual model. *But what do students need to develop this conceptual understanding of situation types while connecting the statements into visual models or symbolic representations?* In this paper, this research question is pursued within a conceptual framework that combines semiotic processes with epistemic processes, as we present in the first section. The design experiments are conducted in design research methodology and analyzed qualitatively with methods introduced in the second section before the analysis is presented and discussed.

Theoretical background: Semiotic and epistemic processes for developing understanding for conditional probability.

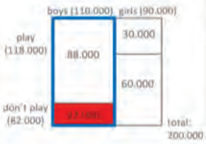
Semiotic processes between multiple representations.

According to the translation models (Lesh, 1979; Duval, 2006), students can develop conceptual understanding by translating between different representations (*e.g., symbolic representation*, *visual models*, *technical language*, *contextual language*). For the case of conditional probabilities, the area model has proven as accessible visual model (Böcherer-Linder & Eichler, 2017) to capture the underlying part-of-part-structure (Leuders & Loibl, submitted), yet most research refers to information processing from *given texts*, not to students' processes of developing conceptual understanding.

In order to overcome well-documented challenges while dealing with multiple representations (Goldin & Shteingold, 2001; Ainsworth, 2006), researchers pointed out that beyond unconsciously *switching* between representations, conscious *translating* activities are crucial (Lesh, 1979, called conversion by Duval, 2006) as well as *explicitly explaining the connections* (Duval, 2006; Marshall et al., 2010). Thus, translating and explaining are connecting processes with a high degree of consciousness about the connection whereas switching involves often low awareness about the change. For clearly explicating how multiple representations are connected, a *meaning-related language* is required to talk about the mathematical structures involved (Pöhler & Prediger, 2015) and how the concept elements occur on the other representation (Renkl et al., 2013). We distinguish contextual language (which can also convey meanings) from a more specific meaning-related language that is decontextualized and suited to *explicate* the *more general structure* behind the part-whole-relationships. In Figure 1, we list the involved representations and distinguish four semiotic processes according to the increasing degree of integration of representations (Post & Prediger, submitted).

Semiotic processes for dealing with multiple representations

no connection switching - - - translating —— explaining connections ====

Given text	Contextual language	Meaning-related language	Visual model	Symbolic representation	Technical language
here: text read and decoded by students *"20% of the boys don´t play the game."*	here: verbalizing referred to context *"group of boys"* *"boys and **thereof** those, who don´t play the game"*	here: verbalizing in a meaning-related but decontextualized way *"whole", "part"* *"part **of** a part"* *"part **of** the whole"*	here: area model 	here: fractions or percentages $\frac{22.000}{110.000} = 20\%$	here: verbalizing referred to symbolic representation *"numerator **divided** by denominator"*

No connection: Juxtaposing multiple representations without connection or only one representation	**Switching** implicitly without awareness of change - - - -	**Translating** by naming the correspondence of the concept/ information ——	**Explaining** connections explicitly by naming the correspondence of elements (and eventually justifying the correspondence) =====
Statement: "the boys and thereof those, who don´t play the game" (here: given text)	It is about all "boys" (here: switch between given text and contextual language)	This group of boys you can see here [*points at rectangle*] (here: translation between contextual language and visual model)	The boys you can see here [*circles around the left part*] and thereof those, who don´t play [*circles around the lower left part*] (here: explanation between contextual language and visual model)

Figure 1: Multiple representations and semiotic processes between representations

Epistemic processes for understanding concepts by unfolding into concept elements.

To characterize students’ development of conceptual understanding, we will draw upon a second perspective, namely an epistemic perspective from cognitive psychology focusing the epistemic processes of students’ mental constructions of new conceptual entities (Aebli, 1981; Drollinger-Vetter, 2011). In this epistemic perspective, learning of concepts is characterized by working with refined concept elements that students have to construct, relate to each other and then to *compact* them into new conceptual entities (ibid.). Characteristic for deep understanding of these new conceptual entities is that they can be *unfolded* back into the contained concept elements.

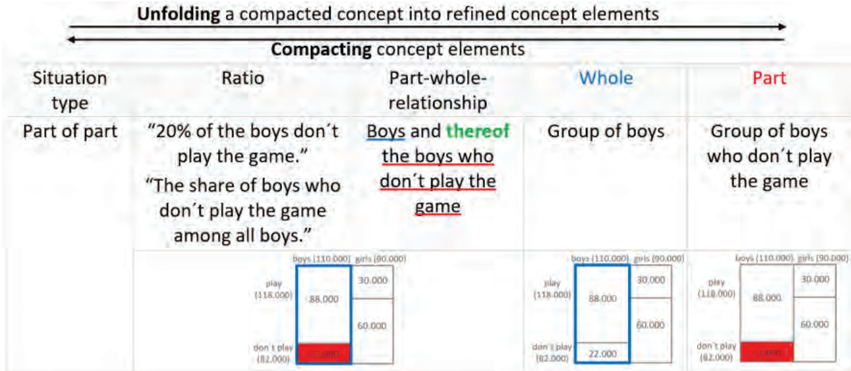


Figure 2: Unfolding and compacting processes for topic-specific concept elements (vertical structure turned into horizontal structure for saving space, here)

Relevant concept elements for our topic-specific conceptual framework were theoretically and empirically identified and are presented in Figure 2. With a focus on statistical ratios, the conditional probability (Question I from above) can be verbalized as “The share of boys who don´t play the game among all boys.”. To differentiate from Question II (joint probability interpreted as “The share of all boys who don´t play the game among all”) these ratios can be conceptualized abstractly as part of different wholes, what we refer to as *situation types*. Question I is a part of a part, whereas Question II corresponds to a part of a whole. Ratio or rather situation types can be unfolded into underlying *concept elements*, namely the *parts*, the *wholes* and the *part-whole-relationships*. For unfolding the ratio, grasping the underlying part-of-part- structure (Leuders & Loibl, submitted) is crucial.

Conceptual framework.

For exploring our research question, we locally integrate the epistemic perspective into our conceptual framework of semiotic processes in Figure 3.

Semiotic processes for dealing with multiple representations

no connection switching - - - translating —— explaining connections ====

Unfolding a compacted concept into refined concept elements / **Compacting** concept elements

	Given text GT	Contextual language CL	Meaning-related language ML	Visual model VM	Technical language TL	Symbolic representation SR
Situation type						
Ratio						
Part-whole-relationship						
Whole						
Part						

Figure 3: Conceptual framework: Navigation space for semiotic and epistemic processes

This allows us to refine the research question: *What concept elements do students address in their pathways of explaining connections between multiple representations for highly compacted conditional probability concepts?* The conceptual framework allows us to trace students' pathways and show how connecting representations requires the unfolding into concept elements.

Method.

Methods of data collection. This case study is part of a larger design research project with two goals, (1) to design learning arrangements and (2) qualitatively investigate learnings processes with the aim of developing local theories on teaching and learning processes (Gravemeijer & Cobb, 2006). In the larger project, design experiments were conducted in seven whole classes (in total n = 150 students) with their regular mathematics teachers. Between the whole class lessons, the first author conducted design experiments with pairs of students to deeply capture their thinking processes. Design experiments were video-recorded and partly transcribed. In this paper, we analyze the transcript of a design experiment with Katja and Justyna from Grade 12 (selected because their intense discussion reveals insights into typical processes found in several cases), restricted to the research question above.

Methods for qualitative data analysis. In order to investigate how semiotic and unfolding/compacting processes interplay, a three-step procedure was conducted: In Step 1, teacher's and student's utterances were coded according to the *concept elements* and *representations* addressed and located in the navigation space of our analytic framework (Figure 3). For analytical clarity and reliability, representations were only coded when references were articulated (e.g., naming the area, quotation from the statement) or gestured, or if the teacher or task demanded it. In Step 2, the *semiotic processes* of relating representations were coded according to their degrees of integrations. In Step 3, the vertical movements up and down were characterized as *conceptual compacting and unfolding processes* and marked in the navigation space by vertical arrows.

Empirical insights into Katja's and Justyna's processes.

In the whole class design experiment sessions, the two girls had already dealt with complex statistical information on parts of different wholes and the area model to visualize part-whole-relationships: For highly compacted statements, students discussed the underlying mathematical structure by explicating the unfolded concept elements whole, part and part-whole-relationship and connecting relevant representations (similar to Post & Prediger, submitted). The teacher had already guided the process of compacting the typical situations into three situation types according to the underlying structure of the statements, called part-of-part-statement, joint statement and simple statement.

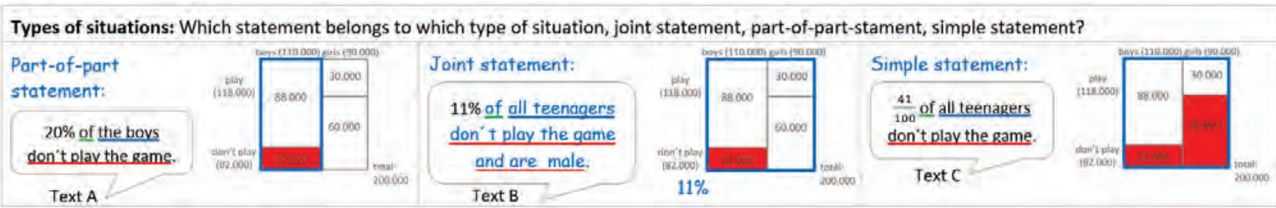


Figure 4: Students' solution for translating compacted situation types to visual model and given text

In the intermediate design experiment in laboratory setting, the design experiment leader went back to the critical step in the learning pathway, connecting highly compacted situation types to visual models and typical given texts. Figure 4 shows their worksheet (translated from German original, blue type marks handwriting, underlining and shaded areas as in original). The students already discussed the given texts by correcting numbers, completing the missing text and coloring in the statements and the area model. The transcript starts when the design experiment (DE) leader pushes them to progress from the successful translating process from texts to visual model to giving titles (Turn 1/2) and explaining these compacted concepts by their explicit comparison (from Turn 3 on).

1	DE leader:	You have got to know these three types of statements. What is this? And this? And this? [*points at the 3 given texts*]	
2	Katja:	That is the simple statement [*falsely points at part-to-part-statement in Text A*]. No, not true, that is the simple [*points at simple statement in Text C*], that is part-of-part [*points at Text A*] and then, that is the combined [*points at Text B; DE writes down titles*].	**2 Situation type** Translate: ML – GT
3	DE leader:	[…] Why? […] What is the difference?	
4	Katja:	Do you want to start?	
5	Justyna:	Um, I cannot explain. Ok, joint is just?	
…		[*hesitates and stumbles, teacher encourages her*]	
12a	Justyna:	Ok, well, here, it is only about the boys who do not play [*unclear on what text she points*] and for the joint, there is one more thing, namely [*2 sec. break*]	**12a Part (Text AB)** Translate: GT – CL
12b		Um, yeah, the boys also take part. But this, there the boys become even more relevant? Thus, these, all persons asked.	**12b Whole (Texts B)** Translate: GT – CL
13	DE leader:	Um, and when you look at the picture again. These are those, aren't they? [*points at the shaded area models*]	
14	Justyna:	Yeah.	
15	DE leader:	This would be part-of-part, this is the joint [*notes titles*]. What exactly, what is the big difference that comes to your eye?	
16	Justyna:	That the persons asked have changed.	**16 Whole (AB)** Switch: CL - - $VM_{A/B}$
…			
25	DE leader:	[*to Katja*] Do you have an idea how to explain it differently?	
26a	Katja:	Um, part-of-part	**26a Situation type** No connection: ML
26b		refers to the subgroup of the whole. And thereof, the other part is the subgroup of the subgroup.	**26b P-W-Relation** No connection: ML
27	DE leader:	Show us, where you are	
28	Justyna:	Oh my god!	
29a	Katja:	Ok, for part-of-part, it refers, it is here, the subgroup, ok, this here, 	**29a Situation type** No connection: ML
29b		the whole [*circles around the left half of the area model*] is the subgroup of the complete, out of the complete persons asked [*circles around the whole area*].	**29b Whole** Translate: VM – ML – ML – CL
29c		And the subgroup is then, um, actually, a part of the part [*circles around the lower left part, then the whole left part*], hence, of the part of the complete.  	**29c P-W-Relation** Explain connection: ML = VM
30	Justyna:	Yes, but said quite complicatedly, no?	
31	Katja:	Yes, though, I understand it [*laughs*].	

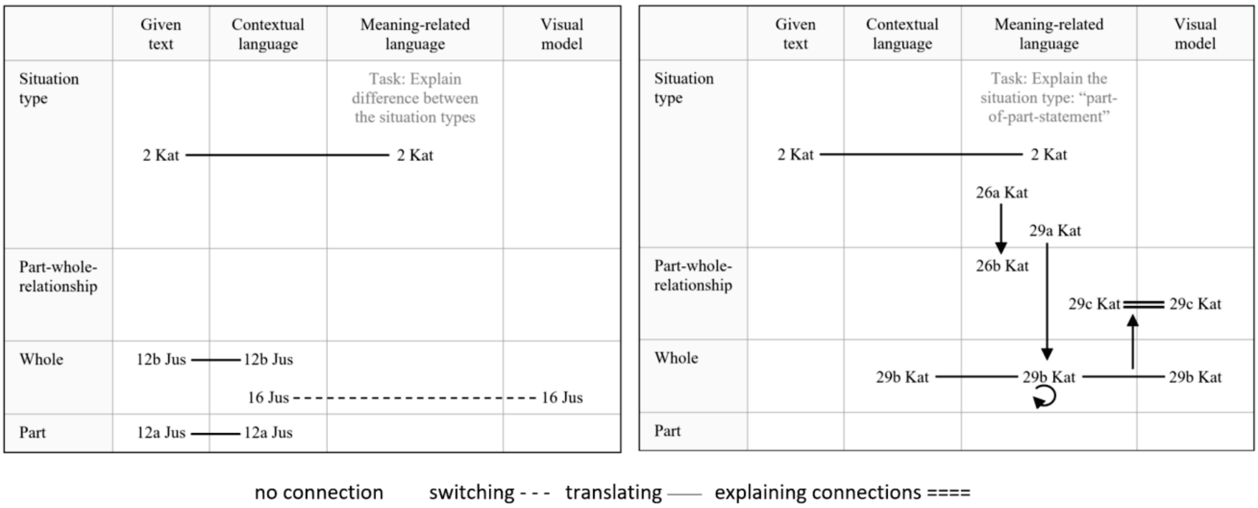


Figure 5: Justyna's and Katja's pathways in the navigation space

In Turn 1, the DE leader asks the girls to assign the given texts (GT) to the situation types (expressed in structural meaning-related language, ML), Katja immediately translates the highly compacted meaning-related notions to the given texts (Turn 2; Translating: ML − GT), yet without explaining her decisions. When the DE leader elicits further explanations (Turn 3), Justyna stumbles (Turn 5-11). Repeatedly encouraged, she starts explaining the connection by unfolding the texts into the part and the whole, and translates both elements into contextual language (Turn 12 ab, Translating: GT − CL), but without explicitly articulating the structure of part and whole. Even when the teachers guides her to comparing the wholes in area model A and B, she unconsciously switches to contextual language ("the persons asked", Turn 16), without explicitly articulating that this is the whole. She keeps on feeling uncomfortable with her explanation and does not even pick up the structural meaning-related language offered by the DE leader in the non-printed Turns 17-24, so the clarification stocks.

In Turn 26, Katja starts her explanation of the meaning of the highly compacted concept "part-of-part statement" and verbalizes the underlying part-whole-relationship in a perfect decontextualized meaning-related language (Turn 26a,b; No connection: ML). In order to strengthen the accessibility of her explanation (for Justyna), DE leader invites Katja to refer to the visual model when continuing the explanation. Katja again starts from the highly compacted concept and addresses the whole (unfolding) by translating between the area model (encircling left half of the area model) and the decontextualized meaning-related language. Moreover, she translates *within* the meaning-related language for expressing the specific character of the whole, before translating to contextual language ("the whole is the subgroup of the complete, out of the complete persons asked", Turn 29b: VM − ML − ML − CL). In Turn 29c, she does not refer separately to the part, but addresses the part-whole-relationship in decontextualized meaning-related language and encircles simultaneously first the lower left rectangle then the left half of the area model. In this way, she explains the connection between the meaning-related language and the area model for the part-whole-relationship (Turn 29c: ML = VM).

For comparing Justyna's and Katja's pathways in the navigation space in Figure 5, the graphical summary makes visible typical phenomena that we could identify also in other cases:

- Both girls do not only switch unconsciously between representations, they connect representations by *translating* or *explaining* with high degree of consciousness.

- Both try to explain the compacted concepts such as “part-of-part-statement” by unfolding some inherent concept elements, part, whole, and part-whole-relationship, and connecting further representations as the visual model or meaning-related language. Whereas a translation can be conducted on each level of compaction, the explanation of connections between these representations always requires the unfolding into more refined concept elements (Renkl et al., 2013).
- Although Justyna works with part and whole separately, she does not have the language to articulate the mathematical structure behind it, so her explanations get stuck. The translations into contextual language alone is not sufficient for her to explain the connection.
- In contrast, Katja successfully unfolds the highly compacts situation types in the texts to the whole, the part, but also to the part-whole-relationship. When she explains the connection between representations, she uses structural decontextualized meaning-related language that allows her to explicitly articulate the structures and the part-whole-relationship that Justyna felt unable to express.

In this way, Katja masters better than Justyna to explicate the connection between these representations in more depth, and to go the way back upwards to the compacted concepts.

Discussion and Outlook.

In this paper, we brought together two theoretical perspectives formerly treated as separate, semiotic processes for dealing with multiple representations (Lesh, 1979; Ainsworth, 2006; Renkl. et al., 2013) and epistemic processes of compacting concepts in conceptual development (Aebli, 1981) or unfolding them back into its constituent concept elements. This act of networking perspectives allowed us to study how two girls deal with multiple representations for complicated concepts such as situation types of conditional or joint probability. When the students’ semiotic processes are disentangled with respect to the concept elements they refer to, we understand better how dealing with representations is connected to epistemic processes of compacting and unfolding concepts.

The qualitative analysis reveals that at the level of compacted concepts, it seems not possible to explain the connection in more depth beyond stating the correspondence of the compacted concepts. For being able to explain the connection more explicitly, the compacted concept must be unfolded into its concept elements, and multiple connections need to be explained, for the part, for the whole and the part-whole-relationship. In this way, the conceptual framework of the navigation space can provide an insightful tool to disentangle what exactly it means that the complex meanings of multiple representations must be negotiated in classrooms (Marshall et al., 2010). Although the vague idea of connecting representations is often mentioned in practical contexts, we will need much more investigations within our conceptual framework to disentangle the complexities of what this entails exactly (Post & Prediger, submitted).

So far, these first empirical findings have to be interpreted with respect to their methodological limitations. Specific limitations are (1) the small sample size, and (2) the restriction to one specific topic and one particular task. In future research, the investigations need to be extended to other tasks and topics to investigate in how far the identified intertwinement of semiotic and epistemic processes cuts across different mathematical topics.

Acknowledgment

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Changes and rigors in systems of mathematical representations within gifted children's problem-solving process

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The paper shows how two gifted primary school children use rules of an individual system of mathematical representations during a problem-solving task. From a research perspective, transformations within and between various representations lead to new mathematical insights and is fundamental for mathematical experiences. But which processes of using mathematical representations do primary school children go through within problem-solving? What happens descriptively before a new mathematical insight in context of representations? For this purpose, two children (3rd grade, 9 years old) were confronted with a mathematical problem-solving task. The case of Fred and Mark shows how they develop the means of representation themselves before a new mathematical insight.

Keywords: Representation, problem solving, gifted, enrichment.

Theoretical Framework

Within a semiotic-pragmatic approach to explaining how mathematicians get new insights, Hoffmann (2005, p. 149; Translation by Schorcht) describes the interplay between representations and representational changes:

> Playing with representations and their [...] transformations are essential to make those associations possible, that we need in order to be able to tap into ideas and conjectures for possible solutions […].

In this quote, he describes how transformations of mathematical representations lead to associations, which in turn allow mathematical insights. Transformations of representations seem to be necessary for mathematical insights. Accordingly, successful problem-solving depends on the ability to transform between representations flexibly. These transformations are one characteristic skill of mathematical gifted children (Käpnick, 1998; Benölken, 2015; Assmus & Fritzlar 2018), who are mostly successful in problem-solving. In order to observe mathematical problem-solving with respect to transformations of representations, the analysis of the problem-solving behavior of mathematically gifted students is particularly suitable. Since this transformation performance is an attribute of mathematical giftedness, transformation processes should be well analyzable in a setting with mathematically gifted children. The resulting choices for the setting are discussed in more detail in the section Methods.

Transformations within or between modes of external representations are grounded on Bruner's (1966) model of different representations of knowledge, which is mostly extended by different researchers (Lesh, Post, & Behr, 1987; Prediger & Wessel, 2011). For the analysis, the semiotic-pragmatic approach will be used, to understand the interplay between these "modes" of representations and new mathematical insights. Representations will be understood as diagrams in the sense of Dörfler (2014, p. 5; Translation by Schorcht): Diagrams contains "inscriptions with a well-defined structure, specified by relationships between parts […], together with rules for reshaping, transforming […] etc." Dörfler understands inscriptions primarily as signs without a visible relation or system of representation. Decompositions and transformations will be possible, if rules of representation are included. Relationships between parts of inscriptions can be transformed, including reference to a rule of the used system of representation.

Dörfler (2014), referring back to Peirce (1976), embeds the actions on diagrams in *diagrammatic reasoning*. Diagrammatic reasoning involves the construction of diagrams: the *creation of a rule-based set of related inscriptions*. Diagrammatic reasoning also involves *performing experiments on the diagrams*: this means transformations, compositions, combinations or decompositions, such as adding, removing or restructuring representations. According to Dörfler (2014), and in special degree also Hoffmann (2004), *observing and recording the experimental results* is a necessary part of diagrammatic reasoning:

> It is not sufficient, thus, that we perform our experiments with diagrams quasi blindly, like a machine. Although a computer might perform experiments quite better than human beings, the essential point of realizing the limits of a selected representational system is self-reflection: *observing what we are doing* when performing an experiment. (Hoffmann, 2004, p. 301; Italic by Hoffmann)

Transformations of representations, such as reshaping, compositions and decompositions of diagrams within certain rules in a system of representation, enable associations that lead to mathematical insights. Through these transformations of representations, new insights become associable for mathematicians by self-reflection on the process. Hoffmann (2004) assumes two possible ways to evolve the diagrammatic reasoning within a certain problem:

> The interesting point, now, is the question *how to get something new* from observing the outcome of diagrammatization. I want to distinguish two possibilities. On the one hand, there is the process of uncovering new *implications* of constructions within a *given* system of representation, and on the other hand, there is the process of *developing the means of representation themselves*. (Hoffmann, 2004, p. 299; Italic by Hoffmann)

In focus on learning mathematics the following research question emerges:

(1) Which processes of self-reflection on diagrammatizations of representations do primary school children use within problem-solving from the first inscription to a subjectively identified mathematical solution?

The paper will briefly show an outline on this research question. In the following, the methodological conditions are clarified and categories for a qualitative content analysis (Mayring, 2008) are deductively determined from the theory just described.

Methods

The survey is affiliated to the project "Mathe für Cracks" (Math for cracks) at Justus-Liebig-University Giessen. The project is an enrichment program for students (3rd to 8th grade; 8 to 14 years old), who are particularly interested in mathematics. Most participants are tested for giftedness while the program accepts children whose interest is mathematics as well. This is due to the origin of the participants, who come through the "Deutsche Gesellschaft für das hochbegabte Kind" (German Society for the Highly Gifted Child) and partly also from "Mensa". For membership in "Mensa" association, the children must be tested. Insights into these test results or the procedure of the tests do not take place for data protection reasons. All participants must present a teacher recommendation in order to participate in the course. Consequently, these are particularly high-achieving students, some of whom have also been tested through their own membership of clubs. In cooperation with "Deutsche Gesellschaft für das hochbegabte Kind" (German Society for the Highly Gifted Child), Mathematikum Giessen and Mathematik-Zentrum Wetzlar, the program provides 40 places for primary school children (8 to 11 years old) and 50 places for secondary school children (10 to 14 years old). "Mathe für Cracks" cooperates with one third of primary and secondary schools in Hesse (State of Germany). Since studies with mathematically gifted children, such as Käpnick (1998) or Assmus and Fritzlar (2018), show that the changes in representation is a characteristic of mathematical giftedness, the framework "Mathe für Cracks" is particularly suitable for recording transformations of representations before new mathematical insights.

The pilot study contains video recordings from 13 interviews with 18 children: 8 children worked individually and 10 children worked together in sets of two. Working on problems alone resulted in not externalizing internal processes or interpretations. Only in collaboration and communication with others did internal processes become visible. Therefore, the setting was adapted in the last 5 interviews. The children operate in these last 5 interviews together with their pencil on one given paper. This was intended to increase the collaboration rate between the two children. 9 settings took place with an accompanying interviewer (7 single interviews and 2 group interview), while 4 settings were only provided with an introductory interviewer (1 single interview and 3 group interview). The setting was changed during the pilot study because the interviewers had too much influence on the students' solving behavior. The children tried to guess a solution via gestures of affirmation instead of solving the problem through mathematical reasoning. For this reason, an accompanying interviewer was not used in the final 4 interviews with 7 children. As Goos and Galbraith (1996) also discussed, working in teams is best suited to avoid influences of the interviewing environment and incomplete and inconsistent verbalizations. For this reason, thinking aloud, interrupting interview questions, or retrospective questions were neglected during the last interviews in the pilot study. One geometric (5 interviews) and one number theory problem-solving task (8 interviews) were provided to the participants. The paper will be focused on the following number theory task:

> A palindrome number can be read forwards and backwards. For example, 1221 and 808 are palindrome numbers. Neighbor palindrome numbers are located next to each other on the number line (for example 121 and 131). Which differences are possible between two neighbor palindrome numbers?

As shown in the student's solution at Figure 1, one good idea to solve this problem is to write down some palindrome numbers in order. By arranging neighbor palindrome numbers, the difference between the numbers can be found. In 6 pupils' solutions are lists of the possible differences as well (circled numbers in Figure 1). These knowledge stores are results of experiments on diagrams.

Distances between two neighbor palindrome numbers are 2, all powers of ten and all powers of ten multiplied by 11.

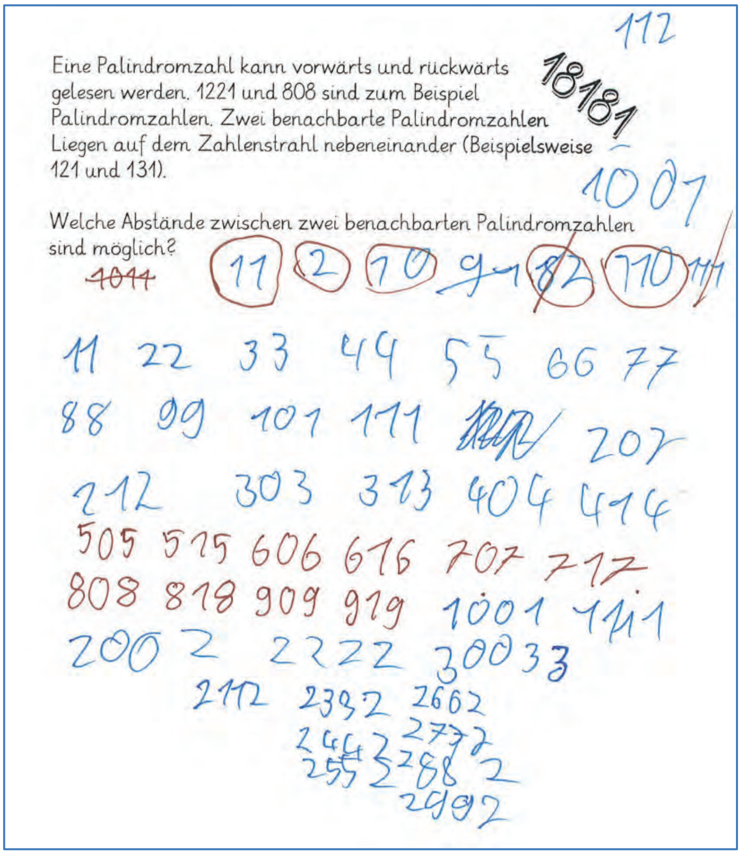

Figure 1: Given task and solution by Fred and Mark (9 years old).

Beside these products of diagramatization, there are transcripts of video recordings. These transcripts were analyzed by qualitative content analysis (Mayring, 2008) with coding categories developed deductively. The coding category *diagrammatization* contains four subcategories:

- *constructing an icon or a diagram*,
- *experimenting upon this icon or diagram*,
- *observing the result of experiments* and
- *determining in general formulae*.

If the children start to use the pencil to make a drawing, this section was coded with *constructing an icon or a diagram*. If the relations were transformed in a new representation, it is an *experimenting upon this icon or diagram*. After an experiment, the *observing the result of experiments* could be identified verbally. *Determining in general formulae* is coded if the results of the experiments for all examples are verified.

The coding category *new mathematical insights* contains two subcategories:

- *uncovering new implications of constructions within a given system*,
- *developing the means of representation themselves.*

Uncovering new implications of constructions within a given system was encoded when students followed the given rules within an experiment or transformation, naming relations that were not included in the previous representation. *Developing the means of representation themselves* was coded when students changed the meaning of the representation by adapting new rules.

In the following, the case study of Fred and Mark is analyzed in order to show the self-reflection process during problem solving and before a new mathematical insight (in the sense of Hoffmann).

Analysis

Mark and Fred (9 years old), who face the palindrome number task, *construct a diagram* by noting the palindrome numbers from 11 to 111. They use representations such as 112 to communicate and negotiate the rules of their own system of representation. They *experiment with the diagram* to identify differences and *observe the results of experiments* by circling the differences. In the following, they note the palindrome numbers from 202 to 919, but only those with 0 and 1 in the hundreds place. After that, 1001 and 1111 follow. By *uncovering new implications of constructions within a given system*, they correctly note all palindrome numbers between 2002 and 3003 (Figure 1). In the following, two moments will be considered in more detail: First, at the moment of establishing a rule to create the diagram, and second at the moment of a new mathematical insight:

In the first case, Mark and Fred are into the process of diagrammatic reasoning. They design a list with all palindrome numbers. To do this, they start at 11 and continue through two-digit multiples of 11 until they reach 101 and 111. From this moment on (line B039 in the transcript below), they discuss different ways of continuing the sequence to satisfy the rule "A palindrome number can be read forwards and backwards" within the given system of natural numbers. Among the numbers discussed are 112, 122 and 220 but these are not included in the list of palindrome numbers (B039 to B053 and B058). Fred and Mark *develop the means of representation themselves.* The rule for construction of palindrome numbers is extended to the rule "same digit in the hundreds place value and ones place value". They construct 202, 212 (B054, B059 to B061):

B039	Fred:	No hundred and twelve is not
B040	Mark:	But there is the one twice
B041	Fred:	Yes, but that means nothing. Look! <<writes separately on the sheet 112>> One, one, two - read backward would be two hundred and eleven
B042	Mark:	True!
B043	Fred:	and that's another one and then the next one is one hundred [twen]-two
B044	Mark:	[twen]
B045	Fred:	Yes, one hundred twenty-two.
B046	Mark:	Noo!
B047	Fred:	Or, yes, yes, yes! Um, because the rest doesn't work.
B048	Teacher:	<<leaves his place>>
B049	Fred:	One hundred and twenty-two <<writes 122>>
B050	Mark:	Yes, but then we now have two hundred, um
B051	Fred:	uhhh, true true true

B052	Mark:	[two hundred and twenty-one] read backwards
B053	Fred:	Oh yes! <<crosses out 122>>
B054	Mark:	So, it does not work. Now it's two hundred and two.
B055	Fred:	Yes, so um
B056	Mark:	Now two hundred and two comes here. <<taps his pencil to the right of the crossed-out 122>>
B057	Fred:	But two hundred, uh, right, the rest does not work. <<writes 202>>
B058	Mark:	Exactly, and then two hundred and twenty.
B059	Fred:	Two hundred and twelve.
B060	Mark:	Right, two hundred and twelve is correct.
B061	Fred:	<<writes 212>> and then three[hundred and three].
B062	Mark:	[hundred and THREE]! <<nods once>>
B063	Fred:	<<writes 303>>
B064	Mark:	We've got it now, right?

Constructing 303 after 212 suggests another rule that both share nonverbally but consensually (B062 to B064). In the following Fred and Mark construct only palindromes with 0 or 1 in the tens place value. An uncommunicated rule might be: "There is 0 or 1 between the largest and smallest place value". Consequently, the digits of the hundreds place value and ones place value are increased by one in the following and noted with 0 or 1 at the tens place value. Fred and Mark construct 313, 404, 414, 505 to 919. Following the rule, that the first and last digit of the number must have the same digit, they construct 1001 and 1111. After Fred notes 2002 and 2222, he compares the representation of four-digit numbers with that of three-digit numbers and stops the further construction of palindrome numbers (B154 to B160). In doing so, he makes a new mathematical insight and disproves the previously nonverbal rule (B162). Digits other than 0 and 1 can be entered in the middle place value. This considerably expands the representation of the palindrome number list. He is *uncovering new implications of constructions within a given system*:

B154	Fred:	Has actually only one zero more than here the two numbers <<points with the pen to 202 and 212>>. Wait, thousand one hundred, there is still one, we can best <<writes 2112>>, ehm, let's note it down according to the size. Here <<points with the pen to 1001 and 1111>> there is also another one (unintelligible), it doesn't work here!
B155	Mark:	<<looks at the task sheet>> (mumbles unintelligibly)
B156	Fred:	Wait! <<points with the pen to the lowest written line>>
B157	Mark:	Truue, true.
B158	Fred:	Paliadrome, <<shakes head>> ahh, no idea. So, it must be <<writes 3003>> three thousand three <<writes 3 and traces the number several times>>.
B159	Mark:	Three thousand three.
B160	Fred:	Hm, moment, I just thought about it for a moment. Then three thousand, there's that possibility again. Three
B161	Teacher:	Can I see what you're doing?
B162	Fred:	Ahh so, here there are still more possibilities <<points with the pen at 2222>>. The whole like here, the whole can make nevertheless also still with the three. There are still thousands

> of possibilities. But we can simply leave that in the small, because that is still the same distances. <<looks at the task sheet>> Namely one hundred and eleven, <<points with the pencil to 2112>> here, these are always, so these are one hundred and eleven again, I'll write under here again briefly the others that work <<writes 2332, 2442, 2552, 2662, 2772, 2882 and 2992>>.

Discussion

Which processes of self-reflection on diagrammatizations of external representations do Fred and Mark use within their problem-solving process? Mathematical insights become possible for them by developing rules within their given system. Palindrome numbers consist of digits whose value is equal in power at the largest and smallest place value, at the second largest and second smallest place value, at the third largest and third smallest place value, and so on. The relation between the digits of a palindrome number Fred and Mark try to capture in a diagram. To do this, they discuss rules for constructing their diagram, which lines up palindrome numbers ordinally. To begin, they start with 11 and correctly continue the series of palindrome numbers to 111. Fred then constructs the number 112 in line B041 to see if it can be read forward and backward. In addressing the construction of the diagram, they also test 122 (B049). 220 is checked mentally (B056). Both develop the means of representation themselves by establishing the rules for constructing their diagram. However, setting the rule "There is 0 or 1 between the largest and smallest place value" to construct the diagram is no longer viable for the task after a certain moment. By the unspoken rule, Fred constructs 1001, 1111, and 2002. He also constructs 2222. Then, in conversation, Fred reflects on his construction by comparing the four-digit numbers 2002 and 2222 with the already constructed three-digit numbers 202 and 212 (B154 to B160). This self-reflection leads to a change in the rule of their representation system. Fred discovers that between 2002 and 2222 the palindrome number 2112 can be formed. In self-reflection, Fred uncovers new implications of constructions within their system. He changes his own rule for constructing the diagram and independently forms a new rule. With this rule he can form new palindrome numbers and thus correctly notates all palindromes between 2002 and 3003.

Conclusions

This one example shows that Hoffmann's idea of self-reflection processes for constructing one's own diagrams for new mathematical insights are also relevant for children. In this case study, one can see particularly well how the self-reflection of the construction of the diagram leads to new mathematical insights. Continuing the diagram of 1001 and 1111 with 2002 and 2222, both children see a new relation that was not visible before. In the comparison of the four-digit numbers, with the three-digit numbers 202 and 212, Fred can uncover a new implication via self-reflection. Both change the set of rules for creating the palindrome list. The question now arises whether this development of rules is typical for new mathematical insights? Do the new insights always manifest themselves in this way in children's problem-solving processes? How can children be encouraged to develop their rules? Is there a way to descriptively capture the timing of children's mathematical insights? Further work on this topic will follow.

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Students’ Perception of Change in Graphs: An Eye-tracking Study

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Changes and dependencies of two different quantities are characteristics of functional relationships, which are often visualized in graphs. Hence, for graph interpretation it is necessary to perceive and interpret change. This paper focuses on how students perceive change in graphs. Since eye tracking is a promising research tool to approach thinking processes, we conducted an exploratory case study with two participants: We used eye tracking and stimulated recall interviews to examine this method’s potential for studying individual processes in perceiving change in graphs. From the observed eye movements and given interpretations, we were able to illustrate that students’ individual approaches in perceiving change in graphs can be related to different levels of covariational reasoning.

Keywords: Eye tracking, eye movements, functions, graphs, covariational reasoning.

Introduction.

Graphs, which are one of the external representations of functions, are pervasive in our lives and therefore also an important topic for mathematics education since they represent functional contexts (Friel et al., 2001). However, the meaning of a graph is not immediately apparent (e.g., Freedman & Shah, 2002). To understand data, represented in graphs, it is important to be able to interpret graphs and especially to perceive the relationship between the values of two quantities.

Some studies have already been conducted on how students reason when working with two different quantities, focusing on how they change in relation to one another. Research focuses, for instance, on different kinds of covariational reasoning (Johnson, 2015). Yet, how students proceed when interpreting empirical graphs and their change is investigated only to a limited extent. Knowledge on this can help to improve the learning and teaching of graph interpretation and in particular of covariational reasoning. To shed light on graph interpretation processes, eye tracking (ET) appears to be a promising research tool. Since ET has—to the best of our knowledge—not yet been used to study students’ empirical graph interpretation, we first want to investigate the method’s potential itself: whether it is possible to draw conclusions from eye movements of students interpreting graphs and to infer students’ approaches of perceiving change in graphs. We first ask a methodological research question RQ1: *Is it possible to infer students’ perception of change in graphs from their eye movements and given interpretations?* If it turns out that such inferences are possible, it is of interest to approach the empirical research question RQ2: *What approaches do the participants use when perceiving change in graphs and what levels of covariational reasoning does this reveal?*

Theoretical Framework.

Graph interpretation and covariational thinking in mathematics education.

Graphs play an important role in everyday life, as they are used, for example, in order to visualize the development of stock market prices, temperature curves, or training processes. Due to this high applicability for real-world phenomena, and because graphs represent also inner-mathematical functional relationships suitably, graphs are central contents in mathematics education. Graphs do not reveal their meaning immediately, which is a general characteristic of mathematical objects, as Duval (2006) describes:

> Mathematical objects, in contrast to phenomena of astronomy, physics, chemistry, biology, etc., are never accessible by perception or by instruments (microscopes, telescopes, measurement apparatus). The only way to have access to them and deal with them is using signs and semiotic representations. (p. 107)

Data visualized in graphs must be understood by inferring their interrelationships, their background, and meaning. To meet these demands, it is crucial for pupils "to identify certain types of changes and dependencies, which are part of common events in the real world, as well as to become familiar with their representations" (Eisenmann, 2009, pp. 73–74). Numerous studies revealed that students may have difficulties dealing with representations of functions and that certain types of errors are common due to a lack of covariation understanding (e.g., iconic interpretations, interval/point confusion, slope/height confusion) (Leinhardt et al., 1990). Working with functions requires dealing with two different quantities, focusing on how these quantities change in relation to one another. Thompson and Carlson (2017) emphasize that students' need to develop an appropriate idea of the relationship between values of two quantities in their work with functions. They argue that "variational and covariational reasoning are fundamental to students' mathematical development" (p. 423). Since the graphs used in our study always consider changes in two quantities (time/distance covered and velocity, or time and filling level), we restrict ourselves to covariational reasoning in this paper. Thompson and Carlson (2017) distinguish six major levels of covariational reasoning, which show a continuum of students' conceptions of covariation: *no coordination, precoordination of values, gross coordination of values, coordination of values, chunky continuous covariation, smooth continuous covariation* (p. 441). The first four levels can be considered as preliminary stages of conceptualizing covariation. Students whose covariational reasoning can be classified in the two highest levels perceive actual covariation, as only then the change in one quantity affects changes in the other quantity. Therefore, our study focuses on chunky and smooth continuous covariation, since we intend to describe students' covariational reasoning when describing change in empirical graphs. In general, *continuous covariation* involves the perception of change in one variable simultaneously with changes in another variable. More specifically, in a *chunky* continuous covariation, the changes are perceived in intervals with a fixed, but not necessarily the same, size. The focus is on the values *at the end* of each interval and how they change as compared to the end of the following interval. In contrast, in a *smooth* continuous covariation the change is perceived as increasing or decreasing. The focus is on how the values change *within* an interval.

Some studies have already investigated how changes and covariation in functions are identified and understood (e.g., Johnson, 2015). Yet, it has hardly been explored how change is perceived in the interpretation of graphs and in how far covariational reasoning is involved. These processes of interpreting graphs and perceiving change within them are complex and may differ between individuals, so that ET appears to be a promising method to investigate these processes on a micro-level. We believe that with the help of eye movements, cognitive processes in the interpretation of graphs and in particular the perception of change in graphs as well as students' approaches and use of covariational reasoning may be inferred: This is why we investigate the use of ET and its potential for this purpose in this paper.

Eye tracking in mathematics education research.

ET describes the capturing of person's eye movements, which can be visualized in a video of participants' field of view, with a wandering dot indicating the gaze. Studies on eye movements have considerably increased over the last years (König et al., 2016). In a mental-oriented view of ET, which we also adopt for our study, ET studies are used to infer cognitive processes from eye movements, i.e., "to use eye movements as a window to cognition" (König et al., 2016, p. 2). The prerequisite for this is the eye mind hypothesis (EMH). The EMH presumes a close relationship between what persons fixate on and what they process (Schindler & Lilienthal, 2019). However, it has been revealed that this assumption cannot easily be taken for granted in mathematics education. For example, Schindler and Lilienthal (2019) found that there are instances in which eye movements cannot easily be mapped to cognitive processes and that even if this is possible, the interpretation of eye movements is often ambiguous. Therefore, they call for domain-specific theories for the interpretation of eye movements. One domain in which ET has rarely been used is graph interpretation. Nevertheless, ET seems to be beneficial for studying students' graph interpretation: Strohmaier et al. (2020) emphasize that ET lends itself to the use of visualizations of mathematical objects since the work with visualizations requires persons to process visualized information with multiple gazes. To investigate the potential of ET for the analysis of graph interpretation processes, we conducted the study that is presented in this paper: It explores the potential of ET for analyzing graph interpretation and perception of change in particular. In addition, we will empirically focus on how change in graphs is perceived by students and on if and how eye movements can be used to infer students' covariational reasoning.

Method.

Sample, task design, and setting.

In our exploratory study, we analyzed eye movements of two university students during graph interpretation tasks, who volunteered to be participants for our study. They were told that they will participate in a study on functions at secondary level. The participants, Gerrit (age 21; engineering and management student with a focus on production engineering; high affinity for mathematics) and Elias (age 28; a teacher student for German and history; low affinity for mathematics) were selected because they have different profiles in terms of their professional background and mathematics affinity. Further, they were not familiar with empirical graphs so that they might show interesting approaches to interpret them. In addition, being university students, they could probably express rich information about their cognitive processes in the interviews.

This paper presents an excerpt from a larger study that examined participants' eye movements while interpreting graphs in different situational contexts (see Figure 1 for two examples) as inspired by the Shell Centre for Mathematical Education (1985)). Each unit consisted of five analogous tasks with different demands. The focus in this paper is on the first task, which asked the students to describe the change in the graph (see Figure 1). The tasks were presented on a screen. There was no time restriction for working on the tasks. The study took place in a quiet room with the participant and the first author of the paper administrating the tasks. The participants sat on a firm chair in front of a table where the monitor was located. Before the questions and the graph were presented, the students were familiarized with the situational context and the graph by presenting them with a digital task sheet on the monitor showing the graph and an introductory text about the situational context.

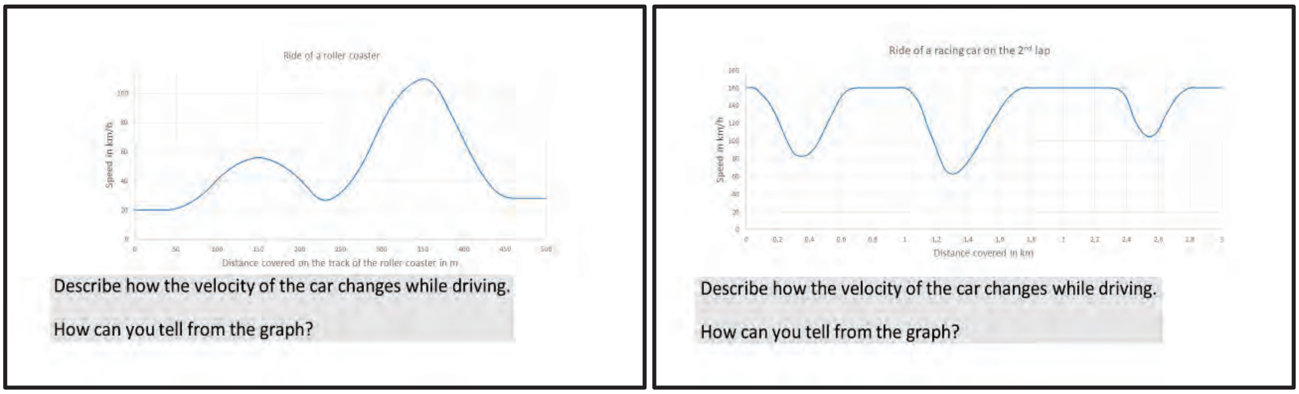

Figure 1: Examples of Task 1 (translated to English)

Eye tracker and ET data.

For data collection we used a wearable eye tracker: Tobii Pro Glasses 2 (50 Hz, binocular, infrared, 45 g, built-in microphone). The tasks were presented on a 24" screen (60 Hz, viewing distance: 60 cm). First, a single-point calibration procedure was performed. Under ideal conditions, gaze estimation is 0.62° (Tobii Pro, 2017). In our study, the accuracy was 1.1° on average, which corresponds to 1.15 cm in the screen. This inaccuracy was taken into account in the task design and data interpretation. Before solving the tasks, Gerrit and Elias passed an additional 9-point calibration verification so that we could later check the measurement's accuracy. Since we wanted to study students' interpretation process and especially their perception of the change in graphs, we considered all eye movements relevant and decided to analyze raw data (Holmqvist & Andersson, 2017), i.e., eye movements as displayed in gaze-overlaid videos.

Stimulated recall interview based on gaze-overlaid video.

In our study, we combined ET with stimulated recall interviews (SRIs) using gaze-overlaid videos, similar to Schindler and Lilienthal (2019) since ET has not been used so far to investigate student's perception of change and therefore it is still unclear how to interpret the eye movements in this context. Stimulated recall is a technique that "gives participants a chance to view themselves in action as a means to help them recall their thoughts of events as they occurred" (Nguyen et al., 2013, p. 2). We used gaze-overlaid videos as stimulus in our study, in which the participants can watch their eye movements as a wandering dot. They were supplemented with students' utterances during task processing. Gaze-overlaid videos represent a strong stimulus because they make eye movements visible, which are usually not conscious (Stickler & Shi, 2017). Thus, it is particularly important to keep the time span between the ET recordings and the SRI short. We therefore conducted the interviews directly, after only a short break for data transfer, so we can assume that they were still very much aware of their thoughts, which were then additionally recalled by the strong stimulus of visible eye movements as trigger. The participants had the possibility to stop the video themselves to explain their thoughts. In addition, the interviewer was able to pause the video and to invite the participants to express their thoughts. The participants wore ET glasses even during the SRI. Here we made use of the scene camera and the built-in microphone of the eye tracker to record the verbal utterances and gestures. This procedure was explained to the participants before data collection.

Data analysis.

To prepare the data analysis, the students' utterances while working on and answering the tasks were transcribed together with a description of the eye movements taking place during this process. In addition, the utterances of the interviewer and the interviewees in the SRI were transcribed. These elements were arranged in one document in neighboring columns to make visible what happened simultaneously and what the SRI refers to. Data analysis followed Schindler and Lilienthal (2019) since this approach is particularly suitable for domains in which ET has rarely been used before and where it is not yet clear how eye movements can be interpreted: The steps of qualitative content analysis (Mayring, 2014) were applied and—due to the explorative and descriptive nature of the research questions—categories were developed inductively. We distinguished between gaze categories that describe gaze patterns and interpretation categories that describe the cognitive processes associated with the respective gazes. Eye movements and cognitive processes, as described by the students in the SRIs, were then mapped for all data.

Results and Discussion.

Research Question 1: Feasibility of inferring students' perception of change in graphs.

Our methodological research question (RQ1) asked whether it is possible to infer students' perception of change in graphs from their eye movements. Results indicate that the participants were able to explain their eye movements in the SRI. Their utterances gave information about their cognitive processes while interpreting graphs and perceiving the change within them. This is the prerequisite, since we have as a basis only utterances and interpretations of the participants' eye movements. We cannot observe the cognitive processes directly, but can only get closer with the help of the participants' eye movements and interpretations. However, like Schindler and Lilienthal (2019) we found that a certain eye movement pattern could not always be clearly assigned to one cognitive process (see Table 1).

Nevertheless, conclusions about approaches can be drawn based on the eye movements that the participants used when perceiving change and interpreted in the SRI. For instance, we were able to observe certain eye movement patterns, consisting of different gazes, that were interpreted by the participants (e.g., following the course of the graph, jumping between a point on one axis and a point on the graph, looking on several different points on the graph in succession). These eye movements were used to perceive the change in the graphs to eventually be able to describe it in order to answer the task. The participants referred to these gaze sequences, which differed between them, in the SRI. Using this information, we were able to infer approaches when perceiving change in graphs.

Eye movement pattern (identified in the gaze-overlaid video)	Cognitive process (described by the student in the SRI)
The gaze follows the course of the graph	Grasp the graph (e.g., the course or properties) Grasp the situational context (e.g., thinking about driving curves with a racing car)
Fixations on turning points of graphs	Focus on prominent parts of the graph (e.g., turning points) Grasp the graph (e.g., the course or properties)

Table 1: Exemplary eye movements from task 1 and students' interpretation

Research Question 2: Students' perception of change and covariational reasoning

Our empirical research question (RQ2) addressed the approaches the participants use when perceiving change in graphs and whether levels of covariational reasoning can be revealed from this. As mentioned above, we were able to observe differences regarding the occurring eye movements between the two participants. Elias followed the course of the graph or a graph section with his gaze particularly often. Afterwards, he interpreted this mostly as trying to grasp the situational context and sometimes the graph (e.g., course or properties). Moreover, he often looked at several different points on the graph in succession. He explained that this eye movement pattern served the same intention (grasping the situational context or graph). Also, Elias used gestures to support his gaze, for example by following a section of the graph with his finger. This was never the case for Gerrit, who also used other eye movements particularly frequently. His gaze often jumped between a point on one of the axes and a point on the graph. He explained that he was reading a particular point or value from the graph. Sometimes he added that he used this to grasp the graph. In addition, he often looked at turning points of the graph. He interpreted this in the SRI as focusing on prominent parts of the graphs. These results indicate that Elias perceived the change in the graphs by looking at the graph and following it with his gaze and, in some instances, his fingers, or by making sense of it by looking at several different points. He explained the change of velocity of a racing car in a car race as follows.

Elias: The car starts. Probably, most likely enters a curve, therefore drives slower here, then comes out of the curve, drives faster again, drives a straight stretch, then drives a steep curve, so narrow, must brake very hard in any case. Then drives relatively quickly out of the curve again, then again a long stretch and then again a small curve. [utterance translated from German to English by first author]

It is clear from the utterance that he focuses on the change of velocity within the intervals, i.e. increasing or decreasing. This is characteristic for smooth continuous covariational reasoning.

Gerrit's perception of change was apparently different. He also said several times that he focused on prominent parts and read values there:

Gerrit: Ehm, so first the velocity decreases to 0.35 km. Then it increases again to the original 160 km/h from 600 m to 1 km. Then it drops again to about 60 km/h. And then increases again to 160 km/h after 1.3 kilometers. From 1.8 km to 2.3 km keeps that, eh the car its velocity. And then drops again to 105 km/h approximately at 2.5 km/h and then rises again to its original 160 km/h until 2.8 km and then keeps that until 3 km. [utterance translated from German to English by first author]

In this utterance, Gerrit defined intervals and read off a value for each endpoint of an interval in order to describe the change of velocity, what is typical for chunky continuous covariational reasoning. Yet, he did not disregard the change of the graph between two of such points: He described whether the graph increased, decreased, or was constant in the respective interval. This hints—at least to a certain extent—also at smooth covariational reasoning.

Discussion and outlook.

In this paper, we have illustrated that it is possible to interpret students' eye movements when interpreting graphs in order to obtain information about their perception of change in graphs. Our tentative results indicate that ET in combination with SRI seems to be a suitable method for studying the perception of graphs. In addition, we were able to infer different approaches regarding the perception of change and different levels of covariational reasoning for Elias and Gerrit. Our results provide evidence that certain eye movement patterns are typical of certain levels of covariational reasoning. For future research, it would be of interest, for instance, whether the relations found between Elias and Gerrit's gaze patterns and approaches can also be found for other individuals. Moreover, we only focused on smooth and chunky covariational reasoning, i.e. the two highest levels of covariational reasoning according to Thompson and Carlson (2017). Thus, it might be valuable to test and further sharpen the insights from this exploratory paper on a larger data set, and with middle school students to maybe find different approaches in interpreting the change in graphs and related levels of covariational reasoning and its preliminary stages. Besides, it should be examined what implications arise from our initial result. Elias, whose approach can be related to smooth covariational reasoning, for example, often referred to the situational contexts and related the change of velocity directly to the circumstances of the situational context (by saying that the car is driving a curve and therefore brakes, etc.). Whether there is a direct connection between the kind of covariational reasoning and the extent to which the situational context is referred to and how this affects the further interpretation of the graph remains to be studied. Even though our case study provided only a glimpse on these aspects, we think that it may be a first step towards investigating students' interpretation of graphs and in particular their perception of change in graphs using eye tracking.

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Teachers’ practice and pupils’ representations in a Grade 3 class

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In this paper we analyze a grade 3 teacher’s practice and how it relates with his pupils’ representations. We focus on teacher-pupils interaction in the classroom while pupils are solving a problem. We aim to understand how the teacher manages his pupils’ use and interpretation of representations. Data were collected through video and audio recording of lessons and later analyzed through content analysis. The results show that the teacher’s representations, actions and questioning types change according to the pupils’ activity and difficulties and that the whole class discussion is the teacher’s preferable moment for a more dynamic change of actions and questioning. That way, to promote his pupils’ use and understanding of representations, the teacher adapts his representations, actions and questioning types.

Keywords: Representations, elementary school teachers, teaching practices, teachers’ actions, questioning.

Introduction

A representation is a mental or physical construct that describes the characteristics of a concept and its relationships with other concepts (Tripathi, 2008). The way teachers deal with representations in their practice has a great influence in how pupils use and understand them (Stylianou, 2010). It is widely acknowledged that teachers must use different kinds of mathematical representation to promote pupils’ understanding of representations (NCTM, 2007; Tripathi, 2008). However, representations are connected to each other in different ways and that can promote several difficulties in the process of pupils’ understanding and learning of representations (Goldin, 2008). Knowing how teachers may use different representations is important to know if that is the source of pupils’ difficulties. Our aim in this study is to understand how a grade 3 teacher explores a task with his class, focusing our analysis on the way he manages to promote the use and interpretation of representations.

Teachers’ practice and representations

Pupils’ activity on a task is determined by the way that teachers explore it in the classroom, specifically the role that teachers assume, their actions and the questions that they ask (Swan, 2007). For Boaler (2003), the process of describing and analysing the practices of teachers is very complex considering that they result from the influence of different aspects like teachers’ knowledge, beliefs and prior experiences. She also states that besides describing teacher practices it is important to know how they are doing these practices, what decisions are they taking and what influences them. Thus, in the classroom, it is important to analyse teachers’ actions and how they are questioning their pupils. In their perspective, McDonough and Clarke (2003) suggest that in the classroom teachers should: (i) talk clearly to their pupils; (ii) explore adequate and challenging tasks with more than one possible solution; (iii) promote the establishment of connections with pupils’ previous knowledge; and (iv) explore learning opportunities that emerge in the classroom. These authors also refer to the importance of using different teaching approaches, materials and representations. Blosser (1975) identifies four question types: (i) managerial, to give operating instructions; (ii) rhetorical, to emphasize an idea; (iii) closed, when there is a limited number of right answers; and (iv) open, when there are several possible answers. In a similar way, for Mason (2000), there are three different main aims in teachers' questioning: (i) focusing, when the teacher questions seek to help pupils focus in detail, using a funneling effect; (ii) testing, in which the teacher analyses the pupils' activity, namely their comprehension, how they articulate ideas and make connections among them; and (iii) inquiring, when the teacher questions the pupils to understand what they are thinking.

Vergnaud (1987) refers to representations as crucial elements for teaching and learning mathematics. Accordingly, NCTM (2007) refers to representations as essential elements in upholding pupils' understanding of mathematical concepts, in mathematical communication, in argumentation and in suiting mathematics into realistic problems situations. Simultaneously, many authors have been categorizing representations in different types. For example, Bruner (1999) indicates that representations may be active, iconic or symbolic, Thomas et al. (2002) refer to pictorial, iconic and notational, and, more recently, Loc and Phuong (2019) indicate visual, symbolic, verbal, contextual and physical representations. It is important to acknowledge that understanding representations is a complex process because a representation may have several meanings and a meaning may have different representations (Goldin, 2008). Concurrently, Duval (2006) refers that for understanding the features of a mathematical object, we need to know how to make changes within a representation (treatment) or to convert it into another representation (conversion).

In the classroom, the role of teacher is crucial in helping pupils to understand and to use this intricate web of representations. In the past years some researchers have been studying teachers' practice regarding mathematical representations. For example, Webb et al. (2008) not only categorize representations as formal, informal and preformal but they also provide suggestions to teachers regarding how to help their pupils in interpreting and using representations. In the pupils' learning process, teacher's representations also have their own role. Bishop and Goffree (1986) indicate that teachers must encourage the establishment of connections among representations while promoting their interpretation. Stylianou (2010) develops these ideas indicating that representations play a crucial part of teachers' explanations as they provide new concepts, drawings regarding problem-solving processes, and create connections among concepts. She also states that a teacher may bring in new representations, connecting them to pupils' previous knowledge and, by doing this, the teacher supports pupils' learning of concepts, procedures and problem solving processes.

Methods

This paper is drawn from larger research about elementary school teachers' practices regarding mathematical representations. During the study, a working group composed of four grade 3 teachers got together in pre and post lesson sessions to analyze their practices and their pupils' work. In this paper, we will only present and analyze the interaction of one of these teachers, Ricardo (teacher and pupil names are pseudonyms), with his pupils, during one lesson. He is a young grade 3 teacher, from a school in the surroundings of Lisbon (where he has been for the past 2 years, since he started to teach). He has twenty pupils (8 to 10 years old) and they have been working together since grade 2. The pupils are used to solving problems as the one reported in this paper. However, the teachers thought that the problem would be interesting to explore with their classes, given their pupils' needs and difficulties in problem solving and in working with whole numbers. That way, we will analyze Ricardo's practices while exploring the following task in his classroom: "*The third graders are planning a field trip. They rent four buses that are fully booked. Each bus has 24 pupils. Knowing that for each group of 10 pupils there must be one teacher, how many children and adults are going in this field trip?*".

Data was gathered by video and audio recording during class observations. Pupils' written work was also collected. Data was analyzed through content analysis (Fiorentini e Lorenzato, 2006), according to the three moments of the lesson indicated by Ponte (2005): (i) introduction of the task, where negotiation of meaning may take place (Bishop & Goffree, 1986); (ii) pupils' autonomous work, individually, in pairs or groups); and (iii) whole class discussion. From the categorizations of Bruner (1999) and Thomas, Mulligan and Goldin (2002) we categorized representations as: active (handling objects or materials), pictorial (drawings really close to context); (iii) iconic (informal symbols – like dots and arrows – diagrams or schemes using different types of representations); verbal (words); and symbolic (mathematical symbols). Based on the framework of Ponte and Quaresma (2016), we defined a new table of analysis (Table 1). Here, we categorize teachers' actions regarding how they promote the understanding of representations, and we relate their actions with their pupils' activity. We assume that it is underlined the mutual influence between the pupils' activity and the teacher's actions. That is, pupils' activity affects teachers' actions and, in turn, teachers' actions promote pupils' activity.

Pupils' activity regarding representations	Teachers' actions
Choosing/Designing	Promoting the free choice of a representation Challenging to choose a different representation Guiding about an adequate representation Providing explicit suggestions or examples
Using	Challenging to use a representation Asking to interpret a representation Guiding about the use or interpretation of a representation Informing pupils about how to interpret or how to use a representation (In)validating a representation chosen by pupils
Transforming	Challenging to establish treatments, conversions and connections Guiding to establish connections Guiding to identify possible treatments and conversions Inform about treatments and conversions
Reflecting	Challenging to systematizations Leading to systematizations Informing about systematizations

Table 1: Teachers' actions in different moments of the pupils' activity

In table 1 there are four categories for pupils' activity, related to teachers' actions: (i) support the pupils' design or selection of a representation; (ii) promote the use or/and interpretation of a representation; (iii) promote the transformation of representations; and (iv) promote pupils' reflection about representations.

Regarding teachers' questioning, based on the general frameworks indicated by Blosser (1975) and Mason (2000) we considered three different types of questions, with some subtypes (table 2).

Type	Subtype	Examples
Focusing	Rhetorical	We saw this already, didn't we?
	Processual	Could you open your books on page 58? What if we look back into the task?
	Orienting	What if you sum it all?
Confirmation	Closed	How many will we have if you add 10?
	Open	Can we solve it in another way?
Inquiring	Open	Do you agree with your colleagues' answer? Why?

Table 2: Different types of teachers' questions

As we move from Focusing to Inquiring, questioning the questions' subtypes become more demanding for pupils and teachers as they enable the possibility of having different answers, strategies and representations.

Results

Introduction of the task

Ricardo reads the statement of the task and his pupils begin a whole group discussion as they try to solve it by using verbal representations. As the discussion gets really confusing, the teacher chooses to help them by informing the class how to interpret the statement of the task. He focuses on data that he finds most relevant ("You must find two different things. How many pupils and how many teachers are there?").

Pupils' autonomous work

While the class is working individually and autonomously, Ricardo notices that some pupils do not understand the statement of the task and he chooses to discuss it again with his class. One of the pupils, Mateus, states that there are 96 pupils. The teacher questions him through closed confirmation questions (How do you know that there are 96 pupils?" and the pupil answers correctly ("That's because it's 24 times 4!"). Additionally, Ricardo brings another representation to the discussion as he converts Mateus' verbal representations into an active representation, by using his own fingers (Twenty-four in one [finger/bus], plus twenty-four in another, and another one, and another! It's 96, isn't it?"). Although Mateus can explain his answer, he is struggling to find the number of adults.

That seems to happen because he’s referring to a real-world situation, as they’re field trip usually has one adult per bus (“There are four adults because each bus takes one adult!”). However, Vanessa opposes him by recalling the statement of the problem (“No! Each group of TEN must have one responsible adult!”).

While the pupils keep trying to solve the problem by using oral verbal representation, Ricardo pressures them to use written representations. When Ana, another pupil, answers “I am solving it! I am thinking about it!”, Ricardo tells her “You are thinking (…) but I want you to explain it here [in your notebook]! I don’t want you to tell me… I want you to show me!”. In saying this, he is not being critical or restrictive about using a specific representation. Much on the contrary, he is trying to promote the use of different types of representation and, at the same time, he seeks that the pupils convert mental representations into written ones. As some pupils are still trying to understand the statement of the problem, Ricardo uses an iconic representation to inform them about its interpretation (Figure 1).

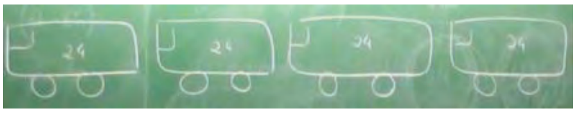

Figure 1: Iconic Representation (with pictorial and symbolic representations) used by the teacher

Whole group discussion

Ricardo transcribes to the board the iconic representation used by Jonas, another pupil, and he asks the pupil to explain it to his colleagues (Figure 2).

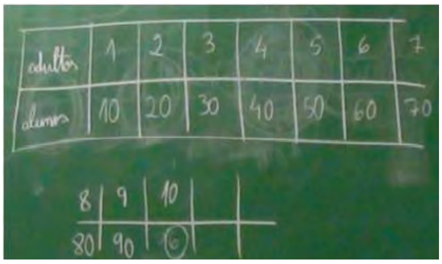

Figure 2: Iconic Representation (table with verbal and symbolic representations) used by Jonas and transcribed by the teacher

The teacher questions Jonas with rhetorical questions (“You did this, didn’t you?”, “For each group of ten pupils there is an adult, right?”), confirmation questions (“One adult… For how many pupils?”, “How did you fill this table?”). In this way, Ricardo almost completes the iconic representation, reaching the 90 pupils square. At this moment, there are six remaining pupils. At first, Ricardo decides to challenge his pupils to interpret Jonas’ representation, questioning them through inquiring questions (“Can I add one more adult?”, “Can we have ten adults?”). When he does this, he triggers a big discussion with two different opinions and several arguments. Most pupils agree that the right answer is nine adults while a few (like Jonas) argue that the right answer is ten adults. However, none of the pupils can give a proper explanation and Ricardo changes his actions and questioning type. This time, he gives some suggestions to interpret Jonas’ table through open confirmation questions (“What about the remaining six pupils? Are they staying at school?”) as he tries to relate to a real-world situation. Now, although pupils seem to notice that leaving behind six pupils it is not a fair thing to do, they still cannot justify the tenth adult. Once again, the teacher changes his actions and questioning type to help the pupils. On a third attempt, Ricardo informs the class about how they can interpret Jonas's table and he questions the pupils with rhetorical questions ("There must be one more adult, doesn't it?", "These six pupils are not going all by themselves, right?", "We need one more adult to go with them, OK?"). Despite Jonas's and Ricardo's explanations, some pupils are confused and they do not accept this solution because they have solved it in a different way. For those pupils the right answer is 96 pupils and 12 adults. Previously, during the teachers' working group session, teachers acknowledged that there could appear two different possible solutions (96 adults with 10 or 12 adults, depending on the assumption made about the requirement for adults accompanying children: is it in the buses or is it for the global group of children). As this task might have two different solutions, they thought that it would be the perfect opportunity to work this type of tasks with their classes. That way, Ricardo knows what is happening and he asks Mauro to present his solution to his colleagues (Figure 3).

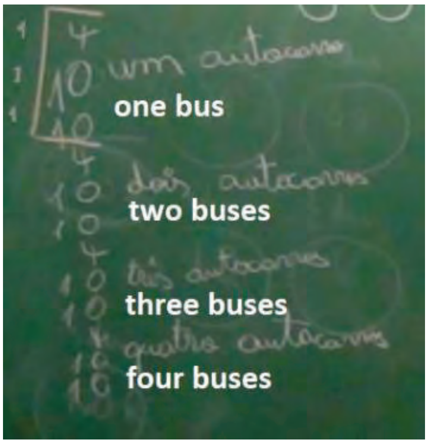

Figure 3: Iconic Representation (with verbal and symbolic representations) used by Mauro.

Mauro explains how he interpreted and used his representation ("There are 24 pupils in each bus… So… Ten pupils go with one teacher, another ten with another teacher and the four children left are going with another one!"). Ricardo informs the class about Mauro's explanation, focusing on the data that he found more relevant (he compares Jonas' and Mario's representations and strategies). For that, he questions the pupils through rhetorical questions ("Before, we add all the pupils to find the number of adults, wasn't it?"). After this, he asks again Mauro to explain his representation as he guides his pupils to establish connections between the two different iconic representations. He questions his class through closed confirmation questions ("How many adults here?"). Ricardo ends the whole group discussion by informing the pupils about the aspects that he found more relevant. That way, he talks about the differences and similarities of the two representations, reinforcing that both representations, strategies and solutions are correct.

Discussion

During the introduction of the task, Ricardo informs his class about the statement of the task, focusing on the information that he finds more relevant. For that purpose, he uses closed confirmation questions as he supervises the whole group discussion that occurs among pupils. Afterwards, during the pupils' autonomous work, the teacher asks the pupils to use and to interpret their own representations, guiding them to convert mental representations into written ones. Although he uses active and pictorial representations (included in the scheme of the buses – figure 1) to help the pupils to interpret the statement of the task, while they are working by themselves he promotes a free choice of representations. By doing this, Ricardo promotes his pupils' awareness of the importance of choosing an adequate representation (Thomas et al., 2002). The whole group discussion is the moment where Ricardo asks the pupils to explain to the class their own representations. For that purpose, he uses rhetorical and closed confirmation questions. Depending on the pupils' answers and difficulties, the teacher: (i) challenges the pupils to interpret a representation, questioning them with inquiring questions; (ii) suggests how to interpret a representation by using open confirmation questions; and (iii) informs the class about how to interpret a representation through closed confirming questions and/or rhetorical questions.

In the three moments of working on the task, Ricardo's actions tend to change according to the pupils' answers and difficulties. When pupils have less difficulties, Ricardo prioritizes actions with a higher level of cognitive demand (Figure 4). When and if pupils start having more difficulties, Ricardo changes his actions and questioning, decreasing its level of cognitive demand. That way, the more difficulties the pupils face, the lower level of cognitive demand his actions have.

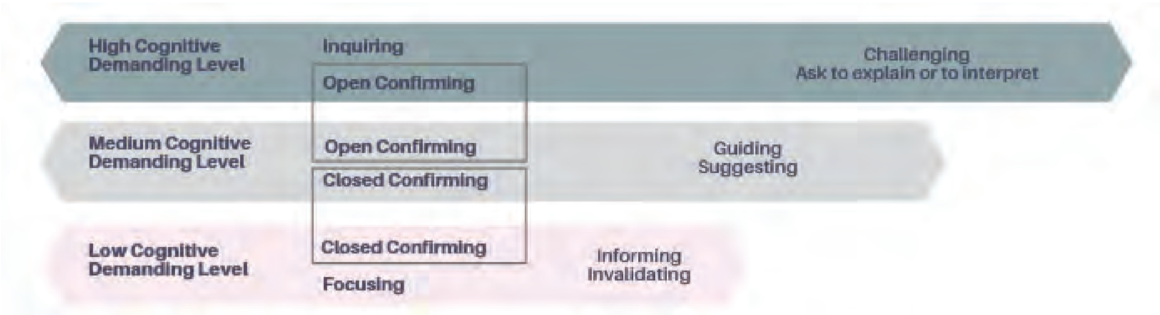

Figure 4: Interrelation teacher's actions and teacher's question

Conclusion

As Duval (2006) states, to understand the features of a mathematical object we need to know how to make treatments or conversions. Ricardo is aware of this as he, at the introduction of the task, informs the pupils about the conversion of the verbal representations (from the statement of the problem) into pictorial and active representations so they can understand it and choose an adequate representation to solve the problem. Additionally, during pupils' autonomous work, the teacher looks up for different representations. Later, during the whole class discussion those representations can be discussed, as the pupils present to their colleagues the two possible solutions for this task. By doing this, Ricardo provides the pupils the opportunity of establishing connections between the two different iconic representations (Bishop & Goffree, 1986). In this way, by using different types of actions, questioning and representations (not only from pupils but also his own), Ricardo strives to promote the pupils' understanding and the use of different representations. From here, in the future, it would be interesting to explore how the teacher's actions influence pupils' representations.

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TWG25: Inclusive Mathematics Education – challenges for students with special needs

Introduction to the papers of TWG25: Inclusive Mathematics Education – challenges for students with special needs

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Keywords: Inclusive mathematics, special education, teacher education, interventions, digital learning.

Introduction

The Thematic Working Group 25 "Inclusive Mathematics Education – Challenges for Students with Special Needs", established for CERME11 in 2019, was running for the second time at CERME12. The scope and focus of TWG25 covers research about special educational needs (SEN) and inclusion, in the intersection of mathematics education research and special education research. Since this scope is broad, the TWG-papers comprised grades 1-12, teacher professionalization and teacher education programs, types of inclusive settings in mathematics, concepts and models for instruction and subject matter didactics, special educational needs and child characteristics, and content related decisions for inclusive mathematics education.

During CERME12, TWG25 had 34 participants from 12 countries (Europe, also Brazil, Canada and the US) who presented 17 papers and 7 posters. The first session was spent on aims and objectives of this TWG and an exchange of overarching issues of the situation of SEN mathematics education in the different represented countries. In the following TWG-sessions, two up to four papers were presented each time, under an overarching theme. The three main fields were:

General papers, conceptual models, research review

Focus on teacher education, pre-service and in-service teachers

Focus on classroom, teaching and learning situations, different school levels

Moreover, poster authors were asked to identify the connection to papers in the initial session in order to include all participants. Posters were about inclusive education and collaborative instruction focusing on pre-service, general and special educators (Dibbs & Boyle) and their different roles (Scherer & Rolka). They also covered specific interventions for supporting students with difficulties in learning mathematics (Larmann & Ludwig), proposed a structure that should facilitate development of teaching materials for inclusive classrooms (Novotná & Moraová), presented ways of supporting special-needs children in the context of probability (Jaschke) and discussed diversity in relation to digitalization (Ludes-Adamy & Viermann), as well as affective and mediational suitability (Blanco et al.) in relation to a specific inclusive program.

In detail, the sessions were arranged according to the following thematic focal points:

Session 2: General papers, conceptual models, research review

This theme includes discussions of prior research regarding conceptual models as well as inclusive models for special educational needs in mathematics.

Helena Roos & Anette Bagger: Explicit instruction and special educational needs in mathematics in early school years

Marzia Garzetti, George Santi, Heidrun Demo & Giulia Tarini: The interplay between theory and practice in the development of a model for inclusive mathematics education

Karine Millon-Fauré, Patricia Marchand, Claire Guille-Biel Winder, Teresa Assude, Jeanne Koudogbo, Laurent Theis & Mathieu Thibault: Preventive support scheme for mathematics learning: possible ways to provide aid before and after the class session

Session 3: Focus on teacher education, pre-service and in-service teachers

This theme includes discussions of how framework conditions such as teacher education, specific teaching, cooperation between professions and steering documents influence inclusive mathematics teaching.

Tabea Knobbe, Christof Schreiber & Michaela Timberlake: Cooperation of mathematics teaching and special education – seminar concept and experiences

Jennifer Bertram & Petra Scherer: Pre-service teachers' beliefs and attitudes about teaching in inclusive mathematics settings

Michael Gaidoschik: "Individual Educational Plans" for "dyscalculic" students in primary schools of South Tyrol: A questionable law, poorly implied

May Ron Ezra & Esther S. Levenson: Perceptions of mathematical creativity among math teachers in special education classrooms

Session 4: Focus on classroom, teaching and learning situations, primary level

This theme includes discussions of specific interventions in the inclusive mathematics classroom to enhance learning of every student. The focus of research could be the learning environments, the students' learning processes, or teachers' acting.

Yola Koch: Working with objects of representation in practical contexts on length in inclusive classrooms

Uta Häsel-Weide & Marcus Nührenbörger: Inclusive math practices in primary school

Marie-Line Gardes, Céline Hugli, Jasinta Dewi, Ludivine Hanssen & Michel Deruaz: Evaluation of a computer-based learning program for students with mathematical learning difficulties

Carina Gander: "Counting with all children from the very beginning": One attempt to promote early arithmetical skills based on part-whole thinking

Session 5: Focus on classroom, teaching and learning situations, primary and lower secondary level

Also in this session, the papers and discussions focused on specific interventions in inclusive mathematics classrooms, with a special view on more advanced content such as problem solving or early algebra.

Raja Herold-Blasius & Benjamin Rott: Low-achieving secondary students learn mathematical problem solving – A longitudinal, qualitative video study

Ángeles Chico, Inmaculada Gómez & Nuria Climent: Problem-solving by students with Asperger's Syndrome

Ann-Kristin Tewes & Marcus Schütte: The mathematical support format reproduction

Francesca Gregorio: The role of examples in early algebra for students with mathematical learning difficulties

Session 6: Focus on classroom, teaching and learning situations

This theme includes discussions of how to create engagement in inclusive mathematics classrooms to enhance learning and participation of every student.

Amanda Queiroz Moura: Inclusive landscapes of investigation in mathematics classrooms with deaf and hearing students

Silvia Baccaro & Annalisa Cusi: A teaching methodology focused on the use of a videogame: analysis of the engagement of students with special educational needs

Introductory discussion – overarching issues of inclusion

Mathematics education needs to embrace the diversity that exists in mathematics classrooms to be able to meet every student. This implies that heterogeneity with respect to language, culture and abilities needs to be acknowledged (Bishop et al., 2015). This requires an accommodation in the mathematics classroom to enhance every student's learning. To be able to embrace the diversity and to meet the needs in a democratic education for all, where students with different languages, cultures, abilities and skills are educated together, inclusion is used as an overarching notion for support (Bishop et al., 2015). Though, often the notion of inclusion has been connected to special education rather than to a democratic education overall (Allan, 2012).

If we define the concept of inclusion in a *mathematics education* context, it implies to look for ways enabling us to meet the diversity in the teaching of *mathematics.* In Europe (and also around the globe) the way "inclusion in mathematics education" is defined and used is very different (Roos, 2019). Some countries and cultures connect inclusion very tightly to special education and disabilities, while others have moved towards meeting diversity on a more overarching level, connected to a democratic and equitable education. This also depends on the national governing documents. Even so, there are many challenges for inclusive education in school and research (Kollosche et al., 2019).

Mathematics education at every school is expected to create inclusive classrooms, with lessons where every student receives appropriate support and challenge. Here, the teachers' competences and those of other professionals are crucial. In turn, this puts pressure on the teacher education to be able to equip teachers for their mission regarding inclusive mathematics teaching (Krainer, 2015; Scherer, 2019). Although there exist big differences in different countries, the need of preparing both pre-service and in-service teachers for inclusive mathematics education is recognized in all countries. The discussions in TWG25 showed the need to recognize different strategies to support pre-service and in-service teachers' processes of inclusive mathematics teaching. If teachers are better prepared for teaching in inclusive classrooms and have models and methods to make it work, education can be able to meet and value diversity (Askew, 2015).

During many sessions in TWG25, both 2019 and 2022, the question of how we understand inclusive education in mathematics was discussed. What does "inclusive education" mean in different countries? What is the role of stakeholders in different countries? How do we apply inclusive education to different cultural and national settings, and what are sufficient strategies on organization, classroom and individual level? These questions are reflected upon in the overarching themes that emerged from the discussions of TWG25, and described below.

Overarching themes of TWG25

Research in the field of inclusive mathematics covers a wide range, and several trends in mathematics education were presented in the papers of TWG25. All papers focus on some aspects of mathematics teaching and learning in relation to SEN and deserve due attention. During the sessions, the following points were discussed, especially how they are linked with each other.

What do we understand as „inclusive maths education", which theoretical perspective do we have/propose?

Within TWG25, it was agreed that it is important to have a broader discussion about inclusion, also from a theoretical point of view. In addition to bringing *solutions*, it is important to ask *questions* about what inclusion means, about the (no) need of labels for students, about the characteristics of students with special needs and other aspects of research not intended to directly provide something applicable in the classroom, but with consequences on classroom practices. Without a focus on theory, it is not possible to have deep enough discussions and propose good enough solutions.

Inclusive mathematics education is a very complex field with manifold perspectives: Student perspective, teacher perspective, teacher training, etc. The question we should ask is who is it in the system that has special needs – the student, the teacher, the system of education?

It follows from the discussions in the TWG that truly "inclusive mathematics education" is such practice in which every pupil in the classroom is welcome regardless of knowledge, skills and background. One way of achieving it is by paying enough attention to the learning environment, to let pupils work together, collaborate, be active. *Participation* and *inclusion* are strictly connected. A way of achieving this is making suitable learning offers and preparing lessons allowing each pupil to be successful and experience the pleasure of achieving. We should look for ways of engaging all students in standards-based, inquiry-based instruction, but provide structures and support so that all pupils will be successful, without diminishing the cognitive demand of the tasks. No doubt that this is not easy to achieve, but it is what we should be aspiring to.

The challenge takes on additional facets when deafness and the lack of a common (sign) language make communication between hearing and deaf children as well as between deaf children and (mathematics) teachers who are not trained in sign language difficult or at times even impossible. In her contribution, Amanda Queiroz Moura pleads for the establishment of “inclusive landscapes of investigations” with the permanent engagement of sign language interpreters, who have the explicit task of facilitating communication also between hearing and deaf children while they work together on challenging problems. In the discussion, however, it was also questioned whether the “dogma” of joint learning of hearing and non-hearing children might not, under certain conditions, lead to a reduction of learning opportunities for the non-hearing, so that in this case *social participation* might happen at the expense of the *content-related participation* of deaf children.

Genuine inclusion always means both (Roos, 2014; Jung & Schütte, 2017) – and obviously requires a great deal of ressources in terms of competent teachers, time, but also technical aids, material, spatial requirements and the like. One important task for future research in inclusive mathematics education might be to provide a more solid empirical basis than we have at the moment for assessing the way in which those inclusive settings, which have been enacted in different nations based on pedagogical convictions and, in the end, *political decisions*, actually prove their worth for the *content-related participation* of children with different learning backgrounds.

What do we understand as „special-needs student“, what (if any) use is it to have „labels“ like „dyscalculic", how helpful/harmful are national laws/policies in that respect, (how) do we/can we have influence on such laws/policies?

Another subject of discussion was the issue of labelling. In many countries, diagnosing a special-needs child is a process that schools have to perform in order to be allowed to adapt the child’s program and start interventions. As a rule, such diagnoses follow a medical-psychological approach, with deficit-oriented definitions of “disorders” and clear cut-off criteria within standardized testing procedures. This seems to be particularly problematic in the case of “dyscalculia”, given the empirical evidence from mathematics education research that learning difficulties in mathematics may at least to some extent be explained by inadequate instruction (Gaidoschik, 2019), and in many other cases be traced back to insufficient fit between the teaching and the learning requirements of a single child. In such cases, it is not the child that suffers from a learning disability, but rather the school system from disability to offer the child a proper learning environment. Attributing “dyscalculia” may therefore result, on the one side, in maintaining such classroom practices that do no good to any child, and on the other side, in single children's learning difficulties being labelled as consequence of *their individual* dispositions (Gaidoschik et al., 2021).

Of course, under the given circumstances, only such a diagnosis may mean the possibility to have fewer pupils in classroom, a special teacher to cooperate, to give the pupil extra time in exams, to have an assistant for the child. Yet, if the label does not lead to *additional pedagogical resources* that could help to overcome the learning difficulties, it rather has the function to formalize the further treatment of a child defined as presumably permanently not able to learn mathematics at a “normal” level. In the following, this labelling may be very unpleasant for the child as they get an official stamp of not being ordinary, of being different from the others. In some cases, it may result in resignation, low aspirations, low self-confidence and low level of motivation. The discussion among the TWG participants showed that there is no simple answer to whether such labelling is the right step in inclusive education. While some argued that the label helps the teacher understand the pupil's needs but also may help the pupil gain self-confidence seeing that they do not perform badly because they would be "stupid", but because of a diagnosis that is not their fault, others argue that a truly inclusive classroom with truly inclusive practices is ready to support every learner whatever their special needs are.

Do we have best practices or proposals for such practices or open questions about how to get to best practices in teacher education for inclusive maths education with a view to collaboration of maths and special education teachers?

A lot of the discussions focused on good as well as bad practices in mathematics classrooms, existing and used in different countries. As Michael Gaidoschik pointed out in his paper, inclusive practices in many countries tend to be tied by demands of the authorities. The school has to create a formal document (which may have different forms in different countries) that defines the methods of work, the objectives, the needs, but all this is rigid and quite formal. In some countries this kind of document or plan is subject to work of the inspection more than what is actually happening in the classroom. This in no way supports good practices at schools. And, as follows from the discussion, research has a responsibility to not just accept what politicians do and how they act and define things, but to always be critical, especially when researching inclusive and special education.

The discussions also focused on collaboration between ordinary mathematics teachers and special teachers, which does not happen in all countries, and special teachers are still very rare in some of the countries. To put it shortly, mathematics teachers tend not to get sufficient training in care for special-needs children and special teachers, in general, do not seem to have sufficient training to understand problems in mathematics. Hence, special teachers need more training in mathematics and mathematics teacher more training in special education. Also the need of close collaboration between special and regular mathematics teachers was part of the discussions and is regarded as one of the topics that should be inquired in more detail in research and on the next CERME.

Following the discussions in TWG25, *collaboration* is at the base of inclusion. This does not refer only to pupil-teacher but also to teacher-teacher and also researcher-researcher as well as teacher-researcher collaboration, where mutual projects of special education and mathematics education will definitely bear fruit. Collaboration and knowledge-sharing between disciplines is what is needed. Effective co-teaching requires sustained professional development in which both (special and mathematics teacher) develop a shared understanding of mathematical development, curricular design and both the challenges and strengths of pupils with special needs. It also requires that teachers have adequate time to plan together. Teacher education programs need to foster collaboration between special education and mathematics education. In turn, research on the mathematical learning of students with disabilities must integrate neurodiversity into research design and implementation.

In the discussions it turned out that a lot of current research focuses on the pupil and his/her activity. However, it is the teacher whose everyday reality is teaching heterogeneous classes with pupils of very different skills and levels. They need to be paid a lot of research attention, looking for ways of helping them grow more self-confident and less anxious in classrooms. The question the research community should ask is “What training and support does the mathematics teacher need?”

Conclusion and further directions of TWG25

Reflecting on the research presented in TWG25 and on the discussions, diverse issues concerning inclusion in mathematics education and challenges for students with special needs in mathematics exist, and further research is needed. One major question is how to cope with the diversity of research directions and cultural as well as national and international differences. Still, it is interesting to see that even if there are very diverse issues concerning inclusion in mathematics education, there are similarities too, and this could be a chance for national and international exchange and cooperations. In our view, research topics should be better connected and, if possible, transferred. Research findings for specific mathematical topics in the sense of best practices should be reviewed for other topics or for similar topics on different school levels or grades. What are the relevant factors or design elements that work in inclusive classrooms? In what way can we consider both students’ perspective and teachers’ perspective? Also on the teacher education level, apart from specific studies with a detailed focus, the different phases of teacher education programs could be considered in a more general way: Understanding professional development of teachers as a life-long-learning process (e. g. Cedefop, 2015), findings with respect to pre-service teacher education programs should be reviewed for in-service teachers, and should be connected to programs for teacher educators. Mathematics programs for special education and programs for regular teacher education should be reviewed in more detail and checked for common objectives. Only in this way will the necessary cooperation in so called multiprofessional teams be ensured.

In summary, there seem to be opportunities to build a common ground regarding inclusive mathematics teaching and challenges for students with special needs. Looking forward to CERME13 the community has a challenge to build further on the work at CERME12 and build a common ground to address issues within the scope of TWG25.

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A teaching methodology focused on the use of a videogame: analysis of the engagement of students with special educational needs

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In this paper we reflect on the effects of a methodology based on the use of a videogame in terms of engagement of students with special educational needs. By referring to theoretical lenses useful to characterize students' in-the-moment engagement, we analyse data collected within a teaching experiment conducted within mixed-abilities lower secondary school classes. The results of our analysis show a general positive trend in the evolution of students' engagement structures.

Keywords: Special educational needs, videogames, inclusion, structures of engagement.

Introduction

The research documented in this paper is part of a wider study on the use of videogames in fostering inclusion in the teaching-learning of mathematics at primary and lower secondary school level. Different studies highlighted the potentialities of the use of videogames in fostering inclusive processes for students with special needs (Durkin et al., 2013), highlighting the key-role of fundamental dynamics that these methodologies trigger, such as identification, gradualness and non-public failure (Gee, 2003). In the case in which the use of videogames supports mathematics teaching, fostering inclusion requires a careful design of the teaching methodologies with the aim of giving to all the students the opportunity to "experience mathematics in ways which make sense to them" (Scherer et al., 2016, p. 641). Research has shown, in particular, the key-role of fostering students' reflections on: (a) the mathematics embedded within the videogame (Jorgensen (Zevenbergen), 2015); (b) the skills they develop when they play (Gros, 2007); (c) their own difficulties and the possible ways to overcome them (Van Eck, 2015).

In tune with these ideas, we designed a teaching methodology which combines the use of a videogame with the activation of reflective practices developed by students at both individual, peer and collective level. In this paper, we reflect on the effects of this methodology in terms of students with special educational needs engagement, by focusing on data from a teaching experiment carried out with mixed-abilities classes of lower secondary school.

Context of the study and teaching methodology

The teaching experiment on which this paper is focused involved 93 students of 5 mixed-abilities lower secondary school classes (grades 6 and 7) and their mathematics teachers. Here, we analyse the case of 25 pupils with special educational needs (in the following, SEN) belonging to these classes.

Before presenting these students, it is important to share some information about the Italian tradition in terms of inclusion. In Italy, differential classes were abolished during the 1970s. The issue of SEN has become central since 2012, when the Ministry of Education introduced a specific regulation that identifies three main categories of students with SEN: (A) students with certified disabilities (sensory, motor or psychic); (B) students with specific developmental disorders (dyslexia, dyscalculia, dysgraphia, dysorthography, attention deficit hyperactivity disorder and limiting or borderline cognitive functioning); and (C) students within a condition of socio-economic, linguistic or cultural disadvantage. The regulation introduces, in particular, a series of benefits for students belonging to these categories, such as the creation of personalized didactical plans that include didactic strategies based on the students' needs and a list of didactic tools to be used to support the students.

21 students, among the 25 on which the study is focused, belong to these three categories: 2 students belong to category (A), 15 students belong to category (B), and 4 students belong to category (C). Other 4 students not belonging to these categories were included within this study, due to their poor performance and the difficulty faced by the teacher in involving them during mathematics lessons.

Table 1: An example of activity that students face within Matematica Superpiatta

Description of the activity "Prime numbers"	**Tools provided to students within the activity**
The activity is divided into 6 tasks of increasing difficulty. The goal of each task is to complete a sequence of integer numbers by inserting the missing numbers. Each number in the sandbox game is represented as a block that can be searched and picked up, placed, or created (crafted) according to some rules. Students can find only prime numbers in the game field. Not prime numbers have to be crafted as products of prime numbers through the tools provided in the game.	"Inventory": a place when the user can collect objects gathered in the game field. "Crafting table": a table where prime numbers can be inserted to craft other numbers as products of prime numbers. "Pick": when used on a block containing a number, the block disappears from the world, appearing in the user inventory. In case the picked block does not contain a prime number, it is "broken" into blocks containing the prime factors, which appear in the inventory.

As stated above, during the teaching experiment we implemented a teaching methodology that combines the use of a sandbox videogame, Matematica Superpiatta (www.matematicasuperpiatta.it) with the activation of students' reflective practices. Matematica Superpiatta (in the following, MS) was designed with the aim of realizing a learning environment within which students could face challenging mathematical activities by interacting with the different tools at disposal. Table 1 summarizes one of the activities that students face within MS ("Prime numbers"), which is aimed at making them reflect on the decomposition of numbers in prime factors. The teaching methodology combines individual interactions with the MS and collective metacognitive reflections on the strategies implemented during this interaction and on the mathematical knowledge on which these strategies are based.

Table 2: Questions from the reflective worksheet related to the activity "Prime numbers"

Questions from the worksheet	**Aims of the questions**
(a) Which of these numbers can you find in the field? Why? (list of numbers written under the question: 11, 14, 27, 31, 59, 75) (b) How do you find the useful numbers to be put in the crafting table?	Questions (a) and (b) are aimed at making students reflect on the mathematical knowledge on which the activity is focused and on the strategies that they adopted during the game.
(c) Does the game become easier or harder when you progress through the levels? Why? (d) What did you learn through this activity?	Questions (c) and (d) are aimed at making students reflect on their own experience of learning through the use of the videogame.

The teaching methodology is characterized by this sequence of phases: individual interaction with MS (phase 1); individual (phase 2) and small groups' (phase 3) reflections on the activities faced within MS; and collective discussions aimed at sharing and comparing ideas, enhancing the contribution of each student in the collective construction of meanings (phase 4). Specific reflecting worksheets have been designed to support students' metacognitive reflections during phases 2 and 3. Examples of questions from the reflecting worksheet proposed after students' individual interaction with the activity "Prime numbers" are presented in Table 2.

Research framework and research questions

Motivation plays a fundamental role, especially in the case of students with special needs and learning disabilities (Sideridis, 2009), since it has the potential to direct students' choice of taking part (or not) in mathematics activity. In the last years, research studies on the issue of motivation in mathematics suggested to shift the focus of the research on motivation in mathematics from the study of longer-term attitudes and beliefs toward the study of in-the-moment engagement (Middleton et al., 2017). *Engagement* is a multidimensional construct, which combines three interrelated components (Fredricks et al., 2004): *behavioral engagement,* which draws on the idea of participation; *emotional engagement,* which refers to students' affective reactions in the classroom; *cognitive engagement,* which incorporates "thoughtfulness and willingness to exert the effort necessary to comprehend complex ideas and master difficult skills" (Fredricks et al., 2004, p. 60).

To study and understand the specific nature of *mathematical engagement*, Goldin et al. (2011) introduce the term *engagement structure*, defined as:

> an idealization involving a characteristic motivating desire or goal, actions including social behaviors toward fulfilling the desire, supporting beliefs, self-talk, sequences of emotional states, meta-affect, strategies, and possible outcomes – a kind of behavioral/affective/social constellation situated in the person, becoming active in social contexts (Goldin et al., 2011, p. 548).

The key-role played by the design of teaching methodologies in structuring students' engagement have been stressed by Jansen (2019), who states that engagement is structured by the opportunities to do mathematics given to students and by students' way of taking up these opportunities to interact with the teachers and peers about mathematics. Goldin et al. (2011) identified 9 main categories of engagement structures: (1) *Get the job done*, related to students' desire of completing an assigned mathematical task correctly following given instructions; (2) *Look how smart I am*, related to the desire of impressing others or him/herself with his/her mathematical ability; (3) *Check this out*, related to the desire of obtaining a reward; (4) *I'm really into this*, related to the desire of experiencing the very activity of addressing a mathematical task with the need of understanding; (5) *Don't disrespect me*, related to the desire of meeting a perceived challenge to the student's dignity, status, or sense of self-respect; (6) *Stay out of trouble*, related to the desire of avoiding interactions that may lead to conflict or distress; (7) *It's not fair*, related to the desire of redressing a perceived inequity; (8) *Let me teach you*, related to the desire of helping another student; and (9) *Pseudo-engagement*, related to the desire of seeming to be engaged while avoiding genuine participation.

The aim of this paper is to reflect on the potentialities of the teaching methodology implemented during our teaching experiment in structuring students' with SEN engagement by providing them with opportunities to both do mathematics and reflect, with the teacher and their classmates, on their experience. The research questions related to this aim are: (1) How could the students' with SEN engagement, during the teaching experiment, be characterized at the behavioural, social, cognitive and affective level? (2) What are the main characteristics of the teaching methodology that structured this kind of engagement?

Research methodology

Middleton et al. (2017) stress on the importance of focusing on multiple methods to investigate students' engagement, combining different techniques, such as, for example, researcher's observations, students' self-reports and teachers' reports.

In tune with this idea, we collected different kind of data: students' written answers to the questions in the reflecting worksheets; videos of classroom discussions; data about students' individual interactions with the videogame (levels of the game that were faced, number of mistakes, number of attempts, …); students' answers to a questionnaire about their experience within the whole teaching experiment; teachers' answers to two different questionnaires proposed at the middle and at the end of the project; teachers' final interview aimed at making them reflect on their students' experiences.

In order to answer our research questions, we developed a qualitative analysis of the collected data that refer to the students who participated in the teaching experiment. For each student, the results of the analysis of each kind of data have been intertwined with the aim of highlighting clues of his/her engagement. Table 3 summarizes the data analysed to characterize each level of engagement.

Table 3: Data analysed to characterise each level of engagement

Level of engagement	Data that have been analysed
Behavioural engagement	- students' interventions during the classroom discussions that highlight their ways of taking part to the collective work that is developed; - students' ways of interacting with others (tone of their voice, kind of gestures they used…); - students' answers to the final questionnaire, in which they share how they perceive their participation within the classroom activity.
Affective engagement	- students' interventions during the collective discussions, answers to the reflecting worksheets and to the final questionnaire, through which they share their emotions toward mathematics and manifest their perceived competence in doing mathematics.
Cognitive engagement	- students' interventions during the classroom discussions that highlight their willingness of understanding and of contributing to the reflective practice developed at a collective level; - students' answers to the reflecting worksheets, in which they make explicit the mathematical reasons subtended to the strategies adopted during their interaction with the videogame.

The data collected through the questionnaires proposed to the teachers and their final interviews were analysed with the aim of confirming (or not) our interpretation of the other data. In particular, during the final interviews, teachers were explicitly asked to reflect about their students' engagement and about the role played by the methodology adopted during the teaching experiment in structuring students' engagement.

Data Analysis

In this section we present two examples of the analysis we performed. We chose to focus on two opposite cases: the case of Antonia, which testifies a positive evolution of the student's engagement throughout the teaching experiment; and the case of Giorgio, which highlights the role played by the student's difficulties in inhibiting his fruitful participation

Example 1: The case of Antonia

Antonia is a female student diagnosed with severe specific learning disorders (dyslexia and dyscalculia; category (B)). She has a problematic relationship with a very close and competing classmate, Elisabetta, who tends to impose herself on Antonia, blocking her attempts to participate in classroom activities and heightening her sense of insecurity. During the different phases of the educational project, Antonia's participation becomes active lesson by lesson.

This evolution of her engagement has been clearly highlighted at the behavioral level. In fact, during the pair activities and the classroom discussions, the tone of her voice and the gestures used to show her involvement in these activities highlight Antonia's ongoing overcoming of her sense of insecurity in interacting with others. Moreover, in different moments during the classroom discussions, Antonia intervenes without the need of the teacher's solicitation, differently from what usually happened in the past. In the final questionnaire, she declares that she was stimulated to participate in an active way when excerpts from her reflecting worksheets were displayed on the interactive whiteboard. This testifies the role played by the design of the classroom discussion (phase 4) in fostering Antonia's positive attitude toward her participation within the discussion itself. The analysis of classroom discussions has also shown that Antonia has become able to overcome the difficulties due to Elisabetta's presence, since she reacts with confidence to her interferences when she intervenes during the discussions. We hypothesize that playing alone with the videogame (phase 1) contributed to strengthen Antonia's self-esteem and the sense of self-efficacy, thanks to the gradualness of the tasks and to the possibility of managing time in an autonomous way.

It was possible to highlight a positive evolution even at the level of cognitive engagement, since Antonia's interventions during the classroom discussions are productive also in relation to the mathematical content, highlighting that she is able to effectively direct her attention at the issues on which the discussion is focused. Moreover, Antonia is able to autonomously complete all the levels of the videogame that the students were asked to face. In the individual reflection sheets (phase 2) and in the final collective discussion (phase 4) Antonia often refers to the videogame as a real concrete experience talking about "numbers that cannot be multiplied", referring to the crafting table. This testifies that: on one side, individually interacting with the videogame (phase 1) contributed to make Antonia's experience with mathematics less abstract; one the other side, working on the reflecting worksheets enabled Antonia to make the mathematics behind the game more explicit and to reflect on the reasons subtended to the effectiveness of the adopted strategies. These factors allowed Antonia to speak more confidently about mathematics during the classroom discussion.

At the level of affective engagement, Antonia's answers to the questions within the reflecting worksheets display her increased perceived competence, as it is testified by this excerpt, which highlights her self-confidence about the competencies she has developed in facing the different levels of the videogame: “The videogame becomes more difficult (level by level), starting from the easiest (levels) to enable us to learn. Since you get better, the game gets harder".

Our analysis highlights different structures characterizing Antonia’s engagement during the project. Although some of her answers show an engagement in tune with the structure “look how smart I am” (testified by Antonia’s declaration that she was proud that her answers were displayed on the IWB, making her protagonist of classroom discussions), we think that the prevailing structure is “I’m really into this”, since Antonia explicitly addresses the idea of learning in her reflections, displaying satisfaction about her increased mathematical understanding through the different levels of the videogame. Our interpretation of Antonia’s engagement has been confirmed by her teacher, who, during the interview, declares that she was positively impressed by the desire to interact that Antonia, usually very shy and insecure, displayed during classroom activities.

Example 2: The case of Giorgio

Giorgio is a male student diagnosed with severe specific learning disorders (dyslexia and dyscalculia; category (B)). He presents serious short-term working memory problems, logical-cognitive difficulties and a strong sense of frustration that often leads to conflicting relationships with peers, teachers and parents. During the different phases of the teaching experiment, Giorgio’s difficulties are so great that they prevent him from deeply and truly participating in the activity at different levels. At the level of behavioral engagement, Giorgio does not participate in the collective discussions, even when one of his classmates solicits his intervention to share some reflections about the phase of small group work. The inconsistency and lack of meaning that characterize Giorgio’s answers to the reflecting worksheet highlight also the difficulties faced by Giorgio in being engaged at the cognitive level. Moreover, the student declares, in his reflecting worksheet, that he felt lost during the activities, testifying the problems related to his engagement also at the affective level. A little evolution of his engagement could be observed only in the last part of the teaching experiment, when, after a collective discussion, Giorgio shows a better sense of perceived competence when answering to the final questionnaire, declaring that he has understood a little better and that he has overcome, albeit with great difficulty, the problems encountered when interacting with the videogame, displaying a more positive attitude towards the activity performed.

Our analysis highlights a main structure characterizing Giorgio's engagement during the teaching experiment, that is “Pseudo-engagement”. In fact, although Giorgio seemed busy when he individually worked on the reflecting worksheet, a sentence within his answers to the final questionnaire testifies that his aim was to pretend to be actively involved in the activities, rather than really reflecting on the mathematics problems he faced and on the possible strategies to face them: “In the worksheet I had no problems because it was enough to write something”. Our interpretation about Giorgio’s engagement has been confirmed by his teacher who, in both the first and second questionnaires and during the final interview declared to be worried about Giorgio’s situation.

Conclusion

In the previous section, we presented two examples of analysis of the data collected during our teaching experiment. The first example, the case of Antonia, testifies the effectiveness of our methodology in fostering the positive evolution of students’ engagement structures toward a structure characterized by students' desire of being involved in the collective construction of mathematical knowledge and of learning, which Goldin et al. (2011) indicate with "I'm really into this". The analysis of the data collected for most of the 25 students with SEN who participated in our teaching experiment has confirmed the positive trend highlighted in Antonia's case. In particular, we observed that most of the students showed a positive evolution of their engagement structures, even at the level of cognitive engagement. 20 of them, in fact, were able to autonomously complete all the activities within the videogame. Among them, 14 students proposed fruitful interventions during the classroom discussions in relation to the mathematical content under scrutiny and 11 students showed awareness of their progress by explicitly reflecting on the improvement of their cognitive engagement. These interpretations were confirmed by the teachers in the intermediate and final questionnaires and in the interview. In particular, teachers stressed on the greater participation that they observed in the case of the students with SEN involved in the study, declaring that they observed a widespread students' desire to interact positively and a sense of gratification, trust and security manifested by most of these students, together with a reduction of fear of making mistakes and a general fun in doing mathematics.

These results allow us to propose some reflections on the characteristics of the teaching methodology implemented during our teaching experiment that played a key-role in structuring students' engagement. In tune with other studies, the phase of individual interaction with MS (phase 1) favors a positive development of students' sense of self-efficacy. Working (individually and in small groups) on the reflecting worksheets (phases 2 and 3) supports students in making the mathematics behind the videogame explicit and in developing argumentative competences, enabling them to effectively contribute to the classroom discussion. Phase 4, in turns, contribute to the refinement of the reflections developed during phases 2 and 3, supporting students' development of awareness about the teaching-learning processes in which they are involved.

The structure "I'm really into this", the one displayed through the analysis of Antonia's data, emerged only in 4 cases. We think that this result shows that it is necessary to devote more time to the implementation of this kind of methodology (in particular, the phases devoted to the activation of reflective practices) in order to foster an engagement characterized by students' will to address the activities guided by a real need of understanding. Our analysis has also highlighted that in some cases (Giorgio and two other students), students tended to strive not to be noticed, adopting an avoidance behavior, or pretended to be involved even if they did not really address the mathematical content under scrutiny. In these cases, students' experience within the teaching experiment was not effective in fostering a positive evolution of the structures of their engagement. Our hypothesis is that, in the case of students like these, it is necessary to plan a targeted intervention to help them overcome the difficulties that prevent them from becoming deeply engaged in the activities. In the case of Giorgio, a first targeted intervention has been carried out, involving him in meetings with his teacher aimed at triggering and supporting his explicit reflections about his experience with the videogame and the other activities around which our teaching methodology is designed. This approach seems to be promising but further experimentation is needed. As a further step of our study, we will also collect further data, throughout a longer time span, with the aim of investigating if the positive trend highlighted by our analysis could be confirmed.

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Pre-service teachers' beliefs and attitudes about teaching in inclusive mathematics settings

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Research on subject-matter specific programs for inclusive teacher education is still of great importance. Beyond concept development and research on cognitive development, pre-service teachers' beliefs and attitudes are of major interest. The paper reports first on the project ProViel ('Professionalisierung für Vielfalt' – 'professionalisation for diversity') and presents its aims and objectives. Afterwards, we focus on pre-service teachers' affective attitudes about teaching in inclusive mathematics settings before and after a course on learning mathematics with substantial learning environments, and compare the findings concerning inclusive teaching in mathematics and in other subjects. The results show that the pre-service teachers in average feel more optimistic and comfortable than pessimistic and uncomfortable while thinking about teaching mathematics in inclusive settings. This neutral to positive position hardly changed having completed the course.

Keywords: Pre-service teacher education, inclusive education, beliefs, attitudes.

Introduction

Coping with heterogeneity in inclusive settings requires specific competencies of teachers (cf. Scherer, 2019a). Teacher education has to prepare prospective teachers for these challenges, and research is needed especially for subject-matter specific programs. Several studies concentrate on conceptual and developmental work with relevant accompanying research. Apart from content-related objectives teacher education programs have to address pre-service teachers' adequate attitudes and beliefs towards inclusion. In connection with this, investigations of pre-service teachers' competencies and competence development as well as their attitudes and beliefs are of great interest. The following paper will focus on the latter one.

Beliefs and attitudes about inclusive mathematics

Knowledge about pre-service and in-service teachers' attitudes about inclusive (mathematics) teaching is of great interest because a successful implementation of inclusion seems to be dependent on teachers having positive attitudes about inclusion (e. g. Avramidis & Norwich, 2002; Ruberg & Porsch, 2017). There are also first hints concerning effects of positive attitudes about inclusion on important practices and strategies in inclusive teaching (cf. Forlin et al., 2011). Research on beliefs and attitudes about inclusion focuses, for example, on factors that might influence teachers' beliefs and attitudes about inclusion (e. g. Kunz et al., 2021; Ruberg & Porsch, 2017) or how these beliefs and attitudes can be captured via questionnaire (e. g. Ewing et al., 2018).

Traditionally, three components of attitudes can be distinguished: cognitive, affective and behavioral attitudes (Rosenberg & Hovland, 1960; Eagly & Chaiken, 2011), whereby attitudes are defined as "a psychological tendency that is expressed by evaluating a particular entity with some degree of favor or disfavor" (Eagly & Chaiken, 2011, p. 11). This tendency or, in terms of Rosenberg and Hovland

(1960), these predispositions, are not directly observable, but the types of response which seem to be indicators of attitudes can be distinguished along the three mentioned categories. While the cognitive and behavioral components of attitudes concentrate on thoughts about and people's actions with the attitude object, "the *affective* [emphasis from original] category consists of feelings or emotions that people have in relation to the attitude object" (Eagly & Chaiken, 2011, p. 10).

The affective component is comparatively less subject of research, although this component seems to be of special interest concerning inclusive teaching (Seifried, 2015). Studies which investigate this component often conceptualize affective attitudes through teacher feelings about the practice of inclusive education or concerns about it (Ewing et al., 2018) as well as teachers' positive expectations of inclusion (Seifried, 2015). First results show that teachers fear being overwhelmed by inclusion or that they might not meet every students' needs (Seifried, 2015). However, teachers also express positive expectations of inclusive education for all students, like their social interaction and learning from each other (Seifried, 2015).

Mostly, this previous research does not consider specific school subjects – a research gap we focused on within our project 'Mathematics Inclusive' (see next section). In different contexts the assumption comes up that it is more difficult to teach mathematics in inclusive settings than other subjects, for example, because of the specific structure of mathematics, but this can exactly be a starting point for individual and joint learning in mathematics for all students (cf. Seitz et al., 2020). Therefore, we are interested in pre-service teachers' affective attitudes about inclusive teaching especially in mathematics but also in comparison to other subjects.

The project 'Mathematics Inclusive' within the project ProViel

The project ProViel 'Professionalisierung für Vielfalt' ('Professionalisation for Diversity'; https://www.uni-due.de/proviel/) at the University of Duisburg-Essen is funded by the Federal Ministry of Education within the frame of a program for teacher education (1st phase: 2016–2019; 2nd phase: 2019–2023). Numerous university departments are involved to ensure the development of a coherent conceptual program for teacher education. One field of action is 'Diversity & Inclusion', and several sub-projects might cover the wide facets and dimensions in this field (cf. Bishop et al., 2015; Good & Brophy, 2008). One of the sub-projects, 'Mathematics Inclusive', aims at implementing subject-specific concepts and modules for inclusive mathematics education, following a design-based research approach (Scherer, 2019b).

The developmental work firstly concentrated on the course 'Learning Mathematics with Substantial Learning Environments (SLEs)' (3rd year, BA-program for primary mathematics). The didactical concept of working with SLEs, and by this realizing a natural differentiation is in line with a constructivist understanding of teaching and learning, and has been proved to be suitable for heterogeneous learning groups in primary mathematics (cf. Krauthausen & Scherer, 2013; Scherer & Krauthausen, 2010). The course lays theoretical foundations with respect to SLEs and natural differentiation and gives examples for planning SLEs as well as analyses of concrete lessons. Moreover, the pre-service teachers have to plan clinical interviews for a chosen SLE, carry out these interviews in school and analyze the interviews according to selected focal points.

The design process for this course started in 2016, and the first course has been running during the winter semester 2016/17, followed by the three repetitions during the winter semesters 2017/18, 2018/19, and 2019/20. All courses were evaluated, and data collection comprised questionnaires (pre-post-design), selected interviews, as well as field notes. The following section will report some former results with respect to pre-service teachers' pre-experiences and competence development with respect to inclusive mathematics. After that the research question focused in this paper as well as the used methods for investigating pre-service teachers' affective attitudes will be reported.

Previous results

For data collection with respect to pre-service teachers' individual pre-experiences with inclusive mathematics an open item "*Which experiences have you made so far with inclusion in mathematics instruction?"* was used within an extensive questionnaire (see also section research question and methods). Written comments showed that only about 50 % of the pre-service teachers have made school-related-experiences whereas the others had no experiences, made experiences out of school or in other fields (see Scherer, 2019a). In many cases, specific experiences for mathematics were related to *differentiated learning offers* and forms of *inner* or *outer differentiation*. Experiences that are of great importance for the course concept 'Learning Mathematics with SLEs', as the pre-service teachers' experiences and classroom observations represent a quite different teaching concept than the university course (Scherer, 2019a).

Moreover, for measuring pre-service teachers' competence development the post-test included six items for a retrospective self-assessment (cf. Nimon et al., 2011). At the end of the course, participants had to rate their competencies 'before the course' and 'today'. These items were designed according to the curriculum objectives focusing on substantial learning environments, clinical interviews and analyses of students' thinking and learning processes. The pre-service teachers had to rate their competencies for these three aspects on the one hand in general, on the other hand concerning the relevance for inclusive mathematics. The differences of the retrospective self-assessment before/after the course showed high significances with all items (for all items $p < .001$; Cohens $d > 0.8$; for detailed analysis see Scherer, 2021). Looking at all items, the results show important developments of pre-service teachers' competencies and beliefs. With regard to the pre-experiences of the majority that differ from the course conception (Scherer, 2019a), these findings are of great importance.

Research question and methods

Beyond central research questions with respect to course design and pre-service teachers' competence development (cf. Scherer, 2019b; 2021), this paper will focus on the following research question:

> Which changes of pre-service teachers' attitudes and beliefs can be identified after they had completed a course that addresses inclusive mathematics?

To answer the research question, a standardized questionnaire was used in a pre-post-design, filled in by about 240 participants. The questionnaire contains items concerning attitudes and beliefs with respect to inclusion and inclusive mathematics (cf. Meyer, 2011), and one of the questions was, if and how pre-service teachers' attitudes and beliefs changed after completing the course. A bipolar scale with five pairs of adjectives (by Meyer, 2011) was used that allows the pre-service teachers to rate how they feel about teaching in inclusive mathematics settings (Figure 1). The participants did answer this question before and after the course with respect to inclusive mathematics teaching, as well as on how they feel about teaching in inclusive settings in other subjects than mathematics.

How do you feel about the thought of teaching MATHEMATICS in an inclusive class? Please mark the box (1 = very anxious to 7 = very relaxed) that describes your feelings best.								
	1	2	3	4	5	6	7	
anxious	O	O	O	O	O	O	O	relaxed
helpless	O	O	O	O	O	O	O	self-confident
burdened	O	O	O	O	O	O	O	unburdened
pessimistic	O	O	O	O	O	O	O	optimistic
uncomfortable	O	O	O	O	O	O	O	comfortable

Figure 1: Question for capturing pre-service teachers' affective attitudes

Results

Pre-service teachers' affective attitudes can be described as follows: They feel neither anxious or relaxed, helpless or self-confident and burdened or unburdened, as they answered in average with a neutral position between the opposite pairs (Table 1).

Table 1: Pre-service teachers' affective attitudes about teaching mathematics in inclusive settings (1 = very anxious and 7 = very relaxed; N = 241)

	1	2	3	4	5	6	7	mean
anxious/relaxed	2 %	5 %	16 %	32 %	27 %	17 %	2 %	4.37
helpless/self-confident	2 %	10 %	19 %	30 %	24 %	13 %	2 %	4.12
burdened/unburdened	3 %	12 %	31 %	27 %	14 %	11 %	2 %	3.79
pessimistic/optimistic	2 %	3 %	8 %	17 %	27 %	29 %	14 %	5.10
uncomfortable/comfortable	1 %	1 %	5 %	29 %	29 %	22 %	11 %	4.96

However, in average they feel more optimistic and comfortable than pessimistic and uncomfortable while thinking about teaching mathematics in inclusive settings. Although it seems as if pre-service teachers feel quite neutral to positive about inclusive mathematics teaching, there are also more than just a few pre-service teachers who seem to be burdened (46 %). This facet of affective attitudes is also the most negatively one in Meyer's (2011) study. The in-service teachers, asked in that study, answered quite similar, with feeling burdened (67 %). Maybe teachers in their everyday practice face the challenges of inclusive mathematics teaching even more than pre-service teachers, as the percentage of teachers being burdened is even higher than the percentage of pre-service teachers feeling burdened. Additionally, pre-service teachers seem to feel more positive than the in-service teachers in Meyer's study did concerning the other facets of affective attitudes.

With respect to our research question (see section research question and methods), we figured out that pre-service teachers' affective attitudes hardly changed over the semester, having completed the SLE-course. In average, at the end of the semester they still feel neutral to positive about teaching mathematics in inclusive settings (Table 2). However, our analysis showed one significant result for the pre-post-comparison in the opposite pair pessimistic/optimistic. Participants still feel optimistic about teaching mathematics in inclusive settings but significantly less optimistic than before the course. Possibly, the pre-service teachers experienced challenges within the course while working with the pupils they did not experience before, which made them feel less optimistic than before.

Table 2: Pre-service teachers' affective attitudes about teaching mathematics in inclusive settings after the course (1 = very anxious and 7 = very relaxed; N = 240 to 241[1]; *p < 0.05)

	1	2	3	4	5	6	7	mean
anxious/relaxed	1 %	6 %	20 %	34 %	26 %	11 %	3 %	4.23
helpless/self-confident	2 %	6 %	17 %	34 %	27 %	12 %	3 %	4.26
burdened/unburdened	3 %	12 %	30 %	29 %	17 %	7 %	2 %	3.72
pessimistic/optimistic	1 %	2 %	10 %	24 %	27 %	25 %	11 %	4.93*
uncomfortable/comfortable	1 %	1 %	10 %	28 %	34 %	20 %	7 %	4.80

Comparing pre-service teachers' attitudes between teaching mathematics and other subjects in inclusive settings showed the following results: Their affective attitudes about teaching other subjects in inclusive settings are slightly more positive than about teaching mathematics when focusing on the average of pre-service teachers' answers (Table 3). This means, that pre-service teachers in average feel more relaxed, more self-confident, more unburdened, more optimistic and more comfortable about teaching other subjects compared to mathematics in inclusive settings. Although this is just a descriptive result (if all numbers were rounded to integers, there would be no visible difference), it seems to be a result consistent with the assumption that it might be more difficult to teach mathematics in inclusive settings than other subjects (see section beliefs and attitudes; cf. Seitz et al., 2020). However, concentrating on the average does not seem to be very informative with regard to differences and similarities in pre-service teachers' attitudes about teaching mathematics and other subjects in inclusive settings. Therefore, we analyzed how often they do feel exactly the same, i. e. how many participants crossed out exactly the same for teaching mathematics and other subjects in inclusive settings. It turned out that between about 73 % (helpless/self-confident) and about 79 % (uncomfortable/comfortable) feel the same about teaching in inclusive settings no matter if they think about mathematics or other subjects. Nevertheless, some pre-service teachers see even big differences between teaching mathematics or other subjects in inclusive settings. Single participants even feel

[1] N = 240 for helpless/self-confident, N = 241 for all other opposite pairs

(very) burdened with respect to teaching mathematics but (very) unburdened with respect to teaching other subjects in inclusive settings. However, also the other way round could be identified, i. e. single participants who feel burdened with respect to teaching other subjects but unburdened with respect to teaching mathematics in inclusive settings.

Table 3: Pre-service teachers' affective attitudes in comparison of teaching mathematics or other subjects in inclusive settings (1 = very anxious and 7 = very relaxed)

	means before the course		means after the course	
	mathematics	other subjects	mathematics	other subjects
anxious/relaxed	4.37	4.46	4.23	4.39
helpless/self-confident	4.12	4.3	4.26	4.40
burdened/unburdened	3.79	3.93	3.72	3.93
pessimistic/optimistic	5.10	5.13	4.93	4.98
uncomfortable/comfortable	4.96	5.01	4.80	4.88

A pre-post-comparison did not show significant changes in pre-service teachers' affective attitudes about teaching other subjects in inclusive settings after they had completed the SLE-course. This might not be surprising because the course on SLEs especially focused on teaching mathematics in inclusive settings.

Discussion and conclusions

In total, results of the project 'Mathematics Inclusive' show that the underlying didactical concept of using SLEs and realizing a natural differentiation is suitable for inclusive classrooms. Moreover, the course concept with the combination of theoretical elements, concrete video examples and pupils' documents (lecture) and practical experiences (interviews at school) with a common reflection (seminar) could reach the mentioned project objectives (see Scherer, 2019b; 2021). However, only a few changes of pre-service teachers' attitudes and beliefs could be identified after they had completed the course (whereby participants still feel optimistic about teaching mathematics in inclusive settings but significantly less optimistic than before the course; see section results). On the one hand, it might be interesting to further develop the concept of the course in a way that pre-service teachers' emotions and feelings are going to be more in focus – especially the ones of pre-service teachers' who associate quite negative feelings about teaching mathematics in inclusive settings. On the other hand, beliefs and attitudes might not change within a course which lasts for one semester. Therefore, it would be interesting to analyze their beliefs and attitudes after a longer period of time.

Future research within the project will focus on a connection between pre-service teachers' pre-experiences with inclusive mathematics (see section previous results) and their beliefs and attitudes. For example, it could be interesting to compare their affective attitudes depending on their experiences (especially, because experiences seem to be an important factor that might influence pre-service teachers' beliefs and attitudes about inclusion (cf. Ruberg & Porsch, 2017)).

At the moment, the used items of the questionnaire for investigating pre-service teachers' affective attitudes take into account teaching mathematics in a class as a whole. It could be interesting to ask for pre-service teachers' affective attitudes in a narrower way, like we did for analyzing their competence development. For example, it could be focused on their feelings concerning interviews with pupils or planning and using SLEs in classroom. As pre-service teachers' feel a bit less optimistic after the course, interviews are planned to gain deeper insights in their affective attitudes. Possibly, the practical experiences within the course did lead to a less optimistic view because they experienced and might assess the specific requirements of inclusive mathematics teaching now more realistic. The differences and similarities concerning teaching mathematics or other subjects in inclusive mathematics could also be analyzed in more detail, if interviews were conducted.

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Affective and mediational suitability of an inclusive mathematics program

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Keywords: Inclusive education, mathematics, didactical suitability and adolescents.

Introduction

One of the main objectives of the current education is to create inclusive teaching-learning processes that guarantee equal opportunities for all students. In recent years, several studies and reports (Renta et al., 2019; Save the Children, 2020) have revealed a worrying increase in adolescents at risk of social exclusion. The term social exclusion encompasses those sectors of society that are in a vulnerable social situation, which causes them to be outside certain rights relating to work, education and culture (Artuch-Garde et al., 2017). The fact of not receiving an adequate education prevents these sectors of society from being formed as citizens capable of functioning within the world that surrounds them, which does not help them to get out of the circle of exclusion.

In this work, we follow the Didactical Suitability Theory (DST) (Godino, 2013) in order to evaluate the didactical suitability of a socio-educational program promoting the mathematical stimulus to adolescents at risk of social exclusion. The DST analyses the didactical suitability of an instructional process through the study of six partial suitabilities: epistemic, cognitive, interactional, mediational, affective and ecological. Concretely, in this study, we focus on the study of both affective and mediational suitability of the program. Affective suitability refers to the degree of the students' involvement (interest, emotions, motivation, attitudes, and beliefs) in the study process; mediational suitability depends on the availability and adequacy of material and temporal resources in the teaching.

Methodology

The program is developed in 14 one-hour sessions addressing the mathematical stimulus by means of integrating the STEAM methodology (Blanco et al., 2018). These sessions were carried out fortnightly outside of school hours in three different secondary schools. The design of the activities carried out in each session contemplates diverse mathematical content from a variety of interdisciplinary contexts. In the "A mix that blows up?" activity, the students have to search through an experiment what is the proportion of vinegar and bicarbonate that establishes the chemical reaction to inflate a balloon as much as possible. The sample was made up of a total of 68 from the first level of the Spanish secondary education and from three different schools. The schools are located in semi-urban areas in which a large proportion of the population consists of families at risk of social exclusion. The students participating in the programme were selected in a joint meeting by the researchers, the mathematics teachers, and the orientation staff of the school, responsible for assessing and attending the educational and family needs of each pupil in his/her learning process. The instruments used for data-collection were the video recordings of the sessions, the researchers' notebook – as a record of participant observation – and semi-structured interviews with the secondary schools' mathematics teachers and orientation staff. In addition, a satisfaction questionnaire was applied for each activity carried out at the end of each session. In order to assess the mediational suitability, the following indicators given by the DST were analysed: the material resources, the number of students, the schedule and classroom conditions, and the duration of the sessions. For the affective suitability, students' interests and needs, their attitudes and emotions are some of the indicators studied.

Results

The results show a high degree of affective and mediational suitability due to the presence of at least 87 % of the indicators in all activities. The use of different materials, individualized work and varied workspaces favour that these students actively participate in the sessions (mediational suitability). The presence of a greater interest in activities leads to a positive attitude towards academic tasks (affective suitability), which has been evidenced, in fact, in better academic results. In conclusion, the role played by mediational suitability in this context is highlighted, with a direct influence on affective suitability, especially in terms of attitude towards the subject.

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Problem-solving by students with Asperger's Syndrome

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This study considers the use of heuristic strategies by pupils with Asperger's Syndrome in a problem-solving workshop. The strategies aimed to mitigate learning difficulties associated with the syndrome as a result of deficits in central coherence and executive functioning by introducing manipulatives and visual organisers to support the processes of understanding and finding solutions to the problems. The different heuristics employed during the comprehension, execution and verification phases, in addition to the visual and tactile nature of the materials, promote diversity in terms of the chosen strategies, and stimulate the pupils' abilities. The limitations imposed by literal thinking and cognitive rigidity, alongside weak attention and organisation, can be reduced by a more schematic, as opposed to mimetic, visual content, and a more explicit sequencing of the problem statement.

Keywords: Problem Solving, Asperger syndrome, inclusion, heuristic, inclusive Mathematics.

Introduction

This study aims to contribute to inclusive education regarding the needs of children with Asperger's Syndrome (AS). Specifically, it considers how a problem-solving approach (PS) can be adapted for use with students with AS, and trials classroom-based strategies that can be applied in any classroom. The mathematical focus targets PS, as this approach allows for the incorporation of tables, diagrams and other visual material, encouraging the organisation of data and developing cognitive flexibility (Liljedahl et al., 2011). This enables the teacher to design an environment conducive to diversity of thinking through the application of different heuristics, generalizable to other contexts.

The study explores how specific pedagogical strategies might be deployed in order to minimise the characteristics of AS which impede mathematical learning, and maximise those that promote it (Roos, 2013). The strategies were trialled in PS workshops with children from the Huelva Asperger's Syndrome Association (AOSA) and the results were analysed. The research questions were the following: What strategies facilitate PS with children with AS? What heuristics do they bring into play when they attempt to solve problems?

Theoretical framework

AS is a pervasive developmental disorder within the autism spectrum. The majority of students with AS have average to above-average intellectual abilities, but present characteristic personality traits and abilities. Research by Happé and Frith (2010) suggests that people with AS present deficits in central coherence, which translates to an incapacity to infer the larger meaning inherent in a communicative situation. The result is that they tend to process meanings without reference to the broader context, which can be manifested in difficulties with understanding non-verbal messages. People with AS tend to prefer fixed routines and patterns of behaviour, usually showing emotional vulnerability and often have a limited tolerance of frustration (Carr & Seah, 2019). Their motor skills are characterised by a low degree of coordination and limited fine motor skills. Many people with AS show weak executive functioning, which limits their flexibility of thinking, the capacity for internal representation, and organisation (Bae et al., 2015). By contrast, they show talent in certain areas, such that they may excel in terms of visual and mechanical memory, auditory perception, the ability to focus on detail, and pattern recognition (de Gimbattista et al., 2019). Into the bargain, those with savant syndrome display astonishing skills in different branches of the arts, and may present hyperlexia, hypermnesia or hypercalculia (Happè & Frith, 2010).

With regards to these characteristics, we consider PS ideally suited to pupils with AS as it provides a narrative context in which they can participate, helping them to assimilate concepts and processes. Finding solutions to problems allows the pupils to try different strategies and ways of thinking. Within the confines of this study, it provides with an excellent opportunity for exploring what it is they need to understand and process from the problem statement in order to proceed to a solution.

Improvements in the PS of pupils with AS through specific strategies have been reported. Among these are interventions aimed at improving sequencing (Klaren et al., 2017), which suggests that structuring the solution into stages would be beneficial. In this regard, the strategies and heuristics described in the work of Polya (1945) can be highlighted for the support they offer in interpreting the information and organising the answers. Various studies illustrate the benefits of simplifying and contextualising problems in terms of pupils' interests, deploying a set of strategies for representing the information and providing stages for solutions (Delisio et al., 2018). Also worth noting are transversal methods, such as the TEACCH programme, which use augmentative and alternative systems of communication to support written language with pictograms (Bruno et al., 2018). Finally, the literature review by Kribbs and Rogowsky (2016) brings together references to PS which foreground visualisation as the key feature, an essential issue in eliminating and/or limiting problems deriving from comprehension of written texts.

Methodology

In collaboration with the AOSA and the INCLUREC Project[1], we organised a two-day mathematics PS workshop featuring problems adapted for an age range of 6 to 18. The AOSA psychologists grouped the participating students according to age and characteristics. There were varying levels of ability; one group included pupils with ADHD (Group 1); two included pupils with limited cognitive capacity (Groups 2 and 4). After problems selection, a set of manipulatives were made, designed to illustrate through pictograms the sequence described in the problems These were accompanied by a visual organiser for recording the information and step-by-step solutions. The materials and problems were first trialled with a control group of schoolgirls not diagnosed with AS, of similar ages to those attending the workshop, aiming to gain experience of using the materials applicable to the workshop itself. The original selection included more problems than were ultimately used, as experimentation

[1] INCLUREC (from the Spanish for 'inclusion resources') is a Teaching Innovation Project (University of Huelva), which maintains a permanent workshop for developing and adapting teaching resources for students with special educational needs.

prompted a degree of reduction and adaptation to the needs of the pupils towards whom the workshop was directed (see Table 1 below, outlining the potential heuristics for each problem, according to each solving stage, drawing on Carrillo, 1998):

GROUP 1	GROUP 2	GROUP 3	GROUP 4
6 students	5 students	2 students	3 students
11, 12, 13 and 15 y.o.	12, 13 and 18 y.o.	6 and 7 y.o.	15 and 16 y.o.

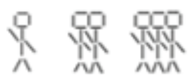
Figure 1: Workshop groups

Content: Those problems that were pre-algebraic in nature were simplified, and those involving fractions were discarded altogether, on the premise that the workshop was oriented towards the use of strategies rather than arithmetic. The topics chosen for the problems was especially significant, and featured situations of interest to the students relating to their everyday lives; they also paid special attention to the characteristic of people with AS regarding specific topics.

Resources: A laminated visual organiser was supplied, which allowed the students to place the illustrations in position with Velcro, with room in a table alongside for them to note down any information that would help them to understand and carry out the problem. The set of illustrations and organiser was complemented by differently coloured tongue depressors to represent the information in tactile form. In the case of the second problem, involving farm animals, the pupils were free to represent the elements involved however they wished, but unlimited numbers of coloured dot stickers were available for representing the animals. In both cases the material allowed different ways of reaching a solution, given out directly after the instructions had been carefully read and sequenced.

Table 1: Problems, workshop materials and heuristics arising at each stage

<table>
<tr><td colspan="2">These figures are made from sticks which are all the same. How many sticks are needed to make 4 figures like these?
And 7 figures?</td><td>1B (Group 3)

1A (Group 1, 2 and 4)
</td><td>Coloured tongue depressors
Instructions with pictograms and illustrations
Visual organiser
Illustrations of 1-7 figures, held in place by Velcro</td></tr>
<tr><td rowspan="3">Heuristics and stages in the solution</td><td>Identification and understanding</td><td colspan="2">Promoting perception: observing the illustrations and the information they represent.
Representation: reproducing the information in the problem using depressors, pictures held in place by Velcro and a table for charting progress.
Exemplification: the illustrations enable the pupils to see examples of how to construct the figures with the right number of sticks.
Organising information: The organiser lets students note down the information alongside the illustrations.</td></tr>
<tr><td>Planning, exploration and execution</td><td colspan="2">Finding regularities: Noting whether the number of sticks varies from one figure to another.</td></tr>
<tr><td>Verification</td><td colspan="2">Generalising: finding a pattern which relates the number of sticks to the figures.</td></tr>
</table>

<table>
<tr><td colspan="3">2 (Groups 1, 2, 3 and a student from group 4)
Juan lives on a farm with some chickens and rabbits. If he counts all the animals, he counts 10 heads and 28 legs. How many chickens are there on the farm? How many rabbits?</td><td>Coloured dot stickers
Question statement with pictograms</td></tr>
<tr><td rowspan="3">Heuristics and stages in the solution</td><td>Identification and understanding</td><td colspan="2">Representation: Extracting the relevant information from the question statement, supported by pictograms and illustrations.</td></tr>
<tr><td>Planning, exploration and execution</td><td colspan="2">Focus on a single variable or condition: Starting from the number of heads, discard the other information, which then allows the pupil to simplify the beginning of the solution.
Trying numbers, randomly or systematically: Assigning a value to the number of chickens or rabbits.
Foreground the progress made along the way: Setting sub-goals helps pupils discover the relationship between the number of legs and heads as the pupils draw them.</td></tr>
<tr><td>Verification</td><td colspan="2">Checking the consistency of the solution and the process: While the pupils are drawing the animals, they check their solution by repeatedly counting of the number of legs.</td></tr>
</table>

Data collection from the PS processes during the workshop was carried out by means of observation and video-recording, along with the students' own comments while they were working their way through the material. The subsequent transcription enabled units of information to be obtained relating to the application of the heuristics, explanations of the strategies and difficulties arising at the comprehension and solution stages linked to the characteristics of the students. These units were then categorised according to the heuristic used and the pedagogical aspect involved, allowing for a comparison between the workshop and control groups.

Results and discussion

Use of heuristics

All those participating in the workshop managed to establish different generalisation strategies in problems 1A and 1B, developed through finding regularities and patterns among figures. Some added 9 sticks for each additional figure, developing a factual generalization (Radford, 2001); and others focused on the shared stick forming part of the heads, multiplying the number of figures by 10 and subtracting from that the number of figures less 1 (10n-(n-1)) through contextual generalization, since they could explain the construction of any given figure but did not manage to apply symbolic language (ibid., p. 83). In some instances, however, pupils had difficulty in generalising the pattern to larger numbers, such as 9 or 10. It was nevertheless very clear that the use of the manipulatives (i.e., the tongue depressors) and the table helped the pupils to sequence each step of the task. Ramón (Group 1) initially rejected the shared stick hypothesis. Instead, he noted down the data in his table as he counted the sticks for each set of figures, in order to establish a count back strategy, ignoring the visual evidence of the figures he had made. Unfortunately, his calculations were incorrect, largely because of his poor concentration skills due to ADHD.

Some students, such as Miguel Ángel and Juan (group 2), worked on a simpler mathematical plane by constructing successive figures and then counted the number of sticks. In this instance, the use of the table for keeping a running total of sticks for each new figure, enabled them to see the numerical pattern involved and anticipate the expected results. The implementation of this problem, albeit under guidance, gave the students the freedom to choose whether to use the organiser or the depressors according to their needs (Figures 2 and 3).

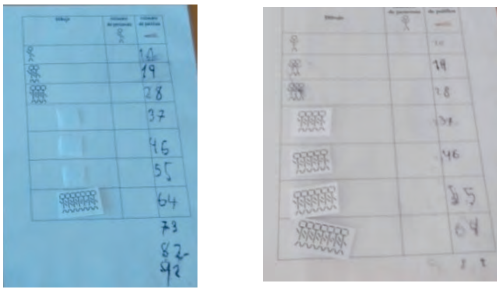


Figure 2: Generalisation in problem 1B

Figure 3: Representation in problem 1A

In Problem 2, the pupils were given free rein to use the materials in any way they chose to represent the animals. The results (Figures 4-7) ranged along a scale from the more literal to the more figurative. What can be noted from these illustrations is that the problem is more clearly understood when the representation is figurative (Kribbs & Rogowsky*,* 2016), as it draws attention to the relationship between the problem data and encourages the pupils to focus on the essential concepts involved rather than a mental image of the animals. The children whose representations approximated to the literality of Figure 5 later found it much harder to substitute rabbits for chickens, and vice versa. This kind of difficulty was no seen in the control group.

Establishing sub-goals promoted the development of pre-algebraic thinking, in that the relationship between two unknown variables needed to be determined. Starting from only one of them (the number of heads), represented by drawings or with dot stickers, the relationship with the number of legs could then be established, through use of the heuristic "trial and error", by which the number of rabbits and chickens was adjusted according to whether the total number was greater or smaller than that given in the problem statement. An unexpected strategy was the systematised trials (Figure 6), since pupils with AS tend to learn sequentially, have weak organisational skills, and lack flexible thinking (Bae et al., 2015), all of which were deployed in this solution.

Figure 4: Representing rabbits by drawing their heads and legs (Problem 2)

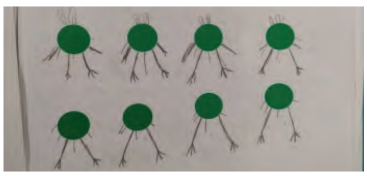


Figure 5: Substituting chickens for rabbits (Problem 2) resulted in 6-legged animals

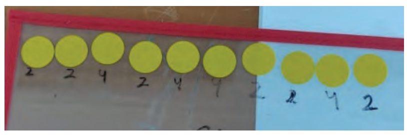


Figure 6: the number of legs is written underneath each sticker (Problem 2), alternating until the student hits the target

Educational aspects

Although the groups varied and there were differences in executive functioning, the key aspects in the teaching-learning process were the use of visual material and the sequencing of the task. Despite the organiser being available in Problems 1A and 1B, which allowed the students to separate out the information into rows and columns, and aids such as the depressors and illustrations on Velcro being supplied, the initial phase of the problem was characterised in 80% of the cases by the wait for instructions or suggestions on how to proceed, such as whether errors could be erased (Gabriel, Group 4), and questions about using the material, and the steps to follow in the solution. The latter example refers to Cristóbal (Group 2), who, despite finding the pattern to follow in the problem, stopped after completing each row to wait for further instructions. This kind of behaviour may have been the result of over-monitoring on the part of tutors, teachers, and family members, by which students with specific needs inadvertently have their range of responsibilities much reduced, disconnecting them from their own learning (Roos, 2013). Mechanisms of self-instruction also occurred, by which pupils attempted to set goals and meet the aims of the problem, although in the majority of cases the pupils followed the steps of their chosen strategy to the point of forgetting why they were doing so, highlighting limitations in their short-term memory. This was the case with Jaime (group 1), who organised the information in Problem 2 into a table but was unable to establish the relationship between the heads of the animals and their legs (Figure 7). Using stickers to represent both animals indiscriminately only increased the degree of difficulty, and he became more frustrated when he found he could not apply his strategy, nor remove the stickers ("I'm very bad at maths because I complicate things a lot"). Although the pupils needed continuous guidance, and were limited by cognitive rigidity the visual organiser and tongue depressors helped to promote a degree of diversity of thinking (Problem 1). Participants constructed the figures and noted down the number of sticks in the table, repeating the process as many times as necessary until they recognised the pattern, some requiring just one construction, others needing more examples to complete or check their calculations.

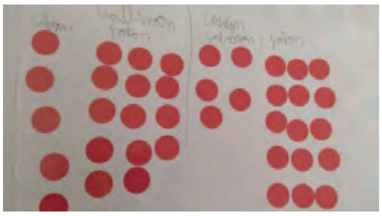


Figure 7: Tabular presentation of chickens and rabbits' heads and feet (Problem 2) using dot stickers

David (4):	It's like it was the 9 times table. You work it out according to the people in the drawing and you add 10 from the first one, which is the odd one out and doesn't follow the pattern.
Ignacio (2):	I'm changing the numbers in the table. To this one (a number in the units column) I take one away and I give it to the other one (a number in the tens column).
Tutor:	What if you didn't know the previous one? 9 people, for example.
Ignacio (2):	Then I take away 2 and I give it to the other one.
Marta (1):	For each figure you add 9 sticks to the others because they share this stick (pointing to the illustration held in place by Velcro).
Tutor:	And how do you then know how many sticks there are in 7 people?
Marta (1):	I count the number of people there are (x10) and I take away the shared sticks.

The acceptability of each strategy proved to be key to combatting frustration, and was a significant motivating factor for a group of students who were used to a kind of mathematics in which there was only one way to find a solution. It is worth noting that during the carrying out of the workshop, the students in Group 4 decided on comparing strategies for themselves, out of curiosity. An atmosphere of expectation was also created by presenting the problems as challenges, graded according to difficulty, such as the progression from 1A to 1B. This intrinsic motivation, together with empathy, proved to be the keys to emotional control and self-esteem (Goleman, 1995).

With respect to the visual nature of the solutions, it was clear that the use of colour made the difference in both understanding and solving Problems 1 and 2 (Figures 8 and 9). Nevertheless, literal thinking once again influenced the pupils' capacity for visualisation: Gabriel (Group 4) rejected the pattern of the shared stick (Problem 1B) because of the thickness of the lines in the illustrations; while Salvador (Group 2) refused to try and solve Problem 2 because the shape of the stickers did not match his mental image of what rabbits and chickens should look like.

Salvador (2): I know that you are referring to that rabbits have four legs and chickens two, but this is not a rabbit (draws a rabbit in great detail on the page).

Figure 8: Using dot stickers and different colour chalks for each animal to improve counting and understanding (Problem 2)

Figure 9: Using differently coloured sticks to distinguish what needed to be added to each figure (Problem 1B)

Conclusions

The visual element and colours helped to reduce the difficulties associated with understanding the problems, by complementing the verbal meaning and reducing the decoding process (Delisio et al., 2018). Nevertheless, literal thinking became an obstacle when the choice of illustration or visual representation did not match the mental image of the pupils. Hence, it is important to transform illustrative representations into schematic ones (Kribbs & Rogowsky, 2016) which form part of the solution itself, and are not limited to decorative additions or demonstrations of artistic ability. Such representations could also help to overcome limitations associated with visuospatial deficits and weak mental flexibility (Liljedahl et al., 2011). On another level, orderliness and organisation were both important and necessary (Klaren et al., 2017), as a dynamic part of the processes of understanding and carrying out the problems and were encouraged by the staged solutions and by heuristic strategies such as representation and the different types of generalisations (1A and 1B). It is necessary, however, to develop the visual organiser and materials which make up for the deficit in the executive function associated with AS (Delisio et al., 2018), ordering each stage of the strategy without losing sight of the objective and the information in the question statement. This was made evident in Problem 2, in which the trial and error and the sub-goal, although applied correctly, ended up becoming disconnected from the process of the problem. We would also highlight the spontaneous use by some of the pupils of heuristics such as best guessing – in some cases systematised – which might on first consideration appear incompatible with the rigidity of thinking and lack of intuition associated with AS, but which was successfully deployed in the workshop.

Based on this experience, we are considering PS workshops as an impetus for the development of student autonomy, whereby the pupils make use of different materials, types of representation and strategies for themselves, and for developing self-regulation and feedback techniques associated to the metacognitive processes.

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Promoting mathematics teaching with real life scenarios in the context of COVID-19 pandemic

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STEM education provides students with 21st century skills considered critical to correspond to the increasing challenges of the real world, and to face an increasing demand for specialized STEM labor. However, there are several issues related to skills shortages in STEM fields, namely related to mathematics. In this regard, it is crucial to highlight the role of the M in STEM in order to innovate and improve the teaching of mathematics. This paper presents an integrated approach of STEM education highlighting the role of mathematics to understand a real-world problem in the context of COVID -19 pandemic. With a qualitative methodology and based on a case study, this research shows mathematical tasks to raise awareness of students of how the virus spreads and to understand the need for isolation measures, among others. Based on this research it is recommended to resort to mathematical tasks based on real world scenarios to promote meaningful learning in students.

Keywords: STEM education, mathematics education, hands-on, COVID-19 pandemic.

Introduction

To face an increasing demand for specialized STEM (Science, Technology, Engineering and Mathematics) labor, it is recommended to implement STEM education to provide students with 21st century skills considered critical to correspond to the challenges of the real world (e.g., Baker & Galanti, 2017). However, there are issues related to skills shortages in STEM fields, namely related to mathematics. For example, it is referred that mathematics contributes to the problem of insufficient STEM graduates (e.g., Beswick & Fraser, 2019). Therefore, there is a need to highlight the role of M in STEM in order to innovate and improve mathematics teaching (Stohlmann, 2018). In particular, a context that integrates M with Science, Technology and Engineering (STE) can provide meaningful connections between mathematics and STE subjects in students (Becker & Park, 2011).

This paper presents an integrated approach of STEM education highlighting the role of mathematics to understand a real-world problem. COVID-19 pandemic has invaded our world and conditioned our way of living (Padmanabhan et al., 2021). For example, students had to be isolated in their homes without going to face-to-face classes for several months. In addition, every day the news reveal the number deaths, of newly infected, and the total number of people infected, among other information related to the disease. The main research question is "How can mathematics be used to make students understand a real-life pandemic scenario?" To answer the research question, a case study of a 6th grade teacher, who participated in a Professional Development Programme (PDP) will be presented. Her example shows the implementation of integrated STEM tasks in class with focus on mathematics.

Literature Review

To face a fast-changing world and the complexity of modern societies, the literature advocates the need for motivating students to learn and gain skills related to STEM subjects (Kelley & Knowles, 2016; Roehrig et al., 2021). In addition, STEM education should make the transition from traditional lecture-based teaching strategies to more inquiry and project-based approaches (Breiner et al., 2012), with the aim of providing students with skills to solve real life challenges and improve their achievement (Beswick & Fraser, 2019; Geiger, 2019). In this regard, the introduction of STEM education, based on real life scenarios, motivates students to learn and promotes the development of 21st century skills (English, 2017; European Schoolnet, 2018; Maass et al., 2019). In Portugal, there are also increasing calls for innovative approaches, having recently emerged several guidelines, namely on Essential Learning (MEC, 2018) in conjunction with the Profile of Students When Completing Mandatory Schooling (ME/DGE, 2017). The last document mentions the need for educational systems that contribute for the development of skills that allow students to respond to the complex challenges of the 21st century, taking into account the evolution of knowledge and technology. In this sense, the curriculum must be interpreted by teachers to explore different themes that must be framed in everyday problems of both the students and the socio-cultural environment where they are inserted in (ME/DGE, 2017). In this regard, meaningful learning should be provided by relating the most important contents of the subject to be taught with relevant aspects of the student's cognitive structure including the environment where the student is inserted in (Ausubel, 2012).

An integrated approach of STEM education should include real-world scenarios with the aim of engaging students and providing them with meaningful learning (Kelley & Knowles, 2016; Maass et al., 2019). In fact, modern world face complex problems that involve interdisciplinary knowledge and skills to solve them, which requires for curricular integration in schools (Roehrig et al., 2021). In addition, mathematics should be more emphasized in STEM integration (Stohlmann, 2018). Moreover, STEM education can be a form of innovation for teaching mathematics (Fitzallen, 2015) and to increase mathematical performance (Stohlmann, 2018). Furthermore, there is a need to develop research to understand how STEM integration can promote mathematics education (Baker & Galanti, 2017). However, the role of mathematics is understated within the STEM field. Therefore, it is necessary to develop more research to make mathematics more meaningful across disciplines, and also to support teachers in this direction (Maass et al., 2019). In fact, teachers have a crucial role on adapting their practices to provide students with the contexts and strategies recommended to provide an effective STEM education in schools (Kelley & Knowles, 2016; Stohlmann et al., 2012). In addition, an increasing number of authors argue about the importance of integrating the four disciplines included in the STEM acronym (Costa et al., 2020; Kennedy & Odell, 2014).

In 2020, COVID -19 pandemic, caused by coronavirus SARS-COV-2, seriously affected all life in the world (Meehan et al., 2020). The pandemic reminded us that the future is uncertain and consequently there is a need for preparing citizens and education systems for what may come, which can include changes that we are not expecting (OECD, 2020). Therefore, virulent communicable diseases are one of the global challenges that must be addressed in the context of STEM education (Maass et al., 2019). More information about modelling COVID-19 pandemic can be provided by Meehan et al. (2019) and Padmanabhan et al. (2021).

Methodology

In this paper, we use a qualitative research methodology and an interpretative approach by resorting to a case study (Cohen, Lawrence, & Keith, 2007). A case study is an empirical research that observes a phenomenon within its real-life context, allowing a generalization of the results obtained, and requiring skills and expertise from the researchers (Yin, 2014).

First authors of this paper designed a PDP that includes several workshops with the aim of providing teachers with knowledge and skills to develop and implement STEM hands-on practices in class. In the school year 2020/2021 the programme occurred exclusively online because of the COVID-19 pandemic. Despite being online, the educators tried to preserve the hands-on practices by exemplifying them through videos and interactive sessions through the Zoom platform. Teacher Elisa (fictitious name) case study was chosen to exemplify how to develop an integrated approach of STEM education based on real life scenarios such as COVID-19 pandemic. She is a Mathematics and Natural Sciences teacher who participated in the referred PDP, aged 55 years old, 30 years of in-service experience and in charge of two 6th grade classes (10 to 11 years old). At the end of the PDP (2021 January), Elisa (like all the other teachers) presented a final report that includes a critical account and her perceptions about the programme, and also proposals of tasks to implement, as well as evidence of the activities developed in class. Data collected include participant observation that occurred during the workshops of the PDP (first author was present in all the workshops, where focus group was promoted) and content analysis of Elisa's final report. In addition, five interviews were conducted with the teacher by phone to better interpret the case.

Data analysis, results and discussion

In this section, it is analyzed Elisa's case study that shows the development and implementation of mathematical tasks within the context of COVID-19 pandemic. Elisa was very participative in the workshops. In addition, in the interviews she mentioned that she wanted to develop activities that were within the curricula she was lecturing and that had meaning for her students. It was not easy to choose a theme that was appropriate for her students and included in the curricula of their grade level. However, after several discussions with the educators, she finally opted by an interdisciplinary theme in the context of COVID-19 pandemic, in order to use mathematics to help interpret and predict the evolution of the pandemic according to various possible contagion scenarios Inspired on Providência (2020) paper, Elisa planned and developed several tasks to be implemented in class.

Table 1: Tasks implemented in the mathematics class

Tasks	Scenario of infection
1	Each student infects two colleagues
2	Each student infects three colleagues
3	Each student only infects one colleague
4	Probability to infect someone is less than one
5	80% of students are vaccinated

In a first stage, she introduced COVID-19 thematic in the Natural Sciences class because the virus context is part of the curricular contents of this discipline, in the field of Environmental Aggressions and Integrity of the Organism. In a second stage, she developed several tasks to be implemented in the mathematics' class. Table 1 describes some possible scenario of infection proposed to the students to help them understand the need for isolation or vaccination measures. It is important to notice that in the 6th grade level it is not possible to resort to advanced mathematics to model the pandemic.

In task 1, Elisa explained the scenario where “Each student infects two colleagues” and that “Sick students are isolated at home and no longer infect colleagues”. She also constructed a grid with 12 lines and 12 columns, where each square represents one student from a school (Figure 1). In addition, a table with the “Day” in the first column, the “Number of students who gets sick” in the second column and the “Total number of sick students” in the third column was constructed (Table 2). After, she asked the students to paint the number of sick students on the grid.

Teacher: Imagine that each square on a grid (Figure 1) represents a student from a school. One day, which we will call day 1, one of the students becomes sick with COVID-19. Represent him, painting a red square on the grid.

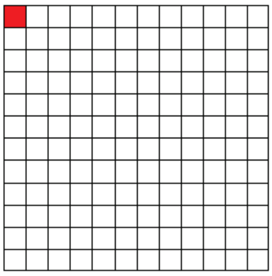

Figure 1: Grid where each square represents one student in a school with 144 students

She also asked students to fill in Table 2 with this information. Next, she gives information about how the sick student infects the others.

Teacher: Let’s now assume that each sick student infects two students on average. To say that it is on average means that it may be that only one person infects one colleague, but there is another that infects three. Therefore, on average, each person infects two colleagues. So, the next day, day 2, there are 2 more students with COVID-19 who were infected by the first student with this disease; in all, there are already 3 sick students. Therefore, paint two more squares on the grid, and fill in the table with this information: on day two, 2 in column two and 3 in column three.

Teacher: Again, sick students are isolated at home and no longer infect colleagues. Now, each of these two students is going to infect two other colleagues, and on the third day there are four more sick students. Therefore, paint four more squares on the grid, and fill in the table with this information.

Table 2: Each student infects two colleagues

Day	Number of students who gets sick	Total number of sick students
1	1	1
2	2	3
3	4	7
4	8	15
5	16	31
6	32	63
7	64	127
8	128	255

Next, she asks several questions:

a) How many new students will be infected on the 4th day?
b) And what is the total number of sick students?
c) In what day is everyone sick?

d) What if the school had 1000 students, after how many days would all students be sick?
e) Can you write the numbers of column 2 in powers of two?

Table 2 is filled with the numbers until the day where all the squares on the grid (Figure 1) are red, which means that all 144 students are sick. As can be seen, on the 8th day all students are sick, in a school with a number of students bigger than 127 and smaller or equal to 255 students. Almost all students from two classes of Elisa presented the results as stated on Table 2.

Only two students, one from each class, considered that the school only had 144 students, and for this reason, in the 8th day only 17 students get sick and consequently the total number of sick students is 144 (Figure 2, table on the left). Below follows the answer of one of these two students.

Student: On the 8th day, only 17 students get sick because in total there are only 144 students in the school and 127 students are already infected, so only 17 students are missing.

The same student also presented the table on the right of Figure 2 to answer that at the 10th day all students from a school with 1000 students would be sick.

dia	n.º de alunos que adoece	n.º total de alunos doentes
0	0	0
1	1	1
2	2	3
3	4	7
4	8	15
5	16	31
6	32	63
7	64	127
8	17	144
9		
10		

dia	n.º de alunos que adoece	n.º total de alunos doentes
0	0	0
1	1	1
2	2	3
3	4	7
4	8	15
5	16	31
6	32	63
7	64	127
8	128	255
9	256	511
10	512	1023

Figure 2: Each student infects two colleagues

n.º de alunos que adoece	Forma de potência	⊛ Continuação	
1	(1) 2^0	64	2^6 ✓
2	(2) 2^1	128	2^7 ✓
4	2^2 ✓	256	2^8 ✓
8	2^3	512	2^9 ✓
16	2^4 ✓		
32	2^5 ✓		

$2^0 - 1$
$2^1 - 2$
$2^2 - 4$
$2^3 - 8$
$2^4 - 16$
$2^5 - 32$
$2^6 - 64$
$17^1 - 17$

Figure 3: Writing the numbers in column 2 in powers of two

In addition, Elisa asked students to write the numbers in column 2 in powers of two. A power of two is a number of the form 2^n where n is an integer, that is, the result of exponentiation with number two as the base and integer *n* as the exponent. Based on this question, she explained to the students that “the growth of students sick with COVID-19 was exponential”. Figure 3 gives examples of answers given by students. On the left, the result given by most of the students and on the right one of the students who considered the total number of students on the school as 144.

After ending task 1, teacher Elisa introduced the following tasks as stated on table 1. During the task 2, where “Each student infects three colleagues” teacher Elisa observed excitement in her students:

> At this stage, a high point of motivation of the students was visible in solving the task. They were excited because they understood the situation they are experiencing in their real life (Final report).

She continues, exemplifying students' discourse:

> "Now I understand, now I understand everything, this is so fast, contagion, ... and we are not counting everyone, families are missing and other people who have contact with them!"

Based on the tasks performed in class, reflection and discussion about them, Elisa recognizes motivation in her students who finally understood “frightening spread of the virus”. Although she already introduced this theme previously, she reinforces that it was with these tasks that students finally understood the need for measures to prevent propagation of the disease:

> I had already taught the content of “Microorganisms” and obviously had contextualized and integrated the whole problem of the pandemic, but it was with this activity that students finally understood the reason for measures to prevent the spread of the disease. (Final report)

This conclusion is in line with her goals, which was to raise awareness and understanding about the need for measures in the context of COVID-19 pandemic, such as social isolation or vaccination. In addition, Table 3 shows STEM contents included in the tasks developed and implemented by Elisa.

Table 3: STEM contents of the tasks implemented by teacher Elisa

Science	Technology	Engineering	Mathematics
Natural sciences Microorganisms Disease spread Pandemic	Computer Internet Wikipedia Power Point Excel.	Planning, designing and performing the activities.	Powers, Exponential growth Mathematical model Variable, Iteration Functions, Graphics Organization of tables and data visualization.

Also, she highlights the interdisciplinary approach provided in the tasks:

> On the other hand, the interdisciplinary aspect of the task allowed the articulation of the disciplines of Natural Sciences, Mathematics and Information and Communication Technologies and, thus, to participate in the Domain of Curricular Autonomy and Flexibility, as well as in the School Educational Project.

Finally, Elisa identifies the importance of implementing this approach to better prepare students to the real-world challenges. Moreover, she intends to keep participating in this type of PDP:

> In today's world, full of complex challenges, the development and integration of multiple literacies, inspired by real situations, will certainly allow for more meaningful learning in which talent, individual qualification, the scientific system and democratic citizenship are strengthened. We are grateful for the opportunity and we await other formative moments of undeniable value.

In summary, teacher Elisa case study exemplifies the implementation of interdisciplinary tasks related to STEM (Table 3), in particular highlighting the role of mathematics to understand real world problems such as the need for isolation measures or vaccination due to COVID-19 pandemic. As can be seen in Table 3, mathematical contents include Powers, Exponential growth, Mathematical model, Functions, Graphics, Organization of tables and data visualization, amongst others.

Conclusions

It was verified that Elisa did develop STEM practices within an authentic context for the purpose of connecting these subjects to enhance students' meaningful learning (Ausubel, 2012; Kelley & Knowles, 2016). In fact, she used the real scenario of COVID_19 pandemic with the aim of introducing tasks with meaning for her students and related to the curricula of Natural Sciences, Mathematics, and Information and Communication Technologies. Moreover, she achieved one of her main objectives, which was to make students understand various virus propagation scenarios and be aware for the need of isolation or vaccination measures to face the COVID_19 pandemic (Table 1). In addition, her example shows the development and implementation of integrated STEM tasks in the class with an important focus on Mathematics (Table 3). Furthermore, Elisa recognized that it was based on this approach that students finally understood the problematic of exponential grow of infection and its impact on real life. Also, she identified critical thinking skills and meaningfull learning as a consequence of this initiative. Her conclusion is in line with Stohlmann et al. (2012), who states that a strategic approach of STEM education provides students with higher levels of critical thinking skills, improves problem solving skills, and also increases learning retention. Moreover, relevant interdisciplinary learning environments were provided as recommended by some authors (Beswick, & Fraser, 2019; Geiger, 2019). In fact, Elisa highlights the interdisciplinary approach provided in the tasks she implemented in class and also the importance of implementing this approach that provides students with meaningful learning and better prepares them to the real-world challenges. Indeed, she promoted mathematics teaching with real life scenarios in the context of COVID-19 pandemic. Based on this research, it is recommended to resort to mathematical tasks based on real world scenarios to promote meaning learning in students.

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Building the bridge for collaborative mathematical instruction: creating a community of practice among pre-service general and special education teachers

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Keywords: Communities of practice, pre-service teachers, special education.

Introduction

A common reason for secondary teachers leaving is challenges with including students with disabilities into general education classrooms (Talmor et al., 2005). Pre-service general education teachers may not receive any instruction about how to effectively adapt their math instruction for students with disabilities (Dibbs et al., 2020). In addition, pre-service special and general education teachers may not be provided with opportunities to practice collaborating on designing lesson plans on math topics for students with disabilities (Trent et al., 2003). This lack of practice can lead to students with disabilities not receiving effective mathematical instruction, potentially leading to lower participation in school, community, and employment (Othman, 2020).

The purpose of this mixed methods study was to investigate how collaborating on lesson plan designs builds awareness of pedagogical content knowledge (PCK) for teaching students with disabilities, guided by the following questions: (1) Does writing collaborative lesson plans throughout the methods course significantly increase preservice teachers' PCK for inclusive mathematics? (2) What is the nature of the knowledge that pre-service mathematics teachers construct about teaching students with disabilities through collaborative lesson plan writing?

Theoretical framework

This study was conducted using situated learning as the lens for inquiry (Lave & Wenger, 1991). The pre-service mathematics teachers were introduced to special education through collaboration with their pre-service special education peers. There were two communities of practice that will occur, one among the pre-service teachers in their content areas, and the other with the pre-service teachers together. By purposefully placing them together, learning will be reinforced theoretically and practically. This study encompasses legitimate peripheral participation by creating a community of pre-service teachers who are familiar with best practices for special education in the general education classroom (Lave & Wenger, 1991).

Methods

A methods course for pre-service secondary teachers course and an upper-level course for pre-service special education teachers were chosen for this study. The pre-service mathematics teachers wrote two student-centered secondary lesson plans. Next, the lesson plans were given to the pre-service special education teachers who adapted each lesson for students with disabilities. Both sets of teachers completed reflection papers, which were coded separately by the authors, and then reconciled. The pre-service mathematics teachers also took a special education PCK instrument (Dibbs, 2021) as a pre- and post-test to measure their knowledge gains on inclusive teaching.

Findings

Three themes emerged from all of the pre-service mathematics teachers following the collaborative lesson plan writing. First, participants noted that the activity helped them to realize how little they knew about special education with respect to inclusive classrooms.

> Maria: "I have no experience with special education and I have very little knowledge of special education. It honestly opened my eyes on how a special education lesson plan is written and how I can adapt my lessons for students with disabilities. I am glad we did this lab because they gave good examples of resources to use for students with auditory, visual, and vocal disabilities."

Second, participants realized that their lesson plans were not as clear as they hoped they were, and much more thought was needed to keep all students engaged in classroom activities.

> Lucinda: "I learned [in the viewing tubes lesson] that students need more instruction, visual organizers, and help with technology than I think they do. It's also important to assign group roles so that all students are actively engaged."

Third, participants noted that the techniques used to support students with disabilities in the classroom could often be applied to all learners to improve the lesson. "Most of the adaptations to the lesson I wrote would actually help all of my students with the activity" Deanna noted in class. Implications of the results, the PCK pre- and post-test, as well as themes from the special education teachers will be discussed further on the poster.

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"Individual Educational Plans" for "dyscalculic" students in primary schools of South Tyrol: A questionable law, poorly implied

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Italian law obliges teachers to draw up Individual Educational Plans (IEPs) for students diagnosed with "dyscalculia". The author analyzed both law and 23 IEPs regarding their compliance with recommendations as stated in current mathematics education literature about how to deal with mathematical learning difficulties (MLDs). Whilst the law clearly mirrors the medical paradigm and largely ignores mathematics education research on MLDs, the IEPs rarely take into account the three content areas that are indicated as crucial for MLDs by educational research. Instead, they tend to lower expectations in terms of reduced number range, continuous granting of compensation measures, and concentration on standard algorithms. The paper outlines key results of a content analysis of the IEPs and discusses their shortcomings with a view to possible consequences on the level of teacher training as well as of desirable changes of the legal and organizational framework.

Keywords: Mathematical learning difficulties/dyscalculia, diagnostic competences, individualization

Focus of the paper

Drawing on mathematics education research on mathematical learning difficulties (MLDs), the paper firstly scrutinizes the reasonableness of the legal and organizational framework for dealing with "dyscalculia" in Italy at present. Italian law foresees that for students with a diagnosed dyscalculia, a council of teachers (including the math teacher of the student) elaborate an "Individual Education Plan" (IEP) to be put into action, evaluated and renewed year after year. By analyzing 23 such IEPs from German speaking primary schools of the Italian province of Bolzano/South Tyrol, the paper secondly aims to gather indications about how adequately the framework is currently put into action in this very province. Finally, against the backdrop of this analysis, alternative and presumably more sensible ways to handle with MLDs are shortly discussed, taking into account the restricted diagnostic and thereto related didactical competences of South Tyrolean primary teachers that emerge from the analysis of the IEPs and maybe comparable to those that can be found in other countries.

Theoretical and empirical framework

MLDs have gained increasing interest from more than one scientific discipline in the last decades (for a current overview, see e.g. Fritz et al., 2019). Psychological and medical research, typically, follow a paradigm in which at least some forms of such learning difficulties are labelled as (symptoms of) a "disorder", nowadays usually termed "dyscalculia". Research in that field is centered on presumed organic, maybe genetic causes of this disorder. As a means for sorting out children afflicted by this disorder (and in order to differentiate them from children with MLDs supposed to be caused otherwise), various tests have been developed and standardized for different countries and age groups (see Kaufmann & von Aster, 2012, for an overview).

From a didactical point of view, as expressed in the recent position paper of the German "Gesellschaft für Didaktik der Mathematik – GDM" (Society for Didactics of Mathematics) (Gaidoschik et al., 2021), the labelling of a subgroup of children with a severe, long lasting MLD as "dyscalculic" is not helpful for pedagogical and didactical purposes and might even turn out to be detrimental. It is not helpful, as for planning remedial measures, we have to understand the concerned child's current stage of mathematical thinking, his or her competences, misunderstandings and shortcomings about fundamentals of arithmetic, in a much more detailed way than standardized tests point at. In addition, with a view to support a single child's mathematical learning, it is irrelevant whether this child belongs to the weakest (or strongest) 5 or 10 or whatever percent of his or her peers (Gaidoschik et al., 2021).

On the downside, being labelled as dyscalculic may have negative effects on the child's self-efficacy expectations and thereby aggravate the MLD (Bishara & Kaplan, 2021). It might also have counterproductive impact on teachers (who could react by just reducing their expectations, thus risking self-fulfilling prophecies, or might think of not being responsible and/or qualified for such "special cases") as well as parents (who could tend to be more interested in compensations for disadvantages than in supporting measures to overcome them) (see e.g. Algraygray & Boyle, 2017, for a broader discussion about possible negative effects of labelling in special education).

As indicated, the above-mentioned position paper of the GDM underlines the importance of process oriented, qualitative assessment of a child's current mathematical thinking. First and foremost, such assessment should focus on those content areas that have been identified by mathematics education research as being crucial for the development of MLDs, namely (cf. Gaidoschik et al., 2021):

Basic number concepts: Severe MLDs characteristically stem from a restricted understanding of natural numbers in that concerned students predominantly conceptualize them as positions within a sequence rather than as "numbers as compositions of other numbers" (Resnick, 1983, p. 114). On that insufficient basis, they struggle to conceive relations between numbers and operations that go beyond their ideas of "forward/backward" within the sequence of numbers, hence tend to stick to counting strategies as their prevailing way to add and subtract (Gaidoschik, 2019).

Place value understanding: Typically, MLDs go with what Fuson et al. (1997) characterized as a "concatenated single-digit conception" of multi-digit numbers. On that basis, children might well learn to perform some calculation algorithms and acquire a certain routine in diverse procedures, yet, lacking conceptual understanding of core principles of the decimal system (bundling and unbundling, multiplicative properties), they will hardly achieve flexibility in computing nor develop sufficient competences in estimating, proportional reasoning, and relational thinking (Gaidoschik et al., 2021).

Conceptual understanding of operations: Children with MLDs, generally, command a limited conceptual understanding of arithmetic operations, in particular of multiplication and division (Gaidoschik et al., 2021), and thus have difficulties to connect them to real-world contexts (Scherer et al., 2016). Subsequently, they often see no chance to solve word problems, if not by deliberately choosing an operation or trying to find a clue by remembering key words, or the like.

As a consequence of the cumulative character of arithmetic learning, if the shortcomings outlined above are not overcome by (remedial) instruction within the first years of schooling, they will nearly inevitably result in even greater difficulties for the concerned students to cope with the curricular content of the following years. Therefore, it is not surprising that we still find the very same problem areas and very similar deficits and misunderstandings as typical also for MLDs in secondary schools, of course amplified by additional deficits and misunderstandings of further content that could not be digested due to the lack of fundamental understanding (Gaidoschik et al., 2021).

From a mathematics education point of view, the high occurrence of such MLDs calls, in the first place, for increased efforts to prepare and enable primary (and already kindergarten) teachers so that they better exploit the considerable potential of preventing MLDs by research-guided instruction (Gaidoschik 2019; Gaidoschik et al. 2021). Then, of course, children who already *have* developed an MLD, need additional measures. The above mentioned position paper of the GDM (Gaidoschik et al., 2021) pleas for a combination of two approaches: On the one hand, teacher resources should be allocated as far as possible to allow for team teaching of mathematics by two well informed teachers. This team should strive to organize themselves so that, while *all* the students of the class are working on what Scherer et al. (2016, p. 641) describe as "substantial and rich learning environments", individual support can be given where needed to those students who otherwise would not be able to participate meaningfully in such mathematical activities, due to missing or deficient prerequisites. On the other hand, for students with MLDs, individualized remedial instruction should be secured to give them a chance to cope with those fundamentals they have not yet been able to acquire. However, this does not have to (and should not) lead to exclusion if class work comprises phases of individualized work for all pupils as a matter of course (Gaidoschik et al., 2021).

The cultural context: An Italian law and its shortcomings from the perspective of mathematics didactics

Of course, the current reality of how MLDs are handled in primary schools seems to be quite different from the recommendations stated above, at least in the Italian province of South Tyrol, to which the present study refers. In fact, though, empirical data about what actually happens in the classrooms are scarce. What can be analyzed and, as a first step, shall be elucidated in the following, is the legal and organizational framework set up in Italy for dealing with "dyscalculia", which is recognized by Italian law as one of several "specific school-related learning disorders" (Gazetta ufficiale, 2010; here and in the following translated by the author).

More accurately, national law 170 determines (article 3) that students diagnosed with "dyscalculia" have the right to receive "individual didactics, tailored to the person" as well as "means to compensate for". The law as a whole declares to aim to "ensure an adequate formation" and foster the "full use of the personal potential" of the student concerned (article 2) (Gazetta ufficiale, 2010). The thereto-related implementation rules of the province of South Tyrol state that the process of diagnosing "dyscalculia" may not be started before "the end of the first semester of year three" of primary school (age 8–9). In case of a diagnosis, to be in accord with law 170, the "team of teachers" of the respective student has to formulate an "Individual Educational Plan" (IEP). This plan has to be drawn up after the first weeks of a school year and is valid until the end of that year. It should consider the report and recommendations of the psychologist who did the diagnosis, and should comprise inter alia "the detailed description of the current level of performance and development" of the student as well as the "planning of the individual objectives" for the student (Brugger-Paggi, 2019; my translation).

Unlike with students diagnosed as "disabled" according to law 104, having a student with dyscalculia does not lead to the assignment of additional teacher's hours to this class. Thus, the mathematics teacher will not receive any support in form of a second teacher, not even at times, nor does the diagnosis entitle the child to receive remedial teaching in or outside the classroom, neither as an individual nor in a group setting (Ianes et al., 2020). Whereas the IEP has to be updated each year, the diagnosis, if made at the end of year 3, is usually reviewed only at the middle of year 5, with a view to the transition from primary to middle school, which in Italy is foreseen after year 5.

Law 170 claims to "consider newest scientific research" (Gazetta ufficiale, 2010). As outlined above, however, at least *mathematics education research* is hardly suitable as an appellate authority for linking *any* special measures for children with MLDs to a diagnosis of "dyscalculia". Secondly, mathematics education research clearly indicates that effective support for children with MLDs should include additional personal resources, which are not foreseen by law 170.

Thirdly, educational *as well as* psychological research shows that MLDs develop on the basis of misconceptions which, often enough, might already be detected in kindergarten and should in any case be in the attention of first grade teachers, so that also counter measures are initiated as soon as possible (Gaidoschik et al., 2021). Of course, law 170 also emphasizes the "duty" of schools and even kindergartens to "take appropriate measures to identify children suspected of having specific learning disorders" (Gazetta ufficiale, 2010). Nevertheless, the law and thereto related regulations link *additional* measures such as drawing up an IEP to the diagnosis of "dyscalculia", and at the same time dispose that such a diagnosis may not be made before year 3 – *against* psychologists who have developed standardized tests for "dyscalculia" already for the end of year 1, and tests that claim to detect "children at risk" already in kindergarten (Kaufmann & von Aster, 2012). If the diagnosis of "dyscalculia" were at all sensible, which is disputed by mathematics education, it would still require explanation (which the legal sources do not provide) why initiating such a diagnosis should be waited for until a child has proven for at least two and a half school years to have severe difficulties in mathematics.

Research questions

As an interim summary of the analysis given above, it has to be stated that the presumably only part of law 170 that may be approved also from a mathematics education point of view is the obligation to draw up an IEP for students diagnosed with dyscalculia. Once again, targeted considerations about how to support students with MLDs should not wait until year 3 or even later, and should not depend on whether the child has received a specific diagnosis. However, an IEP drawn up in a proper way, based on a process-oriented assessment of the current mathematical competences and shortcomings of the child, of course could form a precious guideline for teaching. At the same time, though, it has to be stated that even the best plan has to be readjusted continuously. Then, the additional "administrative load" (Scherer et al., 2019, p. 4622) connected with law 170 might also be seen as deduction of time resources that might better be invested in directly working with the concerned child.

Against this backdrop, the second part of this paper seeks answers to the following research questions:

(1) When South Tyrolean teachers have to state within an IEP the current stage of mathematical competences and shortcomings of the respective student, do they (and if, *how* do they) refer to the content areas that are held crucial for mathematical learning difficulties in mathematics education research and have been elucidated above, i.e. deficient number concepts and thereto linked

predominant use of counting strategies for addition and subtraction, fundamental deficits in understanding the decimal place value system, and lack of conceptual understanding of operations?

(2) Are the learning goals and remedial measures formulated in the IEPs in line with what mathematics education research advices for responding to severe mathematical learning difficulties?

If both questions could be answered with "yes", then, at least, it could be assumed that South Tyrolean schools have found sensible ways to deal with the legal requirements related to "dyscalculia". Of course, in this case it should further be checked what actually happens in the classrooms on the basis of the IEPs. The latter gains even more urgency if already the IEPs should prove to lack appropriateness. However, the present study is limited to the analysis of the IEPs themselves.

Methodology

According to statistical data provided by the educational authority of the province, in 2017/2018 a total of 46 students (or 0.2 %) of all students attending German speaking primary schools of South Tyrol were officially registered as "dyscalculic" and thus addressee of an IEP. As the IEPs underlie strict privacy rules, to analyze them for research purposes, the author had to ask the headmasters of the school districts to transmit anonymized versions of the IEPs. This was done for 23 students. It was not possible to identify the reasons why the other IEPs were not made available but may be assumed that this was due to the fact that anonymizing the IEPs by blackening sensitive data meant some work, and having free choice to participate, some schools might have preferred to avoid this work. Of course, it cannot be excluded that this contributed to a bias of the sample.

All 23 IEPs were submitted to a "qualitatively orientated category-based content analysis" (Mayring, 2020). The main categories were assigned deductively from educational research literature as summarized above, and complemented inductively based on recurring features of the IEPs. In the following, due to the limited space, only the results for those (all deductively formed) categories that are most important in view to the research questions are shortly elucidated. The <u>categories</u> are underlined. Quotes from the IEPs have been translated from German into English by the author.

Results

<u>Overall characteristics of the IEPs, range of elaborateness:</u> The 23 IEPs differ considerably in extent and detail (between 1 and 9 pages; mean: 3.8, median: 3). As explained, the regulations call for a *detailed* description of the student's current level followed by an *individualized* plan of *goals* and thereto related *measures*. Yet, if we take into account only those parts of the IEPs that actually refer to mathematics, we find that 10 out of 23 IEPs dedicate less than one page to both current level and goals/measures. 5 of them, in fact, refer to mathematics only with 2 or 3 lines, typically within a table with tick boxes for briefly formulated measures, such as "use of a calculator". In 7 IEPs the math part takes about one page, in 4 IEPs about 2, and in the two most comprehensive IEPs about 4 pages.

As for the <u>level of detail in which the current mathematical status of the student is described</u>, 3 types of IEPs can be identified, as follows: In 10 IEPs, the authors give a *relatively comprehensive list of curricular contents* (see below for more detail) that the student either has or has not yet mastered. In 6 other cases, only v*ery few content-related competences are named, but at least these clearly refer to mathematics,* such as "Masters the standard algorithms of addition and subtraction". 7 other IEPs practically do not provide *any* information about the *mathematical* competences of a child. One typical example: In one IEP, the child's description with regard to the "mathematical-logical area" is restricted to the following: "Needs much time to comprehend new content. Needs additional explanations and prompts and extended practice. Hardly grasps logical connections."

As for the level of detail with which goals and related measures are formulated, we find the analogous 3 types. As a rule, extent and detailedness of the description of current competences correspond to the attention given to goals/measures. In two cases, yet, a rather detailed status-quo-description is followed by very scarce, mainly general goals/measures, such as: "use of tables and collections of formula, use of specific didactic material, granting of additional working time, personal dedication".

Reference to the crucial content areas: As outlined, only 10 IEPs go into some detail about the concerned child's current mathematical competences and shortcomings. However, the fundamental question whether the child is still dependent from counting strategies for adding and subtracting, is explicitly addressed and answered in only 3 out of 23 IEPs (positively, as has to be expected according to research on MLDs). What's more, not a single IEP explicitly states that the elaboration of non-counting calculation strategies will be a goal of remedial instruction. One IEP, though, foresees "automatizing" basic facts of addition. On the other hand, 5 IEPs explicitly envisage that the child should constantly use material as a help to solve additions and subtractions, arousing the suspicion that the authors regard arithmetic material not as a means to elaborate non-counting strategies, but rather as a device to facilitate computing by counting.

As for the child's understanding of the decimal system, the second crucial area according to research, only 2 IEPs explicitly state problems with bundling/unbundling, in one case in a contradictory manner (the child would understand bundling but not be able to apply it). 6 other IEPs do explicitly indicate difficulties with multi-digit numbers, yet in the unspecified way of "problems to orientate herself with numbers". Accordingly, the related remedial measures stated in these IEPs remain in the vague, such as "consolidate numbers up to 1000". 6 IEPs state as a goal "identify tens, hundreds, thousands", but without explicitly indicating that this should include the elaboration of bundling/unbundling or trading, respectively, as the core principles underlying the place value system.

Finally, as for the child's conceptual understanding of arithmetic operations, only one IEP indicates difficulties with the basic concept of multiplication. Another states problems to connect operations with actions and situations. However, 8 IEPs record severe difficulties with word problems, but without any reference to operation sense as a fundamental prerequisite for solving such problems.

As an interim conclusion, it has to be stated that even those IEPs that, when describing *the current learning status* of a child, refer to mathematical content in some detail, hardly refer to those fundamental content areas that have been identified by research as being crucial for MLDs. Instead, if difficulties are specified on concrete mathematical content level, they mostly refer to *curricular content of year 3 and upwards*; content that can only be learned in a sustainable manner if a child has already acquired the basics which are more or less ignored in almost all IEPs.

As for the individual goals that are set up for the student in the IEP, a common feature of all 23 IEPs is a *reduction of expectations* as compared to the grade level. Only in 8 cases, though, this is combined with explicitly naming *goals also in the area of basic competences* such as single digit multiplication or working on place value in the way characterized above. 7 IEPs stipulate a *concentration on numbers up to 1000*, whereas the grade expectation would go beyond that number range. 12 IEPs explicitly state that the *difficulty and/or complexity of word problems* should be reduced, often in addition with the allowance of "individualized help". 6 other IEPs advocate a *"reduction of demands"* without specifications. The two explicit goals that are set up most frequently are *mastering the standard algorithms* (12 cases) and *automatizing single digit multiplication* (11 cases). On the other hand, 8 IEPs envisage the use of tables to look up single digit multiplications, often linked to the otherwise unrealistic expectation that the child, without having yet mastered *basic* multiplication, should nonetheless solve *multi-digit* multiplication using the standard algorithm.

Discussion and final remarks

Of course, as the only information we have got about the children is what we can read in the IEPs, the appropriateness of these plans can be judged only indirectly and tentatively, by comparing them to what kind of information they *should* comprise, according to mathematics education research, if the concerned child actually does have developed an MLD. Then, we can assess their *internal coherence* and their *compliance with recommendations* given by current mathematics didactics.

In this context, the high weight that many IEPs give to standard algorithms, whereas current literature clearly stresses the importance of mental strategies based on conceptual understanding, further contributes to the overall impression raised by the analysis of the 23 IEPs that may be summarized as follows: On the one hand, with a few selective exemptions, the authors seem to lack the extended didactical competences needed to assess, in a meaningful *qualitative* way, the mathematical status of a child with an MLD, as well as to decide upon appropriate learning goals that could and should be reached for in the next step. On the other hand, they tend to settle for a general reduction of the level that should be attained by the child, by limiting the number range, by allowing for permanent compensation measures, and by concentrating on the training of standard algorithms, notwithstanding the fact that such measures contribute to perpetuating the difficulties rather than overcoming them.

As stated, the present study is narrowly limited, based solely on a sample of 23 IEPs drawn up by teacher teams of South Tyrolean primary schools. Due to space restrictions, only parts of the analysis and very few examples of these plans could be presented. The aim is, of course, not to blame teachers for what they do to the best of their ability. As analysed in the first section of this paper, a law that is questionable in more than one respect obliges teachers to work out IEPs. The qualitative analysis presented in this paper indicates that, as a rule, South Tyrolean teachers are currently overstrained with this obligation. As a consequence, responsible school policies should strive to give support, such as targeted teacher training and the possibility to consult expert teachers specialized on MLDs. Of course, such efforts to extend the didactical and thereby diagnostic competences in primary education are likely to benefit children to an even greater extent if they could be implemented within a legal and organizational framework that allows for additional support of children with MLDs, also in terms of team teaching, starting from the first signs that a child is in danger of developing such difficulties.

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“Counting with all children from the very beginning”: One attempt to promote early arithmetical skills based on part-whole thinking

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Currently, it seems to be difficult for teachers to create substantial learning environments of high quality to manage the arithmetical heterogeneity in First Grade. The 13 substantial learning environments based on part-whole thinking of the large-scale project “Counting with all children from the very beginning” might be one attempt at that. Following design-based research approaches, the substantial learning environments were first developed and then implemented in five mainstream classes with a total of 96 students in the Beta-cycle over a ten-week period. Data from the second cycle is the basis for the results presented in this paper. Preliminary results indicate that not all teachers succeeded in identifying problematic areas in their children’s arithmetical learning, especially in the instances of children who have underachieving mathematical skills and children with special educational needs (SEN).

Keywords: Early arithmetic education, part-whole thinking, computation without counting, inclusive mathematics education, concepts for mathematics education

Preliminary remarks

Every child has the right to access education of good *quality*, as international organisations such as UNICEF, UNESCO or the European Union affirm. Thus, the focus regarding inclusion of all children in developed countries is now on promoting the quality of the teaching and learning situation (Florian, 2008). However, there seems to be uncertainty in German-speaking countries about how to create inclusive learning environments of high quality for *all* children in mainstream classes, not only, but especially in early arithmetical education (Korff, 2016). Hence, the present research project seeks to design substantial learning environments based on *part-whole thinking* for First Grade.

In order to allow others to make comparisons between inclusive school education in different nations, the author of this paper draft the current situation that teachers might face regarding inclusive practices in early arithmetical education in *German-s*peaking countries, with a focus on Austria, where the research of this paper was conducted.

With the passage of time, the Austrian education system, like that of many other countries, has made great progress as far as the issue of inclusion is concerned. In particular the ratification of the UN Convention on the Rights of Persons with Disabilities in 2008 sparked debate about *inclusive* education (as in many countries) and the necessary measures for the common compulsory school system. Nonetheless, *even today*, Austria, like many other countries in the world, still focuses on a multi-track approach: Parents are free to decide whether to send their child to a (local) mainstream school or a special school. In total, in 2020-21, approximately five percent of the children in compulsory education belong to the group denoted as having special educational needs. (Statistik Austria, 2021). It should be made clear that these data refer *merely* to the positioning-oriented definition of inclusion (see below).

By the time children enrol in compulsory school in German-speaking countries (between the ages of six and seven), they have received considerably different targeted *early mathematical education*, depending inter alia on different educational plans in kindergarten and the mathematical knowledge of the kindergarten staff. The educational plans in kindergarten differ within and between German-speaking countries – in Austria, the educational document consists of a few pages, and early mathematical education is dealt with in a few words. The year before school enrolment all children will already have attended kindergarten. By the end of First Grade, it seems to be widely accepted within the relevant German literature, that all students should learn how to *compute without counting* fluently with numbers up to 20 (see also Häsel-Weide & Nührenbörger, 2013). Therefore, it seems to be helpful if teachers first get children to grasp that numbers (at least up to 10, if not up to 20) are composed of other numbers and then, based *merely* on that knowledge, initiate their understanding of addition and subtraction (see also Schipper et al., 2015; Gaidoschik, 2019a; 2019b).

The author of this paper is aware that in the field of *international* mathematics education, designing substantial learning environments of high quality in early arithmetical education is a central element for colleagues from other countries, where there are also likely to be challenges for teachers to teach mathematics inclusively. Thus, the following considerations concerning the implementation of part-whole thinking in early arithmetical teaching may apply to other nations as well.

Considerations on inclusive education in early arithmetical teaching

In the international inclusive education community, there is *no* clear working definition of inclusion. Consequently, in many nations inclusive practices are handled very differently, and these practices can differ also between schools in one country, or even within the same school (see also Booth & Ainscow, 1998). To specify inclusion, it has been defined as both a one-dimensional and/or a multidimensional oriented approach. The former means physical positioning and teaching of children in an ordinary class (Vislie, 2003), the latter represents a wide plurality with other qualities (for example a positive learning environment to promote all children's learning development) in addition to positioning. Many authors agree that inclusion in developed countries is now *not* merely about physical positioning, but has to comprise several aspects. Empirical research has also shown that positioning alone is *no guarantee* of meeting the needs of all children (see also Mitchell, 2014; Vislie, 2003).

The international *mathematics* education community will certainly respect the multidimensional oriented approach of inclusion. To understand how pupils learn, the theory of constructivist learning is vital. The students actively construct knowledge, and of course, there is huge variation amongst children with regard to how quickly and how sophisticatedly they are able to solve mathematical problems: "Children with different developmental backgrounds may well be able to get the same answers on an arithmetical task, but how they do so might differ significantly" (for more details of the discussion between constructivist views and other views on mathematics education theory and research, see Steffe & Kieren, 1994, p. 719).

Thus, in recent decades, some considerations on general inclusive (mathematics) didactics and *few* substantiated didactical concepts for inclusive mathematics education in primary school have been developed *in the mathematics education community*, but I am not aware of any (apart from the following), that *explicitly* deal with the utilization of the concept of numerical part-whole thinking in First Grade.

Considerations on the TIGER concept (Gaidoschik, 2019a)

All children in First Grade classes should be aware of "numbers as compositions of other numbers" (Resnick, 1983, p. 114), also termed the "part-whole schema" (Resnick, 1983, p. 115). Resnick describes this goal as "probably the major conceptual achievement of the early school years" (1983, p. 114), therefore a preferable alternative to computing by counting. According to this concept, children should learn to divide numbers in terms of part and whole relationships for all numbers at least up to 10. Children thus can be led to understand that any whole number can in fact be divided into smaller numbers, and so the number 7 for example, can be divided into 2 and 5. When put together, the parts 5 and 2 fulfil the requirement of being equivalent to the whole number 7. With that understanding, they might be able to solve addition (2+5; 5+2) as well as subtraction problems (7-5; 7-2) – without counting.

The *utilization* of the concept of numerical part-whole thinking in First Grade seems to be the standard in international mathematics education as in for example the "Number framework" (New Zealand Ministry of Education, NZME, 2008) and the "Mathematics programme of study" (UK Department of Education, 2013). Quite when this concept is taught in the academic year can differ across nations. Part-whole thinking is incorporated as *early as possible* in the school programme in these guidelines for teachers' actions. The New Zealand Number framework explicitly states:

> It is important for you to recognise that part-whole thinking is seen as fundamentally more complex and useful than counting strategies. One reason is that counting methods are strictly limited, whereas part-whole methods are more powerful. Counting strategies are an inadequate foundation for these ideas, and this means that for counters, many advanced number ideas are inaccessible. Therefore, your major objective is to assist students to understand and use part-whole thinking as soon as possible. (NZME, 2008, p. 7)

Current models and learning trajectories of these and other guidelines name different numerical stages for counting strategies, and give recommendations for teachers to follow and to work with their classes from one stage to the next. However, Gaidoschik (2019a) describes in detail a structure-genetic didactical attempt as an alternative approach to the already existing ones. The concept he has named TIGER (Gaidoschik, 2019a) attempts to show how to teach children addition and subtraction on the basis of *part-whole thinking*. The author of this paper will provide a more detailed explanation of the TIGER concept and list some examples of the learning situations of the current project.

The TIGER concept of part-whole thinking

The TIGER ("Teile im Ganzen Erkennen und damit Rechnen") concept, created by Gaidoschik (2019a), focuses on solid number concepts in early arithmetical teaching. The 13 learning situations of the current research project are based on these ideas. Drawing on this concept, the learning situations consist of the following three fundamental aspects, whereby the researcher (Gaidoschik, 2019a, p. 424) recommends teachers "work with children in all three Fields A to C more or less *concurrently*" and "it would not help to stick to one single theme for too long in a row […] the sites relate to each other":

Firstly, (A) a child needs to acquire a solid understanding of counting competences and should thus master the significance of cardinality as a means to understand "how many of whatever". This includes teaching the “counting principles”, defined by Gelman and Gallistel (1987). The substantial learning environments that have been designed try to allow all children to consolidate their *individual* counting competences in the counting activities, for example by counting the number of children present in the class. Counting is used almost every day: Is anyone absent today? Is the number different from yesterday and if so, how has it changed? Of course, First Grade classes will be quite heterogeneous in their counting competences: *Many* children will already have no problems with counting (forwards and/or backwards), while some children will need more counting activities than others. As a benchmark, all children should (at an earlier or later stage) be able to count easily up to at least 10, both forwards and backwards (Gaidoschik, 2007).

Secondly, (B) it is important that a child is able to judge very small sets without counting – through direct pattern recognition - and uses *subitizing* for smaller quantities (up to 4) and uses *perceptual subitizing* for quantities greater than 3 or 4. Perceptual subitizing is only possible for children and indeed us adults if bigger sets are presented in a structured way, so that counting can be avoided (see also Clements & Sarama, 2009). The learning situations that the author of this paper has created include activities to teach part-whole thinking, including in particular activities *using fingers*, recommended by Gaidoschik (for more details of fingers as a very useful “material”, see Gaidoschik, 2007). Thus, for example, children should learn at the beginning to show the right number of fingers without actually counting them, and without extending them one by one. As a starting point we should focus on (at least) all numbers up to 5. In the following activities, structured dot patterns, dice, then ten-frames are useful materials in the learning situations. All in all, the use of materials in the current research project is important for children to acquire *mental pictures* and finally automatized knowledge but the use of materials is limited to a *few* structured arithmetical materials, as mentioned above. Note that all the activities with material should be presented in a *structured* way so that children are able to identify quantities (first for quantities up to 5 and later for all numbers at least up to 10) at a glance. The focus of such early activities is to teach children the interpretation of these structures, their relations to 5 and 10; and especially their part-whole *compositions* (5 as consisting of 2 and 3, or of 4 and 1, etc.).

Thirdly, (C) a child should *compare* quantities and numbers. On the basis of one-to-one matching up, children should learn and/or consolidate that no counting is needed for pairing the items for number comparison such as identifying without counting (Gaidoschik, 2007). The learning situations focus on activities that foster children’s ability to compare quantities. Thus, inter alia in the learning activity “Throwing Tiles” they use one-to-one matching without needing to count the tiles, they create rows and then compare in pairs their quantities. In “Finger use” they should “show the right number of fingers without counting them” and then compare the fingers with other quantities and/or numbers.

Organising the created substantial learning environments in the classroom

To achieve a *balance* between children learning individually and learning together in (early arithmetical) education, which is the general consensus within the mathematics education community in German-speaking countries (see for example, Häsel-Weide & Nührenbörger, 2015), the current learning situations are divided into: (I) Whole-class instruction; (II) Individual Work; (III) Teamwork. Each teacher decides, depending on the class situation, which organisational structures are best and for which children. Presumably, for some children it might be useful at some point to be given additional or different activities to practice and/or consolidate their arithmetical knowledge. Nevertheless, as Feuser argues (1997), all pupils should, as much as possible, be working on the same activity even if it is being carried out on a different level. The learning situations in the current paper respect these ideas and try to give specific ideas for teacher's action in classes.

The research project, research method and research question

To be able to examine the effectiveness of the learning situations created, collective case studies following the framework of design-based research approaches (Euler, 2014) have been conducted. In the Alpha-cycle, conducted in the school year 2019-2020 with 45 First Grade students in two Tyrolean mainstream classes, video-recorded conclusions were drawn for the implementation and further development of the substantial learning environments. These results were processed in an additional cycle. The Beta-cycle, which is relevant for the present data analysis, was carried out in five Tyrolean primary school classes in the school year 2020-2021 with 95 students over a 10-week period in First Grade in inclusive mainstream classes. Before starting the classroom implementation, all teachers were instructed in using and adapting the learning situations. To generate insights into children's learning and their mathematical thinking, each lesson was video recorded, and framed as the basis for data analysis using "Qualitative Content Analysis" (Mayring, 2015).

The pairs of children analysed are, as far as possible, unchanged, so that conclusions can be drawn regarding the progress of their learning. Furthermore, "Assessment of Teaching-Learning-Situations in Mathematics of the Early Grades" (Steinweg, 2010) is used for data analysis, as an additional level of analysis. Steinweg's idea of dimensions focuses on the *teachers'* possibilities for action in First and Second Grade. The combination of both gives indications for (further) development of the substantial learning environments.

The questions guiding the paper are: (1) Do the analysed transcripts of the substantial learning environments based on part-whole thinking indicate any difficulties of understanding for many children and/or a particular group of children with certain arithmetical knowledge? (2) Are there difficulties in teaching the substantial learning environments based on part-whole thinking; and if so, are these observed in *several* scenes?

Preliminary results

It should be made clear that it is not the intention of the author to draw conclusions from specific scenes or class observations. The idea is not that a learning activity should be done just once in a class or that *all* children should be able to solve the arithmetic task (at the same time). Thus, the dimensions of Steinweg (2010) – consistent with the author's intention – are considered 'competence-oriented', that is to say, the (analysing) focus is more on the teacher's achievements than on their failings. When Steinweg's dimensions are used in a way that *several* scenarios of teachers' actions in classes are analysed, conclusions can then be drawn to give teachers targeted indications for the work in daily classes.

Drawing on these ideas, preliminary outcomes of the analysis of *some* transcripts using "Qualitative Content Analysis" (Mayring, 2015) indicates content related problematic areas in early arithmetic education: These are problematic areas of arithmetical knowledge that might be common to many children and/or a particular group of children. To illustrate by means there are some examples. In "Throwing Tiles" children take turns throwing 10 reversible tiles, which have a red side and a blue side. Then together they have to match one red with one blue tile, using one-to-one matching without counting. They create two rows to identify who has more tiles, and how many more there are of one colour compared to the other. Regarding Steinweg's idea of mathematical dimensions (2010), *not* all teachers were able to foster children's arithmetical competences and not all teachers were *consistently* able to identify these problematic areas of arithmetical knowledge. In some sequences, for example in this sequence "Throwing Tiles", some teachers did not realize that children require a solid understanding for one-to-one-matching up as a basis for further arithmetical strategies. Especially for underachieving children in mathematics and children with SEN the understanding and utilization of the one-to-one-matching process was *consistently* difficult in several instances on several days.

Again, it is the same group of children that seem to have difficulties in the following scenarios: One of the advanced activities of "Finger Use" is to work out the 5-plus-x structures of the numbers 6 to 10. The children play in pairs, and one says a number, for example 7, and the other child, with hands underneath the table, without looking at their fingers, has to show the right number of fingers without actually counting them, and without extending their fingers one by one. Their partner then checks the solution. In further exercises, they have to describe what they would do, without actually doing it. For example, "Seven. I have to show one full hand and two fingers on the other hand; in total that makes seven". Drawing on Steinweg's idea of mathematical dimensions (2010), even in this activity not all teachers were able to foster the use of fingers. For these children it might be worthwhile to give more thought-provoking stimuli to acquire part-whole thinking. Thus, the base numbers 5 and 10 (5 fingers on one hand, 10 fingers on both hands together) should be focused on first.

Thus, a *preliminary indication* of the analysed transcripts is that particular attention should be paid to the mathematical thinking of underachieving children and to children with SEN, and it is evident that teachers seem to find it especially difficult to meet the learning needs of those children.

Closing remarks

The current research project on early arithmetical teaching in First Grade, presented in this article, is *one* attempt to face challenges in teaching (arithmetic) inclusively. As has been outlined, the substantial learning environments that have been created are based on the TIGER concept (Gaidoschik, 2019a). The author of this paper agrees with the ideas of many researchers in the relevant mathematics education literature, that children who need special support in learning (mathematics) do *not* learn differently, nor need completely different *concepts* than are already recommended for all children (see Gaidoschik, 2019a, 2019b; Moser Opitz, 2008; van de Walle, 2004). Thus, their learning behaviour is not completely different from that of their classmates. Of course, some mathematical content might be acquired by some children at a later stage than other children, but all children should learn them at *some* point during their mathematical education. From the structure-genetic didactical analysis approach this idea seems logical because the mathematical content is the same for all children.

Nevertheless, empirical evidence (see for example, Pfister et al., 2015), as well as *some* interim results of this research project indicate how difficult it is for teachers to implement already *existing* concepts of mathematics education in their classes, even for those who participate voluntarily in in-service development programmes.

Thus, one conclusion of this research is that it is essential to enhance teachers' knowledge of relevant pedagogical methods *and* content, so that teachers can then fulfil all pupils' differing learning requirements. The other conclusion is that data analysis has also shown that for teachers to get a collective picture of their students' arithmetical knowledge, it might be worthwhile to identify problematic areas of arithmetical knowledge. The awareness of these problematic areas of arithmetical knowledge is crucial for *quality* teaching in inclusive mathematics education. Of course, this places high demands on the teacher's abilities.

Yet, in further data analysis something that needs to be evaluated is whether such problematic areas of arithmetical knowledge involve many children and/or a particular group of children in *different* classes that participated; and whether it was *consistently* difficult for the teachers to identify these problematic areas of arithmetical knowledge and to foster children's competences. This would allow us to develop further indications for the work with teachers and children.

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Evaluation of a Computer-based Learning Program for Students with Mathematical Learning Difficulties

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A study was conducted in 2021 in Switzerland (Canton of Vaud) to evaluate two computer-based learning programs Calcularis[1] and Matheros[2] to help students improve their performance in calculation. Calcularis was especially designed to support students with mathematical learning difficulties. Then, our study focuses on the evaluation of the effect of Calcularis on the mathematical learning skills for students with mathematical learning difficulties in ordinary school context. We observed that Calcularis could allow a better progression compared to Matheros in calculation, number-line estimation task for students with mathematical learning difficulties.

Keywords: Mathematical learning difficulties, computer-based education, primary school, calculation.

Context of the study

Following the results of the PISA and TIMSS, some governments place the teaching of mathematics at the heart of their educational policy. This is for example the case of the canton of Vaud in Switzerland, which has set up a "mathematical mission" in 2018-2019 in order to improve the knowledge and skills in mathematics of elementary school students. One of the recommendations of this mission (Dias, 2019) is to propose a computer-based learning program to improve the performance of elementary school students in calculation. In order to choose this computer-based program, a study, leaded by the DGEO[1], was conducted to specifically compare two programs: *Calcularis*[2] and *Matheros*[3]. These programs were chosen by the DGEO because already used in some schools in the canton during the Covid-19 pandemic. One difference between these two programs is that *Calcularis* was designed for students with mathematical learning difficulties (MLD). This paper is focused on evaluating this computer-based program for students with and without MLD.

State of art

Mathematical learning difficulties

For several years, research on mathematical learning difficulties has been developing both in cognitive neuroscience and in mathematics education. Two recent literature reviews (Deruaz et al., 2020; Lewis & Fisher, 2016) report on the following issues in this fields of research: heterogeneity

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[1] DGEO (Direction Générale de l'Enseignement Obligatoire) General management of compulsory school.

[2] https://dybuster.com/fr/calcularis/

[3] https://matheros.fr/

of definitions and characterizations (e.g., mathematical learning disabilities, dyscalculia, mathematical learning difficulties), a diversity of explanatory models, a variety of identification and support tools, a non-consideration from a didactical point of view.

Within the research group “Riteam” (http://riteam.ch/fr/), we are specifically interested in examining the mathematical learning difficulties in the school context and through didactical filters. That seems necessary and complementary to current studies, rather conducted out of class and within cognitive psychology. The aim is to take a more focused look at the teaching and learning processes.

According to Scherer and colleagues (2016), our prior research highlighted the need, for research in mathematics education, to broaden the definition of mathematical learning difficulties used most widely in cognitive sciences. However, in the school context, it is necessary to also consider students with difficulties that are not identified by a medical diagnosis. Thus, in mathematics education, research focuses on students with clinically diagnosed disorders, but also on students with severe difficulties in mathematics, identified by teachers (Deruaz et al., 2020). We also noted that research in mathematics education focuses more on interventions with students with mathematical learning difficulties. These are mostly case studies, outside the school context, evaluating the impact of a remediation intervention on a specific mathematical content. These interventions focus more on teaching strategies than on students' cognitive abilities. Among these teaching strategies, some are mediated by digital technology.

Computer-based intervention for students with mathematical learning difficulties

In 2006, Wilson and colleagues proposed a pioneering study in the design and evaluation of a digital tool for remediation (Wilson et al., 2006). They evaluated the impact of an adaptive digital game (Number Race) on the mathematical learning of students with dyscalculia. Their results show positive effects on student performance in basic mathematical tasks such as subitizing, non-symbolic comparison, number comparison, and subtraction (of two one-digit numbers). However, these effects do not generalize to counting or other arithmetic tasks (two-digit addition or subtraction). Regarding the *Calcularis* program, several studies have evaluated its impact on the mathematical performance of students with or without mathematical learning difficulties (Käser et al., 2013; Kucian et al., 2011; Rauscher et al., 2016). Kucian and colleagues (2011) conducted an initial evaluation of the *Calcularis* program for a group of students with mathematical learning difficulties. The results indicate that training with *Calcularis* leads to improved spatial representation of numbers and modulation of neuronal activation, both of which facilitate the processing of numerical tasks. More recently, Rauscher and colleagues (2016) conducted a study with control groups. Their results show significant progress of the *Calcularis* group compared to the control group, especially with regard to subtractions and spatial representation of numbers. In addition, five months after the end of the use of the application, their level of performance was stable. The authors conclude that *Calcularis* not only leads to short-term improvement, but also allows students to use these improvements to succeed in the long term.

To conclude, current research on mathematical learning difficulties in mathematics education focuses mainly on case studies evaluating the impact of a remediation intervention on specific mathematical content with students with mathematical learning difficulties. These interventions are often based on digital tools. Several meta-analyses have examined the effects of these interventions with digital tools and reveal positive effects (Kulik, 1994; Li & Ma, 2010). In particular, Li and Ma (2010) highlight their contributions to students with special educational needs in primary school. With regard to the *Calcularis* program, current studies highlight its impact on improving the spatial representation of numbers and solving simple subtractions in primary school students, with or without mathematical learning difficulties.

The current study

The current study aims to contribute to the evaluation of *Calcularis* on the performance of elementary school students with and without mathematical learning difficulties. Indeed, the aim is to evaluate *Calcularis* as a computer-based learning program for students with difficulties, in ordinary classrooms in the Vaud school context. The research question focuses on the impact of *Calcularis* on the learning progress of students with or without difficulties. For that, we compare the use of *Calcularis* to another computer-based program (*Matheros*).

Materials and Method

Compter-based learning program: *Calcularis* and *Matheros*

Calcularis 2.0 is a highly adaptative computer-based training program (von Aster et al., 2016). *Calcularis* is based on theoretical neurocognitive foundations of numerical cognition, such as the triple-code model (Dehaene,1992), the four-step developmental model (von Aster and Shavel, 2007) and further theoretical advancements (Kucian and Kaufmann, 2009). *Calcularis* has been designed according to insights on the typical and atypical development of mathematical abilities (Käser, 2013). The program aims to automatize the different number representations, to supports the formation and access to mental number line and to train arithmetic operations as well as arithmetic fact knowledge in expanding number ranges from 0-10 until 0-1,000 (Kohn et al., 2020). *Matheros* is one of the digital tools offered by the Monecole.fr site created in 2011 by L.Walter, a teacher and digital trainer in France. *Matheros* allows students to progress in mental calculation according to the principle of belts (as in martial arts) of skills (Monecole, 2021).

Sample

Our sample is composed of seven classes of third grade and eight classes of fourth grade (aged 8-9 years) in the canton of Vaud. This comprised 15 teachers and 260 students. The classes have been choice by the DGEO. There are 143 students in the *Calcularis* group (classes using the *Calcularis* program) and 117 students in the control group (classes using *Matheros* program). Students with MLD were defined as those whose score in two out of three tasks (see below) was situated in the bottom 20% for a given grade. This resulted in 18 students (10 in *Calcularis* group and 8 in *Matheros* group) in the third grade and 12 students (8 in *Calcularis* group and 4 in *Matheros* group) in the fourth grade.

Measures

We used three tasks to assess students' performance in mathematics.

Numeracy Fluency: we assessed the numeracy fluency with the subtest 6 of the Woodcock-Johnson Test III (Woodcock, McGrew, & Mather, 2001).

Number line estimation: we assessed the spatial representation of numbers with a number line estimation task, adapted to the subtest of the Zareki (Dellatolas & Von Aster, 2006).

Calculation: we assessed the calculations (with multi-digit numbers) with a task designed by our team to evaluate the expectations of the curriculum in the third and fourth grades. This task aligned with the swiss curriculum has four categories: addition, subtraction, multiplication and half/double. For example, in third grade: 87+9=; 4x3=; 140-52=; double of 55; and in fourth grade: 254+2005+17=; 1060-69=; half of 120. Each category is declined according to the different variables (e.g., for the addition: number of terms of the addition, size of the numbers at stake; presence of retention or not; possibility of complement to the top ten/hundred; associativity; commutativity; special procedure (+9)). Students can put the operation on the sheet if they want.

Procedure

The procedure was carried out in four stages. The first step refers to the pre-tests: teachers administered the three mathematical tasks to all their students in their own classrooms. The second step concerns the use of computer-based learning programs: for six weeks, teachers proposed to their students to use *Calcularis* (in *Calcularis* group) or *Matheros* (in control group) at least two times 20 minutes per week, individually, in class or at home. The logbooks kept by the teachers during the study attest to the achievement of this weekly time by the students. The third step is related to the post-tests: teachers administered the (same) mathematical tasks one week after the computer-based learning programs were completed. The fourth step refers to the delayed post-tests: teachers administered the (same) mathematical tasks again, 4.5 months after the computer-based learning programs were discontinued.

Results

Results for all students

In this part, the effect of using *Calcularis* on the three tasks (numeracy fluency, number-line estimation, and calculation) will be evaluated and compared to the effect of using *Matheros*, and the evaluation will be carried out for all students.

Pre-tests

In order to examine whether there was a group difference in the pre-tests, we carried out a one-way ANOVA4 on the pre-test score for the three tasks and the two grades separately. The results showed that in the third grade, compared to *Matheros* group, *Calcularis* group had significantly higher score in calculation test (18.0 vs 16.0, $F(1, 120) = 4.82$, $\eta 2p = .04$, $p = .03$) and marginally lower score in

[4] Analysis of variance (ANOVA) is an analysis tool.

fluency (37.8 vs 42.4, $F(1, 120) = 3.74$, $\eta 2p = .03$, $p = .06$) test. In the fourth grade, *Calcularis* group had significantly lower score than *Matheros* group in fluency test (53.6 vs 58.5, $F(1, 134) = 4.03$, $\eta 2p = .03$, $p = .047$). For the other tasks (number line tasks in third grade and fourth grade and calculation task in fourth grade), there are not significant differences between the two groups.

Post-tests and delayed post-tests

In order to examine the effect of group and pre-test score on the scores of post-tests and delayed post-tests, we ran a multiple linear regression for the three tasks and the two grades separately. Our analysis, that are presented in Table 1, revealed that in general, there was a significant effect of pre-test score on post-test and delayed post-test scores. The exceptions were found for number-line task during post-test for *Calcularis* group in the third grade and for *Matheros* group in the fourth grade. Furthermore, in the post-test, *Calcularis* had an advantage over *Matheros* in the third grade for calculation task ($t(114) = 2.88$, $p = .005$) but a disadvantage for number-line task ($t(102) = –2.44$, $p = .02$). In the delayed post-test, in the fouth grade, there was a difference to the disadvantage of *Calcularis* for fluency task ($t(126) = –3.06$, $p = .003$).

Table 1. The slopes of post-test and delayed post-test as a function of pre-test

Grade	Task	Group	Post-test		Delayed post-test	
			Slope	*p*	Slope	*p*
3rd	Calculation	Calcularis	0.80	< .001	0.60	< .001
		Matheros	0.48	< .001	0.55	< .001
	Number line	Calcularis	0.05	ns	0.18	.06
		Matheros	0.42	< .001	0.21	.051
	Fluency	Calcularis	0.85	< .001	0.89	< .001
		Matheros	0.65	< .001	0.75	< .001
4th	Calculation	Calcularis	0.70	< .001	0.53	< .001
		Matheros	0.75	< .001	0.29	.03
	Number line	Calcularis	0.09	.003	0.22	.03
		Matheros	0.19	ns	0.43	.002
	Fluency	Calcularis	0.84	< .001	0.78	< .001
		Matheros	0.92	< .001	1.19	< .001

Note. ns: non-significant differences

Analyses for students with mathematical learning difficulties

In this part, the effect of using *Calcularis* on the three tasks (fluency, number-line estimation, and calculation) will be evaluated and compared to the effect of using *Matheros*, but the evaluation will be carried out only for students with MLD. The performance of students with MLD was evaluated in the term of predicted and real observed scores of both post-test and delayed post-test. To obtain the predicted scores, we ran a multiple linear regression based on the scores of the students who were not categorized as students with MLD. The obtained slopes for a given tasks and a given grade were then used to calculate the predicted scores of students with MLD. These predicted scores were then compared to the observed scores of these students by means of a one-sample t-test.

The results for students with MLD are presented in Table 2. In the phase of post-test, there was no difference between the observed and predicted scores, except for calculation task in *Matheros* group of the third grade, in which the observed score was significantly lower than the predicted score. In fact, the gap obtained in this group was significantly larger than in *Calcularis* group (t(16) = 2.11, p = .051). In the phase of delayed post-test, the observed score in the *Matheros* group of the third grade was significantly lower than the predicted score for calculation and number-line tasks. Again, these gaps were significantly larger than in *Calcularis* group (t(16) = 2.93, p = .01 for calculation and t(15) = 3.22, p = .006 for number-line task).

Table 2. The difference between observed and predicted scores of post-test and delayed post-test for students with MLD

Grade	Task	Group	Post-test		Delayed post-test	
			Difference	*p*	Difference	*p*
3rd	Calculation	Calcularis	–0.72	ns	0.66	ns
		Matheros	–3.82	.03	–6.16	.02
	Number line	Calcularis	–0.24	ns	0.95	ns
		Matheros	–1.43	ns	–1.41	.02
	Fluency	Calcularis	–0.54	ns	0.47	ns
		Matheros	7.33	ns	7.99	ns
4th	Calculation	Calcularis	1.16	ns	–0.47	ns
		Matheros	–1.94	ns	–0.25	ns
	Number line	Calcularis	–0.22	ns	–1.56	ns
		Matheros	1.41	Ns	0.30	ns
	Fluency	Calcularis	0.77	Ns	–3.67	ns
		Matheros	–7.87	Ns	–4.86	ns

Note. Positive differences imply higher observed score than predicted score. ns: non-significant differences.

Discussion

Concerning the impact of *Calcularis* on the learning progress of student with and without difficulties, our results show that *Calcularis* allows all students to progress in calculation and numeracy fluency significantly after a 6-week use (post-test) and at a deadline of 4.5 months (delayed post-test). For the spatial number representation, the results are more nuanced. *Calcularis* allows fourth grade students to progress significantly but not for third grade students. Concerning the comparison between *Calcularis* and the other computer-based program, *Calcularis* is more efficient than the other computer-based learning program (*Matheros*) for third grade calculation. On the other hand, the other computer-based learning program (*Matheros*) seems more efficient in third grade. Concerning the students with MLD in the *Calcularis* group, they progress more than students with MLD in the Control group for the calculation in third grade. We also see better efficiency for third grade students on the number line estimation task.

These results obtained for students with MLD can be explained by two reasons. First, the specific design of *Calcularis* is based on theoretical neurocognitive foundations of numerical cognition (Kohn et al., 2020). Initially, the aim and conception of *Calcularis* is to support students with MLD. In *Calcularis*, the use of machine learning make takes more progressive being slightly challenging and thus may foster the development of new skills (Kohn et al., 2020). This technology is another major difference with the other computer-based learning program *Matheros*. In contrary, *Matheros* offers a linear path, without the possibility of playing on the difficulty of the proposed calculations. It is possible to change the time given to perform the calculations but the teachers in our control group did not use it. Secondly, the finding that *Calcularis* allows third graders to make more progress in numeracy, especially for students with MLD, is consistent with our didactical analysis of the content of this computer-based learning program. Indeed, by comparing the contents of this program with the curriculum, we have highlighted a more appropriate calibration of *Calcularis* for third grade students, in particular due to machine learning.

The results obtained for students with MLD confirm the results of previous studies in numeracy fluency and number line test (Käser et al., 2013, Kohn et al., 2020, Rauscher, 2016). Rauscher (2016) demonstrates that *Calcularis* can be used effectively to support children in their numerical development and to enhance subtraction and spatial number representation. Käser (2013) found significant results in subtraction for students who used *Calcularis* for 6-8 weeks. This involved subtraction of two single-digit numbers, to be performed mentally. As far as we know, studies concerning posed calculations (with multi-digit numbers) are not very frequent. Therefore, our study provides new results on the effectiveness of *Calcularis* in improving numerical skills in students with MLD in the school context.

In sum, our findings are in line with previous research however they provide a complement because the study was conducted in ordinary classes. *Calcularis* seems particularly interesting as a computer-based learning programs to help students with MLD. In future research, we will analyze the data collected in the application (game duration, skills acquired, number of exercises performed correctly, etc.) to see if the application could be used as a tool for identifying students with MLD by teachers.

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The interplay between theory and practice in the development of a model for inclusive mathematics education

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We present the first results of the OPEN-MATH project that aims at the accomplishment of inclusive mathematics learning environments via the construction of the Open Activity Theory Lesson Plan (OATLP). We built a theoretical framework for inclusion that stems from the networking of differentiation for all, featured as open learning and the theory of objectification. We describe the interplay between theory and practice. Its outcome is a conceptual framework for inclusive mathematics and the development of the OATLP model across 3 implementations involving an Italian middle school class.

Keywords: Inclusion, Differentiation, Open learning, Theory of objectification.

Introduction

The debate about inclusion in mathematics education has broadened in recent years, polarizing on two complementary aspects: those related to the questioning of the term itself, and its meaning in relation to mathematics education, and those related to classroom practices (Roos, 2019). In the project OPEN-MATH, funded by the Free University of Bozen, we tried to build a link between these two polarizations, defining inclusion from the perspective of mathematics education and trying to arrive at a set of design principles that can enhance the participation and learning of each one in the classroom. For that purpose, we decided to combine two approaches, that of the theory of objectification (Radford, 2021), which serves as background theory to define learning in mathematics, and the didactical differentiation (Tomlinson, 2014), which instead provides the inclusive perspective regarding classroom management. We need to recall that in Italy, students with and without SEN attend the same classrooms since 1977. Therefore, the perspective taken in the project starts from the assumption that inclusion does not only mean that all students can attend the same class, but that the educational environment must allow meaningful learning and participation for each student, according to her own characteristics (Aiscow, 2016). In the next pages we present the theoretical background of the project and the developing of the Open Activity Theory Lesson Plan (OATLP) that allowed us to define the first set of design principles for inclusive mathematics education in the classroom combining the theory of objectification and didactical differentiation. What we want to show, is how these two theoretical approaches can shape classroom activity to foster the inclusion of each student according to his or her characteristics.

Theoretical Framework

Inclusive Education

Inclusive education has been conceptualized in several ways. In literature, we find a certain consensus on a general distinction between narrow and broad definitions (Nilholm & Göransson, 2017). Narrow definitions focus on students with disabilities, their presence in mainstream schools and classes and the needed support. Broad definitions are about school systems and school communities and their commitment and capacity of welcoming all students with all their individual differences, granting participation and effective learning processes. Tomlinson (2014) encourages a view that assumes difference as the norm in learning and that places differentiation in the normality of instructional design for all. A broad idea of inclusion poses a great challenge to the way learning processes can be supported in schools both considering all students' differences and granting their participation to a common learning project. Differentiation has been discussed by several authors as a tool that can contribute to tackle the challenge. Within the broad understanding of inclusion that we advocate in our work, we will follow Tomlinson's approach to differentiation. Open education (Demo, 2016) is a learning strategy that accomplishes differentiation for all by promoting students' opportunity to organize the learning process for their own, working on different tasks at the same time in the same space: this is consistent with the broad idea of inclusion presented in Tomlinson's Differentiation framework. Students are expected to be active in their learning processes, aware about the way they learn and to take decisions according to the activities they are exposed to. We can describe three possible areas of decision-making for pupils in terms of learning: 1. organization (spaces, times, learning partners), 2. methodology (how to solve a task), 3. aims and objectives (content and goals). In the project, we used open learning in relation to organization and methodology using learning stations.

The Theory of Objectification

According to the Theory of Objectification, thinking is a *praxis cogitans* (Radford, 2021). Conceptual objects, thinking, learning, and meaning in mathematics are intertwined in reflexive mediated activity that unfolds as joint labour. Learning is a specific *praxis cogitans* termed *process of objectification* that allows the student to notice, find and encounter the cultural object. The artifacts that mediate reflexive activity and accomplish the objectification processes are called *semiotic means of objectification*: objects, tools, linguistic devices, and signs that individuals intentionally use to carry out their actions and attain the goal of their activities. Radford, resorting to a dialectic materialistic stance, conceives embodiment as a sensuous cognition, that is, a multimodal sentient form (perceptual, sensible, and imaginative) of responding to the world sprouting from cultural and historical activity. In this respect, objectification unfolds as a materialization of mathematical knowledge in the student's sensuous cognition and can happen in various ways, according to the individual characteristics of the student. The dialectic interplay between a cultural-historical environment, the individual and reflexive activity gives rise to a double-sided construct: objectification-subjectification. Subjectification, the counterpart of objectification, is related to the production of subjectivities as they engage in the reflexive mediated activity. The Theory of Objectification outlines a dialectical co-production between individuals and their cultural and historical reality. The individual, according to Radford, is continuously inscribing herself in the social world, producing her subjectivity according to the possibilities given by her environment.

A conceptual framework for inclusive mathematics learning

According to the combining-coordinating strategy (Bikner-Ahsbahs & Prediger, 2014), we networked the Theory of Objectification and Differentiation, featured as open learning, to draw a conceptual framework for inclusive learning in mathematics with the following elements (Demo et al., 2021):

Definition of inclusion. Inclusion is conceived as the dialectical and critical positioning (subjectification) of all students in the cultural-historical practice of mathematics, who act, feel, and think according to their individual distinctive traits to pursue their project of life. The process of subjectification described is equated with meaningful participation and learning, which can be defined only with respect to a cultural practice.

Mathematical activity. Mathematical reflexive mediated activity, in its multimodal acceptation, is the meeting point of the social and individual dimension of mathematical learning. The notion of sensuous cognition allows us to keep together social interaction and individual self-determination. Semiotics means of objectification allow multimodal activity both as open learning, making available to different students different learning path, and joint labor with respect to common learning goals.

Teaching-learning model. Starting from the Theory of Objectification we have developed an inclusive lesson plan that intertwines social interaction and individual self-determination. We have added to the original Activity Theory design (Radford, 2021), which alternates phases of small group work and of whole classroom discussion, elements of individual differentiated work, stations work, made according to Open Learning. Multimodality characterizes every aspect of the cycle, giving the possibilities to different students to experience different ways to reach the same learning goal. The outcome is what we have called Open Activity Theory Lesson Plan (OATLP) and it is presented in Figure 1.

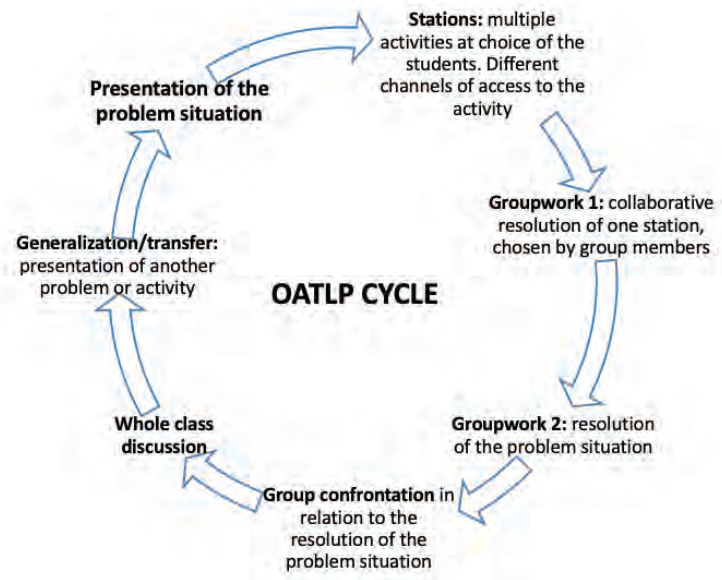

Figure 1: The OATLP Cycle and its different phases.

Stations (Demo, 2016; Tomlinson, 2014) are one of the possible strategies related to the implementation of open education and represent a way to put differentiation into practice by developing classroom environments in which learning processes are multimodal, decentralized and plural. Different learning activities related to a main didactical objective are structured in different stations. Students can move from station to station and choose which ones to complete and with whom. Decision making is enhanced with respect to organization of times (1) and to methodology (2): in fact, students can decide how long to work on each station, and to avoid one or more activities. The activities are connected to the same learning goal but exploit different means to reach them. A "passport" is used for the student to take note of the stations completed, the difficulties encountered, and what enabled them to learn in the most effective or enjoyable way, but it also allows the teacher to keep track of and understand individual differences in learning mathematics.

The evolution of the OATLP model during the research project.

In this section we show how the structure of OATLP cycle has gone through subsequent changes when applied to different topics, with the same classroom during the school year 2020/2021. The changes have been made accordingly to the analysis of the processes of objectification-subjectification, of the levels of participations and of the level of self-determination experienced by the students. The process of reworking and reflection on the OATLP cycle was carried out in accordance with the principles of Educational Design Research (McKenney & Reeves, 2019) in order to understand and develop how the educational intervention could be adapted to a real classroom context maintaining a connection with the theoretical principles defined above. The model has been implemented in a lower secondary school class of 17 students, among which 4 students have special educational needs, during the mathematics lessons. Throughout the experimentation, the OATLP model has been modified both in its structure and in its time scheduling. The set of every cycle's analysed data consists in videorecording of groupwork, collected students' materials and interviews with six chosen student and with the teacher.

In this contribution we analyse the main transformations of the OATLP model across 3 of the 5 cycles we implemented in school, each cycle is related to a different topic, and we addressed each topic only once. The addressed topics are, from cycle number 1 to cycle number 5: Ratios and Proportions, Circle and Circumference, Pythagorean theorems, Area estimation, Quadrilaterals. The first three cycles introduce a new topic, the 4th and the 5th work more on problem solving in relation to already faced topic.

Circle and circumference

Design of the activities

<u>Objectives in the national indications for mathematics</u>: The student is required to know definitions and properties of the circle and to calculate the area of the circle and the length of the circumference, knowing the radius.

<u>Stations</u> (120 min): Focus on the definition of circle and circumference and the elements that characterise them. The activities are designed differentiating according to different approaches to knowledge (Sousa & Tomlinson, 2011). E.g., in station 1 students have to draw a circle using a rope and a pen, in station 4 they make a drawing representing circle and circumference and in station 5 they are asked to define the figures verbally.

Groupwork (80 min): the students are exposed to a problem that contains 3 questions. The aim is to distinguish the definitions of circle and circumference and to calculate the area of the circle. In the last question they are asked to calculate an area corresponding to ¾ of the area of the circle.

Changes made compared to the previous intervention

Four different roles are assigned to students for the groupwork (mediator, designer, verbalist and controller); a procedure is established to handle the request for help during station activity (three coloured cards representing independent work, need for a classmate, need for the teacher); Stations are made shorter, and their objective becomes more precise and defined. Introduction of a framework defining different individual attitudes to learning that have informed the design of the stations.

Retrospective analysis

Strengths: stations are more effective and differentiated; the cards used to ask for help are appreciated both by the teacher and the students. In group work, according to the teacher, students begin to self-organize and divide tasks among them independently.

Critical issues: tight timeframes scheduled for activities hinder the completion of the OATLP cycle and do not acknowledge the students' need to express their different attitudes to learning; the connection between stations and group work needs to be refined; in groupwork, more attention needs to be given on division of roles and collaboration aimed at learning and not just at completing the task.

Pythagorean Theorem

Design of the activities

Objectives in the national indications for mathematics: The student is required to know the Pythagorean Theorem and its applications in mathematics and concrete situations.

Stations (150 min): Focus on the statement of the Pythagorean theorem, from a 'geometric' point of view (equivalence between areas). Some examples are given in Figure 2

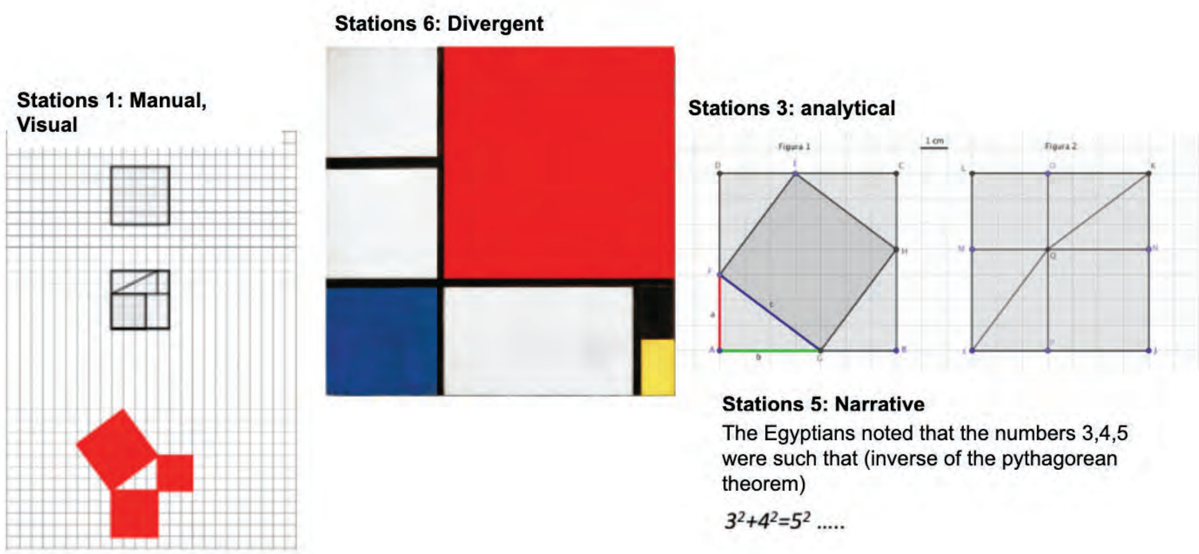

Figure 2: Parts of the stations proposed in relation to the Pythagorean Theorem.

Group (220 min): A double task is proposed: the application of the theorem to a given problem and an activity of problem posing related to the theorem. The groups then exchange the invented problems, solve them, and give feedback to the group that invented it. The feedback must be about the mathematical correctness, clarity, beauty, and interest of the problem. Invention of another problem that can be solved with the Pythagorean theorem is requested at the end of the confrontation.

Changes made compared to the previous intervention

We expanded the timeframe of the OATLP cycle, 2 weeks instead of one; greater attention is given to aspects related to the confrontation among students and a moment of confrontation between groups is introduced; the task for groupwork presents open questions that can adapt to different solving strategies and cognitive approaches; introduction of problem posing activities in the task for the groupwork.

Retrospective analysis

Strengths: Stations are appreciated by students for the variety of attitudes they encompass, and both students and the teacher notice the potentialities of problem posing in relation to the understanding of the Pythagorean Theorem.

Critical issues: insufficient connection between stations and group work, in this OATLP cycle the geometric aspects (stations) and the algebraic aspects (groupwork) of the Pythagorean theorem that needs to be made explicit.

Problems solving on area estimation

Design of the activities

Objectives in the national indications for mathematics: The student is required to determine the area of simple figures by breaking them down into elementary figures, e.g., triangles, or by using the most common formulas and to estimate the area of a figure, including curved lines, by default and by excess.

Stations (150 min): The stations try to emphasize different moments of problem solving: reading and understanding the text, choosing a strategy, implementing it, checking the results, etc. For instance, one station requests to invent a problem starting from the image of a polygon, another to rewrite the text of a given problem before solving it, to make a drawing of the situation, or simply trying to solve a problem and justifying the chosen procedure.

Group (90 min): A problem on the estimation of the area is proposed. Students have to find two different methods, justifying their choices and to explain differences between methods (Figure 3).

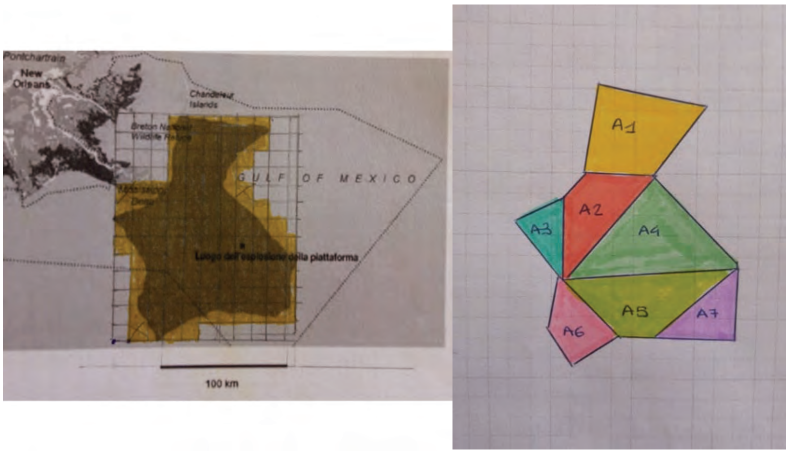

Figure 3: Two different methods implemented by students to calculate the area of the figure.

Changes made compared to the previous intervention

OATLP is designed specifically to work on the ability of problem solving and not in relation to a specific mathematical object like the circle or the Pythagorean Theorem. In this respect, stations and groupwork have a stronger relationship, which can be found in the metacognitive reflection on the different strategies of problem solving.

Retrospective analysis

Strengths: During groupwork students recall the activities done during the stations and build on them to face the new task.

Critical issues: we decide to introduce in the next OATLP cycle a moment of collective resolution (in group) of one of the stations, chosen by the members of the group. This allows students to compare different solving methods and overcome difficulties encountered. Furthermore, we acknowledge the need for a stronger connection between the stations and the groupwork.

Discussion and conclusions

The OATLP cycle is the result of its subsequent implementations with the lower secondary school class involved in the OPEN-MATH project. It is intended as a model that promotes inclusive mathematics education in school. The research project developed in a constant dialogue between the theoretical design principles and classroom activities in a real learning context. In their confrontation, both the theoretical principles and the structure of the model modified: the Theory of Objectification allowed us to focus on multimodality of mathematics education and the relation between subject and mathematical culture. From the theory of objectification come the focus on problem solving and the design of moment of interaction where students are asked to justify their choices and strategies, but also the attention in the choice and role of artifacts in station and groupwork. Didactical differentiation, through Open learning methodology, allowed us to focus on the individual in the process of encountering mathematical culture. In particular, Open Learning allows us to consider the specific learning approaches and bring to the fore strategies to manage the differentiated classroom, for instance through stations, the use of the passport, or the introduction of specific roles during groupwork. These aspects go beyond the specificity of OATLP model and allow to define a set of general design principles that will be a reference for further research on the efficacy of OATLP:

- Alternate modes of work (group, individual, whole class).
- Provide a plurality of semiotic means of objectification and of activities starting from the needs of students in the classroom.
- Work on explication and justification of solving processes and argumentative practices.
- Explain and discuss with students how to help and how to request for help during an activity.
- Encourage effective ways of cooperation and collaboration, for which explicit work with the class is necessary.

The design principles listed and the OATLP cycle are the result of the first phase of the project OPEN-MATH and are the starting point of an interdisciplinary dialogue among the two involved theories: the reciprocal relation of the design principles and the link between each one of them with inclusion in mathematics, and with the specific constructs defining it operationally, will be object of further studies.

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The role of examples in early algebra for students with Mathematical Learning Difficulties

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Recent years have been marked by a growing research interest in students with Mathematical Learning Difficulties (MLD, acronym which denotes specific and/or severe difficulties in mathematics). Most research on MLD has focused almost exclusively on the arithmetic domain, but in recent years, research has begun to taking into consideration other mathematical domains, for example algebra. The present research aims at answering the following research question: What is the role of examples in algebraic thinking for students with MLD? Using Balacheff's typology of proofs, different types of examples are identified: naïve empiricism, crucial experiment, example to spot the regularity and generic example. These examples can be used as tools for observing algebraic thinking in students with MLD and they support the occurrence of algebraic thinking in this population.

Keywords: MLD, mathematical learning difficulties, mathematical learning disabilities, early algebra, role of examples.

Introduction and literature review

Recent years have been marked by a growing research interest in *Mathematical Learning Difficulties* (MLD). This interest, which until a few years ago belonged mainly to the psychological domain, now also concerns mathematics education. An example of this growing interest is the recent creation of *TGW25 Inclusive Mathematics Education – Challenges for Students with Special Needs* for CERME11 in 2019.

The relative youth of this field is reflected in a lack of unanimity on the meaning of the acronym MLD. Some researchers speak of *Mathematical Learning Difficulties*, others of *Disabilities* and others of *Disorders* (Baccaglini et al., 2020). The acronym is therefore used in mathematics education in a polysemic way and to identify different populations (Deruaz et al., 2020; Lewis & Fisher, 2016; Scherer et al., 2016). According to Deruaz et al. (2020), it can refer to students who have been diagnosed with a learning disorder specific to mathematics through a standardised test, usually psychological, and through defined criteria (such as cutoff, etc.). The same acronym can be found to refer to students who have been diagnosed with another learning disorder, not specific to mathematics, but which may have an impact on their learning (e.g., dyslexia or dyspraxia). The term MLD is also used to refer to students who have severe difficulties in mathematics without ever having been diagnosed. The latter category is designated through non-standardised tests, or through teacher assessment.

Despite the wide variety of definitions, they have in common the focus on students with specific and/or severe difficulties in mathematics (Deruaz et al. 2020). This "inclusive" vision of MLD which is not necessarily linked to a medical certificate attesting the difficulties of the students seems to be the most appropriate for mathematics education. Indeed, this research field aims to take charge of all students and their difficulties, regardless of whether or not a diagnosis has been obtained.

Most research on MLD has focused almost exclusively on the arithmetic domain, on number sense and basic arithmetic calculations (Deruaz. et al., 2020; Lewis & Fisher, 2016). In recent years, research has begun to broaden its scope, taking into consideration other mathematical domains, for example equations are included by Karagiannakis et al. (2016) in a battery of tasks designed to detect the causes of difficulties for students with MLD. Furthermore, recent studies in cognitive science have shown that learning difficulties in mathematics are heterogeneous (Fias et al., 2013) and affect several aspects of mathematical skills (Kaufmann et al., 2013). Although there are early signs of interest in other areas of mathematics, these are very rare. This finding shows the need for further research concerning students with MLD, that considers other areas of mathematics. And, as Lewis and Fisher (2016, p.365) say, "in particular, it is critically important that researchers begin to explore MLD in algebra, given its role as an educational gatekeeper".

The last decades of research in mathematics education on algebra have been marked by an interest in *early algebra*, a specific area of teaching identified as a bridge between arithmetic and algebra (Malara and Navarra, 2018). It is a meta-discipline that links arithmetic and algebra and can use mathematical tasks and problems traditionally presented in the arithmetic domain to highlight algebraic processes and algebraic reasoning necessary for a good understanding of algebra (in the traditional sense, Malara and Navarra, 2018). This approach favours the development of a certain way of thinking, *algebraic thinking*, which does not necessarily need the standard algebraic symbolism to be addressed and which is necessary for a proper learning of traditional algebra (Kieran et al., 2016). Malara and Navarra (2018) identify some main language constructs that are fundamental in order to generate new ways to see arithmetic and thus algebraic thinking. One of them is argumentation. Argumentation and its verbalisation are crucial in the approach to generalisation and early algebra. In fact, it fosters students to explicit ideas and procedures of which they were not fully aware before trying to express them. Argumentation and justification make it possible to make explicit an algebraic reasoning that would otherwise remain implicit.

Early algebra therefore seems to be a mathematical domain that is particularly well suited to research about students with MLD because it allows the research to be taken up where it has been left off, at arithmetic, and to bridge the gap with the new mathematical discipline, algebra. Furthermore, it is ideal with students with MLD because it allows algebraic concepts to be tackled without standard algebraic symbolism (which not all students with MLD encounter in their schooling). Although this topic is of great scientific interest, we are currently unaware of any research publications concerning students with MLD in early algebra.

Theoretical framework and research question

The literature review described above led us to become interested in the behaviour of students with MLD in early algebra, wanting to tackle the problem of how we can describe the algebraic thinking of students with MLD. In particular, as we see in the previous section, argumentation and justification are fundamental for algebraic thinking. This consideration leads us to the research question: *What is the role of examples in algebraic thinking for students with MLD?* Our hypothesis is that examples are an opportunity for producing argumentation and justifications and thus showing algebraic thinking of students with MLD. Focusing on examples, we can observe students with MLD using algebraic thinking. On the one hand, therefore, examples serve to study the algebraic reasoning of students with MLD. On the other hand, students with MLD are a good population to study the role of examples in mathematical reasoning since they use examples often. It is important to note that the present research focuses on studying the reasoning of students with MLD, without having the ambition to propose a teaching intervention. This could be an idea for future research that would build on the results of current research to construct the intervention.

To answer the research question, we relied on the theoretical framework of Balacheff's (1987), who created a typology for proofs. The typology offers a classification on the basis of the knowledge involved and the nature of the underlying rationality. From this perspective, the proof is understood as an explanation accepted at a certain point in time by a certain community. There are two main types of proof: pragmatic proofs and intellectual proofs. Pragmatic proofs are action-related and carried out by the students themselves to establish the truth of a certain proposition. If access to this realisation is not possible and the action must be abandoned, we speak of intellectual proofs.

1. *Naïve empiricism* is the first stage of pragmatic proofs. It occurs when the validity of a statement is proved from one or a small number of cases.
2. The *crucial experiment* proves a statement by presenting an example that the student recognises as being as non-specific as possible. If the proposition is true in this case, then it must necessarily be always true. Crucial experiment, which remains a pragmatic proof, differs from naïve empiricism in that the generality of the proposition is taken into account and made explicit.
3. A *generic example* lies on the borderline between pragmatic and intellectual proofs. A proposition can be proved by means of the generic example when a specific case is not treated in its particularity but as representative of a certain family of objects with an argument that can be extended to a whole class of objects. This type of proof consists of proving the validity of an assertion by performing operations or transformations on a particular case, but at the same time use is made of the properties and structure that characterise the class that this particular case represents.
4. *The thought experiment* allows proving by internalising the action and detaching it from its concretisation on a particular representative. By remaining linked to anecdotal temporality, it abandons the treatment of a particular case, as was the case for the generic example.

This typology should not be understood as a tool to assign each student a possible level of knowledge or to identify the cognitive level they are at. It is not a set of successive stages that students must reach in a given order, it is simply a tool for describing students' actions in a certain context in a given mathematical task.

Method

Context and participants

The data presented in this paper has been selected from more extensive research in which more students (19 in total) and more mathematical problems (8 in total) were taken into account.

Data were collected in the canton of Vaud, in the French-speaking part of Switzerland. In this region, interest in school inclusion has grown in recent years. This is evident for example from the creation of the *Concept 360°* (DFJC, 2019), a project which aims to establish the principles of a school that responds to the specific needs of all students. In spite of this growing interest, school organisation is divided into different types of schools (ordinary schools and specialised schools) or school levels depending on pupils' abilities and grades.

The participants were selected based on their severe and/or persistent difficulties in mathematics, as defined by Deruaz. et al. (2020) and described in the previous paragraphs.

In particular, for the research presented in these pages, the students belonged to different classes of lower secondary school (7^{th}-9^{th} grade in Switzerland, 12-14 years old) and have different profiles.

1. Student A (9^{th} grade) is enrolled in an ordinary school, in the level for students with the best grades. Student A has good results in all subjects except mathematics, in which she is considered to be in severe difficulty by her teacher.
2. Student B (8^{th} grade) is schooled in a special teaching class. Student B has a diagnosis of dysphasia and dyslexia and has severe difficulties in all subjects, including mathematics.
3. Student C (8^{th} grade) is enrolled in an ordinary school, in the level for students with low grades. Student C has a diagnosis of dyscalculia and dyspraxia.

Procedure

Data were collected through clinical interviews. This is a semi-directive, open-ended interview between a researcher and a student, whose aim is to encourage the manifestation and observation of mathematical thinking (Ginsburg, 1981). The student interviewed had the task of solving the assigned problem by explaining their procedure. The researcher intervened to ask questions requesting clarification of the procedure used or to unblock the situation in case of difficulty. The aim of the interview was to get the students to show a large number of examples and to progress in their mathematical reasoning. The researcher therefore tried to create the ideal contextual conditions for this objective, by relaying the students' statements to allow them to show their full potential, but without replacing them in finding the answer.

The interviews are filmed and the audio transcribed. The unit of analysis consisted of students' oral and written contributions. These have been analysed on the basis of the typology of proofs (Balacheff, 1987) with a particular focus on the use of examples in algebraic thinking: each sentence said by the students and each written production produced were read and, when relevant, categorised according to a category of the typology.

Mathematical problem

The interview was about solving the following mathematical problem: *If I add two odd numbers, do I always get an even number?*[1]

The chosen problem develops algebraic thinking as it allows us to work on a property of numbers (being odd or even) and on the structure of ℤ (there is a regular alternation of odd and even numbers) by providing evidence for the reasoning carried out. The chosen problem is particularly well suited to answering the research question because the understanding of the statement and the result are facilitated by the mediation of the examples, and examples are not provided directly from the statement but must be created independently by the student, according to his needs.

Results

The analysis through Balacheff's (1987) framework of the clinical interviews allowed us to identify different types of examples created by the students and used to solve the problem.

The first type of example, the *naïve empiricism*, is what the student produces right after starting to work on the problem, it is the first example which lets her begin tacking the problem. For instance, student B starts the problem in the following way:

Researcher: Yes, it requires taking...
Student B: Multiples of two.
Researcher: Two odd numbers. If I take two odd numbers and add them together, do we get an even or an odd number?
Student B: Yes… even… we can… if you do 3+3, it gives 6.

The examples enable the students to start reasoning and to approach the problem that otherwise would remain unreachable.

After the first example, students give other examples, and then other examples, until generality is taken into account and explicitly evoked. For example, student B continues his reasoning:

Researcher: What do you think, if I add an even number plus another even number, how will the result be?
Student B: Even.
Researcher: How do you know?
Student B: Well, if you add 12 plus 12 that's 24 and if you add 14 plus 14 that's 28 and it's always even, otherwise 4 plus 4 is 8, 6 plus 6 is 12, 8 plus 8 is 16, it's always even.

This list of examples ensures the generality of the statement. There are so many examples that, for the student, this is enough to support the generality of the statement. We call this the *crucial experiment*. The crucial experiment may also be given by a single example which in the view of the students is so unspecific that if the statement is true in that case, then it is always true. What is important for the crucial experiment is that the generality of the situation is evoked, in this aspect the crucial experiment is different form the naïve empiricism.

[1] The mathematical problem and transcriptions were translated into English by the author of the text from the original French version.

This iteration of examples leads the student to spot the regularity of the situation and to understand that this regularity has a motivation. The student evokes the fact that the regularity can be identified, understood and generalised. Student A says:

Researcher: We want to understand whether if we add two odd numbers, we always get an even number. Do you remember what an even number is, and an odd number?

Student A: Yes, of course I do. Well... by giving examples. For example, 3 plus 3 is 6. 3 plus 7 is 10. 3 plus 11 is 14. So yes, I think it will always be like that, because there is something logical behind it. In any case, I don't have an example that comes to mind where it wouldn't be possible.

Here the students take into consideration the generality of the situation and she does something more: the example let her see that the regularity has motivations and can be understood and explained. It is “logical”, as the student says. The example generates in the student the need for regularity. With respect to Balacheff’s (1987) typology, his example, *example to spot the regularity*, is not a new category but it is between the crucial experiment and the generic example. It has some characteristics of the pragmatic proofs (it is linked to the action on a limited number of particular cases) and some other characteristics of the intellectual proofs (not only is regularity evoked as in the crucial experiment, but the existence of a logical structure behind this regularity is also evoked).

Examples can also be used to generalise the regularity through a *generic example*. In this case, the example is not treated in its particularity, but as representative of a certain family of objects. The generic example uses the characteristic properties and structures of a certain class of objects by relying on one of its representatives for the implementation of the reasoning. For instance, student C writes $7 + 7 = 14 : 2 = 7$ (Figure 1) and says:

Researcher: Okay, so the question here is... When you add two numbers that are odd, is the result even or odd?

Student C: Yes, okay.

Researcher: Do you have any ideas?

Student C: Well, I could take, for example, 3 which is odd plus 3. That's 6, which is even because if you do 6 divided by 2, it's 3. Then if I take 5 plus 5 which is odd, it's 10. Which is also divided by 2. 7 plus 7, 14, which is also divided by 2. Well, yes, because each time you make an odd number plus another odd number, the same one (indicating 7 on his example), it gives this number (indicating 14) and this number is even because each time, you can divide it by 2 to give 7... well, to give the (indicating 7)…

Here the student is proving that every time that we add two times the same odd number, the result can be divided by 2, so it is even. He solves a particular case of the given problem. He uses the example “7 plus 7”, to support a general reasoning and this particular example is necessary to produce a reasoning that otherwise he couldn’t have had.

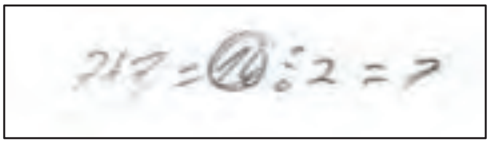

Figure 1: Generic example given by student C

It is interesting to note that this example falls into the generic example category because in the words used by student C we can find references to the particular case (“it gives *this* number”). And also, the gestures (e.g. "*indicating* 14") show that the reasoning is based on the particular case reported in Figure 1. A less thorough analysis could have suggested a thought experiment.

Discussion

The results provide some first answers to the research question: *What is the role of examples in algebraic thinking for students with MLD?*

First of all, the results show that different examples can be used for different purposes (Table 1). They can be used for the student herself, with an *internal purpose*. Examples are essential for the exploration phase: understanding the statement and getting an idea of the results, taking ownership of the problem, entering in reasoning (naïve empiricism), conjecturing (crucial experiment), to spot the regularity of the situation. Examples are also used for solving and proving the conjecture (generic example).

Table 1: The typology of examples in algebraic proofs

Naïve empiricism	Crucial experiment	Example to spot the regularity	Generic example
The first example which lets the students in the problem and in reasoning	The generality of the situation is evoked	The situation is recognised as regular	The example is used as representative of a family of objects

In addition to the internal purposes, examples can also be used in *interaction with others*, to communicate the results obtained.

Table 1 shows how the examples generated by the argumentation promote the generalisation of the mathematical problem by inducing a tendency for students to think algebraically.

Examples can have different status and different roles in algebraic thinking and proving; they can be used as tools for reasoning and for producing algebraic thinking. Examples are particularly important for students with severe difficulties as students with MLD because in this case they support students' thinking, algebraic thinking, which would not be possible without them.

The results show that students with MLD showed traces of algebraic thinking, despite their severe difficulties in mathematics. This is particularly interesting taking in consideration that our sample is also composed by a student who attends a special class, where students rarely encounter certain advanced mathematical topics as algebra.

With students with severe difficulties, pushing towards simplification and meaninglessness in favour of technique is not indispensable, nor is it always fruitful. The results of this research show that proposing problems that make use of *algebraic thinking* is possible.

The study presented here is part of an ongoing research project which macro-objective is to understand if and how students with MLD manifest algebraic thinking. We will carry out further analyses in order to describe the algebraic thinking of these students in more detail.

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Inclusive math practices in primary school

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In the last 10 years inclusive settings in mathematics have increased in German primary schools. In essence, inclusive mathematics education means giving all students equal opportunities for mathematical education, regardless of their subjective learning requirements and learning potential. In our research project we reflect on the one hand central design principles for developing substantial teaching units for inclusive classes. On the other hand, we analyze the emergence and development of practices focusing on the teaching processes of thematizing and negotiating mathematical contents. For this purpose, 22 class discussions from lessons were examined with regard to the reconstructable practices.

Keywords: Inclusion, practices, natural differentiation, multiplication, discourse.

Introduction

"Inclusive schools aim to involve all learners in quality learning experiences which empower them to become active participants in a more equitable system" (Scherer, et al., 2016, 640). Inclusive education in math classes aims at breaking down barriers and creating universal approaches to math learning. Regarding mathematics education, the main aim is to enable all children developing basic mathematical competences in interaction with others.

Even though the primary school in Germany traditionally sees itself as a school for all, the teachers (children, parents, …) are shaped by a system that is traditionally and currently geared towards segregation and is changing very slowly to an inclusive school. In subject didactics, an understanding of inclusion seems to be established in everyday teaching that is based on a one-sided deficit conceptualisation of the concept of inclusion and reduces inclusion to "compensatory support measures" for precisely those children who seem to have deficits. But inclusive education does not mean that only few children need special support. Rather, inclusion means paying attention to each individual person with his or her individual prerequisites and potentials. Different children cannot and do not have to achieve the same goals. Inclusive education demands from all children exactly what they can achieve. Inclusive teaching engages children to bring in their own needs and interests. They are also allowed to set their own priorities within a certain framework and develop a personal educational profile through their chosen topics and subjects. The diversity of children is also reflected in the fact that they learn differently — at different speeds, in different ways and with different prerequisites. Therefore, inclusive mathematics teaching aims to offer learning opportunities that are individually adaptable and enable mathematics learning in community with others, regardless of competences, learning requirements, interests, and development potentials (Scherer et al., 2019).

In our paper we focus on the ongoing practices in inclusive math education with respect to the research question addressed: “Which practices characterise the plenary in introduction and reflection and how far do they correspond to the normative aims?”

Theoretical framework

In many cases of everyday math teaching in inclusive classes, however, teachers cope with the heterogeneity of the children mainly by using individualised working materials organized in station work or plan work. This organisation at first glance promises differentiated learning progress for each child, as well as optimal development of potential: each child works on a prescribed (sometimes also designed by the child) learning plan at different places and with different levels of support. The children's learning activities only take place side by side. This poses the risk that the learning quality of the lessons is reduced and that the children cannot optimally develop their potential. But mathematical learning processes are particularly dependent on content-related social negotiation processes. Therefore, children need to share mathematical discoveries and to present one´s own reflections as well as to communicate and explain them to others (Steinbring, 1997). This is not possible, if children have to learn more or less on their own.

Consequently, there is a need for the design of substantial teaching materials and for the research of practices of math teaching and collaborative math learning processes in inclusive settings.

Principles and learning situations of inclusive math education

The intertwine dimensions of personhood, sociality, and complexity link subject-specific and social participation. In recent years, different guiding principles have emerged for the design of inclusive mathematics teaching (Häsel-Weide & Nührenbörger, 2021). We refer to four design principles that aim to consider the different potentials of the learners and to connect to them in a targeted way:

(1) accessibility to the common subject matter for all children,

(2) subject learning at different levels,

(3) active exploration of content connections and finally

(4) initiation of common learning phases for all children with social negotiation processes of communicating, representing, and arguing.

Reflecting these four principles substantial learning environments and the task formats on which they are based (Wittmann, 1995) are a basis for the design of mathematical learning situations in primary schools. The environments consider the idea of differential sensitivity, i. e. reflective perception of the heterogeneous competencies of children in a concrete learning situation. The influence of academic and content language for the understanding of mathematical concepts is highlighted as well as the use of appropriate material.

But fundamental for successfully initiated joint learning processes are not only substantial task formats, which focus on the basis staff. The support of teachers seems to play a special role for the learning processes. Especially children with difficulties in learning mathematics seem to be dependent on impulses that support the solution process (Korten, 2020). These challenge them to think about mathematical patterns and structures as well as their own and others' thought and solution processes. But not all impulses do this in the same way. Pfister et al. (2015) investigate the scaffolding processes of teacher in inclusive classes. They see differences in the way teachers stimulate the interaction and active the children regarding to the mathematical subject. The video study of Krähenmann et al. (2019) shows, that teachers have difficulties in creating common and at the same time differentiating learning situations. Teachers either provide common but not differentiating learning opportunities for a very heterogeneous learning group or differentiating learning opportunities where joint learning did not take place.

Social and subject-matter practices

Practices are established as a "theoretical construct that allows us to talk explicitly about collective mathematical development" (Cobb, 1998, p. 34). Characterising inclusive practices, a distinction can be made between normative pedagogical practices and the practices that can actually be reconstructed in the observation of teaching. Normativ inclusive practices are e.g. formulated in the Index for Inclusion and describe how practices should be. The dimension of "developing inclusive practices" contains the following three aspects: (1) "Lessons are planned with the diversity of students in mind", (2) "Lessons strengthen the participation of all students" or even (3) "Students learn together" (Booth & Ainscow, 2002). "Mathematical practices include problem solving, sense making, reasoning, modeling, abstracting, generalizing, and looking for patterns, structure, or regularity" (Moschkovich, 2015, p. 1068).

These practices can be distinguished from concrete teaching practices, which Hirschauer (2016) describes as "ways of doing". He defines 'practice' as a physical consummation of social phenomena such as "types of activities, ways of acting, patterns of behavior, forms of interaction" (Hirschauer, 2016, S. 46; transl. by the authors). Practices are analytical units and describe structures, customs, or things themselves which influence teaching and shape learning.

Design of the Study

In our project IGEL M (Inclusive practices in shared learning opportunities in mathematics) we develop existing learning environments further by conducting didactic teaching-learning experiments for inclusive learning settings and by analyzing mathematical learning processes from a qualitative perspective (Häsel-Weide & Nührenbörger, 2021). We focus on the mathematical practices in inclusive lessons. The analysis follows an interactionist perspective, focusing on the classroom microculture and mathematical practices.

We accompany classes during their time in primary school and visit them 4-5 times a school year. In cooperation between the teacher of the class and the authors of the paper we plan learning environments, select material and develop further. The lessons, carried out by the teacher, are videographed and used to a) answer the research questions as well as b) to reflect jointly the quality of the learning. Actually 22 lessons are videographed, each subject was realized in two consecutive lessons. In detail, we tackle three research questions in the project, but in the following, we focus on the first question:

RQ1) Which practices characterise the plenary in introduction and reflection?

RQ2) Which practices characterise the collaborative work of children?

RQ2) Which mathematical understanding can be reconstructed? How far differences the understanding?

The reconstruction of the practices bases on the videographed teaching and learning situations. Corresponding transcripts were interpreted by a group of researchers finding different typifications. The interpretation of the negotiation processes follows four analytical steps: (1) Video scenes are transcribed. (2) In discursive analyses of the researchers involved, these are paraphrased and interpreted by means of turn-by-turn analysis. (3) In discourse, plausible typifications of practices are elaborated by categorising patterns of interaction or activities. (4) The practices are compared comparatively with other analyses of other scenes and examined in a generalised way (Voigt, 1994).

Analysis of an episode

Learning environment

This paper is about a common task to explore the distributive relationships between so-called easy and more difficult tasks. In essence, the students first must recognise the doubling tasks (multiplier 2) and solve the neighboring task (the multiplier 2 is increased by 1) with the help of this (see Figure Fig. 1).

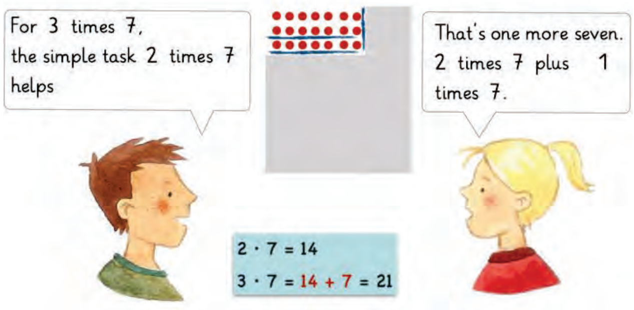


Fig 1: Neighboring tasks. Deriving a solution for a simple task for calculating a difficult task (Nührenbörger et. al., 2017; Illustration by K. Mosen, PIKAS)

The basic material for all students is therefore not limited to recognising simple multiplication tables, but to exploring the structural relationships. Children with mathematical learning difficulties have great issues in seeing the structure, understanding the operation, and deriving results. According to the idea of sensitivity of different competences, this must be considered in inclusive classes. For this purpose, both representation-sensitive and language-sensitive supports are offered to enable access to the common learning object. For example, students can show the simple painting angle task on the 100 field and then move the angle down one line. Moving the painting angle down represents increasing the multiplier by 1, so that the product increases by the multiplicand once.

This multiplication relationship can be supported linguistically by picking up on speaking in groups (see example of students) and by verbalising the move of the angle. If not all students in an inclusive class are already working in the number range up to 100 or if, for example, the whole multiplication table still seems too complex, the distributive relationships can also be explored in qualitatively differentiated tasks with a structurally reduced field (the 25 field, consisting of 5 times 5 points, see Figure 2).

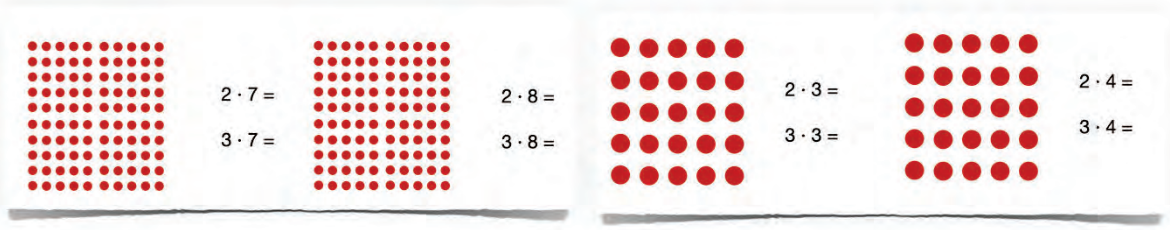


Fig 2: Neighboring tasks at the 100 field and 25 field

In an inclusive class, these tasks can be worked on individually. On the other hand, the students can also work cooperatively in pairs on such a task: For example, one child looks for and shows the easy task, the other notes it down. Then the more difficult task is derived from it by shifting the painting angle and noting and calculating the task in relation to the previous one.

Plenary practices

At the beginning of the lesson, the teacher highlights the aim of the lesson and links it to previous related topics (see transcript). She reminds the students that simple tasks have already been used in addition. The teacher also points out that the result should be found with the help of simple tasks, not by counting.

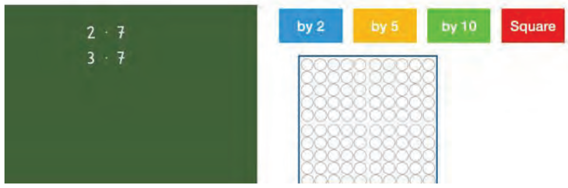


Fig 3: Blackboard picture for the neighboring task

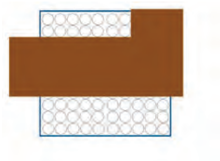
Teacher: Today we calculate difficult multiplication tasks. You may remember from the plus tasks, we did that too. If you can do a simple multiplication task and you know it well, it will help you to solve a difficult task. Then you don't have to start counting or calculating all over again. Yes? And today, I want to show you how this simple task can be a help. Ok? (*goes to the blackboard and points to the tasks 2 - 7 and 3 - 7*). If you have two tasks, think for yourself which of the two tasks is a simple task. Josie.

Josie: Two times seven

Teacher: Why? (*she marks the task 2 · 7 with an x*)

Josie: Because it is a task by two

Teacher: Ok. Good.



Josie: Shall I tell the result?

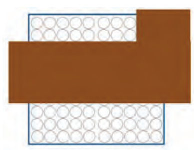
Teacher: I will mark first and then you can tell me the result, ok? (*she takes the multiplication-angle and a pen*). How many are in a row? Frederic

Frederic: Seven

Teacher: (*puts the multiplication-angle on the dot field)* Josie, now.

Josie: Fourteen.

Teacher: So, now we have to think. The next task is called what? The difficult multiplication task. Lea.



Lea: Three times seven

Teacher: What do I have to do to get three times seven?

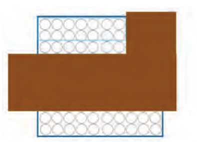
Lea: One to the right

Teacher: (*moves the angle; a couple of children raise both arms*)

Lea: (names a child)

Child: Now the tasks are called two times eight. You have to move one to the left and one down.

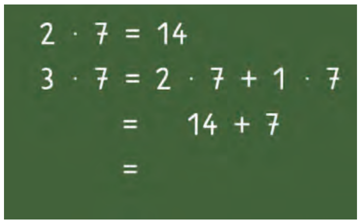
Teacher: So you mean I should go back first and then down. I'll use a different colour, I think you can see it a bit better then. What has been added? (*points on the third row of the field*). Doreen.



Doreen: Seven once again

The teacher always switches between different the symbolic and verbal form of task presentation and the iconic dot field. Josie must wait before she is allowed to give the answer, because the task should first be presented in dot field. Even after the difficult tasks has been solved, she asks the children to look again to the added quantity on the dot field. In doing so, the teacher highlights the connection between the representation. This explicit connection important especially for children with difficulties. The teacher asks the children how the angle must be shifted, too. Leading the conversation, the teacher picks the children. Most of them put up their hands. With this practice she decides which children in participate in the classroom conversation. But the practice “veto” differs from this routine. If the children do not agree with an answer, they rise both hands. Then the student (Lea), who gave the disputed answer chooses someone, to put forward their argument. Similarly, the practice “help” works, which can be reconstructed if children could not solve the tasks or answer questions. Asking for help they pick up a classmate who answers for them (Jonas):

After both tasks are solved and both results are called, the teacher initiates a discussion how to calculate 14 + 7 without counting. Jonas probably remembers the solution already given but had difficulty formulating his way of calculating.



Teacher:	Ok. How do you calculate? You want to tell me the solution right away, don´t you? How much is that? Do you know, Jonas?
Jonas:	Twenty-one
Teacher:	Can you tell me how you can calculate this cleverly without counting. (..)? How far do you calculate first? (..). How did you calculate it? (...) Do you know? Would you like to get some help?
Jonas:	Ahmad.
Ahmad:	First I calculated fourteen and then I added seven.

The teacher seems to be aware that there may be a difficulty in adding the multiplicand to the product of the simple task. So, she asks for a clever, non-counting strategy to solve the addition. She picks up Jonas, who mentioned the result, but did not explain the process. Probably he remembers the solution, which has already been mentioned in the interaction. Jonas asks Ahmad for help, who himself mentions the task without explaining a clever strategy for calculating. Bruno explains later: “You take fourteen plus six, then it's twenty. Then you have to add one more because there was one left and then it's twenty-one”.

In summary, the plenary is characterised by the teacher's effort to connect easy and difficult tasks in an understanding-oriented way. The teacher also addresses skillful calculation strategies in addition. The new subject is connected with past subject. These practices address children with difficulties and allow them to participate without making the address explicit. But, the teacher uses (only) an example for the class discourse with fits to the regular stuff and probably overtaxes children, who work in the reduced field. However, the simple task chosen is one with 2, which should already be familiar to all children as a doubling task.

Results

The practices that are evident in this selected scene are typical of inclusive math practices that could be reconstructed in plenaries at the beginning of inclusive class (Häsel-Weide & Nührenbörger, 2021). Those discourses are characterized by the following practices:

Presentation of the subject

- *Explication of the subject*: The subject and the aims of the lesson are explicated and, in many times, classified as simple, basic or elaborated. This may give orientation for all children in an inclusive class, but could also categorise different children regarding to the used material.
- *Presentation of the subject in different forms*: The didactic-normative inclusive practice of presenting a mathematical object on different levels of representation is established as a basic practice for all pupils and all subjects.

Initiation and moderation of subject-related discourses

- *Initiation and decision of participation by the teacher:* The participation of different children is initiated and controlled by the teacher, picking up students in connection with explicitly formulated questions and impulses.
- *Self-responsible participation practices*: The practices of “veto” and “help” were agreed upon in the class between teacher and children as conversation practices and now moderated by the children themselves. So, they take responsibility for the joint learning process. These practices include the opportunity to make alternative interpretations or to pick up and explain mistakes. Children with good mathematical understanding thus remain involved in the conversation and children with difficulties have the opportunity to decide for themselves when to ask for help. All learners in the inclusive class are involved, empowered and supported.
- *Creating subject-related discourses*: The conversation is condensed and directed towards mathematical aspects that the teacher considers important for all learners. Nevertheless, the children are asked to argue and to verbalise freely.

The reconstructed inclusive mathematical practices move in the field of tension between multi-layered-structural explorations and discursive discussions on the one hand, and condensed, focussed treatments on other. In the analysed scene, no obvious assignments of level are made, nevertheless, a closer analysis shows corresponding to other studies (e.g. Straeler-Pohl, et. al. 2014) that higher-performing children are asked for explanations and help by the teacher, but also by their classmates. Lower-performing children are just only asked to help with routine tasks or tasks to secure the basic material. In this sense, the aim that “all students are empowered to engage meaningfully in mathematical practices, for such engagement is the source of agency and identity” (Schoenfeld, 2020, p. 1173), has not yet been fully achieved.

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Low-achieving secondary students learn mathematical problem solving – A longitudinal, qualitative video study

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Mathematical problem solving is hard to learn and difficult to teach – esp. for classrooms including low-achieving (LA) students. We investigate what difficulties LA students encounter when solving problems and how their problem-solving behavior develops within a period of nine months. Within a broader project, three pairs of LA students have been videotaped four times over a period of nine months. For this contribution, the video data of the first and fourth collection were analyzed. To do so, each problem-solving process was coded using four different codings: episodes, heuristics, self-regulatory activities, and success. We identified three main difficulties that students encounter: basic knowledge, stagnated development of heuristics, and a lack of self-regulation.

Keywords: heuristics, problem solving, resources, strategy keys, strategy prompts.

Introduction and theoretical background

In many countries around the world (as in Germany), becoming competent in solving problems is an integral goal of mathematics curricula (Liljedahl & Cai, 2021). Students are supposed to acquire strategies, apply those flexibly and, thus, master solving problems (NCTM, 2000). Yet, problem solving itself is a very complex process which is hard to learn (Schoenfeld, 1985). This applies especially for low-achieving (LA) students and students with learning difficulties (LD) (Montague et al., 2011). The situation in schools becomes even more delicate as teachers report that "teaching large heterogeneous groups of students with varied academic level" is stressful (Shernoff et al., 2011, p. 65). Again, if students show difficulties within "regular" mathematics lessons, they can be expected to show even more differences when working on non-routine problems.

Our review of the literature reveals a lack of studies regarding problem solving of LA students and students with LD with the exception of word problems for which it is known that "[s]tudents with LD are characteristically poor problem solvers. They typically lack knowledge of problem-solving processes, particularly those necessary for representing problems and, therefore, need to be taught those processes explicitly" (Montague et al., 2011, p. 263). At the same time, these students lack proper strategy use; instead, they "infrequently abandon and replace ineffective strategies, rarely adapt previously learned strategies, and typically do not generalize strategy use across domains" (Montague et al., 2011, p. 263). Yet, word problems often do not initiate non-routine problem-solving processes (Liljedahl, 2021, p. 25).

Thus, if it is that difficult for LA students and students with LD to master word problems, it will be even more difficult for them to work on non-routine problems. At the same time, these students should learn to solve problems as this is an essential skill for learning in the 21st century.

Research interest and questions

Here, we refer to problem solving as working on *non-routine* problems. We report about a longitudinal, qualitative study that addresses the learning of mathematical problem solving within regular math lessons including LA students, that is students who “experience mild but persistent learning difficulties” (Geary et al., 2012, p. 206). We investigate the following questions: (1) What difficulties do LA students encounter when solving mathematical problems? and (2) How does their problem-solving behavior develop over a period of nine months?

Design of the study

We conducted an intervention study involving 162 students (aged 11 to 14 years, grades 5 to 7) in Germany. All students are from a regular comprehensive school with all ability levels; hence, there is a representative selection of the early secondary students – also inclusive and integrative students. We gathered the students’ math and German grades and used these to group the students into high, middle, and low achievers. We asked their teachers to verify this grouping based on their experience. A standardized maths test was not carried out within this study.

The design of our study is based on three current and well-established approaches for supporting teachers and students within very heterogeneous classrooms: (1) “allow students to work on mathematical problem solving tasks in pairs or in small groups”, (2) “provid[e] all students with high-quality mathematics instruction [by] present[ing] the whole class with the same low-floor, high-ceiling, open-ended, problem-solving task”, and (3) “teach mathematics through challenging task, and use enabling and extending prompts to differentiate instruction” (Russo et al., 2020, p. 49).

Over a period of nine months, the students encountered mathematical problems on a regular basis in their math lessons (approach 2). This is not only part of the national curriculum but also part of the intervention. To support the students, they were introduced to so-called *strategy keys* – these are cards in the shape of keys with one general heuristic written on each key, used to trigger the use of heuristics (approach 3: strategy prompts). We refer to heuristics as problem-solving strategies that students can implement when working on a problem, including “mental operations typically useful for the solution of problems” (Pólya 1945, p. 2). That strategy keys are effective in activating the use of heuristics and fostering self-regulatory activities has been shown empirically by Herold-Blasius (2021) in a project with mathematically interested 3rd and 4th graders. In the study at hand, the keys were introduced to the students within one introductory lesson; thereafter, they were allowed to use the keys whenever they “got stuck” within their problem-solving processes.

Within this project, qualitative data was collected at four different points of time. Thus, 30 pairs of students (approach 1) were videotaped and asked to think aloud while working on non-routine mathematical problems. After finishing the problem-solving processes, they were interviewed and their solution sheets were gathered. The chosen problems were structurally similar throughout the entire study, mainly arithmetic and sometimes geometric. This way, we aimed at comparing the processes more objectively than by using different problems each time.

Due to different reasons only eight of those 30 pairs were constant in their composition at all points of data collection. For the contribution at hand, we filtered the three low-achieving pairs of students.

Their problem-solving processes of the first and last video collection as well as the matching worksheets are focused on in this paper. Using this data, we illustrate how the encountered difficulties and if problem-solving skills develop in 9 months.

Two mathematical problems of the study are used as basis for the paper at hand:

- *Seven Gates*: A man picks apples. On his way to town, he has to pass seven gates. At each gate stands a guardian claiming half of the apples and one apple extra. At the end, the man has only one apple left. How many apples did he have at the beginning? (Bruder et al., 2005)
- *Traffic Jam in Space*: In order to reduce the amount of traffic between galaxies in space, the Intergalactic Council has decided to impose customs duties on commercial travelers. Each time a trader changes galaxies, he must now give up a quarter of his goods, plus one more piece. Plutix distributes toxic-green paint. Arriving at his destination, he still has 23 buckets after having to stop at the intergalactic customs three times and hand in goods. How many buckets did he have loaded at the beginning of his journey? (Collet, 2009)

Both problems are similar in their construction: Someone has to pass a certain number of steps (gates or duties) at which part of an initial amount of something has to be paid – either half or a quarter and one extra. In both problems, the amount at the end is given, so working backwards is the most appropriate strategy. However, other heuristics can either support the working backwards (e.g., aid to systemize or a figure) or open other approaches (e.g., working forward).

Methods – Coding of the data

Each problem-solving process is coded in four different ways in order to describe the problem-solving process as detailed as possible: 1) episodes within the process, 2) heuristics used in the process, 3) self-regulatory activities within the process, and 4) success of the process. Codings 1), 2), and 3) are process-oriented (i.e., using video data), whereas coding 4) is product-oriented (i.e., using the students' written worksheets). Each coding will be explained in the subsequent sections.

Process coding – Episodes: Schoenfeld (1985) defined an episode as period of time during which an individual does the same. He identified six types of episodes (see Table 1 for types relevant here).

Process coding – Heuristics: Appearances of heuristic techniques, such as drawing a figure, generating examples, or trying systematically, were coded using a manual based on ideas by Koichu, Berman, and Moore (2007) (cf. Rott, 2013). Codes relevant for this paper are specified in Table 2.

Process coding – Self-regulation: To code self-regulatory activities within problem-solving processes, we used Schoenfeld's (1985) differentiation between local and meta assessment. "*Local assessment* is an evaluation of the current state of the solution at a microscopic level". (Schoenfeld, 1985, p. 299) So, whenever students check their results, local assessment occurs within a problem-solving process, (see Table 3). *Meta assessment* did not occur within the given data and will not be addressed any further here. Additionally, we identified self-regulatory activities is the *interaction with the strategy keys* (cf. Herold-Blasius 2021, showing that using keys can be interpreted as an act of self-regulation). This is because students realize that they need help when they get stuck and they know where to get this help from – the strategy keys. Also, several pre-, while- and post-actional activities of self-regulation could be identified when interacting with the keys (Zimmerman, 2000).

Code	Description
Reading or rereading	The problem is read or re-read. It also contains the silent period after reading.
Analyzing the problem	The problem solver tries to understand the problem, e. g. by thinking about what is sought and what is given.
Exploring	The problem solver investigates the problem which might be more or less chaotic.

Table 1: Codes to identify episodes

Code (Abbreviation)	Description
Sought and given (S & G)	Checking for information given in the problem and what is searched for
Aid for systematization (AfS)	Introducing structuring elements that help to carry out activities
Working backwards (WoBa)	Focusing the aim of the problem/the sought and working from there to the starting point; sometimes mathematical operations must be reversed

Table 2: Codes to identify heuristics

Code (Appreviation)	Description
Local assessment	Evaluation of the current state of the solution, e. g. check results
Interaction with strategy keys	Students read, touch, discuss about the keys or choose a specific key

Table 3: Codes to identify self-regulatory activities

Product coding: Solution

The students' solution sheets, thus, products, were graded in four categories of success:

1. *No access*, when they showed no signs of understanding the task properly or did not work on it meaningfully.
2. *Basic access*, when the pupils mainly understood the problem and showed a basic approach.
3. *Advanced access*, when they understood the problem properly and solved it for the most part.
4. *Full access*, when the pupils solved the task properly and presented appropriate reasons, if necessary. (Rott, 2013, pp. 115–116)

Results

Here, we present the problem-solving processes of three pairs of LA students that worked on the problems Seven Gates and Traffic Jam in Space (1st and 4th video recording, resp.).

K & M: Seven Gates (1st video collection)

During K & M's work on the problem Seven Gates, they spent most of their time in exploration phases. This indicates that both girls did not follow a specific plan.

Concerning heuristics, M & K *reread* the Seven Gate problem and applied solution methods as if the problem were a *routine task*. This means they used all given numbers and put them in some (possibly random) mathematical relation:

M: Can we not just do seven times seven because it's seven gates and then plus seven because another apple for each? [writing: $7 \cdot 7 = 49; 49 + 7 = 56; 56 + = 57;$ *"The man picked up 57 apples."*]

Self-regulatory activities could not be identified and their solution was coded as basic access.

Combining, it seems as if K & M did not understand the problem properly. They understood that there are seven gates and seven guardians and that there is one apple left. However, they could not bring this information together and, thus, treated the problem as if it were a routine task.

K & M: Traffic Jam in Space (4th video collection)

During the problem Traffic Jam in space, both worked within an exploration phase. Concerning their heuristics, they *reread* the problem – as they did in the Seven Gates problem. Yet, they also used an *aid to systemize* as they first drew and later numbered the gates (1, 2, 3). Also, they talked about how they want to proceed and decided that *working backwards* to be an expedient approach. Again, self-regulatory activities could not be identified and their solution was coded as basic access.

Concluding, K& M understand the problem much better, talk about possible heuristics to use and find helpful mathematical operations to work on the problem. Still, to solve the problem correctly they need knowledge about fractions – which they should have but did not show or seem to have. So, when solving mathematical problems, it is not sufficient to understand the problem and to select an appropriate heuristic. It is also necessary to bring along adequate mathematical knowledge.

M & F: Seven Gates (1st video collection)

M & F read the problem and analyze it. M has an idea to calculate straight away (implementation):

M: There is one [apple] left. If they. Double of one is two. Then an extra apple is three.
F: That's the second gate.
M: If we had six and one it would be seven. So, we had two gates.

M has an idea; F follows it and both carry out their plan. Concerning their heuristics, they *work backwards* and use an *aid to systemize* the number of gates by numbering them. At half of the process, F checked the results and figured that doubling the number of apples and then adding one does not work when working forward. However, this self-regulatory activity does not lead to a change of calculation. Still, with 255 apples as their final result, they have a solution coded as advanced access.

Summing up, M & F talk with and understand each other. It seems to be sufficient if one of them has an idea because the other one follows. They are able to select appropriate heuristics and even show a self-regulatory activity. Yet, only recognizing that something might be wrong does not make them change their solution path. Hence, they might be skilled with a seed of self-regulation, that might not have been developed fully yet (Herold-Blasius, 2021).

M & F: Traffic Jam in Space (4th video collection)

After reading this problem, both recognize that they have worked on a similar problem before. This might be one reason why M decides to *work backwards* and to draw the customs stations (*aid for systematization*) right at the beginning. Their first idea is to add one to 23 and then to add one quarter of 24 to 24 which is 30. However, they check their results (self-regulatory activity) and recognize that their calculation does not work – having 30 and taking away one quarter and one extra. At some point (after 11 min) M decides to just do the calculations – without thinking about it anymore. At the end (after 17 min), they explain their strategy use explicitly:

M:	I have calculated 62.5 divided by 4.
Interviewer:	62.5 divided by 4. And why?
M:	Because when you work backwards you can also do it forwards. This way you can check it.

To conclude, both are able to select heuristics and to check their result (self-regulatory activity) – in both processes. Yet, they cannot use this to change their procedure and do not find the correct result. Within the nine months, they did not learn how to change their working backwards in a productive way. Possibly, explicit teaching could have helped here to develop this self-regulatory seed.

F & N: Seven Gates (1st video collection)

F & N start their work by analyzing the problem and roughly estimating a possible result.

N:	There must be an awful lot of them.

They then try to solve the problem within an exploration phase. In terms of heuristics, they looked for the *sought and given* first and *reread* the problem after about five minutes. Shortly after, they interacted with the strategy keys (self-regulatory activity) which led them to a new strategy – *working backwards*. Later they combined this heuristic with an *aid to systematize* to keep track of the number of gates. Their entire process took 14 minutes and F & N's solution was coded as advanced access.

In F & N's process it was noticeable that they extensively discussed the order of operations – first addition and then multiplication or vice versa. At the end, the writer determined the order. Also, after interacting with the strategy keys, they employed the heuristic *work backwards*.

F & N: Traffic Jam in Space (4th video collection)

After reading the problem they analyze it and then work on it. Concerning heuristics, they *reread* the problem, looked for the *sought and given* and used an *auxiliary element* by marking relevant information in the given text. After 7 minutes, they started *working backwards* and later drew some *figures* visualizing the quarter. After 34 minutes, they again used an *aid to systematize* to keep track of the number of gates. Their process lasted ca. 42 minutes and their solution was coded as advanced access.

Both students interact with the strategy keys at three points within the process (self-regulatory activity). However, they did not find a productive way of integrating them.

F & N's problem-solving process is characterized by showing seeds of purposeful problem-solving behavior: First, they mark relevant information but then do not really use it within their solution process. Second, they talk about the order of operations; yet, F – the writer – determines it without N arguing against it. Third, they use circles as visualization of the quarters. Still, this is not used for their calculations or any progress within the problem-solving process. It is remarkable that both calculate with approximated numbers. So, it seems as if their conception of numbers and fractions is suitable for everyday use. Hence, their mathematical knowledge seems to be at a sufficient level.

In conclusion, F & N show similar behavior e.g. by working backwards (key: work backwards). At the same time, they use new behavior such as marking information (key: use different colours) or drawing figures (key: draw a picture). Possibly, both students have internalized the strategy keys and can now employ heuristics without even looking at the keys. They have expanded their heuristics repertoire and internalized heuristics, although this does not lead them to correct solutions yet.

Discussion

For this contribution, we observed the problem-solving behavior of three pairs of LA students over a period of nine months. We showed what kind of difficulties LA students encounter when solving mathematical problems and how their problem-solving behavior develops over the given period.

Three obstacles have been identified: lack of knowledge, stagnated development of heuristics, and lack of self-regulation. First, students (see K & M) need a certain level of mathematical knowledge to properly engage in problem solving. This result is not surprising as Schoenfeld (1995) had already stated this. Yet, there is no concept on how a lack of mathematical knowledge could be compensated using aids without revealing too much – and this is crucial especially for LA students. Students need sufficient knowledge or to develop "the ability to retrieve basic facts from long-term memory" (Geary et al., 2012, p. 206) in order to experience a non-routine problem-solving process.

Second, the strategy keys aimed at activating known heuristics. F & N know heuristics and are able to select appropriate ones. Still, this is not sufficient to successfully work on mathematical problems, as they do not find the correct result. Their heuristic development has stagnated within our study.

Third, F & N interacted with the strategy keys and M & F checked their results within the problem-solving process. Thus, both pairs showed self-regulatory activities. Yet, this did not result in a correct solution. We claim that regarding self-regulation, these students dispose of only seeds rather than a full self-regulatory competence (Stein, 1995) – so they know a self-regulatory activity and they know how to use it, but they cannot integrate it in their processes in a goal-oriented manner.

In the future, it should be investigated how LA students can be enabled to successfully solve mathematical problems. Possibly the introduction phase of a problem solving lesson could be a fruitful resource to support especially LA students (see Häsel-Weide & Nührenbörger, 2022).

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Participation of students with learning disabilities in the context of probability

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Keywords: Participation, inclusion, probability.

Motivation

The inclusion of students with learning disabilities into the general educational system has substantially grown in the last decades (Häsel-Weide & Nührenbörger, 2021). One central challenge for this inclusion is to realize participation in classroom interactions (Jung & Schütte, 2017). With respect to mathematics education, participation comprises social and content-related aspects. Precisely because both, social and content-related participation, are necessary for learning mathematics. Understanding successful conditions for allowing students with learning disabilities to participate in heterogeneous classes is necessary for the factual inclusion of these students. Participation can also be understood as the option for everyone to engage in society (Jung & Schütte, 2017, p. 1500). Participation in society includes to practice basic rights, which also requires a sustainable understanding of mathematics, particularly chance and probability. Therefore, inclusive mathematics education and instruction should also focus on these fields.

The aim of my PhD project is to investigate social and content-related participation of students with learning disabilities in courses on chance and probability. My focus is on negotiations among students who use and develop different conceptions of probability. Guiding questions are (1) how every student can participate in emerging negotiations in inclusive classes and (2) how they can develop their individual ideas while interacting with other students.

This study provides both empirical and theoretical evidence that a content-related participation of students with learning disabilities is possible in negotiations of the concept of probability in inclusive classes towards the end of lower secondary school.

Theoretical framework

Jung and Schütte (2017) distinguish three types of participation in inclusive classes: (1) spatial, (2) social, and (3) content-related participation. Content-related participation refers to students' engagement in content-related negotiations, their interaction with supplied material and delivered content by the teacher (Jung & Schütte, 2017, p. 1502), for example, contributing the Laplacian probability to roll a five with a regular dice is equal to one sixth into the collective negotiation. From an interactionist perspective of learning, both spatial and social participation are necessary for content-related participation.

Using social negotiations to support individual learning processes is a key approach in inclusive mathematics education (Häsel-Weide & Nührenbörger, 2021, p. 50). Content-related participation in those negotiations requires that all students can contribute relevant content and extend their individual conceptions. In heterogenous groups, this might require an extra effort in collective argumentations when sharing different ideas and perceptions for students with learning disabilities.

Research project and methodology

In my research project a learning environment on probability was designed for grade nine classes with the above goals in mind. The students were meant to work in groups of three to four students so that exchange and negotiations of meaning could emerge. Each group had students of different levels in their mathematical competences. While every group was recorded both auditory and visual, groups with students with learning disabilities are the main focus. From those transcripts, the content-related participation, the use of visualizations and development of conceptions of probability are reconstructed. The following research questions guide this study:

- What fosters negotiations in class that allow all students to participate in negotiations?
- What are typical characteristics of negotiations in inclusive settings? How do contributions on different levels support a development of shared meanings of probability?

The tasks and the learning environment

Each task was designed with the intent to allow every student to grasp a concept of probability. Further, the learning environment was designed to engage students at different levels to interact with each other. Every student was supposed to participate both in group and class discourses. To support this, (1) different representations, (2) empirical and theoretical approaches and (3) subjective and objective perspectives for the concept of probability are provided. The design allows different approaches so that negotiation processes would be initiated and individuals could add their point of view. For example, one task engages dyadic groups to examine the repeated drawing of three distinguishable balls. After the groups collected empirical data, interpreted the data qualitatively and constructed different diagrams, every student has to deduce probabilities and reason those independently. Afterwards, the students negotiate both a probability and an associated reasoning collectively, providing every student the opportunity for content-related participation.

So far, my data show that students with learning disabilities can participate in negotiations about chance and probability, even if different perspectives emerge. Students with learning disabilities are generally able to either contribute their own ideas or adopt other students' meanings and reason from these points of view. Addressing on group differences, there are significant deviations. Some groups are able to construct fertile circumstances while others are not. The poster demonstrates the connection between the task, anticipated negotiation processes and the preliminary results.

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Cooperation of mathematics teaching and special education – seminar concept and experiences

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In cooperation with the Institute for Special Education and Inclusive Learning and the Institute for Mathematical Education, different seminars were conceptualized in which students studying elementary and special needs education examined the use of digital media in an inclusive classroom setting. We will now introduce the starting situation for such seminars and describe one of the seminar concepts in more detail. To conclude, we will consider impressions from students.

Keywords: Inclusive settings, multi-professional teams, teacher education, digital media.

Starting Situation

Nationally and internationally, political actors place demands on teacher training in the areas of inclusion and digital media:

- In the UN Convention on the Rights of Persons with Disabilities (2006), in addition to the right to education for all people (ibid., Article 24), access to information, etc. (ibid., Article 4) is required in order to enable full and effective participation in society.

- The "Equity Principle" of NCTM Standards (2003) demands that teachers must be professionalized in order to be able to deal with the heterogeneous starting positions of the students (ibid., p. 13f.). Learning mathematics with electronic technology is presented there as essential. It also offers many opportunities to support learning, especially for students with physical challenges (ibid., p. 25) and other handicaps.

- The conclusions on inclusion in diversity with the aim of high-quality education for all call for the basic and advanced training of teachers, "and foster their motivation and competences [...] to deal with diversity" (European Union, 2017, C 62/5). This also includes "systematic incentives and training to allow teachers to experiment with digital pedagogies" (ibid.).

- The Conference of Ministers of Education and Cultural Affairs (KMK) for Germany calls on the universities in particular to anchor media education in teacher training. The aim is to address the media experiences of the learners in the classroom. In addition, they should be able to analyze available media and use them as required for teaching and fostering (KMK, 2012). Work in multi-professional teams in teacher training should also be intensified in order to enable a multi-perspective view of the child and the interaction between teacher and child (KMK, 2015).

At the Justus Liebig University in Giessen there are no joint events for students studying elementary and special education due to the study regulations. A collegial cooperation with the aim of joint planning of lessons, however, requires knowledge to be acquired about the other profession, and this as early as possible (Hattermann et al., 2014).

Building on existing experience with interdisciplinary activities (Rudinger et al., 2018), new seminars were designed that address the use of digital media in inclusive settings and enable students studying primary and special education to learn together. These students are usually in their penultimate semester. The elementary teacher students study mathematics as a subject, while the special education students usually do not, moreover, their prior knowledge of mathematics didactics usually relates to only one compulsory module.

Digital media is seen as a special aid for differentiation and support in cooperative, inclusive teaching (Bonow et al., 2019). Digital media seems to have particular potential here, since, for example, direct control via input gestures on the screen of tablets eliminates the need for the computer to coordinate hand, eye and mouse (Walter, 2017). However, digital media also offers new possibilities for representing mathematics, e.g., by focusing on written and graphic communication in projects for chatting about mathematics (Schreiber, 2013) or on the oral presentation of mathematics when creating audio podcasts (Schreiber & Klose, 2017). In mathematics didactics, the possibility of creating or using videos is currently seen as a way of learning mathematics (e.g., Leinigen, 2020). In particular, the potential of synchronously linked representations (Schulz & Walter, 2019) can be used in the creation and use of videos in terms of mathematics didactics. Since the individual requirements of the learner can be addressed, there is potential here for use in an inclusive setting (Fehrmann, 2019).

Seminar Concept

The seminar “Explain! Videos in inclusive math lessons” is attended in equal parts by students studying to become teachers in primary and special education. The course has been offered since the winter semester 2020/21, and 42 students participated in the first two course offerings. The seminar program is divided into three different phases. First, basic theory is worked out, then work is done on a specific (Hirt & Wälti, 2012) substantial learning environment (SLE) (Scherer, 2019). Third, video sequences are created for this SLE. Working together in mixed teaching groups plays a central role in order to give the students their first experience of multi-professional cooperation.

In the first phase, important theoretical basics for the use of videos in mathematics lessons are developed. Particularly addressed is the point that explanation should not be given exclusively in the sense of transferring knowledge (Kiel, 1999) through the video, but rather the students should be encouraged by the video to negotiate or develop knowledge (ibid.) by explaining something themselves. The concepts of an SLE and natural differentiation (e.g., Scherer, 2019) are used as the theoretical groundwork for inclusive math lessons, and form an important basis for further work in the seminar.

Natural differentiation is characterized by having the same learning opportunity for all students. This learning offer is mathematically rich and complex, so that the processing can take place at different, naturally resulting complexity levels. Different learning requirements can also be catered for by giving the pupils the greatest possible freedom in the way they work on and solve problems or in the way they present them, and in the notation they use. This also enables social learning from and with each other in a natural way (Scherer, 2019).

A substantial learning environment (SLE) uses the principles of natural differentiation to teach slow and fast learners together. SLE is guided by core content, goals, and principles of mathematics instruction that have mathematical substance. Tasks should have a high cognitive activation potential that allows accessibility and independent activity by all learners (Hirt & Wälti, 2012).

In the next phase, the students work on an SLE for primary school students as described in Hirt and Wälti (2012). The SLEs address different mathematical topics: arithmetic, measurement and modelling, and geometry. The students should first analyze the SLE in terms of content and mathematics by working on the tasks themselves and making a well-founded decision as to which video formats can be used to support the subtasks.

A method developed by Schreiber and Schulz (2017) is used to create the film (see Figure 1). This is a method that, in addition to being used in teacher training, can also be used in schools to create films on mathematical content together with students (Schreiber & Schulz, 2017; Leinigen, 2020; Fehrmann, 2019). In addition to the products created, the seminar should also provide students with suggestions for using the method in school.

Figure 1: Process of film creation (Schreiber & Schulz 2017)

The video creation phase takes up most of the semester. Individual steps of the method can already be worked on in the theoretical phase of the seminar. **(1) Determine the Content**: The students decide on one of the proposed learning environments when they are divided into groups. **(2) Factual Analysis**: The students first work on the learning environment themselves in order to grasp the depth of the mathematical subject. The mathematical richness of a learning environment allows the students to make interesting discoveries beyond the content and learning objectives of the primary level. Dealing with the learning environment should also lead to considerations as to which subtasks could be supported by individual video sequences and what a meaningful, cognitively stimulating video application could look like. **(3) Script I:** Video sequences are planned in the form of scripts. **(4) Peer Review:** These scripts are discussed in groups in the seminar and students edit their scripts based on feedback from fellow students and instructors. **(5) Script II:** Only completely revised scripts are approved for video production. **(6) Film Creation:** The type of technical implementation of the video sequences is up to the students. For example, videos may be created using the laying technique, stop-motion technique, screencasts, etc.

The learning environment should be supported by at least two video sequences between 30 seconds and 2 minutes in length. Additional materials, such as worksheets, haptic material, etc., are also necessary for the implementation of the learning environment. Due to the corona virus pandemic, the students were not able to try out all the videos themselves in classes. However, the video sequences were occasionally used by teachers in alternating or online lessons, so the students received useful feedback from the teachers on their products. For a time after the pandemic it is planned that the students can also try out the videos themselves.

Seminar Focus and Goals

The goal of the seminar is for participants to get ideas for designing videos for inclusive mathematics classes. The guiding principle for this is the assumption that inclusive teaching can only succeed together – in multi-professional collaboration.

One focus of the seminar is to **work in multi-professional teams**. This is promoted by the mixed teaching groups in which the students work together during the entire seminar. There is a social exchange about experiences and competencies, which leads to different perspectives and roles in the classroom becoming clear. Particularly when creating the videos, different expertise and competencies become clear. The primary education students bring expertise in mathematics didactics, the special education students can accompany the planning from the perspective of special education. In this way, differentiation levels are built in, or special education needs are explicitly considered in order to make the videos accessible to the target group. Thus, the primary education students can achieve more understanding of individual support needs. In the peer review of the video scripts, the students benefit again from the different perspectives and previous experiences of fellow students, so that a wide range of feedback can be given. This intensive exchange about the scripts serves to build up knowledge about the qualifications and areas of responsibility of future colleagues and is intended to facilitate later cooperation (e.g., Rudinger et al., 2018; Bonow et al., 2019).

The second focus of the seminar is to **work with digital media**. The media skills of the students are first strengthened (KMK, 2012) by creating videos themselves using different techniques. Promoting media literacy also includes providing a meaningful rationale for the video sequences in terms of subject didactics (ibid.). Possible uses of videos in (inclusive) education are also reflected upon. For school practice, there is also a focus on the media competence of the students. The video creation method carried out in the seminar is also suitable for school practice (Schreiber & Schulz, 2017; Leinigen, 2020). Through the guided assignment, students can acquire media skills.

Results of the Seminar

To use videos within a naturally differentiated learning environment, care is taken that the videos are not only intended for passive consumption, but also cognitively stimulate the students. In the end, there should be an opening through a follow-up task or something similar. This can be implemented using different video formats. There are also many possible uses within a learning environment. The video can give a general introduction, it can provide additional, differentiated help, explain a task or a task format, or provide an impulse to discover a mathematical structure (e.g., Kristinsdóttir et al., 2018).

In the following, videos on the learning environment “rolling the die” for grade 4 are discussed as an example. This learning environment (Hirt & Wälti, 2012, p. 240ff.) is about recognizing and using symmetries and patterns by rolling a die with a colored side on a playing surface with square divisions (like a chessboard), until the marked surface is facing up again. In the middle of the game board is the starting square. The number of rolling movements is noted on the target square. If care is taken to reach the squares with as few rolling movements as possible, a symmetrical pattern results. The learning environment is enacted with the help of the videos. Since it naturally contains differentiating tasks, it should also be applicable in inclusive settings. All pupils in a class should be able to use the videos, as these videos should enable discoveries at different levels. Potentials for special education needs are also considered in the videos. The students use a mixture of discussed and animated PowerPoint slides with symbolic representations of the playing field and self-made video sequences in which they can be seen rolling a large cube on a playing field. The students start with a video sequence in which the “rules of the game” are explained.

This video has great potential in many ways. For example, the video is suitable for beginners. The basic rules of the game and the task format are explained here. Students can watch the video several times to internalize all the rules. This is a cognitive relief, as the children do not have to read the rules for themselves. In addition, the combination of image and sound enables several levels of representation to be combined (Schulz & Walter 2019). In this example, understanding can be ensured by explicitly pointing to the edge. The fact that the die rolling is demonstrated directly on the real model reduces the cognitive burden, as there is no change to the symbolic level here.

The following video explains that the number of rolling movements of the die is considered. Two examples are used to determine squares that can be reached in 4 moves. The two protagonists take turns rolling the die. The main rule is to always begin on the starting square, with one student improving upon the other. The two squares reached are marked with the number 4 and colored in blue. The number 7 is given to the square directly below the start square. The other student protests that 3 is the correct answer. The video ends with the task: “Which girl is right? Check it. Try to find out the number of rolling movements for as many fields as possible.”

The video, created by students for the fourth grade, shows that mathematical learning videos can also be an explicit stimulus for learning through discovery: here a problem is posed, which the students should solve independently and in an action-oriented manner. Occasions for mathematical discussions may also arise and communication about mathematical phenomena becomes possible. Explanations need not necessarily be given exclusively through the video, but can also be developed and negotiated by the students amongst themselves based on the impulses of the video.

Coming back to the above posed question: Both persons are right. The square can be reached with 3 and 7 movements. An unambiguous answer is only possibly if the condition of using as few movements as possible is included. Furthermore, there are different possibilities to reach the square with 3 or 7 movements. This can also be discovered and become an occasion for discussion for the students. The further expansion of the task of finding out the number of rolling movements for as many squares as possible also allows processing on different mathematical levels. In this way, "simple" squares can be reached first, or the criterion of minimal movements can be ignored. The task is also solved if the playing field is not completely labeled. It is also possible for high-performing children to find different paths for other squares. The videos address the idea of natural differentiation and thus enable learning on the same subject. Both videos are designed in the sense of natural differentiation and are therefore intended for all children. Further differentiation is not necessary here, but the videos are only one part of the learning environment. In the implementation of the learning environment in class, there are further tasks or social learning opportunities, which must also be adapted to the inclusive setting.

Voices from the Seminar

The evaluation of the seminar refers to results of a pre-post questionnaire and to informal, oral statements by the students on seminar reflection. Using questionnaires, the students were asked about attitudes, experiences, and self-efficacy regarding inclusive teaching with digital media. The results from two course offerings are available from the beginning and the end of the semester. They show that the students mostly trust themselves to work in a team with teachers from other professions. This positive self-assessment of the students increased again at the end of the semester. While 3 people stated at the beginning of the semester that they did not trust themselves to work with teachers from other professions, after one semester all 42 students questioned agreed or completely agreed that they could. The students commented on this, for example, that working together in mixed teaching groups was “an enrichment because you could think outside the box”. It was noted that the group often did not know each other, but it was precisely this what enabled other perspectives to be taken throughout the intensive exchange.

The detailed feedback regarding the scripts, which was given during the peer review, was also rated positively. It could be observed that the different fields of expertise ensured that the scripts were dealt with in great detail and intensively, and that there were helpful tips on how to design the video sequences.

Regarding the use of digital media, the students report learning successes in terms of video creation and the beneficial use of videos in the classroom. For example, one student said, “I believe that using videos can lead to more motivation because it is rather unusual. In any case, this shows that math does not have to be taught in such a dry way!” It was also noted that the videos are “very suitable for natural differentiation” and thus offer opportunities for inclusive teaching.

In addition to successes, the students also report challenges on the technical level: the process of video creation was more complex than initially thought, and lighting conditions, camera position, speed, etc. must be considered when the video is produced. Moreover, time required to create the videos was also critically reflected upon. Therefore, this task was reduced for the second run. This feedback underpins the claim that the teachers themselves should acquire certain media-technical skills (cf. KMK, 2012).

Overall, the seminar not only enabled students to feel more confident when using and creating videos, but the results of the evaluation also show that the self-efficacy assessed by the students themselves in relation to the general use of different digital media in the classroom during the semester increased. The proportion of students who feel (rather) unsafe about using digital media was reduced from 10 to 4 people over the course of the semester, while the proportion of students who (rather) trust themselves to use digital media increased from 33 to 38 people. These positive results indicate that students benefit from their experiences using and producing digital media.

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Working with objects of representation in practical contexts on length in inclusive classrooms

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Due to the growing number of children with special educational needs in German primary schools, research on inclusive settings in mathematics has increased in recent years. This research primarily focuses on arithmetic settings. Little attention has been paid to measurement processes, especially for children with learning or mental difficulties. The following paper gives an example for inclusive education on lengths to gain social and content-related participation. The developed learning environment is based on the learner's interaction and takes into account the use of objects for representation and learning in practical contexts.

Keywords: Inclusion, participation, objects of representation, practical contexts.

Introduction

Since the UN Convention on the Rights of Persons with Disabilities, the German government seeks to implement an inclusive school system. This makes it necessary to think about how all children can participate by considering empirical research on teaching mathematics. The focus in this work is on the learning processes of children who need special support in learning and/or their cognitive functions because in inclusive teaching children with cognitive disabilities often work only on different subjects and are less involved in common learning situations (Korff, 2015). According to the guidelines of the Ministry for Schools and Education of North Rhine-Westphalia (an area in Germany) *learning difficulties* are severe, persist over longtime and effect several aspects of the children's development (§4 AO-SF NRW). There is a need for support in the *mental development* if cognitive learning and the development of the personality are permanently and severely impaired (§5 AO-SF NRW). Nevertheless, even if the focus is on children with special educational needs (SEN), all other diversity aspects are also respected in the underlying understanding of inclusion in this work. It can be assumed that the resulting indications for teaching will be important for *all* children.

A change of perspective has taken place in mathematics education in recent years: children should not only solve tasks by using fixed procedures, but they should instead understand mathematical structures and learn to use them flexibly. Although studies suggest that all children, also those with SEN, can benefit from learning actively and by exploring mathematical structures (Scherer, 2009), this is often still not taken into account in inclusive classrooms. Learning mathematical structures can be supported by working with objects and tasks in practical situations. Therefore, this study integrates those aspects.

Firstly, the concepts of participation, learning with objects of representation and in practical contexts are discussed. In a second step, the design of the study is described and first results will be presented.

Theoretical background

Several theoretical concepts are relevant for the study and are, thus, briefly described below.

Participation in inclusive classrooms

The term "participation" is closely related to inclusion. Whilst there are different understandings of inclusion, one central goal is to achieve participation (Booth & Ainscow, 2002). According to the *Index of inclusion*, a guideline for the development of inclusive educational institutions,

> Participation means learning alongside others and collaborating with them in shared learning experiences. It requires active engagement with learning and having a say in how education is experienced. More deeply, it is about being recognised, accepted and valued for oneself. (Booth & Ainscow, 2002, p. 3)

This definition includes especially various social aspects. In addition, Roos (2014) emphasizes that content-related aspects are also important for participation and she generally distinguishes spatial, social and didactical (content-related) inclusion. In the following, both *social* and *content-related participation* will be discussed in more detail.

Social participation

Koster et al. (2009) distinguish four aspects of social participation: friendship/relationships, positive contacts/interactions, acceptance by classmates and students' social self-perception. These aspects seem to be related to positive interactions. There are different ways of interacting, depending on the children's abilities. It "cannot be for all children to participate actively in class in the same way" (Jung & Schütte, 2017, p. 1503). According to Krummheuer and Brandt's (2001) participation schema one can be included by being a productive participant (e.g. by initiating a conversation, repeating what someone has said - verbally/nonverbally), or by being a receptive participant (e.g. by listening to what others say) (Krummheuer, 2007). Hence, different manifestations of social participation seem to exist. In this study, different forms of *involvement* will be mentioned. Involvement means interacting with others and can be described by different interactional reactions.

Content-related participation

Content-related participation means "being included in mathematical practice of the classroom" (Roos, 2014, p. 39). Interacting with others is important for social as well as for content-related participation: by interacting, social participation can be realized and content-related participation is influenced by the interaction with others. This understanding is based on a social-constructivist perspective on learning. According to this perspective, the experiences of a person influence the individual construction of knowledge (e.g., Miller, 2002). In this context it is important to define learning processes more precisely. Based on the epistemological perspective, learning is defined as an increasingly differentiating process of interpretation. New knowledge is the extension of prior knowledge by exploring mathematical structures and generalizations. This is a highly individual construction process which is influenced by the interpreter's subjective characteristics and the way mathematical knowledge is presented in these various situations (e.g., Nührenbörger & Schwarzkopf, 2016; Steinbring, 2005; Voigt, 1994). Among others, Steinbring (1997) emphasizes the role of signs for mathematical knowledge. Signs are used for describing and coding mathematical knowledge.

> In elementary teaching, these [signs or symbols] are mainly arithmetical number signs. These signs, on their own, do not have a meaning. The meaning of a mathematical sign has to be constructed by the student. (Steinbring, 2005, p. 21)

Referring to the semiotic triangle of de Saussure and Luhmann's concept of communication, Steinbring (1997) worked out a triangular scheme, the "epistemological triangle", for describing and analyzing children's comprehension. The sign/symbol contains the new, still unknown mathematical knowledge. It has to be interpreted by the individual person by referring to probably familiar reference contexts. In the interplay between sign and reference context, the new concept is established. In this process of negotiation, the three aspects are balanced and support each other (Steinbring, 2005).

Learning with objects of representation and in practical contexts

In inclusive settings it is important to enable students to find different approaches to a task and to explore mathematical structures (Häsel-Weide & Nührenbörger, 2015). This can be supported by working with objects and tasks in practical situations.

For teaching numbers, Schipper (2009) distinguishes different functions of *manipulatives*, which can be transferred to other mathematical contents as well: manipulatives for solving a task, for understanding abstract mathematical concepts and as assistance for communication, argumentation or reflection. In inclusive classrooms manipulatives are often used only as a tool for solving a task (Korff, 2015), but not for communicating or demonstrating the learners' ways of thinking. But this exactly could be important when, for example, working with children who have difficulties in verbally expressing thoughts. Therefore, it is helpful that the researcher may be able to also analyze the children's understanding by observing how they use the objects. Regarding Schipper's (2009) second aspect – manipulatives for understanding abstract mathematical concepts – there is consensus that "the purpose of using manipulatives in mathematics is to help the learner understand abstract concepts" Kosko et al., 2010, p. 80). In this context, understanding the interrelationship of representations is of great importance and needs to be practiced. However, using concrete objects in teaching does not guarantee a meaningful learning process. Referring to the epistemological perspective described above, Söbbeke (2005) showed that the interpretation of manipulatives is not obvious and cannot be predefined by teachers. How objects and the structures they represent are interpreted by the learner depends on the context and the individual experiences. This active process can result in different interpretations. In the process of negotiation, the concrete objects can represent a new sign/symbol that has to be negotiated or it can be used as familiar reference context for negotiating other mathematical concepts (Söbbeke, 2005).

The word "*context*" can be defined differently. Van den Heuvel-Panhuizen (2005) distinguishes between contexts referring to the learning environment and contexts such as "a characteristic of a task presented to the students" (van den Heuvel-Panhuizen, 2005, p. 2). In the following, context is understood to be the task context, in particular a situation relating to everyday life. It is not possible to draw the conclusion that using context tasks will automatically improve learning. However, some positive effects of using contexts are reported in different studies. The context can increase the students' motivation, so that they are possibly more willing to solve the task (Scherer, 2009). Moreover, "while working on context problems, the students can develop mathematical tools and understanding" (van den Heuvel-Panhuizen, 2005, p.2). For example, Selter (1998) shows that in context tasks, students can solve tasks with requirements that had not yet been discussed in class up to that point. On the other hand, contexts can distract and complicate the learning process (Scherer, 2009), especially when it is an unknown context for students. In this study, a familiar context is included in order to use the context as a learning opportunity. However, it is important that not only the context is relevant here, but that the students can explore mathematical structures while working on the tasks (Korff, 2015).

Design of the study

The study is based on Design Science. This implies designing, implementing and analyzing a learning environment to get information about the involved mathematical structures and the students' ways of thinking (Nührenbörger et al., 2016). In this study an interactive learning environment on length (this will be further clarified below) and money was conducted, three lessons each on length and money, respectively. The three sessions on length/money were structured similarly: first, the children's prior knowledge of length/money and of the context situation was assessed and the objects of representation were introduced. This introduction was particularly important so that the children could use the objects in subsequent sessions, when working on two to three context tasks.

The design of the learning environment builds on the basic principles of individual goal-differentiated learning on a common subject (based on Feuser, 1995), learning with objects of representation, learning in practical contexts, and in interaction. These principles are closely interrelated (Figure 1).

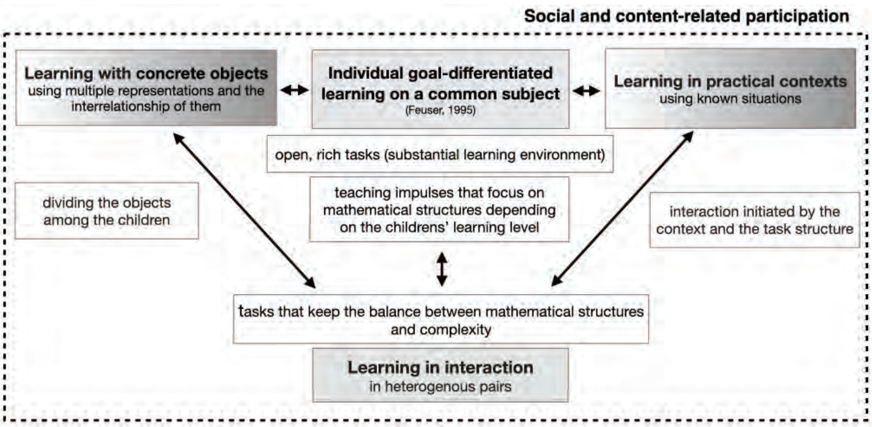

Figure 1: Design principles of the study

The learning environment was tested with different pairs of children (one child with SEN; the other child without SEN) in grades three and four at German inclusive primary schools. The pairs were formed in such a way to take into account the high (expected) range of heterogeneity in inclusive classrooms and to be able to focus on the participation of all children. The children were videotaped and observed while working on the tasks. Additionally, follow-up-questions were asked to find out more about the children's understanding or to intensify their interaction. Aims of the study are to look at the handling of the concrete objects and the influence of the task's context on the development of competences regarding length/(money). Aspects of social involvement are to be worked out and linked with content-related aspects.

Design of the first task on length

The long jump was the subject of the length tasks. Therefore, a long jump field was simulated in the classroom. As objects of representation wooden sticks of different lengths were used, which serve as representatives of standard units (1m, 10cm, 1cm) and of other length (50cm, 20cm, 5cm, 2cm) that can be set in relation to standard units (e.g., 50cm as half as long as 1m). With the sticks, the children compare, estimate and measure length in a practical context, whereby various skills for measurement can be trained. The relation between units and sub-units (especially m and cm) can, for example, be vividly addressed, which is important for the measurement process (Nührenbörger, 2004).

As a first approach to the task a pair of sneakers was put in the jumping area. Every child had sticks of different length (the sticks of each length are multiple available) at their disposal. The children were asked to find out the given jumping distance indicated by the position of the sneakers by using their sticks and to present and discuss their ideas. Then, the children took turns to do the long jump and measure each other's jumping distance.

First Results

The study at hand uses a qualitative methodology. To get a first impression of the nature of the participation in the learning environment, a short scene is analyzed below. A descriptive approach was chosen to describe social participation. The interrelated reactions of the children were examined with regard to possible triggers and reactions. For content-related participation, an attempt was made to analyze the children's understanding with the epistemological triangle.

In the scene two boys around nine years old, Nic (without SEN) and Jan (with mental difficulties of no specific reason, SON IQ range: 47-64), are working together. A pair of sneakers stands at around 2m 3cm in the jumping field. Nic already worked out the jumping distance with his sticks. Meanwhile, Jan has been looking at his sticks but has not yet started measuring the jumping distance. Then, the children start working out the distance with Jan's objects. They have already put two one-meter sticks in the jumping area, starting at the take-off bar. Nic tries to explain to Jan that they cannot use another one-meter stick anymore and that they need "small things". Jan then fetches a two-centimeter stick.

1	Nic:	Then let's take a few (*takes the 2cm stick from Jan*). Look. Then we put it there now (*puts the 2cm long stick against the already lying 1m sticks*). And now it's still like that, so now we have to look (*points to the gap between the lying sticks and the heel of the shoe*) how far does it go? And I don't think a two will fit there anymore, so we need a very small one (*indicates a small distance between thumb and forefinger*).
2	Jan:	(*turns to his supply of sticks*) Very small.
3	Nic:	Very, very small.
4	Jan:	(*takes a 1cm long stick and holds out the stick to Nic*)
5	Nic:	Let's take one of these, put it here (*puts the 1cm stick in the gap*) and then it fits. So now we have two meters and then we have two and one (*holds up two fingers, looks at Jan*). What is this? I have two plus one (*holds up a third finger*). How much is that?
6	Jan:	(*looks at Nic's fingers*) One, two, three.
7	Nic:	So, it's two meters and three centimeters.

Nic explains his approach while acting. He extends the two meters (consisting of two one-meter sticks) with another two-centimeter stick and points out the remaining gap between the end of the track and the shoe. He seems to refer to a strategy for measuring with the help of the sticks because he suggests that another two-centimeter stick will not fit anymore. Nic concludes that an even shorter stick is now needed (line 1). Jan takes up Nic's description "a very small" and repeats part of the statement (line 2). The comments of the two boys refer to each other linguistically. Nic specifies his statement by adding another "very" (line 3). With this description, he classifies the difference between two centimeters (which was previously called "small") and one centimeter ("very, very small"). Jan identifies a one-centimeter stick and holds it out to Nic. After Nic has put the stick down next to the other sticks (as their extension) he points out: "And then it fits" (line 5). He turns to the next step in the process: naming the result of the measurement, and distinguishes between meters and centimeters. Nic names the two meters with the corresponding measurement attribute without explicitly naming the calculation one plus one. In the case of centimeters, a calculation seems appropriate to him. He asks Jan: "Two plus one. How much is that?" (line 5) and holds up the result with his fingers. Jan counts "one, two, three" by looking at Nic's fingers. Nic integrates Jan's answer in his solution: "So, it's two meters and three centimeters" (line 7).

In the scene Jan participates socially, in particular. Nic takes over the role of a teacher who dictates Jan what to think and do. By asking questions he guides him through the solution process and thus integrates him socially. In the process, Nic implements his own thoughts and solution process, but seems to try to adapt his explanations to Jan's abilities. Jan seems to be able to go along with that and to participate by answering questions or taking action (e.g., identifying, handing sticks to Nic).

In terms of content-related participation, one concept that Jan might express in this scene is analyzed in figure 2 (other concepts are not presented due to the limited number of pages in this article).

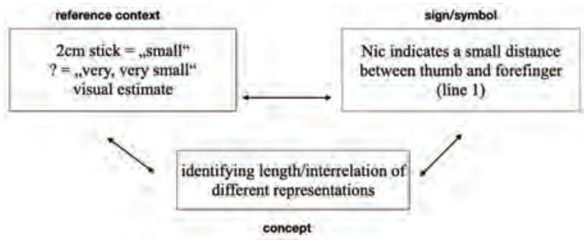

Figure 2: Jan's interpretation

The sign Jan has to interpret is Nic's indication of a small distance with his fingers. As reference context Jan probably refers to the description used previously for the two-centimeter stick as "small" and identifies a smaller one with the strategy of visual estimate. Thus, identifying length and the interrelationship of representations ("very, very small" – indication of the distance with the fingers – wooden stick) seem to be the concept that Jan expresses indirectly.

Conclusion

The scene shows that there are different aspects of social involvement. The interrelated reactions establish a joint working process in which both children are actively involved. In terms of content-related participation, it can be stated that the environment accommodates the children's individual learning process. While Nic probably has the procedure of measurement in mind and oversees the entire situation, Jan deals with different aspects concerning the measurement process: identifying length, the interrelationship of representations and counting (line 7). Thus, for Jan, single aspects are picked out from the complexity of the context and the requirements. It must be noted that even Jan's "small" actions (analyzed in figure 2) include an interpretation, even if Nic gives many instructions. Jan has to interpret the interrelationship of the representations on his own. He therefore participates in terms of content, although he is hardly involved in a common solution process. In the scene, using concrete objects serves different functions (Schipper, 2009): the sticks are especially used, as part of the given task, for solving the task but at the same time they represent abstract mathematical concepts, which the children have to interpret. Moreover, using concrete objects seems to support the communication and participation.

This scene only gives a short impression of participation. More scenes will be transcribed and further analyzed to draw conclusions with respect to social and content-related participation, especially in terms of learning with objects and in practical contexts.

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MathCityMap and supporting learners with exceptional difficulties in learning mathematics

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Keywords: Mathematical competences, digital tools, learning difficulties.

Exceptional difficulties in learning mathematics

The term “exceptional difficulties in learning mathematics” is used to describe serious and persistent difficulties in learning mathematics (Gaidoschik et al., 2021). It therefore follows the more widely known terminology of “mathematical learning difficulties” (Mazzocco, 2008). This circumscription, which captures both diagnosed (e.g., dyscalculia) and undiagnosed problems on the learners' side, is used for two central reasons. First, to avoid a one-sided view that attributes the causes of difficulties only to individual factors. Second, the plural “difficulties” is intended to indicate that a wide range of problems can manifest themselves in a variety of forms in an individual learner (Gaidoschik et al., 2021). Typically, exceptional difficulties are found in the three central mathematical basic skills: understanding natural numbers, understanding the decimal place-value system, and understanding arithmetic operations. If those deficits are too serious, subsequent topics (e.g., decimals, algebra, etc.) can hardly be understood and, in the best case, can only be learnt by rote (Prediger et al., 2019).

MathCityMap – theoretical background and concept

The idea of mathematical learning trails already emerged in the 1980s. A math trail is a walk on which mathematical tasks are implemented at various interesting objects along the way in order to make mathematics directly experienceable in the real world (Shoaf et al., 2004). The EIS principle according to Bruner, which has already proven itself in special and regular schools alike (Ozdem-Yilmaz & Bilican, 2020), can be mentioned as the basic theoretical idea of this concept. On the trails, learners gain important mathematical experiences at different levels of representation of mathematics. They work enactively on real objects (e.g., measuring objects with tools), iconically (e.g., drawing, modelling the objects), and symbolically (e.g., calculations). Especially lower-performing learners benefit from this type of outdoor learning and can better understand content and procedures based on the concretely experienced examples (Buchholtz & Armbrust, 2018). The MathCityMap project has succeeded in developing this original concept further and bringing it into the digital age as well as embedding it in school contexts. Tasks are now found via GPS data in an app, which already provides specific and graded help. On the tasks themselves, the learners work in small groups, whereby they can stay in contact with the teacher via a chat function. The solutions found can already be checked on site in the app, where a detailed sample solution can be retrieved (Gurjanow et al., 2017).

Research question

From the theoretical background on which the concept of MathCityMap is based and its potential for exceptional difficulties, the following research question arises: How can MathCityMap be used to support students with exceptional difficulties in learning mathematics? This question is supported by criteria of successful support, which MathCityMap seems to fulfil. For example, learning on MathCityMap tasks always happens in discourse with other learners, and on learning objects that make natural differentiation possible (Gaidoschik et al., 2021, p. 12). An examination of the suitability is to be carried out with this investigation for the first time.

Outlook

In order to answer this question, a qualitative study will be conducted within the framework of the support project Mathe.Kind, which will be implemented at Goethe University Frankfurt in 2022 and is founded by the researchers that are also working on MathCityMap. Here, students with exceptional difficulties in learning mathematics will be individually supported in reducing their deficits and building up their competencies. MathCityMap will be used as a learning environment in different support sessions. Afterwards, it will be checked if certain difficulties could be reduced. For this purpose, task-supported interviews will be conducted with the students before and after the support sessions. To increase the accessibility of MathCityMap, the app will be equipped with a read-aloud function and an “easy language” setting option before the study is conducted.

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DigiVersity – Thinking digitization and diversity together – from primary school to university and back

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Keywords: Digitization, diversity, inclusive education, primary school.

Introduction

Digitization and inclusive education are two of the most complex challenges that contemporary teaching practice must deal with. In our view, inclusive education is more than just disabled and non-disabled pupils learning together, but mirrors the complexity of inclusion in society as a whole. (Ainscow 2007, p. 3). The two challenges are often dealt with separately or they are put into a charged relationship. The complexity and diffuseness of both discourses can be seen as the reasons for this (Lücke, 2021). In academic research only a few theoretical and empirical efforts can be found that try to think these two domains together (Viermann & Meyer, 2022). Our research efforts try to combine digitization and diversity in the mathematics classroom into one interwoven concept to design subject teaching with both challenges in mind. Furthermore, we try to specify opportunities and challenges of digitized subject teaching in mathematics to realize dealing with diversity.

Learning in a digitized environment

When dealing with digitization in the context of schools and especially in primary schools, people often act cautiously because the common idea of a digitized learning environment is, that children sit alone in front of a computer. Digitized learning settings are much more than solving problems with the help of a computer. Brandhofer et al. (2018) describe four different perspectives on learning in a digitized environment: 1. Education with digital media, which describes using digital media to shape learning. 2. Education about digital media, which describe learning about e.g., the functionality of different media. 3. Education in spite of digital media, which thinks of the possible distractions digital media offer. 4. Education through digital media, which describes digital media as the enabler of education. To overcome the dualism of inclusive education and digitization, digitization has to be seen as a chance to realize inclusive education and an inclusive education-oriented perspective has to be understood as a standpoint for discourse on digitization. The questions that arise from these thoughts are *How can digitization help to extend and improve inclusive mathematical education?* and *How can education* ***about*** *digital media connected to mathematics be designed?* Firstly, we want to expand the model introduced above with the dimension of diversity. Therefore, we try to see mathematics learning in an inclusive education environment as 1. Inclusive education with digital media. 2. Inclusive education about digital media. 3. Inclusive education despite digital media. 4. Inclusive education through digital media.

Challenges

One of the challenges in the conception and design of inclusive digitized mathematics education is the development of appropriate learning methods. Learning environments with mathematics tasks that a multitude of students with different abilities and needs can work on have to be designed with this diverse audience in mind. The concept of natural differentiation (Krauthausen & Scherer, 2014) is a fundamental tool to address different abilities with one task. Natural differentiation follows five characteristics: same task for all students, free choice of aids and means of presentation, social learning, educational framing, and certain amount of complexity. Especially the last characteristic is important as the task has to offer students the opportunity to develop different types of answers with their different approaches on the task itself. For digitized inclusive mathematics teaching, we try to adapt these five characteristics of natural differentiation regarding the conditions of digitality.

Examples

The example presented in this poster is a further development of the learning environment *Muster im Kreis (Patterns in a circle)* from Hirt & Wälti (2010). We use the app GeoGebra to realize the digital enhancement of the task. The students have different representations of a circle (geoboard, on paper, digital). They will have access to an interactive digital worksheet that can be manipulated on different levels. Task 1 and 2 are mainly a digital version of the original tasks. Task 3 introduces a possible addition that cannot be realized with a geoboard or the paper worksheet used in the original task. The seamless scaling of the circle provides the opportunity to change complexity to the student's abilities. The main idea is not to replace the haptic materials from the original task but to introduce the digital version as a possible enhancement and one further way to make the task accessible.

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Preventive support scheme for mathematics learning: possible ways to provide aid before and after the class session

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Since 2013, we have studied a support scheme for students who have difficulties solving mathematical problems. We call this a "preventive support scheme" and are interested in modelling its functions. These interventions take place before and after a specific classroom session, in order to facilitate the students' introduction to the planned learning situation, their involvement in the work required and their appropriation of the target knowledge. The work proposed in this scheme can take various forms and the aim of this communication is to illustrate this point by presenting, for a given situation, different tasks which can be proposed either before or after the classroom session. This study will allow us to highlight the precautions to be taken to design such devices.

Keywords: Didactic system, preventive support scheme, students in difficulty, fractions.

Introduction

Our research focuses on students who have difficulties solving mathematical problems (they fail to engage the situation, hesitate, do not know how to start, wait for their peers to find the solution…). Since 2013, we have studied the effects of a support scheme for them which we call "preventive support schemes" (Millon-Fauré et al., 2021). Unlike other support initiatives implemented after classroom sessions as part of remedial strategies, the proposed preventive strategy is implemented both before and after mathematical problem-solving sessions involving all students: a supervisor (teacher, remedial education specialist, *etc*.) works, before and after the classroom session, with a small group of students deemed (by the teacher) to have difficulties in problem solving. The "preventive" aspect of this scheme is linked, not to the learning difficulties that these students may encounter in a problem-solving context, but rather to the difficulties they demonstrate in taking their place as learners in such a context (Assude & Millon-Fauré, 2021; Tambone, 2014).

The choice of tasks which can be proposed within the framework of this scheme is sometimes not a clear one since it is necessary to facilitate the students' participation in class while not overly advancing in the actual problem-solving. It is for this reason that we want to present in this paper, for a specific classroom session on fractions, different options which could be proposed in the preventive support scheme, either before or after the classroom session. To do this, we will first quickly describe the different functions of the preventive support scheme.

Theoretical framework: the preventive support scheme

To model this preventive support scheme, we draw on the notion of the didactic system and the (structure; functions) coupling. The didactic system comprises three elements (knowledge, student and teacher/supervisor) and all interrelationships between these. Chevallard (1999) distinguishes between two types of didactical systems: the principal didactical system (PDS), essentially the classroom, and the auxiliary didactical systems (ADS), which include some, but not all students of the PDS and are peripheral to the class. The ADS follows the same objectives as the class but does not have its own program. Thus, the class, which guides the study of the target knowledge, consists of the PDS, for which two ADS are provided: one before (pre-ADS) and one after (post-ADS) the classroom session. Supervising a small group of students, the pre-ADS aims to create favorable conditions so that students who are deemed to be "in difficulty" can better integrate the study during the PDS (therefore this device is part of an inclusive perspective). The post-ADS, held after the classroom session, is a further work session with the same group of students, with the purpose of revisiting the knowledge presented in the PDS. The aim of this scheme is to enable students in difficulty to catch up with their classmates and it is therefore not relevant for all the students in the class. These two ADSs depend on the PDS, which determines what the student is supposed to know and do. In summary, we have the following structure:

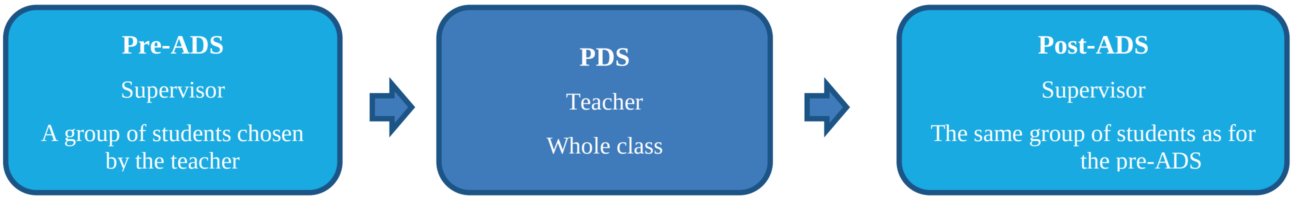

Figure 1: Structure of the preventive support scheme

The pre-ADS

Through our analyses of different potential iterations of preventive support schemes we were able to identify several functions associated with the pre-ADS (Assude, Koudogbo et al., 2016; Assude & Millon-Fauré, 2021; Theis et al., 2014).

The *mesogenetic function* relates to the introduction, during the pre-ADS, of different objects which will make up the PDS milieu. These objects may vary, as they depend on the nature of the situation studied in the PDS. Certain pre-ADSs present the instructions for the problem which will be worked on later in the PDS. In others, objects of formerly acquired knowledge, useful for the work to be done in the PDS, are introduced. Generally speaking, this mesogenetic function enables struggling students participating in the ADS to become familiar with certain objects of the PDS milieu before the other students in the class.

The *chronogenetic function* plays out on two levels: not only do the students involved in the pre-ADS have more time to become familiar with certain objects of the milieu but above all this additional work is set up before the session so that they have a little bit of advance on their classmates. It is nevertheless important to note that, the didactic time (Chevallard & Mercier, 1987) does not progress. In practice, the aim of the pre-ADS is to prepare students to fully engage in solving the problem set out in the PDS, but not to solve it before the other students.

The *topogenetic function* has been demonstrated through numerous observations made during the various experiments conducted within the preventive support scheme, since the students in difficulty have been shown to be capable of fully assuming their role as students in the PDS. We were able to observe that they are very often able to engage as fully as the other students in the PDS situation, and even to contribute to the progress of didactic time, despite having been deemed to have difficulties. Moreover, the teachers were able to confirm that this level of involvement of these students in difficulty was unusual.

The post-ADS

This auxiliary didactic system presents three functions (Assude & Millon-Fauré, 2021; Morin et al., 2019; Theis et al., 2016).

First of all, it presents a *memory function*, since the aim is to have the students recall the session and revisit the essential steps of the lesson. This work seems particularly formative insofar as the description of the various events and their chronology make it possible for students to better understand the articulations between the various concepts approached and the path which led to the construction of the new knowledge. Assude and Paquelier (2005) similarly emphasize the benefits of this recollection and highlight three of its effects: the expression of a personal time in the classroom, the reconfiguration of the student's lived experience and a time recalled as a shared time. The researchers nevertheless specify that this exercise can prove delicate for most of the students to perform and requires specific guidance by the supervisor, which underlines the value of this memory function in the post-ADS.

This ADS also presents a *renewal of the institutionalization*, wherein the supervisor asks the students what should be remembered following the PDS session. This question will lead students to explain the knowledge involved in different ways. These successive reformulations are intended to help students in difficulties to better understand the processes of decontextualization and depersonalization involved in the taught knowledge, which can enable them to make up for the delay they often experience at this point. Indeed, we observed that some students needed more time than their peers to access the targeted knowledge: implementing these post-ADSs sometimes enables certain students to complete a process of learning that had only begun in the classroom (Assude & Millon-Fauré, 2021). This time of recalling the lesson can also lead the students present to ask questions about certain aspects they had not dared to raise in class. These reformulations can furthermore facilitate knowledge.

We also observe a *reinvestment function*, as the post-ADS helps the students to transfer the knowledge discovered in a given situation, to use it in another context. Indeed, it is not only a matter of the student having understood and memorized the problem taught during the PDS: they must also be able to contextualize this knowledge and use it in a new situation, which may sometimes require the support of the supervisor. In order to facilitate the implementation of this articulation between taught knowledge and operational knowledge in different contexts, the supervisor will propose, during the post-ADS, to solve similar tasks or a reflection on the types of tasks in which the targeted techniques could prove useful. However, care should be taken to ensure that the tasks proposed are not too different from those worked on during the PDS, otherwise the student may not be able to build on the knowledge acquired in the PDS.

With these theoretical underpinnings, the purpose of this paper is to consider different pre- and post-ADS actions which can be implemented for a given class session.

Methodological aspects

Our work contributes to the recent current of collaborative research (Bednarz, 2013). The data collected during this project consisted of all ADSs and PDS plans, video recordings of pre- and post-ADS sessions, PDS sessions, student productions, and recordings of interviews with teachers and supervisors before and after each preventive support scheme session. We conducted an *a priori* analysis of the sessions and then viewed the sessions, annotating the verbatim with respect to the potential functions of pre- and post-ADS of the scheme and their effects. We were particularly interested in the students who had participated in the two ADS sessions: we observed the manifestations of the functions, the way that these students engage the situation, and analyzed the proposed techniques and their possible difficulties in order to study their evolution during the sessions. Examples of analyses are presented in our previous articles (Assude, Koudogbo et al., 2016; Assude & Millon-Fauré, 2021; Theis et al., 2014).

To present the potential actions to be implemented before (pre-ADS) and after (post-ADS) the classroom session, which emerged during the implementation of this scheme and the reflections of the team, the focus of this communication is on a specific class session targeting the concept of fractions. The main task of the PDS was to find the whole from one or several parts (Van de Walle & Lovin, 2005).

A priori analysis

The first targeted task is to reconstitute the whole from its quarter, represented by a small cardboard square. The main technique here is to draw the outline of the square four times on the paper to form a whole. The main potential erroneous techniques are the following: performing the reverse of the task or drawing the original square and reproduce it four more times to have a total of five parts instead of four. In the second task, students can reconstitute the whole from its third, represented by a small Cuisenaire© rod. The techniques required for this task are similar to the first, but here are aided by the use of rods. The final targeted task is to reconstruct the whole from its three-fifths, represented by a cardboard rectangle. Examples of incorrect techniques associated with understanding the role of the denominator or numerator of the fraction would be to reproduce the cardboard five times since it is one-fifth, or to reproduce it three times since there are three.

To illustrate the nature of the work proposed in the preventive support scheme, we will now describe different pre- and post-ADS which could be implemented for this particular session. These proposals are based on analyses of the experiments we have conducted, several of which have been effectively implemented.

Propositions for potential pre- and post-ADS

From these three tasks proposed in the PDS, several pre- and post- ADS can be outlined. These are presented hereafter by analyzing them in terms of their potential functions, always with the aim of making problem solving accessible to these students and enhancing their participation in class.

Potential pre-ADS

A first action could be to **reactivate certain objects of formerly acquired knowledge** which are required during the PDS session to ensure that the students will be able to use them *(mesogenetic function).* In the particular case of the classroom session observed here, the supervisor could for instance decide to ask the students questions about the concept of fractions: what do they remember about fractions? Could they give examples of fractions? How can a given fraction be represented? It is also possible here to return to the lexicon linked to this mathematical concept, such as the terms 'numerator' or 'denominator'. It should be noted that this vocabulary is not really essential for understanding the PDS instructions or for solving the proposed problem, but it can facilitate the description of the techniques that will be used in the class.

A second possibility consists of **presenting to the students the instructions of the problem subsequently worked on** and to **focus the questioning on the understanding of the task**, while taking care not to make the students actually perform the task (*mesogenetic* and *chronogenetic* functions). The instructions of the problem chosen being quite limited, this choice can prove difficult or even irrelevant for the supervisor to implement.

Along the same lines, it could also be useful to **have the students work on a similar type of task** to that of the PDS but with different numbers, for example, more simple fractions such as a half or third (*mesogenetic* and *chronogenetic* functions). However, the supervisor must be mindful not to advance on the didactic time by working on techniques which are too similar to those required in the PDS.

Another option is to **work on techniques which are relevant to the PDS tasks** by asking students to identify a part of a whole using a material support (*mesogenetic* function): for example, find the quarter from a whole represented by a rectangular piece of paper. The corresponding technique consists of sharing the whole into four equal parts (by folding, cutting or marking), then identifying the unit fraction one quarter of the whole (by hatching, coloring or superimposing). By isolating the quarter of the whole, it is possible to observe that the parts are all the same size and that what remains (three quarters) is the complement to constitute the whole (four quarters). This technique helps to convey the "part of a whole" meaning of a fraction as well as the relationships between the parts and the whole. This will be useful in solving the PDS tasks (reconstructing the whole from a part of the whole).

Regarding techniques involving the Cuisenaire© rods, it could also be useful to take a moment to **discuss the material** with regards to fractions (*mesogenetic* function). Indeed, it is very likely that the students had already used this material in the context of counting and the first operations on integers. They therefore know that each colored rod has, in the context of natural numbers, a specific length and represents that number (for example, the red rod has a length of 2 cm and represents the number 2). However, when used in the context of fractions, the measurement of the rods differs depending on the unit chosen. Thus, it is necessary for students to choose the rod that represents the whole (or the unit) and this rod can be different to the situations illustrated. These differences can then be pointed out during the pre-ADS, either in the context of a more open discussion based on a simple task of representing a fraction from a whole, or in a more direct way by the supervisor.

Potential post-ADS

The supervisor can first of all ask the class to **recount what happened in the PDS session** *(memory* function). The supervisor's questions will encourage the students to enrich certain passages of their narration by expanding on points that were perhaps covered too quickly. It is also possible to use elements from the PDS to facilitate this recollection: for example, the supervisor can present photos of the board taken during the classroom session or certain student productions. It is also possible to make a video recording of the session and then to extract some key segments which the students must then comment on. It should be noted that, for this type of preventive support scheme, the fact that the post-ADS supervisor is not the class teacher can be an advantage. Indeed, in the students' eyes, questions about the session will seem much more justified than if they are asked by someone who attended the session.

A second type of post-ADS consists of **questioning the students on what they should retain from the session**. It is then a question of activating the didactic memory of the class in order to make the pupils reiterate the knowledge stated in class, possibly after certain reformulations guided by the teacher (*memory* and *recollection of the lesson* functions).

The supervisor can **propose a similar type of task** to that proposed in the PDS and which can be solved in the same way, while being mindful not to propose a more complex task (at the risk of losing the benefit of the confidence acquired by the students during the pre-ADS/PDS). This enables the student to apply the technique acquired during the PDS (*reinvestment* function). One variant consists of **asking students to propose a similar type of task** to that given during the PDS, being nevertheless mindful that the task should not be too different to the original task (the supervisor no longer controls the didactic variables in this case).

A fourth type of post-ADS could take the form of **revisiting a difficulty observed during the PDS** (*memory* and *recollection of the lesson* functions)**.** For example, the final task related to reconstituting the whole from its three fifths represents an obstacle for students who struggle. It might therefore be interesting to return specifically to this task and ask students to comment on the strategies, correct or otherwise, which may have been employed by their peers.

Conclusion

The purpose of this communication is to consider what preventive measures can be put in place to support students identified as having learning difficulties before and after a class session on mathematical problem solving. It is true that the students are taken apart before and after the class session, but this measure allows them to become more involved in the mathematical task and to contribute to it like all the other students. This is a device that gives them back their power to learn in synchrony with the other students in the class.

Thus, after having specified the functions of the scheme, we have, from an experimentation based on the concept of fractions, presented the tasks proposed in the PDS and suggested other possible interventions. Below is a schematic summary of the pre- and post-ADS associated with the chosen classroom session (PDS) (Table 1). Such analyses can be made for other mathematical topics as measure of length (Theis et al., 2014), notion of area (Assude, Millon-Fauré et al., 2016), volume (Marchand et al., 2021) or construction programs (Millon-Fauré et al., 2021).

Possible actions – pre-ADS	PDS	Possible actions – post-ADS
- Reactivate certain objects of formerly acquired knowledge required to solve the PDS tasks - Question the students on their understanding of the PDS instructions - Work on formerly acquired techniques relevant to the PDS - Discuss the material of the PDS	T_1: Reconstitute the whole from its quarter (cardboard) T_2: Reconstitute the whole from its third (rod) T_3: Reconstitute the whole from its three-fifths (cardboard)	- Recount what happened in the classroom sessions - Question the students about what they should retain from the session - Propose a similar task to the students - Revisit a difficulty observed in the PDS

Table 1: Overview of potential actions for the pre- and post-ADS

The result is a range of possible tasks available to supervisors involved in the system. The choice of one of these tasks depends on the students, their difficulties, the PDS teacher, the ADS supervisor and the situation itself.
This study thus illustrates the richness of the preventive support scheme, which can be adapted by the teachers/supervisors in various ways. However, these choices directly influence the effects of the pre-ADS on the PDS and in turn the PDS on the post-ADS and can sometimes call into question the possible gains in terms of student engagement. This is particularly true when tasks that are too similar to those of the PDS are dealt with in the pre-ADS, or when tasks that are too complex are introduced in the post-ADS, which shifts away from the aims of the classroom session and generates certain effects related to the topogenetic function. For this reason, we believe it is essential to inform teachers/supervisors of the different options available, so that they can then choose the one which will prove most beneficial for their students.

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Inclusive landscapes of investigation in mathematics classrooms with deaf and hearing students

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This paper aims to discuss possibilities for educational mathematics practices addressing the inclusion of deaf students in regular schools. In order that I present reflections on the meetings amongst deaf and hearing students in mathematic classes in a Brazilian context. My reference for such discussions is the results of my doctoral research whose focus was interaction in a mathematics classroom in which students speak different languages – Portuguese and Brazilian Language of Signs. First, I present the context of deaf education in Brazil. Following, I present the design of the study and my reflections concerning the dialogical interaction pattern and aspects of Inclusive Landscapes of Investigation with reference to mathematics classes where deaf and hearing students are together. I conclude that educational practices based on Inclusive Landscapes of Investigation can support mathematics classes in the context of inclusion of deaf students in regular school.

Keywords: Inclusive mathematics education, meetings amongst differences, dialogue.

Introduction

Inclusive Education has been based on concepts related to human rights and progresses towards an ideal of equity, which recognizes the different means and conditions for different groups or people to have access to the same rights (Moura, 2021; Skovsmose, 2019). According to Kollosche et al. (2019), most investigations involving this theme in the context of Mathematics Education have been presented through concerns related to students with disabilities or special needs. For the authors, the main motivation for that has been established through legislation, international agreements, and educational practices in schools.

However, the notion of inclusive education, especially in the field of mathematics education, has been discussed as something that challenges the diversity we find in classrooms in general (Bishop et al., 2015; Skovsmose, 2019). In this sense, researchers have been discussing the different obstacles to inclusive education, as well as the possibilities for dealing with them through an agenda of equity and inclusion in mathematics education (Skovsmose, 2019; Moura, 2021; Roos & Bagger, 2021).

The discussions presented in this paper aims to contribute to this agenda through an experience in Brazil's educational context, where most of the concerns of inclusive education are related to schooling for people with disabilities. Since 2008, the introduction of new educational policies determines that these students become enrolled in regular schools[1]. Thus, the proposed discussion

[1] Regular schools are public or private schools which do not include specially assigned special education or student behavioral intervention programs within or outside the enrolled schools

here is directed to the inclusion and permanence of deaf students, as well as the need to provide effective inclusive practices for these students.

The present paper aims to discuss possibilities for mathematics practices addressing the inclusion of deaf students in mathematics classrooms in regular schools. Such discussions are based on the results of a doctoral research that sought to understand how interactions happen in mathematics classes where deaf and hearing students are together, and the tasks are organized according to the perspective of Landscapes of Investigation (Moura, 2020).

The doctoral research data were generated in mathematics classes of a fifth-grade class of elementary school (10 years old) of a municipal public school in the state of São Paulo, Brazil. This school is recognized in the city for the work it has been doing for the inclusion of deaf students by adopting a pedagogical proposal based on a bilingual perspective of education, in which the first language of communication and schooling for deaf students is the Brazilian Sign Language (Libras) and the Portuguese language is used in its written form, which is teaching and learned based on the experiences obtained through sign language.

With a total of 22 participants, including three interpreters, two teachers, and 17 students, of whom five were deaf and 12 were hearing, data production involved the planning and development of the tasks were based on the proposal of Landscapes of Investigation, according to the perspective of Skovsmose (2000). In which follows I present the theoretical framework relate to the context of the research.

Theoretical Background

Although the presence of sign language interpreters doesn't guarantee the inclusion of deaf students in the teaching and learning practices, having this professional in the classes where deaf and hearing students are together is the first step to the promote equity. The role of sign language interpreters has been redefining over the last few years, in view of the demand for the classroom and the implementation of different practices. With this, the relationship between professors, interpreters, and students, as well as the ways of communication have been shown to be distinct (Moura, 2020).

Regarding the mathematics classes, research have been indicating that there is no direct interaction between deaf and hearing students and neither between teachers nor deaf students (Borges & Nogueira, 2019). Thus, usually teachers communicate with hearing students and interpreters. Interpreters mediate the interaction between deaf students and the teacher and with other students when there is an opportunity (or need). Deaf students communicate directly only with each other and/or with the interpreter, and the answers to the deaf student's questions are almost always given by the interpreter and not the teacher, causing the isolation of these students in the classroom.

Based on this fact, Moura (2020) suggests thinking of an interaction pattern like the one described in Figure 1.

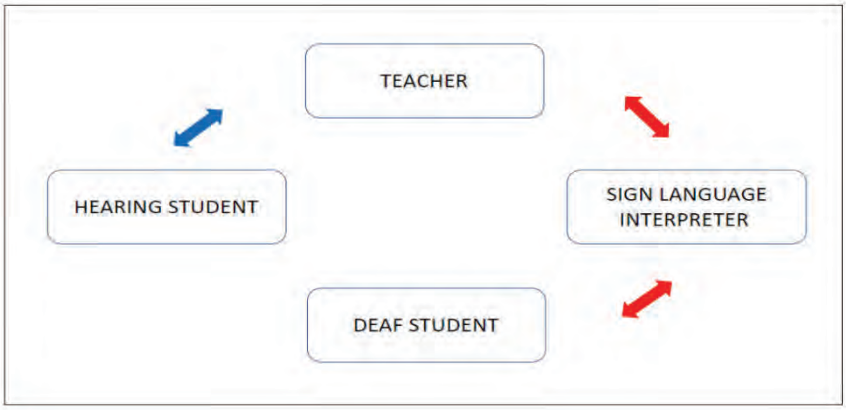

Figure 1: Interaction pattern in classes with deaf students and sign language interpreter

In the classes in which this communication pattern is common, is also possible to realize the little interaction between teachers and interpreters. In this sense, it is believed that this interaction pattern contributes for the interpreter to assume the responsibility for teaching the contents to students. According to Silva and Oliveira (2016), situations like this are common in classrooms where are together deaf and hearing students, and in addition to giving the interpreter a role that does not concern him/her, they end up further distancing deaf students and teachers, since the reference of these students becomes the interpreter and not the teacher. In other words, they only share the same space with the hearing students, there is not an effective inclusion.

In mathematics classes, the concerns about these conditions are bigger, since the sign language interpreters don't have enough mathematics knowledge to teach the contents to deaf students (Pinto & Segadas, 2019). Usually, they teach what they understand from the explanations of the teachers, but certainly, this understanding can be incomplete or insufficient, causing inequality of opportunities of mathematics learning to deaf students.

In addition, this pattern of interaction also reinforces the practice of teaching methodologies designed and directed to listeners, based on the belief that the performance of the interpreter is enough to make the deaf understand the content (Borges & Nogueira, 2019). Although deaf students are educated through Libras, authors (Borges & Nogueira, 2019; Moura, 2020) emphasizes the importance of thinking about methodological strategies that consider the presence of deaf people in the classroom where the majority are hearing students.

Hence, the use of methodological strategies for each student could follow his/her own path towards knowledge, has been expanding. This new way of thinking methodological strategies that favor the learning of students with disabilities, Fernandes (2017) calls construction. According to this author, this proposal involves the construction of a context that allows for the sharing of perspectives and the negotiation of meanings, also enabling the development of autonomy and understanding of mathematical learning processes.

Thus, the ways of teaching and learning mathematics can be explored in a shared space, making the knowledge of this subject be understood in different ways. For this, it is essential to create learning environments that accommodate the differences of students and allow learning across them.

Design of the Study

This investigation used a qualitative approach and, as a methodological procedure, participant observation, which allowed me to investigate the object of study in its context from my insertion in the group of students (Queiroz et al., 2007). This method favored an understanding of how interactions happen in mathematics classes where deaf and hearing people are together, and also the tensions that exist in the process of inclusion of deaf students in regular schools.

These classes were recorded in audio and video and notes were made by the researcher. From these records, episodes were constructed in which the interactions between participants are described. To help structure the episodes, we sought to analyze the videos along with the audios, inspired by the analysis phases presented by Powell et al. (2004).

After, these episodes were analyzed in view of the aims of the research, where some aspects concerning the proposal of Landscapes of Investigation in a context with deaf and hearing students could be identified. In the following, we present results and discussion where we reflect on aspects that support inclusive mathematics practices addressing deaf students in classrooms in regular schools.

Results and Discussions

Seeking to create mathematics learning environments that accommodate the differences of students, in my doctoral research I presume that the Landscapes of Investigation proposal could provide an inclusive learning environment. Usually, in mathematics classes, the teaching is organized based on some theoretical explanations followed by exercises involving techniques and procedures presented previous by teachers. Landscapes of investigation break with this exercise paradigm through dialogical communication, allowing the students to be responsible for their own learning process and encouraging experimentations. The students are invited to engage in investigative tasks, which makes room for the discovery of mathematical facts, refutations, production of new meanings, critiques, reflections, hypothesis testing, and cooperation (Skovsmose, 2000).

Landscapes of Investigation played a central role in the research. The main pattern of communication is dialogue, in which the teacher leaves the centrality of the class and the perspectives of everyone else start to gain importance. Thus, through investigation, deaf and hearing students have the opportunity to interact. Whether directly or with the mediation of the interpreter, students get to know the perspectives of each other in a cooperative manner on an equity condition. The valorization of sign language encourages hearing students and teachers to learn the language to communicate with deaf colleagues, expanding the possibilities of interaction between students and conducting the investigation in different ways. It is possible to experience other ways of learning.

In the episodes of the classroom that made part of my doctoral research, I identified an interaction pattern in math classes with deaf and hearing students that I named Dialogical Interaction Pattern as illustrated in Figure 2.

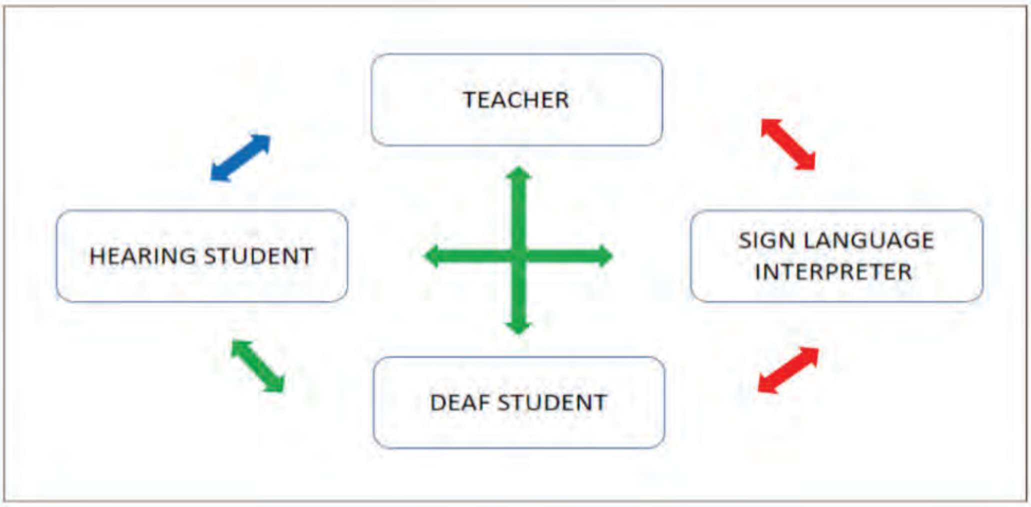

Figure 2: Dialogical Interaction Pattern

In this new pattern, the teacher leaves the centrality of actions enabling new interactions to emerge, namely, interactions between teachers and deaf students, interactions between deaf students and hearing students, and interactions between interpreters and hearing students.

When seeking to understand the perspectives of students and interpreters, the teacher leaves of being the only one to have something to teach. Active listening, as well as teacher displacement to understand the student's perspectives, contributes to overcoming the vertical relationship between professor and students. The teacher is still responsible for planning, mediation, decisions on methodologies appropriate to the various mathematical contents and by assessment strategies. However, the teacher is no longer the one who only teaches, and the student is no longer the one who only learns.

In this learning environment, teachers have the opportunity to learn too with deaf students and sign language interpreters. This movement towards overcoming the vertical relationship during classes allows deaf students to share their perspectives, their way to understand the world, and their language. For this, they rely on the help of a sign language interpreter who, after the deaf, is the main disseminator of deaf culture and identity. The partnership between teachers and interpreters is strengthened in order to contribute to the cooperation among deaf and hearing students. By moving to understand the perspectives of students, the teacher is open to the redefinition of ideas and positions, respecting the different ways of learning and paths to be followed. Thus, the planning and methodological assessment strategies start to also consider the perspectives of deaf students and how they experience the world, which is essential for the construction of equity.

To emphasize the potential of learning environments based on Landscape of Investigation for the inclusion of people with different skills or backgrounds, Skovsmose (2019) defines Inclusive Landscapes of Investigation. A proposal based on a universal design for learning perspective, which seeks to eliminate the need for adaptations through the construction of pedagogical strategies accessible to all students, highlighting the particularities of students as possibilities and not as obstacles to carrying out the tasks.

In Inclusive Landscapes of Investigation, dialogue is assumed as a pattern for interaction favoring cooperation and the construction of equity, in other words, is an environment of learning in which the Dialogical Interaction Pattern is established. From this idea, Skovsmose (2019) highlights three aspects concerning Inclusive Landscapes of Investigation: facilitate investigations; enable an environment of learning accessible to students; favoring for collaboration among students.

During the analyses of the classroom episodes, beyond these three aspects highlighted by Skovsmose (2019), was possible to identify more five as aspects essential which emerged from the Dialogical Interaction Pattern in this Inclusive Landscapes of Investigation: contribution to the construction of equity through the appreciation of sign language; recognition of the sign language interpreter as someone who can contribute to the learning of all; the opening of spaces for the negotiation of signs of mathematical terms that still do not have a defined sign in Brazilian Sign Language; possibility of other learning (and investigations) that support students' mathematical learning; opportunity for students to experience new ways of learning, contributing to the critical and reflective knowledge. Thus, we can say that there are eight important aspects when considering the proposal of Inclusive Landscapes of Investigation mathematics classes with deaf and hearing students.

The facilitation of investigation is an aspect that must be considered from the planning of tasks. Doing investigation is an open task that does not include solving an exercise list. Thus, it is important to seek to develop tasks with different possibilities for answers and to invite students to an investigation.

Enabling an environment of learning accessible to all students is an aspect that must be considered from the planning of tasks. Doing investigation is an open task that does not include solving an exercise list. Thus, it is important to seek to develop tasks with different possibilities for answers and to invite students to an investigation.

The favoring for collaboration among students is mainly due to the dialogic interaction between them. Include sign language in the curriculum of subjects, is a way to appreciate both the deaf culture and the sign language, supporting hearing students to learn sign language in order to interact with the deaf colleagues.

The recognition of the sign language interpreter as someone who can contribute to the learning of all is related to the relationship between teachers, students, and interpreters. The sign language interpreter beyond teaching sign language for all students, can encourage students to use this language for the interactions, as well as can engage in the investigation with the students, sharing the mediation tasks with the teacher.

Opening of spaces for the negotiation of signs of mathematical terms that still do not have a defined sign in sign language is extremely important in classes with the presence of deaf students, as it contributes to the understanding of mathematical concepts. Generally, these negotiations are not foreseen in the planning of tasks but having a learning environment that allows them to be carried out, as in the proposed Inclusive Landscapes of Investigation, is an essential support for the learning process of mathematics for deaf students.

The possibility of other learning (and investigations) that support student´s mathematical learning concerns learning concepts or ideas that are not directly related to the subject investigated, but such understanding is necessary for mathematical concepts or problem situations understood or resolved.

The opportunity for students to experience new ways of learning, contributing to critical and reflective knowledge. This type of knowledge is related to the knowledge that allows students to develop their own mathematical strategies and conjectures.

Finally, I emphasize that Inclusive Landscapes of Investigate are constructions. Thus, the development of tasks based on this proposal can take on different characteristics and formats. There is no instruction guide for such constructions, however, the eight aspects presented are fundamental when considering a pattern of dialogic interaction and can be seen as aspects that contribute to the composition of these landscapes. In other words, these aspects are only possible through a change from the interaction pattern that usually happens in the mathematics classroom (Figure 1) for the new interaction pattern (Figure 2), which has dialogue communication as the protagonist. The inclusion of task proposals based on this format can create new possibilities for deaf and hearing students to learn mathematics together.

Concluding Remarks

Seeking to contribute to the construction of a culture of mathematics education that values differences and diversity, Skovsmose (2019) proposes that Inclusive Education be thought of as a new way of providing meetings amongst differences. Therefore, I understand these encounters as a human ability to be with the other (or others) experiencing a mutual relationship, which suggests a movement, an action of discovering, an act of being aware of new things.

Thinking inclusive education in terms of meetings amongst different people is to think of an education that recognizes and values the difference of each one and that envisions the possibility of learning from what is different. In other words, is an education that brings concerns about all students, whether this student with disability or not, with special needs or not. In this perspective, each student is unique and has their own way to handle mathematics. Knowing different ways of relating to mathematics also contributed to the development of a critical view that allows us to look beyond stereotypes and/or pre-established conceptions about the other.

In this regard, I see the meeting between deaf and hearing people at school as a meeting between students who speak different languages, who recognize and value this linguistic difference, as well as the learning possibilities that can emerge through cooperation and equity.

In this paper, I bring Inclusive Landscapes of Investigation as a possibility for mathematics practices addressing the inclusion of deaf students in mathematics classrooms in regular schools, in view of the meetings amongst deaf and hearing students in mathematics classes. Through dialogical interaction pattern, new meetings are facilitated in the classroom. Teachers meet deaf students and sign language interpreters, sign language interpreters meet hearing students, hearing students meet deaf students. These meetings do not only refer to sharing the same classroom, but to a movement of seeing the other, of wanting to be together, favoring cooperation and the construction of equity, crucial elements for the inclusion of deaf students in mathematics classes.

Finally, I understand that the reflections and conclusions presented in this paper may support mathematics education practices that focus on diversity. The feasibility of dialogic interaction between two culturally different groups – deaf and hearing people – shows us new possibilities for the participation of all students in conditions of equity and cooperation during the mathematics learning process.

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CLIL in teaching pupils with special needs

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Keywords: Inclusive education, pupils with special needs, CLIL (Content and Language Integrated Learning), materials for teaching.

Introduction

Education of pupils with special needs (PSN) has been studied for several decades (see e.g. Farrell, 2001). In many countries, it is an exception to find a homogeneous classroom where PSN are not present, which increases the demands on the teacher. At present, the traditional understanding of PSN is challenged by the presence of pupils with different mother tongues and low level of understanding the language of instruction, which is in most cases the mother tongue of the majority of pupils in the classroom. These children are in some countries considered as PSN while other countries do not classify them as such, despite the fact that they require similar support.

The poster presents results of a study of implementing CLIL[1] in classrooms with PSN. Attention is paid to the rules for creating suitable CLIL materials for this environment. Three pillars originally developed for materials in linguistically and socio-culturally heterogeneous classes are considered as the basis; the reported study suggests that they are appropriate if principles for teaching pupils with special needs are added.

Theoretical background

In the Czech Republic, there is an increasing demand for using non-traditional teaching strategies that develop pupils' language skills and thus their preparedness for future career, like e.g. CLIL. There are studies about the pros and cons of using CLIL, the majority of which present CLIL as a useful teaching strategy. However, taking into account the introduction, there are very few studies about using CLIL with PSN. We found only one work focusing on this issue (Karlíková, 2020). Karlíková studies CLIL teaching at primary school with a focus on PSN, specifically on pupils with learning disabilities and pupils from socially disadvantaged backgrounds. Her research shows that CLIL has no negative impact on the sample of pupils from socially disadvantaged backgrounds. At the same time her research neither confirms nor refutes the potential of teaching CLIL to pupils with learning disabilities. There are many questions that still need to be paid attention to and validated.

In Novotná, Moraová and Ulovec (2021) and Novotná, Ulovec and Moraová (2020), three pillars for a successful creation of materials for teaching in culturally and socio-culturally heterogeneous primary and secondary classrooms are presented and discussed: *Topics of interest for all pupils, Using*

[1] CLIL – Content and Language Integrated Learning refers to the teaching of a non-linguistic subject through a foreign language. CLIL works with an equilibrium between content and language learning.

cultural differences as funds of knowledge, and *How (seemingly) simple things can be very different (and difficult) in other places and cultures.*

Research question

The pillars were developed for the work in linguistically and socio-culturally heterogeneous classrooms, not for teaching PSN in inclusive classrooms. Although we see a lot of similarities when working with PSN in an inclusive classroom, it is not clear if the three pillars are sufficient for preparing materials for CLIL lessons in classrooms where PSN are integrated. Our study presented on the poster focuses exactly on this question.

Results and discussion

In cooperation with practising teachers we observed CLIL lessons in inclusive classrooms and analysed materials prepared by teachers for PSN in their classes. In accordance with Karlíková's results (2020) we found that using CLIL with them based on appropriately designed materials is not an obstacle for majority of PSN that participated in our study. However, the three above mentioned pillars are not sufficient for creating a good CLIL material for PSN. It is not surprising that the fourth pillar, *Respecting PSN's individual differences*, must be added. For example for pupils with dyslexia it is good to shorten all texts, add colours and illustrative pictures. Considering the presence of different types of learners, a matching exercise may be text – text as well as text – picture. The materials should give enough opportunities for pair work and group work allowing each pupil use their strengths in collaborative activities.

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Perceptions of mathematical creativity among math teachers in special education classrooms

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Encouraging mathematical creativity is one of the aims of mathematics education. The present study examined teachers' perceptions of encouraging mathematical creativity in special education classrooms (SEC). Three teachers of mathematics in SEC were interviewed regarding their perceptions of mathematical creativity and their role in encouraging mathematical creativity in SEC. Findings indicated that the teachers believe in the importance of fostering mathematical creativity among their students. In general, they indicated that their role was to create a supportive environment that encouraged students to think on their own. Professional development is needed specifically to introduce teachers to tasks that can promote mathematical creativity.

Keywords: Mathematical creativity, teachers' perceptions, special educational needs.

Introduction

Encouraging mathematical creativity is an important objective of mathematics education (Levenson, 2013). Research suggests that promoting mathematical creativity can strengthen the connections between different topics both within and outside of mathematics, and extend prior knowledge (Leikin, 2009). Alongside increased awareness of classroom mathematical creativity, there is growing awareness of the importance of offering equal learning opportunities for all students (DeSimone & Parmar, 2006). Teaching that exposes students with special education needs (SEN) to a range of strategies can encourage flexibility and creativity, along with a deeper understanding of mathematics, leading to improved accuracy and foster performance among students with SEN (Peters et al., 2014).

The teacher has a significant role in fostering mathematical creativity in both, general and special education classrooms (SEC). Previous studies have shown that mathematics teachers' beliefs and perceptions can affect their teaching methods as well as decisions made in the classroom (Schoenfeld, 2011). Likewise, beliefs teachers hold about creativity can influence what they do in class. Some mathematics teachers regard creativity as an acquired skill which students can develop (Lev Zamir & Leikin, 2012), while others believe that only some students have the ability for creativity (Shirki & Lavy, 2012). The current study examines the perceptions of three mathematics teachers who teach in SEC within general education schools, regarding encouraging mathematical creativity among students with SEN. SEC have a small number of students, where each student learns according to an individual learning plan, and yet may be mainstreamed in certain subjects according to ability.

Theoretical background

Our theoretical perspective of mathematical creativity is in line with Silver (1997), who viewed mathematical creativity as "an orientation or disposition toward mathematical activity that can be fostered broadly in the general school population" (p. 75). Mathematical creativity is commonly assessed according to three criteria: *fluency*, the number of distinct solutions, possibilities, or ways of solving a given problem (Leikin & Lev, 2013); *flexibility*, breaking away from familiar and fixed patterns, posing ideas from different fields of mathematics, examining a problem from different angles, expressing solutions by means of different representations (Leikin, 2009; Levenson, 2013); and *originality*, finding a new or unusual way to interpret an idea or solution (Silver, 1997).

Promoting mathematical creativity in the classroom is the teacher's responsibility, beginning with choosing appropriate tasks. Many researchers recommend engaging students with open tasks that have many answers (e.g., Levenson, 2013). Leiken (2009) recommends engaging students with multiple-solution tasks that have one final answer but many ways to teach that answer. Another type of task is problem posing, often associated with promoting flexibility (Silver, 1997). Another responsibility of the teacher is creating an environment where mathematical creativity can thrive. For example, when a teacher relinquishes some authority as the primary source of knowledge, that teacher creates fertile ground for experimentation and investigation, encouraging students to pose questions and draw conclusions. By creating a climate that makes the classroom a safe environment for posing new and original ideas, and navigating the mathematical discourse, the teacher can encourage the development of mathematical creativity among students (Levenson, 2013).

Previous studies investigated teachers' perceptions of creativity in general education classrooms. For example, elementary mathematics teachers in general education were found to perceive mathematics as a subject with limited opportunities for creativity (Bolden et al., 2010). Although they believed that creativity is important in general, they regarded mathematics as a subject characterized more by logic than by creativity, unlike subjects such as art, music, and language (Panaoura & Panaoura, 2014). Thus, in practice, they failed to create a climate that fostered mathematical creativity in the classroom. Bolden et al. (2010) indicated that teachers believe mathematical creativity manifests itself in teaching that utilizes a range of resources (e.g., technology) and examples from daily life. Other studies found that some mathematics teachers associate mathematical creativity with tasks that are different or unusual, and tasks that have multiple answers (Levenson, 2013). At times, although teachers choose appropriate tasks that have the potential to occasion mathematical creativity in the classroom, there is a gap between the potential of the chosen task, and the way it is implemented in the classroom (Lev-Zamir & Leikin, 2013).

In contrast to the abundance of studies on teachers' perceptions of mathematical creativity in general education, few studies have investigated the encouragement of mathematical creativity among students with SEN. However, studies have examined teachers' views on teaching mathematics for these students, revealing a dispute regarding appropriate teaching methods. Some argue that for students with SEN, mathematics education should focus only on a handful of problem-solving strategies and on achieving an optimal level of proficiency in just a few calculation procedures (Geary, 2003). Others maintain that teaching procedural knowledge based on memorization and retrieving facts is difficult for some students with SEN, and therefore, those students should be exposed to a variety of strategies and encouraged to develop flexibility and creativity (Peters et al., 2014). Creating learning opportunities that encourage students to tackle mathematical challenges and find different ways to solve problems can help even cognitively less proficient students develop mathematical competence (Jonsson et al., 2014). Regarding the issue of equal opportunities in mathematics classrooms, some educators argue that just exposing students with SEN to the same content and topics as those in general classes is insufficient, and that it is necessary to provide teachers with appropriate training and support, so that they can make adjustments for their students by, for example, extending the time of learning and practice (DeSimone & Parmar, 2006).

The purpose of this study was to examine the perceptions of teachers who teach mathematics in SEC within general schools, towards promoting mathematical creativity among students with SEN. The research questions are: How do teachers of students with SEN perceive mathematical creativity? How do teachers of students with SEN perceive their role in encouraging mathematical creativity in a special education classroom?

Methodology

The study included three teachers from three different schools in Israel, who teach mathematics in SEC within general mainstream schools. As can be seen in Table 1, their training background and teaching experience were quite different, representing the reality in Israel. Some teacher colleges offer additional mathematical content training for prospective special education teachers. Ravit had such training, while Irit did not. Rachel had no formal training to teach students with SEN, but instead had a stronger mathematical content and mathematical pedagogical knowledge than Ravit and Irit.

Table 1: Background of the research participants

Teacher	Pre-service education	Years of experience teaching …			
		School, students	Mathematics (non SEC)	Special education (not math)	Mathematics in a SEC
Ravit	Special education + mathematics	Primary students with learning disorders	–	12 years	4 years
Irit	Special education	Primary students with autism spectrum disorder	–	–	12 years
Rachel	Computer science + mathematics	Middle school students with learning disorders	19 years	–	6 years

The study was conducted using a semi-structured interview. The main interview questions were: (1) Is mathematics a creative discipline? (2) Should mathematical creativity be encouraged in SEC? Why? (3) How can mathematical creativity be encouraged among students with SEN? (4) Can tasks that has been shown to encourage mathematical creativity be used for teaching mathematics to students with SEN? (5) Do you implement such tasks in your classroom and if so, how?

Each interview with the researcher lasted between 25 and 45 minutes. The teachers were asked to answer the questions based on their teaching experience, both in mathematics and in special education. For interview question (4), teachers were presented two multiple-solution tasks, a type of task that has been shown to encourage mathematical creativity (Leikin, 2009; Silver, 1997). For example, the first task showed a diagram of 25 circles organized in the form of a diamond, where the learner is required to identify how many circles are in the diagram and then find as many ways as possible to count them. The interviews, which were conducted via Zoom in the afternoon hours after the end of the school day, were recorded and transcribed by the researcher.

Findings and Discussion

Inductive analysis of the data led to three main themes: what is mathematical creativity; the teacher's role in encouraging mathematical creativity; and tasks that occasion mathematical creativity.

Mathematical creativity from the perspective of teachers who teach mathematics in SEC

To the question, "Is mathematics a creative discipline?" only Rachel responded in the affirmative, although she found it difficult to explain why. The other two teachers did not respond at all. Instead, Irit and Ravit, and later also Rachel, responded by relating how they themselves teach mathematics. For example, Irit defined mathematical creativity in the following way: "Creativity (pause) [means] bringing something different, illustrating [the mathematics] in a concrete way, and making it come alive." For Irit and the other teachers, mathematical creativity had more to do with the way they teach, than the way the students learn. Specifically, they all mentioned the use of manipulatives, as related to mathematical creativity. This finding is consistent with Bolden et al.'s (2010) study of general elementary school teachers' who believed that using a variety of methods and examples from daily life is an expression of mathematical creativity.

To understand why the teachers in this current study associated mathematical creativity with the use of manipulatives, we consider their pedagogical knowledge in the realm of special education. Students with SEN often struggle to draw connections based on previously acquired knowledge, and need mediation and curriculum adjustments in order to properly establish new knowledge (Hunt et al., 2016). In the interviews, the teachers frequently expressed those students with SEN need a lot of "manipulatives," "visualization", and "repetitiveness". The teachers' reference to manipulatives may have stemmed from their need and desire to make the mathematical content more interesting, or from wanting to illustrate the content in a way that students would be able to understand. For example, Irit stated: "[Mathematics] can be either very dull or [by using concrete manipulatives] very interesting". Rachel, who was not trained as a special education teacher, illustrates the second viewpoint: "Their (students with SEN) ability to read (understanding underlying meanings), to teach them mathematics … one has to understand that the pace is different … to encourage them to learn in a different way."

Nevertheless, when the researcher delved deeper and the teachers were asked directly whether students with SEN ever exhibit mathematical creativity, they did refer to flexibility and originality, although not necessarily using those terms. For example, Rachel related to flexibility and originality thus: "Some students solve questions in such a creative way that I'm simply stunned … in motion problems, [they] don't use the familiar formula (velocity × time = distance); they solve it in an entirely different way." Rachel is hinting at original thinking. She added, "creative students are students with a different, not rigid way of thinking". This refers to flexible thinking. Rachel also referred to unique representations of solutions, which is another characteristic of flexibility (Leikin & Lev, 2013). She said: "In geometry there is room for creativity … using building blocks, folded paper. We do a lot of creating." Ravit also related to flexibility, in the sense of breaking away from a familiar and fixed pattern and combining ideas from different fields (Levenson, 2013). She attempted to define what creativity is, saying: “using what was learned in one mathematics topic, when solving a problem in another mathematics topic.”

Interestingly, the teachers also believed that creativity in mathematics can emerge through students’ mistakes. For example, Irit said: “Even if someone (a student) says something that is incorrect but explains his reasoning and says why he thinks it’s right, in my opinion it’s even more creative.” Rachel also referred to learning from mistakes as part of the process of mathematical creativity: “If a student does something and makes a mistake, I allow these mistakes. I let them express mistakes. I think mistakes are part of the learning process, and I give a lot of credit (in the positive sense) to mistakes.” Analysing these comments from the standpoint of mathematical creativity shows that the teachers create a safe environment for their students and encourage them to raise new and original ideas (Levenson, 2013), without fear of failing.

Throughout the interview, teachers also referred to difficulties in teaching mathematics to their students. For example, Ravit said that “the students are largely set in their patterns”. The phrase “set in their patterns” was used by the two other teachers as well, hinting that the teachers believe it is difficult for their students to adopt a variety of methods for solving problems. This is in contrast to educators’ suggestions of having students solve problems using different strategies and methods, promoting fluency and flexibility (Levenson, 2013).

The teacher’s role in encouraging mathematical creativity

When the three teachers were asked if and how mathematical creativity can be encouraged in their classes, all three teachers mentioned the types of questions they ask during mathematics lessons. For example, Irit said that she asks: “How did you reach this [solution]? What did you do and how did you know to do it that way?” Ravit asked more general questions which by their generality may be said to encourage fluency and flexibility: “I ask them, ‘How can this be solved? In which ways can this be solved?’” This type of discourse encourages students to think of more than one solution method, which promotes mathematical creativity, and assists in constructing knowledge. By asking open and guiding questions, the teacher raises the level of thinking, opens new channels of thought for the students, and encourages mathematical creativity. On the other hand, none of the teachers stated that they ask individual students to solve a problem in more than one way (Levenson, 2013).

Teachers also mentioned their role as mediators in the learning process, which they believe leads to the encouragement of mathematical creativity. Rachel described what happens in class after she gives students a task to work on individually: “First of all, the student thinks, he constructs some knowledge. I give him the feeling that he’s not alone. I guide him. It’s a learning process.” Rachel continued to describe how the lesson develops and how different students describe different solutions to their classmates: “The students explain … sometimes their classmates manage to understand it better [than the teacher’s explanation].” It appears that the teachers believe that mediation and working together with the students, is an important aspect of promoting expressions of mathematical creativity, as well as constructing the subject matter.

Teachers also believed that their role is to motivate students and that mathematical creativity can be a means to increase motivation. Phrases such as “fear”, “passiveness”, “(low) self-confidence”, “challenge”, “emotional aspect” were very common among the teachers’ description of the climate in mathematics classes. Rachel suggested that the added value of encouraging mathematical creativity is a way to boost motivation: “You give students the feeling that they are doing it on their own … a sensation of learning. Rather than acquiring and ‘regurgitating’ the material … [that way] they explain how they arrived at the solution …”. This is in line with Peters et al. (2014) who suggested that encouraging mathematical creativity creates a challenge for the students and thus students are more active and involved.

As seen above, the teachers appreciate their roles in encouraging mathematical creativity in special education classrooms. They encourage students to solve problems in different ways and avoid dictating only one correct right way, thereby allowing expressions of flexibility and originality.

Incorporating tasks that occasion mathematical creativity among students with SEN

Unlike when teachers were asked about their roles in encouraging mathematical creativity, and they were immediately able to offer several responses, when asked about incorporating tasks that could promote mathematical creativity, there was quiet. Specifically, Ravit and Irit struggled to give examples of tasks that have the potential to occasion mathematical creativity. When they were shown the open tasks, they thought for quite some time if such tasks could be implemented in their classrooms and for which students, making it seem that they were unfamiliar with such tasks. That being said, they were interested to find out how their students with SEN would respond to those tasks. In contrast, Rachel gave an example of a multiple-solution task in geometry (see Figure 1) that she had implemented in her class, in which students are asked to present different ways of finding the area and circumference of a certain polygon.

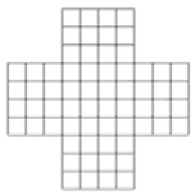
What is the area of the shape? In how many ways can you find the area?	

Figure 1: Rachel’s example of a multiple-solution task in geometry

There are several possible explanations for the differences between the teachers. First, Irit and Ravit were trained as special education teachers and gained most of their professional experience in that field, rather than specializing in mathematics. Rachel, on the other hand, had many years of experience as a mathematics teacher, and had only been teaching in special education classes in recent years. Second, the teachers had experience with different age groups. Irit and Ravit taught primary school. Irit attempted to explain why she thought it would be difficult to incorporate such tasks with young students: “Because of the gaps [in knowledge] that are created, I feel that something is always missed somehow … we are in a rush to close those gaps … we are always ‘in a war’ to make it meaningful and truly develop [students’] thinking, and on the other hand, to complete the curriculum.” Irit’s thoughts may reflect primary school teachers’ perception that primary school mathematics is about acquiring and perfecting basic mathematical skills rather than encouraging mathematical creativity. By contrast, Rachel teaches middle school, where students have more mathematical knowledge, insight, and personal experience, which – alongside the challenge of solving a problem – can encourage mathematical creativity (Silver, 1997). The differences between the teachers can also be explained as a manifestation of the different approaches to teaching mathematics to students with SEN: the notion that focusing on a handful of calculation procedures and perfecting them is the optimal method of teaching mathematics to students with SEN (Geary, 2003), compared to the approach which states that encouraging students to deal with mathematical challenges and find multiple solution strategies can help even cognitively less proficient students to develop mathematical competence (Jonsson et al., 2014).

Summary and Conclusions

This study investigated the perceptions of three teachers who teach mathematics in SEC regarding mathematical creativity and their roles in fostering mathematical creativity in the classroom. All three teachers expressed the importance of incorporating creativity in SECs. In line with mathematics educators (e.g., Levenson, 2013), they expressed the need to foster an environment which encourages discussion and questioning, and allows students to solve problems in their own ways (Leikin & Lev, 2013). Their responses hinted at their recognition of flexibility and originality.

Like general primary school teachers (Bolden et al., 2010), during the interviews it became apparent that the teachers connected mathematical creativity to students' mathematics comprehension and the way they themselves teach mathematics. Building on this perception, we recommend professional development that would introduce mathematics teachers of students with SEN to various types of tasks that have the potential to occasion mathematical creativity. Teachers can then work together, using their pedagogical knowledge and experience, to integrate such tasks during mathematics lessons. For example, teachers described the importance of using manipulatives to enhance learning. Students with SEN can be encouraged to solve a problem using more than one type of manipulative or even the same manipulative but in different ways, fostering both flexibility and a deeper conceptualization of the topic being learned.

While we acknowledge that this study cannot be generalized, we see our findings as a window into the possibility of fostering mathematical creativity in SEC, offering students with SEN equal opportunities to experience mathematics as a creative domain. Further research might investigate how mathematics teachers in SEC would implement creativity promoting tasks in their classes and how students engage in such tasks, enabling teachers to understand how such tasks can be made accessible to students with different needs.

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Explicit instruction and special educational needs in mathematics in early school years

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This paper is an exploration of how an educational method, Explicit Instruction (EI), is used and reflected upon in research in relation to special educational needs in mathematics (SEM) in early school years. The current research front is put in relation to the Swedish curricula in mathematics and the Swedish school act (steering documents) to explore potential possibilities and challenges of the research findings in a Swedish context. The analysis was done in three steps: 1) a systematic search of research literature, 2) a content analysis of the literature 3) a relation of the results of to the steering documents. The exploration of the possibilities of EI displays that EI can contribute to every student's learning by providing a distinct structure in relation to the mathematical content. It displays challenges for special education in mathematics regarding collaboration between teachers, the competence of teachers, and time to plan the EI and time for the students working with EI.

Keywords: Explicit instruction (EI), early years, special educational needs in mathematics.

Introduction

In a systematic literature review of research on special educational needs in mathematics (SEM) explicit instruction (EI) emerged as a common educational method to support and accommodate primary school teaching in mathematics focusing SEM (Bagger et al., 2019). SEM is in this paper interpreted as a socially situated and context-dependent need of educational adjustments in the mathematics education (Bagger & Roos, 2015). In the context of this paper this implies that even though some of the research found in this study has a different perspective on SEM, we view it in the lenses of being socially situated and context dependent. Regarding the notion of EI, it is a notion used in different fields of educational research and in different educational contexts and hence defined somewhat differently depending on where it is used. Though, at its core is a systematic instructional approach of pedagogical instructions. In mathematics education this systematic instruction is used to promote students' conceptual and procedural understanding with fluency (Doabler et al., 2018). In this paper EI is interpreted as a systematic instructional approach of pedagogical instructions in mathematics education to promote students' conceptual and procedural understanding with fluency. EI could be of special interest for SEM since it focuses the need to adapt the education according to every student to promote mathematical understanding. Following, the overall aim of this paper is to explore and discuss EI in relation to SEM, and to contribute with knowledge regarding EI's possible application in a school system, in this case the Swedish school system. To be able to do this, this paper puts the current research front of EI in relation to the Swedish curricula in mathematics (The Swedish National Agency, 2019) and the Swedish School Act (SFS 2010:800). Here an exploration of the potential possibilities and challenges of research findings in this specific context is made. This exploration is made with the help of the research questions of the paper: a) How is EI depicted in research in relation to SEM? b) What are the challenges of using EI in relation to meet SEM students' needs in the Swedish school system?

Setting the scene – Swedish mathematics education

This section describes the Swedish setting in which this study is situated. Compulsory school in Sweden starts at the age of 6 when children start preschool class and start being taught mathematics. There is a national curriculum, Lgr 11 (The Swedish National Agency, 2019), from preschool class and all through compulsory school (students aged 6-15). The Swedish mathematics curriculum does not in detail regulate the teaching and learning content, materials, or methods. Rather, it regulates what areas in mathematics are to be covered in the teaching, how to approach and understand students' learning, and what knowledge, competencies, and abilities education should stimulate: "Teaching should aim at helping students develop knowledge of mathematics and its use in everyday life and different subject areas." (The Swedish National Agency for Education, 2019, p. 55) According to the curriculum, the teaching in mathematics shall give the students opportunities to develop five abilities: Formulate and solve problems using mathematics and also assess selected strategies and methods, use and analyze mathematical concepts and their interrelationships, choose and use appropriate mathematical methods to perform calculations and solve routine tasks, apply and follow mathematical reasoning, and use mathematical forms of expression to discuss, reason and give an account of questions, calculations, and conclusions. (The Swedish National Agency for Education, 2019, p. 56)

Looking at how much time students are spending in mathematics education in Sweden according to the Swedish School act (SFS 2010:800), students in lower primary school shall have 420 hours of mathematics education during their three years, and in upper primary school 410 hours.

Most mathematics teachers in primary school are general teachers. Many schools have a special teacher in mathematics employed. The special teacher works as a consultant to teachers, works with smaller groups or individual students, and sometimes co-teaches in the classroom.

Special educational needs in mathematics (SEM)

In this paper, EI is investigated in relation to SEM and the Swedish steering documents. Hence, this section describes how SEM is defined and used in a Swedish context. In relation to SEM and the context, it is important to consider that the interpretation of special educational support is always situated in culture and time, meaning the interpretation and use of the term "special needs" itself "depend[s] ultimately on value judgments about what is important or desirable in human life and not just on empirical fact" (Wilson, 2002, p. 61). The writings in the Swedish School Act (SFS 2010:800) that regulates special educational support, show that it is highly situational and relative if a student is considered to need special support or a milder form of support that is basically extra adjustments of the regular learning environment. When connecting special support to the school subject of mathematics, the judgments of special support concern how the education can help the student get access to knowledge development in mathematics (Roos, 2019).

SEM is a notion hard to define and is defined differently depending on in which situation it is used and from what epistemological field it derives (Bagger & Roos, 2015). For example, if it is used within the psychological field most often it is referred to as an individual deficiency (e.g., Pitchford, et al., 2018) but if it is used as a notion describing a specific need in mathematics education to meet diversity, the institutional environment in where it is situated is most of interest (Roos, 2019). Bagger and Roos (2015) suggest using the notion of "special educational needs in mathematics” to apply a social-relational perspective, which is adapted from Magne (2006) and used in this paper. This implies a socio-relational perspective on learning when we explore IE as a teaching strategy for the benefit of SEM students studying under the Swedish curricula.

When looking at the Swedish curriculum (The Swedish National Agency for Education, 2019) and Swedish School Act (SFS 2010:800) regarding SEM and inclusion, even though the word inclusion is not used, there are governing functions of inclusion (Roos, 2021). These functions can be described as discourses, Discourse of Democracy and citizenship, Discourse of equity, Discourse of Possibilities for participation and access, and Discourse of Knowledge and assessment in mathematics. When looking at the described Discourses, their functions are mostly ideological, and there appears to be a gap to how to actually do inclusion in the education (Roos, 2021) and how to actually work with SEM.

The analysis process

This section describes the analysis in this paper, which was done in three major steps. The first step was a systematic search of research literature to find how the current research front in SEM uses EI. The second step consisted of a content analysis of the selected research literature and the third step was to put the results of the content analysis in relation to the writings in the Swedish national curriculum (The Swedish National Agency for Education, 2019) and the Swedish school act (SFS 2010:800).

In step one, to not miss out on research in SEM using EI we searched broadly in four databases that focus on pedagogical and educational research both international and national: ERIC, Scopus, Web of science, and MathEduc. The keywords were chosen to have a broad focus on SEM and education: disabilit* and math* and learning* and education* and difficult. From here we could see which of them had research concerning some systematic instructional approach of pedagogical instructions in mathematics. The search resulted in 90 articles with no overlaps between the databases. To limit the result, we added a timespan of the last ten years to focus on the current research front. Also, only articles written in Swedish, or English were selected since we are fluent in those languages. Another limitation was that the articles needed to be published in peer-reviewed journals and focus on students from 6 to 12 years old. This left 58 articles to review. 21 out of these articles somehow discussed or used EI in relation to SEM and thus were selected.

In step two, in the reading of the selected research articles, we applied a content analysis inspired by Feucht and Bendixen (2010) to identify and categorize how EI is used in the context of SEM. Here statements about EI in the research texts were identified and summarized. This identification and summarization allowed for themes to emerge. Through the identification, categorizations were made which resulted in four themes: *systematic instructional approach of pedagogical instructions in mathematics education, word problems, mathematical representations, and working memory*.

In step three we studied the themes that emerged from the content analysis and the writings in the Swedish national curriculum (The Swedish National Agency for Education, 2019) and Swedish School Act (SFS 2010:800), the Swedish steering documents. The themes were investigated in relation to the Swedish steering documents. This made it possible to explore EIs possibilities and challenges to meet SEM students' needs in their learning process and to put it in relation to national learning targets and special support.

Results

In the two sections below the results are presented by answering the two research questions of the paper. In the first section, the focus is on how EI is used in relation to SEM, and the second section focuses on the possibilities and challenges of using EI in relation to SEM in a Swedish context.

Explicit instruction in mathematics in research

Answering the first RQ of this paper, "How is EI depicted in research in relation to SEM?", we have in the content analysis in step two identified four different themes. The first theme is how EI in mathematics can be defined as a *systematic instructional approach of pedagogical instructions in mathematics education to* promote students' conceptual understanding and procedural fluency (Doabler et al., 2018; Archer & Huges, 2011). This type of definition and usage of EI is often used in interventions in relation to SEM (e.g., Swanson et al., 2015; Kong & Orosco, 2016). The instruction is conducted by validated principles of instruction, focusing on mathematical notions and abilities the individual student needs to develop. Doabler (2018) describes that in general, EI concerns three overarching principles: (1) Instructional scaffolding; here the teacher presents the mathematical content and often a procedure. This procedure is often a model that the student can follow regarding the specific type of task. This type of guidance often uses multiple representations, and the tasks are carefully selected. "Because instructional scaffolding is intended to be temporary, the support is gradually withdrawn as students become more independent in their mathematical learning." (Doabler, et al., 2018 p. 98) (2) Student practice opportunities; here the student gets time to engage and work with the content presented and guided by the teacher in step 1. In this step, the focus is to create opportunities for the SEM student to display and explore his or her thinking. This is done through discussions and visual representations of the current mathematical content worked with. The teacher must be active and give academic feedback to support the development of mathematical proficiency. (3) Judicious review; here the focus is to take time to go back and review the concepts and procedures worked with. This is made consciously and frequently. Well-designed reviews over time open opportunities for SEM students to build knowledge with critical topics in mathematics (Doabler et al., 2018).

The second identified theme is *word problems*, which is a reoccurring object of study when investigating EI in relation to SEM (e.g., Kong & Orosco, 2016; Hord & Xin, 2013; Swanson, et al., 2013; 2015). The reason for this might be that word problems contain several steps and procedural skills, as well as reading and language skills, which often is a problem for SEM students (Powell et al., 2010). A way to go about these problems is to implement EI, with specific instructions in relation to word problems (e.g., Morin et al., 2017; Swanson et al., 2013; 2015). These studies show enhanced learning for students when teachers use EI. Although, important to point out from these studies is that EI in relation to word problems demands didactical teacher knowledge, as well as teacher knowledge about the individual learning needs of the students, to really be able to be explicit in the mathematical instructions (Kong & Orosco, 2016). Furthermore, it demands of the teacher to be able to identify different types of word problems and be able to sequence them over time to help students with structures (Powell, et al., 2010).

A third theme emerging in the analysis of studies with EI is the importance of different *mathematical representations* and helping the students to translate between representations (e.g., Kong & Orosco, 2016; Morin et al., 2017). The purpose of using representations is to help SEM students to develop conceptual knowledge in mathematics and eventually be able to translate between representations themselves (Acar, 2012). The purpose is also to help SEM students to see a connection between the conceptual and abstract strands of mathematics (Kong & Orosco, 2016). It is suggested that by using mathematical representations in the form of, for example, place value blocks in the EI students will be helped to grasp general understandings of a concept, for example, subtraction (Kong & Orosco, 2016).

The fourth and last theme identified is *working memory*. Working memory can be explained as concurrent storage and manipulation of information necessary to perform mental tasks (Acar, 2012). An argument made in research for using EI with SEM students is that the structure and explicitness of EI relieve the working memory, which has been identified as a weakness for many SEM-students (e.g., Swanson et al., 2013; 2015). Also, it is suggested that visual strategies rather than verbal strategies were more forceful in supporting SEM-students with low working memory doing problem-solving. (Swanson et al., 2015). Swanson et al. (2015, p.355) argue that "to be effective, instruction should be designed in alignment with the learners' cognitive architecture" indicating the importance for the teacher to know their students.

EI and learning mathematics for SEM students

Answering the second RQ, "What are the challenges of using EI in relation to meet SEM students' need in the Swedish school system?", we elaborate on the possibilities and challenges to meet SEM students' needs by relating the themes identified in research on EI to the Swedish context.

The way EI is thought upon in research, as a systematic instructional approach of pedagogical instructions in mathematics education, does not intervene with the Swedish steering documents negatively. If looking at the abilities in the Swedish curriculum it is stated that the education "shall give the students opportunities to develop" (The Swedish National Agency for Education, 2019, p. 56) their ability in mathematics. EI can contribute to that development for every student by providing a distinct structure in relation to the curriculum goals. Though, for EI to be able to contribute to the teaching of mathematics, teachers need to be able to connect the abilities written in the curriculum to EI and the systematical instructional approach of pedagogical actions. This to put the abilities into action in terms of EI. This is the first challenge of EI since it is not explicitly written in the curriculum how to teach so that the students develop the five abilities. The second challenge is that EI demands time, both time for the teachers to plan the EI, but also time for the students to go through the three steps of EI (Instructional scaffolding, student practice opportunities, judicious review). This could be challenging to frame within the hours of mathematics education regulated by the Swedish School act (SFS 2010:800). Another challenge is the competence of the mathematics teachers, which in primary school often are general teachers. They need to be able to plan the instructional scaffolding and carefully choose tasks in relation to the mathematical concept or the procedure to be learned. Also, they need to be able to support with feedback and give students time to think and have time to work through the steps of EI. This is connected to yet another challenge factor for meeting the SEM students' needs, the knowledge and competence of teachers and school organizations regarding SEM, and multiple representations to support learning (Roos & Gadler, 2018).

Implications

This section discusses the implications of the findings in this paper.

The specific knowledge of SEM, and knowledge of how to support learning with for instance representations is needed to be able to pay attention to every students' knowledge, prerequisites, and needs. This knowledge is needed both in school organizations and for individual teachers. If there is a lack of this knowledge, there is a potential risk for SEM students not getting the right support ending up with an education that can actually hinder learning. Here school organizations must create spaces for collaboration between the special teacher in mathematics and the regular mathematics teacher to be able to plan the EI according to the specific needs of the SEM students and the specific context. Finding these spaces to collaborate and plan for the education has been proven to be a challenge in Swedish schools (Roos, 2015). If these spaces for collaboration would be offered, there is a potential for opportunities for collaboration between teachers. This collaboration might bridge the identified gap between ideological functions of inclusion described in the Swedish curriculum and the Swedish school act about how to do inclusion in the education (Roos, 2021) and support SEM-students in an inclusive classroom. Also, since the Swedish mathematics curriculum does not in detail regulate the teaching and learning content, materials, or methods, how the EI is planned and executed is much up to the schools and teachers. This can be both a possibility and a challenge depending on the knowledge of the teachers involved and the knowledge residing in the school organization. Moreover, a challenge for the schools and teachers can be not to end up in a deficit construction of SEM students where EI is the only solution. Instead, as shown by Lambert (2018), EI can be seen as one way among others to promote mathematics learning for SEM-students. In doing so, it is core to be aware of and recognize the value of students' own strategies, knowledge, and potential to engage in mathematics using multiple strategies. As argued by Lambert (2018), this can be achieved by adopting a neurodiversity perspective, which allows for the understanding of disability as diversity than a deficit.

Important to recognize when talking about EI in relation to SEM is that there are not any shortcuts. Even though overarching principles are guiding the EI, you must take the specific context and the individual student(s) and the situation in which the EI is conducted into consideration when planning, carrying through, and evaluating these lessons with EI. Although EI may take time before, during, and after the lessons, this research shows that in a Swedish setting it has the potential to diminish difficulties and enhance learning in a way that fosters a safe and fruitful learning environment. EI also has the potential to stimulate sustainable learning for every student. Though, it is important to find the appropriate explicit instructions in relation to the mathematical content being taught, appropriate tasks, and the students taking part in the education. Then, instead of taking time from the teacher searching for different methods and tasks and taking time from the students struggling to understand or remember procedures, EI can actually save time instead. Last, but not least, when connecting this teaching method to a neurodiversity perspective on disabilities, students' equity and inclusion is more likely to be fulfilled.

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Facilitator educators’ professional development for inclusive mathematics – addressing their different roles

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Keywords: facilitators, facilitator educators, professional development, inclusive mathematics.

Facilitator educators’ professional development and professional roles

Within the complex field of teacher education, facilitators and their professional development (PD) are of great importance. Facilitators as such are an extremely heterogeneous group: They have different backgrounds, tasks and functions in different working contexts, teach different subjects and types of learners (Dengerink et al., 2015). We focus here on persons responsible for providing PD courses for teachers (facilitators), and providing PD courses for facilitators (facilitator educators).

For qualifying facilitators, the Three-Tetrahedron Model (3T-Model, see Figure 1, Prediger et al., 2019) is of major importance. For the professionalization process, all three levels of the 3T-Model are relevant (Prediger et al., 2021): The tetrahedron on the classroom level – comprising students, teachers, classroom mathematical content, and classroom resources – will be transposed to the teacher PD level as well as to the facilitator PD level. On the facilitator PD level, facilitators are learners, the facilitator PD content comprises the whole tetrahedron on the teacher PD level which, again, comprises as teacher PD content the complete tetrahedron on the classroom level. The main level of acting concerns the facilitator PD level, but teacher PD level as well as classroom level are also touched.

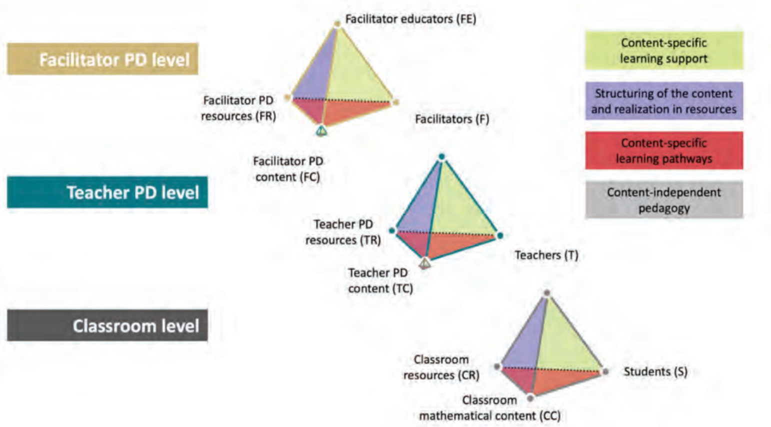

Figure 1: Three-Tetrahedron Model for content-related PD research (Prediger et al., 2019, p. 410)

Subject specific PD course ‘Coping with heterogeneity in inclusive settings’

Within the German Centre for Mathematics Teacher Education (DZLM) in cooperation with a statewide agency for teacher education, a subject specific PD course ‘*Coping with heterogeneity in inclusive settings*’ has been developed, aiming at qualifying a group of 15 facilitators (working in primary and lower secondary mathematics and special education). Exchange and cooperation of the different professions was intended for the whole course. Planned as a scaling up process, the qualified facilitators should function as facilitator educators and offer the course to other facilitators or teachers for qualifying them as facilitators. The research-based course design was grounded on various evaluated course concepts, done by the authors before. The mathematics modules were embedded in a broader concept, starting with a module, focusing on the general role of facilitators and adult learning, and followed by a module, given by the statewide agency, representing their general objectives and strategies with respect to subject specific PD. The four mathematics modules included a basic module as well as three thematic modules for inclusive mathematics, each module lasting 2.5 days. In particular: '*Basic module for deepening didactics of mathematics in the context of inclusion', 'Learning difficulties/learning disabilities/learning potentials in mathematics', 'Diagnosis and support in mathematics instruction'*, and '*Learning mathematics in inclusive settings'*. Beyond the meaningful content selection and consideration of design principles (cf. Prediger et al., 2019), the course concept put the main stress on addressing the role as facilitator educator, for example by integrating reflections as well as simulations and at the same time connecting facilitator PD level, teacher PD level, and classroom level (Prediger et al., 2019). Connecting the levels was also realized with different forms for evaluating the course quality and acceptance. One of these evaluations asked for addressing the different roles on the different levels: At the end of the mathematics modules, the facilitators had to rate the relevance of each module on a six-point likert scale, differentiated according to their role as teacher, facilitator, and facilitator educator (1 indicated 'not at all relevant' and 6 'very relevant').

Exemplary evaluation results

The overall evaluation showed that all mathematics modules mainly were assessed as very relevant or relevant for their various roles (teacher, facilitator, facilitator educator), and that the three levels (classroom, teacher PD, facilitator PD) were thus addressed (cf. Prediger et al., 2019). The primary course goal was the qualification as facilitator educator. Hence, it was expected that this level would be addressed in particular. However, all levels were addressed throughout the course: Exemplary for the module '*Learning mathematics in inclusive settings*', the participants assigned greater relevance to their role as facilitator educator and as facilitator (12 out of 13) than to their role as teacher. This is plausible and also desirable insofar as their qualification as facilitator educator and their subsequent activity as facilitator educator was intended. Nevertheless, for 9 out of 13, this module was also relevant or very relevant with regard to their role as teacher. Among other things, facilitator educators' ratings could be attributed to their individual prerequisites and acquaintance of a specific content.

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The Mathematical Support Format Reproduction

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The study presented in this paper focusses on the mutual learning of mathematics in interaction for all children in inclusive mathematics classroom. Therefore, the concept of Support Systems will be adopted in order to reconstruct different Mathematical Support Systems in the mathematics classroom and to describe the possibilities for students to participate in mathematical negotiation processes. To describe mathematical learning processes, the interaction theory of mathematical learning and the cultural-historical theory, which both place social interaction at the center of (mathematical) learning, are used. This article in particular presents the Mathematical Support Format Reproduction, which develops between a teacher and a student (with special needs).

Keywords: Mathematical support systems, inclusive mathematics education, zone of proximal development, interaction theory, mathematical support format.

Introduction

With the ratification of the UN Convention on the Rights of Persons with Disabilities, the state parties have committed themselves to ensuring equal access to an inclusive school system for all people. As a result of this ratification, in Germany, more and more children and young people who were previously taught in special schools are learning in regular schools (Vock & Gronostaj, 2017). To ensure equal participation in inclusive school and the educational success of the individual, which is a fundamental principle for the realization of inclusion in the school system, teachers with different professions increasingly work together in the classroom. In this context, the different professions are assigned different areas of responsibility in the classroom: the special education teacher is more responsible for supporting and promoting individual children, while the general education teacher is assigned responsibility for the learning success of the entire class (Melzer et al., 2015; Neumann, 2019). The question now arises as to whether the different responsibilities of the teachers result on the one hand in different ways of dealing with the children and on the other hand in a different view of the mathematics classroom and mathematics learning. As learners interact with different teachers in the mathematics classroom, different *Mathematical Support Systems* and thus different learning conditions emerge between them. The aim of the broader research project[1] is to reconstruct these different Mathematical Support Systems in order to make assumptions about the multi-professional cooperation in inclusive mathematics classrooms. In this article, first results of the analysis are presented and therefore a *Mathematical Support Format – Reproduction* is illustrated as an example. For this purpose, the following sections first present the view of learning as an interactional process that underlies this study, and then derive the concept of Mathematical Support Systems and *Mathematical Support Formats (MSF)*.

[1] The research project mentioned here is from the PhD thesis of Ann-Kristin Tewes.

Learning through social interaction

Humans are dependent on or oriented towards social contacts and their environment from birth. Only through mutual interactions with others humans can develop (Vygotskij, 2003; Steffens, 2020; Miller, 1986). Through shared activities in interactive exchanges and the shared experience of these, collective meanings are increasingly built between individuals. Such social development situations have a dynamic character and are subject to constant change. They arise when individuals encounter each other in intersubjective space and are continuously renegotiated collectively. From this it can be deduced that humans always acquire (learning) objects and meanings through exchanges with other individuals, and thus learning and development processes are always social and cultural in origin (Steffens, 2020; Vygotskij, 2003). The area in which the social exchange and thus also the learning and development processes of an individual take place in relation to his environment is called the "Zone of proximal development (ZPD)" by Jantzen (2008) with reference to Vygotskij (2002) (see Figure 1). The ZPD is a special part of the intersubjective space and creates a *Space of negotiation* between intrasubjective space of the individual and intersubjective shared space of several individuals, in which individuals mutually refer to and interact with each other (Jantzen, 2008).

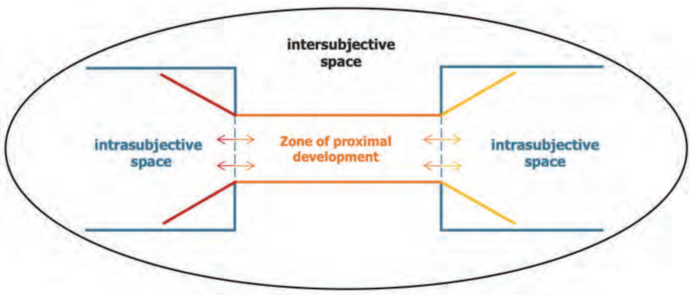

Figure 1: Social development situation between individuals

The individuals acting together in the intersubjective space mutually shape the ZPD through collective negotiation of meaning for themselves and for others (Vygotskij, 2003; Jantzen, 2008; Steffens, 2020). Once a learning or development process transcends from the ZPD – in the form of individual constructions – into an individual's intrasubjective space, Vygotsky (2002) refers to it as the "Zone of current development". The new zone of current development and the changes in the individual are then a starting point for negotiation processes in the Space of negotiation and thus also the basis for new development and learning processes within the ZPD. From this perspective, social interaction and the collective have a constitutive meaning for each subject-oriented learning process.

Mathematics learning from an interactionist perspective

Looking at mathematical learning processes in school with such an understanding of the development of learning processes, a complementary theoretical framework is needed that places interaction in relation to mathematics learning at the center of consideration. With such an interactionist perspective, mathematics learning can be understood as a process that takes place dialectically both within an individual and is located in the interaction processes in which individuals participate (Schütte et al., 2021; see also Voigt, 1995). Since the mid-1980s such interactionist approaches of Interpretative Teaching Research have been taken up and combined with subject didactic theories of mathematics learning in order to better describe everyday learning processes of mathematics (for an overview, see Krummheuer, 1992). Such interactionist approaches to mathematics learning take their learning perspective from the ideas of symbolic interactionism (Blumer, 1969; Mead, 1934) and understand social interaction as the constituting starting point of learning processes. Collectively structured learning processes give the learner the opportunity to systematically transcend his or her own mathematical abilities through exchange with others, which is then subsequently reflected in an individual process of cognitive processing. Such individualized learning then takes place in moments of reflexive deepening of what was originally collectively negotiated or learned (Miller, 1986; Schütte et al., 2021). But how does such a process of mutual learning work in specific terms? When learning in a group, such as in mathematics classes, the individual always creates his or her own interpretations of a situation based on individual experience and knowledge and in anticipation of the interpretations of others. These so-called definitions of the situation are provisional and are rejected, adjusted or stabilized in the process of the mathematical negotiation of meaning (Schütte et al., 2021). Thus, the participants in the interaction enter a process of negotiation and there continuously coordinate their mathematical interpretations of the respective situation or the respective mathematical object of other participants. From an interactionist perspective, (mathematics) learning is thus understood as a social act of interpretation in which meanings are constructed through interactive negotiation processes (Krummheuer, 1992). This negotiation process can take the form of collective argumentation (Miller, 1986; Jung, 2018), leading to the mutual production of an interpretation that is considered as taken-as-shared meaning or an *interpretive interim* (Schütte et al., 2021).

Mathematical Support Systems for participation in mathematical negotiation processes

According to the two theoretical perspectives presented above, social interaction and mutual communicative exchange are fundamental to learning mathematics. Individuals learn mathematics through social exchange in which mathematical meanings are constructed. This exchange or negotiation of meaning (Krummheuer, 1992) takes place between individuals in the Space of negotiation in the ZPD (Jantzen, 2008). Children in school also learn mathematics in interaction with others, whereby in the optimal case support in the mathematical learning process results from the interaction. Bruner (1983) developed the concept of LASS (Language Acquisition Support System) to describe such a Support System in children's language acquisition, which offers the child support in learning to use the language. A Support System not only helps the child to learn the language on the linguistic level, but also conveys the cultural context in which the language is used. According to Bruner, this Support System for the child's language acquisition is established in standardized patterns of interaction between adult and infant called formats. „A format is a standardized, initially microcosmic interaction pattern between an adult and an infant that contains demarcated roles that eventually become reversible." (Bruner, 1983, pp. 120–121) Formats are therefore routine processes with fixed roles of the interaction participants, which represent the Support System. However, by shifting roles within this system, the child can increasingly participate autonomously in the interaction (Bruner, 1983). Based on the above interactionist approaches Tiedemann describes support between mother and child as a unity of two processes. A Support System between mother and child is thus a dialectical process that is jointly shaped in interactions and is characterized by intersubjective processes (Tiedemann, 2012). Regarding the theoretical approaches of Vygotskij (2002) and Jantzen

(2008), Support Systems can be located as a dialectical negotiation process between people in the field of the ZPD (Figure 2).

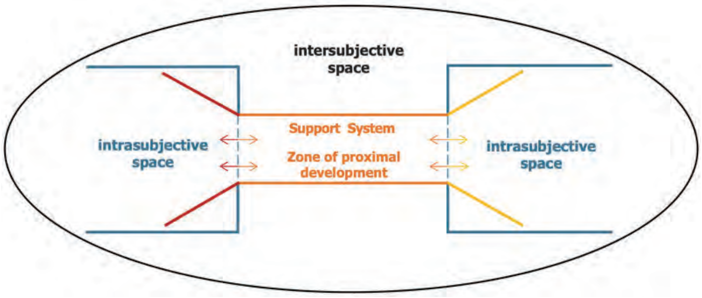

Figure 2: Support Systems as intersubjective processes in the ZPD

The involvement in a social environment and the resulting cooperation is decisive for the development of support. Statements and actions in school are thus established as a Support System when the child and the adult orient their interpretations mutually on it. Just like the ZPD, Support Systems are not fixed constructs that are developed once and then established. Support Systems are subject to constant change and constantly evolve depending on the situation. A Support System is negotiated in interaction and cannot be seen as an activity of a single person (Tiedemann, 2012; 2013). This means that there can be support efforts from the teacher, but what is established as a Support System is negotiated in the interaction in the Space of negotiation. A support system is thus a dialectical process that is shaped together. Accordingly, this also applies to *Mathematical Support Systems*. In the analyses of the collected data material, it became apparent that some Mathematical Support Systems, that could be reconstructed in the mathematics classroom, show recurring characteristics. They follow a certain recurring pattern with regard to increasingly autonomous participation and are therefore described in this paper with the term *Mathematical Support Format (MSF)* in the sense of an abductive theory development with reference to the concept of format (Bruner, 1983; Krummheuer, 1992). Accordingly, the term MSF is to be understood as an umbrella term for Mathematical Support Systems which, in the sense of the term format, occur repeatedly and are structurally similar.

Research Focus, Methodology and Procedures

The diversity of the students implies different opportunities for participation of each student in the mathematics classroom (Tewes, 2020) and thus different Mathematical Support Systems for learning mathematics between the members of the interaction in the ZPD. For this reason, the greater research project, which is located in qualitative social research following a reconstructive-interpretative methodology (Bohnsack, 2007), tries to examine different Mathematical Support Systems between students and the different professions in primary mathematics lessons. The situation from the transcript below is recorded in an inclusive primary school mathematics classroom. Thus, children with and without special educational needs learn together in this class. The diagnosis of special educational needs in Germany is carried out by a special education teacher[2]. There are two teachers working in the class – a primary school teacher and a special education teacher. In the following

[2] Whether and to what extent the diagnosis is accurate may well be debatable, but will not be discussed here.

scene, the primary school teacher works with Niklas, who has a diagnosed special educational need. To analyze the transcript concerning the Mathematical Support System and to reconstruct the mathematical negotiation process, we are doing at first an interaction analysis (Krummheuer, 1992; Schütte et al., 2019). Afterwards, the Mathematical Support System, which establishes itself between the participants of the interaction, will be reconstructed (Tiedemann, 2012).

The Mathematical Support Format – Reproduction

In the analyzed scene, Niklas and the teacher are working with a material on which colorful cards with the 'number of dots' of the numbers one and two on the left side are to be matched to the white cards with the 'number of dots' of the numbers one and two on the right side. Niklas and the teacher have already assigned the yellow card with one dot on the material on the right:

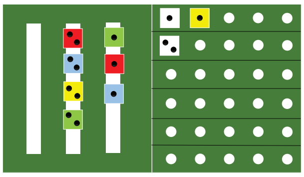

Figure 3: Material 'number of dots'

Transcript: matching 'number of dots'

08		T	[*stands up, leans over Niklas*] look at this one < [*points to the left to the red card with two dots]* > how does this look like this one or like that one\ [*points at first to the white card with one dot on the right side and after that to the white card with two dots*]
09	<	Niklas	[*looks at the card in his hand*]
10	>	Niklas	two
11	<	Niklas	[*takes the red card with two dots and attaches it to the right side of the second line*]
12	<	T	aha
			Conversation with another teacher and Niklas looks around in the classroom
15		T	*[leans over Niklas and points with her index finger at the material 'number of dots'*] (take) the next one [t*eacher removes herself]*
16		Niklas	[*takes the green card with one dot in his left hand and looks at it*] one [*attaches it to the first line on the right. He solves the red card with one dot on the left subsequently, looks at it and attaches it next to the green card with one dot on the right side, afterwards he looks at the 'number of dots' and solves the blue card with two dots on the left and attaches it next to the red card with two dots in the second line on the right side. After that, he looks at the 'number of dots' and spins the blue card with one dot on its place. He takes it with his left hand, then he takes it in both hands and looks around.*]
17		T	[*looks in the direction of Niklas*] Niklas you have to look **at** it < and then think#
18	<	Niklas	[*looks at the card in his hands*]
19	#	Niklas	one *[looks at the 'number of dots' then looks back to the teacher and takes the card and attaches it next to the blue card with two dots in the second line on the right side]*

Interactional Analysis (Summary of the Interpretation)

The teacher leans over Niklas and points with the index finger on the left to the red card with two dots. She supports the pointing gesture with the request to look at the numbers and then to make a comparison between the white cards on the right <08>. Niklas complies with this request and looks at the card in his hand <09>. He seems to be able to determine the number of dots, since he states the number without being asked <10>. Following the naming of the number, he sticks the card to the box <11>. The teacher acknowledges this action <12>. After the teacher was involved in another conversation, she asks Niklas <15> to take a next card. Niklas again complies with this request <16> and picks up the green card with one dot. Then he looks at the card in his hand and names the number of dots. Then, he attaches the card next to the yellow card with one dot in the first row on the right side. He repeats this sequence of actions – but without naming the number of dots – with the red card with one dot, which he sticks next to the green card with one dot, and the blue card with two dots, which he sticks next to the red card with two dots in the second row on the right. After Niklas picks up the blue card with one dot, the teacher asks him again to look closely at the card <17> and then to consider where to assign it. There seems to be an interruption in Niklas' sequence of actions, because he now assigns the blue card with one dot to the second row. It may be assumed that the teacher's request might have disturbed his routine. Niklas could possibly conclude that he made a mistake before and now fixes it by naming the card again, differently from the previous cards, and placing it in the other (second) row <19>.

Support Analysis

In this scene, Niklas takes over the idea of the matching from the teacher and carries out the sequences of action increasingly independently. While the matching of the red card with two points is still more closely guided by the teacher, he independently carries out the action sequence of matching with the green card with one point, the red card with one point and the blue card with two points. The looking and active action of matching the ‘number of dots’ is taken over by Niklas in the interaction and establishes itself here as a Mathematical Support System between the teacher and Niklas. The interruption described in line <17> seems to disrupt Niklas' established routine. Niklas subsequently makes a different matching that does not seem to fit the subject content. This could suggest that this Mathematical Support System is more focused on social (organizational) participation than on professional participation. The Mathematical Support System seems to support Niklas in carrying out the matching process more and more independently. Nevertheless, it does not seem to support him subject-specific, since Niklas carries out the action sequence but makes an inappropriate matching. The teacher does not address the different matching after this scene. This suggests that the focus is more on the matching process than on the mathematical content of the matching exercise. Niklas is not given much space to make his own mathematical discoveries, but learns *to reproduce* a sequence of actions in this situation. There is no evidence of a ZPD developing between Niklas and the teacher at the subject level. The Mathematical Support System seems to effect more in the Zone of current development from the subject-oriented perspective.

Conclusion

Further analyses have shown that the structures of the Mathematical Support System, between Niklas and the teacher, could be reconstructed several times. Based on these structural similarities and with the theoretical elaborations above, it is possible to speak of an MSF. The reconstructed MSF that emerges in this interaction is intended to be an example of the *MSF – Reproduction.* The MSF - Reproduction is characterized primarily by the fact that the Mathematical Support Systems that comprise it tend to support increasingly autonomous participation at the organizational level in mathematical negotiation processes. These Mathematical Support Systems support the participants in learning or performing a procedure or an algorithm. Another characteristic of the MSF – Reproduction is that the participants of the interaction are involved in a social negotiation process in the ZPD, but subject-specific the MSF – Reproduction can be classified in the Zone of current development. Also, in the specific case of Niklas, it is not clear whether the Mathematical Support System – which is established in an interactive social negotiation process in the ZPD – enables him to learn new mathematical content or to make mathematical explorations. As the Support Analysis has shown, the focus of the scene seems to be more on the matching process than on the mathematical background of the matching exercise. Thus, the Mathematical Support System can be assigned to the MSF – Reproduction. In summary, the MSF – Reproduction supports student participation in mathematical negotiation processes, but no increasingly autonomous participation is evident at the subject-specific level. In the further course of the study, it has to be clarified how this MSF has an overall effect on the participation possibilities of the students and which different consequences result from it for students with and without special educational needs and thus for the inclusive mathematics classroom. Another question that needs to be answered is whether different MSF are person-bound to specific teachers and how this affects multi-professional teamwork.

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TWG26: Mathematics in the context of STEM education

Introduction to the papers of TWG26: Mathematics in the context of STEM education

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TWG26 met for the third time, this time virtually, at CERME12 and we continued the work started at CERME11 and the virtual Pre-CERME12 Event. At CERME12, 14 papers and 4 posters coming from 10 countries were presented and discussed in order to make STEM (Science, Technology, Engineering, Mathematics) subjects more relevant to students and teachers. The papers and posters were grouped under four themes: (1) M in STEM/STEAM, (2) designing for students learning, (3) processes in STEM/STEAM, and (4) STEM/STEAM professional development. Since the themes are intertwined, each paper could be assigned to multiple themes. Therefore, the assignment of papers to themes was guided by a "best fit" approach as well as practical considerations. As some of our headings below suggest, we sought research on the broader category of STEAM with little success. Thus, the headings represent the thematic areas of our conference call but with very little representation of the Arts in each category. The ideas and issues in the papers and posters will be presented under these four themes in the upcoming parts.

Thematic areas

M in STEM/STEAM

Within this theme three papers were presented. The first paper by Larsen, Kristensen, Seidelin and Svabo examined the role of mathematics in 19 developed STEM activities within the context of the framework developed by Kristensen et al. (2021). The framework based on Kristensen et al.'s review of 37 papers indicates that mathematics can be applied as a tool or can be regarded as a goal in STEM activities in different ways. The investigation of the activities reveals that the role of mathematics as a tool in these activities is to help the students develop an understanding of science or technology or help them in engineering and design processes. The role of mathematics as a goal in these activities is about the development of students' mathematical skills and knowledge. Actually, both need to be considered together as they work together in some way. They cannot be separated from each other. Mathematics sometimes can be a tool or a goal depending on the complexity and the aim of the task, a teacher's (and/or researcher's) goal, and so forth. Mainly, it is important to understand what happens in each moment in a complex task. The discussion of the papers raised several questions moving toward CERME13:

- What are the roles that mathematics plays in STEM activities?
- How can we make learning mathematics as a goal besides using mathematics as a tool in STEM activities?
- What can STEM activities do for the subject of mathematics?
- What can mathematics do for STEM teaching?

The second paper by Bennett and Ruchti analyzed students' interactions in one classroom in which they watched and discussed a video report from the National Oceanic and Atmospheric Administration on the winter outlook for the region. They also analyzed field notes from lesson planning meetings and notes from post-lesson discussions on the base of three of the nine commonly accepted perspectives for defining STEM outlined by Bybee (2013). These three perspectives are 1) simultaneous infusion, 2) temporary shift in discipline, and 3) lateral concept connection. Simultaneous infusion uses mathematical and scientific habits of mind and practices, such as the importance of using data and communicating ideas (e.g., seeing connections between the ways behavior scientists and mathematicians engage as they investigate and attempt to make sense of real-world phenomena). A temporary shift in discipline is a shift from one discipline (e.g., science) to conceptually explore a skill in another discipline (e.g., mathematics), and then a shift back to the original discipline (science) to bring new understandings to the primary science focus. Lateral concept connection is a purposeful movement between two seemingly unrelated core ideas of the same discipline (e.g., the earth science study of weather and the biological science study of birds). The discussion addressed several important questions moving toward CERME13:

- Where is the place for appropriate STEM integration? (curriculum, assessment, instruction)?
- What are the frameworks for appropriate STEM integration?
- How best to link and leverage cross-curricular learning for authentic STEM integration?
- What are the factors contributing to teachers' STEM integration? (e.g., shifting from teacher-directed pedagogy to the more student-directed nature of an integrated curriculum, the structures of school schedules, finances, strict-level or course-level curriculum, school and teacher's knowledge about what STEM is, conception of STEM education, state of anxiety and insecurity, etc.)?

The last paper by Costa and Domingos aimed to examine an experienced Mathematics and Natural Sciences teacher development and implementation of mathematical tasks (e.g., each student infects two colleagues, each student infects three colleagues, etc.) in a mathematics classroom within the context of the COVID-19 pandemic in order to highlight the role of mathematics to understand them. These tasks raised awareness and understanding about the need for measures in the context of the COVID-19 pandemic, such as social isolation or vaccination. Regarding mathematics, powers, exponential growth, variables, iteration, functions, graphics, organization of tables, and data visualization needed to be used and understood. The discussion of this paper in the TWG raised several important issues moving toward CERME13:

- Where should we leave the STEM integration? Are we going to force this "integration" into the mathematics classroom? Into the science classroom? Into the disciplinary course?
- What are the nuances, challenges, and affordances for teachers in developing/implementing STEM tasks for/in their classroom?
- Was there anything that was more difficult?
- What is the role of M in STEM in order to innovate and improve mathematics teaching?

Designing for students learning

The second theme of the submissions involved research related to designing STEM learning environments for students. Two papers focused on the meaning of mathematical models in supporting students' learning while the third investigated students' understanding of already-created models: graphs of linear functions. Regarding the modelling theme, Haier, Siller, and Vorhölter presented a framework of criteria to guide curriculum design involving Education for Sustainable Development. They introduced the notion of socio-critical modeling which refers to the activity of reflecting on one's world critically, organizing social problems mathmatically, and recognizing the role of mathematics in making the world a better place. By merging the criteria of two similar modeling design traditions and paying particular attention to socio-critical aspects, they suggested a set of eight modeling design criteria.

For their part, Just and Siller used literature on models as black boxes in mathematics, science and the work place to develop meaning for black boxes in mathematics. The term *black box* generally refers to a system of relationships, say mathematical, that is often unseen or unknown by the user. Using their carefully crafted definition, they explored ways to integrate modeling with black boxes in mathematics education using a chemistry context. They argued that designing activities to support the opening of black boxes in mathematics education is integral to the modeling process.

Finally, Knippertz, Becker, Kuhn, and Ruzika explored the ways in which students make sense of graphical models and the implications this has for designing instruction on linear function. They used eye-tracking technology to investigate what characteristics of kinematic and mathematical graphs students pay attention to and if their gaze is drawn to different areas depending on the graph type. Their findings confirmed other studies that conclude interpreting kinematic graphs is more difficult than those that are more mathematical in display.

These three papers left us with important questions moving toward CERME13.

- What are design principles specific to supporting students' modeling and socio-critical modeling more specifically?
- How do we leverage real world situations that span the S-T-E- and M disciplines to design students' STEM learning?

Processes in STEM/STEAM

The third theme of the submissions involved research related to understanding the learning processes of students in STEM/STEAM environments. Two papers and one poster explored a variety of ideas within geometrical and spatial sense making. One paper presented a theoretical argument for an extended definition of spatial thinking within STEM, and a final paper challenged current research that suggests reading and mathematical understandings are related.

The first three research projects within the geometry domain shared findings from work with students situated within classroom teaching environments involving geometrical and spatial reasoning. Eckert and Sjödahl, for example, described the tension between providing elementary students simple coding with too much structure (i.e., pre-made codes) and not enough structure, in geometry tasks, with the aim of promoting computational reasoning. Those students who arrived at a solution fairly quickly did not engage in the process of formulating problems whose solutions could be manifested in code. Students who got stuck in their unhindered exploration did not have the supports they needed to decompose the problem situation into more manageable subproblems. Eckert and Sjödahl's findings suggest that tasks need more built-in supports to promote breaking problems into smaller, more manageable pieces.

Ubuz and Aydınyer, for their part, used a Project Based Learning (PBL) approach to support students' development of geometrical reasoning as the students engaged in the engineering design process. They presented the results of a study in which 13–14 years old students re-designed a local neighborhood by first defining the engineering problem, exploring solutions by interviewing family and community members, selecting solutions and creating scaled drawings of the buildings, streets, and other structures. Ubuz and Aydınyer concluded that designing a two-dimensional scale plan of a neighborhood through PBL can strengthen students' knowledge of the engineering design processes while also developing their spatial reasoning. Furthermore, throughout designing a neighborhood plan, they learned how to design a place, the different types of professions and their duties, how to use a protractor to draw geometrical shapes, how to solve the challenges and difficulties as a group, the importance and value of geometry in real life, how the elements of a neighborhood are placed in it, and the importance of every detail such as accuracy and precision in drawings.

Lasa et al. presented a poster that documented a STEM project in which 13–14 years old students must build and calibrate an electronic weighing machine to contextualize the concept of linear functions. They found that the use of a dynamic, geometry software program was a powerful instrument for supporting students' modelling in STEM contexts because students have the opportunity to test any number of attempts before they move to a definitive physical construction. They concluded that it is not only possible, but powerful to use technology to engage students in mathematical reasoning as a primary activity rather than simply as a tool to do the work of the other disciplines (science, technology and engineering).

Zöggeler presented a more theoretical paper exploring the meaning of spatial reasoning and its location within the STEM curriculum. Critiquing current conceptualization of spatial research, they argued that most research has explored students' spatial sense in purely psychometric terms with little attention to problem solving contexts. They introduced an extended model of spatial thinking in STEM that includes two overarching facets, *spatial problem solving* and *spatial* *memory*, with six more elaborate characteristics within those. They argued for the promotion of extended spatial thinking through mathematical, physical and technical contents as well as the promotion of spatial thinking in STEM subjects.

Finally, Cascella shared the results of a study that delved deeper into the relationship between students' reading and mathematics ability. While research has shown that there is a strong relationship between students' reading and mathematical reasoning, studies rarely account for the possible intersectionality between such a relationship and other contextual variables and/or students' personal characteristics. Cascella's research results confirmed that an integrated, interdisciplinary teaching approach is necessary and can be instrumental and powerful to fight educational inequalities across gender, socio-economic status, and citizenship.

These papers left us with important questions moving toward CERME13.

- If there is something called STEM-thinking, what would it encompass?
- What are the possibilities and challenges in talking about STEM-abilities and which abilities are addressed?

STEM/STEAM professional development

The fourth theme of the submissions involved research focused on STEM/STEAM Professional Development. The four papers and the three posters leading to this theme explored a number of intertwined aspects of STEM/STEAM Professional Development, including (1) providing a critical overview on the characteristics of STEM professionalism (Møller), (2) exploring the usability of 3D modelling and printing in STEAM education in primary school (Anđić, Ulbrich, Dana-Picard & Laviza), (3) analysing teachers' views on innovative learning activities (Erbasan & Çakıroğlu), (4) documenting mathematics teachers' experience in teaching STEM (den Braber, Mazereeuw, Krüger & Kuiper), (5) developing a STEAM professional development program for training in-service teachers and exploiting the role of mathematics within a secondary STEAM context (Diego-Mantecón, Laso, Diamantidis, Kynigos), (6) detailing the relationship between STEM practices and the development of 21st century skills (Amado & Carreira), and (7) understanding what knowledge promotes the development and implementation of mathematical interdisciplinary practices within the context of STEM education (Costa & Domingos).

In particular, Møller pointed out that in Denmark, curriculum descriptions of STEM competencies do not exist. To fill this gap, Møller developed a 'concept map', based on the review of both academic and grey literature (see Monash University, 2022). So far, Møller identified three main categories to describe STEM competences: (1) computing and visualizing 'everyday' data with computers, (2) finding and solving STEM-related problems, and (3) innovative STEM thinking.

Anđić et al. discussed the usability of modern tools, such as 3D printers, by reporting on STEAM teachers' opinions and attitudes about 3D modelling and printing. Results suggested, on the one hand, that teachers from different subjects can understand differently the usability of these tools and, on the other hand, that 3D printers require a high level of computer knowledge in order to be used effectively. These results call for a reflection about the potentials and limitations of modern tools in integrated teaching approaches, and for specific training.

In line with this, Diego-Mantecón et al. described an Erasmus+ project aimed to examine and overcome the main issues obstructing the implementation of STEAM education in secondary education, by involving teachers and scholars from Spain, Austria, Finland, Greece, and Hungary.

Research reporting on teachers' experience from other countries confirmed the need to focus on STEM/STEAM Professional Development. For example, from Turkey, Erbasan & Çakıroğlu reported on teachers' experience in teaching mathematics within an integrated (STEM) approach: even though subjects integration is explicitly mentioned in the national curriculum, teachers do not consider the integrated approach as the ordinary one, and, as with other papers presented in the same section, teachers complain about a number of obstacles (such as time constraints, lack of teamwork experiences, lack of knowledge and experience, lack of equipment for activities, and so on) hindering the success of integrated teaching.

Similarly, from Netherlands, Der Braber et al. reported on mathematics teachers' experience in teaching STEM. They discussed the appropriateness of mathematics teachers' training to teach Nature, Life and Technology (NLT). According to Der Braber et al., (teaching/teachers') freedom is, at the same time, the keyword to teach NLT successfully but also a risk if/when mathematics teachers are not aware of the learning goals of such courses, or when their background and work experience causes sharp differences (between teachers) in dealing with interdisciplinary objectives. Those who plan and develop professional training should thus be aware of this, in order to better support teachers in dealing with interdisciplinarity and exploiting the role of mathematics in such an interdisciplinary context.

Finally, Amado and Carreira presented and critically discussed the effect on a group of Mathematics teachers' attitudes and perceptions about the integrated approach after attending a professional development program committed to the innovation of teaching practices. Results showed that offering teachers practical tools, ideas, and guidelines to develop an integrated approach positively affects teachers' attitudes towards integrated teaching approaches and their willingness to engage their students in solving a real-world problem, thus confirming the importance, actually the need, to focus on professional development.

These papers left us with important questions moving toward CERME13.

- What are the key competencies for future STEM teachers?
- How do we prepare teachers to teach STEM in ways that are not isolated S, T, E and M?
- How do we help teachers discuss the role of mathematics in STEM?

Conclusion

Each of the papers and posters in the TWG26 were critically discussed in small and whole groups and captured specific aspects of "STEM" from a mathematics education perspective.

From both the small groups and the plenary discussions, STE(A)M seems to be defined as both a stand-alone subject (broader than just the sum of S+T+E and M) and as a "learning environment" within which teachers (and students) can work at the intersection of different (but sharply intertwined) subjects to construct a new learning environment within which actors can develop new, innovative, and critical thinking.

Results from the research presented seem to suggest that working with real problems is the way to combine S, T, E and M. To think about STEM as a unique subject, given by the combination - rather than just the sum - of its components, we should move from an interdisciplinary to a transdisciplinary approach. In this sense, a transdisciplinary approach can help students to develop both knowledge and competence (both theoretical and empirical/practical) not just in S, T, E and M, but in STEM, conceived as a unique subject.

Teachers in different countries employ very different teaching approaches, but the perception of STEM as an integrated subject is rare everywhere. Obstacles hindering a transdisciplinary teaching approach have been reported in some of the research presented, but also emerged in the TWG26 discussions. Among these 'hindering factors', in addition to the lack of time, experience, equipment and appropriate teachers' training, scholars from different countries (such as Portugal, Spain and Italy) also mentioned the absence of enough knowledge, in the public opinion, about STEM conceived as an integrated discipline, thus raising concerns among students' relatives/parents. All these reflections confirm that teachers from different countries experience similar difficulties and call for a prompt answer/intervention from educational policy-makers, possibly in an international, common perspective.

Research about STEM, from a mathematics education perspective, should thus focus on these criticalities in the attempt to understand, for example, what STEM education is (if there is such a thing), with emphasis on the M; what the characteristics of instructional materials, teaching practices, STEM programs (e.g., classroom implementation and/or school-wide approach) are, and thus what STEM preparation is and how it can support teachers in developing a proper and effective transdisciplinary approach.

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Teachers realising the role of STEM practices for the development of 21st century skills

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The 21st Century Skills and Competences for New Millennium Learners have been shaping curriculum reforms and educational trends around the world, including in Portugal. In this article we discuss a theoretical perspective on integrated STEM education in which practices have a central place. We consider the current importance of developing professional development programs for teachers, including mathematics, based on a conceptual framework for promoting scientific, technical, and technological knowledge and practices. Our aim is to know how mathematics teachers experience STEM activities and how they evaluate the results of their implementation in the classroom. The study uses a qualitative methodology, in the context of a professional development program. The results show that teachers performed STEM practices in a proposed task and positively evaluated the effects of their students' work on the task, namely in promoting 21st century skills.

Keywords: STEM education, teachers, practices, competences.

Introduction

The 21st Century Skills and Competences for New Millennium Learners in OECD Countries (Ananiadou & Claro, 2009) have had a strong influence in Portugal and were decisive in shaping the overall curricular design of the so-called student's profile at the end of compulsory school (Ministério da Educação, 2017). STEM education is seen as a challenging opportunity to promote the development of such skills and competences. To the public and to most of the educational community, STEM education is seen as something that is new or not yet completely known, especially regarding the ways of putting it into action. On the one hand, STEM education entails connecting science and mathematics knowledge, and their teaching in an integrated way. On the other hand, it includes scientific and engineering practices. Despite the many open questions and unknown terrain, STEM teaching and learning seems to represent a great motivational opportunity to both teachers and students. In particular, the perspective of crafting solutions to real-world problems tends to boost students' interest and curiosity.

There is no more important single factor influencing the quality of a student's educational experience in the classroom than the quality of the teaching and learning practices. Integrated STEM education is particularly relevant to the development of students' skills and competences as it involves solving real-world problems and dealing with complexity.

> Teachers make a difference. The success of any plan for improving educational outcomes depends on teachers who carry it out and thus on the abilities of those attracted to the field and their preparation (National Research Council, 2010, p.1).

Professional development programs must offer opportunities for mathematics teachers to engage in meaningful dialogues about integrated STEM education and to develop learning environments for engaging students in STEM activities (Uttendorfer, 2014). Studies have already shown that the STEM initiative is being generally well received by the teachers (e.g., Kang, 2019). In terms of increasing teacher capacity to teach integrated STEM lessons, studies found that teacher professional development courses increased teachers' recognition of the STEM approach and confidence in teaching STEM. Kang (2019) has found that there is a lack of research on the connections between teachers' perceptions of STEM and their classroom practices. In this paper, we focus on mathematics teachers' views and experiences concerning the development of a STEM activity in their mathematics classes, within the context of a professional development program.

Theoretical Framework

Moore et al. (2014) defined integrated STEM education as "an effort to combine some or all of the four disciplines of science, technology, engineering, and mathematics into one class, unit, or lesson that is based on connections between the subjects and real-world problems" (p. 38). We propose a close idea, by assuming integrated STEM education as the approach to teaching the STEM content of two or more STEM domains, bound by STEM practices within an authentic context, for the purpose of connecting these subjects to enhance students' learning.

Kelley and Knowles (2016) advocate that most content in STEM education can be grounded within the situated cognition theory (Lave & Wenger, 1991). Foundational to this theory is the concept that understanding how knowledge and skills can be applied is as important as learning the knowledge and skills itself. Situated cognition theory recognizes that the contexts, which means both the physical and social elements of a learning activity, are critical to the learning process. When a student develops a knowledge and a skill base around an activity, the context of that activity is essential to the learning process (Putnam & Borko, 2000). Often when learning is grounded within a situated context, learning is authentic and relevant, therefore representative of an experience found in actual STEM practices. When considering integrating STEM content, engineering design can become the situated context and the platform for STEM learning.

Engineering design can be an ideal entry point to work in STEM activities or projects. An engineering design approach creates an opportunity to apply science knowledge and inquiry as well as it provides an authentic context for learning mathematical reasoning for informed decisions during the design process (English, 2019). The analytical element of the engineering design process allows students to use mathematics and science inquiry to create and conduct experiments that will inform the learner about the function and performance of potential design solutions before a final prototype is constructed. Both engineering design and scientific inquiry accentuate learning by doing.

Scientific inquiry prepares students to think and act like real scientists, ask questions, hypothesize, and conduct investigations using standard science practices. Engineering and technology are closely related, and if taught in articulation with technology education can promote technological literacy. In fact, engineering and science are closely related to the way in which the technology is used.

STEM education offers students the opportunity to think through technology as a vehicle for change in culture, society, politics, economics, and environment. Studies show that students are more motivated and perform better on mathematics when teachers use a STEM integrated approach by engaging them in learning activities that include engineering design and prototyping. Incorporating STEM practices that include mathematical analysis necessary for evaluating design solutions provides a powerful basis for students to learn mathematics and to see the connections between what is learned in school and what is required in STEM careers. In this way, students develop their mathematical thinking (Kelley & Knowles, 2016).

According to Kelley and Knowles (2016), the efforts to integrate mathematics and science should be founded, in part, on the idea that knowledge is organized around big ideas, concepts, or themes, and that knowledge is advanced through social discourse. When engaging students into a community of practice, the learning outcomes can be a result of a common and shared practice. A community of practice can provide an opportunity to engage local community experts as STEM partners such as practicing scientists, engineers, and technologists who may help focus the learning.

Leung (2019) argues that teachers need to integrate the correlated STEM disciplines in ways that prevent losing the disciplines' unique characteristics, depth, and rigor. Some STEM models give mathematics and science central roles while others put engineering as the major component of STEM.

Kelley and Knowles (2016) suggest that the key to preparing STEM educators is to first begin by grounding their conceptual understanding of integrated STEM education by sharing key learning theories, pedagogical approaches, and building awareness of research results of current STEM educational initiatives. Furthermore, professional development experiences for in-service teachers may also contribute to a strong conceptual framework of an integrated STEM approach and build their confidence in teaching from an integrated STEM perspective. Kennedy and Odell (2014) claimed that STEM education programs of high quality should include (a) integration of technology and engineering into science and math curriculum at a minimum; (b) promote scientific inquiry and engineering design, include rigorous mathematics and science instruction; (c) develop collaborative approaches to learning, connect students and educators with STEM fields and professionals; (d) provide global and multi perspective viewpoints; (e) incorporate strategies such as project-based learning, provide formal and informal learning experiences; and (f) incorporate appropriate technologies to enhance learning.

Context and method

In the following, we describe and analyse mathematics teachers' experiences concerning the development of a STEM activity in their classes, within the context of a professional development program, and their views about a STEM task. The program was developed over three academic years and involved 243 middle-school (grades 7 to 9) mathematics teachers. All the participants had a professional teaching experience of more than 5 years. The aim of the program was to promote innovative classroom practices and was proposed by the regional educational authorities with the aim of reducing the rate of school failure in mathematics. It was designed according to the recommendations presented in the previous section, namely by integrating STEM practices.

In Figure 1, we present the task proposed: *Customizing paint colour*. The task was solved in small groups, where materials were made available to allow all participants performing experiments related to the re-creation of customized paint colours. The task involves some central ideas/concepts, such as: colour, mixture, ratio, volume, measurement, formulae, computation, experimentation, technological system, customer, and customizing.

MEMO

From: Sales / Orders

To: Laboratory

Subject: Recreating custom colour

We received an order of custom colour acrylic paint. It is asked to recreate the colour of the curtain sample to paint a wall of a room. The manufacturing section needs to know the composition and quantities of pigments to be used for programming the system to produce various amounts of paint. The quantity of the order is not known (1, 5, 10 liters?).

From the experience ...

You will have to find the colour as close as possible to the sample provided by the customer and obtain its composition. White base and two liquid pigments, measuring syringes and cups to make mixture trials are available.

.… to the model

From the closest possible colour obtained, find out how to manufacture any desired amount of paint, using the white base and the two primary pigments. (Consider the automated paint production system named *tintometric* system, in which a dispensing machine releases exactly the amounts of base and pastes needed to make the client's desired amount of paint).

… and to the final product…

Prepare a report of the work performed, explaining all the processes carried out, the reasoning and the conclusions obtained. You may use the proposed report template.

Figure 1. The task: Customizing paint colour

In Figure 2, we present some of the questions raised to steer discussion and reflection among the teachers, after solving the task. The reflection was intended to include the recommendations by Leung (2019) about how to pedagogically integrate the four STEM disciplines.

1. How does the task involve mathematics and the mathematical modelling process?
2. How do you see the student's learning in carrying out the task? Which ideas, contents, practices, competences are aimed?
3. Which key elements of the Educated Student's Profile do you identify in developing the task?
4. How would you integrate this task in your mathematics classes, namely what would you do before proposing the task and how would you foster subsequent learning after the task is completed?

Figure 2. Questions to guide reflection

The data collection was based on observation and documental analysis. The observation took place in some of the program sessions, in which the teachers solved the STEM task and where discussion periods with the whole group took place. After the program sessions, the teachers were challenged to propose the task to their students, in the class. The teachers were also invited to share their teaching experiences in the following sessions and to present some written reflections. Some excerpts from the teachers' reports shared in a program session will be presented in highlighting some of the results.

Results

The sessions dedicated to STEM education started with an overview and discussion of theoretical and conceptual perspectives and proposals, as suggested by the literature. Although STEM was a novel idea and concept for the participating teachers and they were not used to performing this type of tasks in their classes, the teachers showed a clear enthusiasm and interest in solving such tasks.

While working on the problem of customizing paint colour and performing experimentation with the materials provided (Figure 3), some teachers verbalized their concerns with the implementation of the task in the classroom. They were especially worried about the practical work involving the use of several materials to produce the desired colour.

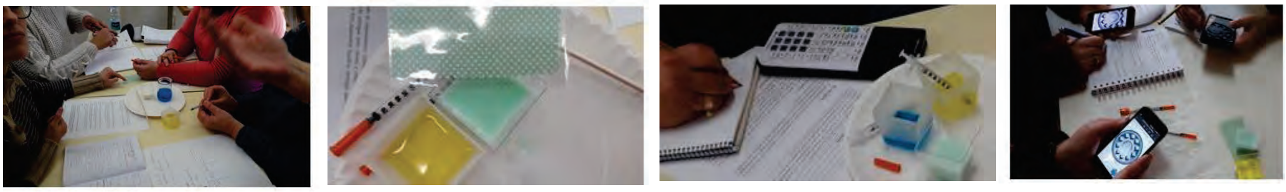

Figure 3. Images of the groups' work using materials and resources

As shown in Figure 3, one consistent finding is that the teachers systematically resorted to the digital technologies they had available. The cell phone calculator and the graphic calculator were always in use in their worktable. Some teachers also used a laptop to do some work with EXCEL. We could observe how the engineering and science practices were closely related to the way in which the technology was used. By challenging the teachers to create a possible simulation of the *tintometric* system, they mainly decided to do it with the spreadsheet (Figure 4).

Figure 4. Example of the use of Excel in a mathematical representation of the problem

Effective STEM teachers need more than just expertise in their subject matter, but they also need to be able to use instructional strategies for integrating science, technology, engineering, and mathematics into their lessons in a way that is both efficient and effective. The teachers showed great enthusiasm and dedication in solving the task, as well as they acknowledged features of the task that they found relevant to an integrated STEM activity. The idea of implementing the task in the classroom was also appealing to most of the participants. Indeed, some time later, several of the teachers used the task in their own classes. Therefore, in a subsequent session, it was possible to organize moments for sharing and discussing the results of that teaching experience.

We selected extracts of two of the teachers' statements regarding their experience with the task proposed to the students in their actual classes.

Teacher 1: Interestingly and to my surprise, the PROFIJ [vocational course] was the class that got the best results in recreating the colour and in the desired accuracy of the amounts of pigments obtained from calculations. I was surprised by the students with more learning difficulties, as they were able to overcome their difficulties with the practical activity of the recreation of the colour.

Teacher 1 proposed the task to students attending a vocational course, where students are aiming to get qualified to perform a job in the workplace. Very often, those students are characterised as highly distracted, usually chatting too much with each other, and often leaving their work unfinished. This is one of the reasons why some teachers refrain from proposing open tasks that require experimentation, collaborative work and using materials. Those students also show little interest on mathematics, have a past of school failure, and may even reject and avoid mathematics. Contrary to what this teacher expected, students' work on the task and the results achieved were unexpectedly positive and the teacher was surprised by the interest, commitment, and enthusiasm of the class in solving the problem.

Teacher 2: The task allowed the development of competencies and skills in **reasoning and problem solving**, as the students had to come up with appropriate strategies to answer the initial questions, generalize conclusions, and create **models to respond to real life situations**. This task has developed **critical thinking and creative thinking**. The students **observed**, **analysed,** and **discussed** ideas and processes by **drawing on evidence**. In groups, the students evaluated the impact of the decisions they made. The collaborative work promoted the interpersonal relationship in which the students learned to **consider different perspectives** and to create consensus. The activity carried out allowed for personal development and autonomy. The students **autonomously** designed, implemented, and evaluated strategies to achieve goals and challenges that they set themselves. The skills associated with **scientific, technical, and technological knowledge** were involved when students had to work with **resources and materials, instruments, and tools, relating technical and scientific knowledge** and identifying the technical requirements, the constraints, and resources for the realization of projects.

The comments from the Teacher 2, namely the aspects highlighted in bold, clearly illustrate the recognition of the task's potential to engage students in a STEM project and the teacher's pride on the work developed. Some of the main positive aspects that she and others highlighted after having implemented the task are here summarised: the development of a sense of responsibility, a reinforcement of collaborative work and team spirit, an increase in student's motivation, the development of students' critical thinking, promoting students' persistence in the search for strategies, and the development of problem-solving skills, mathematical communication, and creativity. Moreover, the teachers considered the work developed on the task as having facilitated and encouraged formative assessment practices aiming at developing students' skills and competences. They asked their students to produce a written report, while encouraging them to search for relevant information on the *tintometric* system. Finally, they were convinced that the work done was important to the students' appreciation of the importance of mathematics in the context of a real professional activity.

The overall results on this activity within the professional development program were quite positive in two aspects. On the one hand, it became evident that teachers formed a positive and attractive idea about the possibility of integrating STEM activities into their teaching practice. On the other hand, when putting the task into practice, many of the teachers were surprised by the way their students got involved, especially the students from vocational courses who are usually little motivated to engage in mathematical activity. In general, to all the participants, the possibilities of the task for the development of 21st century competences and skills have become salient.

Final Remarks

Our study focused on a group of mathematics teachers who attended a professional development program committed to the innovation of teaching practices. Our research methodology assumed a qualitative nature and aimed, above all, to collect evidence about the way they experienced and understood the effect of developing a STEM activity in their mathematics classes.

Regarding their involvement and interaction with the task on the creation of customized colour paint in the course sessions, we have observed that the teachers effectively assumed the scientific, technical, and technological practices involved in the solution to the problem. They have identified and worked on key concepts, namely, mathematical ones, including ratio and proportion. Moreover, they were able to integrate technological resources in their work, including the use of the graphic calculator or the spreadsheet to simulate the various trials they were performing with mixing pigments and the white base, using the materials given. Technical skills involved the need to measure the coloured liquids and to make decisions on the best ways to create the mixture. They have also engaged in discussing ideas about how colour is produced. This indicates that teachers need to be aware of the resources available, both real-world materials and digital tools, to assist them with effective instructional strategies. Using techniques such as project-based and problem-based learning show to be effective methods in STEM approaches.

In what concerns their views on the impact of the STEM activity on the students' learning, we may conclude that the teachers were not only motivated to do practical work in their classes but also to engage their students in solving a real-world problem. Their reflections reveal that after implementing the task the teachers corroborated the positive impact of the activity in their students' development of several skills and competences that are strongly connected with STEM practices. They also reported on the motivational effect observed in lower achievers, namely in mathematics, and on their engagement in finding accurate and sound solutions to the problem.

Our results, although on a limited scale, are relevant and reinforce the conclusions from other studies (Kelley & Knowles, 2016; Kennedy & Odell, 2014; Leung, 2019) in that it is possible to develop consistent frameworks for integrated STEM teacher education. Emphasising the development of 21st century skills and competences along with STEM practices in addressing the design of solutions to real-world problems appears to be a promising way of promoting integrated STEM education in our schools, at different teaching levels and with diverse students.

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Usability of 3D modelling and printing in STEAM education: primary school teachers perspective

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3D printing in STEAM education

With the increasing availability of 3D printers, their impact on various human activities from production to education began to increase (Jiang et al. 2017). In the field of STEAM (science, technology, engineering, the arts, and mathematics) education, there are various approaches to implementing 3D modeling and printing, from printing geometric shapes, molecules and cells to buildings and various buildings and similar. 3D printing has developed as an innovative pedagogical tool for enhancing STEAM learning in a number of contexts and for a variety of objectives (Hasen et al. 2020). So far, research has shown that the use of 3D modelling and printing has the following benefits in teaching: increases the active participation of students in teaching, contributes to better quality and sustainability of student knowledge, increases visualization in teaching, motivating students to learn, increases mathematical and technical skills (Leduc-Mills and Eisenberg, 2011). On the other hand, there is research that emphasizes that if students and teachers are not trained and prepared to apply 3D modelling and printing in teaching and learning, they may experience more failures in this process that can lead to frustration (Nemorin and Selwyn, 2017; Ulbrich et al. 2020). One of the last systematic literature reviews in this area (Hansene et al. 2020) suggests there is a necessity for additional research which will provide a closer insight into the benefits of 3D modelling and printing in STEAM education. This research aims to contribute to the knowledge in this area.

Methodology

We prepare our study in the form of exploratory and descriptive research. Our study uses a qualitative approach based on grounded theory to investigate teachers' perceptions of the usability of 3D modeling and printing in STEAM subject teaching. For data gathering, surveys were used. Data were collected from 200 STEAM teachers who had the opportunity to undergo training in the application of 3D modelling and printing in teaching. Grounded theory was used for data processing. In this process, open coding, axial coding and selective coding were implemented. By recommendation Strauss and Corbin (1998) line-by-line open coding was used to breaking, examining and conceptualizing the data, and gets the researcher off the empirical level by fracturing the data. In the process of open coding, we tried to give an answer on what, how and why is understood by teachers abut 3D modelling and printing. As a result of open coding, the initial codes are produced directly from the participant’s narrative. The constant comparative method (Strauss and Corbin, 1998) was employed in axial coding and selective coding in aim to develop categories and themes. In the process of axial coding, the relationship between the initials codes are established and they are classified into categories. In the process of selecting coding developed categories are refined and integrated into the theme. Examples of initial codes, categories and themes are given in the table 1.

Table 1: Initial codes, categories and themes

Theme:	Categories:	Initial codes:
Teaching contribution	Contribution to the students learning outcomes	Increasing the motivation
		Increase visualization
		Increase active learning
		Can prevent students' misconceptions

Results

STEAM teachers' opinions and attitudes about 3D modelling and printing are classified into 5 main categories named: Time, Software characteristics of 3D modelling, technical characteristic of 3D printers, teaching and learning influence, and teacher's skills. The results of our research indicate that STEAM teachers believe that 3D modelling and printing would contribute to the realization of the principles of STEAM education practice. However, teachers of different subjects within STEAM differ in what they see as difficulties in realizing the full benefits of applying 3D modelling and printing. For example, teachers of engineering and informatics have difficulty linking the implementation of 3D modeling and printing with other subjects in order to realize STEAM principles. On the other hand, teachers of mathematics, biology, chemistry, physics and art state that 3D printers require a high level of computer knowledge in order to be used effectively. Teachers agreed the successful implementation of 3D modelling and printing in STEAM education require at least basic knowledge from mathematics such as geometry– which influences students abilities to manipulate with the plane and solid, during 3D modelling, algebra– which influence students abilities to understand relations between the objects, fractions – which helps students to understand on how many parts they need to divide the figure (cube, square, circle) and which part to use for their model. Most teachers felt that good knowledge of the above mathematical areas would allow more efficient use of 3D modelling and printing in STEAM education. These and similar results will be discussed in more detail in this poster presentation.

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Designing Intentional STEM Connections: An Initial Case Study in an American Curriculum School

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Students benefit from the intentional integration of science, technology, engineering and mathematics (STEM) in ways that support deep conceptual learning in all disciplines. Elaborating on Bybee's (2013) nine perspectives of STEM, the article describes three specific and intentional methods in which integration, specifically in mathematics and science, can be effectively implemented and integrated. Each of the methods of integration has its purpose based on the context of the investigation, the intended learning outcomes, and the students' learning needs during the investigation. These methods of integration are discussed within the context of one lesson from a broader investigation on weather as it relates to the interaction of earth's major systems.

Keywords: STEM, Integration, Primary Grades, Project-Based Learning.

Introduction

Supporting science, technology, engineering, and mathematics (STEM) in the classroom has gained much momentum over the last two decades. Unfortunately, there is still no clear definition of what STEM is or means, and more importantly, what it actually looks like in the classroom (Bybee, 2013; Ring-Whalen et al., 2018). In fact, Bybee (2013) and Brown (2012) posit that a clear understanding is required because the acronym has been used in many ambiguous ways and because these ambiguities do not help educators within the reality of their practice. This means there is a need to develop methods for appropriate STEM integration, which would include such areas as curriculum, assessment or instruction so that educators can appropriately implement and integrate meaningful STEM experiences.

Review of Literature

The acronym STEM is frequently used in place of science, but how the other three content areas (e.g. engineering, technology and mathematics) are integrated varies, is inconsistent, and are often not even evident in experiences that are labelled STEM (Bybee, 2013). Mathematics, in particular, plays a fundamental and structural role in authentic science learning as seen by the description of behaviours that scientists engage in as they investigate the natural world and thus should be integrated (Michaels et al., 2008; National Research Council, 2012). Furthermore, there is an obvious overlap of the Science and Engineering Practices within the Next Generation Science Standards [NGSS] (NGSS Lead States, 2013) and the Standards of Mathematical Practice (National Governors Association, Center for Best Practices, & Council of Chief State School Officers, 2010) to promote technical ways of thinking. This overlap and alignment to technical habits of mind and interactions suggest a natural area for integration (Bennett & Ruchti, 2014).

In addition to practices, science provides multiple opportunities for students to simultaneously learn important mathematics content, especially as it relates to data analysis. However, science curricular resources rarely highlight these connections or leverage them in grade-level appropriate ways (Morris et al., 2015). Even so, educators can be well positioned to determine how best to link and leverage cross-curricular learning for authentic STEM integration. For some situations, this may mean understanding how to integrate STEM practices and habits of mind but in others it can mean purposefully injecting other STEM disciplines so that the learning can become more meaningful and applicable. This means that teachers and teacher leaders need to be aware of the specific nature of integration and how it purposefully builds students' understandings across STEM disciplines (Kelley & Knowles, 2016). However, while technical habits of mind is a good place to start integrating STEM content, the deliberate integration of STEM content at the classroom level can be more challenging.

When integrated, and when it involves a more contextual application, STEM experiences can have greater meaning and thus allow for deeper connections and more authentic learning for students (Sandall et al., 2018). Several variables in the school context can contribute to unsuccessful STEM integration. For example, many teachers struggle shifting from teacher-directed pedagogy to the more student-directed nature of an integrated curriculum. In addition, district-level or course-level curriculum alignment is often inflexible as are the structures of school schedules and finances (Margot & Kettler, 2019). Schools and teachers often do not know what STEM looks like, especially when it comes to modifying and integrating their current curriculum (Portz, 2015). This confusion leads to very different conceptions of STEM education which can cause a "state of anxiety and insecurity and, in some cases, to reject the implementation" of STEM in the classroom (Aguilera et al., 2021, p. 597).

In order to create such learning experiences, teachers often want to understand the nature of integration of the STEM disciplines. Or rather, to understand what qualifies as quality STEM learning; the knowing when and how to integrate disciplines to support greater learning. Thus, the purpose of this study is to understand how an elaboration of Bybee's (2013) integration is actualized within a primary classroom.

Theoretical Framework

Bybee (2013) outlines nine commonly accepted perspectives for defining STEM, but only six of these nine are applicable at the classroom-level. Of these six, the research team examined only those STEM perspectives that best supported a more complex integration at the classroom level; specifically, STEM means coordination across disciplines and STEM means complementary overlapping across disciplines numbers four and six above. Then, the team elaborated on these two definitions and created three specific methods for STEM integration which are referred to as: 1) *Simultaneous Infusion*, 2) *Temporary Shift in Discipline*, and 3) *Lateral Concept Connections*. In this project, *Simultaneous Infusion* uses mathematical and scientific habits of mind and practices, such as the importance of using data and communicating ideas and understandings in STEM, and are considered simultaneously during the science lesson. When using *Temporary Shift in Discipline*, is a shift from one discipline (e.g. science) to conceptually explore a skill in another discipline (e.g. mathematics), and then a shift back to the original discipline (science) to bring new understandings to the primary science focus. And lastly, with *Lateral Concept Connections* the focus of the learning moves from a primary science concept to a new science concept. The purpose of this shift is to give students new content to reinforce a mathematical concept.

In order to provide clarity and consistency, the research team established a working definition of STEM. For this project, STEM learning is understood to be opportunities, within technical content areas, to understand, create, and discuss ideas and concepts that supports students' critical thinking, analysis skills, and connections to larger concepts across and within STEM disciplines (Li et al., 2019). This focus also follows the National Council of Supervisors of Mathematics and the National Council of Teachers of Mathematics joint position paper on STEM learning (2018). Namely, that students have a strong mathematics foundation and that any STEM program or curriculum should enrich the mathematics program and also address mathematics with integrity.

Methods

The purpose of this qualitative exploratory case study (Løkke & Sørensen, 2014) was to understand how an elaboration of Bybee's (2013) integration is actualized within a primary classroom. Examining specific cases is important as it allows for the testing of theory by using multiple data collection methods in varied, yet similar, contexts to provide a more accurate and rich description of the unique, dynamic, and complex nature of classroom learning environments. With a purposeful focus for integration in mind, the research team and classroom teachers examined how the strategic integration of scientific and mathematical experiences, using the three aforementioned elaborations, allowed for a more dynamic and purposeful ebb and flow of STEM learning. That is, by using these three elaborations, the research team wanted to better understand how this elaborated framework supports the teaching and learning of integrated STEM in an upper primary classroom; this includes both challenges and affordances with respect to planning and implementation.

Participants

The primary participant in this case study was one second-year grade five teacher in their classroom, which included 23 students, from a school located in the Intermountain-west of the United States. This school serves a population with substantial social needs and financial insecurities and thus many students also often receive additional behavioural and/or social-emotional support and services. There are approximately 500 students across all grades and most teachers have between 20 and 28 students in each class and are responsible for teaching all subjects except for physical education and music, both of which are offered on a limited rotational basis. It is common to have students several years behind in reading and/or mathematics and annual test scores show that, while slightly higher than average scores across the state, only about 45% of students score at proficiency levels in mathematics.

Instructional Setting

The interdisciplinary activities described herein were developed as part of a larger, upper primary science project examining ways the geosphere, biosphere, hydrosphere and the atmosphere interact. These lessons were co-created with classroom teachers along with science education and mathematics education researchers around the intended elaborations. In the activities described below, which is a subset of the larger project on integrating STEM learning through project-based learning, the lesson objectives were to analyse and interpret weather data in order to develop understandings between weather and climate in different regions of the world.

Data Collection & Analysis

Data collected included classroom observations of students' interactions and comments, field notes from lesson planning meetings, and notes from post-lesson discussions with the teacher. Emphasis was placed on the interpretations and meaning that the researcher and the teachers had towards integrating STEM. Data were independently read and analysed to identify common themes (Paton, 2002) and were then re-categorized as needed to appropriately capture the emergent patterns.

Limitations

A primary limitation in this study revolves around the fact that it includes only one classroom, within one school. Another limitation centres on the fact that data comes from the first and only iteration of the elaborated integrations with teachers. As such, this greatly reduces the richness of the data and thus the interpretations that can be made despite promising findings for future research are limited.

Project Implementation & Findings

At the start of the lesson, students watched a video report from the National Oceanic and Atmospheric Administration on the winter outlook for the region. This was to help students contextualize how weather data can be used in their geographical area. This video report included various types of graphical data, numerical data, and a verbal discussion of the current snowpack, projected snowpack, and what this data means in terms of spring flooding forecasts. Students spent some time discussing what they heard, and what they believed were the implications of the data in their daily lives. Students were also asked to consider what knowledge and information they needed to interpret the data, based on what the meteorologist was telling them. Lastly, students were asked to consider the ways in which scientists and mathematicians think and behave in a similar fashion as they watched the video.

Next, students explored data from their own weather station. This raw data showed the daily mean temperature (F°), high and low temperatures (F°), heat and cool degree days (days when buildings needed to be heated or cooled), time, rain and average wind speed (mph) with data reported to one decimal place. Students were asked to describe what the weather was like for a week, using the raw data from one week. In small groups, students were allowed to choose any variable they wished to consider (i.e. High Temperatures, Heat Degree Days, or Average Wind Speed) and then they discussed what they noticed, wondered about, and questions they had about the data with their group to decide the "story" this data was telling. Next, they were asked to develop a statement about what their variable was "generally like for the week" and then to find a way to support the accuracy of their statement; this led to the first elaborated method of integration.

Simultaneous Infusion

Simultaneous Infusion is when the learning goals and outcomes of both content areas are well aligned and sometimes identical. This is often the case when supporting students' STEM habits of mind and is why this kind of integration was used initially to help set the context for learning. For the first lesson, the intent was for students to see the connections between the behaviours scientists and mathematicians engage in as they investigate and attempt to make sense of real-world phenomena. Thus, the local National Oceanic and Atmospheric Administration report for snowfall and the discussion on the ways in which scientists and mathematicians think and behave in a similar fashion.

Again, getting students to recognize how scientists and mathematicians use data and statistical thinking (Lane-Getaz, 2006) in similar ways is important in helping them make connections to the importance of mathematics within science.

One student indicated “All those numbers come from somewhere and they have to know what they mean before they can give a report,” highlighting the importance of keeping the context in mind when analysing and interpreting data. Another student commented, “Mathematicians make graphs and other ways to share what they know. It looks like scientists do this, too,” which further highlights similarities in how mathematicians and scientists communicate their understandings. This prompted a discussion about the relationship between science and mathematical practices.

During the whole class discussion that followed, students often described the data in more general and relative terms. Such as for High Temperature, students said that the week was warm. This created a great opportunity to discuss objectivity with the statements as the word warm alone actually cannot be used to describe the data as it is both relative and imprecise. After further discussion about how to make precise and objective statements, one student eventually recognized that they should “first try to be exact with the numbers and then figure out what those numbers mean.” This statement was agreed upon by the class and allowed the teacher to focus their attention on how mathematics can strengthen STEM learning and how both scientists and mathematicians need precision in their work. This led to the second elaboration of STEM integration.

Temporary Shift in Discipline

The Temporary Shift in Discipline pauses the learning from one discipline (e.g. science) to focus on learning a specific skill or concept in another discipline (e.g. mathematics), which will be necessary for further exploring the concepts in the initial discipline. This shift in focus allows for “just in time” learning to occur rather than teaching larger and discrete lessons on content that may seem unrelated or irrelevant to students if taught in isolation.

This part of the lesson shifted to a reasoning talk (Bennett, 2018), a process by which students reason about mathematical relationships and structures, in order to focus on key mathematical concepts that were important in exploring the science concepts. Namely, how compensation strategies can aid them in multiplying and dividing decimals. That is, by shifting to a mathematical concept, in this case the relationship between whole number operations and operations on decimals, students were able to quickly shift back to exploring the science using the mathematics in purposeful and relevant ways.

Because the intent was to help students recognize strategies for making sense of the data within a science exploration, we did not move into a formal mathematics lesson about operating on decimals but highlighted how understandings of place value used to transform the decimals into whole numbers in order to work with the data in an easier manner. Temporary Shifts in Discipline is about accessing or learning concepts from another discipline, in this case mathematics, at the right moment to make the learning in the intended discipline more meaningful, in this case the science. At this point, the final elaboration for STEM integration was used to help the students examine the weather data.

Lateral Concept Connections

Lateral Concept Connections is a purposeful movement between two seemingly unrelated core ideas of the same discipline (e.g. the earth science study of weather and the biological science study of birds). While this may seem like a *Temporary Shifts in Discipline*, it does not shift disciplines in a major way and the secondary discipline is also not a major focus. Rather, *Lateral Concept Connections* help students understand a concept within a discipline because of tangential relationships, skills, or concepts within the larger discipline that require a secondary discipline. In the weather study, numbers were presented as a mean, and students had not yet worked on measures of central tendency, there were gaps in their understanding of how to interpret the data.

For example, to help students consider how to describe the general nature of the weather for the week with their chosen variable, students were asked to consider an alternate data set on a given number of birds observed in the morning over a four-day period. This prompt was:

> "Mr. Lopez counted the number of birds he saw on his bird feeder each morning. During the 4 days he counted 8 birds the first morning, 7 birds the second morning, 9 birds the third morning and 8 birds on the last morning. About how many birds came each day? Create a model to show your thinking."

Students discussed what "about how many" meant and described it in two distinct ways. Some students used a sharing approach and talked about how "evening out" of all the birds for all of the days would tell the average while others pointed out that "we are kind of looking for the tipping point. The place where the number of birds on one side is the same as the other." For the first way of thinking about the mean (sharing), students wanted to find how to "share the birds" for the whole week. Through a purposefully orchestrated discussion, students agreed that "putting all of the birds together is liking adding them all up," which then lead another student to say that "sharing them all is kind of like dividing." All of which ultimately allowed the students to understand the process needed to describe their weather data. Students agreed that, in order to "share the weather data correctly," they needed to add up the values for their variable and then divide it by seven, as there were seven days in the week, in order to "share the data" evenly.

At this point, students moved back into the science portion of the lesson to further analyse, interpret, and report on their weather variable for the week. Note that in this case, the lesson content deviated significantly from the primary science activities in order to support a deeper understanding of the concept of mean and measures of central tendency. Again, these concepts were fundamental to their success in describing the weather for the week.

Discussion of Results & Implications

This project found that by deliberately integrating mathematics and science content through the use of Simultaneous Infusion, Temporary Shift in Discipline, and Lateral Concept Connections students were better able to develop deep conceptual knowledge in both disciplines. Attention to these elaborations, and how they appropriately scaffold learning, explicitly attend to understandings in both content areas at strategic times, and create opportunities to apply these understanding to authentic real-world problems are the heart of natural and purposeful STEM integration.

However, the initial planning was substantial and the primary teacher often relied on the other research team members to understand the nuances in the integration. This suggests that the initial planning process was not necessarily easy for the teacher, which leads to other complications with respect to implementation. Namely, if the planning is too challenging for teachers, then the elaborations may not actually help with mathematics integration in STEM learning as intended. On the other hand, and is the case with many new pedagogical approaches when initially adopted, it may be that more experience and time thinking within this framework is needed. This indicates the elaborated integration process needs to be further studied to understand the specific challenges teachers may face when implementing this elaborated integrated approach and the extent to which these challenges are a result of the conceptualization, design, or newness of the framework when implementing in classrooms.

With respect to the student learning, the data indicates that students were able to move fluidly between the different transitions to make the deliberate connections intended during the planning phase. This was even the case for shifts between disciplines (i.e. Temporary Shift in Discipline). Such results are encouraging as it suggests, from a learning perspective, the elaborated integrations support STEM learning outcomes and do so in an organic, natural manner. What is not clear at this point is the extent to which this occurs. That is, given the limitations of this case study, it is unclear if these elaborated integration experiences would be evident in other areas of mathematics and science. This is especially true given the easily accessible nature of statistics and working with data in science. It may be that other mathematical content does not lend itself to the elaborated integrations in a meaningful way.

In a time dominated by talk of STEM, and where students' ability to do STEM is critical for their success in a tech-driven globally competitive society (National Research Council, 2012), a deliberately integrated approach can provide opportunities for "more relevant, less fragmented and more stimulating experiences for learners" (Furner & Kumar, 2007, p. 186). It is time for STEM to be more than just any isolated discipline or an arbitrary accumulation or tangentially related experiences. It is time for STEM to be a transformational process to explore learning in a manner that may not otherwise be realized; a dynamic process that intentionally builds connections for students across and within disciplines through a more organic process.

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Improving students’ reading skills to improve their performance in math: empirical evidence from Italy towards the development of integrated approach to math education

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Understanding the role of Math in STEM and designing an approach to teach math in a STEM context is a timely topic. Nonetheless, research has to understand what STEM means before planning any teaching approach and identifying any prerequisite students need to have/develop to avail themselves of such an integrated approach. In the present paper, I focused on the latter aspect. Starting from previous studies claiming that the higher the students’ reading skills the better their performance in S, T, E, and M, I analysed Italian data to explore the relationship between students’ reading skills and math performance, and how such a relationship interplays with students’ characteristics. Results showed that reading skills predicts math performance and mediate the negative effects of some students’ sociodemographic characteristics, thus calling for the development of integrated teaching approach that also focuses on the improvement of students’ reading skills.

Keywords: STEM integration, Math, Reading and Text Comprehension.

Introduction

Studying STEM (Science, Technology, Engineering, and Mathematics) from a mathematics education perspective has received an increasing attention from both scholars and teachers. Some of the most frequently asked questions are about the definition of STEM and about the role of math in STEM. For example, is STEM a unique subject that is somehow “more” than just the sum of its components, to be taught as a stand-alone topic? Or is it a “learning context” within which math can be used as a tool to teach S, T and/or E, or is it a “learning context” within which math can be find new perspectives and stimulate new lines of reasoning by interacting with real (scientific, technological, engineering) problems? Moreover, what kind of abilities/skills students need to have or to develop to benefit from such an interdisciplinary or transdisciplinary teaching approach?

Answering these (research) questions is not easy at all and calls for more research, also aimed to understand the possible *interaction effects* between students’ performance in math and their characteristics, such as their reading skills, their personal characteristics and the characteristics of the context students inhabit (here included peers’ characteristics). Nonetheless, little research has been carried out to date to understand if (and, if yes, to what extent) the relationship between students’ performance in math and their characteristics and skills can interplay with, and thus being (positively or negatively) affected by the characteristics of the school and the classroom attended by students. The current paper aims to fill this gap.

The relationship between students’ characteristics and their performance in math

There are several factors affecting students’ performance in math, such as schools’ characteristics (facilities, didactical protocols, etc.) (e.g., Fowler & Walberg, 1991) that can enhance or hindering learning, or students’ background characteristics, such as their sex (e.g., Cascella, 2020), citizenship, (Cascella & Giberti, 2020), and socioeconomic status (SES) (Coleman et al., 1966), or other skills, such as reading skills (e.g., Cascella, 2021) that can predict their performance in math. In the educational literature, there is a full agreement that students’ socioeconomic status (SES) can significantly affect their performance and, in particular, that low-SES students develop performance more slowly compared with higher-SES students, across countries (e.g., Organization for Economic Cooperation and Development, 2019; Reardon et al., 2006).

Another background factor significantly associated with students’ performance (both in math and reading) is sex: girls outperform boys in reading and text comprehension, in all the countries around the world. Such a superiority in reading counterbalances female relative underachievement in math (Ajello et al., 2018), that is sharper in some countries (such as Italy, Spain, etc.) (Guiso et al., 2008). Similarly, foreign students’ performance is different (often, but not always, lower) compared with native students and, according to results from the Programme for International Students Assessment (PISA), their relative disadvantage in math compared with native students is primary due to their disadvantage in reading (Organisation for Economic Co-operation & Development, 2016).

In addition, recent research has shown that all the relations described here above interplay with the characteristics of the learning contexts (schools and classrooms) students live in. Peers’ characteristics affect individual academic performance, sometime even more than individual characteristics. Such a phenomenon is more frequent in secondary education (Kessels, 2005). Previous studies have shown for example that, in secondary schools, “the social composition of the student body is more highly related to performance, independent of the student’s own social background, than is any school factor” (Coleman et al., 1966) and, more precisely, that students’ individual performance is more strongly associated with classroom and/or school SES than with student’s individual SES.

Research aims and questions

Even though previous studies have identified some factors affecting students’ performance in math, it seems that these associations can dramatically change depending on the characteristics of the (learning) context students attend. In absence of research aimed at quantifying the effect of external factors, designing innovative teaching approach is taught and may lead to unexpected results.

In the current paper, I investigated the mediating role of students’ reading skills on the relationship between students’ SES and students’ performance in math, and used students’ sex, citizenship status, regularity throughout the academic pathway as control variables. Census data collected in Italy at Grade 10 in 2017 have been used to answer the following research question: Does the relationship between students’ performance in math and their background factors and skills interplay with the characteristics of the school and the classroom students attend? And, if yes, to what extent?

Methodology

Data

In 2017, the Italian national institute for the evaluation of educational system (*Tr. Istituto nazionale per la valutazione del Sistema di istruzione e formazione* - INVALSI hereafter) administered two achievement tests to the Italian students' population to measure their competence in (i) reading and text comprehension, and in (ii) math.

In this paper, I analyzed data collected at Grade 10 (on average, 15 years old students), at census level (427,465 students in 24,870 classrooms, in 3,986 schools). Italian data are particularly suitable for the purposes of the present study because, at Grade 10, there are three school types (i.e., Lice, Tecnici, and Professionali[1]): all of them are embedded in the same educational system and have the same math curriculum, but they are usually attended by students with very different characteristics in terms of socioeconomic status, gender, citizenship, and regularity of academic pathway. (For a description of the sample analysed, please see the 'Data description' section).

The differences between different school types allowed for the investigation of the possible interaction between personal and contextual factors. Results based on such a sample are thus of help to answer my research questions and can be used as an example - of interest for the international reader - of how the mediating role of reading skills on the relationship between students' performance and students' characteristics change depending on contextual factors.

Analytical strategy

A multilevel model (MLM) of students' performance in math (estimated via the Rasch model - Rasch, 1960) against students' characteristics (i.e., sex, socioeconomic status, citizenship, regularity throughout the scholastic pathway, reading skills) and contextual factors (peers' socioeconomic status aggregated at classroom and school level), has been estimated by using MLwiN, a software for multilevel analysis (Rasbash et al., 2017), used to account for data hierarchy (i.e., students nested in classrooms and schools).

Multilevel modeling is very adequate for the purposes of the current study as it models the interdependency within levels under the theoretical assumption that students from the same context share something more as compared to students from other contexts (Hox, 2010).

In the present study, multilevel models have been estimated with random intercepts and fixed slopes (Hox, 2010). School type has been added as a predictor to verify that the difference between school types was statistically significant. In order to observe how the relationship between students' characteristics and reading skills and students' performance in math varies by school type, I

[1] Although all school types allow access to university, Tecnici and Professionali offer a specific education/training and a direct access to the job market in a variety of sectors that do not require an academic degree (but that are more focused, for example, on technological competences or on the development of manual abilities), whereas Licei offer a broader education preparatory for university.

performed three separate analyses[2]: one for Licei (224,791 students), one for Tecnici (132,954 students), and one for Professionali (65,120 students).

Such a decision is statistically robust as the INVALSI national sample is statistically representative of the Italian students' population at regional level and by school type. Moreover, each school type shows unique characteristics: each of them thus represents a unique learning environment (See 'Data description'), and the differences between school type worth a separate analysis.

Results

Before performing the multilevel analysis, in the next sub-section, I presented a description of the data analysed in the present study. Tables and figures have been saved in a separate file, available online[3].

Data description

Students' characteristics sharply vary by 'school type' thus characterizing Licei, Tecnici and Professionali as different, highly segregated learning contexts.

Schools' composition varies in terms of the proportion of enrolled boys and girls, the proportion of enrolled native compared with foreign students, in terms of students' socioeconomic status and performance levels, both in reading and in math, and in terms of the proportion of 'regular' students (i.e., those attending the expected grade) compared with 'retained' and 'in advance' students.

In 2017, at Grade 10, 51.3% of students attending Licei is female. Such a percentage decreases in Tecnici (30.3%), and in Professionali (15%). Moreover, around 85% of students attending Licei are Italian citizens. Such a percentage is slightly lower in Tecnici but drops down to less than 80% in Professionali. In addition, the proportion of retained students over the total is less than 10% in Licei, around 20% in Tecnici and more than 35% in Professionali.

Students' socioeconomic status and students' performance (both in reading and in math) sharply vary by school type. Students' SES is higher in Licei (mean = 0.31, SD = 0.97) than in Tecnici (mean = -0.19, SD = 0.95) and in Professionali (mean = 0.54, SD = 0.96). Students' performance in math is above the national mean in Licei, in line with the national mean in Tecnici, and sharply below the national mean in Professionali. As with it, students' reading skills are above the national mean in Licei, in line with the national mean in Tecnici, and below the national mean in Professionali (Figure 1, in the Supplementary File) (INVALSI, 2017).

[2] In each model, all continuous variables (i.e., 1. students' reading skills, 2. individual, 3. classroom, and 4. school SES) were centered on the grand mean (Hox, 2010). For categorical variables (i.e, sex, citizenship, regularity, and school type), 'Boy', 'Second generation student', 'In advance' and 'Tecnici' have been used as reference category.

[3] To view the Supplementary File, please use the following link:

https://www.dropbox.com/scl/fi/ds8ysg6tj9qchmbatf0oc/Supplementary-file-CERME-12.docx?dl=0&rlkey=wtz0h0hdv83akomac17th1g87

Multilevel analysis

Before performing the multilevel analysis, I estimated the null-model, i.e. a model without predictors used to calculate the variance of students' performance in math (the dependent variable) at different levels of the data hierarchy. The variance calculated at each hierarchical level is also used to calculate the variance partition coefficient (VPC - Hox, 2010) (Table 1, in the Supplementary File).

The model intercept expresses the overall mean of math performance (measured by using the Rasch model (Rasch, 1960) in Licei, Tecnici, and Professionali, and thus serves as a benchmark with which other models are compared with[4].

The proportion of variance in math performance explained by individual factors sharply changes by school type, and it is higher in more disadvantaged contexts (more than 40% in Licei and 50% in Tecnici, and around 60% in Professionali) (Table 2, in the Supplementary File).

To better understand if and, if yes, how the relationship between reading skills and math performance changed depending on the learning context, I analyzed data collected in Licei, Tecnici and Professionali separately, as reported in the next three sections.

Licei

Consistently with previous literature (Ding & Homer, 2020), the regression analysis indicates that boys outperform girls in math, and that female disadvantage increases after having accounted for reading skills, thus confirming the hypothesis that reading skills mediate the relationship between gender and math performance. As with this, regular students are slightly advantaged compared to 'in advance' students whereas retained students are sharply disadvantaged. Nonetheless, these differences decrease after accounting for reading skills.

Reading skills also mediate the relationship between citizenship and math performance. More precisely, higher reading skills are associated with higher performance in math and, after having accounted for reading skills, the retained students' disadvantage significantly decreases. Such results are stable across hierarchical levels but different in magnitude because the proportion of variance explained by individual factors decreases when the hierarchical data structure is accounted for: at classroom and school level, the magnitude of effects related to individual variables is smaller than that at individual level. Such a result is not surprising as the individual level model does not account for data hierarchy, and thus wrongly attributes to the individual level the variance actually explained by factors at the higher hierarchical levels (Hox, 2010).

[4] The difference in the -2*loglikelihood between models, in each school type, is statistically significant - as assessed via a Chi-Square test -, thus confirming that the model accounting for data hierarchy fits better than that does not (i.e., the 1-level model): 1-level model captured the variance actually explained by variables at the higher hierarchical level (i.e., at classroom (models 2) and at school (models 3) levels), and thus wrongly attribute such a variance to individual students' characteristics (Table 1, in the Supplementary File), biasing results interpretability.

In addition, the effect of individual SES decreases when peers' SES (aggregated at classroom and school level) is added to the model. Reading skills also mediate the effect of peers' SES on individual math performance. In particular, the effect of individual SES is statistically significant but very low, thus confirming that peers' SES affects individual performance in math even more than individual SES (Cascella, 2020; Coleman, 1966), and showing that reading skills mediates the effect of both individual and aggregated SES on math text score.

Tecnici

In Tecnici, where students' ability both in text comprehension and in math is lower than in *Licei*, reading skills explain most of the variance in students' performance in math, thus suggesting that even small increases in reading skills can help students to perform better in math. Moreover, reading skills mediate the effect of all students' background factors and aggregated SES. In contrast with Licei, aggregated SES (and, in particular, classroom SES) explains a low proportion of variance in the dependent variable as shown by its regression coefficient that is statistically significant but low.

Professionali

Reading skills mediates both individual and aggregated SES. Moreover, individual SES effect drops down when aggregated SES is added into the model thus suggesting that peers' SES (aggregated at classroom level) is more strongly associated with individual math performance than individual SES. Such an association is statistically significant but less strong than in Licei. Such a result may suggest that the association between contextual factors is stronger in higher-SES contexts (i.e., Licei) than in low-SES contexts (Professionali).

Conclusion

In the present paper, I presented an empirical study based on Italian data and aimed at investigating how the mediating role of reading skills on the relationship between students' characteristics and their performance in math changes depending on the characteristics of the "learning context" students attend. Therefore, even though I showed that students' reading skills have to be taken as a prerequisite to learn and perform well in math, the results presented in the current paper also suggest that the effect of reading skills on students' performance in math does not work for all students exactly in the same way: the association between contextual factors and individual performance in math is stronger in higher-SES contexts than in low-SES contexts.

So, if by a side the results presented here suggest that teachers and/or researchers, in developing an integrated teaching approach, should not ignore the importance of reading skills but rather should include the improvement of students' reading skills as part of such an integrated approach, on the other side, they should (i) be aware of how the relationship between students' reading skills and their performance in math interplays with the external context, and (ii) work to tailor their teaching approach to their own students.

The study presented in the current paper shows some important limitations. First, I used students' performance in math as measured via INVALSI achievement tests as a proxy of students' attainment in math. As with this, reading skills as measured by INVALSI tests can capture just a part of students' reading skills. More investigation about how and why reading skills can affect students' performance in math is necessary to advance the existing knowledge.

Nonetheless, results based on INVALSI data allowed for a comparative analysis between different school types that represent very different learning environments, suitable to understand how the mediating role of reading skills on the relationship between students' characteristics and their performance in math changes by context.

The current paper can thus contribute to the ongoing debate about the construction of an integrated teaching approach by providing an empirical investigation of the possible intersectionality between the relationship between reading skills and math performan with students', classrooms', and schools' characteristics (Aikens & Barbarin, 2008; Caldas & Bankston, 1997; Marks et al., 2006; Perry & McConney, 2013). Understanding how the effect of factors affecting students' performance change depending on the context students inhabit is necessary in order to design an effective teaching approach.

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Specialized knowledge to develop interdisciplinary tasks in the context of STEM education

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Keywords: Teachers' knowledge, professional development, STEM education, hands-on.

Despite increasing recommendations for implementing STEM (Science, Technology, Engineering and Mathematics) education, the literature identifies difficulties about its implementation by teachers (English, 2017; European Schoolnet, 2018). Appointed reasons are related to the need for teachers to acquire a robust Content Knowledge about the subject matters to be integrated (English, 2017). Therefore, it is necessary to offer them with a Professional Development Programme (PDP) that provides them with knowledge and skills to implement this approach in class. Recently, several authors refer the need for highlighting the role of the M in STEM, which also brings added challenges to teachers (Beswick & Fraser, 2019; Stohlmann, 2018). Therefore, it is crucial to understand what knowledge promotes the development and implementation of mathematical interdisciplinary practices within the context of STEM education. Concerning teachers' professional knowledge, the literature presents several studies such as Shulman (1986) that distinguishes three categories of Content Knowledge (CK): Subject Matter Content Knowledge (SMCK), Pedagogical Content Knowledge (PCK) and Curricular Knowledge (CuK). Other authors refer to the subject matters to teach such as mathematics (Ball, Thames and Phelps, 2008), science (Park & Oliver, 2008) and technology (Mishra & Koehler, 2006). However, there is a lack of research about the necessary knowledge for promoting STEM education, in particular by highlighting the role of the M. Therefore, our research question is: what knowledge is necessary for teachers to develop an integrated approach of STEM education with a focus on mathematics? Because the characterization of this knowledge is missing in the literature, this study is crucial for teachers, researchers, and also teacher educators who need to design PDP related to this approach.

This research is inserted in a broader project that includes a PDP, targeted to primary and middle school teachers, with the aim of providing them with knowledge and skills to develop and implement hands-on practices in class that promote an integrated approach of STEM education (Costa et al., 2020). To answer the research question, an empirical study was developed in the framework of the referred PDP for three school years. This training context brings added challenges for teachers to innovate their practices in class, namely mathematical practices related to STEM. With a qualitative methodology and an interpretative approach (Cohen, Lawrence, & Keith, 2007), data collected for three school years include participant observation during PDP and teachers' portfolios developed during the programme. Based on this data and teachers case study, it is concluded that there exists specialized knowledge dimensions that are crucial for teachers to be able to develop and implement the required approach. First, there is theoretical knowledge for example associated with some science themes such as electricity, sound or astronomy: TheoCK_S (Theoretical Content Knowledge to teach

Science). But to be able to implement hands-on experiments related to Science, teachers also need to know how to handle the equipment and materials to implement the hands-on tasks, which requires TechCK_S (Technical Content Knowledge to implement Science experiments). In addition, teachers need to have PCK to transform the specialized theoretical and technical knowledge in order to make them understandable to students, which leads to new dimensions: TheoPCK_S and TechPCK_S. Extending these new dimensions to tasks that integrate all the STEM subjects we propose the following dimensions of knowledge as presented in Table 1.

Table 1: Knowledge necessary to implement hands-on STEM experiments in class

SMCK		PCK	
TheoCK_STEM	TechCK_ STEM	TheoPCK_ STEM	TechPCK_ STEM

Based on our research, it is concluded that to effectively implement hands-on STEM integrated tasks in class, teachers need to acquire specialized knowledge as stated in Table 1. Finally, we suspect that the same knowledge is necessary for other grade levels such as secondary school, but more research needs to be developed in this matter.

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What can be learned from mathematics teachers' experiences who teach an interdisciplinary STEM course

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In the Netherlands, it is common that teacher training courses for upper secondary education educate for a single school discipline, such as mathematics. At the same time, teachers with a mathematics qualification are also formally competent to teach an interdisciplinary school course called Nature, Life and Technology (NLT), regardless whether it has been addressed in the teacher training programmes. However, teaching NLT is substantially different from teaching mathematics with respect to, for instance, pedagogy and objectives. In this paper we report on a study on experiences of mathematics teachers who, in addition to teaching mathematics, teach NLT. The experiential knowledge is meant to inform mathematics teacher educators who prepare students for interdisciplinary courses such as NLT.

Keywords: mathematics teachers in STEM, secondary education, teacher education, interdisciplinary

Introduction.

In the Netherlands, most teachers in secondary education are trained to teach a specific school discipline, such as mathematics. Although cooperation between and integrating of school courses are frequently encouraged in the Netherlands (Folmer, Koopmans-van Noorel & Kuiper, 2017), teachers collaborate mainly with colleagues of their own discipline, unless there is a need for cross-curricular collaboration within a school, due to thematic project work or a specific course. One course in which this is different is the interdisciplinary STEM course Nature, Life and Technology (NLT).

Nature, life and Technology is an elective course in upper secondary education. The objectives of the course are to strengthen the cohesion between science and mathematics and to increase the attractiveness of science education (Stuurgroep NLT, 2007). In this course, students work on complex real-life problems for which they need knowledge from mathematics, physics, chemistry, biology and physical geography to solve these. The course has a modular structure. With each module, students are working on new problems around a topical issue using teaching materials developed through collaboration between teachers and universities or organizations specialised in the topic. Teachers work within a team of teachers with different competences and backgrounds, mainly mathematics, physics, chemistry, biology and geography. Such team of teachers is called an NLT-team.

Teachers who participate in these kinds of activities are confronted with other learning and teaching situations as in their disciplinary course and as a consequence will have to relate to these experienced differences. For example, NLT requires attention for overarching ideas such as the interplay between the natural sciences and technology, and not merely attention for specific disciplinary concepts, such as finding an exponential formula given two points. The usual pedagogical approaches in NLT are also different from those in mathematics school courses, since a lot of coaching of group work is done in NLT. In addition, NLT often uses different assessment formats than the exercises that are customary in mathematics (Den Braber, 2020). Knowing that there are such differences between the courses, it can be argued that mathematics teacher training should prepare future teachers for teaching NLT and for dealing with these differences. Especially, when these differences should indeed be dealt with according to mathematics teachers that teach NLT and knowing that it is considered important that mathematic teachers participate (Stuurgroep NLT, 2007).

However, it has become clear that preparation for NLT is marginal in most mathematics teacher training programmes. Consequently, dealing with the differences will in most cases have to be learned after the teacher training course, during working life, if a teacher chooses to participate in NLT. However, opting for teaching NLT requires some lived experiences in NLT and experiencing the differences with teaching mathematics. However, it is known that mathematics teachers do not seek such experiences in working life easily. Mathematics teachers participate less in NLT than teachers of the other related disciplines such as physics or biology (Den Braber et al. 2020). Mathematics teachers indicate that they do not always see a role for themselves in NLT. Teachers who are unfamiliar with NLT may find it difficult to envision what is required to teach NLT and compare teacher requirements to their own knowledge, skills and attitude to determine whether participation is even feasible, let alone appealing. Teachers who have taught NLT will have formed an image of what is required to teach a course such as NLT and will have found ways to act accordingly.

Despite the fact that mathematics teachers currently participate less in NLT than teacher of sciences, there are some mathematics teachers who are successfully part of an NLT-team. The fact that it is possible and can be successful raises the question whether we can learn from the experiences of these mathematics teachers to help prepare future mathematics teachers. It is this question that led to a study with the underlying research question: How do qualified mathematics teachers shape their role within a NLT- team and deal with mathematics in NLT?

In this paper we will describe results regarding to what the experiences teach us about the possibilities for mathematics teacher training programmes to enable lived experiences in teaching NLT for future mathematics teachers.

Conceptual framework

Analysing the experiences of a teacher within NLT requires a framework that describes a learning process of a teacher in his or her work in the broadest sense. It is not only about the knowledge of the teacher but also about the actions, feelings, motives and foresight within the context of a person's working environment. Roth & Lee (2007) indicate that cultural-historical activity theory (CHAT), because of its focus on processes and individual and collective agency, can be the analytical lens for learning processes in lifelong learning, both in formal and informal (work) situations as well as in educational settings.

It is for this reason that we make use of a CHAT-based analytical perspective on learning processes while engaged in work processes (Mazereeuw, 2020). This perspective depicts a process in which experiences at the workplace are internalised (Zittoun & Gillespie, 2015) and, depending on the interpersonal dialogue are stimuli for envisaging new action possibilities and actions within the work process. Acting on these new perspectives and action possibilities through experimenting can also create new experiences. Hence, experiential learning processes are not linear. Because of the reciprocal nature of the interpersonal process and the social and material consequences in context it is much more dynamic. To emphasize the reciprocal nature of the process we included insights of the Dialogical Self Theory (Hermans, 2001).

The idea of the dialogical self is that a person mirrors experiences with ones knowledge, perspectives, norms and values in an internal dialogue and that these experiences can be re-lived and re-framed beyond the actual experience itself. So, the internal dialogue is an interpersonal process -within the mind- in which a person relates to the lived and re-lived experience. CHAT emphasizes that an internal dialogue can result in the desire to act, especially when frictions are experienced. For instance, a person perceives that contributing to the work calls for knowledge and skills that the person does not feel confident about. When that person wants to continue to participate in the workplace, new action possibilities will be sought out. Therefore, frictions, or the commonly used term contradictions, are often a source of change and development as described by Engeström (2011). We tried to capture the internal dialogical process in four compact questions someone asks aware or unaware. The first question is ‘How do I understand what is going on (with me, others in the environment)?’. Understanding includes ones own perceptions of knowledge of the workplace, division of labour and workplace goals, norms, emotions and values. The second question is ‘How do I value this?’. Here a person mirrors this conception of the experience with what matters to the person, what the person finds important. The third question is ‘Should I do something about this or not?’. This question refers to the motive to become an actor and to contribute to the workplace. The fourth and final question reads ‘What can I do to contribute and/or to engage the friction?’. This question refers to the way a person depicts his or her own role in the situation and how that person envisages action possibilities that might help to contribute to a solution or to engage the friction.

The questions above show that this process may well be influenced by previous work and teaching experiences. That is probably also the case for NLT teachers who have experiences in another school course or have worked in different fields before entering the educational system. Figure 1 schematically represents the dialogical processes when interpreted for teaching NLT.

In order to provide insight in the possible frictions that initiate internal dialogue in teachers we conceive teaching of NLT as an activity within an activity system. To portray the activities we make use of the commonly used model of Engeström (1987), which builds on the ideas of Vygotsky and Leont'ev. The model shows that human action can be seen as an interaction between a person (*subject*), a certain goal (*outcome*) and cultural artefacts that support and shape this action (*mediation*). Furthermore, the goal-directed actions of an individual cannot be separated from its social and material context. An activity system consists of relevant others (*community*), with values and norms (*rules*) and with a division of labour.

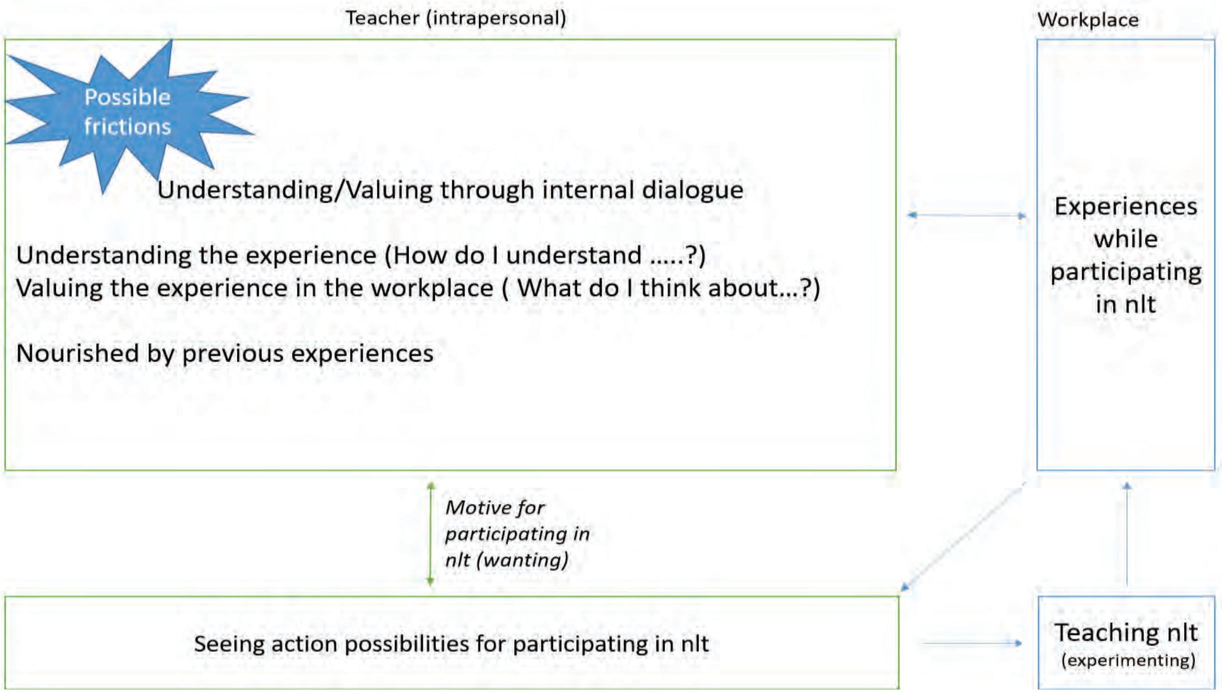

Figure 1: Processes of learning through experiences, based on Mazereeuw (2020) exemplified for teaching NLT

Unlike Engeström, we use the model as a means for portraying a teachers' thinking about the activities in which they participate, shown in figure 2. As indicated above, a person ones social and material work environment and places oneself within it. In this study Engeström's model is used to guide the data collection and analysis and find out how teachers think about their own discipline and NLT and the frictions therein. Teaching NLT in a secondary school can be conceived as a goal-oriented activity in an activity system in secondary education. This is done by using certain teaching materials and supportive (digital) resources (*instruments*) which are in accordance with a national examination program and a plan for assessment, etc. (*rules*). Teachers are part of a professional group of teachers. They have to deal with parents, colleagues within the school who all want NLT to be taught (*community*). Mathematics teachers can be part of an NLT-team which is another community with another set of (informal) rules, and another division of labour compared to the activity of teaching mathematics.

The outcome of the activity deserves a closer look because the outcome of the activity system NLT is still ambiguous. Even though there is an examination programme, it leaves a lot of room for teachers and NLT-teams to formulate their own objectives, objectives that they feel are fitting for NLT. Moreover, Braber et al. (2019) made clear that the general aims of NLT are still debated on and when it comes to mathematics in NLT are not clear to both mathematics teachers and students. So, looking at the way teachers view NLT we can assume that the outcomes pursued are diverse. It is therefore of interest to find out what mathematics teachers view as the outcome of teaching NLT especially with regards to the role of mathematics. In school mathematics learning mathematical skills is seen as the outcome of the activity system. Within NLT mathematics can be described as an instrument, a means to an end. William and Roth (2019) claim that the more disciplinary courses are a goal by itself, the more difficult it becomes to work interdisciplinary.

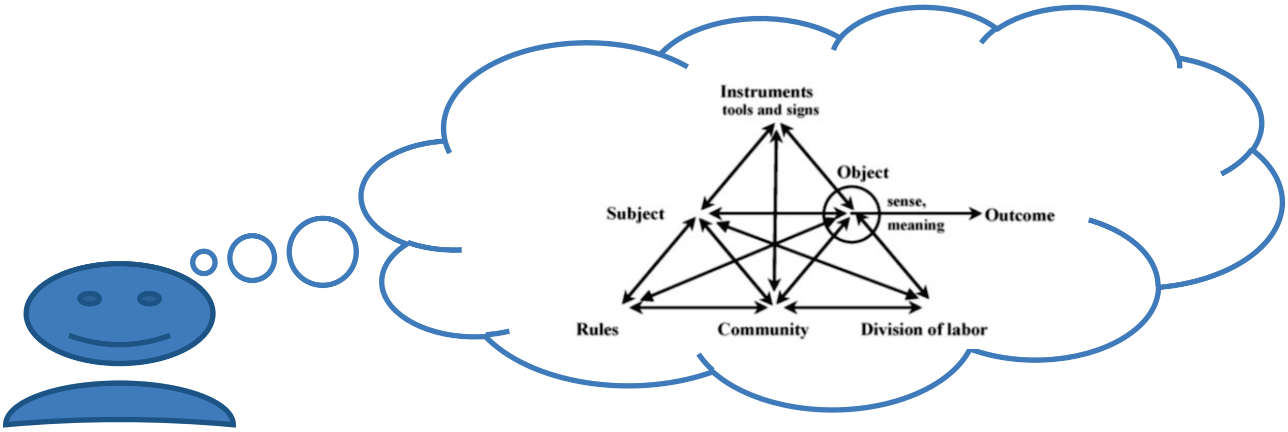

Figure 2: The envisaged activity system by the NLT teacher of the activity in which the teacher participates.

We started with the idea that through an internal dialogue a person mirrors experiences with ones knowledge, perspectives, norms and values. We portrait the envisaged activity using the model of an activity system where the teacher is the subject.

Data collection and analysis

In order to find out how qualified mathematics teachers shape their role within a NLT- team and deal with mathematics in NLT six male and four female qualified mathematics teachers all teaching NLT were interviewed. The teachers varied in age, work and teaching experience and work in different schools around the country. Following the first interview, an additional interview was conducted with a second member of the same NLT-team to triangulate the data of the first interview and evaluate the interview set-up. This led to small changes in the preferred order of the interview questions.

The interviews were meant to provide insight into how mathematics teachers conceive their role within NLT and what thoughts existed about the role of mathematics within NLT. Questions were asked about frictions that teachers experience when teaching NLT, and whether or not these were due to possible differences with the teaching of mathematics. We also asked how they have dealt with these frictions.

The interviews had the character of a conversation about NLT in which the interviewer also provided information about NLT and gave background information when needed. For example, sometimes the teachers asked questions about the organisation of NLT in other schools. The interview can best be characterized as a semi-structured interview. The interview set-up contained a number of topics and questions related to the conceptual framework. The order in which the questions were asked depended on the course of the interview. However, it always started with a few practical questions about background, teaching experiences, current teaching tasks and past work experiences. This was followed by questions about the experience with NLT, their role in the NLT-team and the possible differences between activities when teaching mathematics and NLT. For example, they were asked whether they experience differences in the use of teaching materials or teaching methods (*instruments*), collaboration with colleagues (*division of labour*) and which agreements concerning NLT have been made within the school (*rules*). When frictions were put forward by the teachers, questions were asked about the teachers' views of knowledge, skills and attitude towards interdisciplinary teaching and NLT. The role in the NLT-team and the connection between curricula of mathematics and NLT were also addressed. The interview ended with questions that invited the teachers to talk about their thoughts on the role of mathematics in NLT.

The recorded interviews were transcribed. Semantic units (Aviv, 2001) were categorised using elements of the activity system, and the labels (personal) beliefs, knowledge and skills and attitude towards NLT based on a division by Ernest (1989). For instance, a teacher stating which teaching materials are used in the classroom is labelled under *instruments*. Frictions experienced by teachers were noted as well as previous experiences when stated by a teacher. For instance, the following semantic unit illustrates an experienced friction concerning the subjects knowledge for teaching NLT compared to teaching mathematics.

Teacher: Mathematics, you can just ask me about that, I know it, but with NLT I sometimes doubt my answers are completely correct. […] makes me less comfortable.

Data analysis further focused on the role of each mathematics teacher within NLT. First, we looked at their view on the role within the NLT- team to see if there are similarities or differences. Then we looked at the relationships between how teachers describe their own teaching in NLT and what they see as objectives for NLT. i.e. the subject-instruments-object/outcome relationship. A third analysis focussed on each teacher's orientation towards the objectives of NLT. The emphasis was the teachers' views on intended objectives of mathematics within NLT and views on the overall objectives of teaching NLT. A fourth analysis concerned the frictions each teacher experienced. All frictions mentioned by the teachers were gathered and positioned in an activity system to see if there were similarities or differences in the experienced frictions. Peer-debriefing was used in the process of coding and interpretation.

Results

When we look at division of labour in the NLT-teams, the role of the mathematics teachers vary from teaching a statistical module by themselves, to teaching a science module with a physics teacher. Some are teaching modules with a strong biological component involving the brain or navigation of birds which they had to make their own. With the exception of one, the teachers seem comfortable with what they do because of earlier experiences in which they have acquired knowledge or skills before teaching NLT. The one teacher that was not comfortable assisted teachers from other disciplines only for a few lessons. She was asked to help students with differential equations and modelling with in a spreadsheet. This teacher felt a friction due to her view of her own knowledge of science and the lack of overview of the teaching module.

NLT is seen as a course where mathematics is applied and where mathematics can be useful, especially with regard to mathematical modelling to help solve the problem at hand. So the view of most teachers seems to be consistent with the objectives of NLT. However, often the teachers thoughts of these objectives themselves and their personally desired outcomes do not always correspond with the intended curriculum of NLT. Teachers seem to formulate their own objectives as the formal objectives of NLT are not really known. The latter is specifically the case when it comes to the use of mathematics or the reflection on the role of mathematics in the examination program.

Most teachers are not familiar with objectives in NLT that focus on the importance of mathematics. The ones that do report difficulties translating this for students.

The frictions teachers experience also seem to vary. Much of these frictions have to do with conditionalities. In some schools it is hard to have meetings with the NLT-team because disciplinary faculty meetings are held at the same time and require a teachers attention. Nevertheless, working with other colleagues is viewed as worthwhile and desirable even though it can take some getting used to. Working with and preparing the teaching materials of lessons in NLT generally takes more time than preparing mathematics lessons. On the other hand, teachers also seem to appreciate the freedom they get as they are free to adjust materials, add or skip parts of the content and assignments. Some experience difficulties with the motivation of students because students seem to give priority to the disciplinary courses for which they also take exams. Some teachers indicated that seeing that they only teach one or two modules a year, building a relationship with students can be more difficult than in mathematics lessons.

What stands out in the analyses is that teachers with previous experiences in actual interdisciplinary work environments or with mathematical modelling experience less frictions teaching NLT. In fact, two of those teachers commented that they experience frictions with the way they have to teach mathematics instead of with teaching NLT as it is so different to what they are used to.

Discussion

The freedom to find ways to teach NLT that align with a teacher's personal point of departure has been important for teachers. Especially, since this freedom has been a way for teachers to deal with frictions they experience while teaching NLT. This may indicate that teacher training should focus on the flexibility of working with teaching materials and to encourage experimenting with their pedagogical choices. Even though this is also encouraged in mathematics teacher training, research shows that mathematics teachers are still mainly focused on the textbook (Woldhuis et al., 2018). This is not possible within NLT if a teacher wants to feel comfortable teaching. NLT offers a lot of freedom to make pedagogical choices and many different roles are possible as a teacher, but a teacher has to be aware of the objectives of NLT. Specifically, when it comes to the role of mathematics and the idea of interdisciplinary work. This may have consequences for teacher training programmes, knowing that we saw differences between teachers in dealing with interdisciplinary objectives depending on their background and work experience. This could mean taking into account the different personas in teacher training and take advantage of the lived experience with applied mathematics when present. Teacher training should also support reflection on the role of disciplinary mathematical knowledge when working on an interdisciplinary problem. Seeing mathematics more as an instrument than a 'product'. Something that the some interviewed teachers found difficult to do. Training programs could therefore pay attention to beliefs on mathematics in relation to the sciences. Starting, for instance, with one discipline that is closely connected to mathematics, like physics (Nguyen & Krause, 2020), but perhaps ending with letting students experience mathematics or mathematical modelling in corporate life or interdisciplinary working environments. This could support teachers in letting their students experience how mathematics can help with real-life interdisciplinary questions and discuss the contributions mathematics can make in solving real-life problems.

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Toward a STEAM professional development program to exploit school mathematics

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Keywords: Mathematics, professional development, secondary education, STEAM education.

Introduction

Following European Union recommendations, many countries have incorporated the competence-based learning approach in their curriculum. This approach aims to prepare citizens for current and future challenges in Science, Technology, Engineering, and Mathematics (STEM, Diego-Mantecón et al., 2021a). To provide citizens with the necessary competences, there exists an increasing interest to integrate Arts into STEM, in the so-called STEAM Education.

STEAM education entails teachers crossing boundary disciplines to adequately integrate content. In countries like Spain and Greece, secondary education teachers are usually qualified to teach a single discipline. In an integrated context, it means that teachers must design and implement activities involving content from disciplines in which they did not receive specific training. In STEAM activities science and engineering disciplines often take a dominant role (Martín-Páez et al., 2019), whereas mathematics appears in a basic and utilitarian manner (Lasa et al., 2020). Recently, Diego-Mantecón et al. (2021b) found disparities in the characteristics of the STEAM activities implemented by in-field and out-of-field mathematics teachers as well as in the manner they exploit mathematics. In-field mathematics teachers, unlike the out-of-field ones, seem to avoid transdisciplinary projects (based on real experiences) because of the difficulty in addressing school mathematics. Even when implementing interdisciplinary projects (contextualized based experiences), mathematics teachers encounter problems to meaningfully exploit mathematics and promote high cognitive demands. As consequence, Diego-Mantecón et al. (2021b) suggest developing training programs focused on reproducing experiences in collaborative environments, where both in-field and out-of-field mathematics teachers cooperate to construct and deliver knowledge.

The study

Following Diego-Mantecón et al.’s (2021b) suggestion, we attempt the so-called STEAMTeach project (https://www.steamteach.unican.es/), which is a Teaching Professionalism European-funded initiative based on the Erasmus+ Programme. The objective is to design a cross-cultural STEAM professional development (PD) program for training in-service teachers to exploit mathematics within a STEAM context at middle and high schools. To design this PD program, we firstly interviewed 25

STEAM trainers in five countries: Spain, Austria, Finland, Greece, and Hungary. The interviews helped to identify issues obstructing the implementation of STEAM activities in school classrooms. In the next, we report a preliminary framework emerging from the Greek and Spanish data.

Results

The analysis revealed that trainers agreed on the importance of introducing teachers to theoretical aspects of the integrated STEAM education approach and to the following four active methodologies: (1) collaborative learning, (2) design-, (3) inquiry- and (4) problem-based learning. Based on these outcomes, we devised a preliminary PD framework comprising three blocks: a theoretical block in which, through different sessions, teachers are introduced to STEAM education and the methodologies listed above; an experimental block where teachers, grouped in teams, have to attack a series of STEAM activities in the same way that their students would do it; and an implementation block where teachers are requested to design and implement activities in their classroom with the trainers' support. After executing the first round of training courses, we observed that this preliminary PD framework allows teachers to gain insights into the meaning of STEAM education and its application in the classroom, as they were forced to experience the difficulties that arise in different STEAM contexts. This in turn increases teachers' confidence to implement their STEAM activities and exploit school mathematics. Importantly, this preliminary PD framework is under an iterative process of implementation and evaluation, and hopefully subsequent training courses will provide us with extra data to refine the initial framework into a more consistent and reliable instrument.

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Abstraction and decomposition in tinkering tasks in visual programming environments

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This study examines how a tinkering task in visual programming environments can provide opportunities for developing problem solving skills. We pursue this by analyzing eleven students in year 8 working with a tinkering task in a visual programming environment during a mathematics class. They were asked to create repeating patterns. Their work and discussions were analyzed through a lens of abstraction and decomposition, two elements of computational thinking. The analysis reveals how some students became thoroughly engaged in problem solving while others had a shallower experience of randomly manipulating the pre-made code. Since it was not the difficulty of the task but rather the random outcome of their manipulation of the code that determined whether they became engaged or not suggests that there is a need for support structures to fully tap into the potential of tinkering tasks to elicit problem solving.

Keywords: Problem solving, tinkering, computational thinking.

Introduction

Coding is now, by the year 2020, incorporated in elementary school curricula in many countries around the world. Sweden, amongst others, has incorporated it into the mathematics curriculum and transforming the curriculum into an integrated science, technology, engineering and mathematics (STEM) curriculum in the process. Feurzeig et al. (2011) envisioned a school system where coding could be viewed as a means toward learning mathematics and developing problem solving skills. This cross-section of coding and mathematics situates coding in a STEM-context in Sweden. Coding has since then become framed within the bigger idea of computational thinking (CT), problem-solving by drawing on the concepts fundamental to computer science (Wing, 2006). The educational research literature does not adhere to one conceptualization of CT, nor how one can operationalize CT in the classroom practice for young pupils (Brennan & Resnick, 2012).

Problem solving is highlighted in both mathematics education research and the classroom practice as an important skill and a means for developing understanding (Hiebert & Grouws, 2007). It can be viewed as moving from the world we live in, to the world of symbols, that can be shaped and manipulated (Freudenthal, 1991). Problems are interpreted in the language of mathematics, creating models that has potential to say something about the real world. CT implies that this can be done by applying technical solutions through a process of breaking down the problem into decontextualized subproblems (abstracting and decomposing) which are interpreted in the language of digital technology (code). There are several similarities between problem solving skills in mathematics and programming, and a potential for mutual benefits in terms of learning (Feurzeig et al., 2011). Bråting and Kilhamn (2021) discuss this by pointing out intersections between algebraic thinking and CT with tasks for young children, e.g., that both contexts use variables, but also points out that they are not necessarily understood in the same ways in both contexts and therefor introduce a potential conflict for learners.

Programming has a steep learning curve as it entails learning a new language to express oneself in. Kotsopoulos et al. (2017) suggests tasks where students modify pre-made code to make the learning experience more accessible and there are also visual programming environments (VPE) where code is represented as building blocks (Figure 4). A recent review of research on CT in the VPE Scratch[1] concluded that there is need for studies that make sense of the quality of understanding that students develop while programming (Fagerlund et al., 2021). However, problem solving that requires creative reasoning is hard and requires careful design of tasks and instruction (Jäder, 2019). There is a need to better understand design-elements of tasks to design instruction to engage primary schools' youngest students in tasks with potential to develop their CT. Our guiding question is *how can tinkering tasks in visual programming environments provide opportunities for developing problem solving skills?*

We will pursue the question by analyzing students' CT in a tinkering task which was a part of a design research project in mathematics education. We will use a lens of abstraction and decomposition to analyze students' actions in a VPE and discuss opportunities to develop CT from a STEM perspective.

Abstraction and decomposition

Solving a problem is a process of analyzing the problem, devising a plan and carrying out the plan (Polya, 2004). A way to approach this process is to make the problem more accessible (ibid.), e.g. by drawing a picture, find an analogous or auxiliary problem or break down the problem into more manageable subproblems and later recombine the solutions to a solution for the original problem. Burke (2012) describe similar processes in how middle school students can be taught to solve programming problems. The whole concept of CT can be thought of as "thought processes involved in formulating problems so their solutions can be represented as computational steps and algorithms" (Aho, 2012, p. 832). Kalelioglu et al. (2016) propose a framework for CT based on a systematic literature review that starts in Abstraction and Decomposition. Abstracting is the ability to focus what is essential and ignoring what is not (Wing, 2008) and thereby making the problem more accessible. What is then abstraction when speaking about young students' ability to abstract? Let us imagine a student that want to include a fish swimming from point A to point B in a VPE project. The student realize that the code needs to include the movement of the fish and the direction of the movement, but it doesn't really matter if the fish is green or purple. The problem to solve is how to get from point A to point B. The student must manage to see what is important and what is not (abstracting), to be able to solve the problem. Decomposing into

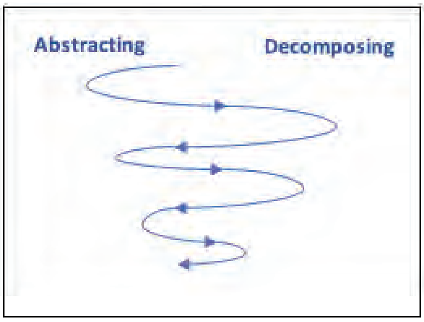

Figure 1. A spiral representing how we conceptualize the computational thinking processes abstraction and decomposing

[1] Scratch is a visual block-based programming environment developed by MIT. It is developed to be suitable for introduction to programming for young students since the pre-made blocks of code lowers the threshold of learning syntax to be able to code.

subproblems is the ability to solve problems by breaking them down into smaller more manageable problems. The student with the swimming fish needs to break this problem down into smaller pieces in order to solve it. One part of the problem is to make the fish move just far enough to get from point A to point B. The other part of the problem is to figure out which direction will get it to the right spot. Two subproblems to solve one problem. Whether one views these two processes as one, e.g. the CT practice (Brennan & Resnick, 2012), or as two, e.g. two tools when attacking large and complex tasks (Wing, 2008), they are closely intertwined. We choose to visualize this in a downward spiral, see figure 1, as we envision it having a direction towards the solution of the problem.

Solving a mathematical problem involves analyzing the problem, devising a plan and carrying out the plan (Polya, 2004). We connect this to our visualization of CT in three practices, visualizing an idea, formulating a plan, and translating into code, see figure 2. Steps of abstracting and decomposing moves the student forward towards a solution of the problem within a computer environment. The student with the fish starts in an idea, wanting to create a scene from the ocean. Acts of abstraction enables the student to figure out what parts of the idea are essential to create the impression of an ocean and every essential part has to be decomposed into solvable sub-problems to devise a plan. This plan, as an intermediary step towards writing an instruction that a computer could understand, is sometimes called a pseudo code (Weintrop et al., 2016). The plan is then carried out within the syntax of the coding environment, i.e., code-blocks in the VPE case.

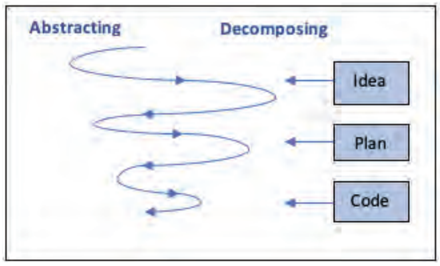

Figure 2. The computational thinking spiral with three stages of problem solving written in, Idea, Plan and Code

This spiral movement visualize the student getting deeper into the problem-solving process, working towards solving subproblems that eventually adds up to a complete solution. The student shift back and forth between the practices, modifying the *idea*, formulating and revising a *plan*, and produce *code*. The student in the example had an idea to create an ocean. The ocean was abstracted into a few essential parts that would make the viewer understand it is an ocean, for example a couple of moving fishes. Those essential parts constitute the plan. As the student start to deal with a problem one at the time, abstracting and decomposing in an iterative process, the processes lead to coding.

The type of subproblems that students engage in depends on the type of task (Eckert & Hjelte, 2021) and whether the subproblem constitutes a problem or not depends on the student. We adhere to the definition of a problem as when the solution method is not known in advance to the learner but require creative reasoning (Lithner, 2008). Tasks where students program for motion, typical to VPEs, tend to put programming subproblems in the foreground and mathematical subproblems in the background (Eckert & Hjelte, 2021). Bråting and Kilhamn (2021) question whether VPEs developed for the purpose of developing students' CT is the future of mathematics education. Even though there are similarities between algebraic thinking and CT, objects such as variables exists in both worlds, how we make sense of those objects differs. Benton et al. (2017) provide evidence for how carefully designed tasks have potential to develop both algebraic thinking and CT, e.g. the idea of the 360° total turn. The risk is though that one or the other gets lost in the process of creating code (Bråting & Kilhamn, 2021; Eckert & Hjelte, 2021).

Method

The present paper focuses on reaching an understanding of how tinkering tasks in VPEs provide opportunities for developing CT. The analytical process of this paper is that of an abductive approach (Alvesson & Sköldberg, 2009). We propose an analytical lens to view the data through, and then allow for the emerging insights into the students' CT process to influence the initial theorization. The idea of the CT spiral is meant to inspire cycles of future design experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) on programming in mathematics classes in the first years of primary school. The CT spiral is used here as an analytical framework in a retrospective analysis using constant comparisons (Gravemeijer & Cobb, 2013). Instances of data are compared to find similarities and differences related to abstraction and decomposition in the data. Looking for indications for when students relabel an object by focusing on certain aspects (abstraction) and when students try solving a subproblem by focusing on one aspect of the object (decomposition).

The data set used in this analysis are video screen captures of student work in the VPE and recordings of their verbal communication, generated within a larger design project concerned with digitally enhanced mathematics education. The tasks utilize the idea of engaging students in actual programming tasks and asking them to reason about the results. The design hypothesis was that the tasks would trigger students to develop their ability to solve mathematical problems using computers. The empirical study was performed with a class of 20 students in year 8 (aged 13/14) in Sweden using a tinkering task. For this particular task, eleven out of those 20 students' screens and discussions were captured through a screen capturing tool for analysis. The students were given a pre-made code in scratch (https://scratch.mit.edu/projects/250218077/) that drew a shape we called a snowflake. Figure 3 and 4 depict the code and result. The inner loop, that repeats twice, contain moves and turns that create the shape of each point of the snowflake. The outer loop, that repeats 8 times, creates the number of points and the angle between them. The task was to create their own repeating pattern based on the pre-made code. This type of task is called a tinkering task (Kotsopoulos et al., 2017) and is an established task design as students immediately are confronted with the intended problem-solving process. Students are meant to make simple modifications to an existing computer program to elicit questions such as "What if I change this part of the code?". Visual programming environments are especially suited for such tinkering because of its low threshold regarding programming syntax (Resnick et al., 2009) and its focus on exploration (Kotsopoulos et al., 2017). The result section in this paper includes a summary of the analysis of all eleven students as well as an in-depth analysis of one student's work. Transcripts are translated from Swedish to English by the authors. Names used are fictional.

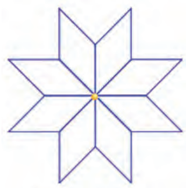

Figure 3. The snowflake

Figure 4. The tinkering task's pre-made code

Results

Our analysis indicates that tinkering tasks in visual programming environments provide both opportunities and challenges for developing CT in terms of abstraction and decomposition.

Reviewing all eleven students showed that in cases where we could detect a formulation of both an idea and a plan in the students' work, the task managed to engage the students in CT during the allotted time. In cases where either an idea, plan or both were missing, there were few indications of CT. The task type seemed to encourage CT in cases where random changes in the code produced new unfinished patterns that interested the students. In cases where the random changes either produced a finished pattern, or a pattern that was not pleasing to the student, the process came to a stop and did not elicit further CT. Following is Nevin's work, who together with Jasper produce interesting unfinished patterns in their random changes that they pursue and try to complete. Their random changes of the turn-blocks of the original code (see figure 3) resulted in a open snowflake shape (see figure 5). Their actions thereafter indicate that they have an idea what they want to accomplish. The idea seems to be to create a closed version of the open new snowflake shape.

Nevin:	It has more repetitions.
Jasper:	You need more repetitions.
Nevin:	I know.
Jasper:	12, pick 12.
Nevin:	Do you think 12 is enough.
Jasper:	I think 12.

Nevin and Jasper analyze their snowflake shape and we interpret Nevin's utterance It has more repetitions as him identifying that the new shape has more repeating shapes than the original snowflake. We interpret Jasper's call for more repetitions as abstracting the notion of points and decompose their idea into the subproblem of filling the gap in the shape with more points. Nevin agrees that this is their plan. Together with their actions on the screen when they place the marker in the outer loop, the plan of adding more points to the snowflake is abstracted into being about repeating particular chunks of code more times, and they settle on twelve. Twelve repetitions are however too much so they try with ten. The change to ten repetitions does not close the shape completely, and this exchange follows.

Jasper:	There is one more needed.
Nevin:	No … we can try but I don't think so.
Jasper:	But, like this. It is one line that is not used. This line. Get it Nevin.
Nevin:	They must connect at the end.
Jasper:	Yes.
Nevin:	But it isn't possible.

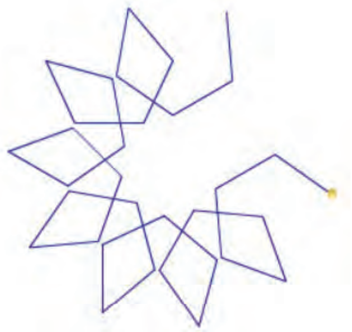

Figure 5. An open snowflake shape that requires more points to close

Jasper is set on their original plan of adding repetitions to close the shape. However, Nevin has spotted that even if they do so, the lines will not line up perfectly. To make the lines connect after one lap, the rotation-block in the outer loop needs to be a devisor to 360. Nevin seems to have spotted this visually from the picture and insinuate that they need to revise the plan.

Jasper:	Maybe have to … do more steps. No, nothing will happen. Maybe more degrees or something.
Nevin:	No, I don't think I will change anything.
Jasper:	But I think so because 65 65 65 does not add up to 360. That's why.
Nevin:	What is it then? How should I change it then?

To revise the plan, Jasper searches for aspects of the idea of a closed snowflake shape to abstract. He rejects the aspect of size and instead abstract the aspect of angles between each point of the snowflake.

Nevin does not agree but Jasper has decomposed the aspect to a subproblem of divisibility. In a previous task with simpler geometrical shapes, the key feature was to make the turn-blocks add up to 360. Their new plan is to use angles that are divisors to 360. Jasper and Nevin start by changing all the turn-blocks to 120 with the argument three times 120 is 360 but change their minds.

Jasper:	120 … you have to split it.
Nevin:	Yes. [whispers: I thought] … but what does 120 become then … 60 60 60. Otherwise it doesn't add up to 120. Does it add up to 120.
Jasper:	No
Nevin:	No. If I do 60. No I need to have the same or.
Jasper:	Yes but this must be higher.
Nevin:	Why.
Jasper:	It must be. What must it be … it must be … 90 it will be then. Wait. It was 45 from the start. Then that one was 135. So I think you need to have it to 180. Yes try 180.
Nevin:	How about this one?
Jasper:	It should be 60.

We see how the CT process elicited by the tinkering task has engaged the students in a mathematical reasoning. Their plan is to use divisors to 360, i.e. 60, 120 and 180 degrees. However, their understanding of the code is lacking which shows when the attempt to abstract elements of the plan into code. Based on the original code, they use 120 and 60 degrees in the inner loop since it adds up to 180 just as in the original code. The turn-block in the outer loop is set to 60 degrees. The end result is a shape very similar to the original snowflake, but at least it is closed which was in line with their idea and the time for the task runs out.

Analyzing the eleven students resulted in three different categories of actions, 1) students who maintained a CT process throughout the activity, similar to that above. Six students ended up in this category. In the other two categories did students not engage in a CT process, but those are difficult to showcase with transcripts. However, they both had in common that the task did not elicit a formulation of either an idea or a plan or both but their reasons why were different. 2) Students who, by mere chance, generated what was deemed by the student as an acceptable repeating pattern never got the chance to formulate an idea to continue working with. Two students were in this category and figure 6 shows an example of how such shape looked like. The engagement with the task stopped when the student achieved this shape from the first manipulation of the code. 3) Students who generated a shape that had potential to start a CT process but did not take time to abstract what were key aspects of their idea of the emerging shape that could be decomposed into manageable subproblems. Those did not formulate plans. Working without plans resulted in aimless manipulation of the code which either made the students give up or resulted in completely different shapes and a reformulation of the idea. Three of the analyzed students were in this category.

Figure 6. A closed shape produced on a student's first try

Discussion

Tinkering tasks in visual programming environments provide both opportunities and challenges for developing CT in terms of abstraction and decomposition. On the one hand, the pre-made code provided limited freedom to create a clear idea of what to achieve and the accessibility encourage a planless revising of code did not elicit abstraction and decomposition. On the other hand, the low threshold of working with a pre-made code provided opportunities for some students to quickly engage in solving subproblems of both programming reasoning and mathematical reasoning. Such opportunities for integrated STEM learning merits further discussion.

Tinkering as a pedagogical activity in a VPE can be summarized as exploration (Kotsopoulos et al., 2017). It aligns with our findings, that students explore different randomly generated shapes. However, there were big differences in their engagement in problem solving. Students who immediately arrived at a shape they were happy with, hindered them from formulating problems which solutions could be manifested in code, essential to CT processes (Aho, 2012). Students who got stuck in unreflected exploration, who never formulated a plan connected to an idea, were not supported enough to decompose their idea into more manageable subproblems. Only students who were lucky enough to arrive at acceptably unfinished shapes were engaged in abstraction and decomposition. Those 6 students in the analysis created subproblems related both to coding and mathematics and thus had similar experiences as those in the Benton et al. (2017) study. The remaining 5 students never engaged in problem solving or mathematical reasoning and thus had the lesser experience pointed out by Bråting and Kilhamn (2021).

This tinkering task's advantage and disadvantage was the low threshold to coding. It was not the difficulty or novelty of the task that usually elicit creative reasoning (Lithner, 2008) but rather the random outcome of the low threshold exploration. The task itself seemed to fulfil the criteria of not having a known solution method and was engaging to the students. It leads us to believe that it is not the task itself that needs the careful designing to elicit creative reasoning proposed by Jäder (2019) in similar settings, or the positioning of mathematics as proposed by Eckert and Hjelte (2021) but rather support strategies that can help those students that for one reason or the other never engages in the problem-solving process. There are more aspects of STEM related problem solving activities in the cross section between coding and mathematics to explore. We invite researchers to use our model of analyzing a CT process to evaluate how support structures connected to tinkering tasks in VPEs can elicit STEM related problem solving experiences to a higher degree in the classroom.

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Investigating middle school mathematics teachers' views on innovative learning activities

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This paper describes a phenomenological study exploring middle school mathematics teachers' views, enablers, and barriers regarding innovative mathematics learning activities by investigating their experiences. The study also aims to determine their suggestions that may lead educational stakeholders to increase the quality of instruction and student learning. Data collected from the interviews and qualitatively analyzed reveal that the enablers mainly were related to receiving support and positive feedback from others. The barriers were associated with time, students' learning habits and classroom learning culture, work environment, and preparing activities. There is a need for providing training and resources for teachers for better mathematics education. Integrating such activities into the curriculum, developing effective training programs, and supporting teachers for successful implementation are the implications based on the findings.

Keywords: Innovative mathematics learning activities, STEM education, mathematical modeling, middle school, teaching practices

Introduction

Today's societies are expecting qualified individuals equipped with 21st-century skills to keep up with the era since technology and information are rapidly produced, developed, changed, and consumed. One of the most crucial factors in achieving the goal of developing individuals with 21st-century skills, identified by educators and economists, is education. However, formal education institutions that mostly use teacher-centered instruction which promotes rote learning and memorization, may negatively affect the need for a qualified workforce and inevitably become insufficient to integrate real-life into the instruction (Akgündüz et al., 2015). In other words, traditional educational practices are insufficient and ineffective to raise individuals as required by the 21st-century (Borich, 2017). As a result of the rapid changes in the globalizing world based on technology, business, and industry, countries have been forced to implement innovative policies in their educational systems as an inevitable result of the change.

There are many approaches in which 21st-century skills can be targetted, including inquiry-based learning, discovery learning, problem-based learning, project-based learning, technology-assisted learning (Westwood, 2008), and STEM education (Barakos, Lujan & Strang, 2012) in the educational literature. The theoretical and practical background of problem-based learning (Hung, Jonassen & Liu, 2008), project-based learning (Condliffe et al., 2017), and STEM education (National Research Council [NRC], 2010) may help students to become skillful at critical thinking, collaboration, communication, creativity, productivity and problem-solving for being up-to-date, having scientific and technological literacy and living in 21st-century.

There is a need for research studies that examine the applicability, strengths, and disadvantages of the activities related to STEM education or mathematical modeling in schools. Approaches that aim to enhance 21-st century skills of students have been implemented by some Turkish middle school teachers. In this regard, exploring their experiences and perspectives by focusing on the enablers and barriers they faced while implementing such activities may provide insights for designing future professional development efforts in many other contexts. Therefore, the current research was designed to investigate the views and experiences of middle school mathematics teachers who have some degree of experience in implementing innovative mathematics activities, such as STEM education or mathematical modeling, in their classrooms.

The current study used the term "innovative mathematics activities" as an umbrella term based on literature review and preliminary informal communications with middle school mathematics teachers about their STEM or mathematical modeling activities. Many of the teachers did not name their implementations as STEM or mathematical modeling as given in the literature and there are blurring boundaries between such relatively new approaches in the Turkish context. In this study, this term referred to non-routine educational activities that emphasize the real-life connections of mathematics and integrate mathematics and other disciplines such as science, technology, and engineering to maximize student learning, help them gain positive attitudes towards mathematics, and develop their 21st-century skills. These integrated activities require planning, implementing, and evaluating student-centered innovative mathematics instruction. We used the term "innovative" to refer to *relatively* new methods in mathematics learning activities rather than teachers discovering "new" educational methods. In summary, we use this term to refer to middle school mathematics teachers' all "relatively new" mathematics learning activity implementations related to non-traditional and non-routine educational approaches. In some educational contexts, the term evolved into including only STEM education and mathematical modeling activities since the participants of the study made a connection between innovative mathematics learning activities and them.

The specific research questions of the study were:

- What views do middle school mathematics teachers have for innovative mathematics learning activities they implement?
- What are the enablers and barriers for the innovative mathematics learning activities implemented by middle school mathematics teachers?
- What are the middle school mathematics teachers' suggestions for implementing innovative mathematics learning activities to other educational stakeholders?

Method

We use a phenomenological research design. The thirteen participants of the study were selected through a snowball approach among middle school mathematics teachers who had knowledge and experience about innovative mathematics activities. Three of them were working in public schools, and ten of them were in private schools. All of them -except one- were female. All of them had teaching experience at all grade levels of middle school, including their internships. Their teaching experiences ranged from 2 years to 20 years.

Data were collected through semi-structured interviews. The interview protocol included 19 questions and follow-up questions for some of them. The interviews included questions related to the participant teachers' thoughts about (i) the nature of their mathematics teaching in general, (ii) their implementation processes, including enablers and barriers to their work, and (iii) reflective interpretations about the activities and recommendations for other teachers.

Data were collected in face-to-face and online environments on a volunteer basis. We followed six steps recommended by Lodico and her colleagues (2010) to analyze data. The interviews were transcribed, redacted, and studied. Then they were analyzed utilizing the content analysis method. The similar views, experiences, enablers, barriers, and suggestions about innovative mathematics learning activities under the codes formed by the researchers. The data analysis process was guided by the research questions.

Examples of innovative mathematics learning activities implemented by the participant teachers

The activity implemented by P2 can be given as a representative example of innovative mathematics learning activities. She formed Caretta Caretta Nest Activity for 6th-graders. The activity included designing a nest for caretta caretta to prevent their egg loss by designing a nest for them after searching their sizes and living conditions. The students needed to use their knowledge related to ratio-proportion, area of polygons, and volume of prisms in the design process. The process required considering using the given whole area effectively, creating nests properly, and placing them productively.

Another example is Oil Spill City Activity formed and implemented by P10 in 6th-grades. She asked her students to design barriers to prevent oil spills after an environmental disaster by using their ratio-proportion and area of polygons-related knowledge. She stated that they used saltwater, olive oil, and bottles filled with to represent seawater, oil, and barriers to experiment with the process. She mentioned that her students connect their knowledge on density and ratio-proportion by changing the amount in these materials to stop oil spills by holding barriers at a certain level.

Findings

Teachers' perceptions regarding innovative mathematics education

In the interviews, the participating teachers were asked what they understood of the term "innovative mathematics education." Their descriptions were mostly based on non-traditional educational approaches. They linked it with student-centered instruction, having real-life connections, emphasizing learning by doing and active participation, involving technology use, and being activity-based and interdisciplinary.

Attributes of implemented innovative learning activities

The participants were asked to explain the characteristic features of the innovative mathematics activities they implemented with their students. The teachers mentioned the grade levels they targeted, the mathematics concepts covered, the physical setting, concepts covered from other disciplines, the average duration, how their students work on the activities, their assessment techniques, and their resources while preparing for the activity. The following table summarizes an overview of the participant teachers' responses to the characteristics of innovative mathematics learning activities.

Table 1: Attributes of innovative mathematics learning activities the teachers implemented

Main Category	Sub-category	Frequency
Grades implemented	5	7
	6	6
	7	4
In-class & out-of-class	In mathematics lesson	8
	In student club	4
	In the "Applications of Mathematics" lesson	1
Mathematics content area	Numbers and operations	9
	Geometry and measurement	7
Related subjects/objectives from other disciplines	Designing a model & modelling	8
	Being able to use educational technology (i.e., Arduino, GeoGebra, Sketchpad, Tinkercad)	8
	Raising (social & environmental) awareness	8
	Optimizing the criteria	6
	Other subjects related to science	4
The average duration of activities	3 class hours	5
	4 class hours	6
	5 class hours	1
	6 class hours	1
Group work	-	13
Means of assessment	Observation	13
	Discussion and questioning	13
	(Student products in response to) Performance task	8
	Peer rating	7
	Teacher-created paper-and-pencil test	5
	Exit card	4
Information sources for teachers	The Internet	12
	Professional development seminars	8
	Books and articles	5

Note. The number of total responses is greater than the number of the participants since they mentioned more than one innovative mathematics learning activity implemented by them, and some teachers' responses include more than one category. So, the frequency represents the number of teachers who point out the given categories. This is valid for the rest of the tables.

Emotion perceptions during innovative mathematics learning activities

Many of the teachers asserted that they have positive emotions such as being satisfied, having fun, motivated, and excited while implementing innovative mathematics learning activities. Almost all of these emotions were related to students' reactions and emotions that were commented as positive by the participants. More explicitly, the participants feel satisfied when their students learn the topics they covered in the activities, and correspondingly when their students feel happy. They felt themselves as enjoyable teachers and did not get bored during the lessons in which such activities were implemented because of their active role and high interaction with their students compared with their traditional lessons. They are motivated more in the lessons they use such activities because of their students' positive reactions such as excitement, happiness, and active participation. Similarly, -but changing roles with their students- they feel excited when their students are motivated to make an effort to learn and participate during the implementations.

Some participants describe their experiences as exhausting, which depends on their role in these activities. According to interview data, this was not a complaint but more of a description of the nature of the workload these activities had. More specifically, they found that planning and implementing such activities requires more effort than their traditional lessons.

Perceived effects of innovative mathematics learning activities

The teachers explained how they perceive the effects of their activities on students based on their observations and experiences. They interpreted these effects as having positive contributions to change students' attitudes and enhancing their 21st-century skills, affective skills, learning of concepts, and psychomotor skills.

All of the participant teachers argued that the students became more aware of their future professional options, social and environmental issues, and gender issues. The participants believed that their students' attitudes towards participating in mathematics lessons and learning mathematics changed positively. Also, they believed in a positive contribution of innovative mathematics learning activities to their students' 21st-century skills. The specific skills they mentioned include collaboration, communication, problem-solving, critical thinking, researching, creative thinking, and curiosity. The positive effects of these activities were stated as motivation, attention, self-confidence, and a sense of achievement. Almost all of the teachers highlighted the improvement in student learning of the concepts and an increase in students' achievement in mathematics due to the interdisciplinary nature and real-life connection of such activities. Lastly, since the teachers implemented mostly modeling or designing activities, they observed the development of their students' psychomotor skills.

Enablers of providing innovative mathematics learning activities

The teachers explained their enablers by connecting to themselves and other people involved in the educational process. Students, colleagues, school managers, parents, and other people contribute to planning their lessons. The following table presents an overview of the enablers of providing innovative mathematics learning activities.

Table 2: Enablers in the process of innovative mathematics learning activities

Enablers	Frequency
Collaborating with colleagues	10
Receiving support from school management	8
Receiving positive feedback from parents	5
Thinking students' possible questions	5
Receiving positive feedback and reactions from students	4
Talking with an expert from a different profession	3

Barriers in the process of innovative mathematics learning activities

The difficulties or barriers the teachers encountered in the process of innovative mathematics learning activities were grouped under four categories: (i) time, (ii) students' learning habits and classroom learning culture, (iii) work environment, and (iv) preparing activities. The following table summarizes these categories and their sub-categories with the frequencies of the difficulties mentioned.

Table 3: Barriers in the process of innovative mathematics learning activities

Main Category	Sub-category	Frequency
Time	Time constraints for covering the curriculum	10
	Need for teaching to test	6
Students' learning habits and classroom learning culture	Being familiar with teacher-centered instruction	7
	Lack of teamwork experiences	6
	Difficulty in classroom management	5
Work environment	Teachers having too much workload	5
	Lack of equipment for activities	4
	Destructive criticism of colleagues	3
Activity preparation	Difficulty in integrating other disciplines into mathematics	5
	Teachers' lack of knowledge and experience	4
	Difficulty in simplifying complex concepts for students	3

Teachers' suggestions about innovative mathematics learning activities

The participants expressed their suggestions about innovative mathematics learning activities by considering their supporting factors and difficulties. When they were asked what would their recommendations to other teachers be, they mentioned keeping themselves up-to-date, knowing students' characteristics and interests, observing and leading students during the activities, becoming persistent in implementing such activities, implementing the well-known basic activities, getting opinions of others, and learning a foreign language. They suggested that the ministry authorities should simplify the mathematics curriculum, provide training and resources for teachers, and put sample activities in the curriculum. They also mentioned that mathematics teacher educators need to contribute by providing teachers resources and professional development opportunities.

Discussion and implications

The participants stated that student-centered instruction, the real-life connection of mathematics concepts, learning by doing strategies, active participation of students, integration of other disciplines (especially science, technology, engineering, and design) with mathematics, and activity-based learning make their mathematics lessons innovative. Usually, the Turkish middle school mathematics curriculum covers and recommends these approaches in Turkish schools (Ministry of National Education [MoNE], 2018). So, these approaches should not be called innovative. However, the teachers perceived themselves as teaching mathematics out of the ordinary when they implemented activities related to such approaches. Although these activities are given a place in the curriculum as regular practices at the policy level, they are seen as non-routine and innovative practices by the teachers. Teachers have restricted educational practices in accordance with policies even though these policies include innovative suggestions. So, policymakers should be careful about how national educational policies are perceived and implemented among teachers.

Although the teachers did not explicitly mention any theoretical point of view, their descriptions and implementations can be interpreted in line with the constructivist and constructionist learning theories, related to STEM and mathematical modeling approaches. It can also be argued that most of the teachers' perceptions of innovative learning activities were in line with the recent rhetoric of STEM education, mathematical modeling, project-based learning, problem-based learning, cooperative learning, or technology-integrated instruction. This indicates that the teachers were up-to-date on the relatively current educational theories and were willing to implement the ideas and approaches in these theories into their classroom practices. It may be because most of the teachers attended professional development seminars and in-service training programs.

The teachers stated one of the most critical factors in maximizing the quality of their activity implementation was their interaction with other people –namely students, colleagues, school managers, parents, and domain experts– who get involved in the activities from beginning to the end by sharing their views, comments, knowledge, and experiences. Therefore, teachers need a collaborative mindset for learning to implement such activities. Regarding the barriers to implementing the innovative activities, the findings of the study confirm the other previous studies on STEM and mathematical modeling (Herro & Quigley, 2017; McMullin & Reeve, 2014; Stohlmann, Moore & Roehrig, 2012).

Since the STEM and mathematical modeling integration into mathematics lessons are relatively new for many teachers, it is considered that there is a need for informing educational stakeholders about the integration for better implementation. The findings of the study demonstrated that some enablers such as collaborating with colleagues and professionals from other disciplines, being supported by colleagues and school management, and receiving positive feedback from students and parents make the process of innovative mathematics learning activities easier. These factors enable teachers to implement well-planned STEM and mathematical modeling activities. Correspondingly, teachers should be encouraged to implement their activities by creating a collaborative working environment and taking moral and material supplies. So, professional development opportunities can be designed to increase communication and collaboration between mathematics teachers and others. On the other hand, the current study put forward that teachers get into several barriers while implementing their activities. They primarily focused on the lack of time for planning and implementing the activities due to their workload and requirement of covering the curriculum in regular classes, the lack of students' familiarity with these types of activities, the lack of equipment required in the implementation of these activities, and the lack of knowledge that teachers experienced in STEM and mathematical modeling activities. The authorities can develop strategies to overcome these barriers for better mathematics education by developing policies about teachers' workload, mathematics class hours, and mathematics curriculum by revising them in accordance with STEM and mathematical modeling activities.

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Criteria for sociocritical modeling tasks in sustainable development contexts

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There is a broad consensus that Education for Sustainable Development (ESD) should be integrated into all core subjects to enable learners to acquire the necessary knowledge and competencies for a SD in a comprehensive way. How ESD can be integrated in mathematical modeling has not been clearly defined yet. Sociocritical modeling can be one possible way for implementing SD in mathematics lessons. In this paper, we develop criteria for sociocritical modeling tasks based on previous empirical findings and theoretical considerations. To conclude, the task textile consumption, that was designed based on these criteria, is presented here.

Keywords: sociocritical modeling, sustainable development, task design

Introduction

In an era of globalization, digitalization, climate change and crises, a growing number of educators and employers are united around the idea, that students need to learn special skills to deal with these kinds of global issues[1]. These issues require a shift in our lifestyles and a transformation towards a more sustainable way we think and act (UNESCO, 2014). According to Bakker et al. (2021) “the big question is what role mathematics education can play in meeting these challenges” (p. 7).

Education for Sustainable Development (ESD) is increasingly demanded from many sides (Nationale Plattform BNE, 2017). This requirement applies to all areas of education and to all subjects, including mathematics education. In terms of educational theory, the relevance of ESD in mathematics education can be justified, among other things, by Winter's basic experiences (Winter, 1995). The first states that "Erscheinungen der Welt um uns, die uns alle angehen oder angehen sollten, aus Natur, Gesellschaft und Kultur, in einer spezifischen Art wahrzunehmen und zu verstehen" [phenomena of the world around us, which concern or should concern us all, from nature, society and culture, are to be perceived and understood in a specific way] (p.17). One goal of ESD is to empower learners with the knowledge, skills, values, and attitudes to address the above-mentioned challenges and take informed decisions and make responsible actions. Therefore, students need to gain competencies such as critical thinking and reflection, both for ESD and in math education (Gutstein, 2006; Skovsmose, 2021).

A great deal has been assumed and researched about modeling in math education to include real-world contexts in teaching mathematics. What has been little studied is the extent to which students are also able to acquire competences that will help them become responsible citizens via working on modeling tasks. This includes, among other things, the competence to make

[1] https://www.unesco.de/sites/default/files/2018-08/unesco_education_for_sustainable_development_goals.pdf

decisions that promote SD (also called *decision-making for SD*). Bakker et al. (2021) published answers from all over the world to the question: "On what themes should research in mathematics education should focus in the coming decade" (p. 2). Many of the responses, especially after the pandemic outbreak, strongly emphasized the importance of teaching mathematics in the context of these global challenges (Skovsmose, 2021). Increased interdisciplinary teaching or problem-based learning to learn competencies such as critical thinking or decision making can be mentioned here as an example. It can be claimed that the demand for a socio-critical orientation of mathematics education is constantly increasing the more crises society has to cope with (Skovsmose, 2021).

This leads to our research question: *What might be useful didactical criteria for designing tasks to establish reference to SD contexts in mathematics lessons?*

Theoretical Framework

The idea of integrating ESD into mathematics education is a more or less young one (UNESCO, 2014). Therefore, it is necessary to briefly address different theoretical aspects from both sides, mathematics education and ESD, to be able to compare and combine later.

Understanding, questioning, and critically reflecting on global issues using sociocritical modeling in mathematics education can be seen as a way to initiate a critical orientation within mathematical modeling. "Modelling is a powerful vehicle for bringing features of twenty-first century problems into the mathematics classroom" (English, 2016, p.10). From this it can be deduced that there is a need for tasks in mathematics education that are oriented towards socio-scientific issues and, in addition to modeling competencies, also offer the opportunity to critical reflect on the context and develop decision making for SD. Because all relevant contexts and issues are composed of content from STEM subjects, this goal cannot be achieved without including STEM (Maass, Geiger, et al., 2019). Furthermore, including SD contexts in mathematical modeling is an important opportunity "to integrate science, technology and engineering in meaningful ways as students tackle problems involving mathematics in relevant settings" (National Council of Supervisors of Mathematics & National Council of Teachers of Mathematics, 2018).

What exactly is meant by socio-critical modeling, how modeling tasks can be designed, and what is understood by contexts of sustainable development is explained in the following sections.

Mathematical Modeling

The goals of modeling in math education can be considered in different ways. A distinction can be made between content-oriented, process-oriented, and general goals. The content-oriented goals include the development of the environment with the help of mathematical means. Process-related goals include general mathematical competencies (e.g., problem-solving skills) and the heuristic strategies. Finally, the general goals include culture-related arguments. Here, the focus is on the image of mathematics as a science and on conveying the importance of mathematics for active participation in society (Greefrath & Vorhölter, 2016).

In addition to the goals of modeling presented here, various theory-based perspectives have emerged over time in the national and international discussion on modeling (Kaiser & Sriraman, 2006). The goals pursued can differ greatly in each case, depending on the aspect under which modeling tasks are developed, selected, and used. The perspective of socio-critical modeling, as one of seven, pursues the intention to promote discourses that are stimulated by mathematics and come from the learners' lifeworld. Barbosa (2006) describes this modeling perspective as a way of critically reflecting on the role of mathematics in society. Here, the goal of modeling is primarily to generate interdisciplinary thinking (special relation to STEM education) and critical discourse among learners. The crucial role, the context of the task can have, is particularly evident in this form of modeling.

Socio-critical modeling can deal, among other things, with socio-scientific issues (e.g., cloning or stem cells) and social issues (e.g., gender-pay-gap). Real problems like these often include ethical, moral, social, or cultural dimensions and thus challenge us to become aware of the subjectivity of modeling processes. This is done by critically reflecting on all steps of a modeling process. However, the outcome of such modeling can lead students to critically reflect on their own behavior and make recommendations based on their own modeling.

Sustainable Development Contexts

The difference between an issue and its context is that the context can be far more complex and considers more perspectives and information (Sadler, 2004). One important question when it comes to realistic contextualized tasks is the amount of reality on the one hand and mathematics on the other hand. So, with SD contexts in modeling, it is rather important to address both. It has been noticed that “it can be difficult to identify and address the mathematics involved in the contexts” (Maass, Doorman, et al., 2019, p.1001). We entirely agree that this difficulty points out the necessity to include such issues and the inherent mathematics. As an example for SD contexts that should be more present in mathematics education, a discourse from science, politics and society led to the identification of central problem areas for the design of SD. According to them, the socioscientific core problems of the 21st century include: *climate change, world food supply, soil degradation, drinking water* and *biodiversity* (Reid et al., 2010; UNESCO, 2014). Of course, there will be many more challenges, especially of a social nature, such as racism or migration, which will not be discussed further here for the time being.

The Guide for embedding ESD in textbooks presents a first orientation to place SD contexts at the core of school subjects through the implementation in textbooks. The authors outlined 15 guidelines for creating mathematics curriculum resources that support ESD which include among others: real contexts, current issues, complexity, values, access to data, interdisciplinarity, opening dialogue, collaboration and courage (MGIEP & UNESCO, 2017).

Modeling task design

To encourage students to engage in mathematical modeling, suitable problems and tasks are required. Several theoretical approaches for the design of modeling tasks exist but will not be further outlined here (Blomhøj & Kjeldsen, 2006; Maaß, 2010).

Another possibility for the conception of good modeling tasks is the orientation at criteria, based on which their quality can be classified. At this point it should be emphasized that there are no "the" criteria of modeling tasks. The goal, which is followed with the processing, is in the focus. Even a good modeling task does not have to fulfill all criteria equally. The criteria according to

Maaß (2007) will be considered here, which she names as "typical" for modeling tasks are *1. open, 2. complex, 3. realistic, 4. authentic, 5. problematic and 6. solvable by carrying out a modeling process*.

For the design of modeling tasks that support decision making for SD, one needs real contexts and current issues. SD is thus considered a new context generator for mathematics education. Therefore, it must be ensured that the SD context is accessible using mathematical methods. As already stated above, scholarly studies and societal experiences have been able to identify some core problem areas in the discourse that are of particular importance for the design of a SD. These core problems now serve as a thematic orientation framework for the task design. Accordingly, the issues to be addressed should be *1. central to SD processes, 2. locally or globally, 3. have potential for action, 4. have long-term significance* and *6. can be dealt with in an interdisciplinary manner*.

Based on these theoretical thoughts we here report our approach for the design of SD context modeling tasks. For this purpose, the process of the criteria merging is shown first. Subsequently, the exemplary task based on these results is presented.

Merging Criteria

Comparing both sets of criteria, some overlap can be found in what seem to be characteristics of good modeling tasks. The design of the tasks should succeed in such a way that modeling competencies are promoted and at the same time interest in the concerns of SD is aroused in the students. Additionally, they should be stimulated to reflect critically or argue concerning SD issues. For this purpose, criteria from the two different domains were compared (see Figure 1). In addition to the criteria for modeling tasks (Maaß, 2007) (green box), the guidelines for embedding ESD in textbooks (MGIEP & UNESCO, 2017) (orange box), were considered. In the process of merging, similarities and differences of both guidelines were worked out. An attempt was made to create a catalog of eight criteria that would include both guidelines in equal parts (see arrows Figure 1). The result of the comparison of the criteria are eight characteristics for modeling tasks in the context of ESD (yellow box).

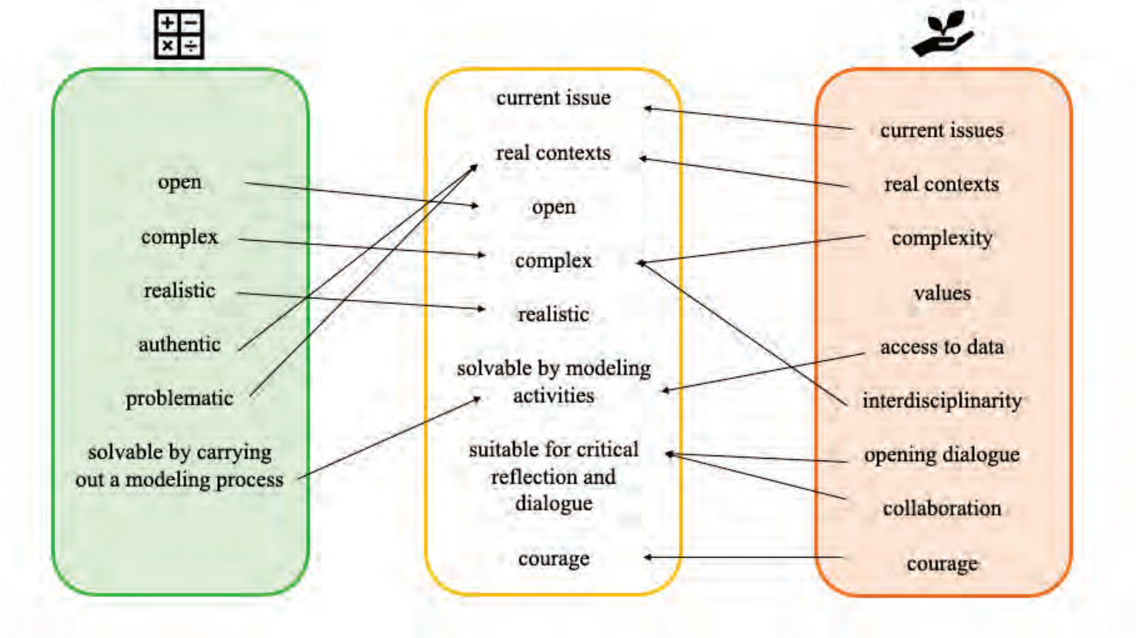

Figure 1: Process of merging of criteria

SD context modeling task design – an example

In the following, we show how these criteria can be applied to a selected current issue for the design of a modeling task. Starting with a current issue like "Fast Fashion" and embedding the problem in a real context of global development like climate change is one possible way to begin with the task design. Based on the problem and context chosen, the corresponding values should be named, and open and realistic question should be formulated, that is solvable by modeling activities and in some way suitable for a critical reflection. If possible, a longer-term potential for action should be able to be defined from the task.

Table 1: Exemplary application of the criteria

current issue	Fast Fashion
real context	Climate Change
open	individual assumptions and solutions
complex	interdisciplinary nature of the problem
realistic	use of real data
solvable by modeling activities	understand/simplify, mathematize, interpret, validate
suitable for critical reflection and dialogue	reflection on one's own consumption behavior
courage	leaves judgment to the students based on their mathematical investigation and knowledge of the relevant context

For the context of the modeling task presented in this paper, the issue "Fast Fashion" has been chosen (see Table 1) which, on the one hand, has a strong connection to the youth reality and, on the other hand, is related to the core issues of SD. Since, in addition to modeling competencies, it is primarily that critical reflection and arguing are to be promoted, the potential for personal action must be apparent to the students. Therefore, the guiding theme *consumer behavior* has been chosen, as it effects every student, has multiple effects on socio-scientific and social issues and provides mathematical content to work with (see Table 2).

Table 2: Textile consumption as an exemplary SD context modeling task

Task	Textile Consumption
Question	How big would your closet have to be in 10 years if you don't sort out or dispose of anything in between and continue to shop the way you have been?
Mathematical Content	Convert units, rule of three, rounding, estimating, calculate with volume
Learning target	Trigger AHA effect about own consumption behavior and effects

Preliminary experience with the modeling task "Textile Consumption" (see Table 2) was gained with 50 grade 10 students. They worked in small groups of 3-5 people, and after completing the tasks, group interviews were conducted for qualitative data collection. The use in a 10th grade mathematics class has shown that the students were able to reach a result with the task through modeling activities (understand/simplify, mathematize, work mathematically, interpret, and validate). In addition, the results of the interviews showed that a critical reflection of the given issue took place at several points. It should be mentioned here that the way the interviews were conducted, and the limited time frame did not allow for further discussion and reflection of the results. Nevertheless, all groups critically reflected on the problem of Fast Fashion and their own consumption. At this point, we would like to provide two quotes from the interviews as examples.[2]

> In any case, the task has made it clear to me that I should perhaps reduce my consumption a bit. My clothes are really a lot, and I might also look in the next few years that I sort things out and give them away...it's always difficult to separate, but that you then get rid of clothes and give them to people who need them. (Diana)

Through this quotation it becomes clear that the student has recognized for herself, due to the work on the task, that she should limit her clothing consumption. However, it can be noted that she does not necessarily aim to buy less new clothing, but rather to pass on existing clothing. Nevertheless, it can be stated that the processing of the task has led to a critical reflection on and examination of the context. This connection becomes even more apparent in the following second quote. In this quote, the student himself makes a direct connection between calculating the amount of clothing and initiating reflection on his own clothing consumption:

> Because you have calculated how many clothes you buy and how big your closet should be, you start thinking that you buy a lot of clothes that you might not even need...not that much. (Enya)

[2] The quotes are translated by the authors and the names of the students have been changed.

These two quotes show that mathematics can certainly be used for critically reflecting on a context in a meaningful way. They show two different kinds of a critical reflection based on the students' experiences while working on the modeling task textile consumption. The students interpret their mathematical solution in the given context and start to substantiate their arguments for reduced textile consumption with their mathematical results.

Summary & Outlook

Based on the question, what role mathematics education can play in meeting global challenges like crises, globalization, and the urgent need of a sustainable development, we started with SD contexts in socio-critical modeling creating content for mathematics classrooms. The research question to be answered here related to possible criteria for the design and implementation of tasks that support arguing for decision making for sustainable development.

The literature research has shown that so far, no criteria for tasks in general as well modelling tasks in particular have been formulated, which are to promote explicitly competencies such as decision making for SD. Thus, the set of criteria elaborated in this work could be usefully applied in further research for the design of SD context modeling tasks. Designing and evaluating further teaching materials based on these criteria is a worthwhile task for future research.

The quotations of students presented make it clear that by working on the task, the students have begun to question their own textile consumption, even though the intentions for action expressed by both are subjective and different. This shows that tasks with an SD context have the potential to stimulate critical reflection and promote arguing even outside of mathematics. Future research is needed to determine how, why, and to what extent this approach can work in general.

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Towards implementing Black Boxes in Mathematical Modeling an epistemological Approach

Adapting models from science and science education

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This article explores the concept of black boxes within mathematical modeling. First, we perform a literature-based clarification of the term black box, the use of a black box, and explore the use of this construct in science, and workplace. A black box is characterized by the following issues: (a) being believed to be distinct, (b) having observable (and relatable) inputs and outputs, (c) being black (that is, opaque to the observer). Afterwards, we demonstrate the uses of black boxes within mathematical modeling. We generalize these uses to a model of how black boxes can be implemented in mathematics education. Based on this model, we derive educationally relevant activities for using and opening such a black box. The application of these activities is illustrated in more details using an example of physical chemistry: the Arrhenius Equation.

Keywords: Black box, mathematical modeling, STEM

Introduction

Mathematical modeling is an important part of mathematics education (Kaiser, 2020). English (2016) described modeling as a powerful tool to implement 21st century skills in mathematic classrooms. This is because real-world problems connect mathematics with aspects of the world, and are often linked with mathematical modeling and mathematical models (s.f. Artigue et al., 2007; Kaiser & Stender, 2013; Lingefjärd, 2006).

In practice, many models use black boxes. For example, they serve as simplifications of phenomena or approximations in modeling tasks (e.g. Bissell, 2004; Buchberger, 1990; Hoijnen et al., 1992; Straesser, 2007; Williams & Wake, 2007). Black boxes are often used, especially in work places and science models (Straesser, 2007; Williams & Wake, 2007). In order to understand how the term black box is used and whether or how it can be utilized for mathematics, we have used an epistemological approach and analyzed tasks of science. Because of these findings, we argue that we should include teaching about black boxes in mathematics education. However, there is no consensus (yet) on how exactly we should do that. Hence, the aim of this paper is to give an overview of possible approaches, to present our own concept, and to deepen existing approaches in the process.

Theoretical background: Black Boxes

Kaiser and Stender (2013) proposed two different directions for mathematical modeling: "*mathematics for applications, models and modeling*" (p. 278), and "*applications, models and modeling for the learning of mathematics*" (p. 278). The second headline emphasizes the goal of learning mathematics. In this paper, we want to implement black boxes into mathematical modeling in the context of STEM. Therefore, we will refer to the first headline. The realistic or applied approach of mathematics puts the solution of the real-world problem in the center. To describe the processes while working out the solution different modeling cycles have been developed (e.g.Blum & Leiß, 2007; Kaiser & Stender, 2013). In the following, we will refer to the modeling cycle proposed by Kaiser and Stender (2013). This corresponds the mathematical modeling cycle in physics education of Redish and Bing (2009) excluding the real-world situations and fits the later presented problem.

Modeling with Black Boxes

As described above assumptions are obligatory during modeling. These can be interpreted as black boxes. In addition, well-accepted mathematical models can work as a starting point of further real-world problems. Therefore the problem solver must not have fully understood the origin model or its emergence, but its input and output behavior. Accordingly, this mathematical model is used as a black box. In our contribution, we give an overview of black boxes in general and in particular in the context of mathematical modeling. We will present a way to open black boxes, but without giving an answer to the question if and when one should open a black box.

The term black box is usually used without a precise definition, but in a kind of intuitive understanding (e.g. Bissell, 2004; Buchberger, 1990; Fojo et al., 2017; Hoijnen et al., 1992; Straesser, 2007; Williams & Wake, 2007). We use the definition of black box provided by Glanville (1982). Following a philosophical approach he characterizes a black box as: "[…] (a) being believed to be distinct, (b) having observable (and relatable) inputs and outputs, (c) being black (that is, opaque to the observer)" (Glanville, 1982, p. 1). We want to emphasize how we understand this definition and give an example. One aspect that we want to enhance is that even though the definition claims observable inputs and outputs, it does not mean that the researcher is already fully aware of its input-output behaviour. Notably, the fact that a black box is both observable and relatable does not necessarily imply that it is also deterministic. Indeed, statistical systems can also be understood as a black box.

State of the art: Black Boxes in Mathematics Education

In mathematics education, black boxes are associated with Computer-Algebra-Systems or other technical systems (Buchberger, 1990; Peschek, 1999; Williams & Wake, 2007). For this usage of black boxes, Peschek (1999) defines a black box as something "[…] in denen operatives Wissen so materialisiert ist, dass es als Ganzes aufrufbar und einsetzbar wird, ohne dass ihre innere Funktions-weise verstanden werden oder auch nur bekannt sein muss" [in which operational knowledge is materialized in such a way that it can be called up and used as a whole without its internal functioning being understood or even having to be known] (p. 1). This mathematical perspective on black boxes is not sufficient. On this account, the focus in the following is not on trivialized mathematics that is taken over by computer systems, but on the usage of black boxes in modeling and STEM education (e.g. Krell & Hergert, 2019; Krell & Krüger, 2017; Upmeier zu Belzen, Krüger, & van Driel, 2019).

Modeling with Black Boxes

In later approaches in science education, a model-based approach is often chosen and the terms *model of* and *model for* are mention in this discussion (Gouvea & Passmore, 2017). The aspect *model for* something means using the model to gain more information about the real world or to make predictions. Therefore they turn into epistemic tools (Gouvea & Passmore, 2017). In this application, it can be useful or even necessary to use a model as a black Box. The idea of black box might emphasize the fact, that a model is a useful thinking tool, but not a precise description of reality. This misconception can otherwise lead to difficulties understanding more complex systems that cannot be explained with the origin model or even prevent scientists to gain new epistemological knowledge (Taber, 2017). But a black box in science education can not only be used to emphasize the model as a tool but is also an approach for modeling itself, by using a black box activity (e.g. Ruebush et al., 2009; Warren, 2001).

Opening a Black Box

The metaphor of opening a black box represents the precise exploration and questioning the inner functioning for which the black box stands for. There are two reasons for opening a black box. The first reason is to gain scientific knowledge about the black box itself. The motivation to gain scientific knowledge about the black box can be research or education. In the context of research, the scientist wants to gain knowledge about the unknown. In science or mathematic education, the incentive to open a black box can be versatile. By opening a black box students can learn something about the model itself, the modeling process and lead to a deeper understanding about the black box, for example, a model that serves as a tool. This can and should prevent application errors. The second reason is that the black box cannot be distinguished.

Black Box Concept

In this section, we explain how we should include black boxes into mathematics education. Therefore, we want to distinguish between a black box that evolved from a white box and a black box that should only represent the unknown. The first, for example, involves mathematical models whose modeling is not apparent to the user, but the user knows about their validity. The second one implies for example phenomena in nature that have not yet been explored in detail. The later criterion of Glanville's definition mentions the user's point of view, and brings both kinds of black boxes together. This view of using a black box is illustrated in Figure 1.

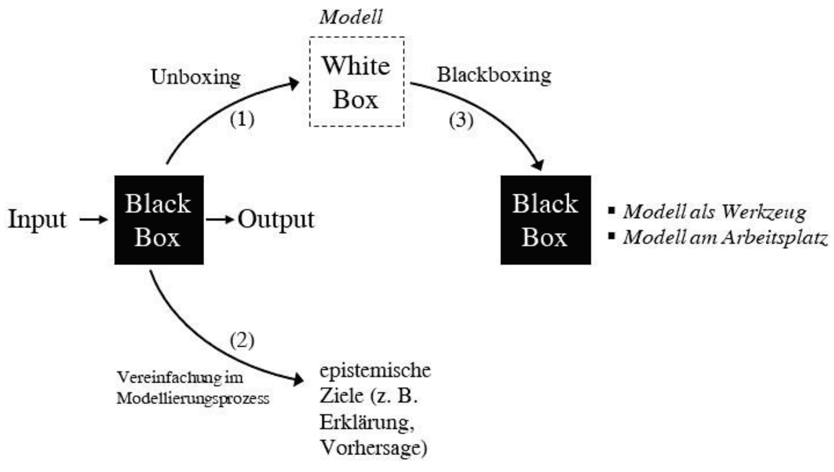

Figure 1: Extended Approach: illustration of the black box, refined from Glanville (1982, p. 1)

We justify this distinction due to their vocational use. By studying unknown phenomena, researchers are trying to unbox the unknown (1) (Glanville, 1982). After unboxing the unknown, it might be useful to turn the white box into a black box (3). Williams and Wake (2007) point out two reasons for black boxes: time and space. However, for mathematics, only time is relevant.

Time represents the black boxes that changed models and/or mathematical work into instruments, and tools (Williams & Wake, 2007). This leads to efficient use of those models but goes along with the leakage of knowledge. For example, the model designer knows the simplifications they made, but the user does not. This is a typical situation at work (Straesser, 2007). CAS-Systems are self-constructed by the designer who is fully aware of the underlying algorithm, the technic, and mathematical knowledge, but it is opaque to its observer/user. The command for differentiation or integration serves as an example. In this application, the software is taking over the mathematical work (Williams & Wake, 2007).

In contrast to opening the black box, it might be useful, not to unbox the black box (2). This may seem to contravene the aim to study the unknown, but "[…] our ability to overcome and cope with ignorance and thus is a primitive of learning and, hence, of science" (Glanville, 1982, p. 1). This can be for example a simplification and is therefore essential in the work of modeling (Bissell, 2004; e.g. Hoijnen et al., 1992). Hoijnen et al. (1992) depict an application for the black box as an approximation in mathematical modeling in a scientific context. He describes the use of a black box to estimate the energy that is gained out of biomass. This enables to calculate auto-and heterotrophic biomass yield the without studying the microorganisms that are not relevant for the modeling (Hoijnen et al., 1992).

One reason for those simplifications is the fact, that exact scientific research is determined by the limits of the measurable aspects (Ortlieb, 2008). Another reason is the intricacy of parts of phenomena that turn out irrelevant compared to the phenomena. Bissell (2004) gives an example in the context of communication and control engineering and points out that:

> In this approach, a complex linear electrical network is represented by its input-output behavior; at this level of abstraction, the precise nature of the interconnections of components inside the black box becomes irrelevant (p. 312).

This leads to the conclusion, that in the scientific context it is necessary to leave some phenomena opaque as a black box to achieve epistemic aims. Therefore, this way of working with models and modeling has to be part of the scientific and mathematic curriculum.

Activities to open a Black Box in Education

The result of an opening process for example by modeling is a more specific model, which still leaves a part of the real word opaque. Latour (1994) describes this endless process as follows: "Each of the parts inside the black box is a black box full of parts" (p. 37). An endless process is incompatible to gain results, efficiency, and education. Therefore, the necessity of a *temporary ending* is essential. In the following, we want to picture relevant activities to work with black boxes, and an option to bring the endless process of opening with a temporary ending together. We try to identify boxes in boxes. This leads to a new modeling cycle, which is illustrated in Figure 2.

The first level represents the black box itself, for example, an equation that is used as a tool. The reasons for this usage have been explained in previous sections. The target itself defines the different underlying levels and can appear in different ways. By opening a black box students can learn something about the model itself, this might be by the interpretation of the model or learning about the modeling process itself. While doing so, assumptions or underlying models appear in this process, which are represented by the smaller boxes below, and defined as new black boxes. Due to this definition, there is a point, where the opening process is finished due to its target, even though not everything is visible to the observer. The most difficult part of this is to know the point of whether to stop the process, or to go further to fulfill the target. This topic needs further research.

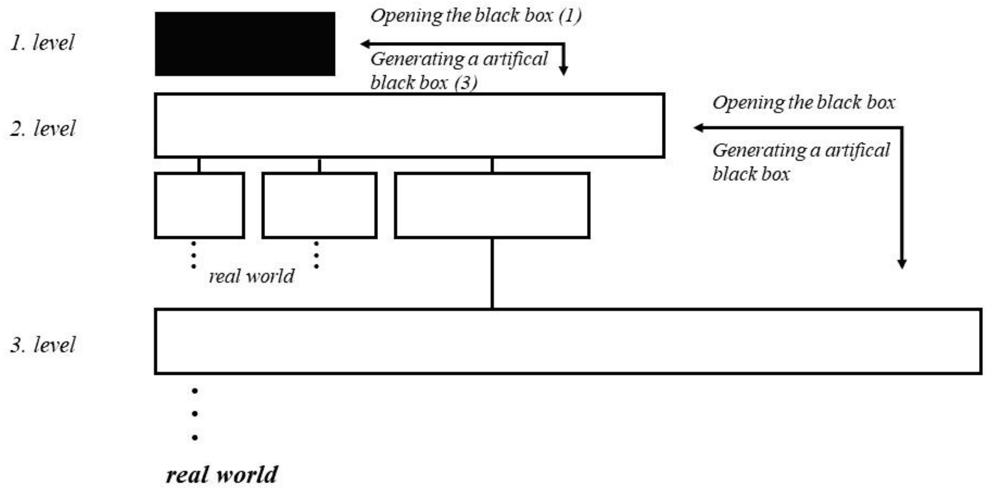

Figure 2: How to open and generate a Black Box.

Another activity to open the black box is that the students interfere it, and make assumptions from the input-output performance. This modeling access is called a black box activity in science education (Krell et al., 2019). This activity can be applied in a different way. It can be linked to a real-world model or applied on a black box for educational reason.[1] The usage of this black box is only to foster students' understanding of models and modeling in science showed positive effects (Krell & Hergert, 2019). Here, modeling serves as a tool for opening a black box.

The described modeling tasks retrieve in the atomistic approach of mathematical modeling, that is besides the holistic approach consensus on the teaching and learning of mathematical modeling (Kaiser, 2020). In the following section, we illustrate the benefit of our model by applying it theoretically to the Arrhenius equation.

Application: The Arrhenius Equation

The Arrhenius Equation (1) is one example in science (education), how a model is used as a black box. The equation is one of the most important equations in physical chemistry (Logan, 1982). For example, it is used in material research for battery technology (Breuer et al., 2015; Ren et al., 2015).

$$k = A \cdot e^{-\frac{E_A}{R \cdot T}} \overset{Linearization}{\Longleftrightarrow} \ln\frac{k}{A} = -\frac{E_A}{R} \cdot \frac{1}{T} \qquad (1)$$

A common application in science education is the ascertainment of the activation energy of the sucrose inversion. To do so the students only need the scientific model as a tool. In this process, the

[1] e.g. Freie Universität Berlin. Modelle der Biologie. https://tetfolio.fu-berlin.de/web/440484 29.08.2021

model is embedded as a black box and will not be opened, which is an example for the first level of Figure 2. The modeling steps for the ascertainment are described in Figure 3 based on the modeling cycle of Kaiser and Stender (2013), in which the real situation is replaced with the black box. In this example the Arrhenius Equation is the black box.

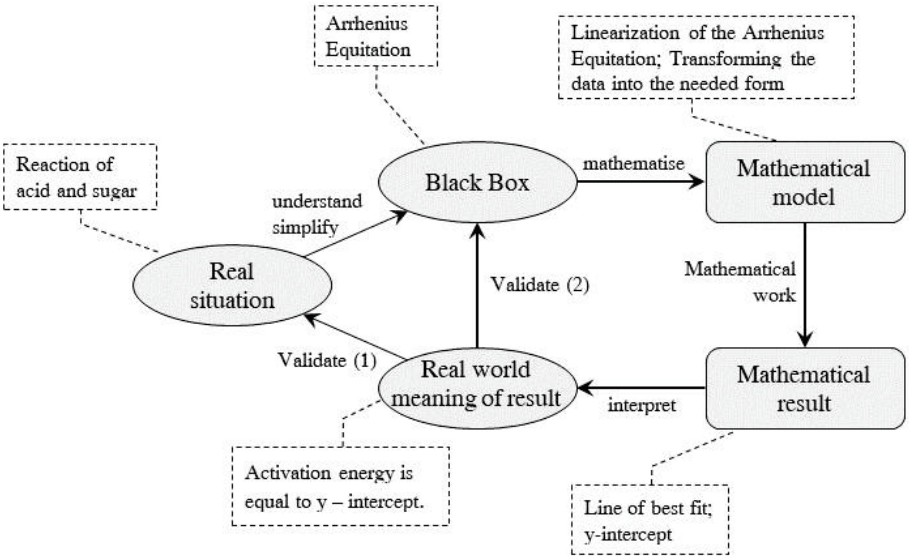

Figure 3: modeling cycle to ascertain the activation energy of the sucrose inversion according to Kaiser and Stender (2013, p. 279)

The modeling task is the ascertainment of the reaction of acid and sugar that cannot be measured directly. The students need to consider, how they can ascertain the activation energy only with measurable data. The Arrhenius Equation is a tool to determine the wanted quantity. To do so the reaction rate is necessary. This one is ascertainable with the changing turning angle of the liquid during the reaction. These results are the base of the following usage of the Arrhenius Equation to determine the activation energy. In the next step, the data is transferred to fit into the mathematical model. This data set is mapped to a coordinate system and the line of best fit is plotted. This leads to the so-called Arrhenius slope (Figure 4). The students have to read off the y-intercept of the slope that corresponds to the activation energy. After this, the results are validated (2) by approximating possible mistakes during the measurement and its analysis. Therefore, every step of the modeling is revised. Following the steps of the modeling cycle can help to structure the propagation of uncertainty. After that the results are compared with the literature value (validate (1)).

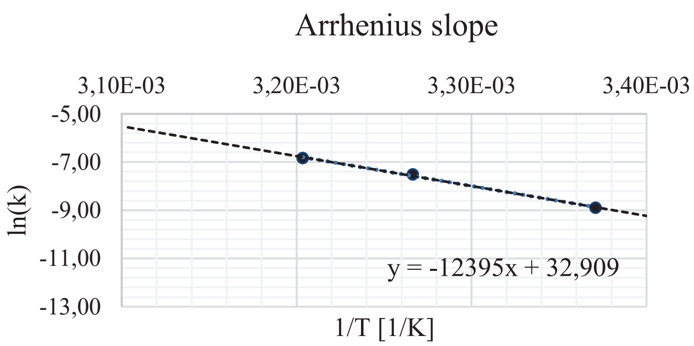

Figure 4: Arrhenius slope

In this described usage, the equation is used as a black box. In the context of education, a reason to unbox the equation is to gain knowledge about the model. This takes us from knowing 'how' to knowing 'why'. The interpretation reveals the connection between the different variables but does not explain the different parameters themselves. This leads to an underlying level. For example, the activation energy and its factors can be interpreted as another black box.

Summary

In this paper, we focused on how we can apply black box techniques in mathematical modeling. By using literature-based normative approach, we developed a model to implement black boxes in mathematical modeling, and argued for its benefits in mathematical education. These benefits show themselves in the necessity for application in science and workplaces, and within the modeling of real-world situations. However, the critical reflection of this process is an open question for future research. This, most importantly, includes deciding when and where to apply these techniques ("critical opening"), both in an educational setting and in the real-world.

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Gaze pattern analysis to reveal student difficulties in interpreting kinematic graphs

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Graphs are an important interdisciplinary and everyday tool for visualizing and interpreting information and communication processes and, thus, an essential part of 21st Century Skills. In particular, linear functions are a fundamental component of school and university education, but students often have difficulties interpreting this type of function, in mathematics as well as in physics – especially in kinematics. This paper presents first results of an eye-tracking study, which compares learners' visual attention during the interpretation of linear graphs in mathematical and physical contexts.

Keywords: Eye tracking, graphs, context, problem solving.

Introduction

Graphs are a typical representation to show the dependencies between different quantities and therefore play an essential role in conveying information in the natural sciences. They are typically introduced in mathematics in secondary school and are part of both the school and university curriculum in different STEM contexts (Glazer, 2011; Leinhardt et al., 1990). Beyond the purely mathematical understanding of the functional relationship, graphs are an important tool, e.g., for visualizing trends in measurement data and are thus an interdisciplinary tool for interpreting quantitative information in the context of 21st Century Skills (National Research Council, 2012).

In recent decades, however, research has shown that many students have difficulties interpreting graphs (Glazer, 2011). This is especially true for kinematic graphs (Beichner, 1993; Ivanjek et al. 2016; McDermott et al., 1987). Since the cognitive processes involved in graph interpretation are closely linked to visual perceptual processes, e.g., extracting relevant information from graphs, eye tracking opens up the possibility of gaining insights into learning or problem solving processes involving graphs (Klein et al., 2021; Susac et al., 2017).

In this work, we use eye tracking to investigate visual strategies of students while solving line graph problems in a mathematical and kinematical context. To this end, we recorded their gaze data to investigate their visual attention while solving item pairs of a test instrument validated by Ceuppens et al. (2019) that require the same mathematical solution procedure. On a selected item pair, we show that gaze behavior differs significantly between the mathematical and kinematical contexts.

State of Research

Students' learning difficulties concerning graphs in the contexts of mathematics and physics can be grouped into three main aspects: First, although mathematics and physics are deeply connected (Redish & Kuo, 2015), students have difficulties linking both disciplines adequately (Ivanjek et al., 2016). Reasons for this are a domain-specific learning (Pollock et al., 2007) and a lack of the ability to transfer knowledge from mathematics to physics (Christensen & Thompson, 2012). Second, students have problems using multiple external representations competently (e.g., Ainsworth, 2006), especially when switching between different representations back and forth (Even, 1998). Third, in particular the high degree of abstraction of graphs poses difficulties since learners need to relate mathematical entities to real world processes (McDermott et al., 1987).

Eye tracking is a method to investigate visual attention by recording eye movements. The eye movement can be described as a sequence of fixations (eye stop points) and saccades (jumps between fixations). Predefined areas in the field of view, so-called areas of interest (AOI), are used to define eye-tracking metrics such as the total visit duration (TVD, cumulated times between first fixation in and first fixation outside an AOI) and transitions (saccades between AOIs). To get an adequate understanding of line graphs, learners have to extract information from the graph and combine this with prior knowledge. Such processes of information extraction and constructing meaning with graphs are described by the Cognitive Theory of Multimedia Learning (CTML, e.g., Mayer, 2009). CTML describes three main processes concerning problem solving and learning: selection (extraction of sensory information from graphs), organization (building a coherent internal representation through information structuring) and integration (combining internal representations such as axis values or axis intervals in graphs with the long-term memory).

The following connection between CTML and eye tracking allows a theory-based interpretation of the eye-tracking data: Gaze durations (TVD, fixation durations) are associated with processes of the selection and organization of information extracted from the processed material, gaze shifts (transitions) are related to integration processes (e.g., Alemdag & Cagiltay, 2018). In summary, eye tracking is a non-intrusive method to obtain information about visual attention and cognitive processing in problem solving processes. So far, to our knowledge, there is no eye-tracking study of the comparison of visual attention processes on linear graphs in the context of mathematics with other contexts, especially with physics in high-school. In general, there is a gap in analyzing mathematical problem-solving processes and representations using eye tracking in secondary school (Strohmaier et al., 2020). This study aims to fill this gap.

Research Questions

For our eye-tracking study, we used the validated test instrument of Ceuppens et al. (2019) because it contains items in both kinematical and mathematical contexts. While Ceuppens et al. (2019) examined 9th grader, we used the test instrument with students in the entry phase of upper secondary school. The aim was to check whether the difficulties in interpreting linear graphs are still present in the upper grades. The background is that the competent handling of graphs in general and of linear graphs in particular is assumed in the upper school and can be seen as a basis for the development of

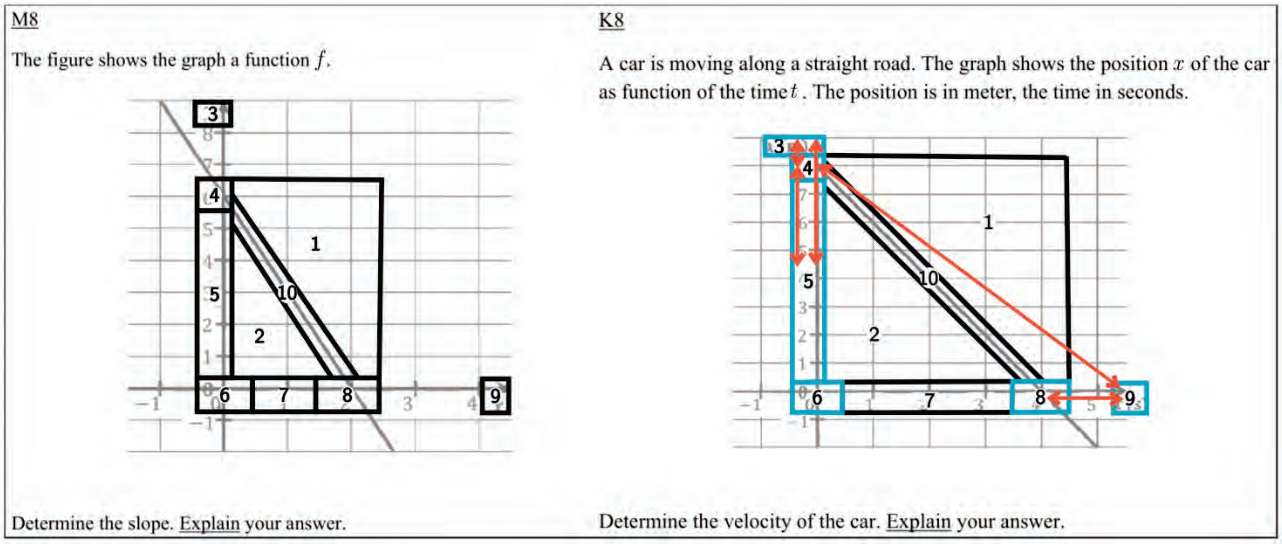

Figure 1: Isomorphic item pair no. 8 from the test instrument used; on the left: mathematical context M8), on the right: kinematical context (K8). The diagram-area of each item is divided into AOIs 1-10. AOIs with significantly longer TVDs than in the isomorphic item are marked in blue. Red arrows indicate transitions between AOIs that appear significantly more often than in the isomorphic counterpart

many new learning contents in the STEM context. The research questions of the study were as follows.

RQ1: Do the difficulties of 9th grader, particularly in solving kinematic items, also occur with students in upper secondary classes?

RQ2: Do selection and organization processes analyzed by gaze behavior in solving isomorphic items differ between mathematics and kinematics contexts?

Methodology

The test instrument consists of 24 items taken from a validated test by Ceuppens et al. (2019) and translated literally into xxx (details redacted for review). The instrument consists of pairs of items in mathematics and kinematics contexts that are isomorphic to each other, i.e., have the same surface features and require the same mathematical solution procedure. In order to avoid sequence effects, all items were presented to the students in arbitrary order and an alternating start either with physics items or mathematics items. In the context of this paper, we focus the eye-tracking based analysis on item pair number 8 (cf. Figure 1). Here, students evaluated the negative slope of a linear graph once for a mathematical function and once for a time-position function.

A total of 35 upper secondary students (14 male, 21 female, all with normal or corrected-to-normal vision) from a secondary school in xxx (details redacted for review) participated in the study. The students voluntarily took part in data collection either in free periods or in regular classes (with teacher permission). At the time the study was conducted, kinematics had already been covered in the courses of all participants. The students were rewarded with a 5 €-voucher for participation. Item pair no. 8 was completed by 24 students only. In the context of this paper, we are interested in why students with mathematical knowledge exhibit problems in the physics context. Therefore, we only consider students who answered the mathematics item correctly and failed solving the physics item (N = 14, 8 female, 6 male).

The study took place in the school's library where two identical eye-tracking systems (Tobii X3-120) were set up. First, the participants answered a short questionnaire about their demographics. After that, a 9-point calibration was performed to obtain a full-customized and accurate gaze point calculation. Subsequently, the 24 items were shown on the computer screen (1920 x 1080 px; refresh rate 75 Hz). If students were ready to give an answer, they pressed a key to move to the next slide. After they had answered, they were asked how confident they were about the correctness of their answer (4-point Likert-type rating scale, ranging from very high confidence to no confidence). The students could take as much time as needed to answer a question. They did not receive feedback after completing a task and could not return to previous tasks.

Performance data: The answers were coded dichotomously (0 for wrong solution, 1 for correct solution) following Ceuppens et al. (2019). Answers for which the participants stated that they had guessed were marked as incorrect. To compare the difficulty of the items, the item difficulty P (proportion of participants who answered the item correctly) was calculated.

Gaze data: The data collection and the definition of AOIs was done with the software Tobii Studio. For the assignment of the eye-movement types, the default I-VT (Identification by Velocity Threshold) algorithm of the software was used (threshold: 30°/s for the velocity; Salvucci & Goldberg, 2000). One participant's results were excluded due to poor quality eye-tracking data. For the gaze data analysis, the items were restricted to the diagram area. The AOIs were chosen such that graph relevant structures are covered by one AOI each (cf. Figure 1). For example, areas of the axis intersection points or areas below and above the linear function are summarized in AOIs. The eye-tracking metrics TVD (selection/organization) and transitions (integration) were considered. A non-parametric Wilcoxon signed rank test was used to test whether the central tendencies of the TVD / number of transitions of the participants in the dependent samples (M8 correct, K8 incorrect) were different. The analyzed datasets met the assumptions required to perform the Wilcoxon test. A threshold of $p = 0.05$ was used to determine the effect significance level within all tests performed. To control the false discovery rate due to multiple testing, the p-values were corrected using the Benjamini-Hochberg procedure. The effect size r (for non-parametric data, cf. Fritz et al., 2012) with 95% confidence interval (calculated using Bootstrapping with 1000 replications) was determined for all Wilcoxon tests with significant results and can be interpreted after Cohen's guidelines (small effect: $0.1 \leq r < 0.3$; medium effect: $0.3 \leq r < 0.5$; strong effect: $0.5 \leq r \leq 1.0$; Cohen, 1988).

Results and Discussion

In this section, descriptive results of all study participants are analyzed first. Afterwards, results of the analysis of the eye-tracking data for item pair no. 8 are presented. This item pair was selected for the eye-tracking based analysis because it shows the greatest difference in terms of item difficulty in the mathematics and physics contexts (cf. Figure 2).

Descriptive results

The results of the item difficulty analysis for all test items are shown in Figure 2. The task description of the item pairs is placed at the top and right-hand side of the diagram. For example, in item pair no. 1 (IP 1), participants were asked to compare the y-intercepts of two linearly increasing functions in mathematics or the initial positions of two objects with linearly increasing $x(t)$-graphs, respectively. Item pair no. 6 required the determination of the slope of a linearly decreasing function or the velocity of an object with a linearly decreasing $x(t)$-graph, respectively. Except for item pair no. 3, the item difficulties of the mathematics items are higher than those of the respective isomorphic kinematics items. Extremely difficult items are kinematical items dealing with negative velocities and with the representation form formula (P_{K4} = 0.15, P_{K8} = 0.08, P_{K9} = 0.20, P_{K10} = 0.10, P_{K11} = 0.15, P_{K12} = 0.05) (Bortz & Schuster, 2010). Whereas the items K2, M2, and M6 are extremely easy items (P_{K2} = 0.89, P_{M2} = 1.00, P_{M6} = 0.83) (Bortz & Schuster, 2010).

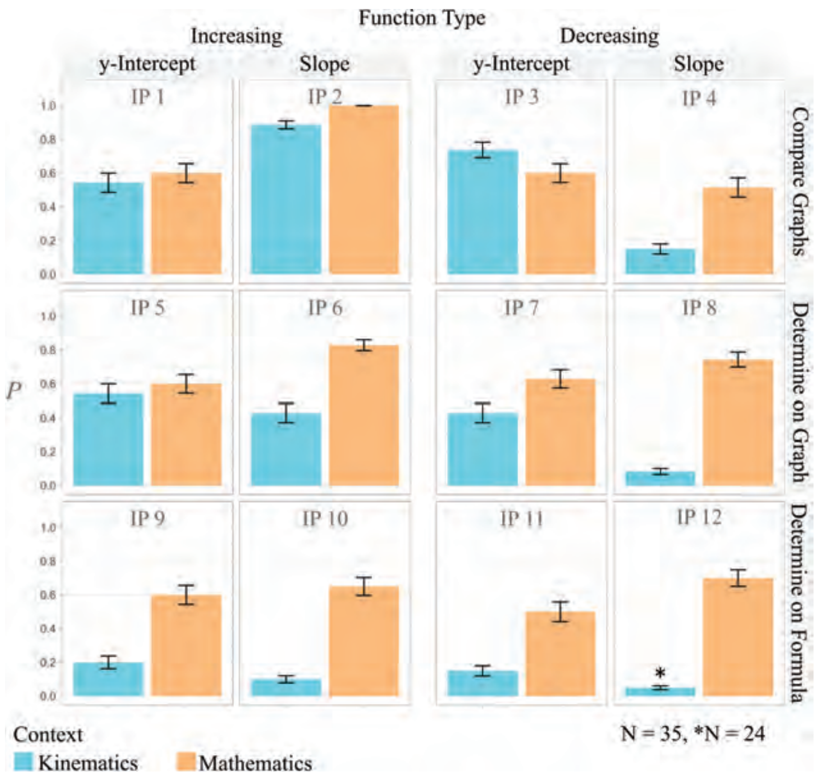

Figure 2: Comparison of item difficulty level *P* of all item pairs (IP) in the test instrument

Item Pair No. 8

The results of the Wilcoxon test (p-values with effect sizes r for significant results) are presented in Table 1 together with the mean TVD for each AOI of item pair no. 8. All significant results relate to longer TVDs in item K8 than in item M8 and are visualized in Figure 1. The two axis labels (AOIs 3 and 9), the x-axis (AOI 5) and the intersections with the axes (AOIs 4 and 8) were viewed significantly longer in item K8.

The number of transitions between AOIs with significant TVDs in both items of item pair no. 8 were analyzed. Table 2 shows the results of the Wilcoxon test (p-values with effect sizes r for significant results) together with the average number of transitions between two AOIs.

Table 1: Average TVD (Mean) per AOI with standard error (SE), adjusted *p*-values of the Wilcoxon Test and effect sizes *r* with 95% confidence intervals are given for all Wilcoxon tests with *p* < .05

AOI	$Mean_M$	SE_M	$Mean_K$	SE_K	*P* (adj.)	*r*
3	.05	.02	2.22	.38	.004	.882 [.877; .882]
4	1.86	.37	4.30	.91	.020	.663 [.441; 1.000]
5	1.24	.19	2.29	.49	.011	.714 [.543; 1.000]
6	.11	.04	.73	.44	.011	.748 [.643; 863]
8	.92	.21	2.65	.69	.020	.663 [.471; 1.000]
9	.03	.02	1.13	.19	.004	.850 [.815; .914]

Table 2: Average number of transitions (Mean) with standard error (SE), adjusted *p*-values of the Wilcoxon Test and effect sizes *r* with 95% confidence intervals for all tests with significant results

AOI	$Mean_M$	SE_M	$Mean_K$	SE_K	*P* (adj.)	*r*
3 | 4	.07	.07	5.53	1.01	.016	.852 [.793; .888]
3 | 5	.00	.00	1.00	.22	.018	.744 [.627; .878]
4 | 5	.87	.41	3.40	.87	.018	.736 [.529; .865]
4 | 9	.00	.00	.67	.27	.044	.653 [.421; .816]
8 | 9	.13	.09	1.73	.51	.018	.772 [.627; .888]

All significant results refer to more frequent transitions in item K8 than in item M8 and are visualized by red arrows in Figure 1. Transitions along the axes that link information between axis labels and other axis content occur significantly more often in item K8 than in item M8. Other gaze shifts along the upper *x*-axis area and gaze shifts between the *x*-axis intercept and the *t*-axis label also occur significantly more often in item K8 than in item M8.

Conclusions and Outlook

In this study, a test instrument validated by Ceuppens et al. (2019) on linear graphs in a mathematical and kinematical context was used with upper secondary students. Our results confirm for the described sample from grade 11 the following main findings of the study by Ceuppens et al. (2019) with students from grade 9: Difficulties in the context of kinematics, with formulas and in the interpretation of functions with negative slopes. This shows that the difficulties also exist with older students and are not remedied by teaching in the classes in between. To gain more insight into the solution process, the eye-tracking data from a selected item were examined in more detail.

Using a selected pair of isomorphic items to quantify a negative slope, it was shown that gaze behavior differs between kinematical and mathematical contexts. In detail, the results of the eye-tracking analysis show a higher dwell time of the gaze from the axes in item K8 than in item M8, which speaks for a stronger focus of the information extraction on the axis sections. In addition, changes of view between time- and position-axis happen more frequently in item K8 than in item M8, which means a stronger linking of the information on the axis sections. The analysis of the gaze data suggests that in the kinematical context mostly the attempt is made to form the quotient of place and time, which leads to the fact that the negative sign of the velocity is not taken into account and the task is thus solved incorrectly. The reason for this could be the physics lessons, in which an algorithmic procedure for calculating the velocity from the location and time data is taught, so that the application of mathematical procedures is no longer taken into account. These assumptions are supported by isolated interviews conducted as part of this study. For example, one student commented as follows in his description of the solution strategy for item K8: “The distance divided by time is the speed, so I just calculated at the distance 8 divided by time 4, 8 divided by 4 and that was 2 meters per second.” To remedy these transfer difficulties, teachers should be sensitized to this, and this should be done early in their curriculum. A targeted linking of mathematics lessons and physics lessons could on the one hand help to promote the transfer of mathematical procedures for solving physical problems, but on the other hand also connect mathematics to an application.

It seems that students struggle interpreting graphs independent of their age. As graphing is one important, ubiquitous and everyday 21st Century Skill, we will study this question in a broader sample. In addition, based on this important preliminary work, we will collect interview data for triangulation in a follow-up study.

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Mathematics as the focal point of STEM teaching

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The role of mathematics in STEM teaching is often described as limited or unclear, and more or less always conceived as a tool. This paper examines the role of mathematics in 19 STEM activities developed in the research and development project LabSTEM, the goal of which includes developing STEM activities where mathematics is the focal point. All 19 activities include mathematics, and they target students from kindergarten to lower secondary school. The analysis contains a categorisation of the different roles that mathematics plays in the 19 STEM activities. The results show that teachers manage to present mathematics not as abstract and decontextualised but as integrated and relevant in most of the 19 activities. Nevertheless, there is a persistent tendency for them to place mathematics learning in the background, relegating mathematics to a supporting role in STEM.

Keywords: STEM education, role of mathematics, 21st-century skills.

Introduction

As STEM teaching has advanced from being mostly a political goal in many countries to gradually becoming a larger and richer component of the classroom. It is important to consider and reflect on the role that mathematics plays or should play in STEM teaching. The acronym STEM does not automatically mean that all four disciplines (science, technology, engineering and mathematics) are included in teaching activities or are included to the same degree. There is no widely accepted agreement on whether STEM education refers to the promotion of knowledge *within* its individual disciplines or an *integrated* interdisciplinary approach. The disciplines can be combined and integrated in various ways. Some scholars argue that even if only two STEM disciplines are included, they are sufficient to constitute STEM teaching (Stohlmann, 2019). Other scholars contend that mathematics is a critical component of any teaching activity labelled STEM teaching (Doğan et al., 2019).

STEM teaching is often described as having a dual purpose. It must both provide students with the skills to perform tasks in complex and interdisciplinary contexts *and* ensure increased competency and skill level in STEM subjects (Maass, 2019). This dual goal poses the risk of a deep understanding of individual subjects being overshadowed by interdisciplinary, context-driven ways of working. Furthermore, a specific concern in the literature is that mathematics, in particular, will not occupy a distinct position because it figures as a background subject in STEM teaching (Shaugnessy, 2013), and STEM approaches seem to have a less positive impact on mathematical outcomes than science outcomes (Honey et al., 2014).

Consequently, the objective of the Danish development and research project, Laboratory for Integrated STEM (LabSTEM), is to develop STEM teaching activities where mathematics is in focus.

The aim is for teachers to balance between addressing real-world problems that students perceive as interesting and relevant and ensure that they do meaningful work with content, skills or methods from mathematics in combination with the other three STEM areas. In this paper, we ask the following research question: RQ: How can mathematics be integrated as the focal point of STEM teaching?

Different ways of integrating mathematics in STEM activities

In the literature, the relation between mathematics and STEM can be understood in various ways (Bybee, 2013). The question that mathematics teachers often ask is what STEM-integrated activities can do for the subject of mathematics. One of the answers is to make mathematics content and teaching more meaningful and relevant by creating a scenario or context for mathematical problem-solving. However, we also need to ask what mathematics can do for STEM teaching. Science provides mathematics with interesting problems to investigate, and mathematics provides science with powerful tools with which to analyse different scientific problems and concepts; thus, the relationship is reciprocal (Fitzallen, 2015).

Pang and Good (2000) reviewed studies that integrated mathematics and science in the 1990s and stated that, at the time, the dominant approach focused primarily on scientific content, with mathematics assuming a supporting role. The authors even posed an interesting question, which remains relevant today: Is the focus on science a much more natural and productive approach for integration? In contrast, Isaacs et al. (1997) suggested that mathematics should form the primary basis of the integrated curriculum because of its inherently logical structure. Other researchers have pointed to engineering as a good starting point for integrating mathematics into STEM activities (Berland & Steingut, 2016). Bennet and Ruchti (2014) suggested the integration of mathematics into STEM activities using mathematical practices as a common framework, and several scholars have also described mathematical modelling as a way of integrating the disciplines, thereby making mathematics the focal point (Auning, 2021; Doğan et al., 2019; Maass et al., 2019).

Kristensen et al. (2021) developed a framework describing the various roles that mathematics can play in STEM activities. The framework is based on the authors' review of 37 papers, all of which included different STEM activities in which mathematics was integrated.

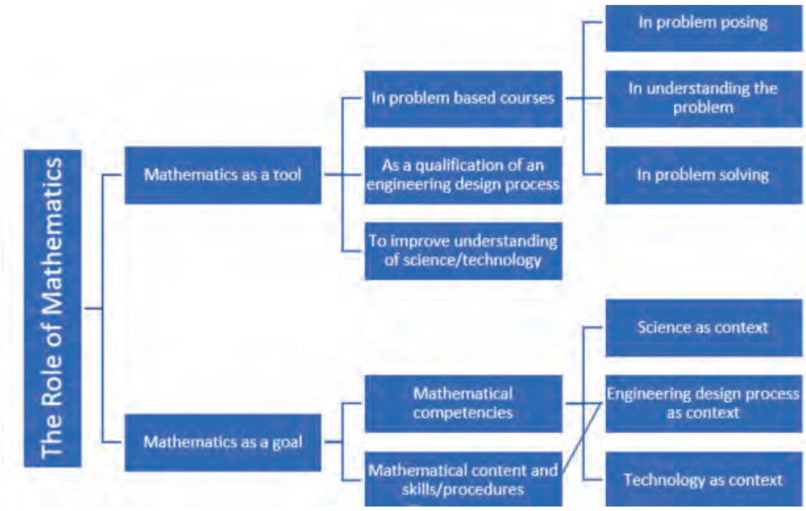

Figure 1: The role of mathematics (Kristensen et al., 2021)

The framework in Figure 1 shows that mathematics can be variously applied as a tool in STEM activities, for example, to qualify an engineering design or improve students' understanding of a specific science. When mathematic is a tool, mathematical skill, contents and competencies are not the focal point of the activity; however, mathematics is included, for example, to solve a problem in science or to develop something in engineering processes. Students do not necessarily learn new mathematics concepts/competences, but they are able to see that their mathematics knowledge can be used in other contexts. Figure 1 also shows that mathematics can be regarded as the primary objective and aim in a STEM activity, for instance, to develop students' mathematical skills and conceptual understandings of mathematics or mathematical competencies (Niss & Højgaard, 2011). When mathematics is the goal, students' learning of mathematics in the activity must be made clear. In these activities, science/engineering/technology is the context for mathematics learning. Mathematics can simultaneously serve as a tool and a goal in specific learning activities, and therefore, the dual roles are not mutually exclusive. More follows on this below.

LabSTEM – mathematics as the focal point of STEM

LabSTEM is a three-year research and development project which began in January 2020. Its purposes are to develop a STEM approach to teaching that is tailored for Denmark and to make it available for teaching and learning practice in order to support sustainable and interdisciplinary STEM teaching from day care to secondary school. Organisationally, temporary communities of practitioners and researchers were established in the form of STEM laboratories. These are not actual, physical laboratories but event-based gatherings of people. In 2021, 17 such laboratories were established, with approximately 250 participants, including kindergarten, primary and secondary school teachers. The aim of these laboratories is for teachers to develop STEM-teaching activities, with mathematics as the focal point. Furthermore, the aim is for these activities to be empirically tested in practice, with guidance from lecturers/researchers from university colleges or the University of Southern Denmark. Throughout the first year, the laboratories developed 19 STEM teaching activities, which we analyse in detail in this paper. These activities explicitly describe how mathematics is included and demonstrate noticeable differences in the ways in which they present mathematics. It is noteworthy that among the participating teachers, some were mathematics teachers, others were science teachers, and many were both.

Methods

The STEM activities were developed in teams of two to six members in the different laboratories. Afterwards, the teachers described each activity, including its title, level, theme, duration, prerequisites, relation to the national curriculum, goals, assessment, the learning process and how the different disciplines (S, T, E and M) operate. The 19 activities were then coded by the first and second authors of this paper in an Excel spreadsheet under different categories: mathematics content from the course description; how mathematics is applied, including a description of student work; a short description of the teaching activity and a description of which STEM discipline was included and its focus. The research group then discussed each activity in relation to the model of Kristensen et al. (2021) (see Figure 1). In this paper, our analysis of the 19 activities is not based on empirical observations from classrooms; it is based exclusively on the teachers' course descriptions.

The 19 activities come from different levels of the school system: two from kindergarten, 12 from primary school and five from lower secondary school. This distribution pertains to the fact that the majority of the STEM laboratories are at the primary school level. The STEM activities deal with myriad contexts, from designing homes for hedgehogs to sorting waste or growing potatoes. A table of the 19 activities can be accessed here: http://kortlink.dk/2dv9k (each of which is numbered; these numberings will be used in this paper). The first two authors of this paper jointly conducted the analyses and categorisations of the different activities. Each activity description was read and discussed with regard to the different categorisations in Figure 1. In the following sections, we first describe one of the activities to clarify our analytical approach, followed by an overall analysis of the 19 activities.

Case – germination and growth of sunflowers and watercress

A first-grade activity called Plants and Germination was developed in one of the laboratories. The focus was on sunflowers and watercress and what their seeds needed in order to grow. The activity began with a walk in the woods, focusing on observing normal conditions for the plants in the forest. Each student was then given a sunflower seed to embed in a cotton ball in a plastic bag, each of which was then taped onto the classroom window, resembling a cardboard greenhouse (see Figure 2a). Each student then kept a schedule of how many millimetres the plant grew from day to day. Additionally, watercress seeds were sown in milk cartons (see Figure 2b), which were positioned in different places in the classroom, with or without light and with more or less water.

Figure 2. Pictures of growing seeds: a) sunflower; b) watercress

In the days that followed, the students observed and documented the seed germination. How many watercress seeds sprouted? By how many millimeters did the sunflower seeds grow per day? The students systematically reported their results on Excel spreadsheets. Clear descriptions were provided that specific focus had to be placed on measuring with a ruler and the concept of measurement. The purpose was to make groups of students find some patterns showing the best conditions for the seeds. At the end of the activity, the students held dialogues in class and then in groups about their results and prepared a short video where they used their data to explain their results while showing the seeds and plants.

The following target outcomes for mathematics are listed in the course description: i) "The student can make a statistical inquiry with simple data". ii) "The student has knowledge of simple methods of collecting, arranging and describing simple data". iii) "The student has knowledge of and can measure units of length" (English translation). Based on the model (Figure 1) of Kristensen et al. (2021), we identified the role played by mathematics in this activity. First, we observed that mathematics was used as *a tool* to develop the students' scientific understanding of photosynthesis. By measuring the sprouts with a ruler, the students learnt more about what the seeds needed to grow. Second, mathematics was a specific *aim* of the activity. As stated in the course description, one of the target outcomes was the development of the students' mathematical skills (by using a ruler) and knowledge about the concept of measurement (as described, e.g. by Lehrer, 2003).

The role of mathematics in the 19 STEM activities developed

Similar to the case discussed in the preceding section, we examined the 19 above-described STEM activities from year one in LabSTEM. In almost all the activities, mathematics was used as a *tool in the STEM activities* (see Table 1). In relation to our RQ, this means that the role of mathematics in these activities was to help the students develop an understanding of science or technology or help them in engineering and design processes. Only in one activity was mathematics not defined as a tool (activity 16). Here, it was the main aim; the focus was on pi and the symbol's history and meaning in everyday life.

Mathematics as a tool and the aim of the mathematics (activity number is in parentheses)					
In problem-based activities	**To qualify an engineering /design process**	**To improve understanding of science**	**To improve understanding of technology**	**To improve understanding of technology <u>and</u> science**	**To improve understanding of engineering <u>and</u> science**
(6,17,18)	**(2,3,4,5,13)**	**(7,8,10,11,12,15)**	**(19)**	**(1)**	**(9,14)**
3	5	6	1	1	2

Table 1: Overview of the role of mathematics in the developed activities

As mentioned earlier, all course descriptions explicitly listed the mathematics-specific goals of the various activities (this was a requirement in the guideline form), but it is important to clarify that we do not know whether the students actually achieved these goals in practice. Moreover, the descriptions were not all clear about how the students would achieve these specific goals.

By studying the mathematics-specific content comprising the explicit aim of the 19 activities, we observed a wide spectrum. It was difficult to judge whether the content was about reviewing known concepts/skills or learning new concepts/skills in mathematics; however, we found a prevalent focus on the application of known mathematics, mostly because the goals described were often included in the curriculum of lower classes. However, in these activities, the students tried to apply the mathematical concepts as tools in other as well as new contexts. Three activities described a specific focus on problem-solving, but in many of the others, mathematical modelling competence was very central, albeit without this being explicitly described. Often, only the mathematical skills were explicitly written, including practising counting, learning or reviewing skills in measuring time, handling data (e.g. with frequency tables) and measuring length with a ruler. Table 2 below presents an overview.

Mathematics as an aim	
Competence	**Skills and concept knowledge**
problem-solving	counting, writing numbers, scale ratio, geometric shapes, spatial figures, curves in the coordinate system, concept of functions (slope), measurement, statistics, spreadsheet, time, data comparison, pi, area and volume, angles, economics

Table 2: Mathematics-specific goals in the developed activities

We include statistics as part of mathematics because it is part of the mathematics teaching curriculum at these grade levels. This is despite the fact that these subject areas can also be viewed as separate from each other (Capaldi, 2019). Table 2 demonstrates that it is not accurate to say that some mathematical topics feature more frequently as aims in STEM courses than others as there is no one-sided focus on specific topics.

Discussion and conclusion

Overall, the LabSTEM teachers developed activities in which mathematics is part and parcel of the aim and where the participating students engage with mathematics as a tool to gain a deeper understanding of science/technology/engineering. It is less clear the extent of the role of mathematics as this is not explicitly stated in the LabSTEM activity descriptions.

In many STEM activities, mathematics has no clear role (Kristensen et al., 2021; Martín-Páez et al., 2019). However, in the 19 activities discussed in this paper, mathematics is not abstract and decontextualised. It is clearly part of the activity, and it is relevant and integrated to a greater or lesser degree. Interestingly, when teachers are assigned the specific task of making mathematics the focal point, they almost always use mathematics as both a tool *and* a goal, which leads to the question of whether it is possible *not* to make mathematics a tool in STEM activities (making it only the aim) and still integrate all the disciplines.

To answer the question of how mathematics can be integrated as the focal point of STEM teaching, we argue that teachers need to have a clear focus on this when planning the teaching activity and need a great deal of support if this is to happen. At the same time, mathematics must figure as both a goal and a tool in the activity itself. The kind of tool that mathematics is, however, is an interesting question, and the precise meaning of *tool* could well be up for further research – for example, is tool a language or a context?

The mathematics-specific goals of the 19 activities are widespread, but they are often characterised by reviews of mathematical concepts, skills training or applications of learned mathematics. Gravemeijer et al. (2016) discussed how mathematics may prepare students for the future and argued for a shift *from* competencies that compete with what computers can do *to* competencies that complement computer capabilities. They suggested a greater focus on developing 21st-century skills such as critical thinking in mathematics, posing mathematical problems and mathematical communication. This is in line with the argument that STEM teaching aims to prepare for 21st-century skills (Maass et al., 2019). There is, however, little connection between the mathematical content that Gravemeijer et al. (2016) proposed and the mathematical content forming the aim of the developed activities. For example, none of the activities have creativity or the posing of problems as aims. However, in addition to the explicitly described mathematical content, other mathematical processes will probably be included when the activities are enacted in practice. Examples include whether students need to make argumentations for their mathematical answers (mathematical reasoning competence) or work with different mathematical representations (mathematical representation competence) (Lim & Seldom, 2010).

An additional theme worth discussing is the clarity of the target outcomes for students. Is it problematic if students are not explicitly told that mathematics competencies are included in the activities? Although we, as researchers in mathematics education, can perceive mathematics as central, it is probably more doubtful that all students will be cognisant of this. In this case, therefore, the activity would not help solve the problem of viewing mathematics as abstract and isolated from students' everyday life (Niss & Højgaard, 2011).

One of the tendencies of working with integrated STEM activities is to relegate the specific subjects to the background and bring the case/problem to the fore (the problem, rather than the subjects, as the aim) (Klausen, 2011). This may form the basis of critical reflection towards our own study since, in one sense, it contains a degree of "silo thinking": focusing on mathematics in integrated activities. Nevertheless, if the most important part of mathematics is invisible to students, there is a great risk that they will not experience the significance of these competencies. Obviously, the teachers in the LabSTEM project do not present mathematics as abstract and decontextualised but as integrated and relevant in most of the 19 activities. In many ways, however, they are still inclined towards pushing mathematics learning to the background, relegating it to a supporting role as a tool in the STEM context rather than important in its own right.

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Construction and calibration of an electronic weighing machine

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We present the structure of a STEM project in which middle school students (i.e., thirteen to fourteen-year-old mandatory secondary school students) must build and calibrate and electronic weighing machine to contextualize the mathematical notion of linear function. During the calibration process, the didactical situation may overcome students' algebraic knowledge; therefore, dynamic software is used to aid what students cannot do for themselves.

Keywords: Secondary school mathematics, STEM education, 8th grade.

Theoretical frame

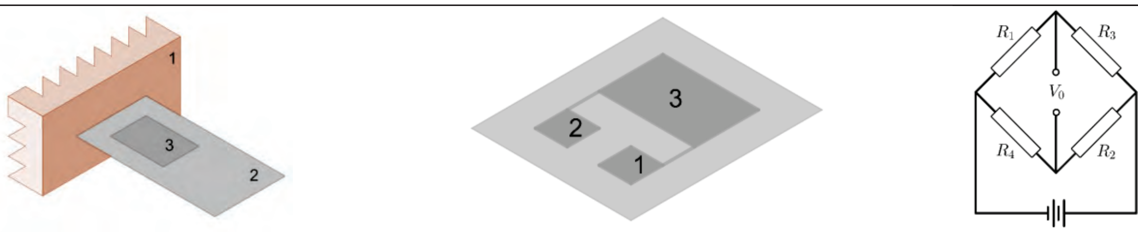
Let's consider a STEM project as a *didactical situation* (Brousseau, 1997) where students apply their personal mathematical knowledge and use a number of basic strategies to solve a particular task. The mathematical knowledge students are required to face in STEM problems mostly regards measurements, elemental geometrical language and arithmetical operations with numbers (Lasa, Abaurrea & Iribas, 2020). Finally, the STEM notion of *technological literacy* in a mathematical context naturally brings us to Instrumental Theory, since dynamic models can operate as antagonistic milieu, giving students the necessary feedback of their performance (Lasa & Wilhelmi, 2013).

Experimentation

Laboratory work (science) involves measurement of quantities in scientific notation and the use of the international unit-system, laboratory work (technology) to design an electric circuit prototype, the calibration of the scale by technological means (engineering) and the modeling of linear functions in real contexts, where students obtain numerical values out of algebraic expressions (mathematics).

Theoretical basis

The experiment is based on the assumption that the scale can be calibrated from a linear relationship between the known weigh of predetermined objects and the variation in the resistance of an electric circuit embedded on the metal piece, which forms the deformation gauge (figure 1).



a) Deformation gauge. 1, structure support; 2, aluminium gauge; 3, extensiometric gauge.

b) Detail of the extensiometric gauge. 1, input terminal; 2, output terminal; 3, electrical circuit.

c) Outline of a Wheatstone bridge

Figure 1: Outline of a deformation gauge and Wheatstone bridge

Experiment

Predetermined experimental phases in both schools avoid phase-order variability:

Global discussion of shapes	→	Software modeling (DGS)	→	Physical construction	→	Mathematical formal study

Figure 3: Project steps (Lasa et al., 2021)

Results

Students have built a personal knowledge regarding the equation of the line, and are receptive to address the mathematical formalization of the line from a theoretical point of view. The mathematics teacher will now assume leadership and institutionalize the notion of linear function.

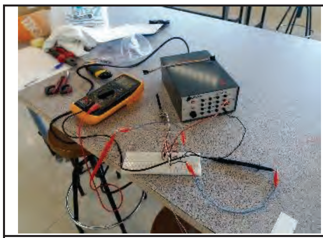
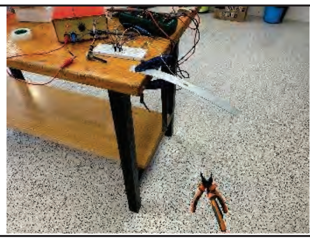
		
a) Staging	b) Calibration	c) Measure

Figure 5: Student working at the lab

Conclusions

This experience confirms that it is possible to integrate formal mathematics in the context of STEM projects, beyond its usual assistant character. Furthermore, the key notion of didactic situation has led us to an adapted structure for STEM project with a final formal mathematical component.

Acknowledgment

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What characterizes STEM professionalism?

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*STEM education and the development of children and young people's STEM competencies are on the political and educational agenda worldwide. In Denmark, curriculum descriptions of STEM competencies do not exist. Before composing such descriptions, it is relevant to investigate what characterizes STEM professionalism (Danish: STEM-*faglighed*). A hermeneutic framework provides the basis for a literature and document review, examining how international research, ministries, and the business community, characterize STEM professionalism. The analysis takes place through a concept map. Three categories characterize STEM professionalism: computing and visualising 'everyday' data with computers, finding and solving STEM-related problems, and innovative STEM thinking. These categories can form a starting point for identifying and describing STEM competencies. A prototype of a STEM competency flower with three petals is presented.*

Keywords: STEM competencies, STEM professionalism, literature review, hermeneutical approach

Introduction: Relevance and research question

The international community aims to ensure that more children and young people acquire competencies in STEM (science, technology, engineering, mathematics) (Bybee, 2018; E. Council, 2006; N. R. Council, 2011; Regeringen, 2018; Zollman, 2012). Competence descriptions can be the starting point for formulating concrete competence goals, which are essential for students to develop specific STEM competencies (Højgaard & Sølberg, 2019). If policy actors, principals, and teaching staff explicitly want to improve students' competencies in STEM, competence descriptions are essential (Sølberg et al., 2015).

In Denmark, academic competence descriptions in the curriculums define the professionalism of all courses in the primary and lower secondary school. For example, six competencies define the professionalism of mathematics (Børne- og Undervisningsministeriet, 2019b), and four define natural science[1](Børne- og Undervisningsministeriet, 2019a). In a Danish curriculum context, no descriptions of STEM competencies exist. With the object of making descriptions of such STEM competencies, it would be relevant to investigate what is understood when talking about STEM professionalism. A well laid-out characterization of STEM professionalism may subsequently identify some STEM competencies which can form the basis for future STEM education in Denmark.

Professionalism in an educational context

What is meant by professionalism in STEM is not clearly defined. Therefore, this paper examines how the community, such as international research, ministries, and the business community,

[1] Natural science includes the courses in nature/technology, physics/chemistry, biology, and geography

characterize STEM professionalism. In that light, the following research question arises: *What characterizes STEM professionalism according to international research and texts from ministries and the business community?*

Methodology

When, in 2020, I completed a Ph.D.-proposal (Møller, 2020) for a scholarship within educational research, I gained knowledge of several texts relating to STEM education and teaching. My search for literature for the project proposal used 'non-systematic' and 'deliberate-random' searches in Google Scholar, with keywords such as 'STEM', 'science', 'mathematics', 'education' and 'teaching' in English and Danish, respectively. Knowledge from this search process forms the basis for my initial thoughts and ideas about my upcoming literature search for STEM professionalism.

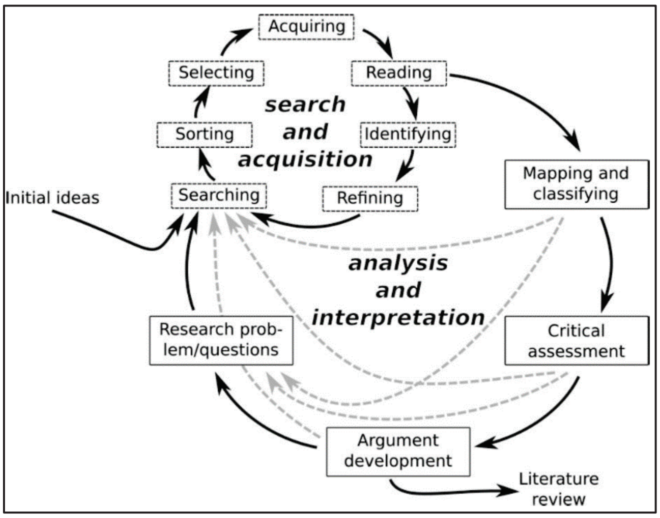

Figure 1. A hermeneutic framework for the literature review process

Based on a hermeneutical view of science, I chose a hermeneutic approach to rule the review process. A hermeneutic approach for a literature review is carried out by iterations between the two circles *search and acquisition* and *analysis and interpretation*, illustrated in Figure 1 (Boell & Cecez-Kecmanovic, 2014, p. 264). The two circles are mutually independent. New knowledge achieved in one part of the process can give a new understanding of the field, and a new understanding of the field can change the perspective of well-known sub-elements. From a hermeneutic viewpoint, you can say that the part gives an insight into the whole, which provides an understanding of the part (Brinkkjær & Høyen, 2018).

The review process

The wording in the research question '*What characterizes STEM professionalism…*' gives reason to be interested in how texts describe and relate to elements that can characterize STEM professionalism. A 'characterization' is a description of a given object's distinctive features or fundamental characteristics. In that light, I am interested in finding characteristics and features of STEM professionalism described in international research and texts from ministries and the business community. The purpose of the review process is, therefore, to assemble a corpus of literature that, through analysis, will develop some arguments for the characterization of STEM professionalism.

Search and acquisition – sorting and identifying core literature

I commenced my review process by rereading the 29 references in my PhD proposal (Møller, 2020). The texts served as *initial ideas* for going in the upper circle in Figure 1. In the texts, I explored keywords and issues that can describe features and characteristics of STEM professionalism; e.g., descriptions of STEM-related competencies, goals for teaching STEM to pupils, and central knowledge and skills in a STEM context. I included the publication if I thought its content could contribute to a characterization of STEM professionalism. An example is Barcelona (2014) because the publication articulates the importance of students' ability to solve STEM-related problems.

I include seven references from my PhD project proposal in the review; see Table 1.

Analysis and interpretation - a concept map as a tool for analysis

As I read the included texts, I shifted to the bottom circle in Figure 1. While I read the individual texts, I organised essential concepts, topics, or content that described scientific elements concerning STEM professionalism in a digital concept map. A concept map can process and structure a collection of concepts and themes with complex interrelationships (Steen et al., 2020). An example of texts that contribute with content in the concept map is the National Research Council (2011) with 'handling a large amount of data' and Holdren et al. (2013) with 'creativity' in the context of solving problems.

As I read more texts and obtained several subject elements, I organised the elements in groups, which related to or impacted each other. I explained the groups' relationship with arrows. When I perceived that the grouping had a certain kind of consistency, I gave them a header. I may have created a header or decided that one of the subject words was descriptive for the grouping. I printed the digital concept map and used analogue Post-It to place new concepts or replace concepts into new groups. I wrote analogue Post-It notes in the digital concept map at natural moments in the analysis process. For example, I would do this at the end of the working day, or if I knew that I could not continue to work on the review process for a few days. Thus, the online concept map was dynamic and continuously changing, and developed as I worked through the first iteration.

Iteration two, three, four …

The purpose of doing more than one iteration is to find more texts that deal with the field and reach a point of saturation (Boell & Cecez-Kecmanovic, 2014). When no new literature or significant arguments arrive in further searches, saturation is reached.

I identified different texts through a snowball effect and citation searches. In the second iteration, I looked back at the seven included texts from my first iteration by noting relevant texts from their lists of references. Currently, I am working with the phases: sorting, selecting, acquiring, reading in the top circle in Figure 1. I also look forward to finding recent texts that refer to the given publication; e.g., Holdren et al. (2013) Carneval et al. (2011). This may also lead to the inclusion of Carneval et al. (2011).

In addition, through iterations three and four, relevant texts will be retrieved by searching in library databases, talking to colleagues and other STEM stakeholders, participating in webinars, and looking into relevant project descriptions; e.g., European project proposals like Erasmus+ (CiSTEM, 2021). The dynamic hermeneutic process leads to a deeper and more wide-ranging overview and understanding of texts dealing with sub-elements to characterize STEM professionalism. I will work in the phases: mapping and classifying, critical assessment and argument development, which belong to the bottom circle in Figure 1. The second, third, and fourth iterations include 13, nine, and 32 texts. The 61 included texts are illustrated in Table 1. In Møller (2021), an exhaustive reference list is available.

Table 1. Included texts in four iterations. In Møller (2021), *a reference list is available.*

Iteration	**Included publications**	**Total**
1: Identify relevant key concepts in STEM professionalism	Re-read publications from my ph.d.-project description – select overview texts (Undervisningsministeriet, 2018) (Erhvervsministeriet, 2018) (E. Council, 2006) (N. R. Council, 2011) (Holdren et al., 2013) (Barcelona, 2014) (Becker & Park, 2011).	7
2: Expand text material	Snowball and citation searches, talk to stakeholders, European projects proposals, webinars (Carnevale et al., 2011) (Arikan et al., 2020) (English, 2016) (Bybee, 2013) (Jensen, 2007) (Wing, 2006) (De Meester et al., 2020) (Halász & Michel, 2011) (Københavns Universitet, 2019) (Jang, 2016) (Johnson et al., 2020) (Carracedo et al., 2018) (Siekmann & Korbel, 2016)	13
3: Systematic search in databases	Search string with keywords found in 1st and 2nd iteration (Baron et al., 1989) (Berns & Erickson, 2001) (Bingman & Stein, 2001) (Harvey & Charnitski, 1998) (Shaw, 2000) (Stull, 1998) (Velez et al., 2015) (White & Berlin, 1985) (Sølberg, 2020)	9
4: Transversely expand text material	Snowball and citation searches, talk to stakeholders, European projects proposals, webinars (Aldron & Soury-Lavergne, 2016) (Augustine et al., 2005) (Berry & Csizmadia, 2016) (Connecticut State Dept. of Education, 1991) (Council on School Performance Standards, Frankfort, 1989) (George, 1996) (Government Office for science, 2017) (Grgurina, 2016) (Howard et al., 2000) (Hõlm et al., 2016) (Hurd, 2000) (Karis & Andersson, 2016) (Kiselova & Gravite, 2017) (Kjelvik & Schultheis, 2019) (Li et al., 2019) (Linney & Walshe, 2016) (N. R. Council, 2014) (Smith & Moore, 2014) (Steele, 2013) (Yeung et al., 2000) …	32
Saturation	**When no new literature or significant arguments arrive in further searches, saturation is reached.**	**61**

Findings

As the review process evolves, groupings in the concept map appear. In the analysis and interpretation of the groups, I use the term ‘category’ for a unit that brings together key concepts and contents that deal with STEM professionalism. I use the word ‘theme’ to describe the analytical units that crystallize within each category. No categories are thus determined in advance but stem from the literature and the documents found through the review process.

Categories and themes found

After four iterations in the review process, I will designate five groups in the concept map. Out of consideration of the limited extent of this CERME-paper, I include arguments for the first three categories with associated themes. I mark the first three categories with a blue oval in Figure 2. Each category is distinct from but overlaps every other category and cover-up several themes. Some themes relate to more than one category.

I take a normative stance on the categories in the descriptions, but I am conscious that the review process aims to characterize STEM professionalism for future STEM education.

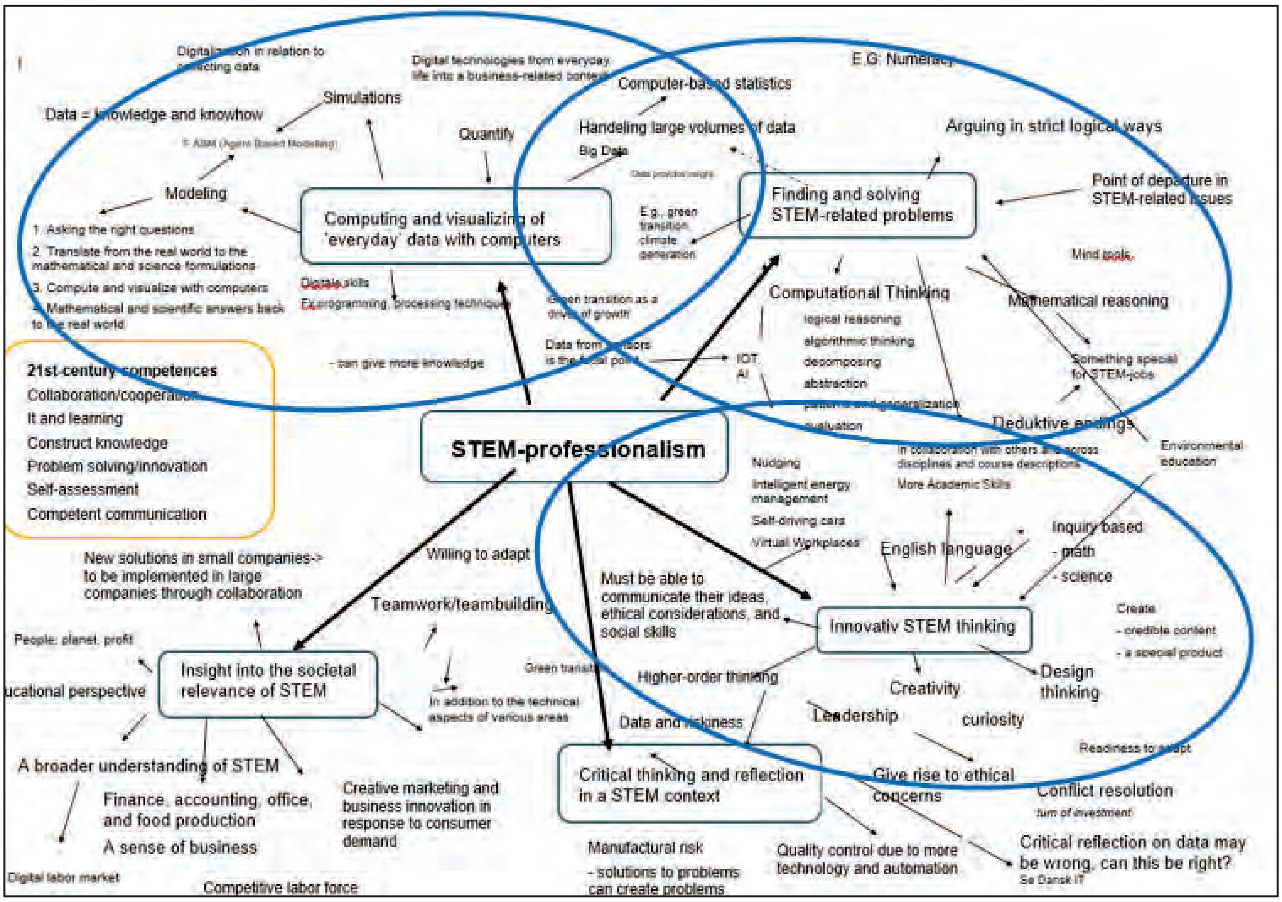

Figure 2. A draft of the concept map of STEM professionalism divided in five groups. Three categories are marked with blue ovals.

Computing and visualising 'everyday' data with computers

The first category I present from the concept map is c*omputing and visualising 'everyday' data with computers*. Computers are necessary to work with and can handle large volumes of data. In a STEM context, authentic data from everyday life is essential (Kjelvik & Schultheis, 2019). Authentic data are accurate, quantitative, or qualitative information collected from real-life phenomena. Computers and associated programs have several advantages to quantify this kind of data (Linney & Walshe, 2016). A way of getting an overview of data is to visualise it with graphs and computer-based statistics (Howard et al., 2000, Hõlm et al., 2016). Modelling is relevant for visualising data in a usable way and becomes a theme in the first category. Modelling in a STEM context is about asking the right questions, translating from the real world into mathematical and science formulations, computing and visualising with computers, and formulating a mathematical and scientific answer back in the real world (De Meester et al., 2020; Jensen, 2007; Steele, 2013, Aldron & Soury-Lavergne, 2016). Simulations can support modelling in STEM by imitating minor or more prominent parts of reality—for example, a cell's interior or how the oceans' ecosystems are affected by acid pollution. An insight into such active representation of reality can contribute to learning and understanding relatively abstract STEM-related phenomes (Karis & Andersson, 2016). Simulations also become a theme. To visualise data with computers, you must manage and handle a large amount of data (E. Council, 2006; Karis & Andersson, 2016) and have some digital skills. These form the last two themes in the first category.

Finding and solving STEM-related problems

Problem-solving skills involve the identification of complex problems based on STEM-related issues and related information required to develop and evaluate options and implement solutions (Bingman & Stein, 2001; Carneval et al., 2011; Carracedo et al., 2018; E. Council, 2006; Halász & Michel, 2011; Howard et al., 2000; Jang, 2016; Shaw, 2000; Steele, 2013; White & Berlin, 1985). This leads to the second category: *Finding and solving STEM-related problems*. One way of solving STEM-related problems includes computational thinking (CT) (Becker & Park, 2011; Wing, 2006). CT becomes the first theme in this category. CT is a problem-solving process that encompasses several sub-processes For example, logical reasoning, algorithmic thinking, decomposing, abstraction, patterns and generalisation, and evaluation (Berry & Csizmadia, 2016). CT, modelling, simulations, and handling a large amount of data with computers can provide new knowledge to find resolutions to STEM-related issues. For example, these include disease control, nutrition, global warming, new energy sources, and understanding the universe (Hurd, 2000). This problem-solving process assumes mathematical reasoning and deductive endings, which becomes the second theme (Augustin et al., 2005). With the language of math, science, and technology, you can argue in strictly logical ways, which is notable for solving STEM problems (Connecticut State Dept. of Education, 1991; Council on School Performance, Frankfort, 1989). Solving complex problems in a STEM context requires an interdisciplinary approach (Arikan et al., 2020; De Meester et al., 2020) and opens the possibility of cooperation between different STEM disciplines (Berns & Erickson, 2001). An interdisciplinary approach is the last theme in this category.

Innovative STEM thinking

When we want to find resolutions to STEM-related issues, thinking innovatively in a STEM-minded way is essential (Barcelona, 2014). *Innovative STEM thinking* is the third category. Innovative thinking is about having an idea and carrying it out. However, it is also about thinking in different ways and involves various perspectives. Creativity, design, and design thinking are themes in this category. Creativity is an important way of thinking in a STEM context (E. Council, 2006; Holdren et al., 2013; Jang, 2016; Regeringen, 2018; Steele, 2013). Design and design thinking can be approaches that encourages different perspectives in viewing and solving problems, and they are vital to creativity and innovation (Li et al., 2019). When you work creatively and innovatively, you need to collaborate with others within and across disciplines. To do that broadly helps communicate ideas to others and to understand inspiration and explanations from international sources, such as the internet (Jang, 2016; Yeung et al., 2000). In this situation, mastering the English language is advantageous (Berns & Erickson, 2001; Bingman & Stein, 2001; Baron et al., 1989; Yeung et al., 2000). Communication and interdisciplinary collaboration are the last two themes in this category.

To summarise, the first category with associated themes is: computing and visualising ‘everyday’ data with computers (modelling, simulations, handling a large amount of data, digital skills). The second category with associated themes is: finding and solving STEM-related problems (computational thinking, mathematical reasoning and deductive endings, interdisciplinary approach). The third category with associated themes is: innovative STEM thinking (creativity, design and design thinking, communication, interdisciplinary collaboration). A complete overview of the included texts related to the categories and themes is registered in Møller (2021).

Concluding remarks

The findings of this paper indicate that it is possible, using a review with a hermeneutical approach, to develop some arguments for the characterization of STEM professionalism. Because of limited space, this paper includes arguments for the first three categories with associated themes. However, it is reasonable to think that the concept map's last two groupings can also be described. When that is properly carried out, the five categories can constitute an identification of STEM competencies. The competencies can be represented in a so-called competency flower, known from mathematics in the KOM framework (Niss & Højgaard Jensen, 2002). In Figure 3, I have designed a prototype of a STEM competency flower with the first three petals: *STEM data handling competency, STEM problem handling competency, and STEM innovative thinking competency*. When the last two competencies are found, it is reasonable to assume that a STEM competency flower with comprehensive descriptions can be a starting point for competency-oriented STEM education.

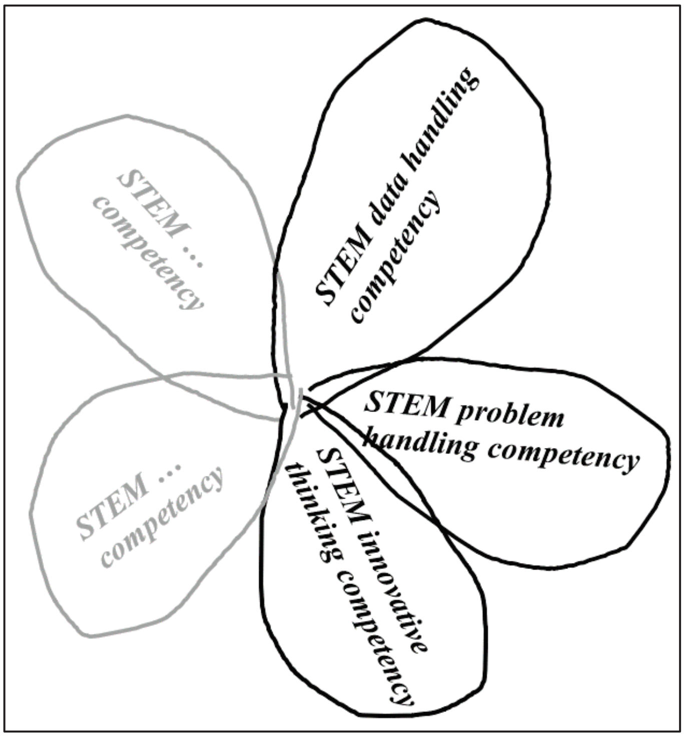

Figure 3 A prototype of a STEM competence flower with the first three petals

Suppose ministries, stakeholders, and teachers take such STEM competence description as a premise. In that case, the STEM competence descriptions can be the gateway for future competence-oriented STEM education, possibly explicitly developing pupils' STEM competencies.

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Exploring the engineering design process on designing a neighborhood in project-based learning environment

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Abstract: In this paper we explore 7th grade students' engagement with engineering design processes while designing a two-dimensional scale plan of a neighborhood in Project Based Learning environment. To do this, verbal protocols throughout classroom observations and interviews were collected from 97 seventh-grade students. We analyze these protocols to document the students' engagement with engineering design processes together with opportunities to learn and apply mathematics. The results show that project-based learning engages students in engineering design processes while designing a two-dimensional scale plan of a neighborhood as a project.

Keywords: Engineering design process, geometry, project-based learning.

Introduction

The project-based learning (PBL) approach engages students actively in pursuing solutions to authentic (driving) question that serves to organize and guide instructional tasks in both the presentation (benchmark lessons) and practice of selected topics (project) (see Ubuz & Erdogan, 2019 for the definition of presentation and practice of selected topics). PBL scaffold learning and build meaningfully powerful Science, Technology, Engineering, and Mathematics (STEM) concepts supported by language, social studies, and art (Capraro & Slough, 2013). PBL builds on engineering design process as the cornerstone (Capraro & Slough, 2013). While engaged in a project, following an engineering design process (EDP) allows systematic learning, simultaneously exposing students to experience the cognitive processes of an engineer (Tate, Chandler, Fontenot, & Talkmitt, 2010).

The key features of PBL (e.g., Markham, Larmer, & Ravitz, 2003) are to encourage students' learning and develop the essential knowledge and skills to engineer a personalized solution to the design problem (Chua, Yang, & Leo, 2014). Even though many works have been conducted on PBL, none has specifically tailored to document students' engagement with EDP. To do this, this paper focuses on documenting students' engagement with EDPs while designing a two-dimensional scale plan of a neighborhood in a PBL environment. EDP model followed in the current paper is composed of the following four characteristics: 1) Defining the problem, 2) generating and selecting between multiple possible solutions, 3) modeling and analysis, and 4) iteration (Berland, Steingut, & Ko, 2014). Specifically, this paper is guided by the following research question: How do students engage with the EDP in the PBL environment?

Methodology

Participants

The participants in the present study included those students for whom Project Based Geometry Learning (see Ubuz & Aydınyer, 2019) were instructed for five 40-min periods per week over the course of six weeks (altogether thirty 40-min periods). A total of 97 seventh-grade students, consisting of 57 females and 40 males, in three intact classes from a private school in Ankara, Turkey were the participants.

Description of the Project in Project-Based Learning Environment

In the PBL environment, students faced a challenging project, including the following problem situation to specify the well-defined outcome and ill-defined task:

> "There is a so-called contest entitled Neighborhood Renewal Project for redesigning a neighborhood replacing old buildings (not historical ones) with new ones. When you begin to design your scale plan, keep in mind that you have some design requirements."

The project was conducted during the last 14 lesson hours. To mirror real-world engineering, the students had to accommodate the following requirements to design their scale plan:

- Designing a two-dimensional scale plan of a neighborhood located on a rectangular smooth surface with actual dimensions of 120 m and 170 m on an empty white cardboard with the corresponding dimensions of 48 cm and 68 cm.
- Considering the needs of the residents and environmental problems encountered by them.
- Including different positions of three lines in a plane representing roads as well as certain polygons with some dimensions representing ground areas covered by buildings and other areas.

Students were expected to design their scale plan mainly using their knowledge of geometry and mathematics. The students were assigned to small groups composing three or mostly four students on the basis of the data from Group Embedded Figures Test. Each group included at least one student from each cognitive style (field dependent, field mixed, and field independent) Once the problem, "How do you design your neighborhood plan?", was posed to the class at the beginning of the PBL, some sample scale plans of different neighborhoods were shown by projecting them on a large screen to discuss the positions of the roads with respect to each other and the types of polygons used for buildings based on their existing knowledge. Then, they were asked to conduct some preliminary research to design their plan, including:

- Finding some scale plans of different neighborhoods to investigate their location, population, climate, economy, industry, history, and natural vegetation; the environmental problems that their residents encounter; and their roads, buildings, and other areas, and possible actual dimensions of them; and
- Investigating different people's involvement in designing a neighborhood.

By reference to information collected through their investigations and discussions throughout their outside classroom work, each group decided their groupmates' professional roles and their group leader. Following this, each group started to construct their scale plan considering the requirements provided above.

Throughout creating their scale plan, the classroom teacher interacted with the groups, answered the students' questions, and prompted them to explain their choices and consider different alternatives. Although the project was set up as a contest between the groups, they were introduced that every project can win the contest as long as it fulfills all requirements.

Data collection and analysis

Classroom observations (abbreviated as O) and interviews (abbreviated as I) with the students depending on their available time during the treatment let us document students' engagement with EDPs. Classroom observations and semi-structured interviews were conducted during and after the treatment. Each class was audio-recorded. Each interview conducted individually was also audio-recorded. There was no time limitation for the interviews. Interview questions are as follows: "What resources did you use while making the outside-classroom search? What information did you find?", "Did you encounter any difficulties while creating your scale plan? If any, what kind of difficulties did you encounter? What did you do to overcome them?", "How did you decide the types of polygons for ground areas covered by buildings and other areas of your scale plan?", "Did you like/dislike creating your scale plan?", and "What did you learn from creating your scale plan?" Please refer to the paper on the PBL environment (see Ubuz & Aydınyer, 2019) for the other details about interviews.

The tapes from the classroom observations and interviews were then transcribed. The transcripts were then segmented into units of text in preparation for coding. Each segment represents one idea. Segmenting was done independently by the two researchers, checked for reliability, and any inconsistencies were resolved. The average reliability for segmenting was 94%. Once the segmenting was completed, data analysis was conducted. Each segment was coded with respect to EDP characteristics and their key aspects in the coding scheme (Berland et al., 2014). This process was fluid—a single utterance could speak to multiple EDP characteristics, and each characteristic could be addressed multiple times throughout the observation and interview.

Findings

The findings are provided around the aforementioned four characteristics of the EDP.

Defining the problem

After the teacher provided the driving question, each student simultaneously communicated with their parents, relatives, social studies teacher, and headman, and conducted research from various sources (e.g., the Internet, books, or other sources). From their parents and relatives, he/she received information regarding their profession (e.g., engineering, architecing, landscape architeching, city planning). From their social studies teacher, he/she got information about the geographical position, population, climate, economy, industry, history, and natural vegetation of some places. From the headman of their neighborhood, he/she got a sample plan of their neighborhood and information about the requests and complaints of the residents. Through searching the Internet, books, or other sources (e.g., atlas, maps, encyclopedias), he/she found (1) some sample scale plans of different neighborhoods, (2) articles on designing a place, (3) the standards of designing a neighborhood (e.g., the number of floors of buildings should be adjusted according to the population of a neighborhood, wider streets are needed to be established if the neighborhood is close to the city center to be reached quickly and easily), (4) the elements of a neighborhod (e.g., roads, buildings and other areas), and (5) actual side lengths of the ground areas of the buildings and other areas as well the width of the roads. Regarding to a particular neighborhood, he/she found its (1) geographical position (e.g., latitude and longitude, neighboring places, whether it is mountanious or by the sea, a town belongs to which city), (2) population, (3) climate (e.g., average highest and lowest temperature each month, continental or Mediterranean climate), natural vegetation (e.g., woodland bush), (4) growing products (e.g., fruit trees, vegetables), economy and main sources of income (e.g., agriculture, tourism, industry), (5) history (e.g., historical and cultural buildings and other areas in it), and (6) environmental problems that the residents encounter.

I-S17: I have learned what professional owners of city planners, architects, engineers, and landscape architects do. We are going to choose black pine for green areas [for our scale plan] because we learned that it produces more oxygen compared to other types of trees.

Upon their investigations, they realized that the main environmental problems were air pollution, noise pollution, lack of green areas, traffic congestion, global warming, and unplanned urbanization; and different elements (e.g., roads, buildings, other areas) needed for the residents of a neighborhood regarding residence, health, education, administration, shopping, transportation, entertainment, doing sports, recreation, eating-out, and religion. Additionally, they realized what professions should be included in a common project to design a place, what they do, and their training process at a university. Upon this, each groupmate decided his/her profession. They mostly chose to be an engineer, an architect, and a city planner. Then, the groups decided where to design a neighborhood. Most of them decided to design it in different places (e.g., big cities, small towns, seaside, island, etc) in Turkey and a few in abroad.

To accommodate a neighborhood they wanted to design that is sustainable, walkable, vibrant, social, and livable, the groups started to make decisions to solve the issues the residents of the neighborhood encounter regarding the location, population, climate, economy, industry, history, natural vegetation, environmental problems, roads, buildings, and other areas.

As emphasized above, the main goal was to improve existing residential communities. The problems raised in the context of this main goal can be listed as follows:

1. What could be the elements of the neighborhood and the places of them?
2. What could be the polygonal shapes of each building and other areas?
3. What could be the side lengths and angle measures of polygonal shapes representing ground areas of the buildings and other areas in real life and on the plan considering the scale together?

4. What could be the positions of roads with respect to each other using positions of three lines in a plane?
5. What could be the width and length of the roads on the plan?
6. How can we draw and place polygons and roads agreed upon on the scale plan using a protractor and a ruler?

The first problem is about deciding the issues related to the scale plan and the following four problems are about deciding geometric representation of the physical environment in the neighborhood, while the last is about drawing the scale plan.

Generating and selecting between multiple possible solutions

Regarding the first problem, the students decided the elements of a neighborhood and the placement of them in it based on their communication, search and investigation emphasized in the previous part. Regarding the elements of a neighborhood, they mostly decided to include houses for residence; buildings of health clinic and pharmacy for health; schools for education; bank, headman's office, fire department, police office and post office for administration; market place and shopping center for shopping; bicycle routes, bus station, parking lot, petrol station, roads and taxi rank for transportation; theatre and cinema for entertainment; areas to do different kinds of sports; green areas and playgrounds for kids for recreation; and restaurant for eating-out. They also included different cultural and religious areas in their plan to be respectful to people from different cultures. They planned to protect existing historic and cultural places as well as natural beauties. Furthermore, depending on the place of the neighborhood, they included other areas such as a harbor if it is by the seaside. It was interesting to observe that some groups decided on cultivating particular fruit trees and vegetables according to the climate of the neighborhood so that the residents can do their own organic farming. They also decided to have buildings that are not too high to solve unplanned urbanization.

In determining the placement of the buildings and other areas in the scale plan, students usually made suggestions within their professional roles. The suggestions of the students in different roles were discussed within the groups and then a common decision was reached. Throughout the presentations of their project to the class, for example, they said the following as a group:

> As a landscape architect and the group leader, I (O-S12) advised to design green areas as large as possible…The reason that we chose this neighborhood was its unplanned urbanization. We tried to give importance to the aesthetic [appearance of it]... As a city planner, I (O-S13) advised to place buildings such as a pharmacy, a fire department, a police office, and schools in the center of the neighborhood so that the residents could be able to reach them easily… As an architect, I (O-S16) advised not to include the two houses very next to each other… We have learned how to design a place while developing this project… As an engineer, I (O-S21) decided the measurements of buildings and roads in real life and on the plan…

Regarding the second problem, they determined polygonal shapes of the buildings and other areas to be placed in the plan by considering the ground areas of the buildings and other areas in reality (e.g., squares and rectangles for football field, houses, and administration buildings). Furthermore, to fit the polygons into the plan and to fill the blanks on it, they considered the polygons' number of sides, angle measures, and positions with respect to each other and with respect to the positions of the roads. They also included regular polygons (e.g., squares for houses which are symmetrical to each other and a regular hexagon for a theatre) to have a pleasing and attractively appearing and architecturally good-style neighborhood. They chose a triangle for having small areas such as a museum and a bus station, and some polygons having more than three sides to have larger areas (e.g., for green areas). They also used nested polygons for buildings used for similar purposes (e.g., for education). They chose trapezoids, parallelograms, and rhombi for buildings which had narrower and wider parts (e.g., for taxi ranks).

Regarding the third problem, they considered the interior and exterior angle measures and the side lengths of polygons representing the ground areas of the buildings and other in real life. They had difficulty with estimating the lengths of the ground areas of the buildings in real life. For example, O-S23 asked, "Is the width of this class 25 meter-long? I want to visualize how long 25 m is." The teacher let her and her groupmate O-S25 measure the length of the classroom using a 1-m ruler and they found it to be approximately 8 meters. Regarding the fourth problem, they decided to include parallel and crossing roads to solve traffic congestion and unplanned urbanization.

Modeling and analysis

Regarding the third problem, they needed to realize that a polygon to be placed on the plan and its shape in real life are similar. Then, first, considering that all corresponding angles of two similar polygons are congruent, they decided to have equal corresponding angle measures of a polygon to be placed on the plan and its shape in real life. Second, considering that all lengths of corresponding sides of two similar polygons are proportional, they decided the scale of the plan (i.e., 1:250) by thinking about the side lengths of the rectangular white cardboard to be used to draw their plan (48 cm by 68 cm), and the side lengths (120 m by 170 m) of the rectangular smooth surface they want to make a neighborhood. To do that, they converted the side lengths of the rectangular smooth surface from meter to centimeter to make the units the same and calculated the ratio of the length of the rectangular white cardboard to its corresponding actual side lengths in the rectangular smooth surface. Third, considering their decisions regarding to the actual side lengths of the polygons representing the ground areas of the buildings and other areas, they calculated the side lengths of the polygons to be placed on the plan using the scale of the plan (1:250) by first converting the actual side lengths of the polygons in real life from meter to centimeter. Regarding the fifth problem, considering their decisions regarding to the actual width of a road (e.g., 10 m), they calculated the width of a road (e.g., 4 cm) to be placed on the plan using the scale of the plan (1:250) by first converting the actual width of the road from meter to centimeter.

Regarding the last problem, to draw their scale plan using a protractor they did not know how to use it to measure a particular angle and/or how to draw an angle having a particular measure. The teacher or groupmates helped the students who did not know how to use it. Besides that, minor mistakes in their drawings regarding to the angle measures or side lengths of polygons made them unable to draw the shapes they wanted to make.

O-S4: We could not draw a parallelogram with 100 and 80 degrees of angles, could you show us?

O-S4: We tried to draw a regular hexagon whose side [lengths] are 2.5 cm. We were careful with [drawing] its angle measures but the sixth [side] length became 3 cm instead of 2.5 cm.

Iteration

Although the students mostly worked in a definite order to create their scale plan (i.e., deciding the issues related to the scale plan, constructing two-dimensional geometric representation of the physical environment, making a rough sketch of the plan, and drawing the scale plan, respectively), they sometimes needed to reconsider previously made decisions and make some adjustments to improve their scale plan. The first remarkable iteration needed when the teacher noticed the errors in students' decisions regarding to the angle measures and side lengths of the polygons representing the ground areas of the buildings and other areas. Based on the teacher warning (e.g., "Be careful about the …"), they needed to reconsider their incorrect decisions regarding to the angle measures and/or side lengths of polygons. This incorrect decision is due to three reasons. The first is the confusion of the exterior and interior angle measures of a polygon (e.g., deciding 60 degrees rather than 120 degrees for each interior angle measure of a regular hexagon). The second is not realizing that the polygons that represent ground areas of the buildings in real life and their corresponding polygons on the plan are similar. For this reason, they calculated the side lengths as well as the angle measures of the polygons on the plan on the basis of the scale. They then noticed that the polygons representing ground areas of the buildings in real life and their corresponding polygons on the plan are similar. That is, the angle measures of polygons representing ground areas of the buildings in real life and their corresponding polygons on the plan are the same. The last is a mistake on converting a unit to another unit (e.g., converting a side length from meter to centimeter by dividing rather than multiplying by 100). The second remarkable iteration occurred when they encountered difficulties while drawing the scale plan (e.g.., not being able to draw the polygon they want to construct, having some polygons not fitting into the plan, having more empty space in the plan). Not being able to draw the polygon they want to construct made the students realize the importance of accuracy and precision in measurements, as mentioned in the previous part. Not fitting into the plan or having more empty space in the plan made them revise their rough sketch and prior decisions regarding the polygonal shapes and their angle measures and side lengths.

I-S20: I realized that drawing is not an easy work, and even a small error can destroy everything if architects do not pay attention to their drawing.

O-S20: We thought this building of the neighborhood as a parallelogram, but it did not fit into the plan. We changed this parallelogram into a trapezoid.

Conclusion

This study has proven that designing a two-dimensional scale plan of a neighborhood in the PBL environment engages students in EDPs and in turn probably deepen their EDP capabilities. This result is expected considering the structured PBL environment developed according to the key features of PBL. To achieve the goal of graduating students who are competent in EDPs, we need to continue to use PBL in schools. In sum, PBL environment builds on EDP as the cornerstone and as the foundation on which students bring their compartmentalized knowledge of science (e.g., types of trees that produce more oxygen compared to other ones), technology (computer, ruler, protractor), mathematics (angle and side properties of polygons, inclusion relationship between the polygons , e.g., "Is square a rhombus?", similarity and concurrence of polygons, angle and side properties of regular and irregular polygons) to bear on solving real-world problem. Furthermore, throughout designing a neighborhood plan, they learned how to design a place, the different types of professions and their duties, how to use a protractor to draw geometrical shapes, how to solve the challenges and difficulties as a group, the importance and value of geometry in real life, how the elements of a neighborhood are placed in it, and the importance of every detail such as accuracy and precision in drawings.

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The movement in spatial thinking in STEM-subjects

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This research project relates to the core of spatial ability linking to the STEM-subjects. Well-developed spatial abilities may lead to better comprehension of scientific and technical matters and to solving current real-world issues by thinking spatially in a variety of ways. The purpose of the study is the application of spatial abilities in the sense of spatial thinking as an entirety, which interacts in a broad way in the fields of science, mathematics, physics and technology. We have developed a new model of the extended Spatial Thinking for STEM, within we have underlined the movement as an essential element of spatial thinking. The various perspectives on movement in spatial thinking are combining elements, which show the interrelation between the subjects of STEM. This is described by selected examples.

Keywords: Experience and space, movement, visual perception, spatial thinking, problem solving.

Spatial thinking – the application of spatial abilities

Spatial thinking as an entire ability

> Spatial thinking is the mental process of representing, analyzing, drawing interferences from spatial relations […] between objects […] or relations within objects, […] analyzing spatial relations and transforming spatial relations. (Uttal et al., 2013, p. 367)

Spatial thinking, the “visual processing and spatial cognition” (McGrew, 2009; p. 5; Buckley et al., 2019, p. 168) is a complex ability, which reveal the various spatial thinking processes. They are used individually and are changed within the solving processes of tasks (Barratt, 1953; Just & Carpenter; 1985; Schultz, 1991).

One of the first researchers, who named various aspects of spatial thinking, is the British scientist and mathematician Francis Galton.

> Much instruction on these matters can be derived from those who possess the power of what is called the visualising faculty, in a high degree. The objects of their memory are conspicuous images; they can retain them for a long time before the eye of their mind, they can dismiss or change them at will, and they can, if they please, subject them to careful examination from every side. (Galton, 1879, pp. 158)

In this description, Galton implied three steps in the process of spatial thinking: the perception, the memory as the possibility to retain and to retrieve a recognized object and the imagining and operating with spatial pictures. This indicates an entire view on spatial thinking, as we can find also in novel definitions, for example the visual and spatial capacity of the intelligence is “the ability to generate, store, retrieve and transform visual images and sensations” (McGrew, 2009, p. 5). Furthermore, McGrew (2009) relates in his description to the perception and imagination of shapes, images and spatial orientation of objects, which change their shapes and positions in space by movement. Also in the actual research, spatial memory is a central element in spatial thinking, which is expressed in the following explanation “largely determined by the capacity to represent and manipulate mental information (spatial cognition) and to hold sufficient amounts of pertinent information (working memory)” (Buckley et al., 2019, p. 166).

Thinking about and with space in STEM

Spatial thinking includes thinking about space and thinking with space (Hegarty & Stull, 2012). The former means the experience in real space, in large-scale environment in which you can only see part of the whole at the same time. Further, it means the mental operating with real and imagined objects in space, in the small-scale. Thinking with space relates to the use of spatial objects and spatial mental processes for the imagining of non-spatial contents, for example the use of symbols and of graphs in coordinate systems, the representation of abstract contents in diagrams and the structure of terms (Malle, 1993).

Spatial thinking needs to be promoted for problem solving in the subjects of STEM. Many studies reveal that students have more success in attending courses and finishing their study on the academic level when they have a good developed ability to think spatially. Also in the further professional working, they can arrive higher aims. (Uttal & Cohen, 2012; Sorby et al., 2018)

Wai et al. (2009) highlight the interaction of the three abilities, spatial ability, mathematical ability and verbal ability. This necessitates a change in mindset in favour of spatial thinking for the curricula. Young adults with well-developed spatial abilities seem to handle better images and shapes in the imagination than those who think in verbal and numerical symbols (Wai et al., 2009; Mix et al., 2016). Probands with less-developed spatial abilities and well-developed verbal skills try to solve mathematical problems with verbal description, while others prefer a spatial representation in the problem solving (Fennema & Tartre, 1985). This relates to the usual structure of the logical thinking: a deductive and verbal process and a structured, global, intuitive spatial and inductive process (Tartre, 1990). Even though “reasoning thinking” with verbal and analytical processes is sometimes elevated above the spatial perception and imagination, spatial thinking is the core of thinking (Duffy et al., 2018, p. 273). Only the interaction of the two mental areas, the abilities of the spatial and of the rational thinking, leads to the expected success in STEM-subjects (Duffy et al., 2018; Sorby, et al. 2018).

Studies reveal that a higher spatial ability combined with a visual working memory supports the success in academic learning and the understanding of subject contents in the beginning of study. However, on a higher level specialist knowledge and experience are more used and successful in problem solving in STEM-subjects. (Uttal & Cohen, 2012; Duffy et al., 2018; Xie et al., 2020)

Spatial thinking extended to problem solving in STEM

In the following, we describe the development of a new model in Spatial Thinking for STEM. The aim of this attempt should be the application of spatial mental processes in mathematical, physical and technical issues.

From visual perception and spatial imagination to further reasoning

Previous research has primarily focused on exploring visual perception (Vernon, 1962; Frostig et al., 1972) and spatial ability (Thurstone, 1938; Linn & Petersen, 1985; Maier, 1994) as two distinct pillars of spatial thinking. Most models on spatial thinking and spatial abilities, and the related tasks have concentrated on artificial, theoretical situations in psychometric tests, and they have rarely considered scientific subject contents. To fill this research gap, there is constructed a model combining neurological findings of visual perception and spatial ability with the application on real problem solving in STEM-subjects. Spatial thinking and acting includes the two areas, visual perception and spatial ability.

Visual perception is the anatomical and neuronal process from the sensory stimulus to the cortical cognition. It is the ability for seeing the world around us recognizing and identifying objects and situations. Other sensory stimuli, especially motoric and auditory ones complete the visual perception, which is the most involved in spatial ability (Nänni, 2009). From the psychological literature (Vernon, 1962; Bak, 2020) and from neurological research (Burgess, 2014) we outline three fundamental components of visual perception: the perceiving of objects, movements and depths.

Spatial ability is the capacity to operate mentally with the images arising from visual perception, for instance, transforming, rotating, cutting and combining them. Sensory impressions are processed and reorganized mentally. We have to work actively on these mental images. (Thurstone, 1938; Maier, 1994)

In addition to the combination of visual perception and spatial ability (Maresch, 2020), a further component is necessary. This relates to the applying of spatial processes in spatial and non-spatial contents, including a higher level of thinking for arguing, reasoning and problem solving on abstract mathematical contents. This level includes expedient analyzing and creating individual solving steps (Figure 1). The "Movable Thinking" (Roth, 2005) in mathematical and geometrical contents can be seen as a further aspect of the higher spatial thinking. It is the capacity to find a movement and movable parts inside a geometrical constellation and to use them for arguing and proving.

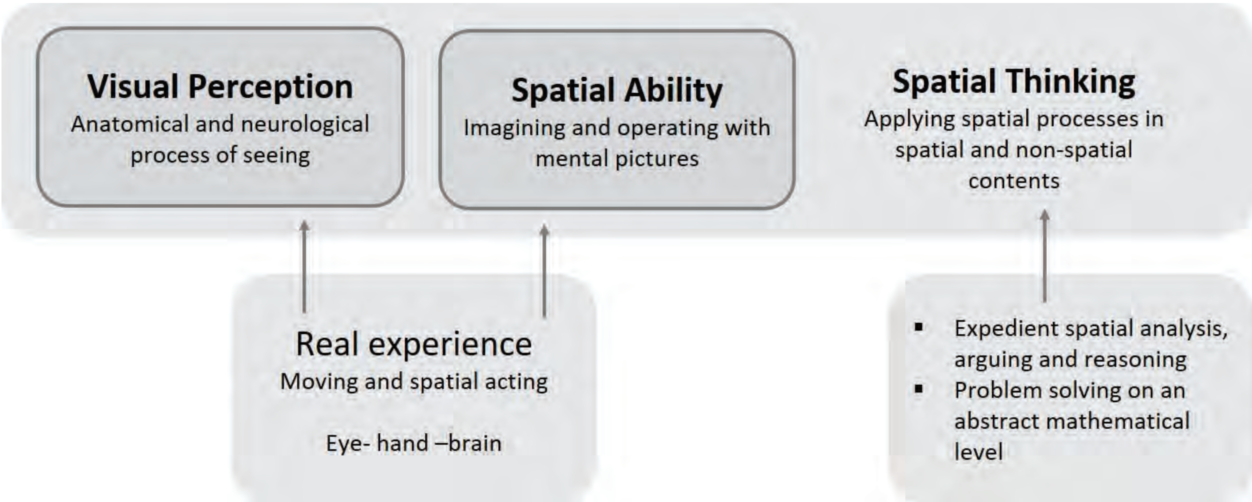

Figure 1: Spatial Thinking in STEM

The extended model of Spatial Thinking in STEM

The model is composed of six fundamental and two overarching facets (Figure 2). They relate to contents of space and take place on every level of spatial thinking, namely the visual perception, the spatial imagining and the expedient thinking.

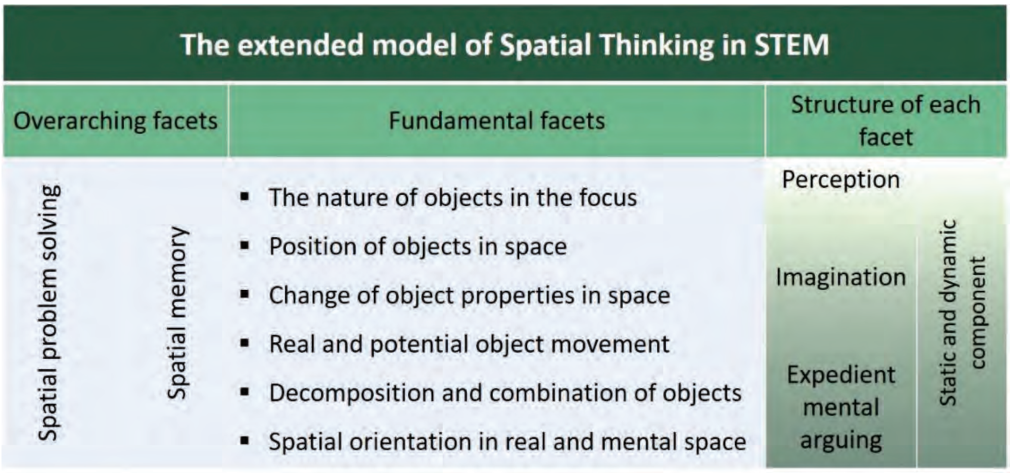

Figure 2: The facets of Spatial Thinking

The nature of objects in the focus relates to the comprehending of a constellation as a whole, from which single objects or parts are distinguished. The objects should be recognized in their relevant properties as constancies for identifying them and removing them from the background. Referring to mathematics, this facet includes arguing with geometrical properties, such as invariances, orthogonality and parallelism.

The position of objects in space deals with the perceiving and imagining of the location and position of one or more objects in a spatial constellation, in relation together and to the viewer. For example, it includes the spatial relations of elements of areas and solids, such as the relation between lines, edges, and angles. The combination between the geometrical imagination and the algebraic representation of straights and planes in a metric space requires this facet using coordinate systems.

The change of object properties in space is the changing of objects and parts of them through mirroring, rotating, shifting and scaling. This happens by the imagining of movements to recognize the changing process between the initial and the final state of the object. As an example, we can name the varying aspect of variable quantities.

The real and potential object movement relates to a real movement in space and a possible movement in a mechanical configuration with movable parts, not to a fictive movement in a mathematical constellation. This facet has a concrete reference to various fields of physics and technology, while it is related indirectly to mathematics via the similarities with the facet of the change of object properties. For imagining the deflection of a moving object, e. g. a rolling ball (Kozhevnikov et al., 2007) or a moving charged particle in a homogeneous magnetic field (Fulmer & Fulmer, 2014), we should be able to follow mentally the path of the movement.

The facet of decomposition and combination of objects involves breaking down an object or a configuration into parts and combining the parts into a whole object or a new configuration through

Boolean operations. In a further sense, it relates also to the structuring of terms and the imagining of fractions, as a part of the whole.

The spatial orientation in real and mental space is about the orientation of a person in the real space and in the mental imagination, whereby it is necessary to put themselves into the space. By changing the perspective, an object or a constellation of objects can be seen from different angles. Referring to physics, for instance, this facet is important for the imagination of movements in astronomy (Cole et al., 2018). As a viewer from different points on earth, as well as from outside the earth we see different movements, which we have to combine for understanding the interrelation, for example the apparent movement of the sun and the real movement of the earth.

The spatial problem solving and the spatial memory flow in into all fundamental facets, in understanding the task, in mental and concrete processing of solving the task, in assessing the solution steps and in finding further possible solutions.

The movement as a central element of spatial thinking

Movement, as the change in location and position of an object in space and as the change of properties of an object, as well as the movability as a possibility for changing, pervades the spatial thinking as a whole. Movement appears in various ways, as imagining and anticipating a sequence of movements, as a mental process of changing an image or as a mental process for solving problems. Based on the real movement and orientation in the space to explore it and on feeling the movement of one's own body occurs the transfer of perception into the imagination. (Stückrath, 1955; Piaget & Inhelder, 1971; Glück et al., 2005; Wolbers & Hegarty, 2010)

We have found four perspectives on movement, which are important in the imagining of mathematical, physical, technological and scientific contents:

- the movement which can actually be experienced in space and which is comprehensible in the imagination,
- the potential movement of a movable part of a mechanical configuration, for example the movement of gearwheels, screws and pulleys,
- the mental movement of an entire object to change its properties,
- the mental movement to change single parts in a mathematical constellation as an internal displacement, for example folding a solid by its surface. This relates also to the movement which is applied in abstract mathematical and geometrical contents, as movable thinking.

In the following, we present two examples, one for the spatial imagination of potential movement in a mechanical constellation (Thurstone, 1938) and one for the spatial imagination of real movement in an astronomical context.

The movement in a mechanical configuration

A rightwing screw engages a gearwheel in the signed direction. The gearwheel rotates on a fix axis. We have to imagine spatially how the rotating movement of the rod is combined with the translation movement. In which direction does the gear rotate when the threaded rod engages the gear in the indicated sense of rotation? In which direction does the gear move, when the worm is put on the rod inversely, from the right side to the left side in this perspective?

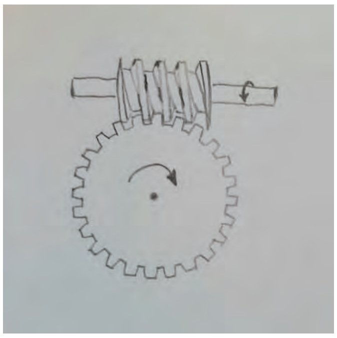

Figure 3: Gearwheel and threaded rod

To solve the task, we fix distinct features in the representation, like the teeth of the gear and the rotating direction of the threaded rod. We can mentally imagine the relation between the rotation and the translation movement of the rod. The picture of a screw turned into a wall can serve as a practical help. When the worm is put inversely on the rod the spatial relations are conserved and the direction of each spiral is unchanged, as the thread remains a right-hand thread. The spatial thinking relates to the recognition of the spatial relation between the parts of configuration and their movement.

The movement in an astronomical task

A further example to improve the imagination of movement in science deals with the path of the sun, seen from diverse places on the earth. The content of the task refers to the daily rotation of the earth around its axes, whereby other movements and astronomical phenomena are neglected. In which direction do you see the path of the sun in Rome on the North Hemisphere and in Cape Town on the Southern Hemisphere? The absolute cardinal points of the sunrise, the sunset and the peaking of the sun have to be recognized, moreover the reference points to the body of the viewer, such as left or right when looking at the sun. This problem can be seen from the earth at rest, the large scale of experience, and from a fictive point in the universe on the moving earth, the small scale of a geometric model. For simplifying the task, two places outside the tropic are chosen, because otherwise the local and seasonal conditions have to be considered. As spatial approaches can be used: putting oneself into the reference frame through mental change of viewer's position and spatial orientation or considering properties of objects and their relation using geometric quantities in an analytical way. As interesting statements of students for explaining their mental processes can be named the following answers: "I have thought about the earth as a sphere with one half in the shadow", "The earth rotates in certain points in the direction of sun and in others away from it", "I have mentally imagined a person living near Cape Town." (Zöggeler et al., 2021).

Space and movement in space are the background of spatial thinking.

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TWG27: The Professional Practices, Preparation and Support of Mathematics Teacher Educators

Introduction to the papers of TWG27: The Professional Practices, Preparation and Support of Mathematics Teacher Educators

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Rationale

Thematic Working Group 27 focuses on research concerning the role of Mathematics Teacher Educators (MTEs) in fostering mathematics teacher learning, both during the pre-service and the in-service periods of teachers' careers. The need for this group emerged in CERME11, when several papers submitted to TWG18 (Mathematics teacher education and professional development), focused on teacher educators. The rationale for opening a new group for this topic stemmed also from the accumulation of research on MTEs witnessed in recent years. This growing interest is reflected in conferences (e.g., Educating the Educators), books dedicated to MTEs (e.g., Goos & Beswick, 2021), journals and special issues on MTEs' professionalization and their role in scaling up sustainable interventions (e.g., ZDM 46(2), 2014; ZDM 47(1), 2015; Int. Jr. of STEM Ed. 4(27), 2017; IJSME (1), 2021), and a host of papers focusing on this issue in leading journals such as ESM, JMB and JMTE. Our goal in creating this TWG is to support further development of this emerging field.

First steps of TWG27: The CERME12 meeting

As TWG27 is new, it met for the first time in CERME12. Twenty-three participants attended the TWG27 sessions, 16 contributions (13 papers and 3 posters) were presented and discussed. Figure 1 shows the distribution of contributions by countries, with about 57% coming from Europe, 25% from North America, 12% from South America and 6% from the Middle East.

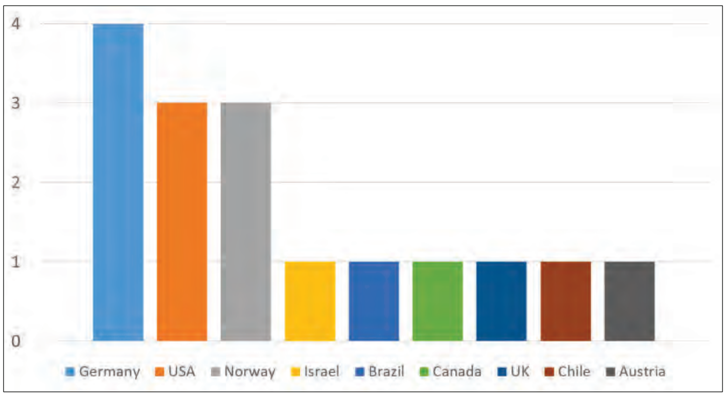

Figure 1: Distribution of contributions to TWG27 by countries

The call for papers proposed the following themes to initially form the scope and focus of the group:

1. Conceptualizing the profession of MTEs: Adapting theoretical frameworks from the teacher level to the MTE level; integrating generic and mathematical aspects within the work of MTEs;
2. Knowledge, beliefs, skills and practices of MTEs: Theoretical models and empirical studies;
3. Preparing MTEs (including formal and informal qualifications): Conceptual frameworks and empirical studies;
4. Designing professional development for MTEs; tools and resources for supporting MTEs;
5. Scaling up programs for mathematics teachers: Building institutional capacity through focusing on facilitators; institutional factors that support or hinder the effectiveness of PD facilitators;
6. The influence of current global issues on the role and practices of MTEs.

The majority of contributions related to themes 2, 3 and 4, one paper related to theme 6, and one to theme 5, while theme 1 was not directly addressed but mentioned in contributions with other foci.

A vocabulary note: MTEs are referred to in the literature by many different terms. For the purpose of coherence of the TWG27 work, we use the terms *facilitators* and *educators* to denote MTEs who support the learning of practicing teachers and prospective teachers, respectively.

Characterizing TWG27 contributions: A brief summary of different aspects

Main issues addressed

Theme 2 was the focus of many contributions, presenting studies on the knowledge, beliefs, skills and especially the practices of MTEs. Several papers focused on facilitators' practices with regard to different PD contents (e.g., conditional probability, Griese et al.; supporting at-risk students, Laschke et al.; algebraic thinking in early years, Ferreira et al.), while others centered on how PD goals are achieved (e.g., enacting norms, Schwarts et al.; modeling, Nolan; theory of change, Eriksen & Solomon). Different tools were suggested to examine MTEs' practices (e.g., scriptwriting tasks, Shure et al.; productive disciplinary engagement, Elliott & Lesseig; disruptive pedagogy, Bjerke & Nolan). Themes 3 and 4 were addressed by several contributions as well: Nieman et al. investigated how a tool, designed to provide insight into mathematics teachers' experiences within a PD course, can be used by facilitators to inquire into their facilitation practices. Rojas et al. examined the impact of "critical friendship" experiences between MTEs on their professional development, and Mayerhofer et al. introduced the idea of personas as yet another means to support MTEs. Bruns et al. investigated the effects of an extensive facilitator PD program, while Opheim et al. discussed the design of PD experiences for MTEs with varied backgrounds. Theme 5 was addressed by Seago and Knotts, who reported on asynchronous online video-based PD modules, which can be flexibly adapted to various facilitation formats, and are therefore highly scalable. Finally, Theme 6 was uniquely represented by Coles, describing MTEs' work with teachers who bring questions of global challenges into their classrooms.

Theoretical and conceptual lenses used

A range of theoretical and conceptual tools or lenses were used across the studies in this TWG. They can be categorized broadly into three groups (with some papers drawing on ideas across more than one group). In the first group are papers which drew on particular notions of teacher expertise, or teacher learning, and used them to reflect on the role of the MTE. For example, Seago and Knotts drew on Ball and Bass's (2003) notion of Mathematics Knowledge for Teaching, as did Elliott and Lesseig (who particularly looked at Specialized Content Knowledge). Griese et al. used a model of teacher expertise in their study of facilitators, while Ferreira et al. used a model labelled Professional Learning Opportunities for Teachers, and Bruns et al. employed Prediger et al.'s (2019) three-tetrahedron model of professional development designed to investigate teacher learning. A second group drew on conceptualizations of MTEs' expertise. Shure et al. used Prediger et al.'s (2021) framework for the expertise of PD facilitators, and Laschke et al. used a model of teacher orientations which they develop into one of MTE orientations. Schwartz et al. analyzed facilitator decisions using a framework of MTEs' resources, orientations, goals and identity (Karsenty et al., 2021). There were fewer papers in this second group than in the first, perhaps reflecting the relative recency of theorizations of MTE expertise. Finally, some papers drew on theoretical frameworks that support the analysis of MTEs' actions and contexts. For example, Eriksen and Solomon used the notion of boundary objects (within Communities of Practice) to conceptualize MTEs' practices, and Elliott and Lesseig mobilized the notion of productive engagement. Rojas et al. conceptualized and analyzed the self-study of MTEs. Sharing a critical perspective, Bjerke and Nolan drew on ideas of disruptive pedagogy to study the practices of MTEs and Coles used ideas from critical mathematics education with the same aim. This diversity can be seen as reflecting how the study of MTEs often draws on ideas from the highly diverse field of the study of teachers. We view it as encouraging that conceptualizations of MTEs' practices are emerging and being shared.

Methodology employed in the studies reported

Most of the presented studies were small-scale studies, following a qualitative research paradigm, sometimes involving only single cases investigated in an in-depth manner, or self-studies (e.g., Nolan). Many of the studies used a variety of data sources to investigate MTEs' work. These included the following: interviews (Bjerke & Nolan; Eriksen & Solomon; Griese et al.; Nieman et al.; Seago & Knotts); facilitators' journal entries (Schwarts et al.); teachers' weekly online logs and teacher community walls (Seago & Knotts); written records from teachers and facilitators (Ferreira et al.); summative memos (Elliot et al.); audio and/or video data from PD sessions (Elliot et al.; Ferreira et al.; Laschke et al.; Schwarts et al.); dialogues from a scriptwriting task (Shure et al.); and video data from stimulated-recall interviews (Schwarts et al.).

As opposed to the aforementioned qualitative studies, Bruns et al. reported on a classical quasi-experimental intervention study, using quantitative methods to investigate the effects of facilitator PD on teachers' learning in the context of early mathematics education.

It can be conjectured that the focus of almost all the contributions on qualitative approaches is another indication that the field of research on MTEs is still emerging, with qualitative studies providing findings that can later be used as springboards to more diversified research approaches.

New emerging issues

Several issues emerged during the course of our group discussions. Here we summarize the most frequent ones.

- *MTEs’ role in bridging between theory and practice.* We considered what conceptual lenses we might use to analyze MTEs’ practices that support the link between theory and practice. One suggested idea for exploring this issue was to conduct a workshop in which participants use different lenses to examine sets of data from multiple projects.
- *The role of reflection as a learning mechanism for MTEs.* We identified several questions for future consideration, for example: How can MTEs use reflection to better attune their practices to teacher learning? How do issues of status and power influence MTEs’ reflection with others such as their peers, researchers, and critical friends? What is the role of criticism or emotional reactions within MTEs’ reflections?
- *Role-modeling as a practice used by MTEs.* We discussed the idea of modeling and the design of research that can address questions such as: How central is role-modeling to the practices of MTEs? To what extent do MTEs model the pedagogical practices that they support in theory? What are possible implications of a lack of coherence between the practices MTEs aim to support and the practices they use?
- *The role that TWG27 might play in moving the field forward.* As pointed out earlier, most studies reported in our group were small-scale, qualitative studies. We recognized the need to develop a more comprehensive picture of the field, for example by identifying areas of research that can benefit from larger-scale studies and an extended range of methodologies.

These issues, as well as other questions raised during the meeting, are reflected in the Call for Paper and Poster Proposals for CERME13.

In summary, the collection of TWG27 papers and posters presented in this volume reflects an effort to capture current themes within the developing research on the profession of MTEs. We look forward to the continuation of this group’s activities in future ERME conferences and aspire to expand this TWG27 in terms of participants, represented countries, topics and types of studies.

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Developing a *disruptive pedagogy* theoretical lens for studying the practices of mathematics teacher educators

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In this paper, we explore the notion of disruptive pedagogy, and propose a disruptive pedagogy theoretical lens to study the practices of mathematics teacher educators (MTEs) in their post-field courses. While much is known about prospective teachers (PTs)' transition from university to field experience, less is known about the under-researched transition from field experience and back to university. We propose that perspectives borrowed from literature focused on disruptive pedagogy can shed light on this transition and help MTEs to better understand their role in unpacking PTs' field experiences. The lens consists of four key areas of focus (two on challenging current practices and two on promoting new practices). We close this paper by exemplifying what these 'challenges' and 'promotions' might look like in the context of MTE post-field practices.

Keywords: Mathematics teacher educator, disruptive pedagogy, theory-practice, theoretical lens.

Introduction

Improving teacher education is a constant concern. In that respect, research on the theory-practice transitions has been extensive (Allen & Wright, 2014; Britzman, 2003; Gainsburg, 2012), including transitions from university (theory) to field experience (practice), as well as transitions from the process of becoming a teacher (university) to the first few years of being a teacher in schools (Nolan, 2014). Another key transition in teacher education programs is the under-researched transition from field experience back to university, where "[l]ittle is known about the way in which teacher educators integrate prospective teachers' actual experiences when they return to university after fieldwork" (Eriksen & Bjerke, 2019, p. 9). This 'unpacking' of field-back-to-university transitions is relevant to the community of teacher educators since teacher education programs, and corresponding field experiences, are frequently critiqued for being steeped in technical-rational approaches (Nolan & Tupper, 2020). Mathematics teacher educators (MTEs) in particular struggle with the tensions implicit in these transitions, as they seek to disrupt dominant 'technique-oriented' discourses of school mathematics and becoming a teacher.

We acknowledge that the use of theory and practice to describe transitions between university teacher education courses and school-based field experiences creates a false binary and hierarchy (Zeichner, 2010). While we draw here on theory-practice language, in reality our interests are positioned within a hybrid space of research where we study the role of MTEs in disrupting and reimagining knowledge constructed in the crucial movement from university to field and back to university. Within this movement, it is the post-field context of teacher education that we focus our attention.

Drawing on Anderson and Justice (2015) and their way of seeing a pedagogy as disruptive if it "requires students to challenge or change their epistemologies and participation in their learning" (p. 400), we propose a theoretical lens of disruptive pedagogies that enables us to better understand the roles and practices of MTEs in post-field contexts and to re-conceptualize post-field possibilities in teacher education. We suggest that such a lens can be applied to study a central question underpinning theory-practice transitions: What are MTEs' roles in unpacking prospective teachers’ (PTs’) field experiences?

Disruptive pedagogies

The use of the term *disruptive* first emerged in juxtaposition with the introduction of new technologies in companies, where “the degree of disruptiveness” was crucial in predicting its success or failure (Christensen, 1997, p. 169). Acknowledging that technological disruptions may provide the opportunity for new ways of thinking, learning, and teaching (despite possible internal resistance to adoption) resulted in Christensen and Raynor (2013) expanding the idea of disruption and introducing the term *disruptive innovation*, where innovations were seen as disruptive if they replaced a dominant current technology.

These lines of inquiry sparked an interest in exploring more generally the impact of new technologies in educational settings (Christensen et al., 2011; Hedberg, 2011; Stevenson & Hedberg, 2011) and also within teacher education contexts, where the possibilities that emerge from the uptake of new technologies were investigated as a *disruptive pedagogy* (DP) (Anderson & Justice, 2015). In this way, DP emerged out of innovation with respect to technologies (Christensen, 1997), not more general pedagogies. To date, however, few teacher education studies have a non-technology focus with respect to DP, and fewer still (if any) turn the lens of DP toward mathematics teacher education and MTEs in particular.

Anderson and Justice (2015) define disruption as “an analytical construct that allows for the investigation of how individual learning and changes in local practice mutually influence the other within a purposefully designed learning context” (p. 401). As with our work, these authors express interest in “disruptive innovations within teacher education contexts” (p. 401); however, they do so with an explicit focus only on pedagogies that engage prospective teachers with technology; that is, they “do not attempt to examine pedagogy independently from technology or technology independent from pedagogy” (p. 401). Thus, our exploration here contributes more generally to the theory and practice of DPs in teacher education as we, in this paper, expand upon that work in DP and propose a theoretical lens to investigate MTEs’ post-field pedagogical practices. We propose that such a DP lens can be used to view post-field contexts to understand the extent to which MTE practices address practice-theory gaps and contribute to a wider goal of improving mathematics teacher education.

Introducing the Research Study

As noted previously, our research interest is in exploring the question: What are MTEs' roles in unpacking PTs’ field experiences? In a recent paper (Nolan & Bjerke, 2021), we discuss and present a list of barriers/challenges encountered in theory-practice transitions (from university to field experience). Primarily based on findings from our own research on theory-practice transitions in teacher education, these barriers/challenges formed the basis for the construction of research interview questions designed to study the roles and practices of MTEs in the next transition – the practice-theory transition (from field experience back to university). The interview questions included asking MTEs to share their own professional challenges with respect to practice-theory transitions, the pedagogical strategies and theoretical tools they draw on to challenge and/or disrupt these transitions in working with PTs in post-field courses, and what they view as their primary role(s) in the post-field context of mathematics teacher education. Hence, we focus here on the work of teacher educators, not so much the actions of PTs, within teacher education.

As MTEs, we know that PTs bring countless stories of success and failure from different mathematics classrooms to their post-field courses, many of which can serve as starting points for reflections – at least for the one who *owns* the story. However, reflection is challenging for PTs (Stiler & Philleo, 2003), and such individual-level reflections are found to have little real value in bringing about lasting change within complex school environments (Forde et al., 2006). Instead, PTs must be skilled at reflecting critically, with a depth and quality that challenges the teacher education community and allows for "disruption" (Anderson & Justice, 2015). Given the shortcomings of a reflection-focused post-field context, we saw a need to develop a lens for analysis in conjunction with collecting our data, as a way to move beyond reflecting on single field experiences, and to move toward a way of detecting those disruptive and transformative practices initiated by MTEs. Here, we offer a theoretical lens that highlights the promising concept of disruption in the context of MTE practices.

Introducing the theoretical lens

Our aim in this section is two-fold: To describe our methods in selecting and reviewing research literature on DP and to present the lens itself.

Once the need to construct a DP theoretical lens was established, we proceeded to locate research texts focusing on DP. Our aim in this engagement with the research literature was to learn about the diverse ways in which authors defined or described disruptive pedagogy. That is, our intention was to return to the roots of DP, not to conduct a comprehensive literature review. Thus, from each of the research texts located, we synthesized key ideas that spoke specifically to how the authors defined/conceptualized the pedagogy, including its aims and examples of what it might look like in practice in classrooms. Through careful study of these synthesized ideas, we noticed that, in some cases, the researchers sought primarily to *challenge* current status quo or traditional practices through DP while, in other cases, the goal was focused more on *promoting* different practices which were intended to disrupt and/or replace these current practices. In this way, it became clear how the literature suggested the existence of certain current pedagogies and practices teacher educators want to shift away from, and also pedagogies they want to promote, or 'shifts towards'. In the end, our synthesis of ideas directed us toward the construction of four key areas of focus, or themes, across the literature on DP: two focused on challenging current practices and two focused on promoting new practices. When we introduce and describe the themes below, we maintain more general teacher educator language; however, after describing each theme with a summary of its focus, we outline a key question that directs our attention toward practices specific to MTEs. The four themes and their accompanying MTE-question constitute our DP lens.

I. Challenge traditional and technical-rational approaches to teaching and learning

The body of research on DP provides us with some helpful insights on 'what currently is'. Acknowledging that something important is lost when inquiry/problem-based teaching and learning are marginalized (Anderson & Justice, 2015), we want to disrupt the conservative nature of traditional teaching cultures (Hedberg, 2011). Instead, we want to push PTs “to challenge ideas, beliefs, and practices that they were not comfortable with, providing them with the opportunity for a re-examination of their own beliefs, values, and practices in the classroom” (Anderson & Justice, 2015, pp. 406-407). This re-examination should include challenging assessment practices associated with traditional, linear ways of “defin[ing] pedagogical content and reduc[ing] learning to the tools which measure it” (Beighton, 2017, p. 114). As well, DP challenges the reducing of “teaching and learning to the application of ‘recipes’ of ‘good practice’” (Beighton, 2017, p. 117). We need to go beyond a focus on simple recall, recognition and reproduction of ideas and performances (Mills et al., 2009, pp. 72-73), which support a deterministic view of learning (Iftody & Sumara, 2010). In this regard, we can learn from Vratulis et al. (2011) who highlight the importance of moving away from those ‘additive’ pedagogies which are “integrated to support existing, often teacher directed, classroom practice” (p. 1180). Taken together, this leads us ask: To what extent do MTEs work toward a (mathematics) teacher education that challenges pedagogies based on traditional and technical-rational approaches to teaching and learning?

II. Challenge practices informed by non-critical approaches

Evidence of teacher educator practices which make knowledge problematic, and demand that PTs critically reflect on their own belief systems and learning experiences, is important due to “the culture of resistance for pedagogical change” that is embedded in schools (Vratulis et al., 2011, p. 1181). To unpack such resistance, Weis and Fine (2001) propose that teacher educators stage “act[s] designed to disrupt the asymmetric relations embedded in a capitalistic economy and racism” (p. 520), and draw attention to how schools “serve to perpetuate and indeed legitimize widespread structural inequalities” (p. 497). Sidebottom (2019) calls for teacher educators to challenge “the neoliberal, performative constraints on our abilities to realise socially-just academic organisations” (p. 233).

To legitimize and promote social justice processes, Mills (1997) proposes a disruptive pedagogy to challenge “the legitimacy of school processes which produce and reproduce oppressive relations of power” (pp. 35-36). Such a critical perspective calls for a pedagogy which moves beyond ‘valuing’ diversity and a culture of tolerance “to a critical understanding of difference that … recognizes micro and macro power relations, and problematizes knowledge about ‘community’” (Mills et al., 2009, p. 75). In this regard, Beighton (2017) offers the idea that “teacher educators can examine how non-diversified practices at the local [micro] level constitute barriers to meaningful student participation and undermine teachers’ responses to issues of equity and social justice” (p. 113). Hence, we ask: To what extent do MTEs work toward a (mathematics) teacher education that challenges current practices from perspectives informed by critical mathematics education?

III. Promote non-traditional and participatory approaches to teaching and learning

While the two first themes were concerned with what DP aims to shift away from, we now turn to two themes that are more concerned with what DP promotes, or ‘shifts towards’. Generally speaking, this third theme is concerned with promoting pedagogies which introduce “new ways of thinking, learning, and teaching” (Anderson & Justice, 2015, p. 405), which frequently imply the “[n]eed for teachers to unlearn traditional teaching beliefs and practices” (Hedberg, 2011, p. 2). The theme embraces a disruptive view of learning which highlights how learning can only be provoked, not predicted (Iftody & Sumara, 2010, p. 105). In addition to provocation, an invitation to PTs to share their insights and stories is also key in this dimension of DP (Sidebottom, 2019); that is, inviting students into teaching and learning conversations is an important step in replacing the "formulaic deficit models" of teaching with "a creative engagement of and crucially *with* learners" (Beighton, 2017, p. 120). Anderson and Justice (2015) advocate for disrupting traditional practices of teacher education courses by creating "a participatory environment that publicly challenges [PTs'] epistemologies and the community practices in the learning process through both their engagement with the content and interactions with their peers" (p. 404).

In sum, a key aim of this DP theme is to mentor/support PTs to "fully engage in transformative, radical educational acts… required to constantly reposition, redefine, and rethink their roles and to deconstruct and redesign their objects of study" (Bastos, 2009, p. 5). Moreover, and above all, this third theme seeks evidence of teacher educator pedagogies that encourage PTs to "experiment with pedagogical approaches learned in their teacher education programs… [even] if they are not evident in schools and may not be supported by sponsor teachers on practicum" (Vratulis et al, 2011, p. 1181).

In light of this theme description, we return our attention to mathematics teacher education, and ask: To what extent do MTEs work toward a (mathematics) teacher education that promotes pedagogies focused on non-traditional and participatory approaches to teaching and learning?

IV. Promote practices informed by equity and social justice aims

Here, we seek evidence of teacher educator pedagogies that promote an agenda grounded in equity and social justice aims. To promote such an agenda, teacher educators are called upon to model pedagogies that "challenge inequities and social injustice rather than… projecting a vision of an ideal school" (Mills, 1997, p. 39). This means teacher educators are called to work toward teaching practices that promote change in the existing relations of power that appear throughout the routines of life within a school Mills, 1997). It also means emphasizing the importance of providing challenging work for students from traditionally underachieving backgrounds (Mills et al., 2009) while paying "more attention to our own agency and responsibility" (Sidebottom, 2019, p. 233) in the face of systemic barriers to social justice. Given the critical and equity-focused aims of this theme, we ask: To what extent do MTEs work toward a (mathematics) teacher education that promotes practices from perspectives informed by equity and social justice aims?

The DP lens in the context of MTE post-field practices

With the four DP inspired themes established, and the way in which they raise questions that directs us to the practices of MTEs, we return to the language of mathematics teacher education and ask: What might these pedagogies—those that 'challenge' and 'promote'— look like in the context of MTE post-field practices? As it would be an impossible task to provide a comprehensive list, we aim here to suggest a few examples. At the same time, we remind the reader that this is a work in progress, and that the next step is to examine our study's data through the four themes of this DP lens with a goal of understanding the pedagogical strategies and theoretical tools MTEs report to draw on as a way to bring about disruptions in their post-field work with PTs; that is, we aim to understand the extent to which there is evidence of disruptions in the post-field approaches of MTEs that set out to effect lasting changes in PTs' practices.

Drawing on the points raised within our theme summaries, a disruptive pedagogy in mathematics teacher education would involve, for example, MTEs *challenging* different instrumentalist technique-oriented approaches, for instance what Skovsmose (2001) refers to as the “exercise paradigm”. DP might also be characterized by MTEs providing opportunities for PTs to challenge their belief systems and learning experiences that have an impact on their ability to consider alternative discourses for what it means to know and learn mathematics. This would necessarily include challenging systems and structures of mathematics education that continue to colonize the learner toward deficit views with respect to who can succeed at mathematics, including misrecognizing the power of mathematics and its uses in schools and in society more generally (Andersson & Nolan, 2021).

In the same manner, a disruptive pedagogy in mathematics teacher education would involve MTEs *promoting*, for example, more experimental task- and inquiry-based mathematics teaching and learning, thus “emphasizing that planning can be about inquiry and joint discovery rather than prediction and transmission” (Beighton, 2017, p. 119). Additionally, in keeping with an equity and social justice focused agenda, MTEs would move “critical mathematics education forward… with the goal of educating critically aware students who have power to question the mathematics that influences and formats their lives” (p. 45). Admittedly, it is no easy task for MTEs to practice disruptive pedagogies; however, Beighton (2017) reminds us, as “teacher educators [we] need to understand that learning, with all its difficulties and complexities, is not a problem to be fixed or a weakness to be confessed, but an ongoing process of engagement with what is becoming” (p. 120).

Implications for the existing research in the area – and for MTEs

Situating their work in the context of PTs working with technological innovations, Anderson and Justice (2015, p. 408) found that the practices of PTs in response to pedagogical disruptions fell into three distinct categories: transformative practices (where PTs took up the disruptive practices), performance of untransformed practices (where PTs participated to ‘get it done’), and practices that were resistant to the disruption (where PTs pushed back) (p. 408). In this way, their findings “provide a starting point for examining the implications of disruptive pedagogical practices within pre-service teacher education programs” (p. 416). Our lens construction is a recent development initiated to engage with MTEs’, rather than PTs’, perspectives on DPs. It remains to be seen whether MTE practices brought forth by our disruptive theoretical lens fall into categories similar to these, or perhaps very different ones.

Our next step of data analysis will provide insights into how we may improve upon and enhance this DP theoretical lens by grounding its application in research data. In addition, presenting our lens for discussion in a community of MTEs (as in this CERME-12 TWG 27 focused on the professional practices, preparation and support of MTEs) will provide opportunities for us to further reflect on our conceptualization of this DP lens, as well as provide a desirable context for sharing this innovative theoretical lens for other MTEs to consider in their own research and practice.

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Effects of Facilitator Professional Development on Teachers' Learning – An Intervention Study in the Context of Early Mathematics Education

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A fundamental question in the context of professional development is whether teacher professional development (TPD) can be scaled up successfully using facilitator professional development (FPD) programs. Studies that link effects of FPD programs to teacher learning are, however, rare. The presented study addresses this research gap by examining effects of an extensive FPD program in the context of early mathematics education (EmMa^M). To examine effects of EmMa^M, a quasi-experimental intervention study with two intervention groups was conducted: While in group A the TPD course was enacted by the program developers, in group B the TPD was enacted by facilitators. Results show that teachers in group B achieved comparable learning gains to teachers in group A. This result supports the idea of scaling up professional development using FPD – if facilitators are qualified accordingly.

Keywords: Facilitator professional development, in-service professional development, teacher learning, early mathematics education.

Facilitator Professional Development Programs and its Effects

Teacher professional development (TPD) is essential to improve teaching quality and support teachers' lifelong learning (e.g., Borko, 2004). Therefore, a shared aim of school administration, developers of professional development (PD) programs, and researchers is to scale up effective PD programs (e.g., Roesken-Winter et al., 2015). This scaling up can be realized by qualifying facilitators to enact TPD at several sites (Borko, 2004). To support facilitators in their (new) role, several authors suggest facilitator professional development (FPD) programs which include PD materials and activities for teachers as well as support materials for facilitators to enact the PD program (Borko, 2004; Koellner et al., 2011; Prediger et al., 2019).

Prediger et al. (2019) developed the *Three-Tetrahedron Model of professional development* "to capture the complexity of PD courses in a multifaceted way and to connect the different levels of (1) teaching and learning on the classroom level, (2) PD on the teacher level, and (3) PD on the facilitator level" (p. 408). The *Three-Tetrahedron Model of professional development* suggests a cascadic approach to scale up PD programs. In cascade models a group of facilitators, who are mostly teachers themselves, is trained to conduct TPD programs (Krainer, 2015).

Different authors question the effectiveness of cascade models: They criticize that cascade models follow a transmissive approach and do not take the context of the individual facilitator into account (e.g., Hayes, 2000; Kennedy, 2014). Additionally, for example Krainer (2015) expects a dilution of expertise with each level of the cascades. In summary, it is assumed, that facilitators might have less developed mathematical expertise and/or less accomplished facilitation practices then their trainers which in turn leads to less expertise at the teacher level. Whether this is the case has yet to be examined.

Therefore, a key question is whether TPD programs enacted by facilitators in FPD programs achieve effects on teachers' learning (Koellner et al., 2011; Perry & Boylan, 2018; Prediger et al., 2019; van Driel et al., 2012). The few research results on the effects of FPD on teachers' learning available in the field of mathematics education indicate that teachers that were trained in a PD course enacted by facilitators increased their knowledge in comparison to untrained teachers (Bell & Higgins, 2010), but show less competence than their facilitators (Koellner & Jacobs, 2015; Lange, 2014; Turner et al., 2017). Additionally, effects of TPD differ with respect to the facilitator (Bell & Higgins, 2010; Carney et al., 2019). In summary, these results support the assumption that the scaling up of TPD using FPD is possible. However, the results do not clarify the extend of these effects and thus do not address the criticism of the dilution of effects with each level of the cascades. This is especially true as there is a lack of research comparing the effects of TPD courses enacted by facilitators and TPD courses enacted by experts in the field with more expertise then the facilitators (i.e. the originators/developers of the PD courses). The present study starts out to address the questions of the dilution of effects with each level of the cascades by examining FPD in the context of early mathematics education exemplified by the FPD program *EmMa*^{M} (Bruns et al., 2021).

The Context of this Study: *EmMa*^{M} – A FPD Program for Early Mathematics Education

A key feature of the *Three-Tetrahedron Model* is to base FDP on TPD. To realize this, we firstly developed a TPD course on early mathematics education called *EmMa - Erzieherinnen und Erzieher machen Mathematik* [EmMa – Early childhood teachers are doing mathematics] (Bruns et al., 2017), secondly examined the effectiveness of this TPD course (Bruns et al., 2017) and thirdly developed the facilitator professional program *EmMa*^{M} (Bruns et al., 2021). The development of the facilitator professional program *EmMa*^{M} was guided by key features of facilitators learning as indicated in the literature (Jacobs et al., 2017; Koellner et al., 2011; Schifter & Lester, 2005):

- **Considering all aspects of teaching and learning on the teacher and the classroom level:** To realize this first key feature, the structure and content of the FPD program *EmMa*^{M} is based on the structure and content of the TPD course *EmMa*. *EmMa*^{M} comprises of an introductory module and four in-depth modules which each lasts two days. Additionally, *EmMa*^{M} integrates various activities from the TPD course *EmMa*, which are firstly carried out by the facilitators themselves and afterwards reflected on with regard to the aims of the activities. *EmMa*^{M} thereby addresses the content of the TPD course *EmMa* from a higher level which also includes typical teacher misconceptions and reflection of teaching strategies.
- **Integrating and modeling activities of the teachers PD program**: As all activities as well as all theoretical aspects from the modules of the TPD course *EmMa* are enacted by the leaders of the FPD course, these leaders also function as a model for the facilitators.

- **Supporting the preparation, implementation as well as the follow-up of the TPD course:** In practical phases between the FPD modules, the facilitators independently lead TPD courses on early mathematical education. Through these practical phases, *EmMa*^{M} realizes an accompanied implementation of the TPD program as advised by Jacobs et al. (2017). Facilitators are supported in the preparation, implementation and follow-up of their TPD course during the FPD.
- **Offering supporting materials to enable facilitators to conduct the PD course in alignment with the intended goals of the PD program:** The fourth key feature of supporting materials is realized by a set of guiding materials. These materials include suggestions for the methodical structuring of the TPD course, a commented set of presentation-slides for each TPD module and templates for several teacher activities. In addition, the leaders of the FPD provided further literature as well as a set of games and play materials used to foster early mathematical learning in kindergarten (games, pattern blocks, etc.) which is used to plan different learning opportunities for children in the TPD program.

All in all, the FPD program extends over a period of 10 months and includes 85 hours of presence time and at least 100 hours of time to prepare and implement the TPD course in the practical phases.

Design of the Study

Research Question and Research Approach

To examine effects of *EmMa*^{M}, a quasi-experimental intervention study with two intervention groups was conducted: While in group A the TPD course was enacted by the program developers (first author of this paper and a colleague), in group B the program was enacted by facilitators. According to the aims of our study and the TPD course *EmMa*, we focus on the effects on teachers' mathematical pedagogical content knowledge (MPCK) and their beliefs. The leading research question is: Are there significant differences in the development of early childhood teachers' MPCK and beliefs between early childhood teachers that undertook the PD course EmMa enacted by facilitators in comparison to early childhood teachers that visited the PD course EmMa enacted by experts (the program developers)?

Sample

The sample of the experimental group A, the expert group, comprises of n = 76 early childhood teachers (n = 65 female, n = 4 male, n = 7 missing) that visited a TPD course *EmMa* on early mathematics education enacted by the program developers between 2014 und 2016. The early childhood teachers were between 24 and 59 years old, on average 43.97 years (SD = 10.81). All early childhood teachers were trained at vocational schools and were working in Germany.

The sample of experimental group B, the facilitator group, comprises of n = 83 early childhood teachers (n = 76 female, n = 7 male) that visited a TPD course *EmMa* on early mathematics education enacted by nine different facilitators. The early childhood teachers visiting these TPD courses were between 21 and 62 years old, on average 42.27 years (SD = 11.08). The early childhood teachers were also trained at vocational schools and are working in five different federal states with comparable frameworks concerning early childhood education.

The developer of the *EmMa* program leading the TPD courses of experimental group A are the first author of this paper, currently a junior professor for mathematics education and a colleague, who is an early childhood teacher himself and currently professor for early childhood education. The nine facilitators leading the PD courses of experimental group B were all visiting the FPD program *EmMa*$^{M.}$. The TPD courses were conducted as a part of this FPD program (practical phase) in 2018. The facilitators were between 31 and 57 years old, in average 43.33 years (*SD* = 9.83). Six facilitators were trained as early childhood teachers themselves and had between 11 to 30 years practical experience as an early childhood teacher. The other three facilitators had a Master's degree in educational studies, one with mathematics as a major field of her studies. Yet, none of the facilitators had been significantly involved with early mathematics education during their training or studies. Most of the facilitators (7 out of 9) additionally reported prior experience in leading PD courses to different topics relevant to early childhood teaching (e.g., language and literacy) but not to early mathematics education. Experience in PD ranged between 3 to 300 days (*M* = 70.13; *SD* = 101.42).

As described facilitators were supported intensively in these practical phases by the leaders of the FPD program. In addition, results of a qualitative study on these practical phases of *EmMa*^{M} using the same sample showed that facilitators all stayed very close to the suggestions for the methodical structuring of the TPD course (Bruns et al., 2021). From this it can be followed, that the TPD courses visited by the early childhood teachers in experimental group A and B were comparable concerning content and structure but not necessarily concerning the thematic depth and mathematical correctness.

Instruments

Early childhood teachers' MPCK was measured by a standardized Rasch-scaled test consisting of 35 items (Blömeke et al., 2015). Each item on MPCK was coded dichotomously (0 = not correct/ not reached; 1 = correct). The Rasch scaling model was applied to the coded data. The z-standardized early childhood teachers' WLEs are used as performance values for any further analyses. In our study, the test showed an EAP reliability[1] of .58 which is comparable to earlier studies (ibid.).

The beliefs towards mathematics in general were assessed by a questionnaire (Blömeke et al., 2017) using 27 items and a 6-point Likert scale. The items distinguish five beliefs facets: (1) a static orientation towards mathematics (7 items, Cronbach's α=.83, (2) a process-related orientation towards mathematics (4 items, Cronbach's α=.82), (3) an application-orientation towards mathematics (6 items, Cronbach's α=.80), (4) gender stereotypes regarding mathematics (5 items, Cronbach's α=.92) and (5) enjoyment of mathematics (5 items, Cronbach's α=.87). Every single item ranges from 1 to 6, mean scores have been computed for the different scales. Higher scores indicated that participants had a higher agreement in the mathematics-related statements.

[1] When calculating IRT reliability using Rasch modeling, each participant is assigned an estimated ability value expressed as a score distribution. The predictive reliability is 1 minus the ratio of the variance of a participant's score distribution relative to the sample variance. EAP represents the mean of such predictive reliabilities in the sample and is thus a measure of the overall reliability of the sample that can be interpreted similarly to Cronbach's α (see e.g., Neumann et al., 2010).

Procedures and Analysis

Data on experimental group A was collected in TPD courses led by the program developers in the years 2014 to 2016. Due to missing data, we have 54 (MPCK) respectively 49 (beliefs) complete data sets for the analysis in group A. The facilitators started their FPD program *EmMa*^{M} in March 2018. As a part of this FPD program the facilitators lead one *EmMa* TPD course with 10 to 12 early childhood teachers in the practical phases (see Figure 1). The early childhood teachers visiting these *EmMa* TPD courses led by the facilitators in training form experimental group B. Due to missing data, we have 82 (MPCK) respectively 81 (beliefs) complete data sets for the analysis in group B.

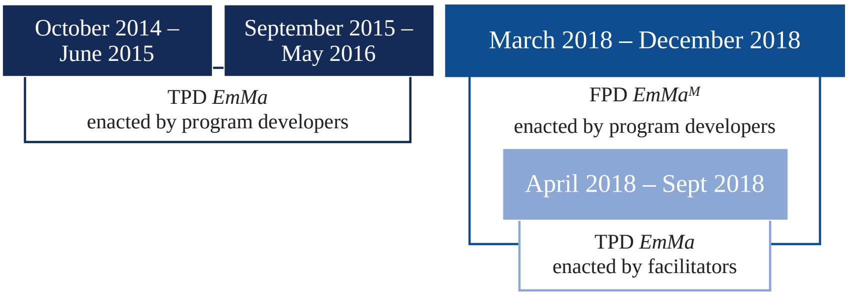

Figure 1: Interconnections between the TPD enacted by the program developers (group A), the FPD and the TPD enacted by the facilitators (group B)

In order to assess whether the early childhood teachers' MPCK and mathematic-related beliefs changed from measurement point 1 to measurement point 2, we used a t-test for dependent samples. Based on these findings, for each competence facet a repeated measure ANOVA was used to determine the extent to which changes differ depending on the experimental group. The absence of significant differences would indicate comparable effects in both groups.

Results

The results of a paired t-test confirm significant increases in MPCK for both groups. The early childhood teachers from experimental group A score a mean of 46.81 points on the pre-test for measuring the MPCK and a mean of 49.89 points on the post-test ($t(53) = -3.03$, $p < .01$, $d = 0.26$). The early childhood teachers from experimental group B score a mean of 47.39 points on the pre-test for measuring the MPCK and a mean of 50.21 points on the post-test ($t(81) = -2.23$, $p < .05$, $d = 0.27$). The results of the repeated measure ANOVA reveal a significant main effect of the measurement point ($p < .01$, $\varepsilon^2 = .08$) but not of the experimental group. There was also no interaction effect. In other words, early childhood teachers in both experimental groups increased their MPCK equally.

Concerning the beliefs, the results of paired t-tests confirm significant increases in the agreement to process (EG A $t(48) = -2.16$, $p < .05$, $d = 0.30$; EG B: $t(81) = -4.41$, $p < .001$, $d = 0.50$) and application orientation (EG A: $t(48) = -4.41$, $p < .001$, $d = 0.55$; EG B: $t(81) = -3.07$, $p < .01$, $d = 0.40$) statements towards mathematics and in the agreement to statements concerning the enjoyment of mathematics (EG A: $t(48) = -5.37$, $p < .001$, $d = 0.76$; EG B: $t(81) = -6.28$, $p < .001$, $d = 0.78$) for both experimental groups. Furthermore, significant decreases in the agreement to static orientation statements towards mathematics (EG A: $t(48) = 6.73$, $p < .001$, $d = 0.86$; EG B: $t(81) = 6.45$, $p < .001$, $d = 0.78$) and in the agreement to gender stereotypes (EG A: $t(48) = 4.69$, $p < .001$, $d = 0.58$; EG B: $t(80) = 4.60$, $p < .001$, $d = 0.42$) statements regarding mathematics are also confirmed for both experimental groups. The results of the repeated measure ANOVA reveal significant main effects of the measurement point but no main effects of the experimental group. There are also not given any interaction effects. In conclusion, early childhood teachers in both experimental groups succeeded equally in positively changing mathematics-related beliefs.

Discussion

Limitations of the study

Our study bears limitations regarding the sample of early childhood teachers in the two groups. The TPD courses in group A and B were time-shifted by about three years. We did, however, find no differences between the groups on the pre-test scores and therefore concluded that the differences in the sample seem to be neglectable for this study. Secondly, limitations can be traced back to the instruments used to measure early childhood teachers' mathematical pedagogical content knowledge. To compare the facilitator and the expert group regarding their learning effects, we used the same instrument to measure teachers' mathematical pedagogical content knowledge (Blömeke et al., 2015). The test score interpretation of this instrument is, however, validated for a sample of pre-service not in-service teachers and the test instrument follows a rather cognitive and broad approach regarding mathematical pedagogical content knowledge. Further limitations occur concerning the sample of the facilitators. This group was very heterogenous regarding their education, experience with TPD in general and their experience with mathematics and mathematics education (in early childhood settings). Likewise, it should be mentioned that facilitators followed the suggested structure, content and activities of the TPD course *EmMa* quite closely (Bruns et al., 2021). It can therefore be assumed that the effects we found can partly be attributed to the concept of the TPD course *EmMa*.

Interpretation and Conclusion

To our knowledge this is the first study that addresses the criticism of expertise dilution with each level of the cascades by comparing effects of TPD courses enacted by facilitators to effects of TPD courses enacted by experts in the field. As did previous research (Bell & Higgins, 2010; Carney et al., 2019; Koellner & Jacobs, 2015; Lange, 2014; Turner et al., 2017), our study found that TPD courses enacted by facilitators can achieve effects on teachers' learning concerning their MPCK as well as their beliefs – if facilitators are qualified accordingly for the job and supported by experts in the field. Adding to the state of research, our study found that these effects are comparable to effects of experts in the field. Our study did not indicate any differences in the effects of the TPD course between the expert group and the facilitator group regarding the development of teachers' MPCK nor their mathematics-related beliefs. This result supports the idea of scaling up PD using FPD (Borko, 2004; Koellner et al., 2011; Prediger et al., 2019) as it reveals that experts and facilitators can achieve the same effects on teachers' learning through TPD.

However, this result must be considered in the light of the FPD concept: The effects found in this study are not only based on quality of the TPD courses lead by the facilitators and the expertise of the facilitators, but also on the extensive resources and specific materials used to support facilitators in leading TPD courses (s. a. Borko, 2004; Koellner et al., 2011; Prediger et al., 2019). It can therefore not be ruled out that there is a dilution of expertise with each level of the cascades. In fact, in an accompanying qualitative study, we found that the facilitators make a lot of incorrect statement during their TPD courses (Bruns et al., 2021). Our results of this study do, however, indicate that this lack of expertise can be compensated by the close monitoring of the facilitators and the high quality of the TPD resources. Our results therefore support those critical perspectives on the cascade model that direct their criticism towards the implementation of the cascade model in practice (Hayes, 2000; Wedell, 2005). Still to be investigated is the extent to which the effects of the TPD courses differ depending on the facilitator (see also Bell & Higgin, 2010; Carney et al., 2019). Nevertheless, we conclude from this that scaling up TPD by qualifying facilitators to enact TPD at several sites can be successful – but probably has to be accompanied not only by extensive facilitator training but also high-quality resources to facilitate the TPD.

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The role of the mathematics teacher educator in supporting engagement with global issues in the mathematics classroom

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The world is in a period of rapid change. Climate chaos is leading to floods and droughts and exacerbating inequality. A recent global survey indicated a majority of young people are 'very' or 'extremely' worried about climate change. This paper asks, in such a context, what is the role of mathematics education and, in particular, what can mathematics teacher educators (MTEs) do, in supporting teachers who want to engage in bringing global issues into their teaching? I report on work taking place in one university in the UK where, for the last 6 years, MTEs have been encouraging teachers to address global issues in their classroom. One role which has emerged for a MTE is in supporting the translation of scientific work on climate change into classroom tasks. In the UK, the spaces for such work are being squeezed in an increasingly politicized education system. The lens of critical mathematics education offers a mechanism for reflection on the role of the MTE.

Keywords: Global challenges, climate change, learning to teach mathematics, role of the mathematics teacher educator, critical mathematics education.

Introduction

The effects of climate change, or the climate emergency, are visible across the world. The year 2021 has seen extreme weather in the form of tornadoes, droughts leading to wildfires, rainfall leading to unprecedented flooding and more. Mathematics is deeply implicated in climate change and other global issues, for example, in how events are communicated and how they are modelled. Mathematical models of climate, pollution, virus spread, the economy have real world effects and so mathematics is also implicated in the creation of some of the global challenges facing humanity. For instance, mathematical models of the economy, which place no value on materials before they are extracted, facilitate exploitation of natural resources and the depletion of environments which have historically sustained communities. A report on a survey, about to be published (but trailed in the media) by Caroline Hickman at Bath University, UK, looked at the views of 10,000 16-25-year-olds across 10 countries and found high levels of eco-anxiety: 60% of young people surveyed said they felt 'very' or 'extremely' worried about climate change. In such contexts, a mathematics education which continues its business as usual, in isolation from the world outside the classroom, seems increasingly jarring. This paper aims to investigate what mathematics teacher educators (MTEs) might do. What role might a MTE take, in working with prospective or in-service teachers, in order to support a re-thinking of the kinds of topics or discussions which take place in mathematics classrooms, to include global issues and challenges such as climate change? My way of approaching such questions is informed by critical mathematics education and I set out this perspective in the next section. I then report on work that has taken place over the last 6 years, at the University of Bristol, in the UK, in which MTEs have been working with teachers to encourage and support tackling global issues in the mathematics classroom.

Critical mathematics education

Critical mathematics education refers to work which brings insights from critical theory into the specific sphere of mathematics education. Andersson and Barwell (2021) offer a summary, that critical mathematics education: "is *driven by urgent, complex questions*; is *inter-disciplinary*; is *politically active and engaged*; is *democratic*; *involves critique*; and is *reflexive and self-aware*" (p.3, italics in original). This characterization says to me that critical mathematics education is particularly appropriate to a consideration of how issues such as climate change might be relevant to the work of a MTE. Global challenges such as the climate emergency, and other ecological disasters are certainly urgent and complex and inter-disciplinary. Addressing such questions as a MTE (in the UK) is a political statement since there is now a mandated curriculum for teacher education which is focused on the techniques and craft of teaching, with no mention, for instance, of our role in preparing citizens for a precarious future. And, as I hope to demonstrate in this paper, the use of critical mathematics education for MTEs can provide a tool for reflexivity.

Andersson and Barwell (2021) identify three broad schools of work within critical mathematics education (acknowledging that such divisions are a simplification): Freirean; Foucauldian; and a Nordic school. They characterize the Freirean school as focused on the use of mathematics for consciousness raising, for "reading the world", in a way that draws parallels to Freire's (1970) work on literacy as consciousness raising. Within this tradition there is often an explicit aim to challenge oppression and effect change (Gutstein, 2006). The Foucauldian school investigates the (often invisible) uses of the discourses of mathematics in the "organization of human affairs" (Andersson & Barwell, 2021, p.9), for instance, the way in which middle class assumptions about what constitutes child development can become accepted as what it means to be "normal", or the way in which differences between girls' and boys' relations to mathematics are constructed (Walkerdine, 1988). In the Nordic school, Skovsmose (1994) explores the use of technology in society and argues for the need for students in school to not only be taught to do mathematics but also to critique how mathematics is used and how it gets embedded in technologies which have real social effects, such as algorithms for welfare payment distributions.

Perhaps surprisingly, there has been relatively little work linking critical mathematics education to issues such as climate change, with exceptions such as Renert (2011) and Barwell (2013); although this is a situation that now appears to be changing (e.g., Barwell & Hiis Hauge, 2021; Steffensen et al., 2021). There also appears to be little work bringing critical mathematics education into discussions of the work of mathematics teacher educators. This article brings both of these strands together. I do not adopt one particular school of thought within critical mathematics education but rather use the range of meanings in order to investigate evidence and possibilities for critical mathematics education in the role of a MTE. I use the broad characterization of the three schools of critical mathematics education as a set of ideas, offering opportunities for reflection and critique of the practices of MTEs. I will be reporting on a long-term project which is on-going at the University of Bristol. This is a project that embodies a process of curriculum innovation within our teacher education course and it has not been guided explicitly by ideas of critical mathematics education. In the next section, I offer a description of this project, before reporting on some of its outcomes via a reflection on three professional journal articles.

The Green Apple Project

The name "Green Apple" came from the internal funding scheme which helped kick-start the project in 2015 at the University of Bristol, UK. The funding was to support innovation in our 1-year teacher education course for secondary school, which leads to a Post Graduate Certificate of Education (PGCE). The idea was to support teachers bringing questions of global challenges into their subject teaching, or their work as form tutors (a pastoral role required of teachers in England). Two (or more) "Green Apple representatives" (reps) are recruited from each of the 8 subjects offered on our PGCE course and these reps meet with a group of teacher educators (including MTEs) on three occasions over the year. The reps are responsible for disseminating outcomes from meetings to the rest of their subject group. Meetings introduce reps to ideas (such as the framing of "wicked problems", or of education for sustainable development) and allow time for discussion between and across subject groups and offer prompts for action between meetings. Classroom resources are shared and previous years' work is available. Over the last 6 years, the project has introduced and run sessions with all prospective teachers, provoking thought on the role of global challenges in their subject teaching.

In order to present some of the outcomes of the project and to focus on the role of the MTE, I report here on three articles, which have been written by mathematics members of the Green Apple Project, and which have appeared in (or are soon to appear in) the professional journal *Mathematics Teaching*. These are the only publications from the group in a professional journal. In what I report here, I focus on the tasks offered in the articles, and reflect on the role of the MTE in each case, linking to ideas of critical mathematics education.

Sustainable futures, from 2018

Karl Bushnell wrote an article for Mathematics Teaching (Bushnell, 2018) based on resources he had developed and trialed for his own classroom. Karl had done his PGCE at the University of Bristol and taken an active role in the Green Apple project. The work he wrote about was conducted when he had taken up a job in a school near Bristol. Karl continued his Master's in Education (with a specialization in Mathematics Education) at the University of Bristol and developed an approach to offering tasks which paralleled standard exercise questions, with questions that led students to answers which told them something about the global environment. An example is given in Figure 1, which shows the worksheet for students, involving a sequence of tasks leading them to calculate the sea-level rise that would be caused by the Greenland icesheet melting.

In Figure 1, tasks are paired, to have one question, or set of questions, with no context and one question drawing on the same skills but in the context of melting ice sheets. For example, the second task (top right of the sheet) has three prisms and asks students to find their volume. Then the prompt is: "Given that the global water [sic] surface area is 361,132,000 square kilometres, and using your answer to Question (1) [which was to convert 50 metres into kilometres], calculate the volume of water needed to cause a 50m rise in sea levels". The final question invites students to write down their thoughts having done the calculations, i.e., to reflect on the implications of the predicted change.

The tasks were developed entirely by Bushnell, i.e., without direct MTE involvement. The role of the MTE, in the case of the work written up here, was one of supporting the raising of issues (via the Green Apple work on the PGCE course) and then subsequent support via a flexible Master's programme that allowed teachers to follow their own classroom interests. Bushnell was an engaged and passionate individual who took the initiative to continue to develop themes from the Green Apple project after he had left the PGCE course as a qualified teacher.

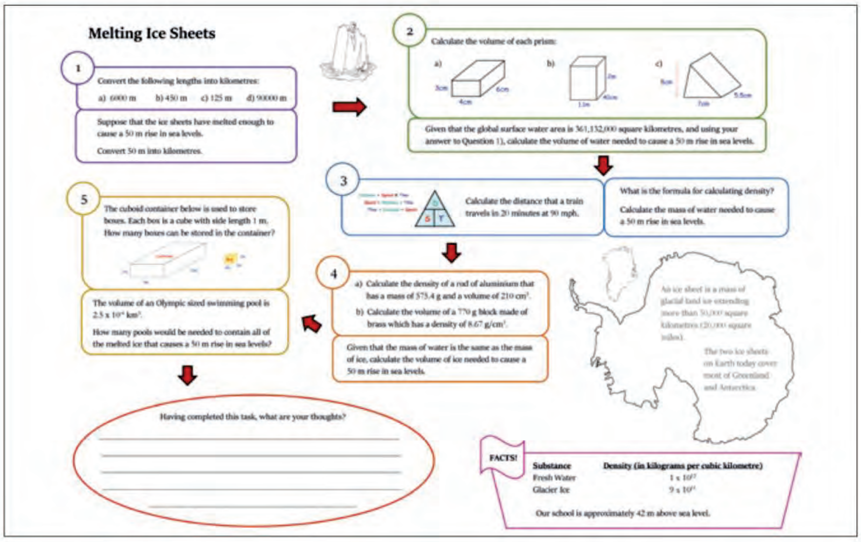

Figure 1: A sequence of tasks calculating sea-level rise from the Greenland icesheet melting

This task, and the MTE support, do not appear to attempt to offer a critique of the mathematics being used (as in the Foucauldian or Nordic schools). The mathematics is presented as a neutral tool and the assumptions behind the model being used are not discussed, for instance. The MTE did not provoke questions about this neutrality, nor prompt reflection on possible links to social action, or oppressions which are entailed in the causes of sea level rise (e.g., consideration of the almost incalculable human suffering which would ensue). There seems to be an aim to raise consciousness (edging towards a Freirean idea); in the classroom task the aim is to raise consciousness of the consequences of the melting of the Greenland icesheet. And the MTE role appears to have raised consciousness, or awareness, for the teacher, of possibilities for the use of global challenges in mathematics teaching and allowed space for reflection on how this could be done, while also maintaining a focus on the curriculum that needs to be taught in secondary school.

Global challenges, from 2019-20

Across two articles (Brown et al., 2021 Part 1; Brown et al., 2021 Part 2) 9 prospective teachers on the PGCE course (in the 2019-20 cohort), supported by their 3 MTEs, each wrote about one task they had either created or found and then used (or planned to use) with classes they were responsible for in their placement schools. By this time, several years into the Green Apple project, global challenges were involved in one of the mathematics programme's Master's assignments on the PGCE course, and so every prospective teacher had to engage, in at least one lesson, with thinking about how to incorporate an issue such as climate change into their classroom teaching. I relay below 5 of the 9 tasks reported on across the articles. Two tasks began with simply stated challenges to provoke discussion, which then developed into more extended activity.

“How does a country’s GPD affect a person’s life?” (Part 1, p.9)

"How many trees could fit in this room?" (Part 2, p.9)

The other tasks got students doing some work or calculations that did not necessarily initially relate to a global challenge and then led into a key question. Here are three examples.

Based on data about incomes in England and Romania: "Which country has a bigger pay gap between male and female workers?" (Part 1, p.10)

Based on drawing paths and associated loci (to model a potential Covid transmission distance of 2m): "assess potential transmission points" (Part 1, p.13)

Having worked out the area of the school grounds and, based on aerial photographs of deforestation in Brazil: "find the area that had been cut down, and how many 'areas of the school' that this represented" (Part 2, p.11)

Several of these lesson ideas came from the prospective teachers' engagement in their Master's studies, where they had to plan a topic (around 5 hours of teaching) which involved at least one global challenge being addressed. For other teachers, these tasks were developed in response to a session given to all prospective teachers on the course, on Green Apple issues. The role of the MTE has shifted here, to one of mandating some exploration of global issues. As MTEs, having seen and been impressed by the work developed by these teachers, my two colleagues and I then proposed some joint writing. As MTEs we helped edit and structure the writing and supported its submission.

As with the previous example, these tasks, and the MTE support for them, appear primarily focused on consciousness raising (Freirean school) without perhaps drawing attention to the uses of mathematics (the Nordic school) and hardly pointing towards more Foucauldian questions around the organisation of society. In other words, the tasks appear focused on raising awareness of issues such as deforestation and the MTE role appears, as before, primarily focused on raising awareness of possibilities within the classroom. The write-ups of the tasks suggest wider implications about society, and links to social activism, were left implicit and that would also be true to say about the MTE support being offered (i.e., consideration of such wider issues was not prompted or provoked).

Climate science, from 2021

In early 2021, I began a collaboration with a scientist (Joseph Darron) from the UK Met Office, after this scientist was given a secondment of a day a week to work at the University of Bristol. Joseph's research centres around questions of how climate models and statistics are communicated (e.g., to politicians). After a few months of intermittent discussions about making his research into something usable in the classroom, we invited a local partner school to see if they would like to join us. One teacher (Barney Rolph) volunteered and worked on adapting and trialing the tasks that we were developing. Barney had been on the PGCE course at the University of Bristol and been involved in the Green Apple group. The task Barney adapted to use in his classroom was the following (Coles et al., 2022, p.7):

Climate models are used to simulate the climate and predict changes. We will be focusing on models that can be used to predict how much wetter or drier a place might be in the future. There are different models, made by different scientists.

> Botswana (in Africa) currently averages around 34mm of rain per month.
>
> Here are 9 projections for the change in rainfall in Botswana (in mm per month) in the coming years:
>
> +17; +13; +6; -2; -2; -6; -6; -6; -14
>
> Your task:
>
> Imagine you are an advisor to farmers in Botswana. You are going to prepare a written summary of this data advising farmers of future risks and changes to the climate. Think about different mathematical techniques that you have learnt that might help analyse and/or present this data.

In the article, Barney offers some of the response of his class to this prompt and further work he went on to do with them using climate data. The MTE role is quite different in this case, compared to the previous two. Here, the MTE acts as a conduit between a climate scientist and a teacher. As the MTE involved, I remember recognizing, in Joseph's presentation of his research, its classroom potential. The questions he was grappling with – to do with what the consequences are of different presentations of data for how they are interpreted – seemed like ones that students in schools could access. I also recognized that the data Joseph was using seemed accessible for quite young students and, in my experience as a teacher, it is not easy to find real data in a form that is suitable for a classroom.

The task here, in contrast to the previous ones, feels more aligned to the Nordic school idea of inviting reflection on the use of mathematics and how complex data might be summarized mathematically. And hence, I also interpret the MTE role in this case as provoking awareness of the use of mathematics. There may be elements of consciousness raising also, in the task and in the work of the MTE, but again little attention to wider societal questions and connections to possibilities for action.

Discussion

Having presented the work of the Green Apple project, as it has manifested in three professional journal articles, and offered a description of the role of the MTE in each case, I now summarise and then reflect further on these outcomes.

My aim in offering the work of the Green Apple project has been to allow reflection on the question: *What role might a MTE take, in working with prospective or in-service teachers, in order to support a re-thinking of the kinds of topics or discussions which take place in mathematics classrooms, to include global issues and challenges such as climate change?*

As alluded to above, I interpret the MTE roles in the first two cases as being most aligned to a Freirean perspective on critical mathematics education, with a focus on consciousness raising. However, having said that, compared to examples offered in Freire (1970), the roles of the MTE in these cases are relatively limited examples of consciousness raising. Nonetheless there is a linking of reflection and action, in the work of the teachers, to implement changes in their classrooms and a hope, on the part of the MTEs, that such work will continue into the future. The final example, of the MTE in the role of conduit or bridge building, edges into more of a Nordic school version of critical mathematics education, in raising questions about the use of mathematics in communicating data about climate change, and the huge uncertainty of current models (despite the certainty that change is happening).

Considering the second example, there seems to be a power in mandating engagement in considering global challenges, as part of a teacher education programme, and supporting further engagement of those who show interest by setting up and coordinating a co-writing opportunity. In the final example, the use of mathematics by politicians, policy-makers and scientists is beginning to be explored in the task, in contrast to the previous two articles.

In all cases, the Foucauldian school of critical mathematics education points to possibilities for the role of a MTE that were not taken. In Bushnell's work, a possible MTE provocation might have been to invite consideration (on the part of the teacher and/or students) of who is in danger from sea-level rise and where do they live? In the case of the 2019-20 writing, similar provocations might have been, who benefits from de-forestation? how fairly spread is a country's GDP? In the climate model scenario, again, we did not address questions of the impacts of drought or flood on difference sectors of society. In a UK context it is unlikely such wider question would be addressed in a mathematics classroom but, in reflecting on MTE roles not taken, these seem like potentially significant provocations for teachers, in forcing a consideration of how global issues link to the organisation of societies and the inequalities which are exposed by global forces such as climate change.

As demonstrated here, the three lenses of critical mathematics education offer a mechanism for recognising what is *not* being done, as a MTE. Questions I might ask myself (e.g., in planning to work with prospective teachers) from each of the different perspectives are:

> What are the global challenges that teachers I am working with care about? What tasks and mechanisms (e.g., as part of a teacher education course) can I offer, to support them acting on their interests, in their own classrooms? (a Freirean perspective).
>
> When does mathematics teaching reinforce, or remain silent about, inequalities embedded in the wider organisation of society? Do I call attention to such absences? What might teachers' lived experiences of inequality be and what sensitivities will be needed to explore this (e.g., the space and safety to express reactions and know they have been heard)? (a Foucauldian perspective).
>
> In any context of mathematics teaching (including my own), do I invite reflection on the uses of the mathematics being learnt? Can I link with a professional, who is working on a global challenge, and explore the reality of their use of mathematics? How might I make resources available for teachers in a form that provokes reflection on the uses of mathematics? (Nordic school).

Despite the developments evident over the six-year period of the Green Apple project, I am left with a sense that the resources developed by teachers and the related work of MTEs remains on the margins. The tasks developed seem to represent likely one-offs in the teachers' practices. Similarly, for the MTEs, a mandated "core content" of initial teacher education precludes any sustained focus on global challenges and hence support for such work also has a "one-off" feel.

For both teachers and MTEs there are demands that are hard to ignore (exam success of their students, for the teachers; and, prospective teachers passing a teaching qualification, for the MTEs) both of which, in the UK, pull away from a focus on global challenges. So, although I have provided some answers to my question about what are possible roles of a MTE in supporting such work, I am struck by the limited nature of the roles I have been able to embody, in comparison to the complexity and depth of the issues facing the planet, and amidst a politicization of educational decisions in the UK that, to take one example, mandates against discussion of “anti-capitalist” perspectives in schools.

One thought that sustains me, however, is that we are developing MTE practices, roles and resources to support a curriculum-in-waiting. COVID has shown how rapidly changes can be made in education systems. Over the six years of the Green Apple project I have sensed prospective teachers increasingly wanting to work on inter-disciplinary and global issues, through the lens of their subject teaching. I suggest that critical mathematics education offers a ready, and perhaps untapped, set of resources to help MTEs think about possibilities for transformation in their roles and practices and, one hope I have is that prospective and in-service teachers will increasingly demand for such change.

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What makes doing mathematics in PD productive: Lifting and adapting a framework for insights on leader PD and facilitation

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Well-prepared facilitators of mathematics PD are essential, yet few studies offer nuanced insights and content-specific practices for how facilitators support teachers' productive mathematical engagement. This study utilizes a framework of productive disciplinary engagement (PDE) to investigate the social and mathematical practices of 48 aspiring facilitators and two expert PD leaders across a two-phase design-research study. We found that the PDE framework allowed us to identify participant and facilitation practices that afforded and constrained productive engagement with disciplinary content and practices. We discuss the use of PDE as an analytic and design framework for facilitator and teacher PD. This study contributes insights on content-specific PD practices applicable to a wide range of mathematics PD models.

Keywords: Facilitators, professional development, mathematics education, facilitator practices.

Introduction

Decades of U.S. policy and international research advance that effective professional development (PD) must actively engage participants in disciplinary work. In mathematics PD, this means offering opportunities for teachers to engage in the mathematics relevant to their classrooms. Previously, we have argued that goals for doing mathematics are underspecified, advocating for PD to attend to teachers' specialized content knowledge (SCK) (Ball et al., 2008; Elliott et al., 2009). Well-prepared facilitators of mathematics PD are essential to realizing these goals and advancing PD research (Karsenty, 2021; Lesseig et al., 2017). Over the past decade, attention has been paid to the growth of PD facilitators (Krainer et al., 2021); however, few studies offer nuanced insights and content-specific practices for *how facilitators support* productive mathematical engagement when working with teachers in PD (Borko et al., 2014). We take up this aim by examining a nested context of facilitator PD focused on teacher PD, in which two expert leaders (ELs) worked with aspiring facilitators. Knowing how and what moves to support teachers' productive engagement is vital to high-quality PD (Tekkumru-Kisa & Stein, 2017).

This study is situated in a two-phased design research project, *Researching Mathematics Leader Learning* (RMLL). The paper examines participants' collective work on one mathematics task across four small groups and two whole group discussions using a framework of *productive disciplinary engagement* (PDE; Engle & Conant, 2002). We use this framework to understand what makes "doing mathematics" in PD effective and ways facilitators might guide collective mathematical activity toward particular goals for teacher learning.

Theoretical perspective

Elsewhere we have argued that teacher learning of mathematics differs from students' learning in at least two specific ways (Elliott et al., 2009). Teachers have experienced the content at least once as students and often have revisited the mathematical ideas as teachers or PD participants. Second, in PD, teachers are working with peers and are often colleagues of the facilitator, resulting in different authority arrangements than those among teachers and students. These differences, coupled with the understanding that teachers come into PD with varied orientations toward mathematics, mathematics teaching, and learning, can create tensions as social positioning is negotiated among colleagues when doing mathematics (Holland & Lave, 2009). As a result, we have argued that a goal for teachers' mathematical learning in PD is to develop SCK (Ball et al., 2008). With this goal, facilitators must anticipate and navigate the range of ways teachers may leverage mathematical concepts while simultaneously holding teachers accountable to the practices of the discipline and supporting their authority by positioning them as competent. Facilitators must also negotiate what constitutes a mathematical explanation, how errors and uncertainty are valued, and how social relations and positioning are mediated. This study utilizes the PDE framework as a lens to consider the dynamic interplay among social and mathematical aspects of the work accomplished by aspiring facilitators (teacher leaders) in a PD setting. Our analysis provides an empirical argument for the utility of the PDE framework addressing how productive disciplinary engagement is afforded or constrained in collective mathematical work in facilitator PD.

Adult learning involves cognitive, social, and cultural dynamics inseparable from contexts and content. Consistent with this situative perspective, we use PDE to examine the activity system - actors engaged with one another, using practices and resources to enact activity - as our unit of analysis (Holland & Lave, 2009). A vital feature of the study context of facilitator PD is that these actors belonged to at least three communities of practice central to doing mathematics - aspiring facilitators, previous participants in mathematics PD, and mathematics teachers. The complexity of these overlapping communities frames the collective work of doing mathematics in this setting and the social positions occupied by participants. Awareness of the practices nested within these multiple communities is vital to evaluate the productivity of PD engagement.

Productive Disciplinary Engagement

Engle and Conant (2002) describe *productive engagement* as the intensity and extent to which learners progress on a problem over time. *Disciplinary engagement* involves working toward goals by individuals and groups using the concepts and practices of that discipline. PDE frames how we make sense of ideas, practices, and discourse when a group of aspiring facilitators solve and share solutions to mathematics tasks meant to support them in developing mathematical knowledge for teaching and leading. Engle and Conant (2002) operationalize PDE as a dynamic system of four coordinated tenets: problematizing–resources and authority–accountability, suggesting that when any tenet is more dominant or missing, PDE is in jeopardy.

Problematizing refers to efforts where disciplinary uncertainties arise. Uncertainty involves questions or indecision regarding what to do, conclude, or justify. Productivity increases as learners reflect on uncertainty and employ *resources* to alleviate it. Resources might be tools, artifacts, or practices needed to do particular kinds of disciplinary work. Problematizing is advanced when learners have sufficient resources and are encouraged to question and make proposals or challenges. As learners assert *authority*, they express their agency to define, address and resolve disciplinary problems. Authority is expanded as learners take control over problematizing the disciplinary content. Learners' authority needs to be balanced by *accountability* to how one's ideas make sense and are relevant to the work of one's peers, others, and to the discipline. PDE was conceived as a set of design principles to understand the quality of learning settings (Engle & Conant, 2002). We extend PDE's application to focus on the supporting conditions of facilitation in the context of PD. We consider the framework's utility as an analytic and design tool across facilitator and teacher PD.

Context

RMLL's five-year design-research project prepared facilitators to lead mathematical tasks in PD using a series of PD videocases to support productive mathematical discussions (Elliott et al., 2009). In six seminars in Phase I, facilitators solved mathematics tasks, discussed solutions, and analyzed facilitation features within PD videocases (c.f., Lesseig, et al., 2017). Math tasks were identical to those in student textbooks or PD curriculum. In eight seminars in Phase II, facilitators again solved mathematics tasks, discussed solutions, and analyzed videocases, but there was an increased emphasis on teacher learning goals to develop SCK. In Phase II, we redesigned mathematical tasks to foster and focus mathematical conversations on concepts and practices critical for teaching (Lesseig et al., 2017). With these "reframed–tasks," we aimed to invoke uncertainty by foregrounding mathematical structure while simultaneously providing facilitators with resources to coordinate representations as they explored mathematical patterns and offered conjectures and justifications.

Methods

We analyzed two small groups in Phase I (three and four members), two small groups in Phase II (four members each), and two whole group discussions (Phase 1, n =11, Phase II, n = 37) as facilitators worked on the *Staircase Task* (Noyce Foundation, 2005). This task is a visual patterning task in which facilitators were asked to determine the number of cubes needed to build the n^{th} staircase (i.e., the sum of consecutive integers). In Phase II, after facilitators worked on the Staircase task, they were given the *reframed* task, which pressed them to coordinate representations (visual models, expressions, and tables) to illustrate solutions to the quadratic relationship that can be generalized as [n(n+1)]/2. We were not looking for evidence of individuals' cognitive resources; instead, we examined patterns of practices and how PDE was afforded or constrained.

First, we viewed the video records to develop a consistent idea unit of leading speaker and focus. Our coding scheme captured the mathematical and social interactions in each idea unit. Disciplinary codes included claims, justifications, and representations. Social codes noted whose ideas were made public and traced how the groups took up these ideas. We also noted how individuals positioned themselves or were positioned by others. Based on this coding, we wrote summative memos of small and whole groups examining how work was accomplished in light of the four PDE tenets to illuminate similarities and differences across groups (Miles & Huberman, 1994). We report on critical ideas across phases to illustrate the interaction of facilitators' mathematical and social work (given pseudonyms initials) and the two ELs (noted as L in the transcript).

Results

Across both phases of small groups, most facilitators revealed their uncertainty with the Staircase task identifying a recursive relationship in their table (figure number and total cubes). Patterns were described as adding the figure number to the previous staircase, the sum of consecutive addends, or a generalized sequence. Facilitators' discussion revolved around various explanations for the recursive pattern and their attempts to find a formula that directly related the figure number to the number of cubes in the staircase. Most facilitators could not coordinate resources within their group to find the summation or move to a closed-form solution. A Phase I small group narrated a part of their uncertainty in the following interchange.

CK:	This might be too long and cumbersome, but I'm thinking n+n-1+n-2 ((GT saying it at the same time))
GT:	I wonder if there's another way to state that?
CK:	I don't have any idea.
LN:	But that's what it is
GT:	Yeah, but isn't there an easier way to put it?
CK:	There's gotta be.
LN:	But how do you show a sequence?
CK:	Well, I started writing out the actual numeric equation, so for 3, it equals 3+2+1, and so for 4, it equals 4+and the quantity 3+2+1. Do you see what I'm saying?…
LN:	I don't know how to show sequencing in algebraic thinking except for parentheses
GT:	Yeah, I don't know either. I can't remember.

Mathematical uncertainty made public

As illustrated above, facilitators willingly made their uncertainty public, pushing papers into the center of the table to show their reasoning and sharing different patterns they noticed. Additional resources were needed to move forward within groups where uncertainty coalesced toward the same strategy (e.g., remembering how to sum a sequence). The compressed knowledge of previous mathematics and the various ways that different mathematical concepts were coordinated perpetuated uncertainty unless further resources were recruited into facilitators' discussions.

When uncertainty was public, facilitators often positioned themselves in terms of mathematics. When attempting to generalize patterns, we heard facilitators suggest that they were "visual thinkers," not "formula queens," and label peers as "the Algebra teacher," which conferred higher status. The positioning of facilitators often provided insights into the uncertainty they faced. We also noticed that this social positioning could result in facilitators *deflecting their authority and hampering the accountability* of the facilitator's ideas to the discipline. This showed up in various ways, such as presenting strategies but not knowing how to continue because of faulty understandings ("remembering") or in compressed solutions that were not pressed on to examine the mathematical claims and unpack justifications.

A counter-narrative to the deflecting of authority and accountability to the discipline emerged when facilitators asked how a strategy works, whether it always works, and how a variable was defined. In Phase I we noted that these questions were potentially powerful, yet justification wasn't normative, and incorrect mathematical responses were not probed. However, we saw the expert leader press for accountability in the whole group by prompting facilitators to define variables and coordinate representations (visual model with expressions) to justify claims. Facilitators, in turn, challenged mistaken ideas pressed for further elaborations of mathematical concepts, and the expert leader continued to press for accountability to the discipline. This created opportunities to slow down conversations and connect resources to the uncertainty that had emerged in small groups.

The following are illustrative interchanges from the Phase I whole group.

NC: ...So first, if you start with a 3 by 3, so when I looked at that, you could just divide this. So, this would be x-squared.
L: And why is it x-squared?
NC: If you complete the whole square…(pointing to the additional squares added to figure 3 staircase).
L: So, what's x?
NC: X would be one dimension, one side. In this case, x would equal 3.

Moments later, in NC’s solution discussion

L: So, I'm curious as to why you subtract the [pointing to quantity ½ x^2 - ½ x]…
SL: Okay, so I don't get, okay, here I am again. I don't get why that's ½ x, those three little chunks.
NC: Well, x is 3 squares, right?
SL: Ok. Yes, yes.
NC: And so, ½ x would be one square and a half square so if this (points to the squares on diagonal that have been cut in half) equals x, then this (points to the portion of squares on diagonal that needs to be included to create staircase) equals ½ x.

In Phase II, pressing questions in small groups like those in Phase I resulted in accountability to one another. Yet, we also found that the reframed task served as a resource to push facilitators toward greater accountability to the discipline. As the excerpt below illustrates, we heard facilitators share solutions, solicit support to co-create solutions and revoice each other’s answers until they felt mathematically confident with the ideas.

SJ: Okay, I’ve got another one, ½ of n^2 + ½ of n. So which one of these models shows half of the square plus half of the number? How can we show that? I know it works. You guys have to help me!
ED: That’s different; let me think that through.
BK: Yea, let me write it down, say it one more time.
SJ: Half of n^2 + half of n
ED: But that just goes back to that is the same formula (pointing out that if you simplify [n(n+1)]/2, the expressions are equivalent).
SJ: But if we look at the square (SJ picks up blocks to recreate the 3 x 3 square). Here’s the square, right? So, half of that would take it to that.
ED: Plus…
SJ: Plus half of the 3 (continues rearranging the cubes to add back on and recreate 3rd staircase).
ED: So, half of the square is that much, and then the half of n is that much (pointing to subset of the blocks)
SJ: Is that what b is showing? (pointing to the square model in the reframed task)
ED: Yea, I think that is what that is showing because that is that half (pointing to the diagonal in figure b).

Here we see how the additional resources of the reframed task supported accountability to the discipline – enabling facilitators to coordinate symbolic and visual representations of the quadratic relationship and move beyond verifying that an expression works to justifying and making sense of expressions coordinated across representations. Small *and* whole group discussions were opportunities to coordinate uncertainty and resources and hold themselves accountable to one another and the discipline. Our analysis documents facilitators unpacking mathematical concepts and reasoning across representations to challenge ideas.

ELs played pivotal roles in both phases

In Phases I and II, ELs drew upon two facilitation practices to support reconciling facilitators’ uncertainty to catalyze resources. One method was to *confer authority on the facilitators to address* *their uncertainty* by inviting facilitators to ask questions, press for justification, and compare reasoning across groups. In the exchange from Phase I, below, the expert leader acknowledged the viability of facilitators' reasoning (a recursive solution) and encouraged them to seek additional resources to address their articulated uncertainty.

GT:	Yeah. We're all stuck here.
L:	Well, what you might want to do is literally go in pairs and eavesdrop on the other two tables because they've actually got two different approaches. So what I would suggest is going and just listening for a couple of minutes and then…
GT:	We have an approach
L:	You do! And I think that you might hear some things that would help you build on your approach.

A second practice the ELs utilized to support facilitators was to *reposition resources generated via facilitators' agency but not fully explored*. This practice allowed facilitators to address uncertainty and hold themselves accountable to the discipline by coordinating the closed-form expression, the pattern in their table, and a visual representation of the algebraic ideas of multiplying the dimension of an n by n+1 rectangle and dividing the figure in half.

In Phase II, the second set of coordinated leader practices emerged, *extending the facilitator's agency* by inviting facilitators to connect representations in the reframed task and *fostering facilitators' rehearsal of partial or incomplete understandings*. For example, when facilitators noticed connections between the visual rectangle model and a table showing Gauss' counting method, the expert leader invited them to explore (extend their agency) and rehearse using the fourth Staircase to articulate the correspondences across the representations.

Although some might suggest that these leader practices are not unique to PD (i.e., similar to classroom practices), we contend that because the goals for learning are geared toward SCK, they are unique from goals for students and, as a result, demand particular sensibilities from ELs that are equally different. We expand upon those differences in the discussion that follows.

Discussion

Our analysis revealed critical insights on facilitators' practices and ELs' practices. One of the necessary conditions for PDE was fostering facilitators' accountability to one another and the discipline when coordinating crucial resources. Persistent accountability amongst facilitators wasn't enough to foster PDE. Facilitators needed access to resources to hold themselves accountable to the discipline. In Phase I, ELs conferred authority and strategically repositioned resources to balance facilitators' uncertainty with resources. With access to these resources, we saw facilitators use a range of socially and mathematically effective practices such as pressing for clarification, mapping correspondences amongst representations, and creating the shared meaning of each other's methods, critical markers of PDE. Similar to Borko et al. (2014), when facilitators took up this kind of social and mathematics work, we saw groups make progress on their uncertainty and productively engage with key concepts and practices in the task.

Our results led us to claim that collective agency to press on uncertainty is a supportive condition to foster PDE. When facilitators collectively asserted their authority to share, add-on, and connect ideas, they could access resources to address their uncertainty. Further, when ELs offered opportunities to slow down conversations via practices of repositioning resources, extending agency, and rehearsing partial understanding, we saw facilitators reveal their errors, refine reasoning, and revoice solutions until they were confident with solutions. These were also spaces for peers to support accountability by asking pressing questions, pointing out insights provided thus far in an explanation, and connecting methods (Tekkumru-Kisa & Stein, 2017).

ELs' practices of conferring authority and repositioning resources were strategically deployed in ways that either re-centered facilitator authority to disrupt status or leveraged facilitator authority in invitations to examine additional resources and share reasoning. These practices pressed facilitators for accountability to one another, and the discipline, and thus were vital to advancing productivity beyond engagement (Engle & Conant, 2002). To consider PDE as a framework for teacher and facilitator PD, we must understand the ways that teachers' and facilitators' social and professional positioning frames how authority and disciplinary uncertainty afford and constrain productive engagement.

The PDE framework allowed us to understand the resources needed in PD, the role of facilitator agency and accountability, and how ELs can foster PDE. Facilitators' engagement with the reframed task highlighted the need for "just enough" uncertainty to create productive struggle alongside resources to accelerate and focus discussions on critical mathematical ideas for teaching. Strategic leader practices of extending agency and fostering rehearsals of understanding were foundational to facilitators expanding their productive engagement with SCK.

Essential to a PDE lens for PD is to consider how mathematics is framed by specialized content central to mathematics teaching and learning (Ball et al., 2008). While related to what Engle and Conant (2012) call the critical content and practice of the discipline, PDE accountability in PD would mean something slightly different. For facilitators to be accountable to the discipline, they need to leverage and connect multiple mathematical structures, examine correspondences across representations, and coordinate various solutions. The EL practices we identified in our analysis fostered this type of facilitator activity and thus advanced these goals.

Conclusion

We used the PDE construct (Engle & Conant, 2002) to examine the dynamic interactions among social and mathematical aspects of doing mathematics in facilitator PD. The study offered empirical evidence for the framework as an analytic tool for PD research. The PDE framework revealed the nuanced ways that aspiring facilitators' authority and accountability were at play as they accessed resources to address uncertainty. Moreover, we identified how ELs supported PDE via a set of strategic practices related to the four tenets of PDE. We offer insights on strategic facilitation practices of doing mathematics that can inform future studies of facilitation (Krainer et al., 2021).

While we did not use PDE as a design framework in RMLL, our emerging research suggests that "lifting" the tenets of PDE to inform PD designs is a viable lens to guide task design and practices. Our analysis of ELs' practices revealed how their support of facilitators taking up resources in response to their uncertainty and fostering authority in balance with accountability was a means for cultivating SCK. Frameworks to support PD design and formative feedback are essential for advancing content-specific facilitation across PD models (Karsenty, 2021). As a potential design framework, we can imagine how the adaptation of the PDE classroom framework could orient PD designs and advance productive social and mathematical norms while doing mathematics in PD. This study contributes insights on content-specific PD practices beneficial to a wide range of PD models where teachers engage in mathematics.

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The mathematics teacher educator as broker: boundary learning

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We analyse how two co-teaching mathematics teacher educators (MTEs) describe and justify the enactment of their theory of change in a course for practicing teachers. Applying concepts from Communities of Practice, we identify a shared view of the key boundary objects highlighted in the design of the course in our two MTEs, alongside divergent but complementary means of brokering learning at the boundary during enactment. Prominent in our analysis is a working relationship in which one MTE brokering through coordination appears to allow the other to work towards radical transformation, by seeking confrontation that allows her to define the problem space. We consider the implications of this dynamic for their emphasis on teaching as a pair.

Keywords: Boundary learning, brokers, mathematics teacher educators.

Introduction

Mathematics teacher educators (MTEs) were oddly absent from earlier research on teacher education, perhaps because they were frequently those doing the research. Although they are now a research subject in their own right (see Goos & Beswick, 2021), as Jaworski (2021) notes in her response to contributions in that volume, in many studies of MTE learning and development, “the course or programme is very much in the background” (p.420). In this paper, we bring the course into focus in an exploration of two MTEs’ theory of change as they intervene in mathematics teachers’ practice.

Theoretical/analytical framework

Many of the studies in Goos and Beswick’s collection share a theoretical orientation towards communities of practice (Wenger, 1998). Jaworski (2021) notes the usefulness of this approach as a way of understanding not only teacher learning and change but also MTE learning and development. In particular, she notes the power of the idea of boundary crossing in analysing the process of professional development, drawing on Akkerman and Bakker’s (2011) seminal review of the literature. Briefly, Akkerman and Bakker identify four mechanisms of learning potential at the boundary between practices – for example, when MTEs are university researchers, the boundary between MTEs (academic expertise) and teachers (practitioner expertise). These mechanisms are:

Identification: The differences between communities' practices are made explicit without attempting to reconcile them.

Coordination: A dialogue is established to translate between the communities, and these can co-exist without noticeable discontinuities.

Reflection: Comparing and contrasting brings about (new) insights into the practices of both communities.

Transformation: Confrontation with a problem triggers reconsideration of existing practices, resulting in recognition of a shared problem space. Sometimes this results in hybridisation, a new practice emerging creatively from the meeting of diverse practices. Finally, crystallisation is the rare phenomenon where the hybrid practice has real consequences and results in new routines.

These learning mechanisms are made possible by *boundary objects*, objects that have different local meanings but enough interpretative flexibility to allow communication across the boundary between communities (Star, 2010): an assignment in teacher education, for example, could be interpreted by students as a requirement that must be met and by the teacher educators as support for future practice.

Literature review

This framework has been used to explore how a diverse group of MTEs (academic researchers and science/mathematics teachers) worked on professional development for teachers which linked authentic workplace situations with mathematics teaching (Bakogianni et al., 2021). Analysis revealed that a range of boundary objects (tasks, course objectives, etc.) enabled a change in MTEs' learning mechanisms over time, as they moved from identification to reflection and coordination. While in this example the boundary learning occurred spontaneously, it can also be fostered by brokers (Wenger, 1998). Goos and Bennison (2018) observe that MTEs frequently enter into explicit brokering between practices, and that this is enhanced by diversity among collaborating MTEs. In their study, MTEs were either specialists in mathematics or mathematics pedagogy, and success was defined as the integration of their respective disciplinary paradigms, mathematics content and pedagogy. In this paper, we focus not on MTE's disciplinary backgrounds but on differences in how they describe enacting the shared goals in their theory of change. Hence, we ask our research question: What changes do two mathematics teacher educators set out to realise through a course for in-service teachers, and which features do they highlight as essential for realising change?

Context of study

The context for this study is a one-year part-time credit-bearing (30 credits, half of a full-time load) course for in-service teachers in a Norwegian university. Many – but not all – are primary teachers lacking the necessary credits to satisfy recent requirements for teaching mathematics. The course aims to introduce teachers to student-centred, inquiry-based mathematics teaching. It promotes a Realistic Mathematics Education (RME) approach to teaching, highlighting a number of key principles, particularly the importance of context in emergent mathematics and the transition from informal to pre-formal to formal models (e.g. Van den Heuvel-Panhuizen, 2003), and guided reinvention (Stephan et al., 2014). In this paper we report on data from two of the MTEs teaching the course, Silje and Daniel (pseudonyms). Both are experienced MTEs (Silje is more senior) who conduct research in mathematics education, and have experience as mathematics teachers in schools.

The course was designed by Silje more than 15 years ago, and has been implemented at this university (cohorts of up to 200 divided in up to 5 classes) for the past 8 years (updating the reading list, modifying the tasks, etc.) by a group of MTEs working in pairs and led by Silje. Over the years, Silje has co-taught the course with a number of MTEs. Currently, she works with Daniel, who at the start of the data collection was beginning his second year teaching the course but had previously taught a course for prospective mathematics teachers that had adopted materials from Silje's course. The two had asked to co-teach the course for the second time, as they consider their collaboration particularly fruitful.

Methodology

This paper presents an analysis of a written statement and interviews with Silje and Daniel, as part of a larger project analysing a community of practice of ten MTEs involved in the course. The idea of theory of change in teacher professional development had been raised at a research group meeting, so we asked Silje and Daniel to write down their version for this course so that we might understand the connections between their goals for teacher change and the design and enactment of the course. They did this exercise together, and we reproduce their account here, translated into English by the first author (Table 1). Silje and Daniel were then interviewed together by the first author. The interview focused on the background to their document, and Silje and Daniel's views on what they believed the teachers brought to the course and how the MTEs built on this to achieve their aims. They were also asked about the nature and extent of teacher change they hoped might happen. A follow-up interview with Silje aimed to clarify points arising in the first interview including her emphasis on having two MTEs in the classroom during teaching. Both interviews were transcribed in Norwegian and translated into English by the first author. In our translations we have aimed to keep as close as possible to our understanding of intended meaning in the original Norwegian.

We analysed the data by first identifying boundary objects in Silje and Daniel's written statement and then classifying the relevant interview extracts in accordance with Akkerman and Bakker's (2011) four mechanisms for learning opportunities. There are at least four potential communities of practice at play here: the teachers-as-students, the teachers as members of the wider community of teachers, the MTEs as educators with a reform purpose, and the course members as a whole, engaged in a joint inquiry. We limit our interest here to examining the change that the MTEs aim to achieve at the boundary between teachers as members of a wider community and their own community of education reformers. In our analysis, we looked for references to these communities and the differences between them (identification), and references to actions taken by Silje and Daniel in terms of the establishment of dialogue which aims to translate between communities (coordination), comparison/contrast between practices (reflection), and presentation of problems which disturb practice (transformation).

Analysis

In this section we report first on Silje and Daniel's written statement, identifying three boundary objects, followed by an analysis of their interview, highlighting their justification of the theory of change and their account of how the three boundary objects support their goals. We notice who introduces new perspectives, and how the other disagreed, supported or elaborated. We focus in the discussion on the relationships between them as brokers on the boundary between practices.

Theory of change (Table 1)

We identified three boundary objects in Silje and Daniel's account: research-based course literature (while only one conceptual tool is referenced [Ulleberg & Solem, 2018], Silje and Daniel draw on technical vocabulary - pre-formal methods, talk moves, learning landscapes etc - from the field of RME and inquiry learning, which they know we are familiar with); teachers' lived experience of being in the classroom led by the MTEs ('gatherings'); and written assignments on engaging with their school students' mathematical thinking ('missions'). There is an emphasis on creating a learning experience for teachers which they will mirror in their classrooms, and on understanding the student point of view. In addition to exposing teachers to close investigation of student work and of their own practice, Silje and Daniel seek to model the practice that they promote; in this sense their "theory of change" suggests that they aspire to transform in their work on the boundary by confronting teachers with new experiences which will lead to reconsideration of their existing practice.

Table 1: Silje and Daniel's "theory of change" document

Our overall goals	How
That teachers experience learning through a reform-based approach, and develop this in their own practice That teachers can facilitate practical, inquiry-based and theoretical work that nurtures and develops students' mathematical knowledge and mathematical thinking To develop teachers' dialogical approach to mathematics so that the use of talk moves and opportunities for oral mathematics increase To develop mathematical and didactic competencies so that discussion of student work is nuanced and teachers are explicit about their didactic tools. To create, as an example of practice that teachers can use in their own classroom, a safe learning environment in mathematics, where we listen to each other, dare to ask questions, dare to make mistakes, learn to persevere and give each other thinking time	By discussing authentic student work and directing attention to what the student can do, what lies on the student's closest learning horizon, and how the teacher can challenge students to develop their thinking by: • discussing the learning landscape / learning trajectories of the students, cf. RME • analysing and discussing the work of their own students in written assignments • becoming acquainted with, work with and be able to account for, informal and pre-formal methods that can eventually be used as teaching tools By using Solem and Ulleberg's model of questioning as analysis and reflection tools to develop conversational features / rich discussion in teaching By getting the teachers to work investigatively in and with mathematics throughout the course That we ourselves have a practice that reflects the overall goals of the course

Learning opportunities: bringing about transformation

The interview analysis suggests considerable complexity in this transformative aim. Asked to explain their theory of change, Silje embeds her account in her personal history as a teacher, and deep convictions about what is involved in learning mathematics. She identifies the core goal of the course as promoting mathematics teaching where students see themselves as sense-makers:

> From the moment I started teaching mathematics and discovered that people found it a very authoritarian subject – they didn't understand anything, they felt stupid – I realized that several generations were deprived of the opportunity to feel [...] that they could think for themselves.

Silje justifies the theory of change primarily based on experience, while the research perspective is secondary. Daniel relates it to both his experience as a school teacher and research:

> I initially taught as I was taught myself ... it was really these [materials] of Malcolm Swan that were a revelation to me…What occurred to me was that the students started talking in a different

> way. It led to a kind of dialogue in the classroom that one could lift up. [The materials] gave me an idea of how I could ask questions and function in a different way in the classroom.

Here, Daniel foregrounds the research (Swan, 2005), spotlighting specific elements of the theory of change (the classroom dialogue). Silje's more holistic perspective emphasises values that she sees as encapsulated in Kierkegaard's writing. Elaborating on this in the follow-up interview Silje explains:

> That's what I see in […] Kierkegaard […], to meet the student where the student *is* and to lead him by the hand ... By 'lead' [I mean] that we go together and the premises are yours. You are the starting point... your path is not the same as hers! Now it's *you* we're talking about.

This idea captures Silje's ethos, and the essential quality of the community of practice she wants teachers to become part of through the course. For her, the shift towards making the student the starting point of teaching entails a radical transformation that must start with a confrontation; she rejects a passing suggestion from Daniel that the course might rely on teachers wanting to change:

> Most of them *don't* want to change. [...] I meet them full of prejudice and assume they will convert, so to speak; [prejudice] that they come with traditional beliefs and experiences of mathematics teaching. And - as far as I can see - it turns out to be largely correct. … As [one teacher] said, it's a paradigm shift. Something happens during the first gathering, they experience something they never experienced before. [...] So, no, I don't think they need to *want to*. On the contrary.

While Silje's strong identification of difference is not at all concerned to reconcile practices, Daniel takes a less radical view, seeing participation in the course as an opportunity for development, a coordination between the teachers' present practice and the goal of the course. He doesn't see the teachers as "necessarily problematic", but concedes that "one wants to develop their ways of teaching mathematics". In this sense, Daniel speaks more readily as a broker concerned to promote dialogue and reflect on practices both old and new. Next, we analyse Silje's and Daniel's justifications of the three boundary objects - missions, course literature and gatherings – as opportunities for learning.

The 'missions'

Silje describes listening to students as a crucial but unfamiliar practice for the teachers:

> We want [...] teachers to learn to listen. Learn to see the student. Understand how this student thinks before going in with my understanding of what I think this student is thinking. That's why we spend so much time on these 'missions'. To get [teachers] to take the student perspective.

The 'missions' create learning opportunities through transformation in which teachers are confronted with a problem space they weren't aware of, valuing and building on students' surprising ideas. She stresses their role as confrontations between reform teaching and their habitual practice:

> I believe it is because of these missions, where they have to sit with the students and have to analyse what they say and think about what to answer, that they discover sometimes – and they write so in the assignments in the start – that they took the answer out of the mouth of the student. This is something they need to experience, too. You need to discover how *not* to do it. And the joy to discover how incredibly lush children are! They think about so many fun things that we've forgotten to think about. I am very happy every time I discover it!

While Silje emphasises missions as opportunities for new discoveries and sheer difference which is not (and cannot be) reconciled with existing practice, Daniel describes them as enabling coordination in the sense of demonstrating the day-to-day 'reality' of theory:

> For the teachers, it's the 'missions' that truly pull together all the threads of the course. In the oral exam they reflect on the missions and [say] that "we read about all these things, and then we did the observations – and the students said exactly the things we had read about!" It was almost a surprise for them that the theory could actually happen in their own day-to-day reality [laughs].

The research-based course literature

The role of the substantial research-based reading list generates similar differences in Silje's and Daniel's accounts of brokering. Although she values the opportunities for reflection that this brings for teachers, Silje believes it is necessary but not sufficient for impacting practice ("it's lethal to assume that literature alone can persuade teachers"), thus justifying the inclusion of missions. This is not to say that Silje thinks that engaging with literature is not important. She recalls her excitement, as a novice MTE, on discovering literature on mathematics pedagogy and its usefulness as a tool for thinking. The RME orientation of the course, too, stems from literature that fits her ethos ("I had read the books from the Netherlands and took in ideas […] – they hit me right in my thinking"). For Silje, rooting the course in research is a given, but it must connect to prior or ongoing experience. Daniel, too, is selective: since joining the course he has consolidated the theoretical aspect of the RME orientation ("I read a lot on RME. [Including] the idea of learning landscapes was a bit my influence") but stresses that the aim is not mere alignment - choices are based on their own assessment ("there is some influence from others, for instance Malcolm Swan … we are free to mix in other things").

Silje returns to missions and Daniel's coordination argument, adding that translation is bi-directional:

> If it's going to be research-based, […] reading is valuable, also to go back and get the theory confirmed. Isn't this what we try to achieve with the missions? They read something - they have the experiences from the gathering - and then go out [to do the 'mission'] and go back and read the theory again?. [...] I claim that the value is in the back-and-forth between theory and practice.

While Silje focuses on the details of this dialogic relationship between reading and experience as part of a process of reflection, Daniel focuses on specifics of what the literature can contribute, pursuing his theme of coordination as translation between practices:

> We take examples of student work ... and try to lift the conversation about what the students did, what they thought - and frame it in the context of theory: both mathematical theory, say associativity and distributivity for multiplication, but also [theory] on development. […] Where they are in a mathematical landscape. [...] Even though the metaphor is limited, it helps! Because I think nobody is in a point in a landscape, but it helps teachers to think "What are the possibilities for this student now? What can he do and how can he develop from here?"

The 'gatherings'

During gatherings, Silje is less concerned with theory, focusing more on defining the problem space by modelling the practices she promotes, with the teachers experiencing the student perspective:

> We go around and listen... It gives us an insight into the dialogue in each group […] and it teaches us an awful lot about how we interact with them - when you jumped in too soon, when you didn't really know what to say, when you were wondering about … when you left that group to make time for another. We can't teach them another practice if we aren't participating and helping them.

However, she is mindful of the danger of the experience amounting to identification rather than transformation and the need to broker the process through articulating her choices:

> We can design a session that [they] experience as very good. And it may well be good as a session where you teach mathematics. But if the goal is for them to learn to be teachers, then they need to know about the decisions behind. They can't see it unless we say it, I think.

Daniel sees the interaction between MTEs and teachers as coordination, an opportunity to translate between theory and practice. Interactions between the teachers are also important as they allow hybridisation, the emergence of new practice as they discuss their responses to students' ideas:

> When we discuss and analyse a case for example […] this dialogue becomes their own, so that when something unexpected happens they will have developed another way of thinking. The dialogue going through their heads and the questions they will ask the student will be different because they had a dialogue with others about that. [...] And then they can use it in their teaching.

Both MTEs value being two in the classroom. For Daniel, it allows identification as decisions and dilemmas become visible in a dialogue between the MTEs where they query each other's choices. Silje elaborates in the follow-up interview that the spontaneity of dialogue is key:

> I love the dynamic […] from Daniel [interrupting me to clarify] – so much fun! [...], to my saying to him "I completely disagree with you and here's why", to someone suddenly jumping in because what I said made them think of something that fits perfectly. But spontaneously! [...] We need to stand together and be in dialogue and talk to each other and talk to the audience.

Discussion: partnering in change

In this paper, we have drawn on concepts from Wenger's (1998) Communities of Practice to understand two MTEs' justification for their practice as they co-teach a course for teachers. Conceptualising the MTEs' roles as brokers (Star, 2010; Wenger, 1998) for teachers' boundary learning, analysis of their written theory of change allowed us to identify three key boundary objects: 'gatherings', research-based course literature, and missions. Interview data showed that the two positioned themselves as members of both communities of practice (mathematics teachers and MTEs) as they shifted between justifying the theory of change from the perspective of practitioners or academics, with Silje foregrounding the first and Daniel the second. Akkerman and Bakker's (2011) framework for boundary learning enabled a distinction between Silje's holistic manner of justifying the theory of change (an aspiration to radical change, to transform the practices of the teachers) and Daniel's more analytical approach picking out specific items from the theory of change (e.g. teacher questioning, Table 1) and unpacking these. The next layer of analysis identified similar contrasts in their accounts of enactment: Silje tended to initiate accounts of pursuing identification and transformation through the three boundary objects, while Daniel concentrated on coordination between practices. This characterisation of the complementary roles they take on as brokers supports their sense that their collaboration is fruitful. To return to our research question, and the issue of Silje and Daniels' justification for their overall goals of the course, these complementary roles and aims perhaps provide some indication of ways forward for understanding mathematics teacher educators' evolving practice in more detail, and the nature and extent of their role in teacher change and development.

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Towards an approach to teachers' professional development: Creating learning opportunities for teachers who teach mathematics

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This paper presents the first intervention cycle of a design-based research on teacher professional development carried out in Brazil. It aims to understand how the formative process helped teachers to understand what algebraic thinking means and how to work with it in the early years of elementary school. Data analysis was based on three principles of design concerning the Role and actions of the teacher educator, Professional learning tasks for teachers and Discursive interactions among participants. The results indicate that the design principles contributed to the teachers' understanding of the meaning of algebraic thinking and how to promote it in elementary students.

Keywords: Professional development, early algebra, teacher educator, professional learning tasks, discursive interactions.

Introduction

The development of teacher knowledge is a condition for improving the quality of education (Borko et al., 2008), and teacher education (initial or continuing) is a cornerstone of this process. Often, the professional education offered to teachers is superficial, disconnected from the ways in which they learn and far from the practice of teaching. This indicates the need to develop research that can contribute to a greater understanding and, consequently, a better organization of teacher education (McDonald et al., 2013). With a design-based research (DBR) approach (Cobb et al., 2016), a teacher education program was developed with the objective of promoting professional learning opportunities for teachers of the early years of elementary school, related to working with algebraic thinking. Thus, the present study seeks to answer the following question: *"How does the teacher educator's role in preparing professional learning tasks and conducting discursive interactions in a formative process, contribute to creating learning opportunities for teachers who teach mathematics in the early grades of elementary school?"* Although the teacher education process focuses on a specific mathematical theme, this study intends to contribute to the construction of general guiding principles for planning and carrying out teacher education processes, based on characteristics of effective professional development (Desimone, 2009) and from a model made to design and assess teacher education processes (Ribeiro & Ponte, 2020).

Theoretical framework

Over the last decades, the introduction of algebra in the school curriculum in the early years of elementary school has been a trend in Mathematics Education (Russel et al., 2011). This is because research has shown that, from an early age, children are already able to think algebraically (Blanton & Kaput, 2005) and that the development of this skill can favor the future learning of abstract algebra (Kieran et al., 2016). In this sense, teachers need to hold knowledge that enables them to work with algebraic thinking. Considering that elementary school teachers have little experience with classroom practice that can support work with algebraic thinking (Hunter et al, 2018), and how recent the inclusion of this topic in the Brazilian national curriculum is (Ferreira, 2017), continuous development programs are critical to creating professional learning opportunities for teachers (Ribeiro & Ponte, 2020). Research points to the need for five characteristics to be present for a teacher education process to be of high quality and thus effective (Darling-Hammond et al., 2017; Desimone, 2009; Kennedy, 2016): (1) focus on knowledge of the subject and on how students learn this content; (2) active learning, in which teachers participate in the knowledge construction process; (3) coherent with the curriculum and school objectives, teachers' previous knowledge and beliefs; (4) duration of meetings with continuous and intensive opportunities; and (5) collective participation, in which groups of teachers participate and build an interactive learning community.

Kennedy (2016), in addition to pointing out the importance of the motivation of teachers to participate in these programs, also suggests that their effectiveness depends largely on the pedagogy adopted (how they are carried out). In this sense, the PLOT model (Ribeiro & Ponte, 2020) (Figure 1) highlights the Role and Actions of the Teacher Educator (RATE), the Professional Learning Tasks for Teachers (PLTT) and the Discursive Interactions Among Participants (DIAP), as domains which, in an integrated way, contribute to the creation of Professional Learning Opportunities for Teachers (PLOT), in a certain context.

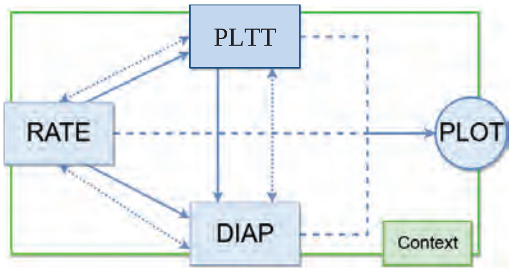

Figure 1: PLOT Model (adapted from Ribeiro & Ponte, 2020)

Although they do not directly refer to high-quality teacher education processes, the domains of the PLOT model consider many of its characteristics. The Role and Actions of the Teacher Educator (RATE), when designing formative processes that consider the characteristics of the local context (characteristic no. 3 of high-quality teacher education programs), takes into account conduction and mediation actions (Stein et al., 2008) through exploratory teaching (Ponte, 2005) in order to create professional learning opportunities for teachers (Ribeiro & Ponte, 2020). The role of the teacher educator, in addition to proposing tasks and creating a collective work environment, has to do with providing adequate and relevant feedback for each situation, ensuring the active and collective participation of all involved (characteristics 2 and 5). The teacher educator's actions and questions should contribute to the reflection on the subject, seeking to establish a relationship between theory, experience and practice (Silver et al., 2007). The Professional Learning Tasks for Teachers (PLTT), which are strongly influenced by practice, focus inseparably on mathematical and didactical knowledge (Ribeiro & Ponte, 2020) (characteristic 1), and can be defined as tasks "that involve teachers in the teaching work, which can be developed in order to meet a specific objective for teacher learning and take into account the prior knowledge and experiences offered by the teachers" (Smith, 2001, p. 8). Smith (2001) points out that materials such as videos of classes (Borko et al., 2008), student work (Kazemi & Franke, 2004) and high cognitive level mathematical tasks (Ribeiro & Ponte,

2020) portray teaching practices and can create opportunities for teachers to analyze and evaluate real classroom scenarios (Silver et al., 2007).

Discursive Interactions Among Participants (DIAP) are directly associated with collective participation (characteristic 5). It assumes that professional learning opportunities can materialize from exchanges between peers, through dialogical communication (Craig & Morgan, 2015). One way to provide discursive interactions is through the exploratory teaching approach, as it presupposes the circulation of mathematical and didactical experiences and knowledge among teachers (Ribeiro & Ponte, 2020). Some authors suggest exploratory teaching should take place in four phases (Stein et al., 2008): introduction, realization in students' autonomous work, whole-class discussion and systematization.

Research Methodology

The present study was carried out in a qualitative-interpretive perspective with Design-Based Research (Cobb et al., 2016), in which data collection from the first design cycle was carried out in a continuing education program developed in the first half of 2019 in a São Paulo State education network, Brazil. Audio and video records of the meetings were collected, as well as written records from the teachers. The participants were 14 teachers from the early years of elementary school. The program lasted 32 hours, with 8 4-hour meetings, in an in-person format, and 32 hours of individual work. The entire teacher education process was video recorded with two cameras, one focused on the teacher educator and the other on the participants. The subgroup discussions were recorded in audio and written records of the teachers and the PLTT used were also collected. The teacher educator, with extensive professional experience at this level of education, is the first author of this paper.

Considering the central aspects of effective teacher education program and the domains of the PLOT model, we established the following factors as design principles (main characteristic of a DBR) (Cobb et al., 2016)): (i) the Role and Actions of the Teacher Educator (RATE), which indicates the importance of considering the articulation between mathematics and didactics in and for teaching, the construction of an exploratory teaching-learning environment (Ponte, 2005) and the orchestration of discussions (Stein et al., 2008); (ii) the Professional Learning Tasks for Teachers (PLTT), which, by involving different practice records, promote the exploration of mathematical and didactical knowledge (Silver et al., 2007); and (iii) the Discursive Interactions Among Participants (DIAP), which, by affirming the importance of collective participation through dialogical communication (Craig & Morgan, 2015), involve teachers in an environment of reflection and discussion. Thus, the hypothesis of this research is that an intervention based on these principles contributes to promoting professional learning opportunities for teachers of the early years of elementary school regarding understanding the meaning and development of work with algebraic thinking. We used the three design principles as analytical lenses to consider the reflections brought by teachers and the ways in which the formative process provided teachers with opportunities for professional learning (Ribeiro & Ponte, 2020).

Results

Design Principle: Discursive Interactions Among Participants (DIAP)

In its design, the PLTT *Generalization* asked teachers to discuss the students' productions, analyzing their justification regarding the veracity of the mathematical sentence (Figure 2). This part of the PLTT focused both on the mathematical knowledge demonstrated in generalizations and on the didactical knowledge, considering how students think, as teachers needed to interpret and give meaning to the different justifications made by students. Regarding the first mathematical sentence in Figure 2, which the students answered is false because "*Calculations are not made, the result never has multiplications*," the teachers made some observations during the whole-class discussion:

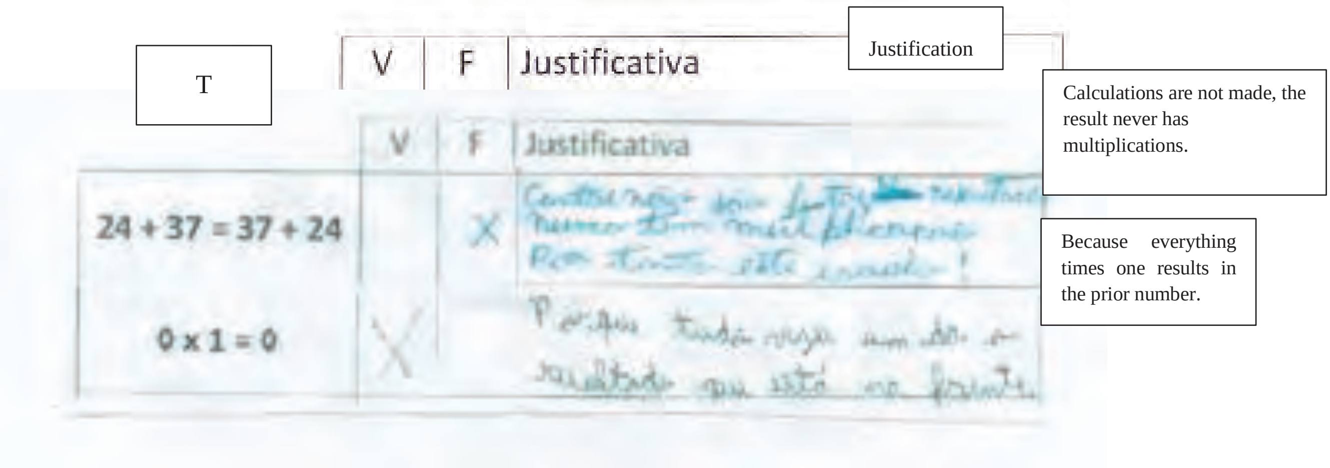

Figure 2: Part of the practice record of the PLT Generalization

Moisés: There's no coherence...
Marina: Usually, they never see calculations with prior numbers. The teacher will give them the expression, not the result. They need to seek the results.
Adriana: Wow, I didn't think of that...

For both Adriana and Moisés, the student's answer made no sense, since the justification was unusual for them. On the other hand, Marina identified that students believed that after the equal sign there could be no other expression, only a result, considering the operational perspective of the equal sign. For Adriana, this discussion provoked a new way of looking at the situation:

Adriana: I didn't think of it like she [Marina] said... in my head she didn't understood [that] if I add 24 and 37, it's the same as adding 37 and 24. She didn't understand what was proposed there, that the result can't have expressions...

By sharing her interpretation with the teachers, Marina contributed so that everyone could look at the student's response from another point of view, motivating the emergence of a professional learning opportunity arising from discursive interactions between participants (DIAP), a situation planned by the teacher educator in the design of the professional development process, from an active and collaborative process, through dialogic communication.

Design Principle: Professional Learning Tasks for Teachers (PLTT)

In the first meeting, the teachers were asked to consider the work of a few students (Figure 3) and analyze the knowledge they demonstrated and what procedures they had adopted to solve the mathematical task: *Pedro was very happy because he finally managed to complete his fourth sticker album! Each album has 225 stickers. How many stickers does Pedro have in all?*"

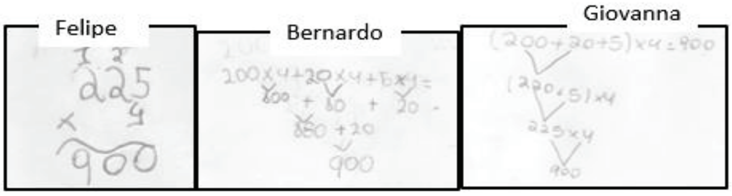

Figure 3: Records of the PLT "Analyzing a multiplication task"

After the analysis of this record of practice, carried out by the subgroups of teachers, the teacher educator asked, during the whole-class discussion, in which of these representations it was possible to perceive algebraic thinking. This questioning sought to survey the teachers' knowledge about the main content addressed by the teacher education program. Teacher Débora selected the Bernardo's representation and justified her choice by saying:

Débora: Because it demonstrates that it is a numerical expression, it has more than one operation, so it is generalizing, it left a specific calculation... See, there [in Bernardo's representation] he used multiplication and he also used sum, so he is using more than one operation to get to the result.

In this excerpt, Débora associated generalization with the presence of more than one operation, in addition to linking it to numerical expressions, expressing your initial knowledge. After working with the PLTT *Generalization* (Figure 2), Débora expressed her concept of generalization, associating it with patterns and regularities:

TE: What is generalization in mathematics?
Débora: It's when you have a situation... that serves not only a specific situation, but for more than one, there is a pattern, a regularity... for example, when they notice the regularity of the multiplication tables…

The PLTT *Generalization* prepared by the teacher educator, using records of practice, that were specially chosen to promote teachers' reflection, promoted the analysis of students' productions and contributed to the emergence of professional learning opportunities, in which Débora redefined her concept of generalization.

Design Principle: Role and Actions of the Teacher Educator (RATE)

In another situation, faced with a difficulty presented by Eliana, the teacher educator, in addition to seeking to contribute to the reflection, offered feedback:

TE: What was easier, what was more difficult, what would you do differently?
Eliana: What I think would be more difficult in this task would be reaching all students.
TE: The groups were made up of how many children?
Eliana: Four.
TE: If you start in pairs, a task for two [students]. Because in groups of four, one does the work, another helps, another pretends to help, and the last one doesn't even pay attention.

The question posed by the teacher educator, promoted reflection because the teachers needed to think about what worked (or not) and what reformulations could be made. In addition, the teacher educator established a direct relationship with the teacher's position when considering her concern, suggesting that, if students were arranged in pairs, this could favor the understanding and performance of the mathematical task. This position of the teacher educator provided adequate feedback to the situation, based on her knowledge of didactic knowledge, establishing a direct relationship with practice.

From the perspective of orchestration of the discussions, the teacher educator proposed reflections when the teachers discussed the division algorithm in a task done by the students:

Débora: [The students thought] of the multiplication tables of the numbers that were in the brackets [divisors], that any number multiplied by number 1 will always result in the dividend.

TE: But... only by number 1? [...]

TE: I would put this on the board and ask the group. Look, this group found this, do you agree, does everyone agree? But just for number 1? Because, for example, when I divided, I had this number [in the quotient], if I multiply it [by the divisor] I have this number [dividend]. This goes for any division.

Débora: I didn't notice that.

In this excerpt, the teacher educator assumes the role of guiding problematizations and calls attention to a rule, which although valid for number 1 would also be valid for other numbers, and that Débora said that she hadn't noticed. The teacher educator's action, articulating mathematical and didactical knowledge, drew attention to a mathematical regularity, causing the teacher to look at the division operation differently.

Discussion

The results show that the professional development process designed and carried out, considering the three design principles, supported the creation of professional learning opportunities in which teachers reflected on the meaning of algebraic thinking and didactics in mathematics classes, advancing in their perspectives. The PLTT *Generalization*, by presenting records of students' work on a mathematical task (Ribeiro & Ponte, 2020), focusing on knowledge of students' reasoning (Desimone, 2009), encouraged the teachers to discuss collectively (Darling-Hammond et al., 2017) the presence of generalizations in the students' justifications (Figure 2). In this case, together with the aforementioned PLTT, the other two domains of the PLOT model, the Role and Actions of Teacher Educator (RATE) when designing the PLTT and promoting the analysis of the students' records, and the promotion of Discursive Interactions Among Participants (DIAP), contributed, as stated by Débora, to reframe her understanding of the meaning of generalizations. The results show us that the design of the PLTTs and, especially, how the teacher educator led them through questions and actions (Stein et al., 2008) created opportunities for teachers' professional learning as they provoked reflections in Débora, who realized that there was a regularity in the division algorithm and how this could be worked with students. In addition, by giving a suggestion on how to proceed to reach all students, the teacher educator provided pertinent feedback (Darling-Hammond et al., 2017) on the situation posed by Eliana establishing a bridge between teachers' theory and practice (Silver et al., 2007).

In addition, during the discursive interactions, Marina presented her interpretation that for the student there can be no other expression after the equal sign. This position by Marina may have contributed to the creation of an opportunity for professional learning, by leading Adriana to make a new interpretation, showing that situations of professional learning can arise from the exchange between peers through dialogic communication (as in Craig & Morgan, 2015).

Conclusion

In addition to the design principles adopted by this study (Role and Actions of the Teacher Educator (RATE), Professional Learning Tasks for Teachers (PLTT), and the Discursive Interactions Among Participants (DIAP) pointing to the creation of learning opportunities, there is a strong interdependence between them, as both PLTTs design and its conduction are the teacher educator's tasks. The teacher educator thus gains a prominent role considering that the professional development process can be more in-depth and interconnected with practice depending on his/her actions. It is up to the teacher educator to carefully plan the teacher education learning processes (Kennedy, 2016), defining objectives, content and strategies organized in PLTTs (Ribeiro & Ponte, 2020), and to orchestrate (Stein et al., 2008) discursive interactions (DIAP), provoking reflections and providing appropriate feedback for each situation (Darling-Hammond et al., 2017). To create professional learning opportunities, it is not enough to place teachers in discussion groups (Kazemi & Franke, 2004); there is a need to carefully plan what to discuss. Thus, the teacher educator's knowledge is emphasized, both with regard to the knowledge of the content discussed in the teacher education program and to how it is led (Desimone, 2009). Although the present study has a mathematical focus on working with algebraic thinking in the early years, we suggest that the design principles that we use can be generalized to other mathematical themes or even other disciplines, which could give rise to further research.

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Facilitators’ views on content goals, learning obstacles, and teaching resources in reference to conditional probability

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Facilitators of teacher professional development (PD) courses mediate the course ideas conceptualized by the PD course designers. It is relevant to research what views facilitators hold in reference to the content goals, the learning obstacles, and the teaching resources that constitute the heart of a PD course. In our current study, we analyze one facilitator’s deliberations in an interview conducted after a PD day on conditional probability, based on a framework of expertise on the classroom and the PD level. The results reveal which elements of the PD course were adopted wholeheartedly, and which remained more or less superficial. The analysis also suggests reasons for this distinction and how to rethink facilitator qualification and PD course material.

Keywords: Professional development, facilitators, expertise, conditional probability.

Introduction: Challenges in teaching conditional probability

With curriculum changes and the growing relevance of data in the modern world, stochastics (statistics and probability calculation) has moved into the spotlight of mathematics education (Batanero et al., 2011). This has led to a growing need for teacher professional development (PD) for stochastics in general, as teachers are the agents in lesson development, just as facilitators are the agents of teacher PD. Stochastics presents a challenge for numerous reasons (Burrill & Biehler, 2011): There are uncertainties to deal with, e.g. when predicting future frequencies from probabilities. And modelling must be taken seriously; the step from reality or real data to the world of mathematics involves an awareness of idealizations, and the necessary interpretation of results requires considering restrictions of the model.

In this paper, the challenges are exemplified by the content of conditional probability, which is connected to the concept of stochastic independence and Bayes’ theorem. The common mistakes respectively misconceptions in this area (Bar-Hillel, 1983; Gigerenzer & Hoffrage, 1995; Kahneman & Tversky, 1973; McDowell & Jacobs, 2017) cover confusing condition and event, misinterpreting stochastic dependence as causality, and underestimating the relevance of the base rate $P(A)$ for the calculation of the conditional probability $P(A|B)$. Substantial knowledge of common mistakes and misconceptions is prerequisite for choosing suitable teaching resources and supporting students in reaching the respective learning goals. Conducting a teacher PD, however, is accompanied by additional challenges. While providing necessary content knowledge or illustrating misconceptions, facilitators need to address aspects which are specific for a PD, like participants’ heterogeneity or their pre-formed opinions on learning pathways. A framework for design of and research on teacher PD, the three-tetrahedron model for content-related PD research (see Prediger et al., 2019) covers these aspects comprehensively: The four corners of a tetrahedron, referring to educator, learners, content, and teaching researches respectively, are specified for three levels, the classroom level, the PD level, and the qualification level. On the PD level e.g., the facilitators are the educators of the learners, who are the participating teachers. This model also illustrates the connections between the different levels. Classroom level issues are nested in the PD level insofar as the content goals and learning obstacles for students are, together with suitable teaching resources, the content at the PD level. Therefore, facilitators should be experts in the content goals, the learning obstacles, and the teaching resources of both the classroom and the PD level. In how far this is the case and what might be the reasons behind, is the focus here.

Theoretical considerations: Expertise for teaching conditional probability

We base our considerations on a situated approach and chose a framework for teaching expertise (Prediger, 2019, adapting Bromme, 1992), which distinguishes between *jobs*, *pedagogical tools*, *categories*, and *orientations*. These concepts allow to describe and explore what teachers or facilitators focus on doing in a specific situation (*jobs*), which thinking *categories* they activate, what they utilize in order to reach their goals (*pedagogical tools*), and which *orientations* influence their choices. The thinking categories, in particular, cover specificities of the content, e.g. the procedural and conceptual learning goals, the possible learning pathways, and the learning obstacles. The framework is tuned towards the actual teaching / learning situation, with its carefully orchestrated resources and its *ad hoc* reactions and decisions.

For this paper, we focus on content goals and learning obstacles (which are parts of the thinking *categories*), and on the *pedagogical tools*, which are closely connected to content goals and learning obstacles – as teachers / facilitators choose their pedagogical tools (e.g. specific tasks or activities, visualizations, software applications) with the aim of supporting their students / the participants in their PD course in reaching the intended content goals, keeping possible learning obstacles in mind, i.e. finding ways to overcome them. In our PD setting, a content goal is to comprehend the relevance of the base rate $P(A)$ for the calculation of the conditional probability $P(A|B)$, which is often underrated when the corresponding calculations are executed with probabilities and Bayes' rule. Using natural frequencies and an easily accessible representation, e.g. a double tree diagram, can help to overcome this learning obstacle (Gigerenzer, 2011; Wassner et al., 2007). Our PD also aimed at generally promoting the use of simulations as a teaching resource, which has been shown to "have the potential to make learning statistics easier" (Lane & Peres, 2006, p. 6). On the one hand, simulations were presented to foster the frequentist view on probability. On the other hand, we introduced simulations as an adequate tool for calculating probabilities where learners' analytic means are insufficient.

The thinking categories at classroom level cover, among other aspects, the content goals that are to be addressed and the learning obstacles that might hinder reaching these goals. In the course section focused here, the content goals can be described as knowing the definition for conditional probability, understanding the sense of this definition, calculating conditional probabilities (via the definition or by using Bayes' rule or other strategies), and being aware of the impact of different base rates. At PD level, knowing how to introduce conditional probability and stochastic independence by utilizing appropriate tasks, activities, simulations, and visualizations, as well as considering misconceptions when planning lessons, would be added to this list.

At classroom level, various learning obstacles should be considered by the teacher, e.g.

- modelling issues as a result of idealizations involved in probabilistic models
- sampling issues, sampling variation and differences between population and samples
- motivation issues that might hinder students from grasping or memorizing the content,
- misconceptions like confusing condition and event, misinterpreting stochastic dependence as causality or underestimating the relevance of the base rate, and
- a high level of abstraction that could present a learning obstacle in itself.

At PD level, additional learning obstacles comprise the heterogeneity of the group of participating teachers (e.g. referring to their individual knowledge or their respective professional learning groups), pre-formed opinions on certain teaching approaches, or previous (positive or negative) experiences when teaching the same or a similar content. For example, teachers might infer from their own learning history that the use of digital technology is not worth the time needed to come to grips with it. Or they might not consider modelling issues as relevant enough to discuss explicitly.

The pedagogical tools comprise teaching and learning resources, which can have a close connection to the content goals. For the content of conditional probability at classroom level, these are: the reflected use of absolute (natural) or relative frequencies, traditional tree diagrams with probabilities, (double) tree diagrams with absolute frequencies, 2x2 tables, linguistic scaffolding, hands-on experiments, digital simulations, data from digital simulations, ideal simulation (gaining natural frequencies by using artificial population sizes), authentic problems, and problems with artificial stories. The resources listed here, taken from the PD course at hand, share the characteristic that they address conceptual understanding, rather than procedural skills (see Binder et al., 2020 for an explanation of the different tools for visualization).

At PD level, these pedagogical tools can also be utilized, in a reconfigured form with a perspective on the PD situation: For example, the content and its associated pedagogical tools from the classroom level can be arranged in possible sequences, to present a range of teaching options. Then, the PD participants can be asked to work on the tasks and materials belonging to the different options, from a student perspective. This, in turn, can be followed by group discussions to reflect upon the teaching options, led by the facilitator who can incorporate his own experience with the material into the discussion, integrating participants' concerns and misgivings.

There are various possible connections between the content goals, the learning obstacles, and the teaching resources; and teachers' or facilitators' more general (and less content-specific) orientations can reveal the underlying reasons for their interpretation and the performance of their jobs: For example, an orientation to actively address misconceptions influences the choice of an activity or a teaching resource; an awareness of modelling issues implies integrating validation considerations; the belief that language matters encourages offering content-specific language learning opportunities. The exemplary connections in Figure 1 also illustrates the interconnection of the classroom and PD level. Using (ideal) simulations in class is an adequate tool to create an awareness for the impact of base rates. Promoting the use of simulations in the classroom implies addressing their advantages in the PD by pointing out their usefulness in e.g., group discussions.

Our research goal is to learn more about facilitators' views connected to the content goals, learning obstacles, and teaching resources – so we aim to answer these questions:

RQ1: Which content goals, learning obstacles, and teaching resources are mentioned by the facilitator and how are they accentuated?
RQ2: Which connections between the content goals, learning obstacles, and teaching resources are mentioned by the facilitator and how are they elaborated upon?

As a perspective, we are interested in exploring in how far the facilitator's notions coincide with the conceptual ideas of the original PD designers. Therefore, our research interest is in knowing which orientations can be inferred from the above, in particular in reference to the facilitator's adaption of the teaching concept. This also includes exploring if and when the facilitator focuses on student learning and / or on teacher PD.

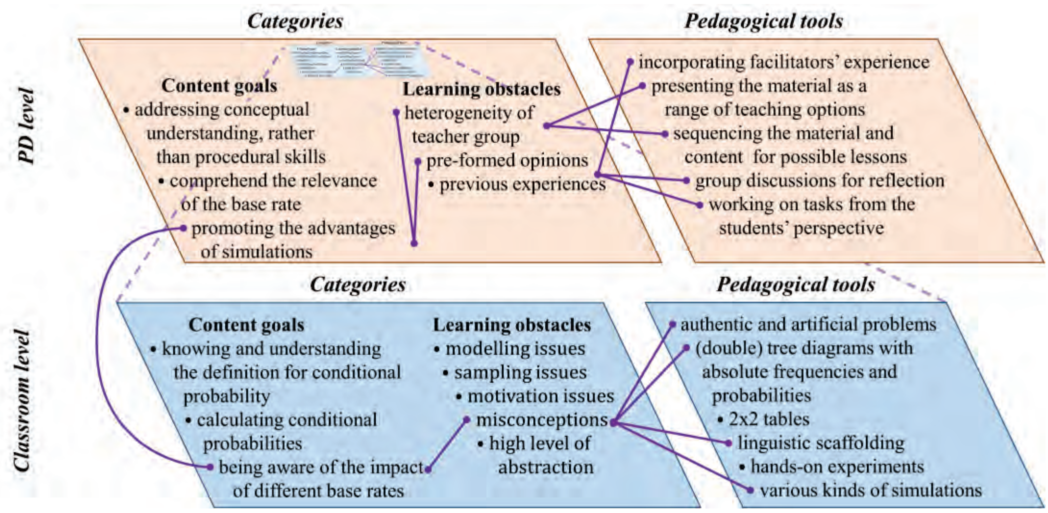

Figure 1: Nested facilitator expertise (categories, pedagogical tools) for teaching conditional probability, with exemplary connections

Context of the PD course

The PD course is part of a five-day PD program on stochastics for upper secondary level, developed at Paderborn University, Germany (Barzel & Biehler, 2017). The PD program envisions a teaching approach based on the principle of consistently promoting concept formation, e.g. via the use of simulations, digital tools, authentic examples and real applications. During the PD course on conditional probability, stochastic (in)dependence, and Bayes' theorem, the teaching recommendations focus on the use of natural frequencies, e.g. gained in simulations, and their use in double tree diagrams, which are regarded as an innovation in the German school context. A more traditional form of representation, 2x2 tables, is mentioned along the way.

The whole PD program was discussed at length and re-designed with four experienced facilitators over a period of three years, in cooperation with a regional education administration. Afterwards, the facilitators moderated the program more than once in teams of two.

Methodology

Directly after each PD day, guided interviews with the facilitators were conducted, audio-recorded and later transcribed. Among other aspects, the interviews covered the PD learning goals (both from the facilitator's and from the course designers' view), possible learning obstacles and how to overcome them, and the teaching resources offered by the PD course for the classroom level. In the course of the interview, facilitators were asked to elaborate on a printed list of PD goals.

In this paper, we concentrate on one facilitator, who we call "Mike", who was involved in the re-design of the PD program, and on the part (on day 2 of the program) on conditional probability, stochastic independence, and Bayes' theorem. Mike is male and has 16 years of experience as teacher, and 13 years as facilitator (mostly for other content than conditional probability). The interview with Mike lasted 70 minutes and has 178 turns; the interviewee's turns ranging in length between short comments of very few words and extensive elaborations of over 450 words.

The transcribed interview was analyzed in three steps: First, the passages relevant for content goals, learning obstacles, and teaching resources were identified, respectively, by the first and second author separately. Second, a consensus was reached between them about which interview passages belonged to which aspect. Third, a qualitative analysis was conducted (Kvale, 2009) in order to dissect the relevant text passages and phrase answers to the research questions.

Results

The research questions (RQ1: Which content goals, learning obstacles, and teaching resources are mentioned by the facilitator and how are they accentuated? RQ2: Which connections between the content goals, learning obstacles, and teaching resources are mentioned by the facilitator and how are they elaborated upon?) can be answered as follows (see Figure 2 for an overview):

Mike mentions the contents conditional probabilities and Bayes' theorem as the most important goals for the PD day, without specifying what exactly is relevant for these topics (turn M_002). He emphasizes that the PD concept is to promote students' understanding and argumentation skills (M_004, M_010, M_014, M_018, M_064) and sees this aspect as an indication for better teaching (M_010, M_012, M_022). There is no mention of procedural skills. Mike connects the advancement of understanding with an awareness of common misconceptions (M_010, M_018, M_111, M_121, M_127, M_165, M_171, M_175) and finds that the most relevant general problem is that "students show very many misconceptions, even with everyday relevance" (M_010), where at the same time he assesses everyday applications as beneficial for students' motivation (M_010). Mike has noticed that PD course participants often hold misconceptions themselves (M_107, M_167), so addressing misunderstandings is an issue both at the classroom and at the PD level (for the connections between the different aspects of expertise, see Figure 2). Mike does not mention simulations, an adequate tool for fostering students' understanding.

The teaching resources Mike specifies can all be located on the classroom level and mostly refer to specific tasks (M_010, M_012, M_127, M_128, M_145, M_157, M_173) that have an authentic background and touch upon the common mistakes. Other pedagogical tools that Mike mentions are double tree diagrams (M_022, M_149, M_151) and 2x2 tables (M_121, M_149, M_151), again with a perspective on classroom teaching, not on teacher PD. He explains at length that he prefers 2x2 tables (M_149, M_151) and gives as reasons that they help students to connect absolute and relative frequencies, and to bridge the transition from a tree diagram to the reversed tree diagram (M_149), therefore connecting a teaching resource to a content goal. Although double tree diagrams, as presented in the PD course, comprise didactic advantages, Mike states he would use this resource only subsequently (M_151). In addition, Mike sees the advantages of using absolute over relative frequencies (M_129, M_131) for the content goal of promoting understanding, but comments on this teaching / learning resource only when hinted by the interview material. He stresses the fact that, in a PD course, the aim is not to offer an ideal teaching approach that works perfectly in every setting (M_127), but to present a range of teaching options (often in the form of tasks, M_052) teachers can choose from. Therefore, a lesson plan is not regarded as an appropriate teaching resource at the PD level, but a collection of tasks and activities is. Mike outlines that he would conceptualize his own lessons following the principle to orchestrate an easy access, stressing connections to previous knowledge elements, and introducing more complex considerations only when students feel secure on the new ground (M_141). The scenario he refers to particularly attends to weaker students and examination situations (M_121, M_145, M_147).

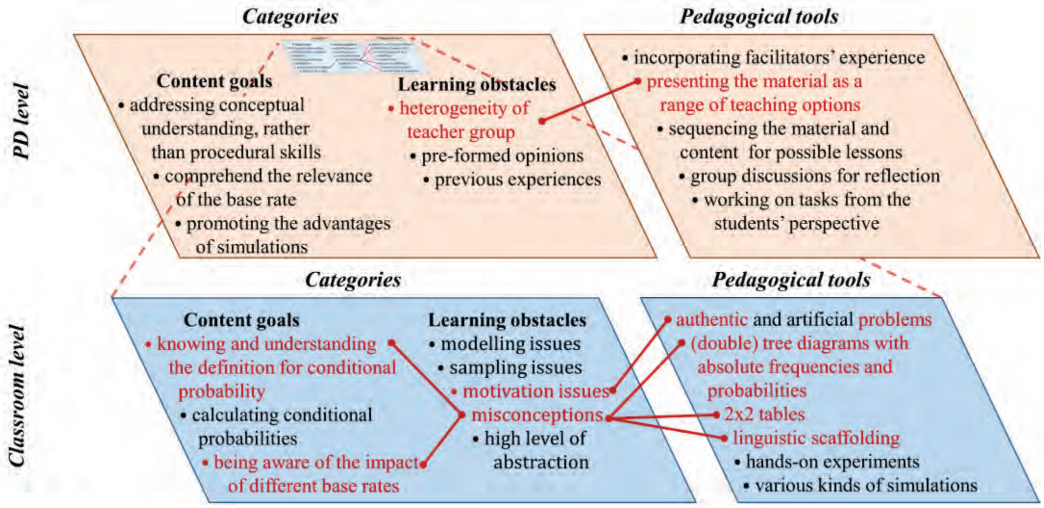

Figure 2: Aspects of expertise and their connections mentioned by facilitator Mike (highlighted)

The use of hands-on experiments or digital simulations triggers questions for Mike, as to when (or if, the German language does not distinguish this) these are helpful (M_109, M_143), and he finds that the result matters, independent of coming from a simulation or from a calculation (M_018). In this context, Mike is keen to refer to hands-on experiments (M_014), thus indicating a certain reserve towards digital simulations. More importantly, Mike always connects digital simulations with the technical skill of handling Graphing Calculators (M_024). As simulations require predetermining the number of overall experiments, Mike does not see the advantage of simulations over 2x2 tables – and he is unaware that these can indeed represent ideal simulations (M_143-145). He would utilize simulations when the probabilistic model is unclear, though (M_018, M_145).

Particularly here, it becomes clear that Mike's argumentation routinely refers to the decisions he has made or would make for his own teaching (M_097, M_145), the PD course participants do not feature in his deliberations as active agents of their own teaching. He visualizes himself teaching, not qualifying the PD participants teaching their respective students (M_141). This is a key point in our analysis, as it not only reveals Mike's self-concept of himself as a facilitator, but also provides a method to spark reflections on this self-concept (via visualizing the prevalent scenes in one's mind when leading a PD course), and categories for facilitator self-concepts (e.g. as teacher, as agent for the PD of the participating teachers, or even as erstwhile learner) in general.

Conclusion

Mike, one of four facilitators, expresses his views openly in the interview. What he does and does not mention in reference to content goals, learning obstacles, and teaching resources allows insights into a facilitator's views on the specific PD course at hand and on teacher PD in general.

Mike's utterances indicate an orientation to stick to teaching strategies that yielded satisfying learning results in the past, e.g. preferring 2x2 tables (a standard form of representation) over double tree diagrams (rather uncommon in German textbooks, but suggested by didactic research and successful teaching experiments). It is remarkable that Mike mentions of his own accord only teaching resources that he either favors (e.g. 2x2 tables), or that are both innovative *and* stand the test of him introducing them into his own lessons (e.g. double tree diagrams). He comments on other resources (like digital simulations) when these are mentioned by the interviewer, but does not introduce them into the conversation himself. This shows that he has remained skeptical of using digital simulations for improving learning in the classroom, and he states that he would only take recourse to them if there is no other way of establishing a probability.

All in all, it becomes obvious that Mike favors the perspective of focusing on the classroom level (M_026, M_125, see Figure 2). He sees the main purpose of PD in teachers discussing and reflecting on concrete teaching situations, himself as *primus inter pares* – albeit acknowledging parallels between the PD course and a mathematics lesson. Mike switches to reflections on the PD level in the later parts of the interview (M_127 onwards), but retains his focus on his own suggested teaching, and on teacher professional development only indirectly via the intended student learning.

Consequently, it remains doubtful how far certain aspects the PD course ideas have been conveyed successfully, in spite of intensive and prolonged cooperation between course designers and facilitators. It seems that orientations are not easily changed, in particular if they are based on previous experience, and addressing them should be planned very carefully. Introducing new teaching resources or pedagogical tools, on the other hand, might be presumed successful up to a certain level. And it we hope that these can impact on orientations in the long term.

What is more, Mike's focus on lessons and students leaves the issue unresolved if the PD courses he leads concentrate on teacher PD in the sense of advancing teacher expertise – which is more than reflections on advancing student learning. Mike emphasizes that teachers are presented with a range of tasks and activities to choose from or to adapt, but does not address the necessary skills for this selection or adaptation process. Ideally, these skills should be promoted during phases of discussion and reflection in the PD course, and the facilitator would disengage from the role of a colleague and view the PD course participants as individuals whose learning processes are also his responsibility.

It will be interesting to explore if this interpretation can be supported by Mike's actions and utterances during the PD course, which was audio-recorded. Although acting as a team, the analyses of the other three facilitators' interviews and moderation will probably reveal different aspects and thus paint a more differentiated picture of facilitators' views.

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Facilitator practices during PD courses in response to teacher orientations for supporting at-risk students

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The orientations that teachers possess impact their instructional decisions. Accordingly, PD facilitators need to be able to identify and respond to these orientations, especially when teachers demonstrate orientations contradictory to PD principles. Facilitators are thus faced with the challenge of balancing PD content goals and atmospheric goals. We first examine teacher orientations at the beginning of a PD. Second, as facilitators' situative goals and orientations become apparent in facilitators' practices, we examine facilitators' practices applied in response to teachers' orientations. Results reflect previous evidence of teachers' procedural, syllabus bound, short-term, and individual orientations. Our research furthermore revealed various practices facilitators use to respond to teachers' various orientations such as confirming teachers' statements, implicitly and explicitly referring to PD principles, and using examples from their own lives or instruction.

Keywords: Professional development, facilitator practices, teacher orientations.

Introduction

Teacher orientations are crucial for instructional decisions in the classroom. Thus, if teachers view, for example, mathematical proficiency as implementing different mathematical procedures, they will highlight different instructional goals in comparison to teachers who accentuate conceptual understanding (e.g. Schoenfeld, 2011; Zohar & Dori, 2003). In the PD program *Mastering Math*, which represents the context of our study, the focus centers on fostering teachers' ability to enhance at-risk students' understanding of mathematical concepts. As orientations such as diagnostic, conceptual, long-term and communicative were identified as decisive to enhance at-risk students' understanding (Prediger et al., 2019), these are the focus of the PD. Accordingly, the facilitators need to identify what orientations teachers possess and consider how they can respond in order to move forward towards the content goal of the PD. In the theory section, we first elaborate on teacher orientations for supporting at-risk students and, second, on goals and orientations as part of facilitator practices. In the empirical part of the paper, we present data from three PD courses to scrutinize teacher orientations and patterns of facilitator practices in response to these orientations.

Teacher orientations for supporting at-risk students

Teacher orientations can have consequences for supporting at-risk students, especially in building conceptual understanding. In expanding upon prior conceptualization of orientations, for instance by Schoenfeld (2011), Prediger (2019) created a framework of mathematics teacher expertise that includes orientations as one of five facets: “Orientations refer to content-related and more general beliefs that implicitly or explicitly guide the teachers’ perceptions and prioritisations of jobs“ (p. 370). Evidence on teacher orientations and practices connected to at-risk students, revealed that teachers often focus on assisting students in developing procedural knowledge and less so on constructing conceptual knowledge (Beswick, 2007; Wilhelm et al., 2017; Zohar et al., 2001). Furthermore, teachers tend to base requirements on those dictated by the school syllabus, highlight short-term mastery instead of working toward long-term goals (Moser Opitz, 2007; Prediger et al., 2016), and emphasize individual student issues (Krähenmann et al., 2019) as opposed to rich discourse including all students and their products (Karsenty et al., 2007). According to these results, the following contrasting orientations can be used to identify the practices teachers show to support at-risk students’ understanding: diagnostic or syllabus-bound orientation; conceptual or procedural orientation; long-term or short-term orientation; and communicative or individualistic orientation. Thereby, empirical studies identified orientations based on a diagnostic, conceptual, long-term and communicative approach as supportive to enhance at-risk students understanding (Prediger et al., 2019).

Goals and orientations as part of facilitator practices

In terms of what (novice) PD facilitators need to know and do to be able to provide successful PD programs for mathematics teachers, researchers have increasingly examined what constitutes necessary PD facilitator knowledge and practices (Borko et al., 2014; Lesseig et al., 2017). In view of diagnostic, conceptual, long-term and communicative orientations as the basis for the PD guiding principles and forming the content goals of the PD within the context of *Mastering Math*, the facilitators need to deal with diverse teacher orientations while targeting these PD content goals. Whether the PD content goal is reached depends on the practices that are “recurrent patterns of a facilitator’s utterances and actions” (Prediger et al., 2021) when conducting a PD. Practices are characterized by pedagogical tools facilitators use such as PD activities and their categories for noticing and thinking that can be based on the content knowledge and pedagogical content knowledge facilitators bring to the PD. Furthermore, the situative goals facilitators pursue, such as content goals or atmospheric goals, depend on facilitators’ orientations, such as goal orientation, participant orientation, or esteem for participants, and are part of facilitators’ practices (Prediger et al., 2021). According to prior case studies on facilitators’ practices in a PD, situations can occur that let different goals compete with one another and challenge facilitators to navigate between them or turn from one to another (Prediger & Pöhler, 2019). In the light of the PD program *Mastering Math*, facilitators may be faced with such a situation, in that teachers show orientations contradictory to the PD guiding principles (for example a procedural instead of a conceptual orientation), challenging facilitators to balance content goals and atmospheric goals. As facilitators’ situative goals and orientations become apparent in facilitators’ practices, we examine the practices applied in response to teacher orientations and the patterns of practices facilitators show. Against the aforementioned theoretical background, we pursue the following research questions:

RQ1: What orientations for supporting at-risk students do teachers show during PD courses related to this topic?

RQ2: What practices do the PD facilitators apply to respond to teachers’ orientations?

RQ3: What patterns of facilitator practices and underlying situative goals can be detected?

Methodology

Participants

Three PD groups (PD 1: N=17, PD 2: N=10; PD 3: N= 9) of the *Mastering Math* PD program participated in the study. The participating teachers possess different mathematical or non-mathematical backgrounds and different experiences in teaching primary students, as some attended a primary teacher education program and completed their internship, and some are teaching out-of-field. For each of the three PD groups, two PD facilitators were responsible to conduct the PD as a tandem. The facilitators are teachers themselves, and teach mathematics either in primary or secondary school. They also draw on different teaching experiences and various educational backgrounds, ranging from primary teacher education programs including an internship, to a major in mathematics or in special education. All facilitators received the same preparation to conduct the PD, then provided similar PD content and used similar methods.

Data collection and analysis

Within each PD group, a group discussion was conducted, videotaped and transcribed. Within this PD activity, teachers were asked to situate themselves in relation to the juxtaposition of all orientations (for example "It is important for at-risk students to find out if they have mastered basic arithmetic skills" for procedural orientation, or "It is important for at-risk students to find out if they have understood basic concepts and representations" for conceptual orientation).

As a first step, the individual teacher utterances were analyzed utilizing a coding system with the aforementioned teacher orientations for supporting at-risk students (see table 1) following a deductive approach. The transcripts were coded by two experts in the field of mathematics education. Intercoder-reliability over all groups showed sufficient consistency (Kappa=0.98) (McHugh, 2012). Absolute and relative frequencies are reported to quantify the orientations. As a second step, all statements of facilitators in response to teachers' statements were coded by two researchers, using an inductive approach. First, the two researches coded the transcript of PD group 1 independently. Second, the identified categories where discussed and revised, resulting in codes seen in the first column of Table 2. Third, the transcripts of the three PD groups were coded, according to the intercoder-reliability with a sufficient consistency (PD group 1: Kappa=1,0; PD group 2: Kappa=0.89, PD group 3: Kappa=0.93) (McHugh, 2012). Additionally, we coded the facilitators' practices in response to the teacher orientations and report absolute frequencies. An inductive approach was used and resulted in four categories (see first column of Table 3, applied by two coders who fully agreed).

Results

According to the analysis of teacher orientations that were visible in the group discussions in the three PDs (RQ1), some teachers had, to an extent, internalized some of the guiding principles of the PD program (see the green marked orientations), including supporting conceptual learning, focusing on long-term mastery, and establishing a communicative atmosphere that involves all students. These orientations were most visible overall in the PD groups (see Table 1). In regard to the individual PD groups, however, differences can be seen in terms of the main orientations.

Table 1: Teacher orientations overall and in the three PD groups in percentages

Teacher orientation	All PD groups (in %)	PD group 1 (in %)	PD group 1 (absolute instances)	PD group 2 (in %)	PD group 2 (absolute instances)	PD group 3 (in %)	PD group 3 (absolute instances)
Conceptual	15.56	22.73	5	0	0	12.50	2
Both	6.67	4.54	1	0	0	12.50	2
Procedural	6.67	9.09	2	0	0	6.25	1
Diagnostic	2.22	4.54	1	0	0	0	0
Both	4.44	4.54	1	0	0	6.25	1
Syllabus-bound	2.22	0	0	14.29	1	0	0
Long-term	15.56	22.73	5	14.29	1	6.25	1
Both	11.11	9.09	2	14.29	1	12.50	2
Short-term	6.67	4.54	1	28.57	2	12.50	2
Communicative	13.33	13.64	3	14.29	1	12.50	2
Both	6.67	4.54	1	0	0	12.50	2
Individualistic	4.44	0	0	14.29	1	6.25	1

While PD group 1 (n = 17) tended to display both a conceptual and a long-term mastery orientation, PD group 2 (n = 9) showed more of a short-term orientation, followed by syllabus-bound, both short-term and long-term, communicative, and lastly, individualistic orientations. PD group 3 (n = 10), in contrast, displayed all orientations with less emphasis on any particular orientation. In all of the PD groups, teachers showed contradictory orientations. Even when a teacher or some of them showed orientations that agree with the guiding principles of the PD program Mastering Math (see the green marked orientations), usually there was also at least one another teacher who revealed orientations that do not agree with the guiding principles (red marked), such as "(…) has mastered the task, even if the background knowledge may not change, but it is a success for them (…)" or who weighed the pros and cons of both contradictory orientations (orange marked) by referring, for example, to the need to take into account the different perspectives students and teachers have, such as "[…] the current material can't be understood, if there are gaps before that are too big (…) for the short-term success of the children this is here [short-term] much better and for the motivation also, but […] as a teacher I have them but then rather the longer-term development in view".

Teachers' statements that, in some cases, point to orientations contradictory to the PD guiding principles represent the starting point of the PD with which facilitators need to deal with when targeting the PD content goals. We examined the practices and patterns of practices facilitators showed in response to teachers' orientations as a next step (RQ 2 and RQ 3). The analysis of the facilitator practices in reaction to the statements made by the teachers in the three PD groups revealed a variety of practices in the three groups (RQ2). When analyzing facilitators' reactions and responses to orientations teachers articulated, the number of statements that agree, disagree or weigh pros and cons were considered (see table 2). Facilitators in PD group 1 most often needed to react to teachers' statements that were in agreement with PD guiding principles. Accordingly, the number of facilitator statements providing confirmation is the highest in that group. However, the facilitators did not provide confirmation to all of the agreeing statements, but instead took note of them, recognizable through verbalized acknowledging.

Table 2: Absolute instances of facilitator practices in response to teacher statements in the PD groups

Facilitator practice in reaction to a teacher statement that	Group 1 facilitators	Group 2 facilitators	Group 3 facilitators
is in agreement with PD guiding principles	(=14 statements)	(=2 statements)	(=5 statements)
• Acknowledging	4	2	2
• Providing confirmation	9	3	2
• Taking a neutral position	0	0	0
• Not commenting	0	1	0
• Invalidating a statement	0	0	0
weighs pros and cons of PD guiding principles	(=5 statements)	(=1statement)	(=7 statements)
• Acknowledging	0	0	1
• Providing confirmation	3	1	1
• Taking a neutral position	0	0	1
• Invalidating a statement	2	0	1
• Reinterpreting a statement	1	0	0
is in disagreement with PD guiding principles	(=3 statements)	(=4 statements)	(=4 statements)
• Acknowledging	0	0	0
• Providing confirmation	1	1	0
• Taking a neutral position	0	0	0
• Invalidating a statement	2	3	1
• Reinterpreting a statement	1	2	0

If teachers' statements were in disagreement with the PD guiding principles or the teachers weighed pros and cons, the facilitators, in most cases, invalidated or reinterpreted teachers' statements, pursuing the content goal, but also provided confirmation when teachers weighed pros and cons, preferring to keep a good atmosphere as a situative goal. The facilitators' practice of providing confirmation, even when a teacher's statement did not agree with the PD principles, is part of a pattern we identified also in PD group 3 (RQ3). Facilitators first showed esteem for participants by providing confirmation as a first step, presumably to maintain a good atmosphere. As a second step, they invalidated or reinterpreted the teacher's statement, to stick to the content goal, by referring to PD guiding principles when highlighting the important role of understanding:

Facilitator: "Absolutely. And exactly it always depends on the person. […] In the short term I totally agree with you that you should build up self-confidence and strengthen the child a bit. But the understanding is […] and according to the concept of being able to do math safely is really the absolute crux. […] I can't understand the small, I can't understand the big – never. Ability. And then this child will always have failures his whole life long in mathematics lessons. […] That we think about it that understanding is really the absolute basis."

In sum, the facilitators in PD group 1 used the chance to refer to PD guiding principles implicitly or explicitly eight times when responding to teachers' statements (see table 3) and additionally illustrated one of the principles by means of driving a car as an everyday example.

The facilitators in PD group 2 pursued another pattern to respond to teachers' statements. Only two teacher statements in this group were in agreement with PD guiding principles. The facilitators did not comment on the two statements directly but acknowledged the statement and returned to it later on to provide confirmation more than one time in order to pursue the content goal of the PD. Furthermore, when all teachers showed nonverbal agreement with the PD guiding principles, the facilitators did not ask them to provide an argument, resulting in a lower number of statements in agreement with the PD principles. Instead of asking teachers for arguments, the facilitator used the chance to praise the orientations of the teachers and additionally to refer to the PD guiding principles implicitly (see Table 3), using an alternative way to target the content goal. In the case of teacher statements in disagreement with PD guiding principles, the facilitators invalided and/or reinterpreted the statement, and, in one case, they provided confirmation. Also, in regard to the one statement that weighed pros and cons, the facilitators provided confirmation of it. In these cases, facilitators pursued an atmospheric goal instead of the content goal.

The facilitators of PD group 3 responded to one of the five statements with the PD guiding principles, agreeing with teachers' orientations by providing confirmation two times, but missed responding to four of them. Instead of directly providing confirmation with PD principles to agreeing statements, the facilitators provided anecdotes from their private lives or from their own teaching experiences. The same occurred in response to disagreeing statements or statements that weighed pros and cons (in sum 7 times, see Table 3). The facilitators referred implicitly or explicitly to PD guiding principles in response to all kinds of statements in six cases. Both contributed to maintaining a good atmosphere in the PD course. In sum, very often, they referred to examples out of their private lives and own instruction, as well as to PD guiding principles, explicitly or implicitly, pursuing an atmospheric goal. Instead of responding to teachers' statements directly, they decided to let the discussion between teachers flow, looking for statements of other teachers that invalidated the disagreeing statement. Furthermore, they avoided direct responses that invalidated or reinterpreted teachers' statements, also contributing to a good atmosphere and, at the same time, preserving the PD content goal.

Table 3: Absolute instances of facilitators reference to practice in the three PD groups

Facilitators use of	Group 1	Group 2	Group 3
• anecdotes or everyday examples	1	0	2
• examples from own instruction	0	0	5
• explicit reference to the PD guiding principles	5	1	4
• implicit reference to the PD guiding principles	3	3	2

Discussion and conclusion

The orientations teachers showed when they started a PD course on fostering the ability to monitor and enhance students' understanding of basic concepts are to some extent in line with previous evidence that points to a more procedural (Beswick, 2007; Wilhelm et al., 2017), syllabus bound, short-term (Prediger et al., 2016), and individual orientation (Krähenmann et al., 2019), instead of a conceptual, diagnostic, long-term and communicative orientation. The latter orientations build the principles of the PD program and inform the content goals facilitators need to target when conducting the PD. The study revealed various practices facilitators use to respond to teachers' various orientations. Facilitators in PD group 1 mostly invited teachers to make a statement or bring an argument to the discussion. In the light of the PD guiding principles, the facilitator succeeded in being a good role model in that he or she exemplified a communicative orientation. Facilitators in PD group 2, in contrast, let many chances pass to invite teachers into a discussion or to ask them to bring arguments for their orientation and therefore missed examining whether teachers actually have conceptual orientations or whether they hold procedural orientations. Facilitators in PD group 3 faced many disagreeing statements as well as statements that weighed pros and cons and they responded to these statements with various examples from their daily lives and their own teaching practices, combined with explicit or implicit references to PD guiding principles.

In regard to the typical facilitators' patterns we identified, first, in many cases, the facilitators avoided reinterpreting or invalidating teachers' statements directly, but started their responses by providing confirmation before invalidating or reinterpreting them, presumably in order to maintain a good atmosphere. An additional pattern was found, also probably aiming to maintain good atmosphere within the PD groups: Instead of responding directly to statements that were not fully in agreement with PD guiding principles, the facilitators in two of the groups often used anecdotes or examples of their own instruction or referred to PD principles. Recourse to examples and principles seems to help the facilitators to pursue the content goal with a simultaneous preservation of a good atmosphere.

The study reveals the variety of facilitator practices used to respond to teachers' orientations and contributes to the understanding of multiple ways and patterns that appear in facilitators' moderation strategies when conducting PD courses. In line with the results of further case studies (Prediger & Pöhler, 2019), the challenge facilitators face when navigating between different PD goals they need to manage becomes obvious; in our study, facilitators' practices in all three of the PD groups demonstrated how facilitators attempted to balance content goals and atmospheric goals. As a next step, we suggest the investigation of whether facilitators' abilities to focus more strongly on the content goal while simultaneously considering atmospheric goals can be fostered by discussing PD incidents in video clubs and probing PD situations in simulated learning environments.

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Developing personas to support professional practices of mathematics teacher educators

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Keywords: Individualized instruction, student centered learning, teacher education.

Introduction

Modern pedagogies place increasing emphasis on individualized learning to address student diversity. This is also required by new educational policies. Providing future teachers with relevant knowledge and fostering the development of their competencies to implement individualized learning settings is a challenge for mathematics teacher educators (MTEs). In this work, we suggest personas as a way to support MTEs in conveying the diversity of characteristics, needs, and conditions of mathematics learners. Personas are concise data-driven descriptions of fictional representatives of learners who share similar characteristics and needs. In our recent work, we have identified the potential of personas to facilitate student-centered teaching and learning of mathematics. We have collected data on students' goals, needs, challenges, joys, fears, and strategies, and have developed personas of students in Austrian academic upper secondary schools to provide teachers with a resource to facilitate individualized learning. In this project, we expand upon that work towards teacher education courses suggesting personas as a new resource for MTEs to facilitate rendering the development of mathematical pedagogies more student-centered and promoting individualized teaching and learning opportunities. We aim at answering the following questions with regard to mathematics teacher education: (a) Which student characteristics should be portrayed by personas to promote decisions on designing individualized teaching and learning settings? (b) What are the scenarios for applying personas?

Theoretical and methodological framework

In Austria, teachers are required to align their teaching to the diversity of students by educational policy documents. In practice, however, teachers tend to consider highly specific and stereotypical student characteristics to decide how to address diversity in lesson design (Larina & Markina, 2019). To reduce subjectivity and bias in design contexts, Cooper (1999) claims that it is beneficial to have one specific person in mind and tune the design to their goals and needs. He suggests to use personas as a portray of the target group and as a communication tool for designers when discussing design ideas and drafts.

Personas are "hypothetical archetypes of actual users […] defined by their goals" with fictional names, fictitious personal details, and a portrait (Cooper, 1999). Originally, they were used in user experience research to present characteristics, goals, skills, and interests of homogenous user groups. While Cooper (1999) introduced personas for product optimization processes with the goal to promote sales, the goal in the context of education is to enhance design processes for more efficient teaching and learning.

There has not been comprehensive research on applying persona development techniques to teacher training contexts although the use of personas has the potential to facilitate preparing teachers to implement individualization in their classrooms.

In recent research, we collected data from mathematics teachers in Austrian academic upper secondary schools about the characteristics and needs of their students and created student personas based on these data. These personas have been utilized in teacher training courses to design materials for teaching mathematics. We plan to refine the personas by conducting a quantitative questionnaire study among school students. Including the teachers' expert views and using standardized questionnaires for surveying students should help minimize subjectivity and bias. To identify potential applications of personas for MTEs, we review current approaches of MTEs to foster individualized mathematics teaching. We then propose applications of personas for MTEs based on the findings of this review.

Results and discussion

Recent literature on individualized teaching and learning states that for successful learning the demands on the students have to match their individual learning conditions (attitude, interests, level of knowledge, misconceptions, skills, self-concept) to avoid a decrease in cognitive activity; therefore, decisions on the demand level of students and on appropriate teaching and learning methods have to be made (Prediger & Aufschnaiter, 2017). As a consequence, when developing personas for use in mathematics teacher education, the learning condition of students should be portrayed.

Personas of such kind can be a valuable resource for MTEs to train how to implement individualized learning in classrooms by promoting well-founded decisions for designing individualized materials and settings for teaching and learning mathematics. In particular, personas can serve as a basis for discussions about design approaches proposed by participants of mathematics teacher education courses. Thereby, personas meet Gueudet et al.'s (2012) demand for resources for MTEs to foster collaborative work. They facilitate including the students in discussions and have the potential to establish individualization as a premise for designing learning settings while reducing the bias stemming from stereotypical beliefs of the discussants.

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Using a tool that assesses teachers' experiences of collaborative professional development to inform and improve facilitation

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Developing the complex practice of facilitation requires participation in a professional community and the use of tools that make key aspects of facilitation concrete. We report on an ongoing empirical analysis of how professional development (PD) facilitators[1] used a tool designed to provide insight into teachers' perceptions of aspects of mathematics PD that prior research has linked to teachers' learning. Findings indicate that the tool, when used in a professional community, can support facilitators to inquire into and make decisions about their facilitation practice, and to assess whether changes in facilitation result in improvement.

Keywords: Facilitators, professional development, tools.

Skillful facilitation of collaborative professional development (PD) focused on ambitious mathematics teaching is complex work (e.g., Prediger et al., 2019; Sztajn et al., 2017). Facilitators must support teachers to engage in authentic inquiry focused on mathematics, students' learning and experiences, mathematics teaching, and relations among these elements (e.g., Jaworski, 1994; Lefstein et al., 2020). Further, they must support teachers to deprivatize their practice for collective inquiry (e.g., Little, 2002), view the PD as relevant to their own instructional contexts (e.g., Putnam & Borko, 2000), and see themselves as valued members of the group (e.g., Grossman et al., 2001).

Given these demands, a critical issue for the field concerns supporting facilitators' ongoing learning and improvement of their facilitation practice (Krainer et al., 2021). Developing complex practice, like facilitation, requires *participatory* as well as *material supports* (Wenger, 1998). Facilitators deepen their practice by co-participating in a professional community with others focused on investigating and experimenting with targeted forms of practice; tools play an important role in making concrete what is valued in the intended forms of practice (van Es et al., 2014).

In this paper, we report on an ongoing empirical analysis of how PD facilitators used a tool our team designed to support facilitators to inquire into and make decisions about their facilitation practice. To our knowledge, few tools currently exist to support facilitators to *assess* and *improve* their ongoing

[1] Our use of the term *facilitator* in this manuscript denotes a mathematics teacher educator who supports the learning of practicing teachers.

practice. We focus on the case of a facilitator in a supportive context to "evoke images of the possible" (Shulman, 1983, p. 495) for using this tool to inform facilitation.

A practical measure of collaborative professional development.

We focus on facilitators' use of a tool that was designed as a *practical measure* (Takahashi et al., in press). Distinct from research or accountability measures, practical measures are intended to support practitioners in quickly gathering data about processes they want to inquire into and improve. Key characteristics of practical measures include that "what is being measured is meaningful to its users," administration of the measure and analysis of resulting data is "minimally burdensome," and "data collection and analysis processes are timely" (p. 9). Users administer practical measures at multiple timepoints as part of inquiry cycles to assess whether deliberate changes to their practice result in desired improvements, and to set goals for their future work.

Our team developed a practical measure of collaborative PD that takes the form of a teacher-facing survey and that assesses teachers' perceptions of aspects of mathematics PD that prior research has linked to teachers' learning. The measure takes teachers three to five minutes to complete and can be used across a range of PD contexts. It was designed to make connections between teachers' experiences and facilitators' practice visible and available for inquiry with others.

To develop the measure, we first identified key aspects of PD that research indicates make a difference for teachers' learning opportunities. One aspect concerns the *discussion practices* employed by a group of teachers. This includes the extent to which teachers feel able to share and revise emergent thinking, press one another for reasoning/evidence, and challenge ideas (Lefstein et al., 2020). A second aspect concerns teachers' *deprivatization of practice*, or the extent to which teachers open their own teaching practice for inquiry and see value in doing so (e.g., Little, 2002). A third aspect concerns *relevance*, or the extent to which teachers experience the PD as responsive to and possible in their own instructional contexts (e.g., Putnam & Borko, 2000). A fourth aspect concerns teachers' sense of their *membership in the community*, or whether teachers feel valued in the group (e.g., Grossman et al., 2001). See our team's annotated measure (Practical Measures, Routines, & Representations, 2021) for elaboration on each of the aspects in relation to research on teachers' learning.

We then generated initial survey items that assess the critical features of each aspect of PD. After generating these items, we engaged in 18 cycles of design, analysis, and revision to ensure that the items assessed what they were designed to measure and that they communicated well and were meaningful to teachers and facilitators. In each cycle, researchers observed a PD session and generated field notes specific to the focus of each item. The then-current measure was administered to teachers at the end of the session. The research team then conducted cognitive interviews with three to five teachers, in which they asked the teacher to explain their response choices and probed the teacher's interpretations of the items. Further, after each session, researchers shared the resulting data with facilitators to understand their interpretations of the items and whether they perceived the data as helpful for informing their practice. They then conducted a qualitative analysis of the various forms of data, resulting in revisions to the survey, including eliminating, adding, and/or modifying particular items. This process resulted in ten survey items (see Table 1).

Table 1: Overview of the practical measure of collaborative PD

Key aspects	Teacher-facing survey items
Discussion practices	1. I feel like I can share a *mathematical idea* I am unsure about with this group of teachers and leaders. ○ *Yes* ○ *No* 2. I feel like I can share an *idea about teaching* I am unsure about with this group of teachers and leaders. ○ *Yes* ○ *No* 3. I feel like I can ask others to elaborate on an idea with this group of teachers and leaders. ○ *Yes* ○ *No* 4. I feel like I can push back on an idea with this group of teachers and leaders. ○ *Yes* ○ *No*
Deprivatization of practice	5. In today's session, I felt like I could share something I'm wondering about my own teaching (examples: a question, a dilemma, a challenge). ○ *Yes* ○ *No* 6. I would be open to sharing the following with this group of teachers and leaders: (Select all that apply.) ☐ *an anecdote about what my students said or did* ☐ *an anecdote about something I said or did when teaching* ☐ *samples of my students' written work (examples: exit tickets; photos of students' work)* ☐ *a math task or activity* ☐ *video of my students solving problems* ☐ *video of my teaching* ☐ *I would not be open to sharing any of the above.* 7. I would be open to inviting members of this group of teachers and leaders to join a lesson of mine. ○ *Yes* ○ *No*
Relevance	8. Today's session was relevant to my work as a teacher. ○ *Yes* ○ *No* If yes, what did you find relevant? If no, why not? 9. I feel ready to try something I learned today in… ○ *All of my math classes* ○ *Some of my math classes* ○ *None of my math classes* If applicable, what are you planning to try? If applicable, in which classes are you hesitant or not ready to try something, and why?
Membership in community	10. In today's session, I felt like my ideas were valued. ○ *Yes* ○ *No*

Methods.

In the 2019-2020 and 2020-2021 school years, facilitators administered the measure in 18 PD sessions across 10 distinct contexts. In this paper, we focus on how one PD facilitator, Reina, used the measure to inform her ongoing work with a middle-grades mathematics department. Specifically, we ask: How does a facilitator use the measure to inquire into and make decisions about their facilitation practice? Reina's use was of special interest because she worked for an extended time with a consistent group of teachers and was engaged in professional inquiry about her practice with other facilitators. Her use provides "images of the possible" (Shulman, 1983, p. 495) for use of the measure in a supportive context to set goals and assess whether deliberate changes resulted in the intended improvements.

Research context.

Reina worked for an organization that provided ongoing, job-embedded mathematics PD to districts and schools around the USA, with a focus on supporting teachers' development of ambitious pedagogical practice and content knowledge (Lampert et al., 2013). The organization's leaders provided ongoing, structured opportunities for facilitators, like Reina, to inquire into and further deepen both their facilitation practice and teachers' learning.

As of 2020-2021, Reina had facilitated teacher PD for three years; two years as an instructional coach in a school district and one year as a facilitator with the PD organization. Prior to this, Reina worked as a secondary mathematics teacher for 28 years. During 2019-2020 and 2020-2021, Reina facilitated PD for a five-person mathematics department at a middle school in the Northwest USA. In 2019-2020, the PD focused on leading whole-group mathematics discussions; in 2020-2021, given COVID-19, the five-session PD sequence focused on how to lead discussions during virtual instruction.

Data sources.

During the 2020-2021 school year, Reina administered the measure at the end of Sessions Two (March) and Five (May). Members of our team attended these sessions, took field notes, and collected artifacts, including teachers' responses to the practical measure.

Interviews with Reina serve as the primary data source for this analysis. After both Sessions Two and Five, members of our team conducted a one-hour semi-structured interview with Reina, in which she interpreted teachers' survey responses. The PD organization's two leaders also participated in the interview. These interviews focused on understanding Reina's interpretation of teachers' responses and modifications she considered making to the facilitation of future sessions. The interview following Session Five involved looking at teachers' responses to the surveys from Session Two and Session Five, side-by-side. In addition, we conducted a follow-up semi-structured interview with Reina two weeks after Session Five that focused on understanding her background, facilitation goals, and perspectives on how, if at all, the measure informed her work.

Data analysis.

Qualitative analysis focused on how Reina interpreted teachers' survey responses and identified goals for her practice. We first generated an initial codebook assessing the range of ways facilitators used the practical measure to inquire into and make decisions about their facilitation practice, based on analysis of Reina's interviews as well as those of another set of facilitators from the broader data set. Our team then used an iterative coding process in which we independently applied codes from the codebook to Reina's interviews, discussed and resolved disagreements in our coding, modified the codebook where necessary, and then returned to the transcripts to update our coding to reflect changes to the codebook. Lastly, we turned to the follow-up interview to further understand Reina's decision-making and to triangulate with what we had identified as the range of uses of the measure.

Results.

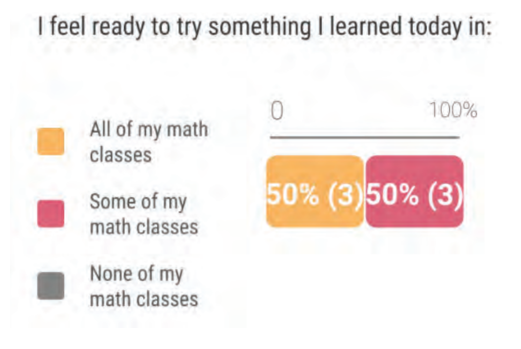
Reina used the practical measure to inquire into and make decisions about her facilitation practice in four ways: to (1) provide insight into critical and otherwise hidden aspects of teachers' perspectives and experiences; (2) prompt reflection on key aspects of the PD; (3) prompt ideas for a change in the preparation for or facilitation of an upcoming session; and (4) consider whether deliberate changes to her facilitation practice resulted in desired improvements. Given space limitations, we focus on two examples selected purposefully to illustrate this range: Reina's interpretation of teachers' responses to Survey Item 9 after Session Two, and to the same item after Session Five.

Example 1: Gaining new insight into teachers' perspectives and experiences, prompting reflection on key aspects of the PD, and prompting change.

In each of the five sessions, Reina facilitated the teachers' engagement in a mathematics task and their discussion of mathematics ideas. She then facilitated their discussion of instructional strategies for facilitating whole-group discussion in their virtual classrooms, and teachers met in small groups to plan for an upcoming lesson. During Session Two, Reina led a 1.5-hour virtual session focused on supporting students to share rough draft thinking by using "discussion frames," sentence starters to scaffold students' sharing in discussion. Reina administered the measure at the end of the session, and all six teachers present (the five department members and a student teacher) completed the survey. The next day, Reina, the PD organization's two leaders, and members of our research team met for an interview.

Reina's interpretation of teachers' responses to Item 9 is illustrative of three uses of the measure. As shown in Figure 1, half of teachers indicated that they only felt ready to try something in some of their math classes following the session. Reina read their responses to the follow-up prompts on the survey (What are you planning to try? In which classes are you hesitant to try something and why?) and said, "that [the use of discussion frames] is not applicable in classes just blows me away, you know?" She emphasized this surprised her given that the teachers eagerly participated in the session. She then reflected on her facilitation, saying:

> Those comments … are concerning to me, because I apparently haven't pressed that this is good teaching and good teaching happens every day … it's not something that we pick and choose. … I need to focus on that with this group – that this is good for all [students]...



What are you planning to try?

Teacher A: I plan to work with discussion frames for my [advanced] class. [My advanced] class works the most collaboratively, so there are opportunities to try the new skills.

In which classes are you hesitant to try something and why?

Teacher B: 4th period … they are so unwilling to participate openly

Teacher C: Algebra … we are taking tests and studying for the final … I'm not sure if it is the right time to learn from each other

Figure 1: Teachers' responses to Item 9 in Session Two (n = 6), with select open-ended responses

As evidenced here, Reina used teachers' responses first to *gain insight into critical and otherwise hidden aspects of teachers' perspectives and experiences*. She interpreted the selected open responses as indicating that some teachers saw ambitious instructional practices as appropriate only for some classes or under some circumstances. Second, she used the data to *prompt reflection on key aspects* *of the PD,* in this case her own facilitation. She focused especially on the extent to which she had supported teachers to connect the PD focus to "good teaching" which "happens every day."

In addition, we see evidence that she used teachers' responses to *prompt change in her planning for and facilitation of a subsequent session* ("I need to focus on that with this group – that [good teaching] is good for all [students]"). Reina described the changes she made in the follow-up interview. One key change concerned engaging teachers in discussing key excerpts from *Principles to Actions: Ensuring Mathematical Success for All* (NCTM, 2014). She described discussion prompts she had posed to teachers: "What are we doing when we don't provide deep rich math conversations and we exclude kids from those? What are we doing to their futures as mathematicians?" Another key change concerned her framing of the mathematical tasks teachers engaged in during the PD sessions as providing access for their students. For example, she described saying the following to the teachers as she introduced a task:

> [This task is] set up in ways that allow students to stop and think privately, share their ideas in a small group, and then come together and share the groups' ideas or individual ideas out loud. It gives [students] more access and more comfort.

Reina modified Sessions Three, Four, and Five to account for what she learned. In what follows, we illustrate Reina's interpretation of the resulting change in teachers' responses.

Example 2: Assessing whether changes in facilitation resulted in improvement.

During Session Five, Reina and the mathematics department met again virtually for 1.5 hours. They focused on learning an instructional routine aimed at supporting students' argumentation. Reina administered the measure at the end of the session, and all five teachers present completed the survey. Two days later, Reina, the PD organization's leaders, and researchers met for another interview.

In this example, we see Reina use teachers' responses in a fourth way: *to assess whether deliberate changes in her facilitation practice result in desired improvements*. Consider Reina's interpretations of the change in teachers' responses to Item 9 from Session Two to Session Five (Figure 2).

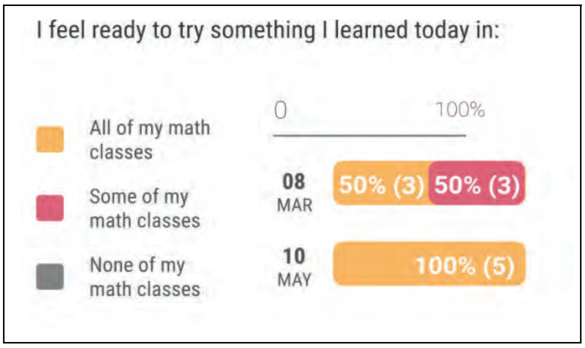


Figure 2: Teachers' responses to Item 9 in Sessions Two (08 MAR; n = 6) and Five (10 MAY; n = 5)

In seeing that in Session Five all five teachers indicated they were ready to try something in all their math classes, Reina said:

> … I'm almost relieved to see that change … It looks like we've reached a little more to [teachers] believ[ing] that they will implement this with all students, in all classes.

Reina used the longitudinal data to assess the changes she had made between Sessions Two and Five: engaging teachers in discussion about their role and responsibility, and carefully framing mathematics tasks as providing access to students. She regarded the shifts in teachers' responses as indicating that the changes she made to her planning and facilitation of the PD resulted in desired improvements, namely, that teachers increasingly viewed ambitious instructional practices as appropriate for all their students.

Discussion and conclusions.

We have provided an image of how a facilitator used a practical measure of collaborative PD designed to provide insight into teachers' perceptions of aspects of mathematics PD that prior research has linked to teachers' learning. Findings indicate that the facilitator used the measure to inquire into her facilitation practice and set goals for her future work. These findings contribute to the burgeoning literature on PD facilitators' learning by highlighting the potential of using a particular tool to inform facilitators' practice, and by association, the improvement of PD.

While these findings suggest the potential value of this tool, it is important to attend to key features of the context in which the facilitator interpreted teachers' responses. An important question for future research concerns the *routine of interacting* with the tool – the "patterned ways of engaging together" (Coburn & Russell, 2008, p. 217) which guide conversations about facilitation practice. For example, it is likely that questions posed in the sense-making interviews ("What do you notice about teachers' responses to this item?" "Why do you think teachers responded the way they did?"), as well as interjections from the PD organization's leaders, impacted the focus and quality of Reina's interpretations of teachers' responses. In future analyses, we plan to analyze the relationship between the role of the PD organization leaders and Reina's use of the data, and the extent to which the range of Reina's uses of the measure is evident in other facilitators' interpretations in other contexts. Understanding a range of uses of the tool will inform the design of supports for facilitators' learning.

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The role of pedagogical modeling in being/becoming a culturally responsive mathematics teacher educator

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Keywords: Mathematics teacher educator, culturally responsive pedagogy (CRP), pedagogical modeling, prospective and practicing teachers.

Introduction

Lunenberg et al (2007) writes, "at present one must have serious doubts about the competence of teacher educators to serve as role models in promoting new visions of learning" (p. 586). Others have suggested that teacher educators are not clear or consistent in what conceptions of teaching are most important to be modeled in teacher education (Montenegro, 2020; Timmerman, 2009). However, just as K-12 teachers cannot teach in ways that they themselves did not experience as learners (Nolan, 2014), teacher educators are also challenged to model pedagogies that they have not experienced as learners or through professional development. For mathematics teacher educators (MTEs), this idea applies to topics of reform and inquiry-based pedagogies as well as other key approaches such as culturally responsive pedagogies (CRP). I draw on Ladson-Billings (1995) to describe CRP as "a theoretical model that not only addresses student achievement but also helps students to accept and affirm their cultural identity while developing critical perspectives that challenge inequities that schools (and other institutions) perpetuate" (p. 469). The research that I describe in this poster begins from the premise that MTEs must develop their *own* CRP as an essential step toward working with prospective and practicing teachers (PTs) to develop *theirs*. That is, MTEs need to grow their own practice as culturally responsive pedagogues (Nolan & Keazer, 2021).

Research Study and Methods

This poster describes a study I conducted while teaching a "CRP in Mathematics Classrooms" course to a group of practicing and prospective teachers (PTs) who were enrolled in a *Teaching Elementary School Mathematics* certificate program. The study was designed to explore PTs' understandings of CRP at various points throughout the one-semester course. In total 38 students took the course over three offerings, with 31 students consenting to participate in the study and allowing their course journal assignment to be used as data; additionally, nine of these 31 participants agreed to participate in a post-course interview. This poster focuses on data gathered from one interview question in the study where I asked about my role as the course instructor and how/if I modeled CRP in the design and teaching of the course: *In addition to teaching about CRP in mathematics, one of my goals is to teach through CRP. Can you think of any ways that I modeled CRP through teaching this course?*

Data and Discussion

Analysis of this one interview question, posed to the nine research participants, yielded several themes which express how (according to the students) I successfully modeled CRP to my students. Examples of these themes are: My use of distributed expertise model (guest speakers invited to present on topics within their realm of experience and/or research); how I positioned myself as a learner in the same field as I am instructing; I conveyed an image of CRP which clearly illustrated that there are many ways to define and enact CRP (what it means and looks like is as varied as the classrooms involved); I made a strong effort to privilege the voices of students throughout the course, from a pre-course survey on student needs/desires through to an opportunity to 'personalize' the course through an open-ended project.

Viewing the data through a critical lens, interesting points can be noticed. For example, the post-course interviews occurred within a few weeks of course completion, so one wonders if these PTs had internalized and processed for themselves what CRP might look like in their own classrooms as teachers, let alone have the capacity to reflect at the level of someone else's classroom practices.

Closing Thoughts

Regrettably, the technical-rational practices teacher educators are expected to model are more widely studied (Aleccia, 2011) than teacher educator pedagogies which promote an agenda grounded in equity, social justice, and culturally responsive aims. To promote such an agenda, teacher educators are called upon to model pedagogies that "challenge inequities and social injustice rather than… projecting a vision of an ideal school" (Mills, 1997, p. 39). Critical analysis of this research study's themes in light of current research on CRP in mathematics teacher education and K-12 schools will be presented in the context of MTEs modeling CRP.

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Designing professional development for mathematics teacher educators

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Keywords: Teacher educators, professional development, higher education.

Two decades ago, Tzur (2001) was surprised to notice that research on development of mathematics teacher educators was almost non-existent. More recently, Lloyd (2020) notes that research on mathematics teacher education still tends to emphasise the prospective teachers or participating teachers more than the mathematics teacher educators. Still, there is little support for teacher educators, and no common education for mathematics teacher educators. MatRIC, which is a Centre for Research, Innovation and Coordination of Mathematics Teaching in Norway, has collaborated with the Norwegian Centre for Mathematics Education to develop a national study programme for mathematics teacher educators. The study programme is designed to be two academic years, part-time and gives 30 ECTS points. This poster presents the design of this programme.

The study programme is designed to include teacher educators with varied backgrounds, so that they together can work and develop their teaching of pre-service teachers in collaboration. The pilot of the programme started in August 2021, with 26 participants from twelve different teacher education institutions in Norway. Some participants have PhD's in applied or pure mathematics, but little or no experience from teaching mathematics in schools. Other participants have taught mathematics in primary or secondary school for many years but have less formal education in mathematics. We brought them all together with the aim of developing their teaching of pre-service teachers in mathematics in collaboration.

The education of mathematics teachers involves mathematics as a scientific field, the practice field, and the mathematics education research field. The goal of this programme is that teacher educators will develop the competence to establish strong(er) connections between these three fields. Participants in this programme will gain insight into central mathematical ideas and central themes in mathematics education. Furthermore, the programme will engage participants in research into a variety of approaches for mathematics teacher education. In collaboration, participants will discuss and research mathematics teacher education, and the goal is for participants to develop their own practice as mathematics teacher educators. Participants will undertake a research and development project with their own pre-service teacher students. This will entail the development of research design, generation and analysis of data, and the production of a research article.

The programme in Mathematics Education for Teacher Educators is divided into four parts.

Part 1: Mathematical thinking in the foreground. Participants will collaboratively:

- investigate "big" ideas in mathematics (e.g., functions, associative property, relation)
- investigate pre-service teacher's understanding of big ideas in mathematics and relate these to pupils' learning
- investigate teaching of big ideas in mathematics teacher education – and relate these to pupils' learning

Part 2: Practice in the foreground. Participants will collaboratively:

- investigate the nature of mathematics teachers' practice and how this may be developed
- investigate practices in teacher education that support the development of meaningful engagement in mathematics for pre-service teachers and pupils – and relate these to pupils' engagement, learning and experience
- analyse, reflect on, and develop teaching and mentoring in mathematics teacher education

Part 3: Research (mathematics education) in the foreground. Participants will collaboratively:

- Investigate some major research themes in mathematics teacher education, for instance through self-study or action research
- Reflect about how research in mathematics teacher education can be applied in participants' own teaching in mathematics teacher education
- Investigate and reflect about how one's own research can be integrated within teaching and supervising in the education of pre-service mathematics teachers

Part 4: Individual research project. Participants will collaboratively:

- Research and undertake systematic inquiry into their own practices
- Author texts that can be further developed into publishable articles
- Discuss and contribute towards each other's research projects and texts

The quality of the study programme – in particular how it was received, and to what extent participants consider it relevant for mathematics teacher educator practice – will be evaluated and explored by conducting interviews and questionaries among the participants, in addition to the reports and research articles written by the participants. The first cohort of the programme is considered a pilot, and it will be evaluated and further improved. However, this is a study programme which is designed to be able to last beyond the extra funding from MatRIC on the pilot, and where the teacher institutions see the value in investing this for their employees.

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Unpacking Critical Friendship between Mathematics Teacher Educators: an opportunity for professional learning

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While the community highly values critical friendship experiences between teacher educators, there is little evidence regarding what happens within such experiences. In this study, we report a Self-Study developed by two mathematics teacher educators (MTE), focusing on their critical friendship sessions on preparing and implementing a sequence of classes by one of them. Data collection occurred by recording three critical friendship meetings conducted at the end of implementing a sequence of classes related to quadratic equations. Results discuss different types of questions focused on understanding MTE's pedagogical reasoning and teaching practice. On the other hand, the analysis of the corpus allowed characterizing their level of depth (descriptive or analytical) and the type of content (professional or didactic-mathematical). These findings enable unpacking the critical friendship relationship and invite to explore in-depth the interactions between MTE and critical friend and the possible impact on both professional learnings.

Keywords: Mathematics teacher educator, critical friendship, reflective practice, pedagogical reasoning, professional development.

Introduction

Several studies, especially those referring to the practice of teaching (Beswick & Goos, 2018; Castro Superfine & Li, 2014), have reported that mathematics teacher educators (MTE) have a crucial role in the development of complex learning in their prospective teachers. Through their teaching practices, MTEs offer prospective teachers opportunities to learn mathematics as their future students should learn that subject (Chapman, 2008). As a result, the quality and effectiveness of mathematics teacher education depend on mathematics teacher educators' expertise (Goos & Beswick, 2021; Ping et al., 2018). However, studies about the development of mathematics teacher educators and the challenges they face in their work are scarce. Masingila & Olanoff (2021) highlight that most mathematics instructors in teacher education programs do not have professional support or training in becoming teacher educators. Although several studies have noted that studying one's teaching practice is a powerful tool for teacher educators to reflect on and improve their practices (Schuck & Brandenburg, 2020), Liang et al. (2019) claim that MTEs have little opportunity to study and develop their teaching practices.

To contribute to this discussion, we present a part of a Self-Study to go deep into the reflection on the practice of a novel MTE promoted by a critical friend (Schuck & Russell 2005), a colleague experienced in mathematics teacher education. This kind of collaboration in self-study is crucial for going beyond their initial background and expertise. In fact, this strategy supports the practice of reflecting on their own teaching as an inherent part of the work of mathematics teacher educators (Chapman, 2008). However, little is known about the practice of being a critical friend, exploring how to manage the questions, tensions, and dilemmas through appropriate support. To understand this collaborative process, we seek to answer the following research questions: what kind of questions does the critical friend promote to elicit the MTE's thinking? From the conversation with the critical friend, what characterizes the reflections of the MTE? This self-study is expected to unpack the reflection generated in a critical friendship between a novel and an experienced MTE and give new instances of reflective practices.

Conceptual Framework

In studies on the professional development of MTE, the notion of reflective practice as a means of linking theory and practice has been relevant (Goos & Beswick, 2021). In this way, different studies state that self-study allows MTEs to improve their teaching practices (Schuck et al., 2008; Schuck & Brandenburg, 2020). For example, Tzur (2001) researched his own professional path, tracing his experience as a mathematics learner, mathematics teacher, mathematics teacher educator, and mentor of fellow mathematics teacher educators, identifying crucial events and experiences that advanced his professional knowledge and practice. Schuck et al. (2008) point out that teacher educators can improve their teaching practices through reflective practice and learning conversations with critical friends. In the literature on self-study, critical friends can be from the same discipline area or different areas. They can be colleagues at different stages of their careers (Schuck & Russell 2005), and commonly reflection on practice involves collegial expert-novice partnerships (Goos & Beswick, 2021). On the other hand, an explicit goal of MTEs' research of their practice should be self-understanding and professional development (Chapman, 2008). Therefore, we need to include in the research how the teacher educator-researchers reflected, inquiring in different kinds of questions reflective practices are developed in the interaction between MTE. To what extent can this practice become unpacking and sharing with other mathematics teacher educators to implement this process as a professional learning practice.

Methods

This paper reports a collaborative research project in two voices based on a critical friendship between Author 1 and Author 2 and is part of a broader self-study. Samaras (2010) states self-study as the critical examination of one's actions to achieve a more conscious mode of professional activity. Hence, it is a self-focused and interactive approach that seeks meaning rather than solutions to a specific problem. In this research, Author 1 who takes the role of critical friend is an experienced university-based MTE. He teaches method courses in primary and secondary mathematics teacher education programs connecting permanently theory and practice in mathematics education, and investigates the role model practices held by mathematics teacher educators. Author 2 is a school-based novel mathematics teacher educator who teaches a method course (Mathematics Teaching and Learning) in a secondary mathematics teacher education program and works as a practicum supervisor of prospective teachers. The double role of Author 2 makes him permanently connect his teaching as a MTE with his practice as a school-teacher.

Data collection occurred by recording three critical friendship meetings before and after implementing a sequence of classes related to quadratic equations and functions teaching. In weekly meetings, Author 1 and Author 2 engaged in critical dialogue around MTE's teaching experience. During the first session, the focus was on establishing critical friendships and exploration of the issues which the MTE wanted to address in this self-study. During the second session, the MTE had to communicate the planning of classes that would be the object of analysis in the study and discuss the same with the critical friend. Finally, the third session addressed implementing the previously analyzed classes, delving more deeply into various topics.

The data analyzed were the transcripts from audio-recorded meetings. We conducted a thematic analysis (Braun & Clarke, 2006) to identify themes related to personal and professional tensions in Author 1's experience of becoming a mathematics teacher educator and his teaching practice. For doing so, a constant comparative method (Corbin & Strauss, 2015) was used to code the data, starting with open coding iteratively. A second coding round was conducted to collapse codes into themes such as recurrent questions, recurrent reflections, and foci of reflection. A third coding round was completed using the refined codes to understand where those themes were expressed or addressed. Several cross-data triangulations were made across this coding scheme, with the collaboration of another researcher (Author 3) to validate codes, themes, and consistency (Cohen et al., 2000). Also, we discuss the main finding with the whole research team.

Results

Based on the analysis conducted, we were able to identify various types of questioning undertaken by the critical friend, and diverse types of answers and reflective practices used by the MTE.

Role of the critical friend: delving further into some assumptions

In general, we observe two major modes of asked questions during dialog and discussion. The first mode, referred to as questions asked to attempt to understand pedagogical reasoning, focuses on the MTE's reasoning about the phenomenon under discussion, promoting explanations regarding adopted decisions. Instead, the second type of questions, identified as questions attempting to understand the MTE's teaching practice, focuses on getting the teacher educator to establish relationships between his experience and decisions made that lie at the basis of his teaching practices, either in a higher education or primary or secondary school context, both at the present time as well as within other time frameworks.

We distinguished three types of questions that focused on pedagogical reasoning: Exploration questions, Discussion questions, and questions aimed at Deepening Insight. Exploration questions sought to open up topics of conversation meant to enable the MTE to begin his reflective practices dealing with said topics. For example, in the first work meeting, the critical friend makes the following statement to get the teacher educator to clarify what he expects to get out of these collaborative work sessions.

CF: Then, the first issue is what it is you would expect to get out of a process of joint reflection such as this self-study. ¿What is it you would expect based on what we have already read and gotten to know of this methodology regarding the meetings with the Critical Friend?

On the other hand, Discussion questions enabled the teacher educator to comment on his reasoning about making decisions, or on the topic under discussion. A clear example is a discussion about the content of classes. The critical friend questions the MTE about the reasons he must incorporate the concept of the didactic variable into his classes.

CF: And why is the didactic variable interesting?
MTE: Why is the didactic variable interesting!? [CF: Yes, why?] In other words, the didactic variable, just as a concept who cares whether it is called didactic variable or whatever it's called.
CF: OK, but why does dedicating a full two-module class to this content seem interesting to you?

Finally, the last type of question was the one aimed at deepening insight, which attempted to get the MTE to reflect on his own explanations or reasoning regarding the phenomenon under discussion. For example, a broad debate about the teacher educator's role as a model develops in the second session, and about what the latter thinks is being modeled in his classes. In this context, the critical friend engages more extensively in reflective practice and formulates questions that enable him/her to delve deeper into the MTE's thinking.

CF: [...] when you are teaching the class on equations or on some other mathematical subject you have taught, what is it you meta-communicate when you teach your classes? You told us you had classes that were more about the content of primary and secondary level math, others that were more about professional performance, such as planning, etc. When you are in classes that are more about primary and secondary level math content, what do you expect to model, or what have you seen yourself modeling?

From the point of view of focus on practice, we distinguish three types of questions: those having to do with Personalization, with Experience, and with Assumption. In the personalization questions, the critical friend seeks to inquire into the MTE's motivations or personal connections with the subject under discussion or into decisions he has made for the class. During the first session, for example, discussions take place in connection with theorizing about certain teaching practices the MTE feels are effective, and the critical friend seeks to understand how those practices affect him.

CF: And how do you feel about that when you are standing in front of our students?
MTE: How do I feel about needing a theory to back me?
CF: Mmm? Do you feel more confident, do you feel calm?

Another type of questions is those aiming to evoke the MTE's practice. These experiential questions made it possible for the teacher educator to bring into the conversation different past experiences or those from other contexts, to be discussed in terms of the phenomenon in question. During the first session, the discussion develops around the modeling role the teacher educator feels he has for his students. To inquire why this is relevant, the critical friend invites the MTE to engage in reflective practice about his experience.

CF: Can you think of an episode in which you were impacted by the lack of coherence on the part of your teacher educators? At what point does [coherence] begin to be important for you?

Finally, the third type of questions focusing on practice is the one having to do with assumptions. These are questions in which the critical friend puts the MTE in situations he/she has not necessarily lived or experienced. The purpose is for the MTE to project a decision or reflection about a phenomenon, in spite of the fact he denies having lived that experience in particular. In the following quote from the third session, we observe an episode where this type of question is being posed, and in which the teacher educator and critical friend discuss what happens to students when subjects are left open in class, which is a habitual strategy of the MTE.

CF: But if you had a different group next year, this could happen to you, right? [MTE: Of course] Because in the final analysis, going beyond the group, your style of teaching leaves things open. So, there could be a group with which that doesn't work … [MTE: How would I address that? what would you do in that case?]

Types of teacher educator reflection practices during critical friendship sessions

It was possible to identify two types of reflection practices dealing with questions and discussions triggered by the critical friend. On the one hand, in terms of depth, we found some reflective practices to be superficial, where the MTE did not engage in introspection with respect to what was being discussed, as opposed to some deeper ones in which the MTE engaged in introspection regarding the phenomenon under discussion. On the other hand, there were reflective practices related to the type of content of the MTE's reflections, some related to professional aspects, and others of a didactic-mathematical type.

With regard to level of depth, the quote below comes from a moment in which the MTE gives an account of what he expects to get out of this critical friendship relationship and of the presence of questions that trigger the surfacing of beliefs. Even though one can observe a certain positioning on the part of the MTE, he does not delve much into the types of questions that he would like to explore.

MTE: No, that seems perfect to me, but I would specifically add to that [...] that if the change with respect to certain beliefs is made explicit, beliefs about which we have no prior idea what they are going to be, but if there is some change or at least a certain…if certain doubts arise in me, at least that. Because I don't expect to clarify here all the doubts, I may have [...] but perhaps, to the contrary, generate more questions.

Nevertheless, when the MTE continued his narrative, we observed an example of increased analysis and depth in his reflective practice, in which he contrasted his vision of the teaching role as a guide with the student's autonomy. This vision reflects the evolution of the MTE's role as a teacher educator resulting from his professional experience, from staking his experiences, expectations, and challenges on that role.

MTE: I believe that I have become aware that, at least during certain moments, independently of classroom contexts, the student is to be the protagonist. I need to be more like a guide [...]. I believe that generating autonomy, I don't know if it can be said that way, but I believe that generating autonomy of thought or freedom of thought [...] I believe in that in my role as a teacher –I am thinking of the classroom – I believe that generating that freedom is of key importance.

On the other hand, in terms of the specific content of the MTE's reflective practice, we observed an increase in statements having to do with professional aspects such as classroom experience related to curricular decision-making, or expectations regarding the vision of teaching that future teachers begin constructing during their formative years. Didactic-mathematical aspects, such as decisions with respect to teaching and learning functions, or a design proposal for a specific mathematical task, were discussed to a lesser extent. The following quote is an example of a MTE's professional response while alluding to the importance of installing decision-making and questioning as a permanent professional practice, especially when working with student teachers:

MTE: Then how that decision-making act has an impact on designing a mathematical task with a specific purpose. The understanding that the fact of posing certain questions conditions your teaching, conditions the way you will be planning your teaching, and what the impact is I am going to have on my students. I believe that is what I would mostly expect to get out of this session with them. And what I would like to remain with, is the knowledge of having evidenced that the above [process] is taking place.

In connection with statements aimed at didactic-mathematical topics, the following quote is extracted from a moment in the second session in which the MTE analyzes the notion of a didactic variable and his role in decision-making:

MTE: That in itself is what is interesting, that one can handle that, manipulate that deliberately with a purpose, depending on the student, and that it makes an impact in the way that student learns. That's the point. So, I understand that the didactic variable is an answer to that process. I believe that it is transcendental. What happens is that if you have not been taught what the didactic variable is, that does not necessarily make you consider it or not consider it. But it is important to make the student aware that this is present in their decision-making in the manner in which they design a mathematical task. It is present, and it is important to consider that.

Discussion and Conclusion

The results obtained show that the critical friend (Author 1) as well as the MTE (Author 2) develops a variety of strategies that allow them to build the critical friendship. On the one hand, the MTE is capable of articulating different levels of depth in his reflective practices and of diversifying their content in order to address professional aspects of a teacher's work, as well as specific aspects of the teaching and learning of mathematics. This makes it possible to enrich Chapman's approach (2008), to the extent of specifying the ways in which to examine the dilemmas of classroom practice. The questions voiced by the critical friend offer the teacher educator an opportunity to consider different points of view and to thus question his beliefs (Schuck & Brandenburg, 2020). At the same time, these results allow us to offer an unpacking of internal dynamics at work in the critical friendship relationship between MTEs. Table 1 summarizes the different types of questions posed by the critical friend and the different types of reasoning displayed by the MTE.

Table 1: Categories of questions and answers tables

Purpose of the Questions (Critical Friend)		Types of reflective practice (Mathematics Teacher Educator)	
Understanding pedagogical reasoning	Understanding teaching practices	Level of Depth	Content
Exploration Discussion Delving into the topic	Personalization Experience Assumption	Descriptive Analytical	Professional Didactic-Mathematical

The relationship between questions and answers, and definitely between the purpose of questions posed by the critical friend and what the MTE displays during the conversation in connection with said questions, becomes a strategy that makes it possible to provide support to the professional learning of both actors within the critical friendship relationship. At the same time, it enables them to be aware of the personal resources they make available during the discussion, which they can project onto their own experience, thereby contributing to the improvement of their teaching practices. In addition, critical friendship is a tool that causes the teacher educators involved to evolve, as happened in this case in which they went from a dynamic based on questions to one in which they shared experiences, both mathematics teacher educator and critical friend having engaged in reflective practices that can potentially impact their teaching practices.

Although this study has contributed to unpacking some crucial aspects of critical friendship, it is important to establish the limitations and scope of the findings. For example, this study focuses on a particular case; therefore, it does not allow it to be generalized to the broader group MTEs. Nonetheless, the findings of this study may contribute to starting new discussions about generating spaces for MTE professional learning supported by collaborative work among them. Based on our experience in this self-study, and particularly in this critical friendship, we consider it necessary to articulate the existence of these relationships between teacher educators of the same program, safeguarding the time and conditions to make them happen. In addition, inclusion of various profiles of MTE has turned out to be an enriching element in the ensuing discussions and their analysis, which leads us to think that critical friendship may be used among other actors (practice tutors and students, teacher educators and students, etc.) for purposes of improving and growing the complexity of their professional learning.

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Ignoring, upholding, redirecting, provoking: Ways of enacting norms in a video-based professional development

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The enactment of norms is a key challenge for professional development (PD) facilitators, particularly in video-based programs where teachers tend to be judgmental. This study follows seven novice facilitators of a video-based PD program that aims to promote reflection on practice while downplaying criticism on the filmed teachers' actions. We describe how the novice facilitators enacted norms and responded to teachers' judgmental comments, in order to unpack this challenge. The findings show that although the facilitators underwent the same preparation, considerable variations were found in their decision-making concerning this issue. We describe possible reasons for the different decisions, and suggest implications.

Keywords: Novice facilitators, PD norms, video-based PD.

Background

All social interactions are built on certain norms, which constitute an implicit or explicit contract by which participants agree upon what is acceptable, what is less acceptable, and what is definitely unacceptable (Forsyth, 1995). In mathematics education, classroom norms have been thoroughly investigated (e.g., Yackel & Cobb, 1996), including teachers' instructional norms (e.g., Herbst & Chazan, 2011). By contrast, norms in the PD context have received less attention. PD courses for practicing mathematics teachers are based on social interactions, and as such, are also conducted according to norms that are meant to be accepted and shared by the PD facilitator and the participants. Facilitators are the dominant actors in introducing and maintaining these norms (Karsenty et al., under review), since they design the sessions, lead them, and have the authority to indicate what courses of action, comments, and directions to pursue are acceptable and valuable for the discussion. They often do so according to guidelines, and perhaps also tools or moves, provided by the PD program they facilitate. Since evaluative comments are prevalent in teachers' talk when watching other teachers' lessons (Coles 2013; Jaworski, 1990), the enactment of norms is a key challenge for facilitators in video-based PD programs (Karsenty et al., 2019). It follows that it is even more challenging for novice facilitators who need to make decisions during unfamiliar situations. When novices facilitate PDs to their colleagues, as often occurs in the upscaling process of a PD, the enactment of norms is also shaped by their sense of credibility and their multiple identities as teachers, colleagues, and facilitators (Knapp, 2017). In this paper, we focus on this challenge by exploring the following research questions:

How do novice facilitators of a video-based PD respond to teachers' judgmental comments? What underlies facilitators' decisions with respect to this issue?

Context

The above questions are investigated in the context of a PD project called VIDEO-LM (Viewing, Investigating, and Discussing Environments of Learning Mathematics), developed at the Weizmann Institute of Science in Israel. The program aims to enhance secondary mathematics teachers' reflective skills, along with their mathematical knowledge for teaching (MKT; Ball et al., 2008), via collective guided analysis and discussions of videotaped lessons (hereafter VLs) of unfamiliar teachers. A six-lens framework is used to focus participants' observations and analysis of VLs (Karsenty & Arcavi, 2017), including: *mathematical and meta-mathematical ideas* in the lesson; the filmed teacher's *goals*; the *tasks* used; the *interactions* in the lesson; the filmed teacher's *dilemmas and decision-making*; the filmed teacher's *beliefs*. To enhance reflections and decenter criticism, the project team determined several core norms for these discussions (adapted from Karsenty & Arcavi, 2017, p. 438-9): (1) Maintain a non-evaluative and respectful conversation about the filmed teachers, assuming that they are acting in the best interest of their students and that they have the knowledge needed for teaching the observed lessons; (2) Instead of criticizing the filmed teachers, practice "stepping into their shoes" in order to understand the goals and beliefs underlying their decisions; (3) Discuss alternative teaching decisions not as better or worse courses of action, but rather as a way to enrich the span of possible options while considering the *gains and losses* involved; (4) Under the assumption that there is no one best practice, observe lessons not as models to imitate but as rich artifacts that are aimed at stimulating discussions on issues of teaching; (5) Substantiate arguments raised in the video-based discussion, for example by using evidence from the VLs. To achieve the project goals as well as the local goals of the groups they work with, the facilitators of VIDEO-LM PDs choose which videos, lenses, and activities to use in each session. It follows that two central roles of the facilitators are *designing sessions* and *leading discussions* around VLs, and that the VIDEO-LM context is a rich setting to explore the research questions posed above: firstly, since discussion norms are central to the program design, and secondly, because facilitators have the latitude to choose how to maintain these norms, and to monitor the extent to which they allow them to be breached.

Method

Participants. This study is part of a broader research project, consisting of a multiple case-study investigating seven novice facilitators, who are also mathematics teachers, in their first year of practice. During the VIDEO-LM project's upscaling, new facilitators who previously participated in the PD as teachers were recruited. They were prepared in a one-year course and were supported by a personal mentor and by facilitators' group meetings during their first year of facilitation. All the facilitators led school-based yearly VIDEO-LM courses in 2016-17 that lasted 21-30 hours, spread over 6-11 sessions. From the seven novice facilitators (named hereafter FacA, FacB, etc.), five facilitated the PD in their school (FacA-FacE) and two were external facilitators (FacF and FacG).

Data collection. To examine facilitators' decisions as well as their own view of them, the following data was used for this paper: (1) journals written by facilitators before and after each PD session, where they responded to guiding questions regarding goals, decisions, challenges, and more; (2) videos of two PD sessions per facilitator, one early in the year and another towards the end of the PD; (3) videos of stimulated-recall interviews (SRIs) held with every facilitator a few days after each of the filmed PD sessions. In these SRIs, the facilitator and the first author jointly watched the PD videos, and the facilitator was asked to stop the video whenever s/he noticed a decision to reflect on.

Data analysis. All the research project's data was coded according to five macro-categories which we named *issues of facilitation*, one of them is "enacting norms", namely, how facilitators introduced norms and responded to judgmental comments. The data presented in this paper are those that were coded under this issue, namely, transcripts of (a) PD discussions where facilitators initiated the enactment of norms and/or responded to their breach, (b) facilitators' reflections on these episodes in SRIs, and (c) reflections in journal entries. The PD sessions were further segmented according to decisions, using the methods presented by Schoenfeld (2010). The criterion for coding a sequence as a "decision" was that it refers to *something the facilitator initiated* that consists of several turns that allow for capturing the context and meaning. To analyze facilitators' decisions we used the ROGI framework (Karsenty, et al., 2021) comprising *Resources*, *Orientations*, *Goals*, (ROG, Schoenfeld, 2010), and *Identity*, as defined by Gee (2000): "Being recognized as a certain 'kind of person', in a given context" (p. 99). By employing these four constructs and pointing to their interplay, we constructed interpretations of the facilitators' decisions. We found four different ways in which facilitators responded to judgmental comments, and possible reasons underlying these decisions. Although facilitators may perform different decisions during one session, we assigned each PD session to one of the four decisions according to the most common decision identified in the session.

Findings: ways to respond to judgmental comments, and their possible reasons

Despite the centrality of discussion norms in VIDEO-LM and the fact that all the facilitators were highly familiar with them (first as PD participants themselves and then in the VIDEO-LM facilitation course they took), considerable variations were found when addressing this issue. Four different decisions were identified: (1) The facilitator **leaves judgmental comments unaddressed**; (2) The facilitator **strictly upholds** the norms; (3) The facilitator **redirects** judgmental comments; (4) The facilitator **deliberately provokes** criticism to stimulate the discussion. In the presentation of each decision we mention which cases were assigned to it: the numbers "1" and "2" refer to early and later sessions, respectively. For example, the notation FacB2 refers to FacB in her later PD session. Three sessions out of 14 analyzed (FacE1, FacC1, FacF2) were omitted since no judgmental comments were raised by the PD participants. Below, each decision is described using examples from different cases, (yet, due to space limitations, we do not represent all of the identified instances in each decision). The notation "I1-4" refers to the facilitator's 1st SRI (interview), line 4. "T2" refers to Teacher 2.

Leaving judgmental comments unaddressed (FacA2, FacD1, FacB1)

This decision, which was identified in three cases, relates to facilitators' non-enforcement of norms and avoidance of dealing with judgmental comments. When such comments appeared, the facilitators either ignored them or tried to move on to a different topic. This decision stemmed from one of the following reasons: (1) limited resources to handle such comments, coupled with ambiguous orientations on how and when to enforce norms ("When I heard these comments, I felt really bad, and I didn't know how to relate to them, how to react", FacD, I1-158); (2) an orientation that the norms should be introduced gradually ("This is only the second session, I want to let them get things out", FacD, I1-178); (3) For a facilitator with a strong *colleague identity*, an aspired goal of maintaining good relationships with the teachers ("I constantly remember that not all teachers participate in the PD with the highest desire and motivation. Since I am their colleague, I want to acknowledge their position", FacB, 2nd Post-session journal); (4) a goal of letting teachers express themselves, in order to ensure their cooperation later on ("I think it was a good decision to let him say whatever he wanted […] afterwards there were some parts where he contributed enormously to the discussion", FacB, I1-59,61); (5) fear of reacting to judgment raised by the teachers by using further judgment towards the teachers themselves, which might lead to an unpleasant atmosphere ("I don't want to be involved in confrontations […] I'd rather avoid such frictions", FacA, I2-6).

Strictly upholding the norms (FacA1)

Interestingly, although norms are essential in the VIDEO-LM design, this decision – in which the facilitator does not allow the violation of norms – was assigned only to one session out of 14. The decision was expressed as follows: a) mentioning the norms explicitly before screening a VL; b) reacting *immediately* when a norm was breached. Both these sub-decisions can be seen in the following illustrative examples from FacA's early PD session:

48	FacA:	Last time I said we have norms of discussion, I said that what guides us is respecting those who stand in front of us [the filmed teachers], but I did not define what "respect" is. When I say "respect" [...] it's the state of mind I want you to get into every time we watch a lesson: assume that whoever is standing in front of us, especially since they knew they are going to be filmed, and they prepared the lesson – assume that they always act in the best interest of their students. And we are not supervisors, we are not instructors, we are not here to evaluate them, we only want to see things that happened in their classes and learn from them.
495	FacA:	[Context: the facilitator responds to a PD teacher that criticized the filmed teacher for only writing the positive solution $x = 13$ for the equation $x^2 = 13^2$] Why do you think, why didn't he [the filmed teacher] correct it? Obviously, he knows that it's [supposed to be] ±.

The main reason underlying this decision appeared to be FacA's goals "to make the teachers assimilate the VIDEO-LM language of gains and losses" (I1-182) and "to have a non-judgmental discussion" (2nd Pre-session journal). These goals, together with Josh's strong adherence to the VIDEO-LM resources and values ("my definition of what a good [PD session] is […] it's whether during the discussion the issues from the Observer's Guide[1] appear", I1-273), have probably caused him to respond immediately to every violation of norms. This decision resonates with the approach that norms should be clear from the outset (Coles, 2013; Jaworsky, 1990) to determine the direction the discussion will take.

Redirecting judgmental comments (FacC2, FacD2, FacE2)

Whereas the previous decision relates to an immediate reaction to non-compliance with the norms, *redirection* is subtler; here the facilitator gradually steers the conversation into ascribing goals to the filmed teacher's actions, using open-ended questions that are directed to the entire group.

[1] The Observer's Guide (OG) is a document linked to each VL in the VIDEO-LM website, that includes a suggestion of how the VL may be analyzed using the six lenses.

Nonetheless, these decisions often go together: the facilitator may reassert the norms and then try to redirect the discussion. The decision to redirect allows for other participants' opinions to be heard, including those which contradict the criticism that was voiced. However, this entails the risk that the non-enforcement of norms may lead to an increasingly judgmental discourse. The following discussion from FacD's later session is an example of a successful redirection:

[Context: The teachers had just observed a teaching episode from an 8th-grade probability lesson that consisted of games. When analyzing the mathematical ideas in one of the games, the filmed teacher asked the students to share their strategies and wrote their answers in a table. In the following PD discussion, Teacher 2 and Teacher 6 criticize this move].

366	T2:	Filling out this table was just a waste of time.
367	FacD:	Filling out this table was a waste of time, what do you think?
369	T1:	No.
370	FacD:	Not a waste of time, why?
371	T1:	Kids love to give their answers. If you were to ask me what my strategy for the game was, I would want to share it very much.
372	FacD:	That was his [the filmed teacher's] consideration, letting all the students share their strategies?
373	T6:	[...] [in a criticizing tone] I think that if he had used technology here, he could have gotten faster and clearer results.
374	FacD:	Why do you think so?
375	T6:	If everyone were sharing their data in a common document [...] he would have seen it [the table of strategies] right away.
376	FacD:	Still, what... what's the gain in what he did?
378	FacD:	What does this allow?
379	T1:	That everyone can share.

The judgmental comments in Turns 366 and 373 were followed by the facilitator's open-ended questions (Turns 367, 370, 374, 376, 378), which encouraged the voicing of a different opinion (Turns 369, 371, 379), even if expressed by only one teacher. The facilitator herself was pleased with the course of the discussion when observing it during SRI-2:

I2-148	FacD:	[Refers to T2's comment in Turn 366] She prepared the groundwork for me [laughs]. Because many teachers think that filling out the table or discussing multiple strategies is a waste of time, and I think that's one of the nicest things in this VL.
I2-149	GS:	This teacher said something that you objected to, so why did it make you happy?
I2-150	FacD:	Because it was an opportunity to see how others feel and let the others think about it too. Also, she herself [T2] could have tried to answer [my question], I don't know if she thought about it that way.

FacD's articulations indicate that she views judgmental comments as a resource for the discussion that enables her to put out feelers about the issue at hand and encourage multiple opinions. Therefore, according to her orientation, such comments should not be immediately rebutted, as the ensuing discussion may provide an opportunity for the critical teacher to change her mind, in light of the other teachers' comments and the open-ended questions posed by the facilitator.

Deliberately provoking judgment (FacB2, FacF1, FacG1, FacG2)

This decision relates to elicitations of judgmental comments, for example by deliberately choosing a controversial VL, by asking judgmental questions, or by probing a teacher who seems dissatisfied with a VL. Here are two examples from FacF's early session:

88	FacF:	[Turns to a teacher who made gestures of dissatisfaction while the group re-watched a 20-minute segment from a VL] Before we talk about the lenses, I want to start with you, Teacher 1. To hear about your experience of watching the same segment for the second time.
92	FacF:	I'm just terribly curious about this question, Teacher 2, I'm looking at you [...] you seem to be a little opposed.

While these excerpts may have different interpretations than the one suggested above, FacF's reflection on his PD video shows that his goal was to stimulate the discussion:

I1-76	FacF:	I was ready for criticism on the VL because my mentor had prepared me. I wanted the teachers to be evaluative, because criticism elicits conversation, and that was needed. Here, I saw Teacher 1 sitting uncomfortably [while watching the VL], and I wanted to get him agitated, I wanted them all to get angry, to get upset, to talk.

A similar stance was expressed by FacG in her final session. In SRI-2 she reflected on a PD episode where the group was 'sleepy' and she used provocative questions (such as "are you sure the students understood what the filmed teacher did there?") to enliven the discussion:

I2-44	FacG:	It was very important to me that they would not just give me the answer that they thought I wanted to hear. I wanted critical thinking. [I asked them if all the students understood] and they answered "yes, they all understood", and I thought, '*are you sure* all these 30 students understood?' That's what's important to me, to constantly elicit their thinking.

This decision lets facilitators control (to some extent) the volume of judgmental comments in the discussion, in the sense that: (1) they will not be caught by surprise when criticism emerges; and (2) the very fact that the facilitators themselves bring up the controversy may reduce teachers' antagonism ("I decided to tell them in advance before watching, 'you are going to squirm in your chairs', because otherwise, [...] they would have been even more judgmental and the discussion would not have been productive", FacB, I2-48). Regardless of this sense of control, facilitators may have little idea as to what to do with such comments when they appear. Thus, a further decision is to prepare for the kind of comments that may emerge, including thinking about possible responses to use in real-time, as FacB described in SRI-2:

I2-62	FacB:	This VL makes people uneasy […]. I wrote in my session plan that the teachers would probably ask "what is she [the filmed teacher] doing this for?". [...] My role as a facilitator is thinking about what criticism could come up, and thinking what answers I can give.

All the above excerpts suggest that as in the previous decision, the facilitators hold the orientation that judgmental comments are a useful resource that provides the spark to kindle a lively and engaging discussion. What distinguishes this decision is that the facilitators are those who initiate or elicit judgment, with the aid of prompts they purposefully chose.

Synthesis and implications

This study set out with the aim of shedding light on how novice facilitators enact norms within a specific video-based PD, in particular how they respond to criticism, which is known to be an inhibitor of teachers' learning and reflection (Coles, 2013; Karsenty & Arcavi, 2017). The four decisions presented above show different responses to judgmental comments that were identified within seven novice facilitators' practices. There are several similarities and differences between the decisions, that can illustrate the complexity of deciding on a course of action: The last three decisions are similar in the sense that they all show how facilitators steer evaluative talk into a discussion about gains and losses which can lead to teachers' reflections. However, they differ in terms of the facilitators' capacity to withstand judgmental discussions, or, the extent to which facilitators see these comments as fruitful. The first decision, i.e., facilitators allow for judgmental comments but do nothing about them, may look on the surface very similar to the decision to redirect judgmental comments: in both cases, criticism towards the filmed teacher is enabled. However, in the *redirection* decision, judgmental comments are used by the facilitators as a resource, thus the liability to the program's norms and the filmed teacher is preserved. In the first decision, in contrast, the criticism is never addressed. Overall, the findings contribute a delineation of different ways to execute the PD's main goal, suggesting that there are multiple "best-practices" to do so, which are shaped by the different contexts. The ROGI analysis assisted to show the complexity of decision-making during facilitation, especially for newcomers who do not have well-established scripts to work by. For each decision, we described various underlying reasons, which are related to the goals of facilitators (e.g., to teach the VIDEO-LM language), their orientations (e.g., criticism can advance the discussion), their resources or lack thereof (e.g., inability to respond immediately to judgment), and their multiple identities (e.g., a facilitator who is also a colleague who prefers to maintain a pleasant atmosphere rather than get into confrontations while enforcing norms). Identifying the complex considerations underlying facilitators' decisions contributes to a better understanding of novice facilitators' practices. Accordingly, this work suggests immediate implications for facilitator educators: (a) to acknowledge that facilitators' practices are shaped by multiple elements. Thus, alongside supplying them with adequate resources for the enactment of norms, it is worthwhile to ponder on their orientations and goals with respect to this issue and to understand what challenges and affordances are generated by their identities as teachers and colleagues; (b) to discuss the idea that there are various ways to accomplish the PD goals, each carries its own gains and losses; (c) to delve on how teachers' criticism can enhance the discussion and turn into a resource for facilitators; (d) to discuss the possible consequences of a non-enactment of norms. In general, it could be of benefit for the field to understand more on how norms may influence mathematics teachers' learning, and on the facilitators' role in this process.

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Designing asynchronous video-based professional development for mathematics teacher educators

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The Video in the Middle (VIM) project is creating forty two-hour video-based professional development modules that can be combined in a variety of ways to form personalized pathways that meet the unique needs of a wide range of professional learning settings and contexts. The VIM asynchronous modules are designed to be used in three flexible facilitation formats: locally facilitated, expert facilitated, or independent/non-facilitated. VIM modules aim to support teacher noticing of student thinking and increase their mathematical knowledge for teaching linear functions. Preliminary research results indicate that teachers appreciated the variety of facilitation formats, found the online modules useful and engaging, and noticed, compared, and analyzed a variety of visual and numeric methods for solving linear function problems.

Keywords: Mathematics teacher educators, facilitation, online professional development, video-based learning, teacher noticing, mathematical content knowledge.

Introduction

Incorporating video within a professional learning environment offers great potential for mathematics teacher educators to support teachers in unpacking the relationships among pedagogical decisions and practices, students' thinking, and the disciplinary content (Borko et al., 2011). With video, teachers can observe and study the complexity of classroom life, reflect on their own instructional decisions, and to integrate multiple domains of knowledge to solve problems of practice (Blomberg et al., 2013). Recent reviews of the literature on video use in professional development (PD) point to the value of video as a tool for improving instructional practice (Gaudin & Chaliès, 2015).

As video technology and online video sharing have become more accessible and widespread, video-based PD is well-positioned to leverage the benefits of digital platforms (Teräs & Kartoglu, 2017). Online platforms can allow teachers access to professional learning resources that may not be available to them locally and can also support those who are reluctant to share ideas in face-to-face settings in becoming more comfortable doing so in digitally mediated interaction. Online PD is considerably more scalable than comparable face-to-face PD, and in many cases is subject to fewer monetary and logistical constraints for teachers (Killion, 2013). Research to date on online PD has shown some positive effects for teachers, even compared to face-to-face formats (O'Dwyer et al., 2010). Most research comparing online, and face-to-face versions of PD has found that well-designed online courses utilizing high-quality learning materials intended for individual use can produce learning outcomes that are like or better than face-to-face options (Fishman et al., 2013).

There is a general recognition of the critical role facilitators play in leading PD and the need for knowledgeable PD facilitators, leaders, and coaches (Bates et al., 2011). As PD shifts to address challenges such as COVID-19, facilitators are increasingly engaging with online platforms. To flexibly respond to teachers' complex and rapidly changing circumstances, new types of facilitation roles will become necessary (Koellner et. al., 2022). This paper reports on the design and preliminary findings from a project that is adapting face-to-face mathematics PD materials to an asynchronous digital format that was pilot tested with mathematics teachers in three facilitation conditions (local facilitated, expert facilitated and non-facilitated) to examine the impact on teacher and student knowledge. The paper will focus on findings related to the three facilitation conditions.

The video in the middle project

The Video in the Middle (VIM) project is designing and researching asynchronous PD modules. The asynchronous format allows participants access to PD at any time, in any location, and can potentially eliminate the often-mentioned roadblocks to participation—lack of scheduling flexibility and geographic distance. The VIM project draws upon the face-to-face *Learning and Teaching Linear Functions: Videocases for Mathematics Professional Development* video and ancillary resources (e.g., lesson graphs, transcripts, mathematical and video commentaries) to develop 40 two-hour modules intended to develop teachers' noticing skills and mathematical knowledge for teaching linear functions. These modules offer flexibility by allowing mathematics educators to design a variety of module sequences to fit their professional learning needs.

Conceptual frameworks

The design and development of the VIM asynchronous modules are conceptually grounded in two main bodies of research related to teacher learning in PD. First, the development of professional knowledge that consists of deep and connected mathematical content knowledge, the knowledge of students' thinking and how students learn the content, and knowledge of pedagogical practices and norms to support student learning. Second, the development of a professional vision that consists of teachers' ability to notice, analyze, and reason about features of student thinking and classroom interactions. In this section, we briefly discuss these two research areas with a focus on how they relate to the design and impact of the VIM asynchronous PD.

Mathematical knowledge for teaching

Ball and colleagues have identified and elucidated "mathematical knowledge for teaching" (MKT) as the professional knowledge that mathematics teachers must have to do the mathematical work of teaching effectively (Ball & Bass, 2002). This conception of knowledge of mathematics for teaching is multifaceted and includes both content and pedagogical content knowledge. MKT includes a sophisticated understanding of effective instructional practices and student thinking related to specific mathematical content and comes into play during all phases of teaching. For mathematics teacher educators, incorporating video within the learning environment supports opportunities for teachers to develop their MKT by unpacking the relationships among pedagogical decisions and practices, students' work, and the disciplinary content (Bloomberg et al., 2013). Collectively viewing and discussing video clips allows for the complexities of classroom practice to be stopped in time, unpacked, and thoughtfully analyzed, helping to bridge the ever-present theory-to-practice divide and support instructional reflection and improvement. The VIM module design incorporates MKT by providing multiple and varied experiences to examine and compare a variety of mathematical methods and representations, and to analyze the complex relations between content, pedagogy, and student thinking.

Professional vision and noticing

One unique aspect of mathematics teacher educators' knowledge is their "professional vision", which refers to their ability to notice and analyze features of classroom interactions, make connections to broader principles of teaching and learning, and reason about classroom events (Seidel & Stürmer, 2014). Over the years, diverse conceptions of noticing have emerged in the literature, but in general most discussions of mathematics teacher noticing involve two main processes: (1) *Attending* to particular events in an instructional setting (i.e., teachers choose where to focus their attention and for how long) and (2) *making sense* of events in an instructional setting (i.e., teachers draw on their existing knowledge to interpret what they notice in classrooms) (Sherin et al., 2011). Sherin et al. (2011) argue that these two aspects of noticing are not discrete, but rather interrelated. Teachers attend to events based on their sense-making, and how they interpret classroom interactions and students' thinking influences where they choose to focus their attention. A noticing conceptual frame is used within the VIM asynchronous module design to support the analysis of classroom interactions and reason about teaching and student thinking within the viewing and analysis probes of the video clips embedded within the modules. In addition, the bridge to practice activities that end each module are designed to connect teachers' learning to their classroom practices.

VIM module design and development

Many, but not all, video-based mathematics PD programs have teachers are designed to engage teachers in specific activities before and after watching the focal video (Borko et al., 2011). For example, prior to watching a clip, PD facilitators may ask the teachers to solve and discuss the math problem shown in the video to develop content knowledge, motivate teachers to notice elements of the content contained within the clip, and attend to specified activities such as a unique solution method or teacher questions that prompt extended student reasoning. After viewing the video, facilitators may guide a discussion and in which the teachers relate what they have seen on the video to their own classroom practices. The discussion and follow-up activities extend teachers' thinking and analysis by probing more deeply into topics or issues presented within the video.

We label this intentional sequencing of video viewing such that it occurs between designated activities with specified learning goals a 'video in the middle' design (Seago et al., 2018). In video-based mathematics PD that incorporates this design feature, video is in the middle of the learning experience, sandwiched between activities such as mathematical problem-solving and pedagogical reflection. Our goal is not to argue that this design feature is new to the field of professional development, but simply to highlight and label it, and consider how the design is likely to support pre- and in-service teachers' learning.

Each VIM module contains the same set of activities embedded in the video in the middle design, placing a video clip at the center, or "in the middle," of professional learning as teachers take part in an online experience of mathematical problem solving, video analysis of classroom practice, and pedagogical reflection (Figure 1). The overall structure of this design is consistent across all VIM modules and is intended to support teachers' professional learning opportunities around mathematical knowledge for teaching (Ball & Bass, 2002) and teacher noticing of student thinking and teacher-student interactions (Sherin et al., 2011).

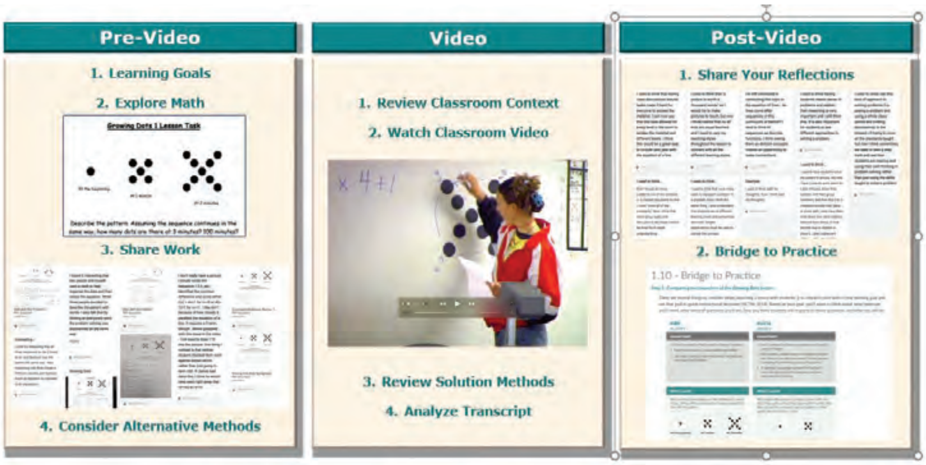

Figure 1: Video in the middle PD activities

The VIM modules are designed to be offered in three asynchronous facilitation formats: (1) locally facilitated, (2) expert facilitated, and (3) independent/non-facilitated. The different formats provide unique affordances for teachers and provides users with both flexibility and choice in their professional learning. Some teachers may prefer to work independently at their own pace and on their own time schedule; others may prefer to work with colleagues at their school with local facilitation from a coach. Or districts may want to offer their teachers the opportunity to participate with other teachers nationally in an expert facilitated experience. VIM's final design will offer a variety of suggested pathways through the modules depending upon goals, grade levels, and mathematics content, with options to personalize a professional learning plan (depending on one's goals) or swap a particular module with another from the bank of VIM modules.

Methodology

During Spring 2020, middle and high school teachers were recruited across the state of California to participate in a pilot efficacy study to address the following research questions:

> *What is the impact of teachers' participation in the three delivery formats on teachers' mathematical knowledge for teaching, their noticing skills, and their teaching practice? What is the impact on their students' performance?*

In this paper, we report on the impact of participation in the three delivery formats on teachers' mathematical knowledge for teaching and noticing skills.

Participants. Mathematics coaches/leaders from two school districts with which researchers had existing relationships were recruited for the locally facilitated condition. The coaches/leaders in each district recruited teachers and then served as the local facilitators for groups in their districts. For the independent /non-facilitated condition and the expert-facilitated condition, teachers were recruited from districts across California and randomized into two groups. Where multiple teachers were recruited from the same district, teachers were split between the two groups. Of the 68 teachers who began the study, 56 (82%) completed all or nearly all study activities, including all four VIM modules

(16 local facilitated, 16 expert facilitated, 24 independent). All three conditions had 80% or higher completion rates—local facilitated (83%), expert facilitated 80%, and non-facilitated 83%.

Intervention. All teachers experienced the same four sequenced, two-hour modules for a total of eight hours of professional development over the course of eight weeks (February-March 2020). While the modules are structured alike and contain a consistent set of activities and resources, each individual module is focused on a set of three unique learning goals (mathematical, pedagogical, and instructional) that are designed around each VIM mathematical task and video clip.

Facilitator training. In January 2020 two project staff designed and facilitated a 90-min zoom facilitator orientation for the two expert and three local facilitators. During the orientation, staff gave an overview of the RCT study and timeline, the VIM module structure, tools (such as Canvas, NowComment and Padlet), and a web-based facilitator guide. Additionally, a video tutorial focused on the journal tool was created for the facilitators to learn how to comment on participants responses and provide feedback to their teacher participants within the asynchronous format.

Measures. A variety of measures were used to gather impact data on teachers and students. Teacher measures included an online pre-post video and student work analysis task, weekly online self-report teacher logs focused on what teachers used in their classroom practice related to the PD, teacher interviews focused on usefulness, engagement and facilitation conditions, classroom observations and PD embedded pre-post community wall posts and comments. A student online quiz was developed to assess shifts in content knowledge. The focus of this paper will be on the analysis and results of the mathematics community wall pre-post data and interview data across the three facilitation conditions.

Analysis and results

COVID's impact on data collection, analysis, and results. Weeks seven and eight of the RCT were impacted by COVID-19. In both facilitated groups there was less interaction among participants in the fourth module than the previous three modules. Typically, there were four-five participants who commented on colleagues' posts. For the fourth module, there were one or two people who completed the module around the same time and interacted with each other. For teachers who completed the fourth module, they completed all the activities and journal entries but didn't comment or interact much with their colleagues. In addition, while ~5000 pre student quiz data was successfully collected, post student quiz data was not able to be collected. Teacher observations were not completed and therefore teaching practice impact data was not collected.

Teacher community walls. Within each of the VIM modules, teachers worked on the mathematical task that the students in the video clip engaged with. After solving the problem, they uploaded an image of their work and colleagues and facilitators commented or asked questions. Two project staff independently examined and categorized the various mathematical methods posted by teachers and analyzed the responses by the teachers on each other's methods. They compared and agreed upon their categories, analysis, and calculations. Community mathematics wall participation was high in all three conditions. In the locally facilitated condition, 80% of participants posted their mathematical work in the first VIM module and 95% posted their work in the final VIM module. In the self-paced group, 88% of the participants posted their mathematical work for the first module and 100% posted in the final module. In the VIM project facilitated group, 100% of the participants posted their work in both the first module and last modules. The VIM project facilitated group had the smallest number of pre-non-facilitator comments, but a similar number of total comments to the other two conditions. The most notable pre-post results emerged in the analysis of the visual versus numerical methods used by teachers. Specifically, by condition:

- *Locally facilitated*: Visual methods increased from 3% of the total methods posted in module 1 to 89% in module 4; numerical methods decreased from 70% of the total methods posted in module 1 to 11% in module 4
- *Expert facilitated:* Visual methods increased from 6% of the total methods posted in module 1 to 94% in module 4; numerical methods decreased from 82% of the total methods posted in module 1 to 6% in module 4
- *Non-facilitated:* visual methods increased from 18% of the total methods posted in module 1 to 85% in module 4; numerical methods decreased from 82% of the total methods posted in module 1 to 6% in module 4

The preliminary results in the analysis pre-post methods not only showed improved MKT with a substantial shift from numerical to visual methods, but their comments indicated an increased appreciation for visual methods in general by mentioning use of color, modeling of expressions, etc.

<u>Teacher interviews</u>. Of the 56 teachers who completed the study, nine were randomly selected for guided interviews in June and July 2020, three from each condition. All interviews were audio-recorded and transcribed. Two project staff identified passages related to teachers' engagement in the PD, the usefulness of module features, the content and resources, their thoughts on the facilitation conditions and the impact on their practice. All nine expressed that they found the VIM PD modules engaging and useful. When asked to comment on features or elements of the VIM modules they found most beneficial, the videos, lesson graphs, and community walls were all mentioned by most teachers. In relation to noticing, many teachers commented that watching a video of a real classroom helped them better understand what teacher moves described in the PD would look like and how 'real' students might respond mathematically. In relation to MKT, teachers mentioned that they learned a variety of ways linear functions tasks can be approached or solved, whether from the analysis of the videos, the solution methods document, or in other participants' work posted on the community walls.

When asked about their experiences, teachers in different conditions described distinctive affordances of each. For example, most teachers in the facilitated groups appreciated receiving feedback from a coach in their district or an expert facilitator, while those in the independent/non-facilitated group enjoyed the flexibility of being able to complete the modules at their own pace. As one independent participant said,

> 'I like this particular experience because I can go at my own pace, and it was still almost like it was facilitated because there were questions that you had to answer.'

Most participants in the facilitated groups felt that the facilitation was supportive and helpful. They appreciated the comments and questions posed on the Padlet wall and said that it helped them reflect on their own learning and perspective of the task. Some shared that it helped them to be accountable and get the work done.

> 'I liked the group I was in because it held me accountable to do a lesson a week, or in the time constraints. I might not have managed my time as efficiently... I wouldn't have gotten as much out of them. I feel I was able to get more out of them by being in the structured setting.'

One person in a facilitated group shared how supportive the facilitator was in helping her understand some of the content of the lesson as well as the postings on the Padlet wall. Almost all the participants in the facilitated groups appreciated being in a facilitated group and said that they would choose that option again. One participant felt ambivalent about the facilitation and said she could be in either a facilitated or independent condition, as the facilitation felt minimal and not very helpful. Participants in the independent condition were divided regarding whether they would prefer having been in a facilitated group instead.

> 'I kind of liked the independent group because I was able to adjust my weekly schedule, but I also like to socialize with colleagues and talk about what we are learning. I would like to have tried the other part, but I don't think I have a preference'.

Discussion and conclusion

The VIM asynchronous video-based PD modules are designed to meet the increasing need for online PD options that include flexibility and choice for teachers and facilitators (Koellner et.al., in press). The preliminary analysis of the community wall and interview data show impact of the three facilitation conditions on teacher noticing and MKT in the teacher's examination of student thinking, classroom interactions, and mathematical representations/methods.

A surprising result was the fact that there were no substantial differences in the RCT study across the three conditions regarding teacher engagement and interaction on the community mathematics task wall. We hypothesized that the facilitated group would be more engaged and post more comments in response to their colleagues' methods and facilitator probes. This did not turn out to be the case, as teachers across all three conditions commented in similar numbers and shifted from numeric to visual methods from pre to post. We wonder if the design of the video in the middle experiences—the opportunities provided to teachers to access multiple perspectives of each other, mathematicians and mathematics educators and engage with their peers within the community wall activities—may have provided teachers with more similar than different experiences across the three conditions. We anticipate learning more as we analyze more data (journals, community wall reflections).

The asynchronous, online nature of the VIM modules makes them highly scalable; unlike many face-to-face and synchronous online PD options, mathematics educators do not need to limit participation due to space or cost concerns. At the same time, the various facilitation options allow for interaction and collaboration among teachers.

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Utilizing a scriptwriting task as a tool to examine facilitator practices in response to teacher orientations for at-risk students

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Teacher orientations influence instructional prioritizations and how teachers attend to students, particularly those at risk of being left behind. In PD programs, facilitators' practices to recognize and respond to such teacher orientations for supporting at-risk students are thus an important aspect of content-related facilitator expertise. Extending the use of scriptwriting tasks to the PD facilitator level, we present two contrasting cases of how facilitators employ practices to respond to teachers' orientations in a PD simulation. One facilitator avoids direct opposition with conflicting teacher orientations, while the second facilitator challenges the teachers' orientations that do not contribute to supporting at-risk students' learning. By discussing the contrasting practices in the facilitators' written scripts, we demonstrate how the scriptwriting task can be used to investigate content-related facilitator expertise in terms of practices in response to teachers' orientations.

Keywords: Scriptwriting tasks, facilitator practices, teacher orientations, professional development.

Introduction

Mathematics represents a cumulative content area with basic concepts laying the foundation for later understanding. It is crucial that students can be supported in gaining knowledge from previous years or units, so they can develop deeper conceptual understanding as they move forward in later grades. The orientations that teachers have in response to less-privileged students who are at risk of being left behind are important to understand and necessary for PD facilitators to be able to identify and respond to. Such facilitator practices are part of their content-related expertise as conceptualized by Prediger et al. (2021). In this study, we draw on the Prediger et al. (2021) framework and utilize a scriptwriting task to examine facilitators' practices for responding to teachers' orientations for supporting at-risk students' learning processes. We examine these practices in the context of a PD for teachers to monitor and enhance at-risk students' conceptual understanding of basic concepts. In the theory section, we first address teacher orientations as part of their teaching expertise. Particularly, we elaborate on four sets of orientations pertinent to the context of fostering at-risks students' mathematics performance. Second, we describe the framework of facilitators' content-related expertise, with a focus on the facets relevant for our study. Third, we expound upon the tool of a scriptwriting task, the instrument we applied. Thereafter, we present and discuss the data we gained in the context of the teacher PD project *Mastering Math*, which is built around supporting at-risk students' mathematics learning. Especially, we report how two facilitators were immersed in a PD simulation by the scriptwriting task and what practices they applied to deal with different teacher orientations.

Theoretical background

Teacher orientations as a part of teacher expertise

Various studies showed how teachers' orientations, respectively beliefs, regarding the nature and meaning of mathematics are manifested in their teaching practices. Schoenfeld (2010) examined the components that impact how a teacher acts in the classroom and developed the theory of goal-oriented decision making, which considers resources, goals, and orientations as essential components. Such orientations and accompanying practices can have consequences for supporting students, especially those in need of additional assistance, if teachers prioritize, for example, procedural fluency over conceptual understanding. In expanding upon this conceptualization of orientations, Prediger (2019) created a framework of mathematics teacher expertise that includes orientations as one of five facets: "Orientations refer to content-related and more general beliefs that implicitly or explicitly guide the teachers' perceptions and prioritisations of jobs (e.g. beliefs about the content or students' learning processes)" (p. 370). Findings from empirical studies revealed four sets of contrasting orientations. First, teachers often focus on the development of procedural knowledge as opposed to conceptual knowledge (Wilhelm et al., 2017). Second, in terms of diagnosing individual student challenges, teachers often follow the school syllabus regardless if students fall behind, and, third, maintain short-term mastery in contrast to emphasizing long-term goal mastery (Prediger et al., 2016). Fourth, teachers concentrate on individual student issues (Krähenmann et al., 2019), instead of facilitating rich discourse with all students. These four sets of orientations are addressed in the PD course *Mastering Math* that constitutes the context of our study.

Facilitator practices as part of their content-related expertise

In terms of what PD facilitators need to know and do to be able to provide successful PD programs for mathematics teachers, researchers have increasingly examined what constitutes necessary PD facilitator knowledge and practices (Borko et al., 2021; Lesseig et al., 2017). In putting different knowledge domains in action, facilitators need to employ practices that are conducive to working productively with adult learners and construct environments in which teachers can collaborate about relevant topics and can feel safe and supported to share information (Borko et al., 2014). Accordingly, Prediger et al. (2021) provided a framework for content-related facilitator expertise. The framework consists of jobs, which are conceptualized as typical and often complex situational demands of facilitating a specific PD, and practices as recurrent patterns of facilitators' utterances and actions for managing the jobs. These practices can be described and analyzed by revealing the underlying facilitator orientations and situative goals on which the facilitator implicitly or explicitly draws:

Orientations: Generic or content-related beliefs and pedagogical attitudes (e.g., about teachers' thinking or about the PD content) that implicitly or explicitly guide the facilitators' perceptions and prioritization of jobs (e.g., participant orientation).

Situative goals: The goals that the facilitators pursue in a respective situation can directly refer to PD content learning goals (in brief, PD learning goals), can address process qualities (e.g.,

> cognitive activation, briefly, process goals), or can be of an atmospheric nature (briefly atmospheric goals). (Prediger et al., 2021, p. 8)

Thus, goals and orientations determine how facilitators act in a specific situation, and it is important to make them visible in facilitators' practices. For that purpose, scriptwriting tasks are a useful tool, which allow for creating a fictional context in which facilitators need to simulate their practice.

Scriptwriting tasks as approximation of PD practices

So far, scriptwriting tasks have been employed on the teacher level as a tool that functions as a bridge between planning and the actual course of action in the classroom by providing classroom situations for teachers to react to learners' utterances and provide possible explanations (Zazkis & Sinclair, 2013). By using scriptwriting tasks, one can draw on a situated approach to enable the approximation of the actual act of supporting students, but instead in a fictionalized situation. Scriptwriting tasks have been accordingly implemented as a means of investigating pre-service teachers' understanding of content and facilitating of learning (Lim et al., 2018); and with in-service teachers (Kontorovich & Zazkis, 2016) to assess how teachers deal with student alternate conceptions. Scriptwriting tasks, however, have not yet been utilized as a tool for assessing facilitators practices in response to teachers' orientations in a situated fictional PD context.

Aims and research questions

To examine PD facilitators' practices in response to teachers' orientations for supporting at-risk students, we employed a scriptwriting task to first provide facilitators with a fictional situation in which teachers in a PD are provided with a student dialogue of three students working on a task and then with a discussion between three teachers concerning how they would continue the conversation with the students. The facilitators were then given the task to first complete the dialogue with the fictional teachers and guide them to in discussing how to support the fictional students and, second, to provide rationale as to why they ended the dialogue at the chosen moment. We pursued the following research question:

RQ: What practices do facilitators apply to respond to teachers' orientations for supporting at-risk students in completing the scriptwriting task, and what situative goals are behind these practices?

Methodology

Instrument: scriptwriting task

The scriptwriting task was developed based on a classroom-level task concerning filling in the tens between 0 and 100 on a number line that contained the following prompt: How can you plot and label numbers on the number line? What can help you? The scriptwriting task included a fictional dialogue of three students with alternate conceptions of the task discussing their solutions followed by a fictional teacher dialogue of three teachers discussing how they could support the students in completing the task (Figure 1).

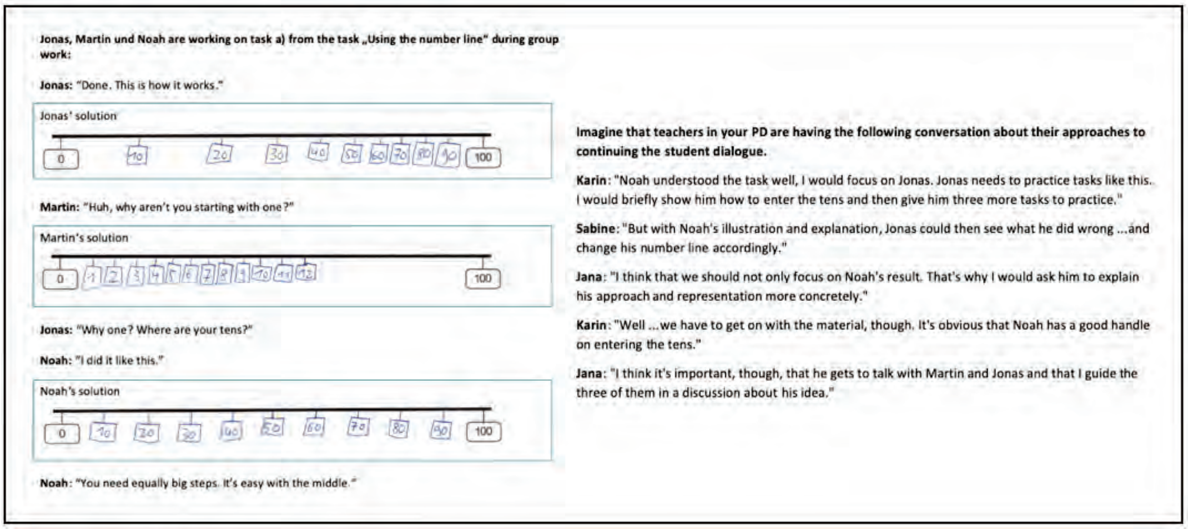

Figure 1: The scriptwriting task student dialogue (on the left) and teacher dialogue (on the right)

After reading the fictional dialogues of the three students and the three teachers, the facilitators were asked to complete the dialogue with the three teachers (Karin, Sabine and Jana), thereby demonstrating how they will address the teachers' orientations for supporting at-risk students and help the teachers to foster student understanding of the basic concepts in the task. The four contrasting sets of teacher orientations for supporting at-risk students were embedded in the scriptwriting task.

Participants, data collection and data analysis

In total, a group of 14 PD facilitators from the federal state of Berlin were asked to participate in this study and were provided with the number line scriptwriting task described in the previous section. Six facilitators agreed to participate. All of the facilitators led PD courses in the program *Mastering Math* for primary school mathematics teachers, which focuses on means of supporting students at risk of being left behind. The *Mastering Math* PD program centers on the four sets of principles: (1) diagnostic vs. syllabus-bound orientation; (2) conceptual vs. procedural orientation; (3) long-term vs. short-term orientation; and (4) communicative vs. individualistic orientation. We selected these unique written scripts from two of the facilitators, in particular, as these two contrasting cases of facilitator practices illustrate how two facilitators react differently in the same fictive PD situation to enhance teachers' learning processes.

Results

The two written scripts demonstrate the facilitators' contrasting approaches in the extent to which they attempt to convince the teachers of the guiding principles of the PD program for supporting at-risk students. While the first facilitator poses questions and encourages the three teachers in the fictional PD situation to discuss the student misconceptions from the task without visibly advancing the guiding principles of the PD, the second facilitator more directly challenges the teachers who are not convinced by the guiding principles and engages them in a discussion surrounding long-term mastery achievement and diagnosing individual student capabilities in order to help move students forward. The two transcripts of the written scripts and the contrasting practices are presented in the following sections.

Case one: A hands-off approach to conveying PD guiding ideas

The facilitator in the first transcript encourages the teachers to provide their opinions concerning the challenges the students in the fictional dialogue encountered by posing questions as to the students' different approaches and different problems they faced. The facilitator responds to the teachers either by demonstrating agreement with the comments concerning Martin's problem that interferes with his understanding of the task and adding to the teachers' ideas with a short commentary that reflects the guiding ideas of the PD, or by remaining neutral by closing the discussion without further comment.

1	Facilitator:	The three students have different bases of understanding. What is the difference between Noah's and Martin's solutions?
2	Jana:	Noah has a strategy. He knows that he first has to look for the middle and then has to record the remaining numbers at equal intervals.
3	Facilitator:	And Martin? […] And what is Jonas' problem?
7	Karin:	Jonas obviously doesn't know what is meant by tens. That is why I would first quickly explain it to him.
8	Jana:	I don't think explaining will help. He could discover it for himself by showing him the work of the other two children. The children could then make comparisons themselves.
9	Facilitator:	Exactly. Martin has a fundamental problem. He may not yet have an understanding of place value. According to MSK, sustainable learning especially for at-risk students should be oriented towards building understanding. One of the guiding ideas of MSK is to promote communication. So what would have to happen for me to catch Martin up?
11	Karin:	Well, with some students it's really hopeless. I wouldn't waste so much time, but rather concentrate on the better-performing students.
12	Jana:	I believe that children have only really understood a subject when they can explain it in their own words.
13	Facilitator:	Thank you very much for your contributions and your assessment. At this point I would like to end the discussion for now and show you a short film.

The facilitator continues the dialogue in such a manner echoing the main elements of the dialogue beginning such that Karin expresses a distinct opinion concerning how to support at-risk students, reiterating that short-term, quick approaches are necessary, especially for such "hopeless" students when support can instead be focused on "better-performing students." It is evident that the facilitator identified the teacher orientations from the scriptwriting task dialogue and extended these to the continuation of the dialogue. The facilitator does not respond to Karin's comments, however, but makes it clear that such orientations can be expressed in the PD. In response to the follow-up question regarding why the facilitator ended the dialogue at that specific moment, the facilitator explained:

Facilitator 1: Karin is obviously still of the old school, according to the motto: Explain quickly and then continue with the material, don't waste any time. It is more important for her to support the better-performing students. The other two participants are more oriented towards the principles of MSK. In the first round of discussion, I would not try to persuade them. It is not about persuasion, but especially such teachers like Karin should be able to make their own discoveries and not be discouraged […]

Thus, the facilitator reiterates her strategy of first assessing the orientations of the participating teachers, before engaging them or encouraging them to think about their viewpoints and their actions in the classroom. The facilitator creates an atmosphere in which the teachers are encouraged to express their opinions, at this stage, however, the facilitator does not push the teachers to reflect on their positions and reasoning behind such viewpoints or have the teachers focus more deeply on the mathematics behind the students' different representations from the student dialogue.

Case two: A direct approach to conveying PD guiding ideas

The second facilitator takes a different approach in the continuation of the dialogue with the three teachers in the written script and provides evaluative comments (see turn 24) of one of the teacher's comments concerning spending too much time on the process of remediation for students considered not capable of doing math.

7	Karin:	But that can take a long time until Martin understands it. What if he doesn't realize that 50 has to be in the middle...then we'll still be sitting together the day after tomorrow.
8	Facilitator:	You're really making an important point here. There are children for whom MSK is ultimately not suitable either. And this would be a good time to find out what Martin's situation is like. If it's not clear to him that 50 is half of 100, and if he's not able to think in tens, then he needs a different kind of support. We would have to take a closer look at that in any case […]
21	Facilitator:	…But why (looking at Karin) is that so important anyway, that we take so much time for this whole process? […]
23	Karin:	But that takes an infinitely long time. And there are simply children where I don't know if that really helps. Math is also a bit like that - either you can do it or you can't.
24	Facilitator:	Yes, that's still a widespread belief. And there really are kids for whom it's very difficult to achieve. But there are also many who just need a little more time and visualization and action. And time for visualization....so...they need to be able to link what they see to their thinking. And when they get that, it clicks pretty quickly. Then they replace their misconceptions with more appropriate ones. If you have experienced this yourself a few times, then you can comprehend it better. You just have to do a few remedial lessons, and then you realize that. But it also takes a bit of time to develop a feeling for which children they really make a difference and for which ones you still have to look for other forms of support. But when you get the children further along, it's totally satisfying for everyone. Can you live with that for now, Karin?
25	Karin:	[nods]
26	Jana:	And luckily we don't have any time pressure in remedial lessons. It just doesn't make sense to keep going if the kids haven't understood it yet.
27	Facilitator:	Exactly, that's the great thing about remedial lessons! But unfortunately we do have a bit of time pressure...we're going to get on with our program now. Anyway, I'm really looking forward to your reports from your first remedial lessons.

The second facilitator responds directly to Karin and even states that the idea that some children cannot do mathematics is "still a widespread belief," and then explains how to support children who struggle with more time and visualizations. While the facilitator does provide an evaluative response to Karin's statements, the facilitator asks Karin if she can try "to live with" the principle of long-term mastery for the time being, and moreover, the facilitator suggests that once Karin has completed a few remedial lessons, she will better understand the process of remediation. Thereby, the facilitator creates an atmosphere in which teachers with opposing orientations to the guiding principles of the PD are not completely discounted. In response to ending the continuation of the dialogue, the PD facilitator explains that such beliefs or orientations cannot be changed in one discussion as they are deeply rooted and will hopefully change with experience and observation:

Facilitator 2:	Jana and Sabine are on the right track already and the essentials for conducting support discussions have been said. Karin still has doubts. But they can't be dispelled with a single conversation. These are deeply anchored beliefs about learning that - hopefully - will gradually disappear once Karin has gained her own experience and realizes that her colleagues are more convinced.

The facilitator thus acknowledges that changing a teacher's orientation to reflect the PD principles will not happen immediately and notes the role of the other members of the PD as relevant for influencing orientations. While the facilitator did respond to Karin's comments concerning some children who just cannot do mathematics, the facilitator does not press Karin further and instead provides an opportunity for another teacher in the PD to express support for long-term mastery.

Discussion and conclusion

The exemplifying cases show how facilitators could react differently in a PD in response to teachers' orientations. Facilitator 1 creates a situation when completing the scriptwriting task in which the facilitator pursues the *content goal* by directing the teachers' focus to students' thinking and challenges in solving the task, to then diagnosing students' learning processes in light of PD guiding principles. Facilitator 1 therefore purses the *content goal* by illustrating one of the PD guiding principles by analyzing the students' potential and challenges. The assertion of one of the teachers that it is important to avoid wasting time on students who are not able to understand is left uncommented. The facilitator thus seems to demonstrate the need to show *esteem for participants* and therefore, especially at the beginning of the PD, purses *atmospheric goals* instead of the *content goal*. In contrast, facilitator 2 discusses and confronts the orientation the teacher shows who insists on avoiding wasting time. Moreover, the facilitator consequently defends the PD guiding principles, thus focusing on the *content goal*, despite repeated objections of the teacher. Moreover, the facilitator seems to try to provoke the teacher to reinterpret or invalidate her own orientation. At the same time, the facilitator engages the teacher in the discussion process and shows esteem for her, presumably to maintain a good atmosphere. The practices of facilitator 1 match facilitator 2 practices to some respect, as both seem to recognize and take teachers' orientations into account. Furthermore, both facilitators refer to the PD guiding principles, with a focus on the *content goal*, albeit to different extents. In addition, both facilitators are anxious to maintain a good atmosphere.

As there have been calls for ways to expand research on expanding the professional growth of facilitators (Borko et al., 2014; Lesseig et al., 2017), this research responds with a research tool that can be utilized to fill this gap. The scriptwriting task as a tool for professional development of PD facilitators of mathematics represents a situated form of practice. First, as a tool for facilitator educators to examine facilitator practices in responding to teachers' orientations, and second, as a means of discussing such responses to help facilitators continue on their path of professional growth.

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